Analysis

Textbook: Elementary Analysis

FRED YUAN

Fall 2023

Homework –> Selected Exercises from Ross.

Last updated on Tuesday 6th August, 2024.

1 Chapter 1 & 2

Exercise 1.1 (Exercise 1.1). *Prove* $1^2 + 2^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$ *for all positive integers n*

Proof. This holds for n = 1. (1 = 1)
Assuming this holds for n, it holds for n + 1.

$$1^2 + 2^2 + ... + n^2 + (n+1)^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$$

□

Exercise 1.2 (Exercise 1.12). *Verify the binomial theorem for a. n = 1, 2, 3.*
b. Show that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

Proof.

- This holds for n = 1, 2, 3.

$$(a+b)^1 = a+b, (a+b)^2 = a^2 + 2ab + b^2, (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

- The trick is to do this: $= \frac{n!}{(n-k)!(k)!} \cdot \frac{(n-k+1)}{(n-k+1)} + \frac{n!}{(n-k+1)!(k-1)!} \cdot \frac{k}{k}$

□

Exercise 1.3 (Exercise 2.1). *Show why* $\sqrt{3}$ *is not a rational number.*

Proof. By contradiction, suppose $\sqrt{3} = \frac{m}{n} \implies 3n^2 = m^2$. But these can't have a common factor, proving that $\sqrt{3}$ is irrational. □

Exercise 1.4 (Exercise 3.6). *Prove* $|a+b+c| \leq |a| + |b| + |c|$ *for all a, b, c.*

Proof. Using the triangle inequality twice,

$$|a+b+c| \leq |a+b| + |c| \leq |a| + |b| + |c|$$

And using induction, this works for all a_n. □

Exercise 1.5 (Exercise 3.7). *a) Show* $|b| < a$ *if and only if* $-a < b < a$
b) Show $|a-b| < c$ *if and only if* $b - c < a < b + c$
c) Show $|a-b| \leq c$ *if and only if* $b - c \leq a \leq b + c$

Proof. a) By definition, if b > 0, |b| = b and b < a.
b) If |a - b| > 0, |a - b| = a - b $\implies$ a < b + c
If |a - b| < 0, |a - b| = b - a $\implies$ b - a < c $\implies$ -a < c - b $\implies$ a > b - c
c) Same as b except instead of < use ≤ □

Exercise 1.6 (Exercise 4.7)**.** *Let S and T be nonempty bounded subsets of* $\mathbb{R}$
a) Prove that if S ⊆ T, then inf T ≤ inf S ≤ sup S ≤ sup T.
b) Prove sup(S ⋃ T) = max{sup S, sup T}.

Proof. We can look at inf(S) := {s : s ≤ M ∀ M ∈ S}. Since S ⊆ T, either inf(T) = {s : s ≤ M ∀ M ∈ T} or there is another element in T such that t < s, meaning inf(T) = min(s, $T \bigcap S$). Thus, inf(T) ≤ inf(S). inf(S) ≤ sup(S) by definition, and by the same logic sup(T) ≥ sup(S)
b) Note: S ⊈ T, so we have 2 cases:
C1: there exists an element s ∈ S such that s ≥ sup(T). Then sup(S ⋃ T) = sup(S).
Do the same for C2. □

Exercise 1.7 (Exercise 4.8)**.** *Let S and T be nonempty subsets of* $\mathbb{R}$ *with the following property: s ≤ t for all s ∈ S and t ∈ T*

(a) Observe S is bounded above and T is bounded below.

(b) Prove sup(S) ≤ inf T

(c) Give an example of such sets S and T where S ⋂ T is nonempty

(d) Give an example of sets S and T where sup S = inf T and S ⋂ T is the empty set.

Proof. (a) Because s ≤ t, we can take any element t ∈ T and t bounds S above. Additionally, we can take any s ∈ S and that bounds T from below.

(b) From (a), we can define a to be the inf of T. And since a ∈ T, t ≥ all s ∈ S.

(c) s:= {-1, 0}, t := {0, 1}

(d) s := $\lim_{n\to\infty}$ { -∞, -$\frac{1}{n}$} which has sup(S) = 0.
t := { $\frac{1}{n} \forall n \in \mathbb{N}$} which has inf(T) = 0.

□

Exercise 1.8 (Exercise 4.14)**.** *Let A and B be nonempty bounded subsets of* $\mathbb{R}$*, and let A + B be the set of all sums a + b where a ∈ A and b ∈ B.*

(a) Prove sup(A + B) = sup(A) + sup(B)

(b) Prove inf(A + B) = inf(A) + inf(B)

Proof. (a) Define a + b ∈ S. Then we have a + b ≤ sup(S).
This gives a ≤ sup(S) - b. $\implies$ sup(A) ≤ sup(S) - b.
We also get sup(B) ≤ sup(S) - sup(A), giving us sup(A) + sup(B) ≤ sup(S).
We can show that sup(A) ≥ a, sup(B) ≥ b, so sup(A) + sup(B) ≥ sup(S).

(b) Replace sup with inf.

□

Exercise 1.9 (Assigned Problem)**.** *Let A and B be nonempty bounded subsets of* $\mathbb{R}$ *and let AB be the set of all products ab where a ∈ A and b ∈ B. Is it always true that sup(AB) = sup(A)sup(B)? Why or why not?*

Proof. No.
Counterexample: a := {-2, -1}, b := { -1, 1}. sup(AB) = 2 ≠ sup(A)sup(B) = -1 □

2 Chapter 3

Exercise 2.1 (Exercise 4.16)**.** *Show sup $\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$*

Proof. By definition, r < a. However, this set contains every element up to a, meaning sup(s) = a. We can prove this by contradiction.

C1: Suppose the sup is a + ϵ, well then I say that a is the real sup.

C2: Suppose sup(s) = a - ϵ. By the archemedian property we can find some $\frac{m}{n}$ such that a - $\epsilon < \frac{m}{n} <$ a. ☐

Exercise 2.2 (Exercise 7.4)**.** *Give examples of*

(a) A sequence (x_n) of irrational numbers having a limit lim x_n that is a rational number.

(b) A sequence (r_n) of rational numbers having a limit lim r_n that is an irrational number.

Proof. (a) We can have a sequence be defined as $X_n \lim_{n\to\infty} = \sqrt{2 - \frac{1}{x}} - \sqrt{2}$ which converges to 0.

(b) Our sequence of numbers can be 3, 3.1, 3.14, 3.141, 3.1415 and this will converge to π

☐

Exercise 2.3 (Exercise 8.1)**.** *(a) Prove lim $\frac{1}{n^{\frac{1}{3}}} = 0$*

Proof. We call this sequence X_n. We want to find some value n > N such that for all $|X_n - 0| < \epsilon$ for any $\epsilon > 0$

Let N = $\frac{1}{\epsilon^3}$, then we have $|X_n - 0| < \epsilon \implies \frac{1}{n^{\frac{1}{3}}} < \epsilon$ ☐

Exercise 2.4 (Exercise 8.2)**.** *What is the limit of $c_n = \frac{4n+3}{7n-5}$*

Proof. Let n > $\frac{1}{7}(\frac{41}{7\epsilon} + 5$

Set $\epsilon > 0$, $|c_n - \frac{4}{7}| < \epsilon \implies \frac{41}{7(7n-5)} < \epsilon$. Since n > N, $\frac{41}{7(7n-5)} < \frac{41}{\frac{41}{\epsilon}} < \epsilon$ ☐

Exercise 2.5 (Exercise 8.4)**.** *Let t_n be a bounded sequence, i.e. there exists M such that $|t_n| \leq M$ for all n, and let s_n be a sequence such that lim $s_n = 0$. Prove lim$(s_n t_n) = 0$*

Proof. $|s_n t_n| = |s_n||t_n| \leq |s_n|M < \epsilon \implies |s_n| < \frac{\epsilon}{M}$

with n > N, $|s_n|M < \frac{\epsilon}{M}M = \epsilon$ ☐

Exercise 2.6 (Exercise 8.5). *(a) Consider three sequences a_n, b_n, and s_n such that $a_n \leq s_n \leq b_n$ for all n and lim a_n = lim $b_n = s$. Prove that lim $s_n = s$.*

(b) Suppose s_n and t_n are sequences such that $|s_n| \leq t_n|$ for all n and lim $t_n = 0$. Prove lim $s_n = 0$.

Proof. (a) By definition, for some $\epsilon > 0$, $|a_n - s| < \epsilon$ meaning $a_n > s - \epsilon$. Since b_n converges to s, there is $N_2 \in \mathbb{N}$ such that for n >N_2, $b_n < s + \epsilon$. Let n = the max of these N's. Then we get $s - \epsilon < s_n < s + \epsilon$. This means s_n converges to s!

□

Exercise 2.7 (Exercise 8.7). *Show that the following does not converge*

$sin(\frac{n\pi}{3})$

Proof. Lets assume this sequence a_n converges to a. Therefore, $|a_n - a| < \epsilon$ given $\epsilon >$ 0. Let's consider the cases for n:

If n % 6 == 0, then this sequence is equal to 0. Therefore, $|-a| < \epsilon$ so $-\epsilon < a < \epsilon$. Additionally,

If n % 6 == 1, then this sequence is $\frac{\sqrt{3}}{2}$ so $-\epsilon < a - \frac{\sqrt{3}}{2} < \epsilon$

Finally, we have a < ϵ < $\frac{\sqrt{3}}{2}$ < a. □

Exercise 2.8 (Exercise 8.8). *Prove lim $\sqrt{n^2+1} - n = 0$*

Proof. $|\sqrt{n^2+1} - n - 0| < \epsilon \implies n^2 + 1 < (\epsilon + n)^2 = n^2 + 2\epsilon n + \epsilon^2 \implies 1 - 2\epsilon n < \epsilon^2$

$n > \frac{1-\epsilon^2}{2\epsilon}$

Formal statement: Let $\epsilon > 0, n > N$ where N = $\frac{1-\epsilon^2}{2\epsilon}$

$n^2 + 1 < (\epsilon + n)^2 \implies 1 - 2\epsilon n < \epsilon^2$. This simplifies to being less than epsilon. □

Exercise 2.9 (Exercise 8.9). *Let s_n be a sequence that converges*

(a) Show that if $s_n \geq a$ for all but finitely many n, then lim $s_n \geq a$

(b) Show that if $s_n \leq b$ for all but finitely many n, then lim $s_n \leq b$

(c) conclude that if all but finitely many s_n belong to [a,b], then lim s_n belongs to [a,b].

Proof. (a) There is a finite term, M such that for all n > M, s_n will converge to s. $lims_n - s < \epsilon \implies |s - lims_n| < \epsilon \implies s < \epsilon + s_n \implies s_n > s - \epsilon$. So as long as we take n > N, $s_n \geq a$ meaning s > a. (Since ϵ converges to 0.

(b) Same as above

(c) If there's an infinite amount of terms belonging to [a,b], we do the same as (a). Choose some n > N such that the limit is within d(a+b/2, r), and then the limit has to be within [a,b]

□

Exercise 2.10 (Exercise 8.10)**.** *Let s_n be a converging sequence, and suppose lim $s_n >$ a. Prove there exists a number N such that n > N implies $s_n > a$.*

Proof. Let s_n converge to s > a. By definition, this means $|s_n - s| < \epsilon \implies -\epsilon < s_n - s < \epsilon \implies s_n > s - \epsilon$. We know s > a, and setting $\epsilon > 0$ such that s - $\epsilon \geq a$ we get $s_n > s - \epsilon \geq a$ □

Exercise 2.11 (Additional Exercise)**.** *Prove that if s_n converges to s, then $|s_n|$ converges to $|s|$.*

b) Is the converse also true?

Proof. a) $|s_n - s| < \epsilon \implies -\epsilon < s_n - s < \epsilon$

$$||s_n| - s| <= |s_n - s| < \epsilon \text{ By triangle inequality}$$

b) We have the sequence s_n = {-1, 1}. $|s_n|$ converges to |1| or |-1|. However, s_n DNC. □

3 Chapter 4

Exercise 3.1 (Exercise 9.1). *(a)* $\frac{n+1}{n} - 1 = 0$

(b) $\frac{3n+7}{6n-5} - \frac{1}{2}$

Proof. (a) $\frac{n+1}{n} - 1 < \epsilon \implies n+1 < (\epsilon+1)n \implies 1 + \frac{1}{n} < \epsilon + 1 \implies n > \frac{1}{\epsilon}$

For all n > N,

$$\frac{n+1}{n} - 1 = 1 + \frac{1}{n} < 1 + \frac{1}{\frac{1}{\epsilon}} - 1 = \epsilon$$

(b) $|\frac{3n+7}{6n-5} - \frac{1}{2}| < \epsilon \implies \frac{6n+14}{2(6n-5)} - \frac{6n-5}{2(6n-5)} < \epsilon \implies \frac{9}{2(6n-5)} < \epsilon \implies (\frac{9}{2\epsilon} + 5)\frac{1}{6} < n$
The rest is trivial

□

Exercise 3.2 (Exercise 9.9). *Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.*

(a) Prove that if lim $s_n = \infty$, then lim $t_n = \infty$

(b) prove that if lim $t_n = -\infty$, then lim $s_n = -\infty$

Proof. (a) Since $t_n \geq s_n$ for all n > N_0, $s_n = \infty \leq t_n = \infty$

(b) $s_n = -\infty \leq t_n = -\infty$

□

Exercise 3.3 (Exercise 9.11). *Show that if lim $s_n = \infty$ and t_n is a bounded sequence, then lim $(s_n + t_n) = \infty$.*

Proof. Since t_n is bounded, $t_n \neq -\infty$. Lets assume t_n converges to t for all n > N. There exists some M such that $s_n > M - t \implies s_n + t_n > M - t + t > M$ □

Exercise 3.4 (Exercise 9.12). *Assume all $s_n \neq 0$ and that the limit $L = lim\ \frac{s_n+1}{s_n}$ exists.*

a) Show that if L < 1, then lim $s_n = 0$.

b) Show that if L > 1, then lim $|s_n| = \infty$

Proof. a) Select a such that L < a < 1. We can let $\epsilon = a - L$. By definition, $-\epsilon < \frac{s_{n+1}}{s_n} < \epsilon \implies \frac{s_n+1}{s_n} < a \implies |s_n + 1| < a|s_n|$.
Step 2: Proving $|s_n| < a^{n-N}|s_N|$ for some n > N

case 1: n = N + 1. This simplifies and solves easily.
case 2: This holds for all n.

$$|s_{n+1}| < a|s_n| < a^{n-N+1}|s_N|$$

And because $0 \le |s_n| \le a^{n-N}|s_N|$ but as n > N and n $\to \infty, a^n = 0$. Therefore, by the squeeze theorem, $|s_n| = 0$

b) L > 1. Let $t_n = \frac{1}{|s_n|}$ then $|\frac{s_{n+1}}{s_n}| \to L \implies \frac{t_{n+1}}{t_n} \to \frac{1}{L}$
Our goal is to show that $\frac{1}{L} = 0 \implies L \to \infty$

Because $\frac{1}{L} < 1$, from a) we know that lim $t_n = 0$. And from 9.10, this means $|s_n| = \infty$

c) Theorem 9.10: lim $s_n = \infty$ if and only if lim $\frac{1}{s_n} = 0$

Proof. Suppose $s_n \to \infty$. Defining M = $\frac{1}{\epsilon}$ for all n > N, $\epsilon > \frac{1}{s_n} > 0 \implies \frac{1}{s_n} \to 0$.
Proving the other way. lim $\frac{1}{s_n} = 0$. Let M > 0, $\epsilon = \frac{1}{M}$.
There is a n > N s.t. $|\frac{1}{s_n} - 0| < \epsilon = \frac{1}{M} \implies \frac{1}{s_n} < \frac{1}{M} \implies s_n > M$. Since this is true for all M, $s_n \to \infty$ □

□

Exercise 3.5 (Exercise 9.15). *Show that* $lim_{n\to\infty} \frac{a^n}{n!} = 0$ *For all* $a \in \mathbb{R}$

Proof. Let's split up $\frac{a^n}{n!}$ into $\frac{a^a a^{n-a}}{a!(n-a)!}$
We know $\frac{a^a}{a!} = M$ for some M > 0. That leaves us with $\frac{a^{n-a}}{(n-a)!} \to 0$. We know that (n-a)! > $(a+1)^{n-a}$. Since a is arbitrary and n $\to \infty$, define N := (n-a).
This leaves $\frac{a^{n-a}}{(n-a)!} > \left(\frac{a}{a+1}\right)^N$. Since $\frac{a}{a+1} < 1, \left(\frac{a}{a+1}\right)^n \to 0$ Because M < ∞, 0M = 0. □

Exercise 3.6 (Exercise Problem 1). *Show directly through the definition of a limit that*

$$\lim_{n\to\infty} a^n = \infty \textit{ if } a > 1.$$

Proof. since a > 1: a := 1 + $\frac{1}{b}$. Let b := $\frac{1}{b}$
$\implies a^n = (1+b)^n$ for some b > 0.
$\implies (1+b)^n \le 1 + nb$. However, by the archemedian property, nb > M for any M > 0. Let n > $\frac{M}{b}$
Therefore, $a^n > nb > M$ □

Exercise 3.7 (Exercise Problem 2)**.** *Suppose $s_n = \infty$ and t_n is a sequence of positive numbers bounded above. Show that $\frac{s_n}{t_n} = \infty$*

Proof. t_n is bounded above by some C > 0.

$\frac{s_n}{t_n} \geq \frac{s_n}{C} > M \implies s_n > CM$

Therefore, $\frac{s_n}{t_n} > \frac{CM}{t_n} > M$ □

Exercise 3.8 (Exercise 10.8)**.** *Let s_n be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + ...s_n)$. Prove σ_n is an increasing sequence.*

Proof. Our goal is to show that $\sigma_n < \sigma_{n+1} \implies \sigma_{n+1} - \sigma_n > 0$

$$\implies \frac{1}{n+1}(s_1+s_2+...s_n+s_{n+1}) - \frac{1}{n}(s_1+s_2+...s_n) = \frac{n}{n(n+1)}(s_1+s_2+...s_n+s_{n+1}) - \frac{n+1}{n(n+1)}(s_1+s_2+...s_n)$$

$$\implies n(s_{n+1}) - (s_1 + s_2 + ...s_n)\frac{1}{n(n+1)}$$

Because $s_{n+1} > s_n > s_{n-1}... \geq s_1 \implies n(s_{n+1}) - (s_1 + s_2 + ...s_n) > 0$

□

Exercise 3.9 (Additional Problem 3)**.** *Define s_n by $s_1 = 1$ and*

$$s_{n+1} = \sqrt{s_n + 1}$$

Show that s_n converges and find its limit.

Proof. From Cauchy, $|s_n - s_m| < \epsilon \implies$ convergence.

n := m + 1

$$|\sqrt{s_n + 1} - \sqrt{s_n}| < \epsilon \implies s_n + 1 < (\epsilon + \sqrt{s_n})^2 = s_n + \epsilon^2 + 2\epsilon\sqrt{s_n}$$

$$1 < \epsilon^2 + 2\epsilon\sqrt{s_n} \implies (\frac{1-\epsilon^2}{2\epsilon})^2 < s_n$$

From this, we know that this sequence converges. Now, since it converges, that means $s_{n+1} = s_n - \epsilon$ for some $\epsilon > 0$.

Solving $x^2 = x + 1$ gives the golden ratio.

□

Exercise 3.10 (Additional Problem 4). *Define t_n by $t_1 = 1$ and*

$$t_{n+1} = \left(\frac{n}{n+2}\right) t_n$$

Show that t_n converges.

Proof. By induction, t_n is a decreasing sequence. $t_2 = \frac{1}{3} < t_1$
$t_n = \frac{n-1}{n+1} t_{n-1} > t_{n+1} = A t_n$ where A < 1

We also note that $0 < \frac{n}{n+2} < 1$ so this sequence is bounded below by 0. Since this sequence is decreasing and bounded below, t_n converges by monotone convergence. □

Exercise 3.11 (Additional Problem 5). *Prove that if s_n is decreasing and bounded below then s_n decreases. [Monotone Convergence]*

Proof. There exists some $\text{C} \leq \{s : n \in \mathbb{N}\}$

Our claim is that s_n converges to C. Since C is the minimum bound, $C + \epsilon$ is not a lower bound. We define $C < s_N < C + \epsilon$. Since our sequence is decreasing, we have $s_n \leq s_N$ for all n > N $\implies s_N$ converges to C. □

Exercise 3.12 (Babylonian Square Root). *a) Let $a > 0$ and $b > \sqrt{a}$ be given. Define a sequence s_n by $s_1 = b$ and*

$$s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$$

Show that s_n is a decreasing sequence that converges to $\sqrt{a}$

b) Let $a = 2$, $b = 2$. Calculus $s_2, s_3, \ldots$ and compare this to $\sqrt{2}$

Proof. a) To show this is decreasing, $s_n \geq \sqrt{a}$ for all n. This is true through induction.

We also need to show this is decreasing. $s_n - s_{n+1} > 0 \implies \frac{1}{2}(-s_n + \frac{a}{s_n}) \geq 0$
$s_n \geq \sqrt{a} \implies -s_n^2 + a < 0$. We have proved the s_n is a decreasing sequence.

Because this is bounded by $\sqrt{a}$ and is a decreasing sequence, we know this converges.

$$s = \frac{1}{2}(s + \frac{a}{s}) \implies s = \sqrt{a} \tag{1}$$

□

4 Week 5

Exercise 4.1 (Additional exercise 1)**.** *Show that $\mathbb{Z}$ is complete and that any Cauchy sequence in $\mathbb{Z}$ converges.*

Proof. By definition, we have $|s_n - s_m| < \epsilon$ for some n > m. and $\epsilon > 0$. Because we are in $\mathbb{Z}$, the only way for this sequence to converge is if for some n > m, $s_n = s_m$. □

Exercise 4.2 (Additional Exercise 2)**.** *Construction of $\mathbb{Q}$ without cuts or decimal expansions.*

Define $p_n \sim q_n$ if and only if $\lim_{n\to\infty} p_n - q_n = 0$

Show that $\sim$ is an equivalence relation on the set of sequences in $\mathbb{Q}$

Proof.

- Does $p_n \sim p_n$ hold? Yes
- $p_n \sim q_n \implies q_n \sim p_n$ Yes
- Holds due to transitivity as well. Suppose $p_n \sim q_n$ and $q_n \sim r_n$

$$\lim_{n\to\infty} p_n - r_n = p_n - q_n + q_n - r_n = 0 \tag{2}$$

□

Exercise 4.3 (Exercise 11.8)**.** *Prove lim inf s_n = - lim sup(-s_n) for every sequence s_n*

Proof. -S := $\{-s : s \in S\}$ and inf(S) = -sup(-S). Let $S_N := \{s_n : n > N\}$

Using the that fact, we can apply the limit and it works as well xd. □

Exercise 4.4 (Exercise 11.9)**.** *Show that the interval [a,b] is a closed set.*

Is there a sequecence s_n such that (0,1) is its set of subsequential limits?

Proof. a) The interval [a,b] is a closed set. We can say a sequence s_n will converge to s. Because $s_n \in [a,b] s_n$ has to converge to some value within [a,b]

b) Because (0,1) is not a closed set: We can have the subsequence $\frac{1}{n}$ which is would converge to 0. □

Exercise 4.5 (Exercise 11.11)**.** *Let S be a bounded set. Prove there is an increasing subsequence s_n of points in S such that lim s_n = sup S.*

Proof. Assuming sup(S) $\notin$ S.
By definition, S is bounded by sup(S) := C. There exists n > N such that $s_n \leq C$. If we can construct an increasing sequence Sup(S) - $\frac{1}{n}$< s_n < sup(S). □

Exercise 4.6 (Exercise 3)**.** *Suppose s_n is a sequence with the property that every subsequence has a further subsequence that converges to s.*

Prove that s_n converges to s.

Proof. Because every subsequence of s_n has another subsequence that converges to s, suppose we have some s_n that doesn't converge to s.

$|s_n - s| > \epsilon$ for some $\epsilon > 0$ and n > N. □

Exercise 4.7 (Exercise 4)**.** *Find an unbounded sequence s_n where every convergent subsequences converges to s but not every s_n converges to s.*

Proof. The sequence 0,2,0,4,0,6 ... has every converging subsequence converging to 0 but s_n does not converge to 0. □

Exercise 4.8 (Exercise 12.4)**.** *Show that lim sup $(s_n + t_n) \leq$ limsup s_n + limsup t_n for bounded sequences s_n and t_n.*

Proof. $s_n + t_n \leq sup(s_n : n > N) + sup(t_n : n > N) \implies sup(s_n + t_n) \leq sup(s_n : n > N) + sup(t_n : n > N)$

$$limsup(s_n+t_n) = limsup(s_n+t_n : n > N) \leq limsup(s_n : n > N)+sup(t_n : n > N) = limsup(s_n)+limsup(t_n)$$

□

5 Week 6

Exercise 5.1 (Exercise 12.8). *Let s_n and t_n be bounded sequences of non-negative numbers. Prove limsup $s_n t_n \leq limsup(s_n)limsup(t_n)$*

Proof. limsup $s_n := limsup\{s_n : n > N\}$ and $s_n \leq sup(s_n : n > N)$

Applying this to t_n as well we get

$$s_n t_n \leq sup(s_n : n > N)sup(t_n : n > N) \implies sup(s_n t_n) \leq sup(s_n : n > Nsup(t_n : n > N).$$

Then taking the limit gives us our desired answer. □

Exercise 5.2 (Exercise 12.12). *Let s_n be a sequence of non-negative numbers. Define $\sigma_n(s_1 + s_2 + ...s_n)$*

a) Show

$$liminf s_n \leq liminf \sigma_n \leq limsup_n \leq limsups_n$$

b)