Stocastic Calculus with an application to Black-Scholes

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ODE's & SDE's

An SDE is an ODE with an additional random noise in the derivative. That is, instead of

$$dX_t = \mu X_t dt \tag{1}$$

we have

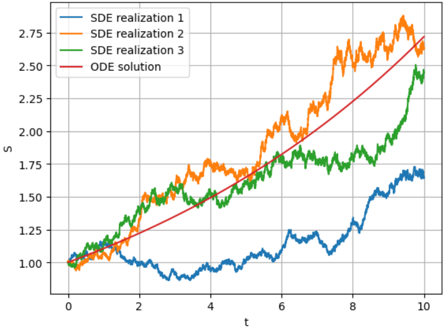
$$dX_t = \mu X_t + \sigma\, dW_t. \tag{2}$$

Heuristically, we want this to mean that, for a short time-step dt that the first order approximation of X_{t+dt} is

$$X_t + \mu X_t dt + \sigma \mathcal{N}(0, dt) \tag{3}$$

where $\mathcal{N}(0, r)$ denotes a mean zero Gaussian with variance r.

ODE vs SDE


SDE realization 1
SDE realization 2
SDE realization 3
ODE solution
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
S
0
2
4
6
8
10
t

Brownian Motion

Definition

A real-valued stochastic process $W.$ is called a Brownian motion if

- $W_0 = 0$
- $W(t) - W(s)$ is $\mathcal{N}(0, t-s)$ for all $t \geq s \geq 0$
- for all times $0 < t_1 < ... < t_n$ the random variables $W_{t_1}, W_{t_2} - W_{t_1}, ... W_{t_n} - W_{t_{n-1}}$ are independent
- W is almost surely continuous

The Need For a New Calculus

If we try to solve

$$dX_t = X_t dW_t, \tag{4}$$

we need a notion of integration with respect to Brownian motion, but simply integrating against its derivative is not possible, as it is not differentiable!

Taking

$$\lim_{h \longrightarrow 0} \frac{W_{t+h} - W_t}{h} = \frac{\mathcal{N}(0,h)}{h} \tag{5}$$

gives us $\mathcal{N}(0, \frac{1}{h})$, which blows up.

The Need For a New Calculus

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Taking

$$\lim_{h \longrightarrow 0} \frac{W_{t+h} - W_t}{h} = \frac{\mathcal{N}(0, h)}{h} \tag{7}$$

gives us $\mathcal{N}(0, \frac{1}{h})$, which blows up.

Idea: Just as we derive integration via the fundamental theorem of calculus, we find a notion of differentiation via a new FTC for a new type of stochastic integration.

Itô's integral

The goal of Itô's integral is to give meaning to stochastic integrals like

$$\int_0^t W_s dW_s. \tag{8}$$

Itô's integral

The goal of Ito's integral is to give meaning to stochastic integrals like

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We do so via Riemann sums.

$$\int_0^t f(W_s) dW_s = \lim_{n \to \infty} \sum_{j=0}^{n-1} f(W_{\frac{jt}{n}})(W_{\frac{(j+1)t}{n}} - W_{\frac{jt}{n}}) \tag{10}$$

Convergence of the Riemann sums

We use the observation that

$$(W_{t_{j+1}}^2 - W_{t_j}^2) - (W_{t_{j+1}} - W_{t_j})^2 = 2W_{t_j}(W_{t_{j+1}} - W_{t_j}) \tag{11}$$

where $t_j = \frac{jt}{n}$. In particular then,

$$\sum_{j=0}^{n-1} W_{t_j}(W_{t_{j+1}} - W_{t_j}) = \frac{1}{2}\sum_{j=0}^{n-1}(W_{t_{j+1}}^2 - W_{t_j}^2) - \frac{1}{2}(W_{t_{j+1}} - W_{t_j})^2. \tag{12}$$

The LHS is telescoping and the RHS converges to $-\frac{t}{2}$ as n goes to infinity so we get

$$\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t. \tag{13}$$

Failure of the FTC

$$\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t. \tag{14}$$

Notice: the standard fundamental theorem of calculus fails to hold!

What does it mean to solve an SDE?

Definition

X solves the SDE

$$dX_t = f(X_t)\,dt + g(X_t)\,dW_t \tag{15}$$

if

$$X_t - X_0 = \int_0^t f(X_s)\,ds + \int_0^t g(X_s)\,dW_S. \tag{16}$$

Itô's rule as a new FTC

Theorem

Suppose X_t solves an SDE

$$dX_t = f(X_t)\,dt + g(X_t)\,dW_t. \tag{17}$$

Then $h(X_t)$ solves the SDE

$$dh(X_t) = h'(X_t)\,dX_t + \frac{1}{2}f''(X_t)\,d\langle X\rangle_t. \tag{18}$$

where $\langle X\rangle_t = \int_0^t g^2(X_s)\,ds$.

A heuristic proof of Itô's rule

Let's Taylor expand $h(X_t)$ to approximate $h(X_{t+dt})$.

$$h(X_{t+dt}) = h(X_t) + h'(X_t)dX_t + \frac{1}{2}h''(X_t)dX_t^2 + O((dX_t)^3) \tag{19}$$

As dt goes to zero, any term comparable to $(dt)^n$ for $n > 1$ can be ignored. Substituting in dX_t we get

$$h(X_t) + h'(X_t)(f(X_t)dt + g(X_t)dW_t) + \frac{1}{2}h''(X_t)(f(x_t)^2dt^2 + 2g(X_t)f(X_t)dtdW_t + g^2(X_t)(dW_t)^2) \tag{20}$$

As $\mathbb{E}[(dW_t)^2] = dt$, we can think of $dW_t \sim (dt)^{\frac{1}{2}}$. Hence asymptotically we have

$$h(X_{t+dt}) = h'(X_t)dX_t + \frac{1}{2}h''(X_t)g^2(X_t)dt + O((dt)^{\frac{3}{2}}). \tag{21}$$

A quick application of Itô's rule

We want to evaluate

$$\int_0^t W_s dW_s \tag{22}$$

an easier way.
We can do a change of variables on W_t by finding some f such that the process $f(W_t)$ involves, by Itô's rule, this very integral.
By Itô's rule,

$$f(W_t) - f(W_0) = \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)d\langle X\rangle_s \tag{23}$$

A quick application of Itô's rule

$$f(W_t) - f(W_0) = \int_0^t f'(W_s)\,dW_s + \frac{1}{2}\int_0^t f''(W_s)\,d\langle X\rangle_s \tag{24}$$

We want $f'(W_s) = W_s$ so that $\int_0^t W_s dW_s$ appears in 24. Therefore we may take $f(X_s) = \frac{1}{2}X_s^2$. $f''(X_s) = 1$ so

$$\frac{1}{2}W_t^2 = \int_0^t W_s dW_s + \frac{1}{2}\int_0^t ds. \tag{25}$$

Rearranging gets us

$$\int_0^t W_s dW_s = \frac{W_t^2}{2} - \frac{t}{2} \tag{26}$$

Solving Black-Scholes from Ito's lemma

Suppose S_t is the price of a stock at time t. $S_0 = s_0$ is the stock's initial value. People commonly model change of price of the stock as an SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{27}$$

This is the so-called Black-Scholes SDE.

Black-Scholes continued

Using a change of variables, and letting $f(x) = ln(x)$ we have

$$df(S_t) = f'(S_t)\,dS_t + \frac{1}{2}f''(S_t)\,d\langle S\rangle_t \tag{28}$$

That is,

$$d(ln(S_t)) = \frac{dS_t}{S_t} - \frac{1}{2}\frac{\sigma^2 S_t^2}{S_t^2}dt = (\mu - \frac{1}{2}\sigma^2)\,dt + \sigma\,dW_t \tag{29}$$

Therefore,

$$S_t = s_0\,e^{\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t} \tag{30}$$

is a solution to

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{31}$$

the Black-Scholes SDE.

Fin

Any questions? Thank you for listening.