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SUM OF INFINITE GEOMETRIC SEQUENCES

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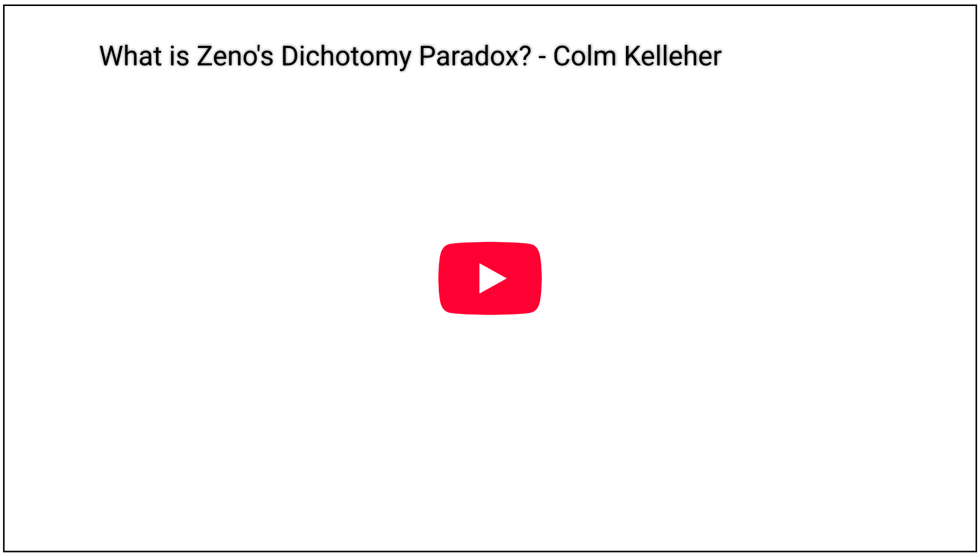
The big picture

In subtopic 1.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-25994/) you learned about geometric sequences where each term is found by multiplying the previous term, u_{n-1} , by r (the common ratio) and the general term is given by $u_n = u_1 r^{n-1}$, where n is the number of terms.

To find the sum of an infinite geometric sequence you need to add up an infinite number of terms.

You might think that the result would be infinitely large, but is it possible that the sum of an infinite number of terms actually turns out to be finite?

For a long time this question intrigued mathematicians and philosophers, including Zeno, the ancient Greek philosopher who formulated the Dichotomy Paradox. Watch the video below to see what this paradox is about.



Concept

In this subtopic you will learn that there are situations in which adding up an infinite number of values produces a finite value. Consider whether the result of adding up infinitely many values

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can be an exact sum or only an approximation of the true sum.

Theory of Knowledge

When you began investigating the sum of infinite series, you may have asked yourself, 'How can an infinite series have a sum?' Good question.

This gets into language as a way of knowing and requires us to expand our concept of 'sum'. In an infinite series, the numbers are indeed headed towards an endpoint, which we call the sum; but, as you know, given the fact the series is stated as 'infinite', we will never reach that endpoint. Thus, can convergent geometric sequences be said to have a sum?

Knowledge Question: To what extent does language limit knowledge production in mathematics?

Infinite geometric sequences

From subtopic 1.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-25994/), you are already familiar with geometric sequences, such as,

$$2,\ 4,\ 8,\ 16,\ \ldots,\ 512,\ 1024,$$

and their sums,

$$2+4+8+16+\cdots+512+1024.$$

Making connections

The n th term of a geometric sequence with common ratio r is:

$$u_n = u_1 r^{n-1}.$$

The sum of the first n terms of a geometric sequence is:

$$S_n = \frac{u_1\left(r^n-1\right)}{r-1} = \frac{u_1\left(1-r^n\right)}{1-r},\ r \neq 1.$$

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Now think about what could happen if the geometric sequence is infinite and you want to find its sum. Do you think that this is possible or impossible? Are there special circumstances in which the sum can be found?

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Activity

Use the applet below to explore what happens to the sum of a geometric sequence as the number of terms, n , is increased.

How does the r value affect the sum as n gets larger? Does your conjecture hold true for all values of r ?

Do you think that your observations extend to values of n that are larger than the ones given in the applet? What about for r values that are larger or smaller than the ones available in the applet?

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27231/print/) Assign

Interactive 1. Explore What Happens to the Sum of a Geometric Sequence as the Number of Terms, n Is Increased.

More information for interactive 1

The interactive graph allows users to explore the behavior of the geometric sequence defined by $u_n = 243(r)^n$. The interface is divided into two sections: the left side displays a dynamic graph, while the right side includes two sliders that let users adjust the common ratio (r) and the number of terms (n). The slider for the common ratio r ranges from -1.5 to 1.5. The range of the n slider, however, is dynamic—it adjusts based on the selected value of r to ensure that the plotted values remain within a meaningful and visible range on the graph.

As users manipulate these sliders, the graph updates in real time to display both the nth term (in pink) and the sum of the first n terms (in blue). This visualization helps users understand how the geometric sequence grows or decays with different values of r, and how both the sum and individual terms behave as n increases. The applet promotes pattern recognition, conjecture-making, and a deeper conceptual understanding of geometric sequences. It emphasizes the impact of the common ratio on sequence behavior and supports visual learning through interactive exploration.

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You can see that when $-1 < r < 1$ the terms of a geometric sequence get smaller as n increases. In this case, smaller and smaller values are added to the running total and the sum can be found for an infinite number of terms.

For example, consider:

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} + \frac{1}{2187} + \frac{1}{6561} + \ldots$$

Even though only the first 9 terms of this infinite sum are shown, you can see that, already, $\frac{1}{6561} = 0.000152 \approx 0$.

Any further terms will not change the sum significantly.

Why does this not happen for a geometric sequence that has $r \geq 1$ or $r \leq -1$?

Be aware

An infinite geometric series, such as

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} + \frac{1}{2187} + \frac{1}{6561} + \ldots$$

can be written in sigma notation.

For example:

$$\sum_{i=1}^{\infty} \left(1 \times \left(\frac{1}{3} \right)^{i-1} \right) = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} + \frac{1}{2187} + \frac{1}{6561} + \ldots$$

To find the value of the sum of an infinite geometric sequence with $-1 < r < 1$, first suppose that $0 < r < 1$ and look at the formula for the sum of the first n terms:

$$S_n = \frac{u_1 (1 - r^n)}{1 - r}$$

For any $0 < r < 1$, the power r^n gets smaller and smaller as n gets larger, so $1 - r^n$ gets closer and closer to 1. This means that the sum approaches $\frac{u_1}{1 - r}$.

Similarly, you can show using the formula $S_n = \frac{u_1 (r^n - 1)}{r - 1}$ that for $-1 < r < 0$, the sum also approaches $\frac{u_1}{1 - r}$ as n increases.

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What about the case when $= 0$?

The conditions $0 < r < 1$, $-1 < r < 0$ and $r = 0$ can be written together in the compact form $|r| < 1$ using modulus notation, where $|r|$ means the modulus, or absolute value, of r .

Important

The sum of an infinite geometric sequence, known as the sum to infinity or the infinite sum, is given by:

$$S_\infty = \frac{u_1}{1-r}, \quad |r| < 1$$

An infinite geometric sequence is convergent if $|r| < 1$, i.e. if $-1 < r < 1$.

If a sequence does not converge, it is divergent.

Example 1

★☆☆

Evaluate $4 - \frac{4}{5} + \frac{4}{25} - \frac{4}{125} + \ldots$

Steps	Explanation
This is an infinite geometric series. $u_1 = 4,\ r = -\frac{1}{5}$ $S_\infty = \frac{4}{1-\left(-\frac{1}{5}\right)} = \frac{4}{\left(\frac{6}{5}\right)} = \frac{10}{3}$	For this series $\lvert r\rvert < 1$, so the sum to infinity can be found by using $S_\infty = \frac{u_1}{1-r}$
Therefore $4 - \frac{4}{5} + \frac{4}{25} - \frac{4}{125} + \ldots = \frac{10}{3}$	

International Mindedness

The questions considered by mathematicians are often too difficult to be solved by one person. The question of whether the infinite sum $1 - 1 + 1 - 1 + 1 - 1 + 1 \cdots$, known as Grandi's series, is convergent or divergent illustrates how mathematicians from all over the world work

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with each other's results. You can do some research to find out what the various mathematicians thought about this infinite sum.

Example 2

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For a geometric sequence, the sum of the first three terms is $\frac{91}{9}$ and the infinite sum of all the terms is $\frac{21}{2}$. Find the common ratio and the first term.

Steps	Explanation
$S_3 = \frac{91}{9} = \frac{u_1\left(1-r^3\right)}{1-r}$ $S_\infty = \frac{21}{2} = \frac{u_1}{1-r}$	Write an equation for each piece of the given information. It will be helpful in the next step if you use the version of S_n where the denominator matches the denominator in the S_∞ formula.
$\begin{cases} \frac{91}{9} = \frac{u_1\left(1-r^3\right)}{1-r} \\ \frac{21}{2} = \frac{u_1}{1-r} \end{cases} \Leftrightarrow \frac{\frac{91}{9}}{\frac{21}{2}} = \frac{\frac{u_1\left(1-r^3\right)}{1-r}}{\frac{u_1}{1-r}}$ $\Leftrightarrow \frac{91}{9} \times \frac{2}{21} = \frac{u_1\left(1-r^3\right)}{1-r} \times \frac{1-r}{u_1}$ $\Leftrightarrow \frac{26}{27} = 1-r^3 \Leftrightarrow r^3 = 1-\frac{26}{27} \Leftrightarrow r = \sqrt[3]{\frac{1}{27}} = \frac{1}{3}$	Solve this system of two equations. The easiest way to solve a system of equations for a geometric sequence is to divide one equation by the other, rather than using substitution or elimination.
$\frac{21}{2} = \frac{u_1}{1-\frac{1}{3}} \Leftrightarrow u_1 = \frac{21}{2} \times \frac{2}{3} = 7$ So the common ratio is $\frac{1}{3}$ and the first term is 7.	Substitute the r value you found into one of the equations, and solve for u_1.

Exam tip

Your IB formula booklet contains formulae for the sums of both a finite geometric sequence and an infinite geometric sequence. Read the questions carefully and apply the correct formula for the information given.

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You can practise more questions, similar to **Example 2**, by using the applet below.

Example 3

★★★

$S_\infty = u_1 + u_1 r + u_1 r^2 + \ldots$

a) Write an expression for $r \times S_\infty$ in terms of u_1 and r .

b) Hence, show that $S_\infty = \dfrac{u_1}{1-r}$.

	Steps	Explanation
a)	$\begin{aligned} r \times S_\infty &= u_1 \times r + u_1 r \times r + u_1 r^2 \times r + \ldots \\ &= u_1 r + u_1 r^2 + u_1 r^3 + \ldots \end{aligned}$	Multiply all terms in the sum by r .
b)	$\begin{aligned} S_\infty &= u_1 + u_1 r + u_1 r^2 + \ldots \\ S_\infty - r \times S_\infty &= u_1 + \quad u_1 r + u_1 r^2 + u_1 r^3 + \ldots \\ &\quad - \left(u_1 r + u_1 r^2 + u_1 r^3 + \ldots\right) \\ &= u_1 \end{aligned}$	All the terms on the RHS of $r \times S_\infty$ also appear in S_∞ , so if you subtract $r \times S_\infty$ from S_∞ then all terms will cancel out except u_1.
	$S_\infty (1-r) = u_1 \quad \Leftrightarrow \quad S_\infty = \dfrac{u_1}{1-r}$	Rearrange to make S_∞ the subject.

4 section questions ⌄

Student view

1. Number and algebra / 1.11 Sum of infinite geometric sequences

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Applications

In this section, you will see examples of applications of the sum of an infinite geometric sequence.

Activity

Squares are drawn following the pattern shown below.

Diagram 1 below shows a square of area 1 unit divided into four equal parts.

The shaded area is $\frac{1}{4}$ units.

In diagram 2, the original square is divided into 16 equal parts, and the total shaded area is $\frac{5}{16}$ units.

This pattern continues as shown in Diagrams 3 and 4.

If this process is continued indefinitely, find the total area for an infinite number of shaded boxes.

Discuss what it means to have an infinite number of shaded boxes in this context.

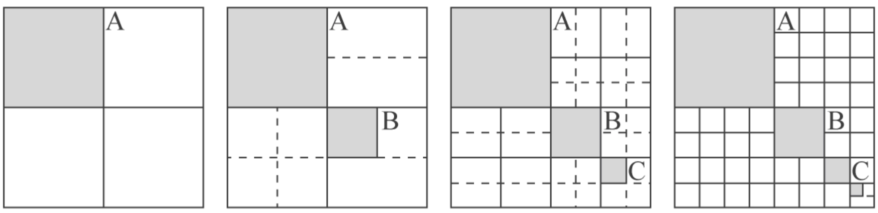

Diagram 1 Diagram 2 Diagram 3 Diagram 4

More information

The image contains four diagrams illustrating the subdivision of a square into progressively smaller sections.

Diagram 1 shows a square divided into four equal smaller squares, labeled as A in the top left. Diagram 2 further divides the right bottom square of Diagram 1 into four smaller squares, with label B appearing on one of them. Diagram 3 further subdivides the right bottom square of Diagram 2, again into four smaller squares, with label C appearing on one of these fourths. Diagram 4 continues this pattern, breaking down the last set's bottom right square into even smaller squares, effectively showing the fractal nature and iterative process of subdivision. This process is used to demonstrate mathematical concepts in topology, showcasing a visual example of an infinite geometric sequence.

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There are a variety of shapes that can be subdivided to produce an infinite number of smaller and smaller regions. The image that is generated in this process is an example of a fractal that is studied by mathematicians in the field of topology. The total of the areas of an infinite number of regions generated in this way is an example of the application of the sum of an infinite geometric sequence.

Example 1

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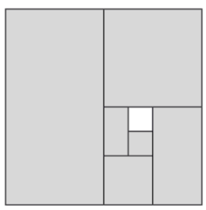
★★☆

a) Draw a square with an area of 1 unit^2 . Divide the square into two equal parts and shade one of them. Then take the unshaded part, divide that into two equal parts and shade one of them. Continue this process until you have six shaded areas.

b) Calculate the total area that is shaded in your drawing.

c) Suppose that the dividing process is continued indefinitely, creating an infinite number of shaded regions. Find the total shaded area.

d) Explain what the total shaded area you found in Part **c** represents in this context.

	Steps	Explanation
a)		Your picture should look similar to this one if you divided the square into two equal rectangles.

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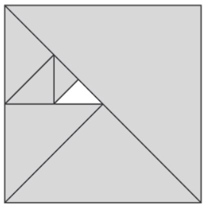
	Steps	Explanation
		You can also divide the square into equal triangles.
b)	$\text{area} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{64}$ $S_6 = \frac{\frac{1}{2} \times \left(1 - \left(\frac{1}{2}\right)^6\right)}{1 - \frac{1}{2}} = 0.984$ $\text{area} = 0.984 \text{ units}^2 \text{ (3 significant figures)}$	Recognise the total shaded area as the sum of 6 terms of a geometric sequence with $r = \frac{1}{2}$.
c)	$S_\infty = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$	Recognise the total shaded area as the sum of an infinite geometric sequence. $-1 < r < 1$ so the sequence converges and you can use $S_\infty = \frac{u_1}{1-r}$.
d)	The total shaded area is 1 unit^2, which is equal to the area of the original square. This is because as you keep subdividing and shading, you shade in more and more of the original square. Doing this an infinite number of times would eventually shade in the whole square.	

Example 2 shows another interesting application. Here you use the infinite sum to rationalise a repeating decimal.

Example 2

★★☆

Write $0.7777\ldots$ as a rational number.

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Steps	Explanation
$0.7777\ldots = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots$ The RHS is an infinite geometric sum with $r = \frac{1}{10}$	A repeating decimal can be written as an infinite sum. Since $\lvert r \rvert < 1$ you can use $S_\infty = \frac{u_1}{1-r}$.
$S_\infty = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{10} \times \frac{10}{9} = \frac{7}{9}$	
So $0.7777\ldots = \frac{7}{9}$	Do not forget to answer the original question. The sum is just a tool for rewriting the repeating decimal as a rational number.

You can also use infinite sums to model the motion of a bouncing ball as in **Example 3**.

Example 3

★★★

A ball is dropped from a height of 3 m.

On each bounce it rebounds to 70% of the previous height.

a) Calculate the total distance covered by the ball on its way up after the first three times that it hits the ground.

b) Suppose the ball continues to bounce indefinitely. Find the total distance covered by the ball when it is moving up.

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	Steps	Explanation
a)	$$\begin{aligned} S_3 &= \frac{2.1\left(1-0.7^3\right)}{1-0.7} \\ &= 4.60\ (3\text{ significant figures}) \end{aligned}$$	Recognise that the (upward) rebound distances form a geometric sequence. After 1st bounce: $$\text{distance} = 3 \times 0.7 = 2.1$$ After 2nd bounce: $$\text{distance} = 2.1 \times 0.7 = 1.47 = 3 \times 0.7^2$$ After 3rd bounce: $\text{distance} = 1.47 \times 0.7 = 1.029 = 3 \times 0.7^3$
	The distance covered is 4.60 m .	
b)	$$S_\infty = \frac{2.1}{1-0.7} = 7$$	You can model the rebound heights of the ball by an infinite geometric sequence where the terms become negligibly small. $r = 0.7$ so the sequence converges.
	The total distance covered is 7 m .	

3 section questions

1. Number and algebra / 1.11 Sum of infinite geometric sequences

Checklist

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What you should know

By the end of this subtopic you should be able to:

- explain why a geometric sequence converges only if $|r| < 1$
- identify convergent geometric sequences
- use $S_\infty = \dfrac{u_1}{1-r}$ to find the sum to infinity of a convergent geometric sequence
- i dentify application questions in which you need to find a sum of an infinite geometric sequence

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- solve application questions involving infinite sums of geometric sequences.

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1. Number and algebra / 1.11 Sum of infinite geometric sequences

Investigation

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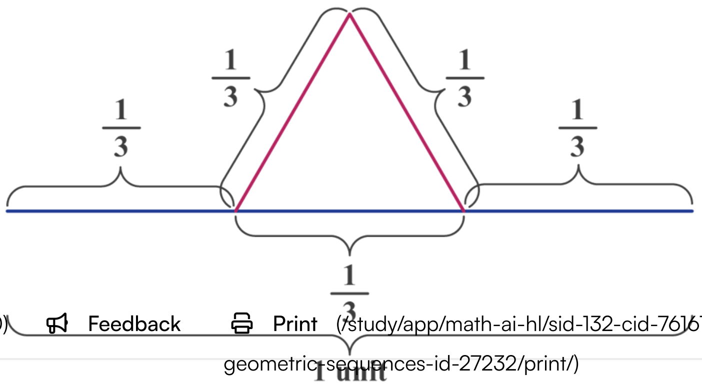
A Koch snowflake is drawn by starting with an equilateral triangle with side length 1 .

Each side is then divided into three equal segments, and an equilateral triangle with side length $\frac{1}{3}$ is drawn so that its base coincides with the middle segment and the opposite vertex lies outside the original triangle, see first diagram below. Then the middle segment of each side of the original triangle is erased to create a 12-sided polygon as seen in the second diagram below. This process of dividing each edge of a polygon into three equal segments and building a triangle on the middle segment can be continued indefinitely to generate more elaborate snowflakes, as you can see in the applet below, which shows the first few iterations.



Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/infinite-geometric-sequences-id-27232/print/) Assign

More information

The image shows a base line labeled as "1 unit" that is divided into three equal parts, each labeled as "1/3". An equilateral triangle with side length "1/3" is constructed such that its base coincides with the middle segment of the original line. The triangle's apex points outwards from the original line, and the segments are labeled appropriately to show the division. This setup demonstrates the initial stage of the fractal creation process used to develop a polygon into a snowflake shape.

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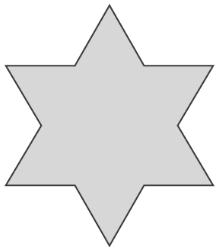

Interactive 1. Building Koch Snowflake from an Equilateral Triangle.

More information for interactive 1

This interactive applet visually demonstrates the iterative construction of the Koch snowflake, a classic example of a fractal. Users can adjust a slider to change the iteration level n, ranging from 0 to 5, and observe how the figure evolves step by step. At $n = 0$, the shape begins as a simple equilateral triangle, outlined in blue with purple vertices. With each successive iteration, every line segment is divided into three equal parts, and the middle segment is replaced with two sides of a smaller equilateral triangle. This process

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creates a progressively more intricate, self-similar pattern. The blue boundary illustrates the changing perimeter of the snowflake, while the purple points highlight key vertices at each stage. This applet offers an engaging way to explore fundamental ideas in fractal geometry, such as recursion, self-similarity, and infinite perimeter with finite area.

Explore what happens to the perimeter and the area of the snowflake as the process is continued indefinitely.

What do your results tell you about how much ink would be needed to colour the inside of the snowflake if you were to continue adding triangles an infinite number of times?

What about the amount of ink needed to draw the boundaries of the snowflake? Comment on these results.

Fractals, of which the Koch snowflake is an example, possess a property of self-similarity, which means that the larger part of the fractal is similar to its smaller components. Where are such images seen in the real-world? Research some practical applications, for example the modelling of coastlines and biological cell structures, using fractal curves.

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