Algebra

Textbook: Real Analysis

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Fall 2024

Homework of our real variables class. Textbook by Folland. Professor: Shukun Wu

Last updated on Monday 9th September, 2024.

1 Week 1

Exercise 1.1. *Prove demorgan's law:* $(\cup_{a\in A}E_\alpha)^c = \cap_{a\in A}E_\alpha^c$

Proof. Let x $\in \cup_{a\in A}E_a$. This means x must be in some $E_{a_0}, E_{a_1}, ...E_{a_n}$. Then $x^c \in E_{a_0}^c, E_{a_1}^c, ...E_{a_n}$ Where $E_{a_n} \in \cap_{a\in A}E_a$
By definition of intersection, x $\in E_{a_0}^c \cap E_{a_1}^c \cap, ...E_{a_n}^c$, x = $\bigcap_{a\in A} E_a^c$ □

Exercise 1.2. *Let X be a finite set. Prove that* $|2^X| = 2^{|X|}$

Proof. Intuitively, we can mark each element as in or out Proof by induction: X = 1. Let X = {1}

P(X) = $\{\emptyset, \{1\}\} = 2$

We WTS: $|2^{X+1}| = 2^{|X+1|}$

$$2^{|X+1|} = P(X) \cup X_{n+1} = |P(X)| \cup X_{n+1}. \text{ You have two choices, include } X_{n+1} \text{ or not, so}$$
$$\implies |P(X)| \cdot 2 = |2^{X+1}|.$$

□

Exercise 1.3. *Let f: X → Y be a mapping. Prove that* $f^{-1}(\cup_{a\in A}E_a) = \cup_{a\in A}f^{-1}E_a$

Proof. 1. Let x $\in f^{-1}\bigcup_{a\in A}E_a \implies y \in \bigcup_{a\in A}E_a$. Then for some a_0, y $\in E_{a_0}$ Where $E_{a_0} \subseteq E_a$ for all a $\in$ A $\implies x \in f^{-1}E_{a_0} \subseteq \bigcup_{a\in A} f^{-1}(E_a)$

2. The proof goes the other way as well. (It's the same) Let x $\in \cup_{a\in A}f^{-1}E_a \implies x \in f^{-1}(E_{a_0}) \implies y \in E_{a_0} \subseteq \bigcup_{a\in A}E_a \implies x \subseteq f^{-1}(\bigcup_{a\in A}E_a)$ □

Exercise 1.4. *Show that there does not exist a linear order of "$\leq$" of $\mathbb{C}$ with the following additional assumptions:*

1. *For any 3 numbers a, b, c, $a \leq b$ implies $a + c \leq b + c$.*
2. *For any two numbers a, b, $a \geq 0$ and $b \geq 0$ impies $ab \geq 0$*
3. *$0 < 1$*

Proof. we must define an ordering for i. Either i $\geq$ 0 or i $\leq$ 0. However, neither of these cases make sense.
If i $\geq$ 0, square both sides to get a contradiction.
If i $\leq$ 0, ^{4} both sides to get a contradiction.

For below, assume i ≤ 0

1. If $i \geq 0$, then a = 0, b = i, c = 0. We have $0 \leq$ b but after squaring both sides, we get $0 \leq -1$. Contradiction.

   If $i \leq 0$, a = i, b = 0, c = 0. Starting with $i \leq 0$, if we ^{4} both sides, we end up with $1 \geq 0$. Contradiction.

2. a = 0a + 1i, b = 0a + ia. ab = -1

3. If 0 < 1, then 0a + 2i < 1a + 0i. However, $(0a + 2i)^4 = 16 > 1^4 = 1$

□

Exercise 1.5. *Let f: $\mathbb{R} \to \mathbb{R}$ be an increasing function (this is, $f(x_1) \geq f(x_2)$ if $x_1 \geq x_2$). Let $E := \{x \in \mathbb{R} : f(x)$ is not continuous in $x\}$. Prove that E is countable*

Proof. Observation: Because f is increasing, lets say we start at arbitrary $-n$. Every f(x_2), f(x_3), ... $\geq$ -n. Lets say f(x_2) = $-n + \epsilon$. And since E is not continuous, this means f(x) must be jumping.

Observation: Since this sequence is increasing, if we can show that there forms either a bijection or injection onto Q, then E is countable.

If ϵ is a rational number, then we are already done.
So if $\epsilon = \pi$ or some irrational value, we need to find some $\frac{p}{q} < \epsilon$ so that we can form a bijection. Because Q is dense in $\mathbb{R}$, we know there always exists a $0 < \frac{p}{q} < \epsilon$.

Thus, we can map E to Q. Since |E| $\leq$ |Q|, |E| is countable. □

Exercise 1.6. *Suppose $E \subset \mathbb{R}$ is a countable set. Prove that there exists $r \in \mathbb{R}$ such that $E \cap (r + E) = \emptyset$. Here $r + E := \{r + x : x \in E\}$*

Proof. Since E is countable, |E| $\leq$ |Q|, meaning E can be listed. E := $\{E_0, E_1, ...E_n\}$. WLOG, assume E is increasing and sorted.

Now if we let x $\in$ E, the most concerning element is x = E_0. Let r = $E_n + 2|E_0|$.

We have two cases: $\begin{cases} E_0 > 0 : r + x = E_n + 2E_0 + x > E_n \\ E_0 < 0 : E_n + 2|E_0| + x > E_n \end{cases}$

Therefore, if we let r := $E_n + 2|E_0|$, all elements r + E will be ">" E. Thus, E $\cap$ (r + E) = $\emptyset$ □

2 Week 2

Exercise 2.1. *Show that a countable union of countable sets is still countable*

Proof. Suppose we have some countable sets, $S_0, S_1, S_2, ...S_n$ and S := $\bigcup_{i \in N} S_i$. Since each of these sets are countable, we can list them all out.
$S_0 = \{S_{0_0}, S_{0_1}, ...S_{0_n}\}$, $S_1 = \{S_{1_0}, S_{1_1}, ...S_{1_n}\}$,.... We can re-arrange these sets in a n x n grid similar to the rationals. Then we can find a mapping from |S| to $|\mathbb{Q}|$ just like cantor's snake.
i.e. There is a mappping f: $|\mathbb{N} \times \mathbb{N}| \to |\mathbb{N}|$ that lists everything in the grid. Therefore, S is countable. □

Exercise 2.2. *Show that* $\mathbb{Q}$ *is a dense subset of* $\mathbb{R}$ *under the euclidian metric.*

Proof. To be dense in R, we must show that for every element r $\in \mathbb{R}$, $\exists q \in \mathbb{Q}$ such that d(q, r) $< \epsilon$ given arbitrary ϵ.
Take q := $\frac{m}{n}$ and suppose r is irrational.
WTF: q such that |r - $\frac{m}{n}$| $< \epsilon$. Assume r > q so $r - \frac{m}{n} < \epsilon$

By the archemedian property, we know there exists a n > N s.t. $\frac{1}{n} < \epsilon$.
Also, for any r $\in \mathbb{R}$, we can find a m such that $m \leq nr < m+1$.

$$= 0 \leq nr - m < 1 \to 0 \leq r - \frac{m}{n} < \frac{1}{n}$$

Finally, since $\epsilon > \frac{1}{n}$, we arrive at $|r - \frac{m}{n}| < \frac{1}{n} < \epsilon$. □