Martingales and Applications

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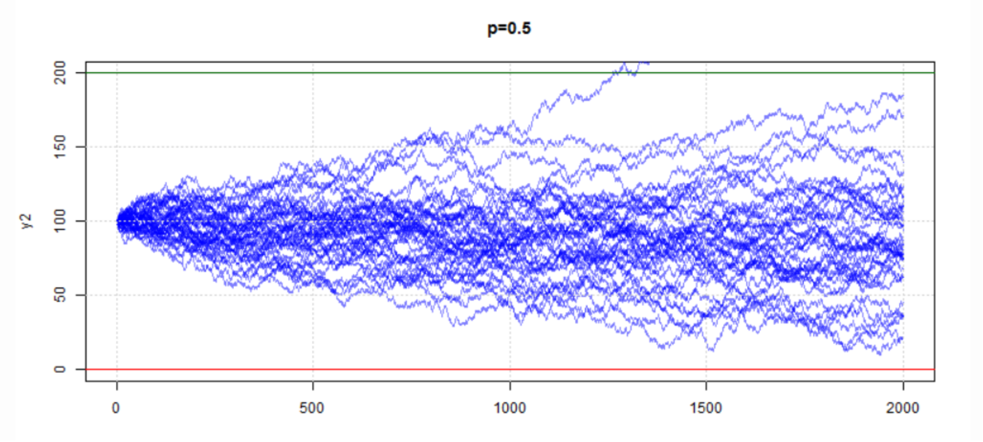
Gambler's Ruin

Lets play a game: You have a 50% chance of gaining \$1 and a 50% chance of losing \$1. Starting with K dollars, what are the odds you hit \$B before hitting 0?



Martingales

Let $X = (X_1, X_2, ..., X_t, ...)$ be a sequence of real-valued random variables.

This history of the sequence X is captured by an ascending family of sigma algebra's $\mathcal{F} = (\mathcal{F}_t)_{t \in I}$ called a filtration. We say that X is adapted to the filtration if $\mathcal{F}_t$ contains the entire history of the process X up to time t.

Definition (Martingale)

Let X be a sequence of random variables and $\mathcal{F}$ a filtration. We say X is a martingale if

- X is adapted to $\mathcal{F}$
- $E(|X_i|) < \infty \quad \text{for} \quad (i = 1, 2, ...)$
- $E[X_{t+s}|\mathcal{F}_t] = X_t$

Gambler's Ruin is a Martingale

For our coinflip game, let Y_n denote the outcome of the nth coinflip and X_n denote the total amount of money we have at time n. A heads is recorded as a 1 and a tails as a -1.

We can show that this is a martingale.

- X is adapted to $\mathcal{F}$
- $E[|X_t|] < \infty$ because $E[X_t - X_0] \leq t$
- $E[X_{t+1}|\mathcal{F}_t] = X_t$

$= E[X_t + Y_{t+1}|\mathcal{F}_t] = X_t + E[Y_{t+1}|\mathcal{F}_t] = X_t$ because $E[Y_{t+1}] = 1 * 1/2 + (-1) * 1/2 = 0$

Note: This is because our flips are i.i.d

Stopping times

Definition (Stopping time)

A stopping time τ is an almost-surely finite time-valued random variable defined by some condition on the process.

Lemma

If τ is a stopping time, then $X_{t\wedge\tau}$ is a martingale. ($t \wedge \tau$ means $min(t, \tau)$)

Proof.

Intuitively, given that X_t is a martingale, then if $t < \tau$, we then our MTG X_t is still walking. However, if $t \geq \tau$ then that indicates that our stopping time, τ has been reached. If the stopping time has been reached, we know in the next step we are at our present position. $\square$

Optional Stopping Theorem

Theorem (Optional Stopping Theorem)

If $(X, \mathcal{F})$ is a martingale, X is pointwise uniformly bounded, and τ is a stopping time, then

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0] \tag{1}$$

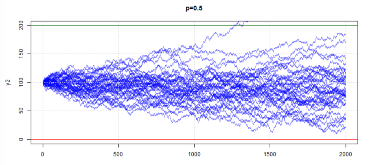
Proof.

$X_{t\wedge\tau}$ is a martingale. Therefore for any time t

$$\mathbb{E}[X_{t\wedge\tau}] = \mathbb{E}[X_{0\wedge\tau}] = \mathbb{E}[X_0]. \tag{2}$$

As t goes to infinity $X_{t\wedge\tau}$ converges to X_τ. Therefore by boundedness the result holds as t goes to infinity. □

Returning to Gambler's Ruin



Returning to our game: You have a 50% chance of gaining \$1 and a 50% chance of losing \$1. Starting with K dollars, what are the odds you hit \$B before hitting 0?
Let τ be the time it takes to hit 0 or hit B. Applying the optional stopping theorem, we have

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0] = K \tag{3}$$

X_τ is equal to B or 0 so

$$K = \mathbb{E}[X_\tau] = B \cdot \mathbb{P}[X_\tau = B] + 0\mathbb{P}[X_\tau = 0]. \tag{4}$$

Therefore

$$\mathbb{P}[X_\tau = B] = \frac{K}{B}. \tag{5}$$

Brownian Motion

Brownian motion is a continuous time analogue of a simple random walk.

It was discovered when R. Brown observed irregular motion of tiny particles.

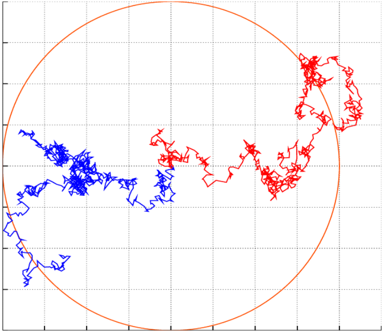
Definition (Brownian Motion)

A real-valued stochastic process $W(\cdot)$ is called a Brownian motion if

- $W(0) = 0$ almost surely
- $W(t) - W(s)$ is $\mathcal{N}(0, t - s)$ for all $t \geq s \geq 0$
- For all times $0 < t_1 < \dots < t_n$, the random variables $W(t_1), W(t_2) - W(t_1), \dots, W(t_n) - W_{t_{n-1}}$ are independent
- $W(t)$ is almost surely continuous in time t

Introduction to the Problem

Let X_t, Y_t be two independent Brownian motions starting at 0. Let T be the first time that (X_t, Y_t) lies on the ellipse $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$.



Solution

Lemma

If W is a Brownian motion, then $W(t)^2 - t$ is a martingale.

Proof.

Let $M(t) = W(t)^2 - t$. Then

$$\mathbb{E}[M(t+s)|\mathcal{F}_t] = \mathbb{E}[(W(t+s) - W_t + W(t))^2 - (t+s)|\mathcal{F}_t] \tag{6}$$

Expanding the square gives

$$\mathbb{E}[(W(t+s) - W(t))^2 + 2(W(t+s) - W(t))W(t) + W(t)^2 - t - s|\mathcal{F}_t] \tag{7}$$

As $W(t+s) - W(t)$ is independent to the history of the process up to time t we have

$$\mathbb{E}[(W(t+s) - W(t))^2] + W(t)^2 - t - s = W(t)^2 - t. \tag{8}$$

□

Solution (2)

Let $M(t) = \frac{X_t^2 - t}{a^2} + \frac{Y_t^2 - t}{b^2}$. Let τ be defined as

$$\inf\{t : \frac{X_t^2}{a^2} + \frac{Y_t^2}{b^2} = 1\}. \tag{9}$$

$M(t)$ is a martingale and τ is a stopping time so $M(t \wedge \tau)$ is a martingale. Applying the optional stopping theorem we get that

$$0 = \mathbb{E}[M(0)] = \mathbb{E}[M(\tau)] = \mathbb{E}[\frac{X_\tau^2 - \tau}{a^2} + \frac{Y_\tau^2 - \tau}{b^2}]. \tag{10}$$

Breaking up the right side:

$$\mathbb{E}[\frac{X_\tau^2}{a^2} + \frac{Y_\tau^2}{b^2} - \frac{\tau}{a^2} - \frac{\tau}{b^2}] = 1 - \frac{\mathbb{E}[\tau]}{a^2} - \frac{\mathbb{E}[\tau]}{b^2} \tag{11}$$

So we're left with

$$E[\tau] = \frac{1}{\frac{1}{a^2} + \frac{1}{b^2}} = \frac{a^2 b^2}{a^2 + b^2} \tag{12}$$

Conclusion

We looked at Martingales, Brownian Motion, and the Optional Stopping Theorem

Definitions (Recap)

- For Martingales: $E[X_{t+s}|\mathcal{F}_t] = X_t$
- For Brownian Motion: $W(t) - W(s)$ is $\mathcal{N}(0, t-s)$ for all $t \geq s \geq 0$
- For Optional Stopping Theorem: $E[X_0] = E[X_\tau]$

These ideas help us compute and understand recursive events directly, and can be extended to games that are not fair.

Fin

Thank You for listening, Any Questions?