Real Variables

Textbook: Real Analysis, Modern Techniques

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Introduction to Measure Theory

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Definition 1.1

Limsup $E_n = \cap_{k=1}^{\infty} \cup_{n=k}^{\infty} E_n$ And

Liminf $E_n = \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} E_n$

Proof. For Limsup $E_n = \{x : x \in E_n$ For infinitely many n$\}$
It is the intersection of all sets formed from n onward. And it there must be infinitely many n's.

Liminf $E_n = \{x : x \in E_n$ For all but finitely many n$\}$
There's a sequecne $E_1, E_2, ...E_n$ where you are not in E_n and then from n onwards you are in. □

Definition 1.2

Symmetric difference: $E \triangle F = (E \backslash F) \cup (F \backslash E)$

Equivalence relations: (From Analysis)
~ is an equivalence relation if

1. x ~ x for all Ω
2. x ~ y if and only if y ~ x
3. x ~ y and y ~ z $\implies$ x ~ z

Definition 1.3

A mapping f: X → Y is a relation from X to Y with the property that for every x ∈ X there is a unique y ∈ Y such that x ~ y. In this case, we can write y = f(x)

Example 1.4

If f: X → Y and g: Y → Z are mappings, we denote f ∘ g their **Composition**
g ∘ f: X → Z
g ∘ f(x) = g(f(x))

If D ⊂ X and E ⊂ Y, we define the image of D and reverse image of E under a mapping f: X → Y by

$$f(D) = \{f(x) : x \in D\}, f^{-1}(E) = \{x : f(x) \in E\}$$

Here, we define f(D) as the mapping of the image x ∈ D over f onto f(x)

If A ⊂ X, we denote f|A the restriction of f to A:

$$(f|A) : A \to Y, \qquad (f|A)(x) \text{ for } x \in A$$

We define a finite sequence to be a map from $\{1, 2, ...n\}$ into X where n ∈ N.

A subsequence is when g(n) < g(m) whenever n < m, then this is called a subsequence.

Definition 1.5

Cartesian Product: $\prod_{\alpha \subset A} X_\alpha$ is the set of maps f: A → $\cup_{\alpha \in A} X_\alpha$ such that f(α) $\in X_\alpha$ for every $\alpha \in A$

1.1 Orderings

Definition 1.6

A partial ordering on a non-empty set X is a relation R on X with the properties (same as equivalence). However, not every partial ordering needs to be comparable.

Example 1.7

Lets say we have $\{0, 1, 2, 3\}$. We can have $\{0\} \le \{0, 1\}$, but $\{1, 3\}$ and $\{2, 3\}$ can't be compared.

Definition 1.8

Well-Ordering.

This is similar to partial ordering, except every element is comparable. Formally, this is known as **Totality**.

For every a, b ∈ S, either $a \leq b$ or $a \geq b$.

Because every set is comparable, we also have the well-foundedness principle where every non-empty subset of S has a least element.

Example 1.9

The set (0,1) is not well ordered because 0 is not in S and from analysis we can show there is no n = inf(S) ∈ S

Linear (total) orderings have one more requirement: Everything can be compared.

E.g.: Fractions, Modulus, $\{nxn\}$ matrixes that can be row-reduced.

If R also satisfies x, y ∈ X, then either xRy or yRx

Two partially ordered sets X and Y are said to be **Order Isomorphic** if there is a bijection f: X → Y such that $x_1 \leq x_2$ iff $f(x_1) \leq f(x_2)$

Fundamental Principle of set theory and some consequences:

Definition 1.10

Hausdorff Maximal Principle

Every partially ordered set has a maximal linearly ordered subset.

Example 1.11

A chain is contained in the maximal chain (because it contains all of the chains)

Zorn's Lemma If X is a partially ordered set and every linearly ordered subset of X has an upper bound, then X has a maximal element

The Well Ordering Principle: Every nonempty set X can be well ordered.

Using Zorn's Lemma, we can say that there is a maximal element in W. And this maximal ordering must exist on the whole set X because otherwise there would be a larger chain.

The Axiom of Choice:

$$\text{If } \{X_\alpha\}_{\alpha \in A} \text{ is a nonempty collection of nonempty sets, then } \prod_{\alpha \in A} X_\alpha \text{ is non-empty} \tag{1}$$

Proof. To prove H to Z: We can note that the maximal element in a maximal chain is still the maximal element of its subsets. Therefore, this maximal element is the maximal element we need from Zorn.

To prove Z to H: We need to look at the powerset of X, P(X).

$$\text{We choose } E \subset P(X) \text{ Such that } c \in E \text{ is in Linear Order}$$

With P(X), this contains every linearly ordered subset and has a maximal element. Therefore, we have partially ordered sets that are bounded by the maximal element X from P(X). □

1.2 Cardinality

Definition 1.12

Injective (one to one): If different elements in the domain A map to different elements in the codomain B.

$$\text{If } f(a_1) = f(a_2) \text{ then } a_1 = a_2$$

Definition 1.13

Surjective (onto): If every element in the codomain B is the image of at least one element of the domain A.

For every b ∈ B, there exists a ∈ A such that f(a) = b.
From this, lets get the simple laws.

card(X) ≤ card(Y) iff card(Y) ≥ card(X)

Definition 1.14

Schroder-Bernstein Theorem: For sets X, Y, either Card(X) ≤ Card(Y) or Card(Y) ≤ Card(X). This implies Card(X) = Card(Y)

Proof. Our gameplan is to bounce back and forth between smaller subsets of A and B. Let f: $X \to Y$ and g: $Y \to X$.

$$Let A_0 = A, B_0 = B$$
$$A_1 = g(B_0), B_1 = f(A_0)$$
$$...$$

We note that

$$A_0 \supseteq A_1 \supseteq A_2 ...$$
$$B_0 \supseteq B_1 \supseteq B_2 ...$$
$$A = A_0 \sim B_1 \sim A_2 ...$$
$$B = B_0 \sim A_1 \sim B_2 ...$$

We now need analogues that are pairwise disjoint. Define $A_n^* := A_n - A_{n+1}$

Now we can rewrite

$$\begin{aligned} A^* &= A_0^* \sim B_1^* ... \\ B^* &= B_0^* \sim A_1^* ... \end{aligned} \tag{2}$$

Now we get

$$A_0^* \cup A_1^* \sim B_0^* \cup B_1^*$$

Taking the Union over all N gives us

$$\bigcup_{n \geq 0} A_n^* \sim \bigcup_{n \geq 0} B_n^*$$

Rewrite the lefthand side as $\tilde{A}$ and the righthand side as $\tilde{B}$. Note that A $\neq \tilde{A}$

So finally we let A = □

Wow! We're finally done with that.

Here's an easier proposition: card(X) < card(P(X)).
I think it's trivial.

Definition 1.15
A set X is countable (denumerable) if card(X) ≤ card(N). In particular, all finite sets are countable, and for these it is convenient to interpret "card(X)" as the number of elements in X.

If X is countable but infintie, card(X) is countably-infinite.

Propositions with countability

(a) If X and Y are countable, so is X x Y

(b) If A is countable and X_a is countable for every $a \in A$, then $\cup_{a \in A} X_a$ is countable.

(c) If X is countable infinite, then card(X) = card($\mathbb{N}$)

Z and Q are countable.

Proof. Z is the union of sets $\{-n : n \in \mathbb{N}\} \cup 0$.
We can make the surjection f: $Z^2 \to \mathbb{Q}$ by f(m, n) = $\frac{m}{n}$ □

A set has the **cardinality of the continuum** if $\mathrm{card}(X) = \mathrm{card}(R)$

Definition 1.16

$\mathrm{card}(P(\mathbb{N})) = \varsigma$

Proof. If $A \in \mathbb{N}$, define $f(A) \in \mathbb{R}$ □

If $\mathrm{card}(X) > \varsigma$, then X is uncountable.

1.3 Well Ordered Sets

Let X be a well ordered set. This set has an infinum and sup.

If x ∈ X, Define an initial segment to be:

$$I_x = \{y \in X : y < x\} \tag{3}$$

The elements of I_x are called predecessors of x.

Principle of Transfinite Induction
Let X be well ordered. If A is a subset of X such that x ∈ A whenever $I_x \subset A$ then A = X

Proof. If X ≠ A, let x = inf(X \ A). Then I_x ⊂ A but x ∉A □

There are elements in X that are not in A. I_x is the set of all elements X that are strictly less than x. If $I_x \subset A$ then x must be in a. We have a contradiction.

Misconception The interval [0,10] is not a well-ordered set. It is a partially-ordered set. A subset of [0,10] is (0, 10] which does not have a least element in the subset.

Definition 1.17
If X is well ordered and A ⊂ X, then $\bigcup_{x \in A} I_x$ is either an initial segment or X itself.

I think this is intuitively easy to understand, and the proof is that let b = inf(X $\setminus J$). We note that J = b.

Definition 1.18
Order Isomorphic means that there is a bijecetive mapping between them that preserves their order.
Formally, Let (A, $\leq_A$) and (B, $\leq_B$) be two ordered sets.

Every countable subset has an upper bound.

1.4 real numbers

$\bar{R} = \mathbb{R} \cup \{-\infty, \infty\}$

Define sup, inf, limsup, liminf to be what you expect:

$$\text{limsup} x_n := inf_{k \geq 1}(sup_{n \geq k} x_n) \text{ liminf } x_n := sup(inf\, x_n)$$

A sequence x_n converges if its limsup = liminf + it is finite. (not infinity)

Right-continuous $f(a^+) := lim_{x \downarrow a} f(x) = f(a)$ Left-continuous $f(a^-) := lim_{x \uparrow a} f(x) = f(a)$

Open sets: (a, b) Not open set: [a, b]

Every open set in $\mathbb{R}$ is a countable disjoint union of open intervals.

Proof. x ∈ U, $I_x := \bigcup\{openintervalscontainingx, containedinU\}$ □

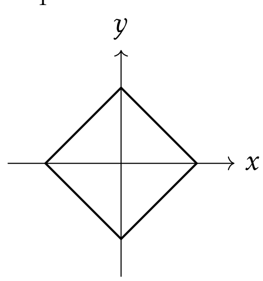
Some notes on the Euclidian norm for n dimensions:

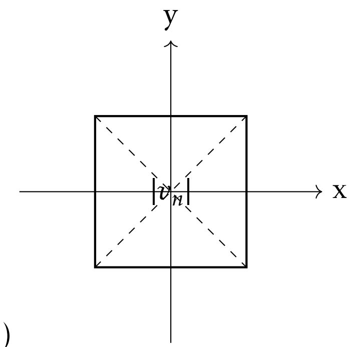
Definition 1.19

For a vector $\mathbf{v} = (v_1, v_2, v_3, ...v_n) \in \mathbb{R}^n$ we define the Euclidian norm,$\|\mathbf{v}\|_3 = \left(\sum_{i=1}^{n} |v_i|^n\right)^{1/n}$

Some examples: The V_2 euclidian norm looks like a circle: ○

The V_1 euclidian norm is |x| + |y| = 1





The $|V_n|$ euclidian norm is max(|x|, |y|)

1.5 Metric Spaces

A metric on a set X is a function ρ: X → X → [0, ∞) such that

1. ρ(x,y) = 0 iff x = y
2. ρ(x,y) = ρ(y,x) for all x, y ∈ X.
3. The distance thing

We can think of ρ as d(x,y)

A set equipped with a metric (ρ) is known as a metric space.

1. Euclidian Distance |x - y|
2. ρ_1(f,g) = $\int_0^1 |f(x) - g(x)| dx$ are metrics on the space of continuous functions [0,1].

Density (I am dense)

E(circle) is the largest open set contained in E $\bar{E}$ is the smallest closed set containing E.

E is dense if $\bar{E} = X$ and is nowhere dnese if $\bar{E}$ has empty interior

X is **Seperable** if it has a countable dense subset. Eg: $\{a + bi\}_{a,b\in\mathbb{Q}}$

Proposition: f: $X_1 \to X_2$ is continuous iff $f^{-1}(U)$ is open in X_1 for every open U $\subset X_2$ (preimage of any open set is open)

Proof. We have x $\in X_1$ and $\epsilon > 0$. The set $f^{-1}(B(\epsilon, f(x)))$ maps us back to x except now we have an open ball around x. Assume we are continuous. Then for y $\in U$ we can make $B(\epsilon_y, y) \in U$ since U is open. And since f is continuous, we let x $\in f^{-1}(\{y\})$ and define $\delta_x > 0$ s.t.$B(\delta_x, x) \subset f^{-1}(B(\epsilon_y, y)) \subset f^{-1}(u)$
Thus $f^{-1}(u) = \bigcup_{x \in f^{-1}(u)} B(\delta_x, x)$ is open. □

For a counterexample, if f is not continuous, then we are unsure how f could map us.

Definition 1.20

A sequence $\{X_n\}$ in a metric space (X, p) is Cauchy if p$(X_n, X_m) \to 0$ as n, m $\to \infty$. A subset is called complete if every Cauhcy sequence in E converges and its limit is in E.

$\mathbb{R}$ is complete, and $\mathbb{Q}$ is not.

A closed subsetof a complete metric space is complete, and a complete subset of an arbitrary metric space is closed.

Proof. If X is complete, E $\subset$ X is closed.
x $\in \bar{E}$ **HEY WHERES THE PROOF** □

True: intersection of finite open sets is open
False: intersection of all open sets is open. B/c $\{0, 1/n\} \to 0$

We can define distances from a point to a set and distances between two sets. Let's say we have E, F $\subset$ X. Then we take the distance between the sets as inf(e $\in$ E, f $\in$ F).

The following are equivalent

1. E is complete and totally bounded
2. Every sequence in E has a subsequence that converges to a point of E.
3. Heine-Borel: If $\{V_a\}_{a \in A}$ is a cover of E by open sets, there is a finite set F $\subset$ A such that V_a covers E.

From (a) to (b): Suppose E can be covered by finitely many balls of radius 2^{-1}. At least one of them must contain X. We can keep making the balls smaller and we can become smaller than any $\epsilon > 0$.

From (b) to (a): If either condition in A fails, then (b) fails. If E is not complete, then there is some subsequence that won't converge in E. No subsequence can converge in E otherwise the whole sequence would converge.
Lets assume E can't be totally bounded: Let $\epsilon > 0$ such that E can't be covered by finitely many balls of radius ϵ. Assume $x_n \in E$ but then $x_{n+1} \notin$ the covered section of E. Then p$(x_n, x_m) > \epsilon$.

From (a, b) to (c): If (b) holds and $\{V_a\}_{a \in A}$ is a cover of E by open sets, **idk write the rest of the proof later**

From (c) to (b): If X_n is a sequence in E with no converging subsequence, then there are finitely many Balls such that $x_n \in$ B(B_ϵ, x). Then $\{B_x\}_{x \in E}$ is a cover of E by open sets.

Definition 1.21
A set that is coverable is compact

If you look very closely then you will see this is true.

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Motivation: For a function to be riemann integrable, it is almost everywhere continuous.

Reminder: Riemann sums are defined as: $\sum_{n=1}^{n=\infty}\left(G_j(x_{n+1}-x_n)\right)=\sum_{n=1}^{n=\infty}\left(g_j(x_{n+1}-x_n)\right)$ where g_n = is the lower end of a function and G_n is the upper end.

A function is defined to be riemann integratable if this lower and upper bound both converge to the same value.

We want to define a Lebesgue integral such that we don't run into these issues.

1. If $E_1, E_2, \ldots$ is a finite or infinite sequence of disjoint sets, then
$$\mu(E_1 \cup E_2 \cup \ldots) = \mu(E_1) + \mu(E_2) + \ldots, \tag{4}$$

2. If E is congruent to F (that is, if E can be transformed info F by translations, rotations, and reflections), then $\mu(E) = \mu(F)$.

3. $\mu(Q) = 1$, where Q is the unit cube
$$Q = \{x \in R^n : 0 \le x_j < 1 \text{ for } j = 1, \ldots n\}$$

TLDR: Sadly, we can't really define a measure like this.

Counterexample: consider a circular set on [0, 1) with equivalence relations being that x ∼ y if x - y ∈ $\mathbb{Q}$. This means x and y must be π away. Construct N_r to be this set transformed to the right by r ∈ $\mathbb{Q}$. We can construct all numbers in [0, 1) through the union of all N_r's. Then we have $\mu([0,1)) = \sum_{r\in R}\mu(N_r)$. But since $\mu(N) = \mu(N_r)$ by (ii) since translations shouldn't add or subtract measure from a set, we get $\mu([0,1)) = \sum_{r\in R}\mu(N)$.
We have a contradiction since $\mu([0,1)) = 1$ by definition of a measure, and if $\mu(N)$ > 0, then the sum goes to infinity, but if it equals 0, then the measure is 0.

Another comment: in higher dimensions with a weaker (i), Banach and Tarski proved that you can multiply a pea until infinite peas.

2.1 σ-algebras

Let's start by defining a $\sigma-$algebra.

Definition 2.1

To be a σ-algebra on X means

1. (Closure under complements) If A $\in X$, then $A^c \in X$
2. (Closure under countable unions) If $A_1, A_2, ...A_n \in X$, then $\bigcup_{n=1}^{\infty} A_n \in X$
3. A is in X.

The intersection of any family of σ-algebras is another σ-algebra. (The textbook said this is Trivial, I do not think so)

Proof. A σ-algebra can't be empty. It must contain $\{\Omega, \emptyset\}$. If A is included in this $\sigma-$algebra, then A must have been included and also A^c. Then, A^c is in our new algebra and also the old ones, and our set now contains A alongside A^c to form another functional algebra. □

Now, using this we have that if ϵ is in any subset of P(X) there must be a unique smallest $\sigma-$algebra M(ϵ) containing ϵ. Specifically, the intersection of all $\sigma-$algebras containing ϵ. M(ϵ) is called the $\sigma-$algebra generated by ϵ.

Proposition 2.1

If $\epsilon \subset M(\mathcal{F})$, then $M(\epsilon) \subset M(\mathcal{F})$

Bring up next class: Should it be $\subseteq$. Example: M(F) = $\{\emptyset, F, F^c, \Omega\}$ $\epsilon = \{\emptyset, F\}$
Then M(ϵ) = M(F).