Index

The big picture

In subtopic 5.5 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25533/) you were introduced to the concept of both the indefinite integral and definite integral. You learned how to use technology to find approximate values of definite integrals, and how definite integrals can be used to find the areas of a region bounded by the x-axis and the graph of a function. In this subtopic you will see how to find exact values of definite integrals using indefinite integrals. You will learn to use the techniques to find indefinite integrals that were introduced in section 5.5.3 (/study/app/math-aa-hl/sid-134-cid-761926/book/antiderivatives-of-power-functions-id-25567/) and further developed in subtopic 5.10 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27807/).

You already encountered the applet below in section 5.5.7 (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-25571/).

The applet shows the graph of a function f and the graph of its derivative, f'. It also lets you move a point on the x-axis, calculating the definite integral of $f'(x)$ over the interval from the origin to this moving point. If we use the notation p for the x-coordinate of this moving point, the applet shows the point $\mathrm{P}\left(p, \int_0^p f'(x)\,\mathrm{d}x\right)$, and the trace of this point.

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Interactive 1. Find Exact Values of Definite Integrals Using Indefinite Integrals.

More information for interactive 1

The interactive allows users to explore the relationship between a function $f(x)$ and its derivative $f'(x)$. The applet displays the graph of $f(x)$ in blue and the graph of its derivative $f'(x)$ in red. There are two options for understanding the interactive. 'Trace point P' and 'Adjust Curve'

When 'Trace point P' is selected users can move a point P along the x-axis, starting from the origin, and the applet calculates the definite integral of $f'(x)$ over the interval from 0 to p, where p is the x-coordinate of P. The point $P\left(p, \int_0^p f'(x)\right)$ traces its path as P moves, highlighted in light blue color on the graph. At $\int_0^2 f'(x)\,dx = 0.81$ the point is $P(2, 0.81)$.

Users can move the red dot along the x axis and note the real time change in the equation and the coordinate of point P changing.

When the 'Adjust curve' is selected, six interactive red points appear on the blue graph, which when dragged, moves the whole curve according to the equation.

This interactive provides a hands-on way to understand the connection between differentiation, integration, and the geometric interpretation of areas under curves, enhancing both algebraic and geometric understanding.

Activity

- Move the red point on the x-axis.
- Is there something you notice about the trace of point P?

Student view

- Change the shape of the curve and repeat the process. Do you need to modify your observation?

Concept

Throughout this subtopic, think about how the **general** statements help you to use the **relationship** between indefinite and definite integrals to find exact areas of specific regions bounded by curves.

Bear in mind that the same integral symbol is used in the **representation** of two different concepts: area and anti-derivative. Why might this be?

Theory of Knowledge

The trapezoidal rule allows for an approximation. This is a knowledge-based approximation, though it is an approximation none the less. Consider other areas of knowledge and whether or not they allow for approximations. Take history, for example. Does history allow for approximation more or less so than mathematics? Does this difference in methodology strengthen or weaken history as an AOK?

Knowledge Question: Can approximation be considered knowledge?

5. Calculus / 5.11 Definite integrals

Definite integral revisited

In section 5.5.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/definite-integrals-id-25566/) you were introduced to the notation $\int_a^b f(x)\,\mathrm{d}x$ to represent the area of the region bounded by the graph of the positive function f above the interval $[a, b]$.

In this section you will learn how to extend the meaning of this notation for functions that are not necessarily positive.

Student view

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Activity

- Use your graphic display calculator to find some definite integrals.
- Sketch the graphs of the functions you used and find the area below the graphs.
- Compare the results you get.
- Experiment with functions that are not always positive.
  For example, find $\int_{-1}^{1} x^3 \, \mathrm{d}x$. What do you notice?

In the previous activity, you should have found that $\int_{-1}^{1} x^3 \, \mathrm{d}x = 0$. This is clearly not the area of the region bounded by the graph and the x-axis. Although there is a connection between area and definite integral, these two concepts are not the same.

The applet below shows a graph of a function f. It also shows a region and lets you modify the lines that bound the region from the left and from the right. The applet gives the area of the region and also the value of the integral of the function between the given limits. Can you find a relationship between the area of the region and the value of the definite integral?

Interactive 1. Graph Showing the Area of a Region and the Value of the Definite Integral.

More information for interactive 1

Student view

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This interactive allows the users to understand the graph of a function $f = f(x)\,dx$. It shows the region covered between $x = 0$ to 5. There are two toggle options, 'Adjust the bounds of the region' and 'Adjust curve'.

When the 'Adjust the bounds of the region' is selected the real time area of the region 3.23 is displayed along with the equation $\int_0^5 f(x)\,dx = -0.6$. There are two points on the graph that can be moved.

The user can modify the points on the x axis between 0 and 5 affecting area and the equation. and the value of the integral of the function between the given limits. Here, if the users adjust the area of bound between 3 and the 4, according to the curve the area will lie between 0.67 and 0.3.

When 'Adjust curve' is selected instead of two points 5 points appear on the screen which can be moved in x and y axis. The equation becomes limited between 3.85 and 1. Moving any point changes the value in the real time of the area and the equation.

Thus, allowing users to find a relationship between the area of the region and the value of the definite integral.

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27908/print/) Assign

Activity

- Move both points to create an interval such that the function is positive between the two points. What do you notice?
- Move both points to create an interval such that the function is negative between the two points. What do you notice?
- What happens if you change the order of the two points?
- How is the value of the integral related to the area of the parts of the region?

The observations above justify the following definition.

Important

- For $a < b$, the **definite integral** (of the continuous function f)
$$\int_a^b f(x)\,\mathrm{d}x$$
is the signed area of the region bounded by the lines $x = a$, $x = b$, the x-axis and the graph of $y = f(x)$.
In the signed area the part of the region above the x-axis counts as positive, the part of the region below the x-axis counts as negative.
- For $a = b$, the value of the definite integral is 0.

Student
view

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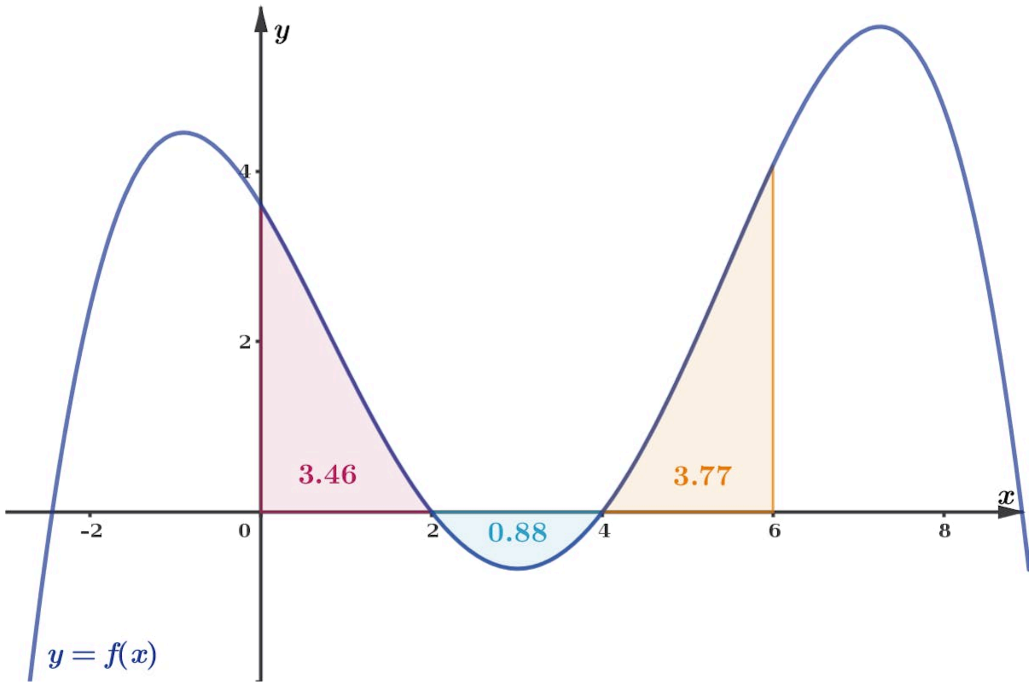
- For $a > b$,
$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$$

Example 1

★☆☆

The diagram below shows part of the graph of $y = f(x)$. There are also three shaded regions on the diagram. The area of these regions are 3.46, 0.88 and 3.77 units squared.



More information

The image is a graph depicting a function y=f(x) with three distinct shaded regions. The X-axis ranges from approximately -3 to 8, and the Y-axis ranges from -5 to 5. The graph shows a curve that peaks at around y=4 when x=-2 and again at x=7. The curve has a minimum point at approximately x=3.5 where the function dips close to the X-axis.

There are three shaded regions under the curve: 1. The first region extends from x=-2 to x=0, ending at the Y-axis. This region is shaded differently and its area is 3.46 units squared, labeled in red. 2. The second region is between x=0 and x=4 and has an area of 0.88 units squared, labeled in blue. 3. The third shaded area spans from x=4 to x=6 and has an area of 3.77 units squared, labeled in orange.

Student view

These regions represent integration segments under the curve of y=f(x).

[Generated by AI]

- Find $\int_0^4 f(x)\,\mathrm{d}x$.
- Find $\int_6^2 f(x)\,\mathrm{d}x$.

- Since the region above the x-axis counts as positive and the region below the x-axis counts as negative,
$$\int_0^4 f(x)\,\mathrm{d}x = 3.46 - 0.88 = 2.58.$$
- Similar to the previous part,
$$\int_2^6 f(x)\,\mathrm{d}x = -0.88 + 3.77 = 2.89.$$
Hence,
$$\int_6^2 f(x)\,\mathrm{d}x = -\int_2^6 f(x)\,\mathrm{d}x = -2.89.$$

In the example above, the areas of some regions were used to find the value of definite integrals. It is more common to do the reverse; to use the definite integral to find the area of a region.

Example 2

★☆☆

The diagram below shows part of the graph of $y = x^3 - 7x^2 + 14x - 8$.

Overview
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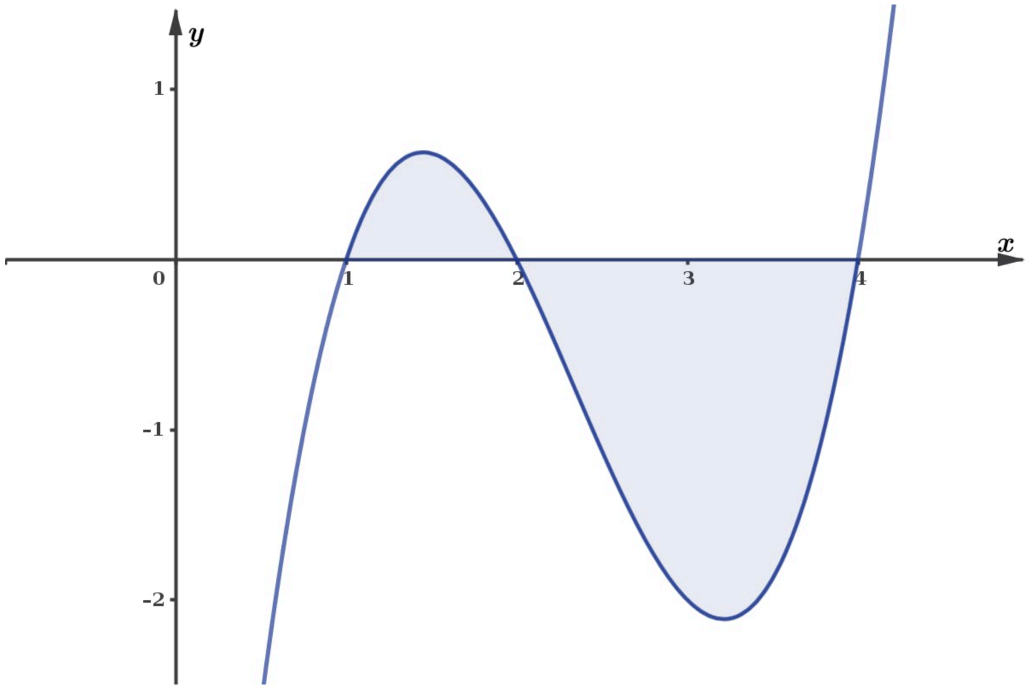

More information

The diagram depicts a graph of the function (y = x^3 - 7x^2 + 14x - 8). The X-axis is labeled from 0 to 4, with intervals of 1, and the Y-axis is labeled from -2 to 1, also in intervals of 1. The graph demonstrates a curve characteristic of a cubic function. Starting at the origin, the curve dips below the X-axis reaching a minimum, then rises above the axis at x=1, peaking, and continues to drop back below the X-axis, forming another minimum between x=3 and x=4, before rising sharply again. The axes intersect at the origin, illustrating the point where (x = 0) and (y = 0). There is shading below the curve in the positive intervals of the Y-axis, emphasizing the area under the curve from the peak to where it intersects with the X-axis towards the value of 4 on the x-axis, highlighting the integral of the function over this interval.

[Generated by AI]

- Find $\displaystyle\int_1^2 x^3 - 7x^2 + 14x - 8\,\mathrm{d}x$.
- Find $\displaystyle\int_2^4 x^3 - 7x^2 + 14x - 8\,\mathrm{d}x$.
- Find $\displaystyle\int_1^4 x^3 - 7x^2 + 14x - 8\,\mathrm{d}x$.
- Find the area of the shaded region.

Student view

Graphic display calculators have applications that can find the definite integrals, which are

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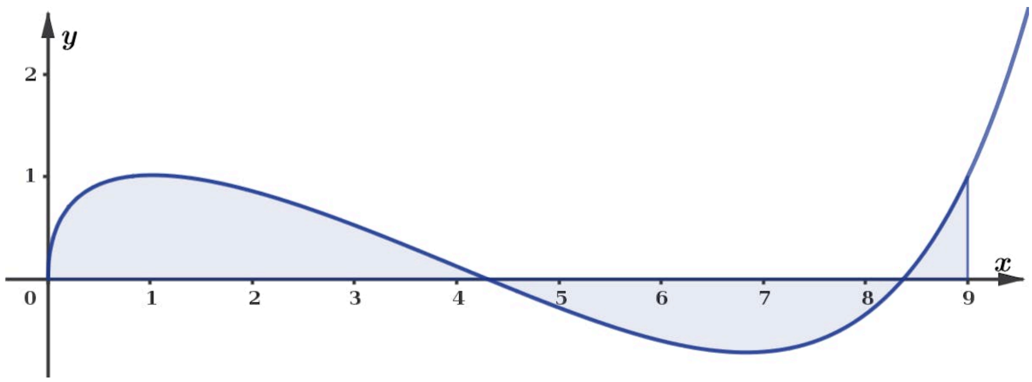
- $\int_1^2 x^3 - 7x^2 + 14x - 8\,\mathrm{d}x \approx 0.41667 \approx 0.417$
- $\int_2^4 x^3 - 7x^2 + 14x - 8\,\mathrm{d}x \approx -2.66667 \approx -2.67$
- $\int_1^4 x^3 - 7x^2 + 14x - 8\,\mathrm{d}x \approx -2.25000 = -2.25$
- The area of the shaded region is the sum of the area of the regions above and below the x-axis. The first integral gives the area of the region above the x-axis. The second integral without the negative sign gives the area of the region below the x-axis. So the total area is

$$0.41667 + 2.66667 = 3.08334 \approx 3.08 \text{ units squared.}$$

Example 3

★★☆

Find the area of the region between x-axis, the graph of $y = 2\sqrt{x} - x + 2^{x-7}$ and the vertical
line $x = 9$.



More information

The image is a graph depicting the function y = 2√x - x + 2^(x-7). It shows a curve that starts at the origin, rises and peaks around x = 1, and then forms a trough near x = 5. The curve rises again sharply as x approaches 9. The x-axis ranges from 0 to 9, while the y-axis extends to slightly above 2. The area between the x-axis, the curve, and the vertical line at x = 9 is shaded, indicating the region whose area needs to be calculated. The curve exhibits a bump near x = 1, dips below the x-axis around x = 5, and then ascends steeply after x = 7 toward x = 9.

Student view

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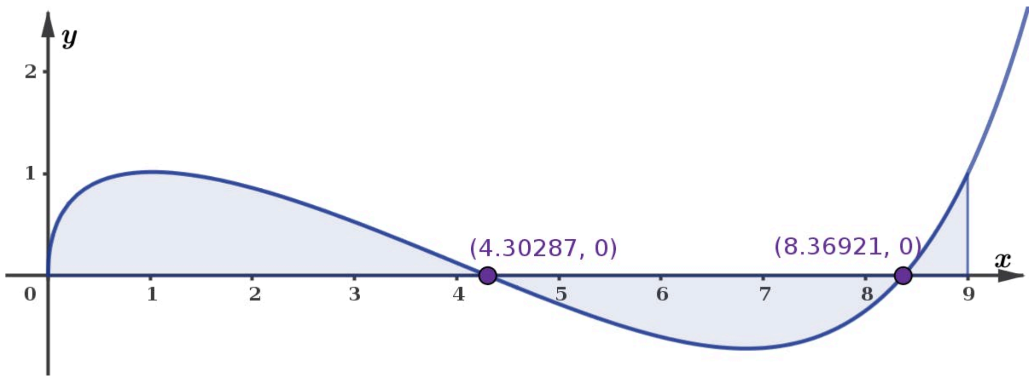
First of all, note that there is no left bound given for the region. This means you will need to use $x = 0$ as the left bound, since the graph starts at the origin.

Method 1

Since there are parts of the graph both below and above the x-axis, you need to find the values where the graph intersects the x-axis and use these values as bounds to find the areas of the regions separately.

The diagram below shows the intersection points.



The area of the first part of the region is approximately

$$\int_0^{4.30287} 2\sqrt{x} - x + 2^{x-7}\,\mathrm{d}x \approx 2.85464.$$

For the second part, we find

$$\int_{4.30287}^{8.36921} 2\sqrt{x} - x + 2^{x-7}\,\mathrm{d}x \approx -1.87848.$$

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So the area of the second part is approximately 1.87848 units squared.

The area of the third part of the region is approximately

$$\int_{8.36921}^{9} 2\sqrt{x} - x + 2^{x-7}\,\mathrm{d}x \approx 0.28335.$$

Hence, the area of the shaded region is approximately

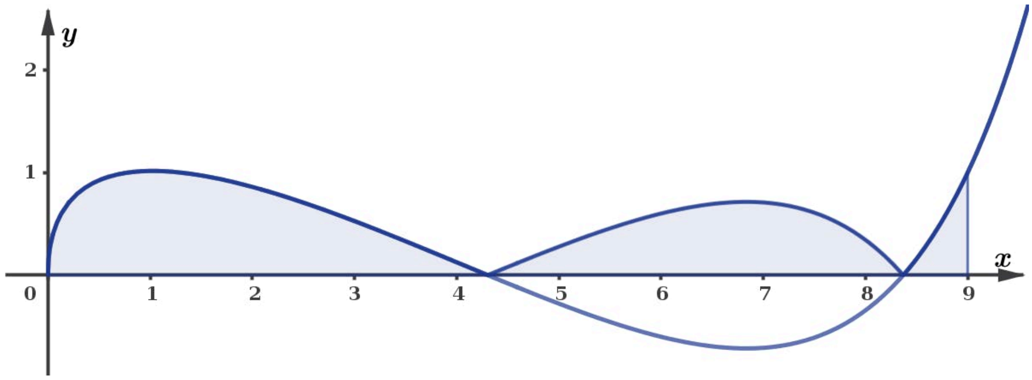
$$2.85464 + 1.87848 + 0.28335 = 5.01647 \approx 5.02 \text{ units squared.}$$

Method 2

When using technology to find the area, you can use a shortcut.

The diagram below shows the graph of

$$y = \left|2\sqrt{x} - x + 2^{x-7}\right|.$$



Note that this graph is entirely above the x-axis, and the area bounded by this graph and the x-axis is the same as the area of the original regions. Hence you can find this area without the need of to find the x-intercepts.

$$\int_{0}^{9} \left|2\sqrt{x} - x + 2^{x-7}\right|\,\mathrm{d}x \approx 5.02$$

Student
view

Exam tip

In the formula booklet, the formula for the area of a region enclosed by a curve and the x-axis is given as

$$A = \int_a^b |y| \, \mathrm{d}x.$$

This represents the area of the region bounded by the x-axis and the graph of $y = f(x)$ over the interval $[a, b]$. Because of the absolute value, this formula gives the correct area even if the graph of $y = f(x)$ has parts below the x-axis.

Note that in **Example 3**, above, you only found an approximate area. To find the exact area is beyond even the capabilities of WolframAlpha (http://www.wolframalpha.com). If you are interested in the answer given by WolframAlpha, type

area below 2sqrt(x)-x+2^(x-7) for 0<x<9

into the search line. It is, however, beyond even the HL syllabus to fully understand the answer.

With the applet below, you can check your understanding.

Interactive 2. Graph To Find the Area of a Region.

More information for interactive 2

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

This interactive enables users to calculate the area of a shaded region using the concept of definite integrals. By applying integration within given limits, users can determine the enclosed area between a curve and the x-axis. Users can practice multiple problems by clicking on the 'Click here for a new question' button and verify their solutions using the 'Show answer' option.

For example, consider the region bounded by the x-axis, the graph of the function

$y = -x^2 + 2.8x + 42.2$

and the vertical lines $x = -5.3$ and $x = 6.5$. The area of this region is approximately 376.1667 square units.

By integrating the function within the given limits, users can compute the enclosed area accurately. This interactive tool helps in reinforcing the application of definite integrals, enhancing conceptual understanding through hands-on problem-solving.

4 section questions

Question 1

Difficulty: ★☆☆

Find $\int_{-2}^{2} \frac{1+x^3}{1+x^2}\, \mathrm{d}x$.

Give your answer as a decimal, accurate to 3 significant figures.

2.21 ✓

Accepted answers

2.21, 2,21

Explanation

Graphic display calculators have applications to find definite integrals.

$$\int_{-2}^{2} \frac{1+x^3}{1+x^2}\, \mathrm{d}x \approx 2.21$$

Question 2

Difficulty: ★★☆

Find the area of the region between the graph of $y = \frac{1+x^3}{1+x^2}$ and the x-axis over the interval $[-2, 2]$.

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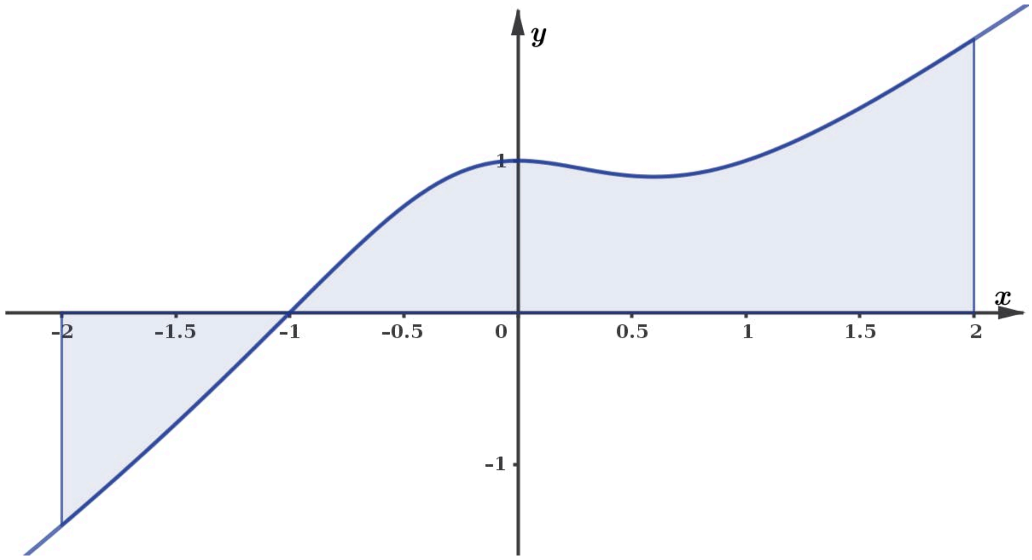
Give your answer as a decimal, accurate to 3 significant figures.



More information

3.65

Accepted answers

3.65, 3,65

Explanation

Method 1

Find, separately, the area of the parts above and below the x-axis.

The x-intercept of the graph is at $x = -1$

$$\int_{-2}^{-1} \frac{1+x^3}{1+x^2}\,\mathrm{d}x \approx -0.720104,$$

so the area of the part below the x-axis is approximately 0.720104.

$$\int_{-1}^{2} \frac{1+x^3}{1+x^2}\,\mathrm{d}x \approx 2.934402,$$

so the area of the part above the x-axis is approximately 2.934402.

Hence, the total area of the region is approximately

$$0.720104 + 2.934402 \approx 3.65.$$

Method 2

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As a shortcut, you can get the same result by integrating the absolute value of the expression:

$$\int_{-2}^{2} \left| \frac{1+x^3}{1+x^2} \right| \mathrm{d}x \approx 3.65.$$

Question 3

Difficulty:

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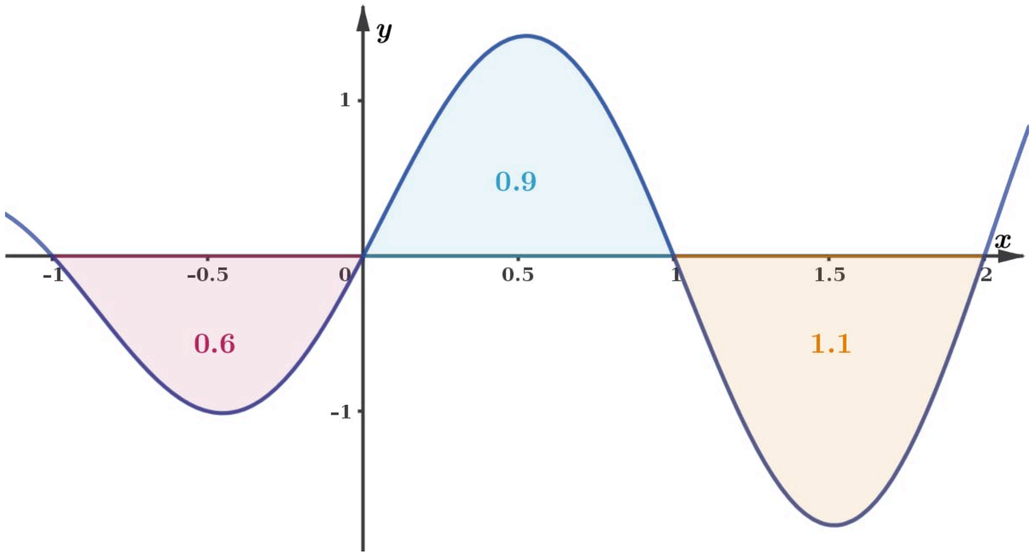
The diagram below shows the graph of $y = f(x)$.



More information

The solutions of $f(x) = 0$ are $x = -1$, $x = 0$, $x = 1$ and $x = 2$.

The area of the three shaded regions on the diagram are 0.6, 0.9 and 1.1.

Find $\int_{-1}^{2} f(x)\,\mathrm{d}x$.

Give an exact answer in decimal form.

-0.8

Accepted answers

-0.8, -0,8, -.8

Explanation

The definite integral is the signed area of the region between the graph of $y = f(x)$ and the x-axis, so it is

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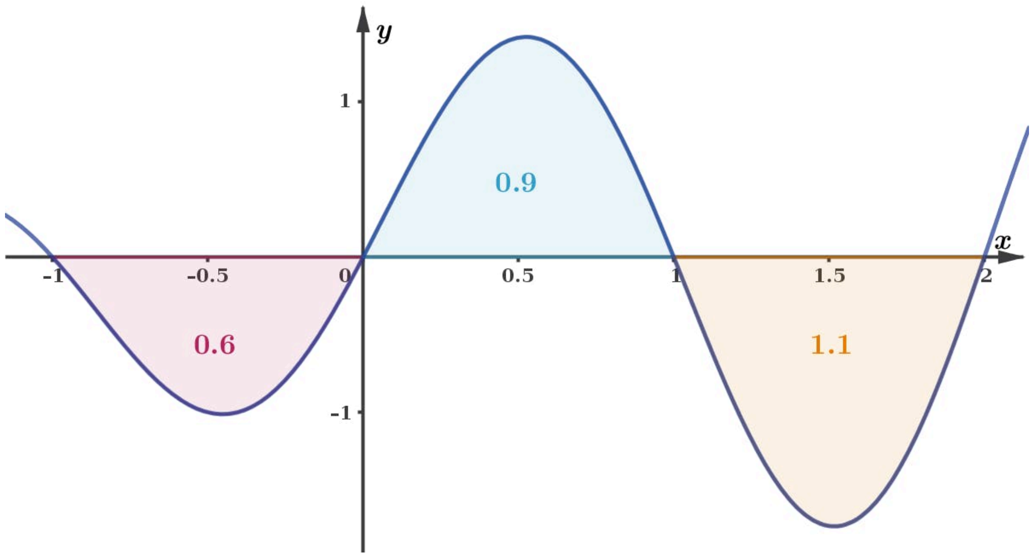
$$\int_{-1}^{2} f(x)\,\mathrm{d}x = -0.6 + 0.9 - 1.1 = -0.8.$$

Question 4

Difficulty: ★★☆

The diagram below shows the graph of $y = f(x)$.



More information

The solutions of $f(x) = 0$ are $x = -1$, $x = 0$, $x = 1$ and $x = 2$.

The area of the three shaded regions on the diagram are 0.6, 0.9 and 1.1.

Find $\int_{2}^{0} f(x)\,\mathrm{d}x$.

Give an exact answer in decimal form.

0.2

Accepted answers

0.2, 0,2, .2

Explanation

The definite integral $\int_{0}^{2} f(x)\,\mathrm{d}x$ is the signed area of the region between the graph of $y = f(x)$ and the x-axis, so

$$\int_{0}^{2} f(x)\,\mathrm{d}x = 0.9 - 1.1 = -0.2.$$

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view

Since

$$\int_2^0 f(x)\,\mathrm{d}x = -\int_0^2 f(x)\,\mathrm{d}x,$$

$$\int_2^0 f(x)\,\mathrm{d}x = -(-0.2) = 0.2.$$

5. Calculus / 5.11 Definite integrals

Newton-Leibniz formula

The following claim connects the concepts of definite and indefinite integrals.

Important

If f is a continuous function defined on the interval $[a, b]$ and F is an anti-derivative of f (so $F'(x) = f(x)$), then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

The proof of this claim goes beyond the syllabus. If you are interested in the claim in full generality and a formal proof, search for **Newton-Leibniz formula** or the **fundamental theorem of calculus** on the internet.

To fully understand the connection between definite and indefinite integrals and the similarity in the notation, let us look at how to apply the claim above. Note that finding the anti-derivative mentioned in the claim involves finding the indefinite integral.

Important

You can find the definite integral $\int_a^b f(x)\,\mathrm{d}x$ in two steps:

1. Find F, an anti-derivative of f by finding the indefinite integral
$$\int f(x)\,\mathrm{d}x.$$

2. Evaluate this anti-derivative at $x = b$ and $x = a$ and find the difference of these values:
$$\int_a^b f(x)\,\mathrm{d}x = \left[\int f(x)\,\mathrm{d}x\right]_a^b = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

Example 1

★☆☆

- Find the exact value of $\displaystyle\int_0^{\pi} \sin x\,\mathrm{d}x$.
- Find the exact value of $\displaystyle\int_0^{1} x^2\,\mathrm{d}x$.
- Find the exact value of $\displaystyle\int_1^{2} \frac{1}{x}\,\mathrm{d}x$.
- Find the exact value of $\displaystyle\int_{-2}^{1} \mathrm{e}^x\,\mathrm{d}x$.

- Since $\displaystyle\int \sin x\,\mathrm{d}x = -\cos x$,
$$\int_0^{\pi} \sin x\,\mathrm{d}x = \Big[-\cos x\Big]_0^{\pi} = (-\cos\pi) - (-\cos 0) = 1 + 1 = 2.$$
- Since $\displaystyle\int x^2\,\mathrm{d}x = \frac{x^3}{3}$,
$$\int_0^1 x^2\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$
- Since $\displaystyle\int \frac{1}{x}\,\mathrm{d}x = \ln x$,
$$\int_1^2 \frac{1}{x}\,\mathrm{d}x = \Big[\ln x\Big]_1^2 = \ln 2 - \ln 1 = \ln 2.$$
- Since $\displaystyle\int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x$,
$$\int_{-2}^1 \mathrm{e}^x\,\mathrm{d}x = \Big[\mathrm{e}^x\Big]_{-2}^1 = \mathrm{e}^1 - \mathrm{e}^{-2} = \mathrm{e} - \frac{1}{\mathrm{e}^2}.$$

Student view

Example 2

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Find the indefinite and definite integrals in the table below.

Indefinite Integral	Definite Integral
$\int \cos(2x-\pi)\,\mathrm{d}x =$	$\int_{\frac{\pi}{3}}^{3\pi} \cos(2x-\pi)\,\mathrm{d}x =$
$\int \frac{2x}{x^2+1}\,\mathrm{d}x =$	$\int_1^2 \frac{2x}{x^2+1}\,\mathrm{d}x =$
$\int x(x-3)^2\,\mathrm{d}x =$	$\int_0^3 x(x-3)^2\,\mathrm{d}x =$
$\int x^2\mathrm{e}^{x^3}\,\mathrm{d}x =$	$\int_{-1}^1 x^2\mathrm{e}^{x^3}\,\mathrm{d}x =$

Indefinite Integral	Definite Integral
$\int \cos(2x-\pi)\,\mathrm{d}x = \frac{1}{2}\sin(2x-\pi)+c$	$\int_{\frac{\pi}{3}}^{3\pi} \cos(2x-\pi)\,\mathrm{d}x = \frac{\sqrt{3}}{4}$
$\int \frac{2x}{x^2+1}\,\mathrm{d}x = \ln(x^2+1)+c$	$\int_1^2 \frac{2x}{x^2+1}\,\mathrm{d}x = \ln\frac{5}{2}$
$\int x(x-3)^2\,\mathrm{d}x = \frac{x^4}{4}-2x^3+\frac{9x^2}{2}+c$	$\int_0^3 x(x-3)^2\,\mathrm{d}x = \frac{27}{4}$
$\int x^2\mathrm{e}^{x^3}\,\mathrm{d}x = \frac{\mathrm{e}^{x^3}}{3}+c$	$\int_{-1}^1 x^2\mathrm{e}^{x^3}\,\mathrm{d}x = \frac{\mathrm{e}^2-1}{3\mathrm{e}}$

With the following applets you can check your understanding.

Student view

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Interactive 1. Definite and Indefinite Integrals.

More information for interactive 1

This interactive allows the users to understand and practice **Newton-Leibniz rule**, it establishes a fundamental connection between differentiation and integration under definite integrals, demonstrating how they are the inverse operations. Users can apply this rule to fully understand the connection between definite and indefinite integrals and the similarity in the notation. To find the area under the curve $f(x)$ from a to b, the users can find the antiderivative $F(x)$, evaluate it at a and b and then subtract the result. For example, $f(x) = x^2$ from $x = 1$ to $x = 3$.

The antiderivative of x^2 is $F(x) = \frac{1}{3}(x^3)$.

On applying Newton Leibniz rule,

$$F(x) = \int_1^3 x^2 dx = F(3) - F(1)$$

$$= \left(\frac{1}{3}\right) \cdot (3^3) - \left(\frac{1}{3}\right) \cdot (1)^3$$
$$= 9 - \frac{1}{3}$$
$$= \frac{26}{3}$$

Users can practice with new questions and check their answers by clicking on the button 'Click here for a new question' on top of the screen. This interactive helps users to understand calculus under curves and solve many other problems in calculus and related fields.

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Interactive 2. Definite and Indefinite Integrals.

More information for interactive 2

This interactive allows users to understand and practice the Newton—Leibniz rule, which establishes a fundamental connection between differentiation and integration in the context of definite integrals. It demonstrates how integration and differentiation are inverse operations and highlights the relationship between definite and indefinite integrals, including their notational similarities.

To calculate the area under a curve $f(x)$ from a to b, users are guided to find the antiderivative $F(x)$, evaluate it at the endpoints a and b, and subtract the results: $F(b) - F(a)$.

For example, if $f(x) = x^2$ from $x = 1$ to $x = 3$.

The antiderivative of x^2 is $F(x) = \frac{1}{3}(x^3)$.

On applying Newton Leibniz rule, $F(x) = \int_1^3 x^2 dx = F(3) - F(1)$

$$= \left(\frac{1}{3}\right)(3^3) - \left(\frac{1}{3}\right)(1^3)$$

$$= 9 - \frac{1}{3}$$

$$= \frac{26}{3}$$

Users can generate multiple problems using the “Click here for new question” option and check their answers with “Show answers.” This hands-on approach helps solidify users' conceptual understanding of integration as accumulation and provides practical experience in solving calculus problems involving areas under curves.

Example 3

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What is the value of $m > 0$ if $\int_m^{m^2} \frac{1}{x}\,\mathrm{d}x = 1$?

For $x > 0$, $\int \frac{1}{x}\,\mathrm{d}x = \ln x + c$. So,

$$\int_m^{m^2} \frac{1}{x}\,\mathrm{d}x = \left[\ln x\right]_m^{m^2} = \ln m^2 - \ln m = \ln \frac{m^2}{m} = \ln m.$$

Hence,

$$\begin{aligned} \ln m &= 1 \\ m &= \mathrm{e}. \end{aligned}$$

Below is a list of some properties of definite integrals. These properties are sometimes useful in evaluating integrals.

Important

If f and g are continuous functions (defined on the given intervals), then

- $\int_a^b f(x) \pm g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$
- $\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$
- $\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x$

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All of these are consequences of the Newton-Leibniz formula.

If $F'(x) = f(x)$ and $G'(x) = g(x)$, then

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$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x &= \Big[F(x)\Big]_a^b \pm \Big[G(x)\Big]_a^b\\ &= \Big(F(b)-F(a)\Big) \pm \Big(G(b)-G(a)\Big)\\ &= \Big(F(b)\pm G(b)\Big) - \Big(F(a)\pm G(a)\Big)\\ &= \Big[F(x)\pm G(x)\Big]_a^b\\ &= \int_a^b f(x)\pm g(x)\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}c\int_a^b f(x)\,\mathrm{d}x &= c\Big[F(x)\Big]_a^b\\ &= c\Big(F(b)-F(a)\Big)\\ &= cF(b)-cF(a)\\ &= \Big[cF(x)\Big]_a^b\\ &= \int_a^b cf(x)\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x &= \Big[F(x)\Big]_a^b + \Big[F(x)\Big]_b^c\\ &= \Big(F(b)-F(a)\Big) + \Big(F(c)-F(b)\Big)\\ &= F(c)-F(a)\\ &= \Big[F(x)\Big]_a^c\\ &= \int_a^c f(x)\,\mathrm{d}x\end{aligned}$$

Example 4

★☆☆

For the functions f and g,

$$\int_0^5 f(x)\,\mathrm{d}x = 4,\ \int_0^2 f(x)\,\mathrm{d}x = 1 \text{ and } \int_5^2 g(x)\,\mathrm{d}x = 6.$$

Find $\int_2^5 2f(x)-7g(x)\,\mathrm{d}x$.

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According to the property

$$\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x,$$

the following is true:

$$\begin{aligned}\int_0^2 f(x)\,\mathrm{d}x + \int_2^5 f(x)\,\mathrm{d}x &= \int_0^5 f(x)\,\mathrm{d}x\\ 1 + \int_2^5 f(x)\,\mathrm{d}x &= 4\\ \int_2^5 f(x)\,\mathrm{d}x &= 4 - 1 = 3\end{aligned}$$

According to the property

$$\int_a^b g(x)\,\mathrm{d} = -\int_b^a g(x)\,\mathrm{d}x,$$

the following is true:

$$\int_2^5 g(x)\,\mathrm{d} = -\int_5^2 g(x)\,\mathrm{d}x = -6$$

Using

$$\int_a^b kf(x)\,\mathrm{d} = k\int_a^b f(x)\,\mathrm{d}x,$$

the following is also true:

$$\begin{aligned}\int_2^5 2f(x) - 7g(x)\,\mathrm{d}x &= 2\int_2^5 f(x)\,\mathrm{d}x - 7\int_2^5 g(x)\,\mathrm{d}x\\ &= 2\times 3 - 7\times(-6) = 48\end{aligned}$$

4 section questions

Student view

Question 1

Difficulty:

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★☆☆

Find the value of $\int_1^2 (2x+1)^3 \, \mathrm{d}x$.

Give an exact answer.

68

Accepted answers

68

Explanation

Expand the expression to find the anti-derivative:

$$\begin{aligned}\int_1^2 (2x+1)^3 \, \mathrm{d}x &= \int_1^2 8x^3 + 12x^2 + 6x + 1 \, \mathrm{d}x \\ &= \left[2x^4 + 4x^3 + 3x^2 + x\right]_1^2 = 78 - 10 = 68.\end{aligned}$$

Question 2

Difficulty:

★★☆

Find the value of p if

$$\int_1^3 \frac{x+1}{x^2+2x} \, \mathrm{d}x = \ln\sqrt{p}.$$

Give an exact answer.

5

Accepted answers

5, p=5

Explanation

Since the derivative of $x^2 + 2x$ is $2x + 2$,

$$\int \frac{x+1}{x^2+2x} \, \mathrm{d}x = \frac{1}{2}\int \frac{2x+2}{x^2+2x} \, \mathrm{d}x = \frac{1}{2}\ln(x^2+2x) + c.$$

Hence,

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$$\int_1^3 \frac{x+1}{x^2+2x}\,\mathrm{d}x = \left[\frac{1}{2}\ln(x^2+2x)\right]_1^3$$
$$= \frac{1}{2}\ln(3^2+2\times 3) - \frac{1}{2}\ln(1^2+2\times 1)$$
$$= \frac{1}{2}(\ln 15 - \ln 3) = \frac{1}{2}\ln\frac{15}{3} = \frac{1}{2}\ln 5 = \ln\sqrt{5}.$$

So, $p = 5$.

Question 3

Difficulty: ★★☆

Find the value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^3 x \cos x\,\mathrm{d}x$.

Give an exact answer as a fully simplified fraction (in the form n/m).

5/64 ✓

Accepted answers

5/64

Explanation

Use the substitution $u = \sin x$ to find $\int \sin^3 x \cos x\,\mathrm{d}x$.

Since $\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$, in the integral replace $\cos x\,\mathrm{d}x$ with $\mathrm{d}u$.

So,

$$\int \sin^3 x\cos x\,\mathrm{d}x = \int u^3\,\mathrm{d}u = \frac{u^4}{4} + c = \frac{\sin^4 x}{4} + c.$$

Hence,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^3 x\cos x\,\mathrm{d}x = \left[\frac{\sin^4 x}{4}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$= \frac{\sin^4\frac{\pi}{3}}{4} - \frac{\sin^4\frac{\pi}{4}}{4}$$
$$= \frac{1}{4}\left(\left(\frac{\sqrt{3}}{2}\right)^4 - \left(\frac{\sqrt{2}}{2}\right)^4\right)$$
$$= \frac{1}{4}\times\frac{9-4}{16} = \frac{5}{64}.$$

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Question 4

Difficulty: ★☆☆

For the functions f and g,

$$\int_{-1}^{3} f(x)\,\mathrm{d}x = 5,\ \int_{-1}^{0} f(x)\,\mathrm{d}x = 2 \text{ and } \int_{3}^{0} g(x)\,\mathrm{d}x = 4.$$

Find $\displaystyle\int_{0}^{3} 7f(x) - g(x)\,\mathrm{d}x$.

Give an exact answer as an integer.

25 ✓

Accepted answers

25

Explanation

According to the property

$$\int_{a}^{b} f(x)\,\mathrm{d}x + \int_{b}^{c} f(x)\,\mathrm{d}x = \int_{a}^{c} f(x)\,\mathrm{d}x,$$

the following is true:

$$\begin{aligned}\int_{-1}^{0} f(x)\,\mathrm{d}x + \int_{0}^{3} f(x)\,\mathrm{d}x &= \int_{-1}^{3} f(x)\,\mathrm{d}x \\ 2 + \int_{0}^{3} f(x)\,\mathrm{d}x &= 5 \\ \int_{0}^{3} f(x)\,\mathrm{d}x &= 5 - 2 = 3.\end{aligned}$$

According to the property

$$\int_{a}^{b} g(x)\,\mathrm{d} = -\int_{b}^{a} g(x)\,\mathrm{d}x,$$

the following is true:

$$\int_{0}^{3} g(x)\,\mathrm{d} = -\int_{3}^{0} g(x)\,\mathrm{d}x = -4.$$

Using

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$$\int_a^b kf(x)\,\mathrm{d} = k\int_a^b f(x)\,\mathrm{d}x,$$

the following is also true:

$$\begin{aligned}\int_0^3 7f(x) - g(x)\,\mathrm{d}x &= 7\int_0^3 f(x)\,\mathrm{d}x - \int_0^3 g(x)\,\mathrm{d}x\\ &= 7\times 3 - (-4) = 25.\end{aligned}$$

5. Calculus / 5.11 Definite integrals

Area and integral

In section 5.11.1 (/study/app/math-aa-hl/sid-134-cid-761926/book/definite-integral-revisited-id-27909/) you studied the connection between area of regions bounded by a graph and the x-axis and the definite integral. You have also seen how to find areas using a graphic display calculator.

In this section you will learn to use the Newton-Leibniz formula to find exact values of areas. Here is an abstract example.

Be aware

Sometimes the form $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$ is used to present the Newton-Leibniz formula.

Example 1

★☆☆

The diagram below shows the graph of $y = f'(x)$ and three shaded regions between the graph and the x-axis. The x-intercepts of the graph are $(1, 0)$, $(3, 0)$, $(6, 0)$ and $(10, 0)$. The area of the three regions are 1.36, 2.68 and 12.11 units squared.

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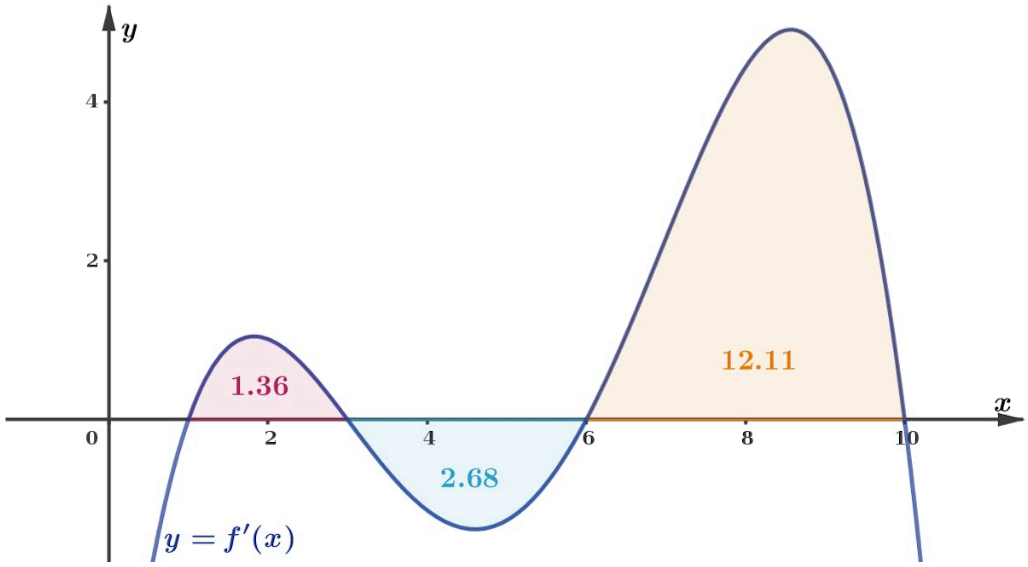

More information

The diagram is a graph of the function (y = f'(x)), showing its curve and the areas under it. The graph has the (x)-intercepts at (x = 1), (x = 3), (x = 6), and (x = 10). There are three distinct shaded regions between the curve and the (x)-axis. The areas of these regions are labeled respectively as 1.36, 2.68, and 12.11 units squared. The graph is displayed on a standard coordinate plane, with the (y)-axis ranging from 0 to 4, and the (x)-axis marked at intervals from 0 to 10.

[Generated by AI]

Given that $f(1) = 5$, find $f(10)$.

Using $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$ for $a = 1$ and $b = 10$,

$$\begin{aligned} \int_1^{10} f'(x)\,\mathrm{d}x &= f(10) - f(1) \\ 1.36 - 2.68 + 12.11 &= f(10) - 5 \\ f(10) &= 15.79. \end{aligned}$$

The next examples show how to use anti-derivatives to find the exact area of specific regions.

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Example 2

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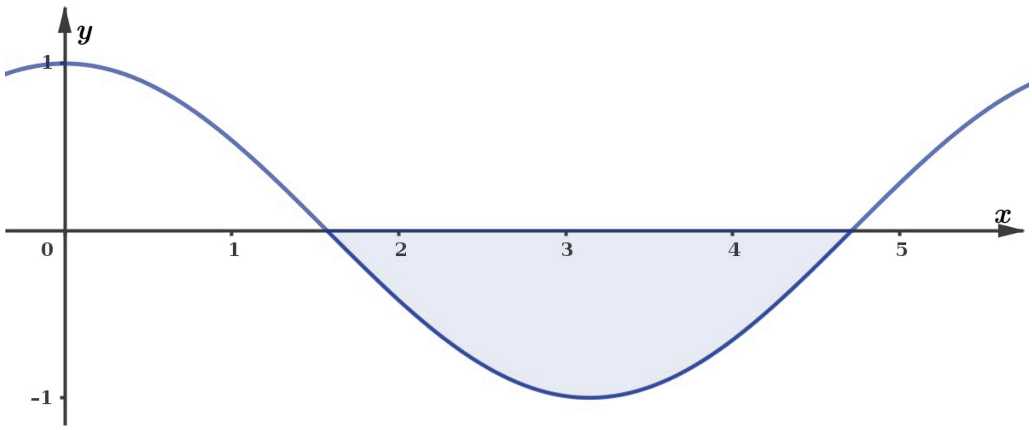
Find the exact area of the shaded region between the graph of $y = \cos x$ and the x-axis.



More information

The graph displays the function y = cos x, showing a cosine wave starting near its peak at y = 1 when x = 0. The wave descends through the x-axis at approximately x = π/2, reaching a minimum of y = -1 around x = π, and climbing back to the x-axis by x = 3π/2. The X-axis, which represents the variable x, is labeled in radians with marks at integral multiples. The Y-axis ranges from -1 to 1, showing the wave amplitude. The shaded area indicates the region between the curve and the x-axis from x = π to x = 2π, representing the area in question.

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The bounds of the region are the x-intercepts of the graph, so $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

The region is below the x-axis, so the area is the absolute value of the definite integral.

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Since $\int \cos x \, \mathrm{d}x = \sin x + c$, the area is

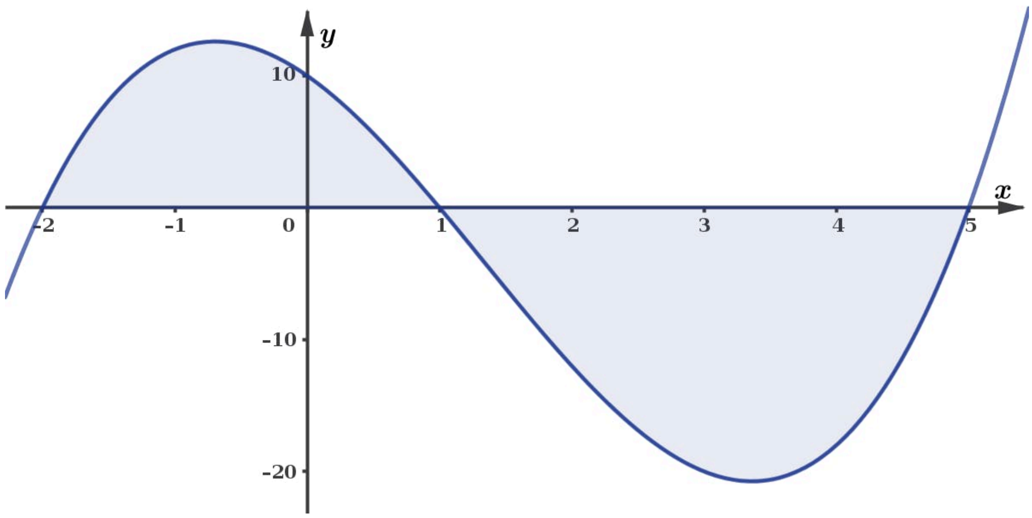
$$\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \, \mathrm{d}x\right| = \left|\left[\sin x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right| = \left|\sin \frac{3\pi}{2} - \sin \frac{\pi}{2}\right| = |-1-1| = 2.$$

Example 3

★☆☆

Find the exact area of the shaded region enclosed by the graph of

$y = (x+2)(x-1)(x-5)$ and the x-axis.



More information

The image is a graph of the polynomial function (y=(x+2)(x-1)(x-5)). The graph is a cubic curve that intersects the x-axis at the points (-2, 0), (1, 0), and (5, 0). The x-axis is labeled with tick marks at intervals from -2 to 5. The y-axis shows values ranging from -20 to 10. The curve starts at a negative value on the far left, rises to a positive peak above the x-axis near x=-1, dips down to a negative trough below the x-axis between x=1 and x=4, and then rises again to a positive value as x increases past 5. The areas above and below the x-axis are shaded differently.

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First expand the given expression:

$$(x+2)(x-1)(x-5) = x^3 - 4x^2 - 7x + 10.$$

To find the area of the shaded region, use the integral

$$\int x^3 - 4x^2 - 7x + 10 \, \mathrm{d}x = \frac{x^4}{4} - \frac{4x^3}{3} - \frac{7x^2}{2} + 10x + c.$$

Since the shaded region has parts above and below the x-axis, find two separate integrals to calculate the two distinct areas.

$$\begin{aligned}
&\int_{-2}^{1} x^3 - 4x^2 - 7x + 10 \, \mathrm{d}x \\
&\quad = \left[\frac{x^4}{4} - \frac{4x^3}{3} - \frac{7x^2}{2} + 10x\right]_{-2}^{1} \\
&\quad = \left(\frac{1}{4} - \frac{4}{3} - \frac{7}{2} + 10\right) - \left(\frac{16}{4} - \frac{-32}{3} - \frac{28}{2} - 20\right) \\
&\quad = \frac{65}{12} - \frac{-58}{3} = \frac{99}{4}
\end{aligned}$$

$$\begin{aligned}
&\int_{1}^{5} x^3 - 4x^2 - 7x + 10 \, \mathrm{d}x \\
&\quad = \left[\frac{x^4}{4} - \frac{4x^3}{3} - \frac{7x^2}{2} + 10x\right]_{1}^{5} \\
&\quad = \left(\frac{625}{4} - \frac{500}{3} - \frac{175}{2} + 50\right) - \left(\frac{1}{4} - \frac{4}{3} - \frac{7}{2} + 10\right) \\
&\quad = \frac{-575}{12} - \frac{65}{12} = \frac{-160}{3}
\end{aligned}$$

Hence, the total area of the two shaded parts is $\frac{99}{4} + \frac{160}{3} = \frac{937}{12} \approx 78.1$ units squared.

Note that, for the exam, you will not be required to work out calculations such as the above without a calculator. However, since the question asked for the exact value, you should use the Newton-Leibniz formula to find this area. Graphic display calculators would produce only an approximate answer. Nevertheless, you can use a calculator to

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check your answer:

$$\int_{-2}^{5}\left|(x+2)(x-1)(x-5)\right|\,\mathrm{d}x \approx 78.083333.$$

Note the modulus signs around the expression within the integral. With the use of the absolute value, indicated by the modulus signs, there is no need to find two integrals to find the area of the region. However, this is only helpful if you use a calculator, because with the absolute value we do not know how to find the anti-derivative.

Be aware

- If the area of a region bounded by the graph of $y = f(x)$ and the x-axis is calculated using the Newton-Leibniz formula, then the parts above and below the
x-axis need to be considered separately.
- If the area is calculated by calculator, then the formula $\int_a^b \left|f(x)\right|\,\mathrm{d}x$ can be used.
The similar formula without the absolute value only gives the correct answer if the entire graph of f is above the x-axis.

Example 4

★★☆

- Show that if $f(x) = \mathrm{e}^x(\sin x - \cos x)$, then $f'(x) = 2\mathrm{e}^x \sin x$.
- Find the exact area of the shaded region enclosed by the graph of $y = 2\mathrm{e}^x \sin x$ and the x-axis for $-\pi \leq x \leq \pi$.

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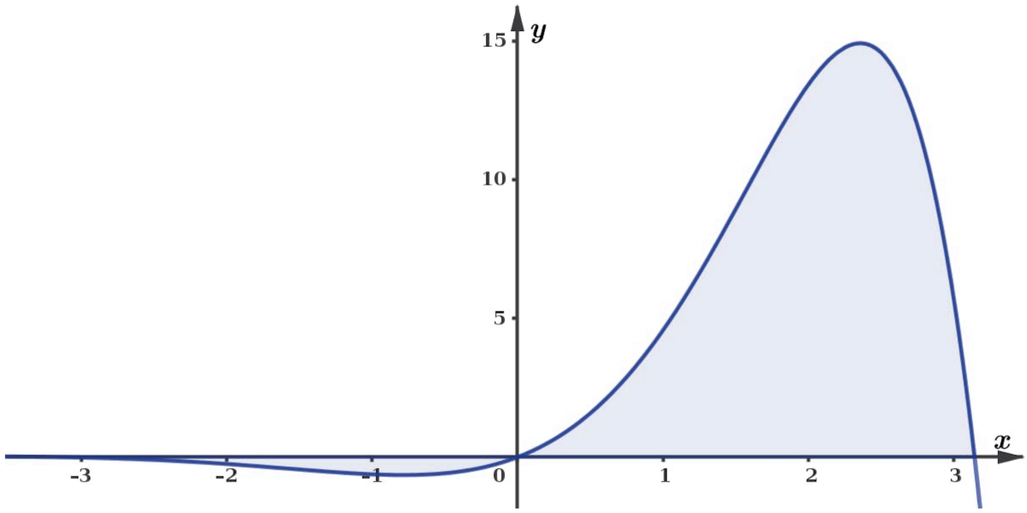

More information

The image is a graph with an X-axis and a Y-axis. The X-axis represents values ranging from approximately -4 to 4, and the Y-axis represents values ranging from 0 to 15, marked at intervals of 5. There is a blue curve on the graph that starts near the X-axis at around X = -4, dips slightly below the X-axis between X = -3 and X = 0, rises sharply after X = 0, and reaches the highest point near Y = 15 around X = 3.5 before descending back. The area under the curve from X = 0 to X = 3.5 is shaded in light blue. The labels "x" and "y" are present for the X-axis and Y-axis, respectively.

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To find the derivative, use the product rule:

$$\begin{aligned} f(x) &= \mathrm{e}^x(\sin x - \cos x) \\ f'(x) &= \mathrm{e}^x(\sin x - \cos x) + \mathrm{e}^x(\cos x - (-\sin x)) \\ &= \mathrm{e}^x(\sin x - \cos x + \cos x + \sin x) \\ &= 2\mathrm{e}^x \sin x. \end{aligned}$$

Since the shaded region has parts above and below the x-axis, find two separate integrals to calculate the two distinct areas.

The x-intercepts of the graph are the points, where $\sin x = 0$, so $(-\pi, 0)$, $(0, 0)$ and $(\pi, 0)$.

Using the previous calculation,

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$$\begin{aligned}
&\int_{-\pi}^{0} 2\mathrm{e}^{x}\sin x\,dx \\
&= \Big[\mathrm{e}^{x}(\sin x - \cos x)\Big]_{-\pi}^{0} \\
&= \Big(\mathrm{e}^{0}(\sin 0 - \cos 0)\Big) - \Big(\mathrm{e}^{-\pi}(\sin(-\pi) - \cos(-\pi))\Big) \\
&= 1 \times (0-1) - \frac{1}{\mathrm{e}^{\pi}} \times (0-(-1)) = -1 - \frac{1}{\mathrm{e}^{\pi}}
\end{aligned}$$

$$\begin{aligned}
&\int_{0}^{\pi} 2\mathrm{e}^{x}\sin x\mathrm{d}x \\
&= \Big[\mathrm{e}^{x}(\sin x - \cos x)\Big]_{0}^{\pi} \\
&= \Big(\mathrm{e}^{\pi}(\sin(\pi) - \cos(\pi))\Big) - \Big(\mathrm{e}^{0}(\sin 0 - \cos 0)\Big) \\
&= \mathrm{e}^{\pi} \times (0-(-1)) - 1 \times (0-1) = 1 + \mathrm{e}^{\pi}
\end{aligned}$$

Hence, the total area of the two shaded parts is $1 + \mathrm{e}^{\pi} + 1 + \frac{1}{\mathrm{e}^{\pi}} = 2 + \mathrm{e}^{\pi} + \frac{1}{\mathrm{e}^{\pi}}$ units squared.

Note that a calculator was not required to find the answer to this question. Nevertheless, if available, it is always a good idea to use a calculator to check the result.

- $2 + \mathrm{e}^{\pi} + \frac{1}{\mathrm{e}^{\pi}} \approx 25.1839$
- $\int_{-\pi}^{\pi} \left|2\mathrm{e}^{x}\sin x\right| \mathrm{d}x \approx 25.1839$

Looking ahead

Note that in **Example 4** it was given that $\int 2\mathrm{e}^{x}\sin x\,\mathrm{d}x = \mathrm{e}^{x}(\sin x - \cos x) + c$.

- $\int 2\mathrm{e}^{x}\cos x\,\mathrm{d}x$ is similar. Can you find it?
- How about $\int x\mathrm{e}^{x}\,\mathrm{d}x$ and $\int x\sin x\,\mathrm{d}x$?

Even though you have not yet learned about the required tools, you can still make a conjecture, checking it with the use of differentiation and modifying as appropriate. You can also take a look at the answer given by WolframAlpha (http://www.wolframalpha.com).

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Type

integral of 2*e^x*cos(x)

(or the other expressions) into the search line.

Be aware

As you have already seen, not all functions have anti-derivatives that you can express using the functions you know. Areas of regions bounded by such curves can only be found using technology. Typical examples of functions like this are the ones defined by $f(x) = \mathrm{e}^{x^2}$ and $g(x) = \cos\left(x^2\right)$. Take a look at the answer given by WolframAlpha when you ask it to find the area bounded by these curves over the interval $[0, 1]$.

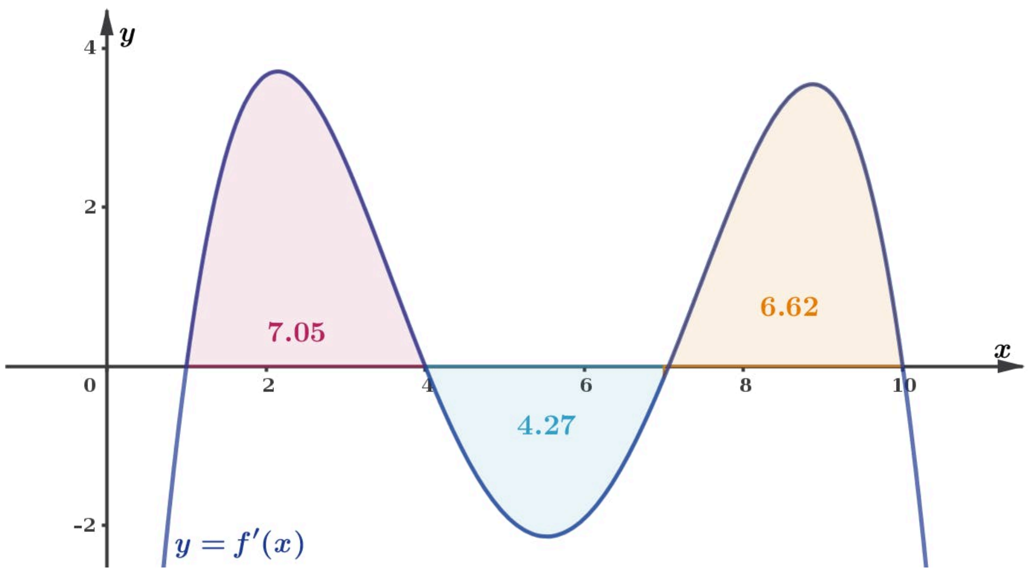
3 section questions

Question 1

Difficulty:

★☆☆

The diagram below shows the graph of $y = f'(x)$ and three shaded regions between the graph and the x-axis. The x-intercepts of the graph are $(1, 0)$, $(4, 0)$, $(7, 0)$ and $(10, 0)$. The area of the three regions are 7.05, 4.27 and 6.62 units squared.



More information

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Given that $f(1) = 2$, find $f(10)$.

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Give an exact answer in decimal form.

11.4

Accepted answers

11.4, 11,4

Explanation

Using $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$ for $a = 1$ and $b = 10$,

$$\begin{aligned}\int_1^{10} f'(x)\,\mathrm{d}x &= f(10) - f(1)\\ 7.05 - 4.27 + 6.62 &= f(10) - 2\\ f(10) &= 11.4.\end{aligned}$$

Question 2

Difficulty:

★☆☆

Find the area of the region enclosed by the graph of $y = (x + 3)(x + 1)(x - 2)$ and the x-axis.

Give an exact answer as a fully simplified fraction (in the form n/m).

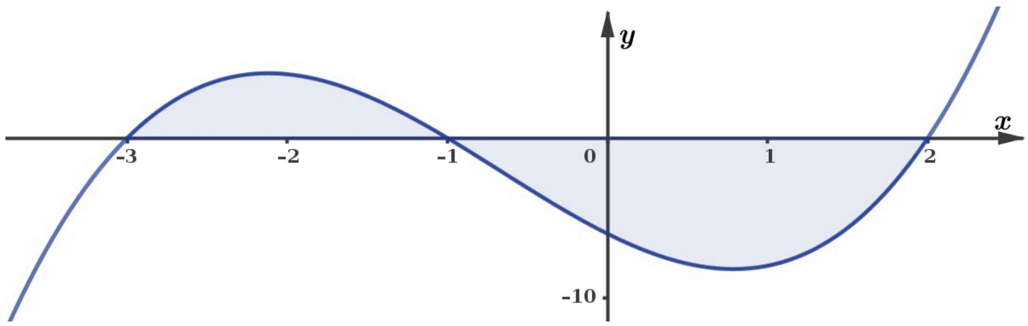
253/12

Accepted answers

253/12

Explanation

The diagram below shows part of the graph and the region.



More information

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First expand the given expression,

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$$(x+3)(x+1)(x-2) = x^3 + 2x^2 - 5x - 6.$$

To find the area of the shaded region, use the integral

$$\int x^3 + 2x^2 - 5x - 6 \, dx = \frac{x^4}{4} + \frac{2x^3}{3} - \frac{5x^2}{2} - 6x + c.$$

Since the shaded region has parts above and below the x-axis, find two separate integrals to calculate the two distinct areas.

$$\begin{aligned}
&\int_{-3}^{-1} x^3 + 2x^2 - 5x - 6 dx \\
&\quad = \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{5x^2}{2} - 6x\right]_{-3}^{-1} \\
&\quad = \left(\frac{1}{4} + \frac{-2}{3} - \frac{5}{2} + 6\right) - \left(\frac{81}{4} + \frac{-54}{3} - \frac{45}{2} + 18\right) \\
&\quad = \frac{37}{12} - \frac{-9}{4} = \frac{16}{3}
\end{aligned}$$

$$\begin{aligned}
&\int_{-1}^{2} x^3 + 2x^2 - 5x - 6 dx \\
&\quad = \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{5x^2}{2} - 6x\right]_{-1}^{2} \\
&\quad = \left(\frac{16}{4} + \frac{16}{3} - \frac{20}{2} - 12\right) - \left(\frac{1}{4} + \frac{-2}{3} - \frac{5}{2} + 6\right) \\
&\quad = \frac{-38}{3} - \frac{37}{12} = -\frac{63}{4}
\end{aligned}$$

Hence, the total area of the two shaded parts is $\frac{16}{3} + \frac{63}{4} = \frac{253}{12} \approx 21.1$ units squared.

Question 3

Difficulty: ★★☆

The area of the region enclosed by the graph of $y = x \cos x$ and the x-axis for $-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$ is $n\pi - 2$.

Find the value of n. Give an exact answer as an integer.

Hint: You can use that if $f(x) = x \sin x + \cos x$, then $f'(x) = x \cos x$.

3

Accepted answers

3, n=3, 3pi-2

Explanation

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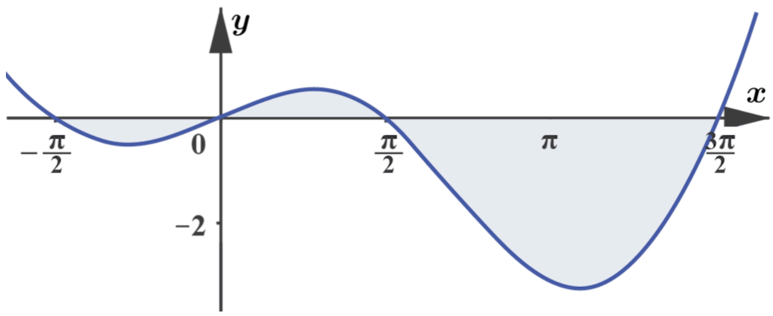
The diagram below shows part of the graph and the region.



More information

The x-intercepts of the graph are the points, where $x = 0$ or $\cos x = 0$, so

$$\left(-\frac{\pi}{2}, 0\right), (0,0), \left(\frac{\pi}{2}, 0\right) \text{ and } \left(\frac{3\pi}{2}, 0\right)$$

Using the hint given in the question,

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} x \cos x \, \mathrm{d}x \\
&\quad = \Big[x \sin x + \cos x\Big]_0^{\frac{\pi}{2}} \\
&\quad = \left(\frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2}\right) - \Big(0 \sin 0 + \cos 0\Big) \\
&\quad = \left(\frac{\pi}{2} \times 1 + 0\right) - (0 + 1) = \frac{\pi}{2} - 1.
\end{aligned}$$

$$\begin{aligned}
&\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x \cos x \, \mathrm{d}x \\
&\quad = \Big[x \sin x + \cos x\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\
&\quad = \left(\frac{3\pi}{2} \sin \frac{3\pi}{2} + \cos \frac{3\pi}{2}\right) - \left(\frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2}\right) \\
&\quad = \left(\frac{3\pi}{2} \times (-1) + 0\right) - \left(\frac{\pi}{2} \times 1 + 0\right) = -2\pi.
\end{aligned}$$

These two integrals give the area of the second and third region. By symmetry, the area of the first and second regions is the same, so the total area of the two shaded parts is

$$\left(\frac{\pi}{2} - 1\right) + \left(\frac{\pi}{2} - 1\right) + 2\pi = 3\pi - 2 \text{units squared.}$$

Hence, $n = 3$.

Student
view

5. Calculus / 5.11 Definite integrals

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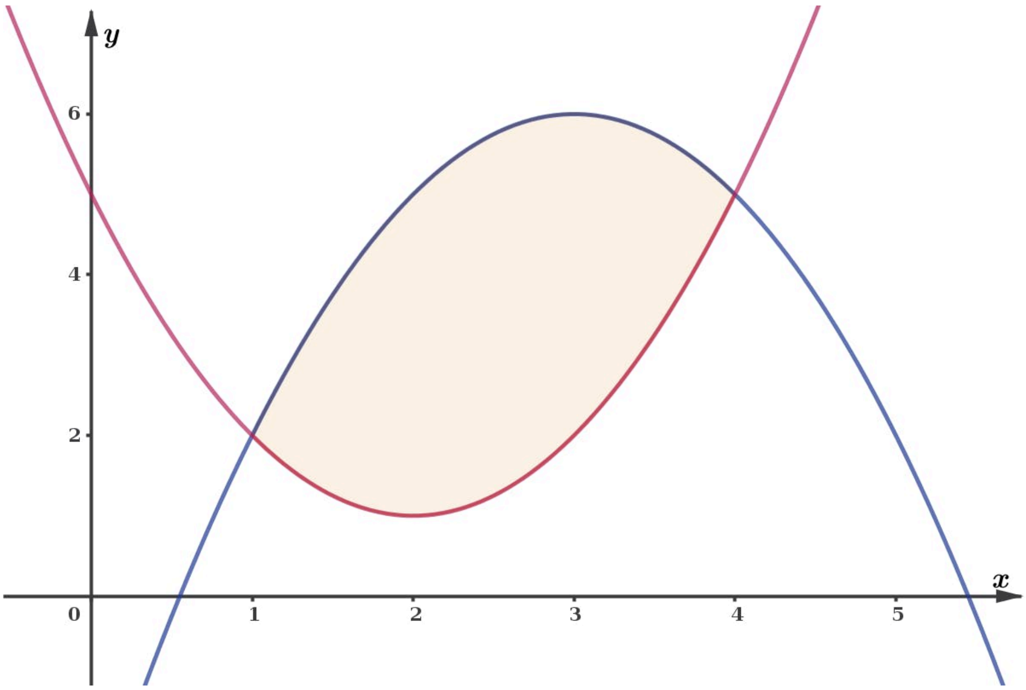
Regions bounded by two curves

Example 1

★☆☆

Find the area of the region enclosed by the graphs of $y = x^2 - 4x + 5$ and $y = 6x - x^2 - 3$.



More information

The image is a graph depicting two quadratic curves. The X-axis represents the variable x and ranges from 0 to 5, while the Y-axis represents the function value y, ranging from 0 to 7. One curve is an upward-opening parabola, labeled as the graph of the equation (y=x^2-4x+5). The other curve is a downward-opening parabola, labeled as the equation (y=6x-x^2-3). The two curves intersect around x=1 and x=4.

The region enclosed between the two curves (the area of interest) is shaded. This region starts at the intersection point on the left and ends at the intersection point on the right, forming an almond shape between the curves. The graph demonstrates classical intersection and inclusion of areas between functions.

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To find the bounds of the region from left and right, find the intersection points of the two curves.

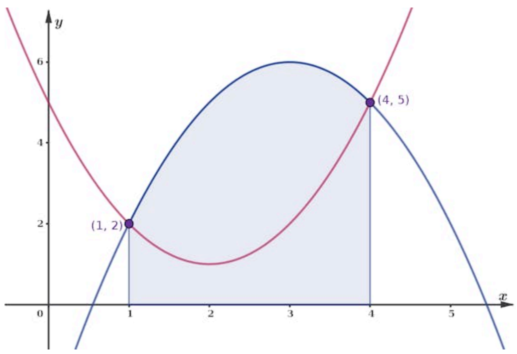
This can either be done using a graphic display calculator or by solving

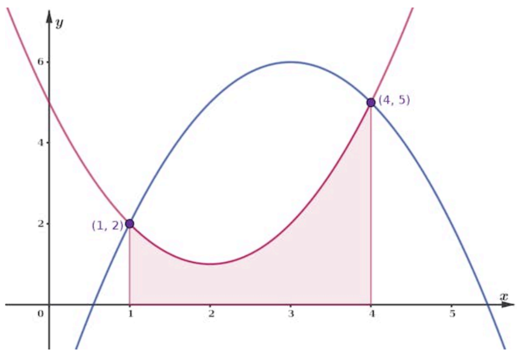
$$\begin{aligned} x^2-4x+5&=6x-x^2-3\\ 2x^2-10x+8&=0\\ x^2-5x+4&=0\\ (x-1)(x-4)&=0. \end{aligned}$$

The solutions are $x=1$ and $x=4$.

The diagrams below show two regions. The difference of the area of these regions gives the area of the region in the question.

Visualizing Area Calculation Using Two Regions





You can find these areas either by calculator or by using integration.

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The area of the region below the graph of $y = 6x - x^2 - 3$ (the one on the left) is

$$\int_1^4 6x - x^2 - 3\,\mathrm{d}x = \left[\frac{6x^2}{2} - \frac{x^3}{3} - 3x\right]_1^4$$
$$= \left(48 - \frac{64}{3} - 12\right) - \left(3 - \frac{1}{3} - 3\right) = 15.$$

The area of the region below the graph of $y = x^2 - 4x + 5$ (the one on the right) is

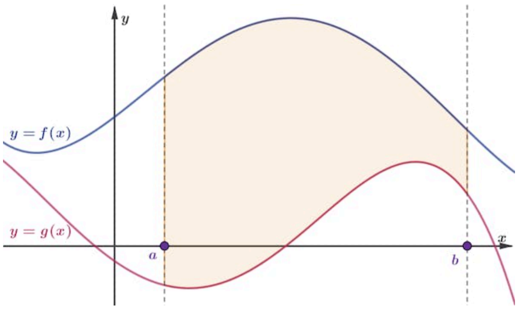
$$\int_1^4 x^2 - 4x + 5\,\mathrm{d}x = \left[\frac{x^3}{3} - \frac{4x^2}{2} + 5x\right]_1^4$$
$$= \left(\frac{64}{3} - 32 + 20\right) - \left(\frac{1}{3} - 2 + 5\right) = 6.$$

Hence, the area of the shaded region in the question is $15 - 6 = 9$ units squared.

Generalising the solution above, consider the following claim.

Important

If $g(x) \leq f(x)$ for $a < x < b$, then the area of the region bounded by the graphs of $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is given by



More information

The image is a graph depicting two curves: y = f(x) and y = g(x). The graph is on a Cartesian plane with labeled axes, where the x-axis represents the variable x and the y-axis represents y. The curve y = f(x) is shown in blue, and the curve y = g(x) is in red. The region between the curves is shaded to indicate the area bounded by y = f(x) and y = g(x) from x = a to x = b. Both “a” and “b” are points on the x-axis, marking the limits of integration for the shaded area. The graph illustrates how the upper curve y = f(x) is above y = g(x) within the interval [a, b]. Dotted vertical lines extend from x = a and x = b to the curves, showing the bound area.

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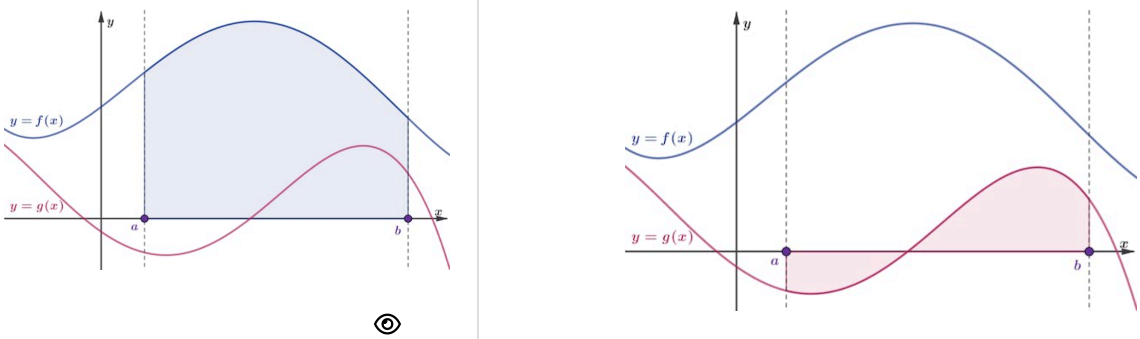
$$\int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x.$$

The same area is also given by

$$\int_a^b f(x) - g(x)\,\mathrm{d}x.$$

Instead of a formal proof, the method with the graphs can illustrate the claim. As in **Example 1**, consider at two regions.

Illustration of Two Regions for Area Calculation



The region on the left illustrates $\int_a^b f(x)\,\mathrm{d}x$ and the region on the right illustrates $\int_a^b g(x)\,\mathrm{d}x$.

Since the area of the part of the region below the x-axis counts as negative in the second integral, when you subtract this integral, this area is added to the area of the region on the diagram on the left.

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The area of the part of the second region that is above the x-axis counts as positive in the integral, so when you subtract the second integral, this area is subtracted from the area of the region on the diagram on the left.

This means that $\int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x$ indeed gives the correct area even when part of the second curve is below the x-axis. There are other cases (for example part of the graph of f can also be below the x-axis), but you will not learn about all cases now.

The proof of the second form of the area is the consequence of the Newton-Leibniz formula.

If $F'(x) = f(x)$ and $G'(x) = g(x)$, then

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x &= \Big[F(x)\Big]_a^b - \Big[G(x)\Big]_a^b \\ &= \Big(F(b) - F(a)\Big) - \Big(G(b) - G(a)\Big) \\ &= \Big(F(b) - G(b)\Big) - \Big(F(a) - G(a)\Big) \\ &= \Big[F(x) - G(x)\Big]_a^b \\ &= \int_a^b f(x) - g(x)\,\mathrm{d}x\end{aligned}$$

Example 2

★★☆

Find the area of the region enclosed by the graphs of $y = x^2 + x$ and $y = x^3 + x^2 - x$.

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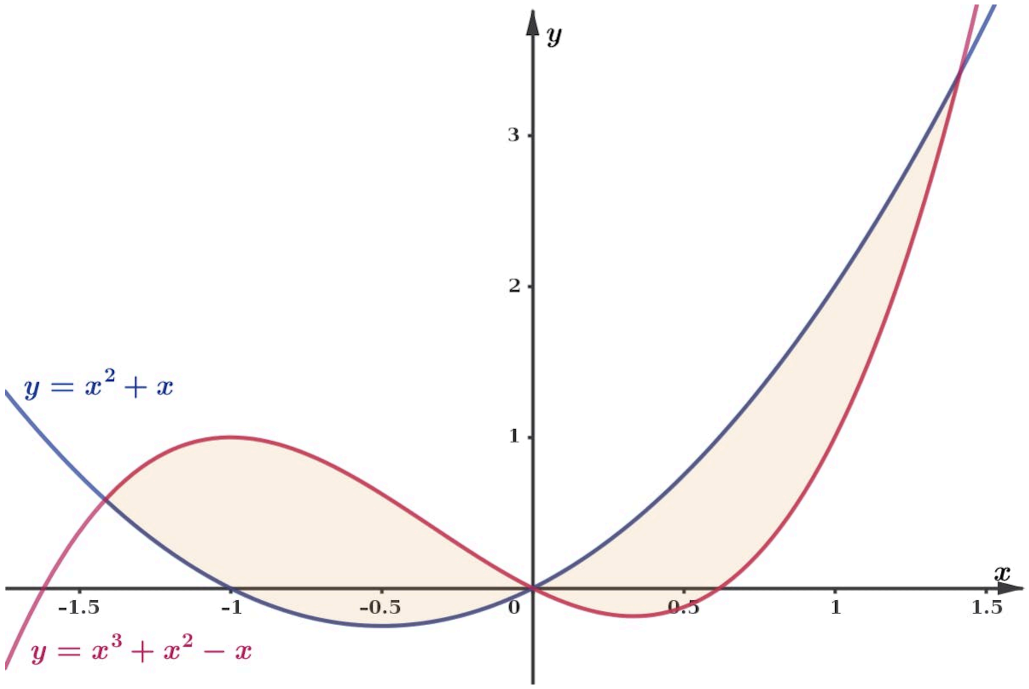

More information

The image is a graph displaying two mathematical functions: y=x^2+x and y=x^3+x^2-x. The graph's X-axis ranges approximately from -1.5 to 1.5 and the Y-axis ranges from -1 to 3. The curves intersect in a region where the area between the curves is shaded. The curve y=x^2+x is a quadratic function, shown in blue, starting at the left, and rising as it moves to the right. The curve y=x^3+x^2-x is a cubic function, shown in red, that starts rising and then dips before rising again, illustrating inflection points. The intersection and area enclosed by these curves are highlighted.

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Method 1 (without calculator)

To apply the claim, you need the bounds of the region and you need to know which is the upper and which is the lower curve.

To find the bounds, solve

$$\begin{aligned} x^3 + x^2 - x &= x^2 + x \\ x^3 - 2x &= 0 \\ x(x^2 - 2) &= 0. \end{aligned}$$

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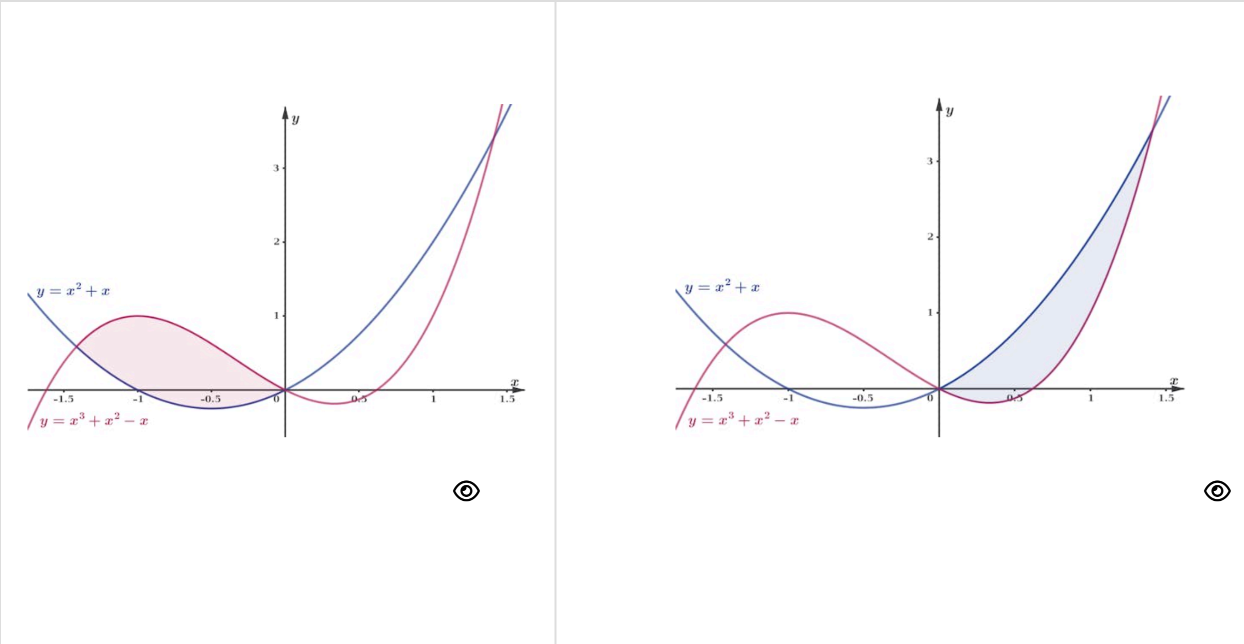
The solutions are $x = 0$ and $x = \pm\sqrt{2}$.

Since the curves cross each other at $x = 0$, find two separate integrals to calculate the areas of the two distinct regions.

Illustration of Two Regions for Area Calculation



For the region on the left, the cubic curve is above the quadratic, so the area is

$$\begin{aligned}\int_{-\sqrt{2}}^{0} (x^3 + x^2 - x) - (x^2 + x)\,\mathrm{d}x &= \int_{-\sqrt{2}}^{0} x^3 - 2x\,\mathrm{d}x \\ &= \left[\frac{x^4}{4} - x^2\right]_{-\sqrt{2}}^{0} \\ &= \left(\frac{0^4}{4} - 0^2\right) - \left(\frac{\left(-\sqrt{2}\right)^4}{4} - \left(-\sqrt{\ }\right.\right. \\ &= 0 - (1 - 2) = 1.\end{aligned}$$

For the region on the right, the cubic curve is below the quadratic, so the area is

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$$\begin{aligned}\int_0^{\sqrt{2}} (x^2+x)-(x^3+x^2-x)\,\mathrm{d}x &= \int_0^{\sqrt{2}} 2x-x^3\,\mathrm{d}x\\ &= \left[x^2-\frac{x^4}{4}\right]_0^{\sqrt{2}}\\ &= \left(\left(\sqrt{2}\right)^2-\frac{\left(\sqrt{2}\right)^4}{4}\right)-\left(0^2-\frac{0^4}{4}\right.\\ &= (2-1)-0=1.\end{aligned}$$

Hence, the area of the region enclosed by the two curves is $1+1=2$ units squared.

Note the interesting fact that the two parts of the region have the same area, even though they have different shapes. Can you explain why this is true?

Method 2 (using graphic display calculator)

Using a graphic display calculator the task is easier, although this way you can only get an approximate value for the area.

Calculators can tell us the intersection points of the curves. These are

$$(-1.41421, 0.58579), (0,0) \text{ and } (1.41421, 3.41421).$$

To eliminate the need to check which is the upper and which is the lower curve, integrate the absolute value of the difference to find that the area of the shaded region is approximately

$$\int_{-1.41421}^{1.41421} \left|(x^3+x^2-x)-(x^2+x)\right|\,\mathrm{d}x \approx 2 \text{ units squared.}$$

Note that the absolute value is important. Without the absolute value,

$$\int_{-1.41421}^{1.41421} (x^3+x^2-x)-(x^2+x)\,\mathrm{d}x \approx 0$$

which is clearly not the correct value for the area. Can you explain the reason?

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Be aware

The condition $f(x) \geq g(x)$ is important. If this is not satisfied, then the integral

$$\int_a^b f(x) - g(x)\,\mathrm{d}x$$

does not give the area bounded by the curves $y = f(x)$ and $y = g(x)$.

Exam tip

Even if the condition $f(x) \geq g(x)$ is not satisfied everywhere, on a calculator paper you can use

$$\int_a^b \left|f(x) - g(x)\right|\,\mathrm{d}x$$

to find the area bounded by the curves $y = f(x)$ and $y = g(x)$ above the interval $[a, b]$.

Example 3

★★★

The area of the region bounded by the graph of $y = x^2$ and the line $y = mx + 1$ is 2 units squared. There are two such values of m. Find the positive one.

The diagram below illustrates the region.

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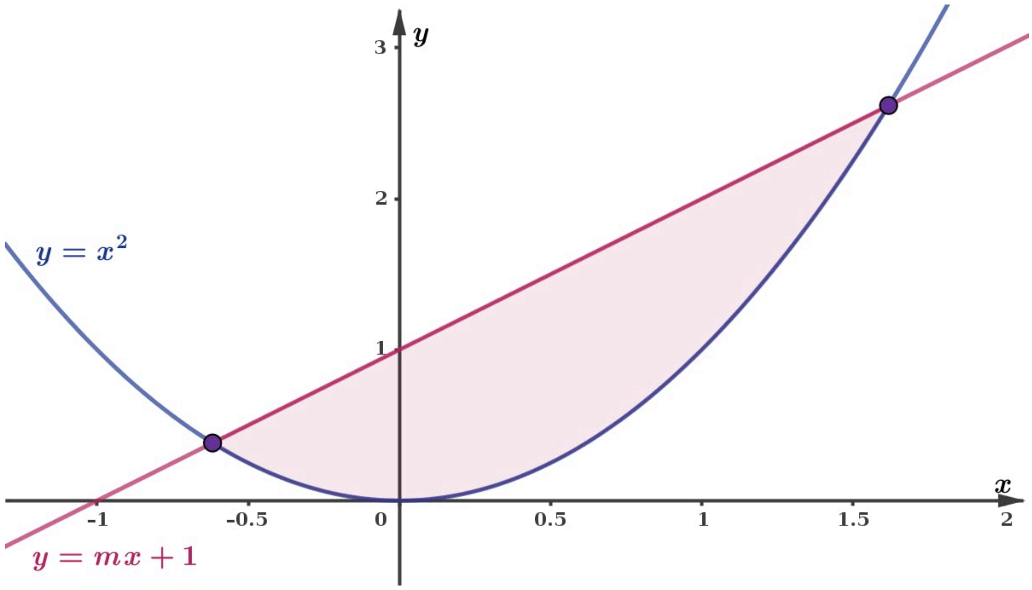

To find the bounds of the region, solve

$$x^2 = mx + 1$$
$$x^2 - mx - 1 = 0.$$

The solutions are $x_1 = \dfrac{m - \sqrt{m^2+4}}{2}$ and $x_2 = \dfrac{m + \sqrt{m^2+4}}{2}$.

Hence, an expression for the area of the region is

$$\int_{x_1}^{x_2} mx + 1 - x^2 \mathrm{d}x.$$

On graphic display calculators you can define the function

$$A : m \mapsto \int_{x_1}^{x_2} mx + 1 - x^2 \mathrm{d}x,$$

so in other words the function, where $A(m)$ is the value of the integral, where the uknown m is both in the expression and in the limits of integration. To see how to do this on a calculator, see the instructions (for some makes of calculator) after the solution.

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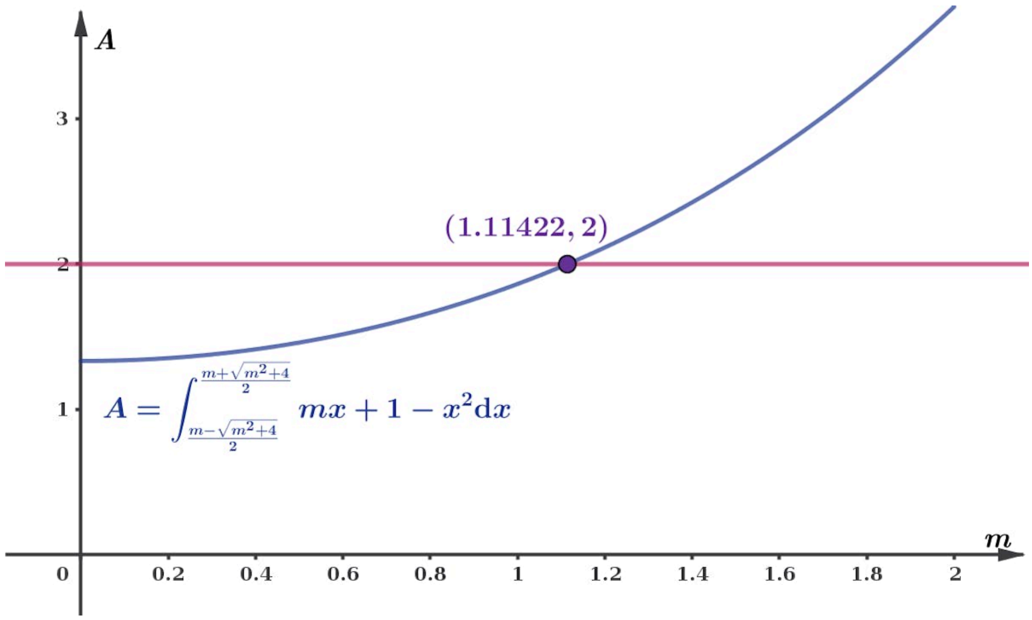
You can then plot the graph and see at which value of m it crosses the line $y = 2$. This value will be the solution to the question.

The diagram below shows the graphs and the intersection point.



Hence, the value you are looking for is $m \approx 1.11$.

Here is some help on how to work out the solution to **Example 3** for different makes of calculator.

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<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>

These instructions show you how to find the value of m, such that

$$\int_{\frac{m-\sqrt{m^2+4}}{2}}^{\frac{m+\sqrt{m^2+4}}{2}} mx+1-x^2\mathrm{d}x = 2.$$

From the main menu select the graph option.

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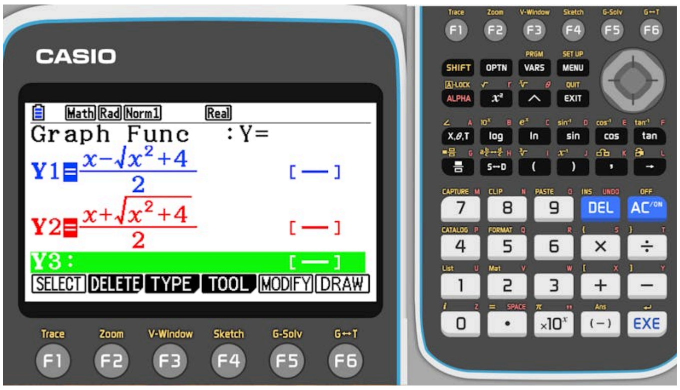
Define two functions, one for each root of the quadratic equation. These will be used in the limit of integration.

</td><td>



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</table>

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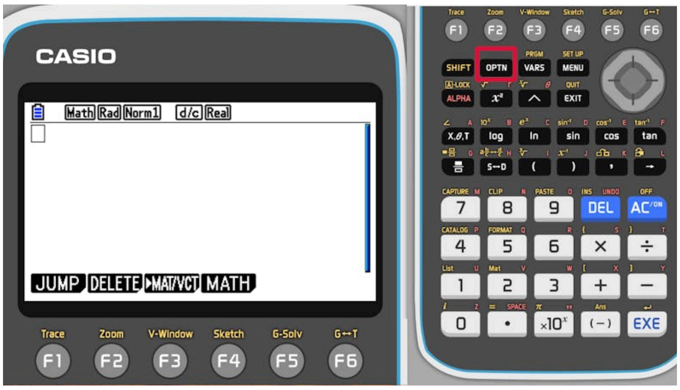
Steps	Explanation
Go back to the main menu and choose the calculator option.	
You will need to find the numerical equation solver (SolveN) option. Press OPTN to start your search ...	

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Steps	Explanation
... press F4 for the calculus related options ...	
... and F5 to open the numerical equation solver (SolveN).	

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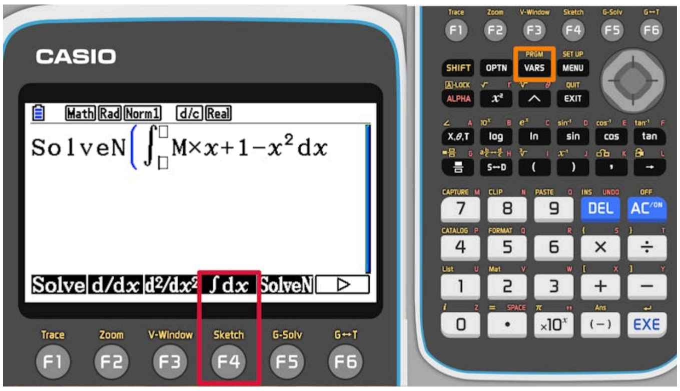
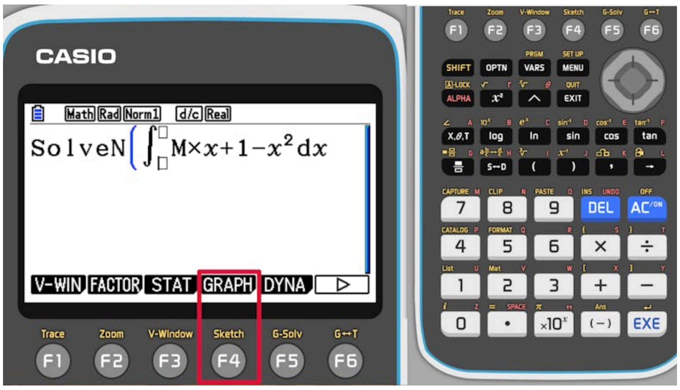
<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>Press F4 to bring up the integral template and enter the function. Note, that you will need a variable name for m. You can enter any letter using the Alpha key.

To use the predefined expressions for the limit, you need to find the function variable names. Press VARS to search for these ...</td><td>
</td></tr>
<tr><td>... and press F4 to access the graph related variable names.</td><td>
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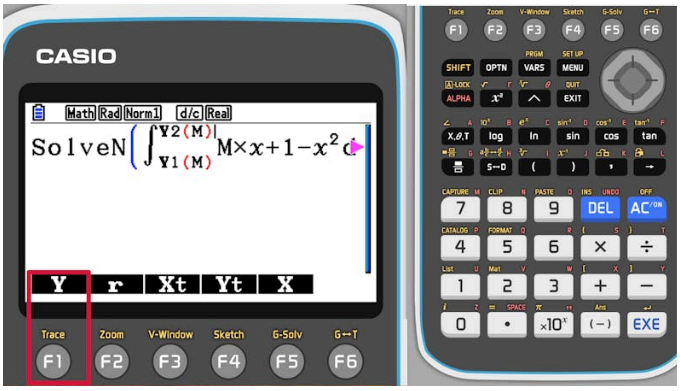
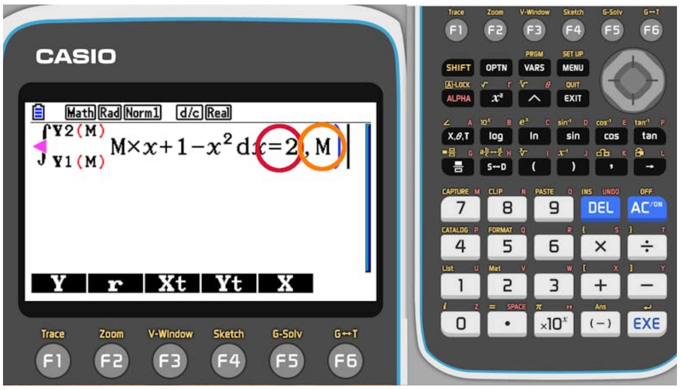
Steps	Explanation
Use F1 for the function variable name. Note, that you need these functions evaluated at m, so use the format on the screenshot. Once done, move to the end of the line to finish the expression.	
You need to tell the calculator that you are looking for the value, where the integral is 2, so set the integral equal to 2. You also need to tell the calculator, that the variable you want the equation to be solved is m, so type a comma and the variable name after the equation. Once done, close the bracket and press enter.	

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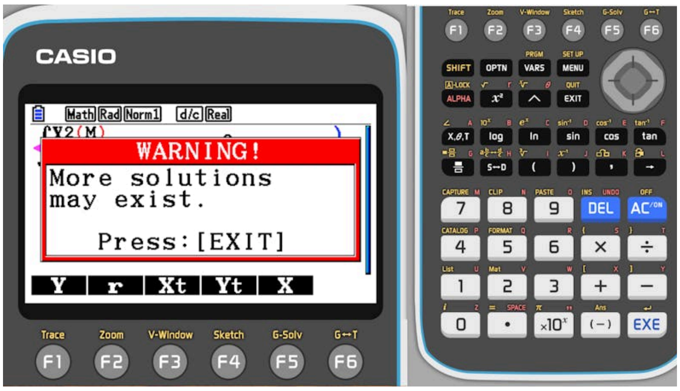
Steps	Explanation
You will see the usual warning that the calculator may not show you all the solutions.	
In this case the calculator shows both the negative and the positive solutions.	

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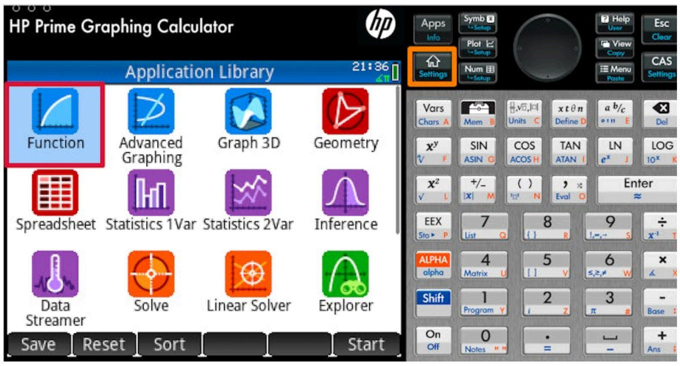
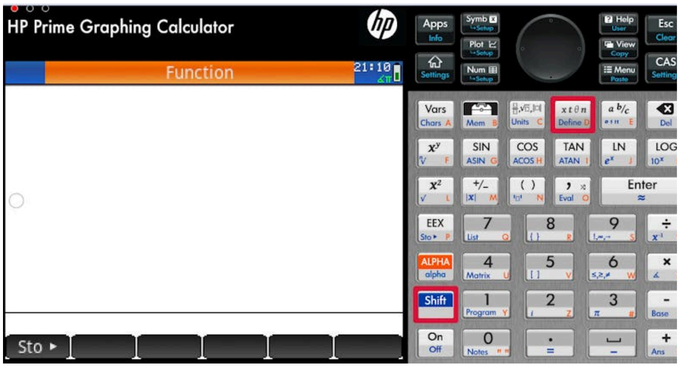
Steps	Explanation
These instructions show you how to find the value of m, such that $$\int_{\frac{m-\sqrt{m^2+4}}{2}}^{\frac{m+\sqrt{m^2+4}}{2}} mx+1-x^2\mathrm{d}x=2.$$ Find the function application and enter the home screen.	
On the home screen select the option to define a user defined function.	

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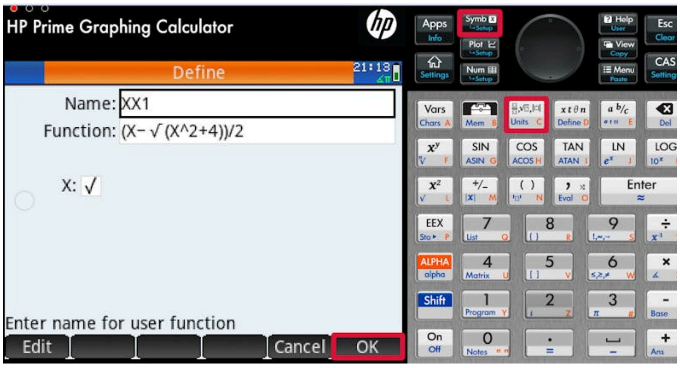
Write the expression for one of the roots. Note that X is used as the variable instead of m in the example above, because this is the name of the variable the calculator uses in graphing functions.

Define also an other function for the other roots of the quadratic equation.

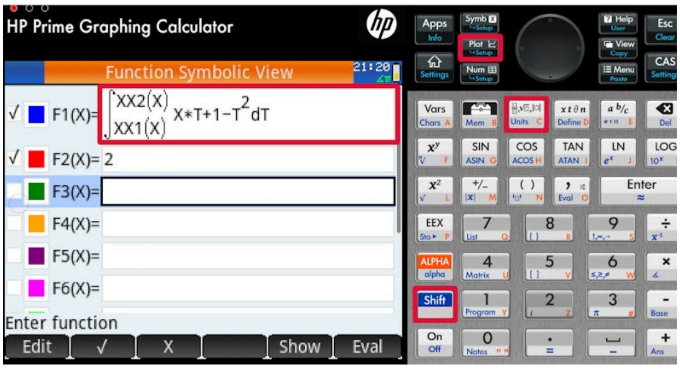
Once done, enter the symbolic view.



In the symbolic view you can define the area of the region between the parabola and the line as a function of the slope. Note that in the example you used m for the slope and x as the integration variable. Here you need to use X for the slope and any other letter as the integration variable (since for the calculator X is the independent variable for the graphs).

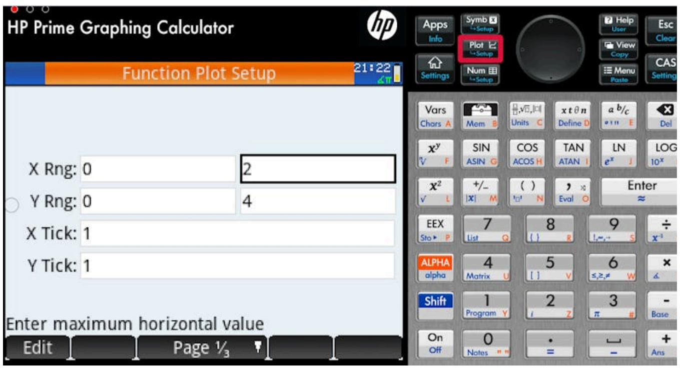
Define also the constant 2 function, because you would like to find the slope when the area is exactly 2.

Once done, enter the setup for the plot.



Enter the range for the plot. Since you want to see the intersection point, 2 should be in the y-range. You may need to experiment with the horizontal range.

Once done, enter the plot view.



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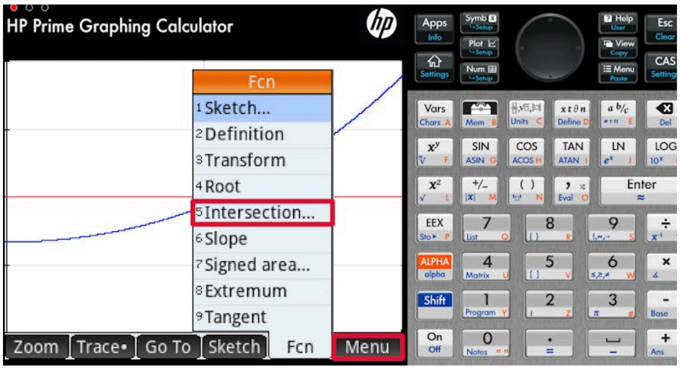
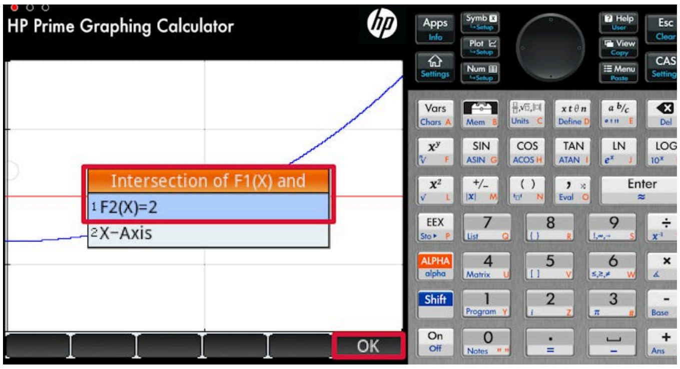
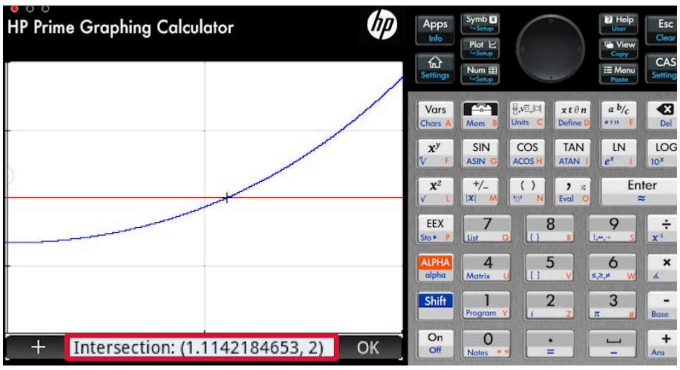
In the plot view select to find the intersection in the function menu.



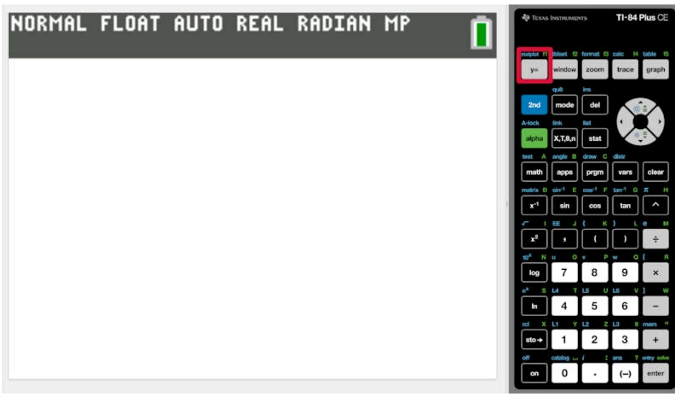
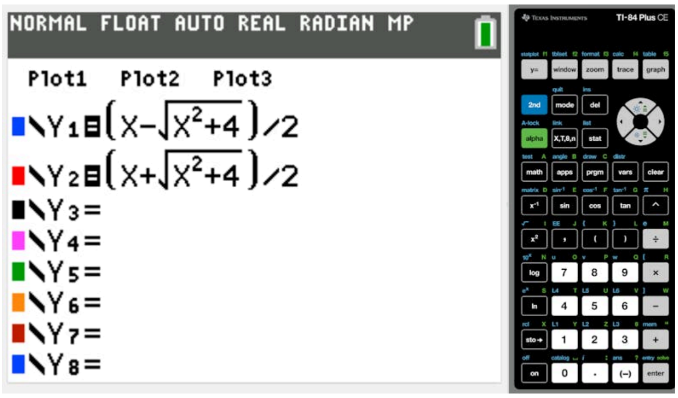
Select the two curves and press OK.



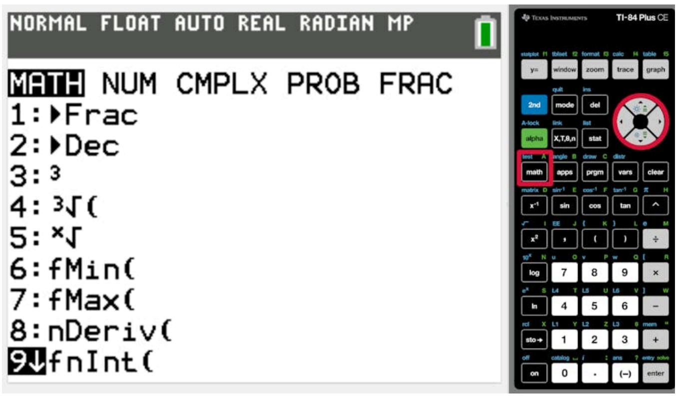
The calculator moves the cursor to the intersection point and displays the coordinates.



Student view

Steps	Explanation
These instructions show you how to find the value of m, such that $\int_{\frac{m-\sqrt{m^2+4}}{2}}^{\frac{m+\sqrt{m^2+4}}{2}} mx+1-x^2\mathrm{d}x=2.$ From the main screen choose the option to enter functions.	
Define two functions, one for each root of the quadratic equation. These will be used in the limit of integration. Once done, move down to the third line.	

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
You will need to write an integral expression for the third function, so press math and choose the numerical integral (fnInt) option.	

Student view

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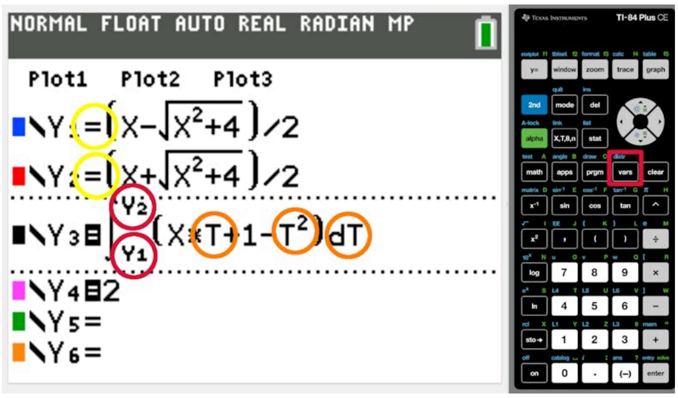
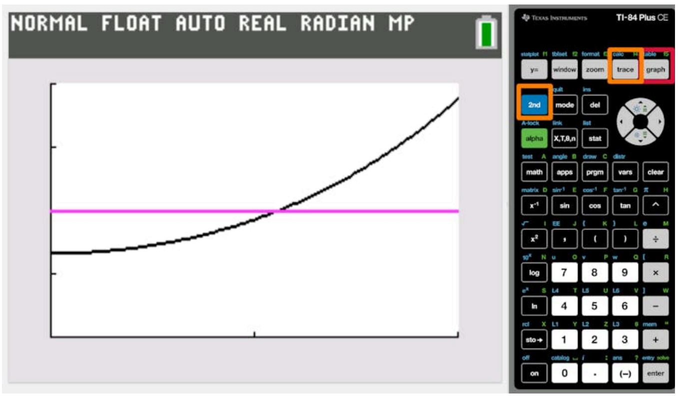
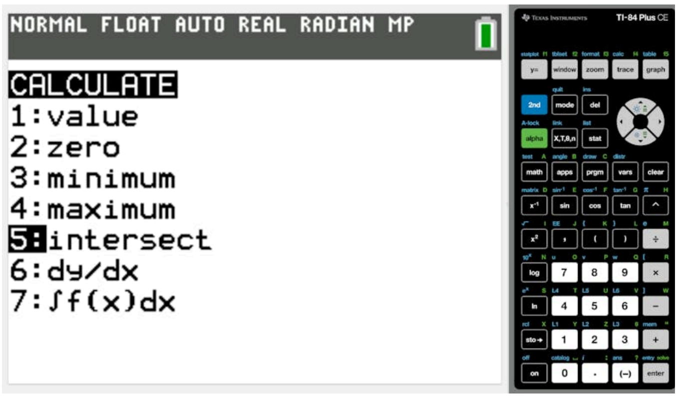
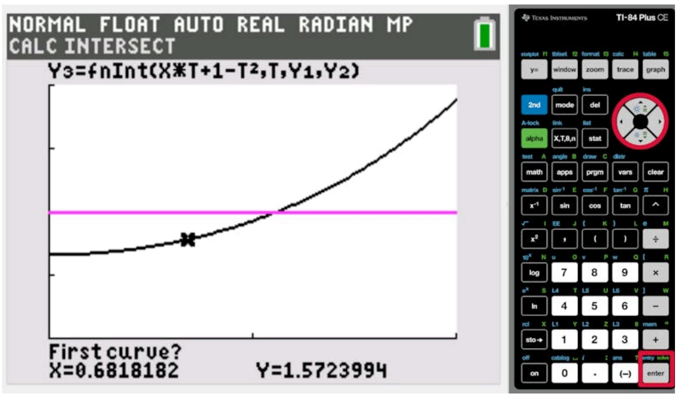
Steps	Explanation
There are several things you need to be careful when you enter the integral expression. • Use the functions you defined previously instead of typing the expressions in the limits of the integration. You can access the names for example by pressing the variable button (and looking for the function variables). • You need to change the variaable names from the one in the expression you have on paper. the calculator has x as the independent variable, so change m to x. Since this way x is already used, you also need to change the integration variable to any letter. Enter also the constant function $y = 2$ (since you are interested in the value of m where the integral is 2). Once all this is done, unselect the first two functions, because you do not need to see the graphs of those.	

Student
view

Steps	Explanation
To set the viewing window, press window. Enter the range for the plot. Since you want to see the intersection point, 2 should be in the y-range. You may need to experiment with the horizontal range.	
Press graph to see the graphs and press 2nd/calc to start the search for the intersection point.	

Steps	Explanation
Select the option to find the intersection points ...	
... select the first curve and press enter to confirm ...	

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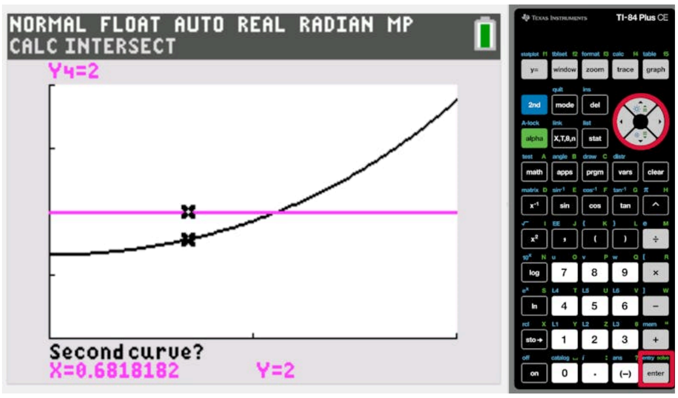
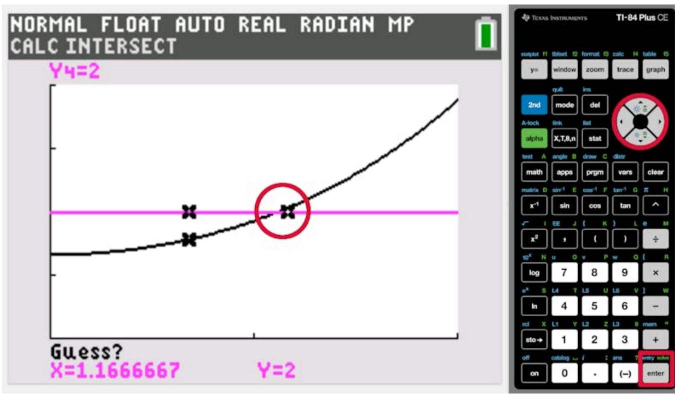
Steps	Explanation
... select the second curve and press enter to confirm ...	
... move the cursor close to the intersection point and press enter to confirm your guess.	

Student
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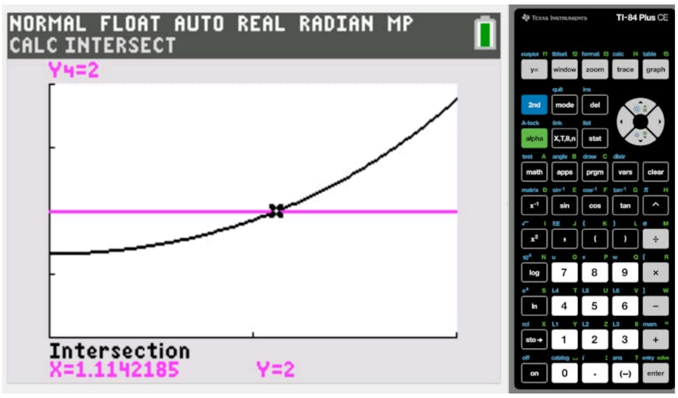
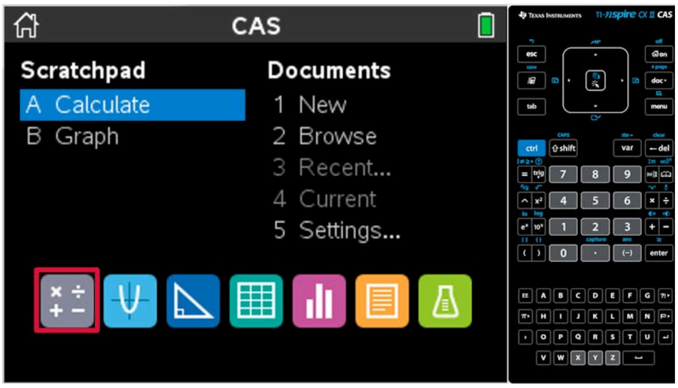
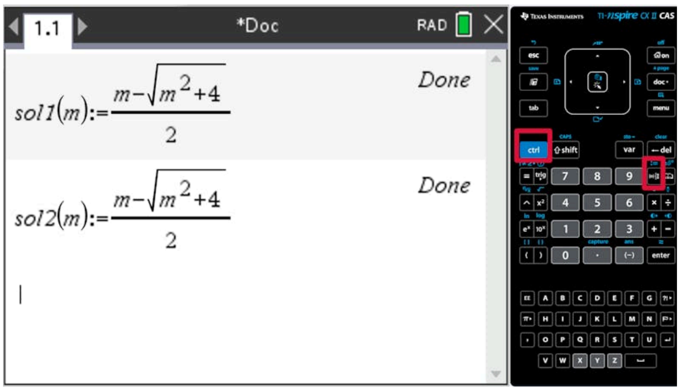
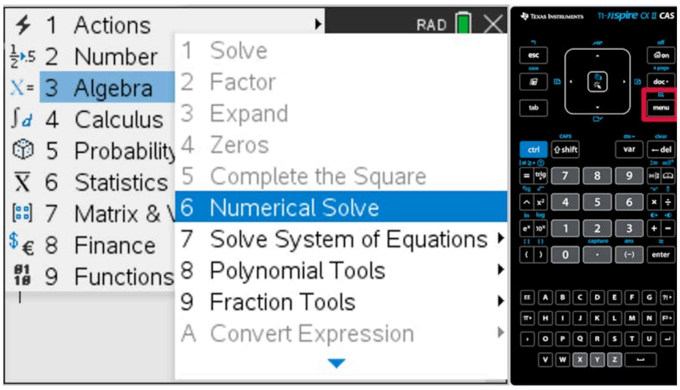
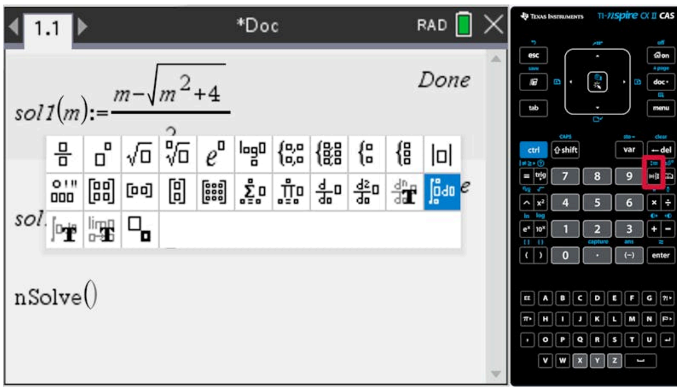
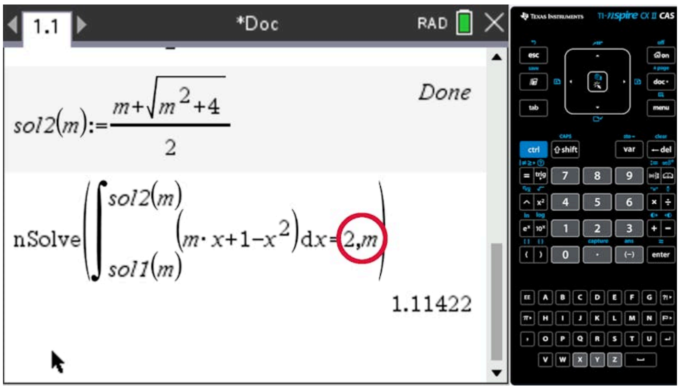
Steps	Explanation
The calculator moves the cursor to the intersection point and displays the coordinates.	

Steps	Explanation
These instructions show you how to find the value of m, such that $$\int_{\frac{m-\sqrt{m^2+4}}{2}}^{\frac{m+\sqrt{m^2+4}}{2}} mx+1-x^2\mathrm{d}x=2.$$ Open a calculator page.	

Student
view

Steps	Explanation
Define expressions for the roots. These will be useful to enter as the limits of the integral. Note, that the colon equal sign is used to indicate a definition. You can give any name to the functions you are defining.	
To find the solution to the question, open the menu and search for the numerical equation solver option.	

Steps	Explanation
You want to solve an equation involving an integral, so open the template menu and choose the integral teplate.	
Type in the expressions. Remember to use the predefined expressions in the limits, this will improve readability. Set the integral equal to 2 (since that is the equation you would like to solve) and add a comma and m to indicate, that you would like the calculator to solve this equation for m. Press enter to get the solution.	

Example 4

★★☆

In an automotive engine the pressure and volume of the gas is periodically changing. This cycle can be illustrated on a so-called PV diagram. Take a look at the video that explains what such a diagram means.

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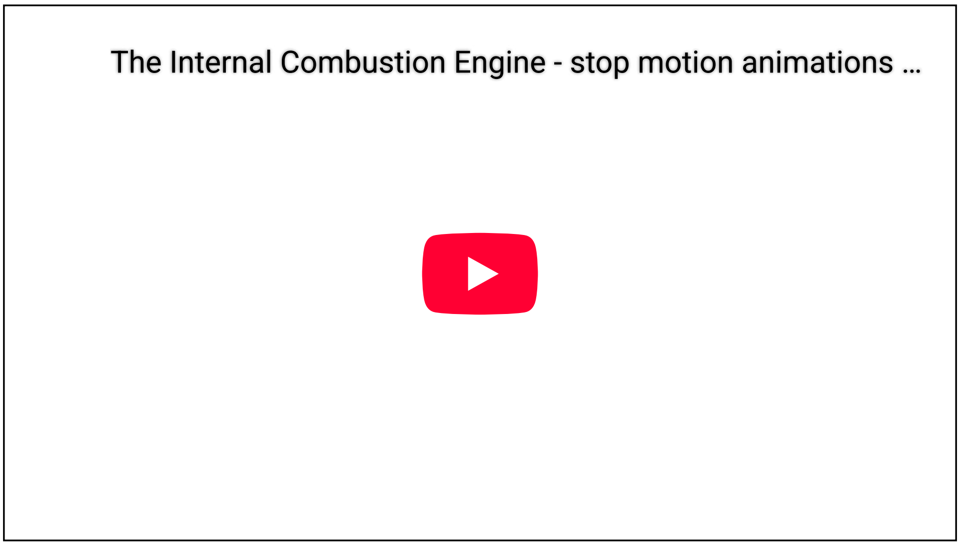

Video 1. The Internal Combustion Engine - stop motion animations and the PV cycle (Otto cycle).

More information for video 1

[man] My class and I had a go at making short stop motion animations

to show how the internal combustion engine works.

This rather nice engine is from a Ferrari Testarossa.

The white particles on the left hand side here

represent the mixture of particles of air and fuel.

The inlet valve opens as the piston moves down.

As the volume of the cylinder increases,

this draws air and fuel mixture

into the chamber at constant atmospheric pressure.

The piston then continues to move upwards,

either forced upwards by the momentum of the crank or the force generated

by the other cylinders in the engine.

This compresses the fuel and the air mixture

in a roughly adiabatic compression.

In a petrol engine, the spark plug at the top of the cylinder ignites

the fuel at this point here.

This is at the point of maximum compression.

This ignition causes a rapid increase in pressure.

After combustion,

I've changed the particles in the animation from whites to blue

to represent the combustion products.

The combustion products are now at a very hot temperature and pressure

and a very high pressure,

and they force the piston back down the cylinder.

This is called the power stroke.

Student
view

Overview
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In the final phase of the cycle, the exhaust valve opens.

As the exhaust products are at a pressure much higher

than atmospheric pressure,

they tend to rush out of the open exhaust port.

Finally, the momentum of the piston continues

to push it back to the starting position

ready for the next inlet stroke.

Some people like to remember the four stages

of the engine cycle.

With these four words suck, squeeze, bang, and blow.

Let's have a look at some animations

that my students created during their recent physics lesson.

Although this first animation is rather simplistic

and doesn't include much detail of the cylinder and valves,

I rather like it for its scientific accuracy.

The students have represented fuel

and air particles with blue and white dots,

and they change to orange to indicate combustion has taken place.

This second animation is rather neat and nicely drawn.

The students have used an orange color

to indicate the high pressure in the cylinder,

and they have changed the color of the combustion products to black.

One slight problem with this animation

is that the number of particles in the cylinder

doesn't seem to stay constant even when the valves are shut.

This animation has been excellently drawn

and also cleverly engineered using some split pins

to show the rotation of the crank.

This nicely illustrates how the momentum of the crank carries the piston upwards

during the compression and exhaust strokes.

I like how this final animation uses paper confetti

from a hole punch to represent the different particles,

white particles for air, yellow for fuel, and orange for the combustion products.

The orange combustion products cleverly spread out

from the point of the spark plug.

The other students in the group affectionately christened

this animation, the rubbish compressor.

I'm now going to explain the stages of the engine cycle

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

in a little more detail.

This diagram on the right is a pressure volume diagram.

P here for the pressure of the gas on the y-axis

and V here for the volume of the cylinder

on the x-axis.

The cycle shown is the idealized cycle

for a petrol engine or auto cycle.

The green dot makes its way around the cycle

roughly in time with the cylinder.

You might be able to spot the different stages of the cycle

on the diagram.

I'm now going to look at each of those stages in a little more detail.

In the inlet or suck stroke,

the volume of the cylinder increases as the valve is opened

to the atmosphere, drawing in air and fuel.

Because the valve is open, the pressure remains at atmospheric pressure.

In the compression stroke, the valves are closed.

As the piston makes a tight fit with the cylinder,

the gas is compressed.

This curved line on the graph is an adiabatic compression,

one in which no heat is exchanged with the surroundings.

This means that the pressure and temperature of the gas both rise.

The ignition happens very quickly in a petrol engine.

In the idealized auto cycle, the volume of the cylinder remains constant,

while the pressure and temperature of the gas increase during combustion.

In reality, of course, the line can't be completely vertical

as the piston never stops moving.

The power stroke is represented in the auto cycle

by an adiabatic expansion,

this second curved line here.

During this expansion, the volume of the cylinder increases

and the pressure and temperature of the gas decrease.

This does work on the piston pushing it down.

The exhaust phase is represented

by another vertical line on the auto cycle.

The pressure reduces at constant volume.

When the exhaust valve opens, the gas is at high pressure

and so it rushes out of the open valve.

Student view

The final exhaust stroke pushes out any remaining gas

at low pressure

and brings the green dot back to the starting position.

Hopefully, you can identify all of those processes

as the green dot makes its way around the auto cycle.

The area enclosed by the graph

represents the amount of work done by the engine per cycle.

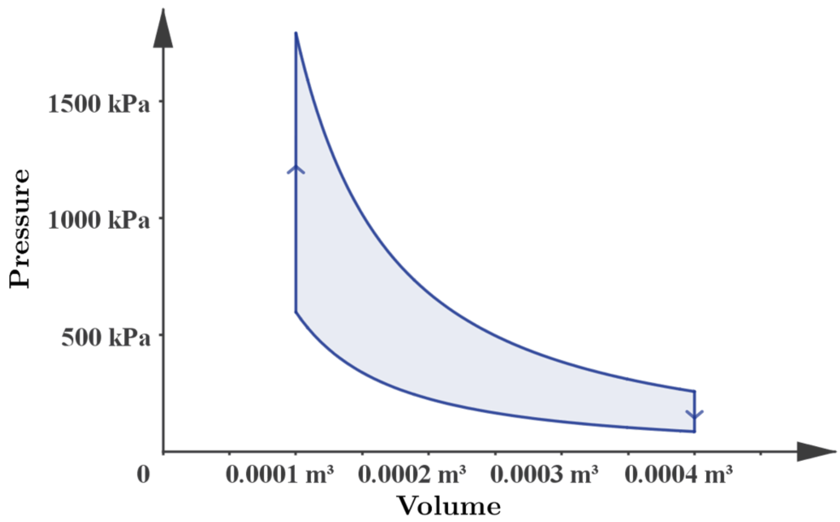
I hope you've enjoyed this video and the animations

that my students and I created.

Why not have a look at some of the other stop motion animation projects

that I've created with some classes by clicking one of the links below?

The diagram below shows multiple steps of a thermodynamic cycle, similar to the one you have seen in the video. The area inside the curve represents the work done during the cycle. On the diagram volume is measured in cubic metres, pressure is measured in kilopascals and the area gives the work done in kiloJoules.



More information

The diagram illustrates a thermodynamic cycle depicted by a curve on a graph, with pressure on the vertical axis and volume on the horizontal axis. The pressure is measured in kilopascals (kPa) and ranges from 0 to 1500 kPa. The volume is in cubic meters, ranging from 0 to 0.0004 m³. The curve represents the boundaries where the equations are (PV^{1.4}=0.0015) and (PV^{1.4}=0.0045). The area inside the curve symbolizes the work done in kiloJoules (kJ) during the cycle.

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[Generated by AI]

The equation of the bounding curves are $PV^{1.4} = 0.0015$ and $PV^{1.4} = 0.0045$.

The vertical boundaries are at $V = 0.0001$ and $V = 0.0004$.

Find how much work is done during the cycle.

The equations of the curves are $P = \frac{0.0015}{V^{1.4}}$ and $P = \frac{0.0045}{V^{1.4}}$.

The area between the two curves is

$$\int_{0.0001}^{0.0004} \frac{0.0045}{V^{1.4}} - \frac{0.0015}{V^{1.4}} \mathrm{d}V \approx 0.127.$$

The work done during the cycle is $0.127 \text{ kJ} = 127 \text{ J}$.

3 section questions

Question 1

Difficulty: ★☆☆

Find the area enclosed by the graphs of $y = x^2 + x - 1$ and $y = x + 3$.

Give an exact answer as a fully simplified fraction (in the form n/m).

32/3

Accepted answers

32/3, 10.7, 10,7

Explanation

The diagram below shows the region.

Student view

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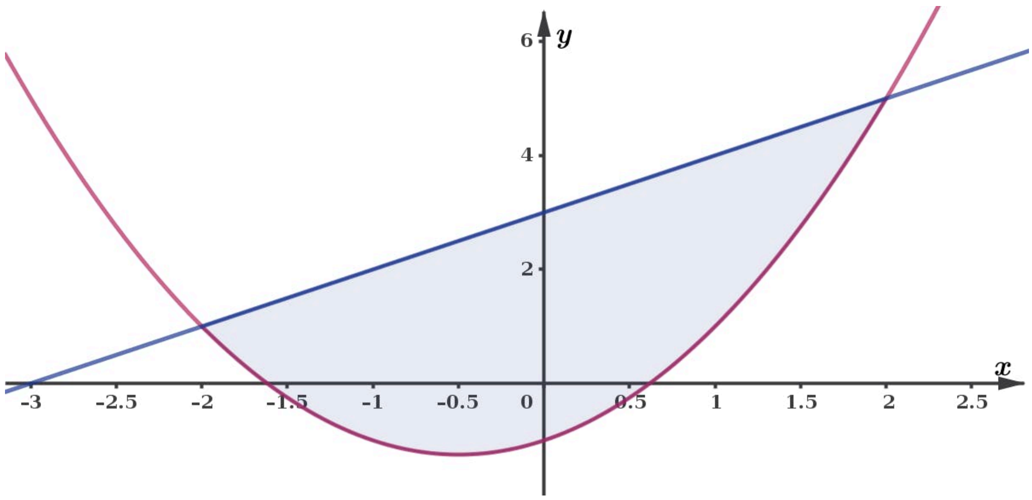

More information

The first coordinates of the points of intersection are the solutions of

$$x^2 + x - 1 = x + 3,$$

which are $x = -2$ and $x = 2$.

The definite integral that gives the area of the enclosed region is

$$\int_{-2}^{2} (x+3) - (x^2 + x - 1)\,\mathrm{d}x = \int_{-2}^{2} -x^2 + 4\,\mathrm{d}x.$$

This definite intetgral evaluates to $\left[-\frac{1}{3}x^3 + 4x\right]_{-2}^{2} = \frac{32}{3} \approx 10.7.$

Question 2

Difficulty:

★★☆

The diagram shows the region enclosed by the curves $y = -x^2 + 2x$ and $y = x^3 - x^2 - x$.

Student view

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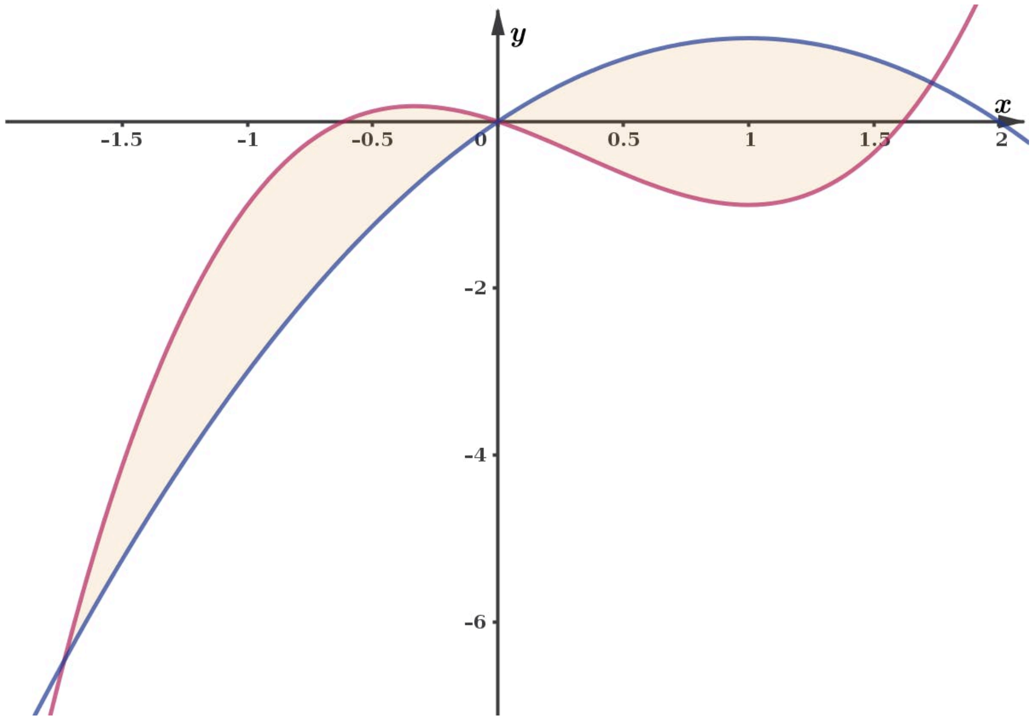

More information

Find the exact area of this region.

Give your answer as a decimal or as a fully simplified fraction (in the form n/m).

9/2

Accepted answers

9/2, 4.5, 4,5

Explanation

The x-coordinates of the points of intersection are the solutions of

$$\begin{aligned}-x^2+2x&=x^3-x^2-x\\ x^3-3x&=0\\ x(x^2-3)&=0.\end{aligned}$$

Hence,

$$x=-\sqrt{3}, x=0 \text{ or } x=\sqrt{3}.$$

Using these values, the area of the region is the sum of two definite integrals.

Student
view

Overview
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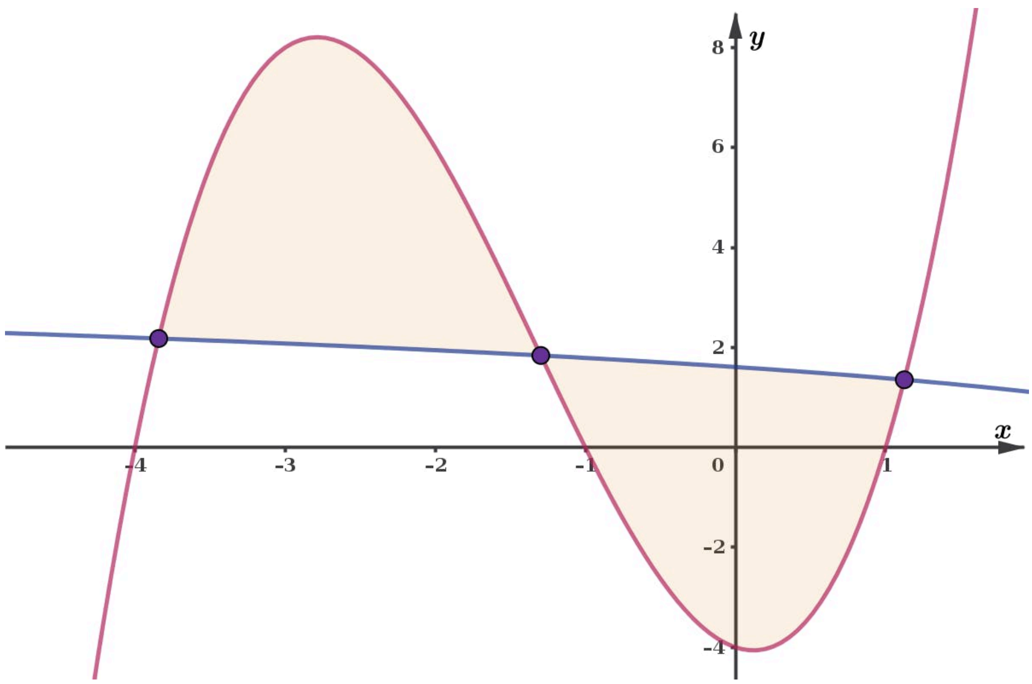
$$
\begin{aligned}
\text{Area} &= \int_{-\sqrt{3}}^{0} \left(g(x) - f(x)\right) \mathrm{d}x + \int_{0}^{\sqrt{3}} \left(f(x) - g(x)\right) \mathrm{d}x \\
&= \int_{-\sqrt{3}}^{0} \left(x^3 - x^2 - x + x^2 - 2x\right) \mathrm{d}x \\
&\quad + \int_{0}^{\sqrt{3}} \left(-x^2 + 2x - x^3 + x^2 + x\right) \mathrm{d}x \\
&= \int_{-\sqrt{3}}^{0} \left(x^3 - 3x\right) \mathrm{d}x + \int_{0}^{\sqrt{3}} \left(3x - x^3\right) \mathrm{d}x \\
&= \left[\frac{x^4}{4} - 3\frac{x^2}{2}\right]_{-\sqrt{3}}^{0} + \left[3\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{\sqrt{3}} \\
&= -\left[\frac{(-\sqrt{3})^4}{4} - 3\frac{(-\sqrt{3})^2}{2}\right] + \left[3\frac{\sqrt{3}^2}{2} - \frac{\sqrt{3}^4}{4}\right] \\
&= \frac{9}{2} \\
&= 4.5
\end{aligned}
$$

Question 3

Difficulty: ★★☆

The graphs of $y = (x-1)(x+1)(x+4)$ and $y = \ln(5-x)$ intersect at three points.



More information

Find the area of the region enclosed by these two graphs.

Give your answer as a decimal, accurate to 3 significant figures.

19.1 ✓

Student view

Accepted answers

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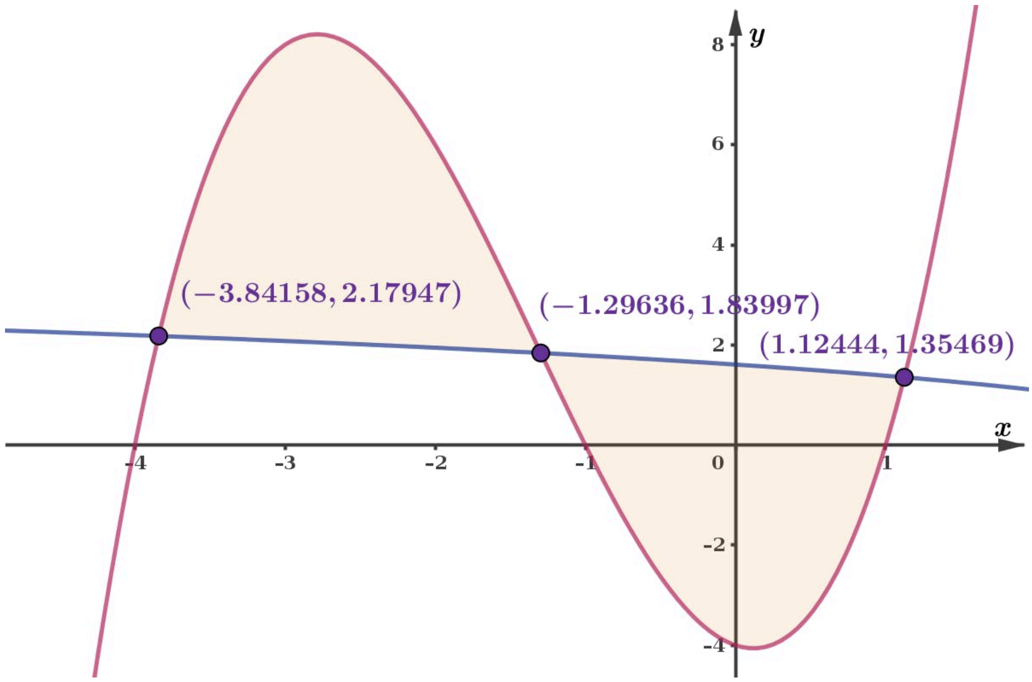
19.1, 19,1

Explanation

Graphic display calculators have applications that can find the points of intersection.



More information

Using the x-coordinate of the first and third points as limits, you can integrate the absolute value of the difference of the expressions to find the area.

$$\int_{-3.84158}^{1.12444} |(x-1)(x+1)(x+4) - \ln(5-x)| \approx 19.1$$

5. Calculus / 5.11 Definite integrals

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-27913/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- understand the connection between indefinite and definite integral
- understand the Newton-Leibniz formula and be able to use it to find exact values of definite integrals

Student
view

- find exact values of areas of regions bounded by the graph of $y = f(x)$ and the x-axis
  - be aware that the regions above and below the x-axis needs to be dealt with separately
- find the area of regions between the graph of two functions
  - be aware that several integrals might be needed in case the graphs intersect
- be aware that technology can also be used if an approximate value of the definite integral is enough to answer the question
  - be aware that in certain cases the techniques you have learned may be insufficient to find exact values, so the use of technology is necessary.

Investigation

Part 1

Use the applet below to investigate how the area of a triangle depends on the coordinates of the vertices.

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Interactive 1. Graph Showing the Area of a Triangle Based on Vertex Coordinates.

More information for interactive 1

This interactive allows users to explore how the area of a triangle is influenced by the coordinates of its vertices. By adjusting the red dots representing the triangle's vertices, users can observe how changes in position affect the area. The graph is set within a coordinate plane where the x-axis ranges from 0 to 50 and the y-axis ranges from 0 to 35. The triangle is outlined with blue sides, and its interior is shaded blue to highlight the enclosed region.

As users move the red points or manually input different coordinates for the vertices, the interactive dynamically updates the calculated area. The area of a triangle with vertices $(x1, y1)$, $(x2, y2)$, and $(x3, y3)$ is determined using the formula: $\text{Area} = \frac{1}{2}|x1(y2 - y3) + x2(y3 - y1) + x3(y1 - y2)|$

For example, if the vertices are at $(20, 4)$, $(8, 30)$, and $(44, 22)$, the area of the triangle will be approximately **420 square units.**

Through this interactive, users will gain a hands-on understanding of the relationship between the coordinates of a triangle's vertices and its area, reinforcing key geometric and algebraic concepts.

Look for a direct relationship between the coordinates and the area. The area given on the applet is precise; it is not an approximate value.

Part 2

In **Example 3** of section 5.11.3 (/study/app/math-aa-hl/sid-134-cid-761926/book/area-and-integral-id-27911/) you saw a region bounded by a line and a parabola. In the 3rd century BC Archimedes proved a certain relationship between the area of a parabolic segment and a related triangle. He used this relationship to find the area of parabolic segments without the use of calculus. The applet below shows you the graph of $y = x^2$ and a chord. It also shows the triangle corresponding to this chord, where the third vertex is below the midpoint of the chord. The applet also shows the area of the parabolic segment and the area of the triangle. Note that the scale on the two axes is not $1 : 1$; the values of the areas are the actual values, not the areas of the shapes on the diagram.

Student view

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Interactive 2. Graph Showing the Area of a Triangle Based on Vertex Coordinates.

More information for interactive 2

This interactive allows users to explore the relationship between a parabolic segment and the area of a triangle formed by a chord of the parabola. The interface displays the graph of y=x2 along with a chord connecting two points on the parabola. Users can manipulate the two red dots, which represent the endpoints of the chord, to observe how the areas of the enclosed regions change dynamically.

The interactive highlights two important geometric areas, The **parabolic segment**, which is the region bounded by the parabola and the chord (shaded in blue). The **triangle**, which is formed by the chord and the x-axis, where the third vertex is positioned at (0,0) (shaded in pink).

As users move the endpoints of the chord, they will observe that both the area of the parabolic segment and the area of the triangle change accordingly. The relationship between these two areas becomes evident—when the area of the triangle increases, the area of the parabolic segment also increases.

For example, users might notice the following approximate relationships: When the area of the triangle is **11.49**, the area of the parabolic segment is **15.32**.

As the area of the triangle changes, the area of the parabolic segment follows a proportional increase.

This visualization helps users investigate and understand the relationship between these two areas, reinforcing key mathematical concepts about parabolic segments, integration, and geometric properties.

- Move the endpoints of the chord and investigate the relationship between the two area values. Creating a scatter plot can be useful. Can you find the relationship?
- Can you prove the relationship you found?

Student
view

Rate subtopic 5.11 Definite integrals

Help us improve the content and user experience.

☆☆☆☆☆