Algebra I

Textbook: Elementary Analysis

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Notes from Algebra I

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1 Chapter 1

Assumptions of sets + notation:

1. A set S is made up of elements. We denote an element of S as $a \in S$
2. Empty Set is defined as: $\emptyset$

We define a subset: $B \subseteq A$ if every element in B is also in A. A subset is strictly smaller than the set.

Improper subset: The set itself A ⊆ B.

The cartesian product: Let A and B be sets. The set A x B = $\{(a, b)|a \in A and b \in B\}$ is the cartesian product of A and B.

Relations between sets:

We define a relation between sets A and B to be a subset R of A x B. We read (a, b) ∈ R as "a is related to b".

Example 1.1

The equality relation = defined on a set S by

$$\{(x, x)|x \in S\}$$

Definition 1.2

If a subset of X x Y is one-to-one function ρ mapping X onto Y, then each x ∈ X appears as the first member of exactly one ordered pair and y ∈ Y appears as the second member.

If we take (y, x) this is the inverse mapping function.

A function R: X → Y is one-to-one if $R(x_1) = R(x_2)$ only when $x_1 = x_2$

Two sets have the same cardinality if there exists a one-to-one function mapping X onto Y.

Example 1.3 The function f: R $\rightarrow$ R where f(x) = x^2 is not one-to-one because f(2) = f(-2) but 2 $\neq$ -2.

However, f(x) = x^3 is both one-to-one and onto.

Definition 1.4
An infinite set can be defined as having the property where the subset has the same cardinality as the "max" subset. An example of this is:

Example 1.5
|Z| = |Z^+| = $\mathcal{N}_0$
Q: Does every infintie subset have the same cardinality as $\mathcal{N}_0$?
A set has cardinality $\mathcal{N}_0$ if all of its elements could be listed in an infinite row, so that we could number them using Z^+

From analysis, we have that $|\mathcal{Q}| = \mathcal{N}_0$.
We also have that $|R| \neq \mathcal{N}_0$.

Partitions and Equivalence Relations

A partition of a set S is a collection of non-empty subsets of S such that every element of S is in exactly one of the subsets. The subsets are the **cells** of the partition.

Points about partitions

1. Non-overlapping: No two subsets (cells) share any elements
2. Non-empty: Must have an element
3. Complete coverage

residue classes modulo n in Z^+ partitions Z into n cells.
Each partition of a set S yields a relation $\mathcal{R}$ on S in a natural way.

Example 1.6
For x, y $\in$ S, let x $\mathcal{R}$ y iff x and y are in the same cell of the partition.

Example 1.7

Let R be on the set Z defined by n R m iff nm ≥ 0, let us determine whether or not R is an equivalence relation.

1. Reflexive: a R a because $a^2 \geq 0$ for all $a \in \mathbb{Z}$
2. Symmetric: If a R b, then b R a. Because ab ≥ 0 $\implies$ ba ≥ 0.
3. Transitive: If a R b, and b R c, does a R c?

$$ab \geq 0, bc \geq 0. \text{ But if b = 0, a < 0, c > 0 yields} ac \leq 0 \implies \text{a R c does not exist.}$$

Definition 1.8

Equivalence Relations and Partitions

Let S be a nonempty set and let ∼ be an equivalence relation on S. Then ∼ yields a partition of S, where

$$\bar{x} = \{x \in S | x \sim a\}$$

Also each partition of S gives rise to an equivalence relation ∼ on S where a ∼ b if and only if a and b are in the same cell of the partition.

2 Chapter 2

Multiplication on circles:

$$\begin{aligned} z_1 &= |z_1|e^{i_1\theta}, z_2 = |z_2|e^{i_2\theta} \\ &= z_1 z_2 = |z_1||z_2|e^{\theta(i_1+i_2)} \end{aligned}$$

Algebra on circles:

Let U = $\{z \in \mathbb{C} | |z| = 1\}$, so U is the circle with center (0,0) and radius 1. The relation $|z_1 z_2| = |z_1||z_2|$ shows that two numbers in U is a number in U. This means that U is closed under multiplication.

For notation, denote addition modulo 2π by $+_{2\pi}$

Let's be a bit more precise:

A binary operation * on a set S is a function mapping S x S into S. For each (a, b) ∈ S x S, we denote the element *(a,b) of S by a x b.

+ is considered a binary operation in the set R. Multiplication is another binary operation.

Let * be a binary operation on S and let H be a subset of S. The subset H is closed under * if for all a, b ∈ H, we also have a * b ∈ H. In this case, the binary operation on H given by restricting * to H is the induced operation of * on H.

By definition, the set S is closed under *, but a subset may not be. Eg: 2 ∈ $\mathbb{R}^*$ and -2 ∈ $\mathbb{R}^*$ but 2 + (-2) = 0 ∉ $\mathbb{R}$. Thus $\mathbb{R}^*$ is not closed under *.

In order to correctly determine if a subset H of S is closed under a binary operation, we have to know what it means to be an element in H.

Example 2.1

Let + and · be usual binary operations of addition and multiplication.
Define H := $\{n^2 | n \in \mathbb{Z}^+\}$

Proof. Is H is closed under addition? No.
Observe H:= $\{1, 4, 9, ...k^2\}$. However 1 + 4 ∈ H = 5 ∉ H.

Is H closed under multiplication? Yes.
Define $H_1 = n_1^2 \forall n \in \mathbb{Z}^+$, $H_2 = n_2^2$
$H_1 H_2 = (n_1 n_2)^2$ and $\mathbb{Z}^+$ is closed under multiplication. □

Binary operations may or may not depend on the order of the given pair. We can define operations in ways such that order does / does not matter. When order does not matter, we call it **commutative** if a * b = b * a

Definition 2.2

Associativity of Composition Let S be a set and let f, g, and h be functions mapping S into S. Then f $\circ (g \circ h) = (f \circ g) \circ h$

Proof. In order to show two functions are equal, we much prove that they give the same assignment for each x ∈ S.

We can see that from the example above, that f(g(h)) = f(g(h)) for both cases. □

Tables

Definition

1. Exactly one element is assigned to each possible ordered pair of elements of S.
2. For each ordered pair of elements of S, the element assigned to it is again in S.

If there is ambiguity or condition 2 is violated, then S is not closed under your operation *.

Eg: Let a * b = $\frac{a}{b}$ on $\mathbb{Q}$. * is not defined on Q because no rational number is assigned to (2,0). But on $\mathbb{Q}^+$ this is defined since $0 \notin \mathbb{Q}^+$ so we are chilling.

Let f * g = h where c is the shortest person in S who is taller than f and g. If f and g are the tallest people in the set, h can't exist. So it's not well-defined.

Example 2.3

1. ∼='='
2. $\mathbb{Z} = \mathbb{N}x\mathbb{N}$ s..t (a, b) ∼ (c, d) if a + d = b + c.
3. $\mathbb{Q}$ has an equivalence class (a, b) = (c, d) if ad = bc.
4. A vector

Lemma: x $\sim x^{'} \implies [x] = [x^{'}]$

Proof. □

Example 2.4

$\frac{p}{q} \to \frac{p+q}{2p+3q}$ is well defined. WTS: f(p, q) = f($p^{'}, q^{'}$) when $\frac{p}{q} = \frac{p^{'}}{q^{'}}$

Permutation on a set X is a bijection f: $x \to X$

1. $\emptyset \to \emptyset$
2. $\{a\} \to \{a\}$
3. Permutations of $\{X\}$ = |X|!

The symmetric group S_x on a set X is the set of permutations on X
If f, g, in $\in S_x$

1. $f \circ (g \circ h)$ = moving parenthases around

2. f $\circ g = h \circ g \implies f = h$

Notice g has an inverse g^{-1} because it is a bijection.