Analysis

Textbook: Elementary Analysis

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This is a self-studied version of analysis. I used resources from Rudin, MIT OCW and Dr Peyam. The notes are somewhat sparse and I'll include the important and interesting facts.

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Contents

1 Reals, Rational, and Natural Numbers

1.1 Natural and Rational Numbers

We define $\mathbb{N}$ to be the set $\{1, 2, 3, ...\}$ of positive integers and $\mathbb{N}$ has the following properties:

- 1 belongs to $\mathbb{N}$
- If n belongs to $\mathbb{N}$, then its successor n + 1 belongs to $\mathbb{N}$
- 1 is not the successor of any element in $\mathbb{N}$
- If n and m have the same successor, then n = m
- A subset of $\mathbb{N}$ which contains 1, and which contains n + 1 whenever it contains n, is equal to $\mathbb{N}$

The set $\mathbb{Q}$ of Rational Numbers is defined as numbers in the form $\frac{m}{n}$. However, it is still not fully satisfactory as we can't solve equations like $x^2 = 2$

We call these algebriac numbers when they satisfy a polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + ... c_1 x + c_0 = 0$$

We define basic algebriac operations on $\mathbb{Q}$ such as

- a + (b + c) = (a + b) + c for all a, b, c
- There are more, but I won't bore me or you

1.2 Construction of Reals

We construct $\mathbb{R}$ through Dedekind cuts

Define a cut to have no upper element within the cut.

And if S and T are cuts, then $S \leq T$ means $S \subseteq T$. With this definition, $\mathbb{R}$ becomes an ordered field

Eg: x: x < 3

1. Definition of Addition: A + B := $\{a + b : a \in A, b \in B\}$

2. Subtraction: A - B := $\{a - b : a \in A \bigwedge b \in (\mathbb{Q} \backslash B)\}$

And the purpose is that with these cuts, it is easy to see we have the LUBP when adding cuts

If we let M be the union of all the cuts $S \in \mathcal{S}$, then we take the max of the individual cuts to find the LUB of M

2 Week 2

2.1 Fields

The difference between $\mathbb{R}$ and $\mathbb{Z}$ is that $\mathbb{R}$ is a Field.

A field $\mathbb{F}$ is a set with addition and multiplication operators

Addition Axioms

- a + b $\in \mathbb{F}$
- associativity
- communicativity
- 0 exists
- There is an inverse in $\mathbb{F}$ such that A + (-A) = 0

  Multiplication Axioms
- ab $\in \mathbb{F}$ closed under multiplication
- associativity
- commutativity
- the element 1 exists such that 1F = F
- There exists an inverse $A^{-1} \cdot A = 1$

Proving some properties:

1. a + c = b + c $\rightarrow a = b$
2. ac = bc $\rightarrow a = b, c \neq 0$
3. a0 = 0
4. (-a)b = -ab
5. (-a)(-b) = ab

1.
$$(a+c)+(-c) = (b+c)+(-c) \rightarrow a = b \tag{1}$$

2.
$$(ac)(c^{-1}) = (bc)(c^{-1}) \rightarrow a = b \tag{2}$$

3. We know 0 + 0 = 0. Technically you are adding 0 a's together.
4. ab + (-a)b = (a + (-a))b = 0
5. Same as 4

2.2 Triangle Inequality

Important!

$$|a+b| \leq |a| + |b| \tag{3}$$

Proof:
If x ≥ 0, |x| = x. and

$$-|x| \leq 0 \leq |x|$$

Otherwise, x ≤ 0, |x| = -x. and

$$x \leq 0 \leq |x|$$

We can combine to make

$$-(|a|+|b|) \leq a+b \leq |a|+|b|$$

If $a+b \geq 0, |a+b| = a+b \leq |a|+|b|$

If $a+b \leq 0, |a+b| = -(a+b) \leq -(-(|a|+|b|))$

3 Week 3

3.1 Max Min, Sup Inf

We can call s_0 a maximum of S if
(1) $s_0 \in \mathcal{S}$
(2) For all $s \in \mathcal{S}, s \leq s_0$

Example: $S = \{n^{-1^n} | n \in N\}$ has no upper bound
However, for the set S = [0,4), this has no max. However, Sup(S) = 4

[Boundedness]

1. S is bounded above by M if $s \leq M$ for all $s \in \mathcal{S}$
2. S is bounded below by reverse (1)
3. S is bounded if it is both bounded above and below.

3.2 Least Upper Bound Property

Least Upper Bound Property

The sup(S) always exists

Very Important Theorem!

inf(S) = -sup(-S)

Proof: Let m = - sup(-S)
Then inf(S) = -sup(-S)$\rightarrow$ inf(S) = m

4 Week 4

4.1 Limits

A sequence is an infinite list of real numbers.

Formally:

Definition 4.1

A sequence $(S_n)_{n\in\mathbb{N}}$ is a function from N to R

Definition of a limit:

For all $\epsilon > 0$ there is N > 0 such that if n > N then:

$$|s_n - s| < \epsilon$$

General Format to prove limits w. Example:

Show

$$\lim_{n\longrightarrow\infty} 3 - \frac{1}{n^2} = 3$$

1. Find N such that n > N.

   In this example, (3-$\frac{1}{n^2}$) - 3$\longrightarrow$ $\frac{1}{n^2} < \epsilon$, so N = $\frac{1}{\sqrt{\epsilon}}$

2. Write up:

$$|s_n - s| = \frac{1}{n^2} \tag{4}$$

   Since n > $\frac{1}{\sqrt{\epsilon}}$, then $\epsilon > \frac{1}{n^2}$

Proving a Limit Doesn't Exist:

$$\lim_{n\longrightarrow\infty} (-1)^n \tag{5}$$

Tool: Contradiction.

Assume

$$|(-1)^n - s| < \epsilon$$

When n is even, this becomes:

$$|1-s| = |s-1| < \epsilon \rightarrow -\epsilon + 1 < s < \epsilon + 1$$

And when n is odd, this becomes:

$$|-1-s| = |s+1| < \epsilon \rightarrow -1-\epsilon < s < -1+\epsilon$$

But if $\epsilon = 1$ then s < 0 < s.

4.2 Archemedian Prop

Definition 4.2 Archemedian Property
If a and b are positive real numbers, then for some $n \in \mathbb{N}$ we have na > b

And this helps show that $\mathbb{Q}$ is dense because there will always be a rational number r between a and b.

A simple example is r:= $\frac{a+b}{2}$

This means that $\mathbb{Q}$ has very few holes compared to something like $\mathbb{Z}$

Proof. Suppose a < b. We want to find a r such that a < $\frac{m}{n}$ < b

Then na < m < nb and by 4.2 $n(b-a) > M$ for any M. □

5 Limit Proofs

Now that we have defined Limits formally, lets try and prove what we intuitively understand about limits.

Definition 5.1
Suppose $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, show that

$$\lim_{n\to\infty} s_n + t_n = s + t$$

Proof. Given $\epsilon > 0$, we can choose an even smaller epsilon $\frac{\epsilon}{2}$

And from the limit definition, $|s_n - s| < \frac{\epsilon}{2}$ for all n > N_s

Also, $|t_n - t| < \frac{\epsilon}{2}$ for all n > N_t
Now when choosing n, set n = MaxN_s, N_t
With the definition of a limit for $s_n + t_n$, we have

$$\lim_{n \longrightarrow \infty} s_n + t_n = |s_n + t_n - (s+t)| = |s_n - s + t_n + t| \leq |s_n - s| + |t_n - t| \tag{6}$$

And for n > MaxN_s, N_t: Equation(6) $\leq \frac{\epsilon}{2} + \leq \frac{\epsilon}{2} = \epsilon$ □

We can do the same for subtraction, multiplication, and division.

Definition 5.2
Suppose $\lim_{n\longrightarrow\infty} s_n = s$ and $\lim_{n\longrightarrow\infty} t_n = t$, show that

$$\lim_{n\longrightarrow\infty} s_n \cdot t_n = s \cdot t$$

Proof. This is the interesting algebriac manipulation:

$$|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \leq |s_n t_n - s_n t| + |s_n t - st| \leq |s_n|(|t_n - t|) + |t|(|s_n - s|) \tag{7}$$

From here, define your s and t to to be $\frac{\epsilon}{2s}$ and $\frac{\epsilon}{2t}$ and you are finished. □

Definition 5.3 Exponents
Prove that $\lim_{n\rightarrow\infty} \frac{1}{n^p} = 0$ for all p > 0
(π)

Example 5.4
If |a| < 1, then $\lim_{n\rightarrow\infty} a^n = 0$

Let $\epsilon > 0$, We have $|s_n - s| < \epsilon \longrightarrow |a^n - 0| < \epsilon \longrightarrow |a|^n < \epsilon$
Denote a := $\frac{1}{1+b} \longrightarrow |a|^n = \left(\frac{1}{1+b}\right)^n$

Expanding $(1+b)^n$ gives (1 + nb + ...) ≥ (1 + nb) > nb
Thus $\frac{1}{(1+b)^n} \leq \frac{1}{nb}$ and we can set n = $\frac{1}{b\epsilon}$

Formally

Proof. Let $\epsilon > 0$. There exists a n $> \frac{1}{b\epsilon}$

We have

$$|a|^n = \left(\frac{1}{1+b}\right)^n < \frac{1}{nb} < \epsilon \tag{8}$$

□

Example 5.5

We want to show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

Let $s_n = n^{\frac{1}{n}} - 1 \to s_n + 1 = n^{\frac{1}{n}}$

Now we get $(s_n + 1)^n = n$ and by binomial thm.

$$n = (s_n + 1)^n \geq 1 + ns_n + \frac{1}{2}n(n-1)s_n^2 \text{ Which is less than } \frac{1}{2}n(n-1)s_n^2 \tag{9}$$

Therefore, n $> \frac{1}{2}n(n-1)s_n^2$ and we set

$$s_n < \sqrt{\frac{2}{n-1}} \tag{10}$$

As n goes to ∞, $0 \leq s_n < 0$

This proof allows us to look at this next theorem and the proof is the same.

Theorem 5.6

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1$$

5.1 Lim goes to ∞

Lets explore what it means for something to approach infinity.

lets see why $\sqrt{n} >$ M for some M. For whatever M, if we define n to be n $> M^2$ then we will be larger than M.

Classic limit laws apply to infinity.

5.2 Limit Theorems

Important Properties of Limits:

Definition 5.7

Limits are Unique!

If $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_n = t$ then s = t

Proof. We can prove this by Squeezing |s - t|, proving that s = t.

$$|s-t| = |s-s_n+s_n-t| \le |s-s_n| + |s_n-t| \le 2\epsilon \tag{11}$$

And by definition |s - t| > 0. This gives $0 \le |s-t| \le 2\epsilon$. As $\lim_{\epsilon\to 0}$, |s - t| = 0 □

5.3 Monotone Sequence Theorem

General Theorems that guarentees a sequence convergence

Theorem 5.8

(s_n) is **increasing** if $s_{n+1} > s_n$ for all n.

(s_n) is **decreasing** if $s_{n+1} < s_n$ for all n.

This means that it is monotonic.

[Monotone Sequence Thm Stated]

if s_n is increasing and bounded above, then s_n converges.

Intuitively, this makes sense.

Just because something is bounded by M doesn't mean it is convering to M.

Your sequence may peak at lets say n = 5 and then converge to 0.

5.4 Limsup

When sequences don't converge, we can still talk about them.

Think back to $(-1)^n$. While this didn't converge, it still has a limsup = 1 and liminf = -1

Definition 5.9

Given N, define the following sequence v_n by

$$v_n = \sup\{s_n | n > N\}$$

Essentially, for some large enough N, we are saying the sequence will converge.

Liminf is the same except in reverse!

And similarly to Inf and Sup,

$$\lim\inf_{n\to\infty} S_n = -\lim\sup(-S_n) \tag{12}$$

Proof: Instead of sup and inf, define it to be $s_n | n > N$ and add a limit $n \to \infty$. Then you get the above.

Now we will prove the limsup squeeze theorem

Theorem 5.10

Suppose

$$\lim_{n\to\infty} inf s_n = \lim_{n\to\infty} sup s_n = s$$

Then this means s_n MUST converge to s

Proof. Let $\epsilon > 0$. Since we know this is bounded by the limsup,

This means we can find some n > N such that $|s_n - s| < \epsilon$

With limsup meaning this is sup $\{s_n | n > N\} < s + \epsilon$

Similarly with liminf there is another n > N_2 so

$$|u_N - s| < \epsilon \rightarrow -\epsilon < u_N - s < \epsilon$$

This gives $u_N > s - \epsilon$ and when liminf'd this turns into $s_n | n > N$.

Now we have $s_n - s < \epsilon$ and also $s_n - s > -\epsilon$ meaning we converge to s. If our sequence doesn't converge, then this would mean that it oscilates and limsup > liminf s_n □

6 Week 6

6.1 Cauchy

Now sometimes a sequence s_n will get closer and closer to itself.

Formally

Definition 6.1

If s_n is a cauchy sequence, then for all $\epsilon > 0$ there exists some n amd m such that $|s_n - s_m| < \epsilon$

This is not the same as convergence!

With convergence, the terms get close to some fixed value s. For cauchy, the terms get close to each other (which could technically go to infinity)

All converging sequences are cauchy, but not all cauchy sequences are converg-erging.

Theorem 6.2

Cauchy boundedness theorem

If a sequence is cauchy, then it HAS to be bounded.

Proof. By definition, our sequence s_n is cauchy, so $|s_n - s_m| < \epsilon$ for some $\epsilon > 0$

We will set N := M + 1. Now we have $|s_{m+1} - s_m| < \epsilon$

Rewriting $|s_m| = |s_m + s_{m+1} - s_{m+1}| \leq |s_m - s_{m+1}| + |s_{m+1}| \leq \epsilon + |s_{m+1}|$

If we set M = max$(|s_1, s_2, ..., \epsilon + |s_{N+1}|)$ then we can see that $|s_n| \leq M$ □

Theorem 6.3

If a sequence is a Cauchy Sequence in $\mathbb{R}$, then s_n converges.

However, if a sequence is a cauchy sequence in $\mathbb{Q}$, then it can fail to converge

Proof. In order to prove that a cauchy sequence will converge, we need to show that limsup = liminf = s. The way we do this is by showing limsup ≥ liminf and limsup ≤ liminf. Then by the limsup squeeze theorem, this converges.

We know that $|s_n - s_m| < \epsilon \implies s_n - s_m < \epsilon \implies s_n < s_m + \epsilon$

This is true for all n > N, and so

$$v_n := sup\{s_n | n > N\} \leq s_m + \epsilon$$

This gives $limsup_{n\to\infty} s_n \leq v_n \leq s_m + \epsilon \implies limsup_{n\to\infty} s_n \leq s_m + \epsilon$
So $limsup s_n - \epsilon \leq s_m$ □

6.2 Subsequences

We can have a subsequence s_n that has the property that $\inf\{s_n | n \in N\} = 0$. Then we can prove that there is always a subsequence in s_n that converges to 0.
Our goal is to construct a subsequence, s_{n_k} such that this converges.

If we can show $0 < s_{n_k} < \frac{1}{k}$ where k > 0, then by the Squeeze Theorem this converges to 0.

Bolanzo-Weierstraus
Every bounded sequence s_n has a convergent subsequence s_{n_k}

To prove, we can technically just use monotonic convergence thm.

Given a sequence s_n, there is a subsequence converging to limsup s_n.

Proof. A sequence s_n is dominant if every term after s_n is less. (It's the peak). Lets say there's a subsequence $s_{n_1}, s_{n_2}, ... s_{n_k}$ will converge to the limsup. (Since s_n will be larger than all $N > n$.

Also lets say s_{n_k} is the last dominant term. Then the limsup is just this dominant term. □

Definition 6.4
A limit point of a sequence s_n means that there is a subsequence s_n that converges to s.

$$s_n = (-1)^n$$

has 2 limit points. 1 and -1

Your limit can be infinity. So something like $\{0, \infty\}$ can exist.

Notes about limit points: sup(S) = $limsup_{n\to\infty} s_n$

Limit Points are closed.

7 MT

Information and Sigma Algebra (Chapter 2 technically, I'll Chuck in Chapter 1 Later)

Definition 7.1

Let Ω be a nonempty set. Let T be a fixed number, and assume that each $t \in [0, T]$ there is a σ-algebra $\mathcal{F}(t)$. Assume further that if s $\leq$ t, then every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$. Then we call the collection of σ-algebras, $\mathcal{F}(t)$, $0 \leq t \leq T$, a filtration

These filtrations tell us the information we will have at future times. So at time t, we will know $\mathcal{F}(t)$.

At time t = 0, we only know two sets $\mathcal{F}(0) = (\emptyset, \Omega)$

8 Dominated Convergence

Dominated Convergence

Suppose $f_n \to f$ as $N \to \infty$ Pointwise

For all x, $f_n(x)$ converges to $f(x)$ as $N \to \infty$

Then, the question is does

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x)dx \text{ Hold?} \tag{13}$$

No, you can not.

Counterexample

Let's look at the (dirchlet) function

$$f_n(x) = N \cdot 1_{(0,\frac{1}{N})} = N \text{ on } (0, \frac{1}{N}) \tag{14}$$

Then $f_n \to 0$ pointwise.

C1: If $x \leq 0, f_n(x) = 0$ for all x

C2: If $x > 0$, Choose a N such that $\frac{1}{N} < x$. We can always find a N, and so $f_n(x) = 0$

However

$$\int_{-\infty}^{\infty} f_n(x)dx = 1 \neq \int_{-\infty}^{\infty} f(x) = 0$$

Theorem 8.1 [DCT / Fatou]

If $f_n \to f$ pointwise,

$|f_n(x)| \leq g(x)$ for all x with $\int g(x)dx < \infty$

And let these all be Lebesgue measurable functions.

Then we can pass the limit into the integral.

Proof. Let $g_n(x) = inf_{k \geq n} f_k(x)$ so that $liminf_{n\to\infty} f_n$ is □

Suppose $\int |g(x)| < \infty$

And $|f'(x)| < C < \infty$ for all X

Does

$$\int g(x)\left(\frac{f(x+h) - f(x)}{h}\right)dx \to \int g(x)f'(x)dx?$$

We can bring the derivative inside becaues of DCT.

Proof. Requirements for DCT: □

9 Topology

Instead of using absolute value, we use d(x,y). So by definition

Definition 9.1
If (S, d) is a metic space and (s_n) is a sequence in S, then s_n converges to s if for all $\epsilon > 0$ there is N such that if n > N, then $d(s_n, s) < \epsilon$

Through this definition, we have balls instead of linear distances.

We can look at a sequence x^n and define it to converge to x. Let's look at the following sequence

$$x^n = (\frac{1}{n}, e^{-n}) \rightarrow x^n \rightarrow (0,0) = x.$$

Our definition for topologies is basically the same. So we can define cauchy as $(x^n, x^m) < \epsilon$. Bolzano is also bounded.

We can start defining open balls being

Definition 9.2
We define a ball as

$$B(x,r) = \{y \in S | d(x,y) < r\}$$

And an open set says that there is a r > 0 s.t. B(x, r) $\subseteq E$
In any metric space B(x,r) is always open. But [a, b] is not open.

Definition 9.3
We can call something an interior point if B(x, r) ⊆ E for some r > 0.

If E = [0,1], then the interior points would be (0,1) . And E is open if and only E = Closed(E)

Closed Sets:

Something is closed if every subsequence in S converges to s and s ∈ E. So (0,1) is not closed.

The set [0, 1) is not closed or open. So the opposite of closed isn't open.

The relation between closed and open is that E is closed if and only if E^c is open.

Proof. Suppose E is not closed. Then we WTS that E^c is not open. Because E isn't closed, there is a sequence in E such that it converges to some $s \notin E \rightarrow s \in E^c$.

This means $d(s_n, s) < r$ implying that $s_n \in B(s, r)$ □

Limit Points: Something s is a limit point if there is a sequence s_n in E that converges to s.
The set of all limit points of E is denoted by E with a squiggle.

$\tilde{E}$ is the set of all destinations of all trains in E.

$(0,\tilde{}1) = [0,1]$

Compactness:

A set E is compact if every open cover U of E has finite subcover v.

Some examples of compact sets:

1. A closed interval [a,b]
2. A box [a, b] x [c, d]
3. A ball B(x,r)

Definition 9.4
Heine-Borel:
If E is closed and a bounded subset of R^k, Then E is compact.

Proof. Suppose E is closed + Bounded. This can be stuck in a box [a1, b1] x [ak, bk]. If we can show that this box is compact, then we show that E is compact. We can cut this box up into 4 boxes.
The diameter of this is big box is d, and so the diameter of the small box is $\frac{d}{2}$

We repeat this forever and □

10 More limits

Partial Sums:

The sum of $\frac{1}{2^n}$ is 1. If the limit exists then it converges. Otherwise, we can say that this diverges.

Cauchy Critereon:

Definition 10.1

A series converges if and only if it satisfies the Cauchy critereon:
For all $\epsilon > 0$ there is N such that if $n \geq m > N$ then

$$\sum_{k=m}^{n} a_k < \epsilon$$

So no matter how small the error, we can find something. We call this part of the sequence the tail and if it converges then the sequence converges.

The divergence test:

If a series converges, then the terms in the sequence $a_n \to 0$. This intuitively makes sense.

Proof. Set $\epsilon > 0$. Since $\sum a_n$ converges, there is N such that if $n \geq m > N$ then $\sum_{k=m}^{n} a_k < \epsilon$. Then by definition we take n = m, and we get the sum $\sum_{k=m}^{m} a_k = a_m$. So this means $a_m = 0$ as $m \to \infty$ □

Limit comparison tests:

If $|b_n| \leq a_n$ and the series $\sum a_n$ converges, then $\sum b_n$ must converge.

Example 10.2

Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

Because $\frac{1}{n^2+1} \leq \frac{1}{n^2}$, we know that $\frac{1}{n^2+1}$ conveges

Proof. Let $\epsilon > 0$. Using Cauchy, there is a tail where $\sum_{k=m}^{n} a_k < \epsilon$. And since our series is smaller, we can say we have found a N where the tail of $\frac{1}{n^2+1}$ converges. □

Corollary: Absolute convergent series converge. So if $|a_n|$ converges, then a_n also converges.

Root Test

Definition 10.3
Since $\lim_{n\to\infty}$ Doesn't Always Exist, we use $\limsup_{n\to\infty}$.

Proof. If a < 1, this converges.

We have lim sup $_{n\to\infty}(a_n)^{\frac{1}{n}}$ which means that for some n > N, $|(a_n)^{\frac{1}{n}}| \leq a$. And then this turns into a geometric series.

Let a < 1, $\epsilon > 0$ and $a + \epsilon < 1$
From limsup, we have limsup $(a_n)^{\frac{1}{n}} = limsup_{N\to\infty}\{|(a_n)^{\frac{1}{n}}|$ conditioned on $|n > N\}$

By def, if this sequence a_n converges to a, we have $|a_n - a| < \epsilon$.
Rewriting gives us $(\epsilon + a)^n | n > N$

$$\text{Now we get} \sum_{n=N}^{\infty} a_n \leq \sum_{n=N}^{\infty} (a+\epsilon)^n \leq \sum_{n=N}^{\infty} 1^n \text{ Which converges}$$

If a > 1, this diverges. See above. □

Ratio Test: Looking at successive terms n, n+1

Definition 10.4
if limsup $|\frac{a_{n+1}}{a_n}| < 1$ then the sequence a_n converges absolutely.
if liminf $|\frac{a_{n+1}}{a_n}| > 1$ then the sequence a_n diverges.

The Root test is strictly better than the ratio test.

Example 10.5
Let's look at the following series:

$$\sum_{n=0}^{\infty} 2^{(-1)^n - n}$$

After applying the ratio test, we get that it is inconclusive.
But by the root test, we see that this converges absolutely.

And then intgral test says if the integral goes to infinty it doesn't converge.

Value of the integral does not tell us what our sequence converges to. It only tells us whether or not we have convergence.

Suppose $f(x) \geq 0$ and is inerasing on $[1, \infty]$. Then if the integral converges, then the summation converges. This gives us the proof for p-series.

Definition 10.6
P-series definition:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

will converge for all p > 1

Proof.

$$\int_1^{\infty} \frac{1}{x^p} dx = \int_1^{\infty} x^{-p} = \frac{x^{1-p}}{1-p} = \frac{1}{p-1} < \infty$$

for all p > 1 □

Alternating Series Test

Definition 10.7
An alternating series test is a series in the form of $\sum_{n=1}^{\infty} (-1)^n a_n$.
In order for this to converge, an alternating series must converge to 0.

Proof. The most common example is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

For some $\epsilon > 0$, we want to find some tail $\sum_{n=n}^{m} a_n < \epsilon$ such that $N \geq M > N$

We can show that when the tail is even or odd, we have our sum $\leq a_m \leq a_n$ □

We can show that a_n is absolutely convergent if $|a_n|$ converges.
Conditionally convergent means that a_n is converging but $|a_n|$ is not.

Definition 10.8
Riemann series theorem: You can rearrange conditional converging series to converge to any number.

Proof. We can have the series $1 + \frac{1}{3} + \frac{1}{5} + ... > a$ for any a. Because this is a conditionally convergent sequence, this sequence reaches infinity.

Whenever we overshoot a, we then subtract terms of a_n to undershoot a.

Proof that these sequences converge: If one doesn't converge and one does, then this wouldn't conditionally converge.

If both converge, then we just have sum of two converging series. □

11 Continuity

How do you show that something is continuous?

lets take the function $f(x) = 2x^2 + 1$

Definition 11.1

Definition of continuity states that if f is continuous at x_0 if whenever x_n is a sequence that converges to x_0, then $f(x_n)$ converges to $f(x_0)$

$x_n \to x_0 \implies f(x_n) \to f(x_0)$

Definition 11.2

f is continuous at x_0 if for all $\epsilon > 0$ there is $\delta > 0$ such that for all x, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$

f is **continuous** for all x_0, f is continuous at x_0

Proof. $|f(x) - f(x_0)| = |2x^2 + 1 - (2(x_0)^2 + 1)| = 2|x^2 - (x_0)^2| = 2|x - x_0||x + x_0|$

And we want to show that this is smaller than epsilon. Suppose $|x - x_0| < 1$. Then

$$|x + x_0| \leq |x| + |x_0| = |x - x_0 + x_0| + |x_0| \leq |x - x_0| + 2|x_0| \leq 1 + 2|x_0|$$

Therefore, our initial equation is $\leq$ to

$$2|x - x_0| \cdot (1 + 2|x_0|) < \epsilon. \text{ Rearranging } \implies |x - x_0| \leq \frac{\epsilon}{2(1 + 2|x_0|}$$

This gives $\delta = \frac{\epsilon}{2(1+2|x_0|}$ conditioned on $|x - x_0| < 1$

Actual Proof

Let

$$\delta = \min\{1, \frac{\epsilon}{2(1 + 2|x_0|}\}$$

And we also want $|x - x_0| < \delta$

This gives $|f(x) - f(x_0)| = 2|x - x_0||x + x_0| \leq 2|x - x_0| \cdot (1 + 2|x_0|) <$ □

Example 11.3
Let f(x) = $x^2 sin(\frac{1}{x})$ if x ≠ 0

Bounded functions
By definition, something is bounded if there is a M such that for all x, $|f(x) \leq M|$
Continuity holds for functions.

Example 11.4
if f and g are continuous at x_0, then f + g is continuous at x_0

Proof. Option 1: (f + g)(x_n) = f(x_n) + g(x_n) = f(x_0) + g(x_0) = (f + g)(x_0)

Option 2: f is continuous at x_0, there is δ_1 such that if $|x - x_0| < \delta_1$ then $|f(x) - f(x_0)| < \frac{\epsilon}{2}$

Same goes for g.

Now to prove this is continuous we have

$$|(f+g)(x) - (f+g)(x_0)| \leq |f(x) - f(x_0) + g(x) - g(x_0)| \leq \epsilon. \tag{15}$$

□

Example 11.5
k is a constant and x_n converges to x_0. Show that kf(x) $\to kf(x_0)$.

Proof. If $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{\epsilon}{|k|}$

$$|kf(x) - kf(x_0)| = |k||f(x) - f(x_0)| < \epsilon \tag{16}$$

□

Additionally, this means subtraction works since replace g with (-g).

Also, |x| is continuous. Meaning if f is continuous at x_0, |f| is continuous at |f(x_0)|

Proof. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$ Using |x| instead of x, we get

$$||x| - |x_0|| \leq |x - x_0| < \delta \implies ||f(x)| - |f(x_0)|| \leq |f(x) - f(x_0)| < \epsilon \tag{17}$$

□

Example 11.6

if f and g are continuous at x, then fg is continuous at x.

Proof.

$$|f(x)g(x) - f(x_0)g(x_0)| = |f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)|$$

We can use triangle inequality and product law for limits to get

$$|f(x)g(x-x_0) + g(x_0)f(x-x_0)| \leq |f(x)||g(x-x_0)| + |g(x_0)||f(x-x_0)| \tag{18}$$

Every term here is fine except for f(x). But because f(x) is continuous, $|f(x) - f(x_0)| < \epsilon$ for some $|x - x_0| < \delta$

Setting $\epsilon = 1$ we get

$$f(x) < 1 + f(x_0)$$

From here, we just have to set the δ for $|g(x-x_0)|, |f(x-x_0)|$. We choose $\delta > \frac{\epsilon}{2(|f(x_0)|+1}, \frac{\epsilon}{2(|g(x_0)|+1}$.

Finally, we get

$$\begin{gathered}|f(x)g(x) - f(x_0)g(x_0)| = |f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)| \\ \leq |f(x)||g(x-x_0)| + |g(x_0)||f(x-x_0)| \leq |1 + f(x_0)||\frac{\epsilon}{2(|f(x_0)|+1}| + |g(x_0)||\frac{\epsilon}{2(|g(x_0)|+1}|\end{gathered} \tag{19}$$

Note that $\frac{g(x_0)}{g(x_0)+1} < 1$

so this whole thing simplifies to $< \epsilon$ □

Example 11.7

Now to show that $\frac{f}{g}$ is continuous.

Proof. We have $|\frac{1}{f(x)} - \frac{1}{f(x_0)|}$ and after simplifying we only need to bound our $|f(x)|$ term.

We can't do the same thing from above because we are trying to bound $\frac{1}{f(x)}$ which means we need f(x) to be large.

Trick: Let $\epsilon = \frac{|f(x_0)}{2}$ so now $|f(x) - f(x_0)| < \frac{|f(x_0)}{2}|$

We want $|f(x)| > M$ so we say $||f(x)| - |f(x_0)|| < |f(x) - f(x_0)|$

$\implies -\epsilon < |f(x)| - |f(x_0)| < \epsilon$

Now we get |f(x)| > $\frac{|f(x_0)|}{2}$. As a result,

$$\frac{1}{f(x)} > \frac{2}{f(x_0)} \tag{20}$$

Now we finally find an $\epsilon > \frac{\epsilon}{2}|f(x_0)|^2$

The rest is the same as above. ☐

f ∘ g is continuous

Definition 11.8

If A, B, C are subsets of ℝ and f: A → B and g: B → C then the composition g ∘ f: A → C is defined by

$$(g \circ f)(x) = g(f(x))$$

If f and g are continuous at x_0, then f ∘ g is continuous at x_0. This is because we can let f converge at some delta, meaning g implies at some delta and then g ∘ f(x) = g(f(x)) and it converges.

12 The Calc Proofs

Extreme Value Theorem

Any continuous function f on [a,b] must have a max and a min.

Proof. We have M := sup(f). This is because we know it is bounded above + below. If you have a closed set, it contains a least upper bound.
We can construct the sequence sup(f) - $\frac{1}{n}$. There is a sequence s_n that will converge to this sup(f). □

Compactneess
If k is compact and f is continuous, then f(k) is compact.

Proof. Let U be an open cover of f(k). U^{-1} is open because U is open. □

Intermediate Value Theorem

Definition 12.1
If f: [a,b] $\rightarrow \mathbb{R}$ is continuous, and c is any value between f(a) and f(b), then there is some x $\in [a,b]$ where f(x) = c.

Proof. Let S := $\{x \in [a,b] \mid f(x) < c\}$
S is bounded by b, and we also know A $\in$ S.
To show that f(x_0) converges to c, we need to show that f(x_0) $\geq c$ and f(x_0) $\leq c$.
Because f is continuous, f(x_n) $\rightarrow f(x_0)$.
But since $x_n \in S$, we have $f(x_n) < c$ meaning the limit is less than c.
Also, we know that $t_n =: x_0 + \frac{1}{n} < b$ which converges to c. □

This holds if you replace [a,b] by any connected set.
Fixed Points:
If f: $[0,1] \rightarrow [0,1]$ is continuous, then f is a fixed point. There is some x_0 in $[0,1]$ such that f(x_0) = x_0.
This means that the line y = x must be crossed.
Applications
We can show that $\sqrt{x}$ exists.
Let 0 < t < 1, 0 < t < x.
Then $t^n < x^n$.
E is bounded above