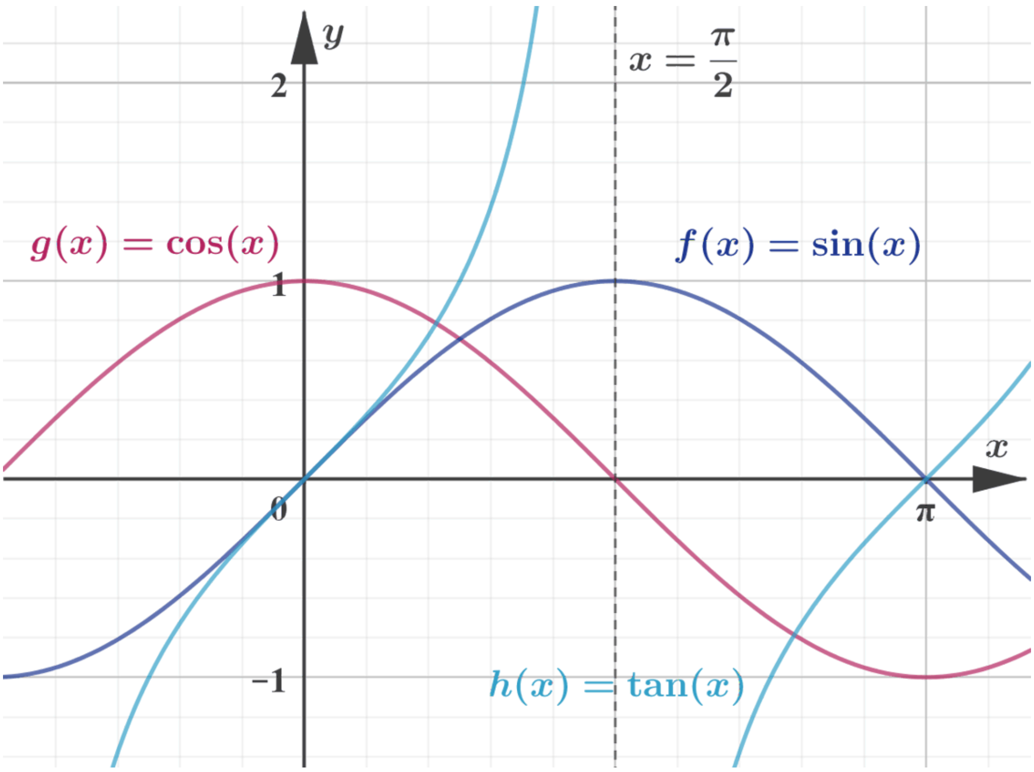
3. Geometry and trigonometry / 3.11 Further circular functions

The big picture

In previous subtopics, you studied properties of trigonometric functions and their graphs. In this subtopic you will have a closer look at the graphs of trigonometric functions and their relationships.



More information

The image is a graph displaying the trigonometric functions sine, cosine, and tangent. The X-axis represents the angle in radians while the Y-axis represents the function values. The range of the Y-axis is from -2 to 2.

1. **g(x) = cos(x)**: This is represented by a magenta curve that oscillates between -1 and 1. It starts from the maximum value of 1 when x = 0.

2. **f(x) = sin(x)**: This function is shown as a blue curve that also oscillates between -1 and 1. It starts from 0 when x = 0 and reaches its peaks (1) and troughs (-1) as x increases.

3. **h(x) = tan(x)**: This is represented by a cyan curve. The tangent function has vertical asymptotes where the function is undefined, which occurs at x = pi/2, 3pi/2, etc., and it crosses the x-axis at multiples of pi.

The graph also includes vertical lines indicating key angles, such as x = pi/2. There are gridlines for easy determination of values.

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Concept

Trigonometric functions allow you to modelcyclic phenomena.

The study of trigonometric functions provides you with the tools to analyse, measure and transform the quantities, movements and relationships of cyclic (periodic) functions.

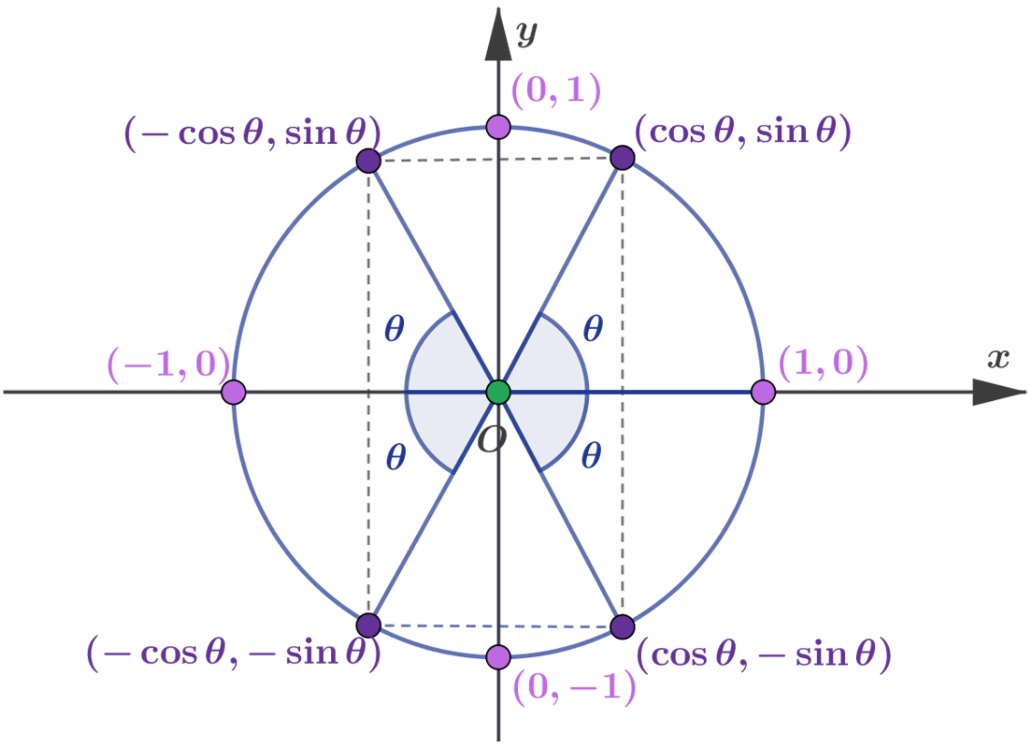
3. Geometry and trigonometry / 3.11 Further circular functions

Circular functions revisit

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

In subtopic 3.5 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27741/), you studied the unit circle and trigonometric ratios beyond acute angles. The symmetry of the unit circle helped you to find trigonometric ratios of any angle using its relationship to the principal angle in the first quadrant.

Making connections



More information

This image is a diagram illustrating the unit circle, a central concept in trigonometry. The circle is centered at the origin (0,0) on a Cartesian coordinate plane, with both x and y axes clearly labeled. Various points on the circle are labeled with their trigonometric identities, including (1,0), (0,1), (-1,0), and (0,-1). Additional points are labeled with expressions of cosine and sine functions: (cos θ, sin θ), (-cos θ, sin θ), (-cos θ, -sin θ), and (cos θ, -sin θ). The diagram illustrates angles θ in standard position, originating from the center of the circle and sweeping outwards. Dotted lines indicate the triangle formed by the radius, helping to visualize sine and cosine as coordinates. The central angle θ is shown multiple times inside the circle. The radius of the circle is consistently marked as 1, which is the defining feature of the unit circle.

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Overview
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In subtopic 3.5 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27741/), you studied the unit circle and used the following identities

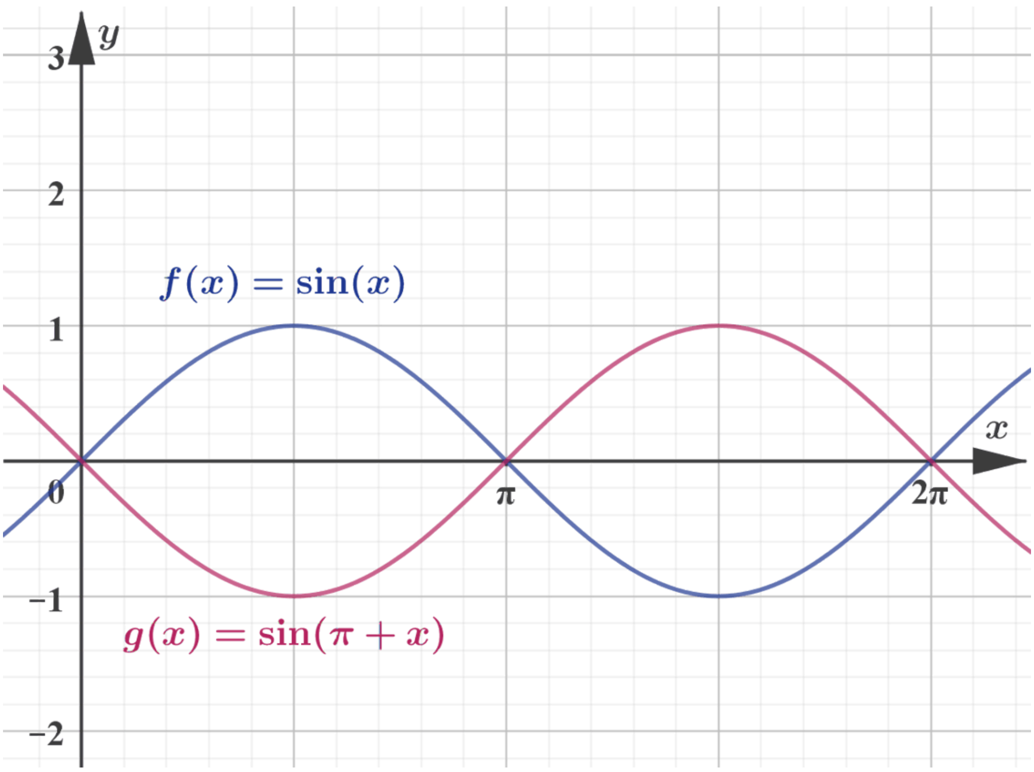
$$\cos(\pi - \theta) = -\cos\theta$$

$$\sin(\pi - \theta) = \sin\theta$$

$$\tan(\pi - \theta) = -\tan\theta$$

Although $f(x) = \sin x$ and $h(x) = \sin(\pi - x)$ are two different functions, when you graph them they would look exactly the same because $\sin(\pi - x) = \sin x$.

Consider the two functions $f(x) = \sin x$ and $q(x) = \sin(\pi + x)$, which are shown in the diagram below. The graph of $y = q(x)$ is the reflection of $y = f(x)$ in the y-axis.



More information

The graph displays the functions (f(x) = \sin x) and (q(x) = \sin(\pi - x)). The X-axis represents values ranging from -(2\pi) to (2\pi), labeled at intervals of (\pi/2), (\pi), and (2\pi). The Y-axis ranges from -1 to 1.

The blue curve represents (f(x) = \sin x), exhibiting a typical sinusoidal wave starting at (0,0), peaking at (\pi/2, 1), crossing zero at (\pi), troughing at (3\pi/2, -1), and completing the cycle at (2\pi, 0).

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The red curve corresponds to (q(x) = \sin(\pi - x)), which is a reflection of the blue curve across the Y-axis. It begins at (0,0), peaks down to (-1) at (\pi/2,), returns to zero at (\pi), peaks up to (1) at (3\pi/2), and then returns to zero at (2\pi).

Both curves overlay such that as one descends, the other ascends, showcasing their reflectional relationship across the Y-axis.

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Example 1

★★☆

Prove that $\cos\left(\frac{3\pi}{2}+x\right)=\sin x$.

Use the compound angle identity for cosine:

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta+\sin\alpha\cos\beta$$

$$\cos\left(\frac{3\pi}{2}+x\right)=\cos\frac{3\pi}{2}\cos x-\sin\frac{3\pi}{2}\sin x$$

$\cos\frac{3\pi}{2}=0$ and $\sin\frac{3\pi}{2}=-1$

$$\cos\left(\frac{3\pi}{2}+x\right)=0\times\cos x-(-1)\sin x$$

Simplify

$$\cos\left(\frac{3\pi}{2}+x\right)=\sin x$$

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Therefore,

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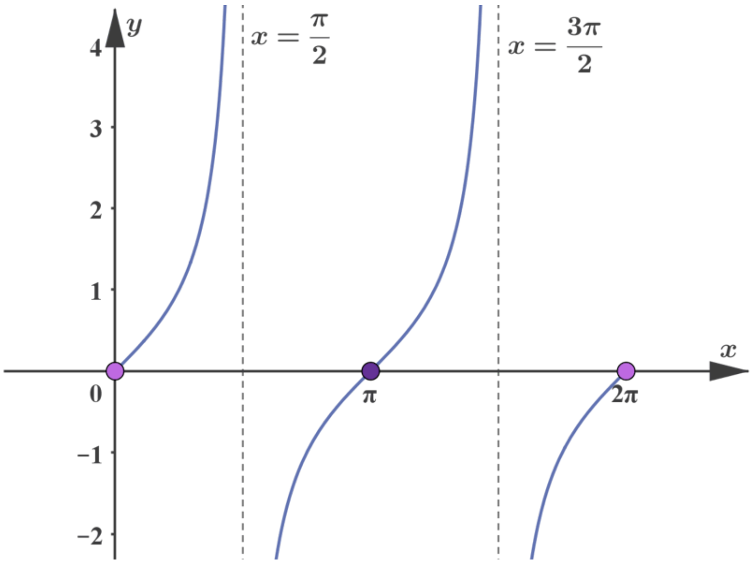
$$\cos\left(\frac{3\pi}{2}+x\right)=\sin x$$

Example 2

★☆☆

Sketch the graph of $y=f(x)$ where $f(x)=\tan x$ for $0\leqslant x\leqslant 2\pi$. Sketch the graph of $y=g(x)$, where $g(x)=f(\pi-x)$. Describe the relationship between the graphs of $y=f(x)$ and $y=g(x)$.

Steps	Explanation
	Sketch the graph for the domain making sure that vertical asymptotes and are marked clearly.

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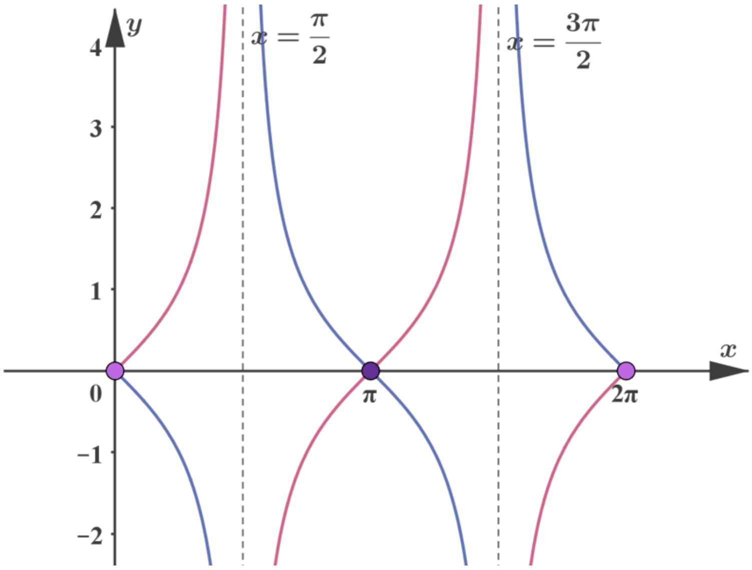
Steps	Explanation
	Use the result that $$\tan(\pi - x) =$$ to sketch the graph of y
The graph of $y = g(x)$ is the reflection of $y = f(x)$ in the x-axis.	

Be aware

In IB examinations, if the question starts with ‘hence’ it means you have to use the results from previous work you have done. If you do not do so, even if your alternative method is correct it will not be accepted.

If the question starts with ‘hence’ followed by ‘or otherwise’ then you can use an alternative method.

Example 3

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Sketch the graph of $y = f(\theta)$ where $f(\theta) = \sin(\pi + 2\theta), 0 \leqslant \theta \leqslant \pi$.

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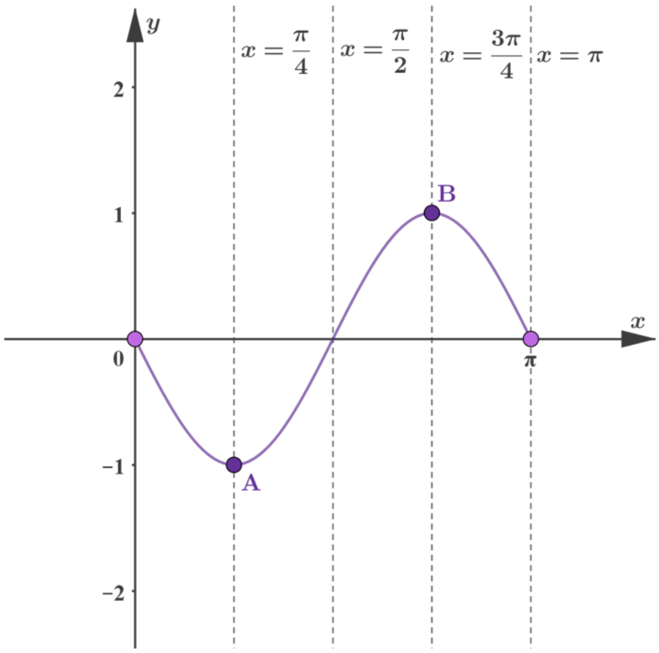
Hence, sketch the graph of $g(\theta) = \operatorname{cosec}(\pi + 2\theta)$, clearly marking the coordinates of any local maximum/minimum points and the equations of any asymptotes.

Steps	Explanation
	$\sin(\pi + 2\theta) = -\sin 2\theta$ which has period of π. T local minimum at $\mathrm{A}\left(\frac{\pi}{4},\right.$ local maximum at $\mathrm{B}\left(\frac{3\pi}{4}\right.$ x-intercepts are $(0, 0)$, ($(\pi, 0)$.

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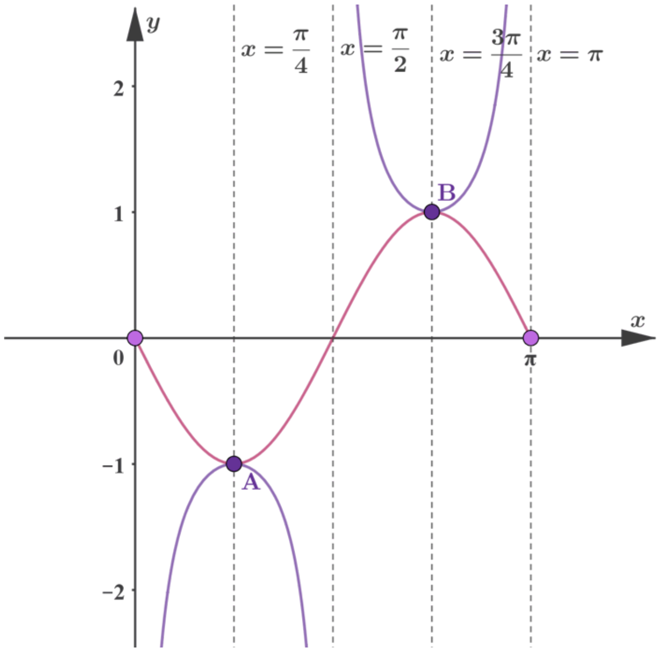

</td><td>
As cosec $(\pi + 2\theta) = \frac{\quad}{\mathrm{si}}$

its graph will have asymp
$f(\theta) = 0$

$y = g(\theta)$ will have a loca
when $f(\theta)$ has a maxim

$y = g(\theta)$ will have a loca
when $f(\theta)$ has a minimu
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<tr><td>
Vertical asymptotes of $y = g(\theta)$ are $x = 0, x = \frac{\pi}{2}$ and $x = \pi$

The local maximum of $y = g(\theta)$ is at $\mathrm{B}\left(\frac{3\pi}{4}, 1\right)$ and the local minimum is at $\mathrm{A}\left(\frac{\pi}{4}, -1\right)$.
</td><td></td></tr>
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Example 4

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Determine whether $f(x) = \cos x^2 - x$is even, odd or neither.

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$$f(-x) = \cos(-x)^2 - (-x)$$
$$f(-x) = \cos x^2 + x$$
$$f(-x) \neq -f(x)$$

And

$$f(-x) \neq f(x)$$

Therefore, $f(-x)$ is neither odd nor even.

Exam tip

In IB examinations, if you are asked to sketch a graph in multiple steps, do not erase your work to show your final answer. You should label your final answer clearly making sure all the critical points like maxima, minima, axes intercepts and asymptotes are marked and their values or equation are written down.

2 section questions

Question 1

★☆☆

Sketch the graph of $y = f(x)$, where $f(x) = \cos(\pi - x)$, for $-\pi \leqslant x \leqslant \pi$.

Hence, find the number of solutions to the equation $f(x) = 0.2$.

Two solutions

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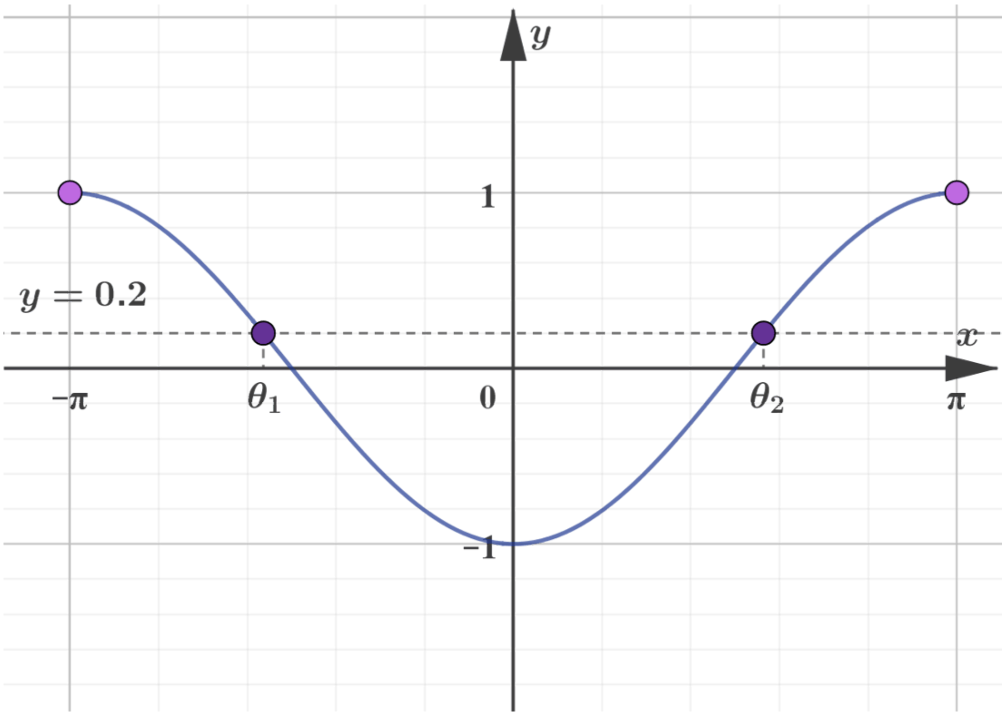
1 

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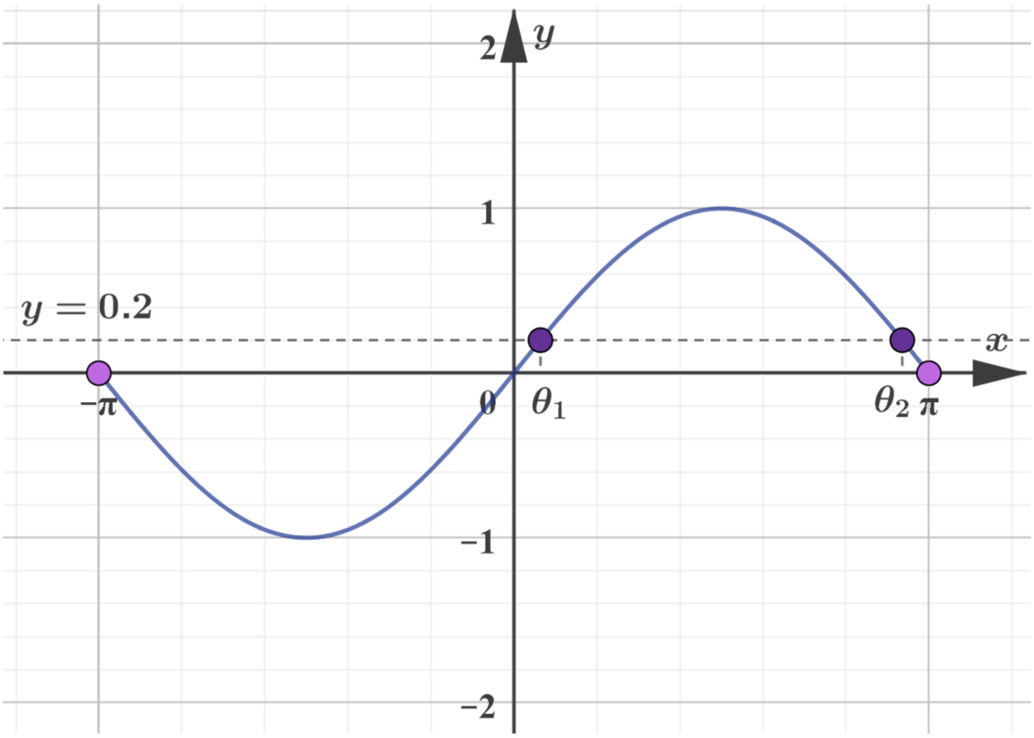
Two solutions

2 

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Two solutions

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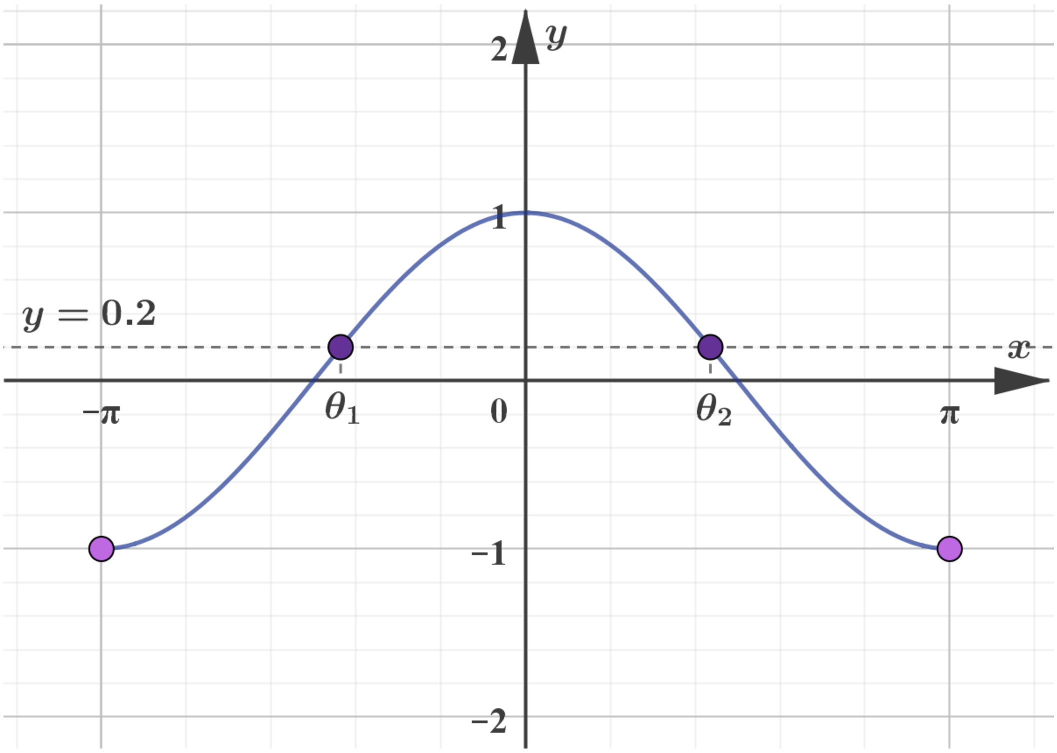
3 

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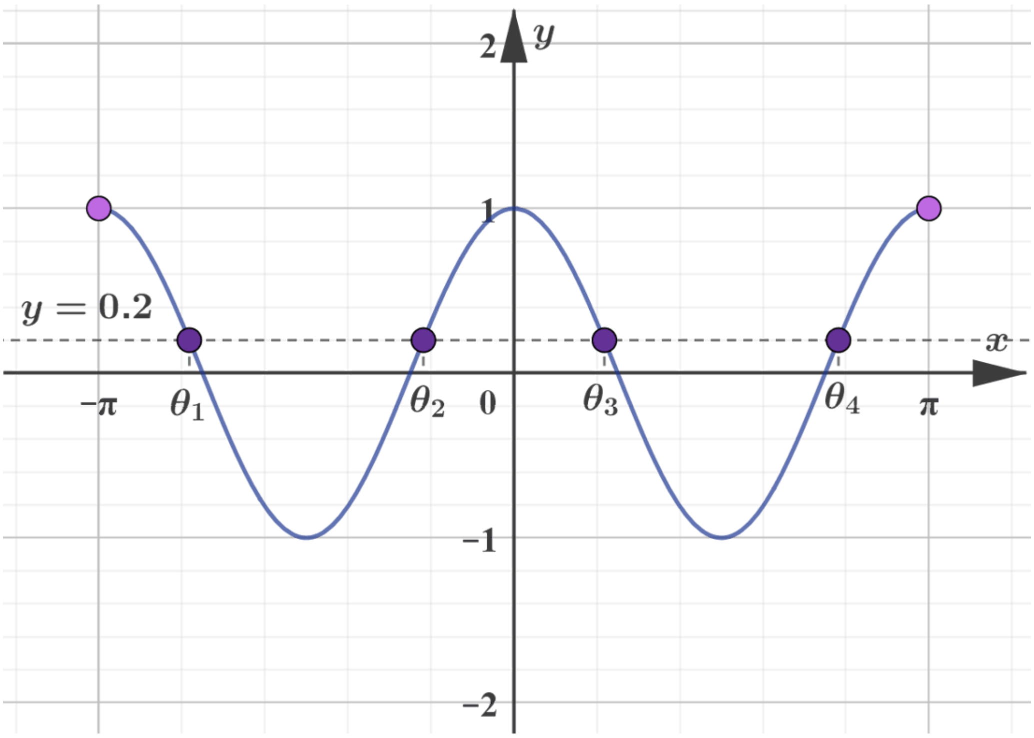
Four solutions

4 

More information

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Explanation

$\cos(\pi - x) = -\cos x$

Sketch the graph, draw the line $y = 0.2$ and mark the intersections with the curve.

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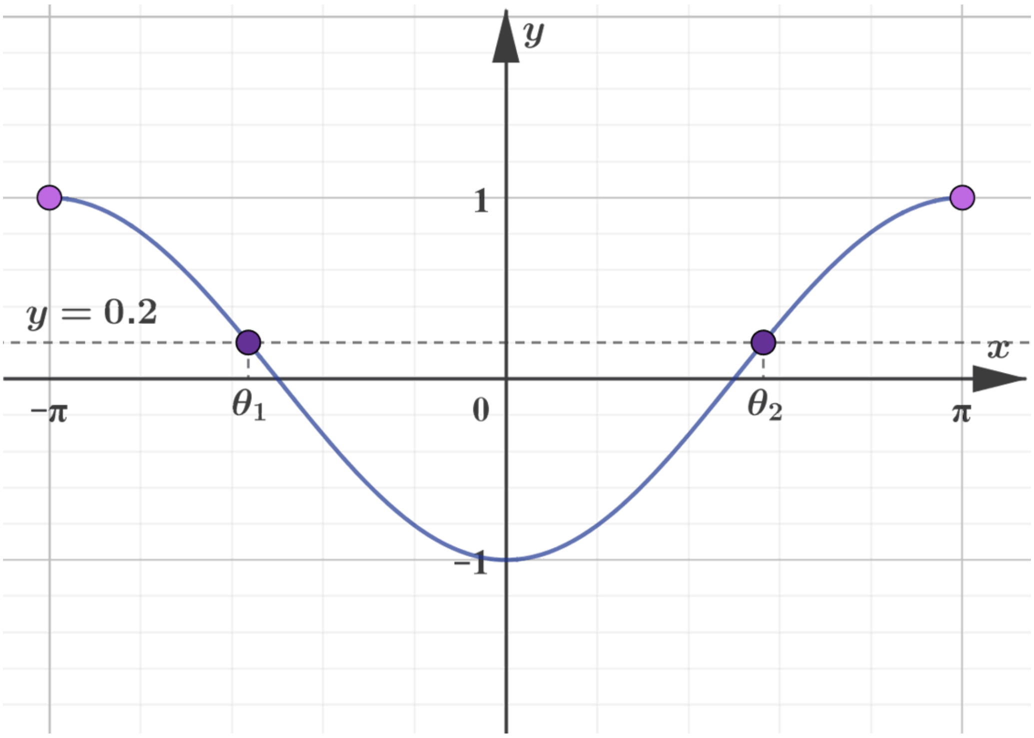

More information

Therefore there are two solutions in the given domain.

Question 2

★★☆

Let $f\left(x\right)=\tan x$, $\frac{-3\pi}{2}<x<\frac{-\pi}{2}$. The graph of $y=f\left(x\right)$ is reflected in the x-axis and then translated by the vector $\begin{pmatrix}\pi\\1\end{pmatrix}$.

Which of the following represents the graph and the equation of the transformed function?

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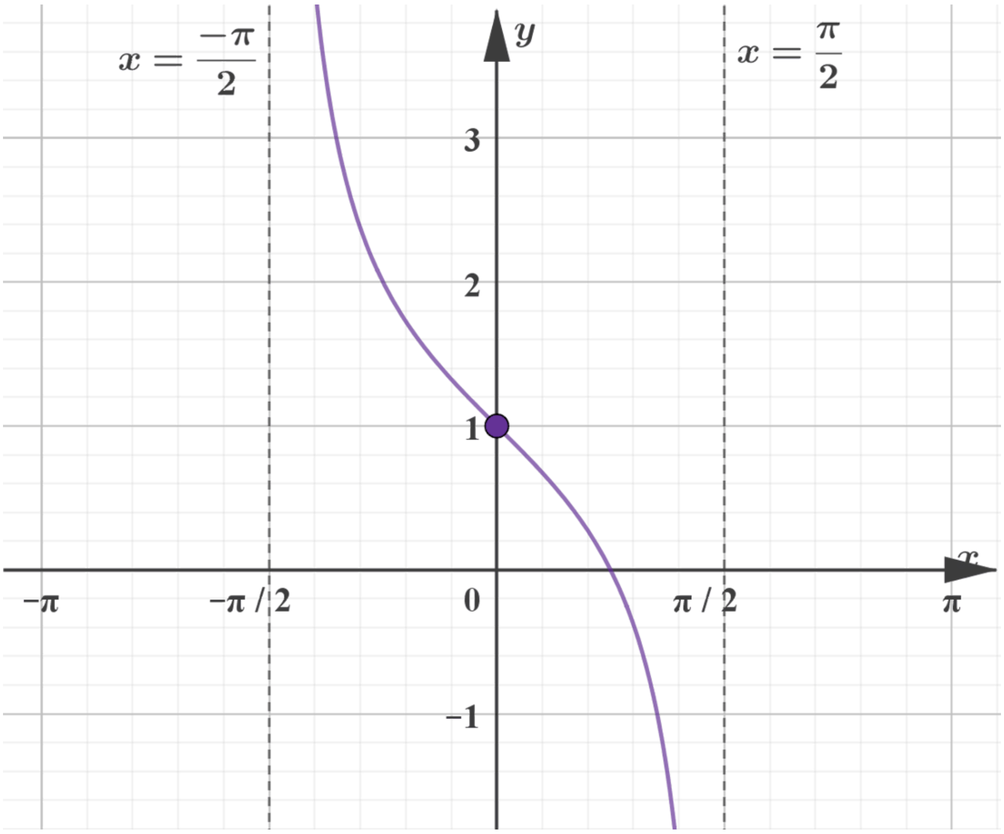

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$h(x) = -\tan x + 1$

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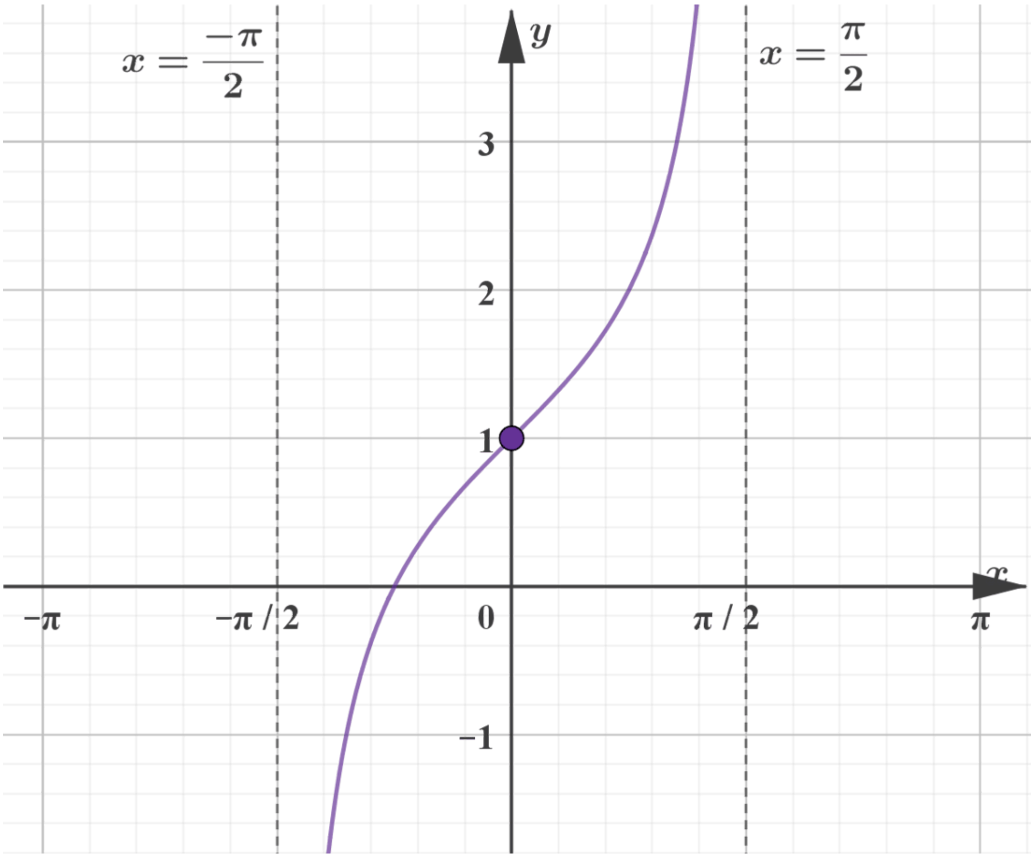

More information

$h(x) = \tan x + 1$

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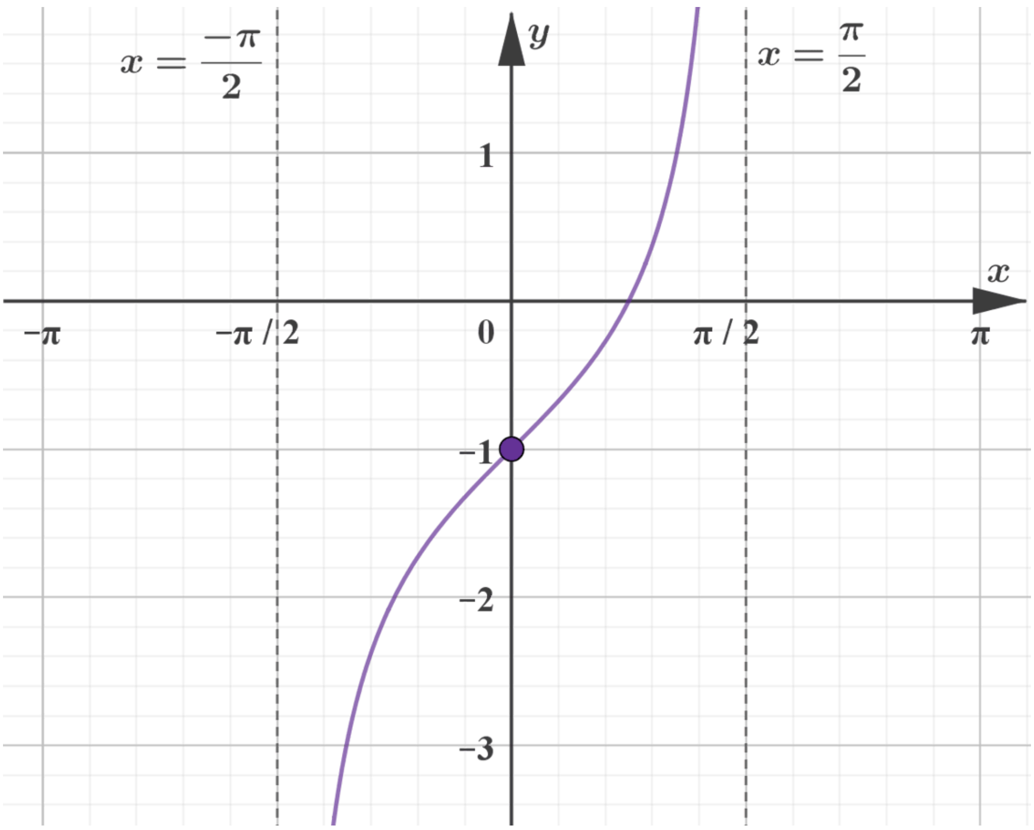

More information

Student view

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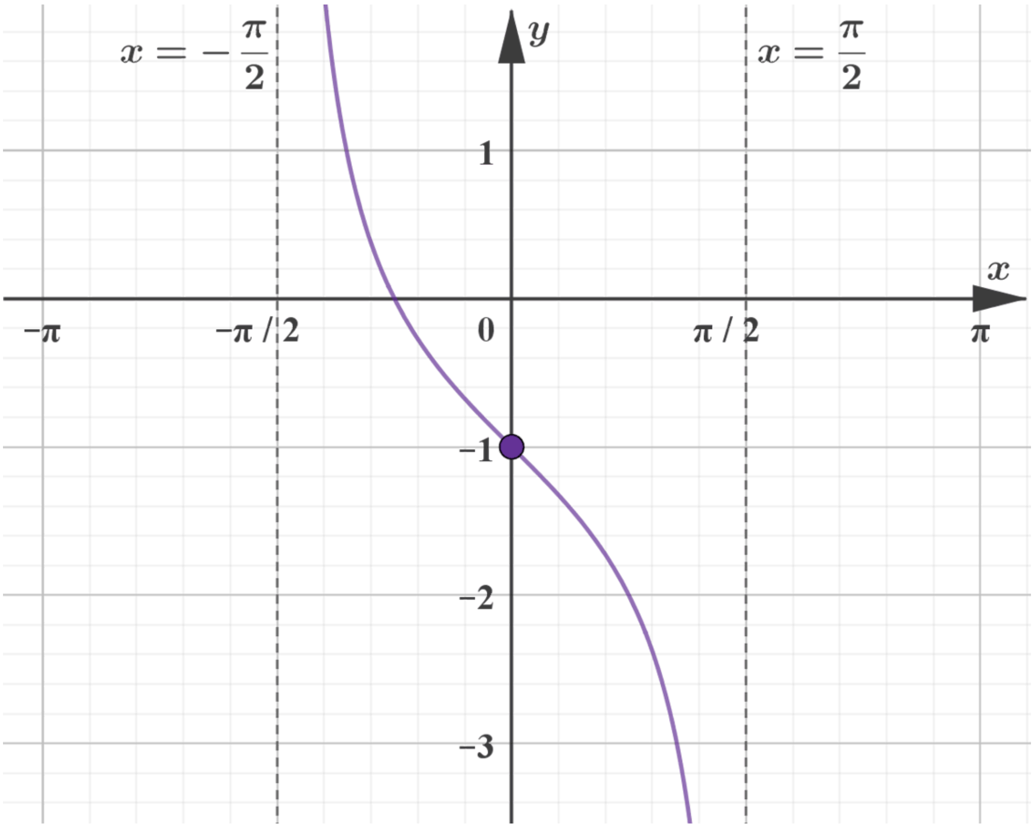
$$h(x) = -\tan x - 1$$

4



More information

$$h(x) = -\tan x - 1$$

Explanation

Reflection in x-axis: $g(x) = -f(x)$

This reflection does not change the domain of the graph.

Translation by vector $\begin{pmatrix} \pi \\ 1 \end{pmatrix}$: $h(x) = g(x - \pi) + 1$

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This translation changes the domain of the graph from $-\frac{3\pi}{2} < x < -\frac{\pi}{2}$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so all answer options are graphed on the correct domain.

Moreover,

$$h(x) = -f(x - \pi) + 1 = -\tan(x - \pi) + 1 = -\tan x + 1.$$

So the correct answer is

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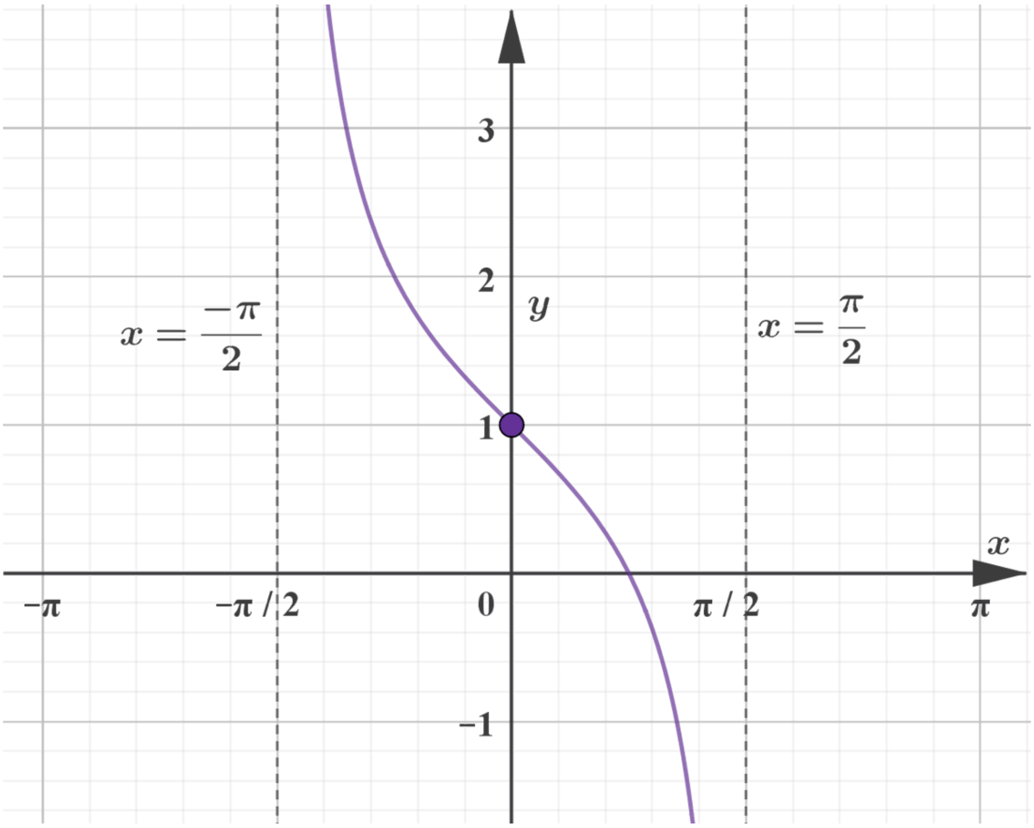

More information

3. Geometry and trigonometry / 3.11 Further circular functions

Symmetries

As trigonometric functions are periodic, you can find many symmetries in each graph.

In this section, you will look into symmetries in relation to even and odd functions.

Making connections

In subtopic 2.14 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26756/), you studied even and odd functions.

A function $f(x)$ is odd if $f(-x) = -f(x)$ for each x in the given domain.

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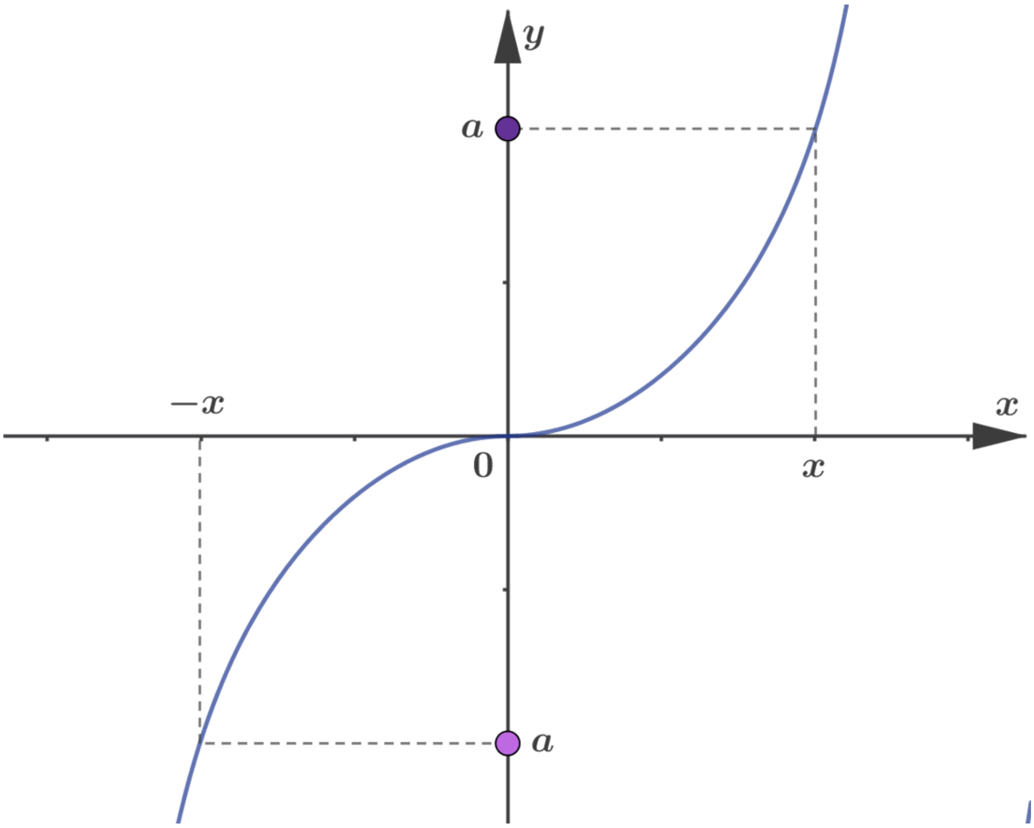

More information

The image shows a graph representing an odd function. The graph is plotted on a Cartesian coordinate system with x and y axes. The x-axis is labeled from negative to positive, and similarly, the y-axis is labeled. The function intersects the origin (0,0) and is symmetrical about this point, demonstrating the property of an odd function where f(-x) = -f(x). Points are marked on the graph: one on the negative side of the x-axis labeled '-x' and a corresponding y value, and one on the positive side of the x-axis labeled 'x' and its corresponding negative y value, emphasizing the function's symmetry. Two points are marked on the y-axis, indicating specific values at 'a' and '-a'.

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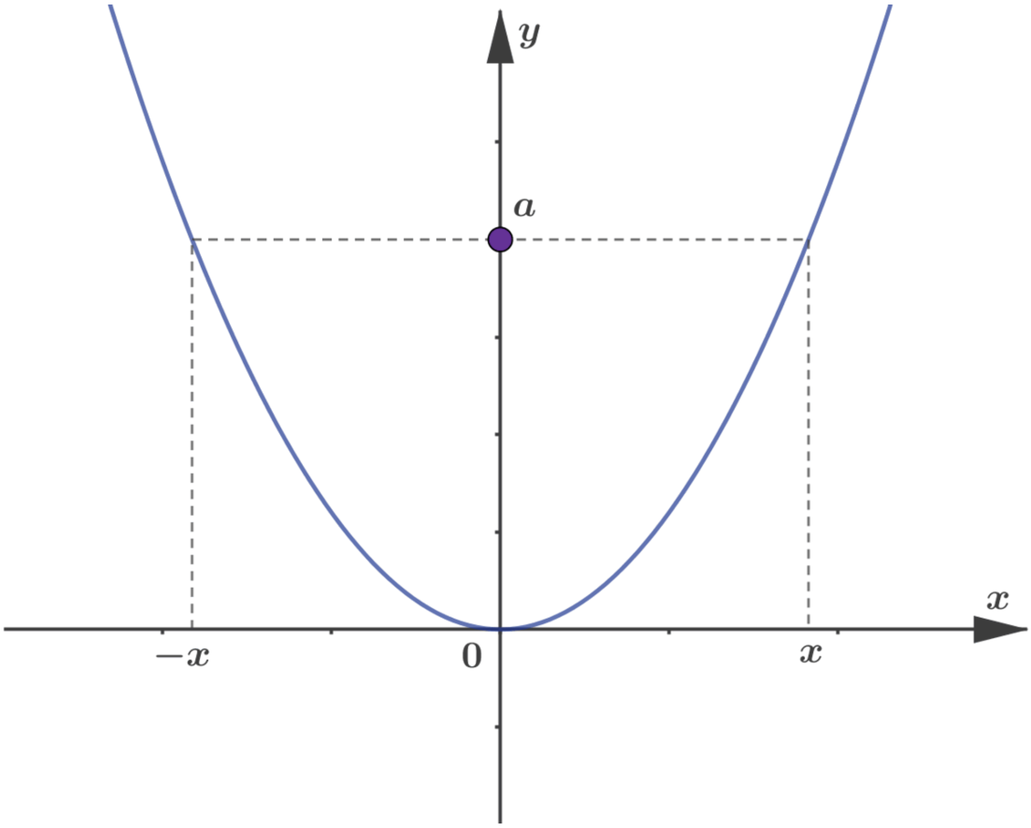
A function $f(x)$ is even if $f(-x) = f(x)$ for each x in the given domain.



More information

A graph depicts an even function demonstrating symmetry about the y-axis. The x-axis and y-axis are clearly labeled. The curve of the function is a parabola, opening upwards, with its vertex at the origin (0, 0). The y-value at the point labeled "a" remains the same for both x = -x and x = x, illustrating the concept of even functions, where f(x) = f(-x). The parabola is symmetric, meaning that for every point on the curve at x, there is a corresponding point at -x with the same y-value.

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Activity

In this activity, you will investigate whether trigonometric functions are even or odd.

For each of the trigonometric functions

$$y = \sin x\ ,\ y = \cos x\ ,\ y = \tan x,\ y = \sec x \text{ and } y = \operatorname{cosec} x$$

1. determine whether it is even, odd or neither

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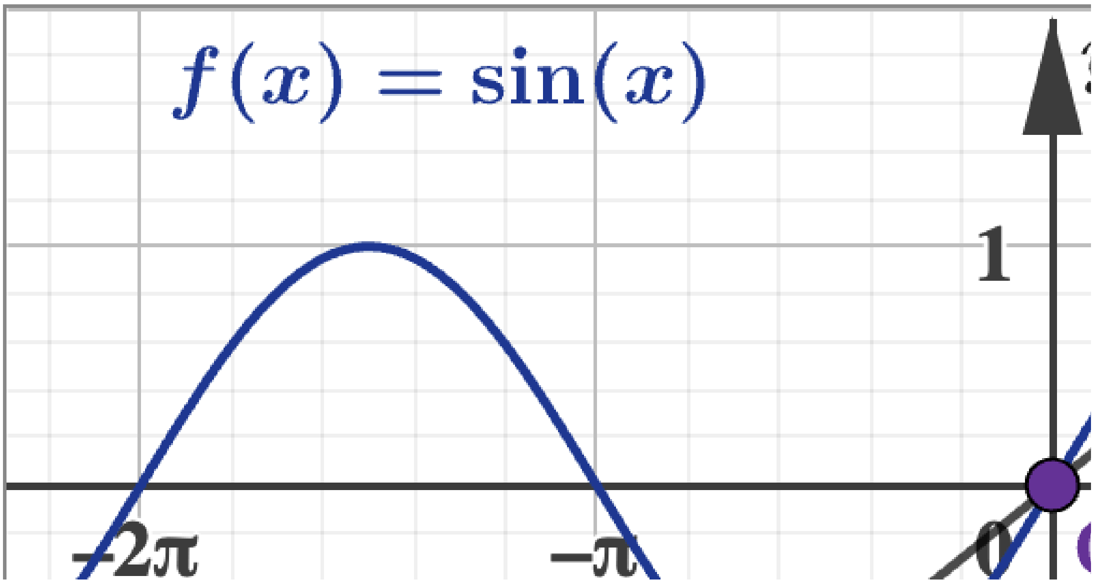
2. by sketching their graphs, describe the symmetries you observe.

The sine graph has rotational symmetry with respect to the origin, as you can see in the applet below, each point on the right-hand side of the origin is rotated π radians.



Interactive 1. Rotational Symmetry of the Sine Graph.

Credit: GeoGebra (https://www.geogebra.org/m/vezpm7qt) Nuriye Sirinoglu Singh

More information for interactive 1

This interactive explores the rotational symmetry of the sine function about the origin. The sine curve maintains symmetrical relationships when rotated by π radians, demonstrating the function's odd symmetry property through visual representation and dynamic point manipulation.

The display shows a graph with xy axes, with x-axis measured in radians ranging from -2 to 2 And y-axis ranging from -2 to 1. A purple sine graph, is centered at (0,0), representing the function $f(x) = \sin(x)$.

Two points A (movable) and B (automatically updated) appear on the curve, connected by a line, with their coordinates visible. Point A's position can be adjusted along the sine wave.

For example, as users drag point A to $(\pi/2, 1)$, point B automatically moves to $(-\pi/2, -1)$, showing the sign change. When A is placed at $(\pi, 0)$, B moves to $(-\pi, 0)$, maintaining the origin symmetry. These movements visually confirm the mathematical relationship $f(-x) = -f(x)$ for sine.

Through this exploration, users gain concrete understanding of odd functions' rotational symmetry, seeing how sine's properties manifest graphically. They learn to predict symmetrical point pairs and verify the function's behavior algebraically through coordinate observation, reinforcing fundamental trigonometric concepts essential for advanced mathematics and physics applications.

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Important

The graph of an odd function is symmetric with respect to the origin (rotational symmetry) and the graph of an even function is symmetric with respect to the y-axis (line symmetry).

Example 1

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Sketch the graph of $y = \cos 2x$ for $-\pi \leqslant x \leqslant \pi$.

Given that $\cos 2x = 0.8$, use your sketch to find

$$\cos(-2\pi + 2x) \qquad \cos(-2x) \qquad \cos(2\pi - 2x).$$

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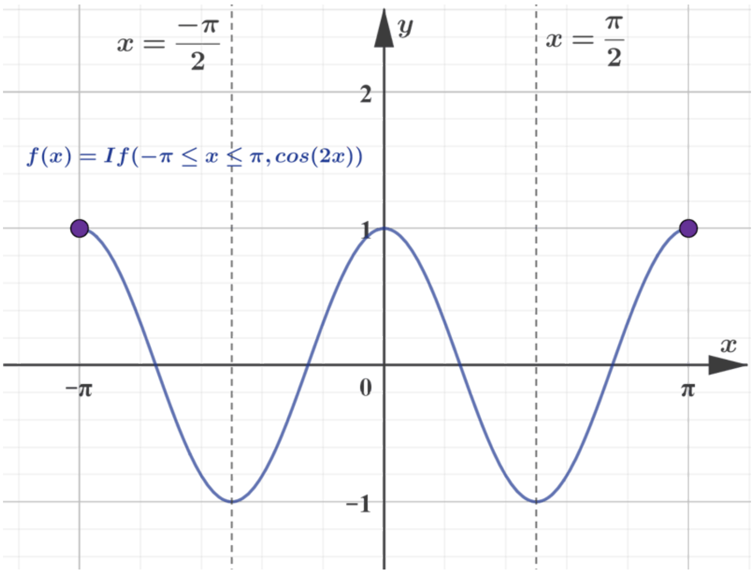
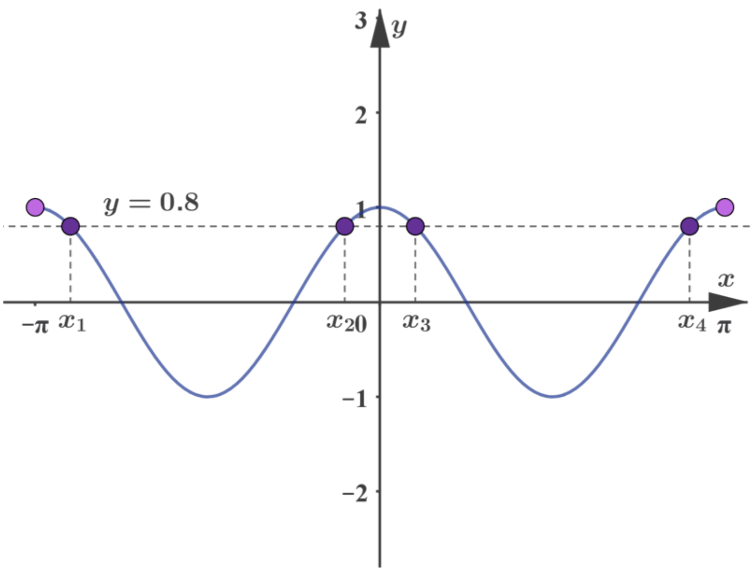
Steps	Explanation
	Sketch the graph of $y =$ marking the end points, and minima.
	Draw the line $y = 0.8$ ar points of intersection wi $y = \cos 2x$. Use the symmetry of the graph to write down the relationship between th coordinates of the point intersection.

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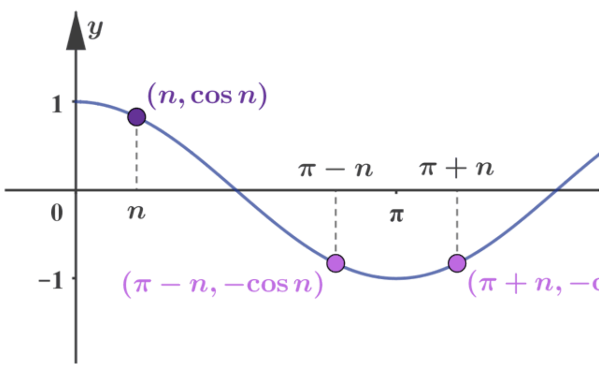
Steps	Explanation
Therefore: $\cos(-2\pi+2x)=0.8$ $\cos(-2x)=0.8$ $\cos(2\pi-2x)=0.8$	

Example 2

★★☆

Let $m=\cos n,\ 0<n<\dfrac{\pi}{2}$

Find the solution(s) of $\cos 2x=-m,\ 0\leqslant x\leqslant\pi$, in terms of n.

Steps	Explanation
$\cos 2x=-\cos n$	Use the symmetry properties of the cosine 

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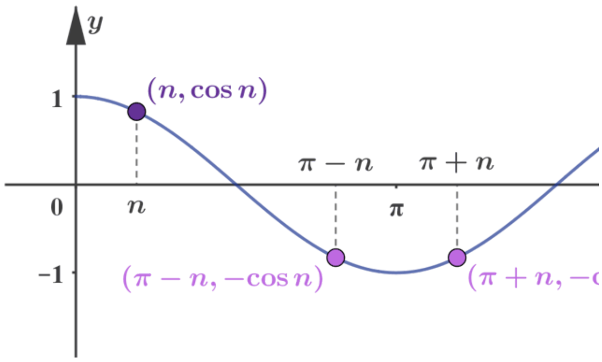
Steps	Explanation
$$\cos 2x = \cos(\pi - n)$$ or $$\cos 2x = \cos(\pi + n)$$	Use the symmetry properties of the cosine 
$$2x = \pi - n \Rightarrow x = \frac{\pi}{2} - \frac{n}{2}$$ or $$2x = \pi + n \Rightarrow x = \frac{\pi}{2} + \frac{n}{2}$$	Solve the equations.
Therefore, the answer is $$x = \frac{\pi}{2} - \frac{n}{2}$$ or $$x = \frac{\pi}{2} + \frac{n}{2}$$	Both results satisfy the initial condition $$0 \leqslant x \leqslant \pi$$

Exam tip

In the IB examination, you will not be told whether a trigonometric function is even or odd, nor will you be given information about the symmetry. You should memorise that, in a given domain,

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odd functions have the property that $f(-x) = -f(x)$, i.e. they have rotational symmetry about the origin

even functions have the property that $f(-x) = f(x)$, i.e. they have line symmetry (with respect to the y-axis)

and be able to sketch the graphs to see the symmetries.

International Mindedness

Each year, as many as 10 000 containers fall off the ships which carry goods across the seas. In 1992, a container full of rubber ducks, which were going to Seattle from China, fell off the ship. After almost a decade they appeared off the coast of New England having travelled the oceans around the globe.

Along with these containers, millions of tons of plastic end up in the oceans, so polluting the water and threatening the sea life. Understanding currents and tides could support cleaning efforts. In this article (https://plus.maths.org/content/os/issue32/features/elwell/index#ref1), Frances Elwell discusses how currents and tides behave and how studying these could help to protect the environment.

3 section questions

Question 1

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Select, from the given options, which of the following is the graph of an even function for the domain $-\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}$.

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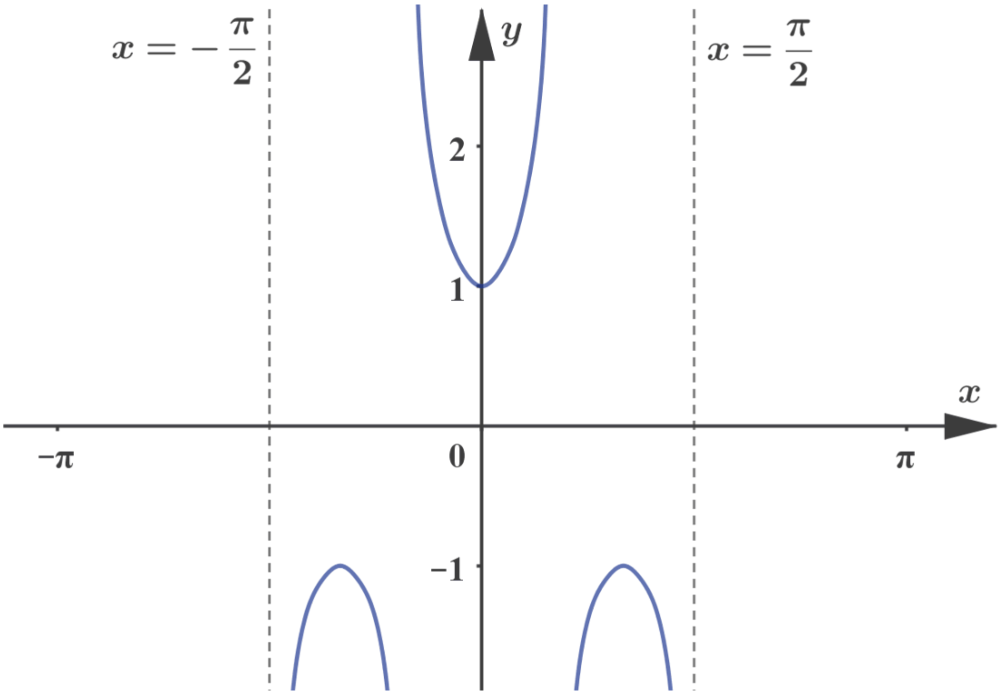

More information

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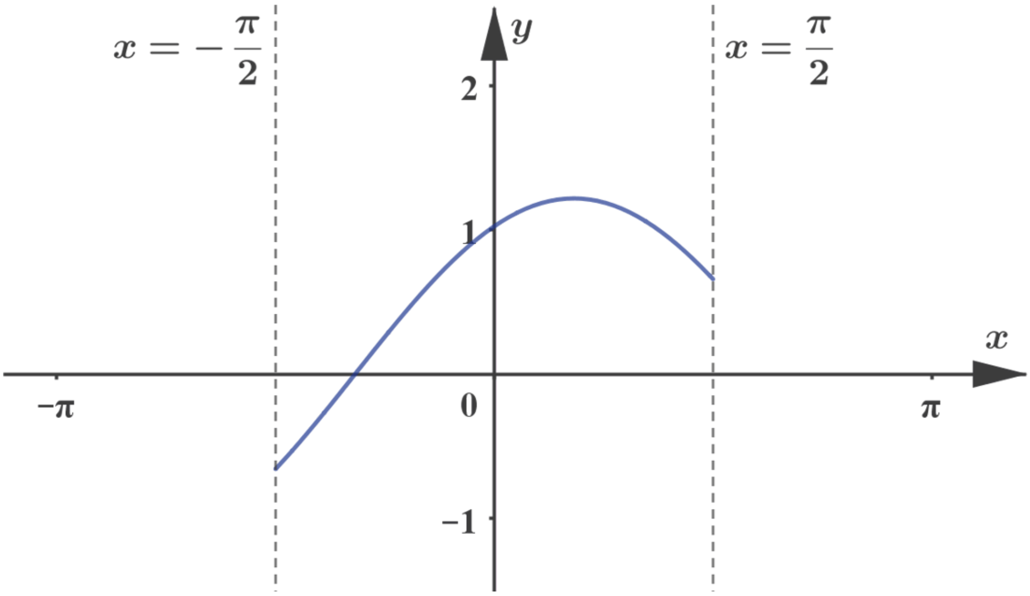

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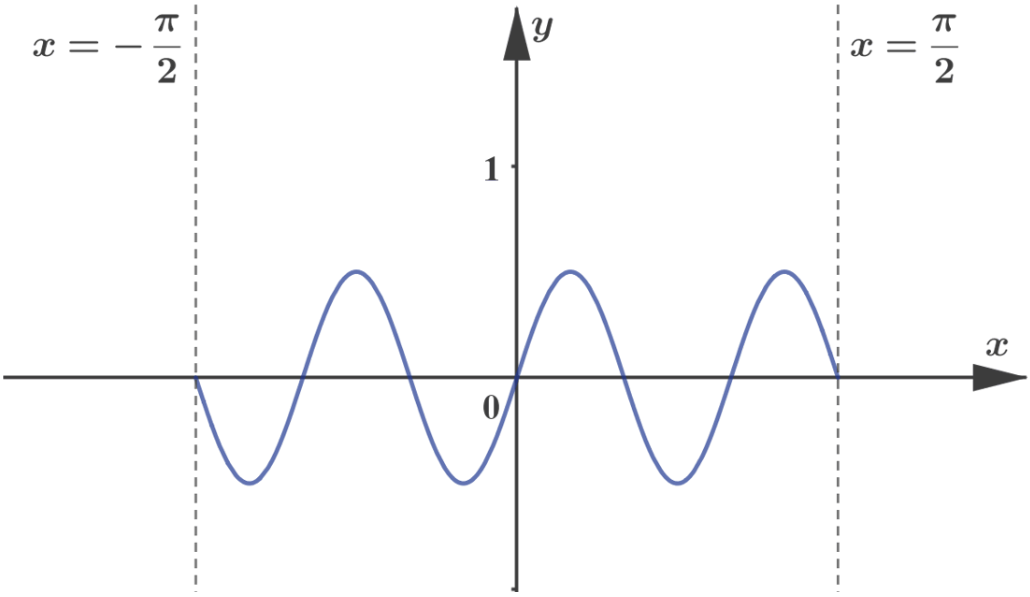

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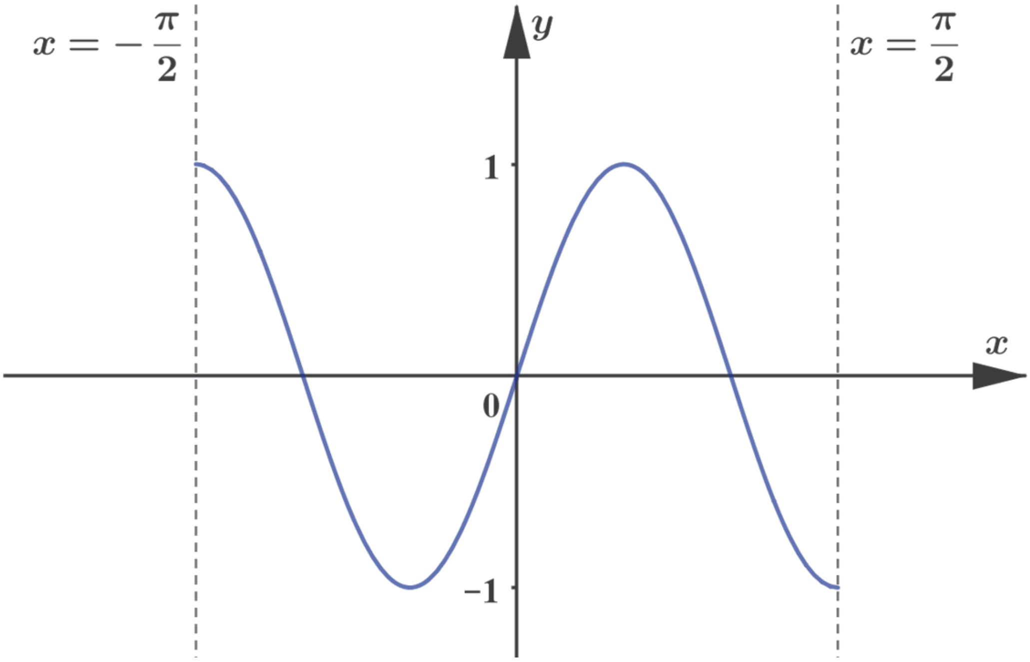

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Explanation

Even functions have line symmetry with respect to the y-axis. Therefore, the correct answer is

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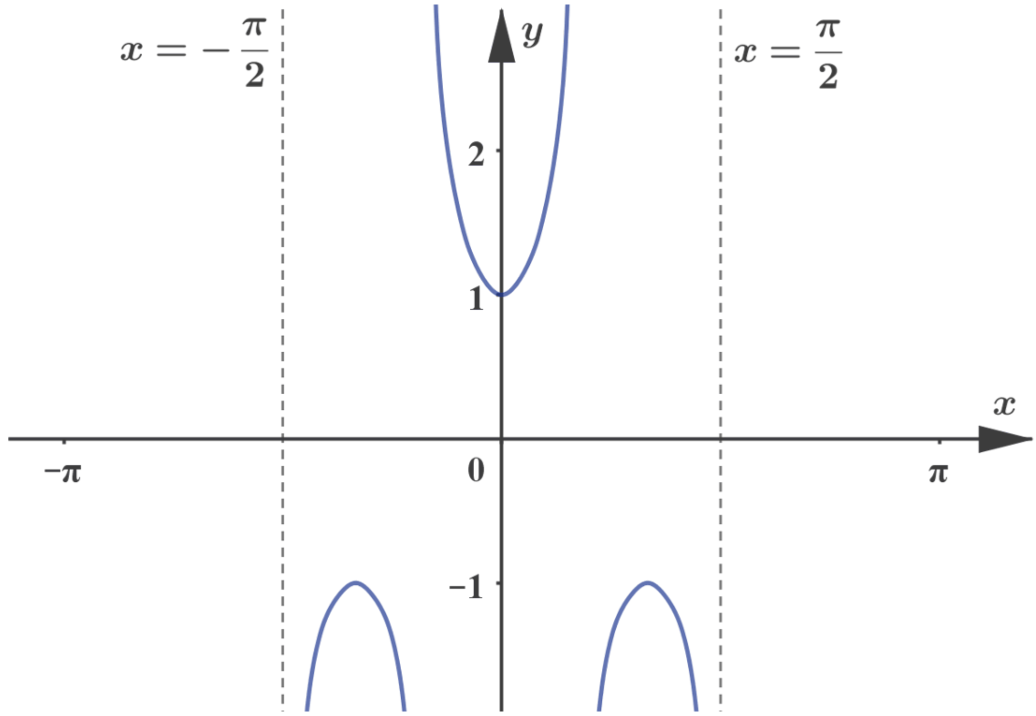

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Question 2

★☆☆

Select, from the given options, an odd function in the domain $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$.

1 $y = \cos x \sin x$ ✓

2 $y = \cos x + \sin x$

3 $y = 2\cos x + \sin x$

4 $y = 2\cos x + 3\sin x$

Explanation

Odd functions have the property that $f(-x) = -f(x)$

$$\cos(-x)\sin(-x) = \cos x\,(-\sin x) = -\cos x \sin x$$

Therefore, the correct answer is

$$y = \cos x \sin x$$

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Question 3

★☆☆

Which of the following is an even function in the domain $-\pi \leqslant x \leqslant \pi$?

1 $y = \sec 2x$ ✓

2 $y = \tan 2x$

3 $y = \operatorname{cosec} 2x$

4 $y = \sin 2x$

Explanation

Even functions have the property that $f(-x) = f(x)$

$$\sec(-2x) = \frac{1}{\cos(-2x)} = \frac{1}{\cos 2x} = \sec 2x$$

Therefore, the correct answer is $y = \sec 2x$.

3. Geometry and trigonometry / 3.11 Further circular functions

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-27918/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- identify whether a trigonometric function is odd, even or neither

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- use properties of even and odd functions to solve equations involving trigonometry.

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3. Geometry and trigonometry / 3.11 Further circular functions

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-27919/print/) Assign

You will investigate the relationship between $y = \sin x$ and $y = \sin\left(\frac{\pi}{2} \pm x\right)$.

Use Geogebra and draw the graphs of $y = \sin x$, $y = \sin\left(\frac{\pi}{2} + x\right)$ and $y = \sin\left(\frac{\pi}{2} - x\right)$.

Describe the relationship between graphs.

Use compound angle identities to prove these relationships.

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