# Products of vectors





# The big picture

You first learned about the four operations of numbers – addition, subtraction, division and multiplication – when you were in primary school. But when you worked with complex numbers, in subtopic 1.12 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27421/) and subtopic 1.13 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27568/) the meaning of these operations changed.

Addition, subtraction, multiplication and division of complex numbers were described in geometrical terms by means of rotations and dilations. You learned that a complex number has real and imaginary parts which are often treated separately and have different physical meaning.

### Making connections

In subtopic 1.13 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27568/), multiplication of two complex numbers

$$z_1 = r_1 \mathrm{e}^{\mathrm{i}\theta_1} \text{ and } z_2 = r_2 \mathrm{e}^{\mathrm{i}\theta_2}$$

was written as

$$z_1 \times z_1 = r_1 r_2 \mathrm{e}^{\mathrm{i}(\theta_1 + \theta_2)}$$

This was described geometrically as a rotation of $z_1$ by $\theta_2$ radians and a stretch by scale factor $r_2$, as seen in the figure below.

Overview
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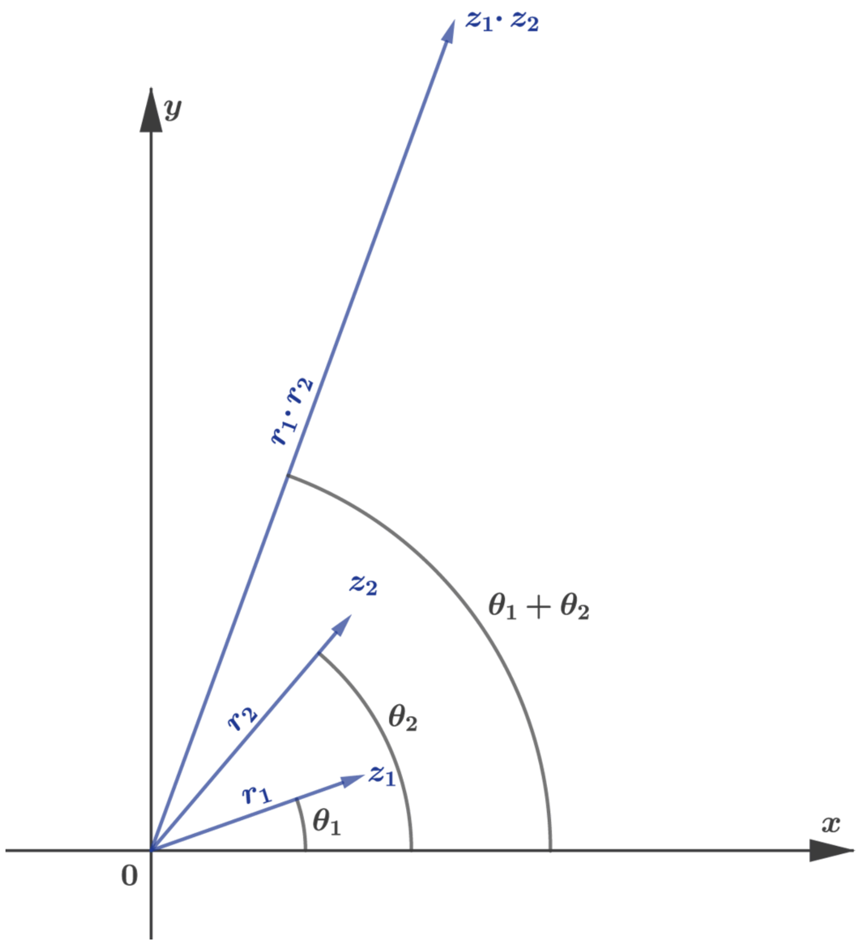


More information

The image is a geometric diagram showcasing the multiplication of complex numbers on a coordinate plane with an x-axis and y-axis. Two complex numbers, represented as vectors, are multiplied: (z_1) and (z_2). The vector (z_1) starts from the origin (0,0) and extends at an angle (\theta_1) with a magnitude (r_1), labeled accordingly. The vector (z_2) also originates from the origin, extending at a different angle (\theta_2) with magnitude (r_2). The result of the multiplication, (z_1 \cdot z_2), is displayed as a vector with a cumulative angle of (\theta_1 + \theta_2) and magnitude (r_1 \cdot r_2), demonstrating the geometric rotation and scaling involved in the operation.

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Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-28348/print/) Assign

Similar to complex numbers, operations involving vectors have a different meaning compared with operations involving numbers (scalars).

In subtopic 3.12 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-28426/) you learned how to add and subtract vectors by considering the two perpendicular components separately and you considered the geometrical meaning of these operations.

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You also learned how to multiply a vector by a scalar and the geometrical interpretation of this.

In this subtopic you will learn that multiplication of vectors has a different meaning to multiplication of scalars.

There are two types of multiplication involving vectors: scalar and vector multiplication. You will learn the definition and geometrical interpretation of both of these.

## Concept

Points in space can be defined using vectors. Vectors can show interactions in space. If an object is acted on by electric, magnetic or gravitational fields, then vectors can be used to predict the effect on an object at a point in space. These forces may speed up or slow down a moving object depending on the direction in which they act. The strength of the interaction will also depend on the magnitudes of the forces.

Scalar multiplication represents directional multiplication. For example, it can show the combined effect of two forces. If two forces are at right angles, how will they interact? If they act in exactly in the same direction how would you calculate the total effect? And what would you do if they are neither perpendicular nor parallel but have components in the same direction?

The vector product can be used to find the area of a parallelogram whose sides are two vectors in a plane. In three dimensions, it can be used along with the scalar product to find the area of a parallelepiped, which is a six-sided solid whose faces are parallelograms. The vector product of two three-dimensional vectors represents a vector that is perpendicular to both of these vectors. Is it possible to generalise the vector product of $n$-dimensional vectors and, if so, what would this represent?

## Theory of Knowledge

Scalar product, sine and cosine are often used to discover unknown knowledge. When you think about it, it is pretty amazing that you can use mathematics to create new unknown knowledge that is also valid!

Student view

Consider the other areas of knowledge. Do they operate in a similar way? Can you identify any other areas of knowledge that are self-perpetuating in regard to knowledge construction in the same way that mathematics is? For example, could a historian use history to create 'new' knowledge? What about an artist?



# The scalar product



When you learned about addition and subtraction of vectors, you saw the geometric interpretation and also an algebraic approach using components. You saw that the sum and difference can be found by adding and subtracting the corresponding components. It would be a natural approach to define the product of vectors similarly. However, as it turns out, this is not a useful definition. Instead, mathematicians agreed on two different ways of defining products of vectors. Both of these are useful in certain applications.

- In one of the definitions the product of two vectors is a number. For this reason, this is called the **scalar product**. The notation used for this product is a dot between the two vectors, $\boldsymbol{v} \cdot \boldsymbol{w}$. This product is also called dot product, this is what you will learn about in this section.
- In the second definition the product of two vectors is a vector. For this reason, this is called the **vector product**. The notation used for this product is a cross between the two vectors, $\boldsymbol{v} \times \boldsymbol{w}$. This product is also called cross product, you will learn about this later.

As with addition and subtraction, the scalar product can also be definied algebraically and geometrically. The formula booklet contains both definitions.

### Important

If $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ then the scalar or dot product of these two vectors can be defined the following two ways.

- $\boldsymbol{v} \cdot \boldsymbol{w} = |\boldsymbol{v}||\boldsymbol{w}| \cos\theta$, where $\theta$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

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- $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

It can be shown that these two definitions are equivalent, but this proof is not presented here. You can find the connection and a motivation behind these definitions in the investigation in section 3.13.6 (/study/app/math-ai-hl/sid-132-cid-761618/book/investigation-id-28368/).

To get used to the concept, let's use the second definition to find the scalar product of two given vectors.

# Example 1

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Find the scalar product of $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

Use the second definition of scalar product.

$$\boldsymbol{v} \cdot \boldsymbol{w} = 1 \cdot 2 + (-1)(-1) + 0 \cdot 2 = 3$$

# Example 2

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Find the scalar product of $\boldsymbol{v} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$. Hence comment on the relationship between the two vectors.

Use the second definition of scalar product.

$$\boldsymbol{v} \cdot \boldsymbol{w} = 2 \cdot 3 + 2 \cdot (-1) + (-1) \cdot 4 = 0$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0$$

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Let's compare this result with the first definition.

As the scalar product is 0, you know that $|\boldsymbol{v}||\boldsymbol{w}|\cos\theta = 0$. This can only happen if $\theta$, the angle between the vectors, is $90°$, so the vectors are perpendicular.

In Example 2 you discovered the following property of the scalar product.

✓ **Important**

Two vectors are perpendicular if and only if their scalar product is 0.

# Example 3

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The vectors $\boldsymbol{v} = \begin{pmatrix} a \\ 1 \\ 3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ are perpendicular.

Find the value of $a$.

If two vectors are perpendicular

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0$$

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{w} &= a \cdot 1 + 1(-1) + 3 \cdot 1 = 0 \\ a + 2 &= 0 \\ a &= -2 \end{aligned}$$

# Geometric property of the scalar product

You saw above that the scalar product of perpendicular vectors is 0. What happens if the scalar product is not 0? Let's rearrange the first definition.

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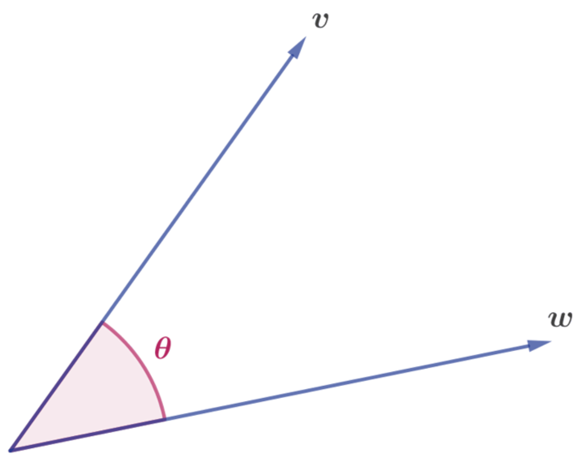
$$\boldsymbol{v} \cdot \boldsymbol{w} = |\boldsymbol{v}||\boldsymbol{w}| \cos\theta$$
$$\cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{|\boldsymbol{v}||\boldsymbol{w}|}$$

If you replace the scalar product with the expression in the second definition, you get the formula that helps you find the angle between two vectors. You can find this formula in the formula booklet.

## Important

If $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ then the angle between the two vectors can be found using the following formula.

$$\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{|\boldsymbol{v}||\boldsymbol{w}|}$$




More information

The image is a diagram depicting two vectors, (\boldsymbol{v}) and (\boldsymbol{w}), originating from a common point. The vectors form an angle labeled (\theta) with the point of origin. The vector (\boldsymbol{v}) is shown pointing towards the top right, and (\boldsymbol{w}) is directed towards the bottom right. The angle (\theta) is highlighted in pink, indicating the space between the two vectors. This visual represents the geometric interpretation of the dot product given by the formula provided: (\cos \theta = \dfrac{v_1w_1+v_2w_2+v_3w_3}{|\boldsymbol{v}||\boldsymbol{w}|}).

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Let's look at some examples.

# Example 4

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Find the exact value of the cosine of the angle between the vectors $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

| Steps | Explanation |
|---|---|
| $\boldsymbol{v} \cdot \boldsymbol{w} = 1 \cdot 1 + 1(-1) + 1 \cdot 2 = 2$ | Use the scalar product $$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3$$ |
| $\lvert\boldsymbol{v}\rvert = \sqrt{3}$ <br> $\lvert\boldsymbol{w}\rvert = \sqrt{6}$ | Find the magnitude of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$. Use $\lvert\boldsymbol{u}\rvert = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}$ |
| $2 = \sqrt{3} \times \sqrt{6} \times \cos\theta$ | Use $\boldsymbol{v} \cdot \boldsymbol{w} = \lvert\boldsymbol{v}\rvert \lvert\boldsymbol{w}\rvert \cos\theta$ |
| $\cos\theta = \frac{2}{\sqrt{3} \times \sqrt{6}} = \frac{2}{3\sqrt{2}}$ | Rearrange and simplify. |
| Therefore, $$\cos\theta = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$$ | |

Student view

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# Example 5

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Consider the two vectors $\boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \boldsymbol{w} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$

Use the scalar product to show that they are parallel.

| Steps | Explanation |
|---|---|
| $\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\| \, \|\boldsymbol{w}\|$ <br> $$\boldsymbol{v} \cdot \boldsymbol{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = 1 \times 3 + 2 \times 6 + 3 \times 9 = 42.$$ <br> Also <br> $$\|\boldsymbol{v}\| = \left\| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$ <br> and <br> $$\|\boldsymbol{w}\| = \left\| \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right\| = \sqrt{3^2 + 6^2 + 9^2} = \sqrt{126}$$ <br> $\|\boldsymbol{v}\| \, \|\boldsymbol{w}\| = \sqrt{14}\sqrt{126} = 42$ <br> So $\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\| \, \|\boldsymbol{w}\|$ and therefore $\boldsymbol{v} \| \boldsymbol{w}$. | If $\boldsymbol{v} \| \boldsymbol{w}$, then $\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\| \, \|\boldsymbol{w}\|$ <br> since $\cos\theta = 1$ when $\theta = 0$. <br> However, since $3\boldsymbol{v} = \boldsymbol{w}$, this conclusion could ha been made without usin the scalar product as $\boldsymbol{w}$ a scalar multiple of $\boldsymbol{v}$. |

# Example 6

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Find the angle between the two vectors $\boldsymbol{a} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Student view

Give your answer in degrees correct to 3 significant figures.

Use the expression for $\theta$ in terms of the components:

$$\theta = \cos^{-1}\left[\frac{(-1)\times 2 + 3\times(-1) + 2\times 3}{\sqrt{(-1)^2+3^2+2^2}\sqrt{2^2+(-1)^2+3^2}}\right]$$
$$= \cos^{-1}\left[\frac{-2-3+6}{\sqrt{14}\sqrt{14}}\right]$$
$$= \cos^{-1}\left[\frac{1}{14}\right] \approx 85.9^\circ (3 \text{ significant figures})$$

## 6 section questions

3. Geometry and trigonometry / 3.13 Products of vectors

# The angle between two lines

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/the-angle-between-two-lines-id-28364/print/) Assign

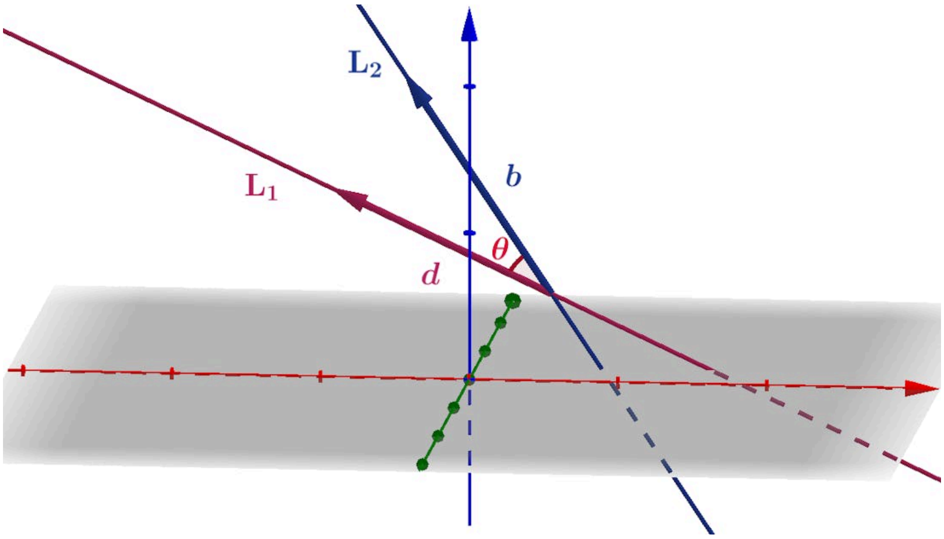
You can find the angle between two lines by using their direction vectors.

Consider the lines

$$\text{L}_1 :\ \boldsymbol{r}_1 = \boldsymbol{a} + \lambda\boldsymbol{b} \text{ and } \text{L}_2 :\ \boldsymbol{r}_2 = \boldsymbol{c} + t\boldsymbol{d}$$

The angle between the two lines is the same as the angle between their direction vectors $\boldsymbol{b}$ and $\boldsymbol{d}$.

Student view




More information

The image is a 3D diagram showing two intersecting lines labeled L1 and L2. The lines intersect at a point where the angle θ is formed. The angle between the two lines is represented by the angle between their direction vectors, labeled as b (for L2) and d (for L1). The line L1 is displayed in a reddish color, while L2 is blue. A flat plane is shown in the background with markings similar to a graph axis. Green dotted lines indicate a perpendicular connection from line L2 to the extension of line L1.

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Using the scalar product,

$$\boldsymbol{b} \cdot \boldsymbol{d} = |\boldsymbol{b}|\,|\boldsymbol{d}| \cos\theta$$

and rearranging gives

$$\cos\theta = \frac{\boldsymbol{b} \cdot \boldsymbol{d}}{|\boldsymbol{b}|\,|\boldsymbol{d}|}$$

So the angle $\theta$ between these lines is given by

$$\theta = \cos^{-1}\left[\frac{\boldsymbol{b} \cdot \boldsymbol{d}}{|\boldsymbol{b}|\,|\boldsymbol{d}|}\right]$$

You can see this relationship in the following interactive graph.

**Interactive 1.** Angle Between Two Lines.

More information for interactive 1

This interactive tool helps users understand and calculate the angle between two lines in 3D space using their direction vectors. It visually demonstrates how vector mathematics can determine the angular relationship between lines, offering a clear and intuitive graphical representation.

The 3D graph displays the Cartesian coordinate axes: **X** (red), **Y** (green), and **Z** (blue). Two distinct lines, $L_1$ (blue) and $L_2$ (magenta), are shown intersecting at a point in space. Each line is defined by its direction vectors: vectors **a** and **b** represent the direction of line $L_1$, while vectors **c** and **d** represent the direction of line $L_2$. These vectors are illustrated as green arrows extending from various points along the lines.

At the intersection of $L_1$ and $L_2$, an angle **θ** is formed, visually represented by a green arc. This arc illustrates the **acute angle between the direction vectors** of the two lines, which is calculated using the **dot product formula:**

$$\cos(\theta) = \frac{v_1 \cdot v_2}{|v_1||v_2|}$$

This visualization makes it easier to grasp the concept that the angle between two lines in space is the smallest angle between their direction vectors when positioned tail-to-tail.

Through this tool, learners can explore vector properties and better understand how the orientation of lines in 3D geometry can be measured and analyzed.

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# Example 1

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Find the angle between the lines $\mathrm{L}_1$ and $\mathrm{L}_2$ defined by

$$\mathrm{L}_1:\ \boldsymbol{r}_1=\begin{pmatrix}1\\-3\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\7\\-3\end{pmatrix}\text{ and }\mathrm{L}_2:\boldsymbol{r}_2=\begin{pmatrix}5\\1\\5\end{pmatrix}+t\begin{pmatrix}-9\\6\\-\frac{15}{2}\end{pmatrix}$$

Write the direction vectors of the lines:

$$\boldsymbol{v}=\begin{pmatrix}1\\7\\-3\end{pmatrix}\text{ and }\boldsymbol{w}=\begin{pmatrix}-9\\6\\-\frac{15}{2}\end{pmatrix}$$

Find the scalar product:

$$\begin{aligned}\boldsymbol{v}\cdot\boldsymbol{w}&=\begin{pmatrix}1\\7\\-3\end{pmatrix}\cdot\begin{pmatrix}-9\\6\\-\frac{15}{2}\end{pmatrix}\\&=1\times(-9)+7\times6+(-3)\times\left(-\frac{15}{2}\right)\\&=\frac{111}{2}\end{aligned}$$

Find the magnitudes of the direction vectors:

$$|\boldsymbol{v}|=\sqrt{1^2+7^2+(-3)^2}=\sqrt{59}$$

$$|\boldsymbol{w}|=\sqrt{(-9)^2+6^2+\left(-\frac{15}{2}\right)^2}=\frac{\sqrt{693}}{2}$$

Student view

Use $\cos\theta=\dfrac{\boldsymbol{b}\cdot\boldsymbol{d}}{|\boldsymbol{b}|\,|\boldsymbol{d}|}$:

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$$\theta = \cos^{-1}\left[\frac{\frac{111}{2}}{\sqrt{59}\times\frac{\sqrt{693}}{2}}\right]$$
$$= \cos^{-1}\left[\frac{111}{\sqrt{59}\times\sqrt{693}}\right]$$
$$= 56.7^\circ \text{ (in degrees, to 1 decimal place)}$$

# Example 2

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Find the acute angle between the lines

$$\mathrm{T}_1 : \boldsymbol{a} = \begin{pmatrix}0\\5\end{pmatrix} + \lambda\begin{pmatrix}-1\\4\end{pmatrix} \text{ and } \mathrm{T}_2 : \boldsymbol{b} = \begin{pmatrix}2\\-1\end{pmatrix} + t\begin{pmatrix}1\\-2\end{pmatrix}$$

Give your answer in degrees correct to 1 decimal place.

Write the direction vectors of the lines:

$$\boldsymbol{v} = \begin{pmatrix}-1\\4\end{pmatrix} \text{and } \boldsymbol{w} = \begin{pmatrix}1\\-2\end{pmatrix}$$

Find the scalar product:

$$\boldsymbol{v}\cdot\boldsymbol{w} = \begin{pmatrix}-1\\4\end{pmatrix}\cdot\begin{pmatrix}1\\-2\end{pmatrix} = (-1)\times 1 + 4\times(-2) = -9$$

Find the magnitudes of the direction vectors:

$$|\boldsymbol{v}| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$$

$$|\boldsymbol{w}| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

So the angle between these vectors is $0$.

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Use $\cos\theta = \dfrac{\boldsymbol{b}\cdot\boldsymbol{d}}{|\boldsymbol{b}|\,|\boldsymbol{d}|}$:

$$\begin{aligned}\theta &= \cos^{-1}\left[\frac{-9}{\sqrt{17}\times\sqrt{5}}\right]\\ &= \cos^{-1}\left[\frac{-9}{\sqrt{85}}\right]\\ &\approx 167.47^\circ\end{aligned}$$

Subtract the obtuse angle from $180^\circ$ to find the acute angle between the lines:

As this angle is obtuse, the acute angle between the lines $\mathrm{T}_1$ and $\mathrm{T}_2$ is $180^\circ - 167.47^\circ = 12.5^\circ$ (in degrees to 1 decimal place).

# Example 3

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Find the angle between the lines given by $\mathrm{L}_1: \dfrac{x+1}{4} = 2-y = z$ and $\mathrm{L}_2: \dfrac{x-2}{3} = y+1 = 2-z$.

Write the direction vectors of the lines:

Line $\mathrm{L}_1$ is parallel to the vector $\begin{pmatrix}4\\-1\\1\end{pmatrix}$ and l ine $\mathrm{L}_2$ is parallel to the vector $\begin{pmatrix}3\\1\\-1\end{pmatrix}$.
You must take care to write $2-z = \dfrac{z-2}{-1}$ so that it is in the form $\dfrac{z-z_0}{n}$.

Use $\cos\theta = \dfrac{\boldsymbol{b}\cdot\boldsymbol{d}}{|\boldsymbol{b}|\,|\boldsymbol{d}|}$:

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$$\theta = \cos^{-1}\left[\frac{\begin{pmatrix}4\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-1\end{pmatrix}}{\left|\begin{pmatrix}4\\-1\\1\end{pmatrix}\right|\left|\begin{pmatrix}3\\1\\-1\end{pmatrix}\right|}\right]$$

$$= \cos^{-1}\left[\frac{4\times 3 - 1\times 1 + 1\times(-1)}{\sqrt{4^2+(-1)^2+1^2}\sqrt{3^2+1^2+(-1)^2}}\right]$$

$$= 44.7^\circ \text{ (1 decimal place)}$$

**Be aware**

Although you may find the angle between the lines to be obtuse, you must give your answer as an acute angle. So if the obtuse angle is $\theta$ then the angle between the two lines is $180^\circ - \theta$, or $\pi - \theta$ radians.

## 4 section questions

3. Geometry and trigonometry / 3.13 Products of vectors

# The vector product

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/the-vector-product-id-28365/print/) Assign

## Vectors with a bit of carpentry

The vector product (which is also known as cross product) represents the vector that is the result of interactions between the components of two vectors in different dimensions, $x$, $y$ and $z$.

When would you need the result of interactions in different dimensions?

Consider a screwdriver. If you turn it in one direction the screw will be tightened (first diagram below) and if you turn it in the other direction the screw will be loosened (second diagram below). You do not need to push or pull the screw as you do when using a hammer

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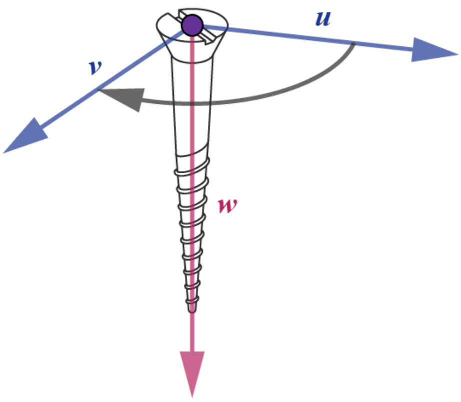
to drive in a nail.




Turning in the direction
from $\boldsymbol{u}$ to $\boldsymbol{v}$ or $\boldsymbol{u} \times \boldsymbol{v}$

More information

The diagram illustrates the concept of a screw being turned in different directions using vectors. The screw is depicted vertically, with its head at the top. Arrows indicate the three vectors: u, v, and w. The arrow u points to the right, v points downward and to the left, and w points directly down along the axis of the screw. The text below the image reads: 'Turning in the direction from u to v or u × v.' This suggests the directions in which the screw can be rotated to either tighten or loosen it. The diagram visually represents how these vectors interact as part of a rotational motion.

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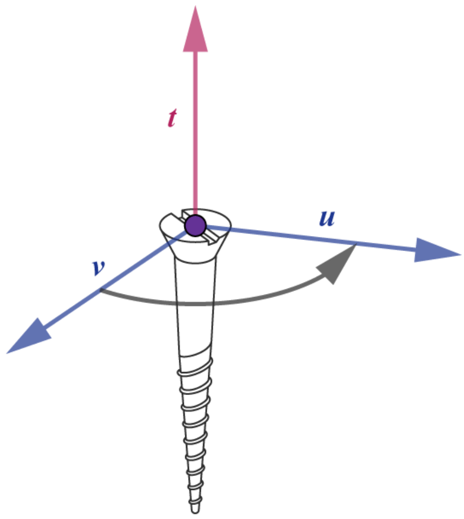


Turning in the direction
from $\boldsymbol{v}$ to $\boldsymbol{u}$ or $\boldsymbol{v} \times \boldsymbol{u}$

More information

The image illustrates a 3D right-handed screw model representing vector relationships. The screw is depicted with three vectors labeled as (\boldsymbol{u}), (\boldsymbol{v}), and (\boldsymbol{t}). Vectors (\boldsymbol{u}) and (\boldsymbol{v}) lie on the horizontal plane, forming an angle with each other. Vector (\boldsymbol{t}), perpendicular to both (\boldsymbol{u}) and (\boldsymbol{v}), points upwards along the axis of the screw. The image demonstrates that the cross product of (\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{t}) and (\boldsymbol{v} \times \boldsymbol{u} = -\boldsymbol{t}). The text around the screw further explains the orthogonal nature of these vectors, illustrating the vector product's directional result via the screw's orientation.

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The resulting vectors, $\boldsymbol{w}$ or $\boldsymbol{t}$, are perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$. These are called orthogonal vectors. These relationships could be represented by the vector product as $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w}$ and $\boldsymbol{v} \times \boldsymbol{u} = \boldsymbol{t}$.

What is the connection between $\boldsymbol{w}$ and $\boldsymbol{t}$?

The direction in which you turn the screwdriver affects the direction of motion of the screw, so $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w}$ but $\boldsymbol{v} \times \boldsymbol{u} = \boldsymbol{t} = -\boldsymbol{w}$

Student
view

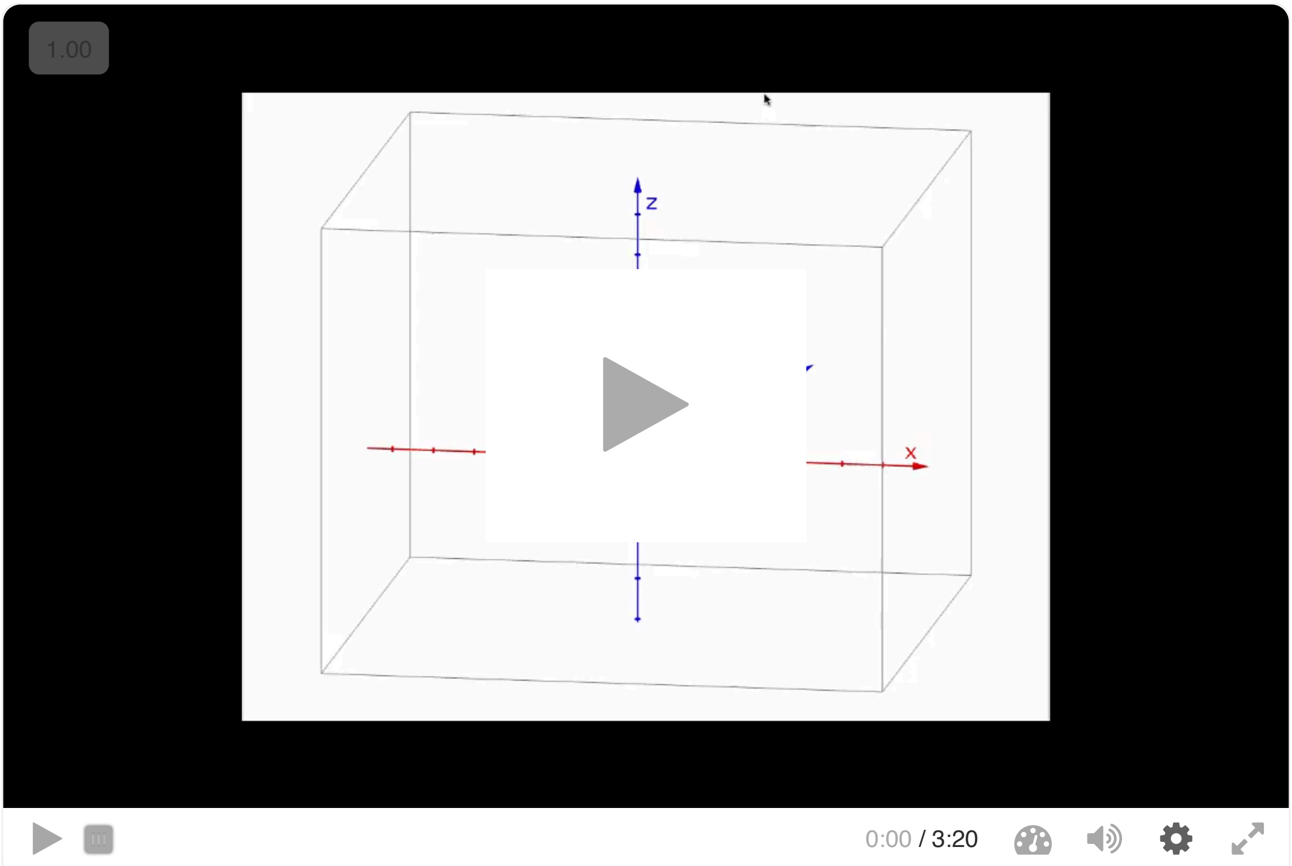
Therefore, the vector product is not commutative.

## Be aware

As the vector product is also called the cross product, the notation used to denote it is $\boldsymbol{a} \times \boldsymbol{b}$.

Recall that the dot or scalar product is denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$.

The following video shows how the vector product is represented in 3D.




**Video 1.** Visualizing Vector Product.

More information for video 1

1

00:00:00,333 --> 00:00:02,067

narrator: So far we've taken two vectors

2

00:00:02,133 --> 00:00:05,400

and asked our ourself

what the scalar product was of those.

3

00:00:06,133 --> 00:00:09,300

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And now we're gonna investigate

the other properties

4

00:00:09,367 --> 00:00:13,133

associated with vectors

and that is the vector product.

5

00:00:13,333 --> 00:00:15,600

Remember, the scalar product

is called the dot product

6

00:00:15,667 --> 00:00:18,767

and this one will be appropriately

called the cross product.

7

00:00:19,633 --> 00:00:22,467

Now here I've taken

the cross product of u and v,

8

00:00:22,533 --> 00:00:24,667

which are two vectors

in three dimensional space

9

00:00:24,867 --> 00:00:28,300

and there you can see the purple

vector is the cross product

10

00:00:28,500 --> 00:00:30,767

of u and v.

11

00:00:30,933 --> 00:00:32,733

Now you can see that it is another vector,

12

00:00:32,800 --> 00:00:35,267

unlike the scalar product,

which of course was a scale.

13

00:00:35,333 --> 00:00:37,333

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So the vector product is a vector.

14

00:00:37,400 --> 00:00:41,600

Now if I plot a plane through

u and v, which I've called π,

15

00:00:41,900 --> 00:00:43,933

then we're going to see

something interesting

16

00:00:44,000 --> 00:00:47,933

about the relative orientation

of u × v,

17

00:00:48,000 --> 00:00:52,667

the vector product with u and v relative

to that plane that goes through u and v.

18

00:00:52,733 --> 00:00:55,933

And you can see here that

it seems to be 90 degrees.

19

00:00:56,000 --> 00:00:59,367

And indeed if I do measure

that angle between the plane

20

00:00:59,733 --> 00:01:02,700

and the cross product of u and v,

21

00:01:02,767 --> 00:01:06,133

then it is indeed 90 degrees.

22

00:01:06,600 --> 00:01:09,267

So that is a property that

we need to keep in mind

23

00:01:09,333 --> 00:01:13,667

that the cross product of two vectors lies

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perpendicular to the plane

24

00:01:13,733 --> 00:01:17,500

through those two vectors

or the plane containing those two vectors.

25

00:01:17,567 --> 00:01:19,567

Alright,

so that's something to keep in mind.

26

00:01:20,233 --> 00:01:22,633

Now here of course I've taken

the cross product

27

00:01:22,700 --> 00:01:25,933

between $u$ and $v$, that is $u \times v$,

28

00:01:26,533 --> 00:01:30,000

and now we're gonna take

the cross product of $v$ and $u$

29

00:01:30,067 --> 00:01:33,333

and you can see that

those are not the same objects.

30

00:01:33,400 --> 00:01:36,767

So as $2 \times 5$ is the same as $5\times$

2 with numbers,

31

00:01:36,833 --> 00:01:41,900

that does not appear to be the case

with the vector product.

32

00:01:41,967 --> 00:01:43,700

However you do see,

33

00:01:43,767 --> 00:01:46,833

I hope that there is a special

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relationship between them.

34

00:01:46,900 --> 00:01:48,300

They lie in a line

35

00:01:48,367 --> 00:01:52,567

and indeed therefore v cross

with u is also perpendicular

36

00:01:52,633 --> 00:01:54,400

to the plane, including v and u.

37

00:01:55,033 --> 00:01:58,267

And v × u is anti-parallel

38

00:01:58,333 --> 00:02:03,400

to u × v as we can clearly

see in these cases.

39

00:02:03,500 --> 00:02:06,900

So in other words, the order in which you

40

00:02:07,300 --> 00:02:11,200

apply the cross product

in which you perform the cross product

41

00:02:11,267 --> 00:02:16,667

does make a difference in terms

of vector product multiplication.

42

00:02:16,733 --> 00:02:19,100

Alright, so this is the other

multiplication

43

00:02:19,167 --> 00:02:20,433

that we can do with two vectors.

44

00:02:20,500 --> 00:02:23,100

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We have the cross product,

this one and a dot product.

45

00:02:23,667 --> 00:02:27,367

Now here is a two dimensional case

of a vector u and v,

46

00:02:27,867 --> 00:02:31,767

and you can now start

to wonder that where is u × v?

47

00:02:31,833 --> 00:02:33,533

I've just asked

this software to create it,

48

00:02:33,600 --> 00:02:35,000

but I cannot see it.

49

00:02:35,200 --> 00:02:38,267

But of course it's obvious that you cannot

see it in a two dimensional case

50

00:02:38,333 --> 00:02:41,600

because the cross product

lies perpendicular

51

00:02:41,733 --> 00:02:43,733

to the plane involving u and v.

52

00:02:43,800 --> 00:02:48,633

So if I project u and v in the xy plane,

53

00:02:48,700 --> 00:02:52,633

then the u × v

is going to be perpendicular,

54

00:02:52,700 --> 00:02:53,800

which is in the z direction,

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55

00:02:53,867 --> 00:02:57,000

which of course I need a three

dimensional plot as we can clearly

56

00:02:57,433 --> 00:02:58,967

see over here.

57

00:02:59,400 --> 00:03:01,733

So in other words, those factors, u and v

00:03:02,867 --> 00:03:05,633

do not have a component

in the z direction.

59

00:03:05,733 --> 00:03:07,200

I can still plot them of course,

60

00:03:07,300 --> 00:03:09,433

in three dimensional space,

kind of boring.

61

00:03:09,733 --> 00:03:11,900

so if I take the cross product,

62

00:03:11,967 --> 00:03:15,900

I need to look at that geometrical object

63

00:03:15,967 --> 00:03:19,300

in three dimensional space

to see that product.

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view

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# Finding the components of the vector product

Consider the two vectors $\boldsymbol{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$, and their vector product $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$.

How would you find the components of the vector $\boldsymbol{w}$ ?

Look at this multiplication table:

| **Multiplication** | $v_x$ | $v_y$ | $v_z$ |
|---|---|---|---|
| $u_x$ | $u_x v_x$ | $u_x v_y$ | $u_x v_z$ |
| $u_y$ | $u_y v_x$ | $u_y v_y$ | $u_y v_z$ |
| $u_z$ | $u_z v_x$ | $u_z v_y$ | $u_z v_z$ |

The entries in the diagonal represent multiplication of the components in the same direction so $\boldsymbol{u} \cdot \boldsymbol{v} = u_x v_x + u_y v_y + u_z v_z$. The other entries represent multiplication in different directions.

Therefore the components of the vector product $\boldsymbol{w}$ are:

$$w_x = u_y v_z - u_z v_y$$

$$w_y = u_z v_x - u_x v_z$$

$$w_z = u_x v_y - u_y v_x$$

To see a more formal proof of this result see solution below .

Write the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in terms of the unit vectors $\boldsymbol{i}$ , $\boldsymbol{j}$ and $\boldsymbol{k}$ .

So $\boldsymbol{u} = u_x \boldsymbol{i} + u_y \boldsymbol{j} + u_z \boldsymbol{z}$ and $\boldsymbol{v} = v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{z}$.

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The vector product is:

$$\begin{aligned}\boldsymbol{u}\times\boldsymbol{v} &= (u_x\boldsymbol{i}+u_y\boldsymbol{j}+u_z\boldsymbol{k})\times(v_x\boldsymbol{i}+v_y\boldsymbol{j}+v_z\boldsymbol{k})\\ &= u_x\boldsymbol{i}\times(v_x\boldsymbol{i}+v_y\boldsymbol{j}+v_z\boldsymbol{k})+u_y\boldsymbol{j}\times(v_x\boldsymbol{i}+v_y\boldsymbol{j}+v_z\boldsymbol{k})+u_z z\times(v_x\boldsymbol{i}+\\ &= u_x\boldsymbol{i}\times v_x\boldsymbol{i}+u_x\boldsymbol{i}\times v_y\boldsymbol{j}+u_x\boldsymbol{i}\times v_z\boldsymbol{k}+u_y\boldsymbol{j}\times v_x\boldsymbol{i}+u_y\boldsymbol{j}\times v_y\boldsymbol{j}+u_y\boldsymbol{j}\\ &\quad u_z\boldsymbol{k}\times v_x\boldsymbol{i}+u_z\boldsymbol{k}\times v_y\boldsymbol{j}+u_z\boldsymbol{k}\times v_z\boldsymbol{k}\\ &= u_xv_x\boldsymbol{i}\times\boldsymbol{i}+u_xv_y\boldsymbol{i}\times\boldsymbol{j}+u_xv_z\boldsymbol{i}\times\boldsymbol{k}+u_yv_x\boldsymbol{j}\times\boldsymbol{i}+u_yv_y\boldsymbol{j}\times\boldsymbol{j}+u_yv_z\\ &\quad u_zv_x\boldsymbol{k}\times\boldsymbol{i}+u_zv_y\boldsymbol{k}\times\boldsymbol{j}+u_zv_z\boldsymbol{k}\times\boldsymbol{k}\end{aligned}$$

But this can be simplified.

When finding the components of the vector product you need to find the cumulative differences between components which are not in the same direction.

$\boldsymbol{i}\times\boldsymbol{i}=0$, $\boldsymbol{j}\times\boldsymbol{j}=0$ and $\boldsymbol{k}\times\boldsymbol{k}=0$ as each pair of vectors are parallel.

Because the vector product gives a vector that is perpendicular to the original vectors,

$\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k}$, $\boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i}$ and $\boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}$

Because the vector product is not commutative,

$\boldsymbol{j}\times\boldsymbol{i}=-\boldsymbol{k}$, $\boldsymbol{k}\times\boldsymbol{j}=-\boldsymbol{i}$ and $\boldsymbol{i}\times\boldsymbol{k}=-\boldsymbol{j}$

So the expression above simplifies to:

$$\begin{aligned}\boldsymbol{u}\times\boldsymbol{v} &= u_xv_y\boldsymbol{k}-u_xv_z\boldsymbol{j}-u_yv_x\boldsymbol{k}+u_yv_z\boldsymbol{i}+u_zv_x\boldsymbol{j}-u_zv_y\boldsymbol{i}\\ &= u_yv_z\boldsymbol{i}-u_zv_y\boldsymbol{i}+u_zv_x\boldsymbol{j}-u_xv_z\boldsymbol{j}+u_xv_y\boldsymbol{k}-u_yv_x\boldsymbol{k}\\ &= (u_yv_z-u_zv_y)\,\boldsymbol{i}+(u_zv_x-u_xv_z)\,\boldsymbol{j}+(u_xv_y-u_yv_x)\,\boldsymbol{k}\end{aligned}$$

### Be aware

You will not be asked to prove this result in your exam.

Student view

### Exam tip

In the IB formula booklet, the components of a vector product are given as

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}, \text{ where } \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ and } \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

You will not be asked to prove this formula.

How do you know the direction the cross product vector will act?

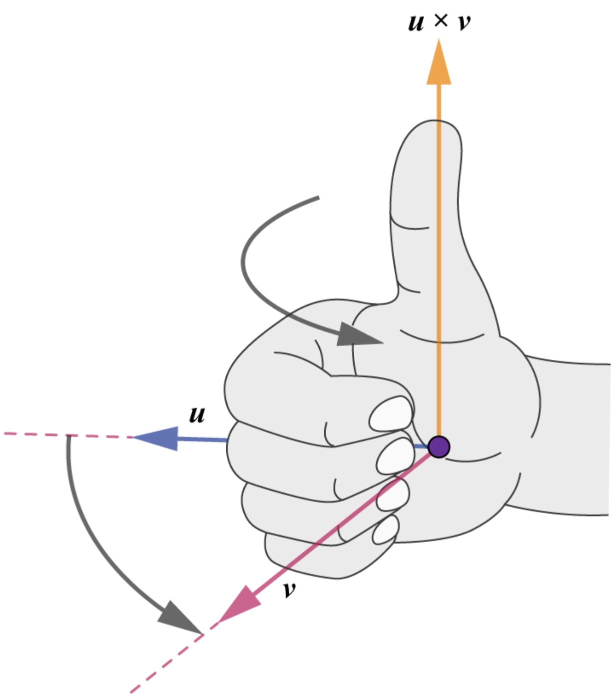
If the two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are in the plane of the page on the screen, the cross product will give a vector that is perpendicular to the plane. But how do you know whether the cross product vector will act into the screen or out of the screen?

You can use the right-hand rule to decide.

**R ight-hand rule**

To find the direction of the vector product $\boldsymbol{u} \times \boldsymbol{v}$ :

- hold your right hand so that your fingers point in the direction of vector $\boldsymbol{u}$
- then curl your fingers so that they point towards vector $\boldsymbol{v}$
- your thumb will now point in the direction of the vector product $\boldsymbol{u} \times \boldsymbol{v}$.



More information

This image illustrates the right-hand rule, a fundamental concept in physics used to determine the direction of magnetic force in relation to current and magnetic field. The diagram depicts a right hand with the thumb, index finger, and middle finger oriented perpendicularly. The thumb points upward with an orange arrow, labeled 'Force (F)', representing the direction of force. The index finger points to the left with a blue arrow, labeled 'Magnetic Field (B)', indicating the direction of the magnetic field. The middle finger points forward and is accompanied by a red arrow, labeled 'Current (I)', describing the direction of electrical current. A purple dot at the intersection of the thumb and fingers marks the origin of the force application. Circular arrows around the hand indicate the rotational direction of the forces. Labels on the arrows and the hand provide guidance for identifying the correct directional relationships between current, magnetic field, and force.

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## Example 1

★☆☆

Let $\boldsymbol{v} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. Find the vector $\boldsymbol{v} \times \boldsymbol{w}$.

$$\begin{aligned} \boldsymbol{v} \times \boldsymbol{w} &= \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 1 - 3 \times (-2) \\ 3 \times 2 - (-2) \times 1 \\ (-2) \times (-2) - 1 \times 2 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ 8 \\ 2 \end{pmatrix} \end{aligned}$$

## Activity

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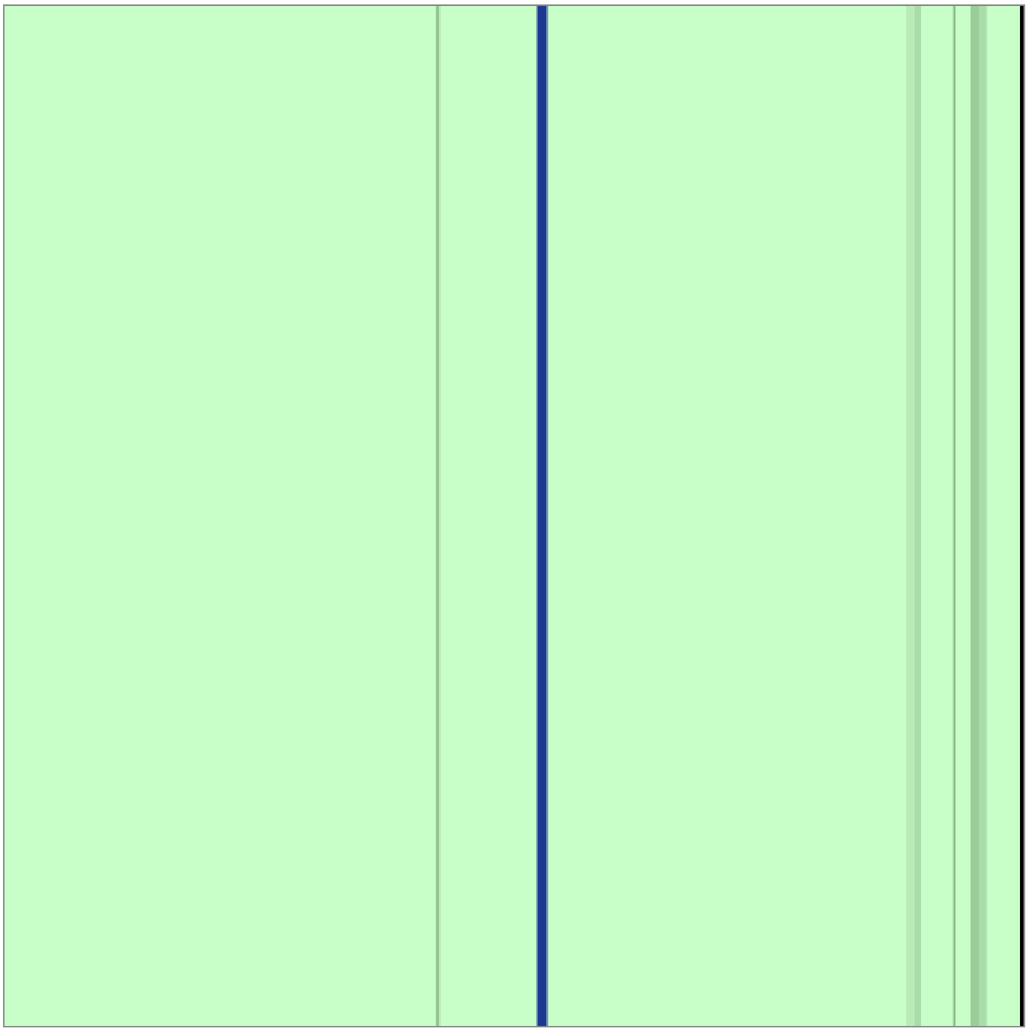
Use the applet below to explore how the cross product of two vectors changes as you change the vectors and the angle between them.You can drag the vectors and also the plane they are on to see the changes.



**Interactive 1.** Exploring the Change in Cross Product of Two Vectors.

Credit: GeoGebra (https://www.geogebra.org/m/RrDv9Wea) Tim Brzezinski

More information for interactive 1

This interactive enables users to explore the cross product of two vectors in 3D space, demonstrating how the resulting vector is perpendicular to both original vectors and how its magnitude relates to the area of the parallelogram they form. Users can manipulate the vectors and observe the dynamic changes in their cross product, gaining a visual understanding of this fundamental vector operation. The visualization helps bridge the gap between abstract mathematical concepts and concrete spatial relationships.

The display presents a 3D coordinate system with x, y, and z axes, where users can see three vectors: input vectors $\mathrm{u}$ and $\mathrm{v}$, and their cross product $\mathrm{w}$. The vectors are represented as colored arrows ($\mathrm{u}$ and $\mathrm{v}$ in distinct colors, $\mathrm{w}$ in a third color) with adjustable positions. Users can drag points A, B, and C to modify vectors $\mathrm{u}$ and $\mathrm{v}$, immediately seeing how these changes affect the resulting cross product vector $\mathrm{w}$. The angle between $\mathrm{u}$ and $\mathrm{v}$ is visibly displayed and updates in real-time, along with the numerical components of all three vectors shown in a panel for reference. For example, with $\mathrm{u} = (0.63, -1.26, -0.43)$, $\mathrm{v} = (1.82, 1.32, -0.46)$, and their cross product $\mathrm{w} = (1.15, -0.49, 3.13)$.

Through this interactive experience, users will develop a solid understanding of the geometric

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significance of the cross product, learning how it generates a vector orthogonal to the original pair and how its magnitude represents the area of the parallelogram formed by u and v. They'll discover the right-hand rule in action and gain intuition about how vector orientation affects the cross product's direction.

### Important

- The vector product of two vectors is a vector and so it has magnitude and direction.
- The vector product of two vectors is not commutative, i.e., $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{w} \times \boldsymbol{v}$.
- $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$ so these vectors are parallel but act in opposite directions.
- The vector product of two vectors, $v \times w$, is oriented perpendicular to the plane containing $\boldsymbol{v}$ and $\boldsymbol{w}$.

## Example 2

★★☆

Find a vector of length 3 that is perpendicular to both $-2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.

You know that the vector product between these two vectors is perpendicular to the plane containing the vectors. Hence the vector product is perpendicular to each of these vectors in turn. Note that you found the vector product of these vectors in the previous example, so let $\boldsymbol{v} = -2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{w} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.

Then

$$\boldsymbol{v} \times \boldsymbol{w} = 7\boldsymbol{i} + 8\boldsymbol{j} + 2\boldsymbol{k}$$

Clearly, this is not a vector of length 3 as $\sqrt{7^2 + 8^2 + 2^2} = \sqrt{117}$.

However, you can find a vector of length 1.

That is, a unit vector in the same direction:

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$$\frac{1}{\sqrt{7^2+8^2+2^2}}(7\boldsymbol{i}+8\boldsymbol{j}+2\boldsymbol{k})=\frac{1}{\sqrt{117}}(7i+8j+2k)$$

This is a vector of length 1 that is parallel to $\boldsymbol{v}\times\boldsymbol{w}$ and therefore it is perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$. Now you can make it the required length of 3 by multiplying by 3.

So the vector $\frac{3}{\sqrt{117}}(7\boldsymbol{i}+8\boldsymbol{j}+2\boldsymbol{k})$ is of length 3 and is perpendicular to both $-2\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k}$ and $2\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}$.

✓ **Important**

A unit vector that is perpendicular to vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is given by $\widehat{\boldsymbol{n}}=\frac{\boldsymbol{v}\times\boldsymbol{w}}{|\boldsymbol{v}\times\boldsymbol{w}|}$.

## Example 3

★☆☆

Consider the three points with coordinates $\mathrm{A}(1,1,1),\ \mathrm{B}(-1,2,1)$ and $\mathrm{C}(-1,3,1)$ relative to a fixed point O.

Find $\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{CB}}$

| Vector Operation | Explanation |
|---|---|
| $\overrightarrow{\mathrm{AB}}=\begin{pmatrix}-1-1\\2-1\\1-1\end{pmatrix}=\begin{pmatrix}-2\\1\\0\end{pmatrix}$ | As $\overrightarrow{\mathrm{AB}}=\vec{\mathrm{B}}-\vec{\mathrm{A}}=\begin{pmatrix}-1\\2\\1\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}$ |
| $\overrightarrow{\mathrm{CB}}=\vec{\mathrm{B}}-\vec{\mathrm{C}}=\begin{pmatrix}-1+1\\2-3\\1-1\end{pmatrix}=\begin{pmatrix}0\\-1\\0\end{pmatrix}$ | As $\overrightarrow{\mathrm{CB}}=\vec{\mathrm{B}}-\vec{\mathrm{C}}=\begin{pmatrix}-1\\2\\1\end{pmatrix}-\begin{pmatrix}-1\\3\\1\end{pmatrix}$ |
| $\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{CB}}=\begin{pmatrix}-2\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\-1\\0\end{pmatrix}$ | |

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| Vector Operation | Explanation |
|---|---|
| $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{CB}} = \begin{pmatrix} 1 \times 0 - 0 \times -1 \\ 0 \times 0 - (-2) \times 0 \\ (-2)(-1) - 1 \times 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ | Using $\boldsymbol{u} \times \boldsymbol{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$ |

### Be aware

In IB examinations, examiner reports often identify that candidates make numerical mistakes when finding direction vectors or the cross product of vectors. It is always good practice to write each step of the calculation clearly to avoid such mistakes, even if you are using a calculator.

### Making connections

You can use determinants to find the components of the cross product of two vectors.

Let $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$. Then

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{i}(v_2w_3 - v_3w_2) - \boldsymbol{j}(v_1w_3 - v_3w_1) + \boldsymbol{k}(v_1w_2 - v_2w_1)$$

Note the minus sign in front of the $\boldsymbol{j}$ component.

# Geometric interpretation of vector product

The cross product of two vectors, $\boldsymbol{a} \times \boldsymbol{b}$, gives a vector that is perpendicular to both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Vectors have both magnitude and direction, so what is the geometrical meaning of the magnitude of the cross product of two vectors?

Go back to the example of a screwdriver that you saw earlier. Consider the diagrams below. What could the magnitude of the cross product represent if you are loosening the screw?

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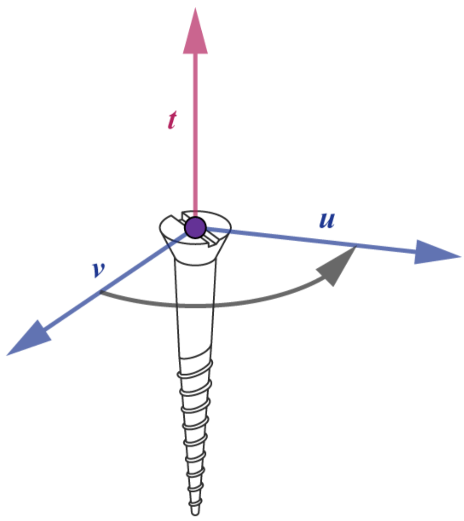


More information

The image is a diagram of a screw with labeled vectors. It shows a screw from a top-down perspective, where three vectors are illustrated: 'v' pointing downward to the left at an angle, 'u' pointing to the right, and 't' pointing straight upwards from the head of the screw. The diagram suggests the representational axes of rotation and possibly torque in context to loosening the screw, with the magnitude of the cross product potentially determining the rotational force. This could be a representation of the right-hand rule, used to understand the direction of angular momentum behaving around these vectors during the operation of turning.

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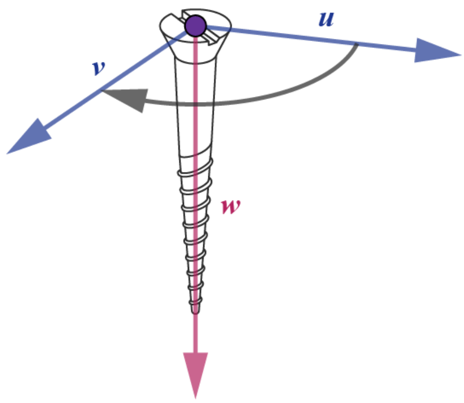



More information

Student view

The image is a 3D diagram showing three vectors, labeled (\mathbf{u}), (\mathbf{v}), and (\mathbf{w}). Vectors (\mathbf{u}) and (\mathbf{v}) lie on the plane and are perpendicular to each other, pointing horizontally. Vector (\mathbf{w}) is perpendicular to both (\mathbf{u}) and (\mathbf{v}), pointing downward, representing the cross product (\mathbf{a} \times \mathbf{b}). It is depicted as a series of segmented lines, resembling a spring or screw, conveying the screw's axis of rotation, which is orthogonal to the plane formed by (\mathbf{u}) and (\mathbf{v}). The labeling implies directionality and the spatial relationship between the vectors. The image illustrates orthogonal vectors in 3D space, emphasizing how (\mathbf{w}) represents the area vector defined by (\mathbf{a}) and (\mathbf{b}).

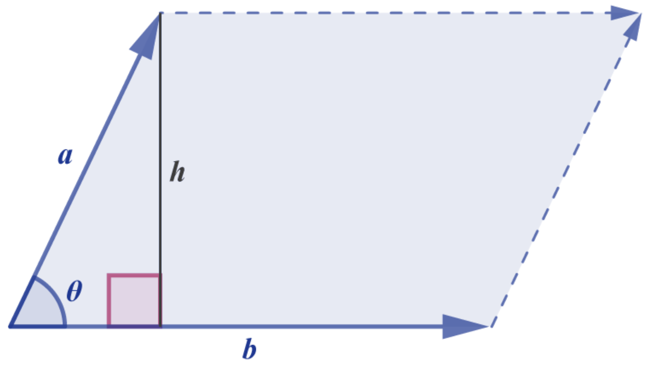
[Generated by AI]

In the figure above, the top diagram $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$. The bottom diagram represents the area covered by both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

You can define the magnitude of the cross product as the area covered by both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. In this context, it will tell you how high the screw will move up.

The diagram below shows that the area covered by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a parallelogram.

What is the formula for the area of a parallelogram?




More information

The image is a diagram showing a parallelogram. The diagram illustrates the sides labeled as 'a' and 'b' and the height 'h' perpendicular to the side 'b'. It features a right angle between the height 'h' and the base 'b', indicating the perpendicular distance. The diagram helps to visually explain how the area of the parallelogram is calculated using

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base 'b' and height 'h'.

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In this case, $area = |\boldsymbol{b}|\, h$ and $\sin\theta = \dfrac{h}{|a|} \Rightarrow h = |\boldsymbol{a}|\sin\theta$.

Combining both equations gives area $= |\boldsymbol{b}|\, |\boldsymbol{a}| \sin\theta$.

Therefore the area of the parallelogram formed by the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\text{area} = |\boldsymbol{b}|\, |\boldsymbol{a}| \sin\theta$$

The magnitude of the cross product $\boldsymbol{a} \times \boldsymbol{b}$ is defined as the area covered by the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, so

$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{b}|\, |\boldsymbol{a}| \sin\theta$, where $\theta$ is the angle between the two vectors.

Note that the direction of the cross product does not matter here as you are finding the magnitude, i.e. the magnitude will be the same in both directions, i.e. $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{b} \times \boldsymbol{a}|$.

How could you adapt the formula for the area of a parallelogram to get a formula for the area of a triangle?

You are likely to have met this formula already in your IB course. How is it usually written?

### Exam tip

The IB formula book gives the components of a vector product as

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}, \text{ where } \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ and } \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}|\, |\boldsymbol{w}| \sin\theta$, where $\theta$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

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## Exam tip

The IB formula book gives the area of a parallelogram as

$$A = |\boldsymbol{v} \times \boldsymbol{w}|$$

where $\boldsymbol{v}$ and $\boldsymbol{w}$ form two adjacent sides of a parallelogram.

## Example 4

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Find the exact value of the sine of the angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, if $|\boldsymbol{v}| = 2$, $|w| = 3$ and $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

| Steps | Explanation |
|---|---|
| $\lvert\boldsymbol{v} \times \boldsymbol{w}\rvert = \sqrt{1^2 + 2^2 + 3^2}$ | Use Pythagoras' theorem to find the magnitude. |
| $\lvert\boldsymbol{v} \times \boldsymbol{w}\rvert = \sqrt{14}$ | |
| $\lvert\boldsymbol{v} \times \boldsymbol{w}\rvert = (2)\,(3)\sin\theta$ | Use $\lvert\boldsymbol{v} \times \boldsymbol{w}\rvert = \lvert\boldsymbol{v}\rvert\,\lvert\boldsymbol{w}\rvert\sin\theta$. |
| $\sqrt{14} = 6\sin\theta$ | Simplify and solve for $\sin\theta$. |
| $\sin\theta = \frac{\sqrt{14}}{6}$ | |

## Important

Defining the cross product as $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}|\,|\boldsymbol{w}|\sin\theta$, where $\theta$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ produces some important results.

- If vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel, then $\boldsymbol{v} \times \boldsymbol{w} = 0$ since $\sin 0 = 0$.
- $\boldsymbol{v} \times \boldsymbol{v} = 0$
- If vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular, then $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}|\,|\boldsymbol{w}|$.

Student view

These results are not given in the formula booklet and you do not need to memorise them. You can deduce them from the formulae that you are given if necessary.

## Activity

You can explore the relationship between two vectors and their cross product using the applet below.

Drag the points and see how the cross product changes.

What is the cross product when the angle between two vectors is $0^\circ$ or $180^\circ$?

**Interactive 2.** Exploring the Relationship Between Two Vectors and Their Cross Products.

More information for interactive 2

This interactive allows users to explore the concept of the cross product between two vectors in three-dimensional space.

The screen displays a 3D coordinate system with a dotted grid and three red arrows representing

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vectors. Two of these vectors, labeled u and v, lie in the horizontal plane and can be adjusted by dragging their endpoints. A third vector labeled "Cross Product" points vertically from the origin, representing the result of the cross product between vectors u and v. As users move the endpoints of vectors u and v, they can observe how the direction and length of the cross product vector change dynamically.

The cross product vector always remains perpendicular to the plane formed by u and v, visually demonstrating a core property of the cross product. Additionally, users can see that when vectors u and v are either parallel or point in exactly opposite directions (an angle of $0^{\circ}$ or $180^{\circ}$), the cross product becomes a zero vector with no direction or magnitude.

This interactive effectively helps users understand both the geometric interpretation and algebraic behavior of the cross product in 3D space.

## Example 5

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Consider the quadrilateral $\mathrm{ABCD}$ with vertices $\mathrm{A}(2,0,4)$, $\mathrm{B}(5,1,1)$, $\mathrm{C}(-1,1,3)$ and $\mathrm{D}(-4,0,6)$. Show that the quadrilateral is a parallelogram and find its area.

$$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 5-2 \\ 1-0 \\ 1-4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$$

$$\overrightarrow{\mathrm{DC}} = \begin{pmatrix} -1-(-4) \\ 1-0 \\ 3-6 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$$

As $\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{DC}}$, it is implied that the quadrilateral $\mathrm{ABCD}$ is a parallelogram with $\mathrm{AB}$ and $\mathrm{BC}$ adjacent sides.

Find $\overrightarrow{\mathrm{BC}} = \begin{pmatrix} -1-5 \\ 1-1 \\ 3-1 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 2 \end{pmatrix}$

The vector product of $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{BC}}$ is

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$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} -6 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 2 - (-3) \times 0 - 3) \times (-6) - 3 \times 2 \\ 3 \times 0 - 1 \times (-6) \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 6 \end{pmatrix}$$

Therefore the area is given by

$$\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \left|\begin{pmatrix} 2 \\ 12 \\ 6 \end{pmatrix}\right| = \sqrt{2^2 + 12^2 + 6^2} = \sqrt{184} = 2\sqrt{46}$$

## Example 6

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The area of a parallelogram formed by the two adjacent vectors of $\boldsymbol{a} = x\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $x > 0$ and $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ is 4 unit$^2$. Find the value of $x$. Give your answer to 3 significant figures.

| Steps | Explanation |
|---|---|
| $\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} x \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 - 2x \\ x - 1 \end{pmatrix}$ | |
| $\lvert\boldsymbol{a} \times \boldsymbol{b}\rvert = \sqrt{9 + (-1 - 2x)^2 + (x - 1)^2}$ | Area of parallelogram $= \lvert\boldsymbol{a} \times \boldsymbol{b}\rvert$ |
| $\sqrt{9 + (-1 - 2x)^2 + (x - 1)^2} = 4$ | |
| $9 + (-1 - 2x)^2 + (x - 1)^2 = 16$ | |
| $(-1 - 2x)^2 + (x - 1)^2 = 7$ | |
| $x = -1.220$ or $x = 0.8198$ | Use your graphic display calculator correct 4 significant figures. |

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| Steps | Explanation |
|---|---|
| Therefore, $x = 0.820$ correct to 3 significant figures | Because $x > 0$. |

## International Mindedness

The vector product has many applications in mathematics and physics, for example, when calculating torque and the current induced when a conductor attached to a circuit moves in a magnetic field.

Another application is the modelling of tornadoes (also known as twisters). These are rotating winds that create funnels between the ground and clouds and cause devastation in the area in which they move. There are many destructive tornados reported each year around the world. Therefore, it is important to model them in order to understand their behaviour and be able to predict when they will occur. The vector product can be applied in more than three dimensions, so modelling can be carried out in higher dimensions.

Watch the video below and follow the link to learn more about the modelling of tornadoes.


**Video 1.** Tornado Simulation of 2011 EF-5.

More information for video 1

A gray background sets the stage for a white wireframe rectangular prism, clearly marked with its dimensions: 120 km in length, 120 km in width, and 20 km in height. The camera angle shifts to provide a direct, head-on view of the prism, revealing a grid pattern formed by thin gray lines filling its interior. Within this structure, a swirling gray mass begins to take shape, resembling smoke or clouds, gradually becoming the focal point of the scene.

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As the camera zooms in, the swirling gray cloud formation grows larger in the frame. Labels appear to identify different parts of the cloud: "Cloud ice" is marked at the top, "Cloud water" in the middle, and "Tornado on ground" is indicated with an arrow pointing to the base of the formation. The camera remains focused on the grayscale cloud, which continues to dominate the view. A time counter in the lower right corner tracks the simulation's progress, displaying the elapsed time in seconds.

The scene transitions to a new background, featuring a gradient that shifts from light green to light blue. The cloud formation, now a darker gray, is positioned prominently in the center. The wireframe prism reappears, with its dimensions once again labeled as 120 km, 120 km, and 20 km. The background briefly reverts to gray before returning to the green-blue gradient. Above the larger cloud formation, the label "supercell" appears, while "tornado" is marked with an arrow pointing to a smaller cloud extension reaching downward.

The perspective shifts to a closer view of the lower portion of the cloud formation as it interacts with a flat, orange-brown plane representing the ground. Wispy extensions begin to emerge from the base of the cloud, reaching downward. The background remains consistent with the light green-blue gradient, providing a contrasting backdrop to the cloud and ground interaction.

The scene changes abruptly to a dark background, overlaid with a three-dimensional grid marked by planes in green, dark blue, purple, and yellow. From the bottom plane, colorful streams of particles—red, green, and yellow—emerge, swirling and twisting as they move upward. The perspective shifts slightly, offering a clearer view of these particle streams. Additional streams, in white and blue, join the initial red, green, and yellow, each originating from distinct sections of the bottom plane.

Another sudden transition occurs, revealing a semi-transparent, grayish-purple cloud interacting with a gridded surface. Within the cloud, green streaks become visible, while yellow rectangular particles are distributed along the grid's surface. A label, "Streamwise Vorticity Current," appears above the yellow particles, providing context for the visualization.

The visuals shift once more, presenting a grayscale representation of the supercell and tornado over a horizontal plane. A spectrum key on the left side of the screen provides a scale labeled "Pseudovolor var-transparant," with values ranging from -1200 to 1200, represented by a range of colors. Additional labels indicate "Max VWR" and "Min VWR" values. Below this, another scale labeled "Volume var-op-shaded" ranges from 0 to -2000, with corresponding "Max" and "Min" values tied to brightness levels. At the bottom of the image, a thin horizontal band displays a gradient of colors transitioning from green to light blue, adding a final layer of detail to the visualization.This simulation provides a **mathematical model** of tornadoes, helping scientists and meteorologists understand their formation, structure, and behavior. Through **vector product calculations**, tornadoes can be analyzed in **higher dimensions**, improving predictions and preparedness for these destructive natural phenomena.

# 7 section questions

Student view

3. Geometry and trigonometry / 3.13 Products of vectors

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# Components of vectors

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/components-of-vectors-id-28366/print/) Assign

In this section, you will find the components of a vector acting in the same direction and perpendicular to another vector.

Consider a boat trying to cross a river. Let the velocity of the boat be $\boldsymbol{w}$ and the velocity of the current in the river be $\boldsymbol{v}$. How would the current affect the velocity of the boat? How could you represent this effect using vectors?

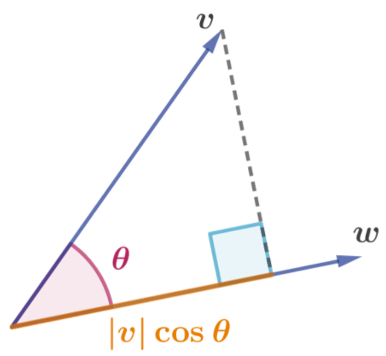
As shown in the diagram, the component of $\boldsymbol{v}$. in the direction of $\boldsymbol{w}$ is $|\boldsymbol{v}|\cos\theta$. Using the scalar product

$$\boldsymbol{v}\cdot\boldsymbol{w} = |\boldsymbol{v}|\,|\boldsymbol{w}|\cos\theta$$

So the component of $\boldsymbol{v}$ in the direction of the $\boldsymbol{w}$ can be written as:

$$|\boldsymbol{v}|\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{|\boldsymbol{w}|}$$

Therefore, while the current is taking the boat away from its original direction, it is also adding to the component of the boat's velocity in the direction in which the current is flowing.




More information

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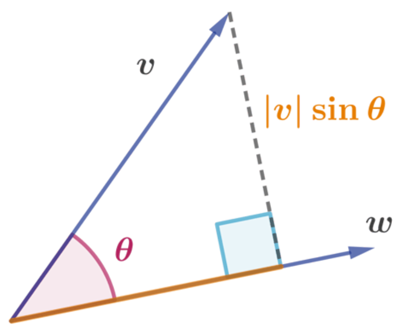
The diagram is a geometric representation of a right triangle formed by two vectors, ( \boldsymbol{v} ) and ( \boldsymbol{w} ), with an angle ( \theta ) between them. The triangle has the vector ( \boldsymbol{v} ) shown as a hypotenuse extending upwards to the right. Vector ( \boldsymbol{w} ) is one of the triangle's legs, extending horizontally to the right. The angle ( \theta ) is marked between the hypotenuse ( \boldsymbol{v} ) and the horizontal vector ( \boldsymbol{w} ). Inside the triangle, the leg opposite to ( \theta ) is marked as ( |\boldsymbol{v}| \cos \theta ). A right angle is indicated between this leg and the hypotenuse. This diagram demonstrates how the component of vector ( \boldsymbol{v} ) can be projected onto ( \boldsymbol{w} ) to find the parallel component.

[Generated by AI]

Using a similar approach, you can find the component of $\boldsymbol{v}$ perpendicular to $\boldsymbol{w}$ , in the plane formed by two vectors. As shown in the diagram below, the component of $\boldsymbol{v}$ perpendicular to $\boldsymbol{w}$ is given by $|\boldsymbol{v}| \sin \theta$.




More information

The diagram illustrates two vectors, ( \boldsymbol{v} ) and ( \boldsymbol{w} ), forming a triangle with the vertex angle ( \theta ) between them. The vector ( \boldsymbol{v} ) is represented by a diagonal arrow, and ( \boldsymbol{w} ) is horizontal at the base. A perpendicular line is drawn from the tip of ( \boldsymbol{v} ) to ( \boldsymbol{w} ), forming a right triangle. The length of the perpendicular from both vectors is labeled as ( |\boldsymbol{v}| \sin \theta ). The angle ( \theta ) is marked at the junction of ( \boldsymbol{v} ) and ( \boldsymbol{w} ), with the right angle marked at the base of the perpendicular line. This diagram visually represents the concept of a vector component perpendicular to another vector in a two-dimensional plane.

[Generated by AI]

Student view

Recall that the vector product (cross product) of two vectors is $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}|\,|\boldsymbol{w}| \sin\theta$, where $\theta$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

Therefore the component of $\boldsymbol{v}$ perpendicular to the $\boldsymbol{w}$ in the plane formed by two vectors is

$$|\boldsymbol{v}| \sin\theta = \frac{|\boldsymbol{v} \times \boldsymbol{w}|}{|\boldsymbol{w}|}$$

### Making connections

In physics, both the scalar and vector products can be used to analyse situations in which forces act on an object. For example, the scalar product can be used to calculate work done or power, while the vector product can be used to find the magnetic force acting on a moving charge in a magnetic field.

# Example 1

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Consider the vectors $\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$. Find the component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ in the plane formed by two vectors. Give your answer correct to 3 significant figures.

| Steps | Explanation |
|---|---|
| $\boldsymbol{a} \cdot \boldsymbol{b} = 0 + 2 + 4$ = 6<br>$|\boldsymbol{b}| = \sqrt{0+4+4} = \sqrt{8}$<br>$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|} = \frac{6}{\sqrt{8}} = 2.12132 = 2.12$ (3 significant figures) | Use $|\boldsymbol{a}| \cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$ |

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# Example 2

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The angle between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $1.2$ rad. If $\boldsymbol{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, find the component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ in the plane formed by two vectors.

| Steps | Explanation |
|---|---|
| $\lvert\boldsymbol{a}\rvert \cos\theta = \sqrt{1+0+1}\cos 1.2$ | The component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\lvert\boldsymbol{a}\rvert \cos\theta$. |
| $= \sqrt{2}\cos 1.2 = 0.512$ (3 significant figures) | Set the mode of your calculator to radians. |

# Example 3

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The angle between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $30^\circ$. If $\boldsymbol{a} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$, find the component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ in the plane formed by two vectors **.**

| Steps | Explanation |
|---|---|
| $\lvert\boldsymbol{a}\rvert \sin\theta = \sqrt{1+9+1}\sin 30$ | The component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ is given by $\lvert\boldsymbol{a}\rvert \sin\theta$. |
| $= \sqrt{11}\sin 1.2 = 1.66$ (3 significant figures) | Set the mode of your calculator to radians. |

Student view

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# Example 4

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Consider vector $\boldsymbol{a} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. The component of $\boldsymbol{a}$ in the direction of vector $\boldsymbol{b}$ in the plane formed by two vectors is $1.3$. Find the angle between two vectors. Give your answer correct to the nearest degree.

| Steps | Explanation |
|---|---|
| $\lvert\boldsymbol{a}\rvert \cos\theta = \sqrt{1+0+1}\cos\theta = 1.3$ | Use $\lvert\boldsymbol{a}\rvert \cos\theta$ |
| $\cos\theta = \frac{1.3}{\sqrt{2}}$ | Rearrange. |
| $\theta = 23.185\ldots$ | Solve for $\theta$. Make sure your calculator is in degree mode. |
| $\theta = 23^\circ$ | Give your answer to the nearest degree. |

## 3 section questions

3. Geometry and trigonometry / 3.13 Products of vectors

# Checklist

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### What you should know

By the end of this subtopic you will be able to:

- define the scalar product as directional multiplication of vectors:

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3, \text{ where } \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ and } \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

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- define the scalar product as the projection of one vector onto another: $\boldsymbol{v} \cdot \boldsymbol{w} = |\boldsymbol{v}|\,|\boldsymbol{w}| \cos\theta$, where $\theta$ is the angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$
- recall that if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular and, conversely, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$
- recall that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, then $\boldsymbol{v} \cdot \boldsymbol{w} = |\boldsymbol{v}|\,|\boldsymbol{w}|$
- recall that the angle between two straight lines is given by the angle between their direction vectors:
  - if $\boldsymbol{b}$ and $\boldsymbol{d}$ are the direction vectors of two straight lines, then the angle $\theta$ between these lines is given using the scalar product as $\theta = \cos^{-1}\left(\frac{\boldsymbol{b} \cdot \boldsymbol{d}}{|\boldsymbol{b}||\boldsymbol{d}|}\right)$
  - recall that the angle between two straight lines is usually given as the acute angle not the obtuse angle
- recall that the vector product of vectors $\boldsymbol{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ is denoted by $\boldsymbol{v} \times \boldsymbol{w}$
- calculate the vector product from the components of $\boldsymbol{u}$ and $\boldsymbol{v}$ using the formula

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

- recall that the vector product of two vectors $\boldsymbol{v} \times \boldsymbol{w}$ is oriented perpendicular to the plane containing $\boldsymbol{v}$ and $\boldsymbol{w}$
- define the vector product as $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}|\,|\boldsymbol{w}| \sin\theta$, where $\theta$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
- recall that a unit vector that is perpendicular to vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is given by

$$\widehat{\boldsymbol{n}} = \frac{\boldsymbol{v} \times \boldsymbol{w}}{|\boldsymbol{v} \times \boldsymbol{w}|}$$

- recall that the area of a parallelogram can be calculated using

$$|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}|\,|\boldsymbol{w}| \sin\theta$$

- recall that the area of a triangle can be calculated using $\frac{1}{2}|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}|\,|\boldsymbol{w}| \sin\theta$
- recall that the component of vector $\boldsymbol{a}$ acting in the direction of vector $\boldsymbol{b}$ is given by

$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|} = |\boldsymbol{a}| \cos\theta$, where $\theta$ is the acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$

Student view

- recall that the component of vector $\boldsymbol{a}$ acting perpendicular to vector $\boldsymbol{b}$ in the plane formed by the two vectors is given by

$$\frac{|\boldsymbol{a} \times \boldsymbol{b}|}{|\boldsymbol{b}|} = |\boldsymbol{a}| \sin \theta$$, where $\theta$ is the acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

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3. Geometry and trigonometry / 3.13 Products of vectors

# Investigation

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In this investigation you will explore the relationship between the two definitions of the scalar product.

Consider a triangle $\mathrm{OVW}$ with vertices $\mathrm{O}(0,0)$, $\mathrm{V}(v_1, v_2)$ and $\mathrm{W}(w_1, w_2)$ in the coordinate plane.

- Express the lengths of the sides of the triangle in terms of the coordinates of the vertices.
- Use the cosine rule to find the angle at vertex $\mathrm{O}$.
- Did you get $\cos\theta = \frac{v_1 w_1 + v_2 w_2}{\sqrt{v_1^2 + v_2^2}\sqrt{w_1^2 + w_2^2}}$?

If you now introduce vectors $\boldsymbol{v} = \overrightarrow{\mathrm{OV}}$ and $\boldsymbol{w} = \overrightarrow{\mathrm{OW}}$, the equation above can be rearranged to the following form.

$$|\boldsymbol{v}||\boldsymbol{w}| \cos\theta = v_1 w_1 + w_2 w_2.$$

This equality justifies that both sides of the equality can be used to define the scalar product.

- In the formula booklet you have the definitions using three coordinates. Repeat the steps above for a triangle in space with vertices $\mathrm{O}(0,0,0)$, $\mathrm{V}(v_1, v_2, v_3)$ and $\mathrm{W}(w_1, w_2, w_3)$ . Did you get a similar equality?

Student view

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