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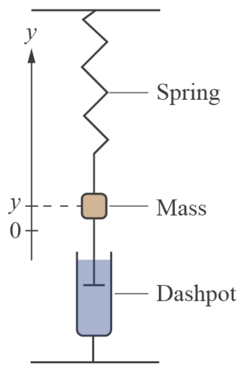
The big picture

Have you ever opened a screen door that has a pneumatic cylinder that prevents it from slamming shut? The mechanical engineering term for this is a dashpot. A dashpot retards force through the use of viscous friction, often in one direction only.

Consider the following simplified system with a mass connected to a dashpot and a spring:



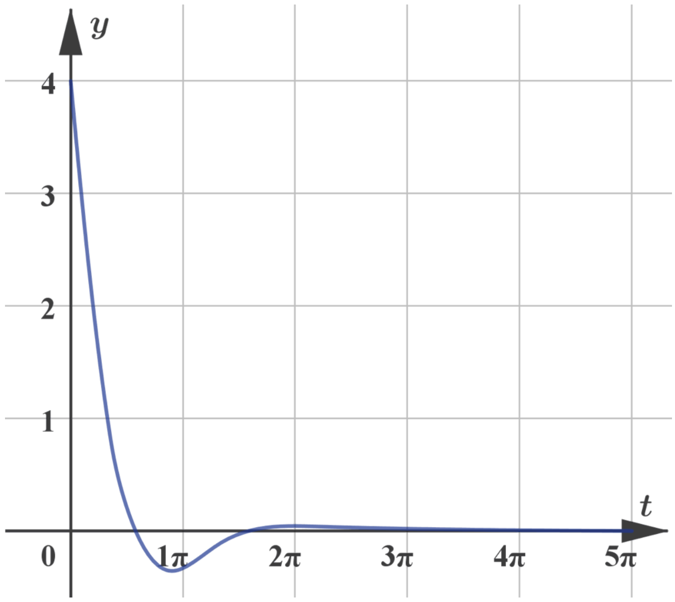
More information

The image is a diagram of a simplified mechanical system consisting of a mass, a spring, and a dashpot. The mass is represented as a square labeled "Mass," positioned centrally in the diagram. It is connected vertically above to a zigzag line labeled "Spring," indicating the spring component. Below the mass is a vertical structure labeled "Dashpot," representing the dashpot component used for damping in the system.

An arrow labeled 'y' points upward on the left side of the diagram, indicating the direction of displacement or position. The bottom of the arrow is labeled 'y = 0,' marking the equilibrium or rest position of the system. The diagram visually illustrates the vertical arrangement and connection of these three components, showing how they interact to form a dynamic system that can be used to model oscillatory motion.

[Generated by AI]

The position of the mass can be modelled as $y = \mathrm{e}^{-t}\cos t$. This can be graphed as:



More information

This is a graph representing the function (y = {{e}^{ - t}}\cos t). The horizontal axis (x-axis) is labeled (t) and represents time, with tick marks at intervals of (\pi) up to (5\pi). The vertical axis (y-axis) represents (y) and is labeled with values from 0 to 4. The curve starts at the point (0, 4) and drops sharply towards (\pi), reaching a local minimum just after (\pi). It then oscillates with decreasing amplitude as (t) increases towards (5\pi). The trend shows a rapid decay in amplitude as (t) increases, consistent with the (e^{-t}) component.

[Generated by AI]

How would you find the average value of y over the first π seconds?

The average value over a closed interval can be found as:

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$

In this specific case, it would be:

$$y_{avg} = \frac{1}{\pi - 0}\int_0^{\pi} \mathrm{e}^{-t}\cos t\,\mathrm{d}t = \frac{1}{\pi}\int_0^{\pi} \mathrm{e}^{-t}\cos t\,\mathrm{d}t$$

Unfortunately, you do not yet know how to find this integral. In this subtopic, you will learn some advanced techniques to enable you to integrate complex functions like this.

Concept

Throughout this subtopic, think about the **relationships** between differentiation and integration. You already know that there are relationships as described in the fundamental theorem of calculus. In particular, pay attention to anti-differentiation of functions produced from derivatives using the chain and product rules.

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Making connections

As described in subtopic 5.12 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26489/), there is a firm relationship between derivatives and integrals. In this subtopic, you will find the relationships associated with the chain and product rules from subtopic 5.6 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27778/).

5. Calculus / 5.16 Further integration

Integration by substitution

Unlike differentiation, there is no general product or quotient rule for integration; that is, there is no rule that allows you to integrate the product or quotient of any combination of functions.

However, by thinking of integration as anti-differentiation and remembering the chain rule for differentiation, there are certain integrands involving products and quotients that you can integrate.

Consider the function $f(x) = \mathrm{e}^{x^2}$. When you differentiate this, your answer using the chain rule would be:

$$f'(x) = \mathrm{e}^{x^2} \times 2x = 2x\mathrm{e}^{x^2}$$

Since the integral is an anti-derivative, you should therefore be able to go the other way and see that:

$$\int 2x\mathrm{e}^{x^2}\mathrm{d}x = \mathrm{e}^{x^2} + C$$

In some special cases like this, it is easy to spot the integral by inspection.

But what if the integral is close to a special case, like $\int 6x\mathrm{e}^{x^2}\mathrm{d}x$? You might be able to see the commonality of these two examples and reason that $\int 6x\mathrm{e}^{x^2}\mathrm{d}x = 3\mathrm{e}^{x^2} + C$.

However, it is more useful to have a formal process.

Consider again $\int 2x\mathrm{e}^{x^2}\mathrm{d}x$

There is a composite function, x^2 , in the integral. You can simplify the integral by substituting $u = x^2$. But if you change the variable, you also need to change $\mathrm{d}x$ (which can be thought of as 'with respect to x') to $\mathrm{d}u$ as you will now be integrating with respect to u .

If $u = x^2$, $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ or $\mathrm{d}u = 2x\mathrm{d}x$.

When this is substituted, the integral becomes:

$$\int 2x\mathrm{e}^{x^2}\mathrm{d}x = \int \mathrm{e}^u\mathrm{d}u$$

Student view

This is simple to integrate:

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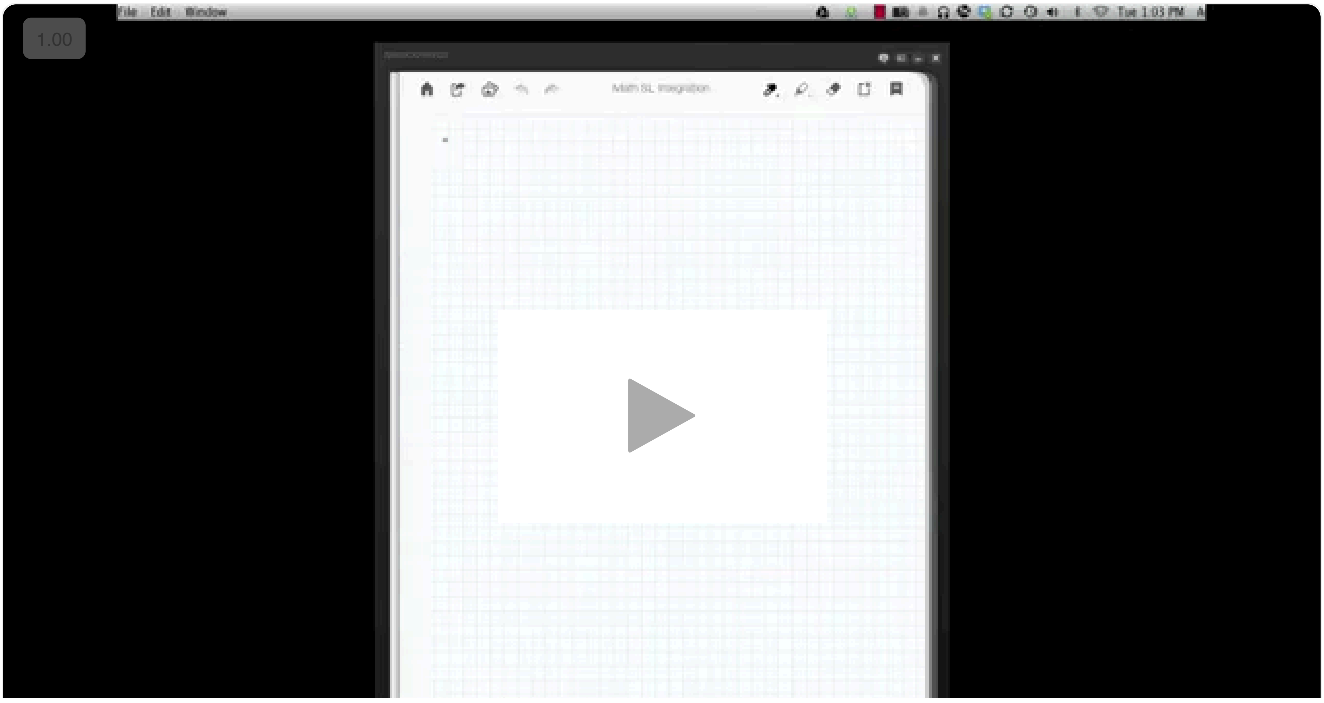
(/study/app/math-aa-hl/sid-134-cid-761926/o

$$\int \mathrm{e}^u \mathrm{d}u = \mathrm{e}^u + C$$

When you substitute back $u = x^2$, you get the result $\int 2x\mathrm{e}^{x^2}\mathrm{d}x = \mathrm{e}^{x^2} + C$

The following video explores the process of integration by substitution and identifies a form of integrand involving a product that you can integrate.



Video 1. Substitution Method in Integration.

More information for video 1

1

00:00:00,467 --> 00:00:01,401

narrator: In this short video,

2

00:00:01,468 --> 00:00:05,038

we're going to have a first look

at integration by substitution,

3

00:00:05,506 --> 00:00:07,741

which can be quite involving.

4

00:00:08,008 --> 00:00:10,377

The first part involves

doing it by inspection,

5

00:00:10,444 --> 00:00:14,147

which once again hinges

upon the idea that integration

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26514/print/) Assign

6

00:00:14,448 --> 00:00:17,918

can be considered as anti differentiation.

7

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

00:00:18,418 --> 00:00:24,124

Let's consider function sin of 5x

and let's differentiate that respect to x.

8

00:00:24,424 --> 00:00:27,895

Now, we already know what we get

is 5 cosine of 5x.

9

00:00:28,595 --> 00:00:31,031

Hence using the idea

of entity differentiation.

10

00:00:31,098 --> 00:00:34,902

We know that if we integrate

5 cosine of 5x,

11

00:00:35,636 --> 00:00:39,239

then we should get sin 5x plus this

integration constant C.

12

00:00:39,573 --> 00:00:43,443

Similarly, if I look at x square plus

2 raised to the power of 6,

13

00:00:43,510 --> 00:00:46,180

and if I differentiate that respect to x,

14

00:00:46,246 --> 00:00:50,884

then what I get is 6 times x square plus

2 to the power of 5

15

00:00:50,951 --> 00:00:53,554

times 2x using the chain rule.

16

00:00:53,620 --> 00:00:57,191

Therefore, if I integrate x square plus 2

17

00:00:57,257 --> 00:00:59,359

to the power of 5 times 2x,

18

00:00:59,860 --> 00:01:02,329

I see that they're using portion

of the right hand side

19

00:01:02,396 --> 00:01:05,666

and therefore I took the left

hand side x square plus 2

20

00:01:05,732 --> 00:01:08,035

Student view

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hl/sid-
134-
cid-
761926/o

to the power of 6 except

that I didn't have 6,

21

00:01:08,101 --> 00:01:10,871

so I need to divide by 6.

22

00:01:11,471 --> 00:01:13,407

So what we have, if the integrand,

23

00:01:13,473 --> 00:01:19,913

what I'm integrating is f as a function

of u of x times u prime of x dx,

24

00:01:20,581 --> 00:01:23,317

then the result of that

will be capital F of u

25

00:01:23,717 --> 00:01:25,018

of u of x plus C,

26

00:01:25,252 --> 00:01:30,591

if capital F of u

is equal to the integration

27

00:01:31,491 --> 00:01:34,895

of f of u du, which looks

perhaps a little bit strange

28

00:01:34,995 --> 00:01:36,964

but of course used just a variable.

29

00:01:37,331 --> 00:01:42,336

So I can say capital F of x is equal

to the integral little f of x dx.

30

00:01:42,669 --> 00:01:45,539

And this allows you

to use the standard integrals,

31

00:01:45,606 --> 00:01:50,444

which you find in the IB formula booklet.

32

00:01:51,712 --> 00:01:53,747

Let us return the original question.

33

00:01:54,014 --> 00:01:58,218

5 cosine of 5x can be written as

cosine 5 of x times 5.

34

Student
view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

00:01:58,352 --> 00:02:01,855

If I integrate that, then I can make the

35

00:02:02,523 --> 00:02:05,993

identification, cos of 5x is f of u,

36

00:02:06,059 --> 00:02:09,763

which is cos of u,

but only if I let u of x equals 5 of x.

37

00:02:10,163 --> 00:02:12,766

Now if I differentiate 5 of x to get 5,

38

00:02:12,866 --> 00:02:15,936

and now I see that

the 5 appears over there.

39

00:02:16,870 --> 00:02:20,240

Now since the integral of cos of u du

40

00:02:20,307 --> 00:02:24,978

equals sin of u plus C

, then I can conclude that

41

00:02:25,045 --> 00:02:31,118

cosine 5 of x times 5 integrated

is sin 5x plus C.

42

00:02:33,654 --> 00:02:37,591

And of course I can check

this using differentiation

43

00:02:37,658 --> 00:02:40,427

to make sure that I get the integrand back

44

00:02:41,862 --> 00:02:43,163

always worth remembering.

45

00:02:43,230 --> 00:02:47,534

Similarly, here,

if I identify x square plus 2

46

00:02:47,601 --> 00:02:49,770

to the power 5 as u to the 5,

47

00:02:49,837 --> 00:02:53,073

I can do that only if I let u equals

x squared plus 2,

48

Student
view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

00:02:53,140 --> 00:02:55,275

which of course differentiate becomes 2x.

49

00:02:55,409 --> 00:02:58,879

And again, I noticed

that I have an f of u times

50

00:02:58,946 --> 00:03:00,581

u prime times dx.

51

00:03:00,747 --> 00:03:06,019

Now if I remember that the integral

of u to the 5 is one sixth u6,

52

00:03:06,286 --> 00:03:10,591

then I know that x square plus 2

all raise to the power 5 times 2x

53

00:03:11,091 --> 00:03:16,630

integrated is one sixth x square plus 2

to the power 6 plus C.

54

00:03:17,231 --> 00:03:20,400

Now remember that this helps us very much

55

00:03:20,467 --> 00:03:23,904

by using an integrand

which has a product rule

56

00:03:23,971 --> 00:03:27,674

in that because unlike differentiation,

there is no such thing

57

00:03:27,741 --> 00:03:30,577

as a straight product rule in integration.

In the video, you saw that an integrand of the following form can be integrated:

$$\int f\left(u\left(x\right)\right)u'\left(x\right)\mathrm{d}x = F\left(u\left(x\right)\right) + C\text{, if } F\left(u\right) = \int f\left(u\right)du\,.$$

In particular, you saw that

1. $\int 5\cos 5x\,\mathrm{d}x = \sin 5x + C$, and
2. $\int \left(x^2+2\right)^5 2x\,\mathrm{d}x = \frac{1}{6}\left(x^2+2\right)^6 + C$.

Student view

Notice the composite functions present in both the integral and the result, $f(u(x))$ and $F(u(x))$. This is important. You can write this as:

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$$\int f(x)\mathrm{d}x = \int \underline{f(g(x))g'(x)\mathrm{d}x} = \int f(u)\mathrm{d}u = F(u) + C = F(g(x)) + C$$

where you substitute $f(g(x)) = f(u)$ and $g'(x)\,\mathrm{d}x = \mathrm{d}u$.

Making connections

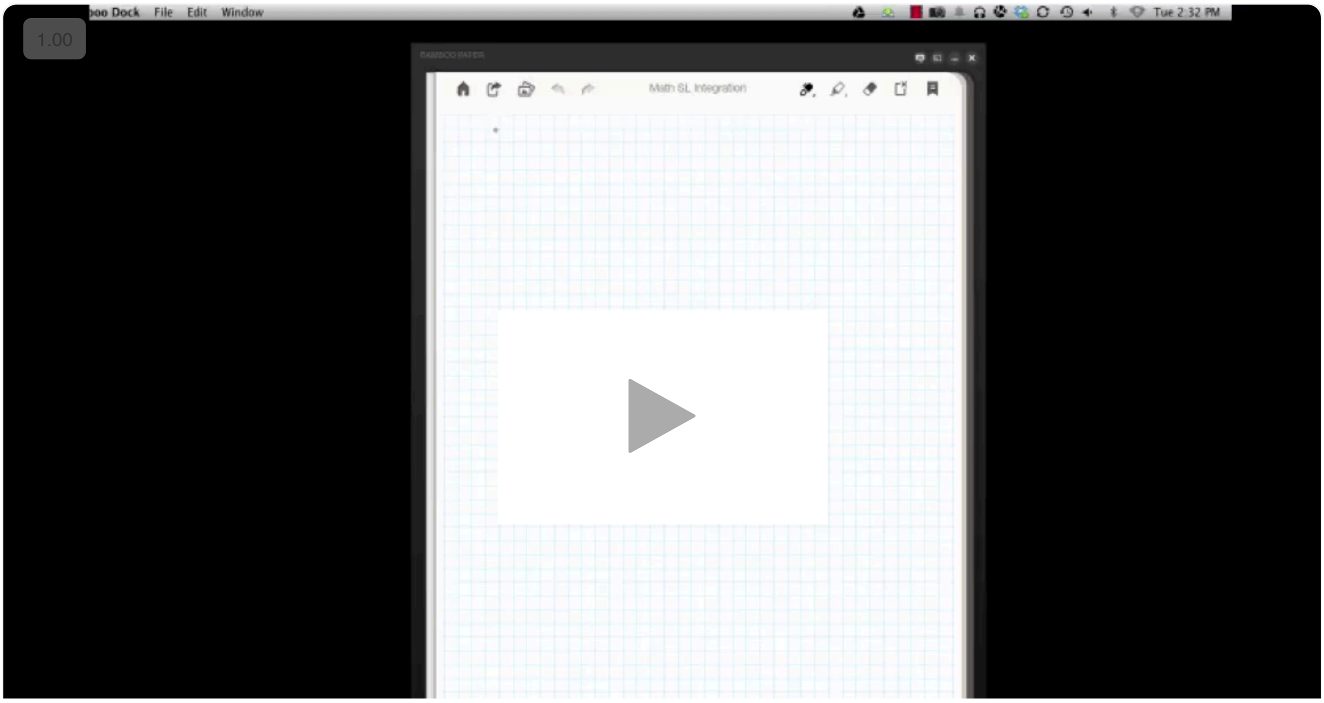
Composite functions were covered in detail in subtopic 2.5 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27692/). If you are not comfortable with this subject, go back and review it before proceeding.

Be aware

Get in the habit of differentiating your integral to make sure that you obtain the original integrand.

Formal process for integration by substitution

When integration by inspection does not work, or you want to more formally represent what you are doing, you can use integration by substitution using a formal change of variable. This is explored in the following video.



Video 2. Integration Using Substitution and the Chain Rule.

More information for video 2

1

00:00:00,433 --> 00:00:03,900

narrator: Now integration by inspection

required you to see the relationship

2

00:00:03,967 --> 00:00:06,200

between the two components

in the integrand

3

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

00:00:06,600 --> 00:00:07,900

and that is not always very easy.

4

00:00:08,000 --> 00:00:10,933

So we're gonna do integration

by substitution proper

5

00:00:11,133 --> 00:00:13,167

by actually changing the variable.

6

00:00:13,633 --> 00:00:15,100

And let's look at an example first.

7

00:00:16,733 --> 00:00:20,633

So we're gonna look

at the integration of x times sin

8

00:00:20,833 --> 00:00:22,033

of x squared dx,

9

00:00:23,000 --> 00:00:24,967

and we're gonna make

a substitution formally,

10

00:00:25,133 --> 00:00:27,067

and I'm gonna say let u be the variable

11

00:00:27,133 --> 00:00:29,067

that makes this complicated,

which is x squared.

12

00:00:29,133 --> 00:00:31,467

And then we're gonna

differentiate u prime,

13

00:00:31,533 --> 00:00:33,400

which is really du by dx,

14

00:00:33,600 --> 00:00:35,633

which is simple enough, it's 2x.

15

00:00:35,733 --> 00:00:40,733

Now first of all, we are going to

rewrite du is 2x times dx,

16

00:00:40,933 --> 00:00:45,767

so that actually we get

1 over 2x times du is equal to dx

17

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

00:00:45,833 --> 00:00:47,100

because what we need to do,

18

00:00:47,167 --> 00:00:49,733

we are actually going

to change all the x's into integration

19

00:00:49,800 --> 00:00:53,000

and that includes the measure dx.

20

00:00:53,467 --> 00:00:55,567

And then we're gonna make

all the substitutions.

21

00:00:55,633 --> 00:00:59,867

So we had x times sin x squared dx,

22

00:01:00,067 --> 00:01:04,167

which now becomes integral of x times sin

23

00:01:04,667 --> 00:01:10,167

of u times du by 2x

and you see that the x is cancel out.

24

00:01:10,233 --> 00:01:14,600

So we left with a half times

integral of sin of u du,

25

00:01:14,800 --> 00:01:16,500

which using our standard integrals

26

00:01:16,567 --> 00:01:20,233

becomes minus a half cosine of u plus C.

27

00:01:20,300 --> 00:01:23,667

Of course, now we need to go back to axis.

28

00:01:23,733 --> 00:01:27,100

So a half cosine of x squared plus C.

29

00:01:27,800 --> 00:01:29,200

Don't forget always to check.

30

00:01:29,667 --> 00:01:33,733

So if we differentiate minus half

cosine of x squared,

31

00:01:35,100 --> 00:01:38,400

if we differentiate that,

Student view

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then you should get

32

00:01:38,700 --> 00:01:41,100

the integrand so minus a half times

33

00:01:41,167 --> 00:01:46,933

minus sin of x squared times

2x using our chain rule.

34

00:01:47,000 --> 00:01:50,033

And you see that we left

with sin x square times x,

35

00:01:50,100 --> 00:01:51,567

which was the integrand.

36

00:01:52,333 --> 00:01:55,633

The other example

is the integral of x divided

37

00:01:55,867 --> 00:01:59,800

by the square root of x squared minus 4.

38

00:02:03,267 --> 00:02:07,400

So let's use as our substitution u equals

39

00:02:07,467 --> 00:02:09,167

x squared minus 4 here,

40

00:02:09,233 --> 00:02:14,467

and then we're gonna differentiate

du by dx is equal to 2x

41

00:02:14,833 --> 00:02:19,033

and that again we are going to

write dx as a function of du,

42

00:02:19,100 --> 00:02:20,367

which is the same as before.

43

00:02:20,433 --> 00:02:21,867

Then we're gonna make a substitution.

44

00:02:21,933 --> 00:02:24,300

So we add the integral

of x divided a square root

45

00:02:24,367 --> 00:02:26,167

of x squared minus 4 dx,

Student view

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46

00:02:26,233 --> 00:02:30,000

which now becomes the integral

of x times u to the minus half.

47

00:02:30,067 --> 00:02:32,767

We write explicitly as a power law times

48

00:02:33,000 --> 00:02:34,933

du divided by 2x.

49

00:02:35,167 --> 00:02:36,300

So the x's cancel

50

00:02:36,400 --> 00:02:38,900

and we left with a half times the integral

51

00:02:38,967 --> 00:02:40,767

of u to the minus a half du,

52

00:02:41,067 --> 00:02:45,933

which we can use as standard integrals

for becomes a half times 2

53

00:02:46,000 --> 00:02:48,600

times u to the plus a half plus C.

54

00:02:48,667 --> 00:02:50,800

And now we substitute back in favor of x,

55

00:02:50,867 --> 00:02:52,833

so it becomes x square minus 4

56

00:02:52,900 --> 00:02:54,833

to the power of one half plus C.

57

00:02:55,067 --> 00:02:56,300

Now of course we are going to check

58

00:02:56,367 --> 00:02:59,200

after all this work,

so we are gonna differentiate

59

00:02:59,400 --> 00:03:01,533

x square minus 4

to the power of one a half,

60

00:03:01,700 --> 00:03:04,567

and then that of course becomes one a half

Student view

61

00:03:04,900 --> 00:03:08,300

x squared minus 4 to the power

of minus a half

62

00:03:08,367 --> 00:03:10,733

times 2x using the chain rule.

63

00:03:10,800 --> 00:03:13,433

And we find that we recover d integrant.

64

00:03:13,500 --> 00:03:14,667

And that was that.

The procedure laid out in the video is as follows:

1. Make the substitution for part of the integrand by letting $u = f(x)$.
2. Differentiate: $\frac{\mathrm{d}u}{\mathrm{d}x} = f'(x)$.
3. Substitute u into the integrand and also substitute $\frac{\mathrm{d}u}{f'(x)} = \mathrm{d}x$. Make any cancellations in the integrand that involve the variable x (there should be no x remaining).
4. Integrate with respect to the variable u.
5. In the final expression, change back to x using $f(x) = u$.

Example 1

★★☆

Find $\int x \sin x^2 dx$.

$\int x \sin x^2 \mathrm{d}x$

1. Determine u : The complex part is $\sin x^2$, and x^2 is the inside part

$u = x^2$

2. Differentiate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right) = 2x \text{ , so } \mathrm{d}u = 2x\,\mathrm{d}x \text{ .}$$

3. Substitute

$$\int x \sin x^2 \mathrm{d}x = \frac{1}{2}\int \sin x^2 2x\,\mathrm{d}x = \frac{1}{2}\int \sin u\,\mathrm{d}u$$

4. Integrate

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$$\frac{1}{2}\int \sin u\, \mathrm{d}u = -\frac{1}{2}\cos u + C$$

5. Convert back

$$\int x \sin x^2 \mathrm{d}x = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos x^2 + C$$

Use this procedure to obtain these results that were shown in the video:

- $\int x \sin x^2 \mathrm{d}x = -\frac{1}{2}\cos x^2 + C$, and
- $\int \frac{x}{\sqrt{x^2-4}}\mathrm{d}x = \left(x^2-4\right)^{\frac{1}{2}} + C = \sqrt{x^2-4} + C$.

The following particularly interesting result can be shown using integration by substitution.

If $f(x) > 0$, then $\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln f(x) + C$.

- In general, $\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln|f(x)| + C$.

Thus, for example,

$$\int \frac{x}{x^2+1}\mathrm{d}x = \frac{1}{2}\int \frac{2x}{x^2+1}\mathrm{d}x = \frac{1}{2}\ln\left(x^2+1\right) + C\,.$$

Because the first step is to assign a value to u, this technique is sometimes called u-substitution.

Exam tip

- When finding integrals that result in a logarithm, in SL exams you are expected to use the form without the absolute value even if the $f(x) > 0$ condition is not given.
- In HL exams, the use of the absolute value is important. Note, that in the example above there is no need to use the absolute value, since $x^2 + 1$ is always positive.

Why is it important that the argument of a logarithm is always positive, i.e. $\ln|f(x)|$? Remember: the domain of any logarithm is $x > 0$.

Be aware

- You must remove all the appearances of x after the substitution, and end up with an integral of only one variable, such as u, which includes the substitution $\frac{\mathrm{d}u}{f'(x)} = \mathrm{d}x$.
- Do not forget to substitute back to the original variable, usually x, after integration for the substituted variable.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Why is it important to convert all x-values to expressions containing u? Remember that an integral is the area under a curve. Total area can be thought of as a summation of infinite rectangular areas, but all of those rectangles have to be built using the same variable. This technique changes the variable from x to u to make life easier. Mixing variables does not work well. Once complete, it is necessary to convert back to the original variable x to give the final answer.

Indefinite integrals

The hardest part for most people in applying this method is deciding what to substitute in for u and u'. You can think of it as substituting u for the inside of the most complex part of the integral.

Example 2

★★☆

Find $\int \cos x \, \mathrm{e}^{\sin x} \mathrm{d}x$.

$\int \cos x \, \mathrm{e}^{\sin x} \mathrm{d}x$

1. Determine u: The most complex part is $\mathrm{e}^{\sin x}$, and $\sin x$ is the inside part

   $u = \sin x$

2. Differentiate

   $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$, so $\mathrm{d}u = \cos x \, \mathrm{d}x$

3. Substitute

   $\int \cos x \, \mathrm{e}^{\sin x} \mathrm{d}x = \int \mathrm{e}^{\sin x} \cos x \, \mathrm{d}x = \int \mathrm{e}^u \mathrm{d}u$

4. Integrate

   $\int \mathrm{e}^u \mathrm{d}u = \mathrm{e}^u + C$

5. Convert back

   $\int \mathrm{e}^u \mathrm{d}u = \mathrm{e}^u + C = \mathrm{e}^{\sin x} + C$

Example 3

★★☆

Find $\int x^2 \sqrt{5 + 2x^3} \mathrm{d}x$.

$\int x^2 \sqrt{5 + 2x^3} \mathrm{d}x$

Student view

Overview
(/study/ap
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761926/o

1. Determine u: The most complex part is $\sqrt{5+2x^3}$, and the polynomial is the inside part

$$u = 5 + 2x^3$$

2. Differentiate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(5+2x^3\right) = 6x^2 \text{, so } \mathrm{d}u = 6x^2\,\mathrm{d}x$$

3. Substitute

$$\int x^2\sqrt{5+2x^3}\mathrm{d}x = \frac{1}{6}\int 6x^2\sqrt{5+2x^3}\mathrm{d}x = \frac{1}{6}\int \sqrt{u}\mathrm{d}u$$

4. Integrate

$$\frac{1}{6}\int u^{\frac{1}{2}}\mathrm{d}u = \frac{1}{6}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + C = \frac{1}{9}u^{\frac{3}{2}} + C$$

5. Convert back

$$\int x^2\sqrt{5+2x^3}\mathrm{d}x = \frac{1}{9}u^{\frac{3}{2}} + C = \frac{1}{9}\left(5+2x^3\right)^{\frac{3}{2}} + C$$

Example 4

★★☆

Find $\int (\ln x)^2 \frac{1}{x}\mathrm{d}x$.

$$\int (\ln x)^2 \frac{1}{x}\mathrm{d}x$$

1. Determine u: The most complex part is $(\ln x)^2$, and $\ln x$is the inside part

$$u = \ln x$$

2. Differentiate

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x} \text{, so } \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$$

3. Substitute

$$\int (\ln x)^2 \frac{1}{x}\mathrm{d}x = \int u^2\mathrm{d}u$$

4. Integrate

$$\int u^2\mathrm{du} = \frac{1}{3}\mathrm{u}^3 + \mathrm{C}$$

Student
view

Overview
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cid-
761926/o

5. Convert back

$$\int (\ln x)^2 \frac{1}{x} \mathrm{d}x = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$$

4 section questions

Question 1

Difficulty:

★★☆

What is the indefinite integral $\int 10x\sqrt{5x^2+2}\,\mathrm{d}x$?

1 $\frac{2}{3}\left(5x^2+2\right)^{\frac{3}{2}} + C$ ✓

2 $\frac{4}{3}\left(5x^2+2\right)^{\frac{3}{2}} + C$

3 $\frac{3}{2}\left(5x^2+2\right)^{\frac{3}{2}} + C$

4 $3\left(5x^2+2\right)^{\frac{3}{2}} + C$

Explanation

Use the substitution $u = 5x^2 + 2$, so

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 10x$$

Thus,

$$\begin{aligned}\int 10x\sqrt{u}\,\mathrm{d}u\,\frac{\mathrm{d}x}{\mathrm{d}u} &= \int 10x\sqrt{u}\,\mathrm{d}u\,\frac{1}{10x}\\ &= \int \sqrt{u}\,\mathrm{d}u\\ &= \frac{2}{3}u^{\frac{3}{2}} + C\end{aligned}$$

Substitute back for u to give: $\frac{2}{3}\left(5x^2+2\right)^{\frac{3}{2}} + C$

Question 2

Difficulty:

★★☆

What is the integral of $\frac{\sin x}{\cos x}$ $(= \tan x)$ for $0 < x < \frac{\pi}{2}$?

1 $-\ln\cos x + C$ ✓

2 $\sin^2 x + C$

3 $\cos^2 x + C$

4 $\ln\cos x + C$

Student
view

Explanation

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Use the substitution $u = \cos x$. As well as replacing $\cos x$ with u, we also need to replace $\mathrm{d}x$ with $\mathrm{d}u$ by differentiating our substitution, obtaining $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$. Substituting both of these expressions into the integral gives:

$$\begin{aligned}\int \frac{\sin x}{\cos x}\,\mathrm{d}x &= \int \frac{1}{u}\left(-\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x \\ &= -\int \frac{1}{u}\,\mathrm{d}u \\ &= -\ln|u| + c \\ &= -\ln|\cos x| + C\end{aligned}$$

On the domain given in the question $\cos x$ is positive, so we do not need the absolute value.

$$\int \frac{\sin x}{\cos x}\,\mathrm{d}x = -\ln\cos x + C$$

Question 3

Difficulty:

★★☆

Calculate the indefinite integral $\int \cos^3 x \sin x\,\mathrm{d}x$.

1 $-\frac{\cos^4 x}{4} + C$ ✓

2 $-\frac{\cos^4 x}{4}\sin x + C$

3 $\frac{\cos^4 x}{4} + C$

4 $-\frac{\cos^3 x}{3}\sin x + C$

Explanation

Use the substitution $u = \cos x$

Then, $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \Rightarrow \mathrm{d}u = -\sin x\,\mathrm{d}x$

Hence,

$$\begin{aligned}\int \cos^3 x \sin x\,\mathrm{d}x &= -\int \cos^3 x(-\sin x)\,\mathrm{d}x \\ &= -\int u^3\,\mathrm{d}u \\ &= -\frac{u^4}{4} + C\end{aligned}$$

and substituting back for u we get

$$-\frac{\cos^4 x}{4} + C$$

Question 4

Difficulty:

★★★

What is $\int x^2\left(x^3+2\right)^3\mathrm{d}x$?

Student view

1 $\frac{1}{12}(x^3+2)^4+C$ ✓

2 $\frac{1}{4}(x^3+2)^4+C$

3 $\frac{1}{12}(x^3)^4+C$

4 $\frac{3}{4}(x^3+2)^3+C$

Explanation

We let $u = x^3 + 2$, such that $\mathrm{d}x = \frac{1}{3x^2}\,\mathrm{d}u$.

Then we get

$$\begin{aligned}
& \int x^2(x^3+2)^3\,\mathrm{d}x = \int x^2(u)^3\frac{1}{3x^2}\,\mathrm{d}u \\
& = \frac{1}{3}\int u^3\,\mathrm{d}u \\
\Rightarrow \quad & = \frac{1}{3}\frac{1}{4}u^4 + C \\
\Rightarrow \quad & = \frac{1}{12}(x^3+2)^4 + C
\end{aligned}$$

5. Calculus / 5.16 Further integration

Definite integrals

So far you have been evaluating indefinite integrals. When evaluating definite integrals, you need to evaluate the result of the integration over the limits of the integral. There are two ways to do this.

- Complete steps 1–5 of integration by substitution and evaluate across the original limits of integration.
- Complete steps 1–4 of integration by substitution, convert the limits from x to u, and evaluate across the new limits of integration.

As with any definite integral, there is no need to include the ‘$+C$’ as the constant will cancel out during this evaluation.

Example 1

★★★

As an example, consider $\int_0^2 x\sqrt{x^2+1}\mathrm{d}x$. Regardless of your choice of techniques, the first four steps remain the same.

$$\int x\sqrt{x^2+1}\mathrm{d}x$$

1. Determine u: The most complex part is $\sqrt{x^2+1}$, and the radical is the inside part

$$u = x^2 + 1$$

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

2. Differentiate

$$x^2 + 1\text{, so } \mathrm{d}u = 2x\,\mathrm{d}x$$

3. Substitute

$$\int x\sqrt{x^2+1}\mathrm{d}x = \frac{1}{2}\int 2x\sqrt{x^2+1}\mathrm{d}x = \frac{1}{2}\int \sqrt{u}\mathrm{d}u$$

4. Integrate

$$\frac{1}{2}\int \sqrt{u}\mathrm{d}u = \frac{1}{3}u^{\frac{3}{2}}$$

5. Convert back

$$\int x\sqrt{x^2+1}\mathrm{d}x = \frac{1}{3}u^{\frac{3}{2}} = \frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}}$$

6. Evaluate

$$\int_0^2 x\sqrt{x^2+1}\mathrm{d}x = \frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}}\Big|_{x=0}^{2} = \frac{1}{3}\left((2)^2+1\right)^{\frac{3}{2}} - \frac{1}{3}\left((0)^2+1\right)^{\frac{3}{2}} = \frac{5\sqrt{5}-1}{3} = 3.39$$

Alternatively, you do not need to convert back to x. After integrating in step 4, you can convert the limits of integration over to u instead, and then evaluate. In this case:

5. Convert limits

bottom: $u = x^2 + 1 = 0^2 + 1 = 1$; top: $u = x^2 + 1 = 2^2 + 1 = 5$

6. Evaluate

$$\int_0^2 x\sqrt{x^2+1}\mathrm{d}x = \frac{1}{3}u^{\frac{3}{2}}\Big|_{u=1}^{5} = \frac{1}{3}(5)^{\frac{3}{2}} - \frac{1}{3}(1)^{\frac{3}{2}} = \frac{5\sqrt{5}-1}{3} = 3.39$$

Both methods have their own pros and cons and yield the same result. The method you choose comes down to personal preference.

Example 2

★★★

Find $\int_0^{\sqrt{\frac{\pi}{2}}} x\cos x^2\mathrm{d}x$.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

$$\int_0^{\sqrt{\frac{\pi}{2}}} x\cos x^2 \mathrm{d}x$$

1. Determine u: The most complex part is $\cos x^2$, and the power is the inside part

$$u = x^2$$

2. Differentiate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right) = 2x, \text{ so } \mathrm{d}u = 2x\,\mathrm{d}x$$

3. Substitute

$$\int_0^{\sqrt{\frac{\pi}{2}}} x\cos x^2 \mathrm{d}x = \frac{1}{2}\int_0^{\sqrt{\frac{\pi}{2}}} 2x\cos x^2 \mathrm{d}x = \frac{1}{2}\int \cos u \mathrm{d}u$$

4. Integrate

$$\frac{1}{2}\int \cos u \mathrm{d}u = \frac{1}{2}\sin u$$

5. Convert back

$$\frac{1}{2}\sin u = \frac{1}{2}\sin x^2\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}}$$

6. Evaluate

$$\frac{1}{2}\sin x^2\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}} = \frac{1}{2}\sin\left(\sqrt{\frac{\pi}{2}}\right)^2 - \frac{1}{2}\sin(0)^2 = \frac{1}{2}\sin\left(\frac{\pi}{2}\right)^2 - 0 = \frac{1}{2}$$

Alternative method

5. Convert limits

$$\text{bottom:} u = x^2 = 0^2 = 0, \text{ top:} u = x^2 = \left(\sqrt{\frac{\pi}{2}}\right)^2 = \frac{\pi}{2}$$

6. Evaluate

$$\int_0^{\sqrt{\frac{\pi}{2}}} x\cos x^2 \mathrm{d}x = \frac{1}{2}\sin u\Big|_{u=0}^{\frac{\pi}{2}} = \frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\sin 0 = \frac{1}{2}$$

Advanced integrals

As with any substitution, you need to recognise what to substitute. However, if the substitution is considered non-standard, meaning too complex to recognise, it will be provided in the question.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

When you studied trigonometric functions, you came across many identities. These will be useful when transforming integrands into those for which you can use standard integrals.

Consider $\int \sin^3 x \, dx$.

Using trigonometric identities, you get

$$\sin^3 x = \sin^2 x \sin x = \left(1 - \cos^2 x\right) \sin x$$

and so the integral becomes

$$\int \sin^3 x \, dx = \int \left(1 - \cos^2 x\right) \sin x \, dx = \int \left(\sin x - \cos^2 x \sin x\right) dx = \int \sin x \, dx - \int \cos^2 x \sin x \, dx$$

The first integral is a standard one, and for the second you use the substitution $u = \cos x$, such that $dx = -\frac{1}{\sin x} du$.

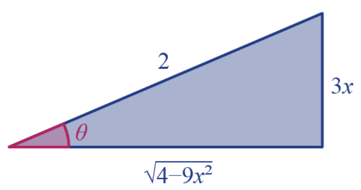
Hence,

$$\begin{aligned} \int \sin^3 x \, dx &= \int \sin x \, dx - \int \cos^2 x \sin x \, dx = -\cos x + C - \int u^2 \sin x \frac{-1}{\sin x} du \\ &= -\cos x + C + \int u^2 du = -\cos x + \frac{1}{3} u^3 + C = -\cos x + \frac{1}{3} \cos^3 x + C \end{aligned}$$

A trigonometric substitution can also be interesting to study.

Consider $\int_0^{\frac{1}{3}} \frac{1}{\sqrt{4 - 9x^2}} dx$.

First, you need a basic substitution. Consider the following right-angled triangle.



More information

The image illustrates a right-angled triangle. The triangle includes three labels for its sides. The hypotenuse is labeled as '2'. One of the legs is labeled as '3x'. The other leg is labeled with the expression '√(4 - 9x²)'. Additionally, there is an angle marked at the base of the triangle with the symbol 'θ', indicating it as a notable angle.

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Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

The side lengths have been chosen so as to use the difference of two squares inside the radical. As the minuend is 4, the hypotenuse can be set to $\sqrt{4}$, or 2. Similarly, the subtrahend is $9x^2$, so one leg can be set equal to $3x$. From Pythagoras' theorem, the final leg is the entire radical.

From this diagram, you can see that $\sin\theta = \frac{3x}{2}$, or $x = \frac{2}{3}\sin\theta$.

Differentiating with respect to θ, you get $\cos\theta = \frac{3}{2}\frac{\mathrm{d}x}{\mathrm{d}\theta}$, or $\mathrm{d}x = \frac{2}{3}\cos\theta\,\mathrm{d}\theta$.

You can now substitute into the original expression and reduce it:

$$\begin{aligned}\int_0^{\frac{1}{3}} \frac{1}{\sqrt{4-9x^2}}\mathrm{d}x &= \int \left(\frac{1}{\sqrt{4-9\left(\frac{2}{3}\sin\theta\right)^2}}\right)\left(\frac{2}{3}\cos\theta\,\mathrm{d}\theta\right)\\ &= \int \left(\frac{1}{\sqrt{4-4\sin^2\theta}}\right)\left(\frac{2}{3}\cos\theta\,\mathrm{d}\theta\right)\\ &= \int \frac{1}{2\sqrt{1-\sin^2\theta}}\left(\frac{2}{3}\cos\theta\,\mathrm{d}\theta\right) = \int \frac{\cos\theta}{3\sqrt{\cos^2\theta}}\mathrm{d}\theta = \int \frac{1}{3}\mathrm{d}\theta\end{aligned}$$

This is easy to integrate:

$$\int \frac{1}{3}\mathrm{d}\theta = \frac{\theta}{3}$$

If you convert back to x and substitute, this becomes:

$$\int \frac{1}{3}\mathrm{d}\theta = \frac{\theta}{3} = \frac{1}{3}\arcsin\frac{3x}{2}\bigg|_{x=0}^{\frac{1}{3}} = \left[\frac{1}{3}\arcsin\frac{1}{2}\right] - \left[\frac{1}{3}\arcsin 0\right] = \frac{1}{3}\left(\frac{\pi}{6}\right) - 0 = \frac{\pi}{18}$$

Alternately, you could convert the limits:

$$\text{bottom: } \theta = \arcsin\left(\frac{3(0)}{2}\right) = 0, \text{ top: } \theta = \arcsin\left(\frac{3(1/3)}{2}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

This yields

$$\frac{\theta}{3}\bigg|_{\theta=0}^{\frac{\pi}{6}} = \frac{\pi/6}{3} = 0 = \frac{\pi}{18}$$

Example 3

★★★

Find $\int \frac{\arctan x}{x^2+1}\mathrm{d}x$.

$$\int \frac{\arctan x}{x^2+1}\mathrm{d}x$$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

1. Determine u: From earlier, you know $\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{x^2+1}$. Using this, the most complex part is the numerator $\arctan x$

   $u = \arctan x$

2. Differentiate

   $\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{x^2+1}$, so $\mathrm{d}u = \frac{1}{x^2+1}\mathrm{d}x$

3. Substitute

   $\int \frac{\arctan x}{x^2+1}\mathrm{d}x = \int \arctan x \frac{1}{x^2+1}\mathrm{d}x = \int u\mathrm{d}u$

4. Integrate

   $\int udu = \frac{1}{2}u^2 + C$

5. Convert back

   $\int \frac{\arctan x}{x^2+1}\mathrm{d}x = \frac{1}{2}u^2 + C = \frac{1}{2}(\arctan x)^2 + C$

Example 4

★★★

Find $\int_0^1 x\sqrt{1-x}\mathrm{d}x$.

$\int x\sqrt{1-x}\mathrm{d}x$

1. Determine u: The most complex part is $\sqrt{1-x}$, and the power is the inside part of the radical, so $u = 1-x$

   $x = 1-u$

2. Differentiate

   $\frac{\mathrm{d}}{\mathrm{d}x}(1-x) = -1$, so $\mathrm{d}u = -\mathrm{d}x$. Alternately, $\mathrm{d}x = -\mathrm{d}u$

3. Algebra

   $\int x\sqrt{1-x}\mathrm{d}x = -\int -x\sqrt{1-x}\mathrm{d}x$

   Substitute

   $= -\int (1-u)\sqrt{u}\mathrm{d}u$

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Algebra

$$= \int (u-1)\sqrt{u}\mathrm{d}u = \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) \mathrm{d}u$$

4. Integrate

$$\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) \mathrm{d}u = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}$$

5. Convert back

$$\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} = \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}}\Big|_{x=0}^{1}$$

6. Evaluate

$$= \left[\frac{2}{5}(1-1)^{\frac{5}{2}} - \frac{2}{3}(1-1)^{\frac{3}{2}}\right] - \left[\frac{2}{5}(1-0)^{\frac{5}{2}} - \frac{2}{3}(1-0)^{\frac{3}{2}}\right]$$
$$= \left[\frac{2}{5}(0)^{\frac{5}{2}} - \frac{2}{3}(0)^{\frac{3}{2}}\right] - \left[\frac{2}{5}(1)^{\frac{5}{2}} - \frac{2}{3}(1)^{\frac{3}{2}}\right] = [0] - \left[\frac{2}{5} - \frac{2}{3}\right] = \frac{4}{15}$$

Alternative method

5. Convert limits

bottom: $u = 1 - x = 1 - 0 = 1$, top: $u = 1 - x = 1 - 1 = 0$

6. Substitute

$$\int_0^1 x\sqrt{1-x}\mathrm{d}x = \frac{2}{5}(u)^{\frac{5}{2}} - \frac{2}{3}(u)^{\frac{3}{2}}\Big|_{u=1}^{0}$$

Evaluate

$$\left[\frac{2}{5}(u)^{\frac{5}{2}} - \frac{2}{3}(0)^{\frac{3}{2}}\right] - \left[\frac{2}{5}(1)^{\frac{5}{2}} - \frac{2}{3}(1)^{\frac{3}{2}}\right] = [0] - \left[\frac{2}{5} - \frac{2}{3}\right] = \frac{4}{15}$$

Example 5

★★★

Find $\int_0^2 \sqrt{4-x^2}dx$.

$$\int_0^2 \sqrt{4-x^2}\mathrm{d}x$$

Student view

1. Determine u: An integrand of the form $\sqrt{a^2 - x^2}$ is common, and the substitution is $x = a\sin\theta$. Note that you are using θ instead of u.

$x = 2\sin\theta$

2. Differentiate

$\frac{\mathrm{d}}{\mathrm{d}\theta}(2\sin\theta) = 2\cos\theta$, so $\mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$

3. Substitute

$$\begin{aligned}\int_0^2 \sqrt{4-x^2}\mathrm{d}x &= \int \sqrt{4-(2\sin\theta)^2}2\cos\theta\,\mathrm{d}\theta \\ &= \int \sqrt{4-4\sin^2\theta}2\cos\theta\,\mathrm{d}\theta = 2\int 2\sqrt{1-\sin^2\theta}\cos\theta\,\mathrm{d}\theta \\ &= 2\int 2\sqrt{\cos^2\theta}\cos\theta\,\mathrm{d}\theta = 2\int 2\cos^2\theta\,\mathrm{d}\theta = 2\int(1+\cos(2\theta))\,\mathrm{d}\theta\end{aligned}$$

4. Integrate

$2\int(1+\cos(2\theta))\,\mathrm{d}\theta = 2\theta + \sin(2\theta)$

In this case, converting back is cumbersome, so just convert the limits.

5. Convert limits

bottom: $0 = 2\sin\theta$, or $\theta = \arcsin 0 = 0$, top: $2 = 2\sin\theta$, or $\theta = \arcsin 1 = \frac{\pi}{2}$

6. Evaluate

$2\theta + \sin(2\theta)|_{\theta=0}^{\frac{\pi}{2}} = \left[2\left(\frac{\pi}{2}\right) + \sin\left(2\left(\frac{\pi}{2}\right)\right)\right] - [2(0) + \sin(2(0))] = [\pi + 0] - [0] = \pi$

Definite integrals by GDC

You can use technology to evaluate integrals quickly. Unlike differentiation, integration can seem difficult to do by hand and even if you can find and evaluate an integral by hand, the time it takes may not be worth it. Whether you are evaluating a definite integral on the IB Paper 2 or for an assignment at work after you graduate, it may be that you just need the final answer. This can be found using a graphing display calculator (GDC). Many other calculators also have this capability.

Example 6

★☆☆

Using a GDC, find $\int_0^2 \sqrt{4-x^2}\mathrm{d}x$.

$$\int_0^2 \sqrt{4-x^2}dx$$

On the TI-84 Plus:

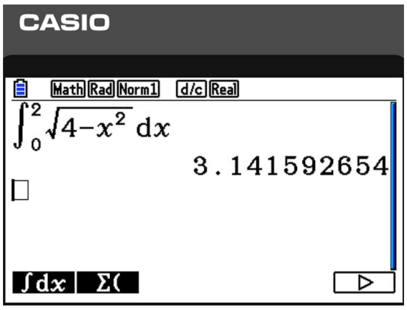
Type MATH, 9

An integral should appear with boxes to type the lower and upper limits and the function.
Fill them in and press ENTER.

The result screens from the calculators

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

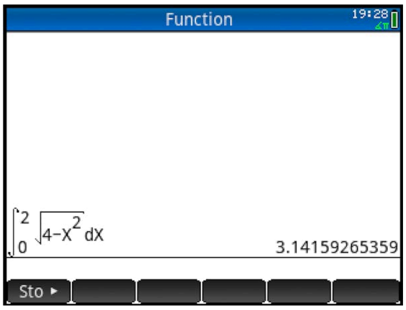
Steps	Explanation



Casio fx-CG50

The image shows a display from a Casio calculator featuring a mathematical calculation. At the top of the screen, there's a series of tabs labeled "Math," "Rad," "Norm1," "d/c," and "Real." Below these tabs, there is an integral displayed: the integral from 0 to 2 of the square root of (4 minus x squared) dx. To the right of this integral, the calculated result is shown as 3.141592654. Below the integral and the result, there is an empty checkbox and buttons labeled with calculus operations, including one shown as a differential operator and another for summation. The display is framed in a black bezel, labeled with the Casio brand.

[Generated by AI]



HP Prime

The image is a screenshot of a calculator displaying an integral calculation. At the top, there is a navigation bar with the word "Function." The main part of the screen shows an integral from 0 to 2 of the function (\sqrt{4-x^2} \, dx). The result of the calculation, 3.14159265359, is displayed on the right side of the screen. The bottom of the screen contains a partial set of buttons or icons, with 'Sto' visible.

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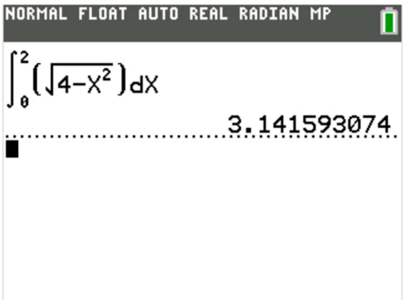
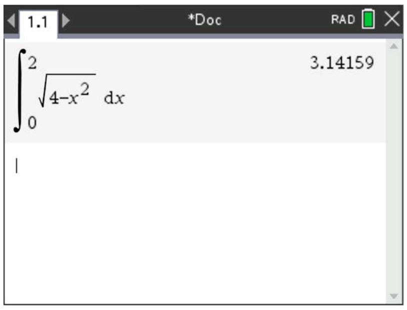


TI-84 plus CE

The image shows a calculator screen displaying the result of an integral calculation. The integral displayed is from 0 to 2 of the square root of (4 minus x squared) dx. The result shown on the screen is 3.141593074, which is an approximation of pi. The top of the calculator screen displays modes: 'NORMAL', 'FLOAT', 'AUTO', 'REAL', 'RADIAN', 'MP'. A small battery icon is visible in the upper right corner, indicating the power status.

[Generated by AI]



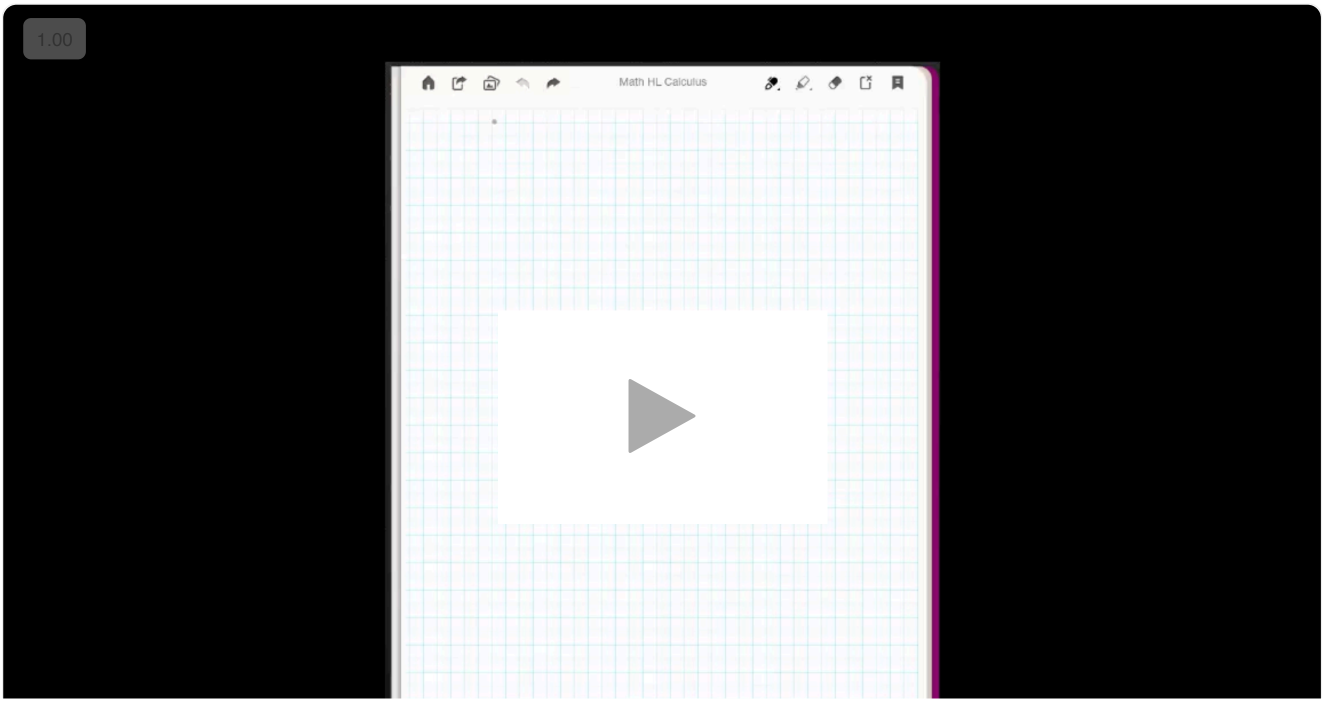
TI-nspire CX

5. Calculus / 5.16 Further integration

Integration by parts

Student view

In the following video, you can explore the challenge of integrating a product of functions in which you will apply the method of integration by parts.



Video 1. Integrating a Product of Functions.

More information for video 1

1

00:00:00,300 --> 00:00:03,367

narrator: In this video we're going to take

a look at the last part of integration,

2

00:00:03,433 --> 00:00:06,067

namely integration by parts.

3

00:00:06,500 --> 00:00:08,633

And this is to address

the following problem.

4

00:00:09,633 --> 00:00:14,633

In integration,

we do not have a product rule

5

00:00:15,100 --> 00:00:17,067

as we do in differentiation.

6

00:00:17,133 --> 00:00:19,633

Therefore, if I have two functions,

7

00:00:19,700 --> 00:00:25,333

let's call 'em u(x) and v(x),

then if I take the product of them

8

00:00:25,400 --> 00:00:27,933

and try to integrate 'em,

there is no general method

9

00:00:28,200 --> 00:00:30,900

for the integral $\int u \cdot v\, dx$.

10

00:00:31,433 --> 00:00:36,167

And many times functions are of course

multiplied by each other.

11

00:00:36,233 --> 00:00:38,800

And in integration per parts,

12

00:00:38,867 --> 00:00:42,400

we're going to use the fundamental result

that we've seen before.

13

00:00:42,500 --> 00:00:45,733

That led to the idea

of anti differentiation for integration.

14

00:00:45,867 --> 00:00:49,867

And that is that if we take

the derivative of a function

15

00:00:50,267 --> 00:00:53,467

and $\int \frac{d}{dx}[f(x)]\, dx = f(x) + c$ if we then integrate

the result of that differentiation,

16

00:00:53,533 --> 00:00:57,000

we get back the original function.

17

00:00:58,900 --> 00:01:02,567

Okay? So you are going

to see what it brings us.

18

00:01:02,633 --> 00:01:04,967

So let $u = u(x)$

19

00:01:05,033 --> 00:01:07,333

and $v = v(x)$ be any function

20

00:01:07,967 --> 00:01:10,133

by which we really mean

that there are nice functions,

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21

00:01:10,200 --> 00:01:12,033

that means they're continuous

and differentiable.

22

00:01:12,333 --> 00:01:14,867

Okay? But we don't worry

too much about that here

23

00:01:14,933 --> 00:01:17,467

unless we look at the calculus option.

24

00:01:18,100 --> 00:01:23,000

Then we know by differentiation

of the product that $\frac{d}{dx}(u \cdot v)$

25

00:01:23,067 --> 00:01:28,300

is $\frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$.

26

00:01:28,467 --> 00:01:30,200

We've seen that before many times.

27

00:01:30,333 --> 00:01:34,200

Now if we integrate left hand side

and right hand side,

28

00:01:34,633 --> 00:01:38,667

then what we get is $u \cdot v = \int \left(\frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}\right) dx$.

29

00:01:38,733 --> 00:01:40,200

We do have the sum rule.

30

00:01:40,267 --> 00:01:45,400

So we've got a sum of $\int \frac{du}{dx} \cdot v\, dx + \int u \cdot \frac{dv}{dx} dx$.

31

00:01:45,567 --> 00:01:48,267

And now we're going to rewrite it in terms of

32

00:01:48,733 --> 00:01:53,500

$\int u \cdot \frac{dv}{dx} dx$ is an equal to $u \cdot v - \int v \cdot \frac{du}{dx} dx$.

33

00:01:54,100 --> 00:01:57,267

So on the left hand side we've got

a product of two functions.

34

00:01:57,333 --> 00:02:02,500

'cause $\frac{du}{dx}$ is a function.

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Of course you see that

on the right hand side,

35

00:02:02,967 --> 00:02:08,633

we still have a product

of two functions as the integrands,

36

00:02:08,700 --> 00:02:10,367

so it still needs to be integrated.

37

00:02:10,433 --> 00:02:14,900

However, the hope is that we actually

can integrate that product,

38

00:02:15,100 --> 00:02:18,233

and therefore you can see

by me saying it's a hope

39

00:02:18,400 --> 00:02:23,267

that this is not actually

general result for any kind

40

00:02:23,533 --> 00:02:26,933

of functions, but for some

functions it does indeed works.

41

00:02:27,000 --> 00:02:29,300

So let's have a look at an example.

42

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00:02:29,567 --> 00:02:34,333

Let's consider taking

the integral of $\int x \cdot \cos(x)\,dx$,

43

00:02:34,400 --> 00:02:37,700

which of course explicitly is a product.

44

00:02:38,167 --> 00:02:41,567

So now we're going to identify u as x

45

00:02:42,000 --> 00:02:45,667

and we are going to identify

$\frac{dv}{dx}$ as $\cos(x)$.

46

00:02:45,733 --> 00:02:48,067

So $\frac{du}{dx}$ is in one

47

00:02:48,200 --> 00:02:54,367

and the integral

Student view

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of $\frac{dv}{dx}$, v is the integral

of $\cos(x)$,

which of course is $\sin(x)$.

48

00:02:54,433 --> 00:02:56,567

We leave C out, we put it in the end.

49

00:02:57,167 --> 00:02:59,267

Now of course we need

to apply the equation,

50

00:02:59,333 --> 00:03:02,000

which fortunately

you do not have to memorize.

51

00:03:02,067 --> 00:03:03,433

It is in information booklet.

52

00:03:03,500 --> 00:03:05,500

So here in information booklet,

53

00:03:05,567 --> 00:03:08,067

integration by parts equation is given.

54

00:03:08,633 --> 00:03:11,367

Now because

of the minus sign, I do advise you

55

00:03:11,433 --> 00:03:12,900

to write it out explicitly.

56

00:03:13,000 --> 00:03:19,000

$\int u \cdot \frac{dv}{dx} dx = u \cdot v - \int v \cdot \frac{du}{dx} dx$.

57

00:03:19,133 --> 00:03:21,233

And now we write down what we add.

58

00:03:21,300 --> 00:03:23,333

So we had the integral

of $\int x \cdot \cos(x)\, dx$,

59

00:03:23,400 --> 00:03:26,400

which we then changed into $u \cdot v$,

60

00:03:26,467 --> 00:03:32,667

which is $\int \sin x \cdot 1 dx = x\sin x + \cos x + c$.

61

00:03:32,733 --> 00:03:37,467

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And now of course you see

that indeed the second product

62

00:03:37,867 --> 00:03:41,133

∫ sin(x) dx is easy to integrate.

63

00:03:41,267 --> 00:03:45,700

Now what we've seen here

that x loses a power

64

00:03:45,767 --> 00:03:47,167

until it become a constant.

65

00:03:47,233 --> 00:03:51,433

So that is a hint of when to apply

the integration of parts.

This gives the formula for integration by parts:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \text{ or } \int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

and applies it to an example to show that

$$\int x\cos x\mathrm{d}x = x\sin x + \cos x + C\,.$$

Important

The formula booklet contains the equation

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \text{ or } \int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

Example 1

★★☆

Find $\int x\sin x\,\mathrm{d}x$

$$\int x\sin x\,\mathrm{d}x$$

$$\begin{matrix} u = x & \mathrm{d}v = \sin x\,\mathrm{d}x \\ du = dx & v = -\cos x \end{matrix} \qquad \int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

$$\int x\sin x\,\mathrm{d}x = -x\cos x - \int -\cos x\,\mathrm{d}x = -x\cos x + \sin x + C$$

Student view

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Example 2

★★☆

Find $\int 3x\mathrm{e}^x\,\mathrm{d}x$

$$\int 3x\mathrm{e}^x\,\mathrm{d}x$$

$$\begin{aligned} u &= 3x & \mathrm{d}v &= \mathrm{e}^x\,\mathrm{d}x \\ \mathrm{d}u &= 3\mathrm{d}x & v &= \mathrm{e}^x \end{aligned} \qquad \int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

$$\int 3x\mathrm{e}^x\mathrm{d}x = 3x\mathrm{e}^x - \int 3\mathrm{e}^x\,\mathrm{d}x = 3x\mathrm{e}^x - 3\mathrm{e}^x + C$$

Example 3

★★☆

Find $\int x\ln x\,\mathrm{d}x$

$$\int x\ln\ x\mathrm{d}x$$

$$\begin{aligned} u &= \ln x & \mathrm{d}v &= x\,\mathrm{d}x \\ \mathrm{d}u &= \frac{1}{x}\mathrm{d}x & v &= \frac{x^2}{2} \end{aligned} \qquad \int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

$$\int x\ln x\,\mathrm{d}x = \ln x\frac{x^2}{2} - \int \frac{x^2}{2}\frac{1}{x}\,\mathrm{d}x = \frac{x^2\ln x}{2} - \int \frac{x}{2}\,\mathrm{d}x = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C$$

Another issue is deciding on how to break up the integral. Typically, it is pretty easy to figure out what the two parts are, but deciding which one is assigned to u and which one is assigned to v can be a challenge. Why, in **Example 3**, is $u = \ln x$ and $\mathrm{d}v = x\mathrm{d}x$? Would $u = x$ and $\mathrm{d}v = \ln x\mathrm{d}x$ work just as well? Finding the derivative of u does not appear to be a problem, but can you find $\int \ln x\mathrm{d}x$? You could just try both ways and go with the one that works. But there is a more efficient way to decide.

There are two mnemonics used to identify the prioritisation.

✓ **Important**

LIPET is a guideline for determining u in integration by parts:

L — Logarithmic functions
I — Inverse trigonometric functions
P — Polynomial functions
E — Exponential functions
T — Trigonometric functions

LIATE is an alternate guideline for determining u in integration by parts:

L — Logarithmic functions
I — Inverse trigonometric functions
A — Algebraic functions

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T — Trigonometric functions
E — Exponential functions

In the mnemonics above, the idea is to remember the word and assign a type of factor to each letter. When working through the types, check for the first on the list. If there is not one present, move on to the next.

Both versions are found commonly in different college textbooks for determining the order. Although the third steps have different names, the first three priorities are essentially the same for both. The only difference is whether your method chooses exponential or trigonometric functions as the fourth priority. You get to decide which one works best for you.

Example 4

★★☆

Find $\int x^2 \ln x \, \mathrm{d}x$

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$\int x^2 \ln x \, \mathrm{d}x$

Logarithmic functions: $u = \ln x$

$$\begin{array}{cc} u = \ln x & \mathrm{d}v = x^2 \, \mathrm{d}x \\ \mathrm{d}u = \frac{1}{x}\mathrm{d}x & v = \frac{x^3}{3} \end{array} \qquad \int u \mathrm{d}v = uv - \int v \mathrm{d}u$$

$$\int x^2 \ln x \, \mathrm{d}x = \ln x \frac{x^3}{3} - \int \frac{x^3}{3}\frac{1}{x}\,\mathrm{d}x = \ln x \frac{x^3}{3} - \int \frac{x^2}{3}\mathrm{d}x = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

Example 5

★★☆

Find $\int \ln x \, \mathrm{d}x$

$\int \ln x \, \mathrm{d}x = \int 1 \; \ln x \, \mathrm{d}x$

Logarithmic functions: $u = \ln x$

$$\begin{array}{cc} u = \ln x & \mathrm{d}v = \mathrm{d}x \\ \mathrm{d}u = \frac{1}{x}\mathrm{d}x & v = x \end{array} \qquad \int u \mathrm{d}v = uv - \int v \mathrm{d}u$$

$$\int 1 \; \ln x \, \mathrm{d}x = x \ln x - \int x \frac{1}{x} \, \mathrm{d}x = x \ln x - x + C$$

Example 6

★★☆

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Find $\int \sin^{-1}x\,\mathrm{d}x$

Remember that $\int \sin^{-1}x\,\mathrm{d}x$ is alternative notation for $\int \arcsin x\,\mathrm{d}x$.

Logarithmic functions: none

Inverse trigonometric functions: $u = \sin^{-1}x$

$$u = \sin^{-1}x \qquad \mathrm{d}v = \mathrm{d}x \qquad \int u\mathrm{d}v = uv - \int v\mathrm{d}u$$
$$\mathrm{d}u = \frac{1}{\sqrt{1-x^2}}\mathrm{d}x \qquad v = x$$

$$\int \sin^{-1}x\,\mathrm{d}x = x\sin^{-1}x - \int x\frac{1}{\sqrt{1-x^2}}\mathrm{d}x = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$$

$\int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$ requires a substitution:

1. Determine u: The most complex part is $\sqrt{1-x^2}$, and the polynomial is the inside part of the radical, so $u = 1 - x^2$
2. Differentiate: $\frac{\mathrm{d}}{\mathrm{d}x}\left(1-x^2\right) = -2x$, so $\mathrm{d}u = -2x\mathrm{d}x$.
3. Substitute: $\int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x = -\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\mathrm{d}x = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\mathrm{d}u$.
4. Integrate: $-\frac{1}{2}\int u^{-\frac{1}{2}}\,\mathrm{d}u = -\frac{1}{2}\left(2u^{\frac{1}{2}}\right) = -\sqrt{u}$.
5. Convert back: $\int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x = -\sqrt{u} = -\sqrt{1-x^2}$

$$\int \sin^{-1}x\,\mathrm{d}x = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x = x\sin^{-1}x + \sqrt{1-x^2} + C$$

Both methods of choosing your u and dv, LIPET and LIATE, yield the same results until the fourth option. At that point, if the function is integrable, the integral can probably be found using either prioritisation. You will see an example of this after you look at repeated integration by parts in the next section (/study/app/math-aa-hl/sid-134-cid-761926/book/repeated-integration-by-parts-id-26517/).

3 section questions ^

Question 1

Difficulty:

★★☆

What is the integral $\int \frac{1}{x^2}\ln x\,\mathrm{d}x$?

1 $-\frac{1+\ln x}{x}+C$ ✓

2 $-\ln x + C$

3 $-\frac{\ln x}{x}+C$

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4 $-x \ln x + C$

Explanation

Using integration by parts:

$$u = \ln x,\ \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x},\ \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x^2} \text{ and } v = -\frac{1}{x}$$

Thus,

$$\begin{aligned}\int \frac{1}{x^2}\ln x\,\mathrm{d}x &= -\ln x \times \frac{1}{x} - \int -\frac{1}{x} \times \frac{1}{x}\,\mathrm{d}x\\ &= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,\mathrm{d}x\\ &= -\frac{\ln x}{x} - \frac{1}{x} + C = -\frac{1+\ln x}{x} + C\end{aligned}$$

Question 2

Difficulty:

★★☆

Find $\int\limits_0^1 9x\mathrm{e}^{3x}\,\mathrm{d}x$.

Enter your answer as a decimal to three significant figures.

41.2

Accepted answers

41.2, 41,2

Explanation

$\int\limits_0^1 9x\mathrm{e}^{3x}\,\mathrm{d}x$

Logarithmic functions: none

Inverse trigonometric functions: none

Polynomial functions (or Algebraic functions): $u = 3x$

$$\begin{aligned}u &= 9x \qquad & \mathrm{d}v &= \mathrm{e}^{3x}\mathrm{d}x\\ \mathrm{d}u &= 9\mathrm{d}x & v &= \frac{1}{3}\mathrm{e}^{3x}\end{aligned}$$

$$\int\limits_0^1 9x\mathrm{e}^{3x}\mathrm{d}x = 3x\mathrm{e}^{3x} - \int\limits_0^1 3\mathrm{e}^{3x}\mathrm{d}x$$
$$= 3x\mathrm{e}^{3x} - \mathrm{e}^{3x}\big|_0^1 = \left(3\mathrm{e}^3 - \mathrm{e}^3\right) - (0 - 1) = 2\mathrm{e}^3 + 1 = 41.2 \text{ (to 3 significant figures)}$$

Question 3

Difficulty:

★★★

Find $\int\limits_0^{\pi/2} 4x\cos(2x)\,\mathrm{d}x$.

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-2

Accepted answers

-2, −2

Explanation

$$\int_0^{\pi/2} 4x \cos 2x \, dx$$

Logarithmic functions: none

Inverse trigonometric functions: none

Polynomial functions (or Algebraic functions): $u = 4x$

$$\begin{array}{ll} u = 4x & dv = \cos 2x dx \\ du = 4dx & v = \frac{1}{2} \sin 2x \end{array}$$

$$\int_0^{\pi/2} 4x \cos 2x \, dx = 2x \sin 2x - \int_0^{\pi/2} 2 \sin 2x \, dx = 2x \sin 2x + \cos 2x|_0^{\pi/2}$$
$$= \left[2\left(\frac{\pi}{2}\right) \sin\left(2\left(\frac{\pi}{2}\right)\right) + \cos\left(2\left(\frac{\pi}{2}\right)\right)\right] - [2(0)\sin(0) + \cos(0)]$$
$$= [0 - 1] - [0 + 1] = -2$$

5. Calculus / 5.16 Further integration

Repeated integration by parts

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/repeated-integration-by-parts-id-26517/print/) Assign

In the last section (/study/app/math-aa-hl/sid-134-cid-761926/book/integration-by-parts-id-26516/), you applied integration by parts once in each question. However, you may need to apply it multiple times.

Consider:

$$\int e^{2x} \sin \frac{x}{3} dx$$

as shown in the following video.

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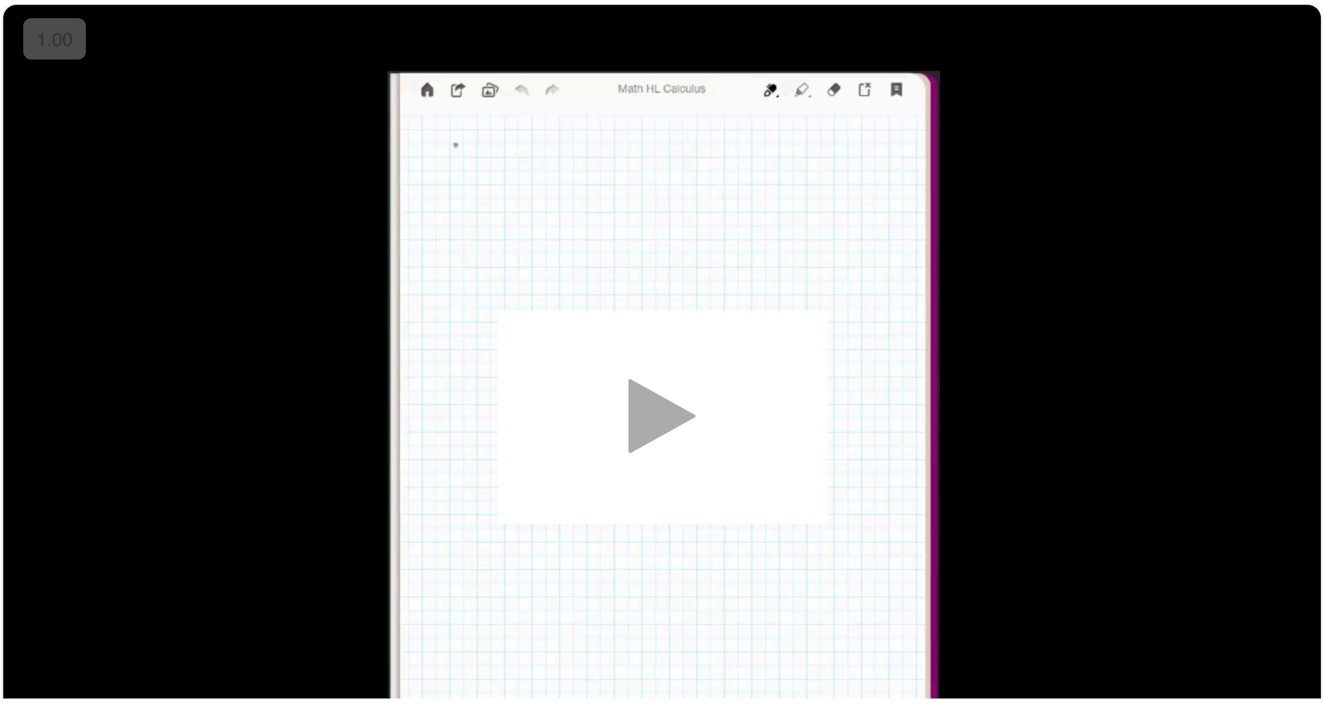
761926/o



Video 1. Applying Integration by Parts Multiple Times.

More information for video 1

1

00:00:00,833 --> 00:00:02,200

narrator: It is possible that sometimes

2

00:00:02,267 --> 00:00:04,967

you need to apply integration

by parts multiple times

3

00:00:05,033 --> 00:00:08,433

and that is what we're going

to explore in this video.

4

00:00:08,600 --> 00:00:12,133

Now before we start, some hints

of when this might happen.

5

00:00:12,500 --> 00:00:14,767

So one hint might be that you have

$x^2 \Longleftrightarrow \quad x^2 \rightarrow x^1 \rightarrow x^0$

6

00:00:14,833 --> 00:00:17,500

one of the functions

in the product is a power,

7

00:00:17,633 --> 00:00:19,433

like second extra power of two,

8

00:00:19,700 --> 00:00:23,533

which then if you differentiate it slowly

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view

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becomes a constant.

9

00:00:23,600 --> 00:00:26,900

The other one is a cyclical function

like $\sin(x)$,

10

00:00:26,967 --> 00:00:29,000

$\sin x \iff \sin x \to \cos x \to -\sin x$

which if you differentiate it

or integrate,

11

00:00:29,067 --> 00:00:34,500

ends up being itself after two

such operations but with a minus sign.

12

00:00:34,733 --> 00:00:38,300

So you may get something the same

13

00:00:38,367 --> 00:00:40,633

up to a constant factor,

14

00:00:40,700 --> 00:00:43,767

a number that is,

and so when you move to the other side,

15

00:00:43,900 --> 00:00:48,233

you actually get something

that is quite easy to then solve.

16

00:00:48,500 --> 00:00:51,467

And of course, instead of $\sin(x)$,

you can also have $\cos(x)$.

17

00:00:51,933 --> 00:00:54,733

So let's look at a particular

example $\int e^{2x} \sin(\frac{x}{3})\, dx$.

18

00:00:57,933 --> 00:01:00,100

So u is gonna be e^{2x}.

19

00:01:00,267 --> 00:01:06,100

So that $\frac{du}{dx} = 2e^{2x}$,

$\frac{dv}{dx}$ is gonna be $\sin(\frac{x}{3})$.

20

00:01:06,300 --> 00:01:09,533

So that integrated v is $-3\cos(\frac{x}{3})$.

21

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00:01:10,633 --> 00:01:12,833

So now we're going

to apply the integration

22

00:01:12,900 --> 00:01:17,500

by parts $\int u\,dv = uv - \int v\,du$.

23

00:01:17,800 --> 00:01:26,667

So we're gonna substitute what we found

$-3e^{2x}\cos(\frac{x}{3}) + \int 6e^{2x}\cos(\frac{x}{3})\,dx$.

24

00:01:26,733 --> 00:01:30,133

Now we still can't integrate

that last one, the one in blue.

25

00:01:30,267 --> 00:01:34,467

So we're gonna do another integration

by parts $u = e^{2x}$,

26

00:01:34,600 --> 00:01:37,300

$\frac{du}{dx} = 2e^{2x}$,

27

00:01:37,533 --> 00:01:39,567

the $\frac{dv}{dx} = \cos(\frac{x}{3})$.

28

00:01:39,700 --> 00:01:43,200

So $v = 3\sin(\frac{x}{3})$.

29

00:01:43,467 --> 00:01:45,700

So now the blue integral becomes

30

00:01:45,767 --> 00:01:53,900

$3e^{2x}\sin(\frac{x}{3}) - \int 6e^{2x}\sin(\frac{x}{3})\,dx$.

31

00:01:54,233 --> 00:01:57,100

So now the entire integral,

32

00:01:57,300 --> 00:01:59,500

the black one plus blue one

33

00:01:59,567 --> 00:02:10,433

becomes $-3e^{2x}\cos(\frac{x}{3}) + 6(3e^{2x}\sin(\frac{x}{3}) - \int 2e^{2x}\cdot 3\sin(\frac{x}{3})\,dx)$.

34

00:02:10,500 --> 00:02:13,333

And now you notice that some portions

35

00:02:13,400 --> 00:02:15,933

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of this right hand side is equal

36

00:02:16,267 --> 00:02:19,433

to the integral on the left hand side,

i.e., the original integral.

37

00:02:19,500 --> 00:02:21,133

So we're gonna bring it to the other side.

38

00:02:21,400 --> 00:02:46,967

So we have $\int e^{2x} \sin(\frac{x}{3})\, dx = -3e^{2x} \cos(\frac{x}{3}) + 18e^{2x} \sin(\frac{x}{3}) - 36\int e^{2x} \sin(\frac{x}{3})\, dx$.

Bringing the integral term to the left side:

$37\int e^{2x} \sin(\frac{x}{3})\, dx = -3e^{2x} \cos(\frac{x}{3}) + 18e^{2x} \sin(\frac{x}{3})$.

39

00:02:47,200 --> 00:02:51,600

And now of course we simply

divide both sides by 37

so that the integral

of $\int e^{2x} \sin(\frac{x}{3})\, dx$ becomes

$\frac{1}{37}e^{2x}(18\sin(\frac{x}{3}) - 3\cos(\frac{x}{3}))$.

40

00:02:51,600 --> 00:02:53,000

And we collect some terms

$\frac{3}{37}e^{2x}(6\sin(\frac{x}{3}) - \cos(\frac{x}{3}))$.

41

00:02:51,600 --> 00:02:59,233

plus of course,

the constant of integration.

So here we've performed

integration by part twice

and solved a rather challenging integral.

After integrating by parts twice, you find that

$$\int e^{2x} \sin\frac{x}{3} dx = \frac{3}{37} e^{2x}\left(6\sin\frac{x}{3} - \cos\frac{x}{3}\right) + C$$

In this case, you use algebra to move the remaining integral to the left-hand side. In other examples, the remaining integrand will get simpler after each iteration until it is possible to find the integral.

Example 1

★★☆

Find $\int 3x^2 e^{2x}\, dx$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$\int 3x^2 \mathrm{e}^{2x}\, \mathrm{d}x$

Logarithmic functions: none

Inverse trigonometric functions: none

Polynomial functions (or Algebraic functions): $u = 3x^2$

$$\begin{array}{lll} u = 3x^2 & \mathrm{d}v = \mathrm{e}^{2x}\mathrm{d}x & \\ \mathrm{d}u = 6x\mathrm{d}x & v = \dfrac{\mathrm{e}^{2x}}{2} & \int u\mathrm{d}v = uv - \int v\mathrm{d}u \end{array}$$

$$\int 3x^2\mathrm{e}^{2x}\mathrm{d}x = 3x^2\frac{\mathrm{e}^{2x}}{2} - \int \frac{\mathrm{e}^{2x}}{2}6x\mathrm{d}x = \frac{3}{2}x^2\mathrm{e}^{2x} - \int 3x\mathrm{e}^{2x}\,\mathrm{d}x$$

The remaining integral is still not suitable for integration and needs to be integrated by parts.

Logarithmic functions: none

Inverse trigonometric functions: none

Polynomial functions (or Algebraic functions): $u = 3x$

$$\begin{array}{lll} u = 3x & \mathrm{d}v = \mathrm{e}^{2x}\,\mathrm{d}x & \\ \mathrm{d}u = 3\mathrm{d}x & v = \dfrac{\mathrm{e}^{2x}}{2} & (\int u\mathrm{d}v = uv - \int v\mathrm{d}u \end{array}$$

$$\int 3x\mathrm{e}^{2x}\mathrm{d}x = 3x\frac{\mathrm{e}^{2x}}{2} - \int \frac{\mathrm{e}^{2x}}{2}3\mathrm{d}x = \frac{3}{2}x\mathrm{e}^{2x} - \int \frac{3}{2}\mathrm{e}^{2x}\,\mathrm{d}x = \frac{3}{2}x\mathrm{e}^{2x} - \frac{3}{4}\mathrm{e}^{2x} + C$$

Putting this all together:

$$\int 3x^2\mathrm{e}^{2x}\mathrm{d}x = \frac{3}{2}x^2\mathrm{e}^{2x} - \left(\frac{3}{2}x\mathrm{e}^{2x} - \frac{3}{4}\mathrm{e}^{2x} + C\right) = \frac{3}{2}x^2\mathrm{e}^{2x} - \frac{3}{2}x\mathrm{e}^{2x} + \frac{3}{4}\mathrm{e}^{2x} - C$$

Exam tip

In the IB exam, applications will include repeated algebraic integration by parts with up to two iterations, and cases of trigonometric or exponential functions for which you can use algebraic manipulation to solve for the integral.

Finally, look at examples where you need considerably more than two iterations of integration by parts. Consider $\int x^6\mathrm{e}^x\mathrm{d}x$.

It would seem that after the first iteration, you have a fifth-order polynomial multiplied by an exponential. After the second iteration, you have a fourth-order polynomial, and so on. After six iterations, many of us would have a hard time keeping up with all of the algebra.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

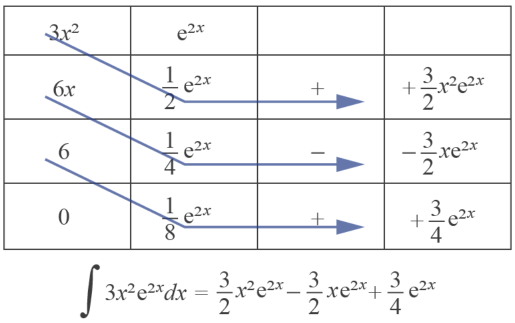
Tabular integration, colloquially called 'Tic-Tac-Toe' integration, helps with the bookkeeping. Before you use it on this one, revisit **Example 1**.

Find $\int 3x^2\mathrm{e}^{2x}\,\mathrm{d}x$

Important

Tabular integration:

- Column 1 starts with the polynomial function in the header row. Each successive row is the derivative of the previous row, ending with '0'.
- Column 2 starts with the remaining components of the integrand in the header row. Each successive row is the integral of the previous row.
- Column 3 contains alternating '+' and '–' signs in each row, starting with '+'.
- Column 4 is the product of the previous row from column 1 and the current row from columns 2 and 3; hence 'Tic-Tac-Toe'.
- The final integral is the sum of these products.



More information

The image is a diagram showing a mathematical integration process for the function $3x^2e^{2x}$. It is displayed in a tabular format with three columns and multiple rows. The first column contains expressions such as $3x^2$, $6x$, 6, and 0. The second column aligns with these expressions with values like e^{2x}, $1/2e^{2x}$, $1/4e^{2x}$, and $1/8e^{2x}$. Arrows point from top to bottom indicating the integration steps. The third column includes results such as $+3/2x^2e^{2x}$, $-3/2xe^{2x}$, and $+3/4e^{2x}$, suggesting the integrated outcomes. Below the table, there is a mathematical equation integrating the entire function: $\int 3x^2e^{2x}\,dx = 3/2x^2e^{2x} - 3/2xe^{2x} + 3/4e^{2x}$. Arrows on the table connect the rows, indicating the sequence of operations.

[Generated by AI]

Applying this to the more complex example, it looks like this:

Find $\int x^6\mathrm{e}^x\mathrm{d}x$

Student view

$\cancel{x^6}$	e^x		
$\cancel{6x^5}$	e^x	$+$	$+x^6e^x$
$\cancel{30x^4}$	e^x	$-$	$-6x^5e^x$
$\cancel{120x^3}$	e^x	$+$	$+30x^4e^x$
$\cancel{360x^2}$	e^x	$-$	$-120x^3e^x$
$\cancel{720x}$	e^x	$+$	$+360x^2e^x$
$\cancel{720}$	e^x	$-$	$-720xe^x$
0	e^x	$+$	$+720e^x$

$$\int x^6e^x dx = x^6e^x - 6x^5e^x + 30x^4e^x - 120x^3e^x + 360x^2e^x - 720xe^x + 720e^x$$

More information

The image is a table used to illustrate the process of solving the integral (\int x^6 e^x \, dx) using the method of integration by parts. The table consists of four columns. The first column lists derivatives of (x^6), starting from (x^6) itself and decreasing in power by one in each subsequent row until reaching 0. The second column contains repeating expressions of (e^x). The third column is used for alternating signs, starting with a positive sign for each iteration. The fourth column presents the resulting terms after applying integration by parts, such as (+x^6 e^x), (-6x^5 e^x), and so on. Each row reflects one step in the integration by parts process. Arrows show the relationships between the terms in the first two columns and the resulting term in the fourth column. Below the table, the completed solution is written as the integral: (x^6 e^x - 6x^5 e^x + 30x^4 e^x - 120x^3 e^x + 360x^2 e^x - 720x e^x + 720 e^x).

[Generated by AI]

As mentioned earlier, in the IB exam, applications will only include repeated algebraic integration by parts with up to two iterations.

Example 2

★★☆

Find $\int x^3 \cos\left(\frac{x}{2}\right)\ \mathrm{d}x$

$\int x^3 \cos\left(\frac{x}{2}\right)\ \mathrm{d}x$

Logarithmic functions: none

Inverse trigonometric functions: none

Polynomial functions (or Algebraic functions): $u = x^3$

x^3	$\cos(\frac{x}{2})$		
$3x^2$	$2\sin(\frac{x}{2})$	$+$	$+2x^3\sin(\frac{x}{2})$
$6x$	$-4\cos(\frac{x}{2})$	$-$	$+12x^2\cos(\frac{x}{2})$
6	$-8\sin(\frac{x}{2})$	$+$	$-48x\sin(\frac{x}{2})$
0	$16\cos(\frac{x}{2})$	$-$	$-96\cos(\frac{x}{2})$

$$\int x^3\cos(\frac{x}{2})dx = 2x^3\sin(\frac{x}{2}) + 12x^2\cos(\frac{x}{2}) - 48x\sin(\frac{x}{2}) - 96\cos(\frac{x}{2})$$

In section 5.16.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/integration-by-parts-id-26516/), you were shown two different mnemonics to help you to decide which function is replaced by u. In the next two examples, you will see that both methods get to the same result in slightly different ways.

Example 3

★★★

Find $\int \mathrm{e}^x \cos x \,\mathrm{d}x$ using LIPET

$\int \mathrm{e}^x \cos x \,\mathrm{d}x$

L ogarithmic functions: none

I nverse trigonometric functions: none

P olynomial functions: none

E xponential functions: $u = \mathrm{e}^x$

$$\begin{array}{cc} u = \mathrm{e}^x & \mathrm{d}v = \cos x \,\mathrm{d}x \\ \mathrm{d}u = \mathrm{e}^x \mathrm{d}x & v = \sin x \end{array} \quad \int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

$$\int \mathrm{e}^x \cos x \,\mathrm{d}x = \mathrm{e}^x \sin x - \int \sin x \,\mathrm{e}^x \mathrm{d}x = \mathrm{e}^x \sin x - \int \mathrm{e}^x \sin x \,\mathrm{d}x$$

The remaining integral is still not suitable for integration and needs to be integrated by parts.

$$\begin{array}{cc} u = \mathrm{e}^x & \mathrm{d}v = \sin x \,\mathrm{d}x \\ \mathrm{d}u = \mathrm{e}^x \mathrm{d}x & v = -\cos x \end{array} \quad \int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

$$\int \mathrm{e}^x \sin x \,\mathrm{d}x = \mathrm{e}^x (-\cos x) - \int -\cos x \,\mathrm{e}^x \mathrm{d}x = -\mathrm{e}^x \cos x + \int \mathrm{e}^x \cos x \,\mathrm{d}x$$

Putting this together gives:

$$\int \mathrm{e}^x \cos x \,\mathrm{d}x = \mathrm{e}^x \sin x - \left(-\mathrm{e}^x \cos x + \int \mathrm{e}^x \cos x \,\mathrm{d}x\right) = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x - \int \mathrm{e}^x \cos x \,\mathrm{d}x$$

Overview (/study/app/aa-hl/sid-134-cid-761926/o

You can solve this using algebraic techniques:

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C$$

Example 4

★★★

Find $\int e^x \cos x \, dx$ using LIATE

$\int e^x \cos x \, dx$

Logarithmic functions: none

Inverse trigonometric functions: none

Algebraic functions: none

Trigonometric functions: $u = \cos x$

$$\begin{array}{ccc} u = \cos x & dv = e^x \, dx & \\ du = -\sin x \, dx & v = e^x & \end{array} \quad \int u dv = uv - \int v du$$

$$\int e^x \sin x \, dx = \cos x \, e^x - \int e^x \ (-\sin x) \, dx = e^x \cos x + \int e^x \sin x \, dx$$

The remaining integral is still not suitable for integration and needs to be integrated by parts.

$$\begin{array}{cc} u = \sin x & ldv = e^x \, dx \\ du = \cos x dx & v = e^x \end{array} \quad \int u dv = uv - \int v du$$

$$\int e^x \sin x \, dx = \sin x \, e^x - \int e^x \ \cos x dx = e^x \sin x - \int e^x \cos x \, dx$$

Putting this together gives:

$$\int e^x \cos x \, dx = e^x \cos x + \left(e^x \sin x - \int e^x \cos x \, dx\right) = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

You can solve this using algebraic techniques:

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$$\int e^x \cos x \, dx = \frac{e^x \cos x + e^x \sin x}{2} + C$$

With the additional step of applying the commutative property of addition, you can see that the results using LIPET and LIATE are identical.

Example 5

★★★

Remember the problem in the big picture? Find $y_{avg} = \frac{1}{\pi} \int_0^\pi e^{-t} \cos t \, dt$

$\int_0^\pi e^{-t} \cos t \, dt$ Omit $\frac{1}{\pi}$ for now but put back at the end

L ogarithmic functions: none

I nverse trigonometric functions: none

P olynomial functions: none

E xponential functions: $u = e^{-t}$

$$\begin{aligned} u &= e^{-t} & dv &= \cos t \, dt \\ du &= -e^{-t} dt & v &= \sin t \end{aligned} \qquad \int u dv = uv - \int v du$$

$$\int_0^\pi e^{-t} \cos t \, dt = -e^{-t} \sin t - \int_0^\pi - \sin t \, e^{-t} dt = -e^{-t} \sin t + \int_0^\pi e^{-t} \sin t \, dt$$

The remaining integral is still not suitable for integration and needs to be integrated by parts.

$$\begin{aligned} u &= e^{-t} & ldv &= \sin t \, dt \\ du &= -e^{-t} dt & v &= -\cos t \end{aligned} \qquad \int u dv = uv - \int v du$$

$$\int_0^\pi e^{-t} \sin t \, dt = -e^{-t} \cos t - \int_0^\pi \cos t \, e^{-t} dt = -e^{-t} \cos t - \int_0^\pi e^{-t} \cos t \, dt$$

Putting this together gives:

$$\int_0^\pi e^{-t} \cos t \, dt = e^{-t} \sin t - e^{-t} \cos t - \int_0^\pi e^{-t} \cos t \, dt$$

You can solve this using algebraic techniques:

$$\int_0^\pi e^{-t} \cos t \, dt = \left. \frac{e^{-t} \sin t - e^{-t} \cos t}{2} \right|_0^\pi$$

Replace the constant $\frac{1}{\pi}$ in front of the integral and evaluate:

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$$
\begin{aligned}
y_{avg} &= \frac{1}{\pi}\int_0^{\pi} \mathrm{e}^{-t}\cos t\,\mathrm{d}t = \left.\frac{\mathrm{e}^{-t}\sin t - \mathrm{e}^{-t}\cos t}{2\pi}\right|_0^{\pi} \\
&= \left[\frac{\mathrm{e}^{-\pi}(0) - \mathrm{e}^{-\pi}(1)}{2\pi}\right] - \left[\frac{\mathrm{e}^{0}(0) - \mathrm{e}^{0}(-1)}{2\pi}\right] \\
&= \frac{1}{2\pi e^{\pi}} + \frac{1}{2\pi} \approx 0.166
\end{aligned}
$$

4 section questions

Question 1

Difficulty:

★★☆

Find $\int x^2\,\mathrm{e}^{2x}\,\mathrm{d}x$

1 $\frac{1}{2}e^{2x}\left(x^2 - x + \frac{1}{2}\right) + C$ ✓

2 $\frac{1}{2}\mathrm{e}^{2x}\left(x^2 + x + 1\right) + C$

3 $-\frac{1}{2}\mathrm{e}^{2x}\left(x^2 - x + \frac{1}{2}\right) + C$

4 $\frac{1}{2}x\,\mathrm{e}^{2x}\left(x - 1\right) + C$

Explanation

$$\int x^2 e^{2x}\,dx \qquad u = x^2 \quad \frac{dv}{dx} = e^{2x} \qquad \frac{du}{dx} = 2x \quad v = \frac{1}{2}e^{2x}$$

$$= \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx \qquad u = x \quad \frac{dv}{dx} = e^{2x} \qquad \frac{du}{dx} = 1 \quad v = \frac{1}{2}e^{2x}$$

$$= \frac{1}{2}x^2 e^{2x} - \left[\frac{1}{2}x e^{2x} - \frac{1}{2}\int e^{2x}\,dx\right]$$

$$= \frac{1}{2}x^2 e^{2x} - \left[\frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x}\right] + C$$

$$= \frac{1}{2}e^{2x}\left(x^2 - x + \frac{1}{2}\right) + C$$

Student view

Question 2

Difficulty:

★★☆

Find $\int x^2 \cos 3x \, dx$

1 $\frac{1}{27}\left((9x^2-2)\sin 3x + 6x\cos 3x\right) + C$ ✓

2 $\frac{1}{9}\left((3x^2-6)\sin 3x + 2x\cos 3x\right) + C$

3 $\frac{1}{9}\left(3x^2\sin 3x + 2x\cos 3x\right) + C$

4 $\frac{1}{27}\left((9x^2-2)\sin 3x - 2x\cos 3x\right) + C$

Explanation

$$\int x^2 \cos 3x \, dx \qquad u = x^2 \quad \frac{du}{dx} = 2x \qquad \frac{dv}{dx} = \cos 3x \quad v = \frac{1}{3}\sin 3x$$

$$= \frac{1}{3}x^2 \sin 3x - \frac{2}{3}\int x \sin 3x \, dx \qquad u = x \quad \frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \sin 3x \quad v = -\frac{1}{3}\cos 3x$$

$$= \frac{1}{3}x^2 \sin 3x - \frac{2}{3}\left[-\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x \, dx\right]$$

$$= \frac{1}{3}x^2 \sin 3x - \frac{2}{3}\left[-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x\right] + C$$

$$= \frac{1}{3}x^2 \sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C$$

$$= \frac{1}{27}\left((9x^2-2)\sin 3x + 6x\cos 3x\right) + C$$

More information

Question 3

Difficulty:

★★☆

Find $\int e^x \sin x \, dx$

1 $\frac{e^x(\sin x - \cos x)}{2} + C$ ✓

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

2 $\frac{e^x(\sin x + \cos x)}{2} + C$

3 $\frac{e^x(\cos x - \sin x)}{2} + C$

4 $2e^x(\sin x - \cos x) + C$

Explanation

Set

$$u = e^x \Rightarrow \frac{du}{dx} = e^x \Rightarrow du = e^x\,dx$$

and

$$\frac{dv}{dx} = \sin x \Rightarrow dv = \sin x\,dx \Rightarrow v = -\cos x.$$

Then

$$\int e^x \sin x\,dx = \int u\,dv = uv - \int v\,du = -e^x \cos x + \int e^x \cos x\,dx. \qquad (1)$$

We repeat the method of integrating by parts for the integral $\int e^x \cos x\,dx$.

$$w = e^x \Rightarrow \frac{dw}{dx} = e^x \Rightarrow dw = e^x\,dx$$

and

$$\frac{dz}{dx} = \cos x \Rightarrow dz = \cos x\,dx \Rightarrow z = \sin x.$$

Then

$$\int e^x \cos x\,dx = \int w\,dz = wz - \int z\,dw = e^x \sin x - \int e^x \sin x\,dx. \qquad (2)$$

Substituting what we found in (2) into (1) above, we have

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx,$$

from which we can see that the integral in question is on both sides of the equation with opposite signs thus,

$$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x,$$

which finally gives

$$\int e^x \sin x\,dx = \frac{e^x(\sin x - \cos x)}{2} + C.$$

Question 4

Difficulty:

★★★

Find $\int_0^{\pi} x^3 \sin x\,dx$.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Enter your answer as a decimal to three significant figures.

12.2

Accepted answers

12.2, 12,2

Explanation

$$\int_0^\pi x^3 \sin x \, dx$$

Repeated integration by parts or tabular integration:

$$\begin{aligned}\int_0^\pi x^3 \sin x \, dx &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x \Big|_0^\pi \\ &= \left[-\pi^3 \cos \pi + 3\pi^2 \sin \pi + 6\pi \cos \pi - 6 \sin \pi\right] - \left[-0^3 \cos 0 + 3(0)^2 \sin 0 + 6(0) \cos 0 - 6 \sin 0\right] \\ &= \left[-\pi^3(-1) + 0 + 6\pi(-1)\right] - [0] = \pi^3 - 6\pi = 12.2 \text{ (to 3 significant figures)}\end{aligned}$$

5. Calculus / 5.16 Further integration

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-26518/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- find indefinite integrals using integration by substitution
- find definite integrals using integration by substitution by either:
  - converting the indefinite integral back to x before applying the limits; or
  - converting the limits over to u before application
- find indefinite integrals using integration by parts
- find definite integrals using integration by parts

$$\int u \frac{dv}{du} = uv - \int v \frac{du}{dx} \text{ or } \int u dv = uv - \int v du$$

- integrate by parts multiple times or use tabular integration.

5. Calculus / 5.16 Further integration

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-26519/print/) Assign

Student view

Part 1

For this investigation, you will work by going back and forth with derivatives of composite functions and their associated anti-derivatives.

First, open up a browser window and go to Symbolab at https://www.symbolab.com/ (https://www.symbolab.com/).

In the ‘Enter a problem’ box, enter the derivative $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(3x^2\right)\right)$ and click ‘Go’. The program will find the derivative as $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(3x^2\right)\right) = \cos\left(3x^2\right) \cdot 6x$. The program will also show the steps used, identifying the use of the chain rule.

Now open a second window with Symbolab and enter $\int \cos\left(3x^2\right) \cdot 6x\,\mathrm{d}x$. The program will find the integral as $\int \cos\left(3x^2\right) \cdot 6x\,\mathrm{d}x = \sin\left(3x^2\right) + C$ (Why do you have the ‘$+C$’?) The program will also show the steps used, identifying the use of u-substitution and $u = 3x^2$.

Part 2

Make up your own composite functions. Experiment with a variety of trigonometric, logarithmic, exponential, and polynomial functions. With each composite function:

- find the derivative using Symbolab
- write down the resulting derivative
- find the integral of that result using Symbolab
- write down the resulting integral
- identify the similarities in the equations.

Can you see the relationship between the chain rule for differentiation and u-substitution for integration?