First Order Logic

March 29, 2021

Contents

1 Exercise 4

Provide expressions to represent the following facts in FOL

1.1 Men are not women

$\forall x\ men(x) \supset \neg women(x)$

1.2 Surgeons are doctors

$\forall x\ surgeon(x) \supset doctor(x)$

1.3 Adults can only be Men and Women

$\forall x \ adult(x) \supset man(x) \vee woman(x)$

1.4 If a person marries another person, this one is also married to the first one

$\forall x \forall y \ married(x, y) \supset married(y, x)$

1.5 Parents have children

$\forall x \exists y \ parent(x) \supset haschild(x, y)$

1.6 Adults are defined as Men and Women who are older than 18

$\forall x \ adult(x) \supset (man(x) \vee woman(x)) \wedge \neg minor(x)$

1.7 Marriage is only allowed between two Adults

$\forall x \forall y \ married(x, y) \supset adult(x) \wedge adult(y)$

1.8 A person cannot be married to two or more different persons

$\forall x \forall y \forall z \ married(x, y) \wedge married(y, z) \supset (z = x)$

1.9 Two persons can only get divorced if they are previously married

$\forall x \forall y \ candivorce(x, y) \supset married(x, y)$

1.10 People can only be given birth by a Man and a Woman

$\forall z \exists x \exists y \ haschild(x, z) \wedge haschild(y, z) \supset man(x) \wedge woman(y)$

2 Exercise 5

Formalize the following sentences as FOL expressions, after identifying function symbols and predicate symbols.

Given a domain, we can construct a FOL knowledge base following these steps

Named individuals $mike$, $mary$

No-named individuals boy_1, $tshirt_1$, $tshirt_2$

Types Boy, $Girl$, $TShirt$, $Person$, $Symbol$

Properties $Color$, $Symbol$, Age

Relationships $Younger(x, y)$, $Wears(x, y)$, $Youngest(x)$

Functions $countWearing(tshirt, color, symbol)$, $age(x)$

2.1 Mike is younger than the boy in the green T-shirt

$Boy(boy_1), TShirt(tshirt_1), Wears(boy_1, tshirt_1), Color(tshirt_1, green), Younger(mike, boy_1)$

2.2 The five-year boy wore a T-shirt with a square symbol

$Age(boy_1, 5)$, $Symbol(tshirt_1, square)$

2.3 Mikes T-shirt is yellow

$TShirt(tshirt_{mike})$, $Wears(mike, tshirt_{mike})$, $Color(tshirt_{mike}, yellow)$

2.4 Marys T-shirt does not bear a square symbol

$Girl(mary), TShirt(tshirt_{mary}), Wears(mary, tshirt_{mary}), \neg Symbol(tshirt_{mary}, square)$

2.5 Square symbols cannot appear in white T-shirts.

$\forall x\ TShirt(x) \wedge Color(x, white) \supset \neg Symbol(x, square)$

2.6 The youngest person cannot wear a T-shirt

$\forall x \forall t\ youngest(x) \wedge TShirt(t) \supset \neg Wears(x, t) \wedge youngest(x) = person(x) \wedge (\forall z\ person(x) \supset younger(x, z))$

2.7 There are three T-shirt symbols: squares, pictures, and circles

$\forall t \forall s\ TShirt(t) \wedge Symbol(t, s) \supset Symbol(t, square) \vee Symbol(t, picture) \vee Symbol(t, circle)$

2.8 There is not a person wearing a T-shirt with a circle if theres another person older than the first one wearing a square

$\forall x \forall t_1\, (\exists y \exists t\, Wears(x,t) \wedge Symbol(t, square) \wedge age(y) > age(x)) \supset \neg(Wears(x,t_1) \wedge Symbol(t_1, circle))$

2.9 Everybody wearing a T-shirt is older than any other not wearing a T-shirt

$\forall x \forall y \forall t_x \forall t_y\, Wears(x,t) \wedge TShirt(t_x) \wedge Wears(y,t_y) \wedge TShirt(t_y) \supset age(x) > age(y)$

2.10 The number of people wearing a T-shirt yellow, are bigger than the ones not wearing a T-shirt with a square

$countWearing(true, yellow, any) > countWearing(any, any, \{picture, circle, none\})$

3 Exercise 6

Given the following description Tony, Mike, and John belong to the Alpine Club. Every member of the Alpine Club who is not a skier is a mountain climber. Mountain climbers do not like rain, and anyone who does no like snow is not a skier. Mike dislikes whatever Tony likes, and likes whatever Tony dislikes.

3.1 Formalize this knowledge as FOL expressions

1. Tony, Mike, and John belong to the Alpine Club $in(tony, aclub)$, $in(mike, aclub)$, $in(john, aclub)$
2. Every member of the Alpine Club who is not a skier is a mountain climber $\forall x\ in(x, aclub) \wedge \neg skier(x) \supset climber(x)$
3. Mountain climbers do not like rain, and anyone who does no like snow is not a skier $\forall x\ climber(x) \supset \neg like(x, rain)$ $\forall x\ \neg like(x, snow) \supset \neg skier(x)$
4. Mike dislikes whatever Tony likes, and likes whatever Tony dislikes $\forall a\ like(tony, a) \supset \neg like(mike, a)$ $\forall a\ \neg like(tony, a) \supset like(mike, a)$

3.2 Find out whether Tony is a mountain climber or not. Is it possible?

We don't have enough knowledge to state if Tony is a mountain climber or not.

3.3 What do you know about John?

- $in(john, aclub) = true$
- $\neg skier(john) \supset climber(john)$
- $climber(john) \supset \neg like(john, rain)$
- $\neg like(john, snow) \supset \neg skier(john)$

3.4 Prove that there is a member of the Alpine Club who is a mountain climber but not a skier

1. Tony dislikes anything that Mike likes (and the other way around)
2. Either Tony or Mike dislikes snow
3. Either Tony or Mike is not a skier
4. Either Tony or Mike is a mountain climber
5. Either Tony or Mike is not a mountain climber (due to point 1)

3.5 Suppose that Mary, a new member of the Alpine Club, likes what Mike and John likes. What can you say about Mary

The fact that Mary is a new member implies all the following

- $in(mary, aclub) = true$
- $\neg skier(mary) \supset climber(mary)$
- $climber(mary) \supset \neg like(mary, rain)$
- $\neg like(mary, snow) \supset \neg skier(mary)$

Remeber that $\forall a\ like(tony, a) \supset \neg like(mike, a)$ and $\forall a\ \neg like(tony, a) \supset like(mike, a)$ In natural language, the phrase `Likes what Mike and John likes` can be interpreted as

Conjunction $\forall a\ like(mike, a) \wedge like(john, a) \supset like(mary, a)$

Equivalence $\forall a\ like(mary, a) \supset like(mike, a) \wedge like(john, a)$

Partial $\exists a\ like(mary, a) \supset \neg(like(mike, a) \wedge like(john, a)) \supset \neg like(mike, a) \vee \neg like(john, a) \supset like(tony, a)$

Inclusive disjunction $\forall a\ like(mike, a) \vee like(john, a) \supset like(mary, a)$

Equivalence $\forall a\ like(mary, a) \supset like(mike, a) \vee like(john, a)$

Partial $\exists a\ like(mary, a) \supset \neg(like(mike, a) \vee like(john, a)) \supset \neg like(mike, a) \wedge \neg like(john, a) \supset like(tony, a)$

4 Exercise 7

Given the relationship $Parent(x, y)$ representing the fact x is parent of y, and $Male(x)$ representing x is male, define in FOL the following family relationships

4.1 Son, Daughter, Brother, Sister, Sibling, Ancestor, Father, Mother, Grandfather, Grandmother, Uncle, Aunt, Cousin, and Nephew.

Son $Parent(x, y) \wedge Male(y) \supset Son(y, x)$

Daughter $Parent(x, y) \wedge \neg Male(y) \supset Daughter(y, x)$

Brother $Parent(x, y) \wedge Son(z, x) \supset Brother(z, y)$

Sister $Parent(x, y) \wedge Daughter(z, x) \supset Sister(z, y)$

Sibling $Brother(x, y) \vee Sister(x, y) \supset Sibling(x, y)$

Ancestor $Parent(x, y) \vee (Parent(x, z) \wedge Ancestor(z, y))) \supset Ancestor(x, y)$

Father $(Son(y, x) \vee Daughter(y, x)) \wedge Son(x, z) \supset Father(x, y)$

Mother $(Son(y, x) \vee Daughter(y, x)) \wedge Daughter(x, z) \supset Mother(x, y)$

Grandfather $Father(x, z) \wedge (Father(z, y) \vee Mother(z, y)) \supset Grandfather(x, y)$

Grandmother $Mother(x, z) \wedge (Father(z, y) \vee Mother(z, y)) \supset Grandmother(x, y)$

Uncle $Brother(x, z) \wedge (Father(z, y) \vee Mother(z, y)) \supset Uncle(x, y)$

Aunt $Sister(x,z) \wedge (Father(z,y) \vee Mother(z,y)) \supset Aunt(x,y)$

Cousin $(Son(x,t) \vee Daughter(x,t)) \wedge Sibling(t,z) \wedge (Father(z,y) \vee Mother(z,y)) \supset Cousin(x,y)$

Nephew $Sibling(y,t) \wedge (Father(t,x) \vee (Mother(t,x)) \supset Nephew(x,y)$

4.2 John has not children. Jon has not siblings.

$\neg(\exists x\ Father(john,x))$, $\neg(\exists x\ Sibling(john,x))$

4.3 Johns parents are Mary (female) and Paul (male).

$Mother(mary,john)$, $Father(paul,john)$

4.4 Johns sister has some children.

$\exists x\ \exists y Sister(x,john) \wedge Mother(x,y)$

4.5 The mother of Mary is the aunt of Michael.

$\exists x\ Mother(mary,x) \wedge Aunt(x,michael)$

5 Exercise 8

Given the simplified set theory in which all the variables are considered sets, and using the predicates $Sub(x,y) =$ "x is a subset of y", $E(e,x) =$ "e is an element of the set x", and the functions $u(x,y) =$ "the union of x and y", $i(x,y) =$ "the intersection of x and y"; provide FOL expressions to represent the following knowledge:

5.1 No set is an element of itself

$\forall e \forall x\ E(e,x) \supset e \neq x$

5.2 A set x is a subset of a set y iff every element of x is an element of y

$\forall x \forall y \forall e\ E(E(e,x),y) \supset Sub(x,y)$

5.3 Something is an element of the union of two sets x and y iff it is an element of x or an element of y

$\forall x \forall y \forall e \; E(e,x) \vee E(e,y) \supset E(e,u(x,y))$

5.4 Something is an element of the intersection of two sets x and y iff it is an element of x and an element of y

$\forall x \forall y \forall e \; E(e,x) \wedge E(e,y) \supset E(e,i(x,y))$

6 Exercise 9

Let $C(x)$ be the statement x has a cat, let $D(x)$ be the statement x has a dog, and let $F(x)$ be the statement x has a ferret. Express each of these statements in first-order logic using these relations. Let the domain be your classmates.

6.1 A classmate has a cat, a dog, and a ferret.

$\exists x \; C(x) \wedge D(x) \wedge F(x)$

6.2 All your classmates have a cat, a dog, or a ferret.

$\forall x \; C(x) \vee D(x) \vee F(x)$

6.3 At least one of your classmates has a cat and a ferret, but not a dog.

$\exists x \; C(x) \wedge F(x) \wedge \neg D(x)$

6.4 None of your classmates has a cat, a dog, and a ferret.

$\neg(\exists x \; C(x) \wedge D(x) \wedge F(x))$

6.5 For each of the three animals, there is a classmate of yours that has one.

$\exists x \exists y \exists z \; C(x) \wedge D(y) \wedge F(z)$

7 Exercise 13

What is the meaning of the following FOL expressions:

7.1 $\forall x \forall y \ Loves(x, y)$

Everybody loves everybody

7.2 $\forall x \exists y \ Loves(x, y)$

Everybody loves somebody

7.3 $\exists x \forall y \ Loves(x, y)$

Somebody loves everybody

7.4 $\exists x \exists y \ Loves(x, y)$

Somebody loves somebody

7.5 $\forall x \forall y \ Loves(x, y) \supseteq \forall z \ Loves(x, z)$

If everybody loves everybody then x loves everybody

7.6 $\forall x \forall y \ Loves(x, y) \supseteq \exists z \ Loves(x, z)$

If everybody loves everybody then x loves somebody

7.7 $\forall x \exists y \ Loves(x, y) \supseteq \forall z \ Loves(x, z)$

If everybody loves somebody then x loves everybody

7.8 $\forall x \exists y \ Loves(x, y) \supseteq \exists z \ Loves(x, z)$

If everybody loves somebody then x loves somebody

8 Exercise 16

For the sentence $\forall x (\forall y (A(x) \land B(x, y) \Rightarrow A(y)))$ state whether it is true or false, relative to the following interpretations. If false, give values for x and y witnessing that.

8.1 The domain of the natural numbers, where A is interpreted as even?, and B is interpreted as equals

$\forall x \forall y \ even(x) \land equals(x, y) \Rightarrow even(y)$ is true

8.2 The domain of the natural numbers, where A is interpreted as even?, and B is interpreted as is an integer divisor of

$\forall x \forall y \; even(x) \wedge divisor(x,y) \Rightarrow even(y)$ is false

8.3 The domain of the natural numbers, where A is interpreted as even?, and B is interpreted as is an integer multiple of

$\forall x \forall y \; even(x) \wedge multiple(x,y) \Rightarrow even(y)$ is false, $x = 6, \; y = 3$

8.4 The domain of the Booleans, {true,false}, where A is interpreted as false?, and B is interpreted as equals

$\forall x \forall y \; false(x) \wedge equals(x,y) \Rightarrow false(y)$ is true

9 Exercise 18

Check for free and bound variables in the following expressions

9.1 $\forall x \; (\exists y \; P(x,y) \Rightarrow \exists z \; (Q(y,z) \Rightarrow R(x,y)\hat{\ } P(x,y)))$

No free variables

9.2 $(\forall x \; (\exists y \; P(x,y) \Rightarrow \exists z \; (Q(y,z)))) \Rightarrow R(x,y)\hat{\ } P(x,y)$

x and y are free in the last part of the expression

9.3 $(\forall x \; (\exists y \; P(x,y) \Rightarrow \exists z \; (Q(y,z)) \Rightarrow R(x,y)\hat{\ } P(x,y)))$

No free variables

9.4 $(\forall x \; (\exists y \; P(x,y))) \Rightarrow (\exists z \; (Q(y,z) \Rightarrow R(x,y)\hat{\ } P(x,y)))$

x and y are free in the second and last part of the expression

10 Exercise 19

Represent in FOL

10.1 Maria is mother of a son and a daughter

$\exists x \exists y\ x = son(maria) \wedge y = daughter(maria)$

10.2 Maria is mother of only one son and only one daughter

$\forall x \forall y \forall z\ x = son(maria) \wedge\ y = daughter(maria) \wedge \neg(z = x) \wedge \neg(z = y) \supset \neg mother(maria, z)$

10.3 Maria is mother of a son or a daughter

$\forall x\ son(x, maria) \supset \neg(\exists y\ daughter(y, maria))\ \forall x\ daughter(x, maria) \supset \neg(\exists y\ son(y, maria))$

10.4 All women are beautiful and some men are beautiful

$\forall x\ woman(x) \supset beautiful(x)\ \exists x\ men(x) \supset beautiful(x)$