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The big picture

'When will I ever use differentiation?'You may have been asking your teachers this question for some time. In this subtopic, you will see how using differentiation can actually help society.

Optimisation is important for a business owner or CEO.

Credit: Luis Alvarez Getty Images

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

As a business owner or CEO, the main idea is to make money. You cannot stay in business long if you lose money. As long as you are making money, you might as well make as much as possible. How many items do you make? How many resources do you procure? These are optimisation problems. Whether you want to maximise profits or minimise time, calculus can come to the rescue.

There are times you need to study how things change over time. For something geometric, like filling a water tank or tracking a moving object, you need to be able to study a variable over time. This can be modelled by a derivative with respect to time. If you can find that variable in some equation, you can find how that changing variable relates to another. For example, how does a change in volume affect the change in height? How does the change in position affect some other variable?

In the real world, formulas are not always presented in the nice, easy format you learn in algebra. The mathematical models may have all of the variables pieced together in a way that makes more sense, but is mathematically more challenging to work with. In this subtopic, you will learn implicit differentiation, a method for finding the derivatives of equations that do not have y as the subject. You will then use these techniques to solve related rates of change and optimisation problems.

Concept

Calculus is an important tool for investigating **relationships** in real life. In this subtopic, you will see some examples of this, but, of course, there are plenty more that are not discussed here. While studying the examples in this book, look out for possible similarities with other applications where you could use calculus as a **modelling** tool to investigate a problem and find an optimal solution.

Student view

5. Calculus / 5.14 Applications of differentiation

Implicit differentiation

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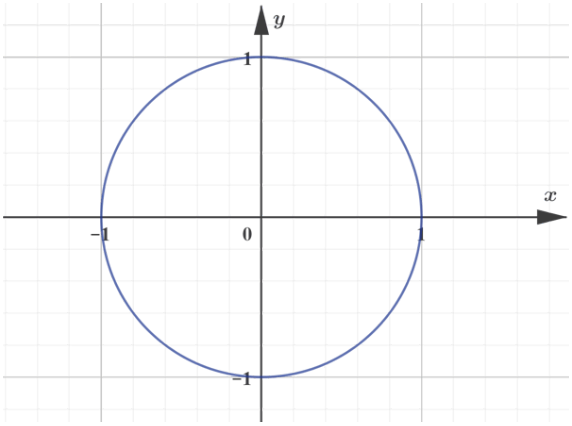
So far, you have been finding derivatives of functions. However, some equations do not satisfy the definition of a function; that is, they do not pass the vertical line test , or to put this another way, there is not a unique value of y for each value of x .



$$x^2 + y^2 = 1$$

More information

The image is a graph showing a circle centered at the origin of a Cartesian coordinate plane. The circle has a radius of 1, as indicated by its intersection with the axes at points (1,0), (0,1), (-1,0), and (0,-1). The x-axis and y-axis are labeled, with numbers marking intervals of 1. A grid facilitates the visualization, and each square in the grid matches a unit length along the axes.

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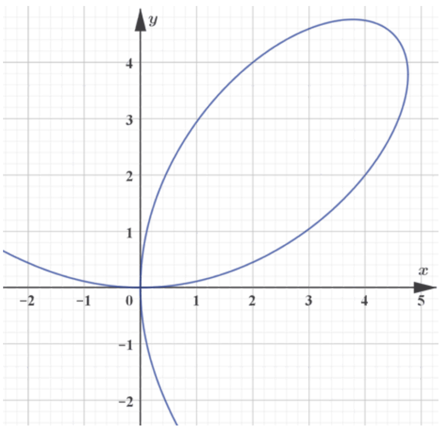

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$$x^3 + y^3 - 9xy = 0$$

More information

The image is a graph of the equation (x^3 + y^3 - 9xy = 0). The graph is plotted on a coordinate grid.

X-axis and Y-axis:

- Both the x-axis and y-axis are labeled with interval marks at each unit.
- The x-axis ranges from -5 to 5.
- The y-axis ranges from -5 to 5.

Graph Details:

- The curve intersects the axes at the origin (0,0), forming a loop-like shape.
- The loop extends across all four quadrants, with a notable direction change suggesting complex behavior in both positive and negative values for x and y.
- In the first quadrant, the curve bends towards the y-axis before curving back towards the x-axis.
- In the third quadrant, the curve follows a similar pattern, curving outwards and then back towards the axes, maintaining symmetry around the origin.

Additional Attributes:

- The graph demonstrates symmetry, reflecting across both the x-axis and y-axis.

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Although these are not functions, you may still be interested in finding the instantaneous slope, or the gradient, at some specific point.

When you can express y on the left-hand side as a function of only the variable x on the right-hand side, you say that y has been expressed explicitly in terms of x. Thus, the following are both explicitly expressed functions:

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- $y = x^2 \sin x$
- $y = \ln(\frac{1}{x} - \mathrm{e}^{-x})$

However, you can also have implicit equations where y is not expressed as a function of x only. The following are examples of implicit functions:

- $x^2y + 2y - 2 = 0$
- $y^2 + \left(2x - x^2\right) y - 2x^3 = 0$
- $\mathrm{e}^{x+y} = x + y$

Sometimes you can manipulate an implicit equation into explicit form. For example:

$$x^2y + 2y - 2 = 0 \Leftrightarrow y\left(x^2 + 2\right) = 2 \Leftrightarrow y = \frac{2}{x^2 + 2}$$

In other cases, you can manipulate an implicit function into two (or more) explicit functions:

$$y^2 + \left(2x - x^2\right) y - 2x^3 = 0 \Leftrightarrow (y + 2x)\left(y - x^2\right) = 0\,.$$

At this point, you can apply the zero-product property from subtopic 2.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25390/) and solve each factor separately: $y + 2x = 0$ and $y - x^2 = 0$.

But there are many cases, such as $\mathrm{e}^{x+y} = x + y$, where there is nothing you can do to convert the implicit function into an explicit function.

So, how do you differentiate implicit functions? You do so implicitly.

Important

Implicit differentiation is the process of finding the derivative of a dependent variable in an implicit function by differentiating each term

Student view

separately.

Exam tip

Implicit differentiation is not difficult provided you remember to think of $\frac{\mathrm{d}}{\mathrm{d}x}$ as an **operator** that operates on a function. Technically, this is the **chain rule.**

Be aware

The chain rule from subtopic 5.6 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27778/) is critical for this process.

In fact, thinking of $\frac{d}{dx}$ as an operator operating on functions renders all of differentiation much simpler. However, when using the operator method, remember that:

1. $\frac{d}{dx}(x) = \frac{dx}{dx} = 1$.
2. $\frac{d}{dx}(y) = \frac{dy}{dx}$. This is the gradient function.
3. $\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}$. This is the chain rule, where $\frac{d}{dy}(f(y))$ is the derivative of $f(y)$ with respect to y . All the standard derivatives from the formula booklet may be used, even though they are written in terms of the variable x because they apply to any variable. So, for example, $\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$.
4. $\frac{dy}{dx} = \frac{1}{dx/dy}$.
5. When the function is implicit, the gradient function will be a function of both x and y. To find the value of the gradient at a particular point, you will need to substitute values for both x and y into the gradient function. You may be

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given these values of x and y in the question, or you may need to calculate y from a given value of x.

Let us start with an easy one first, the gradient of $x^2 + y^2 = 4$.

Example 1

★★☆

Rearrange $x^2 + y^2 = 4$ into an explicit function. Hence find the gradient of $x^2 + y^2 = 4$ at the point $(-1.2, -1.6)$.

First, solve the literal equation for y:

$$y = f(x) = \pm\sqrt{4 - x^2}$$

You can graph this as:

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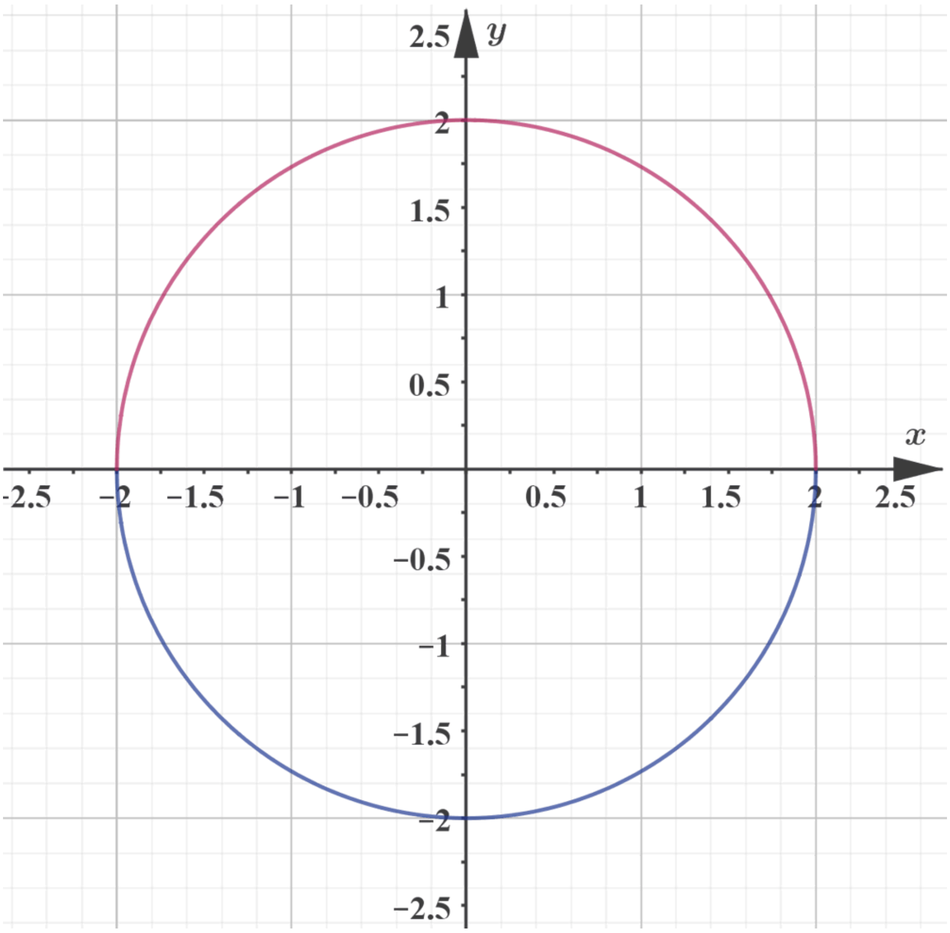

By looking at the point $(-1.2, -1.6)$ and noticing that it is in quadrant three, you know that you need to analyse the bottom half of the graph – the blue curve, $f(x) = -\sqrt{4-x^2}$. Using the chain rule:

$$f'(x) = \frac{x}{\sqrt{4-x^2}}$$

You can now evaluate the gradient where $x = -1.2$:

$$f'(-1.2) = \frac{-1.2}{\sqrt{4-(-1.2)^2}} = -\frac{1.2}{1.6} = -\frac{3}{4}$$

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view

As long as you can convert an implicit function into an explicit form, this method will work, although deciding on which half of the circle you were interested in and completing the chain rule might not have been easy. Let's now try using implicit

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differentiation instead.

Example 2

★★☆

Using implicit differentiation, find the gradient of $x^2 + y^2 = 4$ at $(-1.2, -1.6)$.

Take the derivative of all terms on both sides of the equation:

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(4)$$

Don't forget the chain rule.

$$2x\left(\frac{dx}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0.$$

In the first term, $\frac{dx}{dx}$ reduces to 1 and is typically not shown .

$$\frac{dy}{dx} = y' = -\frac{x}{y}$$

This now describes the gradient at every point along the equation.

You can now evaluate:

$$y'|_{(-1.2,-1.6)} = -\frac{-1.2}{-1.6} = -\frac{3}{4}$$

Notice that the answers for **Examples 1** and **2** match and the maths is not noticeably harder in **Example 2**. Which method do you prefer?

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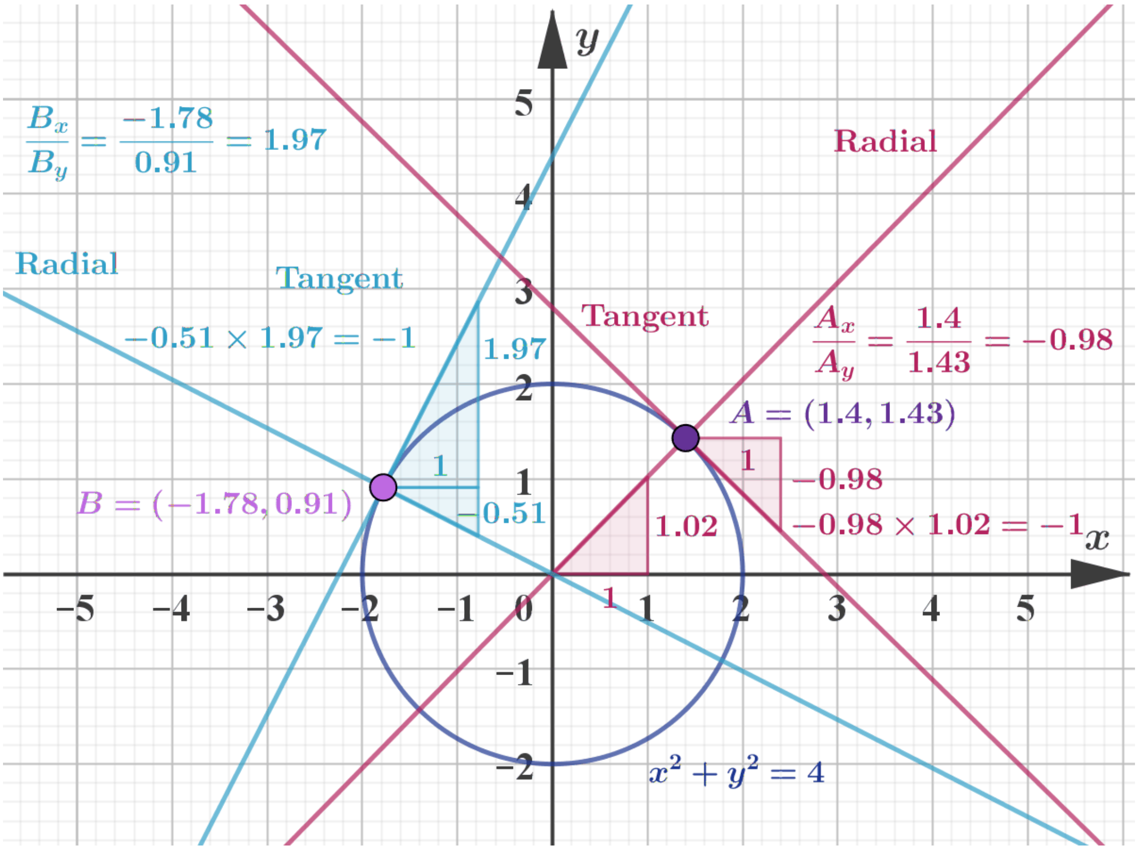
It is worth noting that the result for the gradient of the tangent to a circle, $\frac{dy}{dx} = -\frac{x}{y}$, implies that the tangent to a circle is always at right angles to the radius. Think about a circle centred at the origin. The slope of the radius would be $m_r = \frac{y-0}{x-0} = \frac{y}{x}$, and the slope of the tangent as found above is $m_t = -\frac{x}{y}$. The product of these gradients is -1 everywhere on the circle. This is shown for $x^2 + y^2 = 4$ in the graph below.



More information

The image shows a graph illustrating a circle of equation (x^2 + y^2 = 4). The graph is overlaid with a grid and features two tangents and their corresponding radial lines, which intersect the circle at specific points. These intersections demonstrate the perpendicular relationship between the radial lines and tangents. The circle is centered at the origin.

The graph includes labeled points A and B, with specific coordinates given in the format (A = (1.4, 1.43)) and (B = (1.78, 0.91)). Radial and tangent lines are labeled and colored differently for clarity.

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For instance: - The point A shows the radial line (A_c = 1.4), and tangent information (0.98 \times 1.02 = -1). - At the point B, the radial is marked as (-0.51 \times 1.97 = -1).

The x-axis and y-axis are shown with their respective increments and scales, demonstrating how the slopes of the radial lines and tangents multiply to give -1, fulfilling the property that they are perpendicular.

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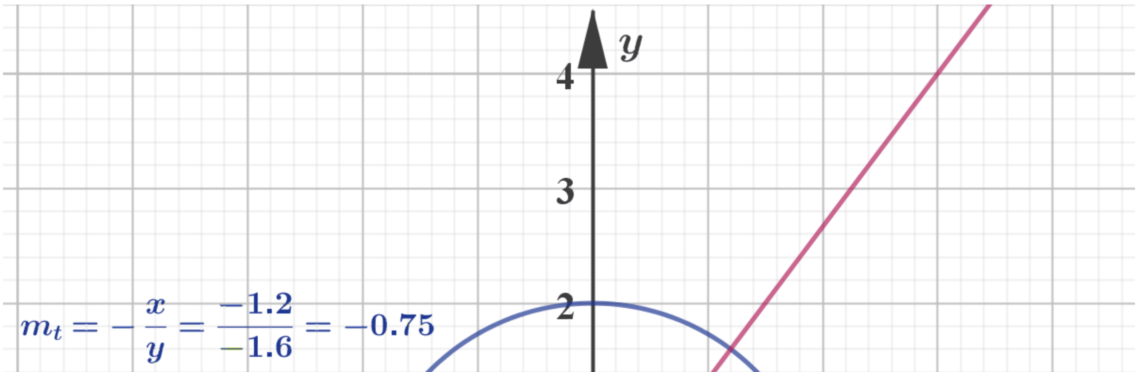

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The image is a graph illustrating a circle described by the equation (x^2 + y^2 = 4). The circle is centered at the origin with a radius of 2, forming a complete closed loop on the Cartesian plane. Two arbitrary points on the circle are emphasized. At one of these points, point (R), tangential and radial lines are drawn. The point (R) has coordinates ((-1.2, 1.6)). The radial line extends from the origin to point (R), effectively acting as a radius of the circle at this point. The tangential line touches the circle at point (R) and proceeds to extend in both directions across the plane. The slope of the tangential line at (R) is determined by the ratio of the (y)-coordinate to the (x)-coordinate of point (R), calculated as (\frac{1.6}{-1.2}), which is equivalent to (-1.33). The tangential and radial lines intersect perpendicularly at point (R), a geometric property evident in the perpendicular slopes that multiply to (-1).

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In the graph above, t wo arbitrary points on the circle given by $x^2 + y^2 = 4$ and the tangential and radial lines at these points. Note how the gradient of the tangential line is given by the ratio of the x- and y-coordinates of the point, and how the gradients of the tangential and radial lines multiply to give -1. Thus, they are perpendicular.

In general, you can carry out implicit differentiation using the following steps.

✓ **Important**

Implicit differentiation steps

1. Differentiate all terms on both sides of the equal sign with respect to x.
2. Collect all terms with $\frac{dy}{dx}$ on one side, and all other terms on the other side.
3. Factorise out $\frac{dy}{dx}$.

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4. Rearrange to find $\frac{dy}{dx}$, or y'.

Here is an example that cannot be solved by explicit differentiation.

Example 3

★★☆

Find the gradient of $2y = x^2 + \sin y$. (undefined)

Steps	Explanation
$2\frac{dy}{dx} = 2x + \frac{dy}{dx}\cos y$	Differentiate all terms on both sides of the equat
$2\frac{dy}{dx} - \frac{dy}{dx}\cos y = 2x$	Collect $\frac{dy}{dx}$ on one side
$\frac{dy}{dx}(2 - \cos y) = 2x$	Factorise out $\frac{dy}{dx}$
$\frac{dy}{dx} = y' = \frac{2x}{2 - \cos y}$	Rearrange

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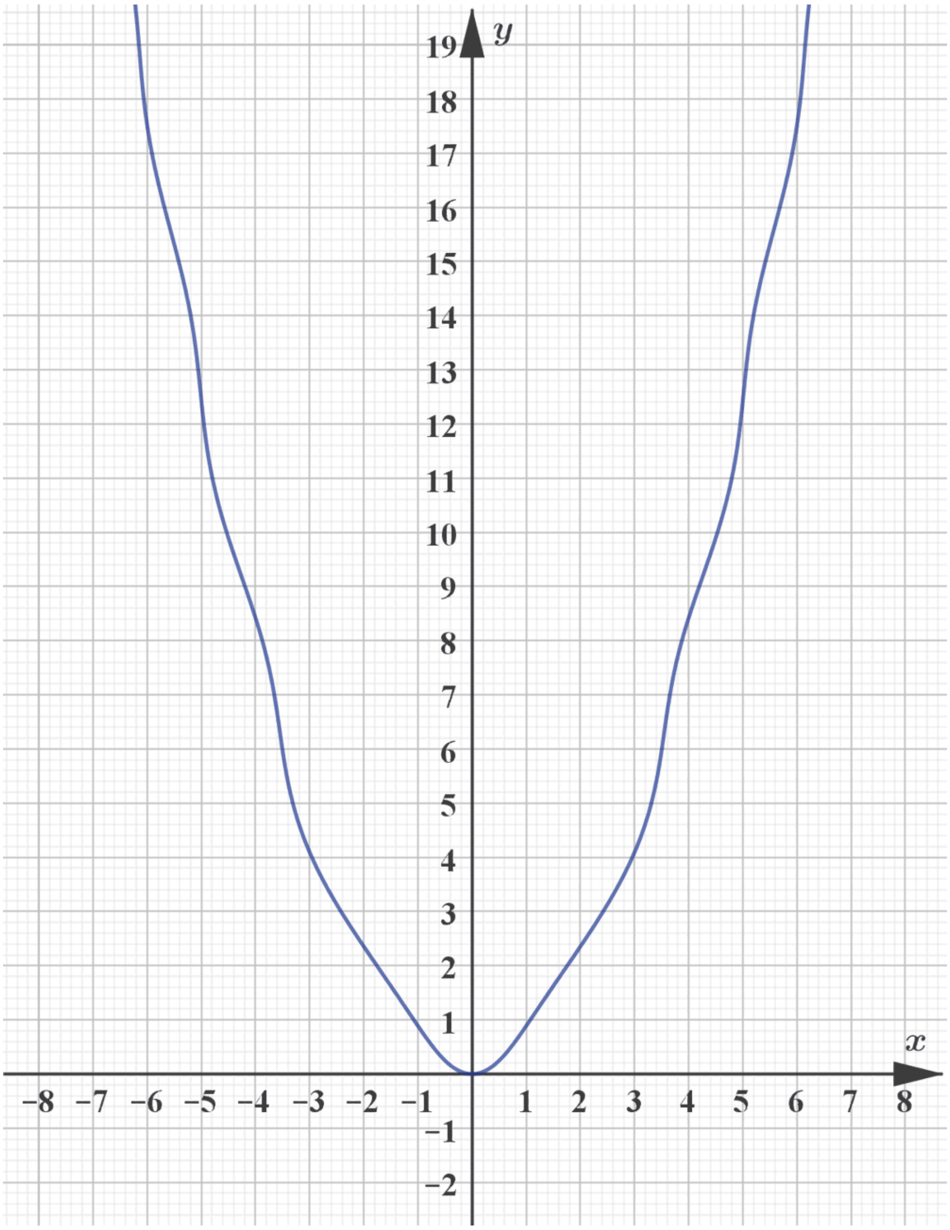

Why would you not be able to solve this explicitly?

$$2y = x^2 + \sin y$$

$$2y - \sin y = x^2$$

With the y in the sine function, there is no way to isolate the y on one side.

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But it is possible to work it out by finding $\frac{dx}{dy}$.

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For $x > 0$ you can follow for example the following route.

$$x = \sqrt{2y - \sin y}$$

$$\frac{dx}{dy} = \frac{2 - \cos y}{2\sqrt{2y - \sin y}}\text{, chain rule}$$

$$\frac{dx}{dy} = \frac{2 - \cos y}{2\sqrt{x^2 + \sin y - \sin y}} = \frac{2 - \cos y}{2\sqrt{x^2}} = \frac{2 - \cos y}{2x},$$

substitute original equation $2y = x^2 + \sin y$

$$\frac{dy}{dx} = \frac{2x}{2 - \cos y}\text{, take the reciprocal of both sides.}$$

For $x < 0$ you can use a similar calculation.

If you compare this with the solution, you can see that there are times when explicit functions can be solved much more easily using implicit differentiation.

Example 4

★★★

Find the gradient of $y = x^{\cos x}$.

Steps	Explanation
$\log_x y = \log_x x^{\cos x}$	Take logarithms to base x on both side
$\log_x y = \cos x$	Simplify using inverse functions
$\frac{\ln y}{\ln x} = \cos x$	Apply the change-of-base formula (see 2.9)

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Steps	Explanation
$\ln y = \ln x \cos x$	Clear fractions
$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{x} - \ln x \sin x$	Take the derivative of both sides (produ
$\frac{dy}{dx} = y' = y\left(\frac{\cos x}{x} - \ln x \sin x\right)$	Rearrange
$y' = x^{\cos x}\left(\frac{\cos x}{x} - \ln x \sin x\right)$	Substitute for y in terms of x

Example 5

★★☆

Find the second derivative, y'', of $y^2 = x^2 - 3$ at point $(2, 1)$.

Steps	Explanation
$2y\,y' = 2x$ $y' = \frac{x}{y}$	Take first derivative
$y'' = \frac{y - xy'}{y^2}$	Take second derivative using the quotient rule subtopic 5.6 for a reminder)
$y'' = \frac{y - x\frac{x}{y}}{y^2} = \frac{1}{y} - \frac{x^2}{y^3}$ $y'' = \frac{1}{1} - \frac{2^2}{1^3} = 1 - 4 = -3$	Substitute $x = 2$ and $y = 1$ and evaluate

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3 section questions

Question 1

Difficulty: ★★☆

What is the gradient function of $\ln{(xy)} = x + y$?

1 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - \frac{1}{x}}{\frac{1}{y} - 1}$ ✓

2 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + \frac{1}{x}}{\frac{1}{y} - 1}$

3 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - \frac{1}{y}}{\frac{1}{x} - 1}$

4 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - x}{y - 1}$

Explanation

Differentiating both sides of the equation with respect to x gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(xy)\right) = \frac{\mathrm{d}}{\mathrm{d}x}(x + y)$$
$$\frac{1}{xy}(y + xy') = 1 + y'$$

Solving this equation for y' gives the gradient function:

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$$\frac{1}{xy}(y+xy') = 1+y'$$
$$\frac{1}{x}+\frac{y'}{y} = 1+y'$$
$$\frac{y'}{y}-y' = 1-\frac{1}{x}$$
$$y'\left(\frac{1}{y}-1\right) = 1-\frac{1}{x}$$
$$y' = \frac{1-\frac{1}{x}}{\frac{1}{y}-1}$$

Question 2

Difficulty:

★☆☆

What is the **exact** value of the gradient of the curve $x^3+y^3-xy^2=5$ at the point $(1,2)$?

1 $\frac{1}{8}$ ✓

2 $\frac{7}{8}$

3 $\frac{1}{16}$

4 $\frac{7}{16}$

Explanation

We find the derivative function first:

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}x}\left[x^3+y^3-xy^2\right] = \frac{\mathrm{d}}{\mathrm{d}x}[5] \\ \Rightarrow \quad & 3x^2+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}-y^2-2xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\ \Rightarrow \quad & \frac{\mathrm{d}y}{\mathrm{d}x}\left(3y^2-2xy\right) = y^2-3x^2 \\ \Rightarrow \quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2-3x^2}{3y^2-2xy} \end{aligned}$$

Then we evaluate for $x=1$ and $y=2$:

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$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 3x^2}{3y^2 - 2xy}$$
$$\Rightarrow \quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1,y=2} = \frac{2^2 - 3 \times 1^2}{3 \times 2^2 - 2 \times 1 \times 2}$$
$$\Rightarrow \quad = \frac{1}{8}$$

Question 3

Difficulty:

★★★

Consider the curve defined by $y^2 = x^2 + 2x$. Find the positive value of y'' at $x = 1$. Give your answer to the nearest thousandth.

0.192

Accepted answers

0.192

Explanation

Find the y-coordinate of the point where $x = 1$

$$y^2 = (1)^2 + 2(1) = 3$$
$$y = \pm\sqrt{3}$$

Find the derivative (see subtopic 5.6 for the quotient rule),

$$2yy' = 2x + 2$$
$$y' = \frac{2x+2}{2y} = \frac{x+1}{y}$$

Find the derivative of the derivative (i.e. differentiate again).

$$y'' = \frac{y - (x+1)y'}{y^2} = \frac{y - (x+1)\left(\frac{x+1}{y}\right)}{y^2} = \frac{y^2 - (x+1)^2}{y^3}$$
$$= \frac{x^2 + 2x - x^2 - 2x - 1}{y^3} = -\frac{1}{y^3}$$

Evaluate

$$y = \sqrt{3} : y'' = -\frac{1}{\left(\sqrt{3}\right)^3} = -0.192$$

Student view

$$y = -\sqrt{3} : y'' = -\frac{1}{\left(-\sqrt{3}\right)^3} = 0.192$$

Related rates of change

Consider filling a glass with water from a slowly dripping tap.

Over time, the rate at which the water enters the glass remains constant. Will the height of the water rise at a constant rate?

Problems about related rates of change involve you using known values and rates to find the rate at which some other variable is changing.

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You can use the following method.

Important

Strategy for solving related rates of change problems.

1. **Understand the problem.** In particular, identify:
   a) What rate of change do you *want*?
   b) At what *time* do you want that rate of change?
   c) What rate of change have you been *given*?
2. **Develop a model.** Often, drawing or studying a diagram is the best way to start.
3. **Write an equation relating what you want and what you know.** This is often a geometric fact, and you may have extra variables that you need to eliminate.
4. **Differentiate both sides with respect to time.** All the basic rules still apply, especially the chain rule.
5. **Substitute any known values.**
6. **Answer the question.** Don't forget to include units.

Example 1

★★☆

An open cone of height $10\,\text{cm}$ and base of radius $5\,\text{cm}$ is being filled with water at a rate of $1.5\,\text{cm}^3\,\text{s}^{-1}$. When the height of the water is $5\,\text{cm}$, what is the rate at which the water level is rising?

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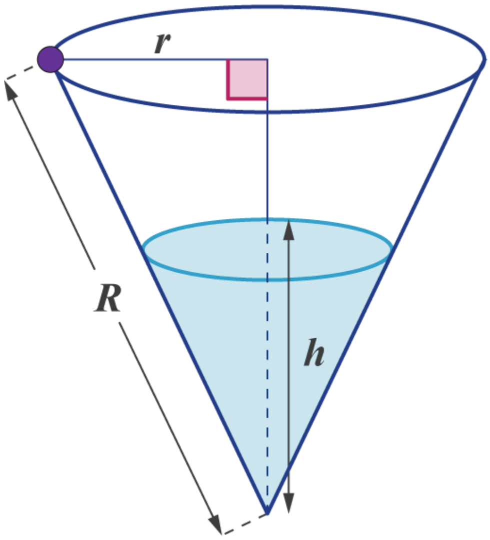

More information

The image is a diagram showing a vertical cross-section of an open, upright cone partially filled with water. The cone has a height labeled as (R) and a base radius labeled as (r). There is a smaller cone of water inside, indicated by a blue shaded area, with its height labeled as (h). The cone's slant height is depicted by a dashed line from the apex to the edge of the base. The diagram also includes a red square indicating a right angle between (r) and (h) lines at the top of the small cone, showing the water level. The external radius and height of the large cone are not detailed in the diagram, but the diagram connects to contextual text that mentions specifics such as the cone's height and base radius.

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For this problem, you would like to find the rate at which the height is increasing, $\frac{dh}{dt}$, when the height of the water is $5\,\text{cm}$. You know that the cone is being filled at a rate of $1.5\,\text{cm}^3\,\text{s}^{-1}$, so $\frac{dV}{dt} = 1.5$. You also know that $h = 5$ at the point in time you are interested in. Finally, you know the dimensions of the cone; specifically, the height is $10\,\text{cm}$ and the radius is $5\,\text{cm}$. All of this is shown in the diagram.

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Since you are working with volume and height, you need a geometric formula that includes both of these, i.e. the formula for the volume of a cone. Then you need to replace r using the ratio given for the cone.

Steps	Explanation
$V = \frac{1}{3}\pi r^2 h$	Volume of a cone
$\frac{r}{h} = \frac{5}{10}$ $r = \frac{5h}{10} = \frac{h}{2}$	From the dimensions of the cone
$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$	Differentiate both sides
$1.5 = \frac{\pi}{4}5^2\frac{dh}{dt}$	Substitute the known values: $\frac{dV}{dt} =$ $h = 5$
$\frac{dh}{dt} = \frac{6}{25\pi} \approx 0.0764$	Rearrange and solve for $\frac{dh}{dt}$.

Exam tip

In practical applications, it is always worth considering whether your result makes sense.

Thus, for the example of a cone being filled at a constant rate, and considering the rate of change of the water level, it is clear that the water level changes at an ever-slower rate as it increases because the surface area increases with increasing water level. You can see that the equation satisfies this because $\frac{dh}{dt}$ is inversely proportional to the square of the height. Therefore, as h increases, $\frac{dh}{dt}$ decreases.

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Example 2

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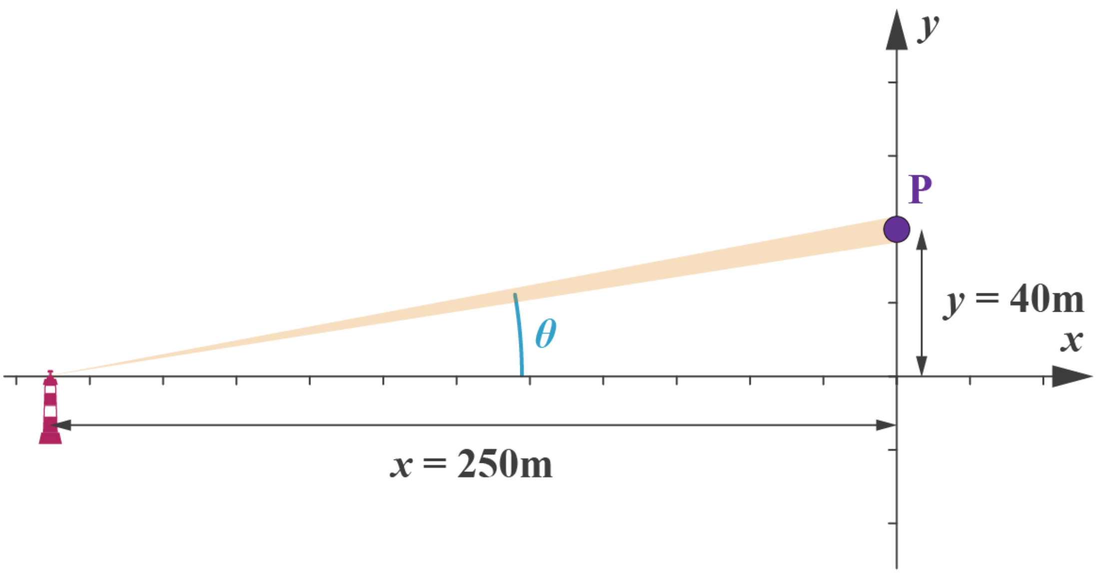
A lighthouse is located 250 m off the coast. Its light rotates at a rate of 2 revolutions per minute. How fast does the light beam pass a point on the coastline that is 40 m from the point directly opposite the lighthouse?

On the diagram below the coastline is the y-axis and the lighthouse is positioned on the x-axis. Angle θ represents the angle between the x-axis and the light beam.



For this problem, you would like to find the rate at which the light beam is moving along the shore. You know that the lighthouse is 250 m from the shore and the light beam is rotating at 2 revolutions per minute. One revolution is 2π radians, so the beam is rotating $2 \times 2\pi$ radians in 60 seconds. Therefore,

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$$\frac{d\theta}{dt} = ()\text{mfrac}\frac{\pi \text{ radians}}{15 \text{ seconds}}, \text{ or } \frac{\pi}{15}\text{rad}\,\text{s}^{-1}.$$

You also know that $y = 40$ at the point of time you are interested in, as the beam is $40\,\text{m}$ along the shoreline. From basic trigonometry, you can find the angle at this time:

$$\tan\theta = \frac{y}{x} = \frac{40}{250}$$
$$\theta = 0.1587$$

Finally, you know you are working with a right triangle as you are in Cartesian space. Using basic trigonometry you can relate θ and y with $y = 250\tan\theta$.

$$\frac{dy}{dt} = 250\sec^2\theta\frac{d\theta}{dt} \qquad \text{Find the derivative o}$$
$$\frac{dy}{dt} = \frac{250}{\cos^2\theta}\frac{d\theta}{dt}$$
$$\frac{dy}{dt} = \frac{250}{\cos^2 0.1587}\frac{\pi}{15} = 53.7 \text{ m s}^{-1} \qquad \text{Substitute the values}$$

Be aware

You will inevitably apply the chain rule when working with related rates, so this gives you a hint for how to relate the information you have been given to what you want to find.

3 section questions

Question 1

Difficulty:

★★☆

Overview
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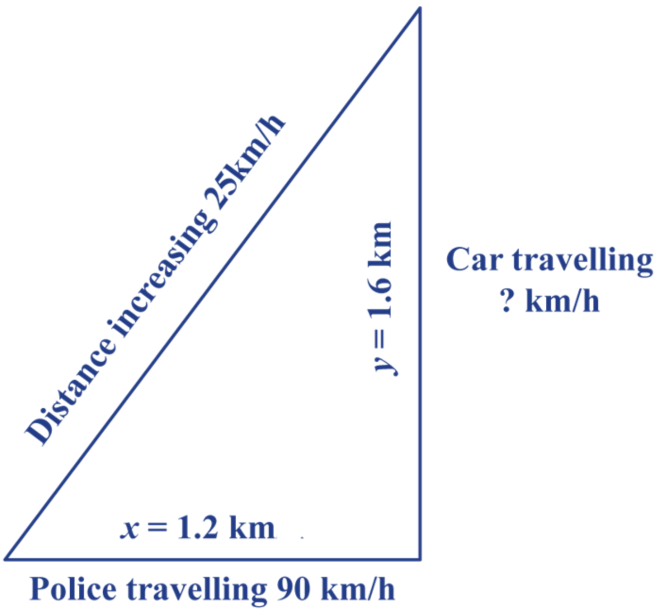
A police car is travelling east and is approaching an intersection at a speed of $90\,\text{km}\,\text{h}^{-1}$. When the police car is $1.2\,\text{km}$ from the intersection, another car is $1.6\,\text{km}$ from the intersection heading north. Radar indicates that the distance between the cars is increasing at $25\,\text{km}\,\text{h}^{-1}$.



More information

How fast is the second car moving? Do not give units with your answer. Round to the nearest hundredth if necessary.

98.75

Accepted answers

98.75, 98,75

Explanation

Given: $\frac{dx}{dt} = -90$, $\frac{dz}{dt} = 25$, $x = 1.2$, $y = 1.6$

x is the distance of the police car from the intersection in km.
y is the distance of the other car from the intersection in km.
z is the distance between the cars in km.
t is time

$\frac{dx}{dt}$ is negative as that car is *approaching* the intersection (so the distance from the intersection is decreasing).

Student view

Overview
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Using Pythagoras' theorem:

$$x^2 + y^2 = z^2$$

Differentiate:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt} \text{ or } x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$

Find z:

$$1.2^2 + 1.6^2 = z^2 \text{ or } z = 2.$$

Substitute:

$$1.2(-90) + 1.6\frac{dy}{dt} = 2(25)$$

$$\frac{dy}{dt} = 98.75\,\text{km h}^{-1}.$$

Question 2

Difficulty: ★★☆

An aeroplane flying at an altitude of $6\,\text{km}$ passes directly over a radar antenna. When the plane is $10\,\text{km}$ from the antenna and heading away, the radar indicates that the distance between the aeroplane and the antenna is increasing at a rate of $400\,\text{km}\,\text{h}^{-1}$. What is the speed of the aeroplane, assuming it is travelling in a straight line at constant speed? Do not give units with your answer. Round to the nearest tenth if necessary.

Student
view

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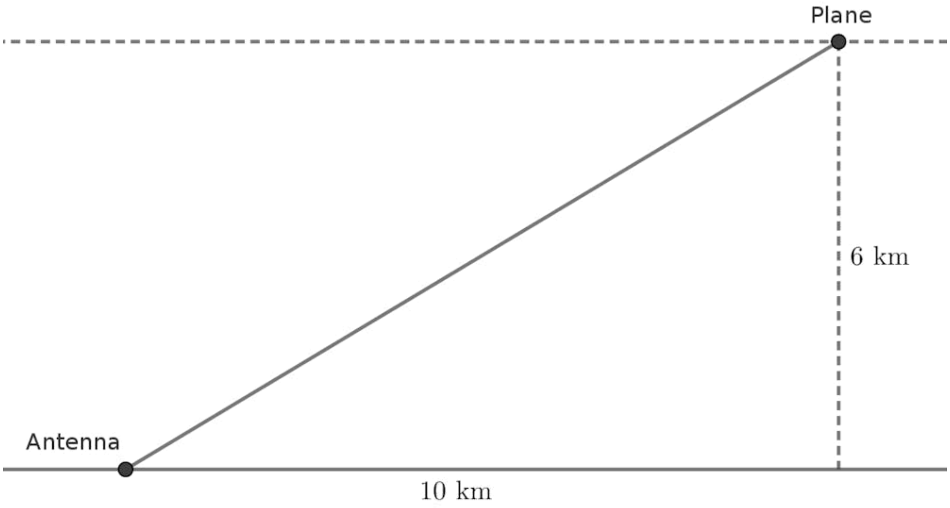

More information

466.5

Accepted answers

466.5, 466,5

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/implicit-differentiation-id-26502/print/) Assign

Explanation

Given: $h = 6, x = 10, \frac{ds}{dt} = 400.$

h is altitude in km

x is the position of the plane relative to the antenna

s is distance between the aeroplane and the antenna

t is time

Using Pythagoras' theorem:

$$x^2 + h^2 = s^2$$

Differentiate:

$$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 2s\frac{ds}{dt}$$

or, after dividing by 2,

$$x\frac{dx}{dt} + h\frac{dh}{dt} = s\frac{ds}{dt}$$

Find s:

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$$10^2 + 6^2 = s^2 \text{ or } s = \sqrt{136} = 11.6619\dots$$

Substitute:

$$10\frac{dx}{dt} + 6(0) = 11.6619\dots \times 400$$

$$\frac{dx}{dt} \approx 466.5\,\mathrm{km\,h^{-1}}.$$

Question 3

Difficulty:

★★★

A spherical balloon is expanding at a constant rate of $10\,\mathrm{cm^3\,s^{-1}}$. Find the rate of change of the total surface area of the balloon when its radius is $25\,\mathrm{cm}$. Enter your answer as decimal without units.

0.8 ✓

Accepted answers

0.8, 0,8, .8, 8/10, 4/5

Explanation

Given: $r = 25$, $\frac{dV}{dt} = 10$.

r is radius
V is volume
t is time

Volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Differentiate:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substitute:

Student
view

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$$10 = 4\pi(25)^2\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{250\pi}\text{ cm s}^{-1}$$

Surface area of a sphere:

$$A = 4\pi r^2$$

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/related-rates-of-change-id-26503/print/) Assign

Differentiate:

$$\frac{dA}{dt} = 8\pi r\frac{dr}{dt}$$

Substitute for $\frac{dr}{dt}$:

$$\frac{dA}{dt} = 8\pi(25)\frac{1}{250\pi} = 0.8\text{ cm}^2\text{ s}^{-1}$$

5. Calculus / 5.14 Applications of differentiation

Optimisation revisited

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/optimisation-revisited-id-26504/print/) Assign

In subtopic 5.8 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27795/), you met the basic concept of optimisation. The idea of optimisation is to find the best possible value of a variable, which occurs mathematically at a maximum or a minimum depending on the problem. In this section, you will revisit this concept and apply it to more challenging problems.

As a reminder, when looking for optimum solutions (maxima and minima), you should consider any point where the gradient is 0, any endpoints from a restricted domain, and any points of non-differentiability. A global optimum is the most extreme of all local optima. The graph below indicates several types of optimum points. The global optima occur at A (maximum) and B (minimum).

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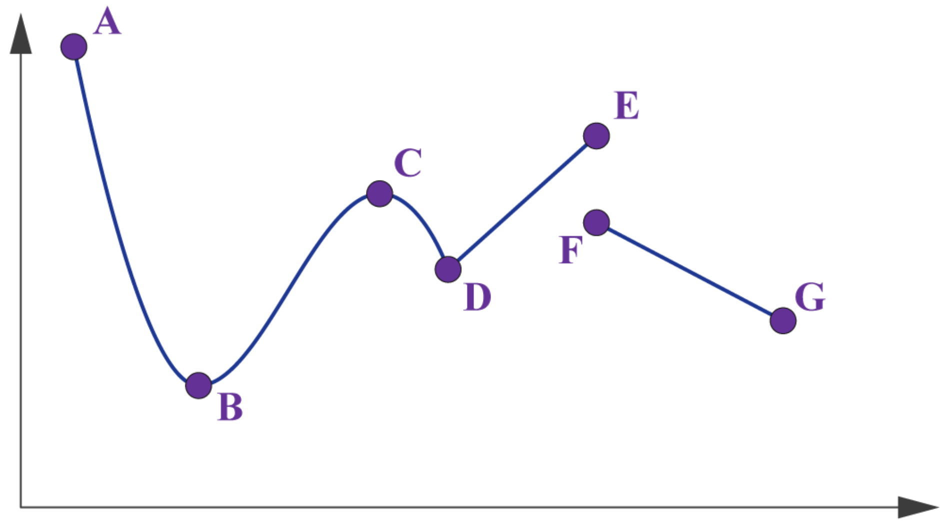

More information

The image is a graph illustrating a curve with several labeled points, from A to G. The X-axis and Y-axis are unlabeled, but the graph shows a continuous curve with peaks and troughs.\n\n- Point A is a high point on the left side of the graph, indicating a local and global maximum. \n- Moving downwards, Point B is the lowest point, indicating a local and global minimum. \n- From B, the curve goes upward to Point C, a local maximum, then down to Point D, a local minimum. \n- The curve rises again to Point E, another local maximum.\n- It then descends to Point F, a local minimum, and continues downward to Point G on the rightmost end of the graph. \n\nThe graph provides a visual representation of various types of optimum points, with global optima highlighted at points A (maximum) and B (minimum).

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Student
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Important

A good procedure for optimisation is:

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1. Define the variables and choose symbols for them.
2. Find the relationship between them, i.e. the function.
3. Differentiate the function and find each stationary point, i.e. set the function for the first derivative to zero and solve the resulting equation. Do not forget to consider the endpoints.
4. Determine whether each stationary point is a local maximum or minimum. A picture or the second derivative concavity test may come in handy.
5. If asked, find the value of the function at the maximum or minimum.

Example 1

★☆☆

A small company receives €7 million from investments every month and has the opportunity to produce widgets to be sold at €6 million per thousand . Therefore, revenue is $r(x) = 6x + 7$, where x represents thousands of units. Cost of production, c , varies based on the amount produced expressed by $c(x) = x^3 - 6x^2 + 16x$, where x represents thousands of units. Determine the production level that will maximise profits.

Step 1: Variables are already given as x, $r(x)$, and $c(x)$.

Step 2: Profit can be found as revenue minus cost, or

$$P(x) = r(x) - c(x) = (6x + 7) - \left(x^3 - 6x^2 + 16x\right) = -x^3 + 6x^2 - 10x$$

Step 3: $P'(x) = -3x^2 + 12x - 10 = 0$

$$x = 2 \pm \frac{\sqrt{6}}{3} \approx 1.184, 2.816$$

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Consider endpoints:

Left endpoint $x = 0$ as you cannot make negative units. There is no natural right endpoint.

Step 4: $P''(x) = -6x + 12$

$$\begin{array}{lll} P'(0) = -10 < 0 & \text{Decreasing} & \text{So maximum at left en} \\ P''(1.184) = 4.896 > 0 & \text{Concave up} & \text{Minimum} \\ P''(2.816) = -4.896 < 0 & \text{Concave down} & \text{Maximum} \end{array}$$

Step 5: You want to maximise profit, so you evaluate both maxima:

$$P(0) = 7$$
$$P(2.816) = 4.089$$

The ideal production level in this case is 0, which means the company should not produce. There are times when the best economic choice is to stay out of the market.

The graph of the function would look something like this:

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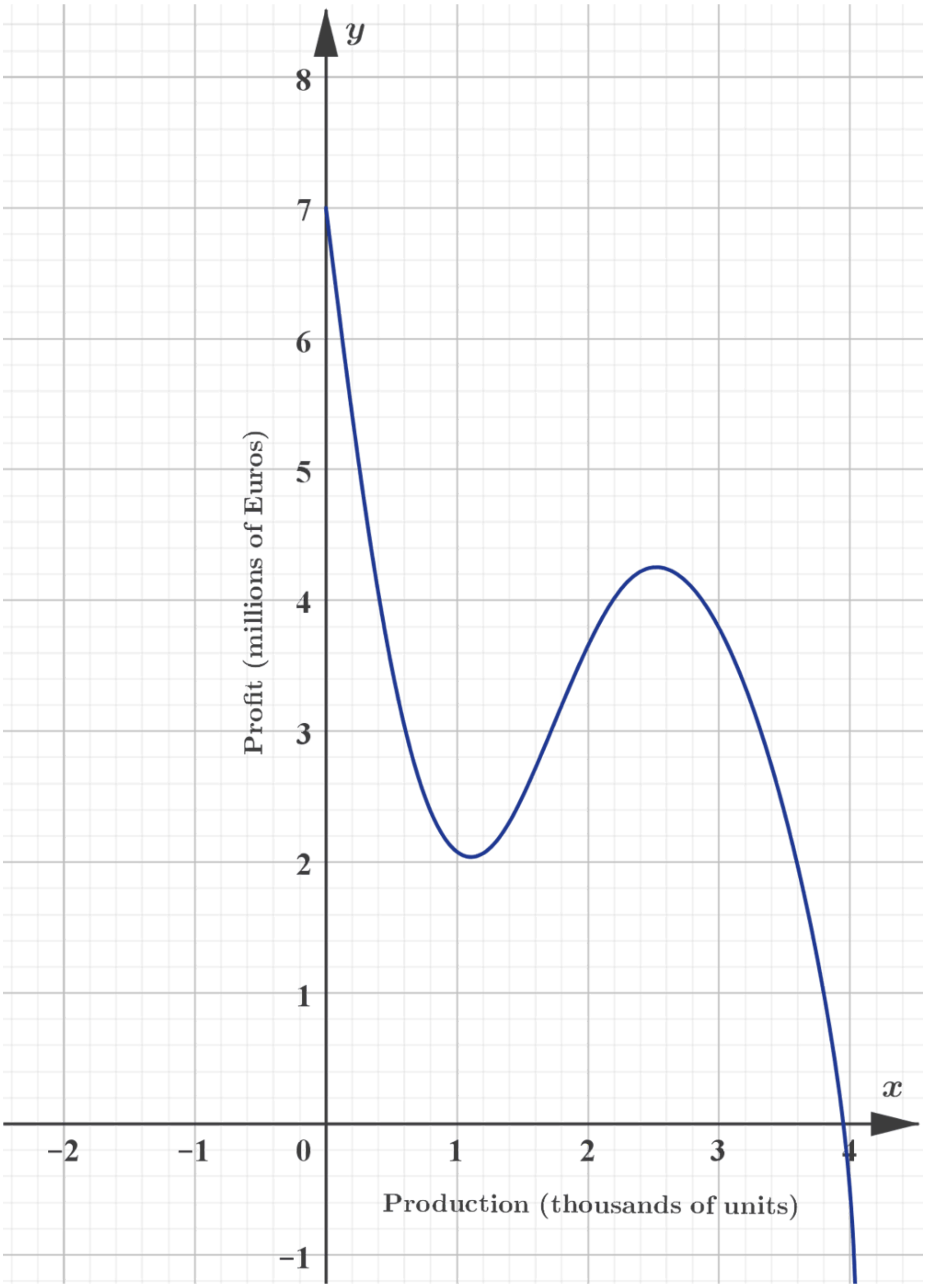

Example 2

★★☆

Consider a cylindrical silo with a hemispherical roof. The volume is $5000\ \mathrm{m}^3$. Find the radius that minimises the surface area.

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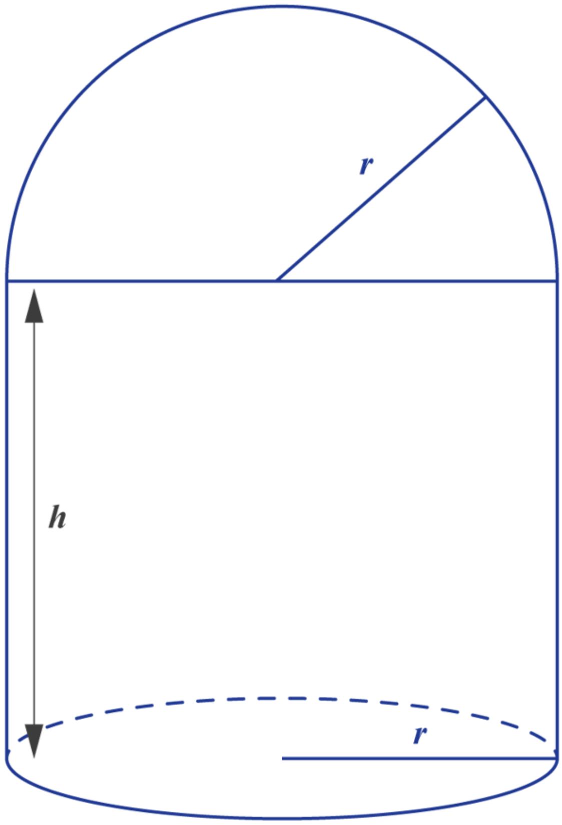

More information

The image is a diagram depicting a cylindrical silo with a hemispherical roof. The diagram shows a vertical cross-section of the silo. The lower part of the silo is a cylinder with a height labeled as 'h' and a radius marked as 'r'. The upper part is a hemisphere with the same radius 'r'. The radius 'r' is drawn from the center of the base to a point on the edge of the base and is represented with a solid line. The height 'h' is shown as a vertical dashed line extending from the base of the cylinder to the bottom of the hemisphere. The diagram's purpose is to understand the geometric relationship between the cylinder and the hemisphere in order to solve for the radius that minimizes the surface area given the volume constraint.

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Student view

Step 1: You can assign variables of v for volume, a for surface area (m^2), r for radius (m), and h for height (m) of the cylinder.

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Step 2: The surface area can be found by adding the area of the floor, the cylindrical wall and the hemispherical roof:

$$A = \pi r^2 + 2\pi rh + \frac{1}{2}(4\pi r^2) = 3\pi r^2 + 2\pi rh$$

Volume can be found by adding the volume of the cylinder and the hemisphere:

$$V = \pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \pi r^2 h + \frac{2\pi}{3}r^3$$

Step 3: You could try to solve for h and substitute $h = \frac{V - \frac{2\pi}{3}r^3}{\pi r^2} = \frac{V}{\pi r^2} - \frac{2}{3}r$ into the surface area equation, but that would become a challenge algebraically. Instead, use implicit differentiation and find the point where $\frac{dV}{dr}$ is 0.

$$\begin{aligned}
A &= 3\pi r^2 + 2\pi rh && \text{Surface area f} \\
\frac{dA}{dr} &= 6\pi r + 2\pi h + 2\pi r\frac{dh}{dr} && \text{Implicit differ} \\
V &= \pi r^2 h + \frac{2\pi}{3}r^3 && \text{Volume formu} \\
\frac{dV}{dr} &= 2\pi rh + \pi r^2\frac{dh}{dr} + 2\pi r^2 && \text{Implicit differ} \\
0 &= 2\pi rh + \pi r^2\frac{dh}{dr} + 2\pi r^2 && \text{Set derivative} \\
\frac{dh}{dr} &= -2\left(\frac{h+r}{r}\right) && \text{Rearrange} \\
\frac{dA}{dr} &= 6\pi r + 2\pi h + 2\pi r\left(-2\left(\frac{h+r}{r}\right)\right) && \text{Substitute for} \\
\frac{dA}{dr} &= 6\pi r + 2\pi h - 4\pi(h+r) = 2\pi r - 2\pi h = 0 \\
r &= h
\end{aligned}$$

Substituting back into the volume equation, you get:

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$$V = \pi r^2 h + \frac{2\pi}{3}r^3 = \pi r^3 + \frac{2\pi}{3}r^3 = \frac{5\pi}{3}r^3$$

$$r = \sqrt[3]{\frac{3V}{5\pi}} = \sqrt[3]{\frac{3(5000)}{5\pi}} = 9.847$$

Therefore, the required radius is 9.85 m (3 significant figures).

3 section questions

Question 1

Difficulty:

★★☆

Consider a box with an open top and a square base. If the outer surface area is 108 cm^2, find the maximum volume. Enter your answer in cm^3 without units.

108

Accepted answers

108

Explanation

Step 1: Assign variables of A for surface area and V for volume. Base is a square x-by-x. Sides have height y.

Step 2: Surface area: $A = 108 = x^2 + 4xy$

Rearrange:

$$y = \frac{108 - x^2}{4x}$$

Volume:

$$V = x^2 y = x^2\left(\frac{108 - x^2}{4x}\right) = 27x - \frac{x^3}{4}$$

Differentiate:

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$$\frac{dV}{dx} = 27 - \frac{3x^2}{4} = 0$$
$$27 = \frac{3x^2}{4}$$
$$36 = x^2$$
$$x = 6$$

$$y = \frac{108 - 6^2}{4(6)} = 3$$ Substitute $x = 6$ to find y

Volume:

$$V = x^2 y = 6^2(3) = 108\,\text{cm}^3$$ Substitute $x = 6$ and $y = 3$

Question 2

Difficulty: ★★☆

Consider a 1 litre cylindrical can. Determine the height of the can that minimises the surface area. Give your answer without units to the nearest tenth of a centimetre.

10.8 ✓

Accepted answers

10.8, 10,8

Explanation

Step 1: Assign variables: A for surface area, V for volume, r for radius and h for height.

Step 2: Volume of cylinder:

$$V = \pi r^2 h = 1000\text{cm}^3$$

Rearrange:

$$h = \frac{1000}{\pi r^2}$$

Total surface area of cylinder:

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

Student view

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Differentiate and set derivative equal to 0:

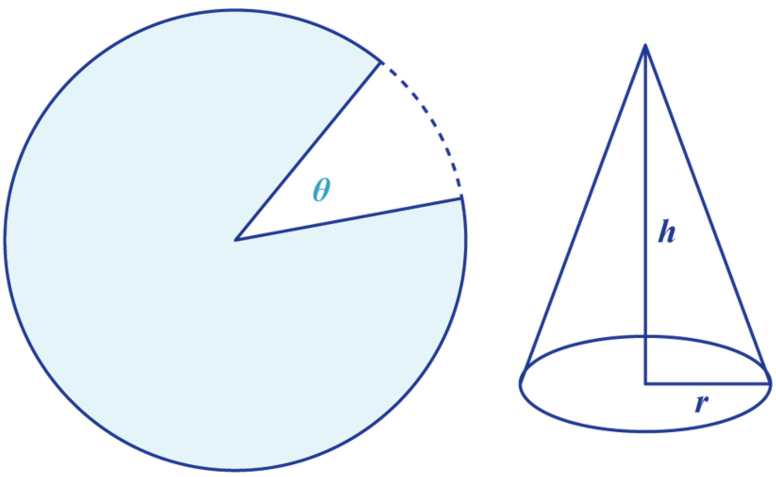
$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2} = 0$$
$$4\pi r = \frac{2000}{r^2}$$
$$4\pi r^3 = 2000$$
$$R = \sqrt[3]{\frac{500}{\pi}} \approx 5.419\ldots$$
$$\text{so } h = \frac{1000}{\pi(5.419\ldots)^2} \approx 10.838\ldots \approx 10.8$$

Question 3

Difficulty:

★★★

The diagram shows a circular piece of card. The radius of the circle is 1 unit. The card is cut along a radius and is then folded to make a cone. Find the arc length of the overlap that results in the cone with the maximum volume. Enter your answer to the nearest hundredth.



More information

1.15

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Accepted answers

1.15, 1,15

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Explanation

Step 1: Assign variables: V for volume, C for circumference, r for radius of cone and h for height of cone.

Step 2: Use Pythagoras' theorem. No dimensions are given, so you can choose, for instance, let the slant height of the cone be 1. This is also the radius of the card.

Therefore, $r^2 + h^2 = 1$

Rearrange: $r^2 = 1 - h^2$

Volume of cone:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(1 - h^2)h = \frac{\pi}{3}\left(h - h^3\right)$$

Step 3: Differentiate and set derivative equal to 0:

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}h} &= \frac{\pi}{3}\left(1 - 3h^2\right) = 0 \\ 1 - 3h^2 &= 0 \\ 3h^2 &= 1 \\ h &= \sqrt{\frac{1}{3}} \\ r^2 &= 1 - \left(\sqrt{\frac{1}{3}}\right)^2 = \frac{2}{3} \\ r &= \sqrt{\frac{2}{3}} \end{aligned}$$

Circumference of cone:

$$C_{cone} = 2\pi r = 2\pi\sqrt{\frac{2}{3}}$$

Arc length of overlap:

$$l = C_{card} - C_{cone} = 2\pi(1) - 2\pi\sqrt{\frac{2}{3}} = 1.15$$

Student
view

5. Calculus / 5.14 Applications of differentiation

Checklist

What you should know

By the end of this subtopic you should be able to:

- apply the rules of implicit differentiation to implicit functions
- solve related rates of change by following these steps:
  - understand the problem
  - develop a model
  - write an equation relating what you want and what you know
  - differentiate both sides
  - substitute any known values
  - solve and answer the question
- translate word problems into functions and apply differentiation to find the optimum solution (maximum or minimum).

Investigation

Take another look at **Example 1** about filling a cone in section 5.14.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/related-rates-of-change-id-26503/).

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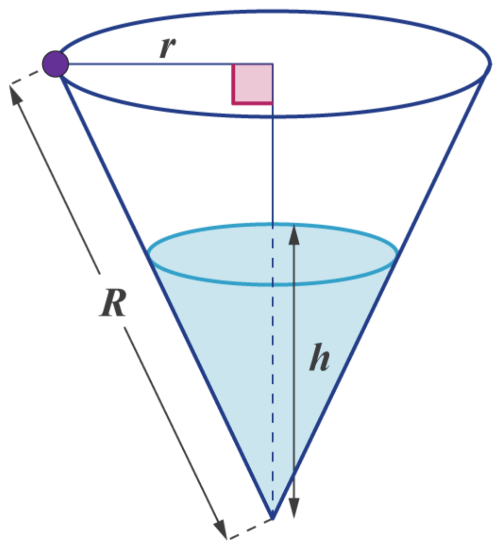

More information

The image is a diagram of a cone, showing it being filled with water. The cone has a circular top and tapers to a point at the bottom. There is a central vertical line indicating the height of the cone labeled 'h'. The top circular edge has a radius labeled 'r', and the slant height from the top edge to the bottom is labeled 'R'. Part of the cone is filled with light blue shading representing water. A purple circle indicates a point on the cone's top, emphasizing the dimensions and connections between 'r', 'R', and 'h'.

[Generated by AI]

This applet allow you to set some of these values and observe the process.

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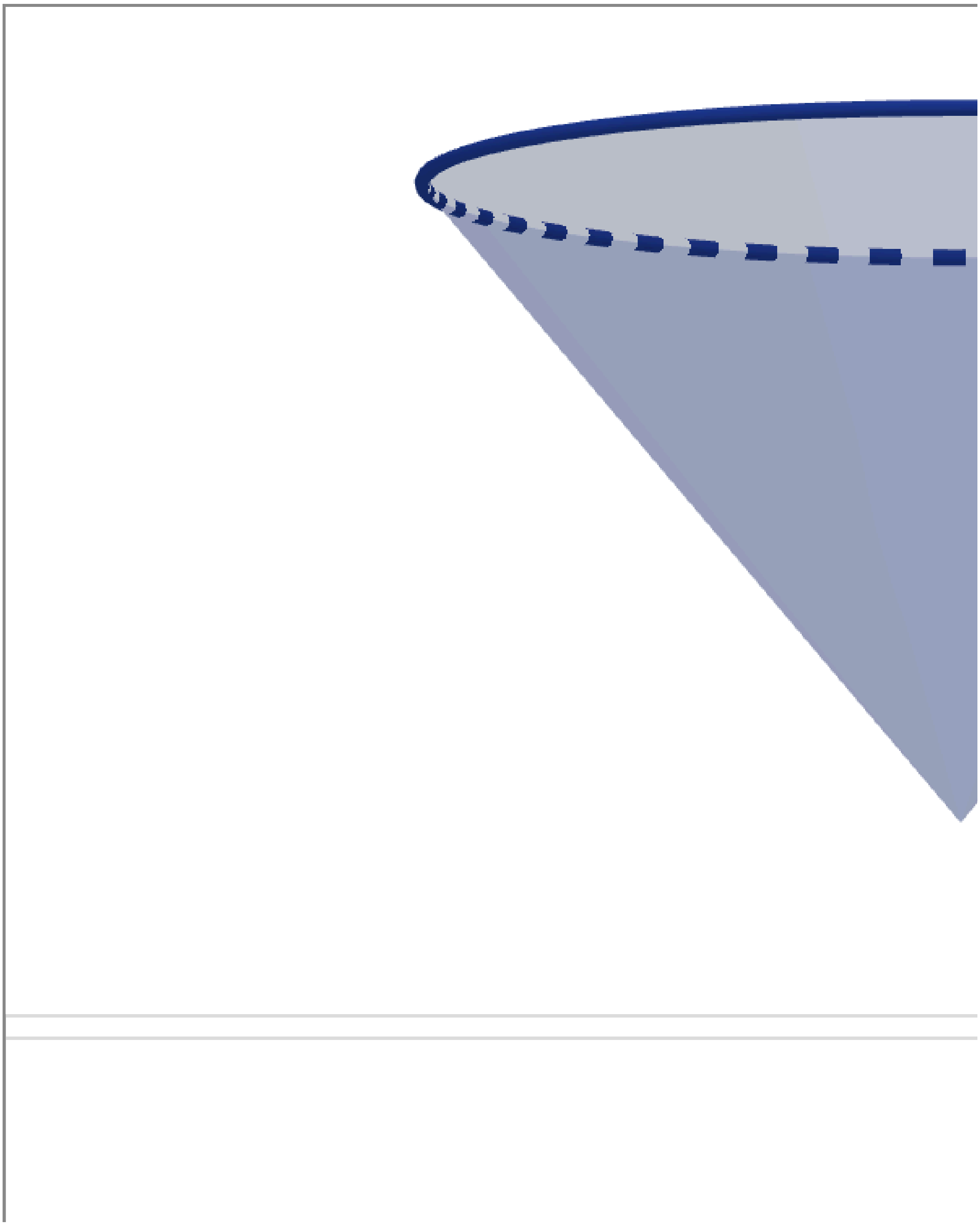

Interactive 1. Observe the Process of Filling a Cone.

Credit: GeoGebra (https://www.geogebra.org/m/MjZjvsKA) Tim Brzezinski

More information for interactive 1

This interactive allows the user to understand the representation of a cone being filled or drained with a liquid and the corresponding rate of change in its dimensions.

The screen is divided into two halves. The top half of the screen, a 3D image of an inverted cone. The cone is translucent, allowing the user to see its overall shape. The cone is filled with a light blue liquid.

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In the bottom half of the screen, users can see a right triangle on the left which illustrates the relationship between the cone's height (h) and radius (r), representing a cross-section of the cone. On the right of the 2D triangle, a horizontal slider is given, representing the fill rate. The users can slide the fill rate slider to choose the values of $\frac{dh}{dt}$ from -4 centimeters per second to 4 centimeters per second. The users can also select the height of the cone using a horizontal slider h from 0 to 50 units. Another horizontal slider users can use to modify the ratio of $\frac{h}{r}$ from 0 to 5.

The interactive also has a "show answer" checkbox using which users can even verify the calculated values of $\frac{dr}{dt}$ and $\frac{dV}{dt}$. Two buttons at the bottom left, "Start filling" and "pause" allow users to control the simulation of the filling or draining process.

Suppose if the users select the fill rate as -4 cm per sec and height $h = 50$ and ratio $\frac{h}{r} = 5$ then clicking on "Show answers" will display $\frac{dr}{dt} = -0.8\text{ cm/sec}$ and $\frac{dv}{dt} = -1256.64\text{ cm3/sec}$. On clicking "start filling" will show that the volume is decreasing.

This interactive helps the users in the graphical visualization of a cone whose dimensions are shrinking or expanding at specified rates.

Part 1

- Set the fill rate in $\text{cm}^3\text{ s}^{-1}$
- Set the ratio of the height over the radius of the cone
- Click the 'Start filling' button

What do you observe? Does the height of the fluid rise steadily? Does it speed up? Does it slow down? How does this fit the equations and the process you learned in this subtopic?

Part 2

Student view

- Set the fill rate in $\text{cm}^3\text{ s}^{-1}$
- Set the ratio of the height over the radius of the cone

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- Set the height of the fluid in cm
- Click the 'Show answer' box

How does changing the given values change the answer for $\frac{\mathrm{d}V}{\mathrm{d}t}$?

Do all of the values make sense?

Rate subtopic 5.14 Applications of differentiation

Help us improve the content and user experience.

Student
view