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The big picture

If you are asked to find the solutions of $x^4 = 16$ you will probably say the answers are 2 and -2. Your answer would be correct but incomplete. Just as a quadratic equation has two solutions, a cubic equation has three, a quartic equation has four and so on. Therefore, $x^n = a$ has n roots.

In the case of $x^4 = 16$ the two missing roots are complex numbers. In this section you will learn how to find these complex roots and see the geometrical patterns that are created when the roots are plotted on the Argand diagram, as seen in the diagram below for the case of $x^4 = 16$. All four of these complex numbers can be considered as the fourth root of 16.

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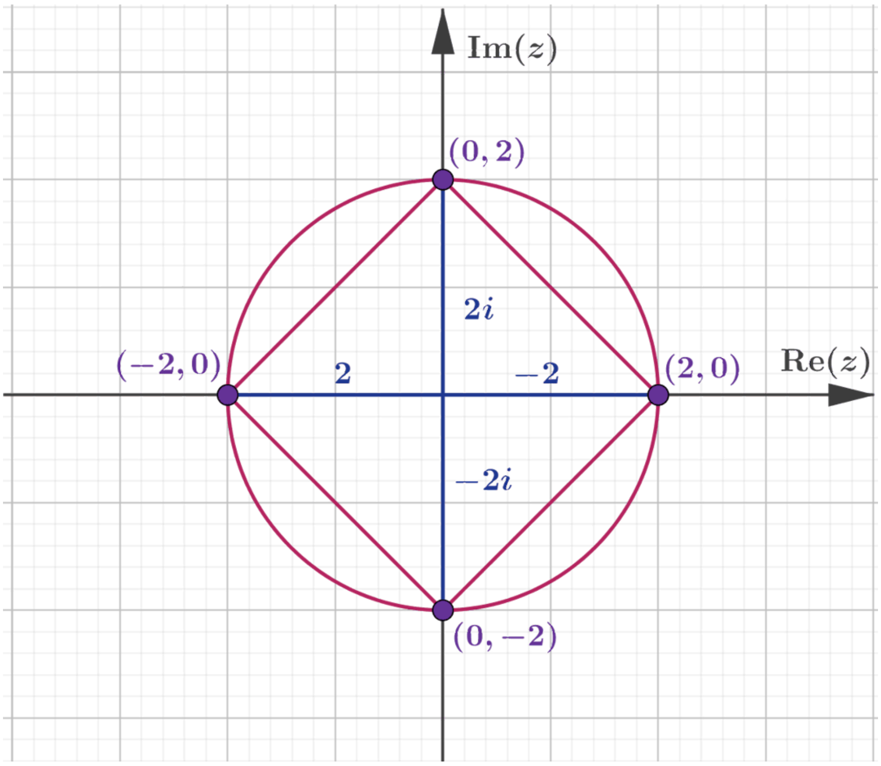

More information

This diagram represents the complex roots of the equation (x^4 = 16) on an Argand plane, which is a type of complex plane graph. The grid background provides a reference for plotting these roots. The structure reveals a circle centered at the origin (0, 0) with a radius of 2 units. Along the circumference of this circle, four points are clearly marked: ((2, 0)), ((0, 2i)), ((-2, 0)), and ((0, -2i)), illustrating the fourth roots of 16.

The horizontal axis is labeled as "Re(z)", representing the real part of the complex numbers. The vertical axis labeled "Im(z)" represents the imaginary component. This setup allows the illustration of both real and imaginary numbers on the same plane.

The symmetry in the diagram showcases the geometric arrangement of these roots around the origin, forming a perfect cross intersecting at the real and imaginary axes at equal intervals, indicating equal magnitude but varying phases of the complex numbers involved. These markers indicate the positions of the roots spaced evenly around the circle, representing the geometric principle underlying the calculation of complex roots.

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Concept

The patternsin the Argand diagrams of roots of complex numbers are both aesthetically pleasing (as seen above) and useful in simplifying and checking calculations. As you work through this subtopic, look for connections between the geometric and the algebraic forms of complex numbers to inform your understanding.

1. Number and algebra / 1.14 Powers and roots of complex numbers

De Moivre's theorem

Activity

Let $z = r \text{ cis } \theta$. Use your knowledge of multiplication in polar form to find:

z^2 z^3 z^4 z^5

Propose a conjecture for the result of z^n .

Important

De Moivre's theorem

For $z = r\ (\cos\theta + \mathrm{i}\sin\theta)$, $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$, where $n \in \mathbb{Z}$.

This can also be written as $z^n = r^n \text{ cis } n\theta$ or $z^n = r^n \mathrm{e}^{\mathrm{i}n\theta}$.

International Mindedness

Abraham De Moivre (1667—1754), after whom the theorem is named, was a French mathematician who worked in London. In addition to his work with complex numbers he also made advances in the field of probability and normal

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distributions. In 1710, he was appointed to a Royal Society commission tasked with deciding whether Isaac Newton or Gottfried Liebniz should be credited with the invention of calculus.

Example 1

★☆☆

Show that if n is a positive integer and $z = r \operatorname{cis} \theta$, then $z^n = r^n \operatorname{cis} n\theta$.

Steps	Explanation
$(r \operatorname{cis} \theta)^n = r^n (\operatorname{cis} \theta)^n = r^n \operatorname{cis} n\theta$	In section 1.15.4 (/study/app/math-aa-hl/sid-134-cid-761926/book/proof-by-induction-continued-id-270(you will see how to use the method of proof by indu to prove that $(\operatorname{cis} \theta)^n = \operatorname{cis} n\theta$.

De Moivre's theorem is very useful when working with powers of complex numbers. Consider the following examples.

Example 2

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Given that $z = 2 + 2\sqrt{3}\mathrm{i}$, find z^3 in $a + b\mathrm{i}$ form.

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Steps	Explanation
Method 1 $z^3 = (2)^3 + 3(2)^2\left(2\sqrt{3}i\right) + 3\,(2)\left(2\sqrt{3}\mathrm{i}\right)^2 + \left(2\sqrt{3}\mathrm{i}\right)^3$ $= 8 + 24\sqrt{3}\mathrm{i} - 72 - 24\sqrt{3}\mathrm{i}$ $= -64$	Using the binomial theorem i Cartesian form, which you m section 1.9.2 (/study/app/mat aa-hl/sid-134-cid-761926/book/binomial-theor id-27689/).
Method 2 $\lvert z\rvert = \sqrt{(2)^2 + \left(2\sqrt{3}\right)^2} = 4$ $\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$ Since z is in the first quadrant, $\arg(z) = \frac{\pi}{3}$. $z = 4\ \mathrm{cis}\frac{\pi}{3}$	Convert to polar form.
$z^3 = 4^3\ \mathrm{cis}\ \pi = 64\ (\cos\pi + \mathrm{i}\sin\pi) = -64$	Use De Moivre's theorem.

Would you want to use the binomial theorem method for a question asking to find z^{10}?

Example 3

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Write $\frac{1+\mathrm{i}}{(1-\mathrm{i})^5}$ in polar form.

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Steps	Explanation
Let $z = 1 + \mathrm{i}$: $$\|z\| = \sqrt{2}$$ $$\theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$$ Since z is in the first quadrant, $\arg(z) = \frac{\pi}{4}$. $$z = \sqrt{2}\,\mathrm{cis}\frac{\pi}{4}$$ Let $w = 1 - i$: $$\|w\| = \sqrt{2}$$ $$\theta = \tan^{-1} - \frac{1}{1} = -\frac{\pi}{4}$$ Since w is in the fourth quadrant, $\arg(w) = -\frac{\pi}{4}$. $$w = \sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$$	Convert to polar form.
$$w^5 = \left(\sqrt{2}\right)^5 \mathrm{cis}\left(-\frac{5\pi}{4}\right)$$	Use De Moivre's theorem.
$$\begin{aligned}\frac{z}{w^5} &= \frac{\sqrt{2}}{\left(\sqrt{2}\right)^5}\,\mathrm{cis}\left(\frac{\pi}{4} - \left(-\frac{5\pi}{4}\right)\right)\\ &= \left(\sqrt{2}\right)^{-4}\mathrm{cis}\frac{6\pi}{4}\\ &= \frac{1}{4}\,\mathrm{cis}\frac{3\pi}{2}\end{aligned}$$	Divide.

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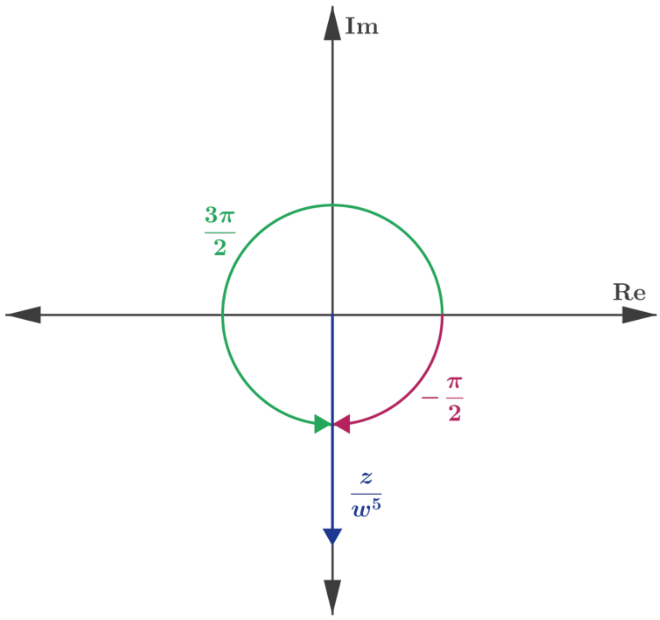
Steps	Explanation
 $\frac{z}{w^5} = \frac{1}{4} \text{ cis} \left(-\frac{\pi}{2}\right)$	The argument of the result is no the range for the principle argur which is $-\pi$ to π . You need to find an angle that is equivalent to $\frac{3\pi}{2}$ in this range.

Exam tip

Unless a question specifies otherwise, you are expected to write the principle argument in your final answers. Always check that in your final answer $-\pi < \theta \leq \pi$. If this is not the case, find the equivalent angle in the range of the principle argument.

Example 4

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If $w = \sqrt{3} - \text{i}$, find $\frac{1}{w^2}$ in polar form.

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Steps	Explanation
Method 1 $\lvert w\rvert = \sqrt{4} = 2$ $\theta = \tan^{-1} - \frac{1}{\sqrt{3}} = -\frac{\pi}{6}$ Since w is in the fourth quadrant, $\arg(w) = -\frac{\pi}{6}$. $w = 2 \text{ cis}\left(-\frac{\pi}{6}\right)$	Convert to polar form.
$\frac{1}{w^2} = w^{-2} = 2^{-2} \text{ cis}\left(\frac{2\pi}{6}\right) = \frac{1}{4} \text{ cis}\left(\frac{\pi}{3}\right)$	
Method 2 $\lvert w\rvert = \sqrt{4} = 2$ $\theta = \tan^{-1} - \frac{1}{\sqrt{3}} = -\frac{\pi}{6}$ Since w is in the fourth quadrant, $\arg(w) = -\frac{\pi}{6}$. $w = 2 \text{ cis}\left(-\frac{\pi}{6}\right)$	Convert to polar form.
$w^2 = 2^2 \text{ cis}\left(-\frac{2\pi}{6}\right) = 4 \text{ cis}\left(-\frac{\pi}{3}\right)$	
$1 = 1 \text{ cis } 0$ $\frac{1}{w^2} = \frac{1}{4} \text{ cis}\left(0 - \left(-\frac{\pi}{3}\right)\right) = \frac{1}{4} \text{ cis}\frac{\pi}{3}$	Divide.

Once De Moivre's theorem is proved for $n \in \mathbb{Z}^+$ it can be extended to negative integers. You can see this in **Example 5**.

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Example 5

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a) Show that if $z = r \text{ cis } \theta$, then $z^{-1} = \frac{1}{r} \text{ cis } (-\theta)$.

b) Hence, show that $z^{-m} = r^{-m} \text{ cis } (-m\theta)$ for $m \in \mathbb{Z}^+$.

	Steps	Explanation
a)	$z^{-1} = \frac{1}{z} = \frac{1 \text{ cis } 0}{r \text{ cis } \theta} = \frac{1}{r} \text{ cis } (0 - \theta) = \frac{1}{r} \text{ cis } (-\theta)$	Remember that $1 = 1 \text{ cis }$ (
b)	$z^{-m} = \left(z^{-1}\right)^m = \left(\frac{1}{r} \text{ cis } (-\theta)\right)^m$	Using the result from part a z^{-1}.
	$\left(\frac{1}{r} \text{ cis } (-\theta)\right)^m = r^{-m} \text{ cis } (-m\theta)$	Using De Moivre's theorem positive integers.

It is tempting to use De Moivre's theorem for non-integer exponents, but this needs to be done with care. The first step is to think about what we mean by non-integer powers of complex numbers. You will see this in the next section.

4 section questions

1. Number and algebra / 1.14 Powers and roots of complex numbers

Complex roots

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Properties of complex numbers in polar form

Before looking at how roots of complex numbers are found, you need to investigate an important property of polar notation for complex numbers.

Activity

Plot each of these complex number on the Argand plane and rewrite in Cartesian form.

Comment on your results.

$$z = 3 \text{ cis} \frac{7\pi}{8} \quad z = 3 \text{ cis} \frac{23\pi}{8} \quad z = 3 \text{ cis} \frac{39\pi}{8}$$

Explain why the pattern you found holds for other complex numbers.

Generalise your findings to $z = r \text{ cis } \theta$.

Important

Due to the periodic nature of sine and cosine functions, if $r_1 \operatorname{cis} \theta_1 = r_2 \operatorname{cis} \theta_2$, then

- $r_1 = r_2$
- $\theta_1 = \theta_2 + 2k\pi$ for some integer k.

Example 1

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Determine which of the following is equivalent to $z = 2 \text{ cis} \frac{-\pi}{3}$.

$$2 \text{ cis} \frac{5\pi}{3} \qquad 2 \text{ cis} \frac{4\pi}{3} \qquad 2 \text{ cis} \frac{-4\pi}{3} \qquad 2 \text{ cis} \frac{-5\pi}{3}$$

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Steps	Explanation
$z = 2\ \text{cis}\dfrac{-\pi}{3} = 2\ \text{cis}\ \left(-\dfrac{\pi}{3} + k2\pi\right)$ where $k = 0,\ 1,\ 2,\ 3,\ \ldots$	Using $z = r\ \text{cis}\ \theta = r\ \text{cis}\ \ (\theta + k2\pi)$.
$z = 2\ \text{cis}\dfrac{-\pi}{3} = 2\ \text{cis}\dfrac{5\pi}{3} = 2\ \text{cis}\dfrac{11\pi}{3} = \ldots$	Using $k = 0$ and $k = 1$.
z is equivalent to $2\ \text{cis}\dfrac{5\pi}{3}$.	You can also draw the choices on the Arg plane to see that only $2\ \text{cis}\dfrac{5\pi}{3}$ is equivale

Finding roots

Important

The nth roots of a number w are the solutions to the equation $z^n = w$.

You also know that for $x^3 = 1$, a solution to this equation is $x = 1$.

However, the fundamental theorem of algebra says that there are n solutions to an equation in the form $z^n = w$.

Therefore, $x^3 = 1$ has two more solutions. There are, in fact, three cube roots of 1.

Example 2 shows you how to find these roots.

Example 2

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Find the three cube roots of 1 in Cartesian form.

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Steps	Explanation
$z = r\,\text{cis}\,\theta$, $1 = \text{cis}\ 0$	Write 1 in polar form. You can look for the cube roo first in polar form.
$(r\,\text{cis}\,\theta)^3 = \text{cis}\ 0$	You need to find the solutions of $z^3 = 1$.
$r^3\,\text{cis}\,3\theta = \text{cis}\ 0$	Using De Moivre's theorem.
$r^3 = 1$, so $r = 1$	If $r_1\,\text{cis}\,\theta_1 = r_2\,\text{cis}\,\theta_2$, then $r_1 = r_2$
$3\theta = 2k\pi$ for some integer k $\theta = \frac{2k\pi}{3}$	If $r_1\,\text{cis}\,\theta_1 = r_2\,\text{cis}\,\theta_2$, then $\theta_1 = \theta_2 + 2k\pi$ for som integer k.
For $k = 0$: $z_1 = \text{cis}\ 0 = 1$ For $k = 1$: $z_2 = \text{cis}\frac{2\pi}{3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ For $k = 2$: $z_3 = \text{cis}\frac{4\pi}{3} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$	There are three roots so you use the first 3 values of You can check each root by raising it to the third po on the calculator.

Be aware

The roots of the number 1 are often referred to as roots of unity.

Example 3

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Find the solution to $z^5 = -1$ in polar form.

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Steps	Explanation
$(r \operatorname{cis} \theta)^5 = \operatorname{cis} \pi$	Write -1 in polar for You can look for the roots in polar form.
$r^5 \operatorname{cis} 5\theta = \operatorname{cis} \pi$	Using De Moivre's theorem.
$r = 1$ and $5\theta = \pi + 2k\pi$ for some integer k $z = \operatorname{cis} \frac{\pi + 2k\pi}{5}$	Using the equality of complex numbers in polar form.
For $k = 0$: $z_1 = \operatorname{cis} \frac{\pi}{5}$ For $k = 1$: $z_2 = \operatorname{cis} \frac{3\pi}{5}$ For $k = 2$: $z_3 = \operatorname{cis} \pi$ For $k = 3$: $z_4 = \operatorname{cis} \frac{7\pi}{5} = \operatorname{cis} \left(-\frac{3\pi}{5}\right)$ For $k = 4$: $z_5 = \operatorname{cis} \frac{9\pi}{5} = \operatorname{cis} \left(-\frac{\pi}{5}\right)$	There are five roots s you use the first 5 va so of k. Don't forget to write arguments as primar arguments (between and $-\pi$).

Exam tip

A question that is written in $z^n = a + b\mathrm{i}$ form should be answered in Cartesian form, and a question that is written in $z^n = r \operatorname{cis} \theta$ form should be answered in polar form, unless you are told otherwise.

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Example 4

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Find the cube roots of 8i.

Steps	Explanation
$(r\,\mathrm{cis}\,\theta)^3 = 8\mathrm{cis}\,\dfrac{\pi}{2}$	Write 8i in polar form ar look for the cube roots i polar form.
$r^3\,\mathrm{cis}\,3\theta = 8\,\mathrm{cis}\,\dfrac{\pi}{2}$	Using De Moivre's theor
$r = 2$ and $3\theta = \dfrac{\pi}{2} + 2k\pi$ for some integer k $z = 2\,\mathrm{cis}\,\dfrac{\pi + 4k\pi}{6}$	Using the equality of complex numbers in pol form.
For $k = 0$: $z_1 = 2\,\mathrm{cis}\dfrac{\pi}{6} = 2\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\mathrm{i}\right) = \sqrt{3} + \mathrm{i}.$ For $k = 1$: $z_2 = 2\,\mathrm{cis}\dfrac{5\pi}{6} = 2\left(-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = -\sqrt{3} + \mathrm{i}\,.$ For $k = 2$: $z_3 = 2\,\mathrm{cis}\dfrac{3\pi}{2} = 2\,(0 - \mathrm{i}) = -2\mathrm{i}.$	

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/de-moivres-theorem-id-26995/print/) Assign

Activity

Plot the roots in **Examples 2—4** on an Argand diagram. Describe any pattern that you notice.

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Use the applet below to see nth roots for more complex numbers. The complex number is represented by the red dot which you can move to generate new complex numbers. You can also change the slider for n. The roots are represented by the purple dots and shown in Cartesian form at the bottom of the screen.

How do your findings from the applet compare with those from **Examples 2—4?**

Explain why this makes sense in the context of the techniques used for finding the roots.

How can these results be used to check your working for questions where you are asked to find roots of complex numbers?

Interactive 1. Finding Roots of Complex Numbers.

More information for interactive 1

This interactive allows users to understand how to find the n^{th} roots for complex numbers. A graph is displayed with xy axes, the x-axis ranges from -4 to 4 and the y-axis ranges from -4 to 3. The complex number is represented by the red dot which users can move to generate new complex numbers. Users can change the value of n using a horizontal slider on the top left corner, ranging from 2 to 6. The roots are represented by the purple dots and are shown in Cartesian form as the solution. The value of n determines the degree of the root being

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calculated; whether it's a square root, cube root, fourth root, etc. As n increases, the number of evenly spaced purple dots increases on the complex plane.

At the bottom of the applet, each n root is shown in Cartesian form: $z_k = a + bi$. For each root $k = 01, 2, \ldots (n-1)$. This connects the visual geometry to algebraic expressions students can compute manually.

If the users take the complex number as $2.1 + 0.6i$. Then, $z^2 = 2.1 + 0.6i$ and the solutions are $1.4 + 0.2i$, and $-1.4 - 0.2i$. In the above case, $n = 2$, therefore there will be 2 roots.

Similarly, users can choose different complex numbers with different values of n to get a better understanding of the concept.

The regular spacing of roots is a direct consequence of Euler's Formula and cis notation for representing complex numbers. The fundamental theorem of algebra tells us there are n distinct complex n^{th} roots of a nonzero complex number. De Moivre's Theorem formalizes the method for finding all roots via powers and angles. The roots naturally arrange themselves in a circle of radius $\sqrt[n]{r}$ evenly spaced by angle $\frac{2\pi}{n}$. This is what the applet animates and displays with purple points.

Hence, this interactive applet gives users a better visualization of the connections between the geometric and the algebraic forms of complex numbers.

✓ Important

If the n roots that are the solutions to $z^n = r \text{ cis } \theta$, are plotted on an Argand diagram, they lie on a circle with radius $\sqrt[n]{r}$ and form the vertices of an n sided regular polygon. The solutions are

$$\sqrt[n]{r}\,\text{cis}\,\frac{\theta}{n}, \sqrt[n]{r}\,\text{cis}\,\frac{\theta+2\pi}{n}, \ldots \sqrt[n]{r}\,\text{cis}\,\frac{\theta+2(n-1)\pi}{n}.$$

You can think of these numbers as $\sqrt[n]{r \text{ cis } \theta} = (r \text{ cis } \theta)^{\frac{1}{n}}$.

Based on the form of the first solution, you can also think of this as an extension of De Moivre's theorem to rational exponents.

If you want more practice of finding roots of complex numbers, you can use the applet above to generate examples by changing the complex number and n.

Exam tip

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You need to be able to work with trigonometric ratios of special angles without the use of calculator. These ratios are studied in subtopic 3.5 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27741/) .

If you are currently unfamiliar with this topic, you may choose to do the section questions with a calculator but you should come back to them and practise answering them without a calculator after you have studied subtopic 3.5 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27741/) .

6 section questions

1. Number and algebra / 1.14 Powers and roots of complex numbers

Conjugate root theorem

There is a very important result in algebra called the **fundamental theorem of algebra**, and one form of it is the following statement.

Important

A polynomial equation $\mathrm{P}(z)=0,\ z\in\mathbb{C}$ of degree $n\in\mathbb{Z}^{+}$, can be written as n linear factors and hence will produce n solutions , i.e. have n roots , of the equation $\mathrm{P}(z)=0$.

A zero of a polynomial, is the value of z such that $\mathrm{P}(z)=0$.

In topic 2 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-24414/), you will study the real solutions of polynomials such as $\mathrm{P}(x)=ax^2+bx+c$ and $\mathrm{P}(x)=ax^3+bx^2+cx+d$. Here it is important to note that a polynomial may also have complex solutions.

Example 1

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Show that $2-7\mathrm{i}$ is a zero of $\mathrm{P}(x)=x^2-4x+53$.

Steps	Explanation
$\begin{aligned}\mathrm{P}(2-7\mathrm{i}) &= (2-7\mathrm{i})^2-4(2-7\mathrm{i})+53\\ &= 4-28\mathrm{i}+49\mathrm{i}^2-8+28\mathrm{i}+53\\ &= 4-49-8+53\\ &= 0\end{aligned}$	If $2-7i$ is a zero of $\mathrm{P}(x)=x^2-4x+$ then $\mathrm{P}(2-7\mathrm{i})=0$.

Example 2

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a) The two roots of a quadratic equation are $3+2\mathrm{i}$ and 4. Write the quadratic equation in $x^2+bx+c=0$ form and comment on the nature of the coefficients.

b) The two roots of another quadratic equation are $1+\mathrm{i}$ and $1-\mathrm{i}$. Write the quadratic equation in $x^2+bx+c=0$ form and comment on the nature of the coefficients.

	Steps	Explanat
a)	$\begin{aligned}x^2+bx+c &= (x-(3+2\mathrm{i}))(x-4)\\ &= x^2-4x-(3+2\mathrm{i})x+12+8\mathrm{i}\\ 0 &= x^2+(-7-2\mathrm{i})x+12+8\mathrm{i}\end{aligned}$ Some of the coefficients are real and some are complex.	A polynomial can as n linear factors Note that if the qu asked you to write quadratic in $ax^2+bx+c=($ would have to wri factorised form as $a(x-(3+2\mathrm{i}))($ ax^2+bx+

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	Steps	Explanat
b)	$x^2+bx+c=(x-(1+\mathrm{i}))(x-(1-\mathrm{i}))$ $=x^2-(1-\mathrm{i})x-(1+\mathrm{i})x+(1-\mathrm{i})(1+\mathrm{i})$ $=x^2-2x+2$ $0=x^2-2x+2$ All of the coefficients are real.	A polynomial can as n linear factors

In **Example 2** you can see that complex conjugate root pairs produce quadratics with real coefficients. Can this result be expanded to other polynomials? What is the significance of the fact that the coefficients are real?

✓ **Important**

Conjugate root theorem:

The complex roots of a polynomial equation with real coefficients occur in conjugate pairs.

Example 3

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Given that $z=-3\mathrm{i}$ is a root to the equation $z^3-2z^2+9z-18=0$, find the other roots.

Steps	Explanation
If $z_1=-3\mathrm{i}$, then $z_2=3\mathrm{i}$.	Coefficients of the polynomial are real Complex roots will occur in conjugate pairs.

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Steps	Explanation
$z^3-2z^2+9z-18=(z-(-3\mathrm{i}))(z-3\mathrm{i})(az+b)$ $=(z^2-9\mathrm{i}^2)(az+b)$ $=(z^2+9)(az+b)=az^3+bz^2+9az+9b$	Write in factorised form. Use $az+b$ fo the unknown facto the coefficient for z 1 you can also use $z+b$ as the unkno factor. Always multiply the conjugate pair tern together to simplify your work. If you w to see the benefits this observation, yo can try to multiply $(z-3i)(az+b)$ f and then multiply b $(z+3i)$.
$z^3=az^3 \ \therefore a=1$ $-2z^2=bz^2 \ \therefore b=-2$ $9z=9az \ \therefore a=1$ $-18=9b \ \therefore b=-2$	Equate the coeffici You only need the f two to find a and b you can use all four check that you are getting consistent values for a and b
$0=z-2$ $\therefore z_3=2$	The last factor (whi also a zero) is $z-2$
The other roots are $z_2=3\mathrm{i}$ and $z_3=2$.	

You can use your calculator to find real and complex solutions for polynomial equations.

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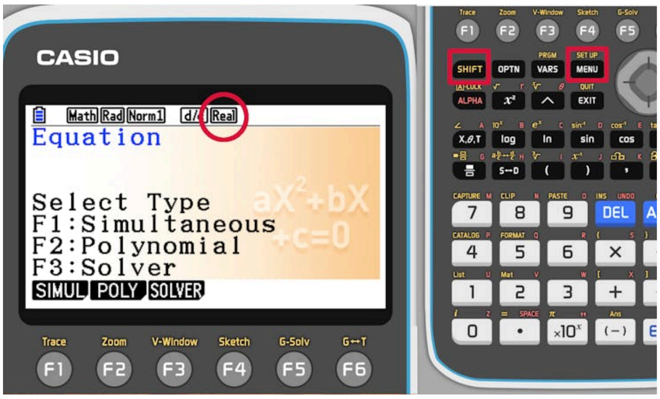
Guidance	Screenshots
To find the solutions of $$z^3 - 2z^2 + 9z - 18 = 0,$$ open the equation solver.	
If the sign on the screen indicates, that the calculator is working with real numbers, you will only see the real solutions. To see all the complex solutions, You need to change the mode, so open set up.	

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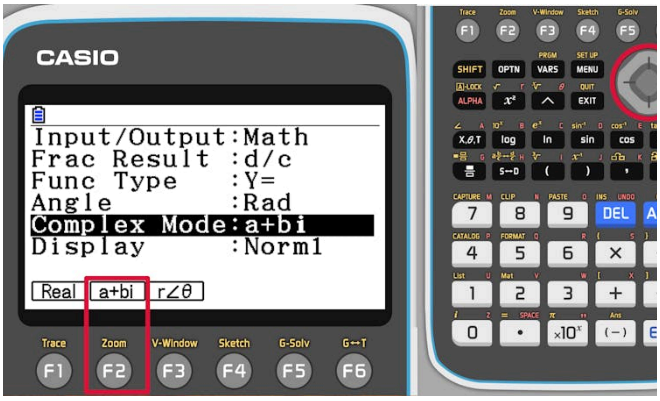
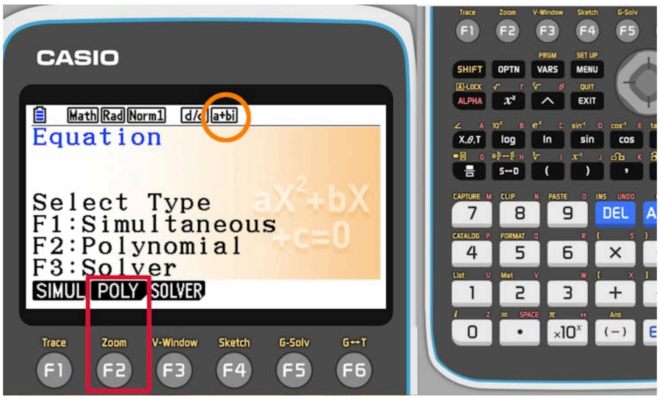
Guidance	Screenshots
Navigate to the line to change the complex mode and press F2 to select the Cartesian form of complex numbers. You can of course press F3 to select the polar mode if you prefer that form. once done, press EXIT to go back to the previous scree,	
Double check that the cakculator is in the correct complex mode, and press F2 to select the polynomial root finder application.	

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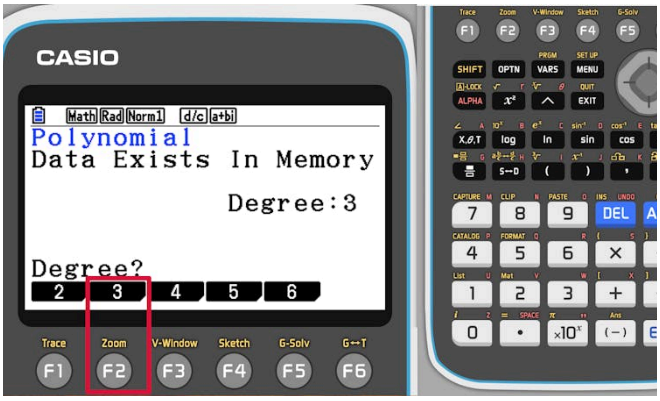
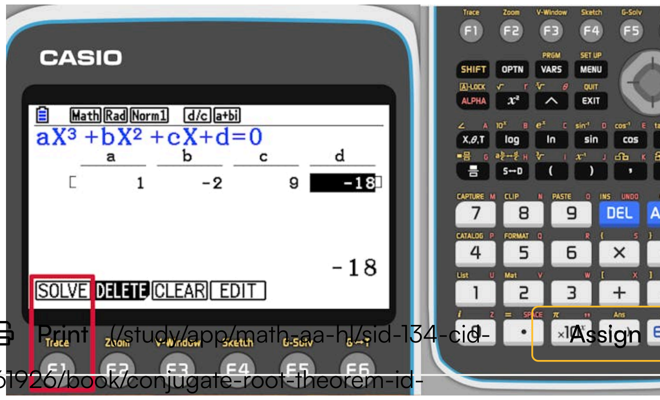
Guidance	Screenshots
Choose the degree of the polynomial (in this case press F2 to set degree 3) ...	
... and on the next screen enter the coefficients. Once done, press F1.	

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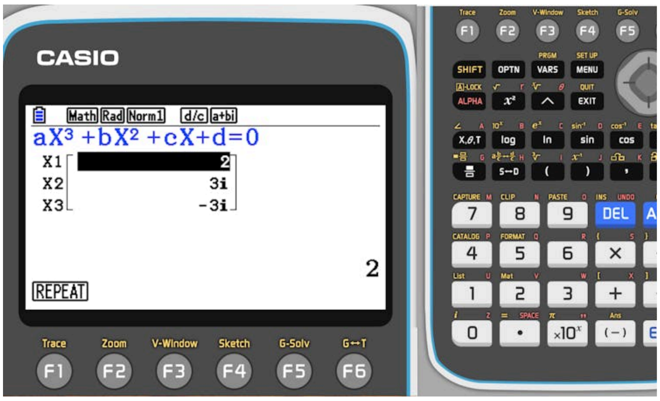
Guidance	Screenshots
The three solutions (one real and two nonreal) are displayed.	

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Guidance	Screenshots
At the time of writing these instructions, the polynomial root finder option is only available in CAS, which is not allowed on exams. Nevertheless, the instructions are given here, because this probably will change in future updates of the operating system. To find the solutions of $$z^3 - 2z^2 + 9z - 18 = 0,$$ enter CAS calculator mode in any application.	

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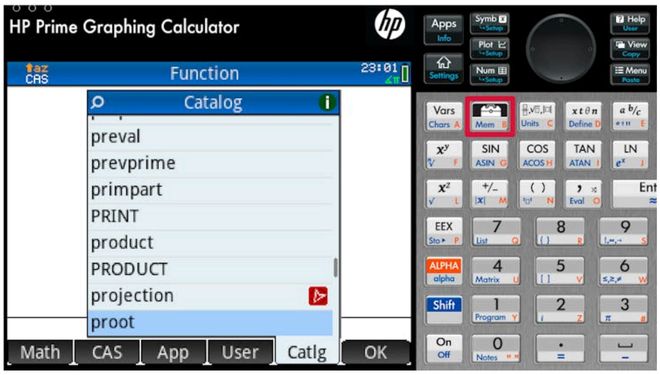
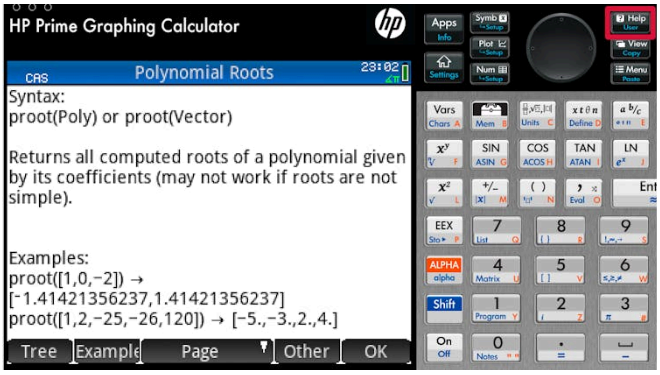
Guidance	Screenshots
Open the toolbox and find the polynomial root finder tool (proot) in the catalog.	
You will need to know how to enter the polynomial. Remember, you can always open a help screen for detailed explanation.	

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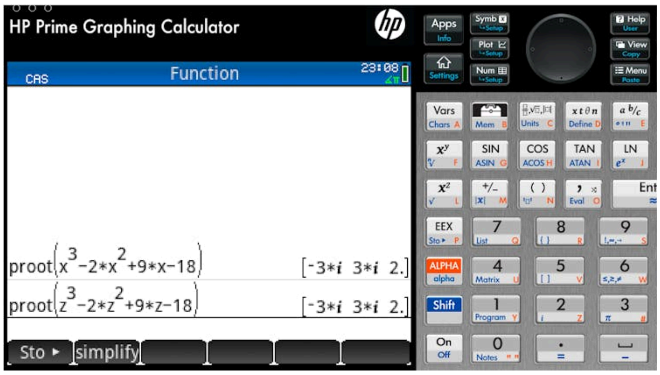
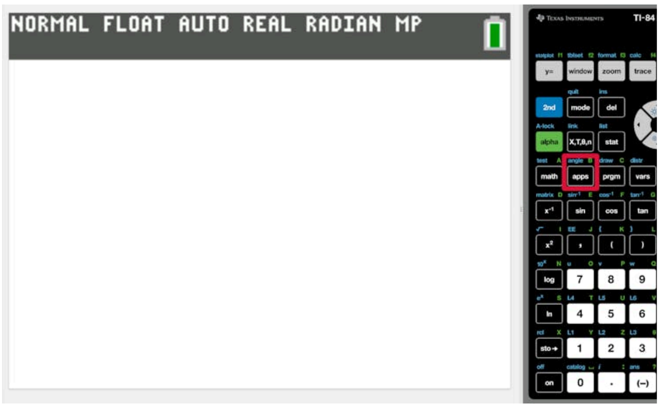
Guidance	Screenshots
The calculator understands any variable name used in the polynomial.	

Guidance	Screenshot
To find the solutions of $$z^3 - 2z^2 + 9z - 18 = 0,$$ open the application list, ...	

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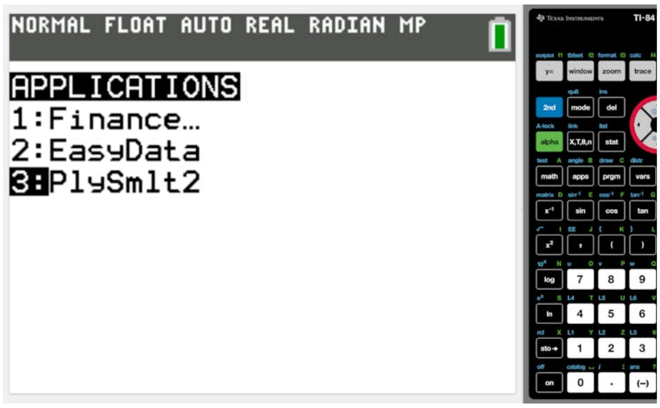
Guidance	Screenshot
... find the solver application (PlySmlt) ...	
... and choose the polynomial root finder option.	

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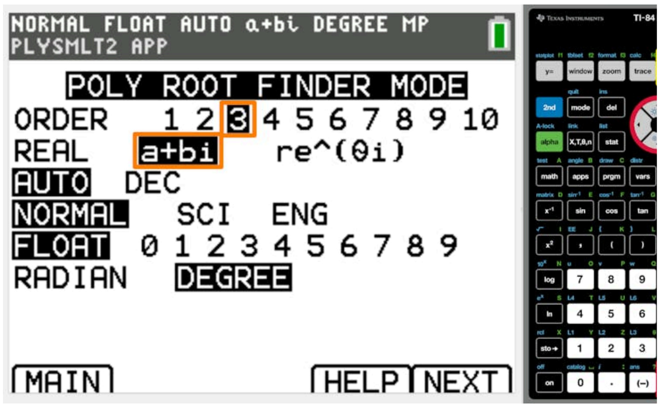
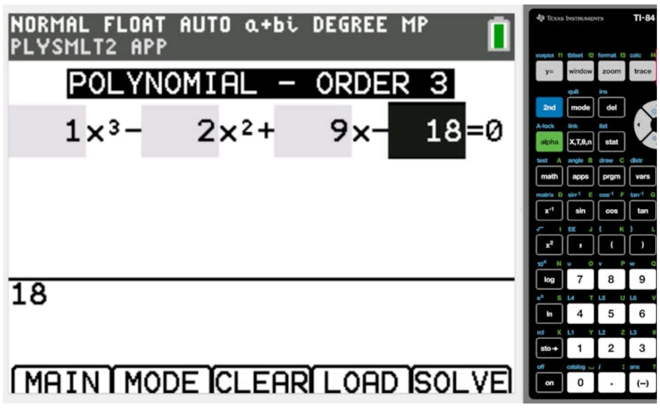
<table>
<tr><th>Guidance</th><th>Screenshot</th></tr>
<tr><td>Set the order of the polynomial and choose the option to find complex roots. You can decide if you want to see the roots in Cartesian or Euler form. You can set these options by using the arrow keys and confirming your choice by pressing enter.

Once done, press the graph button (the button below the option NEXT on the screen) indicating that you want to move to the next step.</td><td>
</td></tr>
<tr><td>Enter the polynomial and press the graph button to solve the equation.</td><td>
</td></tr>
</table>

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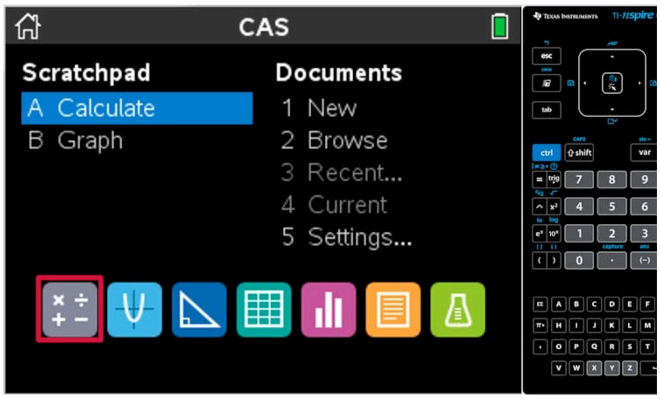
Guidance	Screenshot
The three solutions (one real and two nonreal) are displayed.	

Guidance	Screenshot
To find the solutions of $$z^3 - 2z^2 + 9z - 18 = 0,$$ open a calculator page.	

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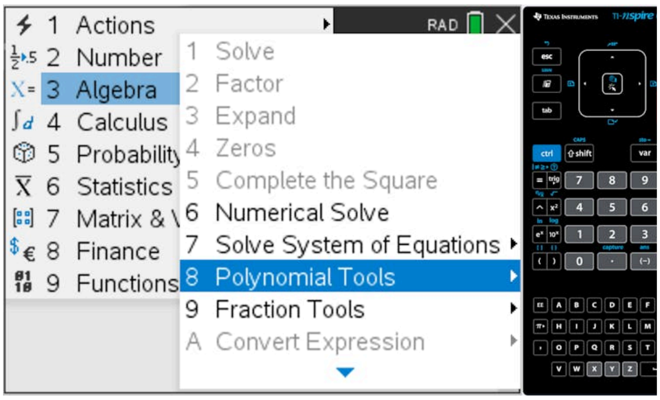
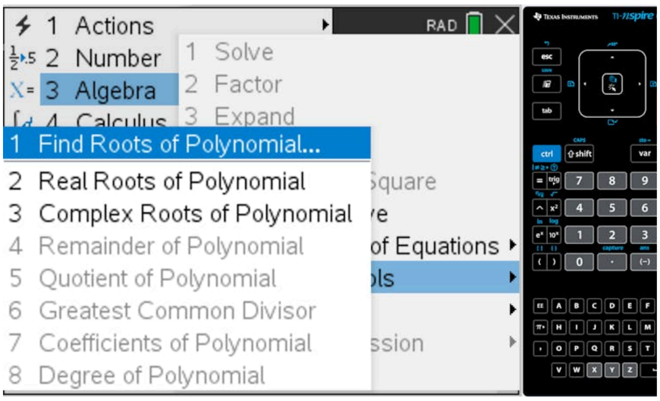
Open the menu, navigate to the polynomial tools ...



... and choose the option to find roots of polynomials.



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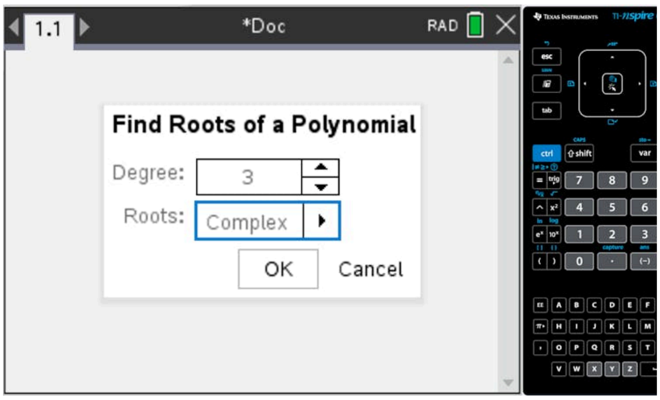
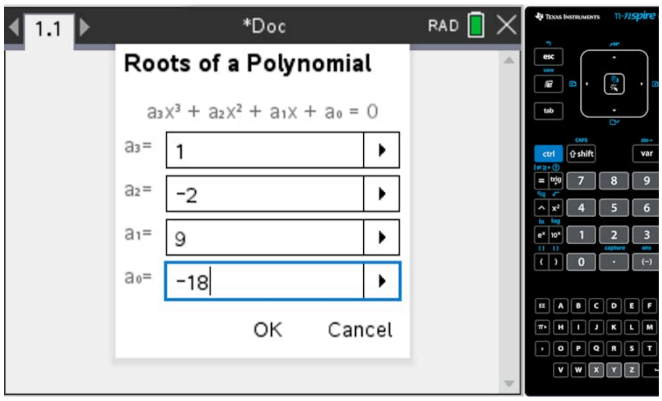
Enter the degree and tell the calculator that you are interested in complex roots.



Enter the coefficients.



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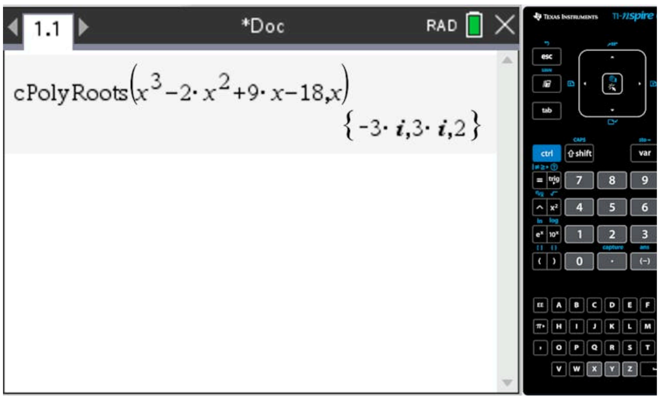
The three solutions (one real and two nonreal) are displayed.

Note, that if you remember the name of the polynomial root finder function (cPolyRoots), then you do not need to go through the menu system. You can type it in directly.



Exam tip

In these types of questions, you will usually be expanding $(z-w)(z-w^*)(az+b)\ldots$. It is useful to remember that the expansion of the terms with the conjugates follows a nice pattern such that,

$$(z-w)(z-w^*)=z^2-2\mathrm{Re}(w)z+|w|^2.$$

You do not have to memorise this pattern, but you should remember that multiplying the conjugate pair terms will simplify your algebraic work.

You will see a similar pattern again when you study the sum and product of the roots of polynomials in [subtopic 2.12.5 (/study/app/math-aa-hl/sid-134-cid-761926/book/sum-and-products-of-the-roots-of-polynomial-id-26608/)](/study/app/math-aa-hl/sid-134-cid-761926/book/sum-and-products-of-the-roots-of-polynomial-id-26608/) .

Example 4

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Given that $1-\mathrm{i}$ is a root of the equation of $z^3+2z^2-6z+k=0$, $k\in\mathbb{R}$, find the value of k.

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Steps	Explanation
If $z_1 = 1 - \mathrm{i}$, then $z_2 = 1 + \mathrm{i}$.	The coefficients, including are real. Complex roots w in conjugate pairs.
$\begin{aligned} z^3 + 2z^2 - 6z + k &= (z - (1 - \mathrm{i}))(z - (1 + \mathrm{i}))(az + b) \\ &= \left(z^2 - 2(1)z + \left(\sqrt{2}\right)^2\right)(az + b) \\ &= \left(z^2 - 2z + 2\right)(az + b) \\ &= az^3 + bz^2 - 2az^2 - 2bz + 2az + 2b \end{aligned}$	You can use $az + b$ for th unknown factor or note th $a = 1$ since the coefficier z^3 is 1 and use $z + b$ as th unknown factor. Using $(z - w)(z - w^*) =$ $z^2 - 2\mathrm{Re}(w)z + \|w\|^2$.
$z^3 = az^3 \quad \therefore a = 1$ $2z^2 = (b - 2a)z^2 \quad \Leftrightarrow \quad 2 = b - 2(1) \quad \Leftrightarrow$ $b = 4$ $-6z = (-2b + 2a)z \quad \Leftrightarrow \quad -6 = -2b + 2(1) \quad \Leftrightarrow \quad b = 4$ $k = 2b \quad \therefore k = 8$	Equating coefficients.

4 section questions ⌄

1. Number and algebra / 1.14 Powers and roots of complex numbers

Checklist

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What you should know

By the end of this subtopic you should be able to:

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- use De Moivre's theorem to find powers of complex numbers
- recall that a complex number in modulus—argument (polar) form can be represented by $z = r \text{ cis } (\theta + k2\pi)$, where $k = 0,\ 1,\ 2,\ \ldots$
- recall that $z^n = w$ has n solutions
- find complex roots
- describe the geometric patterns created by plotting nth roots on the Argand plane
- use the conjugate root theorem to find roots of polynomials with real coefficients.

1. Number and algebra / 1.14 Powers and roots of complex numbers

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-26999/print/) Assign

Part 1

Let $z = a + b\text{i}$.

Find

$\left(z^2\right)^*$ and $\left(z^*\right)^2$

$\left(z^3\right)^*$ and $\left(z^*\right)^3$

$\left(z^4\right)^*$ and $\left(z^*\right)^4$

Generalise your findings for $\left(z^n\right)^*$ and $\left(z^*\right)^n$. Show that your findings are true.

Part 2

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Solve $z^3 = 1$. Plot the roots on an Argand diagram and draw line segments connecting one of the roots to each of the remaining ones. Calculate the length of each line segment and multiply these lengths. (Consider using graphing software such as GeoGebra for this activity.)

Extend your work to other powers of z.

Describe the patterns that you notice and generalise your findings to $z^n = 1$.

Part 3

Let $z = r \text{ cis } \theta$. How does the Argand diagram of z compare to $z^2,\ z^3,\ \ldots$. Describe the geometric relationship between z and z^n