TOPIC 3
GEOMETRY AND TRIGONOMETRY

SUBTOPIC 3.6
VORONOI DIAGRAMS

Overview (/study/ap 122- cid- 754029/

Teacher view

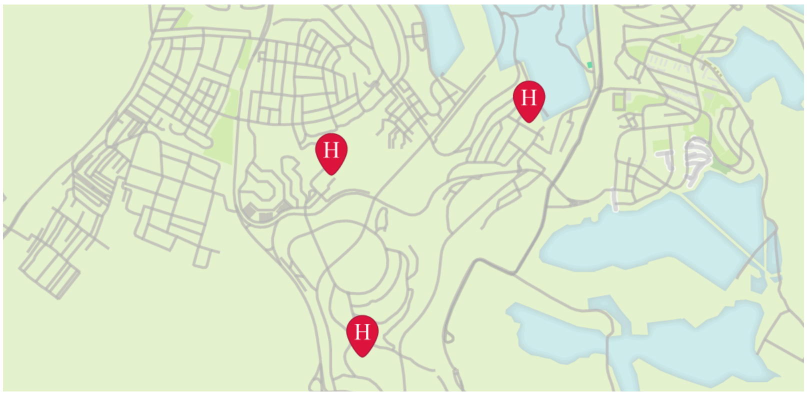
3. Geometry and trigonometry / 3.6 Voronoi diagrams

The big picture

Nowadays most fast food and restaurant chains have delivery options. You can call their number or use the internet to place an order and the food will be delivered to your door. However, how do they know what area each restaurant serves? How do they choose where to locate their restaurants?

Meteorologists collect data about the weather all year around. They position their weather stations strategically, so each covers a certain area. How do they position them?

Many similar strategical location problems can be solved with the help of geometry, especially using loci of points. If you are looking for a locusof points that are equidistantfrom a fixed point, then you are looking for a circle. If you are looking for a locus of points that are equidistant from two points, then you are looking for a perpendicular bisectorof the segment between them. What if you want a locus that is equidistant from three points? Or more?



More information

The image is a map showing part of an urban area with various roads and water bodies. Three red pins labeled with the letter 'H' are positioned at different locations on the map, suggesting important or significant points. The map shows intricate street details with curves and intersections, likely representing residential or urban planning design. Water bodies are present in the top right and bottom areas of the image. The 'H' markers could indicate hospitals or other significant landmarks equidistant from each other or certain points, relating to the discussion about loci of points being equidistant from multiple locations.

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Concept

Voronoi diagrams are tools for analysing spaces, locations and paths with respect to established points, known as sites. They can identify optimum locations and the relationship between positions.

How would you identify the optimum location for a new hospital? A new school? Or a new restaurant?

How does analysing the territories of animals help to preserve wildlife?

Voronoi diagrams

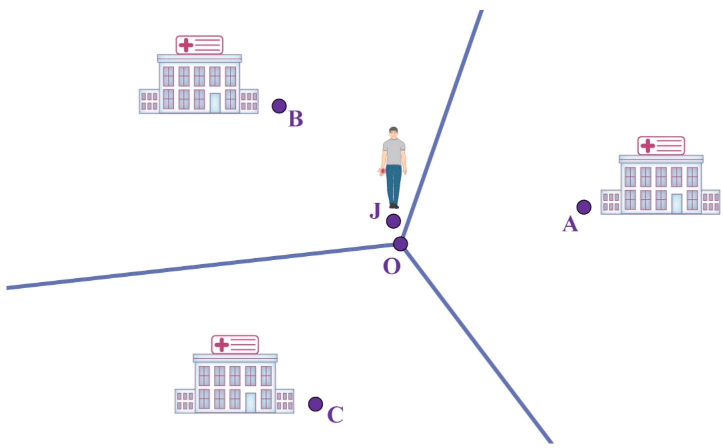
Dividing a plane into regions

A Voronoi diagram of a set of sites (points) is a collection of regions that divide up a plane. Each site has a surrounding region, such that the points in the region are closer to the site than to any other site. In this section you will divide a plane into convex regions.

Equidistant from three points

In subtopic 3.5 (/study/app/m/sid-122-cid-754029/book/the-big-picture-id-27404/), you used the perpendicular bisector of a line segment connecting two points to find the locus of points equidistant from the two points. How can you find the locus of points that are equidistant from three points?

Consider an emergency call made to a central call centre. If there are three hospitals in the area, which one of the emergency teams should respond to the call? If the hospitals are sites, then a Voronoi diagram of the area divides it into regions that are closest to each hospital, A, B or C, as shown below. As James is in the region around B, that is the closest hospital to him so he will receive an ambulance from hospital B.



More information

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The image is a Voronoi diagram illustrating three hospitals labeled A, B, and C and their respective regions. Each hospital's region is determined by proximity, with the diagram divided into three sections by lines that represent boundaries equidistant between two hospitals. The intersecting point of these boundaries is labeled O. A person, labeled J, is located in the region closest to Hospital B, indicating that Hospital B is the most suitable choice for sending an ambulance. The diagram visually demonstrates how divisions in a Voronoi diagram determine the nearest hospital based on the patient's location. These divisions are defined by edges or boundaries, and intersection points like O are referred to as vertices. Each section is known as a Voronoi cell, with point O equidistant from all three hospitals.

Geometrically speaking, the lines between two points are equidistant from the two points on either side, and the point O is equidistant from all three points. The line segments dividing cells are called edges or boundaries and the intersection points of edges, such as point O, is called a vertex. Each region is called a Voronoi cell.

Important

All points in a Voronoi cell are closest to the site in the cell.

Activity

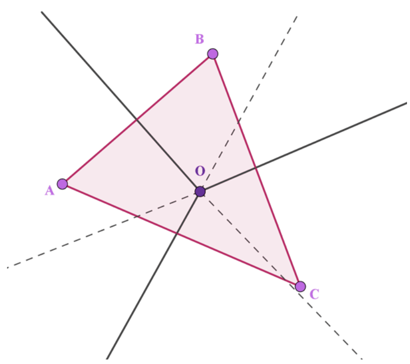
In this activity, you will be investigating how to divide a plane into three regions.

1. Use diagram-drawing software, such as GeoGebra, to draw a triangle ABC .
2. Construct the perpendicular bisectors of each side of the triangle.
3. Mark the intersection of the perpendicular bisectors as O .
4. Draw a circle with centre O and passing through one of the vertices of triangle ABC .
5. Drag the vertices of the triangle and observe how the shape of the circle changes.

What do you notice?

When you draw the perpendicular bisectors of the sides of a triangle ABC, they intersect at one point, O, see the diagram on the left.

This point is the centre of the surrounding circumscribed circle or circumcircle. It is called the circumcentre, as shown in the diagram on the right.



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More information

The image is a diagram showing a triangle with three vertices labeled A, B, and C. Lines are drawn from each vertex and intersect at a point labeled O, which appears to be the centroid or an important center of the triangle. Each vertex is marked with a circle. The triangle is filled with a shaded area, and additional dashed lines are shown extending from each vertex, possibly indicating medians or altitudes. The relationships between point O and the triangle's vertices are a central focus.

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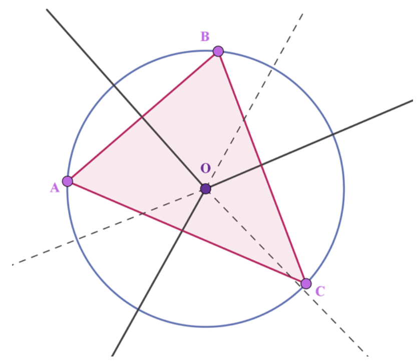

More information

The image is a geometric diagram featuring a triangle labeled A, B, and C, inscribed inside a circle. The circle has a center labeled O. The triangle ABC is highlighted with a reddish shade and lines extend from the points A, B, and C to intersect at the circle's circumference. The circle, drawn in blue, intersects at the triangle's vertices. Additionally, dashed lines are drawn from the center O to each vertex of the triangle, probably indicating the radius of the circle. The diagram shows relationships between the circle's center, the triangle's vertices, and the enclosing circumcircle.

[Generated by AI]

If you would like to learn how to construct the perpendicular bisectors of the sides of a triangle and also the circumscribed circle, follow the link here (https://www.mathsisfun.com/geometry/construct-trianglecircum.html).

The circumcentre O is the vertex of the Voronoi diagram for three points, see the diagram on the left, which shows the perpendicular bisectors of the sides of triangle ABC .

The edges of the diagram start at the vertex and extend to the end of the plane, see the diagram on the right, which shows the Voronoi diagram for three points. Each edge of a Voronoi diagram lies on the perpendicular bisector of the line segment joining two sites.

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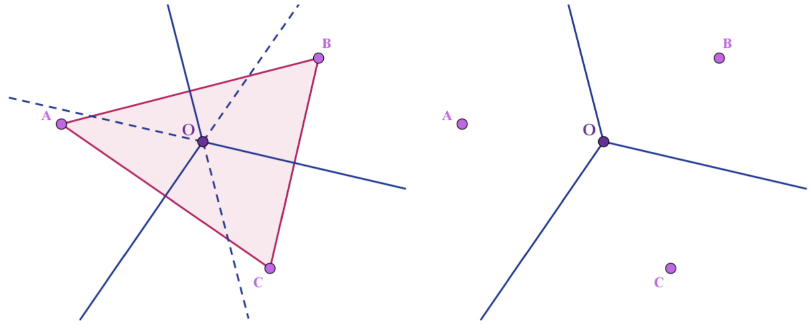

More information

The image displays two diagrams side by side. On the left, a triangle is constructed with vertices labeled A, B, and C, forming a closed shape filled with a pink area. The point labeled Q is located inside this triangle, with several dashed lines extending from it to form the perpendicular bisectors of the triangle's sides. On the right, a Voronoi diagram shows the perpendicular bisectors extending outwards from a central point labeled O. Points A, B, and C are placed around O, with lines dividing the plane into regions corresponding to the proximity to these three points. The diagram illustrates the concept of Voronoi partitions based on equidistant perpendicular bisectors between the sites.

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Example 1

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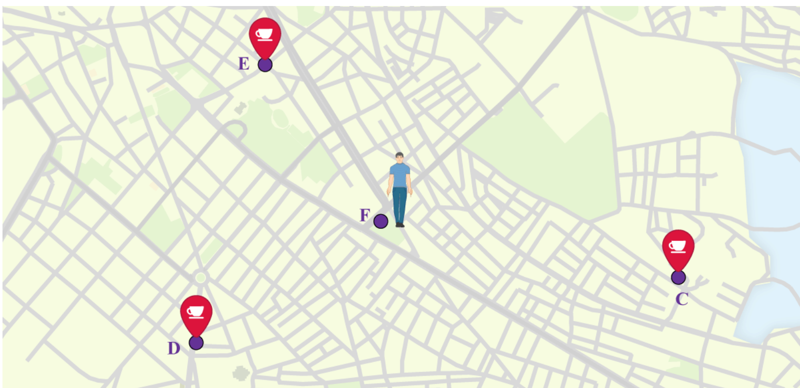
Alicia is visiting a city. After visiting all the historical sites, she wants to have a cup of coffee and something to eat. From her map, she can see that three coffee shops are nearby, located at C, D and E. She is at F. Using a Voronoi diagram, find which of the coffee shops is closest to her.

Note: Either copy the map into GeoGebra, and do your work there, or print it out and construct the diagram using a compass and ruler.



More information

The image is a map with various roads and intersections. The map includes icons in the shape of a coffee cup, each marked with a red pin and labeled with letters. Label 'C' is located towards the bottom right, near a body of water. Label 'D' is more centrally located towards the bottom of the map. Label 'E' is placed towards the top part of the map, surrounded by a grid of streets. Label 'F' is slightly off-center towards the middle of the image, and is

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accompanied by the figure of a person standing on the road. The map seems to represent a city layout with various locations marked for reference or points of interest. There are no units of measurement or scale provided.

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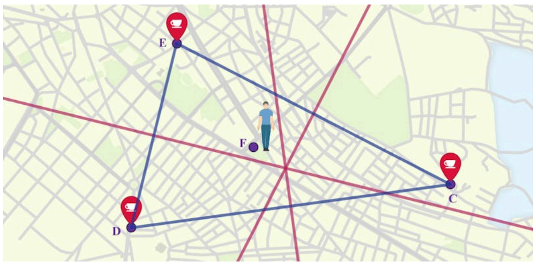
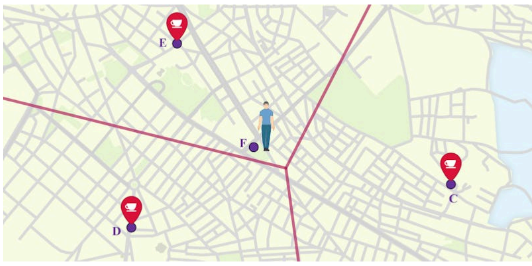
Steps	Explanation
	Draw the perpendicular bisectors of the sides of the triangle CDE.
	Draw the Voronoi diagram.
Coffee shop E is closest to her.	She is in the same region as E.

Example 2

★★☆

The graph shows a partially drawn Voronoi diagram for points $\mathrm{A}(2,1)$, $\mathrm{B}(4,4)$ and $\mathrm{C}(7,1)$. If $\mathrm{D}(4.5,1.5)$ is the vertex of the Voronoi diagram, find the equation of the third edge of the diagram and complete the diagram.

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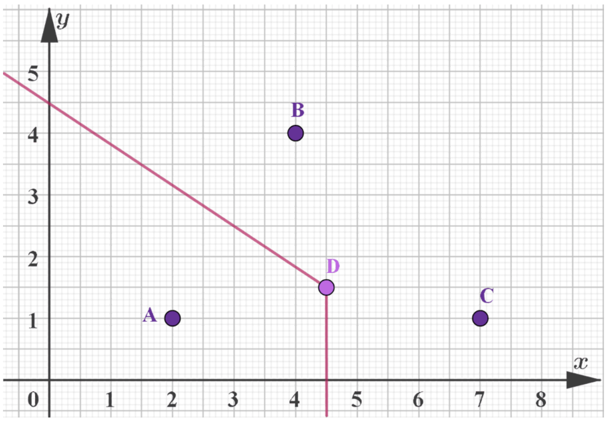

More information

The image shows a partially drawn Voronoi diagram on a grid. The diagram includes four labeled points: A(2, 1), B(4, 4), C(7, 1), and D(4.5, 1.5). Point D is identified as the vertex of the Voronoi diagram. There are two visible edges that intersect at point D and extend towards the other points, creating linear boundaries between the regions. The grid background suggests a coordinate plane for precise positioning of the points. The task involves finding the equation of the third edge and completing the diagram to section the plane correctly based on the distances to these points.

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Steps	Explanation
$m_{BC} = \dfrac{4-1}{4-7} = -1$	You need to find the gradient of BC as the third edge is the perpendicular bisector of BC.
$m = 1$	The gradient of the perpendicular bisector is the negative reciprocal of the gradient of BC.
$y - 1.5 = 1\,(x - 4.5)$	Because point D(4.5,1.5) lies on the perpendicular bisector of BC.
$y = x - 3$	Rearrange.

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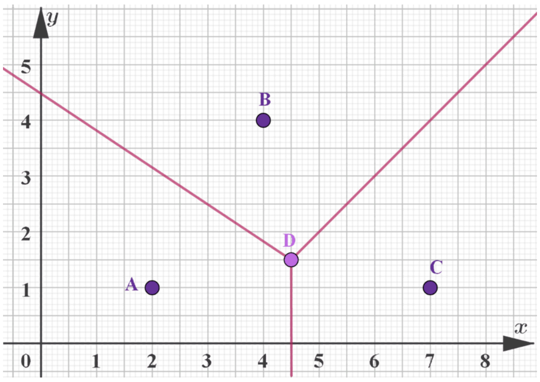
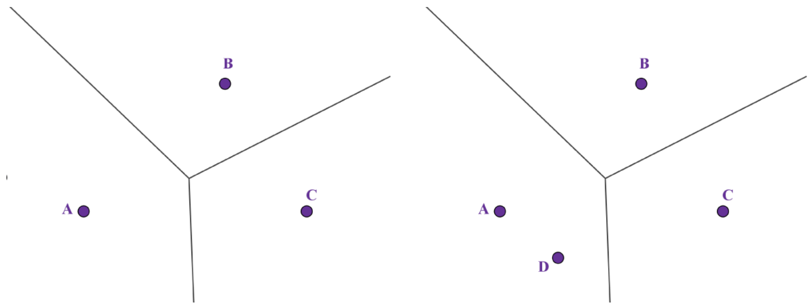
Steps	Explanation
	Complete the diagram.

Four or more points (sites)

If you have four points and wanted to divide the plane into four regions, how would you do that using the method for constructing a Voronoi diagram?

The first step is to divide the plane into three regions using three of the points, as shown in the first diagram below. How would you divide the plane into four Voronoi cells using the fourth point D, as shown in the second diagram below?



More information

The image contains two diagrams. The first diagram shows a plane divided into three regions using three points labeled A, B, and C. Each region is bounded by lines that maintain equal distance from the neighboring points, forming separate cells around each labeled point. In the first diagram, point A is in the lower left, point B is at the top, and point C is in the middle.

In the second diagram, a fourth point labeled D is introduced. The plane is now divided into four Voronoi cells, each enclosing one of the points A, B, C, and D. Point D is placed in the lower right section of the plane. The cells are adjusted to keep all bounding lines equidistant from the neighboring points. The introduction of point D changes the arrangement of boundaries and the shape of the Voronoi cells. The position of all points are adjusted accordingly to maintain equidistance, and the boundaries between cells shift to reflect the changes.

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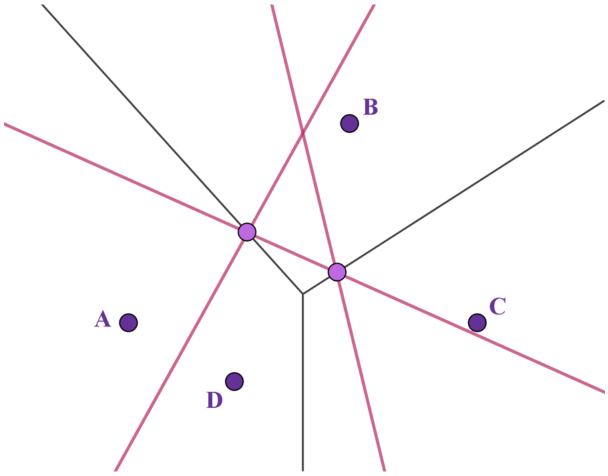
There are various algorithms that can add a site to a Voronoi diagram. Here you will use the incremental algorithm, which builds a Voronoi diagram recursively by adding one site at a time.

Incremental algorithm to add sites

As shown below, to add site D to the diagram, you first draw the perpendicular bisectors with all the sites nearest to D, which here are A, B and C. Then you mark the points where these new perpendicular bisectors intersect the existing edges of the Voronoi diagram. The intersections occur on existing edges as they are circumcentres.



More information

The image is a Voronoi diagram illustrating the addition of site D. It shows a set of labeled points A, B, C, and D. Perpendicular bisectors are drawn from site D to the nearest existing sites A, B, and C. These bisectors intersect the existing edges of the Voronoi diagram, forming circumcenters at the points of intersection. The bisectors are shown as pink lines crossing over the black lines that represent the existing edges of the Voronoi cells corresponding to sites A, B, and C. The diagram visually demonstrates how the inclusion of a new site can alter the structure of the Voronoi cells, depicted by the intersections of the new and old bisectors.

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To find the edges:

- Site A is the closest site to the new site D, so we begin with the perpendicular bisector for A and D. The edge starts at the edge of the plane and extends to the intersection point with the perpendicular bisector of AB.
- Draw the edge between sites A and B.
- Draw the edge between sites B and D, from intersection to intersection.
- Draw the edge between sites B and C.
- Finally, draw the edge between the sites D and C.

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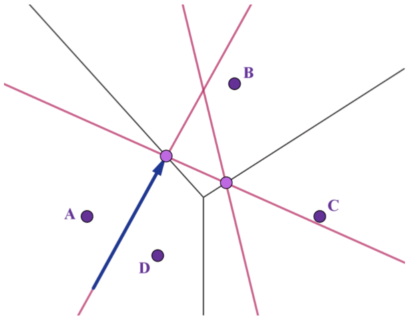

Drawing the edge between sites A and D

More information

The diagram illustrates a network of lines intersecting at a central point. There are labeled points A, B, C, and D, each represented by a dot. Point A is located to the left of the central junction, point B is above, point C is to the right, and point D is below and to the left. The lines connecting these points converge at the central point, forming a star-like pattern. An arrow is marked along the line connecting points A and D, indicating directionality from A to D.

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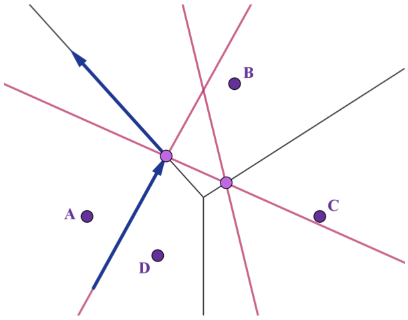

Drawing the edge between sites A and B

More information

The image is a geometric diagram showing several lines intersecting at different points, labeled as A, B, C, and D. Each line is drawn in different directions and intersect at a central node, marked with a purple circle. From this central node, arrows indicate direction along the lines. The lines are represented with different styles and may imply different relationships or equations. The diagram appears to explore relationships between these points and their connected lines, potentially illustrating angles or vectors.

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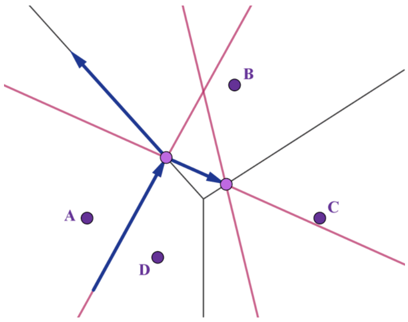

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Drawing the edge between sites B and D

More information

This diagram illustrates a geometric structure with four labeled points: A, B, C, and D. The central focus is a network of lines converging at a central point. Point A is connected to the central point with a line featuring an arrow pointing toward the center. Similarly, point D is linked to the central point with an arrowed line. Points B and C are also displayed, each connected by lines without arrows extending towards the center. The overall arrangement suggests a system of directional relationships among the points, signified by the arrows on lines. The layout does not detail specific metrics but emphasizes directional flow in a network form.

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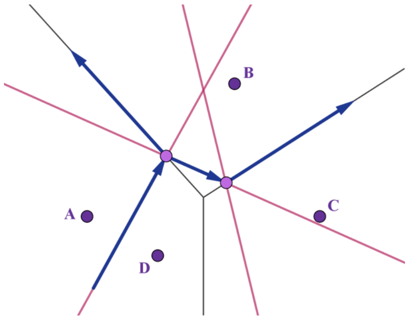

Drawing the edge between sites B and C

More information

The diagram shows a network of interconnected points with arrows indicating directions. There are four labeled points: A, B, C, and D. The arrows are originating and ending at these points, illustrating a flow of movement between them. The connections between the points form a crisscross pattern, indicating multiple pathways between the points. The arrows are labeled with letters representing the points they connect. Each pathway is clearly marked, showing the direction of flow towards the associated points, which are marked with circular dots on a plane.

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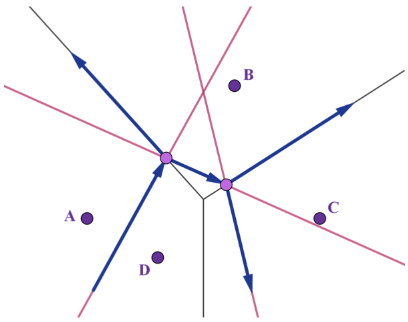

Drawing the edge between sites D and C

More information

The diagram represents a network of interconnected nodes labeled A, B, C, and D. Each node is connected with directed arrows indicating flow or directionality. The central point has arrows going towards nodes B and C, while nodes A and D have arrows pointing towards the central node. The arrows are color-coded, likely to differentiate between types or categories of connections. The nodes and arrows are arranged in a way that suggests a hierarchical or prioritized flow between multiple sites.

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Important

When you add a new site (point) to a Voronoi diagram, it will be in the current region of the nearest site. First, draw all the perpendicular bisectors between the new site and the other sites. Then, draw the edges, starting with the edge between the new site and the nearest site.

Activity

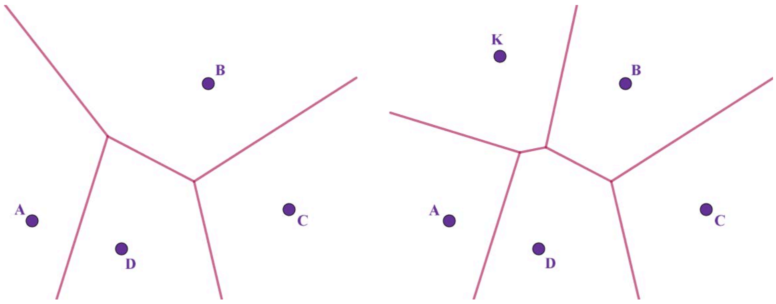
Use this link
(http://www.netlogoweb.org/launch#http://www.netlogoweb.org/assets/modelslib/Sample%20Models/Mathematics/Vo
to investigate how changing the number of sites, and moving a site, changes a Voronoi diagram.

1. Change the number of sites using the number button.
2. Drag a point (site) using your mouse and watch the diagram slowly change.

What do you notice? Think about it.

The diagram on the left, below, is the Voronoi diagram for the four sites, A, B, C and D.

When site K is added, you need to draw only the perpendicular bisectors between K and its three nearest sites A, B, and D.



More information

The image consists of two diagrams side by side. In the left diagram, four points labeled A, B, C, and D are connected by lines forming convex regions, showing perpendicular bisectors among these points. In the right diagram, a new point K has been added. The lines have been adjusted to show perpendicular bisectors between K and its three nearest sites A, B, and D, as described in the text provided before the image. The diagram illustrates how the addition of point K alters the spatial relationships, reorganizing the divisions formed by the bisectors.

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Be aware

Note that site C is not within the circumscribed circles for triangles ABK or ADK, so the perpendicular bisector for C and new site K is not relevant.

Activity

The applet below is a Voronoi diagram for four sites A, B, C and D.

Add one more site, E, to the diagram and investigate how the location of the new site changes the Voronoi diagram.

Interactive 1. Find How the Location of the New Site Changes the Voronoi Diagram.

More information for interactive 1

This interactive illustrates a Voronoi diagram for four initial sites: *A*, *B*, *C* and *D*. Each site has a corresponding Voronoi cell, which consists of all points in the plane that are closer to that site than any other. The black edges represent the boundaries between regions, where points are equidistant from two neighboring sites. The vertices (intersection points of edges) indicate locations that are equidistant from three or more sites.
Also users can add a new site, *E*, and observe how its position influences the Voronoi diagram. When site *E* is introduced, new perpendicular bisectors are drawn between *E* and its nearest neighboring sites, altering the existing boundaries and creating a new Voronoi cell for *E*. Users can drag points to dynamically modify the diagram, exploring how site locations affect the overall structure.

Example 3

★★★

In the figure, $\mathrm{B}(7,-2)$, $\mathrm{C}(5,2)$ and $\mathrm{D}(8,2)$ divide the plane into three regions, which contain the points closest to each of the sites.

Copy the diagram. Add one more site $\mathrm{A}(1,0)$ and then identify the Voronoi cell for A.

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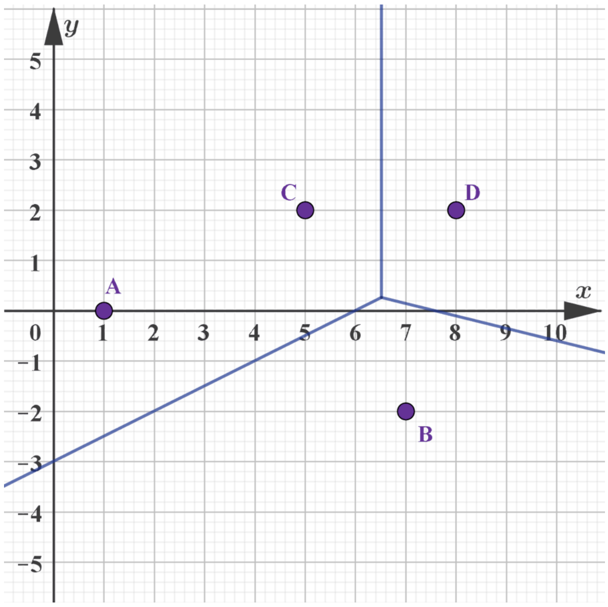
Hence, identify the closest site to point $\mathrm{K}(4, -1.5)$.



More information

The image shows a 3D grid graph with labeled points marked on it. The graph has three axes intersecting at the origin. Each axis is labeled with a scale from 1 to 5. Four purple circular points are marked on the graph, labeled as A, B, C, and D. These points seem to represent coordinates in a 3D space, similar to positions within a coordinate grid, possibly aligned with specific coordinates defined relative to axes scales. The point closest to K(4, -1.5) seems to be one of the displayed points, and you are tasked with identifying it based on their positions.

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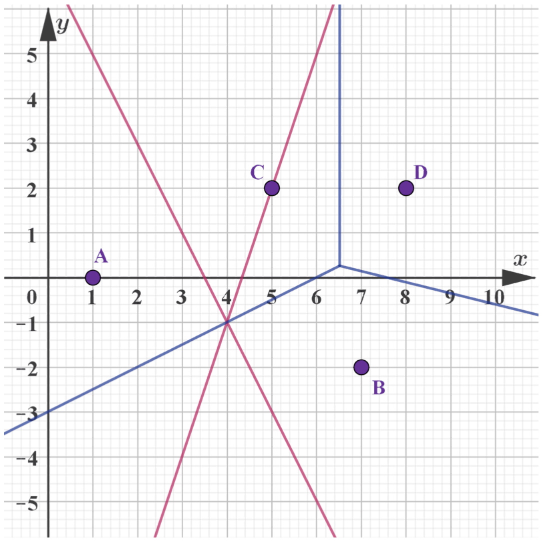
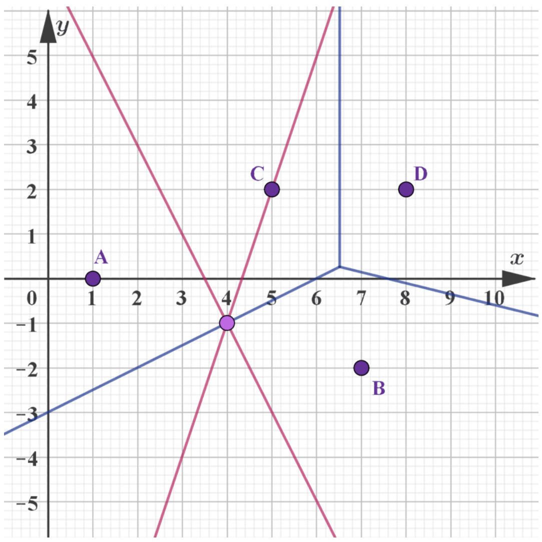
Steps	Explanation
	Draw the perpendicular bisectors of AC and AB. Ignore site D as it is outside the circumscribed circle of ABC. In this case, the perpendicular bisector for AB passes through C.
	Mark the intersection of the new perpendicular bisectors with the existing Voronoi edge.

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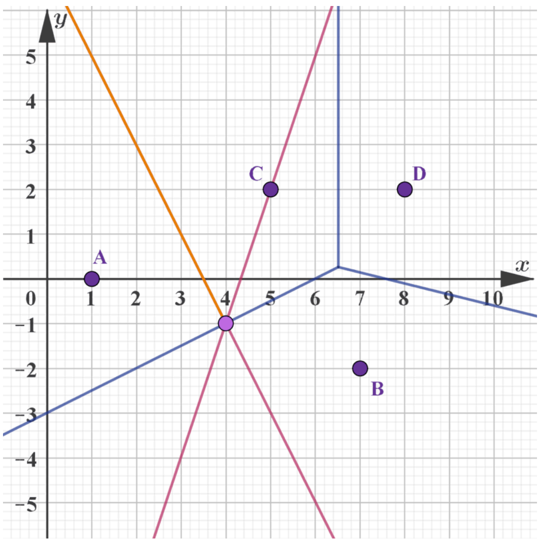
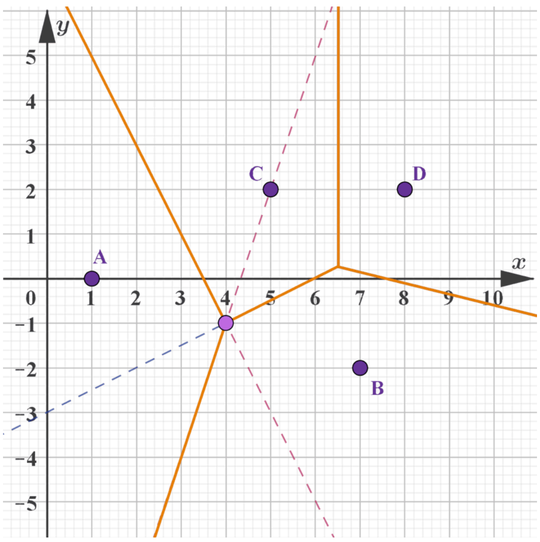
Steps	Explanation
	Start with the edge between A and C, halting at the new vertex.
	Continue with the edges between A and B, B and C, C and D, and D and B.

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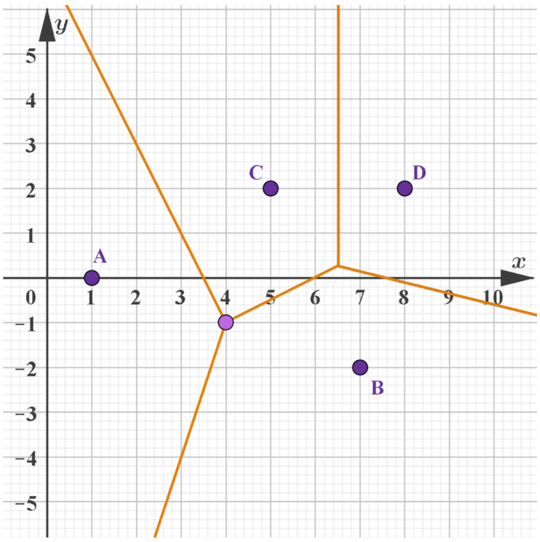
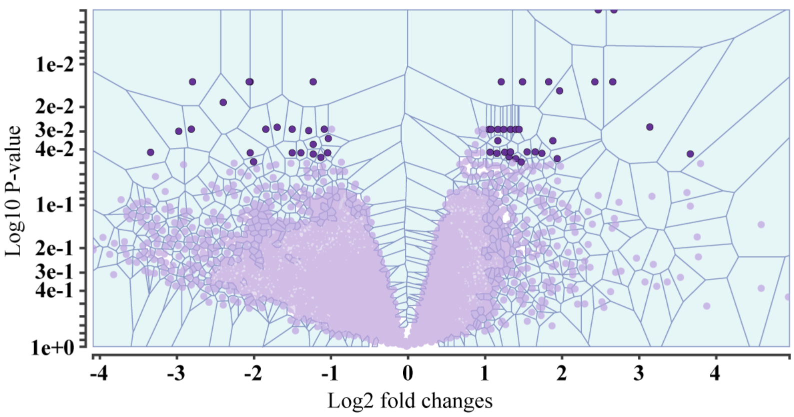
Steps	Explanation
 The site closest to K is B.	As K is in the same Voronoi cell as B.

Making connections

In life sciences, volcano plots, a type of scatter plot, are used to identify quickly changes in large data sets. They represent a single condensed view of a large set of points. As labelling each and every data point on the scatter plot would make the graph confusing and impractical, Voronoi diagrams are used to represent each data point in relation to the points around it. Creating such graphs requires some coding, but the results are beautiful and meaningful.



More information

The image is a scatter plot, commonly known as a volcano plot, with an overlay of a Voronoi diagram. The plot features an X-axis, representing statistical significance, and a Y-axis, representing fold change. The data points, shown as dots, vary in density with more concentration towards the center, forming a V-like shape — typical of a volcano plot. The Voronoi diagram segments the plot into regions around each data point, illustrating the spatial relationships between points. The background shows an intricate network of lines dividing the plot into polygonal regions surrounding each individual data point.

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International Mindedness

“The pure mathematics lectures captivate me more and more. I prefer Professor Sokhotsky’s lectures in special course on higher algebra to all the others. ... The main thing that concerns me is whether I have enough talent.” Georgy Voronoy

Although initially Voronoy had doubts about his ability, eventually he became one of the most prominent mathematicians studying number theory.

A Voronoi portrait (click to enlarge) of Georgy Voronoy (1868—1908).
Credit: parameter_bond

Exam tip

In IB examinations, the formula booklet has formulae for

- coordinates of the midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) :

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- gradient of a line:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- equations of a straight line:

$$y = mx + c, ax + by + d = 0 \text{ and } y - y_1 = m(x - x_1)$$

- If two lines are perpendicular, their gradients are negative reciprocals of each other:

$$m_1 \times m_2 = -1$$

3 section questions

3. Geometry and trigonometry / 3.6 Voronoi diagrams

Applications

Student
view

Overview
(/study/app/m/sid-122-cid-754029/

As mentioned earlier, there are numerous applications of Voronoi diagrams. The points in each cell are closest to the site in the cell. This can be very helpful when you are looking for the nearest hospital or the nearest airport when landing a plane, or if you are looking for somewhere to buy a cup of coffee.

Here are some applications of Voronoi diagrams:

- colouring graphics
- delivery areas
- ecology
- epidemiology: the spread of a disease from a contaminated site
- meteorology: estimating regional rainfall
- urban planning: sites are a resource such as a school, police station or fire station and cells are areas that utilise that resource
- zoology: modelling animal territories.

Making connections

Voronoi diagrams are used in analysing and finding the source of an epidemic.

This article (http://plus.maths.org/content/uncovering-cause-cholera) explains how it was used by John Snow in a cholera epidemic in 1854.

'London, September, 1854. A cholera outbreak has decimated Soho, killing 10% of the population and wiping out entire families in days. Current medical theories assert that the disease is spread by "bad air" emanating from the stinking open sewers. But one physician, John Snow, has a different theory: that cholera is spread through contaminated water. And he is just about to use mathematics to prove that he is right.'

Luckily, nowadays you do not need to draw all the cells in a diagram by hand, making it faster to identify the source of the epidemic.

In general, you could group applications under three headings:

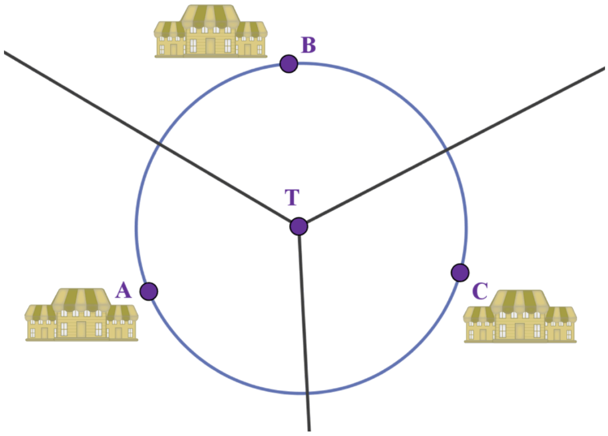
- distances
- areas
- function interpolation.

Distances

Toxic dump problem: largest empty circle

In the past, most toxic waste ended up in rivers, lakes or the sea, or it was buried underground where it contaminated underground water and the soil. In the past few decades, countries have enacted regulations on how to handle toxic waste. One common method of handling toxic waste is to bury it in sealed containers. The diagram shows an area with three towns. If you had to dump toxic waste in that area, where would you do so?

Student view

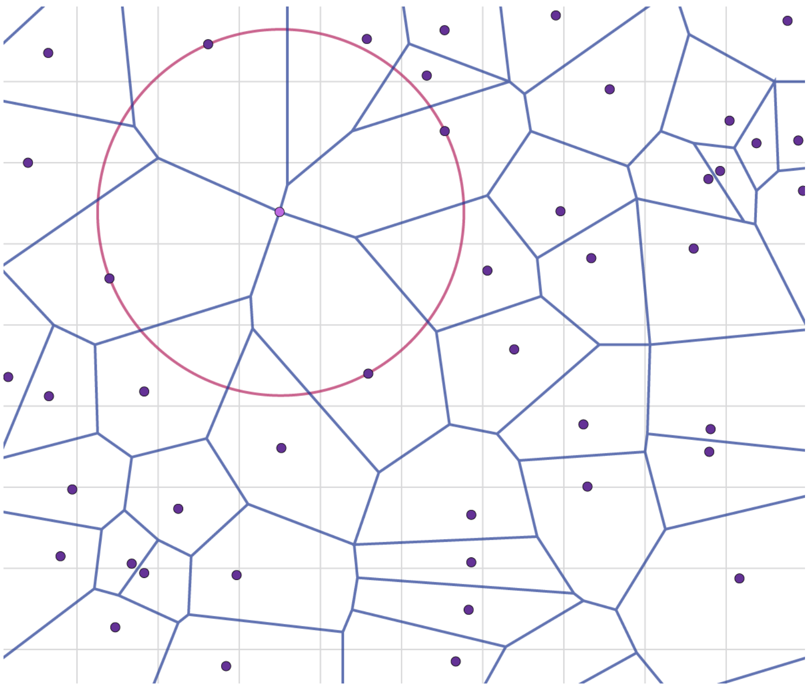


More information

The image is a diagram showing three towns labeled as A, B, and C, each represented by an icon of buildings and positioned around a circle. The circle is the circumcircle of the triangle formed by the towns. At the center of the circle is point T, marked as the equidistant point from all three towns. This circumcircle is noted as the largest empty circle, meaning it does not include any of the sites, representing the ideal location for a toxic dump site. Connecting lines from T to each point A, B, and C are shown, illustrating the equal distance from T to the towns. The explanation mentions that any location other than point T will be closer to one of the towns, making T the ideal location for dumping toxic waste.

[Generated by AI]

The circumcircle of triangle ABC is the largest empty circle since it does not include any of the sites. The centre of the circle is equidistant to all three towns. Any location other than the centre point T will be closer to one of the towns. Therefore, point T is the ideal location for the toxic dump site. With more than three sites, the centre of the largest empty circle will always be one of the vertices of the Voronoi diagram, as shown below.



More information

Student
view

The image is a Voronoi diagram featuring a circumcircle surrounding the triangle ABC. The circumcircle is drawn in pink and highlights a region of the diagram. Blue lines form the Voronoi cells, which are polygonal in shape. Each cell surrounds a purple dot representing a site. The center of the pink circumcircle, labeled as point T, is equidistant from the vertices of triangle ABC within the diagram. The grid in the background indicates a coordinate reference system, helping locate the various sites and lines accurately.

[Generated by AI]

Activity

In this activity, you will pick a location for a toxic dump using the applet. A , B , C , D and E are five towns. The vertices in the Voronoi diagram, J , M and L , are possible sites for a toxic dump. See how changing the location of the new town E changes the best place for the dump site.

Interactive 1. Application of Voronoi Diagrams: Pick a Location for a Toxic Dump.

More information for interactive 1

This interactive represents five towns and their Voronoi diagram, which divides the area into regions based on proximity to each town. The vertices of the Voronoi diagram are potential locations for a toxic waste dump, as they are equidistant from three or more towns.

The purple points A, B, C, D, and E are placed irregularly across the plane, representing the five towns. The black points J, M, and L represent the Voronoi vertices, points equidistant from three surrounding towns. Three black circles are shown centered at points J, M, and L. Each circle passes through three surrounding town points. The red lines meet at the vertices, J, M, and L, reinforcing them as equidistant intersection points.

Using this, users can explore how the placement of the dump site changes by adjusting the position of town *E*. Moving town E will reshape the Voronoi diagram, altering which vertex is the most central or optimal dump site.

The goal is to find a fair and safe location for the dump by maximizing its distance from all towns while keeping it centrally positioned. This ensures that no single town is disproportionately affected.

Overview
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Example 1

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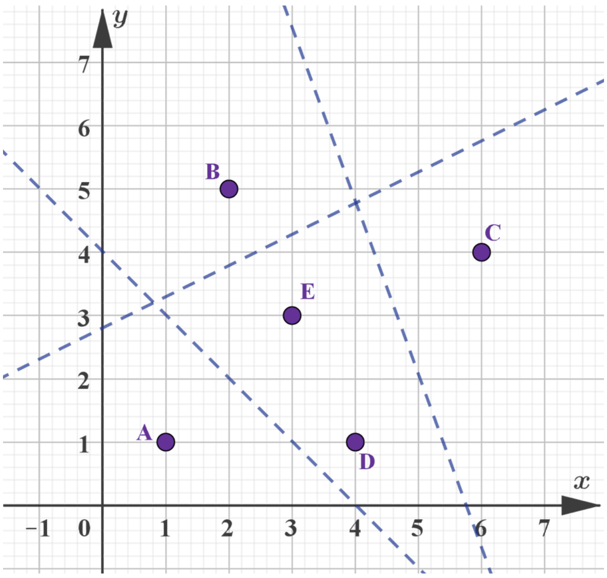
A hiker on a mountain trail marks the water sources $\mathrm{A}(1,1), \mathrm{B}(2,5), \mathrm{C}(6,4), \mathrm{D}(4,1)$ and $\mathrm{E}(3,3)$ on a map. He starts drawing the perpendicular bisectors, as seen in the incomplete Voronoi diagram below.

a) Draw the perpendicular bisector to complete the Voronoi cell that encloses site E and find its equation.

b) In the context of the problem, explain the meaning of the Voronoi cell E.

c) Complete the Voronoi diagram.



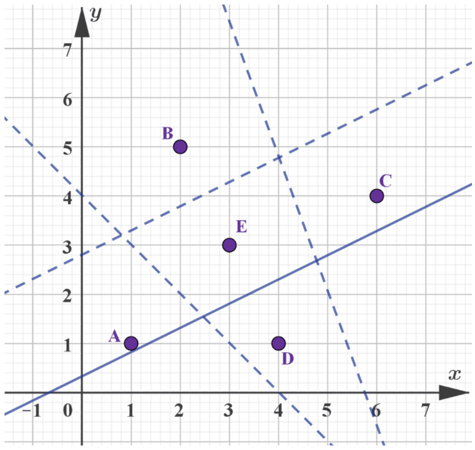
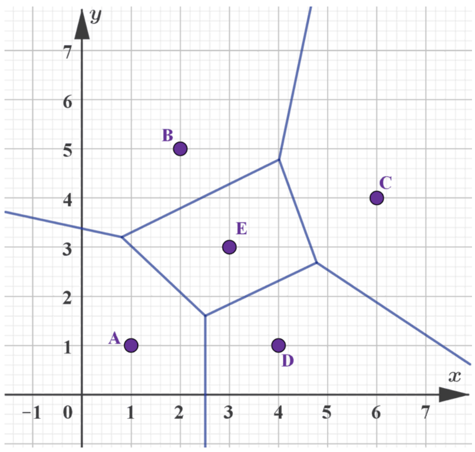
More information

The image is a Voronoi diagram on a grid. It shows a coordinate plane with grid lines and several labeled points, including A, B, C, D, and E, marked as purple dots. The axes are marked with numbers, with the x-axis showing ranges from 0 to 7, and the y-axis from 0 to 9. Dashed blue lines extend across the grid to divide the plane into regions around each labeled point. These lines represent the boundaries where any location within the region is closer to its respective point than to any other point. The diagram helps to demonstrate the concept of Voronoi regions where proximity to the points is depicted through these boundaries.

[Generated by AI]

Student
view

Overview (/study/ap 122-cid-754029/

	Steps	Explanation
a)		The line enclosing the site E will be the perpendicular bisector of ED.
	$m = \frac{3-1}{3-4} = -2$	Gradient of the line segment ED.
	$\left(\frac{3+4}{2}, \frac{3+1}{2}\right) = (3.5, 2)$	Midpoint of the line segment ED.
	$y - 2 = \frac{1}{2}(x - 3.5)$	Use the straight-line equation.
	$x - 2y = -0.5$	Rearrange.
b)	For all the points in the Voronoi cell, E is the closest water source.	
c)		

Student view

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Areas

Delivery regions

Example 2

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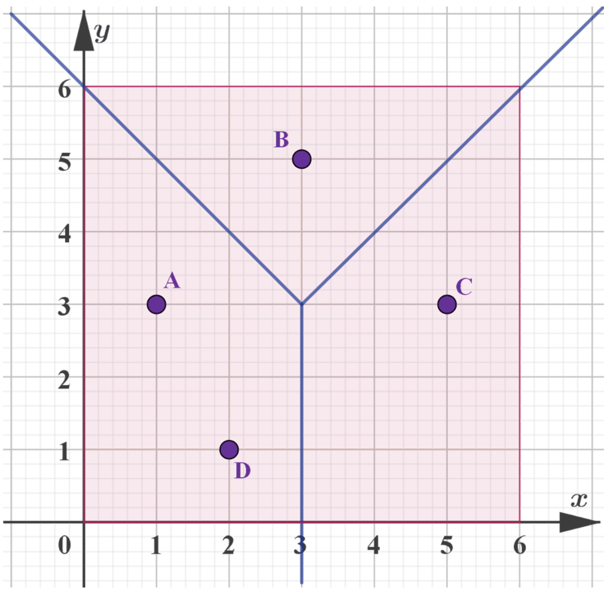
Mr PizzaMan had divided his town into three regions so that each house in the town would have pizza delivered from the closest of his three pizzerias A, B and C. The town lies within the shaded square shown. He has just opened a fourth pizzeria at site D, so now he needs to divide the town into four regions. If the original sites are $\mathrm{A}(1, 3)$, $\mathrm{B}(3, 5)$ and $\mathrm{C}(5, 3)$ and the new site is $\mathrm{D}(2, 1)$, then:

a) Draw the Voronoi cell for site D.

b) Find the equation of the line on which the edge of the cell between sites A and D lies.

c) If 1 unit represent $1\ \mathrm{km}$, find the site which has the largest delivery area, calculating to the nearest square kilometre.



More information

This is a Voronoi diagram displayed on a grid, depicting a planar subdivision into regions based on distance to a specific set of points. The grid has coordinates ranging from 0 to 7 on both axes. There are four highlighted points labeled A, B, C, and D, which represent different sites. Each of these sites is surrounded by a region, outlined with blue lines, showing the area closer to that site than to any other site. The regions are shaded in a magenta color to differentiate them. The large square grid has squares that are labeled with numbers at regular intervals, indicating their position relative to the x-axis and y-axis starting from zero. This diagram is typically used to represent proximity or influence areas.

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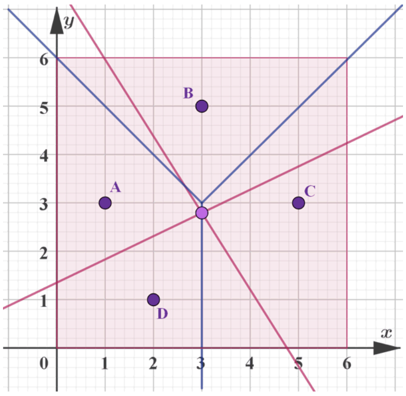
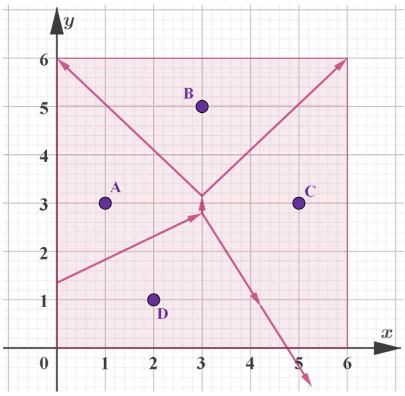
Student view

Overview (/study/ap 122- cid- 754029/

<table>
<tr><th></th><th>Steps</th><th>Explanation</th></tr>
<tr><td>a)</td><td>
</td><td>Draw the perpendicular bisectors of AD and CD.

Mark the intersection point of the new perpendicular bisector with the perpendicular bisector of AC.

Note that these three perpendicular bisectors meet at one point.</td></tr>
<tr><td></td><td>
</td><td>Starting from the edge between sites A and D, draw the edges of the regions.</td></tr>
<tr><td>b)</td><td>$m = \frac{3-1}{1-2} = -2$</td><td>The equation of the edge between A and D will be on the perpendicular bisector of A and D .

Gradient of the line segment AD.</td></tr>
<tr><td></td><td>$m_{\text{P.bisector}} = \frac{1}{2} = 0.5$</td><td>Gradient of the perpendicular bisector is negative reciprocal.</td></tr>
<tr><td></td><td>Midpoint of the line segment.
$$\left(\frac{1+2}{2}, \frac{3+1}{2}\right) = (1.5, 2)$$</td><td></td></tr>
<tr><td></td><td>$y - 2 = 0.5\,(x - 1.5)$</td><td>Use the equation for a straight line.</td></tr>
</table>

Student view

Overview
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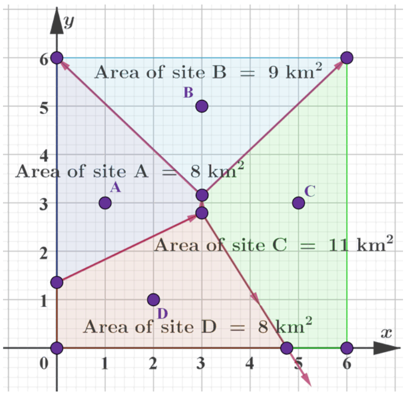
	Steps	Explanation
	$0.5x - y + 1.25 = 0$ Or $x - 2y + 2.5 = 0$	Rearrange.
c)	 Site C has the largest area with approximately $11\,\text{km}^2$.	You can divide all regions to rectangles and right triangles and use the area formula for these shapes to work out the areas. Besides the equation you found in part (b), you will need the equations of all lines on the diagram to work out the vertices of the regions. $x + y = 6$ $x - y = 0$ $x - 2y = -2.5$ $3x + 2y = 14.5$ The intersection points of these lines on the diagram are $(3, 3)$ and $(3, 2.75)$. The points on the boundary of the square are $(0, 1.125)$ and $(4.833, 0)$. The area of the regions are: 7.5, 9, 10.979 and 8.521 square kilometres. Note that on an exam you will not be asked to do all these calculations, but it is reasonable to ask the area of any of these regions.

Wolf packs in a National Park

Example 3

★★★

In a National Park, there are four territories A , B , C and D in which wolves live. The table shows the number of wolves in each territory.

	A	B	C	D
Number of wolves	27	10	12	9

The park has rangers, who manage the fauna and flora. Although pack sizes vary, the rangers are concerned that there are too many wolves in territory A. They plan to relocate 14 of the wolves to a new territory. The new den will be at one of the intersection points of the current Voronoi diagram.

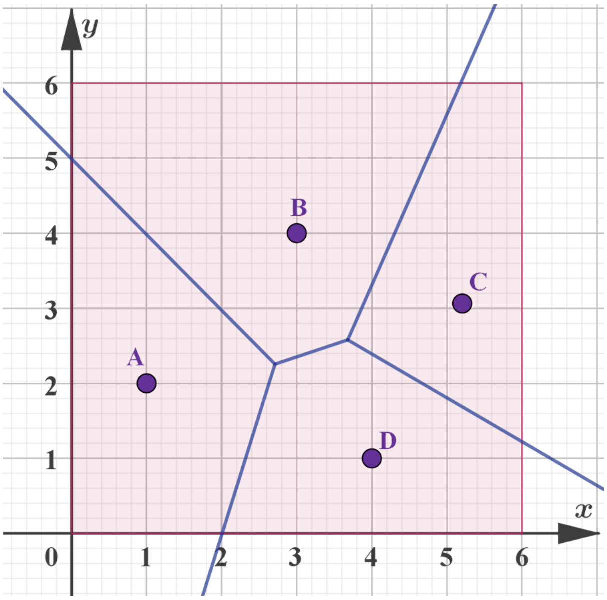
a) If a visitor is at $(3, 2)$ before the relocation, what territory are they in and how many wolves are in that territory?

Student
view

Overview (/study/ap 122-cid-754029/

b) If the sites are $\mathrm{A}(1, 2)$, $\mathrm{B}(3, 4)$, $\mathrm{C}(5.16, 3.05)$ and $\mathrm{D}(4, 1)$, then determine the best position to relocate the wolves so they are as far as possible from the other wolfpacks.

c) If the visitor is still at $(3, 2)$ after the relocation, what territory are they now in and how many wolves are in that territory?



More information

The image is a graph depicting a Voronoi diagram on a grid. The diagram includes four labeled points: A, B, C, and D. The graph has a grid with coordinates along the X and Y axes ranging from 0 to 6. The points are positioned as follows: A is near (1,2), B is near (3,4), C is near (5,3), and D is near (4,1).

Lines are drawn to divide the space into regions, each region corresponding to the area closest to a particular labeled point. These lines intersect forming a network of polygons around the points. The points demarcate different territories on the graph, illustrating how the space is divided based on proximity to the labeled points.

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	Steps	Explanation
a)	9 wolves in the area.	As the visitor is in the area of site D with 9 wolves.

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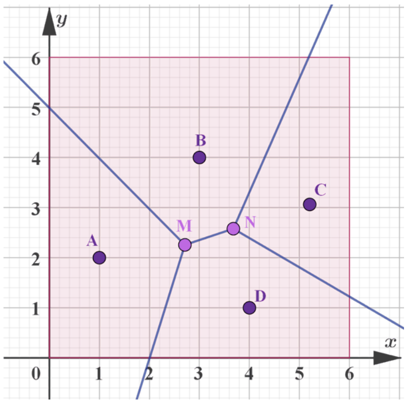
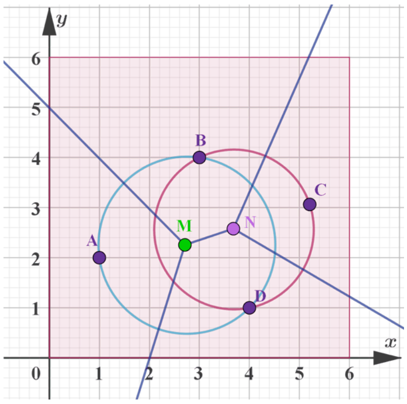
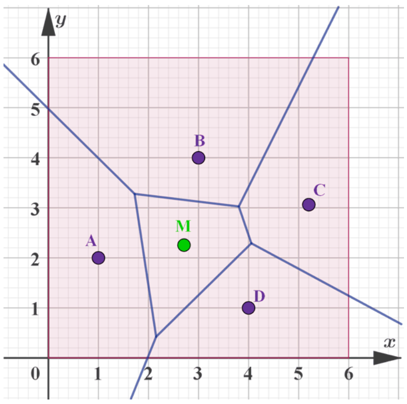
	Steps	Explanation
b)		Label the intersection points of the edges as vertices M and N.
	The perpendicular bisector of AB is $x + y = 5.$ The perpendicular bisector of AD is $3x - y = 6.$	M is the intersection point of the perpendicular bisectors of AB and AD.
	$x = 2.75 \text{ and } y = 2.25$ So $\mathrm{M}\,(2.75, 2.25)$.	Using a graphic display calculator.
	The perpendicular bisector of BC is $-2.16x + 0.95y = -5.45.$ The perpendicular bisector of CD is $1.16x + 2.05y = 9.45.$	N is the intersection point of perpendicular bisectors of BC and CD.
	$x = 3.65$ and $y = 2.55$ So N $(3.65, 2.55)$.	Using a graphic display calculator.

Student view

	Steps	Explanation
		The pink circle is centred at N, and B, C and D are on its circumference (it is the circumcircle of triangle BCD). The green circle is centred at M, and A, B and D are on its circumference (it is the circumcircle of $\Delta\mathrm{ABD}$). You are looking for the largest empty circle, so the centre of whichever circle has the larger radius will be the location for the new site.
	$\mathrm{MB} = \sqrt{(2.75-3)^2 + (2.25-4)^2} = 1.77$ $\mathrm{NB} = \sqrt{(3.65-3)^2 + (2.55-4)^2} = 1.60$	The radii of the circles.
	 So, the new pack should be relocated as close as possible to $\mathrm{M}\,(2.75, 2.25)$.	
c)	There are 14 wolves.	The visitor is now in Voronoi cell M, which has 14 wolves.

Be aware

Note that in examinations, the solution point will always be the intersection of three edges of a Voronoi diagram.

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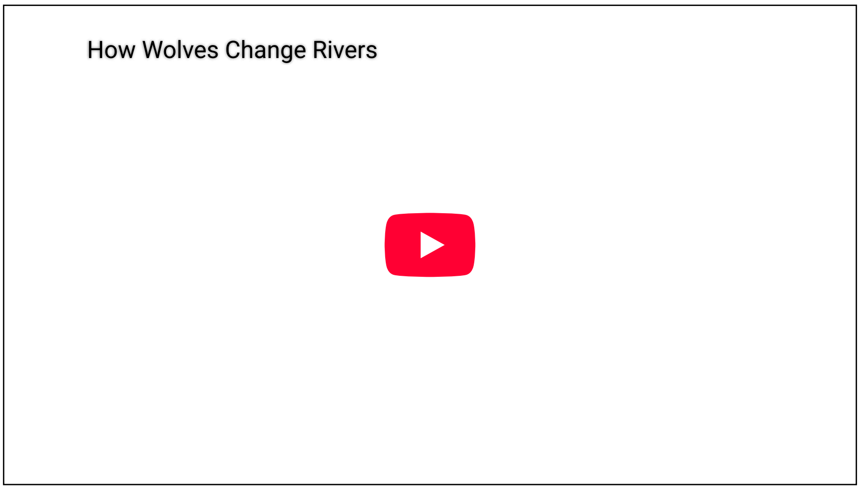
Important

Here, we associated the number of wolves with each cell. In other cases, the number may be associated with only the site, for example rainfall measured at a gauge. If we wanted an estimate of rainfall at other places, we could use the value at the nearest site as an approximation. This is called the nearest neighbourhood interpolation.

International Mindedness

It is important for ecologists to understand how food chains work and also how the absence or extinction of one or more species can impact the ecosystem. They can use this information to identify areas in danger and species that need to be protected. Ecologists use Voronoi diagrams in a territorial analysis to estimate the population of species and their distribution across regions. In some cases, introducing a non-native species can be disastrous, like when Burmese pythons escaped or were released into the wild after Hurricane Andrew in Florida in 1992. The pythons threaten the local wildlife, including birds, coyotes and panthers. In contrast, the reintroduction of wolves resulted in changes to the landscape of Yellowstone Park, United States. Wolves were reintroduced to the park in 1995 and the results were amazing. See the video below for more details.



3 section questions

3. Geometry and trigonometry / 3.6 Voronoi diagrams

Checklist

Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/checklist-id-27581/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- draw a Voronoi diagram for a given number of points
- Identify the site closest to a given point
- add a site to an existing Voronoi diagram and draw the new diagram.
- apply the 'toxic dump' problem to different situations
- find the equation of an edge of a cell

Student view

- calculate the area of a region.

Investigation

The air quality index (AQI) is used by government agencies around the world to measure how polluted the air is. It is based on the amount of particulate matter ($PM_{2.5}$ and PM_{10}), ozone (O_3), nitrogen dioxide (NO_2), sulphur dioxide (SO_2) and carbon monoxide (CO) in the air.

The WAQI website (https://waqi.info/) shows a colour-coded map of AQI readings from around the world. It is obvious that most locations for which the AQI is known are coded yellow or higher, which is bad for the health of the people living in those regions.

In this investigation, you will investigate the overall AQI for a specific region. Note that AQI is calculated for a volume of air. For simplicity, you will consider only the area of the region being discussed.

Part A: Using three sites

- Choose the three AQI readings nearest to your location.
- Using Google Maps, find the longitude and latitude of each of these AQI readings and put them in a table.
- Convert these values to UTM coordinates and add these to your table.
- Create a Voronoi diagram for these three sites in Geogebra. What is the best boundary for your region?
- Using an average weighted by area, calculate the mean AQI for the entire region containing these three sites. You may wish to research methods for calculating weighted averages. Why is the boundary important?

Part B: Adding sites

- Choose at least another 15 of the AQI readings nearest to your location. If there are not enough, choose an area that you would like to investigate. Choose the best boundary for the region.
- Follow the steps in **Part A** to calculate the weighted average AQI for the sites you have chosen.

Part C: Report your findings

- Create a report describing your findings on average AQI in the area you choose and the impact of hazardous air pollutants on humans, animals and plants.