Universidade de Aveiro
Departamento de Matemática

Cálculo II - Agrupamento 4 **2021/22**

Folha 5: *Transformadas de Laplace e aplicações às EDO*

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1. Para cada uma das funções seguintes, determine $F(s) = \mathcal{L}\{f(t)\}$:

   (a) $f(t) = 2\,\mathrm{sen}\,(3t) + t - 5e^{-t}$;
   (b) $f(t) = e^{2t}\cos(5t)$;
   (c) $f(t) = te^{3t}$;
   (d) $f(t) = \pi - 5e^{-t}t^{10}$;
   (e) $f(t) = (3t-1)\,\mathrm{sen}\,t$;
   (f) $f(t) = (1 - H_\pi(t))\,\mathrm{sen}\,t$;
   (g) $f(t) = (t-2)^2 e^{2(t-2)} H_2(t)$.

2. Para cada uma das funções seguintes, determine $\mathcal{L}^{-1}\{F(s)\}$:

   (a) $F(s) = \dfrac{2s}{s^2-9}$; (b) $F(s) = \dfrac{4}{s^7}$; (c) $F(s) = \dfrac{1}{s^2+6s+9}$;
   (d) $F(s) = \dfrac{1}{s^2+s-2}$; (e) $F(s) = \dfrac{1}{s^2+4s+6}$; (f) $F(s) = \dfrac{3s-1}{s^2-4s+13}$;
   (g) $F(s) = \dfrac{4s+e^{-s}}{s^2+s-2}$; (h) $F(s) = \dfrac{s}{(s^2+4)^2}$.

3. Calcule o valor dos seguintes integrais impróprios, usando transformadas de Laplace:
   (a) $\displaystyle\int_0^{+\infty} t^{10}\, e^{-2t} dt$; (b) $\displaystyle\int_0^{+\infty} e^{-3t} t\,\mathrm{sen}\,t\, dt$.

4. Seja $f : \mathbb{R} \to \mathbb{R}$ uma função diferenciável. Sabendo que $f'(t) + 2f(t) = e^t$ e que $f(0) = 2$, determine a expressão de $f(t)$.

5. Calcule:

   (a) $\mathcal{L}\{(t-2+e^{-2t})\cos(4t)\}$;
   (b) $\mathcal{L}^{-1}\left\{\dfrac{2s-1}{s^2-4s+6}\right\}$;
   (c) $\mathcal{L}^{-1}\left\{\dfrac{2s}{(s-1)(s^2+2s+5)}\right\}$.

6. Usando transformadas de Laplace mostre que

$$t^m * t^n = \frac{m!\,n!}{(m+n+1)!}\, t^{m+n+1} \qquad (m, n \in \mathbb{N}_0).$$

7. Determine a solução da equação

$$y'(t) = 1 - \mathrm{sen}\,t - \int_0^t y(\tau)\, d\tau$$

   que satisfaz a condição $y(0) = 0$.

8. Resolva cada um dos seguintes problemas de Cauchy usando transformadas de Laplace.

   (a) $3x' - x = \cos t\ , \qquad x(0) = -1;$

   (b) $\dfrac{d^2y}{dt^2} + 36y = 0\ , \qquad y(0) = -1\ , \quad \dfrac{dy}{dt}(0) = 2;$

   (c) $y'' + 2y' + 3y = 3t\ , \qquad y(0) = 0\ , \quad y'(0) = 1;$

   (d) $y''' + 2y'' + y' = x\ , \qquad y(0) = y'(0) = y''(0) - 1 = 0;$

   (e) $y'' + y' = \dfrac{e^{-t}}{2}\ , \qquad y(0) = 0 = y'(0).$

9. Resolva o seguinte problema de valores iniciais recorrendo às transformadas de Laplace:
$$y'' + y = t^2 + 1\ , \qquad y(\pi) = \pi^2\ , \quad y'(\pi) = 2\pi.$$
(Sugestão: Efetuar a substituição definida por $x = t - \pi$).

10. Usando transformadas de Laplace, resolva o seguinte sistema de EDOs sujeito às condições indicadas (onde x e y são funções da variável independente t):
$$\begin{cases} x' = 2x - 2y \\ y' = -3x + y \end{cases}, \qquad x(0) = 5, \quad y(0) = 0.$$

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Folha 5: *Soluções*

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1. (a) $\dfrac{6}{s^2+9}+\dfrac{1}{s^2}-\dfrac{5}{s+1}\,, \quad s>0;$

   (b) $\dfrac{s-2}{(s-2)^2+25}\,, \quad s>2;$

   (c) $\dfrac{1}{(s-3)^2}\,, \quad s>3;$

   (d) $\dfrac{\pi}{s}-\dfrac{5\cdot 10!}{(s+1)^{11}}\,, \quad s>0;$

   (e) $\dfrac{6s}{(s^2+1)^2}-\dfrac{1}{s^2+1}\,, \quad s>0;$

   (f) $\dfrac{1}{s^2+1}+\dfrac{e^{-\pi s}}{s^2+1}\,, \quad s>0;$

   (g) $e^{-2s}\dfrac{2!}{(s-2)^3}\,, \quad s>2.$

2. (a) $2\cosh(3t)=e^{3t}+e^{-3t}\,, \quad t\geq 0;$

   (b) $\dfrac{t^6}{180}\,, \quad t\geq 0;$

   (c) $t\,e^{-3t}\,, \quad t\geq 0;$

   (d) $\dfrac{1}{3}e^{t}-\dfrac{1}{3}e^{-2t}\,, \quad t\geq 0;$

   (e) $\dfrac{e^{-2t}}{\sqrt{2}}\,\mathrm{sen}\,(\sqrt{2}t)\,, \quad t\geq 0;$

   (f) $e^{2t}\Big(3\cos(3t)+\dfrac{5}{3}\mathrm{sen}\,(3t)\Big)\,, \quad t\geq 0.$

   (g) $\dfrac{4}{3}\,e^{t}+\dfrac{8}{3}\,e^{-2t}+\dfrac{1}{3}\,H_1(t)e^{t-1}-\dfrac{1}{3}\,H_1(t)e^{-2t+2};$

   (h) $\dfrac{1}{4}\,t\,\mathrm{sen}\,(2t).$

3. (a) $\dfrac{10!}{2^{11}};$ (b) $\dfrac{3}{50}.$

4. $f(t)=\dfrac{1}{3}e^{t}+\dfrac{5}{3}e^{-2t}.$

5. (a) $\dfrac{s^2-16}{(s^2+16)^2}-\dfrac{2s}{s^2+16}+\dfrac{s+2}{(s+2)^2+16}\,, \quad s>0;$

   (b) $e^{2t}\left(2\cos(\sqrt{2}\,t)+\dfrac{3}{\sqrt{2}}\,\mathrm{sen}\,(\sqrt{2}\,t)\right)\,, \quad t\geq 0.$

   (c) $\dfrac{1}{4}\,e^{t}-\dfrac{1}{4}\,e^{-t}\,\cos(2t)+\dfrac{3}{4}\,e^{-t}\,\mathrm{sen}\,(2t)\,, \quad t\geq 0.$

6. –

7. $\left(1-\dfrac{t}{2}\right)\operatorname{sen} t.$

8. (a) $x(t)=\dfrac{3}{10}\operatorname{sen} t-\dfrac{1}{10}\cos t-\dfrac{9}{10}\,\mathrm{e}^{\frac{t}{3}};$

   (b) $y(t)=\dfrac{1}{3}\operatorname{sen}(6t)-\cos(6t);$

   (c) $y(t)=t-\dfrac{2}{3}+\dfrac{2}{3\sqrt{2}}\,\mathrm{e}^{-t}\operatorname{sen}(\sqrt{2}\,t)+\dfrac{2}{3}\,\mathrm{e}^{-t}\cos(\sqrt{2}\,t);$

   (d) $y(x)=\dfrac{1}{2}\,(x^2-4x+8)-2e^{-x}(x+2);$

   (e) $y(t)=\dfrac{e^{-t}}{2}\left(e^t-t-1\right).$

9. $y(t)=(t-\pi)^2+2\pi(t-\pi)+\pi^2-1+\cos(t-\pi)=t^2-1-\cos t.$

10. $\begin{cases} x(t)=2e^{-t}+3e^{4t} \\ y(t)=3e^{-t}-3e^{4t} \end{cases}.$

1 – **a)** $\mathcal{L}\{2\operatorname{sen}(3t)+t-5e^{-t}\}(s) = 2\mathcal{L}\{\operatorname{sen}(3t)\}(s)+\mathcal{L}\{t\}(s)-5\mathcal{L}\{e^{-t}\}(s) = 2\times\underset{(s>0)}{\frac{3}{s^2+3^2}}+\underset{(s>0)}{\frac{1}{s^2}}-5\times\underset{(s>-1)}{\frac{1}{s+1}} = \frac{6}{s^2+9}+\frac{1}{s^2}-\frac{5}{s+1}\,,\ s>0$

b) $\mathcal{L}\{e^{2t}\cos(5t)\}(s) = \mathcal{L}\{\cos(5t)\}(s-2) = \mathcal{L}\{\cos(5t)\}(s-2) = \underset{(s-2)>0}{\frac{s-2}{(s-2)^2+5^2}} = \frac{s-2}{(s-2)^2+25}\,,\ s>2$

c) $\mathcal{L}\{\underbrace{t}_{g(t)}e^{3t}\}(s) = \mathcal{L}\{t\}(s-3) = \mathcal{L}\{t\}(s-3) = \frac{1}{(s-3)^2}\,,\ s>3$

d) $\mathcal{L}\{\pi-5e^{-t}t^{10}\}(s) = \pi\,\mathcal{L}\{1\}(s)-5\,\mathcal{L}\{e^{-t}\underbrace{t^{10}}_{g(t)}\}(s) = \pi\times\underset{(s>0)}{\frac{1}{s}}-5\,\mathcal{L}\{t^{10}\}(s+1) = \underset{(s>0)}{\frac{\pi}{s}}-5\times\underset{(s+1>0)}{\frac{10!}{(s+1)^{11}}} = \frac{\pi}{s}-\frac{5\times10!}{(s+1)^{11}}\,,\ s>0$

e) $\mathcal{L}\{(3t-1)\operatorname{sen}(t)\}(s) = \mathcal{L}\{3t\operatorname{sen}(t)-\operatorname{sen}(t)\}(s) = 3\,\mathcal{L}\{t\underbrace{\operatorname{sen}(t)}_{g(t)}\}(s)-\mathcal{L}\{\operatorname{sen}(t)\}(s) = 3\times(-1)^1\left(\mathcal{L}\{\operatorname{sen}(t)\}(s)\right)'-\mathcal{L}\{\operatorname{sen}(t)\}(s) = -3\times\underset{(s>0)}{\left(\frac{1}{s^2+1^2}\right)'}-\underset{(s>0)}{\frac{1}{s^2+1^2}} =$

$= -3\times\underset{(s>0)}{\frac{-2s}{(s^2+1)^2}}-\underset{(s>0)}{\frac{1}{s^2+1}} = \frac{6s}{(s^2+1)^2}-\frac{1}{s^2+1}\,,\ s>0$

f) $\mathcal{L}\{(1-H_\pi(t))\operatorname{sen}(t)\}(s) = \mathcal{L}\{\operatorname{sen}(t)-H_\pi(t)\operatorname{sen}(t)\}(s) = \mathcal{L}\{\operatorname{sen}(t)\}(s)-\mathcal{L}\{H_\pi(t)\operatorname{sen}(t)\}(s) = \underset{(s>0)}{\frac{1}{s^2+1^2}}-\mathcal{L}\{-H_\pi(t)\operatorname{sen}(t-\pi)\}(s) = \underset{(s>0)}{\frac{1}{s^2+1}}+\mathcal{L}\{H_\pi(t)\operatorname{sen}(t-\pi)\}(s) =$

$= \underset{(s>0)}{\frac{1}{s^2+1}}+e^{-\pi s}\mathcal{L}\{\operatorname{sen}(t)\}(s) = \underset{(s>0)}{\frac{1}{s^2+1}}+e^{-\pi s}\times\underset{(s>0)}{\frac{1}{s^2+1^2}} = \frac{1+e^{-\pi s}}{s^2+1}\,,\ s>0$

g) $\mathcal{L}\{\underbrace{(t-2)^2e^{2(t-2)}}_{g(t)=t^2e^{2t}}H_2(t)\}(s) = e^{-2s}\mathcal{L}\{\underbrace{t^2}_{g(t)}e^{2t}\}(s) = e^{-2s}\times\mathcal{L}\{t^2\}(s-2) = e^{-2s}\frac{2!}{(s-2)^3}\,,\ s>2$

$g(t)=t^2e^{2t}$

$\downarrow$

$g(t-2)=(t-2)^2e^{2(t-2)}$

2 – **a)** $F(s)=\frac{2s}{s^2-9} = 2\times\frac{s}{s^2-3^2} = 2\,\mathcal{L}\{\cosh(3t)\}(s)$

$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\{2\,\mathcal{L}\{\cosh(3t)\}(s)\}(t) = 2\mathcal{L}^{-1}\{\mathcal{L}\{\cosh(3t)\}(s)\}(t) = 2\cosh(3t) = 2\times\frac{e^{3t}+e^{-3t}}{2} = e^{3t}+e^{-3t}$

b) $F(s) = \frac{4}{s^7} = 4\times\frac{1}{s^7} = \frac{4}{6!}\times\frac{6!}{s^7} = \frac{4}{6!}\,\mathcal{L}\{t^6\}(s)$

$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\{\frac{4}{6!}\,\mathcal{L}\{t^6\}(s)\}(t) = \frac{4}{6!}\,\mathcal{L}^{-1}\{\mathcal{L}\{t^6\}(s)\}(t) = \frac{4}{6!}t^6 = \frac{t^6}{180}$

c) $F(s) = \frac{1}{s^2+6s+9} = \frac{1}{(s+3)^2} = \frac{1}{1!}\times\frac{1!}{(s-(-3))^2} = \mathcal{L}\{e^{-3t}t\}(s)$

$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\{\mathcal{L}\{e^{-3t}t\}(s)\}(t) = e^{-3t}t$

d) $F(s) = \frac{1}{s^2+s-2} = \frac{1}{(s-1)(s+2)} = \frac{\frac{1}{3}}{s-1}-\frac{\frac{1}{3}}{s+2} = \frac{1}{3}\times\frac{1}{s-1}-\frac{1}{3}\times\frac{1}{s+2} = \frac{1}{3}\,\mathcal{L}\{e^{t}\}(s)-\frac{1}{3}\,\mathcal{L}\{e^{-2t}\}(s)$

	1	1	-2
1		1	2
	1	2	$\boxed{0}$

$\frac{A}{s-1}+\frac{B}{s+2} = \frac{A(s+2)}{(s-1)(s+2)}+\frac{B(s-1)}{(s-1)(s+2)} = \frac{As+2A+Bs-B}{(s-1)(s+2)} = \frac{(A+B)s+2A-B}{(s-1)(s+2)}$

$s^2+s-2=(s-1)(s+2)$

$$\begin{cases} A+B=0 \\ 2A-B=1 \end{cases} \Leftrightarrow \begin{cases} A=-B \\ -3B=1 \end{cases} \Leftrightarrow \begin{cases} A=\frac{1}{3} \\ B=-\frac{1}{3} \end{cases}$$

$$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{3}\mathcal{L}\{e^{t}\}(s) - \frac{1}{3}\mathcal{L}\{e^{-2t}\}(s)\right\}(t) = \frac{1}{3}\mathcal{L}^{-1}\{\mathcal{L}\{e^{t}\}(s)\}(t) - \frac{1}{3}\mathcal{L}^{-1}\{\mathcal{L}\{e^{-2t}\}(t)\}(s) = \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}$$

e) $F(s) = \frac{1}{s^2+4s+6} = \frac{1}{(s+2)^2+2} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{(s+2)^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\mathcal{L}\{e^{-2t}\operatorname{sen}(\sqrt{2}t)\}(s)$

$$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{2}}\mathcal{L}\{e^{-2t}\operatorname{sen}(\sqrt{2}t)\}(s)\right\}(t) = \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\{\mathcal{L}\{e^{-2t}\operatorname{sen}(\sqrt{2}t)\}(s)\}(t) = \frac{1}{\sqrt{2}}e^{-2t}\operatorname{sen}(\sqrt{2}t) = \frac{e^{-2t}}{\sqrt{2}}\operatorname{sen}(\sqrt{2}t)$$

f) $F(s) = \frac{3s-1}{s^2-4s+13} = \frac{3s-1}{(s-2)^2+9} = \frac{3(s-2+2)-1}{(s-2)^2+9} = \frac{3(s-2)+6-1}{(s-2)^2+9} = 3\times\frac{s-2}{(s-2)^2+3^2} + 5\times\frac{1}{(s-2)^2+3^2} = 3\times\frac{s-2}{(s-2)^2+3^2} + \frac{5}{3}\times\frac{3}{(s-2)^2+3^2} =$

$= 3\,\mathcal{L}\{e^{2t}\cos(3t)\}(s) + \frac{5}{3}\mathcal{L}\{e^{2t}\operatorname{sen}(3t)\}(s)$

$$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\left\{3\,\mathcal{L}\{e^{2t}\cos(3t)\}(s) + \frac{5}{3}\mathcal{L}\{e^{2t}\operatorname{sen}(3t)\}(s)\right\}(t) = 3\,\mathcal{L}^{-1}\{\mathcal{L}\{e^{2t}\cos(3t)\}(s)\}(t) + \frac{5}{3}\mathcal{L}^{-1}\{\mathcal{L}\{e^{2t}\operatorname{sen}(3t)\}(s)\}(t) =$$

$$= 3e^{2t}\cos(3t) + \frac{5}{3}e^{2t}\operatorname{sen}(3t) = e^{2t}\left(3\cos(3t) + \frac{5}{3}\operatorname{sen}(3t)\right)$$

g) $F(s) = \frac{4s+e^{-s}}{s^2+s-2} = \frac{4s}{s^2+s-2} + \frac{e^{-s}}{s^2+s-2} = \frac{\frac{4}{3}}{s-1} + \frac{\frac{8}{3}}{s+2} + e^{-s}\times\frac{1}{(s-1)(s+2)} = \frac{4}{3}\times\frac{1}{s-1} + \frac{8}{3}\times\frac{1}{s+2} + e^{-s}\left(\frac{\frac{1}{3}}{s-1} - \frac{\frac{1}{3}}{s+2}\right) =$

$= \frac{4}{3}\mathcal{L}\{e^{t}\}(s) + \frac{8}{3}\mathcal{L}\{e^{-2t}\}(s) + \frac{1}{3}\times e^{s}\times\frac{1}{s-1} - \frac{1}{3}\times e^{-2s}\times\frac{1}{s+2} = \frac{4}{3}\mathcal{L}\{e^{t}\}(s) + \frac{8}{3}\mathcal{L}\{e^{-2t}\}(s) + \frac{1}{3}\mathcal{L}\{H_1(t)e^{t-1}\}(s) - \frac{1}{3}\mathcal{L}\{H_1(t)e^{-2t+2}\}(s) =$

$= \mathcal{L}\left\{\frac{4}{3}e^{t} + \frac{8}{3}e^{-2t} + \frac{1}{3}H_1(t)e^{t-1} + \frac{1}{3}H_1(t)e^{-2t+2}\right\}(s)$

$\frac{A}{s-1} + \frac{B}{s+2} = \frac{A(s+2)+B(s-1)}{(s-1)(s+2)} = \frac{As+2A+Bs-B}{(s-1)(s-2)} = \frac{(A+B)s+2A-B}{(s-1)(s-2)}$

$$\begin{cases} A+B=4 \\ 2A-B=0 \end{cases} \Leftrightarrow \begin{cases} 3A=4 \\ B=2A \end{cases} \Leftrightarrow \begin{cases} A=\frac{4}{3} \\ B=\frac{8}{3} \end{cases}$$

$$\begin{array}{c|ccc} & 1 & 1 & -2 \\ 1 & & 1 & 2 \\ \hline & 1 & 2 & \boxed{0} \end{array}$$

$s^2+s-2 = (s-1)(s+2)$

$$\mathcal{L}^{-1}\left\{\mathcal{L}\left\{\frac{4}{3}e^{t} + \frac{8}{3}e^{-2t} + \frac{1}{3}H_1(t)e^{t-1} + \frac{1}{3}H_1(t)e^{-2t+2}\right\}(s)\right\}(t) = \frac{4}{3}e^{t} + \frac{8}{3}e^{-2t} + \frac{1}{3}H_1(t)e^{t-1} + \frac{1}{3}H_1(t)e^{-2t+2}$$

h) $F(s) = \frac{s}{(s^2+4)^2} = \frac{1}{2}\times\frac{2s}{(s^2+4)^2} = -\frac{1}{2}\times\frac{-2s}{(s^2+4)^2} = \frac{1}{2}\times(-1)^1\times\left(\frac{1}{s^2+4}\right)' = \frac{1}{2}t\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}(t) = \frac{1}{2}t\,\mathcal{L}^{-1}\left\{\frac{1}{2}\times\frac{2}{s^2+2^2}\right\}(t) = \frac{1}{4}t\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}(t) =$

$= \frac{1}{4}t\operatorname{sen}(2t)$

3 – a) $\int_0^{+\infty} t^{10}e^{-2t}\,dt = F(2)$ se $F(s) = \int_0^{+\infty} t^{10}e^{-st}\,dt$, logo:

$F(s) = \mathcal{L}\{t^{10}\}(s) = \frac{10!}{s^{11}}$, logo $F(2) = \frac{10!}{2^{11}}$

b) $\displaystyle\int_0^{+\infty} e^{-3t}\, t \operatorname{sen}(t)\, dt = F(3)$ se $\displaystyle F(s) = \int_0^{+\infty} e^{-st}\, t \operatorname{sen}(t)\, dt$, logo:

$$F(s) = \mathcal{L}\left\{ t \operatorname{sen}(t) \right\}(s) = (-1)^1 \times \left(\mathcal{L}\left\{ \operatorname{sen}(t) \right\}(s) \right)' = -\left(\frac{1}{s^2+1^2} \right)' = -\left(\frac{-2s}{(s^2+1)^2} \right) = \frac{2s}{(s^2+1)^2}, \text{ logo } F(3) = \frac{2\times 3}{(3^2+1)^2} = \frac{6}{10^2} = \frac{6}{100} = \frac{3}{50}$$

4 – $f'(t) + 2f(t) = e^t \Leftrightarrow \mathcal{L}\left\{ f'(t) + 2f(t) \right\}(s) = \mathcal{L}\left\{ e^t \right\}(s) \Leftrightarrow \mathcal{L}\left\{ f'(t) \right\}(s) + 2\mathcal{L}\left\{ f(t) \right\} = \frac{1}{s-1} \Leftrightarrow s\mathcal{L}\left\{ f(t) \right\}(s) - f(0) + 2\mathcal{L}\left\{ f(t) \right\}(s) \Leftrightarrow$

$$\Leftrightarrow s\mathcal{L}\left\{ f(t) \right\}(s) - 2 + 2\mathcal{L}\left\{ f(t) \right\}(s) = \frac{1}{s-1} \Leftrightarrow (s+2)\mathcal{L}\left\{ f(t) \right\}(s) = \frac{1}{s-1} + 2 \Leftrightarrow \mathcal{L}\left\{ f(t) \right\}(s) = \frac{1}{(s-1)(s+2)} + \frac{2}{s+2} \Leftrightarrow$$

$$\Leftrightarrow \mathcal{L}\left\{ f(t) \right\}(s) = \frac{\frac{1}{3}}{s-1} + \frac{-\frac{1}{3}}{s+2} + \frac{2}{s+2} \Leftrightarrow \mathcal{L}\left\{ f(t) \right\}(s) = \frac{1}{3} \times \frac{1}{s-1} + \frac{5}{3} \times \frac{1}{s+2} \Leftrightarrow \mathcal{L}\left\{ f(t) \right\}(s) = \frac{1}{3}\left(\frac{1}{s-1} + \frac{5}{s+2} \right) \Leftrightarrow$$

$$\Leftrightarrow f(t) = \mathcal{L}^{-1}\left\{ \frac{1}{3}\left(\frac{1}{s-1} + \frac{5}{s+2} \right) \right\}(t) \Leftrightarrow f(t) = \frac{1}{3}\left(\mathcal{L}^{-1}\left\{ \frac{1}{s-1} \right\}(t) + \mathcal{L}^{-1}\left\{ \frac{5}{s+2} \right\}(t) \right) \Leftrightarrow f(t) = \frac{1}{3}\left(e^t + 5\mathcal{L}^{-1}\left\{ \frac{1}{s+2} \right\}(t) \right) \Leftrightarrow$$

$$\Leftrightarrow f(t) = \frac{1}{3}\left(e^t + 5e^{-2t} \right) \Leftrightarrow f(t) = \frac{1}{3}e^t + \frac{5}{3}e^{-2t}$$

$$\frac{A}{s-1} + \frac{B}{s+2} = \frac{A(s+2) + B(s-1)}{(s-1)(s+2)} = \frac{As + 2A + Bs - B}{(s-1)(s+2)} = \frac{(A+B)s + 2A - B}{(s-1)(s+2)}$$

$$\begin{cases} A + B = 0 \\ 2A - B = 1 \end{cases} \Leftrightarrow \begin{cases} A = -B \\ -3B = 1 \end{cases} \Leftrightarrow \begin{cases} A = \frac{1}{3} \\ B = -\frac{1}{3} \end{cases}$$

5 – **a)** $\mathcal{L}\left\{ (t - 2 + e^{-2t}) \cos(4t) \right\}(s) = \mathcal{L}\left\{ t\cos(4t) - 2\cos(4t) + e^{-2t}\cos(4t) \right\}(s) = \mathcal{L}\left\{ t\cos(4t) \right\}(s) - 2\mathcal{L}\left\{ \cos(4t) \right\}(s) + \mathcal{L}\left\{ e^{-2t} \underbrace{\cos(4t)}_{g(t)} \right\}(s) =$

$$= (-1)^1 \left(\mathcal{L}\left\{ \cos(4t) \right\} \right)' - 2 \times \underset{(s>0)}{\frac{s}{s^2+4^2}} + \mathcal{L}\left\{ \cos(4t) \right\}(s+2) = -\underset{(s>0)}{\left(\frac{s}{s^2+4^2} \right)'} - \underset{(s>0)}{\frac{2s}{s^2+16}} + \underset{(s+2>0)}{\frac{s+2}{(s+2)^2+4^2}} = -\left(\frac{(s)'(s^2+16) - s(s^2+16)'}{(s^2+16)^2} \right) - \underset{(s>0)}{\frac{2s}{s^2+16}} + \underset{(s>-2)}{\frac{s+2}{(s+2)^2+16}} =$$

$$= -\left(\frac{-s^2+16}{(s^2+16)^2} \right) - \underset{(s>0)}{\frac{2s}{s^2+16}} + \underset{(s>-2)}{\frac{s+2}{(s+2)^2+16}} = \frac{s^2-16}{(s^2+16)^2} - \frac{2s}{s^2+16} + \frac{s+2}{(s+2)^2+16}, \quad s>0$$

b) $\mathcal{L}^{-1}\left\{ \frac{2s-1}{s^2-4s+6} \right\}(t) = \mathcal{L}^{-1}\left\{ \frac{2(s-2+2)-1}{(s-2)^2+2} \right\}(t) = \mathcal{L}^{-1}\left\{ \frac{2(s-2)+3}{(s-2)^2+2} \right\}(t) = 2\mathcal{L}^{-1}\left\{ \frac{s-2}{(s-2)^2+2} \right\} + 3\mathcal{L}^{-1}\left\{ \frac{1}{(s-2)^2+2} \right\} = 2e^{2t}\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}\mathcal{L}^{-1}\left\{ \frac{\sqrt{2}}{(s-2)^2+(\sqrt{2})^2} \right\} =$

$$= 2e^{2t}\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}e^{2t}\operatorname{sen}(\sqrt{2}t) = e^{2t}\left(2\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}\operatorname{sen}(\sqrt{2}t) \right)$$

c) $\mathcal{L}^{-1}\left\{ \frac{2s}{(s-1)(s^2+2s+5)} \right\}(t) = \mathcal{L}^{-1}\left\{ \frac{\frac{1}{4}}{s-1} + \frac{-\frac{1}{4}s + \frac{5}{4}}{s^2+2s+5} \right\}(t) = \frac{1}{4}\mathcal{L}^{-1}\left\{ \frac{1}{s-1} \right\}(t) - \frac{1}{4}\mathcal{L}^{-1}\left\{ \frac{s-5}{s^2+2s+5} \right\}(t) = \frac{1}{4}e^t - \frac{1}{4}\mathcal{L}^{-1}\left\{ \frac{s+1-1-5}{(s+1)^2+4} \right\}(t) =$

$$= \frac{1}{4}e^t - \frac{1}{4}\left(\mathcal{L}^{-1}\left\{ \frac{s+1}{(s+1)^2+2^2} \right\}(t) - 6\mathcal{L}^{-1}\left\{ \frac{1}{(s+1)^2+2^2} \right\}(t) \right) = \frac{1}{4}e^t - \frac{1}{4}\left(e^{-t}\cos(2t) - \frac{6}{2}\mathcal{L}^{-1}\left\{ \frac{2}{(s+1)^2+2^2} \right\}(t) \right) = \frac{1}{4}e^t - \frac{1}{4}\left(e^{-t}\cos(2t) - 3e^{-t}\operatorname{sen}(2t) \right) =$$

$$= \frac{1}{4}e^t - \frac{1}{4}e^{-t}\cos(2t) + \frac{3}{4}e^{-t}\operatorname{sen}(2t)$$

$$\frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5} = \frac{A(s^2+2s+5) + (Bs+C)(s-1)}{(s-1)(s^2+2s+5)} = \frac{As^2 + 2As + 5A + Bs^2 - Bs + Cs - C}{(s-1)(s^2+2s+5)} = \frac{(A+B)s^2 + (2A-B+C)s + 5A - C}{(s-1)(s^2+2s+5)}$$

$$\begin{cases} A + B = 0 \\ 2A - B + C = 2 \\ 5A - C = 0 \end{cases} \Leftrightarrow \begin{cases} A = -B \\ -3B + C = 2 \\ -5B - C = 0 \end{cases} \Leftrightarrow \begin{cases} A = -B \\ -8B = 2 \\ C = -5B \end{cases} \Leftrightarrow \begin{cases} A = \frac{1}{4} \\ B = -\frac{1}{4} \\ C = \frac{5}{4} \end{cases}$$

6 – $\mathcal{L}\left\{t^m * t^n\right\}(s) = \mathcal{L}\left\{t^m\right\}(s) \times \mathcal{L}\left\{t^n\right\}(s) \Leftrightarrow \mathcal{L}\left\{t^m * t^n\right\}(s) = \frac{m!}{s^{m+1}} \times \frac{n!}{s^{n+1}} \Leftrightarrow \mathcal{L}\left\{t^m * t^n\right\}(s) = \frac{m!\,n!}{s^{m+n+2}} \Leftrightarrow t^m * t^n = \mathcal{L}^{-1}\left\{\frac{m!\,n!}{s^{m+n+2}}\right\}(t) \Leftrightarrow$

$\Leftrightarrow m!\,n!\,\mathcal{L}^{-1}\left\{\frac{1}{s^{m+n+2}}\right\}(t) \Leftrightarrow t^m * t^n = \frac{m!\,n!}{(m+n+1)!}\ \mathcal{L}^{-1}\left\{\frac{(m+n+1)!}{s^{m+n+2}}\right\}(t) \Leftrightarrow t^m * t^n = \frac{m!\,n!}{(m+n+1)!}\ t^{m+n+1}$

7 – $y'(t) = 1 - \operatorname{sen}(t) - \int_0^t y(\tau)\, d\tau \Leftrightarrow \mathcal{L}\left\{y'(t)\right\}(s) = \mathcal{L}\left\{1 - \operatorname{sen}(t) - \int_0^t y(\tau)\, d\tau\right\}(t) \Leftrightarrow s\,\mathcal{L}\left\{y(t)\right\} - y(0) = \mathcal{L}\{1\}(s) - \mathcal{L}\left\{\operatorname{sen}(t)\right\}(s) - \mathcal{L}\left\{\int_0^t y(\tau)\, d\tau\right\} \Leftrightarrow$

$\Leftrightarrow s\,\mathcal{L}\left\{y(t)\right\}(s) - 0 = \frac{1}{s} - \frac{1}{s^2+1^2} - \frac{\mathcal{L}\left\{y(t)\right\}(s)}{s} \Leftrightarrow s\,\mathcal{L}\left\{y(t)\right\}(s) + \frac{1}{s}\,\mathcal{L}\left\{y(t)\right\}(s) = \frac{1}{s} - \frac{1}{s^2+1} \Leftrightarrow \left(s + \frac{1}{s}\right)\mathcal{L}\left\{y(t)\right\}(s) = \underset{(s>0)}{\frac{1}{s}} - \underset{(s>0)}{\frac{1}{s^2+1}} \Leftrightarrow$

$\Leftrightarrow \frac{s^2+1}{s}\,\mathcal{L}\left\{y(t)\right\}(s) = \frac{1}{s} - \frac{1}{s^2+1} \Leftrightarrow \mathcal{L}\left\{y(t)\right\}(s) = \frac{s}{s^2+1}\left(\frac{1}{s} - \frac{1}{s^2+1}\right) \Leftrightarrow \mathcal{L}\left\{y(t)\right\}(s) = \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2} \Leftrightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}\right\}(t) \Leftrightarrow$

$\Leftrightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}(t) \Leftrightarrow y(t) = \operatorname{sen}(t) + \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\}(t) \Leftrightarrow y(t) = \operatorname{sen}(t) + \frac{1}{2}(-1)^1\,\mathcal{L}^{-1}\left\{\left(\frac{1}{s^2+1}\right)'\right\}(t) \Leftrightarrow$

$\Leftrightarrow y(t) = \operatorname{sen}(t) - \frac{1}{2}\, t \operatorname{sen}(t) \Leftrightarrow y(t) = \left(1 - \frac{t}{2}\right)\operatorname{sen}(t)$

8 – **a)** $3x' - x = \cos(t) \Leftrightarrow \mathcal{L}\left\{3x' - x\right\}(s) = \mathcal{L}\left\{\cos(t)\right\}(s) \Leftrightarrow 3\,\mathcal{L}\left\{x'\right\}(s) - \mathcal{L}\left\{x\right\}(s) = \frac{s}{s^2+1^2} \Leftrightarrow 3\left(s\,\mathcal{L}\{x\}(s) - x(0)\right) - \mathcal{L}\{x\}(s) = \frac{s}{s^2+1} \Leftrightarrow$

$\Leftrightarrow 3s\,\mathcal{L}\{x\}(s) + 3 - \mathcal{L}\{x\}(s) = \frac{s}{s^2+1} \Leftrightarrow (3s-1)\,\mathcal{L}\{x\}(s) = \frac{s}{s^2+1} - 3 \Leftrightarrow \mathcal{L}\{x\}(s) = \frac{s}{(s^2+1)(3s-1)} - \frac{3}{3s-1} \Leftrightarrow x = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(3s-1)} - \frac{3}{3s-1}\right\}(t) \Leftrightarrow$

$\Leftrightarrow x = \mathcal{L}^{-1}\left\{\frac{-\frac{1}{10}s + \frac{3}{10}}{s^2+1}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{\frac{3}{10}}{3s-1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{3}{3s-1}\right\}(t) \Leftrightarrow x = -\frac{1}{10}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) + \frac{3}{10}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) + \frac{3}{10}\,\mathcal{L}^{-1}\left\{\frac{1}{3s-1}\right\}(t) - 3\,\mathcal{L}^{-1}\left\{\frac{1}{3s-1}\right\}(t) \Leftrightarrow$

$\Leftrightarrow x = -\frac{1}{10}\cos(t) + \frac{3}{10}\operatorname{sen}(t) - \frac{27}{10}\,\mathcal{L}^{-1}\left\{\frac{1}{3s-1}\right\}(t) \Leftrightarrow x = \frac{-1}{10}\cos(t) + \frac{3}{10}\operatorname{sen}(t) - \frac{9}{10}\,\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{1}{3}}\right\} \Leftrightarrow x = \frac{-1}{10}\cos(t) + \frac{3}{10}\operatorname{sen}(t) - \frac{9}{10}e^{\frac{t}{3}}$

$$\frac{As+B}{s^2+1} + \frac{C}{3s-1} = \frac{(As+B)(3s-1) + C(s^2+1)}{(s^2+1)(3s-1)} = \frac{3As^2 - As + 3Bs - B + Cs^2 + C}{(s^2+1)(3s-1)} = \frac{(3A+C)s^2 + (-A+3B)s - B + C}{(s^2+1)(3s-1)}$$

$$\begin{cases} 3A + C = 0 \\ -A + 3B = 1 \\ -B + C = 0 \end{cases} \Leftrightarrow \begin{cases} C = -3A \\ -A + 3B = 1 \\ B = C \end{cases} \Leftrightarrow \begin{cases} C = -3A \\ -10A = 1 \\ B = -3A \end{cases} \Leftrightarrow \begin{cases} C = \frac{3}{10} \\ A = \frac{-1}{10} \\ B = \frac{3}{10} \end{cases}$$

b) $\frac{d^2y}{dt^2} + 36y = 0 \Leftrightarrow y'' + 36y = 0 \Leftrightarrow \mathcal{L}\left\{y'' + 36y\right\} = \mathcal{L}\{0\}(s) \Leftrightarrow \mathcal{L}\left\{y''\right\}(s) + 36\,\mathcal{L}\{y\}(s) = 0 \Leftrightarrow s^2\mathcal{L}\{y\}(s) - s\,y(0) - y'(0) + 36\,\mathcal{L}\{y\}(s) = 0 \Leftrightarrow$

$\Leftrightarrow s^2\,\mathcal{L}\{y\}(s) + s - 2 + 36\,\mathcal{L}\{y\}(s) = 0 \Leftrightarrow (s^2+36)\,\mathcal{L}\{y\}(s) = -s + 2 \Leftrightarrow \mathcal{L}\{y\}(s) \Leftrightarrow \frac{-s+2}{s^2+36} \Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{-s+2}{s^2+36}\right\}(t) \Leftrightarrow$

$\Leftrightarrow y = -\mathcal{L}^{-1}\left\{\frac{s}{s^2+6^2}\right\}(t) + 2\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+6^2}\right\}(t) \Leftrightarrow y = -\cos(6t) + \frac{2}{6}\,\mathcal{L}^{-1}\left\{\frac{6}{s^2+6^2}\right\}(t) \Leftrightarrow y = -\cos(6t) + \frac{1}{3}\operatorname{sen}(6t) \Leftrightarrow y = \frac{1}{3}\operatorname{sen}(6t) - \cos(6t)$

c) $y'' + 2y' + 3y = 3t \Leftrightarrow \mathcal{L}\left\{y'' + 2y' + 3y\right\}(s) = \mathcal{L}\{3t\}(s) \Leftrightarrow \mathcal{L}\left\{y''\right\}(s) + 2\,\mathcal{L}\left\{y'\right\}(s) + 3\,\mathcal{L}\{y\}(s) = 3\,\mathcal{L}\{t\}(s) \Leftrightarrow$

$\Leftrightarrow s^2\,\mathcal{L}\{y\}(s) - s\,y(0) - y'(0) + 2\left(s\,\mathcal{L}\{y\}(s) - y(0)\right) + 3\,\mathcal{L}\{y\}(s) = 3 \times \frac{1!}{s^2} \Leftrightarrow s^2\,\mathcal{L}\{y\}(s) - 1 + 2s\,\mathcal{L}\{y\}(s) + 3\,\mathcal{L}\{y\}(s) = \frac{3}{s^2} \Leftrightarrow$

$\Leftrightarrow (s^2+2s+3)\,\mathcal{L}\{y\}(s) = \frac{3}{s^2} + 1 \Leftrightarrow \mathcal{L}\{y\}(s) = \frac{3+s^2}{s^2(s^2+2s+3)} \Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{3+s^2}{s^2(s^2+2s+3)}\right\}(t) \Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{3}{s^2(s^2+2s+3)}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+3}\right\}(t) \Leftrightarrow$

$\Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{-\frac{2}{3}s + 1}{s^2} + \frac{\frac{2}{3}s + \frac{1}{3}}{s^2+2s+3}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2}\right\}(t) \Leftrightarrow$

$\Leftrightarrow y = -\frac{2}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) + \frac{2}{3}\,\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+2}\right\}(t) + \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2}\right\}(t) + \frac{1}{\sqrt{2}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}\right\}(t) \Leftrightarrow$

$\Leftrightarrow y = -\frac{2}{3} + t + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2+2}\right\}(t) + \frac{1}{3\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}\right\}(t) + \frac{1}{\sqrt{2}}e^{-t}\operatorname{sen}(\sqrt{2}t) \Leftrightarrow$

$\Leftrightarrow y = -\frac{2}{3} + t + \frac{2}{3}\left(\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2}\right\}(t)\right) + \frac{1}{3\sqrt{2}}e^{-t}\operatorname{sen}(\sqrt{2}t) + \frac{1}{\sqrt{2}}e^{-t}\operatorname{sen}(\sqrt{2}t) \Leftrightarrow$

$\Leftrightarrow y = -\frac{2}{3} + t + \frac{2}{3}\left(\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+(\sqrt{2})^2}\right\}(t) - \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}\right\}(t)\right) + \frac{4}{3\sqrt{2}}e^{-t}\operatorname{sen}(\sqrt{2}t) \Leftrightarrow$

$\Leftrightarrow y = -\frac{2}{3} + t + \frac{2}{3}\left(e^{-t}\cos(\sqrt{2}t) - \frac{1}{\sqrt{2}}e^{-t}\operatorname{sen}(\sqrt{2}t)\right) + \frac{4}{3\sqrt{2}}e^{-t}\operatorname{sen}(\sqrt{2}t) \Leftrightarrow$

$\Leftrightarrow y = -\frac{2}{3} + t + \frac{2}{3}e^{-t}\cos(\sqrt{2}t) - \frac{2}{3\sqrt{2}}e^{-t}\operatorname{sen}(\sqrt{2}t) + \frac{4}{3\sqrt{2}}e^{-t}\operatorname{sen}(\sqrt{2}t) \Leftrightarrow$

$\Leftrightarrow y = t - \frac{2}{3} + \frac{2}{3\sqrt{2}}e^{-t}\operatorname{sen}(\sqrt{2}t) + \frac{2}{3}e^{-t}\cos(\sqrt{2}t)$

$$\frac{As+B}{s^2} + \frac{Cs+D}{s^2+2s+3} = \frac{(As+B)(s^2+2s+3)+(Cs+D)s^2}{s^2(s^2+2s+3)} = \frac{As^3+2As^2+3As+Bs^2+2Bs+3B+Cs^3+Ds^2}{s^2(s^2+2s+3)} = \frac{(A+C)s^3+(2A+B+D)s^2+(3A+2B)s+3B}{s^2(s^2+2s+3)}$$

$$\begin{cases} A+C=0 \\ 2A+B+D=0 \\ 3A+2B=0 \\ 3B=3 \end{cases} \Leftrightarrow \begin{cases} C=-A \\ 2A+D=-1 \\ 3A=-2 \\ B=1 \end{cases} \Leftrightarrow \begin{cases} C=\frac{2}{3} \\ D=\frac{1}{3} \\ A=-\frac{2}{3} \\ B=1 \end{cases}$$

d) $y''' + 2y'' + y' = x \Leftrightarrow \mathcal{L}\{y'''+2y''+y'\}(s) = \mathcal{L}\{x\}(s) \Leftrightarrow \mathcal{L}\{y'''\}(s) + 2\mathcal{L}\{y''\}(s) + \mathcal{L}\{y'\}(s) = \frac{1!}{s^2} \Leftrightarrow$

$\Leftrightarrow s^3\mathcal{L}\{y\}(s) - s^2y(0) - sy'(0) - y''(0) + 2\left(s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0)\right) + s\mathcal{L}\{y\}(s) - y(0) = \frac{1}{s^2} \Leftrightarrow$

$\Leftrightarrow s^3\mathcal{L}\{y\}(s) - 1 + 2s^2\mathcal{L}\{y\}(s) + s\mathcal{L}\{y\}(s) = \frac{1}{s^2} \Leftrightarrow (s^3+2s^2+s)\mathcal{L}\{y\}(s) = \frac{1}{s^2} + 1 \Leftrightarrow \mathcal{L}\{y\}(s) = \frac{1+s^2}{s^2(s^3+2s^2+s)} \Leftrightarrow \mathcal{L}\{y\}(s) = \frac{1+s^2}{s^3(s^2+2s+1)} \Leftrightarrow$

$\Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{1+s^2}{s^3(s^2+2s+1)}\right\}(x) \Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{4s^2-2s+1}{s^3} + \frac{-4s-6}{s^2+2s+1}\right\}(x) \Leftrightarrow$

$\Leftrightarrow y = 4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(x) - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(x) + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}(x) - 4\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\}(x) - 6\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(x) \Leftrightarrow$

$\Leftrightarrow y = 4e^0 - 2x + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}(x) - 4\mathcal{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2}\right\}(x) - 6e^{-x}x \Leftrightarrow y = 4 - 2x + \frac{1}{2}x^2 - 4\left(\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2}\right\}(x) - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(x)\right) - 6e^{-x}x \Leftrightarrow$

$\Leftrightarrow y = 4 - 2x + \frac{1}{2}x^2 - 4e^{-x}\cos(0) + 4e^{-x}x - 6e^{-x}x \Leftrightarrow y = 4 - 2x + \frac{1}{2}x^2 - 4e^{-x} - 2e^{-x}x \Leftrightarrow y = \frac{1}{2}(x^2-4x+8) - 2e^{-x}(x+2)$

$$\frac{As^2+Bs+C}{s^3} + \frac{Ds+E}{s^2+2s+1} = \frac{(As^2+Bs+C)(s^2+2s+1)+(Ds+E)s^3}{s^3(s^2+2s+1)} = \frac{As^4+2As^3+As^2+Bs^3+2Bs^2+Bs+Cs^2+2Cs+C+Ds^4+Es^3}{s^3(s^2+2s+1)} =$$

$$= \frac{(A+D)s^4+(2A+B+E)s^3+(A+2B+C)s^2+(B+2C)s+C}{s^3(s^2+2s+1)}$$

$$\begin{cases} A+D=0 \\ 2A+B+E=0 \\ A+2B+C=1 \\ B+2C=0 \\ C=1 \end{cases} \Leftrightarrow \begin{cases} D=-4 \\ E=-6 \\ A=4 \\ B=-2 \\ C=1 \end{cases}$$

e) $y'' + y' = \frac{e^{-t}}{2} \Leftrightarrow \mathcal{L}\{y''+y'\}(s) = \mathcal{L}\left\{\frac{e^{-t}}{2}\right\}(s) \Leftrightarrow \mathcal{L}\{y''\}(s) + \mathcal{L}\{y'\}(s) = \frac{1}{2}\mathcal{L}\{e^{-t}\}(s) \Leftrightarrow$

$\Leftrightarrow s^2\mathcal{L}\{y\}(s) - s\,y(0) - y'(0) + s\mathcal{L}\{y\} - y(0) = \frac{1}{2} \times \frac{1}{s+1} \Leftrightarrow s^2\mathcal{L}\{y\}(s) + s\mathcal{L}\{y\}(s) = \frac{1}{2s+2} \Leftrightarrow (s^2+s)\mathcal{L}\{y\}(s) = \frac{1}{2s+2} \Leftrightarrow$

$\Leftrightarrow \mathcal{L}\{y\}(s) = \frac{1}{(s^2+s)(2s+2)} \Leftrightarrow \mathcal{L}\{y\}(s) = \frac{1}{2s^3+4s^2+2s} \Leftrightarrow \mathcal{L}\{y\}(x) = \frac{1}{2s(s^2+2s+1)} \Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{1}{2s(s^2+2s+1)}\right\}(t) \Leftrightarrow$

$\Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{1}{2s} + \frac{-\frac{1}{2}s-1}{s^2+2s+1}\right\}(t) \Leftrightarrow y = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t) \Leftrightarrow$

$\Leftrightarrow y = \frac{1}{2}e^0 - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2}\right\}(t) - e^{-t}t \Leftrightarrow y = \frac{1}{2} - \frac{1}{2}\left(\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t)\right) - e^{-t}t \Leftrightarrow$

$\Leftrightarrow y = \frac{1}{2} - \frac{1}{2}e^{-t}\cos(0) + \frac{1}{2}e^{-t}t - e^{-t}t \Leftrightarrow y = \frac{1}{2} - \frac{1}{2}e^{-t} - \frac{1}{2}e^{-t}t \Leftrightarrow y = \frac{e^{-t}}{2}\left(e^t - t - 1\right)$

$$\frac{A}{2s} + \frac{Bs+C}{s^2+2s+1} = \frac{A(s^2+2s+1) + (Bs+C)2s}{2s(s^2+2s+1)} = \frac{As^2+2As+A+2Bs^2+2Cs}{2s(s^2+2s+1)} = \frac{(A+2B)s^2+(2A+2C)s+A}{2s(s^2+2s+1)}$$

$$\begin{cases} A+2B=0 \\ 2A+2C=0 \\ A=1 \end{cases} \Leftrightarrow \begin{cases} 2B=-1 \\ 2C=-2 \\ A=1 \end{cases} \Leftrightarrow \begin{cases} B=-\frac{1}{2} \\ C=-1 \\ A=1 \end{cases}$$

9 – $y'' + y = t^2+1 \Leftrightarrow \mathcal{L}\{y''+y\}(s) = \mathcal{L}\{t^2+1\}(s) \Leftrightarrow \mathcal{L}\{y''\}(s) + \mathcal{L}\{y\}(s) = \mathcal{L}\{t^2\}(s) + \mathcal{L}\{1\}(s) \Leftrightarrow$

$\Leftrightarrow s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) + \mathcal{L}\{y\}(s) = \frac{2!}{s^3} + \frac{1}{s}$

Considerando $x = t - \pi$, então :

$s^2\mathcal{L}\{y\}(s) - s\,y(x) - y'(x) + \mathcal{L}\{y\}(s) = \frac{2+s^2}{s^3} \Leftrightarrow (s^2+1)\mathcal{L}\{y\}(s) = \frac{2+s^2}{s^3} + sx^2 + 2x \Leftrightarrow \mathcal{L}\{y\}(s) = \frac{2+s^2}{s^3(s^2+1)} + \frac{sx^2}{s^2+1} + \frac{2x}{s^2+1} \Leftrightarrow$

$\Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{2+s^2}{s^3(s^2+1)} + \frac{sx^2+2x}{(s^2+1)}\right\}(t) \Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{2+s^2}{s^3(s^2+1)}\right\}(t) + x^2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) + 2x\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) \Leftrightarrow$

$\Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{-s^2+2}{s^3} + \frac{s}{s^2+1}\right\}(t) +$

$$\frac{As^2+Bs+C}{s^3} + \frac{Ds+E}{s^2+1} = \frac{(As^2+Bs+C)(s^2+1)+(Ds+E)s^3}{s^3(s^2+1)} = \frac{As^4+As^2+Bs^3+Bs+Cs^2+C+Ds^4+Es^3}{s^3(s^2+1)} = \frac{(A+D)s^4+(B+E)s^3+(A+C)s^2+Bs+C}{s^3(s^2+1)}$$

$$\begin{cases} A+D=0 \\ B+E=0 \\ A+C=1 \\ B=0 \\ C=2 \end{cases} \Leftrightarrow \begin{cases} D=1 \\ E=0 \\ A=-1 \\ B=0 \\ C=2 \end{cases}$$

10 — $\begin{cases} x' = 2x - 2y \\ y' = -3x + y \end{cases} \Leftrightarrow \begin{cases} \mathcal{L}\{x'\}(s) = \mathcal{L}\{2x-2y\}(s) \\ \mathcal{L}\{y'\}(s) = \mathcal{L}\{-3x+y\} \end{cases} \Leftrightarrow \begin{cases} s\,\mathcal{L}\{x\}(s) - x(0) = 2\,\mathcal{L}\{x\}(s) - 2\,\mathcal{L}\{y\}(s) \\ s\,\mathcal{L}\{y\}(s) - y(0) = -3\,\mathcal{L}\{x\}(x) + \mathcal{L}\{y\}(s) \end{cases} \Leftrightarrow \begin{cases} s\,\mathcal{L}\{x\}(s) - 2\,\mathcal{L}\{x\}(s) = -2\,\mathcal{L}\{y\}(s) + 5 \\ s\,\mathcal{L}\{y\}(s) - \mathcal{L}\{y\}(s) = -3\,\mathcal{L}\{x\}(s) \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} (s-2)\,\mathcal{L}\{x\}(s) = -2\,\mathcal{L}\{y\}(s) + 5 \\ (s-1)\,\mathcal{L}\{y\}(s) = -3\,\mathcal{L}\{x\}(s) \end{cases} \Leftrightarrow \begin{cases} (s-2)\dfrac{(s-1)}{-3}\,\mathcal{L}\{y\}(s) = -2\,\mathcal{L}\{y\}(s) + 5 \\ \mathcal{L}\{x\}(s) = \dfrac{s-1}{-3}\,\mathcal{L}\{y\}(s) \end{cases} \Leftrightarrow \begin{cases} \dfrac{-(s-2)(s-1)}{3}\,\mathcal{L}\{y\}(s) + 2\,\mathcal{L}\{y\}(s) = 5 \\ \mathcal{L}\{x\}(s) = \dfrac{s-1}{-3}\,\mathcal{L}\{y\}(s) \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} \left(\dfrac{-(s-2)(s-1)+6}{3}\right)\mathcal{L}\{y\}(s) = 5 \\ \mathcal{L}\{x\}(s) = \dfrac{s-1}{-3}\,\mathcal{L}\{y\}(s) \end{cases} \Leftrightarrow \begin{cases} \mathcal{L}\{y\}(s) = \dfrac{15}{-(s-2)(s-1)+6} \\ \mathcal{L}\{x\}(s) = \dfrac{s-1}{-3}\,\mathcal{L}\{y\}(s) \end{cases} \Leftrightarrow \begin{cases} \mathcal{L}\{y\}(s) = \dfrac{15}{-s^2+3s+4} \\ \mathcal{L}\{x\}(s) = \dfrac{-5(s-1)}{-s^2+3s+4} \end{cases}$

$\mathcal{L}\{y\}(s) = \dfrac{15}{-s^2+3s+4} \Leftrightarrow y = 15\,\mathcal{L}^{-1}\left\{\dfrac{1}{-s^2+3s+4}\right\}(t) \Leftrightarrow y = 15\,\mathcal{L}^{-1}\left\{\dfrac{1}{-(s+1)(s-4)}\right\}(t) \Leftrightarrow y = -15\,\mathcal{L}^{-1}\left\{\dfrac{-\frac{1}{5}}{s+1} + \dfrac{\frac{1}{5}}{s-4}\right\}(t) \Leftrightarrow$

$\Leftrightarrow y = -15\left(-\dfrac{1}{5}\,\mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\}(t) + \dfrac{1}{5}\,\mathcal{L}^{-1}\left\{\dfrac{1}{s-4}\right\}(t)\right) \Leftrightarrow y = 3e^{-t} - 3e^{4t}$

$\dfrac{A}{s+1} + \dfrac{B}{s-4} = \dfrac{A(s-4) + B(s+1)}{(s+1)(s-4)} = \dfrac{As - 4A + Bs + B}{(s+1)(s-4)} = \dfrac{(A+B)s - 4A + B}{(s+1)(s-4)}$

$\begin{cases} A + B = 0 \\ -4A + B = 1 \end{cases} \Leftrightarrow \begin{cases} A = -B \\ 5B = 1 \end{cases} \Leftrightarrow \begin{cases} A = -\dfrac{1}{5} \\ B = \dfrac{1}{5} \end{cases}$

$\mathcal{L}\{x\}(s) = \dfrac{-5(s-1)}{-s^2+3s+4} \Leftrightarrow x = -5\,\mathcal{L}^{-1}\left\{\dfrac{s-1}{-(s+1)(s-4)}\right\} \Leftrightarrow x = 5\,\mathcal{L}^{-1}\left\{\dfrac{\frac{2}{5}}{s+1} + \dfrac{\frac{3}{5}}{s-4}\right\}(t) \Leftrightarrow x = 2\,\mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\}(t) + 3\,\mathcal{L}^{-1}\left\{\dfrac{1}{s-4}\right\}(t) \Leftrightarrow$

$\Leftrightarrow x = 2e^{-t} + 3e^{4t}$

$\dfrac{A}{s+1} + \dfrac{B}{s-4} = \dfrac{A(s-4) + B(s+1)}{(s+1)(s-4)} = \dfrac{As - 4A + Bs + B}{(s+1)(s-4)} = \dfrac{(A+B)s - 4A + B}{(s+1)(s-4)}$

$\begin{cases} A + B = 1 \\ -4A + B = -1 \end{cases} \Leftrightarrow \begin{cases} A = 1 - B \\ -4 + 5B = -1 \end{cases} \Leftrightarrow \begin{cases} A = \dfrac{2}{5} \\ B = \dfrac{3}{5} \end{cases}$