Teacher view

Overview
(/study/app
aa-
hl/sid-
134-
cid-
761926/o

Table of
contents

Notebook

Glossary

Reading
assistance

3. Geometry and trigonometry / 3.4 The circle

The big picture

A circle is a collection of points that are equidistant from a fixed point. The distance is called the radius and the fixed point is called the centre of the circle. Although a circle appears to have no parts, we can divide one into various sections using chords, arcs and central angles, as shown in the diagrams below.
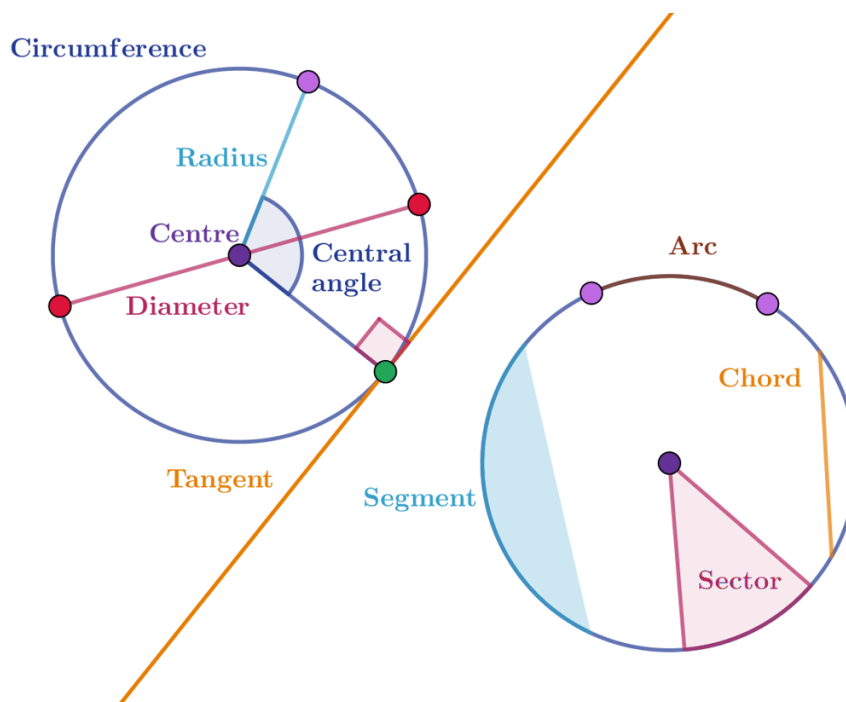
👁 More information

The image consists of two diagrams illustrating various parts of a circle.

Student
view

The left diagram shows a circle with labeled parts: 'Circumference', 'Radius', 'Centre', 'Diameter', 'Tangent', and 'Central angle'. The circle is divided by a central angle into two segments. The radius is depicted from the centre to the circumference, while the diameter is labeled as a line passing through the centre across the circle. A tangent touches the circle at a single point on the circumference.

The right diagram displays another circle, highlighting different sections: 'Arc', 'Chord', 'Segment', and 'Sector'. The chord spans across a segment of the circle, while the arc is the curved line on the circumference. The sector is a pie-shaped area between two radii of the circle.

The image visually explains the geometrical terms associated with a circle.

[Generated by AI]

⊕ International Mindedness

Why are there $360°$ in a circle?

The early Mesopotamians used base 60 as their number system. They passed this system on to the Egyptians, who were fascinated by perfect (equilateral) triangles. They could fit six perfect triangles into a circle, so they divided the circle into 360 equal parts to give six central angles of $60°$ each.

The Egyptians divided the day into 24 hours. Each hour is 60 minutes. A year was calculated as 360 days, which is only 5.25 days off the length of a solar year. The Gregorian calendar system, which is the most widely used calendar system in the world, combines the Mesopotamians' number system, the Egyptians' practice and the Greeks' documentation of all these practices.

There are lunar, solar, religious and national calendars each with different systems and reasons. Which calendar is used where you live? What year is it in other calendar systems?

⚷ Concept

In this subtopic, you will study the relationships between parts of a circle and their corresponding angles, lengths and areas. Why are all lengths and areas in circles approximate ? Why do we use the irrational number π ?

3. Geometry and trigonometry / 3.4 The circle

Radian measure of angles

Radius, arc length and angle

So far you have used the degree unit of measure, where the circle is divided into 360 equal sectors and the central angle of each sector is $1°$. This unit of measure does not relate to the circle itself; it is only dividing it into 360 equal parts.

Have you ever wondered why the circumference of a circle is $2\pi r$ or the area is πr^2 ? There is another unit of measure for angle and it is derived from the relationship between the arc length and the radius of the circle. This unit of measure is called the radian. The radian is the SI unit of angle. It is the measure of a central angle subtended by a circular arc which has the same length as the radius.

The following applet shows some angles measured in radians.
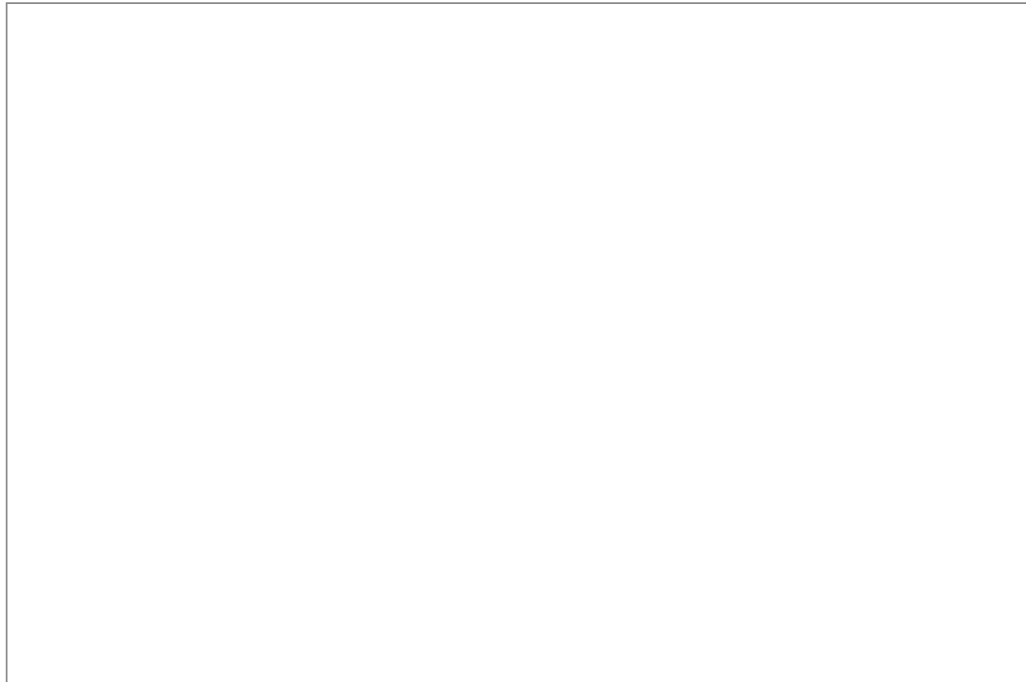
Interactive 1. Angles Measured in Radians.

👁 More information for interactive 1

This video provides a systematic visual explanation of radian measurement by demonstrating its fundamental relationship with circular geometry. The presentation begins by establishing the definition of one radian through clear animation: when an arc length precisely equals the radius of its circle, the subtended angle at the center measures exactly one radian. This geometric foundation shows how radians offer a direct, proportional way to quantify angles based on the circle's inherent properties, unlike the arbitrary division of degrees.

The video then scales this basic unit to reveal the complete angular system. Through animated progression, it illustrates how a full 360° rotation corresponds to exactly 2π radians, a relationship derived from the circle's circumference formula $C = 2\pi r$. This visualization effectively bridges the conceptual gap between linear and angular measurement, showing that 2π radians represent the angle subtended by one complete circumference. The systematic presentation methodically builds understanding from the basic radian definition to its full circular application, emphasizing why radians serve as the natural angular unit in mathematical and scientific contexts.

✓ Important

When the arc length is equal to half of the circumference, the corresponding angle is π radians. If it is a complete turn then it is 2π radians.

So $2\pi \text{ rad} = 360°$ and $\pi \text{ rad} = 180°$.

🔗 Making connections

Why is π such a mysterious number?

To start with, it is an irrational number. This means you cannot write it as a fraction of two integers and its decimal places do not repeat or end. Computer scientists have calculated π to billions of digits. It starts with $3.141592653\ldots$.

Apart from being irrational, there are many other reasons why people are fascinated by π. It appears whenever there is a curve and straight line involved, such as in, the spiral of DNA, waves in the physical world, the meandering of rivers and many other places. It also emerges in unlikely places: for example, in a group of random whole numbers, the probability of any two numbers being coprime (also known as relatively or mutually prime) is $\dfrac{6}{\pi^2}$.

You can watch a beautiful visualisation of the number π in the following video.

Pi is Beautiful - Numberphile

The ratio of the river's actual length to the distance from its source to its mouth approaches π

Credit: Dan Reynolds Photography / Getty Images

⊙ **Exam tip**

In examinations, if you are asked to find an exact value, always leave your answer as a multiple of π. If you are finding approximate values, do not approximate π as 3.14 or $\dfrac{22}{7}$. Use the π button on your calculator to avoid approximation errors.

⌂

Example 1

★☆☆

An angle is given as $40°$.

Find its value in radians, giving your answer accurate to 3 significant figures.

Step	Explanation
$\dfrac{R}{40°} = \dfrac{\pi}{180°}$	As π radians is $180°$, you can use this ratio to fin the radian measure. Call the angle in radians R .
$R = \dfrac{40\pi}{180} = \dfrac{2}{9}\pi$	Rearrange the equation and simplify.
$R = \dfrac{2}{9}\pi$	For an exact answer, leave as a multiple of π .
$R \approx 0.698 \text{ rad}$	Approximate to 3 significant figures $(\pi \approx 3.14159\ldots)$ as requested in the question.

Example 2

★☆☆

An angle is given as $\dfrac{3}{4}\pi$ radians. Find its value in degrees.

Step	Explanation
$\dfrac{\dfrac{3}{4}\pi}{A} = \dfrac{\pi}{180°}$	As π radians is $180°$, you can use this ratio to find the value in degrees. Call the angle in degrees A .

Step	Explanation
$A = \dfrac{\frac{3}{4}\pi}{\frac{\pi}{180°}} = 135°$	Rearrange the equation and simplify.
$A = 135°$	So the angle in degrees is $135°$.

ⓘ **Exam tip**

The conversion between degrees and radians is not in the IB formula booklet.

Always remember $180°$ is π radians. This means that $1\,\mathrm{rad} = \dfrac{180}{\pi}$ degrees $\approx 57.3°$.

It is worth having an idea of the value of 1 radian when you are converting between the two forms.

4 section questions ⌃

Question 1   🖩⊗

★☆☆

What is $120°$ in radians?

1	$\dfrac{2}{3}\pi$	✓
2	$\dfrac{3}{2}\pi$	
3	$\dfrac{1}{3}\pi$	
4	3π	

Explanation

We have that $\theta° = 120°$ and hence the conversion

⌂

$$\frac{120}{180} \times \pi = \frac{2}{3}\pi.$$

Question 2

★☆☆

Which of the following represents the conversion of 1.2 radians to degrees?

1. $\dfrac{1.2 \times 180}{\pi}$ ✓

2. $\dfrac{1.2 \times 360}{\pi}$

3. $\dfrac{1.2 \times 180}{2\pi}$

4. $1.2 \times 180\pi$

Explanation

Using

$$\frac{\text{radian measure}}{\pi} = \frac{\text{degree measure}}{180}$$

$$\frac{1.2}{\pi} = \frac{D}{180}$$

$$D = \frac{1.2 \times 180}{\pi}$$

Question 3

★☆☆

What is $0.15 \, \text{rad}$ in degrees?

1. $8.59°$ ✓

2. $0.116°$

3. $4.30°$

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Student
view

4. $17.2°$

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Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

Explanation

We have that $\theta = 0.15\,\text{rad}$ and hence, the conversion

$$\frac{0.15}{\pi} \times 180 \approx \frac{0.15}{3.14159} \times 180 \approx 8.59°(\text{3 significant figures})$$

Question 4

★☆☆

Convert $73°$ to radians.

Give your answer accurate to 3 significant figures with no units.

✏ 1.27   ✓

Accepted answers

1.27, 1,27

Explanation

$$\frac{\text{radian measure}}{\pi} = \frac{\text{degree measure}}{180}$$

$$\frac{R}{\pi} = \frac{73}{180}$$

$$R = \frac{\pi \times 73}{180}$$

$$R = 1.27 \ (\text{3 significant figures})$$

3. Geometry and trigonometry / 3.4 The circle

Length of an arc

Ratios of length

An arc is part of a circle's circumference. The diagram below shows a circle with centre O and radius r . The points A and B on the circumference form the arc AB , which subtends the central angle θ . Arc AB is called a <u>minor arc</u> as it is smaller than the half of the circle, and remaining arc is called <u>major arc</u> as it is more than the half of circumference of the circle.
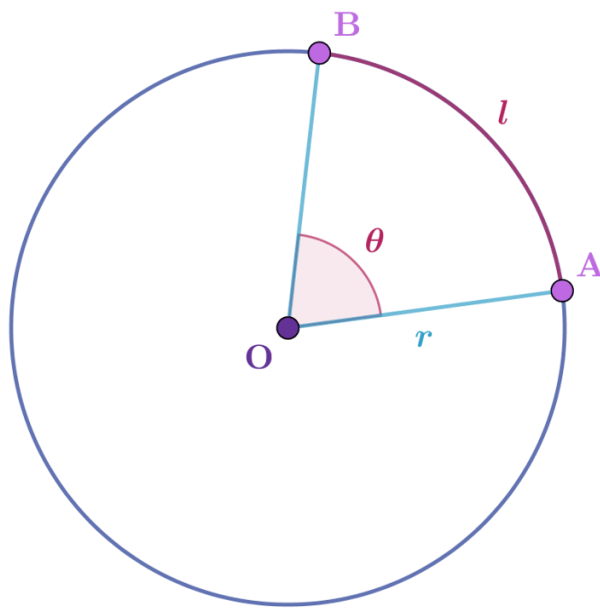
👁̸ More information

The diagram illustrates a circle centered at point O with radius r. Points A and B are located on the circumference forming a minor arc AB. This arc subtends a central angle θ at point O. The segment OA and OB are radii, and the line between them is the arc length l. The shaded sector represents angle θ, which is less than a semicircle, therefore forming a minor arc while the remaining opposite arc is the major arc.

[Generated by AI]

✓   **Important**

The ratio of θ to the full rotation, 2π , is the same as the ratio for the arc length l to the circumference, $2\pi r$.

Thus,

$$\frac{\theta}{2\pi} = \frac{l}{2\pi r}$$

Rearranging and simplifying, you get the length of the arc in terms of the angle and radius as

$$l = r\theta$$

⚠   **Be aware**

The formula $l = r\theta$ is only true when the angle θ is given in radians.

If the angle is given in degrees, either convert to radians or use the ratio

$$\frac{l}{2\pi r} = \frac{\theta}{360°} \ .$$

ⓘ   **Exam tip**

In IB examinations, the formula booklet will give the formula for the length of an arc as $l = r\theta$, where r is the radius and θ is the angle measured in radians.

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Example 1

★☆☆

A circle with radius 5 cm has an arc which subtends an angle of $60°$.

Find the length of the arc accurate to 3 significant figures.

Steps	Explanation
$\dfrac{l}{2\pi r} = \dfrac{60°}{360°}$	Formula $l = r\theta$ is only true when angle is in radians. So you need to use the ratio of the angle to $360°$.
$\dfrac{l}{10\pi} = \dfrac{1}{6}$	Substitute $r = 5$ and simplify.
$l = \dfrac{5}{3}\pi$	Rearrange and simplify.
$l \approx 5.24$ cm	Approximate to 3 significant figures as request in the question.

Example 2

★☆☆

A circle with a radius of 5 cm has an arc of length 8 cm which subtends an angle θ.

Find the size of θ in radians.

Steps	Explanation
$\theta = \dfrac{l}{r}$	Rearrange the formula $l = r\theta$.
$\theta = \dfrac{8}{5}$	Substitute $l = 8$ and $r = 5$.
$\theta = 1.6$ rad	Perform the calculation.

3 section questions ⌃

Question 1

★☆☆

A circle has a radius of $8\,\text{cm}$. What is the length of an arc with a central angle of $80°$?

1    $\dfrac{80}{180} \times \pi \times 8\,\text{cm}$ ✓

2    $\dfrac{80}{360} \times \pi \times 8\,\text{cm}$

3    $\dfrac{80}{\pi} \times 8\,\text{cm}$

4    $\dfrac{80}{2\pi} \times 8\,\text{cm}$

Explanation

Since $80° = \dfrac{80}{180} \times \pi\,\text{rad}$, we have the length of the arc of $l = \theta \times r = \dfrac{80}{180} \times \pi \times 8\,\text{cm}$.

Question 2

★☆☆

Given that a circle has a radius of $7\,\text{cm}$ and an arc of length $9\,\text{cm}$ is subtended by a central angle θ, find the measure of θ in degrees. Give your answer to 3 significant figures without the symbol $°$ or the word 'degrees'.

✎   73.7   ✓

Accepted answers

73.7, 73,7, 73.7°, 73.7 degrees, 73.7degrees, 73.7°

Explanation

From the arc length formula, we have $l = \theta r$ or $\theta = \dfrac{l}{r}$ in radians.

Thus,

$$\theta = \frac{9}{7}\,\text{rad.}$$

Converting to degrees,

$$\frac{9}{7}\frac{180°}{\pi} \approx 73.7°.$$

Question 3

Li Wei and his friend are conducting an experiment on pendulum motion.

If their pendulum is 45 cm and traverses an arc of 45 cm, find, to the nearest degree, the rotation of the pendulum swing.

1     $57°$ ✓

2     $1°$

3     $180°$

4     $45°$

Explanation

Using the formula $l = r\theta$

$$45 = 45\theta$$

$\theta = 1$ radian

Converting radians to degrees:

$$D = \frac{1 \times 180}{\pi}$$

$D = 57°$ correct to the nearest degree

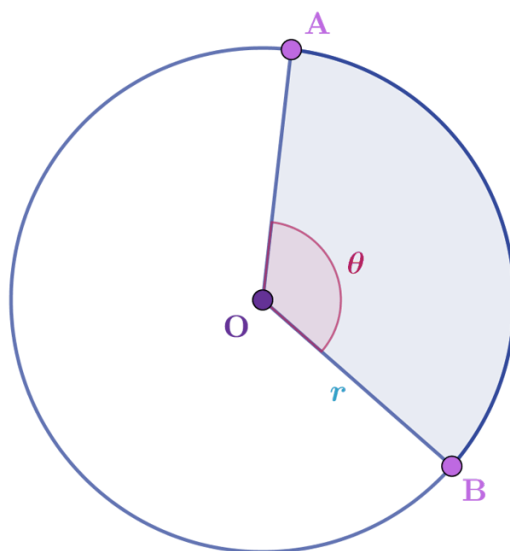3. Geometry and trigonometry / 3.4 The circle

Area of a sector

Ratios of area

Just as the length of an arc is a fraction of the circle's circumference, so the area of a sector is a fraction of a circle's area.

👁 More information

The image is a diagram of a circle with a sector highlighted in blue. The circle is centered at point O. The radius extending from point O to point A is labeled as 'r', and the radius extending from O to point B is also labeled as 'r', denoting that both are radii of the circle. The arc connecting point A to point B on the circumference forms the boundary of the sector.

| Section | Student... (0/0) | 📢 Feedback | 🖨 Print (/study/app/math-aa-hl/sid-134-cid-761926/book/length-of-an-arc-id-27737/print/) | Assign |

Within the sector, the central angle formed by radii OA and OB is labeled as 'θ'. Point O, the center of the circle, is marked, as well as points A and B along the circle's boundary.

The diagram represents the concept of a sector of a circle, which is defined by the two radii OA and OB and the arc AB, and is used to illustrate how the area of a sector is a portion of the circle's total area, indicated by the fraction determined by angle θ.

[Generated by AI]

Writing this as ratios gives:

$$\frac{\text{area of the sector}}{\pi r^2} = \frac{\theta}{2\pi}.$$

Rearrange to get

$$\text{area of the sector} = \frac{\pi r^2 \theta}{2\pi}$$

and simplify to get

$$\text{area of the sector} = \frac{1}{2} r^2 \theta.$$

Of course, the angle θ here is in radians.

ⓘ **Exam tip**

In IB examinations, the formula booklet gives the formula for the area of a sector as

$$A = \frac{1}{2} r^2 \theta$$

where r is the radius and θ is the angle measured in radians.

Example 1

★☆☆

The radius of a circle is 5 cm.

Find the area of a sector of the circle with a central angle of 1.3 radians.

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Overview
(/study/app
aa-
hl/sid-
134-
cid-
761926/o

Step	Explanation
$A = \dfrac{1}{2}\theta r^2$	As the angle is given in radians you can use the formula $A = \dfrac{1}{2}\theta r^2$.
$A = \dfrac{1}{2}1.3 \times 5^2$	Substitute the values for the angle and the radius.
$A = 16.25 \text{ cm}^2$	Calculate the area of the sector.

Example 2

★★☆

The diagram shows a circle with centre O and radius 4 cm.

If the measure of the central angle is $\dfrac{3\pi}{4}$, find the area of the shaded segment. Give your answer accurate to 3 significant figures.
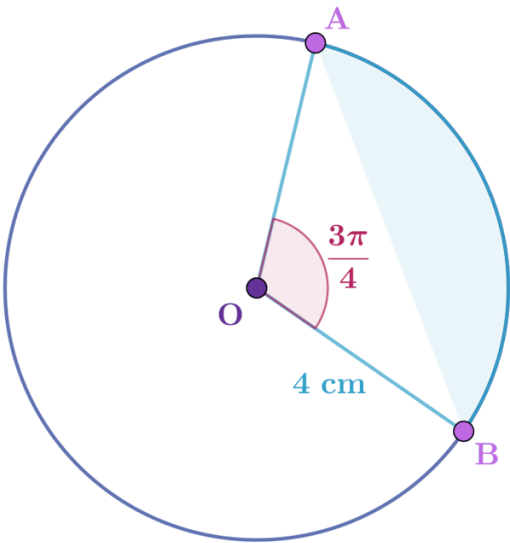
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Student
view

👁 More information

This image is a diagram illustrating a circle with a segment shaded in blue. The circle is centered at point O. There are two radii, OA and OB, forming the central angle (\dfrac{3\pi}{4}), which is depicted in purple. Point A and point B are on the circle's circumference.

The angle (\dfrac{3\pi}{4}) is highlighted in the diagram, showing the central part of the circle with a label indicating the angle size. The radius of the circle is marked as 4 cm.

The shaded segment of the circle, bordered by the two radii OA and OB and the circular arc AB, is the focus of the accompanying question about finding its area.

[Generated by AI]

Step	Explanation
$\text{area of segment } = \text{ area of sector } - \text{ area of triangle}$	To find the area of the shaded segment, find the area of the sector AOB and subtract the area of the triangle AOB.
$\text{area of sector} = \dfrac{1}{2} \times 4^2 \times \dfrac{3\pi}{4}$	As the angle is given in radians you can use the formula $$A = \dfrac{1}{2}r^2\theta$$ to find the sector area. Substitute $r = 4$ and $\theta = \dfrac{3}{4}\pi$.
$\text{area of sector} = 6\pi \approx 18.8496$	Do not round your answer here. Or use at least 4 decimal places to avoid rounding errors.

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

Step	Explanation
$$\text{area of triangle} = \frac{1}{2}4^2 \sin\left(\frac{3\pi}{4}\right) = 4\sqrt{2} \approx 5.6569$$	Use the formula for the area of a triangle $$A = \frac{1}{2}ab\sin C$$ In the triangle AOB, a and b are both 4 cm and C is $\frac{3\pi}{4}$.
$$\begin{aligned}\text{area of segment} &= \text{ area of sector } - \text{ area of triangle}\\ &= 6\pi - 4\sqrt{2}\\ &\approx 13.1927\end{aligned}$$	Now subtract the area of the triangle from the area of the sector.
$$\text{area of segment } = 13.2 \text{ cm}^2$$	Give your answer to 3 significant figures as requested in the question.

⚙⚙ **Activity**

Using A3 paper, make a cone with base radius 9 cm and perpendicular height of 12 cm.

- Draw the net of a cone.
- What is the shape of the slanted (curved) surface?
- Calculate the dimensions of the parts of the net.
- Derive a general formula for the surface area.

4 section questions ∧

Question 1

★★☆

A is the centre of a circle with radius 12 cm.

BC is perpendicular to AE, and angle $BAC = \frac{\pi}{4}$.
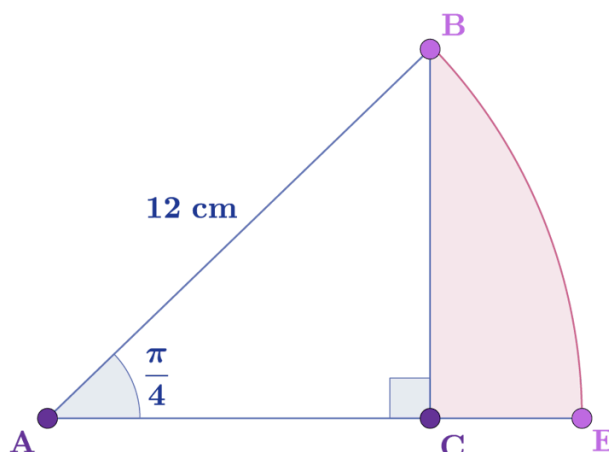
What is the area of the shaded region?

👁 More information

1    $18\pi - 36 \text{ cm}^2$ ✓

2    $32\pi - 9 \text{ cm}^2$

3    $32\pi - 36 \text{ cm}^2$

4    $18\pi - 32 \text{ cm}^2$

Explanation

area of the shaded region = area of the sector − area of $\triangle ABC$

$$BC = 12\sin\left(\frac{\pi}{4}\right) = 6\sqrt{2}$$

Since the angle at A is $\frac{\pi}{4}$, the right triangle ABC is isosceles, so AC is also $6\sqrt{2}$.

area of $\triangle ABC = \dfrac{1}{2}6\sqrt{2} \times 6\sqrt{2} = 36$ area of the sector $= \dfrac{1}{2} \times \dfrac{\pi}{4} \times 12^2 = 18\pi$

area of the shaded region $= 18\pi - 36 \text{ cm}^2$

Question 2

🖩

❌

Student
view

★★☆

A circle has a radius of $8\,\mathrm{cm}$. What is the area of the sector with a central angle of $80°$?

1    $\dfrac{128}{9}\pi\,\mathrm{cm}^2$ ✓

2    $\dfrac{128}{18}\pi\,\mathrm{cm}^2$

3    $\dfrac{16}{9}\pi\,\mathrm{cm}^2$

4    $\dfrac{16}{18}\pi\,\mathrm{cm}^2$

Explanation

Since $80° = \dfrac{80}{180} \times \pi\,\mathrm{rad}$, we have a sector with an area of $A = \dfrac{1}{2}\theta \times r^2 = \dfrac{1}{2}\dfrac{80}{180} \times \pi \times 8^2 = \dfrac{128}{9}\pi\,\mathrm{cm}^2$.

Question 3   🧮

★★☆

Given a circle of radius of $6\,\mathrm{cm}$ and a central angle of $130°$, find the area of the sector of this central angle. Give your answer to 3 significant figures with no units.

✏️ 40.8   ✓

Accepted answers

40.8, 40,8, 13pi

Explanation

We know that the $130°$ is $\dfrac{130°}{180°}\pi = \dfrac{13}{18}\pi\,\mathrm{rad}$. Hence, using the equation for the area of a sector, we obtain:

$$A = \dfrac{1}{2}\theta r^2 = \dfrac{1}{2} \times \dfrac{13}{18}\pi \times 6^2 = 13\pi\,\mathrm{cm}^2 \approx 40.8\,\mathrm{cm}^2.$$

Question 4   🧮

★★☆

On a car, one windshield wiper is $45\,\mathrm{cm}$ long, and is fixed to a swing arm which is $76\,\mathrm{cm}$ long from pivot point to wiper-blade tip.

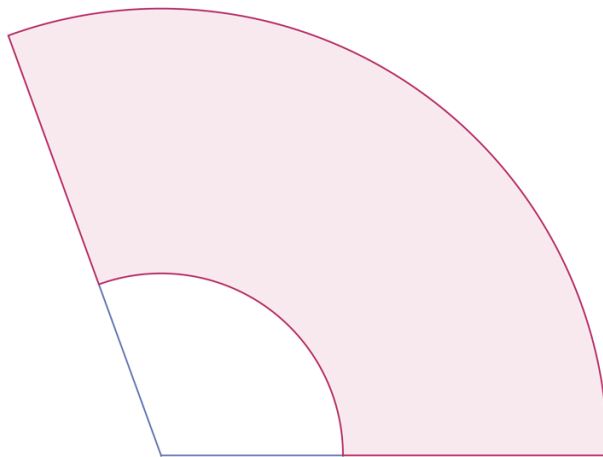Find the area of the windshield swept, if the swing arm turns around $110°$.

Give your answer, without units, to 3 significant figures.

👁 More information

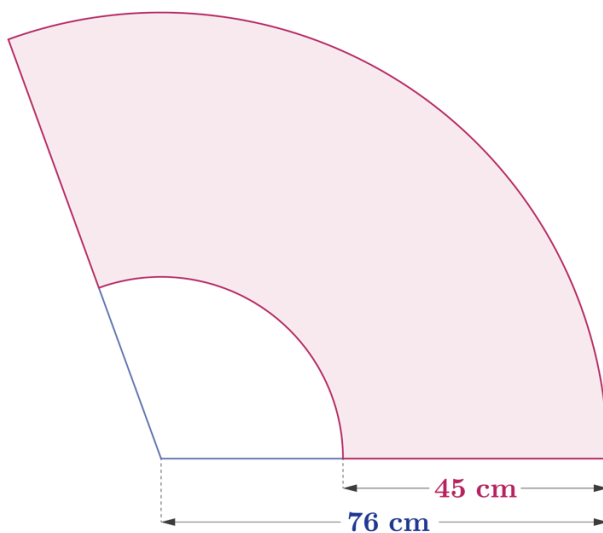✏ 4620   ✓

Accepted answers

4620

Explanation

👁 More information

The difference between the two sectors gives the area swept.

The radius of the smaller sector is $76 - 45 = 31$cm.

Convert the degrees to radians

$$R = \frac{\pi \times 110}{180} = \frac{11\pi}{18}$$

Do not round the answer at this point.

Using the formula $A = \frac{1}{2}r^2\theta$ for each sector.

For two different radii:

$$\text{shaded area} = \left(\frac{1}{2}76^2 \times \frac{11\pi}{18}\right) - \left(\frac{1}{2}31^2 \times \frac{11\pi}{18}\right)$$

$$\text{shaded area} = 4620 \text{ cm}^2 \text{ (3 significant figures)}$$

3. Geometry and trigonometry / 3.4 The circle

Checklist

📋 **What you should know**

By the end of this subtopic you should be able to:

- convert degrees to radians

- convert radians to degrees

- find the length of an arc using the formula $l = r\theta$

- find the area of a sector using the formula $A = \frac{1}{2}r^2\theta$.

3. Geometry and trigonometry / 3.4 The circle

Investigation

Archimedes used the method of exhaustion to approximate π using the ratio of the diameter of the perimeter of the inscribed polygons to the diameter of the circle. Later, other mathematicians used the same method with areas. The method approximates the area of a shape by inscribing it inside of a sequence of polygons whose area converges to the area of the containing shape.

In this investigation you will follow the steps of these mathematicians. You will use the areas of the inscribed and circumscribed polygons to find upper and lower bounds for the area of a circle with radius 6 cm.

1. Start by using regular pentagons. Use the applet below to find:

- the area of the inscribed regular pentagon
- the area of the circumscribed regular pentagon
- the average of the two areas.
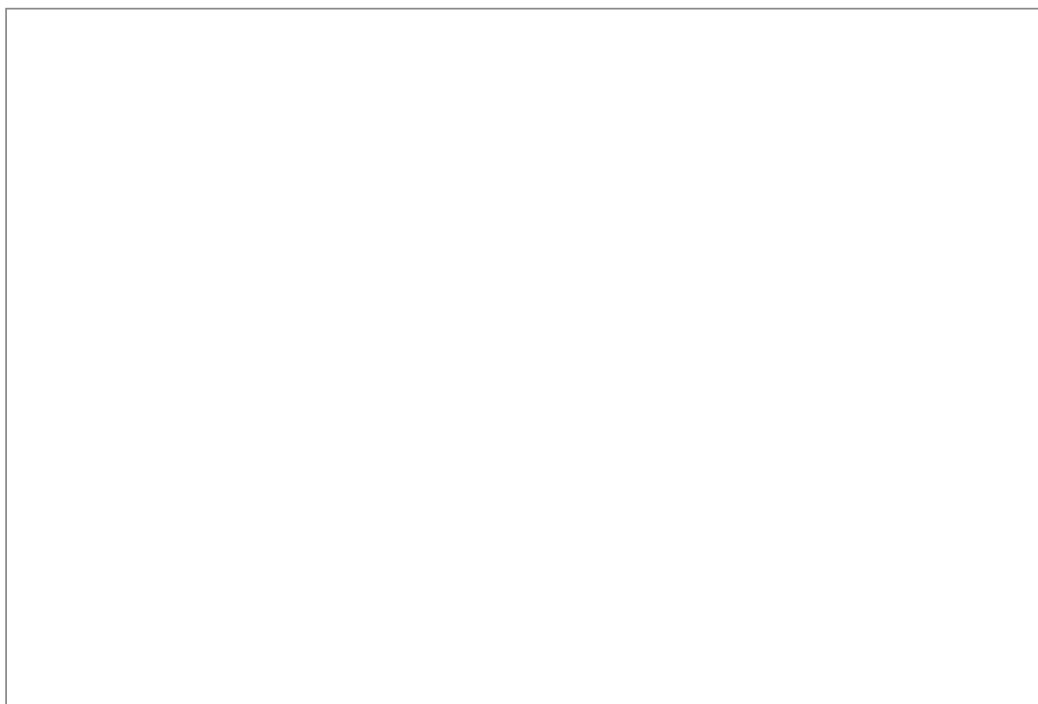
Interactive 1. Upper and Lower Bounds for Circle Area Using Polygons.

👁 More information for interactive 1

This interactive tool demonstrates Archimedes' ancient technique for estimating π by comparing polygons to circles.

Users adjust two sliders: one for the circle's radius r (ranging from 4 to 8 units) and another for the number of polygon sides n (from 3 to 50). As these values change, the tool instantly calculates and displays the areas of both inscribed (in blue color) and circumscribed (in red color) regular polygons, visually showing how they increasingly

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

approximate the circle's true area (πr^2) as more sides are added.

By calculating the average of the two polygonal areas, users obtain increasingly accurate estimates of the circle's area. When this average is set equal to πr^2 and solved for π, the approximation improves dramatically with higher side counts. Starting from rough estimates like π ≈ 3.0 for hexagons, it approaches 3.1416 for 50-sided polygons, illustrating how Archimedes bounded π between these polygonal measurements.

The simulation beautifully connects classical geometry to fundamental calculus concepts. Through the visual demonstration of the "method of exhaustion," it shows how infinite-sided polygons perfectly match a circle's area, making abstract limit concepts concrete while showcasing Archimedes' remarkable mathematical insight that laid the foundations for modern calculus.

2. Now use regular hexagons:

- Find the area of the inscribed regular hexagon.
- Find the area of the circumscribed regular hexagon.
- Find the average of the two areas.

3. Use a spreadsheet program to create a table similar to the one below.

Complete extra rows in the table for polygons with more sides.

Number of sides	Area of inscribed polygon	Area of circumscribed polygon	Average of the areas of polygons (AP)	(AP)-Area of circle using πr^2
5				
6				

When does the average of the areas of the inscribed and circumscribed polygons seem to be equal to the area of the circle? As this method is an approximation, why do you think the difference reaches zero?

In the video below, you will discover how the ancient mathematicians used the methods of exhaustion and limits to approximate π .

A Brief History of Pi

▶

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