Log IU

Textbook: Ross

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Work from LogIU project from the IU math dept. Mentor: Max N. This will be problem writeups.

simulations can be found at https://github.com/zeegy99/Log-iu-

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Exercise 1.1 (Exercise 1)**.** *We have i.i.d random variables* X_n *with* $\begin{cases} P(X_n = 1) = \frac{1}{2} \\ P(X_n = -1) = \frac{1}{2} \end{cases}$

Denote $S_n := X_1 + X_2 ... X_n$ *and stopping time* τ *to be inf(*$l : S_l = 0$ *or* 20)
Let $a_k :=$ *k starting dollars.*

What is the expected time to ruin from a_5*?*

Simulation: https://github.com/zeegy99/Log-iu-/blob/main/ranodm/main.py

Proof. Specific Case P = .5:

$$a_2 - a_1 = a_1 - 2, \text{ and generally, } a_k - a_{k-1} = a_1 - (k-1)2$$
$$a_{19} - a_{18} = a_1 - 36, \text{ and adding } a_{19} - a_{18} \text{ with } a_{18} - a_{17} ... a_2 - a_1 \text{ gives}$$
$$a_{19} - a_1 = 18a_1 - (2 + 4 + 6 + ...36) = 18a_1 - (18 \times 19)$$

By symmetry, we can note that $a_{19} = a_1$ which finally gives $a_1 = 19$. Finally, using the same trick we can write $a_k - a_1 = (k-1)a_1 - (k)(k-1)$. (Proof by induction)

This gives our final equation to be $a_k = k(a_1 + 1 - k) = k(20 - k)$

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Proof. General Case: We know $a_k = Pa_{k+1} + Qa_{k-1} + 1$ where P = P, Q = 1-P.

In our question specifically, P = .5, but assume P = $\frac{x}{y}$ where x $\leq$ y. So Q = $\frac{y-x}{y}$

Then
$$a_1 = \frac{x}{y}a_2 + 1, \implies ya_1 = xa_2 + y \implies a_2 = \frac{ya_1 - y}{x}$$
$$a_2 - a_1 = \frac{ya_1 - y}{x} - a_1 = \frac{(y-x)a_1 - y}{x}$$
$$a_2 = \frac{y-x}{y}a_1 + \frac{x}{y}a_3 + 1 \implies ya_2 = (y-x)a_1 + xa_3 + y$$
$$a_3 = \frac{ya_2 - ya_1 + xa_1 - y}{x}$$
$$a_3 - a_2 = \frac{ya_2 - ya_1 + xa_1 - y}{x} - \frac{ya_1 - y}{x} = \frac{ya_2 - 2ya_1 + xa_1}{x}$$

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Solving this through Martingales: Our definitions were

- X_n := value of our i.i.d random variable at timestep n.
- S_n := total value of money at time step n.

We will use this martingale: $S_n^2 - n$. To see if this is a martingale we need to show
$$E[S_{n+1}^2 - (n+1)|\mathcal{F}_n] = S_n^2 - n$$

Distributing:
$$= E[(M_n + X_{n+1})^2 = E[(M_n)^2 + 2M_nX_{n+1} + (X_{n+1})^2 - (n+1)|\mathcal{F}_n]$$

Breaking up the expectations we verify that this holds.
$$= (M_n)^2 + 0 + 1 - n - 1 = S_n^2 - n$$

By the Optional Stopping Theorem (OST), we know:
$$E[M_\tau] = E[M_0].$$

At $n = 0$, $S_0 = 5$, so:
$$E[M_0] = S_0^2 - 0 = 25.$$

At the stopping time τ, the gambler either reaches 0 or 20, so:

$$E[M_\tau] = \frac{1}{4}(400 - \tau) + \frac{3}{4}(0 - \tau),$$

where $\frac{1}{4}$ is the probability of reaching 20, and $\frac{3}{4}$ is the probability of ruin at 0.

Setting $E[M_\tau] = 25$ gives:

$$\frac{1}{4}(400 - \tau) + \frac{3}{4}(-\tau) = 25.$$

Simplifying:

$$100 - \tau = 25,$$

which leads to:

$$\tau = 75.$$

Thus, the expected time to ruin or reach 20 is $\tau = 75$, as intended.

Exercise 1.2 (Exercise 2)**.** *Percolation Question: Assume we have a n x m grid. Each node will have edges connecting it to its neighbors. These edges will be uniformly distributed N(0,1). Fix p ∈ (0, 1) and remove all edges with values smaller than p.*

Proof. `https://github.com/zeegy99/Log-iu-/blob/main/ranodm/problem_2.py`

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