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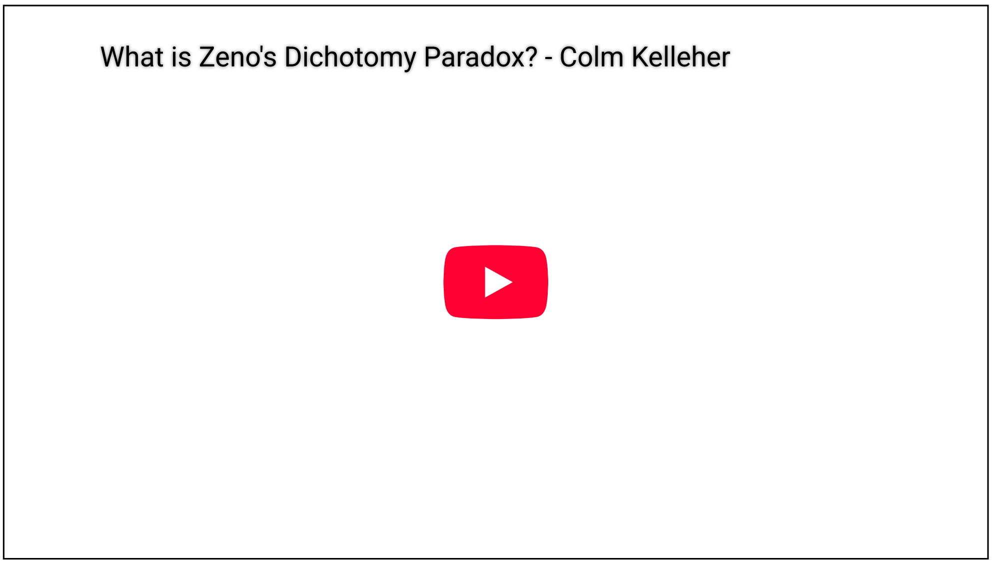
The big picture

To find the sum of an infinite geometric sequence you need to add up an infinite number of terms.

You might think that the result would be infinitely large, but is it possible that the sum of an infinite number of terms actually turns out to be finite?

For a long time this question intrigued mathematicians and philosophers, including Zeno, the ancient Greek philosopher who formulated the Dichotomy Paradox. Watch the video to see what this paradox is about.



Concept

In this subtopic you will learn that there are situations in which adding up an infinite number of values produces a finite value. Consider whether the result of adding up infinitely many values can be an exact sum or only an approximation of the true sum.

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Theory of Knowledge

When you began investigating the sum of infinite series, you may have asked yourself: 'How can an infinite series have a sum?' Good question.

This gets into language as a way of knowing and requires us to expand our concept of 'sum'. In an infinite series, the numbers are indeed headed towards an endpoint, which we call the sum; but, as you know, given the fact the series is stated as 'infinite', we will never reach that endpoint. Thus, can convergent geometric sequences be said to have a sum?

Knowledge Question: To what extent does language limit knowledge production in mathematics?

Infinite geometric sequences

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/infinite-geometric-sequences-id-27683/print/) Assign

From subtopic 1.3 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25043/), you are already familiar with geometric sequences, such as

$$2,\ 4,\ 8,\ 16,\ \ldots,\ 512,\ 1024,$$

and their sums,

$$2+4+8+16+\cdots+512+1024.$$

Making connections

The n th term of a geometric sequence with common ratio r is:

$$u_n = u_1 r^{n-1}.$$

The sum of the first n terms of a geometric sequence is:

$$S_n = \frac{u_1\left(r^n-1\right)}{r-1} = \frac{u_1\left(1-r^n\right)}{1-r},\quad r \neq 1.$$

Now think about what could happen if the geometric sequence is infinite and you want to find its sum. Do you think that this is possible? Are there special circumstances in which the sum can be found?

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Activity

Use the applet below to explore what happens to the sum of the terms of a geometric sequence as the number of terms, n, is increased.

How does the r value affect the sum as n gets larger? Does your conjecture hold true for all values of r?

Do you think that your observations extend to values of n that are larger than the ones given in the applet? What about for r values that are larger or smaller than the ones available in the applet?

Interactive 1. Sum of the Terms of a Geometric Sequence.

More information for interactive 1

This interactive tool allows users to explore the behavior of a converging geometric sequence defined by $u_n = 243(r)^n$, where the common ratio r is always less than 1. By adjusting the ratio ‘r’ and the term number ‘n’, users can observe how each term's value (shown as a red dot) decreases exponentially toward zero, while the partial sums (marked by crosses) approach a finite limit. The visualization clearly demonstrates that smaller values of ‘r’ cause the sequence to converge more rapidly - both in terms of individual terms shrinking toward zero and partial sums stabilizing near their limiting value of $\frac{243}{1-r}$. For example, with $r = 0.5$, the sum quickly approaches 486 within about 10 terms, while with $r = 0.9$, users will need to examine nearly 100 terms to see the sum get comparably close to its limit of 2,430. The tool provides an intuitive way to understand the relationship between the ratio's magnitude and the convergence rate, showing why mathematicians emphasize the condition $|r| < 1$ for geometric series convergence. Through experimentation, users gain concrete insights into how infinite series with diminishing terms can nevertheless sum to finite values, a fundamental concept in calculus and mathematical analysis.

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You can see that when $-1 < r < 1$ the terms of a geometric sequence get smaller as n increases. In this case, smaller and smaller values are added to the running total and the sum can be found for an infinite number of terms.

For example, consider:

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} + \frac{1}{2187} + \frac{1}{6561} + \ldots$$

Even though only the first 9 terms of this infinite sum are shown, you can see that already $\frac{1}{6561} = 0.000152 \approx 0$. Any further terms will not change the sum significantly.

Why does this not happen for a geometric sequence that has $r \geq 1$ or $r \leq -1$?

Be aware

An infinite geometric series, such as

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} + \frac{1}{2187} + \frac{1}{6561} + \ldots,$$

can be written in sigma notation. For example:

$$\sum_{i=1}^{\infty} \left(1 \times \left(\frac{1}{3}\right)^{i-1}\right) = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} + \frac{1}{2187} + \frac{1}{6561} + \ldots.$$

To find the value of the sum of all terms of an infinite geometric sequence with $-1 < r < 1$, let's first suppose that $0 < r < 1$ and look at the formula for the sum up to n terms:

$$S_n = \frac{u_1 \left(1 - r^n\right)}{1 - r}$$

For any $0 < r < 1$, the power r^n gets smaller and smaller as n gets larger, so $1 - r^n$ gets closer and closer to 1. This means that the sum approaches $\frac{u_1}{1 - r}$.

Similarly you can show using the formula $S_n = \frac{u_1 \left(r^n - 1\right)}{r - 1}$ that for $-1 < r < 0$, the sum also approaches $\frac{u_1}{1 - r}$ as n increases.

What about the $r = 0$ case?

The conditions $0 < r < 1$, $-1 < r < 0$ and $r = 0$ can be written together in the compact form $|r| < 1$ using modulus notation, where $|r|$ means the modulus, or absolute value, of r.

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Important

The sum of the terms of an infinite geometric sequence, or infinite sum , is given by

$$S_\infty = \frac{u_1}{1-r}, \quad |r| < 1.$$

An infinite geometric sequence is convergent if

$|r| < 1$, which means $-1 < r < 1$.

If a sequence does not converge, it is divergent.

Example 1

★☆☆

Evaluate $4 - \frac{4}{5} + \frac{4}{25} - \frac{4}{125} + \ldots$

Steps	Explanation
This is an infinite geometric series. $u_1 = 4,\ r = -\frac{1}{5}$ $S_\infty = \frac{4}{1-\left(-\frac{1}{5}\right)} = \frac{4}{\left(\frac{6}{5}\right)} = \frac{10}{3}$	For this series $\|r\| < 1$, so the infinite sum can be found by using $S_\infty = \frac{u_1}{1-r}$
Therefore $4 - \frac{4}{5} + \frac{4}{25} - \frac{4}{125} + \ldots = \frac{10}{3}$	

International Mindedness

The questions considered by mathematicians are often too difficult to be solved by one person. The question of whether the infinite sum $1 - 1 + 1 - 1 + 1 - 1 + 1 \cdots$, known as Grandi's series, is convergent or divergent illustrates how mathematicians from all over the world work with each other's results. You can do some research to find out what the various mathematicians thought about this infinite sum.

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Example 2

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For a geometric sequence, the sum of the first three terms is $\frac{91}{9}$ and the infinite sum of all the terms is $\frac{21}{2}$. Find the common ratio and the first term.

Steps	Explanation
$S_3 = \frac{91}{9} = \frac{u_1\left(1-r^3\right)}{1-r}$ $S_\infty = \frac{21}{2} = \frac{u_1}{1-r}$	Write an equation for each piece of the given information. It will be helpful in the next step if you use the version of S_n where the denominator matches the denominator in the S_∞ formula.
$\begin{cases} \frac{91}{9} = \frac{u_1\left(1-r^3\right)}{1-r} \\ \frac{21}{2} = \frac{u_1}{1-r} \end{cases} \Leftrightarrow \frac{\frac{91}{9}}{\frac{21}{2}} = \frac{\frac{u_1\left(1-r^3\right)}{1-r}}{\frac{u_1}{1-r}}$ $\Leftrightarrow \frac{91}{9} \times \frac{2}{21} = \frac{u_1\left(1-r^3\right)}{1-r} \times \frac{1-r}{u_1}$ $\Leftrightarrow \frac{26}{27} = 1-r^3 \quad \Leftrightarrow r^3 = 1-\frac{26}{27}$ $\Leftrightarrow r = \sqrt[3]{\frac{1}{27}} = \frac{1}{3}$	Solve this system of two equations. The easiest way to solve a system of equations for a geometric sequence is to divide one equation by the other, rather than using substitution or elimination.
$\frac{21}{2} = \frac{u_1}{1-\frac{1}{3}} \quad \Leftrightarrow u_1 = \frac{21}{2} \times \frac{2}{3} = 7$ So the common ratio is $\frac{1}{3}$ and the first term is 7.	Plug the r value you found into one of the equations and solve for u_1.

Exam tip

Your IB formula booklet contains formulae for the sums of the terms of both a finite geometric sequence and an infinite geometric sequence. Read the questions carefully and apply the correct formula for the information given.

Student view

You can practise more questions like Example 2 by using the applet below.

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Interactive 2. Geometric Sequence Solver.

More information for interactive 2

This interactive tool allows users to strengthen their understanding of geometric sequences by solving problems involving the sum of terms. Given the sum of the first n terms (S_n) and the sum to infinity (S_∞) of a geometric sequence with positive terms, users must determine the **common ratio** (r) and the **first term** (u_1) .

Users can attempt the problem, check their answers for accuracy, and generate **new questions** for continuous practice. This hands-on approach reinforces key concepts such as the formula for the sum of a finite geometric series and the sum to infinity of a convergent sequence.

Example:

For a geometric sequence with positive terms:

$S_4 = \frac{280}{27}\ S_\infty = \frac{21}{2}$.

The correct values are:

Common ratio: $r = \frac{1}{3} = 0.33$

First term: $u_1 = 7$

By engaging with this interactive, users develop a deeper intuition for geometric sequences and improve their problem-solving skills in an active and dynamic way.

Example 3

★★★

$$S_\infty = u_1 + u_1 r + u_1 r^2 + \ldots$$

a) Write an expression for $r \times S_\infty$ in terms of $u_1$and r.

Student view

b) Hence, show that $S_\infty = \dfrac{u_1}{1-r}$.

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	Steps	Explanation
a)	$r \times S_\infty = u_1 \times r + u_1 r \times r + u_1 r^2 \times r + \ldots$ $= u_1 r + u_1 r^2 + u_1 r^3 + \ldots$	Multiply all terms in the sum by r.
b)	$S_\infty = u_1 + u_1 r + u_1 r^2 + \ldots$ $S_\infty - r \times S_\infty = \left(u_1 + u_1 r + u_1 r^2 + u_1 r^3 + \ldots\right)$ -$\left(u_1 r + u_1 r^2 + u_1 r^3 + \ldots\right)$ $= u_1$	All the terms on the RHS of $r \times S_\infty$ also appear in S_∞, so if we subtract $r \times S_\infty$ from S_∞ then all terms will cancel out except u_1.
	$S_\infty (1 - r) = u_1 \quad \Leftrightarrow \quad S_\infty = \dfrac{u_1}{1 - r}$	Solve for S_∞.

4 section questions

1. Number and algebra / 1.8 Sum of infinite geometric sequences

Applications

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/applications-id-27684/print/) Assign

In this section you will see examples of applications of the sum of an infinite geometric sequence.

Activity

Squares are drawn following the pattern shown in the diagram below. Diagram 1 shows a square of area 1 unit divided into four equal parts. The shaded area is $\frac{1}{4}$ units. In Diagram 2 the original square is divided into 16 equal parts, and the total shaded area is $\frac{5}{16}$ units. This pattern continues as shown in Diagrams 3 and 4.

If this process is continued for ever, find the total area for an infinite number of shaded boxes.

Discuss what it means to have an infinite number of shaded boxes in this context.

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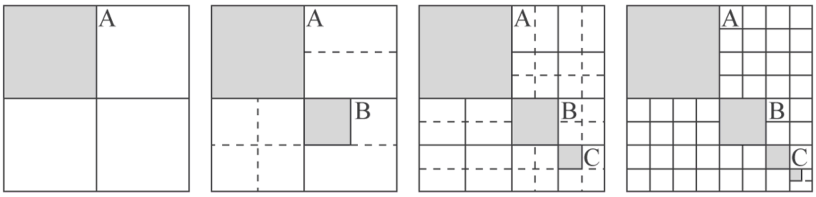

Diagram 1 Diagram 2 Diagram 3 Diagram 4

More information

The image comprises four separate diagrams, each illustrating a process of subdividing a square into smaller regions.

Diagram 1: - The square is split into four quadrants. - The top left quadrant is shaded and labeled 'A'.

Diagram 2: - The square is further subdivided. - The top left region remains 'A'. - The bottom right quadrant is divided again, creating a new smaller shaded region labeled 'B'.

Diagram 3: - Further division occurs, maintaining region 'A'. - Region 'B' is subdivided again, creating additional lines and shading. - A new region is labeled 'C'.

Diagram 4: - Maximum division, showing 'A', 'B', and 'C', along with smaller regions created by continuous division.

These diagrams demonstrate the subdivisions forming an infinite series of smaller regions, illustrating a geometric sequence.

[Generated by AI]

There are a variety of shapes that can be subdivided to produce an infinite number of smaller and smaller regions. The total areas of an infinite number of regions generated in this way are all examples of application of the sum of an infinite geometric sequence.

Example 1

★★☆

a) Draw a square with an area of 1 unit^2. Divide the square into two equal parts and shade one of them. Then take the unshaded part, divide that into two equal parts and shade one of them. Continue this process until you have six shaded areas.

b) Calculate the total area that is shaded in your drawing.

c) Suppose that the dividing process is continued indefinitely, creating an infinite number of shaded regions. Find the total shaded area.

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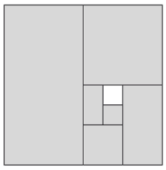
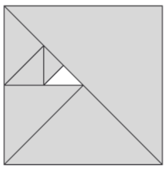
d) Explain what the total shaded area you found in part **c** represents in this context.

	Steps	Explanation
a)		Your picture should look similar to this one if you divided the square into two equal rectangles.
		You can also divide the square into equal triangles.
b)	$\text{Area} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{64}$ $S_6 = \frac{\frac{1}{2} \times \left(1 - \left(\frac{1}{2}\right)^6\right)}{1 - \frac{1}{2}} = 0.984$ $\text{Area} = 0.984 \text{ units}^2$ (3 significant figures)	Recognise the total shaded area as the sum of 6 terms of a geometric sequence with $r = \frac{1}{2}$.
c)	$S_\infty = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$	Recognise the total shaded area as the sum of an infinite geometric sequence. $-1 < r < 1$ so the sequence converges and you can use $S_\infty = \frac{u_1}{1 - r}$.

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	Steps	Explanation
d)	The total shaded area is $1\ \text{unit}^2$, which is equal to the area of the original square. This is because as you keep subdividing and shading, you shade in more and more of the original square. Doing this an infinite number of times eventually shades in the whole square.	

Example 2 shows another interesting application. Here you use the infinite sum to rationalise a repeating decimal.

Example 2

★★☆

Write $0.7777\ldots$ (which you may also know as $0.\dot{7}$ or $0.\bar{7}$) as a rational number.

Steps	Explanation
$0.7777\ldots = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots$ The RHS is an infinite geometric sum with $$r = \frac{1}{10}$$	A repeating decimal can be written as an infinite sum. Since $\lvert r \rvert < 1$ you can use $S_\infty = \frac{u_1}{1-r}$.
$S_\infty = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{10} \times \frac{10}{9} = \frac{7}{9}$	
So $0.7777\ldots = \frac{7}{9}$	Do not forget to answer the original question. The sum is just a tool for rewriting the repeating decimal as a rational number.

You can also use infinite sums to model the motion of a bouncing ball as in **Example 3**.

Example 3

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Student view

A ball is dropped from a height of 3 m. On each bounce it rebounds to 70% of the previous height.

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a) Calculate the distance covered by the ball on its way up from the first three times that it hits the ground.

b) Supposing the ball continues to bounce indefinitely, find the total distance covered by the ball when it is moving up.

	Steps	Explanation
a)	$S_3 = \frac{2.1\left(1-0.7^3\right)}{1-0.7}$ $= 4.60$ (3 significant figures)	Recognise that the (upward) rebound distances form a geometric sequence. After 1st bounce: $\text{distance} = 3 \times 0.7 = 2.1$ After 2nd bounce: $\text{distance} = 2.1 \times 0.7 = 1.47 = 3 \times 0.7^2$ After 3rd bounce: $\text{distance} = 1.47 \times 0.7 = 1.029 = 3 \times 0.7^3$
	The distance covered is 4.60 m.	
b)	$S_\infty = \frac{2.1}{1-0.7} = 7$	You can model the rebound heights of the ball with an infinite geometric sequence where the terms get negligibly small. $r = 0.7$ so the sequence converges.
	The total distance covered is 7 m.	

3 section questions ⌄

1. Number and algebra / 1.8 Sum of infinite geometric sequences

Checklist

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What you should know

By the end of this subtopic you should be able to:

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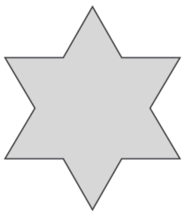
- explain why a geometric sequence converges only if $|r| < 1$
- identify convergent geometric sequences
- use $S_\infty = \frac{u_1}{1-r}$ to find sums of convergent geometric sequences
- identify application questions in which you need to find a sum of an infinite geometric sequence
- solve application questions involving infinite sums of geometric sequences.

1. Number and algebra / 1.8 Sum of infinite geometric sequences

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-27686/print/) Assign

A Koch snowflake is drawn by starting with an equilateral triangle with side length 1. Each side is then divided into three equal segments, and an equilateral triangle with side length $\frac{1}{3}$ is drawn so that its base coincides with the middle segment and the opposite vertex lies outside the original triangle. Then the middle segment of each side of the original triangle is erased to create a 12-sided polygon as seen in the diagram below.



More information

The image depicts a step in the creation of a Koch snowflake, specifically showing a grey six-pointed star shape against a black background. This shape is formed by modifying an equilateral triangle. Initially drawn as a 12-sided polygon, each side is segmented into three equal parts, and a smaller equilateral triangle is added to the middle segment of each side. This process creates an intricate snowflake pattern that looks like a six-pointed star, symbolizing the first iteration of Koch's fractal snowflake formula.

[Generated by AI]

Student view

This process of dividing each edge of a polygon into three equal segments and building a triangle on the middle segment can be continued indefinitely to generate more elaborate snowflakes, as you can see in the applet below, which shows the first few iterations.

Interactive 1. Koch Snowflake.

Credit: GeoGebra (https://www.geogebra.org/m/J6PHUJWs) Len Brin

More information for interactive 1

This applet visually demonstrates the iterative construction of the Koch snowflake, a well-known fractal curve. The user can adjust the slider to change the iteration level (n) from 0 to 5, revealing how the shape evolves with each step. Initially, at $n = 0$, the figure is a simple equilateral triangle with blue lines and purple vertices. With each successive iteration, each side is divided into three equal segments, and a smaller equilateral triangle is added to the middle segment, creating a progressively more intricate, self-similar pattern. The purple points highlight key vertices, while the blue boundary represents the evolving perimeter. This interactive helps learners visualize recursive geometric constructions and explore the fascinating properties of fractals.

Explore what happens to the perimeter and the area of the snowflake as the process is continued indefinitely.

What do your results tell you about how much ink would be needed to colour the inside of the snowflake if you were to continue adding triangles an infinite number of times? What about the amount of ink needed to draw the boundaries of the snowflake?

Comment on these results.

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