TOPIC 2
FUNCTIONS

SUBTOPIC 2.7
COMPOSITE FUNCTIONS AND INVERSE FUNCTIONS

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2. Functions / 2.7 Composite functions and inverse functions

The big picture

A weather balloon carries instruments up in the air to send back information on atmospheric pressure, temperature, humidity and other variables. The data is collected by small, expendable measuring devices. For example, a device called a radiosonde can measure variables such as wind speed and wind direction at high altitudes of various geographical locations.

Temperature gauge

Credit: JADEZMITH Getty Images

Data informs us that the temperature T in the atmosphere is a function of the altitude, or the height h of a weather balloon above the ground. The height h of the weather balloon is a function of the time t since the balloon was launched. A change in time will result a change

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in height, which in turn will produce a change in the air temperature. There are many real-world phenomena in which one quantity is a function of second quantity, which is in turn a function of a third quantity.

In this subtopic you will learn about the concepts of:

- composite functions
- inverse function

Concept

Functions are often used to describe relationships between quantities. There are also **relationships between functions** that can help us to model and understand more complex phenomena. A composite function is the result of combining two functions so that the output of the first function becomes the input of the second function. As you learn algebraic methods of finding composite functions in this subtopic, think about real-life applications in which composite functions can be used as **mathematical models.** An inverse function 'undoes' the effect of a given function. As you learn how to find the formula of an inverse function, think about ways in which you might verify that two functions are inverses of each other.

2. Functions / 2.7 Composite functions and inverse functions

Composition functions

Consider the function $f(x) = x^2$. It squares any input x, so

$$f(3) = 3^2 \quad \text{and} \quad f(x-1) = (x-1)^2.$$

In the second case, the input for f is another function of x. If we let $g(x) = x - 1$, the second result can be written as

$$f(g(x)) = f(x-1) = (x-1)^2.$$

So we have combined two functions

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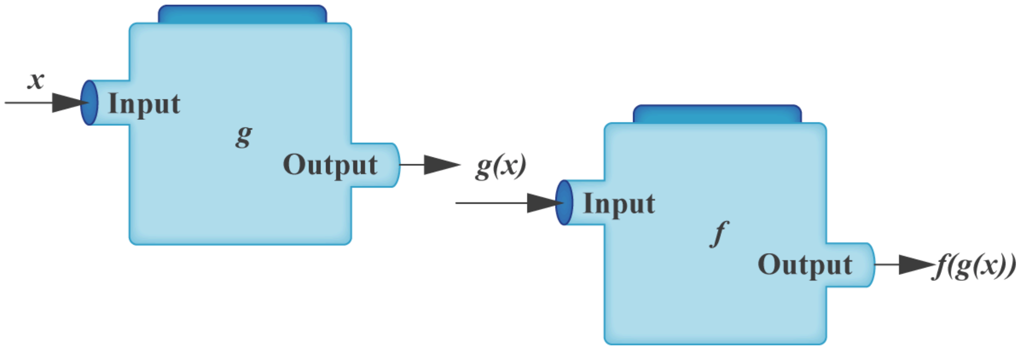
$$f(x) = x^2 \quad \text{and} \quad g(x) = x - 1$$

to obtain the composite function $(f \circ g)(x) = f(g(x))$. This process of turning the output of one function into the input for another function is called composition. Notice that the function g is applied first and the function f is applied second. The concept of composite function is illustrated as a function machine in the below figure.



More information

The diagram illustrates the concept of composite functions using a function machine analogy. It consists of two sections, each representing a function. On the left, there is a machine labeled 'g' with an arrow pointing into it from the left labeled 'x' for input and an arrow coming out on the right side labeled 'g(x)' for output. This output then feeds into the second machine on the right labeled 'f', again with an arrow pointing into it, marked as 'g(x)' for input. The final output arrow on the right side is labeled 'f(g(x))'. This visual setup demonstrates the process where the function 'g' is applied first, taking input 'x' and outputting 'g(x)', which then becomes the input for the function 'f', resulting in the final output 'f(g(x))'. Labels 'Input' and 'Output' are also present near each function machine to clearly indicate the direction of data flow.

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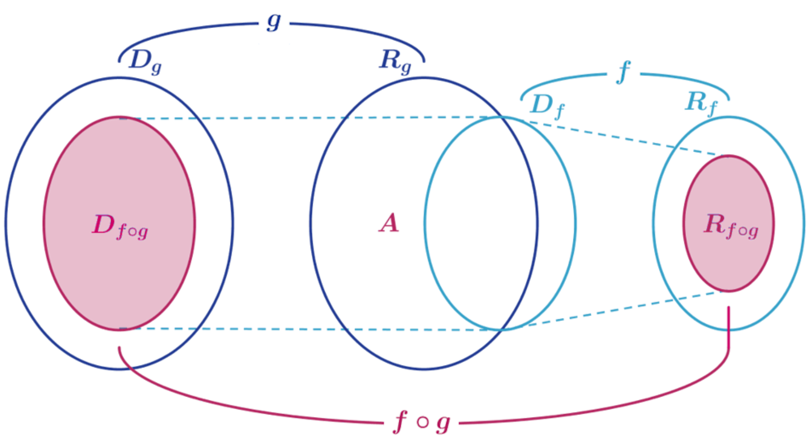
Important

A **composite function** of two functions f and g such that function g is applied first and function f is applied second is written as $f \circ g$ and maps x to $f(g(x))$. So

$$(f \circ g)(x) = f(g(x)).$$

The domain of the function $f \circ g$ is the set of all x values in the domain of g such that the corresponding $g(x)$ values belong to the domain of f.

The illustration below shows the mapping for the composite function $f \circ g$. The domains and ranges of the functions are denoted by D_{function} and R_{function}, respectively. The domain $D_{f \circ g}$ of the composite function $f \circ g$ is made up of all the x values in D_g whose images $g(x)$ belong to the domain of D_f. These images are represented by the area A in the diagram. They are then mapped by the function f into its range R_f, producing the range $R_{f \circ g}$ for $f \circ g$, which is a subset of R_f.



More information

The diagram illustrates the mapping process of the composite function f∘g. It consists of multiple labeled circles and arrows, indicating the domains and ranges involved in the function composition. On the left, there is a circle labeled Dg, representing the domain of function g. The image of Dg, labeled with g(x), is shown to map onto another circle labeled Rg, indicating the range of g. The domain Df of function f is depicted as a subsequent circle that intersects with Rg in an area labeled A. This intersection represents the set of values from Rg that belong to Df. The function f then maps these values into its own range, denoted as Rf. The final circle on the right is labeled Rf∘g, which is the range of the composite function f∘g, shown as a subset of Rf.

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You write	You say
$f \circ g$	f following g
$f(g(x))$	f of g of x

Be aware

It is important to apply the functions in the correct order:

$$(f \circ g)(x) \quad : x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$$

and

$$(g \circ f)(x) \quad : x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x))$$

Example 1

★☆☆

Let $f(x) = x^2$ and $g(x) = x + 4$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Are they the same?

Step	Explanation
$(f \circ g)(x) = f(g(x)) = f(x+4) = (x+4)^2$ $= x^2 + 8x + 16$	Remember the correct order applying the two functions. For $f \circ g$ you apply g first anc then feed the output of g into .
$(g \circ f)(x) = g(f(x)) = g\left(x^2\right) = \left(x^2\right) + 4 = x^2 + 4$	For $g \circ f$ you apply f first anc then feed the output of f into .

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Step	Explanation
$(f \circ g)(x) \neq (g \circ f)(x)$	This is generally the case for composite functions.

Be aware

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

Because $(f \circ g)(x) \neq (g \circ f)(x)$ in general, usually $D_{f \circ g} \neq D_{g \circ f}$. For the composite function on the left, the range of g will be the new domain of f, while for the composite function on the right it will be the opposite. **Example 3** below shows this difference.

Moreover, since in the composite function $f(g(x))$ the range of g is the new domain of f, this may affect the range of the final execution under the rule f. For example, the function $f(x) = 2x$ without any restrictions on x has range $-\infty < y < +\infty$. But if we consider $f(g(x))$ where $g(x) = \sqrt{x}$, then the range of $f(g(x)) = 2\sqrt{x}$ will be only $0 \leqslant y < +\infty$ because the range of $g(x) = \sqrt{x}$ is only $g(x) \geqslant 0$. You can see this kind of range restriction from the graph in **Example 2**.

Exam tip

The solutions in the examples are presented in detailed steps. Although there is no need to give so much detail in an exam, you should make sure that the steps you take are clear in your mind and also easy to follow for the examiner reading your solution.

Example 2

★★☆

Consider the functions $f(x) = 2x$ and $g(x) = x^2 + 2$. Find the domain, the range and the formula of the composite function $f \circ g$.

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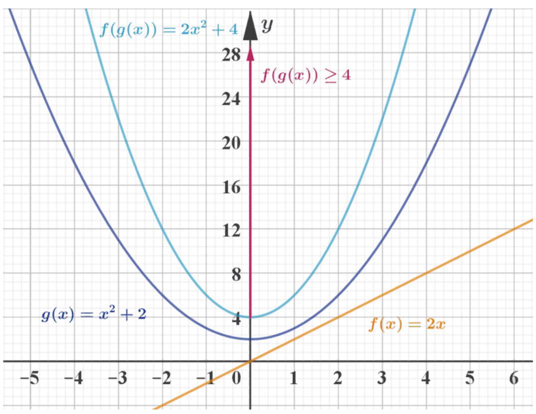
Step	Explanation
Let D_f, D_g and $D_{f\circ g}$ be the domains of f, g and $f \circ g$, respectively. Then $$D_f = \mathbb{R},\ D_g = \mathbb{R}$$ and $$\begin{aligned} D_{f\circ g} &= \{x \mid x \in D_g,\ g(x) \in D_f\} \\ &= \{x \mid x \in \mathbb{R}, x^2 + 2 \in \mathbb{R}\} \\ &= \mathbb{R} \end{aligned}$$	First write down the domains of the giver functions. Recall the set notation you met earlier: $\{x \vert \ldots\}$ means 'the set of all x such that $\ldots$'. $\mathbb{R}$ means the set of all real numbers; so $x \in \mathbb{R}$ is the same as $-\infty < x < +\infty$.
For the range of $f \circ g$: $$\begin{aligned} x &\in D_{f\circ g} \\ x &\in \mathbb{R} \\ x^2 &\geqslant 0 \\ x^2 + 2 &\geqslant 2 \\ g(x) &\geqslant 2 \\ 2g(x) &\geqslant 2 \times 2 \\ f(g(x)) &\geqslant 4 \\ (f \circ g)(x) &\geqslant 4 \end{aligned}$$ So the range of $f \circ g$ is $\{y \mid y \geqslant 4\}$.	To find the range of a composite functior it is helpful to go step by step. Take care with the order in which you apply the functions: For $f \circ g$, g is applied first and f is applie second..
The formula for $f \circ g$ is $$(f \circ g)(x) = f(g(x)) = 2\left(x^2 + 2\right) = 2x^2 + 4$$.	The graphs of f, g and $f \circ g$, as well as tl range of $f \circ g$, are shown in the following figure: 

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Example 3

★★★

Consider the functions $f(x) = x^2 - 1$ and $g(x) = \sqrt{x}$. Find the domain, range and formula of the functions $f \circ g$ and $g \circ f$.

Step	Explanation
$D_f = \mathbb{R}$ and $D_g = \{x \mid x \in \mathbb{R},\ x \geqslant 0\}$. $$\begin{aligned} D_{f \circ g} &= \{x \mid x \in D_g, g(x) \in D_f\} \\ &= \{x \mid x \geqslant 0, \sqrt{x} \in \mathbb{R}\} \quad \} \\ &= \{x \mid x \in \mathbb{R}, x \geqslant 0\} \end{aligned}$$	Write down the domains of the given functions. Why does the range of g have to be restricted to $x \geqslant 0$?
For the range of $f \circ g$: $$\begin{aligned} x \in D_{f \circ g} x &\geqslant 0 \\ \sqrt{x} &\geqslant 0 \\ g(x) &\geqslant 0 \\ (g(x))^2 &\geqslant 0 \\ (g(x))^2 - 1 &\geqslant -1 \\ f(g(x)) &\geqslant -1 \\ (f \circ g)(x) &\geqslant -1. \end{aligned}$$ So the range of $f \circ g$ is $\{y \mid y \geqslant -1\}$.	As in **Example 2**, we write this out step by step. For $f \circ g$, g is applied first and f is applied second.
The formula for $f \circ g$ is $$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= (\sqrt{x})^2 - 1 \\ &= x - 1\ (as\ x \geqslant 0) \end{aligned}$$	
The domain of $g \circ f$ is $$\begin{aligned} D_{g \circ f} &= \{x \mid x \in D_f, f(x) \in D_g\} \\ &= \{x \mid x \in \mathbb{R},\ x^2 - 1 \geqslant 0\} \\ &= \{x \mid x \in \mathbb{R}, x \leqslant -1 \text{ or } x \geqslant 1\} \\ &= \ ]-\infty, -1]\ \cup\ [1, +\infty[\end{aligned}$$	Recall interval notation: $]a, b]$ means the interval $a < x \leqslant b$ and $[a, b[$ means the interval $a \leqslant x < b$.

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Step	Explanation
For the range of $g \circ f$: $$\begin{aligned} x &\in D_{g\circ f} \\ x \leqslant -1 \text{ or } x &\geqslant 1 \\ x^2 &\geqslant 1 \\ x^2 - 1 &\geqslant 0 \\ f(x) &\geqslant 0 \\ \sqrt{f(x)} &\geqslant 0 \\ g(f(x)) &\geqslant 0 \\ (g\circ f)(x) &\geqslant 0 \end{aligned}$$ So the range of $g \circ f$ is $\{y \mid y \in \mathbb{R}, y \geqslant 0\}$.	
The formula for $g \circ f$ is $$(g\circ f)(x) = g(f(x)) = \sqrt{x^2-1}$$	

Be aware

A composite function can be made up of more than two functions. The idea remains the same: the correct order of execution is from right to left. The function closest to the input variable x is the first function to be executed and the result of this is passed on to the next function to the left, and so on.

International Mindedness

Nicolas Bourbaki (the collective name of a group of mathematicians based mainly in France) first used the notation $f \circ g$ with the interpretation $(f\circ g)(x) = f(g(x))$ in 1949 (*Fonctions d'une variable réelle: Théorie élémentaire*). It is certainly conceivable that this notation for composite functions was invented by someone from the Bourbaki group, which was very concerned with good mathematical notation.

However, the symbol ○ appears not only in the context of composite functions, and interpretation of the notation $f \circ g$ varies across different areas of knowledge such as category theory, group theory and computer science, as well as across different regions of the world.

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Making connections

In the biological sciences there are many functional relationships where one quantity depends on another, and the second quantity is a function of a third quantity. For example, data suggests that the number of bacteria in food kept in the refrigerator can be described by a function $n\left(T\right)$, where T is the temperature of the food in degrees Celsius. After the food is removed from the refrigerator, the temperature of the food is given by a function $T\left(t\right)$, where t is the time in hours that the food has been out of the refrigerator.

Discuss with your fellow students the meaning of the composite function $n\left(T\left(t\right)\right)$ in this context. Explain how this composite function could be useful for a biologist or food scientist.

Theory of Knowledge

One of the fundamental TOK ‘takeaways’ is the ability to reconsider what you know and how you know it. This epistemological mindset works well in challenging your own ways of thinking (i.e. personal knowledge), and hopefully these changes in thinking are long lasting, affecting you as a learner through your university and professional studies. In many cases, this same mindset will help you to challenge the thinking of others as well (i.e. shared knowledge). One interesting leap, from the above Be aware box in 2.1 about the order of operations for composite functions, to TOK is the following video, which examines and challenges what was most likely the first mathematical order of operations and its acronym that you probably learned — PEMDAS, PEDMAS, BEDMAS, BIDMAS, BODMAS, etc. — depending on the region where you were taught.

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This sample knowledge question (KQ) may help, as will the video: 'To what extent does analogical thinking help link various areas of knowledge (AOKs) and/or ways of knowing (WOKs)?' Consider specifically language and sense perception as two WOKs related to this video.

3 section questions

2. Functions / 2.7 Composite functions and inverse functions

Inverse functions

Identity function

Consider a very special function that does **nothing** . What is meant by that? First, let us look at some mathematical operations we are familiar with. What is the number that I can add to any number without changing its value? Zero: $a + 0 = a$ for any a. What is the number that I can multiply by any number without changing that number's value? One: $a \times 1 = a$ for any a. We call 0 the identity under addition, and 1 the identity under multiplication. By the same token, there is an identity function that takes any value x and maps it to x itself.

Important

The **identity function** is the function defined by $I(x) = x$, which assigns any number in its domain to the number itself.

The domain D_I and the range R_I of the identity function I are the same set: $D_I = R_I$.

The inverse function f^{-1}

In subtopic 2.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26018/) we have seen that the inverse function f^{-1} 'undoes' what function f does; that is, if f maps x to y, then f^{-1} maps y back to x.

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For example, consider the function $f(x) = 2x$, which doubles its input. We can 'undo' the action of f by applying the function $g(x) = \frac{1}{2}x$, which halves its input, so that $x \xrightarrow{f} 2x \xrightarrow{g} x$. Observe that the composition of functions f and g, in any order, gives the identity function:

$$g(f(x)) = \frac{1}{2}(2x) = x = I(x)$$

$$f(g(x)) = 2\left(\frac{1}{2}x\right) = x = I(x)$$

Important

Let f and g be two functions with respective domains D_f and D_g. If $(f \circ g)(x) = x$ for every $x \in D_g$ and $(g \circ f)(x) = x$ for every $x \in D_f$, then the function g is called the **inverse function of** f and is denoted by f^{-1}. Thus, we write:

$$\left(f \circ f^{-1}\right)(x) = \left(f^{-1} \circ f\right)(x) = x$$

You also explored how to find the formula for an inverse function f^{-1} either by reversing the order of operations of the function f or by using the graphical relationship between f and f^{-1}. The video shows another method for finding the equation of the inverse function f^{-1}.

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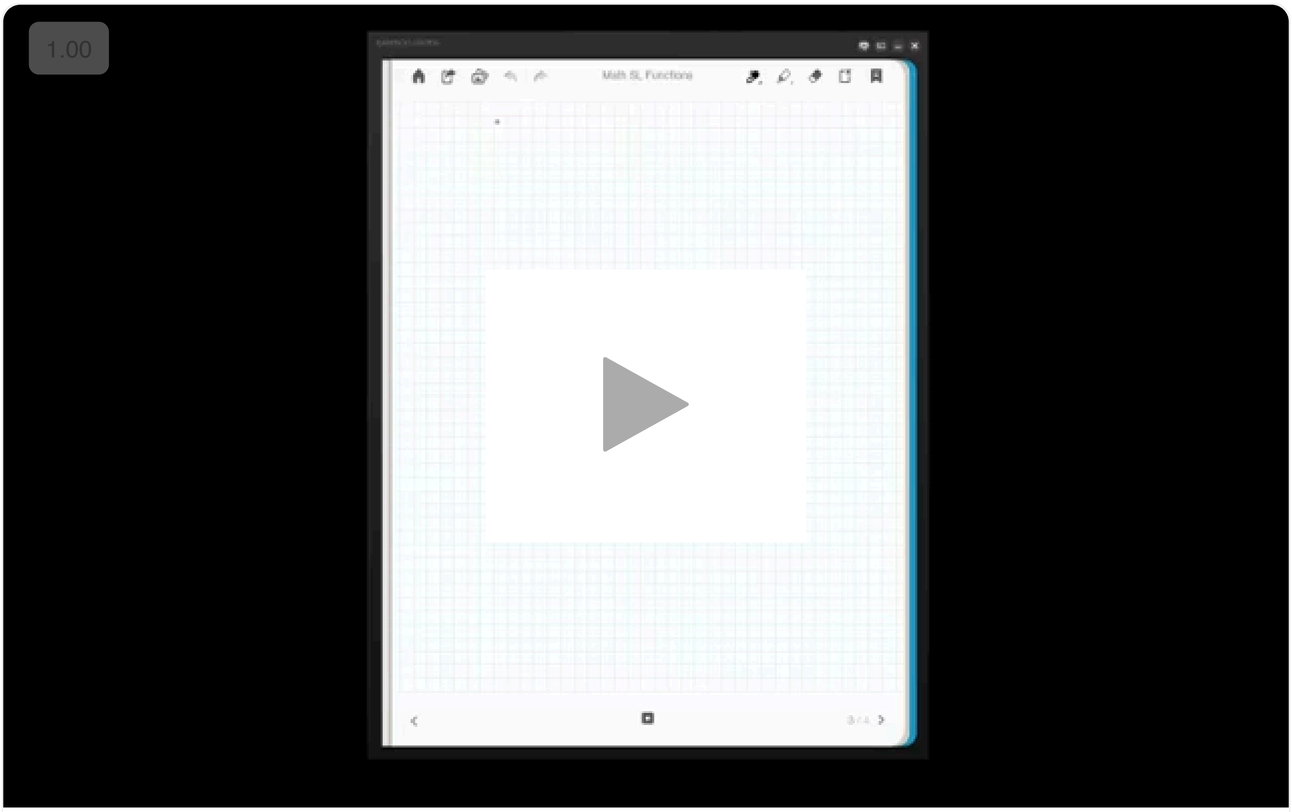

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Video 1. Graphical Implications of the Inverse Function.

More information for video 1

1

00:00:00,534 --> 00:00:03,403

narrator: In this video, we're gonna

investigate inverse functions

2

00:00:03,470 --> 00:00:06,039

and remember what inverse function are.

3

00:00:06,139 --> 00:00:07,975

They undo what another function does.

4

00:00:08,041 --> 00:00:09,643

So if x goes through f

5

00:00:10,010 --> 00:00:11,912

and the image is y equals f of x,

6

00:00:11,979 --> 00:00:14,648

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if put that through f inverse,

I get x back,

7

00:00:14,715 --> 00:00:16,183

or of course the other way around.

8

00:00:16,250 --> 00:00:19,586

If x goes through f inverse

producing y equals f inverse

9

00:00:19,653 --> 00:00:22,289

of x, then bring it through f,

I get x back.

10

00:00:22,623 --> 00:00:26,627

In other words, x through f,

the result through f inverses x

11

00:00:26,693 --> 00:00:29,129

and the opposite directions through two .

12

00:00:29,296 --> 00:00:30,831

So what does it mean in terms of graphs?

13

00:00:30,931 --> 00:00:32,733

Well let's you take a general graph,

14

00:00:33,000 --> 00:00:38,739

then x gets mapped to y,

so it goes through point x, f of x.

15

00:00:39,273 --> 00:00:41,008

Now if I take it y value,

16

00:00:41,375 --> 00:00:45,679

then if that goes

through f inverse, it should produce x.

17

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00:00:45,746 --> 00:00:49,149

So the point y, f inverse of y,

18

00:00:49,449 --> 00:00:53,086

but f inverse of y

is really f inverse of f of x.

19

00:00:53,153 --> 00:00:56,356

And that should of course

produce that value x.

20

00:00:56,557 --> 00:00:59,660

Now here we have a line

f of x is 2x plus 1.

21

00:00:59,726 --> 00:01:01,428

Now let's investigate this.

22

00:01:01,795 --> 00:01:06,233

So if I take a point 1.0,

then it gets mapped

23

00:01:06,633 --> 00:01:08,335

through f of x.

24

00:01:08,669 --> 00:01:11,305

Now, if I take a value 3,

if I put that through

25

00:01:11,371 --> 00:01:14,641

and the inverse function,

it should produce a value 1.

26

00:01:14,775 --> 00:01:18,011

Now let's take one more

point minus 2 and 0.

27

00:01:18,078 --> 00:01:20,013

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That's get mapped through f

28

00:01:20,214 --> 00:01:22,883

to the point y equals minus 3.

29

00:01:23,250 --> 00:01:25,886

So if I take minus 3

through the inverse function,

30

00:01:25,953 --> 00:01:28,288

that should get mapped to minus 2,

31

00:01:28,589 --> 00:01:30,991

'cause that's what I started

with in the blue mapping.

32

00:01:31,225 --> 00:01:32,659

Well now we've got two points

33

00:01:32,826 --> 00:01:36,797

and inverse function of a line

is another line, and here it is.

34

00:01:36,864 --> 00:01:39,766

The inverse of 2x plus 1

is a half x minus a half.

35

00:01:40,400 --> 00:01:44,171

Now let's take one more point of interest

and I'm gonna take minus 1,

36

00:01:44,238 --> 00:01:46,139

which get mapped to minus 1

37

00:01:46,206 --> 00:01:49,543

through blue, but it also gets

mapped to minus 1 through

38

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00:01:49,877 --> 00:01:50,978

the inverse function.

39

00:01:51,211 --> 00:01:53,380

It goes, it is an intersection point.

40

00:01:53,747 --> 00:01:56,817

Now very great importance is the fact that

41

00:01:56,884 --> 00:02:01,522

f and f inverse are reflections

in the y equals x axis.

42

00:02:02,122 --> 00:02:05,526

That will certainly help you to plot them

and it also helps you that

43

00:02:05,592 --> 00:02:09,696

if f intersects y equals x,

44

00:02:09,763 --> 00:02:13,800

then f inverse will intersect

y equals x at exactly the same point

45

00:02:13,901 --> 00:02:15,269

as seen here.

Using the procedure explained in the video, we found that if $f(x) = x^3 - 2$, then $f^{-1}(x) = (x+2)^{\frac{1}{3}}$. The graphs of the two functions are shown in the figure below.

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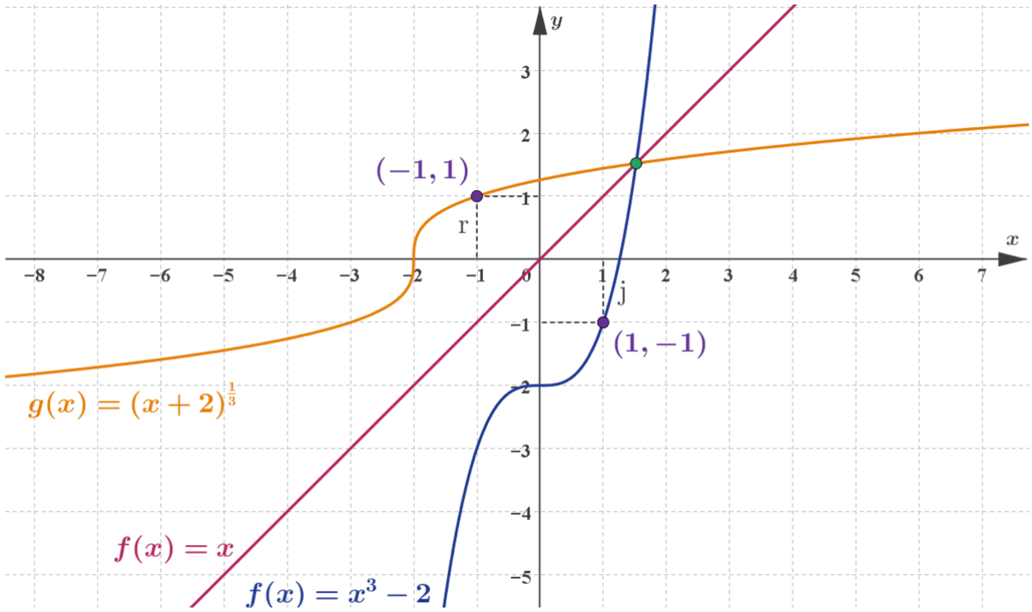

More information

The image is a graph displaying two functions and their inverse relations. (f(x) = x^3 - 2) is shown in blue, and its inverse (g(x) = (x+2)^{1/3}) is depicted in orange. The graph is plotted on a coordinate grid with both x and y axes labeled without specific units. The x-axis ranges approximately from -8 to 8, while the y-axis ranges from -8 to 8.

The pink line (y = x) acts as the line of reflection, where both functions intersect. This pink line exhibits a linear and diagonal increase from the bottom left to the top right of the grid.

Points are marked where the blue curve crosses the orange curve at roughly the coordinates (1, -1) and (-1, 1), confirming the inverse relationship. The functions appear as mirror images, with symmetry about the line (y = x). The annotation texts "(g(x) = (x + 2)^{1/3})", "(f(x) = x^3 - 2)", and point coordinates are included to assist with understanding the graph.

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Notice how the two functions are reflections of each other in the line $y = x$ (pink line). The point of intersection of f and f^{-1} lies on the line $y = x$, as expected.

The procedure for finding the expression for an inverse function is summarised as follows:

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Important

Procedure for finding f^{-1} :

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1. Write $y = f(x)$.
2. Replace all the x's with y's and the y's by x's.
3. Solve the equation until you obtain $y = g(x)$, where $g(x)$ is an expression that does not contain y.
4. Then $g(x)$ is the formula for the inverse function; that is, $f^{-1}(x) = g(x)$.

Example 1

★★☆

Find the inverse of the function defined by $f(x) = \dfrac{1+5x}{3x-2}$.

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Step	Explanation
The domain of f is $\{x \mid x \neq \frac{2}{3}\}$.	Why does the domain of f not include all real numbers?
$y = \dfrac{1+5x}{3x-2}$	Write $y = f(x)$.
$x = \dfrac{1+5y}{3y-2}$	Swap the x's and y's and solve for y.
$(3y-2)x = 1+5y$	Multiply both sides by $(3y-2)$.
$3yx - 2x = 1+5y$	Expand brackets.
$3yx - 5y = 1+2x$	Gather all y's on LHS of the equation.
$(3x-5)y = 1+2x$	Factorise y out and solve for y.
$y = \dfrac{1+2x}{3x-5}$	This formula is valid for $x \neq \dfrac{5}{3}$.
Thus, $f^{-1}(x) = \dfrac{1+2x}{3x-5},\ x \neq \dfrac{5}{3}$	

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Existence of an inverse function

In the Investigation of subtopic 2.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/investigation-id-26023/), you explored conditions for a function to have an inverse.

The following activity revisits this investigation.

Activity

This applet shows the inverse of a given function. Follow the steps and discuss the questions with your fellow students.

Interactive 1. A Graphical Representation of the Inverse of a given Function.

More information for interactive 1

This interactive applet enables users to explore the concept of inverse functions through dynamic visualization and manipulation.
The screen features a coordinate plane where the graph of a function $f(x) = 2x(x - 1)(x - 2)$ is plotted in blue. A diagonal line $y = x$, which acts as the mirror line for inverse functions, is also

Student view

displayed passing through the origin. Users can analyze how the graph of a function and its inverse relate to one another by reflecting points across this line.

Users can modify the function rule using the input box at the bottom of the screen, allowing for real-time updates to the graph.

Below the function input, users can restrict the domain by entering custom bounds. This is particularly useful for ensuring that the inverse of the function is also a function. The inverse relation can be displayed by selecting the “Show inverse relation” checkbox, which reflects the original graph across the line y = x, rendering the inverse curve in orange.

To explore individual values and their mappings, users can activate the “Show points” checkbox. A draggable red point appears on the x-axis, and as users move this point left or right, the corresponding image on the function is shown, along with its reflected point on the inverse. As this red point is dragged, it gradually traces out the inverse curve, making it easier to see how the inverse is formed point by point.

The “Vertical line test” checkbox visually checks whether the inverse graph passes the test, confirming whether it qualifies as a function. Navigation buttons labeled “Zoom +” and “Zoom —” in the lower-right corner allow users to zoom in and out of the graph, while a reset icon at the top-right corner returns the display to its original view.

This interactive offers an engaging way to understand the symmetry between a function and its inverse, how domain restrictions influence invertibility, and how inverse graphs are constructed through reflection.

1. Enter a rule for a function in the \$f(x) ='$ box.
2. Click ‘Show points’ to display a point on the x-axis and the point(s) corresponding to $f^{-1}(x)$.
3. Drag the point to change x. What do you observe as you move x along the axis?
4. Click ‘Show inverse’ to display the graph of the inverse $f^{-1}(x)$. Is it a function? Click ‘Vertical line test’ to show a vertical line which may help you decide.
5. How could you predict from the graph of the original function whether it has an inverse function? Formulate a general rule for a function to have an inverse function.
6. Can you restrict the domain of f to make its inverse a function?

In the activity you discovered that a function has an inverse function if and only if the reflection of its graph about the line $y = x$ passes the vertical line test. This is the same as saying that the graph of the original function passes the **horizonal line test**: any horizontal line intersects the graph of the function at most once.

Making connections

Recall that a **one-to-one** function is any function such that:

- for each value of the input variable x there is only one value of the output variable y
- for each value of y there is only one value of x.

One-to-one functions satisfy both the vertical and the horizontal line tests.

Important

A function f has an inverse function if and only if f is a one-to-one function. The graph of f passes both the vertical and the horizontal line tests.

Example 2

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Consider the function $f(x) = x^2 + 1$.

a) Explain why this function does not have an inverse function.

b) Restrict the domain of f such that it has an inverse function.

c) Find f^{-1} if the domain of f is restricted (as in part b).

	Step	Explanation
a)	f has domain $x \in \mathbb{R}$ and is a many-to-one function (it does not pass the horizontal line test). Therefore it does not have an inverse function.	f maps two different x values to th same y value. For example, $f(x) =$ when $x = -2$ and when $x = 2$.
b)	If the domain of f is restricted to $x \geqslant 0$or $x \leqslant 0$, the function becomes one-to-one and so has an inverse function.	The graph of f passes the horizont line test when the domain is restricted to either $x \geqslant 0$ or $x \leqslant 0$.

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	Step	Explanation
c)	*Case 1:* f is defined by $y = x^2 + 1,\ x \geqslant 0$ To find f^{-1}: $x = y^2 + 1,\ y \geqslant 0$ $y^2 = x - 1,\ y \geqslant 0$ $y = \pm\sqrt{x-1},\ y \geqslant 0$ $y = \sqrt{x-1}$ So $f^{-1}(x) = \sqrt{x-1}$.	There are two cases to consider, depending on how you restricted th domain of f.
	Case 2: f is defined by $y = x^2 + 1,\ x \leqslant 0$ To find f^{-1}: $x = y^2 + 1,\ y \leqslant 0$ $y^2 = x - 1,\ y \leqslant 0$ $y = \pm\sqrt{x-1},\ y \leqslant 0$ $y = -\sqrt{x-1}$ So $f^{-1}(x) = -\sqrt{x-1}$.	

4 section questions

2. Functions / 2.7 Composite functions and inverse functions

Restricted domains

In the example below, you will find the inverse of a quadratic function using the method described in the previous section.

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Example 1

★☆☆

Find the inverse of the function $f(x) = (x-1)^2 - 5$

$$\begin{aligned} y &= (x-1)^2 - 5 \\ y+5 &= (x-1)^2 \\ x-1 &= \pm\sqrt{y+5} \\ x &= 1 \pm \sqrt{y+5} \\ y &= 1 \pm \sqrt{x+5} \\ f^{-1}(x) &= 1 \pm \sqrt{x+5} \end{aligned}$$

In this example, the quadratic function was already given in vertex form (see section 2.5.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/vertex-form-of-a-quadratic-function-id-27841/)). How could you find the inverse if it is given in general form $ax^2 + bx + c$? You can convert it into vertex form and use the above method to find the inverse.

✓ **Important**

It is important to note that you should **not** try to move the constant to the left-hand side. For example, if $f(x) = x^2 + x - 5$:

$y = x^2 + x - 5$ [replacing $f(x)$ by y]

$y + 5 = x^2 + x$ [it is now difficult to isolate x if you do this. A better approach is to convert it to vertex form and proceed as in Example 1.]

Alternatively, you can use the quadratic formula in order to isolate x (see section 2.5.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/standard-form-of-quadratic-functions-id-27840/)) and as shown in the example below.

Example 2

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Find the inverse of $f(x) = x^2 + 2x - 1$.

Change $f(x)$ to y, then subtract y from bith sides to form a quadratic equation in x:

$$y = x^2 + 2x - 1$$
$$x^2 + 2x - 1 - y = 0$$

Using the quadratic formula,

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(-1-y)}}{2} && \text{[note that } c = -1 - y \text{ in this ca} \\ x &= -\frac{-2 \pm \sqrt{4+4+4y}}{2} \\ x &= \frac{-2 + \sqrt{8+4y}}{2} \\ x &= \frac{-2 \pm 2\sqrt{2+y}}{2} \\ x &= -1 \pm \sqrt{2+y} \\ y &= -1 \pm \sqrt{2+x} && \text{[interchanging } x \text{ and } y\text{]} \\ f^{-1}(x) &= -1 \pm \sqrt{2+x} && \text{[replacing } y \text{ by } f^{-1}(x) \end{aligned}$$

The two functions and their inverses that we just found are given below:

$$f(x) = (x-1)^2 - 5 \implies f^{-1}(x) = 1 \pm \sqrt{x+5}$$

$$f(x) = x^2 + 2x - 1 \implies f^{-1}(x) = -1 \pm \sqrt{2+x}$$

You may have noticed that both of these functions had two inverse functions because of $\pm$ before the radical . Does this make sense? Can we have two inverse functions for a function? If not, which one is the right inverse for $f(x)$?

Recall what you learned in section 2.2.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-concept-of-an-inverse-function-id-26021/):

1. A function and its inverse are reflections in the line $y = x$.
2. In order to have an inverse, a function should be one-to-one.

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view

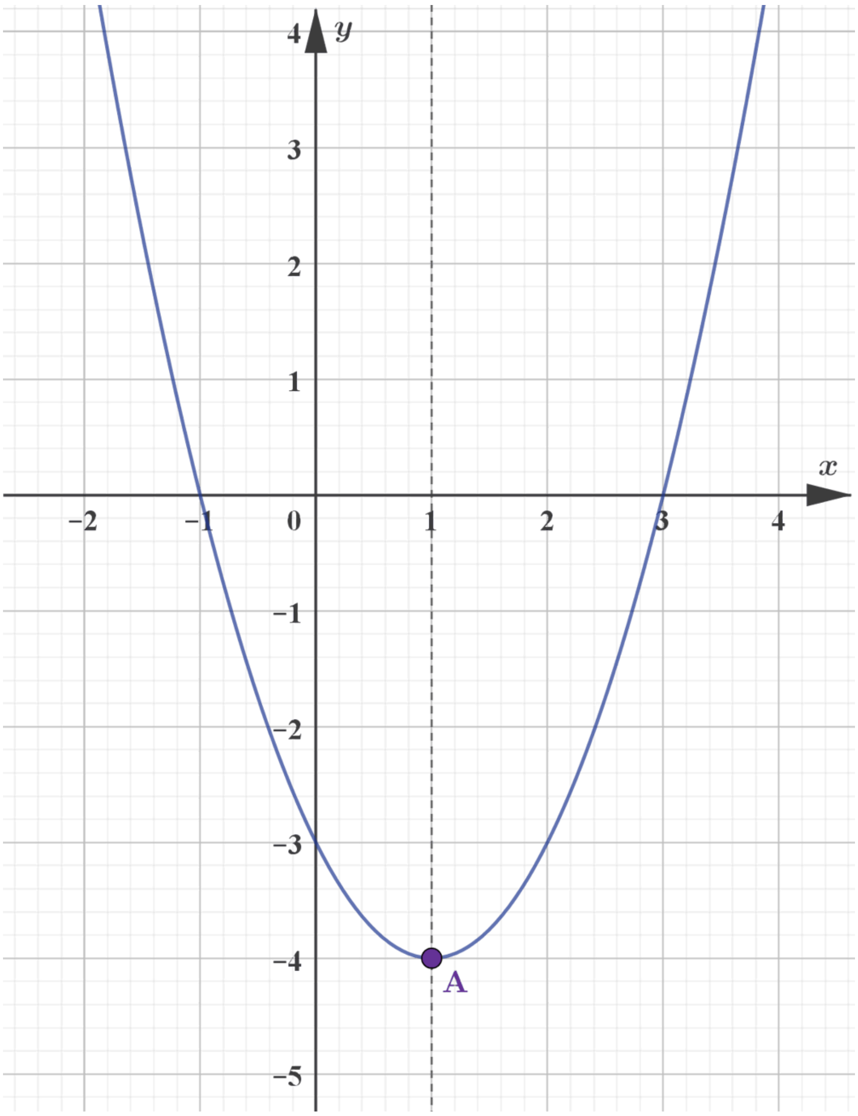
Which of the two inverse functions is a reflection of the given function $f(x)$ in $y = x$?

Is a quadratic function one-to-one?

For every value of y, there are two values of x. Hence it is not one-to-one. Therefore, it is important that you first make $f(x)$ a one-to-one function.

Is there a y-value on the quadratic graph with one x-value?

In order to make it into a one-to-one function so that you can find its inverse, you can restrict the function to a part of the graph as shown below:



More information

The image is a graph displaying a parabolic curve intersecting the y-axis at its vertex. The x-axis and y-axis are both labeled with integers, with the x-axis ranging from -2 to 2 and the y-axis ranging from -5 to 5. A point labeled 'A' is marked on the curve at the vertex, indicating the lowest point of the parabola. The curve appears symmetric around the y-axis and is positioned on a grid for reference. This graph illustrates how restricting the graph to one side of its line of symmetry results in a one-to-one function.

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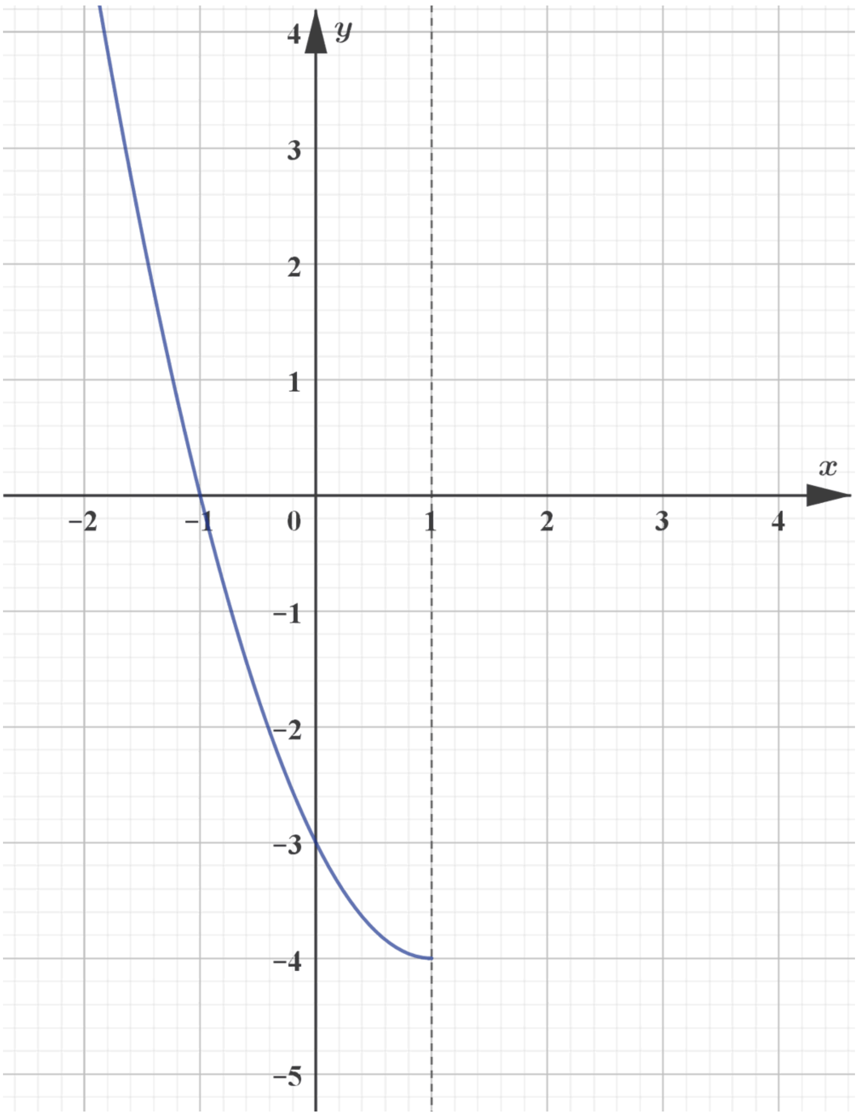
The graph is cut into two parts along the line of symmetry . If you consider one part, it becomes a one-to-one function.

For example, the part which is to the left of the axis of symmetry (red dotted line) will have only one x-value mapped to one y-value. In order to make this partition, you restrict the domain of the function to $(-\infty, x_{min}]$ or $[x_{min}, \infty)$, so that the graph would look as shown below.

The function considered here is $g(x) = x^2 - 2x - 3$.



More information

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The image is a graph that illustrates the function (g(x) = x^2 - 2x - 3). The X-axis represents the input values of (x) and spans from -2 to 2. The Y-axis represents the function values (g(x)), ranging from -5 to 5. The graph displays a curve typical of a quadratic function, opening upward. The function's minimum point, where the value of (x) is 1, corresponds to (g(x) = -4). From the description, the domain of the function can be restricted to either ((-\infty, 1]) or ([1, \infty)). The grid marks the coordinates, with tick marks at regular intervals, aiding in visualizing and plotting the exact points along the curve.

[Generated by AI]

The minimum point is $(1, -4)$ and the domain can be restricted to either $(-\infty, 1]$ or $[1, \infty)$.

For simplicity, take the domain to be $(-\infty, 1]$

The range is $[-4, \infty)$. You may recall that the range of a function is equal to the domain of its inverse (see section 2.2.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-concept-of-an-inverse-function-id-26021/)). Therefore, the domain of the inverse, in this case, is $[-4, \infty)$.

Finding the inverse of the above function using any of the methods stated above, you get:

$$g^{-1}(x) = 1 \pm \sqrt{x+4}$$

In order to identify which of the two options is reasonable, you need to apply the restricted domain $(-\infty, 1]$ of $g(x)$, because this is the range of $g^{-1}(x)$.

⚠ Be aware

Always keep in mind both the domain and range of the original function and be careful to apply them to the inverse function as required.

Hence, the above inverse function $g^{-1}(x) = 1 \pm \sqrt{x+4}$, should have its y-values less than or equal to 1.

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This is possible only when you choose, $g^{-1}(x) = 1 - \sqrt{x+4}$, since $1 - \sqrt{x+4} \leqslant 1$.

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Exam tip

Always use the range of the inverse function to eliminate the unwanted part before you give the final answer. This process will have a 'reason' mark and only one right answer will be considered for the final answer mark in the examination.

Example 3

★★☆

1. $f(x) = (x-1)^2 - 5 \Longrightarrow f^{-1}(x) = 1 \pm \sqrt{x+5}$
2. $f(x) = x^2 + 2x - 1 \Longrightarrow f^{-1}(x) = -1 \pm \sqrt{2+x}$

Give the correct inverse functions for the above two functions by restricting their domains.

1. $f(x) = (x-1)^2 - 5 \Longrightarrow f^{-1}(x) = 1 \pm \sqrt{x+5}$
   Minimum point $(1, -5)$
   Restricted domain: $(-\infty, 1]$
   All y values of $f^{-1}(x) = 1 \pm \sqrt{x+5} \leqslant 1 \Rightarrow f^{-1}(x) = 1 - \sqrt{x+5}$.
2. $f(x) = x^2 + 2x - 1 \Longrightarrow f^{-1}(x) = -1 \pm \sqrt{2+x}$
   Minimum point $(-1, -2)$
   Restricted domain: $(-\infty, -1]$
   All y values of $f^{-1}(x) = -1 \pm \sqrt{2+x} \leqslant -1 \Rightarrow f^{-1}(x) = -1 - \sqrt{2+x}$

Note: If the restricted domains in the above were given as $[1, \infty)$ and $[-1, \infty)$, your answer should be $1 + \sqrt{x+5}$ and $-1 + \sqrt{2+x}$ respectively.

Activity

Consider $f(x) = 3x + 2$ and $g(x) = x^2 + 1$.

Find

(1) $f^{-1}(x)$

(2) $g^{-1}(x)$

(3) $f \circ g(x)$

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(4) $(f \circ g)^{-1}(x)$

(5) $\left(f^{-1} \circ g^{-1}\right)(x)$

(6) $\left(g^{-1} \circ f^{-1}\right)(x)$

What do you infer from the answers to questions (4), (5) and (6)? Explain your answer with more examples.

International Mindedness

Imagine studying graphs and inverses of a function without the notation that we are using now ($f(x)$).

How complicated it would have been to find the domain and range, or to find out whether the function is odd or even? When was this notation developed? Who discovered it? How did it become internationally accepted?

Theory of Knowledge

Is there any connection between the notation used in mathematics and culture or history? Can mathematics exist without the effect of culture? Justify your answers with examples.

3 section questions

2. Functions / 2.7 Composite functions and inverse functions

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/checklist-id-27974/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- find the composition of two functions
- find the domain and range of a composite function

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view

- verify whether a function is the inverse of another function
- determine whether a given function has an inverse function, and appropriately modify a function so that it has an inverse function
- find the formula for the inverse function of a given function
- restrict domain to find an inverse.

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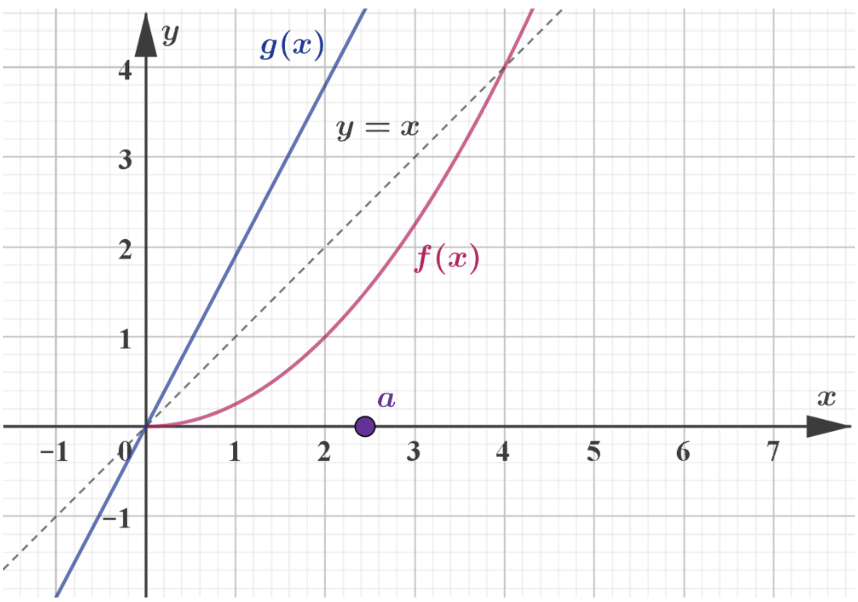
2. Functions / 2.7 Composite functions and inverse functions

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/investigation-id-27975/print/) Assign

In this subtopic we have explored how to find the composite function of two functions using algebra. The composition of two functions can be expressed graphically as well.

The figure shows the graphs of functions f and g, the line $y = x$ and an input value a. Think about how you could use the figure and the ideas of this subtopic to locate the point $(a,\ g(f(a)))$ that lies on the graph of the composite function $g(f(x))$. Explain your process step by step, together with the reasoning behind it.



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More information

The image shows a graph with a grid background, depicting three functions across a Cartesian coordinate system. The functions are labeled as (f(x)) and (g(x)), along with the line (y = x). The X-axis represents the input values, while the Y-axis represents the output. The graph of (f(x)) is in magenta, (g(x)) in blue, and (y = x) in gray. An input value is marked as (a) with a purple point on the X-axis.

- The graph of (f(x)) is a curve that begins at a point close to the X-axis and curves upwards steeply as you move from left to right.
- The graph of (g(x)) is another curve, which also increases, but more gradually compared to (f(x)), and it intersects the line (y = x) at approximately (1, 1).
- The line (y = x) is a diagonal line starting from the origin (0,0) and extending through the grid diagonally, where the value of y equals the value of x.
- The input value (a) is situated around 2 on the X-axis.

This setup illustrates a scenario for finding and explaining a point on the graph of the composite function (g(f(x))).

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For a hint, explore the applet below: enter some different formulas for functions f and g, choose a few different input values on the x- axis and push the slider slowly to the right.

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Interactive 1. A Graphical Representation of the Composite Function of Two Functions.

More information for interactive 1

This interactive tool enables users to explore composite functions through dynamic, graphical visualization. The screen is divided into two sections. On the right, a coordinate plane displays labeled x- and y-axes, a dotted line representing $y = x$, and two function graphs — one in red and one in blue. A draggable point labeled $(a, 0)$ sits on the x-axis, and a "Center (0,0)" button repositions the origin to the center of the screen for easier viewing.

On the left side, users can input and modify two functions: $f(x)$ (displayed in blue) and $g(x)$ (displayed in red). Two buttons — "Show $f(g(a))$" and "Show $g(f(a))$" — allow users to visualize the respective compositions. A slider animates the process, gradually revealing how the input value a is transformed through the inner and then the outer function.

For instance, when $f(x) = 0.5x$ and $g(x) = 0.25x^2$, the red graph shows a parabola opening upward, while the blue graph is a straight line through the origin. If the "Show $f(g(a))$" checkbox is clicked, then as users move the slider, a sequence of lines illustrates the transformation: a red vertical segment rises from $(a, 0)$ to $(a, g(a))$, then a horizontal segment moves to $(g(a), 0)$ followed by a blue vertical rise to $(g(a), f(g(a)))$, clearly tracing the full composition.

A "Reset" button in the bottom left corner resets the graph to its initial state, allowing users to try new combinations.

This dynamic visualization deepens users' conceptual understanding of function composition by showing how inputs are transformed step by step through multiple functions, and how the output of one function becomes the input of another.

Rate subtopic 2.7 Composite functions and inverse functions

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