Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

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Table of contents

Notebook

Glossary

Reading assistance

TOPIC 1
NUMBER AND ALGEBRA

SUBTOPIC 1.14
MATRICES

Student view

Show all topics

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

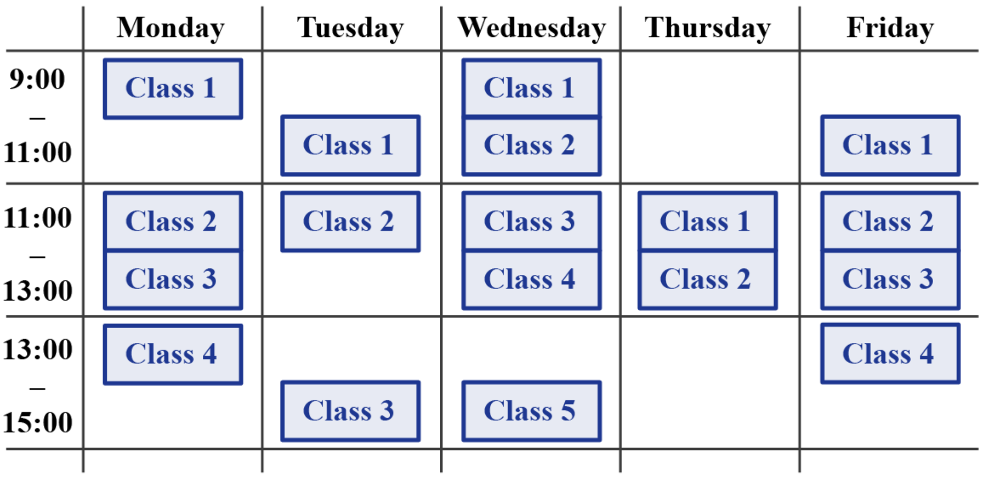
Teacher view

Index

1. Number and algebra / 1.14 Matrices

The big picture

A matrix is a tool used to organise data using columns and rows. Even though you might not be aware of this, you use matrices in your daily life. One example is when you look at a schedule such as the one shown below.



More information

The image displays a week-long schedule formatted as a matrix from Monday to Friday, with time slots at 9:00-11:00, 11:00-13:00, and 13:00-15:00. Each day of the week is represented by a column, and each time slot by a row. Blue blocks fill certain cells to indicate scheduled activities. On Monday, all three time slots have activities. Tuesday has activities at 9:00-11:00 and 13:00-15:00. Wednesday has activities in all slots, similar to Monday. Thursday has activities at 11:00-13:00 and 13:00-15:00. Friday has activities in all slots as well. The schedule indicates that some time periods are occupied across multiple days, potentially indicating recurring activities or blocks of time set aside for specific tasks or meetings.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

[Generated by AI]

The information in this schedule can be written in matrix form as follows:

$$\begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

The columns represent the day of the week, the rows represent 2-hour time intervals, and the entries represent the number of classes in each time interval.

Organising the numerical information in the form of a matrix allows you to manipulate it efficiently using matrix algebra that you will learn in this subtopic.

Concept

Matrix multiplication shares some but not all of the properties of multiplication of numbers. This means that matrix algebra will also have some unique properties. How does the format of representationof information in a matrix shape the properties of matrix algebra? How does this form of representation allow for efficient manipulation of large amounts of data?

1. Number and algebra / 1.14 Matrices

Definitions and equality

Definitions

A matrix is a collection of data that is represented as entries organised in rows and columns. A matrix can be described by stating its order which tells you the number of rows and columns.

Student
view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

The order is given in $m \times n$ format, where m is the number of rows and n is the number of columns.

Example 1

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State the order of $\begin{pmatrix} 1 & 7 & -5 \\ 3 & 6 & 2 \end{pmatrix}$.

Steps	Explanation
This is a 2×3 matrix.	The matrix has two rows ($m = 2$) and three columns ($n = 3$). If you had to describe the order of this matrix in words you would say: 'a two by three matrix'.

A matrix is often represented by a letter. The convention is to use a capital letter which is often shown in bold. For example:

$$A = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}.$$

The entries in a matrix are called elements . The elements in matrix A will be labelled as a_{ij}, where a_{ij} is the entry that corresponds to the ith row and jth column.

Example 2

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27428/print/) Assign

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Write down b_{23} in $B = \begin{pmatrix} 3 & 5 & 0 & 4 \\ 0 & 2 & -15 & 8 \\ 1 & -5 & 1 & 9 \end{pmatrix}$.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$b_{23} = -15$	b_{23} is the element in the second row and third column.

A matrix such as, $C = \begin{pmatrix} 3 & 1 \\ 5 & 0 \end{pmatrix}$, where the number of rows is equal to the number of columns $(m = n)$ is called a square matrix. A matrix with one column $(n = 1)$ is called a column matrix and a matrix with one row $(m = 1)$ is called a row matrix.

Matrix equality

If $a_{ij} = b_{ij}$ for all i and j, then matrix A is equal to matrix B.

For matrices to be equal all corresponding elements must be equal. Is it possible for matrices with different orders to be equal?

Example 3

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Given $A = \begin{pmatrix} x^2+1 & 2 \\ y & x \end{pmatrix}$ and $B = \begin{pmatrix} -17y & 2 \\ y & 4 \end{pmatrix}$, find x and y if $A = B$.

Steps	Explanation
$\begin{pmatrix} x^2+1 & 2 \\ y & x \end{pmatrix} = \begin{pmatrix} -17y & 2 \\ y & 4 \end{pmatrix}$ Therefore: $x^2 + 1 = -17y$ $2 = 2$ $y = y$ $x = 4$	If the matrices are equal all the corresponding elements are equal.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$x^2+1=-17y \quad \Leftrightarrow \quad (4)^2+1=-17y$ $\Leftrightarrow \quad 17=-17y \quad \Leftrightarrow \quad y=-1$	
$x=4,\ y=-1$	

Example 4

★☆☆

The order of matrix A is 2×8 and the order of matrix B is $2\times 2x^2$. Find x if $A=B$.

Steps	Explanation
$8=2x^2 \quad \Leftrightarrow \quad 4=x^2 \quad \Leftrightarrow \quad x=\pm 2$ reject $x=-2$ $x=2$	For two matrices to be equal their order must be the same.

3 section questions

1. Number and algebra / 1.14 Matrices

Addition, subtraction, multiplication by a scalar

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Addition and subtraction

A trainer of two athletes compiles the following information about the number of hours each one spent on endurance training and weightlifting in two consecutive weeks. The data is shown in the tables below.

First week of training	Endurance training (hours)	Weightlifting (hours)	Second week of training	Endurance training (hours)	Weightlifting (hours)
Athlete 1	17.8	4.3	Athlete 1	21.5	3.5
Athlete 2	12.5	10.5	Athlete 2	8.0	14.4

How would you find the training time in each category for each athlete over the 2-week training period?

Adding each corresponding entry in each table will tell you the training time over the 2-week period. This is an example of matrix addition.

✓ **Important**

To add or subtract matrices you need to add or subtract the corresponding elements. The order of the matrices must be the same.

Example 1

★☆☆

The price of each of three products sold by two stores is shown by matrix $A = \begin{pmatrix} 21.3 & 22.5 \\ 45.0 & 43.0 \\ 15.9 & 14.5 \end{pmatrix}$. The prices are in USD. Element a_{ij} describes the price of product i in store j. Each of the stores is running a sale. Matrix $B = \begin{pmatrix} 0.5 & 1.4 \\ 1.4 & 2.1 \\ 3.0 & 1.0 \end{pmatrix}$ shows the amount by which each product is discounted, where b_{ij} is the amount of discount (in USD) for item i in store j.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Write a matrix that represents the final price of each product in each store.

Steps	Explanation
$\begin{pmatrix} 21.3 & 22.5 \\ 45.0 & 43.0 \\ 15.9 & 14.5 \end{pmatrix} - \begin{pmatrix} 0.5 & 1.4 \\ 1.4 & 2.1 \\ 3.0 & 1.0 \end{pmatrix} = \begin{pmatrix} 20.8 & 21.1 \\ 43.6 & 40.9 \\ 12.9 & 13.5 \end{pmatrix}$	The final price of each product is found by taking the initial price an subtracting the discount. This can be done by finding $A - B$.

Example 2

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Given that $\begin{pmatrix} 2 & 3 & a+1 \\ a & -2 & 4 \end{pmatrix} + \begin{pmatrix} a & -1 & 2 \\ 0 & 1 & a \end{pmatrix} = \begin{pmatrix} 2b & 2 & b \\ a & b & c \end{pmatrix}$, find the values of $a,\ b,$ and c.

Steps	Explanation
$\begin{pmatrix} 2 & 3 & a+1 \\ a & -2 & 4 \end{pmatrix} + \begin{pmatrix} a & -1 & 2 \\ 0 & 1 & a \end{pmatrix} = \begin{pmatrix} a+2 & 2 & a+3 \\ a & -1 & a+4 \end{pmatrix}$	Add the matrices by adding corresponding elements.
$\begin{pmatrix} a+2 & 2 & a+3 \\ a & -1 & a+4 \end{pmatrix} = \begin{pmatrix} 2b & 2 & b \\ a & b & c \end{pmatrix}$ Therefore: $a+2=2b$ $a+3=b$ $-1=b$ $a+4=c.$ Since $b=-1$, $a+2=2(-1) \quad \Leftrightarrow \quad a=-4.$ Since $a=-4$, $-4+4=c \quad \Leftrightarrow \quad c=0.$ The answer is: $a=-4,\ b=-1,\ c=0.$	Use matrix equality.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Activity

Addition of real numbers is commutative and associative. That is $a+b=b+a$ and $a+(b+c)=(a+b)+c$ where $a,\ b,\ c\in\mathbb{R}$.

Do the same properties hold true for matrix addition? Use examples to justify your answer.

Important

Matrix addition is commutative, so

$$A+B=B+A$$

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and associative, so

$$A+(B+C)=(A+B)+C$$

for all matrices A, B, and C that have the same order.

Multiplication by a scalar

Important

To multiply a matrix by a scalar you multiply all elements in the matrix by the scalar.

For example, $k\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}k\times a & k\times b\\ k\times c & k\times d\end{pmatrix}$, where $k\in\mathbb{R}$.

Example 3

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Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

The amount of toxic emissions produced by two factories are shown in matrix $A = \begin{pmatrix} 24.5 & 107.2 \\ 17.0 & 12.3 \end{pmatrix}$ where element a_{ij} represents the amount of toxin i released by factory j. The factories are required to reduce their emissions of all toxins by 10%.

Write a matrix that represents the new emission amounts of each toxic at each factory.

Round all elements to 3 significant figures.

Steps	Explanation
$0.9 \times \begin{pmatrix} 24.5 & 107.2 \\ 17.0 & 12.3 \end{pmatrix} = \begin{pmatrix} 22.1 & 96.5 \\ 15.3 & 11.1 \end{pmatrix}$	A reduction by 10% can be represented by multiplication by 0.9.

Example 4

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Given that $k\begin{pmatrix} x^2 & -2 \\ 3 & 2y \end{pmatrix} = \begin{pmatrix} -2y+1 & x-1 \\ -x & -10 \end{pmatrix}$, find the values of $k,\ x,\ y$.

Steps	Explanation
$k\begin{pmatrix} x^2 & -2 \\ 3 & 2y \end{pmatrix} = \begin{pmatrix} kx^2 & -2k \\ 3k & 2ky \end{pmatrix}$	Multiply by k.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$\begin{pmatrix} kx^2 & -2k \\ 3k & 2ky \end{pmatrix} = \begin{pmatrix} -2y+1 & x-1 \\ -x & -10 \end{pmatrix}$ Therefore: $kx^2 = -2y+1$ $-2k = x-1$ $3k = -x$ $2ky = -10$ Solving for k and x: $\begin{cases} -2k = x-1 \\ 3k = -x \end{cases} \rightarrow \begin{cases} -2k = (-3k)-1 \\ -3k = x \end{cases} \rightarrow \begin{cases} k = -1 \\ -3(-1) = x \end{cases} \rightarrow \begin{cases} k = -1 \\ x = 3 \end{cases}$ Finding y: $2ky = -10 \quad \Leftrightarrow \quad y = -\frac{10}{2k} = -\frac{10}{2(-1)} = 5$ The answer is: $k = -1,\ x = 3,\ y = 5.$	Use matrix equality.

3 section questions

1. Number and algebra / 1.14 Matrices

Matrix multiplication

Defining matrix multiplication

Before looking at the definition of matrix multiplication, consider **Example 1**.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Example 1

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A teacher gives a three-part test to her students. A correct answer on the first part is awarded 1 point, on the second part 1.5 points, and 2 points on the third part. The number of correct questions answered by four students is shown in the table below.

	Part 1	Part 2	Part 3
Student A	18	2	4
Student B	10	8	4
Student C	5	10	2
Student D	16	6	3

Determine which student earns the highest combined score on the test and state their total number of points.

Steps	Explanation
Student A: $$18\,(1)+2\,(1.5)+4\,(2)=29$$ Student B: $$10\,(1)+8\,(1.5)+4\,(2)=30$$ Student C: $$5\,(1)+10\,(1.5)+2\,(2)=24$$ Student D: $16\,(1)+6\,(1.5)+3\,(2)=31$	You can find the answer to this question by finding the total number of points for each student.
Student D has the highest total score of 31 points.	

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Example 1 can be solved more efficiently using matrix multiplication which is defined below.

Important

The elements of the product matrix $P = A \times B$ are found by adding the products of elements in row i of matrix A and the elements in column j in matrix B.

For example:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} \times b_{11} + a_{12} \times b_{21} + a_{13} \times b_{31} & a_{11} \times b_{12} + a_{12} \times b_{22} + a_{13} \times b_{32} \\ a_{21} \times b_{11} + a_{22} \times b_{21} + a_{23} \times b_{31} & a_{21} \times b_{12} + a_{22} \times b_{22} + a_{23} \times b_{32} \end{pmatrix}$$

This definition of multiplication means that **Example 1** can be solved as follows:

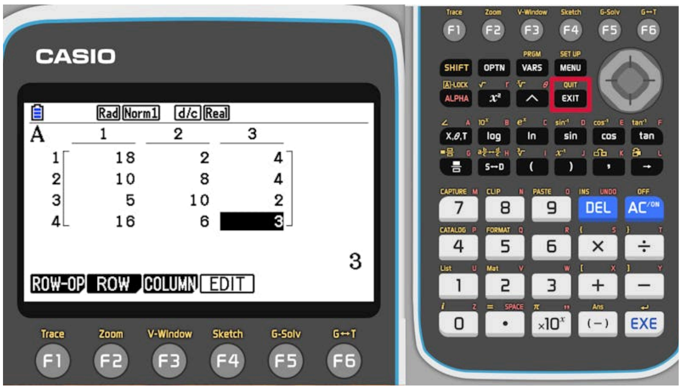
$$\begin{pmatrix} 18 & 2 & 4 \\ 10 & 8 & 4 \\ 5 & 10 & 2 \\ 16 & 6 & 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1.5 \\ 2 \end{pmatrix} = \begin{pmatrix} 18 \times 1 + 2 \times 1.5 + 4 \times 2 \\ 10 \times 1 + 8 \times 1.5 + 4 \times 2 \\ 5 \times 1 + 10 \times 1.5 + 2 \times 2 \\ 16 \times 1 + 6 \times 1.5 + 3 \times 2 \end{pmatrix} = \begin{pmatrix} 29 \\ 30 \\ 24 \\ 31 \end{pmatrix}$$

Using matrix multiplication in **Example 1** is more efficient when you use your calculator for the calculations. Carrying out matrix multiplication using a calculator allows a teacher with a more realistic class size of 20 or more students to find the total scores very quickly.

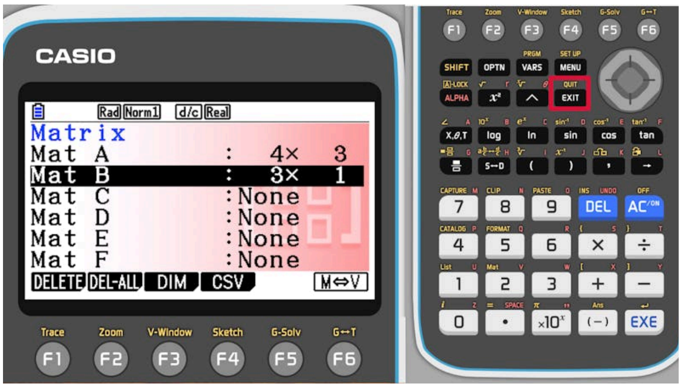
Student view

Steps	Explanation
These instructions show you how to store matrices in the memory of the calculator and how to multiply them. To work with matrices, choose the calculator mode.	
To store matrices in the memory, press F3 to access the matrix editor.	

Steps	Explanation
Choose any of the matrices and press F3 to set the dimensions.	
m is the number of rows, n is the number of columns.	

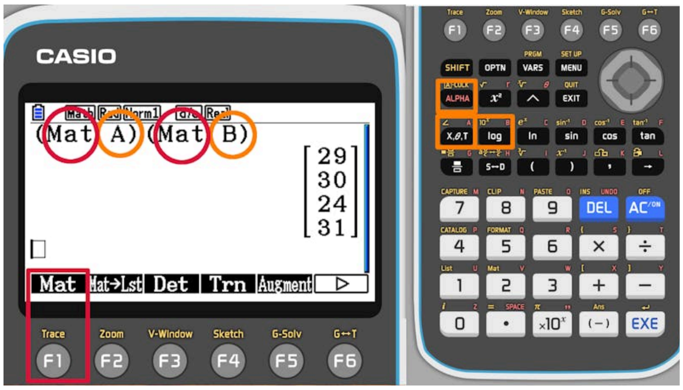
Steps	Explanation
Enter the entries of the matrix and press EXIT when done.	
Do the same for the other matrix.	

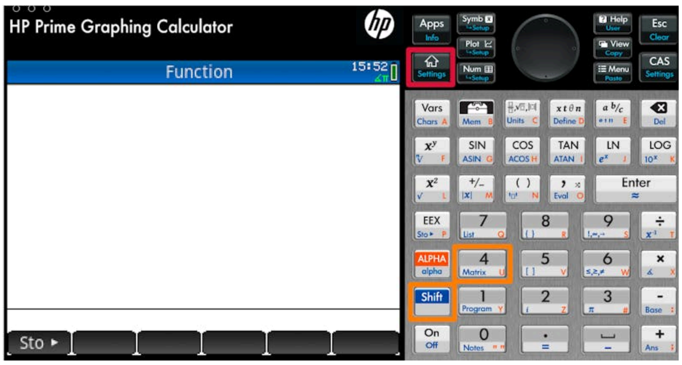
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
	
Once you stored both matrices, press EXIT to go back to the calculator screen.	

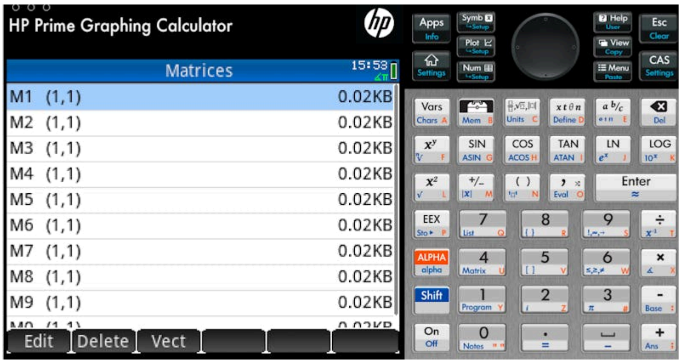
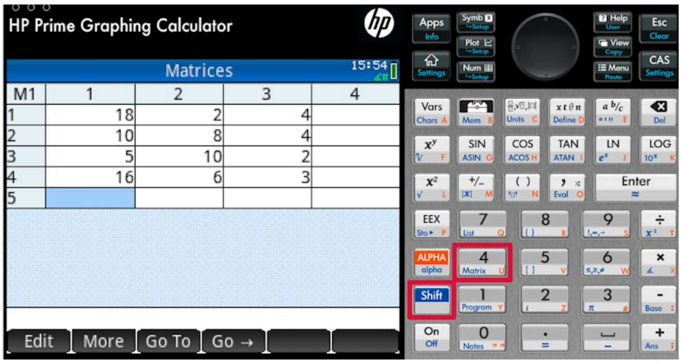
Student view

Steps	Explanation
This time, press OPTN ...	
... and press F2 to see the options related to matrices.	

Steps	Explanation
You can use multiplication (and addition, subtraction, integer powers) like when you work with numbers. Use F1 to tell the calculator that you want to work with matrices and use the ALPHA key to enter the names. Press EXE to see the product matrix.	

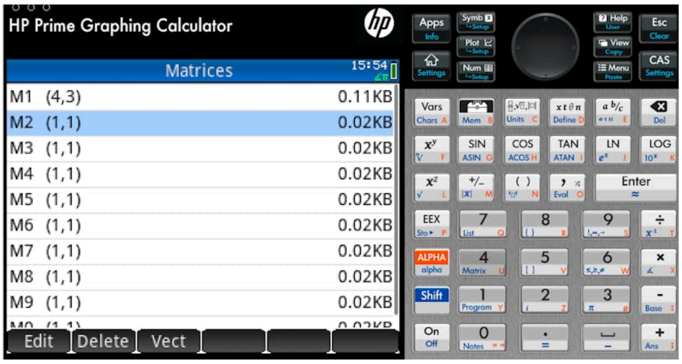
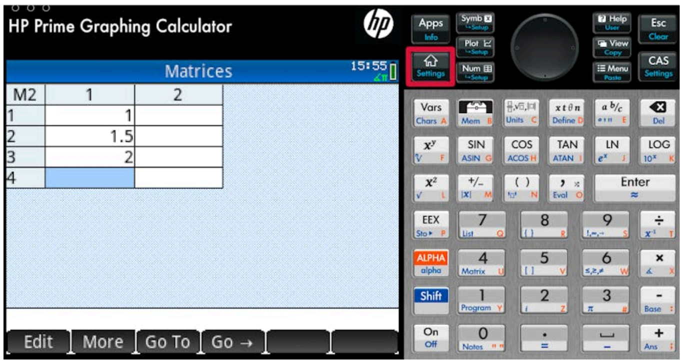
Steps	Explanation
These instructions show you how to store matrices in the memory of the calculator and how to multiply them. To work with matrices, choose the home screen of any application. To store matrices in the memory, open the matrix editor.	

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
Choose any of the available matrices ...	
... and enter the entries of the matrix. You do not need to specify the dimensions, the matrix editor table automatically adjusts the size. Once done, open the editor again ...	

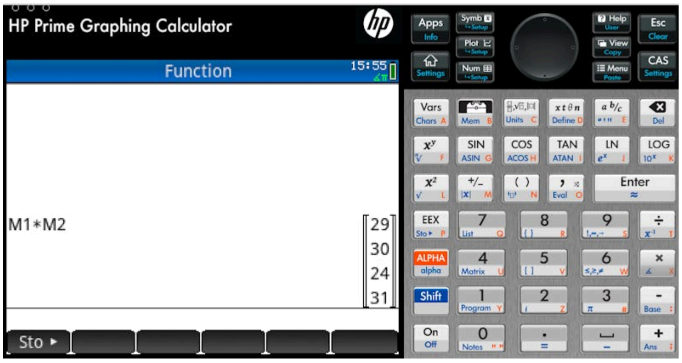
Student view

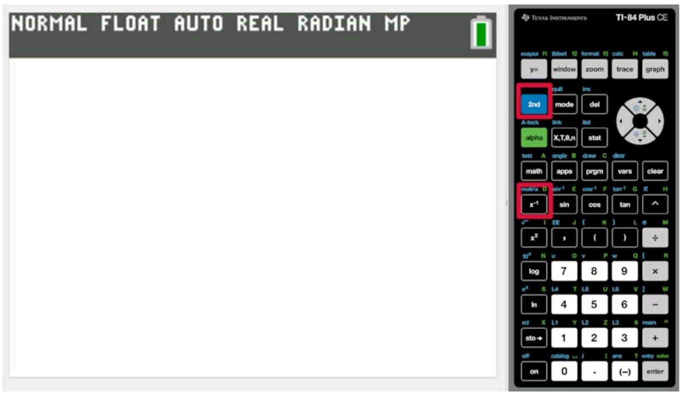
Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Steps	Explanation
... and repeat the process to store the second matrix.	
Once both matrices are stored, go back to the home screen.	

Student
view

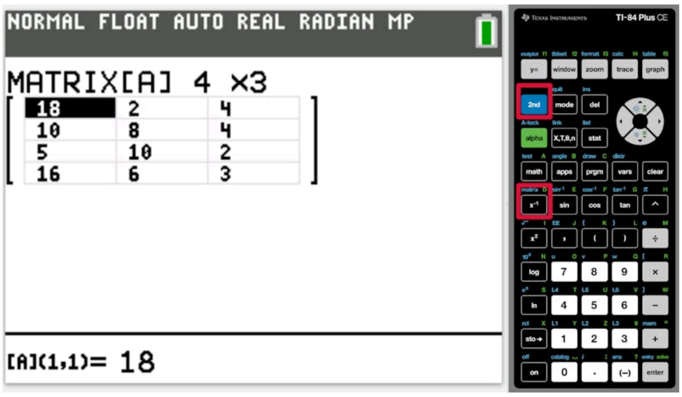
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
You can use multiplication (and addition, subtraction, integer powers) like when you work with numbers. Use the names of the stored matrices and press enter to see the product.	

Steps	Explanation
These instructions show you how to store matrices in the memory of the calculator and how to multiply them. To work with matrices, choose the matrix option.	

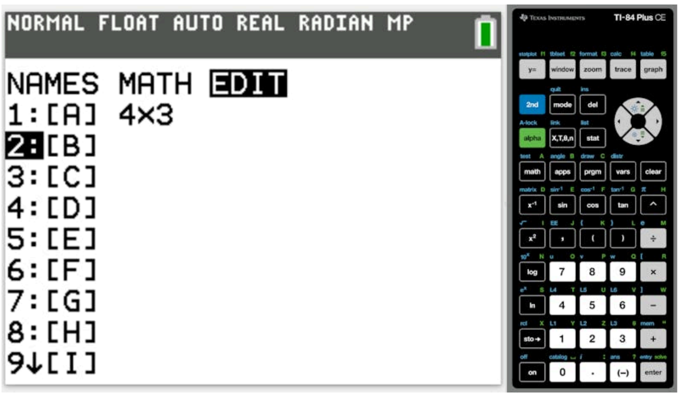
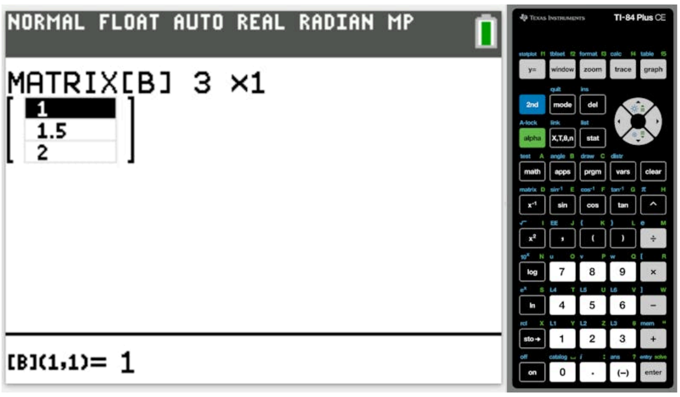
Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Steps	Explanation
To store the matrix in the memory, choose to edit any of the available matrices ...	
... and enter the dimensions and the entries of the matrix. Once done, open the editor again ...	

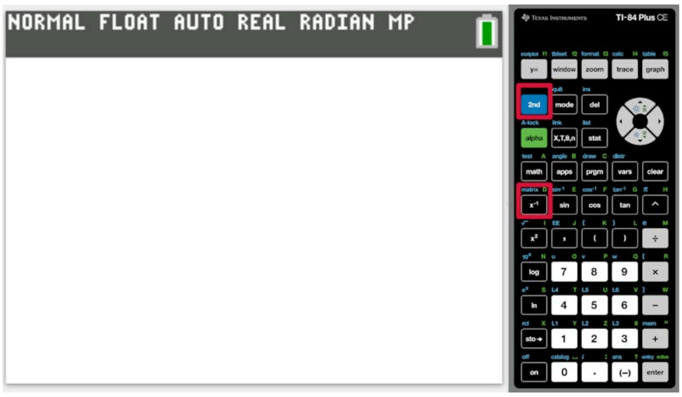
Student
view

Overview (/study/ap ai- hl/sid- 132- cid- 761618/ov

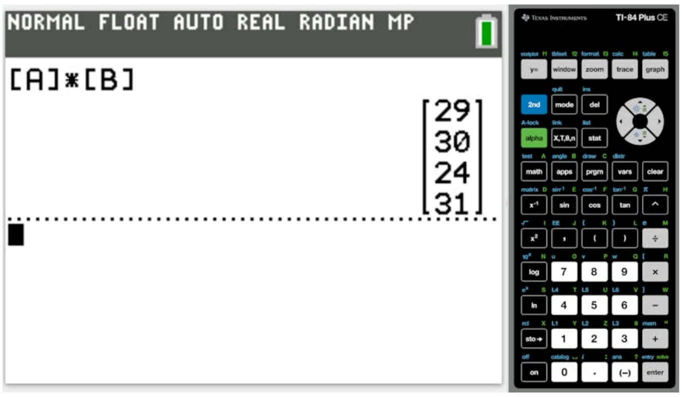
Steps	Explanation
... and repeat the process to store the second matrix.	
	

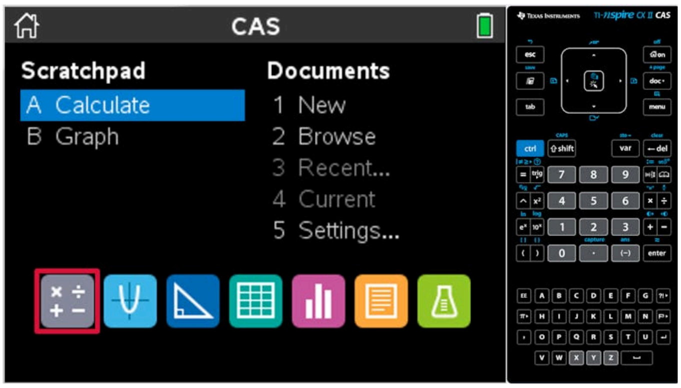
Student view

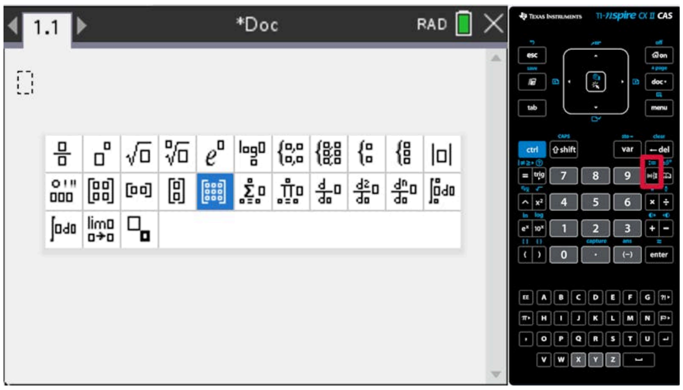
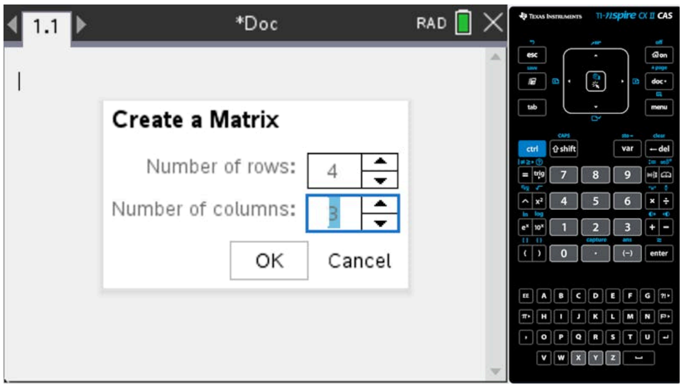
Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>Once the second matrix is stored, go back to the main calculator screen and open the matrix options yet again.</td><td>
</td></tr>
<tr><td>Choose the name of the first matrix, ...</td><td>
</td></tr>
</table>

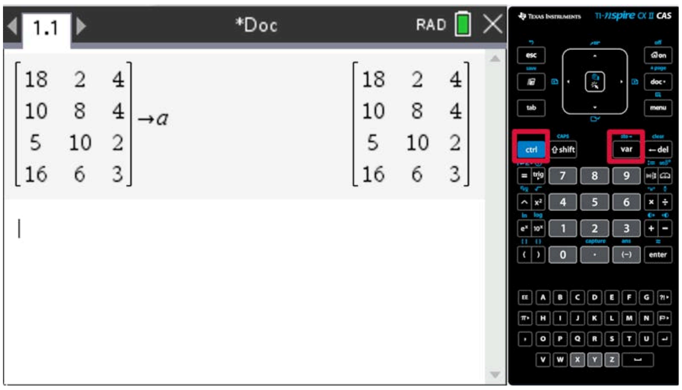
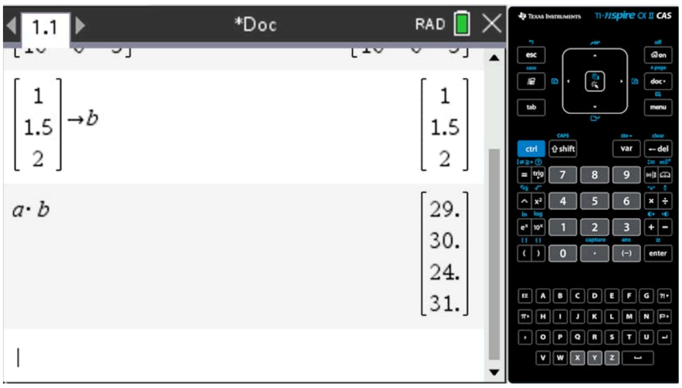
Student
view

Steps	Explanation
... press the multiplication button and (through the matrix menu again) choose also the name of the second matrix. Press enter to see the product. You can use multiplication (and addition, subtraction, integer powers) like when you work with numbers.	

Steps	Explanation
These instructions show you how to store matrices in the memory of the calculator and how to multiply them. To work with matrices, open a calculator page.	

Steps	Explanation
To edit a matrix, open the templates and find the template for entering a matrix ...	
... set the dimensions, ...	

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Steps	Explanation
... and enter the entries of the matrix. You can store the matrix in the memory. You can choose any name.	
Once you store also the second matrix, you can use the names to find the product. You can use multiplication (and addition, subtraction, integer powers) like when you work with numbers.	

You can practice matrix multiplication questions by using the applet below.

Student
view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Interactive 1. A Practice Exercise on Matrix Multiplication.

Credit: GeoGebra (https://www.geogebra.org/m/FpCnqqXC) Terry Lee Lindenmuth

More information for interactive 1

This interactive allows users to practice matrix multiplication by generating new problems. Users are presented with two matrices and must perform the multiplication to find the resulting matrix. The "new question" button allows users to load a fresh set of matrices for continued practice.

After attempting the problem, users can click the "Show answer" button to reveal the correct solution and compare it with their own work. This button toggles to "Hide answer" to allow users to focus on solving without distraction. The interactive provides immediate feedback, helping users identify and correct any mistakes, while reinforcing their understanding of matrix multiplication.

For example:

Let $A =$

$$\begin{bmatrix} 0 & 7 & 3 \\ 9 & 14 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -16 & -8 \\ -6 & -3 \\ 10 & 5 \end{bmatrix}$$

Product $C = AB$ will be a $2X\,2$ elements C_{ij} calculated as the dot product of the i-th row of A and the j-th column of B.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

c11 = (0)(−16) + (7)(−6) + (3)(10) = 0 − 42 + 30 = −12

c12 = (0)(−8) + (7)(−3) + (3)(5) = 0 − 21 + 15 = −6

c21 = (9)(−16) + (14)(−6) + (−5)(10) = −144 − 84 − 50 = −278

c22 = (9)(−8) + (14)(−3) + (−5)(5) = −72 − 42 − 25 = −139

Thus, the resulting matrix C is:

C =

$$\begin{bmatrix} -12 & -6 \\ -278 & -139 \end{bmatrix}$$

Properties of matrix multiplication

Activity

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -7 \end{pmatrix}, B = \begin{pmatrix} 2 & -3 \\ -1 & 8 \\ 0 & 4 \end{pmatrix}, C = \begin{pmatrix} 1 & 5 & 2 \end{pmatrix},$$

$$D = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} \text{ and } E = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & -1 \end{pmatrix}.$$

Find each of the following products by hand and confirm your results on the calculator:

$$BA \qquad CB \qquad CD \qquad DE \qquad BD$$

Explain your results using the definition of matrix multiplication.

Important

The product $P = AB$ of an $m \times n$ matrix A and a $k \times l$ matrix B exists only if $n = k$.

The order of matrix P is $m \times l$.

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Activity

Multiplication of real numbers has the following properties:

Associative: $a \times (b \times c) = (a \times b) \times c$

Distributive: $a \times (b + c) = a \times b + a \times c$

Commutative: $a \times b = b \times a$

Let $A = \begin{pmatrix} 1 & 0 \\ 4 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix}$.

Determine whether matrix multiplication follows all of the properties described above.

Do your findings hold for non-square matrices?

Important

Matrix multiplication is associative, distributive, but non-commutative.

Is it possible to think of an example where matrix multiplication is commutative?

Example 2

★☆☆

Let $A = \begin{pmatrix} 1 & x \\ 3 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} x \\ 3 \end{pmatrix}$.

Find, if possible:

a) AB

b) BA

a) $AB = \begin{pmatrix} 1 & x \\ 3 & 7 \end{pmatrix} \begin{pmatrix} x \\ 3 \end{pmatrix} = \begin{pmatrix} 4x \\ 3x + 21 \end{pmatrix}$

Student view

b) BA cannot be found. The number of columns in B does not match the number of rows in A.

Example 3

★★☆

Given that $\begin{pmatrix} x^2 & 2y \\ y & 2 \\ 1 & x \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} a & b \\ 4 & 4 \\ c & 33 \end{pmatrix}$ find the value of a.

Steps	Explanation
$\begin{pmatrix} x^2 & 2y \\ y & 2 \\ 1 & x \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} x^2+6y & 5x^2+14y \\ y+6 & 5y+14 \\ 1+3x & 5+7x \end{pmatrix}$	Simplify the left side by matri multiplication.
$\begin{pmatrix} x^2+6y & 5x^2+14y \\ y+6 & 5y+14 \\ 1+3x & 5+7x \end{pmatrix} = \begin{pmatrix} a & b \\ 4 & 4 \\ c & 33 \end{pmatrix}$ $x^2+6y=a$ $y+6=4$ $1+3x=c$ $5x^2+14y=b$ $5y+14=4$ $5+7x=33$	Use matrix equality.
$y+6=4 \quad \Leftrightarrow \quad y=-2$ $5+7x=33 \quad \Leftrightarrow \quad 7x=28 \quad \Leftrightarrow \quad x=4$	Solve for x and y.
$x^2+6y=a \quad \Leftrightarrow \quad (4)^2+6(-2)=a$ $a=4$	Find the value of a.

Example 4

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In one month, Factory A produced 452 wallets and shipped 270 of them. Factory B produced 720 wallets and shipped 514 of them in the same amount of time.

a) Write a 2×2 matrix to represent this information where each row represents one of the factories.

b) It costs each factory $\text{USD}12.74$ to produce one wallet and $\text{USD}2.39$ to ship it.

Write a matrix that represents the amount of money spent by each factory on production and shipping in one month. Round your answers to the nearest cent.

	Steps	Explanation
a)	$\begin{pmatrix} 452 & 270 \\ 720 & 514 \end{pmatrix}$	The first column shows the number of walle produced and the second column shows th number shipped.
b)	$\begin{pmatrix} 452 & 270 \\ 720 & 514 \end{pmatrix} \begin{pmatrix} 12.74 \\ 2.39 \end{pmatrix} = \begin{pmatrix} 6403.78 \\ 10401.26 \end{pmatrix}$	The total shipping and production cost can be found using matrix multiplication.

4 section questions

1. Number and algebra / 1.14 Matrices

Matrix algebra

Matrices can be used in writing equations and algebraic expressions. You need to know how to manipulate these expressions using algebra rules that are specific to matrices.To work with matrix algebra rules, it is helpful to define a zero matrix, a negative matrix , and an identity matrix. A zero matrix is one where all the elements are 0. For example, a 2×2 zero matrix looks like $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

A zero matrix is usually denoted as O. A negative matrix such as $-A$ is simply $-1 \times A$. If

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$A=\begin{pmatrix}1 & -3\\ -2 & 4\end{pmatrix}$, then $-A=\begin{pmatrix}-1 & 3\\ 2 & -4\end{pmatrix}$. The identity matrix for multiplication is a square matrix where the elements on the principle diagonalare 1 and all other elements are 0. Here is an example of a 3×3 identity matrix $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$. The identity matrix is usually denoted as I.

What does the term identity mean in non-matrix algebra? What does this tell you about the properties of the identity matrix? Can the identity matrix for multiplication also be an identity matrix for other operations? Why do you think the identity matrix needs to be a square matrix?

Activity

As you have seen in previous sections, matrix addition is both associative and commutative and matrix multiplication is associative and distributive (but not commutative).

Use at least two examples to show that the following properties are true:

$$\begin{aligned}A+O&=A\\ A+(-A)&=O\\ AO=OA&=O\\ AI=IA&=A\quad \text{(where }\mathbf{I}\text{ is the identity matrix under multiplication)}\\ O^2&=O\\ I^2&=I\end{aligned}$$

You can use these properties to manipulate and simplify algebraic expressions that contain matrices.

Example 1

★★☆

Write in expanded form:

a) $(A+B)^2$

b) $(3I+A)^2$ (where $\mathbf{I}$ is the identity matrix under multiplication)

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Overview
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hl/sid-
132-
cid-
761618/ov

	Steps	Explanation
a)	$(A+B)^2 = A^2 + AB + BA + B^2$	Matrix multiplication is nc commutative, $AB \neq BA$ so the middle terms cannot be combined.
b)	$(3I+A)^2 = 3^2I^2 + 3IA + 3AI + A^2 = 9I + 6A + A^2$	To simplify, remember that: $$\begin{aligned} I^2 &= I \\ IA &= A \\ AI &= A \end{aligned}$$

⚠ Be aware

Since matrix multiplication is not commutative, it is very important to write matrix multiplication in the order that it is carried out in the question.

Example 2

★★☆

Given that $A(B+C) = A$ show that
$A(B+C-I) = O,\ \text{(where } \mathbf{I} \text{ is the identity matrix under multiplication)}.$

Steps	Explanation
$$\begin{aligned} A(B+C) &= A \quad \Leftrightarrow \\ A(B+C) - A &= A - A \quad \Leftrightarrow \end{aligned}$$	Add $-A$ to both sides of the expression.
$AB + AC - A = O$	Use: $$\begin{aligned} A + (-A) &= O \\ A(B+C) &= AB + AC \end{aligned}$$
$AB + AC - AI = O$	Rewrite $-A$ as $-AI$ since $-AI = A$.

Student
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Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$A(B+C-I)=O$	Factorise out A.

When you work with non-matrix algebraic equations you often use additive and multiplicative inverses. For example, to solve for x in $3x+2=5$, you do the following:

Steps	Explanation
$3x+2-2=5-2$	Add the additive inverse of 2 to both sides.
$3x \times \frac{1}{3} = 3 \times \frac{1}{3}$	Multiply both sides by the multiplicative inverse of 3.
$x=1$	

The additive inverse of matrices has already been defined. For matrix A it is $-A$.

Due to the nature of matrix multiplication, the multiplicative inverse is a little less simple.

✓ **Important**

The multiplicative inverse of A is A^{-1}. This means that $AA^{-1}=I=A^{-1}A$. For a 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where $ad \neq bc$. This can also be written in terms of the determinantof matrix A so that $A^{-1}=\frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, where $\det A = |A| = ad-bc$.

Multiplication of A and A^{-1} is commutative.

How does this property imply that the inverse is only defined for a square matrix?

Example 3

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Find the inverse of $A = \begin{pmatrix} 1 & 5 \\ 3 & -2 \end{pmatrix}$ and confirm that $AA^{-1} = I$.

Steps	Explanation
$A^{-1} = \frac{1}{1(-2) - 15} \begin{pmatrix} -2 & -5 \\ -3 & 1 \end{pmatrix}$ $= \begin{pmatrix} \frac{2}{17} & \frac{5}{17} \\ \frac{3}{17} & -\frac{1}{17} \end{pmatrix}$	Use: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where a
$AA^{-1} = \begin{pmatrix} 1 & 5 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} \frac{2}{17} & \frac{5}{17} \\ \frac{3}{17} & -\frac{1}{17} \end{pmatrix}$ $= \begin{pmatrix} \frac{2}{17} + \frac{15}{17} & \frac{5}{17} - \frac{5}{17} \\ \frac{6}{17} - \frac{6}{17} & \frac{15}{17} + \frac{2}{17} \end{pmatrix}$ $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $= I$	

✓ **Important**

If $\det A = 0$, then matrix A is called a singular matrix, which means that it does not have an inverse.

Why can't the inverse of a 2×2 matrix be found if the determinant is 0?

Example 4

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Find the values of x such that $A = \begin{pmatrix} x^2 & 2 \\ 4 & 2 \end{pmatrix}$ is not a singular matrix.

Overview
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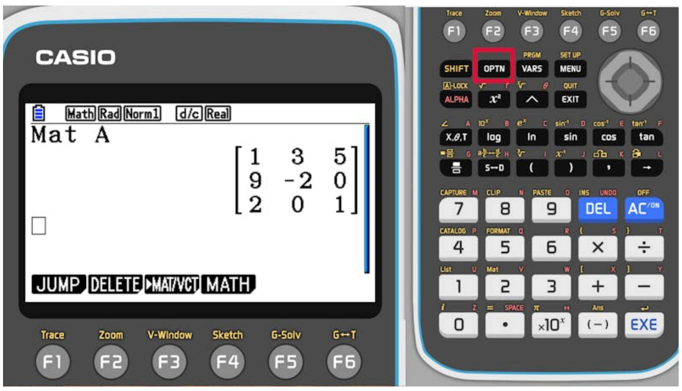
761618/ov

Steps	Explanation
$\det A \neq 0$ $\det A = 2x^2 - 8$	
$2x^2 - 8 = 0 \quad \Leftrightarrow \quad x^2 = 4 \quad \Leftrightarrow \quad x = \pm 2$	Find values of x for which the determinant is .
Matrix is not singular for: $x \neq \pm 2$	

Exam tip

The formula for the inverse and determinant of a 2×2 matrix is given in your IB formula booklet. You will be expected to find both the determinant and the inverse of a 2×2 matrix by hand. Inverses and determinants of other $n \times n$ matrices can be found using the calculator.

Steps	Explanation
These instructions will show you how to find the determinant and the inverse of a matrix that is already stored in the memory of the calculator. If you need help in how to store the matrix in the memory, see the instructions in the previous section. On the calculator page, open the options ...	

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Overview
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hl/sid-
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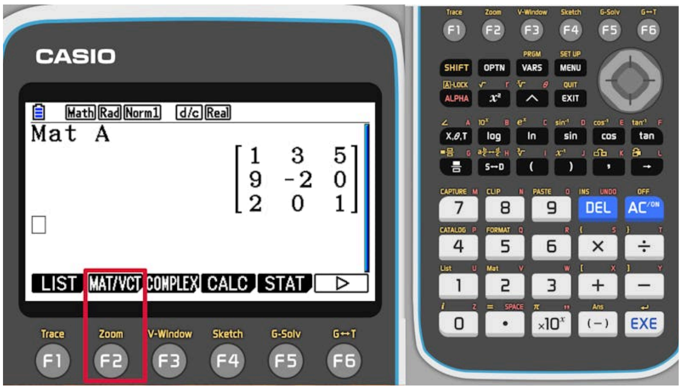
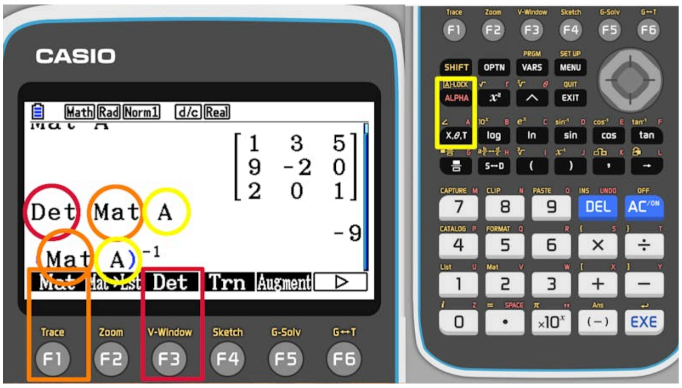
761618/ov

Steps	Explanation
... and press F2 to bring up the options related to matrices.	
To find the determinant, press F3. To tell the calculator that you are working with matrices, press F1. Use the ALPHA key to enter the matrix name. You can find the inverse like you find powers of numbers. You need -1 in the exponent.	

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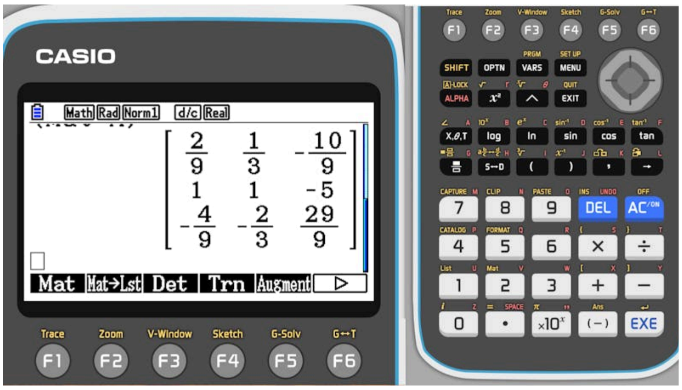
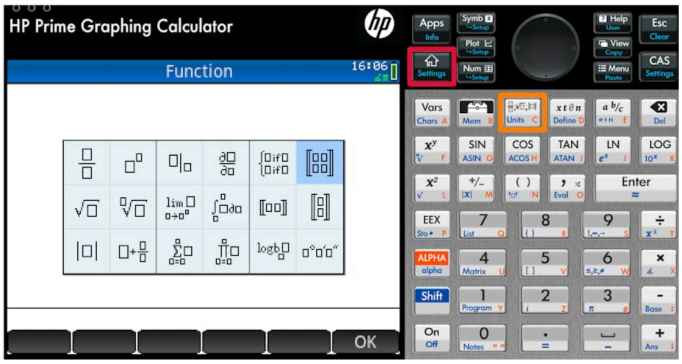
Steps	Explanation
The inverse matrix is displayed.	

Steps	Explanation
These instructions will show you how to find the determinant and the inverse of a matrix. Apart from using the matrix editor, you can also enter matrices directly using the matrix template on the home screen of any application.	

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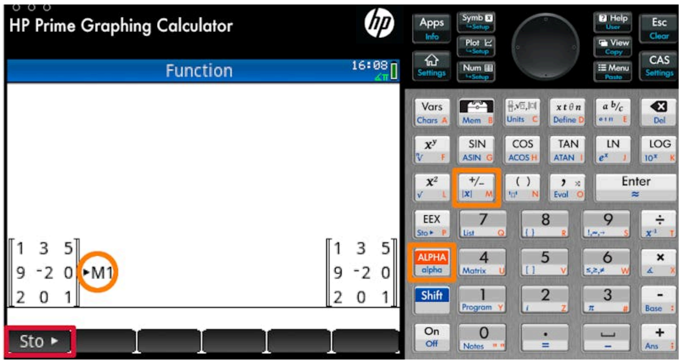
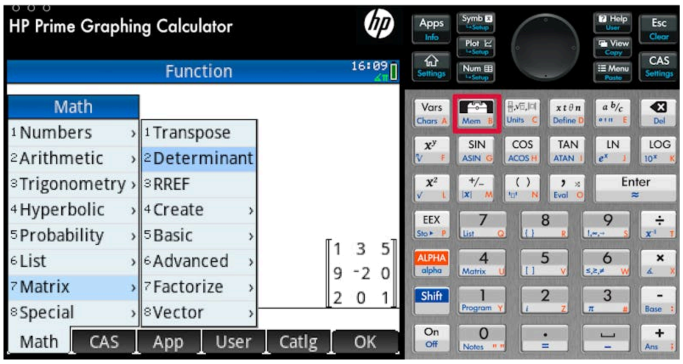
Steps	Explanation
You can store a matrix entered this way. You need to make sure you use a matrix memory name, otherwise you will get an error message. Matrix memory names are M1, M2, ...	
To find the determinant, open the toolbox and find the appropriate tool.	

Overview
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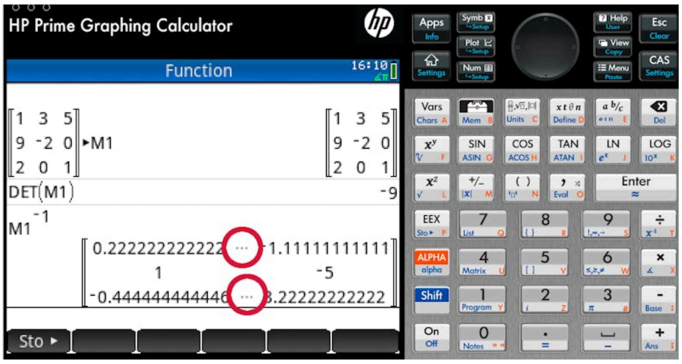
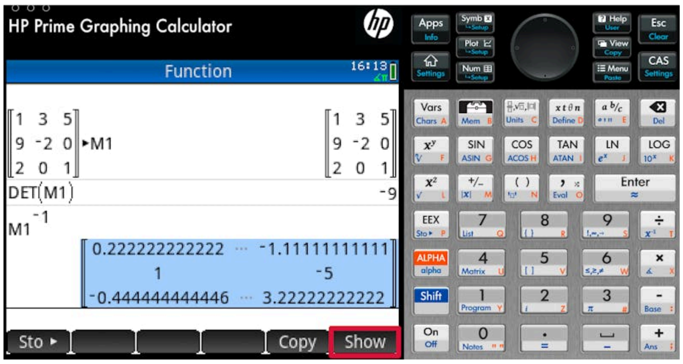
Steps	Explanation
You can find the inverse like you find powers of numbers. You need -1 in the exponent. Notice, that the three dots indicate, that not the full inverse matrix is displayed.	
To see the full inverse matrix, tap on the matrix and ask the calculator to show the details.	

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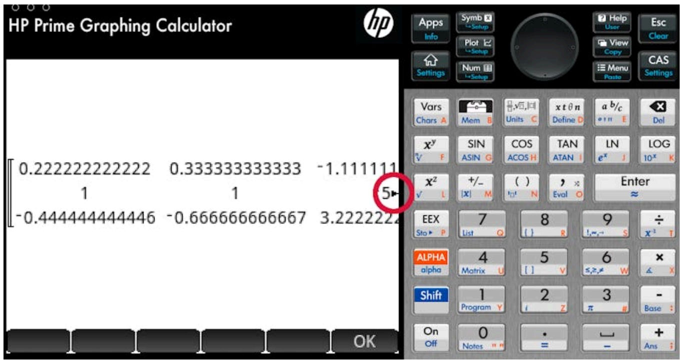
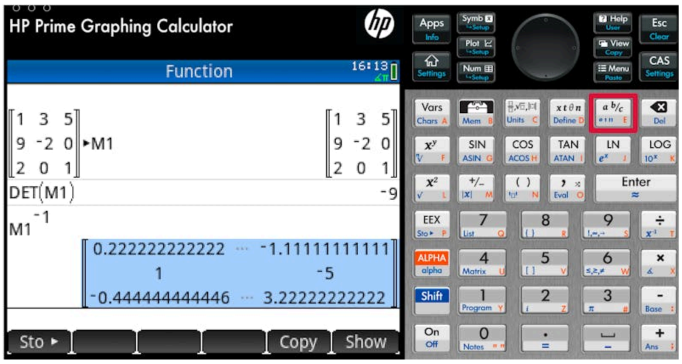
The arrow still indicates, that you need to scroll the screen to see the full matrix.



In this case the calculator also knows the fraction form of the entries of the inverse matrix. To see this form, press the button that converts between different forms of a number.



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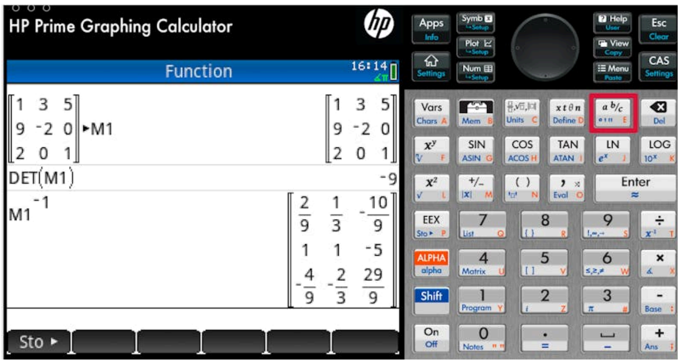
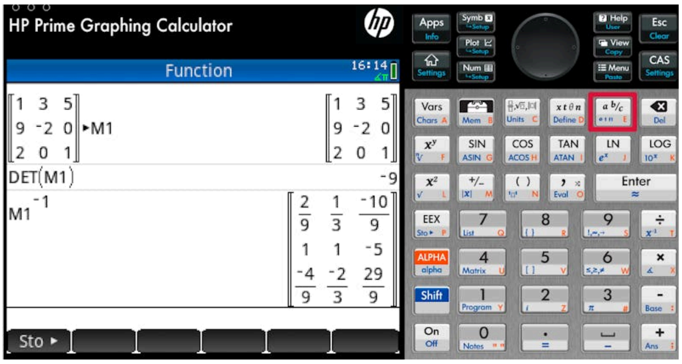
Steps	Explanation
Presssing the button again will give the entries in different forms.	
	

Student view

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132-
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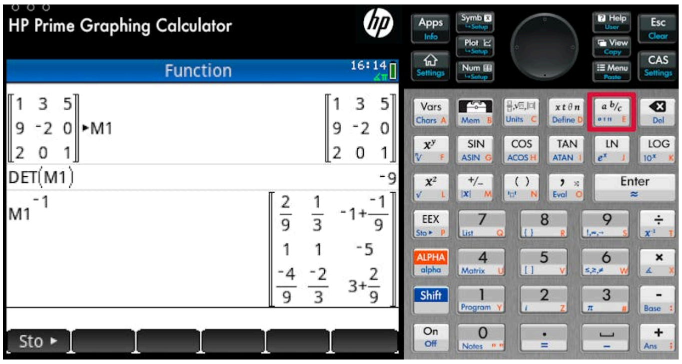
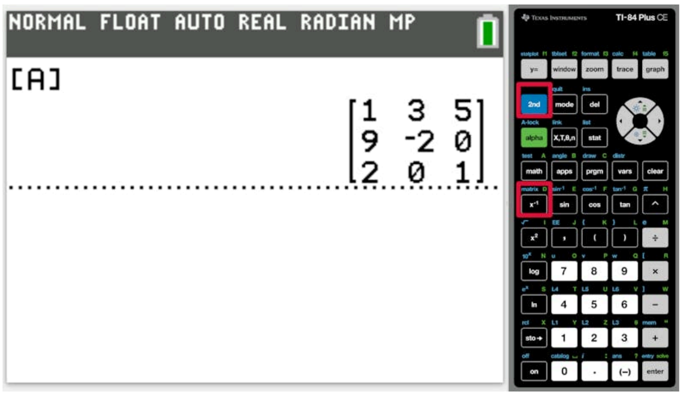
761618/ov

Steps	Explanation
	

Steps	Explanation
These instructions will show you how to find the determinant and the inverse of a matrix that is already stored in the memory of the calculator. If you need help in how to store the matrix in the memory, see the instructions in the previous section. On the calculator page, open the matrix options ...	

Student
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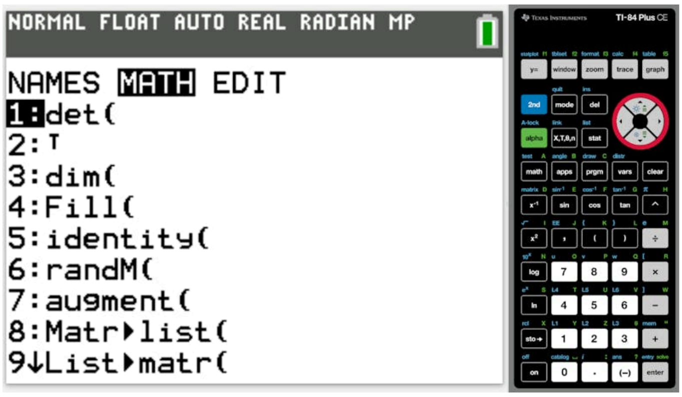
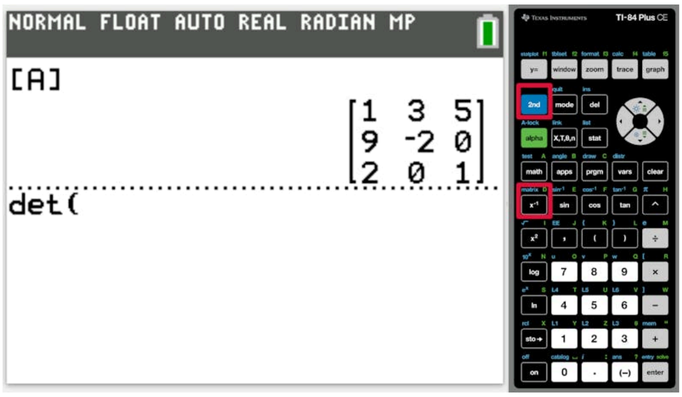
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>... and look for the tool that finds the determinant of a matrix.</td><td>
</td></tr>
<tr><td>You need to tell the name of the matrix you are interested in, so open the matrix tools again ...</td><td>
</td></tr>
</table>

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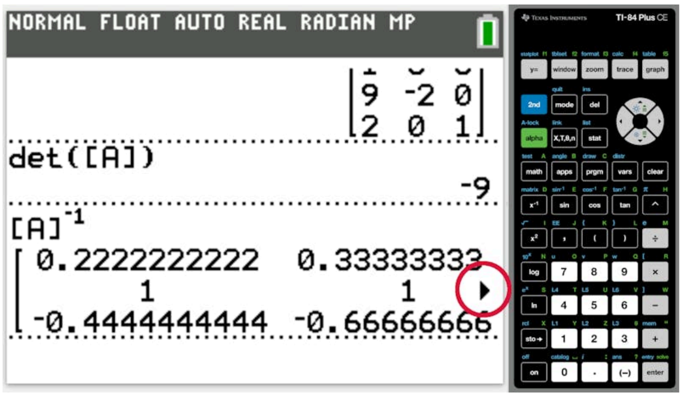
... and choose the name, where you stored your matrix.

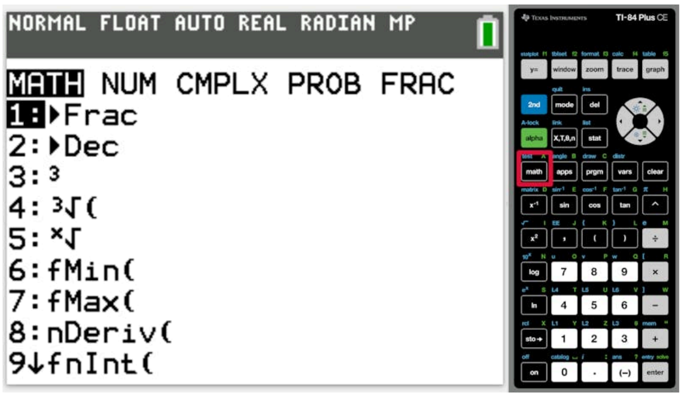
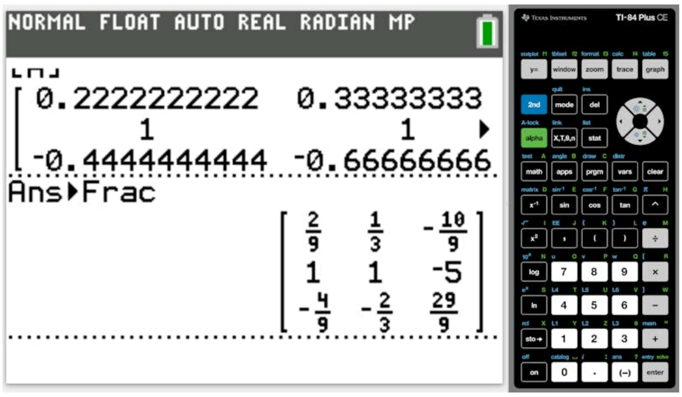
You can find the inverse like you find powers of numbers. You need -1 in the exponent.

Notice, that the arrow indicates, that not the full inverse matrix is displayed. You can scroll left/right to see the full matrix.



Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Steps	Explanation
In this case the calculator also knows the fraction form of the entries of the inverse matrix. To see this form, open the math menu and choose the option to convert decimal numbers to fractions.	
	

Student
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Overview
(/study/ap
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hl/sid-
132-
cid-
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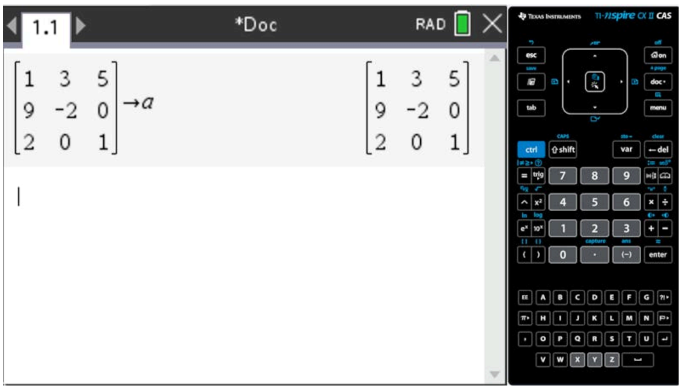
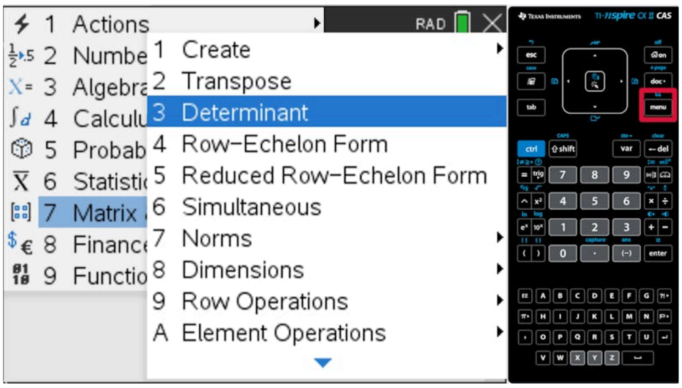
Steps	Explanation
These instructions will show you how to find the determinant and the inverse of a matrix that is already stored in the memory of the calculator. If you need help in how to store the matrix in the memory, see the instructions in the previous section.	
To find the determinant, open the menu and look for the appropriate tool.	

Student
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Overview
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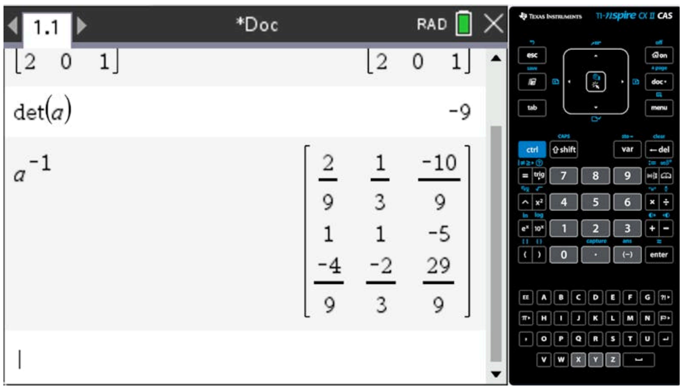
Steps	Explanation
You can find the inverse like you find powers of numbers. You need -1 in the exponent.	

Example 5

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Find the inverse of $\begin{pmatrix} 1 & 3 & 5 \\ 9 & -2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

Steps	Explanation
The inverse matrix is: $$\begin{pmatrix} \frac{2}{9} & \frac{1}{3} & -\frac{10}{9} \\ 1 & 1 & -5 \\ -\frac{4}{9} & -\frac{2}{3} & \frac{29}{9} \end{pmatrix}$$	Since this is 3×3 matrix you should use your calculator

Student view

You can practise finding determinants and inverses of matrices using the calculator and by hand for 2×2 matrices using the applet below.

Overview
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Interactive 1. Find the Determinants and Inverses of Matrices.

More information for interactive 1

This interactive applet allows users to practice finding the determinants and inverses of square matrices. The activity presents a randomly generated square matrix on the screen, with the dimension selectable by the user via a horizontal slider at the top, ranging from 2x2 to 5x5. Users are tasked with calculating the determinant and, if it exists, the inverse of the displayed matrix.

For example,

Let the given matrix be $A = \begin{bmatrix} 2 & -14 \\ 10 & -5 \end{bmatrix}$

Step 1: Find the determinant of the matrix A.

For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

the determinant is given by $ad - bc$.

For matrix A, $a = 2$, $b = -14$, $c = 10$, and $d = -5$.

Determinant of A, denoted as $|A|$ or $\det(A)$, is:

$|A| = (2)(-5) - (-14)(10)$

$|A| = -10 - (-140)$

$|A| = -10 + 140$

$|A| = 130$

Step 2: Find the inverse of the matrix A.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

with a non-zero determinant, the inverse is given by

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Using the values from matrix A and its determinant:

$$A^{-1} = \frac{1}{130}\begin{bmatrix} -5 & -(-14) \\ -10 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{130}\begin{bmatrix} -5 & 14 \\ -10 & 2 \end{bmatrix}$$

To write the inverse with individual elements:

$$A^{-1} = \begin{bmatrix} \frac{5}{130} & \frac{14}{130} \\ \frac{10}{130} & \frac{2}{130} \end{bmatrix}$$

Simplify the fractions to get the inverse of matrix A

$$A^{-1} = \begin{bmatrix} -\frac{1}{26} & \frac{7}{65} \\ -\frac{1}{13} & \frac{1}{65} \end{bmatrix}$$

The final answer is Determinant= 130, Inverse = $\begin{bmatrix} -\frac{1}{26} & \frac{7}{65} \\ -\frac{1}{13} & \frac{1}{65} \end{bmatrix}$

To aid in learning and self-assessment, the applet features two buttons at the bottom: "Show determinant," which reveals the correct determinant value, and "Show inverse," which displays the inverse matrix (when applicable). For generating new practice problems with the current matrix dimension, users can click the "new question" button located at the top right.

International Mindedness

The word matrix is derived from the Latin word related to mother and womb. The name was chosen by James Joseph Sylvester who described matrices as the source of determinants much like a womb is a source of children.

You can use inverse matrices to manipulate algebraic expressions.

Example 6

Student view

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Overview
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a) Solve $AX = B$ for matrix X .

b) Show that your solution is consistent with the orders of B and X for any non-singular square
matrix A.

	Steps	Explanation
a)	$AX = B \quad \Leftrightarrow \quad A^{-1}AX = A^{-1}B \quad \Leftrightarrow$ $IX = A^{-1}B \quad \Leftrightarrow \quad X = A^{-1}B$	Multiply by A^{-1} using the same order. Do not write: $A^{-1}AX = BA^{-1}$. This statement is not true because matrix multiplication is not commutative.
b)	Let A be an $n \times n$ matrix. If $AX = B$, then X must be an $n \times m$ matrix and B is the product of A and X so it must be $n \times m$. A^{-1} is also $n \times n$. In $X = A^{-1}B$, $A^{-1}B$ is the product of an $n \times n$ and a $n \times m$ matrix which has the order $n \times m$. The order of this product is consistent with the order of X.	

Example 7

★★☆

You are given that $AB + C = D$ and that matrices A, B, C and D are 2×2.

Solve the equation for matrix A .

Student
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Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$AB+C-C=D-C \quad \Leftrightarrow$ $AB+O=D-C \quad \Leftrightarrow$ $AB=D-C$	Use the additive inverse of C.
$ABB^{-1}=(D-C)\,B^{-1} \quad \Leftrightarrow$ $AI=(D-C)\,B^{-1} \quad \Leftrightarrow$ $A=(D-C)\,B^{-1}$	Multiply by B^{-1} on both sides. Keep the same order of multiplication on both sides. You should not write: $$ABB^{-1}=B^{-1}\,(D-C)$$ because matrix multiplication is not commutative

Example 8

★★★

Show that $(AB)^{-1}=B^{-1}A^{-1}$.

$$AB(AB)^{-1}=I \quad \Leftrightarrow$$

$$A^{-1}AB(AB)^{-1}=A^{-1}I \quad \Leftrightarrow$$

$$IB(AB)^{-1}=A^{-1} \quad \Leftrightarrow$$

$$B^{-1}B(AB)^{-1}=B^{-1}A^{-1} \quad \Leftrightarrow$$

$$I(AB)^{-1}=B^{-1}A^{-1} \quad \Leftrightarrow$$

$$(AB)^{-1}=B^{-1}A^{-1}$$

5 section questions

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Applications

Systems of linear equations

Making connections

In subtopic 1.8 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27833/) you learned how to solve systems of linear equations using your calculator. In this section you will learn how to solve the same problems using matrices.

Consider the following system of linear equations:

$$\begin{cases} 2x+3y=7 \\ 3x-y=-5 \end{cases}$$

This system can be written as a matrix equation as follows:

$$\begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 7 \\ -5 \end{pmatrix}$$

Example 1

★★☆

Show that $\begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 7 \\ -5 \end{pmatrix}$ is equivalent to $\begin{cases} 2x+3y=7 \\ 3x-y=-5 \end{cases}$.

Steps	Explanation
$\begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2x+3y \\ 3x-y \end{pmatrix}$	Use matrix multiplication.

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$\begin{pmatrix} 2x+3y \\ 3x-y \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$ Therefore: $\begin{aligned} 2x+3y &= 7 \\ 3x-y &= -5 \end{aligned}$	Use matrix equality.

You can write $\begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$ in a more general form as $AX = B$ and solve for X using matrix algebra as follows:

$$A^{-1}AX = A^{-1}B \quad \Leftrightarrow \quad IX = A^{-1}B \quad \Leftrightarrow \quad X = A^{-1}B.$$

This result can be used to solve a system of equations as shown in **Example 2**.

Example 2

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a) Write $\begin{cases} 3x+2y-z=4 \\ x+y+z=0 \\ -2x+y=1 \end{cases}$ in the form $AX = B$.

b) Hence, solve the system of equations.

	Steps	Explanation
a)	$\begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ -2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$	

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	Steps	Explanation
b)	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ -2 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} \frac{1}{10} & \frac{1}{10} & -\frac{3}{10} \\ \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ -\frac{3}{10} & \frac{7}{10} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} \frac{1}{10} \\ \frac{6}{5} \\ -\frac{13}{10} \end{pmatrix}$	Write in $X = A^{-1}B$ form. Use your calculat to find $\begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ -2 & 1 & 0 \end{pmatrix}^{-1}$ and to multiply.
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{10} \\ \frac{6}{5} \\ -\frac{13}{10} \end{pmatrix}$ $x = \frac{1}{10}, y = \frac{6}{5}, z = -\frac{13}{10}$	Use matrix equality to read the solutions.

Example 3

★☆☆

Solve $\begin{cases} 3x - y = 5 \\ -x + 4y = 7 \end{cases}$ using matrix equations.

Steps	Explanation
$\begin{pmatrix} 3 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$	Write in terms of $AX = B$.

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Steps	Explanation
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{4}{11} & \frac{1}{11} \\ \frac{1}{11} & \frac{3}{11} \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{27}{11} \\ \frac{26}{11} \end{pmatrix}$	Rearrange into $X = A^{-1}B$ form. Find the inverse and multip using your calculator.
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{27}{11} \\ \frac{26}{11} \end{pmatrix}$ $x = \frac{27}{11},\ y = \frac{26}{11}$	

Activity

What does it mean if an inverse of A in $AX = B$ can't be found when you use matrix equations to solve a system of linear equations? Support your explanation with $1—2$ examples.

One of the properties of determinants is that $\det A = 0$ if two rows of matrix A are identical or in proportion. Explain how this property of the determinant is consistent with the graphical interpretation of a solution to a system of two linear equations.

Example 4

★★☆

The members of a chess club are ranked based on points awarded for winning games in various competitions. The table shows the number of each type of competition won and the total number of points for the top three members of the club.

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	Practice matches	City tournament	Regional tournament	Total points
Jordi	27	9	2	289
Amanda	24	8	1	218
Zinha	35	5	1	195

Find the number of points awarded for winning a game in each of the three competitions.

Steps	Explanation
Let A represent the number of wins for each player. $$A = \begin{pmatrix} 27 & 9 & 2 \\ 24 & 8 & 1 \\ 35 & 5 & 1 \end{pmatrix}$$ Let B represent the number of points for winning each of the three games. $$B = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$ Let C represent the total number of points for each player. $$C = \begin{pmatrix} 289 \\ 218 \\ 195 \end{pmatrix}$$ Therefore: $$AB = C \quad \Leftrightarrow \quad B = A^{-1}C$$	Let $a,\ b,\ \text{and } c$ be the number of points for winning a game in each type of competition. Recognise that for Jordi the information can be written as: $$27a + 9b + 2c = 289$$ If you write out all three equations yc will have a system of linear equation: with unknowns $a,\ b,\ \text{and } c$ which ca be solved using matrices.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 27 & 9 & 2 \\ 24 & 8 & 1 \\ 35 & 5 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 289 \\ 218 \\ 195 \end{pmatrix} = \begin{pmatrix} 2 \\ 15 \\ 50 \end{pmatrix}$ The players are awarded 2 points for winning a game in practice matches, 15 points for a game in a city tournament, and 50 points for a game in the regional tournament.	

Example 5

★★☆

Participants in a nutritional study are given four kinds of energy bars that they can eat for one of the days in the study. The number of bars consumed by each participant and the total number of calories are noted in the table below.

	Peanut	Vanilla	Cherry	Apricot	Total calories
Participant A	3	8	2	0	1700
Participant B	5	4	1	3	1930
Participant C	2	3	3	4	1800
Participant D	4	4	3	5	2410

Find the number of calories in one apricot flavour bar.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
Let A represent the number of each type of bar consumed by each participant. $$A = \begin{pmatrix} 3 & 8 & 2 & 0 \\ 5 & 4 & 1 & 3 \\ 2 & 3 & 3 & 4 \\ 4 & 4 & 3 & 5 \end{pmatrix}$$ Let B represent the number of calories in each type of bar. $$B = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$ Let C represent the total number of calories consumed by each participant. $$C = \begin{pmatrix} 1700 \\ 1930 \\ 1800 \\ 2410 \end{pmatrix}$$ Therefore: $$AB = C \quad \Leftrightarrow \quad B = A^{-1}C$$	This question can be solved by representing the information as a system of four linear equations with four unknowns.
$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 & 8 & 2 & 0 \\ 5 & 4 & 1 & 3 \\ 2 & 3 & 3 & 4 \\ 4 & 4 & 3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1700 \\ 1930 \\ 1800 \\ 2410 \end{pmatrix} = \begin{pmatrix} 180 \\ 100 \\ 180 \\ 150 \end{pmatrix}$$ One apricot bar provides 150 calories.	

Cryptography

Matrix multiplication can be used to encryptinformation such as a password or a text message.

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Overview
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If you want to encrypt a password that such as *mypassword12* that uses numbers and letters you will start by assigning a number to each of the possible characters, as seen in the table below.

a	b	c	d	e	f	g	h	i	j	k	l	m	n	c
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
s	t	u	v	w	x	y	z	1	2	3	4	5	6	7
19	20	21	22	23	24	25	26	27	28	29	30	31	32	3

Now you can write *mypassword12* in terms of the assigned numbers. This will be:

$$13, 25, 16, 1, 19, 19, 23, 15, 18, 4, 27, 28$$

You can write this message in a matrix. Since there are 12 characters, a 3×4 matrix is convenient. The password can be written as matrix P:

$$P = \begin{pmatrix} 13 & 25 & 16 & 1 \\ 19 & 19 & 23 & 15 \\ 18 & 4 & 27 & 28 \end{pmatrix}.$$

Let $E = \begin{pmatrix} 1 & 5 & 0 \\ -2 & 1 & 3 \\ 4 & 8 & 2 \end{pmatrix}$

Then $EP = \begin{pmatrix} 1 & 5 & 0 \\ -2 & 1 & 3 \\ 4 & 8 & 2 \end{pmatrix} \begin{pmatrix} 13 & 25 & 16 & 1 \\ 19 & 19 & 23 & 15 \\ 18 & 4 & 27 & 28 \end{pmatrix} = \begin{pmatrix} 108 & 120 & 131 & 76 \\ 47 & -19 & 72 & 97 \\ 240 & 260 & 302 & 180 \end{pmatrix}.$

The product matrix is the encoded matrix. You can call it matrix M.

The encoding of the password can be written as a matrix equation $EP = M$.

To decode the message, you must solve $EP = M$ for P.

$$E^{-1}EP = E^{-1}M \quad \Rightarrow \quad P = E^{-1}M.$$

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Example 6

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Given that the encoding matrix is $E = \begin{pmatrix} 1 & 5 & 0 \\ -2 & 1 & 3 \\ 4 & 8 & 2 \end{pmatrix}$ and the encoded message is $M = \begin{pmatrix} 108 & 120 & 131 & 76 \\ 47 & -19 & 72 & 97 \\ 240 & 260 & 302 & 180 \end{pmatrix}$, show that the matrix for the original message is $P = \begin{pmatrix} 13 & 25 & 16 & 1 \\ 19 & 19 & 23 & 15 \\ 18 & 4 & 27 & 28 \end{pmatrix}$.

Steps	Explanation
$P = E^{-1}M$ $P = \begin{pmatrix} 1 & 5 & 0 \\ -2 & 1 & 3 \\ 4 & 8 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 108 & 120 & 131 & 76 \\ 47 & -19 & 72 & 97 \\ 240 & 260 & 302 & 180 \end{pmatrix}$ $= \begin{pmatrix} -\frac{11}{29} & -\frac{5}{29} & \frac{15}{58} \\ \frac{8}{29} & \frac{1}{29} & -\frac{3}{58} \\ -\frac{10}{29} & \frac{6}{29} & \frac{11}{58} \end{pmatrix} \begin{pmatrix} 108 & 120 & 131 & 76 \\ 47 & -19 & 72 & 97 \\ 240 & 260 & 302 & 180 \end{pmatrix}$ $= \begin{pmatrix} 13 & 25 & 16 & 1 \\ 19 & 19 & 23 & 15 \\ 18 & 4 & 27 & 28 \end{pmatrix}$.	Use your calculator to find E^{-1}. If you remember the order i which the letters were placed in matrix P you can read them back using the table that tells you the number of each character. $13 = m,\ 25 = y,\ \ldots$

Can you follow the same procedure to encode and decode the message if you use $PE = M$, where P is the matrix for the unencoded password, E is the encryption matrix, and M is the encrypted message? How would you modify matrix E and your calculations?

Example 7

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Student view

Use the table and encoding matrix $E = \begin{pmatrix} 2 & 1 \\ -3 & 7 \end{pmatrix}$ to encode *hlmath1*.

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761618/ov

a	b	c	d	e	f	g	h	i	j	k	l	m	n
1	2	3	4	5	6	7	8	9	10	11	12	13	14
t	u	v	w	x	y	z	1	2	3	4	5	6	7
20	21	22	23	24	25	26	27	28	29	30	31	32	33

Steps	Explanation
$P = \begin{pmatrix} 8 & 12 & 13 & 1 \\ 20 & 8 & 27 & 37 \end{pmatrix}$	Since E is a 2×2 matrix and there are 7 characters in the message yo can use a 2×4 matrix for P. This allows for multiplication EP. The 8 character can be defined as space.
Let M be the encoded matrix. $$\begin{aligned} M &= EP \\ &= \begin{pmatrix} 2 & 1 \\ -3 & 7 \end{pmatrix} \begin{pmatrix} 8 & 12 & 13 & 1 \\ 20 & 8 & 27 & 37 \end{pmatrix} \\ &= \begin{pmatrix} 36 & 32 & 53 & 39 \\ 116 & 20 & 150 & 256 \end{pmatrix} \end{aligned}$$	

Activity

Create your own message that you want to encrypt and a table that tells you the number associated with every possible character in the message.

How will you modify the table of characters for a case sensitive message? What about a message that contains symbols?

Create an encoding matrix and encode your message. Check that your message can be successfully decoded.

What characteristics must the encoding matrix possess? Does each message that needs to be encoded have a unique encoding matrix? Does each message that needs to be decoded have a unique encoding matrix?

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4 section questions

Checklist

What you should know

By the end of this subtopic you should be able to:

- write data in matrix form
- interpret matrix notation
- add and subtract matrices
- multiply a matrix by a scalar
- calculate the product of matrices and state the conditions where this is possible
- find the determinant and the inverse of a 2×2 matrix algebraically and with a calculator and understand the conditions where this is possible
- find the determinant and the inverse of an $n \times n$ matrix with a calculator and understand the conditions where this is possible
- know the properties of matrix algebra and use them to simplify or solve matrix equations
- write systems of linear equations in terms of matrix equations and solve using matrix algebra
- understand how matrix multiplication can be use in cryptography
- use matrix multiplication to encrypt and decrypt messages.

Investigation

Part 1

Investigate the properties of determinants of 2×2 matrices. Find an expression for $\det(kA)$ where $k \in \mathbb{R}$ and for $\det AB$.

What happens to the determinant if the columns or the rows of a 2×2 matrix are swapped?

Part 2

In non-matrix algebra you often use the null factor law to solve quadratic equations. For instance, you might find that $(x-a)(x-b)=0$ and therefore conclude that $x-a=0$ and $x-b=0$.

Does the same law apply to matrices?

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. How do your findings about the null factor law apply to finding the solution for A in $A^2 = A$?

Part 3

Diagonal matrices are of the form $\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$ where the elements outside the principle diagonal are 0.

Investigate the properties of the powers of this kind of matrix. Do these properties extend to other types of matrices

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