Index

The big picture

Functions are fundamental mathematical tools that help you organise information and model the real world. They are used by mathematicians, scientists, and engineers to study phenomena and predict behaviour. In this subtopic you will study the properties of various functions and learn how to build and use function models so that you may obtain precise information about the situations that are being modelled. A solid understanding of functions is essential for the study of the topic of calculus.

Making connections

Gottfried Wilhelm Leibniz was a German mathematician in the 17th century, who in 1673 was the first to use the term 'function' to denote the dependence of one quantity to another. His work on calculus made it necessary to formalise the rules governing how variables change in relation to each other, and this in turn led to developments in mathematics without which modern technologies, such as computing, would not exist.

In this subtopic, you will study six families of functions – listed below – and explore their unique characteristics and behaviour.

- Linear functions
- Quadratic functions

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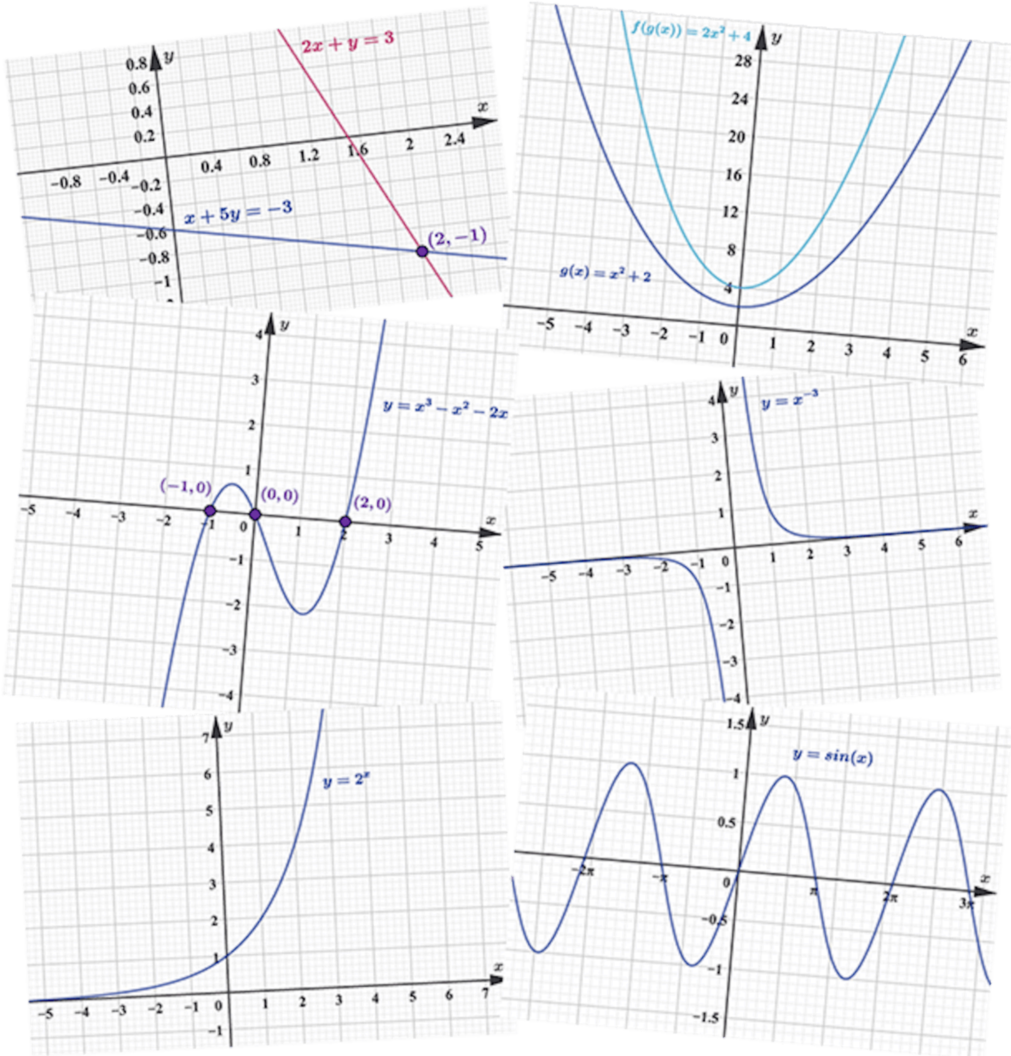
- Cubic functions
- Power functions
- Exponential functions
- Sinusoidal function



More information

The image is a collage of six distinct mathematical graphs, each represented on a Cartesian plane with both X and Y axes.

1. **First Graph (Top Left):**
2. A red line representing the equation "2x + y = 3," intersecting the axes at (1.5, 0) and (0, 3).
3. A blue line representing "x + 5y = -3," intersecting the axes at (-3, 0) and (0, -0.6).
4. Intersecting point at (2, -1).

5. **Second Graph (Top Center):**

6. A blue parabola showing the function "g(x) = x^2 + 2."

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7. The vertex at (0, 2) and opens upwards.
8. Another blue function "f(g(x)) = 2x^2 + 4."
9. **Third Graph (Center Left):**
10. A cubic function "y = x^3 - x" with three key points labeled: (-1, 0), (0, 0), and (2, 0).
11. Shows typical cubic curve behavior with a point of inflection.
12. **Fourth Graph (Top Right):**
13. Graph of the function "y = x^-3" shown with an asymptotic behavior as x approaches zero.
14. **Fifth Graph (Bottom Left):**
15. Exponential growth function "y = 2^x," smoothly increasing as x increases.
16. **Sixth Graph (Bottom Right):**
17. Sine wave of the function "y = sin(x)", showing periodic behavior with peaks and troughs.

Each graph is on a grid background, and numerical values are presented on both axes marking units of measurement.

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Theory of Knowledge

You can test the validity of a quadratic function by graphing it and seeing if it ‘works’. This provides one key element of knowledge — falsifiability — and brings up an important knowledge question with ramifications both in mathematics and other areas of knowledge.

Knowledge Question: Must knowledge be falsifiable in order to be considered valid?

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Concept

Functions come in all different shapes, and there is a great value in being able to recognise a function **pattern.** While exploring the unique characteristics of families of functions, reflect on the different forms that each family of functions represents. Think about what information different **forms** reveal about the properties of functions and how they can be used to **model** real-life phenomena.

2. Functions / 2.5 Introduction to modelling

Linear functions

Constant rate of change

A linear function describes the relation between two variables where there is a constant rate of change, and, as the first four letters of the name suggest, the graph of a linear function looks like a straight line. For example, it has been observed that under specific meteorological conditions, as dry air moves upward, it expands and cools at a constant rate. The figure below shows the graph of the air temperature T (in $^\circ\mathrm{C}$) as a function of the height (in metres) above the ground.

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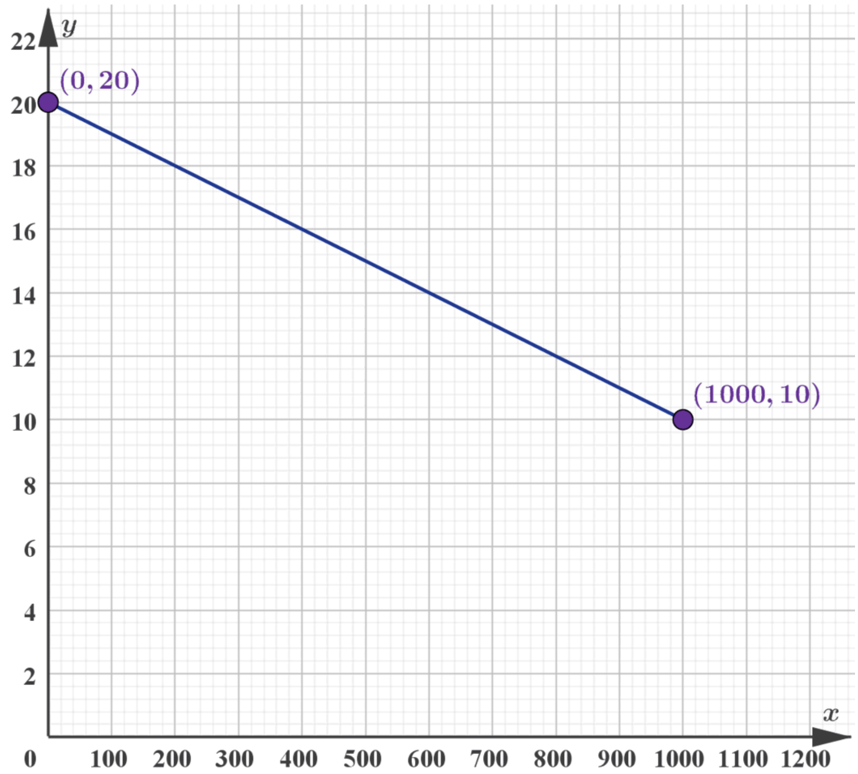

More information

The image is a graph depicting a linear function. The X-axis represents the height above ground in meters, ranging from 0 to 1200. The Y-axis represents the temperature in degrees Celsius, ranging from 8 to 22. There is a straight line connecting two data points: (0, 20.5) and (1000, 15), indicating a constant rate of change in temperature as height increases. The graph shows that as the height increases, the temperature decreases linearly.

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- Identify the independent and dependent variable in this context?
- How does the temperature change as the air moves upwards? By how much does the temperature change for every 100 metres?
- What is the temperature of the air at ground level?

Important

The graph of a **linear function** is a straight line and has the general form $f(x) = mx + c$ where m is the gradient and c the y -intercept.

In a **linear model** m and c are constants.

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The linear formula that expresses the air temperature T (in $°\text{C}$) as a function of the height x (in metres) above the ground is:

$$T(x) = -0.01x + 20$$

- What does the gradient represent in this context?
- What does the y -intercept represent?

Important

The **rate of change** of a linear function is the increase/decrease in the dependent variable $f(x)$ (or y) for every unit by which the independent variable x increases.

- An **increasing** linear function has a positive rate of change.
- A **decreasing** linear function has a negative rate of change.

Notice that the rate of change of a linear model is represented by the gradient of the corresponding linear function.

Making connections

Recall from subtopic 2.1 (/study/app/m/sid-122-cid-754029/book/the-big-picture-id-26160/) that a line can be expressed in three forms:

- Gradient—intercept form: $y = mx + c$, where m is the gradient and c the y -intercept.
- Gradient—point form: $y - y_0 = m(x - x_0)$, where m is the slope and (x_0, y_0) is a point on the line.
- Standard form: $Ax + By + C = 0$, where A, B and C are constants.

Example 1

Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/the-big-picture-id-27457/print/) Assign

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Taxi company A charges $\$0.80$ for every km travelled while taxi company B charges $\$0.45$ for every km travelled plus a fixed amount of $\$7$.

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a) Suggest a linear function that models the cost C_A (in \$) as a function of the number of kilometres travelled with company A.

b) Suggest a linear function that models the cost C_B (in \$) as a function of the number of kilometres travelled with company B.

c) Which company would you choose for a ride of 15 km? Explain your answer.

d) Graph both linear functions on the same axes.

e) Find the coordinates of the point of intersection of the graphs and interpret their values.

	Steps	Description
a)	$C_A = 0.8x$	The rate of change of the cost function is constant and equal tc $\$0.8 \text{ per km}$ and thus $m = 0.8.$ Notice that $c = 0$ in this context, as there is no fixed fee.
b)	$C_B = 0.45x + 7$	The rate of change of the cost function over the kilometres travelled is constant and equal tc $\$0.8 \text{ per km}$. Therefore, $m = 0$ The fixed fee is represented by tl y -intercept $c = 7\$$.
c)	Company A: $C_A(15) = 0.8(15) = \$12$ Company B: $C_B(15) = 0.45(15) + 7 = \$13.75$ Therefore company A is cheaper for a ride of 15 km.	Substitute $x = 15$ into the formulae of each cost function.

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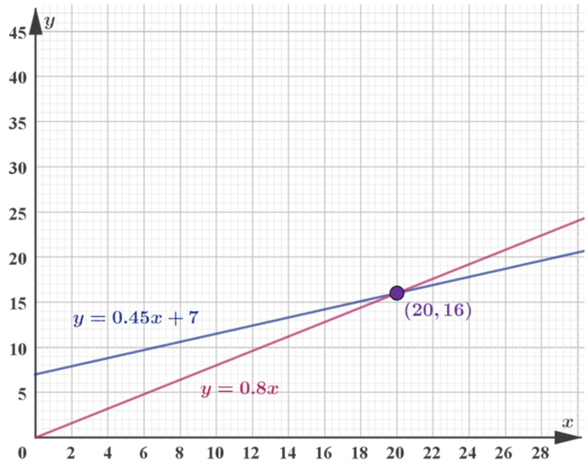
	Steps	Description
d)	 After 20 km company B is cheaper, as its graph lies below the graph of company A.	Use a GDC to obtain the graphs both functions. The two companies charge the same amount of money when th ride is 20 km.
e)	Point of intersection $(20, 16)$. The two companies charge the same amount of money when the ride is 20 km. They both charge $\$16$.	Use your GDC to find the point c intersection. The point of intersection of the graphs represents the number o kilometres for which the two companies charge the same amount of money.

Be aware

In a linear model of the form $f(x) = mx + c$, the **independent variable** is x and the **dependent variable** is y . In the equation, m and c are called the parameters of the linear model.

The independent variable is the variable being changed to test the effect on the dependent variable. The dependent variable is the variable being tested and measured.

Student view

Think about how the parameters m and c of a linear model help you to interpret the model?

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What is the difference between the parameters and variables of a linear model?

Activity

The water pressure at sea level is 1.01325 bar (where 'bar' is an SI unit of pressure), or 1 atmosphere as it is equal to the air pressure above the water. Below sea level, there is an increase in water pressure of 1 atmosphere for every 10 metres of depth. Devise a formula that expresses the water pressure in bars as a function of the depth in metres below sea level. Estimate at what depth the water pressure is 400 bars .

International Mindedness

Across the world, temperature is measured in different units. For example, in Europe, temperature is measured in degrees Celsius ($^\circ\mathrm{C}$), named after the Swedish astronomer Anders Celsius (1701—1744). This is also the temperature scale used by the International System of Units (SI). In other countries, such as the USA, the Bahamas, Belize, the Cayman Islands and Liberia, temperature is measured in degrees Fahrenheit ($^\circ\mathrm{F}$), named after the Polish-born Dutch physicist Daniel Fahrenheit (1686—1736).

Activity

To convert degrees Celsius to degrees Fahrenheit you can use the formula $F = \frac{9}{5}C + 32$. Sketch a graph of this function. What is the slope of the graph and what does it represent? What is the y -intercept and what does it represent?

Making connections

The nth term of an arithmetic sequence is given by the formula:

$$u_n = u_1 + (n-1)\,d$$

Where n is a positive integer, u_1 is the first term of the sequence and d is the common difference. Explain why this relation can be described as a linear model.

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Piecewise linear functions

Piecewise functions can be used to describe situations in which quantitative relationships are defined by different formulae in different intervals within the domain of the function.

Important

A **piecewise linear function** is a function that is defined on a sequence of intervals and each part of the function is a linear model.

In the applet below, you can visualise a car and a motorcycle as they travel over an interval of time.

Activity

Click the boxes ‘Road’, ‘Photos’, ‘Markers’, ‘Graph’ and press Play. What do you notice? Identify the variables in this context. Describe the graphs obtained on the right of the screen. Suggest two formulae that can be used to represent the graphs of the functions.

Interactive 1. A Graphical Representation of Piecewise Linear Function.

More information for interactive 1

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This interactive simulates the motion of two vehicles—a bike and a truck—along a straight road, and visually represents their movement over time through multiple coordinated views. Users can use the 'Play', 'Pause', and 'Reset' buttons to control the animation and interact with four checkbox options: 'Road', 'Photos', 'Markers', and 'Graph', allowing for a customized visual experience.

The distance-time graph plots time (seconds) on the x-axis and distance along the road (metres) on the y-axis. The bike's path is represented by a pink line with bike icons appearing at each second, and the truck's path is represented by a blue line with corresponding truck icons. These lines show how far each vehicle has traveled from the start over time.

The slope of each line indicates the speed: a steeper slope means a higher speed. For example, during the interval from $t = 4$ to $t = 6$, the bike's line becomes steeper, indicating that it was moving faster than the truck. The graph markers and photos show snapshots of each vehicle's position at different times, while the 'Road' view provides a side-by-side layout simulating their positions along an actual road.

At $t = 10$ seconds, the bike has traveled a significantly greater distance than the truck, as seen from the graph and the photo trail. This supports a conceptual understanding of constant vs. changing speed and reinforces how motion can be analyzed graphically.

This tool is ideal for teaching core kinematics concepts, helping learners interpret and connect graphical data to real-world motion, and visualize comparative speed and distance with respect to time.

In the above applet, the time t (in seconds) is the independent variable. The dependent variable is the distance d (in metres) of the vehicles measured from an initial point. Notice that the car's distance, d , from the initial point is increasing at a constant rate and the distance–time graph is a straight line. The linear function that models the car's distance from the initial point as a function of time is:

$$d\left(t\right) = 6t + 60$$

- What does the gradient of the function represent in this context?
- What does the y -intercept represent?

The linear model that describes the motorcyclist's distance from the initial point as a function of time is:

$$D\left(t\right) = \begin{cases} \frac{40}{3}t, & if\ 0 \leq t \leq 3 \\ 25t - 35, & if\ 3 < t \leq 7 \\ \frac{40}{3}t + \frac{140}{3}, & if\ 7 < t \leq 10 \end{cases}$$

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- How is the rate of change of the dependent variable represented on the graph and in the formula for the motorcyclist's function?
- How do you interpret the gradient of a linear function in the time interval $3 < t \leq 7$?
- Reflect on what the value of $\frac{140}{3}$ in the function formula represents.
- What will be the gradient of a linear function that models the distance–time relationship of a car that is coming from the opposite direction?

Example 2

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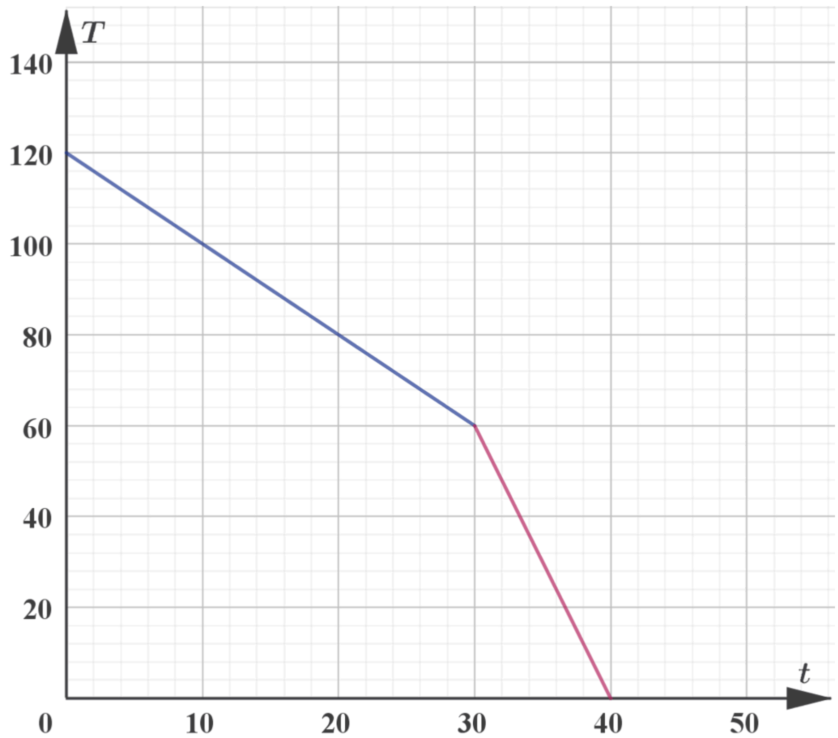
The graph below shows the relationship between the temperature of a liquid, T , measured in $^\circ\mathrm{C}$, over time, t , measured in minutes.



More information

The graph illustrates the relationship between the temperature of a liquid, T, in degrees Celsius (°C), and time, t, in minutes. The X-axis represents time, t, in minutes, ranging from 0 to 50 minutes. The Y-axis represents temperature, T, in degrees Celsius, ranging from 0 to 140°C. The graph displays two linear segments: the first segment shows a

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decreasing trend from approximately 120°C at 0 minutes to about 50°C at 30 minutes. The second segment further decreases from approximately 50°C at 30 minutes to 0°C at 40 minutes. The graph has a consistent downward slope, indicating a steady decrease in temperature over time.

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a) What is the initial temperature of the liquid?

b) State the time required for the liquid temperature to reach half of its initial temperature.

c) State the time required for the liquid temperature to reach 0 .

d) Estimate the rate of change of the temperature over time.

e) Suggest a piecewise function that models the temperature T of the liquid in terms of time t .

	Steps	Description
a)	$120°\text{C}$	The graph shows that the temperature is $120°\text{C}$ at $t = 0$.
b)	30 minutes	This can be read off the graph by looking at the point at which the temperature is equal t $60°\text{C}$. The corresponding t -value is 30 minutes.
c)	40 minutes	This can be read off the graph by looking at the point at which the temperature is equal t $0°\text{C}$. The corresponding t -value is 40 minutes.

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	Steps	Description
d)	$m = -2$	In the time interval between $t = 0$ and $t = 3$ the rate of change is the same as the gradien of the corresponding line. The line passes through the points $(0, 120)$ and $(30, 60)$ and thus the gradient is equal to $m = \frac{60 - 120}{30 - 0} = -2$.
	$m = -6$	In the time interval between $t = 30$ and $t = 40$ the rate of change is the same as the gradient of the corresponding line. The line passes through the points $(30, 60)$ and $(40,$ and thus the gradient is equal to $m = \frac{0 - 60}{40 - 30} = -6$.
e)	$T(t) = \begin{cases} -2t + 120, & 0 \leq t \leq 30 \\ -6t + 240, & 30 < t \leq 40 \end{cases}$	The temperature is modelled with linear functions of the form $T = mt + c$, in the corresponding time intervals. In the time interval between $t = 0$ and $t = 3$, $m = -2$ and the y -intercept equals to $12($. In the time interval between $t = 30$ and $t = 60$, $m = -6$. Also, the line passes through the point $(40, 0)$ so you find the y -intercept as follows: $0 = -6(40) + c$ $c = 240.$

4 section questions

2. Functions / 2.5 Introduction to modelling

Standard form of quadratic functions

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Standard form and features of a quadratic function

Quadratic functions are a special class of function that are very useful in modelling real-life phenomena. The graph of a quadratic function is a type of curve called a parabola, which is of great interest and utility in mathematics and many other fields. Parabolic curves and their special features appear in nature as well as in design and engineering.

Credit: visualspace Getty Images

Source: " L'Oceanografic, Valencia, Spain 1 - Jan 07 (https://fr.wikipedia.org/wiki/Fichier:L%27Oceanografic,_Valencia,_Spain_1_-_Jan_07.jpg) " by Diliff is licensed under CC BY-SA 3.0 (https://creativecommons.org/licenses/by-sa/3.0/deed.fr)

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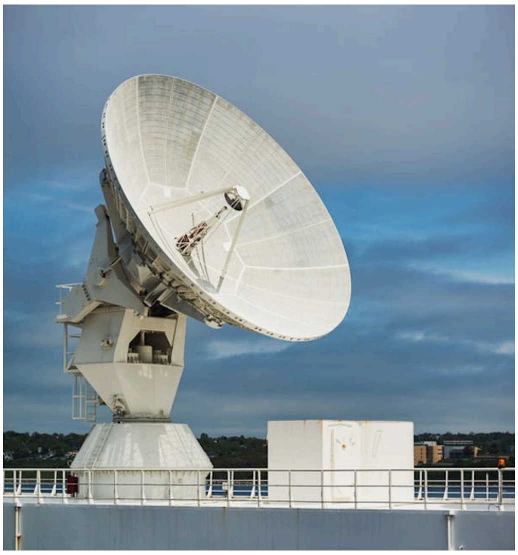
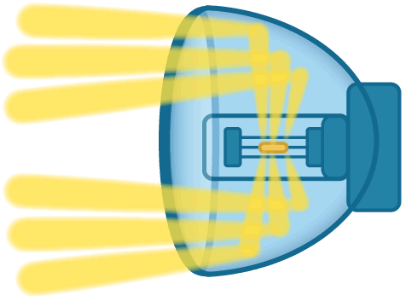


Credit: shaunl Getty Images



More information

This image is a diagram of a car headlamp design. It shows a cross-sectional view of the headlamp, including the bulb and reflector. The bulb is depicted in the center, supported by a mount, and emits yellow beams of light. The beams are shown spreading outward, indicating the direction of light projection. The surrounding reflector is shaped to direct the light rays forward in a focused manner. The headlamp assembly is shown in blue, with the bulb and light rays in contrasting yellow to highlight their function. This diagram visually explains the components and light direction within a headlamp.

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Parabolas in the real world: trajectory of a ball, architectural arches, satellite dish and headlight

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Standard form of quadratics and features of a quadratic function

The standard form of a quadratic function f is

$$f(x) = ax^2 + bx + c\,,$$

where a , b and c are real numbers and $a \neq 0$. The numbers a, b and c are called the **parameters** of the quadratic function. The graph $y = f(x)$ of a quadratic function is called a parabola, and its shape is determined by the parameters a, b and c. In the applet below you can visualise the graph of a quadratic function in its standard form.

Interactive 1. Graph of a Quadratic Function in Its Standard Form.

More information for interactive 1

This interactive allows users to explore quadratic functions by adjusting the parameters aaa, bbb, and ccc, each ranging from -5 to 5. The function is represented as $f(x) = ax^2 + bx + c$, where $a \neq 0$. The graph of this function is a parabola—a symmetric curve whose shape and position depend on the parameter values. When $a > 0$, the parabola opens upward with a minimum vertex; when $a < 0$, it opens downward with a maximum vertex. The vertex is marked by a pink point, and the y-intercept, located at (0,c), is shown by an orange point. The dashed vertical line indicates the axis of symmetry. The blue points indicate the zeros (or roots) of the function, where the parabola intersects the x-axis. By using the sliders to vary aaa, bbb, and ccc, users can observe how each parameter affects the orientation, width, and position of the parabola and its key features. For example, when $a = 0.3$, $b = -1.4$

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, and c = -2, the parabola opens upward, the y-intercept is at -2, and the axis of symmetry passes through approximately x = 2.33 helping users visualize how these values influence the graph. This dynamic visualization helps build intuition about how quadratic functions behave.

Activity

The applet above shows the graph and equation of a quadratic function in its standard form $y = ax^2 + bx + c$. Using the sliders 'a', 'b' and 'c' you can change the values of the parameters a , b and c , leading to changes in the graph of the function.

- Using slider 'a', observe the graph of the function for positive and negative values of a . What do you notice?
- Compare the parabolas for different positive values of a . What do you notice?
- Using slider 'c', observe the graph of the function for various values of c . What do you notice?
- Use slider 'b' to explore how the parameter b affects the parabola.
- What type of function do you obtain if $a = 0$?
- State any differences between linear and quadratic graphs.
- What special features does a quadratic graph have?

Important

A quadratic function has standard form $f(x) = ax^2 + bx + c,\ a \neq 0$. Its graph $y = ax^2 + bx + c,\ a \neq 0$ is a parabola, a symmetric curve with the following features:

- If $a > 0$, the parabola opens upwards; its vertex is the **minimum turning point** and the curve is concave up .
- If $a < 0$, the parabola opens downwards; the **vertex** is the **maximum turning point** and the curve is concave down .
- The parabola crosses the y -axis at the point $(0,\ c)$, called the y **-intercept** .
- The vertex of the parabola is $\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$.

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- The **axis of symmetry** of the parabola is the vertical line $x = -\dfrac{b}{2a}$, which passes through the vertex.

- Reflect on how can you find the domain and range of a quadratic function by plotting the graph on your GDC.
- How can you find the points where the graph crosses the x -axis using your GDC?

✓ Important

The x -intercepts are the points where the graph of $y = f(x)$ crosses the x -axis. The x -intercepts are also called the **zeros or roots of the function**, because they are the solutions of $f(x) = 0$.

Notice that the parabola is a symmetric curve and thus the x -intercepts are equidistant from the axis of symmetry.

⚠ Be aware

The equation of the axis of symmetry of a parabola is

$$x = \frac{x_1 + x_2}{2}$$

where x_1, x_2 are the x -coordinates of the x -intercepts.

Exam tip

When you are asked to sketch the graph of a quadratic function you should:

- show the overall shape of the graph
- label the coordinates of the x - and y -intercepts
- label the coordinates of the vertex
- show and label the axis of symmetry.

For example, the quadratic function

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$$f(x) = -x^2 + 8x - 12$$

has $a = -1$, $b = 8$ and $c = -12$. The x -coordinate of the vertex is

$$-\frac{b}{2a} = -\frac{8}{2(-1)} = 4$$

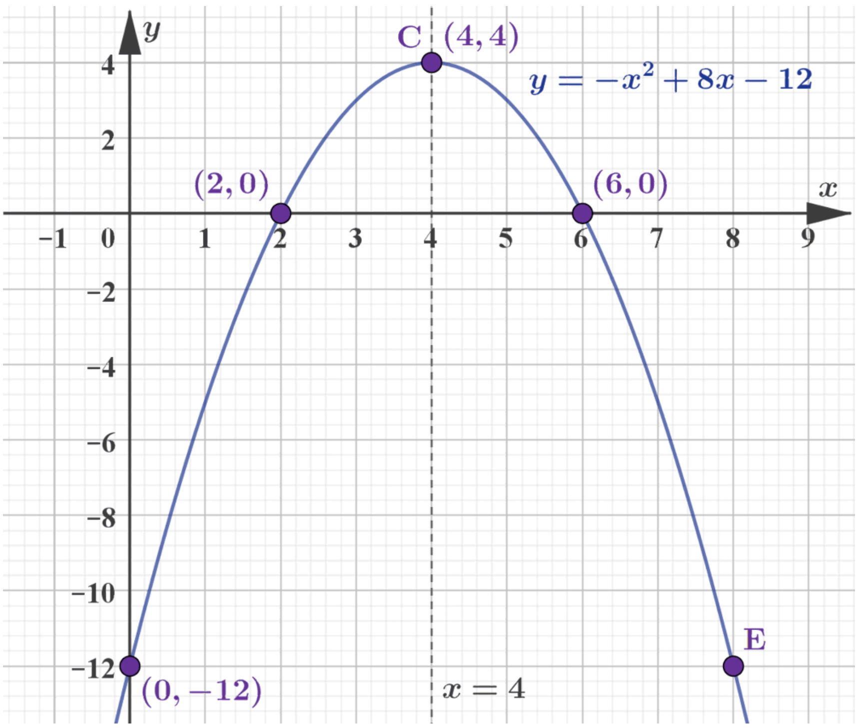
and the corresponding y -coordinate is

$$f(4) = -4^2 + 8(4) - 12 = -16 + 32 - 12 = 4.$$

Since $a < 0$, the parabola is concave down and the vertex $(4,\ 4)$ is the maximum turning point. Thus, the range of the function is $\{y|y \leq 4\}$.

The axis of symmetry of the parabola passes through the vertex and therefore has equation $x = 4$.

Finally, $c = -12$ means that the y -intercept of the graph is the point $(0, -12)$. You can graph the function on a GDC and verify these properties. The parabola for the quadratic function $f(x) = -x^2 + 8x - 12$ is shown below.



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The image is a graph of the quadratic function (f(x) = -x^2 + 8x - 12). It features a parabola opening downwards. The X-axis ranges from (-2) to (10), and the Y-axis ranges from (-14) to (6). Key points on the graph include ((0, -12)), ((2, 0)), ((4, 4)), and ((6, 0)), which are marked and labeled. The vertex of the parabola is at ((4, 4)). The function's equation is written on the graph. The Y-intercept at ((0, -12)) is where the graph intersects the Y-axis.

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Reflect on the properties of the parabola and explain the coordinates of point E .

Be aware

In general, the domain of a quadratic function is the set of real numbers , unless otherwise stated.

Example 1

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Sketch the graph of the quadratic function $f\left(x\right)=x^2-4x$ and state its domain and range.

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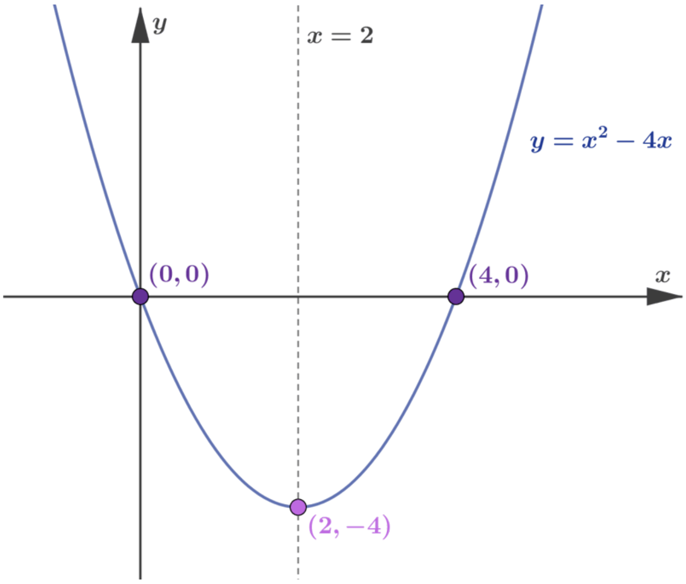
Steps	Explanation
The sketch of the parabola is shown below. 	Use a GDC to plot the function. Transfer the graph onto paper and l abel the axes, show the coordinates the axes intercepts and vertex, show the axis of symmetry and state its equation.
The domain of the quadratic function is the set of real numbers $\mathbb{R}$.	
So, the range of the function is $\{y \| y \geq -4\}$.	The vertex is $(2, -4)$, and since $a > 0$ the parabola is concave up. Hence, the vertex is the minimum point.

Example 2

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Sketch the graph of the quadratic function $y = 2x^2 - 6x + 1$ and state its domain and range.

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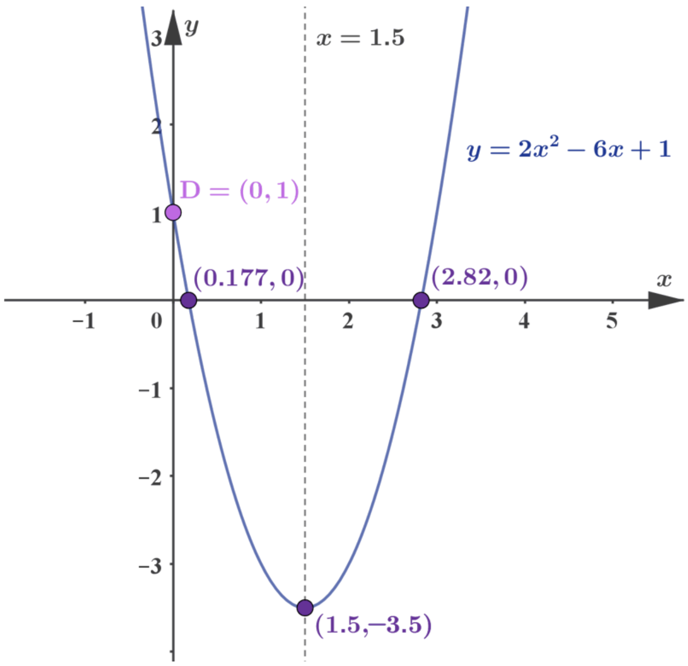
Steps	Explanation
	Use a GDC to plot the function. Transfer the graph onto paper and l abel the axes, show the coordinates of the axes intercepts and vertex, show the axis of symmetry and state its equation.
The domain of the function is all real numbers.	
The range of the function is $\left\{ y \vert y \geq -\frac{7}{2} \right\}$.	The vertex is a minimum point.

Example 3

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The graph of $y = ax^2 + bx + c$ passes through the points $(-2, 8)$ and $(0, -4)$. Given that the axis of symmetry has equation $x = \frac{1}{2}$, find the values of parameters a, b and c .

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Steps	Explanation
$c = -4$	Point $(0, -4)$ is the y -intercept of the function, which is represented by parameter c in the formula.
The graph also passes through point $(1, -4)$.	The horizontal distance of the y -intercept from the axis of symmetry is $\frac{1}{2}$. Thus, there is a second point on the graph with horizontal distance $\frac{1}{2}$ from the axis of symmetry.
$8 = a(-2)^2 + b(-2) - 4$ $12 = 4a - 2b$ and $-4 = a1^2 + b1 - 4$ $a + b = 0$	Substitute the coordinates of points $(-2, 8)$ and $(1, -4)$ into the formula.
$12 = 4a - 2b$ $a + b = 0$ $a = 2$ and $b = -2$.	Solve the equations simultaneously,
Therefore, $a = 2,\ b = -2$ and $c = -4$.	

4 section questions

Vertex form of a quadratic function

In this section and the next, you will learn about different forms for representing quadratic functions and how to transform a quadratic function from one form to another.

Exam tip

The vertex form and factorised form of a quadratic function is not in the syllabus. The content of this section and the next can be useful to enhance understanding, but these can be safely skipped for exam preparation.

The vertex form of a quadratic function f is

$$f(x) = a(x - h)^2 + k.$$

The applet below allows you to visualise parabolas when their functions are expressed in vertex form.

Activity

Move the red point labelled 'Change parabola' around to adjust the position and shape of the parabola, and observe the formula displayed for the corresponding quadratic function. What do you notice? Based on your observations, can you formulate a rule about the vertex form of a quadratic function? Why do you think this is called the vertex form?

Interactive 1. A Parabola Representing a Quadratic Function in Vertex Form.

More information for interactive 1

Student
view

This interactive helps users explore quadratic functions in vertex form, $y = a(x - h)^2 + k$. By dragging the red point labeled “Change parabola,” users can adjust the shape and position of the parabola. The graph and equation update in real time to reflect changes in the parameters a, h, and k.
The vertex of the parabola, shown on the graph, corresponds directly to the values (h, k) in the equation. Users can observe how these changes affect the graph’s orientation, width, and position. This interactive supports a deeper understanding of how the vertex form relates to a parabola’s graph and helps develop intuition for analyzing quadratic functions.

Important

A parabola representing a quadratic function in vertex form has equation

$$y = a(x - h)^2 + k.$$

- The **vertex** of the parabola is the point (h, k) .
- If $a > 0$ the parabola is concave up, and if $a < 0$ the parabola is concave down.
- The **axis of symmetry** has equation $x = h$.
- The y **-intercept** is $\left(0,\ ah^2 + k\right)$.

For example, the quadratic function

$$y = -3(x - 2)^2 - 1.$$

has vertex $(2, -1)$. Because $a = -3 < 0$, the parabola is concave down and so the vertex is the maximum turning point. The axis of symmetry passes through the vertex and has equation $x = 2$. The y -intercept has y -coordinate $-3 \times 2^2 - 1 = -13$. These features are shown in the figure below.

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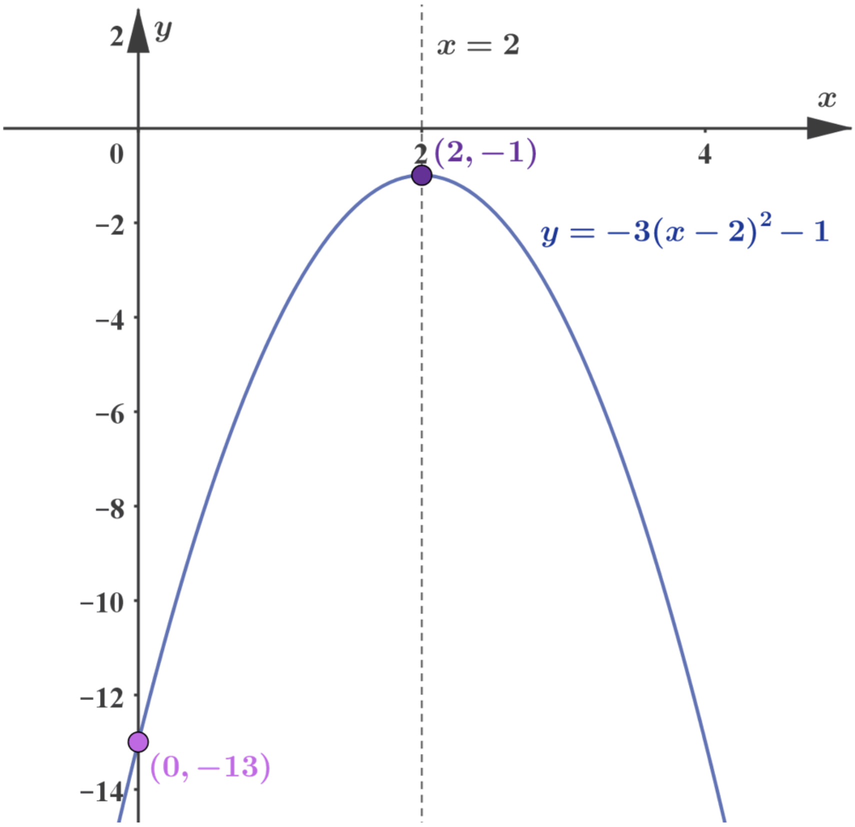

More information

The image is a graph illustrating a downward-facing parabola described by the equation (y = -3(x - 2)^2 - 1). The graph includes labeled axes, with the X-axis representing input values (range approximately from -2 to 4) and the Y-axis showing output values (range approximately from -14 to 2). The key points on the graph are the vertex at ((2, -1)), which is the highest point due to the negative coefficient of (a = -3), and the Y-intercept at ((0, -13)). A dashed vertical line labeled as the axis of symmetry passes through x = 2. The parabola opens downwards, confirming that (a < 0).

[Generated by AI]

Example 1

★☆☆

Use the vertex, axis of symmetry and y -intercept to sketch the graph of

$$y = -\frac{1}{2}(x+1)^2 - 2\,.$$

Student view

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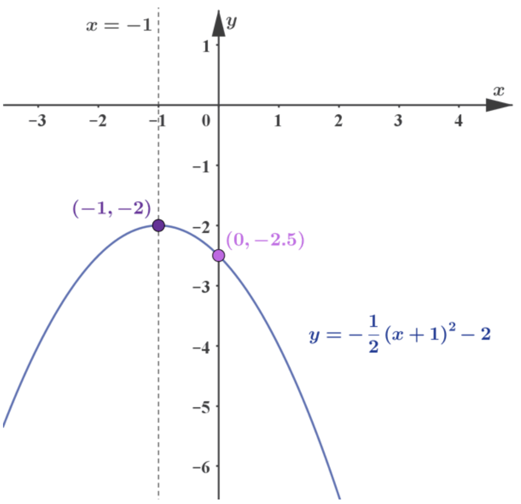
Steps	Explanation
The vertex is $(-1, -2)$. The axis of symmetry is $x = -1$. $a < 0$ so the parabola is concave down and the vertex is the maximum turning point.	The parameters are $$a = -\frac{1}{2},\ h = -1 \text{ and } k = -2.$$ Be careful with the sign of h : the squared term can be written as $(x - (-1))^2$.
For the y -intercept, when $x = 0$, $$y = -\frac{1}{2}(0+1)^2 - 2 = -\frac{5}{2}.$$ So the y -intercept is $\left(0, -\frac{5}{2}\right)$.	You can also use the formula $ah^2 + k$ to fir the y -intercept.
The graph of the function is as follows.  	Show the vertex, axis of symmetry and y -intercept clearly on your graph.

Example 2

★★☆

Student view

A parabola has vertex $(2, 3)$ and passes through the point $(-2, -1)$.

Overview
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a) Find the equation of the quadratic function in vertex form.

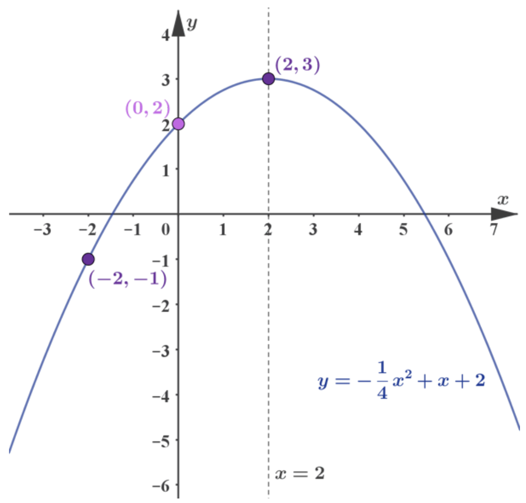
b) Express the quadratic function in standard form.

c) Find the y -intercept of the parabola using the standard form.

d) Hence, sketch the parabola.

	Steps	Explanation
a)	$y = a(x-2)^2 + 3$	Substitute the coordinates of the vertex into the vertex form of a quadratic function
	$-1 = a(-2-2)^2 + 3$ $-1 = 16a + 3$ $16a = -4$ $a = -\frac{4}{16}$ $a = -\frac{1}{4}$	Substitute the coordinates $(-2,\ -1)$ into the equation. Then solve for a .
	Therefore, the quadratic function has vertex form $y = -\frac{1}{4}(x-2)^2 + 3$.	
b)	$y = -\frac{1}{4}(x-2)^2 + 3$ $= -\frac{1}{4}\left(x^2 - 4x + 4\right) + 3$ $= -\frac{1}{4}x^2 + x + 2$	Transform the vertex form to standard form by expanding the brackets and simplifying.
	Hence, the quadratic function has standard form $y = -\frac{1}{4}x^2 + x + 2$.	
c)	The y -intercept is $(0, 2)$.	In the standard form $c = 2$.

Student
view

	Steps	Explanation
	The graph of the function is shown below. 	Show the vertex, intercept and poi $(-2, -1)$ on your graph.

Completing the square

In **Example 2**, you saw that to transform a quadratic function from vertex form to standard form, you just expand the brackets and simplify. In the opposite direction, to transform from standard form to vertex form, one approach is **completing the square**.

International Mindedness

The method of completing the square was described extensively by the great mathematician al-Khwarizmi (ca. 780—850 CE) in his book *The Compendious Book on Calculation by Completion and Balancing.* Al-Khwarizmi was a Persian mathematician and astronomer who lived in Baghdad and is considered to be the father of algebra. His name, formerly Latinised as *Algorithmi* , is the origin of the term 'algorithm'. Some of al-Khwarizmi's work was based on Babylonian, Indian and Greek mathematics.

The method of completing the square gets its name from the fact that the vertex form of a quadratic function contains a perfect square.

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Activity

The applet below demonstrates a geometric interpretation of completing the square, as described by al-Khwarizmi. Al-Khwarizmi showed how a quadratic expression of the form $x^2 + bx$ can be transformed into a perfect square, by using a geometrical approach.

Interactive 2. Geometric Interpretation of Completing the Square.

More information for interactive 2

This interactive illustrates how to complete the square for quadratic expressions of the form $x^2 + bx$ using a geometric method inspired by al-Khwarizmi. Users can enter positive values for b and use the slider to visualize how the expression transforms into $(x + \frac{b}{2})^2 - (\frac{b}{2})^2$.

The process is represented using geometric shapes—squares and rectangles—that show how adding and subtracting $(\frac{b}{2})^2$ completes the square. This visual model helps users connect algebraic steps with area-based reasoning, deepening their understanding of the method and its underlying structure.

- In the input box, enter a positive value for coefficient b .
- Drag the slider to visualise al-Khwarizmi's method of completing the square.
- Try different values of b and find the value of c that should be added to $x^2 + bx$ to make a perfect square.
- Generalise your observation by formulating a rule on how to complete the square for $x^2 + bx$.
- Explain al-Khwarizmi's interpretation of x and b in his method.

Student view
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Overview
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- What are the steps for completing the square?

The method described by al-Khwarizmi deals only with quadratic functions $ax^2 + bx + c$ where $a = 1$ and $c = 0$. So how do you complete the square for more general quadratic functions? Look at an example: completing the square for the quadratic function $y = 2x^2 - 4x + 7$.

The first step is to factorise out any coefficient of x^2 (namely, a) that is not 1. The second step is to add and subtract the square of half the coefficient of x (namely, b) within the brackets. Can you see why you need to do this? Remember; you are trying to make the expression in the brackets become a perfect square.

Steps	Explanation
$y = 2x^2 - 4x + 7$ $= 2\left(x^2 - 2x\right) + 7$	Write the function in standard form and factorise 2 out of any terms containing x.
$= 2\left(x^2 - 2x + 1 - 1\right) + 7$	Add and subtract $\left(-\frac{2}{2}\right)^2 = 1$ within the brackets.

Next, regroup the terms so that the three terms remaining within the bracket form a perfect square. In this case the -1 is moved outside the brackets; note that it needs to be multiplied by the factor 2 in front of the brackets.

Steps	Explanation
$y = 2\left(x^2 - 2x + 1\right) - 2(1) + 7$	Regroup terms and move -1 outside the brackets, remembering to multiply it by 2.
$= 2\left(x^2 - 2x + 1\right) - 2 + 7$ $= 2(x-1)^2 + 5$	Write the expression in the brackets as a perfect square and simplify the constant terms.

From the vertex form, you can see that the parabola $y = 2(x-1)^2 + 5$ opens upwards because $a = 2 > 0$ and its vertex is$(1, 5)$. The axis of symmetry has equation $x = 1$ and the y-intercept is$(0, 7)$. These features are shown below.

Student
view

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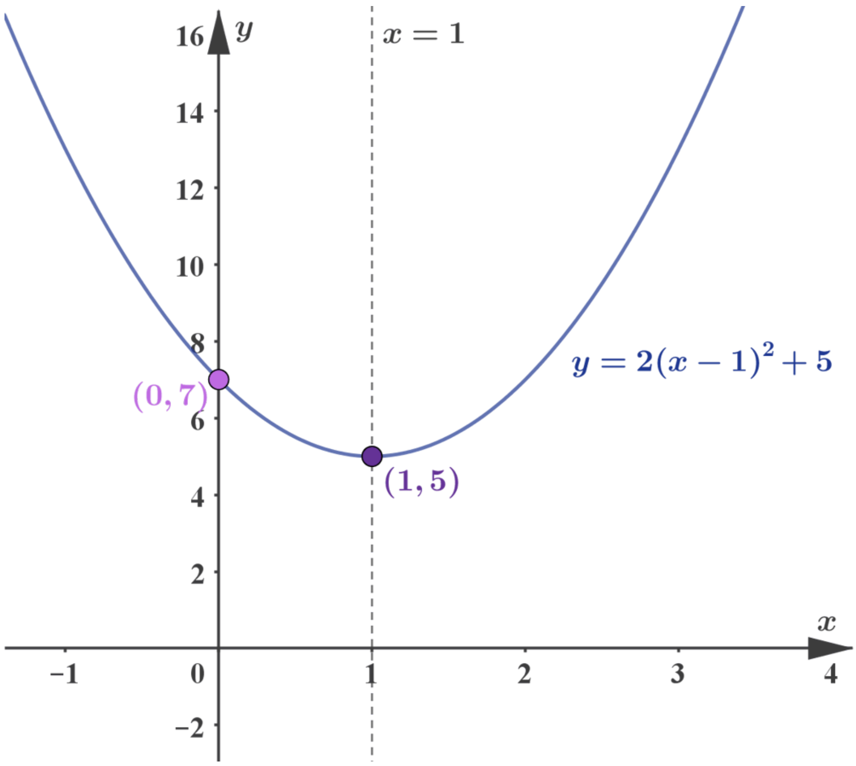

More information

The image is a graph illustrating a parabola with the equation (y = 2(x - 1)^2 + 5). The graph is plotted on a Cartesian coordinate system with the X-axis ranging from -1 to 4 and the Y-axis from -2 to 16. The parabola is opening upwards and its vertex is marked at the point (1, 5). Another point on the graph, the Y-intercept at (0, 7), is also labeled. The axis of symmetry is a vertical dashed line along (x = 1). Both the vertex and the Y-intercept are marked with small dots and are labeled respectively. The equation of the parabola is printed on the graph for reference.

[Generated by AI]

Example 3

★★☆

Write $y = x^2 + 4x + 5$ in the form $y = (x - h)^2 + k$ by completing the square.

Hence, state the coordinates of the vertex and sketch the function.

Student
view

Steps	Explanation
$y = x^2 + 4x + 5$	Write the function in standard form. Check whether the coefficient of is 1. In this case it is, so you don't need to take out any factor.
$y = x^2 + 4x + 2^2 - 2^2 + 5$	Add and subtract $\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 2^2$.
$y = x^2 + 4x + 4 - 4 + 5$	Evaluate the squared constants.
$y = (x + 2)^2 + 1$	Write the first three terms as a perfect square, and combine the remaining constant terms.
$y = (x - (-2))^2 \ + 1$	Express the perfect square in the vertex form of a quadratic function.
The vertex is $(-2, \ \ 1)$.	

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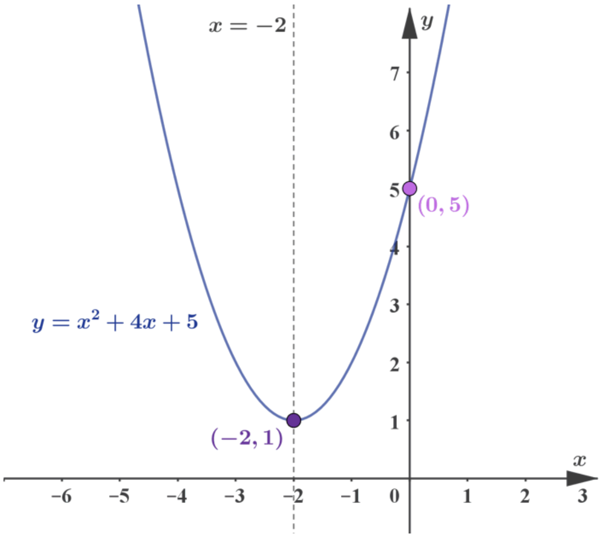
Steps	Explanation
The graph is sketched below. 	From the standard form, the y -intercept is 5 . Use this and the vertex to sketch the parabola.

Example 4

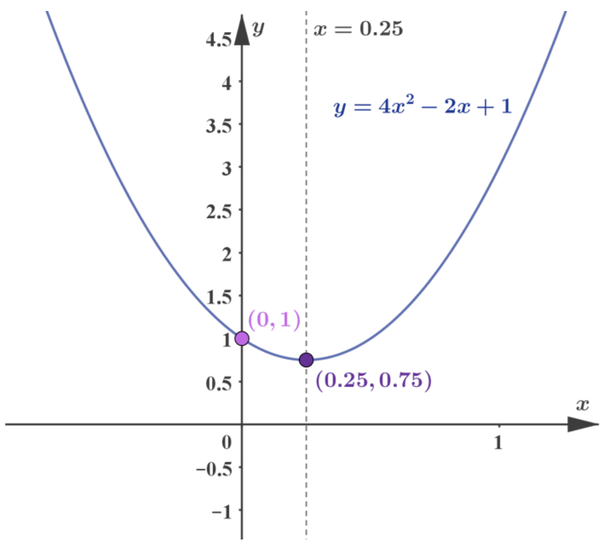
★★★

Write $f(x) = 4x^2 - 2x + 1$ in vertex form by completing the square.

Hence, state the coordinates of the vertex and sketch the function.

Steps	Explanation
$f(x) = 4x^2 - 2x + 1$	Write the function in standard form.
$= 4\left(x^2 - \frac{1}{2}x\right) + 1$	Factorise 4 out of the x terms.

Student
view

Steps	Explanation
$= 4\left(x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2 - \left(-\frac{1}{4}\right)^2\right) + 1$	Add and subtract $\left(-\frac{\frac{1}{2}}{2}\right)^2 = \left(-\frac{1}{4}\right)$ within the brackets.
$= 4\left(x^2 - \frac{1}{2}x + \frac{1}{16} - \frac{1}{16}\right) + 1$	Evaluate the squared constants.
$= 4\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) - 4\left(\frac{1}{16}\right) + 1$	Move $-\frac{1}{16}$ outside the brackets, multiplying it by 4.
$= 4\left(x - \frac{1}{4}\right)^2 - \frac{1}{4} + 1$	Write the expression in brackets as a perfect square.
$= 4\left(x - \frac{1}{4}\right)^2 + \frac{3}{4}$	Combine the constant terms.
The vertex is $\left(\frac{1}{4}, \frac{3}{4}\right)$.	
The graph of the function is sketched below. 	From the standard form, the y -intercep is 1 . Since $a = 4 > 0$, the parabola opens upwards. Use this information and the vertex to sketch the graph.

Factorised form of quadratic functions

Exam tip

The factorised form of a quadratic function is not in the syllabus. The content of this section can be useful to enhance understanding, but this section can be safely skipped for exam preparation.

The factorised form of a quadratic function is

$$f(x) = a(x - p)(x - q)$$

Activity

The applet below allows you to visualise a parabola whose equation is given in the factorised form $y = a(x - p)(x - q)$.

- Use the sliders to adjust the values of a, p and q . What do you notice about the resulting parabola?
- What is the effect of the parameter a on the graph?
- What do the parameters p and q represent on the graph?
- Can you formulate a rule involving a, p and q about the y -intercept of the parabola?
- Can you formulate a rule involving a, p and q about the axis of symmetry or the vertex of the parabola?

Student view

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✓ **Important**

A parabola representing a quadratic function in factorised form has equation $y = a\left(x-p\right)\left(x-q\right)$

- The x **-intercepts** of the parabola are $\left(p, 0\right)$ and $\left(q, 0\right)$.
- The y **-intercept** of the parabola is $\left(0, apq\right)$.
- The **axis of symmetry** has equation $x = \dfrac{p+q}{2}$.
- The **vertex** of the parabola is $\left(\dfrac{p+q}{2}, f\left(\dfrac{p+q}{2}\right)\right)$, where $f\left(\dfrac{p+q}{2}\right) = -\dfrac{a(p-q)^2}{4}.$

Example 1

★★☆

For the graph of the quadratic function $f\left(x\right) = 2\left(x+1\right)\left(x-2\right)$, find the x -intercepts, y -intercept and coordinates of the vertex. Hence, sketch the graph.

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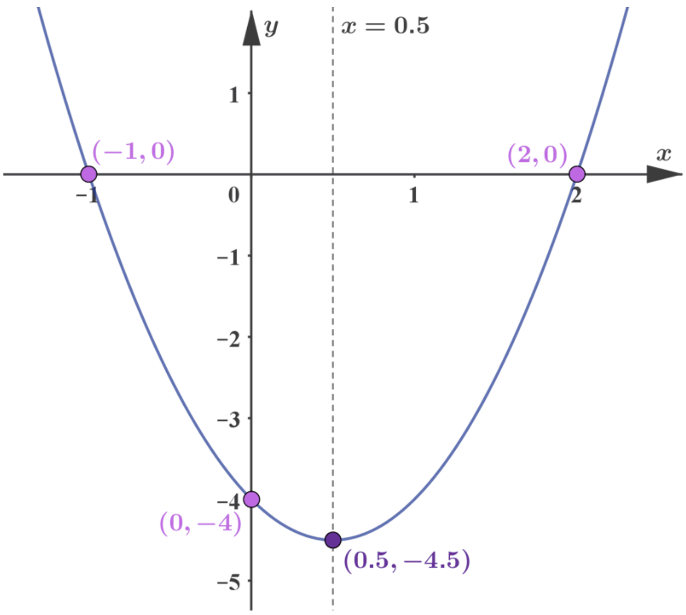
Steps	Explanation
The x -intercepts are $(-1,\ 0)$ and $(2,\ 0)$.	The function can be expressed as $f(x) = 2(x-(-1))(x-2)$, so $p=-1$ and $q=2$ (or vice versa).
$2(-1)(2) = -4$ The y -intercept is $(0,\ -4)$.	The y -coordinate of the y -intercep is $apq = 2(-1)(2)$.
$$\frac{p+q}{2} = \frac{-1+2}{2} = \frac{1}{2}$$ $$f\left(\frac{p+q}{2}\right) = f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}+1\right)\left(\frac{1}{2}-2\right) = 2\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right) = -\frac{9}{2} = -4.5$$ So the vertex is $\left(\frac{1}{2},\ -\frac{9}{2}\right)$.	The x -coordinate of the vertex is $\frac{p+q}{2}$, and the y -coordinate is $f\left(\frac{p+q}{2}\right) = -\frac{a(p-q)^2}{4}$
	The axis of symmetry has equation $x = \frac{1}{2}$. Use all the information you have fou to sketch the parabola.

Student
view

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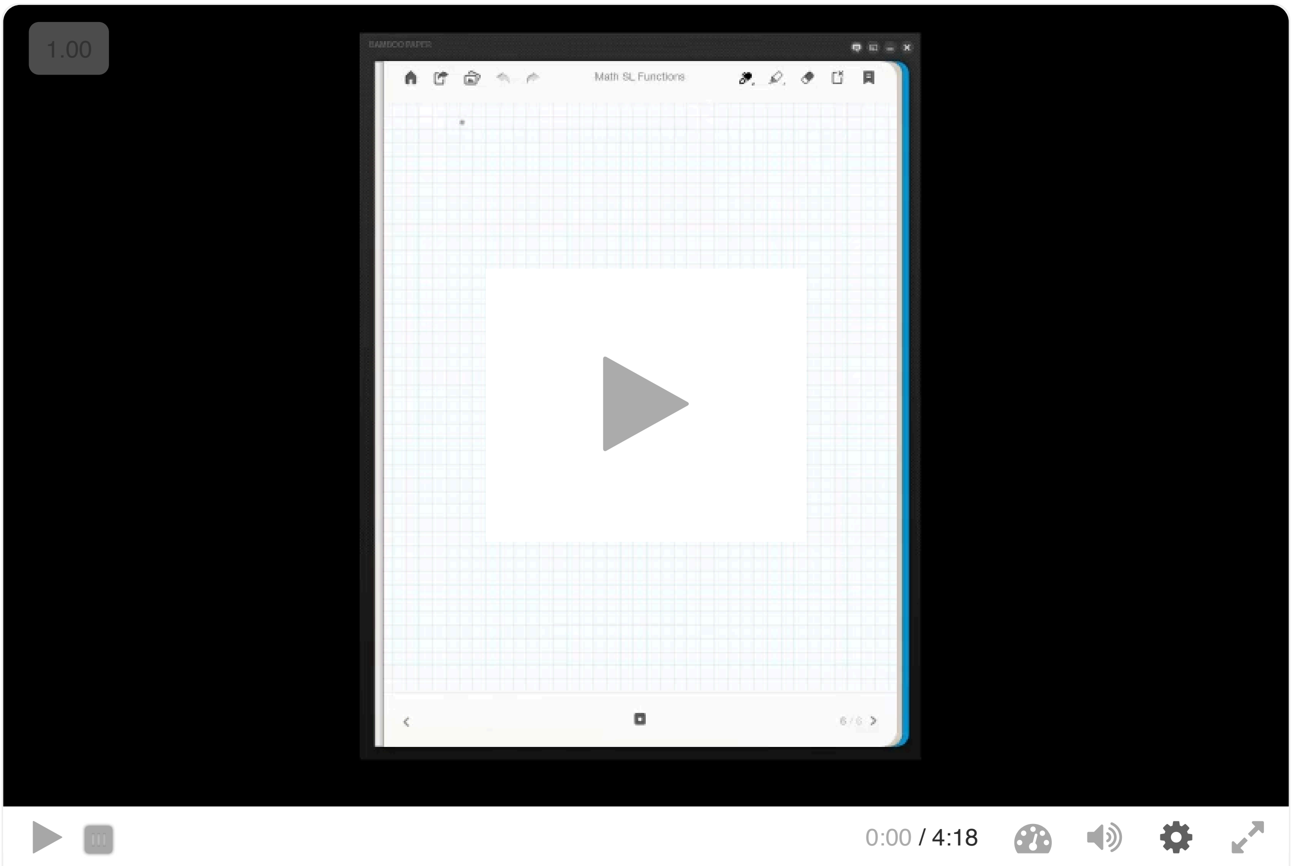
To transform a quadratic function from factorised form to standard form, just multiply out the brackets.

The video below shows how to transform a quadratic function from standard form to factorised form.



Video 1. Quadratic Function: Standarized Form to Factorized Form.

More information for video 1

1

00:00:00,633 --> 00:00:03,933

narrator: In this video we're going to

look at quadratics as graphs,

2

00:00:04,000 --> 00:00:06,433

both equations and functions.

3

00:00:08,500 --> 00:00:11,433

Now, let's remind ourselves

what a quadratic looks like.

4

Student view

Overview (/study/ap 122- cid- 754029/

00:00:11,500 --> 00:00:14,800

$ax^2 + bx + c$,

where $a \neq 0$.

5

00:00:15,000 --> 00:00:17,200

So for starters, let's distinguish

between the two

6

00:00:17,267 --> 00:00:19,867

major types of quadratic as a graph,

7

00:00:19,933 --> 00:00:23,500

and that is the one that looks

like a ball where a is positive,

8

00:00:23,567 --> 00:00:24,800

and we call that a concave up.

9

00:00:24,933 --> 00:00:27,933

And one looks like a mountain

a is negative concave down.

10

00:00:28,667 --> 00:00:31,433

Now all quadratics

have a line of symmetry,

11

00:00:31,500 --> 00:00:35,533

which means that if you take two points

that are the same y value,

12

00:00:35,600 --> 00:00:38,533

then they are equidistant

from the symmetry line.

13

Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/factorised-form-of-quadratic-functions-id-27461/print/) Assign

00:00:38,900 --> 00:00:40,300

It is useful to remember.

14

Student view

00:00:40,900 --> 00:00:44,333

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Now, the first thing we're gonna concern

ourselves with are the roots,

15

00:00:44,400 --> 00:00:47,467

and these are places

where the function evaluates the zero,

16

00:00:47,533 --> 00:00:49,767

so it crosses or touches the x axis.

17

00:00:50,133 --> 00:00:51,633

And there are a few possibilities.

18

00:00:51,767 --> 00:00:55,133

So either the x axis

doesn't encounter the curve,

19

00:00:55,233 --> 00:00:59,333

it can touch the curve where it can

actually cross the curve at two places.

20

00:00:59,400 --> 00:01:01,533

So if $y = ax^2 + bx + c$,

21

00:01:01,600 --> 00:01:03,533

we've already seen

that the discriminant is important

22

00:01:03,667 --> 00:01:06,033

for solving the equation

of that equals zero.

23

00:01:06,100 --> 00:01:09,333

For positive discriminant,

we have two real distinct roots.

24

00:01:09,400 --> 00:01:10,767

Discriminant being equal to zero.

Student
view

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754029/

25

00:01:10,833 --> 00:01:14,167

I've got one repeated root,

and if the discriminant is less than zero,

26

00:01:14,233 --> 00:01:18,100

there are no real solutions,

meaning that the graph does not

27

00:01:18,600 --> 00:01:21,300

come to touch across the x axis.

28

00:01:21,367 --> 00:01:23,733

So that's the graphical

importance of the discriminant.

29

00:01:25,067 --> 00:01:27,433

Now let's look at another

form of the quadratic function.

30

00:01:27,500 --> 00:01:30,933

And we often see

this $ax^2 + bx + c$,

31

00:01:31,000 --> 00:01:33,900

but we can also look at it in another form.

32

00:01:34,000 --> 00:01:36,967

So here I've got three

important points, c,

33

00:01:37,033 --> 00:01:39,433

p, and q, where p and q are the roots.

34

00:01:39,533 --> 00:01:43,367

Now I can immediately write

the quadric function as

35

Student
view

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754029/

00:01:43,867 --> 00:01:47,033

$a(x - p)(x - q)$,

36

00:01:47,100 --> 00:01:50,167

and $(x - p)$

and $(x - q)$ are called factors,

37

00:01:50,233 --> 00:01:51,600

so this is the factorized form.

38

00:01:51,667 --> 00:01:53,167

Now be very wary

39

00:01:53,233 --> 00:01:56,533

that if p or q is a negative number,

40

00:01:56,600 --> 00:01:59,033

we still write $x - p$.

41

00:01:59,100 --> 00:02:02,933

So this then becomes $x - (-2)$

is $x + 2$ is a factor.

42

00:02:03,667 --> 00:02:06,433

Also, it's important to realize that

43

00:02:06,500 --> 00:02:11,167

y is positive sort of functionalize above

the x axis in this case

44

00:02:11,233 --> 00:02:14,133

for $x < p$

and $x > q$

45

00:02:14,400 --> 00:02:17,100

and it lies below for x between p and q.

Student
view

46

00:02:17,667 --> 00:02:20,933

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Now if a is negative, then it would be

the exactly the other way around.

47

00:02:21,000 --> 00:02:25,300

So keep in mind that there are sets

of values involved here.

48

00:02:26,033 --> 00:02:29,167

If I have this repeated root at p,

49

00:02:29,633 --> 00:02:33,867

then $(x-p)^2 \cdot a$ is a way

50

00:02:33,933 --> 00:02:36,000

of writing the quadratic equation.

51

00:02:38,400 --> 00:02:41,667

Now let's look at the symmetry properties

of the quadratic function.

52

00:02:41,733 --> 00:02:44,667

So we plot a general

quadratic function here

53

00:02:45,200 --> 00:02:47,933

and you can see that it has a definite

54

00:02:48,000 --> 00:02:50,000

turning point or a vertex.

55

00:02:50,333 --> 00:02:52,800

And of course here

the zeros, p and q again.

56

00:02:53,267 --> 00:02:56,367

Now turning point

we can give co-ordinates h and k

Student
view

57

00:02:56,433 --> 00:02:59,433

and then $x = h$ is the axis

of symmetry

58

00:02:59,500 --> 00:03:02,400

around which the quadratic function

is distributed.

59

00:03:03,267 --> 00:03:04,567

And now we can ask the question,

60

00:03:04,633 --> 00:03:08,533

how are h and k related

to our previous coefficient?

61

00:03:08,633 --> 00:03:10,833

So for $y = ax^2 + bx + c$

62

00:03:10,933 --> 00:03:13,567

the general form $h = -\frac{b}{2a}$.

63

00:03:14,200 --> 00:03:19,133

And for $y = a(x - p)(x - q)$,

64

00:03:19,200 --> 00:03:22,200

h is given by $\frac{p+q}{2}$,

65

00:03:22,267 --> 00:03:25,767

which as you can see

is the arithmetic mean of the zeros

66

00:03:25,833 --> 00:03:27,300

to roots p and q.

67

00:03:27,367 --> 00:03:29,633

And that has all to do

with this wonderful symmetry

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68

00:03:29,767 --> 00:03:31,100

property of the quadratic function.

69

00:03:32,467 --> 00:03:35,467

Now in the x value of the coordinates

and we can also ask ourselves

70

00:03:35,533 --> 00:03:37,900

what is the y value of the vertex?

71

00:03:40,100 --> 00:03:42,333

Now we can substitute, for example,

72

00:03:42,733 --> 00:03:45,733

$x = -\frac{b}{2a}$

73

00:03:45,800 --> 00:03:51,433

into the general quadratic function,

$y = ax^2 + bx + c.$

74

00:03:51,500 --> 00:03:53,467

And this then gives us a solution

75

00:03:54,033 --> 00:03:58,233

$y = -\frac{b^2}{4a} + c$

76

00:03:58,300 --> 00:04:02,267

as the y coordinate of the vertex

in terms of a, b, and c.

77

00:04:02,333 --> 00:04:06,533

So the vertex given by h, k

78

00:04:06,633 --> 00:04:11,100

we can write the quadratic equation

as $a(x - h)^2 + k$.

Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/vertex-form-of-a-quadratic-function-id-27460/print/) Assign

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79

00:04:11,167 --> 00:04:12,467

And this is called the vertex form

80

00:04:12,533 --> 00:04:16,033

so that you can see when x = h, y = k.

81

00:04:16,100 --> 00:04:18,367

And that is a look at quadratic functions.

Example 2

★★☆

A parabola has x -intercepts $(1, 0)$ and $(4, 0)$ and passes through point $(6, -12)$. Find the equation of the quadratic function and express it in standard form.

Steps	Explanation
$y = a(x-1)(x-4)$	You are given the x -intercepts, so first write the function in factorised form with $p = 1$ and $q = 4$ (o vice versa).
	To find the value of a , use the fact that the point $(6, -12)$ lies on the graph.
$-12 = a(6-1)(6-4)$ $-12 = a(5)(2)$ $10a = -12$ $a = -\frac{12}{10}$ $a = -\frac{6}{5}$	Substitute the x - and y -coordinates of the given point into the equation. Then solve for a .
The equation of the quadratic function in factorised form is $$y = -\frac{6}{5}(x-1)(x-4)$$	

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Steps	Explanation
$$y = -\frac{6}{5}\left(x^2 - 5x + 4\right) = -\frac{6}{5}x^2 + 6x - \frac{24}{5}$$ So in standard form the quadratic function is $y = -\frac{6}{5}x^2 + 6x - \frac{24}{5}$	Expand the brackets to get the quadratic in standa form.

Being able to appreciate the different forms in which quadratic functions can be represented, both as equations and as graphs, will be of great benefit throughout this course.

Reflect on what information each quadratic form gives you.

Discuss what makes one quadratic form better than another when modelling with quadratic functions.

3 section questions

2. Functions / 2.5 Introduction to modelling

Modelling with quadratic functions: change and motion

Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/modelling-with-quadratic-functions-id-27462/print/) Assign

Applications of quadratic functions

Various real-life situations, such as a rocket launch or the design of a product, can be modelled with quadratic functions. In this section, you will use your GDC, and what you learned in the previous sections, to graph quadratic functions and answer questions about quadratic models.

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Making connections

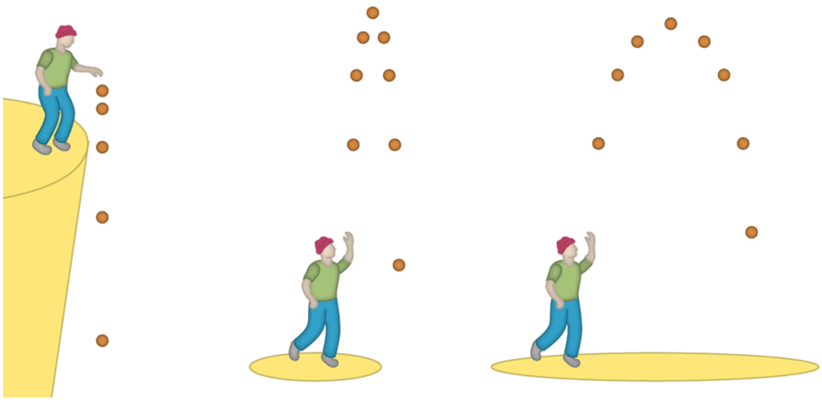
Recall from section 2.5 (/study/app/m/sid-122-cid-754029/book/the-big-picture-id-27457/) that a quadratic function can be expressed in various forms:

- Standard form: $ax^2 + bx + c = 0$, where $a \neq 0$.
- Vertex form: $y = a(x - h)^2 + k$, where the vertex has coordinates (h, k) .
- Factorised form: $y = a\left(x - p\right)\left(x - q\right)$, where $(p, 0)$ and $(q, 0)$ are the x -intercepts.

Reflect on the different forms of quadratic functions and discuss whether one form is more useful than another in different contexts

Projectile motion

A projectile is an object that is launched up in the air and falls under the influence of gravity. If an object is thrown straight up, dropped down from a certain height, or thrown up at an angle, a quadratic function can be used to describe the height of the object as a function of time. Factors such as the height from which the object was launched, the initial velocity of the object and the force of gravity are incorporated into the quadratic model.



More information

The image is a diagram illustrating the path of a ball thrown from a high point and following a parabolic trajectory toward a lower point where two people stand. The ball is depicted at multiple points along its curved path, showing its motion through the air. This trajectory demonstrates a typical projectile motion, influenced by gravity, where the

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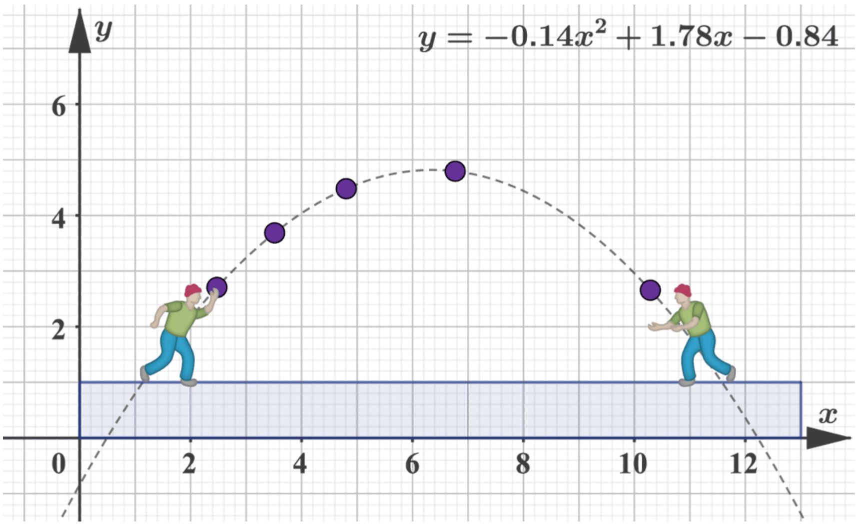
ball ascends to a peak and then descends. The initial person is positioned at the top of a platform, launching the ball, and the two other figures are on the ground, illustrating the ball being caught. The diagram visually represents the concepts discussed about projectile motion in the text.

[Generated by AI]

The figure below shows the path of a ball as thrown from one person to another.



More information

The image is a graph depicting the parabolic path of a ball as it is thrown from one person to another. The graph is overlaid on a grid background with X and Y axes marked. The Y-axis ranges from 0 to 9, indicating height, while the X-axis goes from 0 to 13, denoting distance or time in increments. The path of the ball is illustrated as a dotted line with purple markers tracking its trajectory, forming an arch or parabola. There is an equation depicted at the upper right corner: g = -0.14t² + 1.85t - 0.34, which models the path of the ball. Two individuals are positioned at either end, one is throwing the ball and the other is receiving.

[Generated by AI]

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The path of the ball can be modelled by the function:

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$$y = -0.14x^2 + 1.78x - 0.84$$

where:

- x represents the horizontal distance, in metres, that the ball travelled through the air
- y represent the vertical height, in metres, of the ball above the ground.

How can you find the distance between the two players?

What is the maximum height of the ball?

To investigate this situation, you could plot the graph of the function that models the path of the ball. By finding certain points on the graph, you could determine information about the path of the ball.

Example 1

★★☆

Suppose that the height, h, in metres, of a projectile above the ground at time t seconds after its launch is given by the quadratic function $h\left(t\right) = 2.25 + 6.55t - 1.75t^2$.

a) From what height was the projectile launched?

b) What is the maximum height reached by the projectile?

c) At what time will the projectile be at its initial height again?

d) At what time after launch does the projectile reach ground level?

⚠ Be aware

You may have noticed that this equation does not model a projectile on Earth. Our model suggests a gravitational acceleration $2 \times 1.75 = 3.5$, which is not the gravitational acceleration on Earth. You can think of this as a hypothetical model or a model on some other planet. You can also think of this as a model where forces other than gravity affect the motion.

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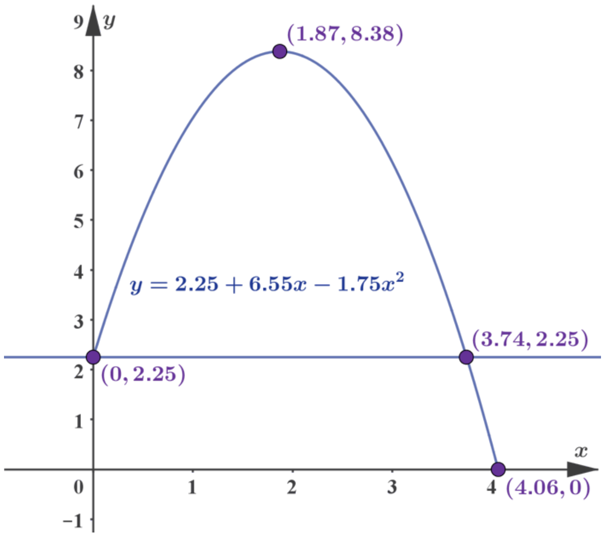
	Steps	Explanation
		Use your GDC to obtain the grapl of $y = 2.25 + 6.55x - 1.75x^2$. Please note that the independent variable of the function is time t . However, on the GDC you use x .
a)	$h(0) = 2.25\text{ m}$.	This is the initial height of the projectile at time $t = 0$.
b)	The maximum height reached by the projectile is 8.38 m .	Using a GDC, you can find the y -coordinate of the maximum point (vertex). It is 8.38 correct to 3 significant figures.
c)	The projectile reaches its initial height again at 3.74 s after launch.	Using a GDC, find the x value of the intersection point between th parabola $y = 2.25 + 6.55x - 1.75x^2$ and the line $y = 2.25$. Rejecting the $x = 0$ solution (which correspond to the initial time $t = 0$) , we are left with the solution $x = 3.74$.
d)	The projectile reaches ground level at 4.06 s .	Using a GDC, you can find the positive x -intercept of the parabola $y = 2.25 + 6.55x - 1.75x^2$ to be 4.06 .

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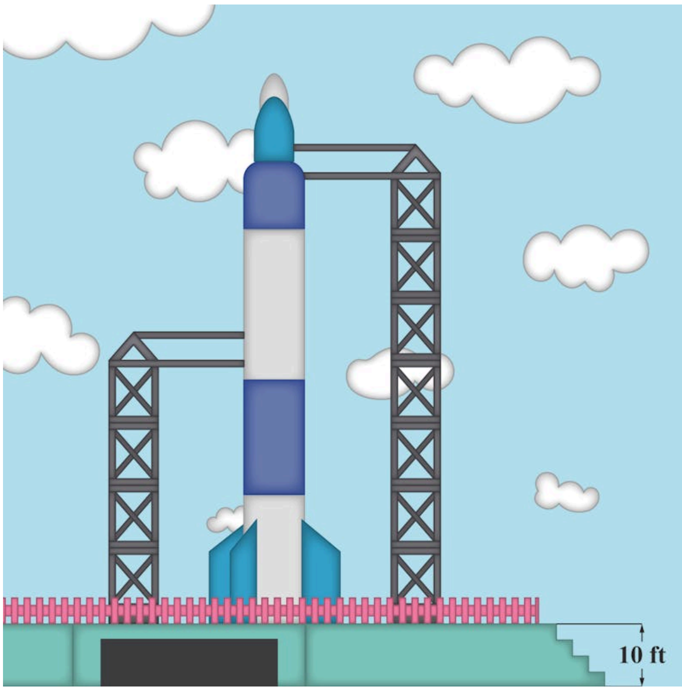
Be aware

The graph of the function might give the impression that the projectile moves in the horizontal direction, but you are not told whether or not this is the case. Remember that the graph of the function actually shows height against time.

Example 2

★★☆

A model rocket is launched straight upwards from a platform 10 feet above ground level. The rocket leaves the platform at an initial velocity of 64 feet per second. The height, h feet, above ground level as a function of time, t seconds after launch, is modelled by the quadratic function $h\left(t\right) = -16t^2 + 64t + 10$. Use the graph of the parabola to determine:



a) the maximum height reached by the rocket

b) the amount of time it took the rocket to reach the maximum height

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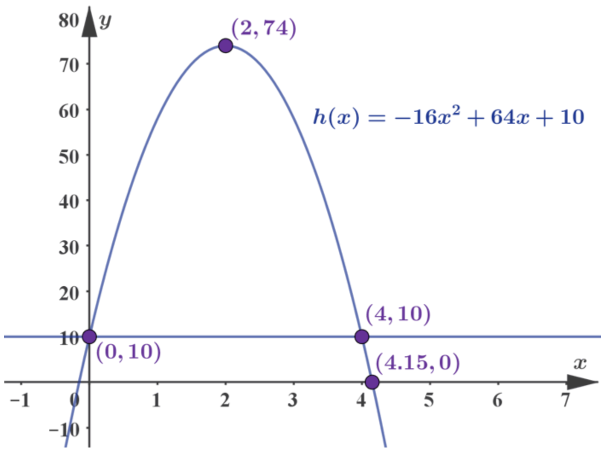
view

c) the amount of time it took the rocket to come back to its initial height again

d) the total amount of time the rocket was in the air.

	Steps	Explanation
		Use your GDC to obtain the g of $y = -16x^2 + 64x + 10$.
a)	The rocket's maximum height is 74 feet above ground level.	The maximum height is the y coordinate of the vertex (max turning point).
b)	The amount of time it took the rocket to reach its maximum height is 2 seconds.	The time taken to reach the maximum is the x -coordinate the vertex.
c)	It took 4 seconds to the rocket to reach its initial height again.	This is the positive x -coordin of the point of intersection between the parabola and the $y = 10$.
d)	The rocket was in the air for 4.15 seconds.	The time at which the rocket reached the ground is the pos x -intercept of the parabola, this is the total amount of time the rocket spent in the air.

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Example 3

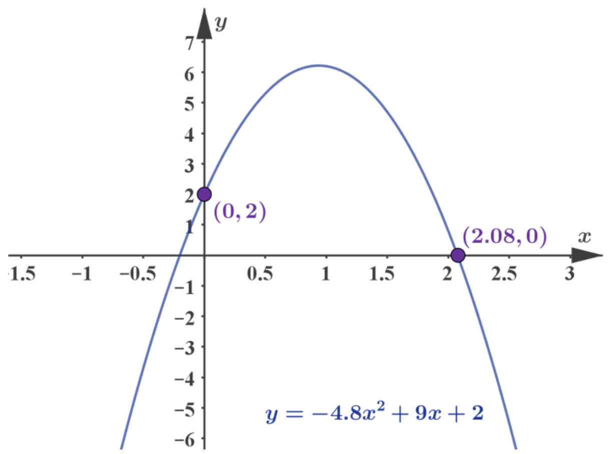
★★☆

A man who is 2 metres tall throws a ball straight up in the air; the ball's height above the ground at time t seconds is $h = 2 + 9t - 4.8t^2$ m .

a) Find the time (in seconds) when the ball reaches its maximum height.

b) Find the maximum height of the ball.

c) After how much time does the ball hit the ground?

	Steps	Explanation
		Use the GDC to obtain the grap of $y = 2 + 9x - 4.8x^2$.
a)	The ball reaches its maximum height 0.938 s after it was thrown.	The time at which the ball reaches its maximum height is the x -coordinate of the vertex.
b)	The maximum height that the ball reached is $y = 6.22$ m above the ground.	The maximum height is the y -coordinate of the vertex.

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	Steps	Explanation
c)	The ball hits the ground after 2.08 s .	The time at which the ball reaches the ground is the positive x -intercept of the parabola.

International Mindedness

Galileo Galilei (1564—1642), who was born in Pisa, Tuscany, was a philosopher and mathematician who a played significant role in the scientific revolution of that period. Galileo experimented with paths of projectiles and attempted to describe falling objects using mathematics.

A portrait of Galileo
Credit: ilbusca Getty Images

Galileo took various measurements of rolling a ball off the surface of a table and marking where it lands according to how fast the ball was going. Carry out research to answer the following questions:

- What experiments did Galileo perform to model projectile motion?
- What did Galileo discover by comparing the motion of a rolling ball that falls off the surface of a table to the motion of a ball that is just dropped vertically from the same height?

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- What quadratic models did Galileo form?

Fluid mechanics

Fluids and their motion play a huge role in our lives. The areas of application range from industrial liquid transport, aircraft control systems and transport ship elevation to everyday uses such as hidden underground car parking with hydraulic lifting. The various ways in which fluids flow are influenced by many factors, and quadratic models are often used to model their behaviour.

Example 4

★★☆

A tank is filled with water. A valve at its base is opened and the tank drains in such a way that the height, h centimetres, of water remaining in the tank at time t hours after the draining began is given by the function $h(t) = (0.12t - 8.25)^2$.

a) What is the initial water level?

b) What is the water level 10 hours after draining began?

c) When, to the nearest hour, will the tank be empty?

d) When, to the nearest tenth of an hour, will the tank be one-third full?

	Steps	Explanation
a)	T he initial water level correct to 3 significant figures is $$h(0) = (0.12(0) - 8.25)^2 = 68.1 \text{ cm}.$$	The initial water level is the heig at $t = 0$.
b)	The water level after 10 hours is $$h(10) = (0.12(10) - 8.25)^2 = 49.7 \text{ cm}$$.	Evaluate function $h(t)$ at $t = 1$.

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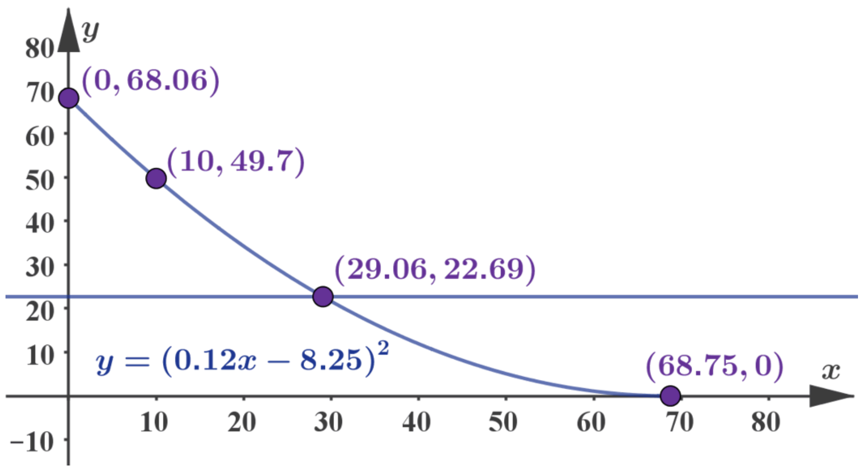
	Steps	Explanation
c)	To the nearest hour, i t takes 69 hours for the tank to drain.	The tank will be empty when the water level, h , is equal to zero. That corresponds to an x -intercept of the parabola. Use your GDC to find the x -intercept, which is at $(68.75, 0)$.
d)	One-third of the initial height is $\frac{1}{3}8.25^2 = 22.69$. Using a GDC gives a time of 29.1 hours (to 3 significant figures) for the tank to drain to $\frac{1}{3}$ of its initial level.	Use your GDC to find the x -coordinate of the point of intersection between the graphs $y = (0.12x - 8.25)^2$ and $y = 29.09$.
	The graph of this parabola and the answers to parts a)—d) are shown below.  The curve $y = (0.12x - 8.25)^2$, which models the draining of a water tank, where y c is the water level, h , and x represents time, t .	

3 section questions

Student view

2. Functions / 2.5 Introduction to modelling

Cubic functions

Polynomial functions

Important

Let n be a non-negative integer. A polynomial function has general form:

$$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0.$$

Each a_i is a coefficient and can be any real number. Each product $a_i x^i$ is a term of a polynomial function.

- Is the standard form of a quadratic function a special case of polynomial function?

Example 1

★☆☆

Which of the following are polynomial functions?

a) $f(x) = 2x^3(3x+4)$

b) $g(x) = -x\left(x^3 - 2\right)$

c) $h(x) = 5\sqrt{x} + 2$

	Steps	Explanation
a)	$f(x)$ is a polynomial function.	f can be written as $f(x) = 6x^4 + 8x^3$.
b)	$g(x)$ is a polynomial function.	g can be written as $g(x) = -x^4 + 2x$.
c)	$h(x)$ is not a polynomial function.	The exponent of variable x is not an integer.

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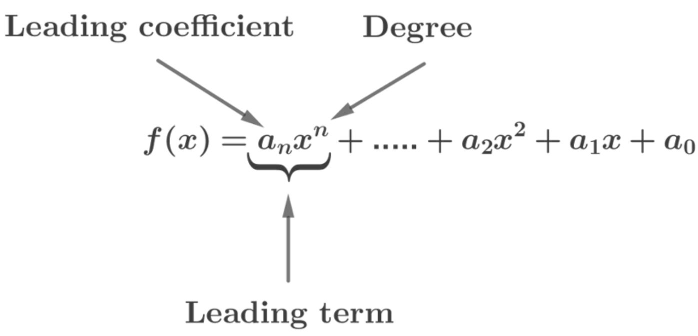
The figure below shows the terminology used to describe polynomial functions.



More information

The image shows a labeled diagram explaining the terminology used in polynomial functions. The polynomial function is written as f(x) = a_nx^n + ... + a_2x^2 + a_1x + a_0. Arrows point to specific parts of the equation: 'Leading coefficient' points to a_n, 'Degree' points to the exponent n, and 'Leading term' points to a_nx^n. Each label helps to identify the components of the polynomial function.

[Generated by AI]

Be aware

- The degree of the polynomial is the highest power of the variable.
- The leading term is the term containing the highest power of the variable.

Cubic functions

Important

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The standard form of a cubic function is $f(x) = ax^3 + bx^2 + cx + d$, where and $a \neq 0$.

Notice that cubic functions are polynomial functions of degree 3 .

In the following applet, you can visualise the graph of a cubic function that is determined by the parameters a, b, c and d .

Activity

You can see below the graphs of cubic functions of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$ and $a \neq 0$.

Use the sliders to adjust the values of parameters a, b, c, and d and observe the graphs.

Interactive 1. Graph of a Cubic Function Determined by the Parameters.

More information for interactive 1

This interactive allows users to explore cubic functions of the form

$f(x) = ax^3 + bx^2 + cx + d$

where *a*,*b*,*c* and *d* are real numbers, and $a \neq 0$. By adjusting sliders for each parameter within the range of -5 to 5, users can observe how changes in these coefficients affect the shape and behavior of the graph.

The parameter *a* determines the steepness and direction of the curve, while *b* and *c* influence its

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curvature and overall slope. The parameter d shifts the graph vertically, affecting the y-intercept. As users manipulate the sliders, they can observe key characteristics of cubic functions, including the number and nature of turning points (up to two), the number of real roots (which may be one, two, or three), and point symmetry about the graph's inflection point. The y-intercept, located at $(0, d)$, provides additional insight into the function's vertical position.
This interactive provides a dynamic, hands-on way to understand how each coefficient contributes to the shape and properties of a cubic function's graph.

Describe how the parameters affect the shape of the graph. You should consider features such as:

- symmetry
- turning points
- number of roots
- y -intercept.

In the activity above, notice that not all cubic functions have a maximum and a minimum turning point. Also, cubic functions can have one, two or three roots.

Exam tip

When sketching the graph of a cubic function, remember to show the following features:

- Draw, label and scale the axes.
- Show the coordinates of the x - and y - intercepts.
- Show the coordinates of the maximum and minimum points, if any.

Example 2

★☆☆

Sketch the graph of $y = (x + 1)^3$ showing all relevant features.

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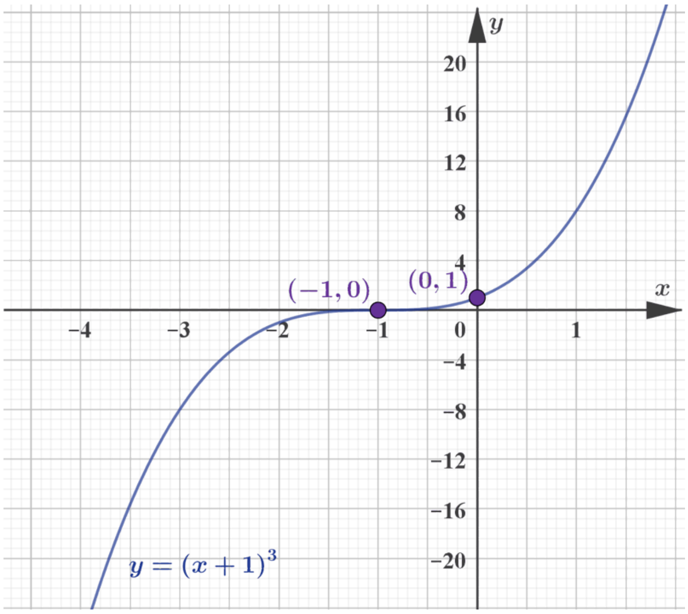
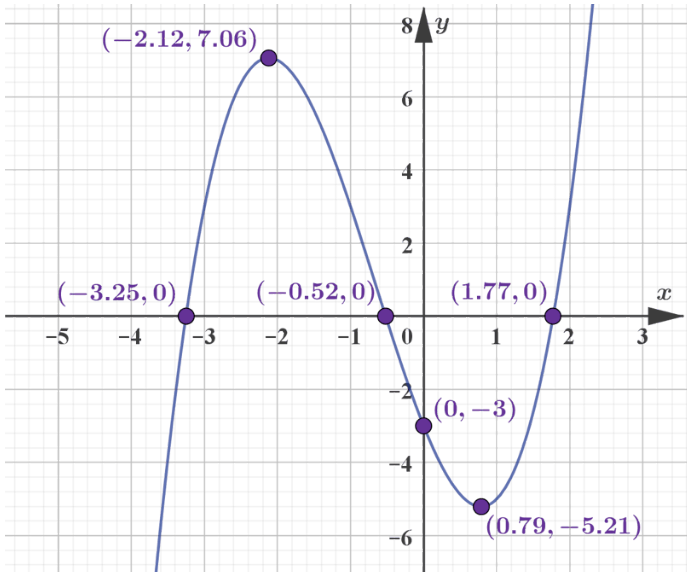
view

Steps	Explanation
	Use your GDC to obtain the graph of $y = (x + 1)^3$. Show the axes intercepts on your graph.

Example 3

★☆☆

Sketch the graph of $y = x^3 + 2x^2 - 5x - 3$ showing all relevant features.

<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>


</td><td>
Use your GDC to obtain the graph of $y = x^3 + 2x^2 - 5x -$.
Show the axes intercepts on your graph and the local maximum and minimum points
Note that on this graph the coordinates are rounded to two decimal places. On your IB exa you will need to round your answers to three significant figures.
</td></tr>
</table>

Power functions

Important

A power function is a function that can be expressed in the form $f(x) = ax^n$, where a and n are real numbers. The constant a is known as the coefficient.

In the applet below, you can visualise the graphs of power functions of the form $y = ax^n$.

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Interactive 2. The Graphs of Power Functions.

More information for interactive 2

This interactive allows users to explore power functions of the form $y = ax^n$, where a and n are adjustable parameters. Users can change the value of a (the coefficient) and n (the exponent) using sliders, which range from 0 to 10. By adjusting these parameters, users can observe how the graph of the power function changes.

For even integer values of n, the graphs exhibit symmetry about the y-axis and share similar shapes. As the exponent n increases, the graphs tend to flatten near the origin (around $x = 0$) and become steeper as x moves away from the origin. For odd integer values of *n*, the graphs are symmetric about the origin and also become steeper as n increases. The parameter '*a*' scales the graph vertically, making it steeper or flatter depending on its value.

This interactive tool helps users understand how the exponent '*n*' and coefficient '*a*' influence the shape, steepness, and symmetry of power functions. By experimenting with different values, users can draw conclusions about the behavior of power functions and their graphical representations.

Activity

Use slider 'n' to adjust the value of the exponent of the power function and observe the corresponding graphs. What do you conclude about the graphs of power functions?

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Notice that all power functions with even integer powers have similar graphs. However, as the power increases, the graphs flatten out close to the origin and become steeper away from the origin.

Important

For power functions with even integer powers and positive coefficient

- as the input increases or decreases, the output values are positive and become very large.

Using mathematical notation, you could write:

as $x \to \pm\infty$, $f(x) \to \infty$.

Also, notice that all power functions with odd integer powers have similar graphs. As the power increases, the graphs flatten close to the origin and become steeper away from the origin.

Important

For power functions with odd integer powers and positive coefficient

- as the input increases, the output values are positive and become very large
- as the input decreases, the output values are negative and become very large.

Using mathematical notation, you could write:

as $x \to +\infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$.

Activity

Consider the following questions:

- Do power functions present any kind of symmetry?

Student view

- What is the effect of changing the value of coefficient a on the graph of a power function?

3 section questions

Exponential functions

Exponential functions and their properties

Have you ever wondered how archaeologists or palaeontologists estimate the ages of archaeological artefacts, or what type of mathematical models are used by geologists to study the evolution of the Earth?

An excavation site

Credit: BertBeekman Getty Images

Exponential functions are a special type of function that have a large number of modelling applications. These range from financial applications such as calculating compound interest, to scientific applications in the context of radioactive decay, the spread of disease,

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the decay of the concentration of a drug and Newton's law of cooling. Exponential functions are also used in geography, where they can be used to model population growth or decline.

Exponential functions are functions where the independent variable, say x , is the exponent of a positive number, i.e. $f(x) = a^x$, $a > 0$.

✓ **Important**

The general equation of an **exponential function** f **with base** a is denoted by $f(x) = ka^x + c$, where $a > 0$, $a \neq 1$ and $x \in \mathbb{R}$.

Examples of exponential functions are: $f(x) = 2^x$, $g(x) = -2 \cdot 3^x$, $h(x) = 25^x + 1$.

⚠ **Be aware**

Note that when the base $a = 1$, the exponential function becomes the constant function and thus it is of no interest here.

The graph of an exponential function

In your exploration of exponential functions of the form $f(x) = ka^x + c$, you will distinguish between two cases:

- $0 < a < 1$
- $a > 1$.

In the applet below, you can visualise the graphs of some exponential functions.

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Interactive 1. The Graphs of Some Exponential Functions.

More information for interactive 1

This interactive allows users to explore exponential functions of the form $f(x) = ka^{x} + c$ and $g(x) = ka^{-x} + c$ by adjusting the parameters a, k, and c, each ranging from -5 to 5.

As users change these values using sliders, the graph updates in real time to show how the function's shape is affected. When the base a is greater than 1, the function $f(x)$ demonstrates exponential growth, rising rapidly as x increases. When the base is between 0 and 1, the function exhibits exponential decay, decreasing as x increases. If a equals 1, the function becomes a horizontal line, since there is no exponential effect. The parameter k controls vertical scaling, determining how steep the curve is, while c shifts the graph vertically, setting the horizontal asymptote at $y = c$. Users can toggle between $f(x)$ and $g(x)$ to observe that $g(x)$ is a reflection of $f(x)$ over the y-axis. This illustrates how reversing the sign of the exponent transforms exponential growth into decay, and vice versa, while keeping the same vertical stretch and asymptote.

For example, setting $a = 2.1$, $k = 2.4$, and $c = 1.5$ produces the function $f(x) = (2.4) \cdot (2.1)^{x} + 1.5$, which grows rapidly and levels off near $y = 1.5$. Toggling to $g(x) = (2.4) \cdot (2.1)^{x} + 1.5$ shows a symmetric decay curve, decreasing from left to right but sharing the same asymptote and steepness.

This interactive tool provides an engaging and intuitive way for learners to understand how exponential functions behave and how their algebraic parameters control growth, decay, and vertical shifts.

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Activity

Use the applet to visualise the graph of $f(x) = ka^x + c$ for different values of a.

Which graphs represent exponential growth? Which represent exponential decay? What happens when $a = 1$?

Click the box to show $g(x) = ka^{-x} + c$. How are the graphs of the functions $f(x) = ka^x + c$ and $g(x) = ka^{-x} + c$ related to each other?

Reflect on your discoveries and discuss how the equation of an exponential function informs you whether the graph is increasing or decreasing.

Important

For the exponential function $f(x) = ka^x + c, \quad a > 0$:

- the domain is $(-\infty, +\infty)$
- the range is (c, ∞) if $k > 0$ and $(-\infty, c)$ if $k < 0$
- the y -intercept is $(0, k + c)$.

If $a > 1$ and $k > 0$:

- the function is a **growing exponential** function; it **continuously increases**
- the $y = c$ line is a **horizontal asymptote** ; $f(x) \to c$ **asymptotically** as $x \to -\infty$.

If $0 < a < 1$ and $k > 0$:

- the function is a **decaying exponential** function; it **continuously decreases**
- the $y = c$ line is a **horizontal asymptote** ; $f(x) \to c$ **asymptotically** as $x \to +\infty$.

Notice that the graph of $f(x) = a^x, \quad a > 1$ is always positive and increasing in the entire domain $(0, \infty)$ of the function and as x increases, the graph gets steeper. Also, for $a > 0$ the graph of $y = ka^{-x} + c$ is a reflection of the graph $y = ka^x + c$ in the y -axis.

Student
view

Making connections

Which type of exponential equation would you choose to represent the future value of an investment or the value of a car after a period of time?

Example 1

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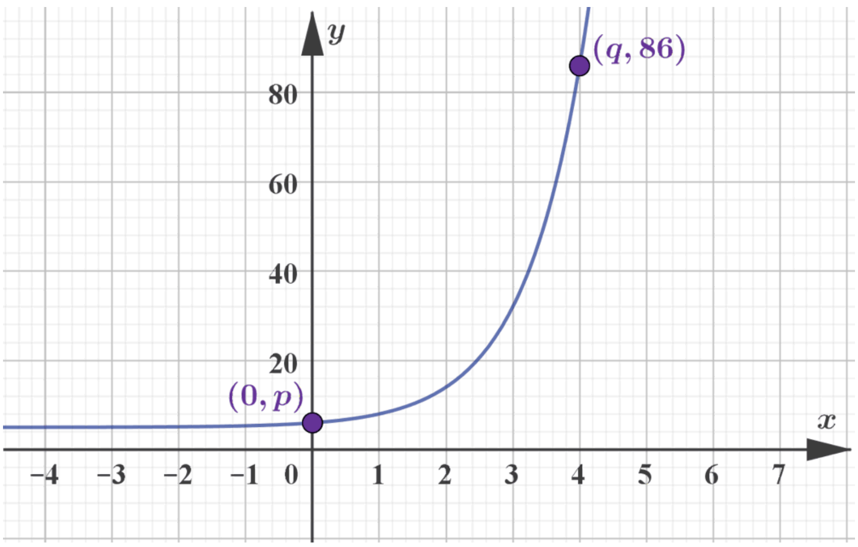
The graph of the function $f(x) = 3^x + 5$ is shown below.



More information

The image depicts a graph of the exponential function (f(x) = 3^x + 5). The graph is plotted on a standard Cartesian coordinate grid with both X and Y axes clearly marked. The X-axis is labeled with values ranging from -4 to 8, and the Y-axis is labeled from 0 to 80 in increments of 10. The curve begins gradually from the left and sharply increases as it moves to the right, demonstrating the exponential nature of the function. Key points are marked on the curve at coordinates (0, p) and (q, 86), where the challenge is to find the value of (p). The point at (q, 86) is specifically highlighted as it lies vertically aligned with a position indicative of significant increase where q approximates 3, indicating a function value close to 86.

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view

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a) Find the value of p .

b) Find the value of q .

c) Write down the equation of the horizontal asymptote.

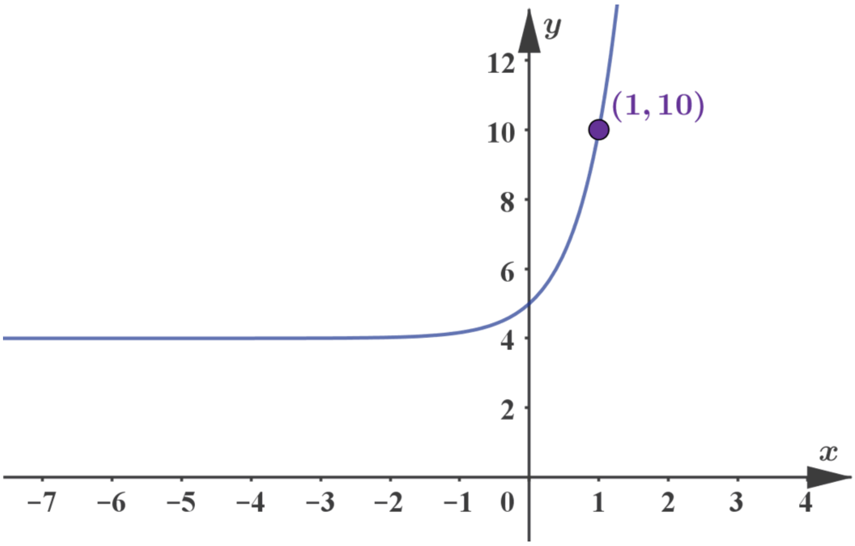
	Steps	Explanation
a)	$p = 6$	p is the y -coordinate of the y -intercept, that is, $f(0) = 3^0 + 5 = 1 + 5 = 6$.
b)	$q = 4$	To find q you need to find x such that $f(x) = 86$. Solve the equation, as follows: $$\begin{aligned} 86 &= 3^x + 5 \\ 81 &= 3^x \\ x &= 4. \end{aligned}$$
c)	$y = 5$	On the graph of the function $y = ka^x + c$, the horizontal asymptote is the line $y = c$.

Example 2

★★☆

The graph of the function $f(x) = c + ka^x$ is shown below. Find the values of $a,\ k$ and c .

Student view



More information

The graph depicts the function (y = c - a^x) plotted on a Cartesian coordinate system. The X-axis is labeled from -7 to 4, while the Y-axis ranges from 0 to 12.

The curve is shown moving from a flat segment on the left and sharply rising upwards as it moves to the right. A specific point on the curve is marked at the coordinates (1, 10), which is highlighted in purple.

This implies that at (x = 1), the function has a value of (y = 10). The points on the curve move upwards more steeply past (x = 1).

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Steps	Explanation
$c = 4$	As seen from the graph, the exponential function has horizontal asymptote $y = 4$ and thus $c = 4$.

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Steps	Explanation
$k = 1$	The y -intercept of the graph is $(0, 5)$ and thus: $$\begin{aligned} f(0) &= 5 \\ 4 + ka^0 &= 5 \\ ka^0 &= 1 \\ k &= 1. \end{aligned}$$
$a = 6$	The graph of the function $f(x) = 4 + a^x$ passes through poi $(1, 10)$ and thus: $$\begin{aligned} f(1) &= 10 \\ 4 + a^1 &= 10 \\ a &= 6. \end{aligned}$$

Making connections

The shape of the exponential graph is very much like the shape of the graph of a geometric sequence. Indeed, like the link between linear graphs and arithmetic sequences, an exponential graph can be thought of as a geometric sequence but without any gaps. The figure below shows the graph of a geometric sequence.

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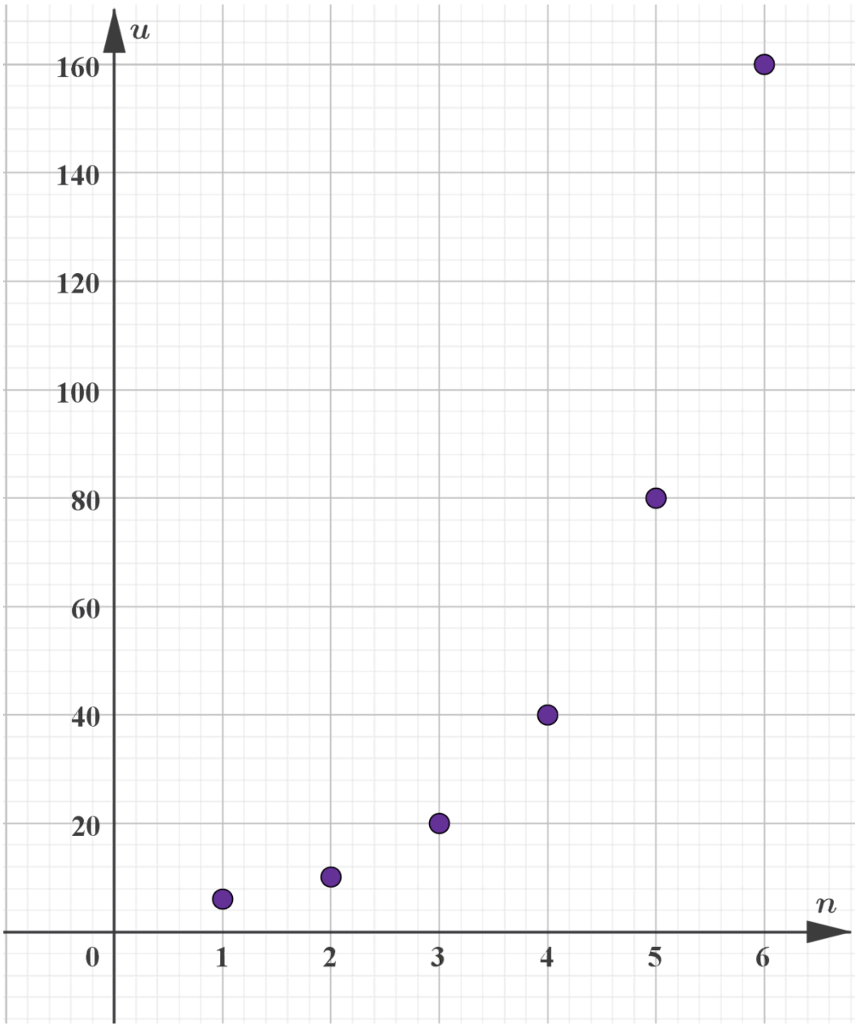

More information

The image shows a graph representing a geometric sequence. The X-axis is labeled with numbers from 0 to 10, with intervals of 1. The Y-axis is labeled from 0 to 160, with intervals of 20.

There are five data points plotted on the graph: - At X=0, Y=10 - At X=2, Y=20 - At X=3, Y=40 - At X=5, Y=80 - At X=6, Y=160

The data points show an exponential increase, illustrating the characteristic shape of a geometric sequence. The alignment of the points reflects the exponential nature of the graph, with each subsequent point showing a doubling or more of the previous value in terms of Y. The background consists of a grid that helps in visualizing the increments along both axes.

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The natural exponential function

When working with the exponential function $f(x) = ka^x + c$, there is one specific value of the base a that is frequently used in different subject areas other than maths (science, engineering, economics, etc.).

International Mindedness

The constant e is a famous number in mathematics and is also known as the natural base e, the **natural number** e **, Euler's number** or **Napier's number.**

The natural base is the number e, whose value (rounded to 5 decimal places) is 2.71828.

Important

The natural exponential function is $f(x) = \mathrm{e}^x$.

Be aware

Euler's number e is an irrational number, meaning that it cannot be written as a fraction. It has an infinite number of decimal places, so you usually give an approximate answer when working with e. However, if you leave your answers in terms of e, they will be exact.

The graph of $f(x) = \mathrm{e}^x$ is one of **exponential growth** (as its base $\mathrm{e} > 1$) and the graph of $f(x) = \mathrm{e}^{-x}$ is one of **exponential decay** (as it can be written $f(x) = \left(\frac{1}{\mathrm{e}}\right)^x$ with its base being $0 < \frac{1}{\mathrm{e}} < 1$).

The figure below shows the graphs of the two functions along with their key features.

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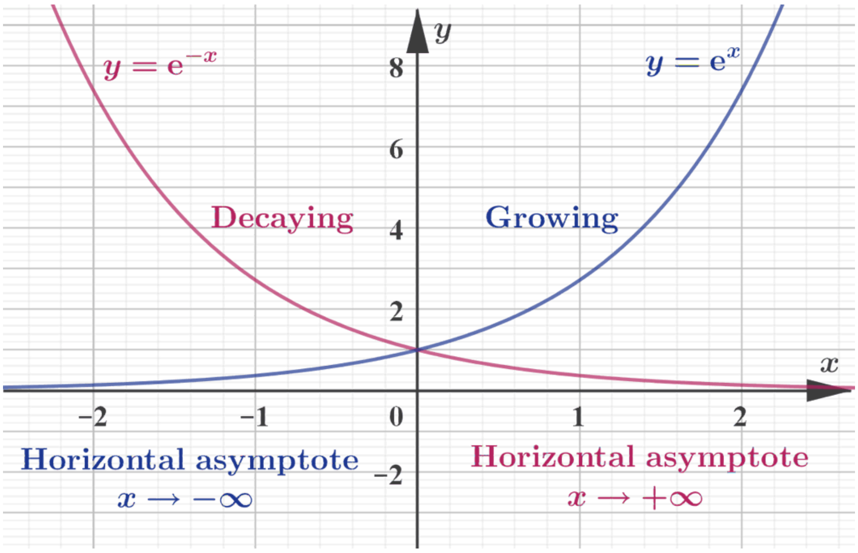

More information

The image is a graph showing the functions y = e^x and y = e^(-x) over a rectangular coordinate system. The X-axis is labeled with values ranging from -3 to 3, marked at intervals of 1. The Y-axis is similarly labeled from 0 to 9, marked at intervals of 1. There are two curves depicted:

1. The red curve represents the function y = e^(-x), described as 'Decaying'. This curve starts from a high value when x is negative and approaches the horizontal asymptote y = 0 as x increases.

2. The blue curve represents the function y = e^x, described as 'Growing'. This curve begins at y = 0 at x = 0, increasing rapidly as x becomes positive.

The graph features are pointed out with dashed lines, showing the horizontal asymptotes where each function approaches y = 0 as x goes to infinity or negative infinity, respectively. The labeled points and direction arrows indicate the overall behavior of each function as x progresses in both directions.

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Activity

For the natural exponential function of the form $f\left(x\right) = k\mathrm{e}^{x} + c$, investigate how the parameters k and c affect the shape of the graph of the function.

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Example 3

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What is the range of the function $f(x) = 2e^x + 2$?

Steps	Explanation
$R_f = \{y \mid y > 2\}$	The function $f(x) = 2e^x + 2$ is increasing and positive with horizontal asymptote $y = 2$. Therefore, $f(x) > 2$ in the entire domain of the function and thus $\{y \vert y > 2\}$.

Important

By introducing a different parameter, the exponential function with base e can also be used as a general exponential model, not just as a special case of the model of the form $ka^x + c$.

Since for any $a > 0$ there is an r such that $a = e^r$, the exponential model $ka^x + c$ can also be written in the form $ke^{rx} + c$.

International Mindedness

Historically, the number e comes into mathematics in a very dubious way in the 17th century . Various mathematicians came close to e many times around this period without directly calculating or recognising it as anything out of the ordinary. The first occurrence of e appears in John Napier's work. He was a Scottish mathematician, physicist and astronomer who, in 1619, in an appendix to his work on logarithms, included a table giving the natural logarithms of various numbers. The number e also makes many appearances in the calculus work of mathematicians such as Dutch physicist and mathematician Christiaan Huygens (1661) and German mathematician Nicolas Mercator (1668).

6 section questions

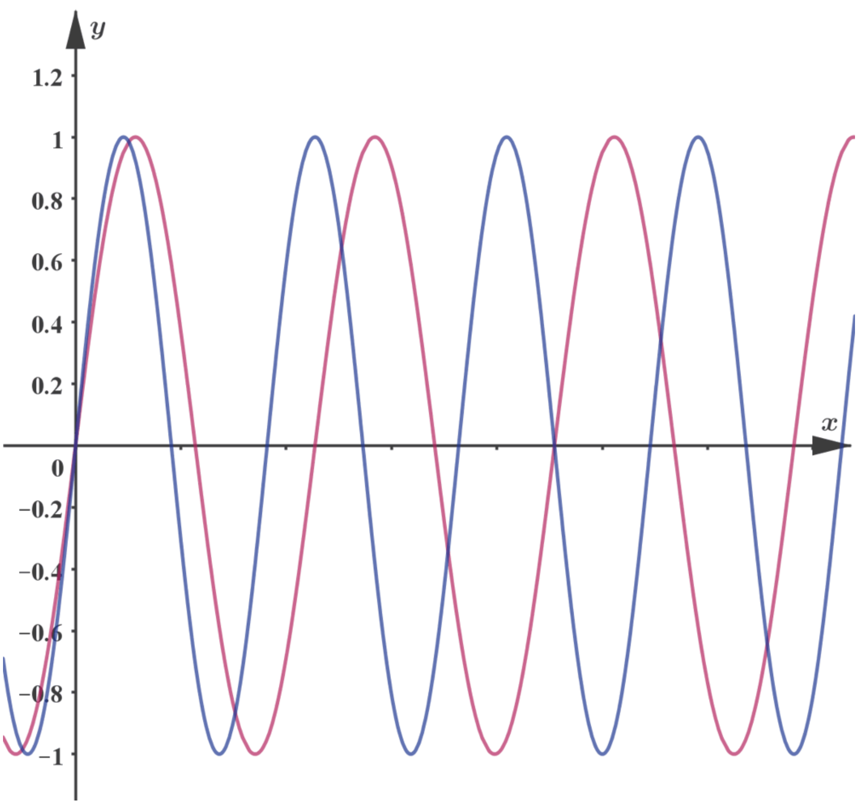
Student view

Trigonometric functions

Periodicity

In the physical world around us, there are many phenomena that seem to go through a repetitive and predictable cycle. Examples of real-life phenomena that repeat after a certain interval of time are radio waves, musical tones, electrical currents or the low and high tides of the oceans.

Periodic functions are very useful for modelling data where the dependent variable repeats its values in regular intervals or periods. For example, when you listen to music or play an instrument, your ear responds to sound waves. The figure below shows the sound waves that you listen to when note A (blue curve) and note C (red curve) are played on a piano, one at a time.



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More information

The image is a graph illustrating the sound waves of musical notes A and C. The X-axis represents time or interval with no specific labels, while the Y-axis represents amplitude, ranging from -1 to 1. Both axes are labeled "x" and "y" respectively. There are two smooth, sinusoidal curves: one blue representing note A, and one red representing note C. These curves exhibit periodic behavior, intersecting periodically over the same time intervals, highlighting their similar frequency but different phase and amplitude characteristics. The curves demonstrate continuous, repeated patterns typical of periodic functions, indicating how sound waves manifest in the auditory experience of music.

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Observe that both graphs are continuous smooth curves that show a periodic behaviour. The functions repeat their values in regular intervals and their graphs are formed by the repetition of congruent shapes.

Important

A **periodic function** is a function that repeats its values at regular intervals or periods.

The **period** of a periodic function is the minimum length required for one full repetition or cycle.

In the activity below you can visualise a cycloid, which is defined as the curve traced out by a point on the circumference of a circle as it rolls along a flat surface.

Activity

Start the applet below and set the radius ‘r’ of the wheel equal to 1 and drag slider ‘t’ to move the wheel along the x -axis.

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Interactive 1. Graphical Representation of Periodic Function Using a Wheel.

More information for interactive 1

This interactive allows users to explore the cycloid, a classic example of a periodic function, by allowing them to manipulate key parameters and observe the resulting curve. Users can set the radius ‘r’ of the wheel and adjust the slider ‘t’ to move the wheel along the x-axis. By starting with the radius $r = 1$ and dragging the slider, the graph as the wheel completes one full rotation. The cycloid forms a series of arches, with point P tracing the path of the curve. As the wheel completes two full rotations, the cycloid pattern repeats, extending further along the x-axis. The coordinates of point P represent its position on the cycloid at any given moment. The cycloid is a periodic function, with its period directly related to the circumference of the wheel. Users can use the 'Zoom' buttons to control the view and better analyze the curve. By experimenting with different radii, users can formulate a rule that the period of the cycloid is proportional to the radius of the wheel, specifically equal to $2\pi r$.
This interactive exploration helps users understand the relationship between the wheel's radius and the cycloid's period, as well as the periodic nature of the cycloid.

- Describe the graph as the wheel completes one full rotation.
- What happens as the wheel completes two full rotations. Use the 'Zoom' buttons to control the view.
- What do the coordinates of point P represent?
- Does a cycloid represent a periodic function?
- Formulate a rule for the relationship between the period of the cycloid and the radius of the wheel.

Example 1

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Identify which of the following graphs show periodic behaviour and find the period.

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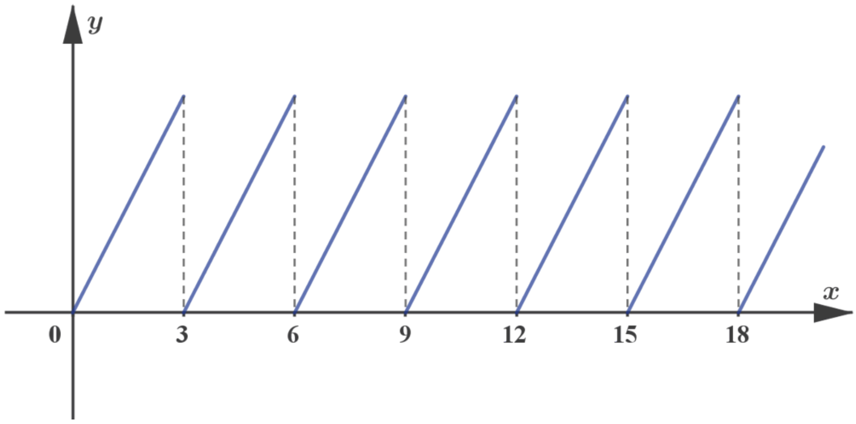

Graph 1

More information

The image depicts a graph illustrating periodic behavior with linear segments. The x-axis ranges from 0 to 18, marked at intervals of 3, 6, 9, 12, 15, and 18. The y-axis does not specify values, but it is marked by periodic peaks with a consistent slope. Each segment starts at the origin and rises linearly to a peak, then drops back to the x-axis at intervals of 3, creating a repetitive triangle wave pattern. This indicates periodic behavior, with each cycle completing every 3 units along the x-axis.

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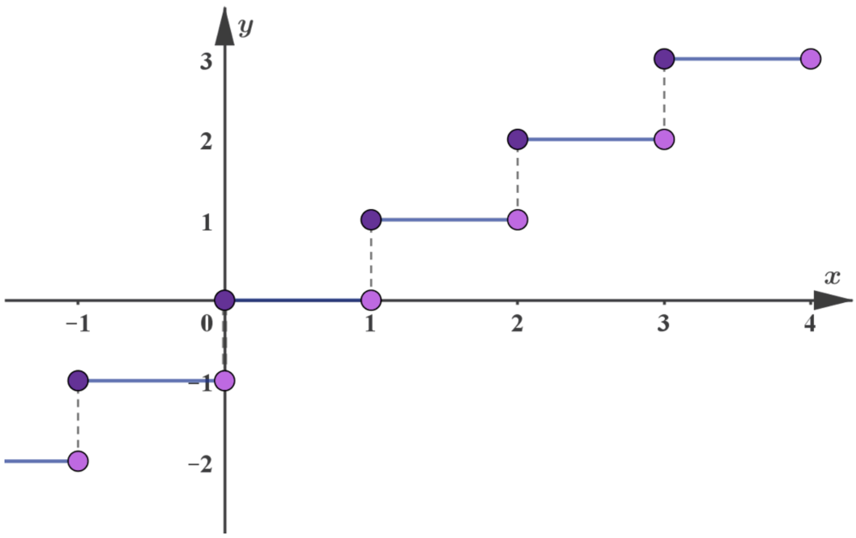

Graph 2

More information

The image is a graph depicting a step function on a coordinate plane. The x-axis ranges from -1 to 4 and the y-axis ranges from -2 to 3. The function features distinct steps or discontinuities.

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- The first step is at
- x = -1 to x = 0, y = -2.
- At x = 0, the function jumps to y = 0, represented by a circle at (0, 0).
- The next step from
- x = 0 to x = 1, y = 0.
- At x = 1, it jumps to y = 1, circle at (1, 1).
- Following this pattern, the function steps are:
- x = 1 to x = 2, y = 1.
- At x = 2, jump to y = 2, circle at (2, 2).
- x = 2 to x = 3, y = 2.
- At x = 3, jump to y = 3, circle at (3, 3).
- x = 3 to x = 4, y = 3.

Each jump is represented with an open circle at the initial point of the next step level, indicating discontinuity in the step function. The pattern reflects a typical ascending step function, with each increment representing a step increase in function value of 1 as x increases.

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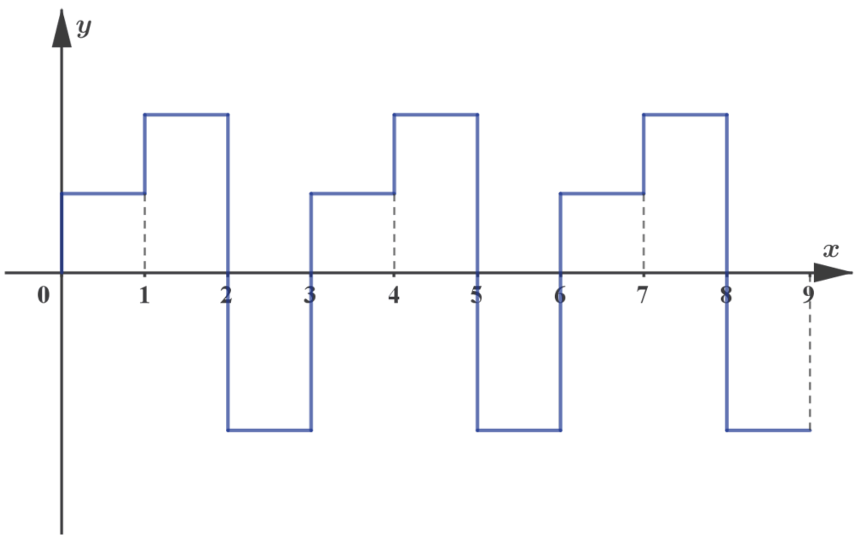

Graph 3

More information

The image is a graph depicting a step function plotted on a coordinate plane. It features horizontal steps alternating between two levels along the X-axis, which is labeled from 0 to 9. The Y-axis represents the function's value, with distinct steps occurring at integral points. The function increases at each odd integer and decreases immediately

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after. Specifically, steps occur between 0 and 1, 2 and 3, 4 and 5, 6 and 7, and 8 and 9 on the X-axis, each step characterized by an immediate rise followed by a plateau until the next integer.

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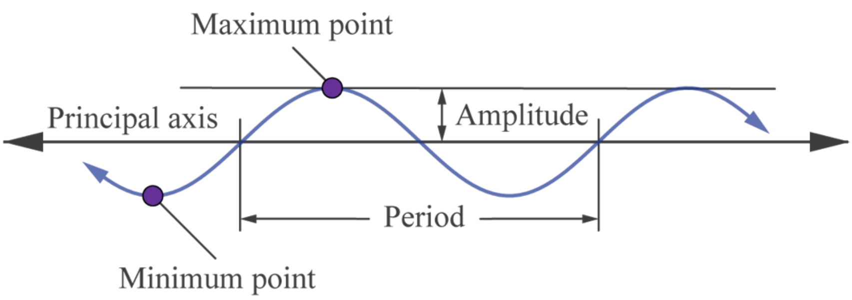
Graphs 1 and 3 are periodic.

The period of Graph 1 is 3 units.

The period of Graph 3 is 3 units.

In your study of periodic phenomena, you will consider trigonometric functions that show a wave pattern. Trigonometric functions have characteristics that render them very useful in modelling systems that display periodic behaviour. Systems with periodic behaviour can be described in terms of their period, principal axis, maximum point, minimum point and amplitude, as shown below.



More information

The image is a graph illustrating a waveform, resembling a sine wave, used to explain concepts related to periodic behavior in trigonometric functions. The waveform has labeled points and lines depicting various aspects:

- "Principal axis" is a horizontal line running through the center of the wave, which acts as a baseline around which the wave oscillates.
- The "Maximum point" is marked at the peak of the upward wave, indicating the highest point on the curve.

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- The "Minimum point" is indicated at the lowest part of the downward wave.
- "Amplitude" is labeled as the vertical distance from the principal axis to the maximum point, showcasing how high above or below the principal axis the wave peaks.
- "Period" is marked as the horizontal distance between two consecutive points on the wave where the wave pattern repeats itself.

The wave continues beyond the points shown, suggesting a continuous periodic pattern. This image is used to illustrate how certain characteristics of waves, such as amplitude and period, are key in modelling periodic functions.

[Generated by AI]

Important

- A **maximum point** ($\max$) occurs at the top of a crest. A **minimum point** ($\min$) occurs at the bottom of a trough.
- The wave oscillates about a horizontal line called the **principal axis** which has equation:
$$y = \frac{\max + \min}{2}$$
- The **amplitude** of a wave pattern is the magnitude of the distance of a maximum point or a minimum point from the principal axis, i.e.
$$\text{amplitude} = |\max - \text{principal axis}| = |\min - \text{principal axis}|$$
So,
$$\text{amplitude} = \left|\frac{\max - \min}{2}\right|$$

In your study of periodic functions, you will explore the trigonometric functions of sine and cosine.

Activity

The figure below shows a point P that lies on the unit circle. The corresponding radius at P forms an angle θ with the x -axis (in the anticlockwise direction). Use trigonometry to express the coordinates of point P in terms of angle θ .

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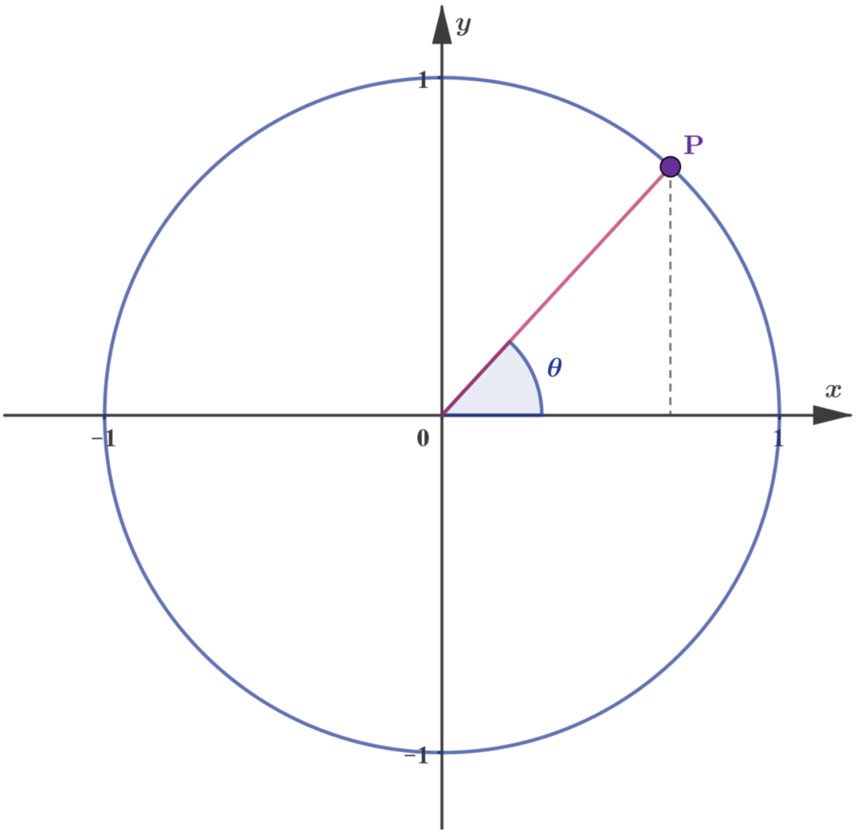

More information

The diagram illustrates a unit circle centered at the origin (0,0) of a Cartesian coordinate system. The circle is plotted with its circumference intersecting the x and y axes at points (1,0), (-1,0), (0,1), and (0,-1). A point "P" is marked on the circumference in the first quadrant, where a radius line from the origin to "P" makes an angle θ with the positive x-axis in an anticlockwise direction. This radius forms the hypotenuse of a right triangle with the x-axis and a vertical line dropping from "P" to the x-axis. The diagram is used to express the coordinates of point "P" in terms of angle θ using trigonometric functions, typically resulting in coordinates (cos(θ), sin(θ)).

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Be aware

The coordinates of any point on the unit circle can be expressed in terms of the angle that the corresponding radius at that point makes with the x -axis in the anticlockwise direction.

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In the following activities, you will investigate the real-life example of the London Eye – the colossal Ferris wheel which is the largest in Europe – and define the coordinates of a capsule as it rotates in terms of sine and cosine functions.

Sine function

Activity

In the following applet, the London Eye is represented by the unit circle. Drag slider ‘θ’ to start rotating a particular passenger capsule (red point) around the unit circle.

Interactive 2. The Graphical Representation of Sine Function Using London Eye.

More information for interactive 2

This interactive allows users to explore the sine function using the real-life example of the London Eye, represented by a unit circle $(r = 1)$. By dragging the slider for the angle θ, users can rotate a passenger capsule (represented by a red point) around the unit circle and observe how its vertical position changes. The points plotted on the coordinate system represent the vertical distance of the capsule from the horizontal axis as it moves, with the independent variable being the angle θ (in degrees) and the dependent variable being the sine of that angle.

The interactive demonstrates that the sine function $y = \sin(x)$ is periodic, with a period of $360°$,

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meaning the pattern repeats every full rotation. The sine values are positive in the first and second quadrants
(0° to 180°) and negative in the third and fourth quadrants
(180° to 360°). The graph of $y = \sin(x)$ forms a continuous wave curve, accurately modeling the periodic motion of the capsule. This gives users a better understanding of the graphical representation of the sine function.

- What do the points plotted on the coordinate system represent?
- Is it reasonable to model the data using a continuous curve?
- Is the trigonometric function $y = \sin x$ periodic? What is the period?

Observe that as the capsule moves around the unit circle, the points plotted on the graph model the extent to which the cabin is above or below the horizontal axis. The independent variable θ represents the angle (measured in degrees) between the corresponding radius of the cabin and the horizontal axis (anticlockwise direction) and the dependent variable represents the vertical distance of the capsule from the horizontal axis.

Notice how the sine values are positive between 0 and 180° , which correspond to the values of the sine in quadrants 1 and 2 on the unit circle, and the sine values are negative between 180° and 360° , representing quadrants 3 and 4.

The graph of the function $y = \sin x$ is a continuous wave curve that shows periodic behaviour after one full rotation. The figure below shows the graph of the function $y = \sin x$.

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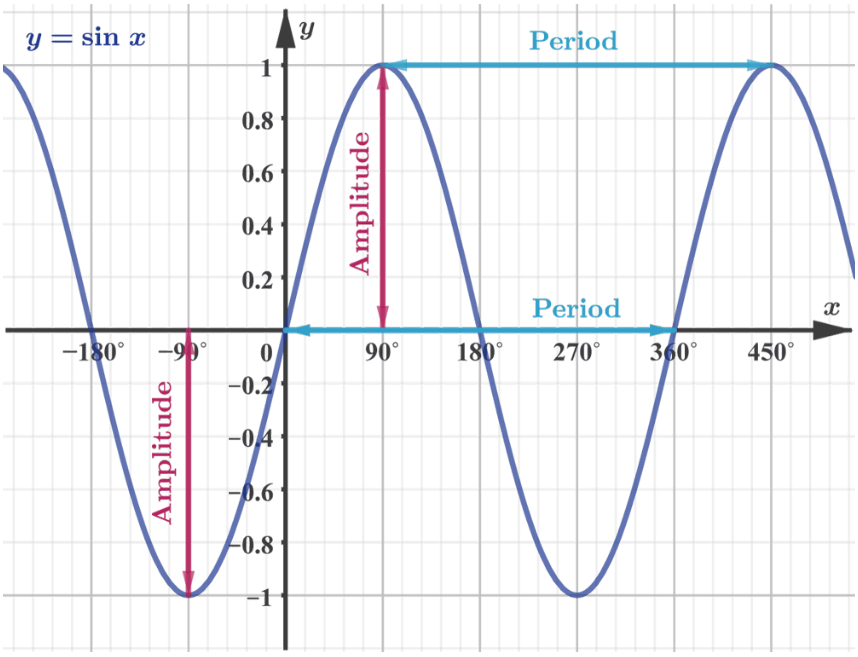

More information

The image is a graph of the function (y = \sin x) plotted on a coordinate grid. The X-axis represents the angle in degrees, labeled from -180° to 450° at intervals of 90°. The Y-axis represents the sine value, with labels at -1, -0.5, 0, 0.5, and 1, indicating the amplitude of the curve. The sine wave starts at -180° reaching 0 at -90°, a peak of 1 at 90°, crossing 0 again at 180°, achieving a minimum of -1 at 270°, and so on. The graph shows a periodic behavior, highlighting a full wave from 0° to 360°, emphasizing the period of the sine function as 360°. An arrow labeled "Period" points from 0° to 360°, and another pointing from peak to trough labeled "Amplitude." The graph is depicted with consistent wave curves crossing the axes, showing the continuous nature of the sine function.

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Important

The trigonometric function $y = \sin x$ has the following characteristics:

- the period is 360°
- the maximum value of the function is 1
- the minimum value of the function is -1
- the principal axis is:

$$y = \frac{\max + \min}{2} = \frac{1-1}{2} = 0$$

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- the amplitude is:
$$\left|\frac{\max - \min}{2}\right| = \left|\frac{1-(-1)}{2}\right| = \left|\frac{2}{2}\right| = 1$$

Cosine function

Activity

In the next applet the London Eye is represented by the unit circle. Drag slider ‘θ’ to rotate a particular passenger capsule (red point) around the unit circle.

Interactive 3. The Graphical Representation of Cosine Function Using London Eye.

More information for interactive 3

This interactive allows users to explore the cosine function through a real-world scenario modeled on the London Eye Ferris Wheel, using a unit circle with a radius of 1. A red point, representing a passenger capsule, moves around the circle as the user adjusts an angle slider. As the angle θ changes, the horizontal position of the capsule—corresponding to $\cos(\theta)$—is dynamically plotted, allowing users to visually and conceptually link circular motion with the cosine function. Three interactive features enhance the experience: the angle slider adjusts the capsule's position and updates both the angle and cosine value in real time; the "Show cosine graph" checkbox

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overlays the standard cosine wave on the graph to connect the capsule's motion to its corresponding wave pattern; and the "Clear trace" button removes all previous plotted points so users can restart their exploration or trace a new path.

The activity demonstrates that the cosine function $y = \cos(x)$ is periodic, repeating every $360°$, and that the cosine value is positive in the first and fourth quadrants ($0° – 90°$ and $270° – 360°$) and negative in the second and third ($90° – 270°$). The wave-shaped graph of $y = \cos(x)$ emerges from the movement of the capsule, showing how cosine values vary smoothly with angle.

Through this dynamic visualization, users gain a deeper understanding of cosine as both a geometric and a periodic function, making the abstract concept more intuitive and memorable.

- What do the points plotted on the coordinate system represent?
- Is it reasonable to model the data using a continuous curve?
- Is the trigonometric function $y = \cos x$ periodic? What is the period?

Observe that as the capsule moves around the unit circle, the points on the graph represent the extent to which the cabin lies to the right or left of the vertical axis. The independent variable θ represents the angle (measured in degrees) between the corresponding radius of the cabin and the x -axis (anticlockwise direction) and the dependent variable represents the horizontal distance of the capsule from the vertical axis.

The graph of the function $y = \cos x$ is a continuous wave curve that shows periodic behaviour after one full rotation. The figure below shows the graph of the function $y = \cos x$.

Student
view

Overview
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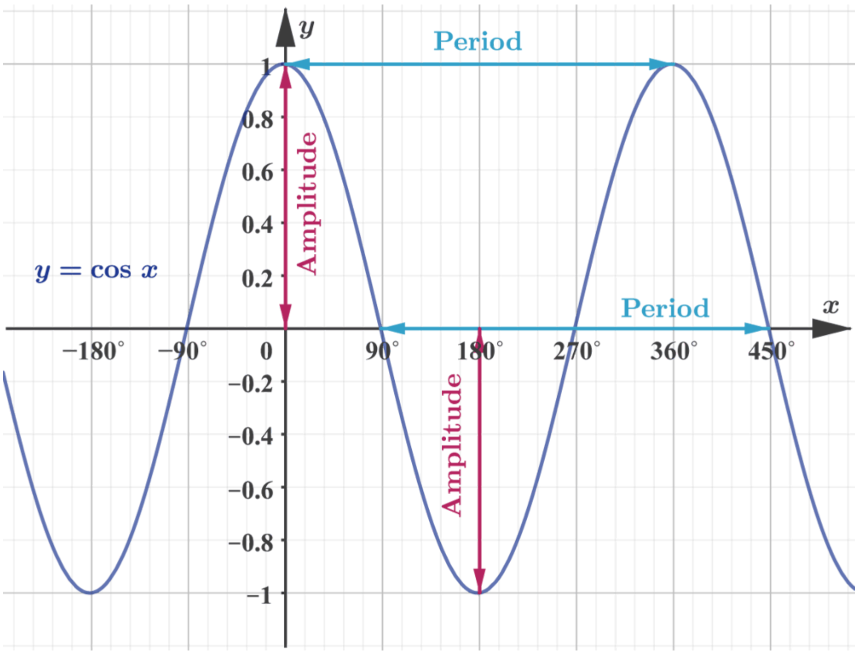


More information

The image is a graph representing the function y = cos x. The X-axis is labeled with angles in degrees, ranging from -180° to 450°, with notable points at 0°, 180°, 270°, 360°, and 450°. The Y-axis represents the amplitude, ranging from -1 to 1. The cosine curve displays a wave pattern illustrating periodic behavior. Annotations indicate the 'Period' along the X-axis and 'Amplitude' along the Y-axis, highlighting the distance between peaks and troughs. The graph illustrates how the function completes one full cycle from 0° to 360°.

[Generated by AI]

Important

The trigonometric function $y = \cos x$ has the following characteristics:

- the period is 360°
- the maximum value of the function is 1
- the minimum value of the function is -1
- the principal axis is:
$$y = \frac{\max + \min}{2} = \frac{1 - 1}{2} = 0$$
- the amplitude is:
$$\left|\frac{\max - \min}{2}\right| = \left|\frac{1 - (-1)}{2}\right| = \left|\frac{2}{2}\right| = 1$$

Student
view

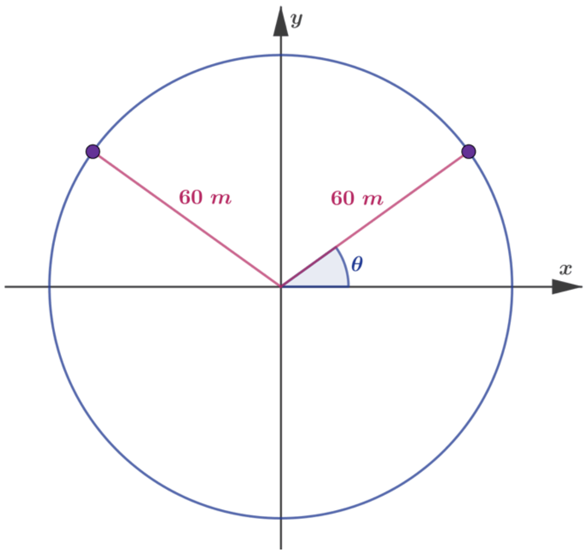
Overview (/study/ap 122-cid-754029/

Compare the graphs of the sine and cosine function and reflect on how you could transform the sine function into the cosine function.

Example 2

★★☆

The radius of a Ferris wheel is 60 m and θ represents the angle measured in the anticlockwise direction from the x -axis. Find all the angles for which a passenger capsule is a distance of 35 m above the x -axis. Give your answer in degrees correct to one decimal place.

Steps	Explanation
 	Make a sketch of the given situation. There are two points on the circle that are 35 m above the x -axis. One has ar acute angle θ, the other is an obtuse angle equal to $180^\circ - \theta$.
Acute angle: $$\sin\theta = \frac{35}{60}$$ $$\theta = 35.7^\circ \text{ (to 1 decimal place)}$$ Obtuse angle: $$\theta = 180^\circ - 35.7^\circ$$ $$\theta = 144.3^\circ \text{ (to 1 decimal place)}$$	The vertical distance is represented by the sine function and is equal to the rat of the opposite side 35 m over the hypotenuse 60 m.

Student view

Overview
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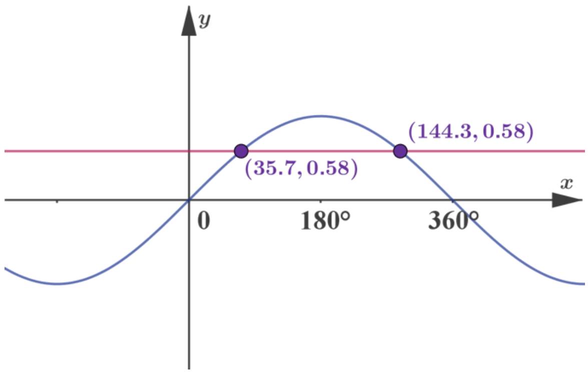
Steps	Explanation
Alternatively, you can g raph the functions $y = \sin x$ and $y = \frac{35}{60}$ on your GDC.	
	Find the x -coordinates of the points o intersection between the graphs.
Within one period, there are two values of θ for which the capsule is 35 m above the x -axis: $35.7°$ and $144.3°$.	

3 section questions

2. Functions / 2.5 Introduction to modelling

Checklist

Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/checklist-id-27466/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- model relationships between two variables that are related with a constant rate of change using linear functions
- find the vertex, axis of symmetry and y -intercept of a quadratic function from its standard form

Student view

Overview
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754029/

- find the vertex, axis of symmetry and y -intercept of a quadratic function from its vertex form
- find the vertex, axis of symmetry and y -intercept of a quadratic function from its factorised form
- determine the domain and range of a quadratic function
- use the method of completing the square to transform a quadratic function from standard form to vertex form
- transform quadratic functions from one form to another
- use a GDC to find the x -intercepts of a parabola
- sketch a parabola, showing all relevant features such as the vertex, the axis of symmetry, the y -intercept and the x -intercepts if there are any
- use a GDC to graph cubic functions and find all relevant features
- find the domain, range, asymptotes and axes intercepts of exponential functions
- sketch the graphs of exponential functions by showing all relevant features
- apply exponential models to real-life situations
- determine whether a function shows periodic behaviour
- find the amplitude, principal axis and period of trigonometric functions.

2. Functions / 2.5 Introduction to modelling

Investigation

Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/investigation-id-27467/print/) Assign

Be aware

Notice that an exponential function $f(x) = a^x$ is a constant raised to a variable power, whereas a power function $g(x) = x^a$ is a variable raised to a constant non-negative power.

Student view

In the following activity, you will investigate the graphs of $f(x) = a^x$ and $g(x) = x^n$, where a and n are constant non-negative numbers.

Compare the graphs of the functions for the same values of a and n . Then change these values. What do you notice when the value of a is not equal to n . Draw some conclusions about the values of a and n that will make the exponential function grow faster than the polynomial. Reflect on the statement, 'Exponential growth is bigger and faster than polynomial growth of any degree.'