Shreeve

Textbook: Elementary Analysis

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This is my notes of Shreeve's calculus for finance II

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2 Information and Conditioning

2.1 Information and Sigma-Algebras

We denote a specific outcome as ω out of our sample space of Ω. We might know some information about ω to narrow it down to a few possibilities.

Example 2.1
Here we will do a coin-toss example for a concrete understanding.

If Ω is the result of 3 coin tosses, then given the first result of the coin toss we can resolve

$$A_H = \{HHH, HHT, HTH, HTT\}, A_T = \{THH, THT, TTH, TTT\}$$

We also know about Ω and $\emptyset$ at time 0. Specifically, $\emptyset$ does not contain ω and Ω contains ω.

We denote $\mathcal{F}_1 = \{\emptyset, \Omega, A_H, A_T\}$. If you tell us whether or not ω is in each of these sets, we will know the result of the first coin toss.

We can denote $\mathcal{F}_2$ and $\mathcal{F}_3$ indexed by time.

Definition 2.2
Let Ω be a non-empty set. Let T be a fixed positive number. Assume for each t $\in$ [0, T] there is a σ-algebra $\mathcal{F}$(t). If s $\leq$ t, then every $\mathcal{F}(s) \subseteq \mathcal{F}(t)$. Then the collection of σ-algebra's $\mathcal{F}(t), 0 \leq t \leq T$ is a filtration.

This filtration tells us information we will have at future times. When we get to t, we will know for each set in $\mathcal{F}(t)$ whether ω is in this set.

Definition 2.3
Let X be a random variable defined on a nonempty sample space Ω. Let $\mathcal{G}$ be a σ-algebra of subsets of Ω. If every set in $\sigma(X)$ is also in $\mathcal{G}$ then we say that X is $\mathcal{G}$-measurable.

X is $\mathcal{G}$ measurable if and only if the information in $\mathcal{G}$ is sufficient to determine the value of X. Naturally, it will also be enough to measure f(X) where f is a mapping.

Definition 2.4
Let Ω be a nonempty sample space with a filtration $\mathcal{F}(t), 0 \leq t \leq T$. Let X(t) be a collection of random variables indexed with $t \in [0, T]$. We say this collection of random variables is an adapted stochastic process if, for each t, the random variable X(t) is $\mathcal{F}$(t) measurable.

2.2 Independence

When a random variable is measurable with respect to $\mathcal{G}$ then the information contained in $\mathcal{G}$ is enough to determine the value of the random variable.

The opposite means information about $\mathcal{G}$ gives no information about this random variable. Then they are independent.

If a random variable X is not measurable by $\mathcal{G}$ not independent, $\mathcal{G}$ is not sufficient to measure X.

Definition 2.5
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{G}$ and $\mathcal{H}$ be sub σ-algebra's of $\mathcal{F}$. (Sets of $\mathcal{G}$ and $\mathcal{H}$ are in $\mathcal{F}$).
These two sigma-algebra's are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for A $\in \mathcal{G}$ and B $\in \mathcal{H}$
Random variables X, Y are defined to be independent if $\sigma(Y)$ and $\sigma(X)$ generate independent sigma-algebras.
Random variable X is independent if $\sigma(X)$ is independent from $\mathcal{G}$

Example: Let p := H, q := T.
Verify that $\mathbb{P}\left\{S_2 = 16 \text{ and } \frac{S_3}{S_2} = 2\right\}$ are independent.

Left Hand Side:

$$\mathbb{P}\left\{S_2 = 16 \text{ and } \frac{S_3}{S_2} = 2\right\} = \{HHH\} = p^3$$

Right Hand Side:

$$\mathbb{P}\{S_2 = 16\}\mathbb{P}\{\frac{S_3}{S_2} = 2\} = \{HHT, HHH\} \cdot \{HHH, HTH, THH, TTH\} = p^3$$

Definition 2.6

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}_1, \mathcal{G}_2, ...\mathcal{G}_n$ be a sequence of sub σ-algebras of $\mathcal{F}$. For a fixed positive integer n, we say that the n σ-algebras $\mathcal{G}_1, \mathcal{G}_2, ...\mathcal{G}_n$ are independent if

$$\mathbb{P}(A_1 \cap A_2 ... \cap A_N) = \mathbb{P}(A_1)\mathbb{P}(A_2)...\mathbb{P}(A_N) \text{ for all } A_1 \in \mathcal{G}_1, ...A_N \in \mathcal{G}_n$$

Similarly, random variables are independent if the sigma-algebra's containing them are independent.

[Infinite Coin Toss Space]
Let $\mathcal{G}_k$ be the σ-algebra associated with the kth toss.

So $\mathcal{G}_k$ is the sigma-algebra of information associated with the kth toss. This means $\mathcal{G}_k$ contains the sets $\emptyset, \Omega_\infty$ and atoms

$$\{\omega \in \Omega_\infty; \omega_k = H\} \text{ and } \{\omega \in \Omega_\infty; \omega_k = T\}$$

In other words, this tells us if ω is a head or tail all the way up until time k.

Independence in terms of sigma-algebras is a hard condition to check in practice.

Theorem 2.7

If X and Y are independent r.v.'s, f(X) and g(Y) are independent r.v.'s if f and g are Borel-measurable in $\mathbb{R}$

Proof. Let A $\in$ f(X). What the fuck just happened. I have no clue. THE EXERCISE IS LEFT TO THE READER SHEESH □

If X and Y are r.v.'s we can measure their joint distribution by

$$\mu_{X,Y}(C) = \mathbb{P}\{(X,Y) \in C\} \text{ for all Borel sets C } \in \mathbb{R}^2$$

The joint cumulative distribution function of (X, Y) is

$$F_{X,Y}(a,b) = \mu_{X,Y}((-\infty,a]x(-\infty,b]) = \mathbb{P}\{X \le a, Y \le b\}, a \in \mathbb{R}, b \in \mathbb{R} \tag{1}$$

A function $f_{X,Y}(x,y)$ is a joint density for the pair of random variables (X,Y) if

$$\mu_{X,Y}(C) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{I}_C(x,y) f_{X,Y}(x,y)\,dy\,dx \text{ for all Borel sets } C \in \mathbb{R}^2 \tag{2}$$

2 only holds when

$$F_{X,Y}(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f_{X,Y}(x,y)\,dy\,dx \tag{3}$$

Marginal distribution measures: