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Introduction

1.1 Why a New Book on Probability?

There exist numerous treatments of this topic, many of which are very good, and others continue to appear. To add one more would certainly be a presumptuous undertaking if I thought in terms of doing something better, and a useless undertaking if I were to content myself with producing something similar to the 'standard' type. Instead, the purpose is a different one: it is that already essentially contained in the dedication to Beniamino Segre

> [who about twenty years ago pressed me to write it as a necessary document for clarifying one point of view in its entirety.]

Segre was with me at the International Congress of the Philosophy of Science (Paris 1949), and it was on the occasion of the discussions developed there on the theme of probability that he expressed to me, in persuasive and peremptory terms, a truth, perhaps obvious, but which only since appeared to me as an obligation, difficult but unavoidable.

> 'Only a complete treatment, inspired by a well-defined point of view and collecting together the different objections and innovations, showing how the whole theory results in coherence in all of its parts, can turn out to be convincing. Only in this way is it possible to avoid the criticisms to which fragmentary expositions easily give rise since, to a person who in looking for a completed theory interprets them within the framework of a different point of view, they can seem to lead unavoidably to contradictions.'

These are Segre's words, or, at least, the gist of them.

It follows that the requirements of the present treatment are twofold: first of all to clarify, exhaustively, the conceptual premises, and then to give an essentially complete exposition of the calculus of probability and its applications in order to establish the adequacy of the interpretations deriving from those premises. In saying 'essentially' complete, I mean that what matters is to develop each topic just as far as is necessary to avoid conceptual misunderstandings. From then on, the reader could follow any other

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book without finding great difficulty in making those modifications that are needed in order to translate it, if such be desired, according to the point of view that will be taken here. Apart from these conceptual exigencies, each topic will also be developed, in terms of the content, to an extent sufficient for the treatment to turn out to be adequate for the needs of the average reader.

1.2 What are the Mathematical Differences?

1.2.1. If I thought I were writing for readers absolutely innocent of probabilistic–statistical concepts, I could present, with no difficulty, the theory of probability in the way I judge to be meaningful. In such a case, it would not even have been necessary to say that the treatment contains something new and, except possibly under the heading of information, that different points of view exist. The actual situation is very different, however, and we cannot expect any sudden change.

> My estimation is that another fifty years will be needed to overcome the present situation, but perhaps even this is too optimistic. It is based on the consideration that about thirty years were required for ideas born in Europe (Ramsey, 1926; de Finetti, 1931) to begin to take root in America (even though B.O. Koopman (1940) had come to them in a similar form). Supposing that the same amount of time might be required for them to establish themselves there, and then the same amount of time to return, we arrive at the year 2020.

It would obviously be impossible and absurd to discuss in advance concepts and, even worse, differences between concepts to whose clarification we will be devoting all of what follows; however, much less might be useful (and, anyway, will have to suffice for the time being). It will be sufficient to make certain summary remarks that are intended to exemplify, explain and anticipate for the reader certain differences in attitude that could disorientate him, and leave him undecided between continuing without understanding or, on the other hand, stopping reading altogether. It will be necessary to show that the 'wherefore' exists and to give at least an idea of the 'wherefore', and of the 'wherefores', even without anticipating the 'wherefore' of every single case (which can only be seen and gone into in depth at the appropriate time and place).

1.2.2. From a mathematical point of view, it will certainly seem to the reader that either by desire or through ineptitude I complicate simple things; introducing captious objections concerning aspects that modern developments in mathematical analysis have definitively dealt with. Why do I myself not also conform to the introduction of such developments into the calculus of probability? Is it a question of incomprehension? Of misoneism? Of affectation in preferring to use the tools of the craftsman in an era of automation which allows mass production even of brains – both electronic and human?

The 'wherefore', as I see it, is a different one. To me, mathematics is an instrument that should conform itself strictly to the exigencies of the field in which it is to be applied. One cannot impose, for their own convenience, axioms not required for essential reasons, or actually in conflict with them.

I do not think that it is appropriate to speak of 'incomprehension'. I have followed through, and appreciated, the reasons *pro* (which are the ones usually put forward), but I found the reasons *contra* (which are usually neglected) more valid, and even preclusive.

I do not think that one can talk of misoneism. I am, in fact, very much in favour of innovation and against any form of conservatism (but only after due consideration, and not by submission to the tyrannical caprice of fashion). Fashion has its use in that it continuously throws up novelties, guarding against fossilization; in view of such a function, it is wise to tolerate with goodwill even those things we do not like. It is not wise, however, to submit to passively adapting our own taste, or accepting its validity beyond the limits that correspond to our own dutiful, critical examination.

I do not think that one can talk of 'affectation' either. If anything, the type of 'affectation' that is congenial to my taste would consist of making everything simple, intuitive and informal. Thus, when I raise 'subtle' questions, it means that, in my opinion, one simply cannot avoid doing so.

1.2.3. The 'wherefore' of the choice of mathematical apparatus, which the reader might find irksome, resides, therefore, in the 'wherefores' related to the specific meaning of probability, and of the theory that makes it an object of study. Such 'wherefores' depend, in part, on the adoption of this or that particular point of view with regard to the concept and meaning of probability, and to the basis from which derives the possibility of reasoning about it, and of translating such reasoning into calculations. Many of the 'wherefores' seem to me, however, also to be valid for all, or many, of the different concepts (perhaps with different force and different explanations). In any case, the critical analysis is more specifically hinged on the conception that we follow here, and which will appear more and more clear (and, hopefully, natural) as the reader proceeds to the end – provided he or she has the patience to do so.

1.3 What are the Conceptual Differences?

1.3.1. Meanwhile, for those who are not aware of it, it is necessary to mention that in the conception we follow and sustain here only *subjective* probabilities exist – that is, the *degree of belief* in the occurrence of an event attributed by a given person at a given instant and with a given set of information. This is in contrast to other conceptions that limit themselves to special types of cases in which they attribute meaning to 'objective probabilities' (for instance, cases of symmetry as for dice etc., 'statistical' cases of 'repeatable' events, etc.). This said, it is necessary to add at once that we have no interest, at least for now, either in a discussion, or in taking up a position, about the 'philosophical' aspects of the dispute; in fact, it would be premature and prejudicial because it would entangle the examination of each concrete point in a web of metaphysical misunderstandings.

Instead, we are interested, on the contrary, in clearly understanding what one means *according to one's own conception and in one's own language*, and learning to enter into this conception and language in its motivations and implications (even if provisionally, in order to be able to make pertinent criticism later on). This is, it seems to me, an inviolable methodological need.

1.3.2. There is nothing more disappointing than to hear repeated, presented as 'criticisms', clichés so superficial that it is not possible to infer whether the speaker has even read the arguments developed to confute them and clear them up, or has read them without understanding anything, or else has understood them back to front. The fault could be that of obscure presentation, but a somewhat more meaningful reaction would be required in order to be able to specify accurately, and to correct, those points which lend themselves to misunderstanding.

The fault may be the incompleteness of the preceding, more or less fragmentary, expositions, which, although probably more than complete if taken altogether, are difficult to locate and hold in view simultaneously. If so, the present work should obviate the inconvenience: unfortunately, the fact that it is published is not sufficient; the result depends on the fact that it is read with enough care to enable the reader to make pertinent criticisms.

I would like to add that I understand very well the difficulties that those who have been brought up on the objectivistic conceptions meet in escaping from them. I understand it because I myself was perplexed for quite a while some time ago (even though I was free from the worst impediment, never having had occasion to submit to a ready-made and presented point of view, but only coming across a number of them while studying various books and works on my own behalf). It was only after having analysed and mulled over the objectivistic conceptions in all possible ways that I arrived, instead, at the firm conviction that they were all irredeemably illusory. It was only after having gone over the finer details and developed, to an extent, the subjectivistic conception, assuring myself that it accounted (in fact, in a perfect and more natural way) for everything that is usually accredited, overhastily, to the fruit of the objectivistic conception, it was only after this difficult and deep work, that I convinced myself, and everything became clear to me. It is certainly possible that these conclusions are wrong; in any case they are undoubtedly open to discussion, and I would appreciate it if they were discussed.

However, a dialogue between the deaf is not a discussion. I think that I am doing my best to understand the arguments of others and to answer them with care (and even with patience when it is a question of repeating things over and over again to refute trivial misunderstandings). It is seldom that I have the pleasure of forming the impression that other people make a similar effort; but, as the Gospel says, 'And why beholdest thou the mote that is in thy brother's eye, but considerest not the beam that is in thine own eye?': if this has happened to me, or is happening to me, I would appreciate it if someone would enlighten me.

1.3.3. One more word (hopefully unnecessary for those who know me): I find it much more enlightening, persuasive, and in the end more essentially serious, to reason by means of paradoxes; to reduce a thesis to absurdity; to make use of images, even light-hearted ones provided they are relevant, rather than to be limited to lifeless manipulations in technical terms, or to heavy and indigestible technical language. It is for this reason that I very much favour the use of colourful and vivid forms of expression, which, hopefully, may turn out to be effective and a little entertaining, making concrete, in a whimsical fashion, those things that would appear dull, boring or insipid and, therefore, inevitably badly understood, if formulated in an abstract way, stiffly or with affected gravity. It is for this reason that I write in such a fashion, and desire to do so; not because of ill-will or lack of respect for other people, or their opinions (even when I judge them wrong). If somebody finds this or that sentence a little too sharp, I beg him to believe in the total absence of intention and animosity, and to accept my apologies as of now.

1.4 Preliminary Clarifications

1.4.1. For the purpose of understanding, the important thing is not the difference in philosophical position on the subject of probability between 'objective' and 'subjective', but rather the resulting reversals of the rôles and meanings of many concepts, and, above all, of what is 'rigorous', both logically and mathematically. It might seem paradoxical but the fact is that the subjectivistic conception distinguishes itself precisely by a more rigorous respect for that which is really objective, and which it calls, therefore, 'objective'.[1] There are cases in which, in order to define a notion, in formulating the problem, or in justifying the reasoning, there exists a choice between an unexceptionable, subjectivistic interpretation and a would-be objectivistic interpretation. The former is made in terms of the opinions or attitudes of a given person; the latter derives from a confused transposition from this opinion to the undefinable complex of objective circumstances that might have contributed to its determination: in such cases there is nothing to do but choose the first alternative. The subjective opinion, as something known by the individual under consideration, is, at least in this sense, something objective and can be a reasonable object of a rigorous study. It is certainly not a sign of greater realism, of greater respect for objectivity, to substitute for it a metaphysical chimera, even if with the laudable intention of calling it 'objective' in order to be able to then claim to be concerned only with objective things.

There might be an objection that we are in a vicious circle, or engaged in a vacuous discussion, since we have not specified what is to be understood by 'objective'. This objection is readily met, however: statements have *objective* meaning if one can say, on the basis of a well-determined observation (which is at least conceptually possible), whether they are either TRUE or FALSE. Within a greater or lesser range of this delimitation a large margin of variation can be tolerated, with one condition – do not *cheat*. To *cheat* means to leave in the statement sufficient confusion and vagueness to allow ambiguity, second-thoughts and equivocations in the ascertainment of its being TRUE or FALSE. This, instead, must always appear simple, neat and definitive.

1.4.2. Statements of this nature, that is the only 'statements' in the true sense of the word, are the object of the *logic of certainty*, that is ordinary logic, which could also be in the form of mathematical logic, or of mathematics. They are also the objects *to which* judgements of probability apply (as long as one does not know whether they are true or false) and are called either *propositions*, if one is thinking more in terms of the expressions in which they are formulated, or *events*, if one is thinking more in terms of the situations and circumstances to which their being true or false corresponds.

On the basis of the considerations now developed, one can better understand the statement made previously, according to which the fundamental difference between the subjectivistic conception and the objectivistic ones is not philosophical but *methodological*. It seems to me that no-one could refute the methodological rigour of the subjectivistic conception: not even an objectivist. He himself, in fact, would have unlimited need of it in trying to expose, in a sensible way, the reasons that would lead him to consider 'philosophically correct' this one, or that one, among the infinitely many possible opinions about the evaluations of probability. To argue against this can only mean,

1 This fact has often been underlined by L.J. Savage (see Kyburg and Smokler (1964), p. 178, and elsewhere).

even though without realizing it, perpetuating profitless discussions and playing on the ambiguities that are deeply rooted in the uncertainty.

At this stage, a few simple examples might give some preliminary clarification of the meaning and compass of the claimed 'methodological rigour' – under the condition, however, that one takes into account the necessarily summary character of these preliminary observations. It is necessary to pay attention to this latter remark to avoid both the acceptance of such observations as exhaustive and the criticism of them that results from assuming that they claim to be exhaustive: one should realize, with good reason, that they are by no means such.

1.5 Some Implications to Note

1.5.1. We proceed to give some examples: to save space, let us denote by 'O' statements often made by *objectivists*, and by 'S' those with which a *subjectivist* (or, anyway, *this author*) would reply.

O: Two events of the same type in identical conditions for all the relevant circumstances are 'identical' and, therefore, necessarily have the same probability.[2]

S: Two distinct events are always different, by virtue of an infinite number of circumstances (otherwise how would it be possible to distinguish them?!). They are equally probable (for an individual) if – and so far as – he judges them as such (possibly by judging the differences to be irrelevant in the sense that they do not influence his judgement). An even more fundamental objection should be added: the judgement about the probability of an event depends not only on the event (or on the person) but also on the state of information. This is occasionally recalled, but more often forgotten, by many objectivists.

O: Two events are (stochastically) independent[3] if the occurrence of one does not influence the probability of the other.

S: I would say instead: by definition, two events are such (for an individual) if the knowledge of the outcome of one does not make him change the evaluation of probability for the other.

O: Let us suppose *by hypothesis* that these events are equally probable, for example with probability $p = \frac{1}{2}$, and independent, and so on.

S: It is meaningless to consider as an 'hypothesis' something that is not an objective statement. A statement about probability (the one given in the example or any other one whatsoever) either *is* the evaluation of probabilities (those of the speaker or of someone else), in which case there is nothing to do but simply register the fact, or it is nothing.

O: These events are independent and all have the same probability which is, however, 'unknown'.

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2 The objectivists often use the word event in a generic sense also, using '*trials*' (or '*repetitions*') *of the same* '*event*' to mean *single events*, '*identical*' *or* '*similar*'. From time to time we will say '*trials*' (or '*repetitions*') *of a phenomenon*, always meaning by event a single event. It is not simply a question of terminology, however: we use 'phenomenon' because we do not give this word any technical meaning; by saying 'trials of a phenomenon' one may allude to some exterior analogy but one does not mean to assume anything that would imply either equal probability, or independence, or anything else of probabilistic relevance.

3 Among events, random quantities, or random entities in general, it is possible to have various relations termed 'independence' (linear, logical, stochastic); it is better to be specific if there is any risk of ambiguity.

S: This formulation is a nonsense in the same sense as the preceding one but to a greater extent. By interpreting the underlying intention (which, as an intention, is reasonable) one can translate it (see Chapter 11) into a completely different formulation, 'exchangeability', in which we do *not* have independence, the probabilities are *known*, and vary, precisely, in depending only on the number of successes and failures of which one has information.

One might continue in this fashion, and it could be said that almost the whole of what follows will be, more or less implicitly, a continuation of this same discussion. Rather, let us see, by gathering together the common factors, the essential element in all these contrapositions.

1.5.2. For the subjectivist everything is clear and rigorous when he is expressing something about somebody's evaluation of probabilities; an evaluation which is, simply, what it is. For that somebody, it will have motivations that we might, or might not, know; share, or not share; judge[4] more or less reasonable, and that might be more or less 'close' to those of a few, or many, or all people. All this can be interesting, but it does not alter anything. To express this in a better way: all these things matter in so far as they determined that unique thing that matters, and that is the evaluation of probability to which, in the end, they have given rise.

From the theoretical, mathematical point of view, even the fact that the evaluation of probability expresses somebody's opinion is then irrelevant. It is purely a question of studying it and saying whether it is coherent or not; that is whether it is free of, or affected by, intrinsic contradictions. In the same way, in the logic of certainty one ascertains the correctness of the deductions but not the accuracy of the factual data assumed as premises.

1.5.3. Instead, the objectivist would like to ignore the evaluations, actual or hypothetical, and go back to the circumstances that might serve as a basis for motivations which would lead to evaluations. Not being able to invent methods of synthesis comparable in power and insight to those of the human intuition, nor to construct miraculous robots capable of such, he contents himself, willingly, with simplistic schematizations of very simple cases based on neglecting all knowledge except a unique element which lends itself to utilization in the crudest way.

A further consequence is the following. The subjectivist, who knows how much caution is necessary in order to remain within the bounds of realism, will exercise great care in not going far beyond the consideration of cases immediately at hand and directly interesting. The objectivist, who substitutes the abstraction of schematized models for the changing and transient reality, cannot resist the opposite temptation. Instead of engaging himself, even though in a probabilistic sense (the only one which is valid), in saying something about the specific case of interest, he prefers to 'race on ahead', occupying himself with the asymptotic problems of a large number of cases, or even playing around with illusory problems, contemplating infinite cases where he can try, without any risk, to pass off his results as 'certain predictions'.[5]

4 With a judgment which is 'subjective squared': our subjective judgment regarding the subjective judgment of others.
5 Concerning the different senses in which we use the terms 'prevision' and 'prediction', see Chapter 3 (at the beginning and then in various places, in particular 3.7.3).

1.6 Implications for the Mathematical Formulation

1.6.1. From these conceptual contrapositions there follows, amongst other things, an analogous contraposition in the way in which the mathematical formulation is conceived. The subjectivistic way is the one that it seems appropriate to call 'natural': it is possible to evaluate the probability over any set of events whatsoever; those for which it serves a purpose, or is of interest, to evaluate it; there is nothing further to be said. The objectivistic way (and also the way most congenial to contemporary mathematicians, independently of the conception adopted regarding probability) consists in requiring, as an obligatory starting point, a mathematical structure much more formidable, complete and complicated than necessary (and than it is, in general, reasonable to regard as conceivable).

1.6.2. Concerning a known evaluation of probability, over any set of events whatsoever, and interpretable as the opinion of an individual, real or hypothetical, we can only judge whether, or not, it is *coherent*.[6] If it is not, the evaluator, when made aware of it, should modify it in order to make it coherent. In the same way, if someone claimed to have measured the sides and area of a rectangle and found 3 m, 5 m and 12 m^2, we, even without being entitled, or having the inclination, to enter into the merits of the question, or to discuss the individual measurements, would draw his attention to the fact that at least one of them is wrong, since it is not true that $3 \times 5 = 12$.

Such a condition of coherence should, therefore, be *the weakest one* if we want it to be the strongest in terms of absolute validity. In fact, *it must only exclude the absolutely inadmissible evaluations*; that is those that one cannot help but judge contradictory (in a sense that we shall see later).

Such a condition, as we shall see, reduces to *finite additivity* (and *non-negativity*). It is not admissible to make it more restrictive (unless it turns out to be necessary if we discover the preceding statement to be wrong); it would make us exclude, erroneously, admissible evaluations.

1.6.3. What the objectivistic, or the purely formalistic, conceptions generally postulate is, instead, that countable additivity holds (as for Borel or Lebesgue measure), and that the field over which the probability is defined be the whole of a Boolean algebra. From the subjectivistic point of view this is both too much and too little: according to what serves the purpose and is of interest, one could limit oneself to much less, or even go further. One could attribute probabilities, finitely but not countably additive, to all, and only, those events that it is convenient to admit into the formulation of a problem and into the arguments required for its solution. One might also go from one extreme to the other: referring to the analogy of events and probability with sets and measure, it might, at times, be convenient to limit oneself to thinking of a measure as defined on certain simple sets (like the intervals), or even on certain sets but not their intersections (for instance, for 'vertical' and 'horizontal' 'stripes' in the (x,y)-plane ($x' \leq x < x''$, $y' \leq y < y''$) but not on the rectangles); and, at other times, to think of it instead as extended to all the sets that the above-mentioned convention would exclude (like the 'non-Lebesgue-measureable sets').

6 See Chapter 3.

1.6.4. In a more general sense, it seems that many of the current conceptions consider as a success the introduction of mathematical methods so powerful, or of tricks of formulation so slick, that they permit the derivation of a uniquely determined answer to a problem even when, due to the insufficiency of the data, it is indeterminate. A capable geometer in order to conform to this aspiration would have to invent a formula for calculating the area of a triangle given two sides.

> Attempts of this kind are to be found in abundance, mainly in the field of statistical induction (see some remarks further on in this Introduction, 1.7.6).
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> In the present case, the defect is somewhat hidden and consists in the following distinction between the two cases of *measure* and of *probability*.
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> To extend a *mathematical* notion (measure) from one field (Jordan–Peano) to another (Borel–Lebesgue) is a question of convention. If, however, a notion (like probability) *already has a meaning* (for each event, at least potentially, even if not already evaluated), one cannot give it a value by conventional extension of the probabilities already evaluated *except for the case in which it turns out to be the unique one compatible with them by virtue of the sole conditions of coherence* (conditions pertaining to the meaning of probability, not to motives of a mathematical nature). The same would happen if it were a question of a physical quantity like mass. If one thought of being able to give meaning to the notion of 'mass belonging to any set of points of a body' (for instance those with rational coordinates), in the sense that it were, at least conceptually, possible to isolate such a mass and weigh it, then it would be legitimate, when referring to it, to talk about everything that can be deduced about it by mathematical properties that translate necessary physical properties, and only such things. To say something more (and in particular to give it a unique value when such properties leave the value indeterminate between certain limits), by means of the introduction of arbitrary mathematical conventions, would be unjustified, and therefore inadmissible.

1.7 An Outline of the 'Introductory Treatment'

1.7.1. The reader must feel as though he has been plunged alternately into baths of hot and cold water: in Section 1.5 he encountered the contraposed examples of the conceptual formulation, presented either as meaningful or as meaningless; in Section 1.6 the mathematical formulations, presented either as suitable or as academic. Following this, a simple and ordered presentation of the topics that will follow may provide a suitable relaxation, and might even induce a return to the preceding 'baths' in order, with a greater knowledge of the motives, to soak up some further meaning.

1.7.2. In Chapter 2 we will *not* talk of probability. Since we wish to make absolutely clear the distinction between the subjective character of the notion of probability and the objective character of the elements (events, or any random entities whatsoever) to which it refers, we will first treat only these entities. In other words, we will deal with the preliminary logic of certainty where there exist only:

- TRUE and FALSE as final answers;
- CERTAIN and IMPOSSIBLE and POSSIBLE as alternatives, with respect to the present knowledge of each individual.

In this way, the range of uncertainty, that is of what is not known, will emerge in outline. This is the framework into which the (subjective) notion of probability will be introduced as an indispensable tool for our orientation and decision making.

The random events, random quantities and any other random entities, will already be defined, however, before we enter the domain of probability, and they will simply be events, quantities, entities, well-defined but with no particular features except the fact of not being known by a certain individual. For any individual who does not know the value of a quantity X, there will be, instead of a unique *certain* value, two, or several, or infinitely many, *possible* values of X. They depend on his degree of ignorance and are, therefore, relative to his state of information; nevertheless, they are objective because they do not depend on his opinions but only on these objective circumstances.

1.7.3. Up until now the consideration of uncertainty has been limited to the negative aspect of *nonknowledge*. In Chapter 3 we will see how the need arises, as natural and appropriate, to integrate this aspect with the positive aspect (albeit weak and temporary while awaiting the information that would give it certainty) given by the evaluation of probabilities. To any event in which we have an interest, we are accustomed to attributing, perhaps vaguely and unconsciously, a probability: if we are sufficiently interested we may try to evaluate it with some care. This implies introspection in depth by weighing each element of judgment and controlling the coherence by means of other evaluations made with equal accuracy. In this way, each event can be assigned a probability, and each random quantity or entity a distribution of probability, as an expression of the attitude of the individual under consideration.

Let us note at once a few of the points that arise.

Others, in speaking of a random quantity, assume a probability distribution as already attached to it. To adopt a different concept is not only a consequence of the subjectivistic formulation, according to which the distribution can vary from person to person, but also of the unavoidable fact that the distribution varies with the information (a fact which, in any case, makes the usual terminology inappropriate).

Another thing that might usefully be mentioned now is that the conditions of coherence will turn out to be particularly simplified and clarified by means of a simple device for simultaneously handling events and random quantities (or entities of any linear space whatever). Putting the logical values 'True' and 'False' equal to the numbers '1' and '0', an event is a random quantity that can assume these two values: the function $\mathbf{P}(X)$, which for X = event gives its probability, is, for arbitrary X, the 'prevision' of X (i.e. in the usual terminology, the mathematical expectation).

The use of this *arithmetic* interpretation of the events, preferable to, but not excluding, the set-theoretic interpretation, has its utility and motivation, as will be seen. The essential fact is that the *linearity* of the arithmetic interpretation plays a fundamental rôle (which is, in general, kept in the background), whereas the structure of the Boolean algebra enters rather indirectly.

1.7.4. After having extended these considerations, in Chapter 4, to the case of conditional probabilities and previsions (encountering the notions of stochastic independence and correlation), we will, in Chapter 5, dwell upon the evaluation of probabilities. The notions previously established will allow us not only to apply the instruments for this evaluation, but also to relate them to the usual criteria, inspired by partial, objectivistic 'definitions'. We will see that the subjectivistic formulation, far from making the valid elements in the ideas underlying these criteria redundant, allows the best and most complete use of them, checking and adapting, case by case, the importance of each of them. In contrast to the usual, and rather crude, procedure, which consists of the mechanical and one-sided application of this or that criterion, the proposed formulation allows one to behave in conformity with what the miraculous robot, evoked in Section 1.5.3, would do.

1.7.5. Chapters 6–10 extend to give a panoramic vision of the field of problems with which the calculus of probabilities is concerned. Of course, it is a question of compromising between the desire to present a relatively complete overall view and the desire to concentrate attention on a small number of concepts, problems and methods, whose rôle is fundamental both in the first group of ideas, to be given straightaway, and, even more, in further developments, which, here, we can at most give a glimpse of.

Also in these chapters, which in themselves are more concerned with content than critical appraisal, there are aspects and, here and there, observations and digressions that are relevant from the conceptual angle. It would be inappropriate to make detailed mention of them but, as examples, we could quote the more careful analysis of what the knowledge of the distribution function says, or does not say (also in connection with the 'possible' values), and of the meaning of 'stochastic independence' (between random quantities), expressed by means of the distribution function.

1.7.6. The last two chapters, 11 and 12, deal briefly with the problems of induction (or inference) and their applications, which constitute mathematical statistics. Here we encounter anew the conceptual questions connected with the subjective conception, which, of course, bases all inference on the Bayesian procedure (from Thomas Bayes,[7] 1763). In this way, the theory and the applications come to have a unified and coherent foundation: it is simply a question of starting from the evaluation of the initial probabilities (i.e. before acquiring new information – by observation, experiment, or whatever) and then bringing them up to date on the basis of this new information, thus obtaining the final probabilities (i.e. those on which to base oneself after acquiring such information).

The objectivistic theories, in seeking to eschew the evaluation and use of 'initial probabilities', lack an indispensable element for proceeding in a sensible way and appeal to a variety of empirical methods, often invented *ad hoc* for particular cases. We shall use the term '*Adhockeries*', following Good[8] (1965) who coined this apt expression, for the methods, criteria and procedures that, instead of following the path of the logical formulation, try to answer particular problems by means of particular tricks (which are sometimes rather contrived).

7 One must be careful not to confuse Bayes' *theorem* (which is a simple corollary of the theorem of compound probabilities) with Bayes' *postulate* (which assumes the uniform distribution as a representation of 'knowing nothing'). Criticisms of the latter, often mistakenly directed against the former, are not therefore valid as criticisms of the position adopted here.

8 Good's position is less radical than I supposed when I interpreted 'Adhockery' as having a derogatory connotation. I gathered this from his talk at the Salzburg Colloquium, and commented to this effect in an Addendum to the paper I delivered there; *Synthese* **20** (1969), 2–16: 'According to it, "adhockeries" ought not to be rejected outright; their use may sometimes be an acceptable substitute for a more systematic approach. I can agree with this only if – and in so far as – such a method is justifiable as an approximate version of the correct (i.e. Bayesian) approach. (Then it is no longer a mere "adhockery".)'

1.8 A Few Words about the 'Critical' Appendix

1.8.1. Many of the conceptual questions are, unfortunately, inexhaustible if one wishes to examine them thoroughly; and the worst thing is that, often, they are also rather boring unless one has a special interest in them.

A work that is intended to clarify a particular conceptual point of view cannot do without this kind of analysis in depth, but it certainly seems appropriate to avoid weighing down the text more than is necessary to meet the needs of an ordinary reader who desires to arrive at an overall view. For this reason, the most systematic and detailed critical considerations have been postponed to an Appendix. This is intended as a reassurance that there is no obligation to read it in order to understand what follows, nor to make the conclusions meaningful. This does not mean, however, that it is a question of abstruse and sophisticated matters being set aside for a few specialists and not to be read by others. It is a question of further consideration of different points that might appear interesting and difficult, to a greater or lesser extent, but which might always improve, in a meaningful and useful, though not indispensable, way, the awareness of certain questions and difficulties, and of the motives which inspire different attitudes towards them.

1.8.2. In any case, one should point out that it is a question of an attempt to view, in a unified fashion, a group of topics that are in general considered separately, each by specialists in a single field, paying little or no attention to what is being done in other fields. Notwithstanding the many gaps or uncertainties, and the many imperfections (and maybe precisely also for the attention it may attract to them), I think that such an attempt should turn out to be useful.

Among other things, we have tried to insert into the framework of the difficulties associated with the 'verifiability' of events in general, the question of 'complementarity' that arose in quantum physics. The answer is the one already indicated, in a summary fashion, elsewhere (de Finetti, 1959), and coinciding with that of B.O. Koopman (1957), but the analysis has been pursued in depth and related to the points of view of other authors as far as possible (given the margin of uncertainty in the interpretation of the thought of those consulted, and the impossibility of spending more time on this topic in attempting to become familiar with others).

1.8.3. Various other questions that are discussed extensively in the Appendix, are currently objects of discussion in various places: for instance, the relationships between possibility and tautology seem to be attracting the attention of philosophers (the intervention of Hacking at a recent meeting, Chicago 1967); while the critical questions about the mathematical axioms of the calculus of probability (in the sense, to be understood, of making it a theory strictly identical to measure theory, or with appropriate variations) are always a subject of debate.

Apart from the points of view on separate questions, the Appendix will also have as a main motive the proposal to model the mathematical formulation on the analysis of the actual needs of the substantive interpretation. Moreover, to do so with the greatest respect for 'realism', which the inevitable degree of idealization must purify just a little, but must never overwhelm or distort, neither for analytical convenience, nor for any other reason.

1.9 Other Remarks

1.9.1. It seems appropriate here to draw attention also to some further aspects, all secondary, even if only to underline the importance that attaches, in my opinion, to 'secondary' things.

One characteristic of the calculus of probability is that mathematical results are often automatically obtained because their probabilistic interpretations are obvious. In all these cases I think it is much more effective and instructive to consider as their proofs these latter expressive interpretations, and as formal verifications their translation into technical details (to be omitted, or left to the reader). This seems to me to be the best way of realizing the ideal expressed in the maxim that Chisini[9] often repeated: '*mathematics is the art which teaches one how not to make calculations*'.

It is incredible how many things are regularly presented in a heavy and obscure fashion, arriving at the result through a labyrinth of calculations that make one lose sight of the meaning, whereas simple, synthetic considerations would be sufficient to reveal that, for those not wishing to behave as if handcuffed or blindfolded, results and meaning are at hand, staring one in the face.

On numerous occasions one sees very long calculations made in order to prove results that are either wrong or obvious. The latter case is the more serious, without any extenuating circumstances, since it implies lack of realization that the conclusion was obvious, even after having seen it. On the other hand, failing to get the result due to a casual mistake merits only half a reproach, since the lack of realization only applies before starting the calculations.

Instead, it is often sufficient to remark that two formulae are necessarily identical for the simple reason that they express the same thing in different ways, since they provide the result of the same process starting from different properties which characterize it, or for other similar reasons. Problems that can, more or less 'surprisingly', be reduced to synthetic arguments arise frequently in, amongst other things, questions connected with random processes (ranging from the game of Heads and Tails to cases involving properties of characteristic functions etc.). Often, on the other hand, it is an appropriate geometric representation that clarifies the situation and also suggests, without calculations and without any doubts, the solution in formulae.

1.9.2. In addition, however, there are even more secondary things which have their importance. These I would like to explain with a few examples so that it does not seem that some small innovation, perhaps in notation or terminology, has been introduced just for the sake of changing things, instead of with reluctance, overcome by the realization that this was the only way of getting rid of many useless complications.

The very simple device, from which most of the others derive, is that mentioned already in 1.7.3. We identify an event E with the random quantity, commonly called the 'indicator of E', which takes values 1 or 0 according to whether E is true or false. Not only can one operate arithmetically on the events (the arithmetic sum of many events = the

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9 Oscar Chisini, a distinguished and gifted pupil of Federigo Enriques, was Professor at the University of Milan where the author attended his course on Advanced Geometry. Chisini's generalized definition of the concept of mean (see Chapter 2, Section 2.9) came about as a result of his occasionally being concerned with this notion in connection with secondary-school examinations.

number of successes; $E - p$ = the gain from a bet for a person who stakes a sum p in order to receive a sum 1 if E occurs etc.) but one operates with a unique symbol **P** in order to denote both probability and prevision (or 'mathematical expectation'), thus avoiding duplication. The 'theorem' $\mathbf{M}(I_E) = \mathbf{P}(E)$, 'the mathematical expectation of the indicator of an event is equal to the probability of the same event', *is rendered superfluous* (it could only be expressed by $\mathbf{P}(E) = \mathbf{P}\ (E)$!).

1.9.3. The identification TRUE = 1, FALSE = 0 is also very useful as a simple conventional device for denoting, in a straightforward and synthetic way, many mathematical expressions that usually require additional verbal explanation. Applying the same identifications to formulae expressing conditions, for instance, interpreting '$(0 \leqslant x \leqslant 1)$' as a symbol with value 1 for x between 0 and 1, where the inequality is *true*, and value 0 outside, where it is *false*, one can simply write expressions of the type

$$f(x) = g(x) \qquad (0 \leqslant x \leqslant 1)$$

(and more complicated forms), which otherwise require verbal explanations, like 'the function $f(x)$ which coincides with $g(x)$ for $0 \leqslant x \leqslant 1$ and is zero elsewhere', or writing in the cumbersome form

$$f(x) = \begin{cases} = 0 & \text{for} \quad x < 0, \\ = g(x) & \text{for} \quad 0 \leqslant x \leqslant 1, \\ = 0 & \text{for} \quad x > 1. \end{cases}$$

It is easy to imagine many cases in which the utility of such a convention is much greater, but I think it is difficult to realize the number and variety of such cases (I am often surprised by new, important applications not previously foreseen).

1.9.4. Other simplifications of this kind, which can sometimes be used in conjunction with the above, result from a parallel (or dual) extension of the Boolean operations to the field of real numbers, coinciding, for the values 0 and 1, with the usual meaning for the events. This natural and meaningful extension will also reveal its utility in many applications[10] (see Chapter 2, Sections 2.5 and 2.11).

1.9.5. A small innovation in notation is that of denoting the three most important types of convergence in the probabilistic field by:
symbol: type of convergence:

$\overset{<}{\to}$	weak	(in probability)	(in measure)
$\overset{>}{\to}$	strong	(almost certain)	(almost everywhere)
$\overset{\bullet}{\to}$	quadratic	(in mean-square)	(in mean (quadratic))

10 The advantages of these two conventions (0 and 1 for True–False, and $\vee$ and $\wedge$ among numbers) are illustrated, somewhat systematically and with concise examples, in a paper in the volume in honour of O. Onicescu (75th birthday): 'Revue roumaine de mathématiques pures et appliquées', Bucharest (1967), **XII**, 9, 1227–1233. An English translation of this appears in B. de Finetti, *Probability, Induction and Statistics*, John Wiley & Sons (1972).

(this could also have value in function theory). The innovation seems to me appropriate not only to avoid abbreviations which differ from language to language but also for greater clarity, avoiding the typographic composition and deciphering of symbols that are either cumbersome or unreadable.

1.9.6. Another device which we will introduce with the intention of simplifying the notation does not have a direct relationship with the calculus of probability. For this reason we were even more hesitant to introduce it, but finally realized that without such a remedy there remained simple and necessary things which could not be expressed in a decently straightforward way.

The most essential is the device of obtaining symbols indicating functions, substituting for the variable (in any expression whatever) a 'place-name' symbol: as such $\square$ would seem suitable; it also suggests something which awaits filling in. The scope is the same as obtained by Peano by means of the notation '$|x$', 'varying x', which, applied for instance to the expression $(x \sin x^2 + \sqrt{(3-x)})/\log(2+\cos x)$ gives $f = \{[(x \sin x^2 + \sqrt{(3-x)})/\log(2+\cos x)]|x\}$, where f is the symbol of the function such that $f(x)$ gives the expression above, and $f(y), f(ax^2+b), f(\mathrm{e}^z)$, … is the same thing in which at each place where an x is found we substitute y, or ax^2+b, or e^z, or whatever. This notation, however, does not lend itself to many cases where it would be required, and where, instead, the notation which puts the 'place-name' for the variable, which is left at our disposal,[11] is very useful. In the preceding example one would write

$$f = \frac{\square \sin \square^2 + \sqrt{\left(3 - \square\right)}}{\log\left(2 + \cos \square\right)}$$

and to denote $f(x), f(y), f(ax^2+b), f(\mathrm{e}^z)$, it would suffice to write on the right, within parentheses, (), the desired variable.

The greatest utility is perhaps obtained in the simplest cases: for instance, in order to denote by $\square, \square^2, \square^{-1}$ the identity function, $f(x) = x$, or the quadratic, $f(x) = x^2$, or the reciprocal, $f(x) = 1/x$, when the f must be denoted as the argument in a functional. For example, $F(\square)$, $F(\square^2)$, might indicate the first and second moments of a distribution F (according to the conventions of which we shall speak in Chapter 6), and then for any others, $F(\square^n)$, $F(|\square^n|)$ and so on.

1.9.7. Finally, a secondary device is that of consistently denoting by K any multiplicative constant whatever and, if necessary, indicating its expression immediately afterwards, instead of writing it directly, in extensive form, in the formulae. Otherwise, it often happens that a function, of x say, has a rather complicated appearance and each symbol, even those in small print or in the exponents and so on, must be deciphered with care in order to see where x appears. Often one subsequently realizes that the function is very simple and that the complexity of the expression derives solely from having expressed the constant in extensive form. We may have a normalizing constant which, at times, could even be ignored because it automatically disappears in the sequel, or can be calculated more

11 In the case of many variables (for instance three) one could easily use the same device, putting in their places different 'placenames'; for example, $\square_1, \square_2, \square_3$, with the understanding that $f(x, y, z)$ or $f(5, -\frac{1}{2}, 0)$ or $f(x+y, -\frac{1}{2}x, 1-2y)$ etc., is what one obtains putting the 1st or 2nd or 3rd elements of the triple in the places indicated by the three 'place names' with indices 1, 2, 3.

easily from the final formula. At times, in fact, it will be left as a 'reminder' of the existence of an omitted multiplicative factor, which will always be indicated by K, even if the value might change at each step: the reader should make careful note of this remark.

1.10 Some Remarks on Terminology

1.10.1. It is without doubt unreasonable, and rather annoying, to dwell at length on questions of terminology; on the other hand, a dual purpose glossary would be useful and instructive. In the first place, it could improve on a simple alphabetical index in aiding those who forget a definition, or remember it only vaguely; secondly, it could explain the motivation behind the choice, or sometimes the creation, of certain terms, or the fixing of certain conventions for their use.[12] For those interested, such an explanation would also provide an account of the wherefores of the choices. Such a glossary would, however, be out of place here and, in any case, the unusual terms are few and they will be explained as and when they arise.

1.10.2. More importantly, attention must be drawn to some generic remarks, like paying attention to the nuances of divergences of interpretation, which depend on differences in conception. The main one, that of registering that an *event* is always a single case, has already been underlined (Section 1.5.1); the same remark holds for a *random quantity* (Section 1.7.2), and for every kind of 'random entity'. Two clarifications of terminology are appropriate at this juncture: the first to explain why I do not use the term 'variable'; the second to explain the different uses of the terms 'chance', 'random' and 'stochastic'.

To say 'random (or "chance") variable' might suggest that we are thinking of the 'statistical' interpretation in which one thinks of many 'trials' in which the random quantity can *vary*, assuming different values from trial to trial: this is contrary to our way of understanding the problem. Others might think that, even if it is a question of a unique well-determined value, it is 'variable' for one who does not know it, in the sense that it may assume any one of the values 'possible' for him. This does not appear, however, to be a happy nomenclature, and, even less, does it appear to be necessary. In addition, if one wanted to adopt it, it would be logical to do so always, by saying: random variable numbers, random variable vectors, random variable points, random variable matrices, random variable distributions, random variable functions, …, random variable events, and not saying random vector, random point, random matrix, random distribution, random function, random event, and only in the case of numbers not to call it number any more, but variable.

With regard to the three terms – '*chance*', '*random*', '*stochastic*' – there are no real problems: it is simply the convenience of avoiding indiscriminate usage by supporting the consolidation of a tendency that seems to me already present but not, as far as I know, expressly stated. Specifically, it seems to me preferable to use, systematically:

- '*Random*' for that which is the *object* of the theory of probability (as in the preceding cases); I will, therefore, say random process, not stochastic process.
- '*Stochastic*' for that which is valid 'in the sense of the calculus of probability': for instance, stochastic independence, stochastic convergence, stochastic integral; more

12 A very good example would be that of the *Dictionary* at the end of the 'book' by Bourbaki (1939).

generally, stochastic property, stochastic models, stochastic interpretation, stochastic laws; or also, stochastic matrix, stochastic distribution,[13] and so on.

- '*Chance*' is perhaps better reserved for less technical use: in the familiar sense of 'by chance', 'not for a known or imaginable reason', or (but in this case we should give notice of the fact) in the sense of 'with equal probability' as in 'chance drawings from an urn', 'chance subdivision', and similar examples.

1.10.3. Special mention should be made of what is perhaps the important change in terminology: *prevision* in place of mathematical expectation, or expected value and so on. Firstly, all these other nomenclatures have, taken literally, a rather inappropriate meaning and often, through the word 'expectation', convey something old-fashioned and humorous (particularly in French and Italian, where 'espérance' and 'speranza' primarily mean 'hope'!). In any case, it is inconvenient that the expression of such a fundamental notion, so often repeated, should require two words. Above all, however, there was another reason: to use a term beginning with P, since the symbol $\mathbf{P}$ (from what we have said and recalled) then serves for that unique notion which in general we call *prevision*[14] and, in the case of events, also *probability*.[15]

1.11 The Tyranny of Language

All the devices of notation and terminology and all the clarifications of the interpretations are not sufficient, however, to eliminate the fundamental obstacle to a clear and simple explication, adequate for conceptual needs: they can at most serve as palliatives, or to eliminate blemishes.

That fundamental obstacle is the difficulty of escaping from the tyranny of everyday language, whose viscosity often obliges us to adopt phrases conforming to current usage instead of meditating on more apt, although more difficult, versions. We all continue to say 'the sun rises' and I would not know which phrase to use in order not to seem an anachronistic follower of the Ptolemaic system. Fortunately the suspicion does not even enter one's mind because nobody quibbles about the literal meaning of this phrase.

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13 The case of *matrices* and *distributions* illustrates the difference well. A random matrix is a matrix whose entries are random quantities; a stochastic matrix (in the theory of Markov chains) is the matrix of 'transition probabilities'; i.e. well-determined quantities that define the random process. A random distribution (well-defined but not known) is that of the population in a future census, according to age, or that of the measures that will be obtained in n observations that are to be made; a stochastic distribution would mean distribution of probability (but it is not used, nor would it be useful).

14 *Translators' note*. We have used *prevision* rather than *foresight* (as in Kyburg and Smokier, p. 93) precisely for the reasons given in 1.10.3.

15 SIn almost all languages other than Italian, the letter E is unobjectionable, and often a single word is sufficient: Expectation (English), Erwartung (German), Espérance mathématique (French), etc. However, the use of E is inconvenient because this is often used to denote an event and, in any case, it can hardly remain if one seeks to unify it with $\mathbf{P}$. It is difficult to foresee whether this unification will command widespread support and lead to a search for terms with initial letter P in other languages (*see footnote above*), or other solutions. We say this to note that the proposed modification causes little difficulty in Italy, not only because of the existence and appropriateness of the term 'Previsione' but also because the international symbol E has not been adopted there.

In the present exposition we shall often, for the sake of brevity, use incorrect language, saying, for example: 'let the probability of *E be* $= \frac{1}{2}$', 'let the events *A* and *B* *be* (stochastically) independent', 'let the probability distribution of a random quantity *X be* normal', and so on. This is incorrect, or, more accurately, it is meaningless, unless we mean that it is a question of an abbreviated form to be completed by 'according to the opinion of the individual (for example You) with whom we are concerned and who, we suppose, desires to remain coherent'. The latter should be understood as the constant, though not always explicitly stated, intention and interpretation of the present author.

This is stated, and explicitly repeated, wherever it seems necessary, due to the introduction of new topics, or for the examination of delicate points—perhaps even too insistently, with the risk, and near certainty, of irritating the reader. Even so, notwithstanding the present remark (even imagining that it has been read), I am afraid that the very same reader when confronted with phrases like those we quoted, instead of understanding implicitly those things necessary in order to interpret them correctly, could have the illusion of being in an oasis – in the 'enchanted garden' of the objectivists (as noted at the end of Chapter 7, 7.5.7) – where these phrases could constitute 'statements' or 'hypotheses' in an objective sense.

In our case, in fact, the consequences of the pitfalls of the language are much more serious than they are in relationship to the Copernican system, where, apart from the strong psychological impediments due to man's egocentric geocentrism, it was simply a question of choosing between two objective models, differing only in the reference system. Much more serious is the reluctance to abandon the inveterate tendency of savages to objectivize and mythologize everything;[16] a tendency that, unfortunately, has been, and is, favoured by many more philosophers than have struggled to free us from it.[17] This has been acutely remarked, and precisely with reference to probability, by Harold Jeffreys:[18]

> 'Realism has the advantage that language has been created by realists, and mostly very naïve ones at that; we have enormous possibilities of describing the inferred properties of objects, but very meagre ones of describing the directly known ones of sensations.'

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16 The main responsibility for the objectivizationistic fetters inflicted on thought by everyday language rests with the verb 'to be' or 'to exist', and this is why we drew attention to it in the exemplifying sentences by the use of italics. From it derives the swarm of pseudoproblems from 'to be or not to be', to 'cogito ergo sum', from the existence of the 'cosmic ether' to that of 'philosophical dogmas'.

17 This is what distinguishes the acute minds, who enlivened thought and stimulated its progress, from the narrow-minded spirits who mortified it and tried to mummify it: those who took every achievement as the starting point to presage further achievement, or those, on the contrary, who had the presumption to use it as a starting point on which to be able to base a definitive systematization.

For the two types, the qualification given by R. von Mises seems appropriate (see *Selected Papers*, Vol. II, p. 544): 'great thinkers' (like Socrates and Hume) and 'school philosophers' (like Plato and Kant).

18 Jeffreys, a geophysicist, who as such was led to occupy himself deeply with the foundations of probability, holds a position similar in many aspects to the subjectivistic one. The quotation is taken from H. Jeffreys, *Theory of Probability*, Oxford (1939), p. 394.

1.12 References

1.12.1. We intend to limit the present references to a bare minimum. The reader who wishes to study the topics on his own can easily discover elsewhere numerous books and references to books. Here the plan is simply to suggest the way which I consider most appropriate for the reader who would like to delve more deeply into certain topics, beyond the level reached here, without the inconvenience of passing from one book to another, with differences in notation, terminology and degree of difficulty.

1.12.2. The most suitable book for consultation according to this plan is, in my opinion, that of Feller:

Willy Feller, *An Introduction to Probability Theory and its Applications*, in two volumes: I (1950) (2nd and 3rd edn, more and more enriched and perfected, in 1956 and 1968); II (1966); John Wiley & Sons, Inc., New York.

The treatment, although being on a high level and as rigorous as is required by the topic, is not difficult to read and consult. This is due to the care taken in abolishing useless complications, in making, as far as possible, the various chapters independent of each other while facilitating the links with cross-references, and in maintaining a constant interplay between theoretical questions and expressive examples. Further discussion may be found in a review of it, by the present author, in *Statistica*, **26**, 2 (1966), 526–528.

The point of view is not subjectivistic, but the mainly mathematical character of the treatment makes differences of conceptual formulation relatively unobtrusive.

1.12.3. For the topics in which such differences are more important, that is those of inference and mathematical statistics (Chapter 11 and Chapter 12), there exists another work that is inspired by the concepts we follow here. Such topics are not expressly treated in Feller and thus, with particular reference to these aspects, we recommend the following work, and above all the second volume:

Dennis V. Lindley, *Introduction to Probability and Statistics from a Bayesian viewpoint*, in two volumes: I, *Probability*; II, *Inference*; Cambridge University Press (1965).

Complementing the present work with those of Feller and Lindley would undoubtedly mean to learn much more, and better, than from this work alone, except in one aspect; that is the coherent continuation of the work of conceptual and mathematical revision in conformity with the criteria and needs already summarily presented in this introductory chapter.

The above-mentioned volumes are also rich in interesting examples and exercises, varied in nature and difficulty.