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2. Functions / 2.14 Odd and even functions

The big picture

In the applet below, type in the functions $x^2,\ x^3,\ \frac{1}{x},\ \frac{1}{x^2},\ \sin x,\ \cos x$. Using the sliders given, identify which of these functions:

- retains its position and shape after reflection in the y-axis
- retains its position and shape only after first reflecting in the y-axis and then reflecting in the
  x-axis.

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Interactive 1. Exploring Even and Odd Functions Through Reflection.

Credit: GeoGebra (https://www.geogebra.org/m/pwYgbEhf) Irina Boyadzhiev

More information for interactive 1

This interactive allows the user to understand how the graph of a function changes when it is rotated along the x-axis and y-axis. The goal is to identify whether functions are **even**, **odd**, or neither by observing how their graphs transform with reflections.

The screen is divided into two halves. The top half has a graph displayed with the XY axis. A curve is projected on the graph representing the function $f(x)$. The curve changes corresponding to the function provided by the users.

On the bottom half, The users can enter the function in the box "Enter a function of *x*". Users can type in mathematical functions like x^2,x^3, $\frac{1}{x}$,$\frac{1}{x^2}$, $\sin x$ and $\cos\ x$ in the box.

A horizontal slider is provided for rotating about the y-axis. After this, another horizontal slider appears for rotating about the x-axis, allowing the user to observe how the graph rotates around these axes.

Functions that retain shape and position after reflecting in the y-axis are even functions satisfying the rule,

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$f(-x) = f(x)$.

Functions that require two reflections (first in the y-axis, then in the x-axis) to retain their original shape are odd functions. These functions satisfy the equation, $f(-x) = -f(x)$.

Odd functions have rotational symmetry about the origin. Reflecting them in the y-axis alone flips them horizontally, changing their direction. Reflecting them again in the x-axis restores the original orientation. Even functions, on the other hand, are symmetric about the y-axis, so reflecting them in the y-axis doesn't change their graph at all.

For example when the users give the input function as $\sin x$, the interactive will show a sinusoidal graph along the x-axis. On sliding "Rotate about the y-axis" users will notice how the graph becomes the reflection of the original graph and is shown by dashed lines. Similarly, on sliding "Rotate about the x-axis" users will notice that the graph remains the same.

Thus, giving the users a better insight of how the graph of function changes when it is rotated along the x-axis and y-axis in different functions.

Why do you think certain functions need to be reflected twice to get back to the original function?

In this subtopic, you will learn about

- odd and even functions
- how to find the inverse of any given function.

Concept

Relationship

Functions represent **relationships** between two variables. They assign to each value of the independent variable (input) one and only one dependent variable (output).

Changing the input values changes the output values. Is this true for all functions? Are there functions that have same output for two or more inputs? How do functions change when only the sign of the input is changed?

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Odd and even functions

In the applet below, you can see three graphs:

- $f(x)$ (blue line)
- $f(-x)$ (green dotted line)
- $-f(x)$ (pink dotted line).

You are going to compare the graph of $f(-x)$ with $f(x)$ and $-f(x)$, in order to fill **Table 1** below.

Interactive 1. Compare the Graph of $f(-x)$ With $f(x)$ and $-f(x)$, in Order to Fill the Table Below.

More information for interactive 1

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view

This interactive helps users to compare the graph of function $f(x)$ with the graph of $f(-x)$ and $-f(x)$.

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A graph of the xy axis is displayed, with the x-axis ranging from -12 to 6 and the y-axis ranging from -3 to 5. The graph of $f(x)$ in a blue line, the graph of $f(-x)$ in a green dotted line, and the graph of $-f(x)$ in a pink dotted line are projected on the graph. The users can enter a function in the box $f(x)$.

For example, when the chosen function is x^2:

$f(x) = x^2$, the graph in the blue line will represent a standard parabola opening upwards with its vertex at $(0, 0)$.

$f(-x) = (-x)^2$, this function is the same as $f(x) = x^2$ because the square of a negative number will give a positive number. Therefore, the graph in the green dotted line is identical to the graph of $f(x)$.

$-f(x) = -x^2$, this function is a reflection of $-f(x) = -x^2$ across the x-axis represented by a pink dotted line. It opens downwards with its vertex at the origin $(0, 0)$.

Similarly, users can give different input functions and can conclude that all even functions are symmetrical about the y-axis and all odd functions are symmetric about the origin.

Table 1

Function	x^2	x	$x+\frac{1}{x}$	$x^2-\frac{1}{x^2}$	$2x^3+3x$	$-x^4+2x^2$
Does $f(-x)=f(x)$ or, $=-f(x)$?	$f(x)$					

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26756/print/) Assign

Type the functions given in the first row of **Table 1** into the $f(x)$ box in the above applet. Fill in the second row with either $f(x)$ or $-f(x)$, by looking at the graphs. An example is shown in red.

Important

The function $f(-x)$ is called an **odd function** if $f(-x) = -f(x)$.

It is called an **even function** if $f(-x) = f(x)$.

Student view

Think about the reflection of functions. You should notice that all even functions are symmetrical about the y-axis and all odd functions are symmetric about the origin.

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Activity

Which of the trigonometric functions are even and which are odd? Use the same GeoGebra applet to find out.

Can there be functions that are neither odd nor even? Give an example if there are such functions.

How do you find out whether a function is odd or even without using graphing software?

Example 1

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Without using a GDC or any graphing software, find whether the following functions are odd, even or neither.

Function $f(x)$	$2x+3$	$x^2+\frac{3}{x}$	$2x^3-x$	$\sin(x)$	$\sec(x)$	$2x-\frac{x^3}{3}$	$\sqrt{x+1}$
Odd or, Even or, Neither							

To find out whether a function is odd or even or neither, it is enough to find $f(-x)$ and check whether the result is equal to $f(x)$ or $-f(x)$ or neither.

Function $f(x)$	$2x+3$	$x^2+\frac{3}{x}$	$2x^3-x$	$\sin(x)$	$\sec(x)$	$2x-\frac{x^3}{3}$	$\sqrt{x}$
Odd, Even or Neither?	Neither	Neither	Odd	Odd	Even	Odd	Nei

For example:

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1. $f(x)=2x+3$

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$f(-x) = -2x + 3$ which is not equal to $f(x)$ nor $-f(x)$.

Hence it is neither odd nor even.

$$f(x) = 2x^3 - x$$

$$f(-x) = -2x^3 + x = -f(x)$$

Hence this is an odd function.

3 section questions

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Inverse of a function

An inverse function is a function that is obtained by reversing the operations with which the original function was defined.

You learned about inverse functions in subtopic 2.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25390/).

Recall this definition and notation. The inverse function f^{-1} of a function f is the function that 'undoes' what f does. The graph of the inverse function f^{-1} is always a reflection of the graph of f in the line $y = x$.

In the previous section, you found the inverse of a function informally. You will now learn how to find the inverse in a more general way.

Follow these steps to find the inverse of $f(x)$:

1. Replace $f(x)$ by y.
2. Solve for x in terms of y.

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3. Interchange the letters x and y in the expression you obtained in step 2.
4. Replace y in step 3 by $f^{-1}(x)$.

Use steps 1–4 to find the inverse of $f(x) = 3x - 2$

1. $y = 3x - 2$ [Replace $f(x)$ by y]
2. $x = \dfrac{y+2}{3}$ [Solve for x in terms of y]
3. $y = \dfrac{x+2}{3}$ [Interchange the letters x and y in the expression]
4. $f^{-1}(x) = \dfrac{x+2}{3}$ [Replace y by $f^{-1}(x)$]

Example 1

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Find the inverse of $f(x) = \dfrac{x+3}{x+2},\ x \neq -2$

Using the method given above,

$$\begin{aligned} y &= \frac{x+3}{x+2} \\ xy + 2y &= x + 3 \\ xy - x &= 3 - 2y \\ x(y-1) &= 3 - 2y \\ x &= \frac{3-2y}{y-1} \\ f^{-1}(x) &= \frac{3-2x}{x-1},\ x \neq 1 \end{aligned}$$

Example 2

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Find the inverse of $f(x) = \dfrac{5}{x-3} + 3$

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Using the method given above,

$$\begin{aligned} y &= \frac{5}{x-3}+3 \\ y-3 &= \frac{5}{x-3} \\ x-3 &= \frac{5}{y-3} \\ x &= \frac{5}{y-3}+3 \\ f^{-1}(x) &= \frac{5}{x-3}+3,\ x \neq 3 \end{aligned}$$

What was special in the above example? The inverse and the original functions are the same. Such functions are called self-inverse functions.

What type of graphs do the self-inverse functions have? Why is this? Graph the function given in **Example 2** and find out how it is related to the line $y = x$.

2 section questions

2. Functions / 2.14 Odd and even functions

Restricted domains

In the example below, you will find the inverse of a quadratic function using the method described in the previous section.

Example 1

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Find the inverse of the function $f(x) = (x-1)^2 - 5$

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$$
\begin{aligned}
y &= (x-1)^2 - 5 \\
y + 5 &= (x-1)^2 \\
x - 1 &= \pm\sqrt{y+5} \\
x &= 1 \pm \sqrt{y+5} \\
y &= 1 \pm \sqrt{x+5} \\
f^{-1}(x) &= 1 \pm \sqrt{x+5}
\end{aligned}
$$

In this example, the quadratic function was already given in vertex form (subtopic 2.7 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27705/)). How could you find the inverse if it is given in general form $ax^2 + bx + c$? You can convert it into vertex form and use the above method to find the inverse.

Alternatively, you can use the quadratic formula in order to isolate x (see quadratic formula in subtopic 2.7 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27705/)) and as shown in the example below.

Example 2

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Find the inverse of $f(x) = x^2 + 2x - 1$.

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/odd-and-even-functions-id-26757/print/) Assign

Change $f(x)$ to y, then subtract y from both sides to form a quadratic equation in x:

$$
\begin{aligned}
y &= x^2 + 2x - 1 \\
x^2 + 2x - 1 - y &= 0
\end{aligned}
$$

Using the quadratic formula,

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$$x = \frac{-2 \pm \sqrt{4 - 4(-1 - y)}}{2} \qquad \text{[note that } c = -1 - y \text{ in this cas}$$

$$x = -\frac{-2 \pm \sqrt{4 + 4 + 4y}}{2}$$

$$x = \frac{-2 + \sqrt{8 + 4y}}{2}$$

$$x = \frac{-2 \pm 2\sqrt{2 + y}}{2}$$

$$x = -1 \pm \sqrt{2 + y}$$

$$y = -1 \pm \sqrt{2 + x} \qquad \text{[interchanging } x \text{ and } y\text{]}$$

$$f^{-1}(x) = -1 \pm \sqrt{2 + x} \qquad \text{[replacing } y \text{ by } f^{-1}(x)\text{]}$$

The two functions and their inverses that we just found are given below:

$$f(x) = (x - 1)^2 - 5 \implies f^{-1}(x) = 1 \pm \sqrt{x + 5}$$

$$f(x) = x^2 + 2x - 1 \implies f^{-1}(x) = -1 \pm \sqrt{2 + x}$$

You may have noticed that both of these functions had two inverse functions because of $\pm$ before the radical. Does this make sense? Can we have two inverse functions for a function? If not, which one is the right inverse for $f(x)$?

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Recall what you learned in subtopic 2.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25390/) .

1. A function and its inverse are reflections in the line $y = x$.
2. In order to have an inverse, a function should be one-to-one.

Which of the two inverse functions is a reflection of the given function $f(x)$ in $y = x$?

Is a quadratic function one-to-one?

For every value of y, there are two values of x. Hence it is not one-to-one. Therefore, it is important that you first make $f(x)$ a one-to-one function.

Is there a y-value on the quadratic graph with one x-value?

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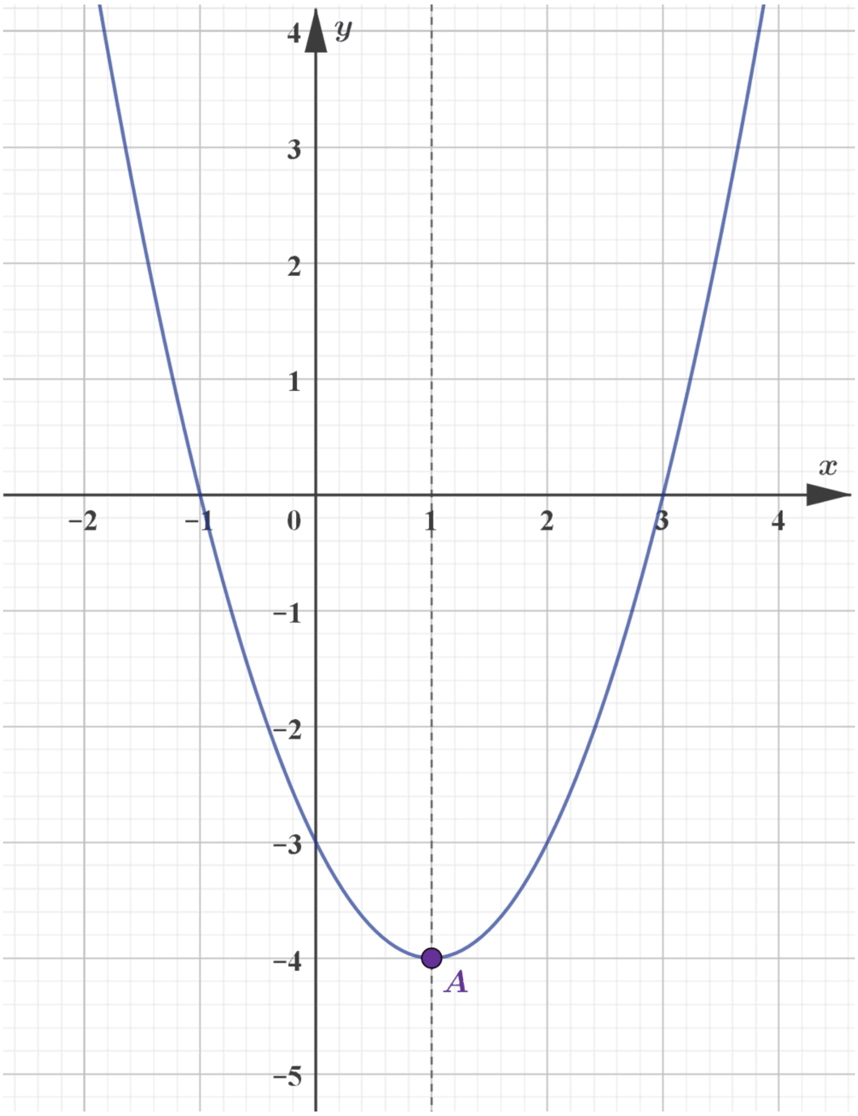
In order to make it into a one-to-one function so that you can find its inverse, you can restrict the function to a part of the graph as shown below:



More information

The image shows a graph with a parabola opening upwards on a grid. The X-axis represents a numerical scale ranging from -2 to 2 in intervals of 1, labeled at 0.5 increments. The Y-axis also uses a numeric scale from -5 to 5, marked at each integer. The graph features a blue curve that forms a U-shape, intersecting the Y-axis at the origin and marking a point labeled 'A' at its vertex. This point is centered at the lowest part of the U, indicating the symmetry along the vertical Y-axis. It illustrates how the graph is divided to make the function one-to-one so that an inverse can be found, as described in the accompanying text.

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view

The graph is cut into two parts along the line of symmetry . If you consider one part, it becomes a one-to-one function.

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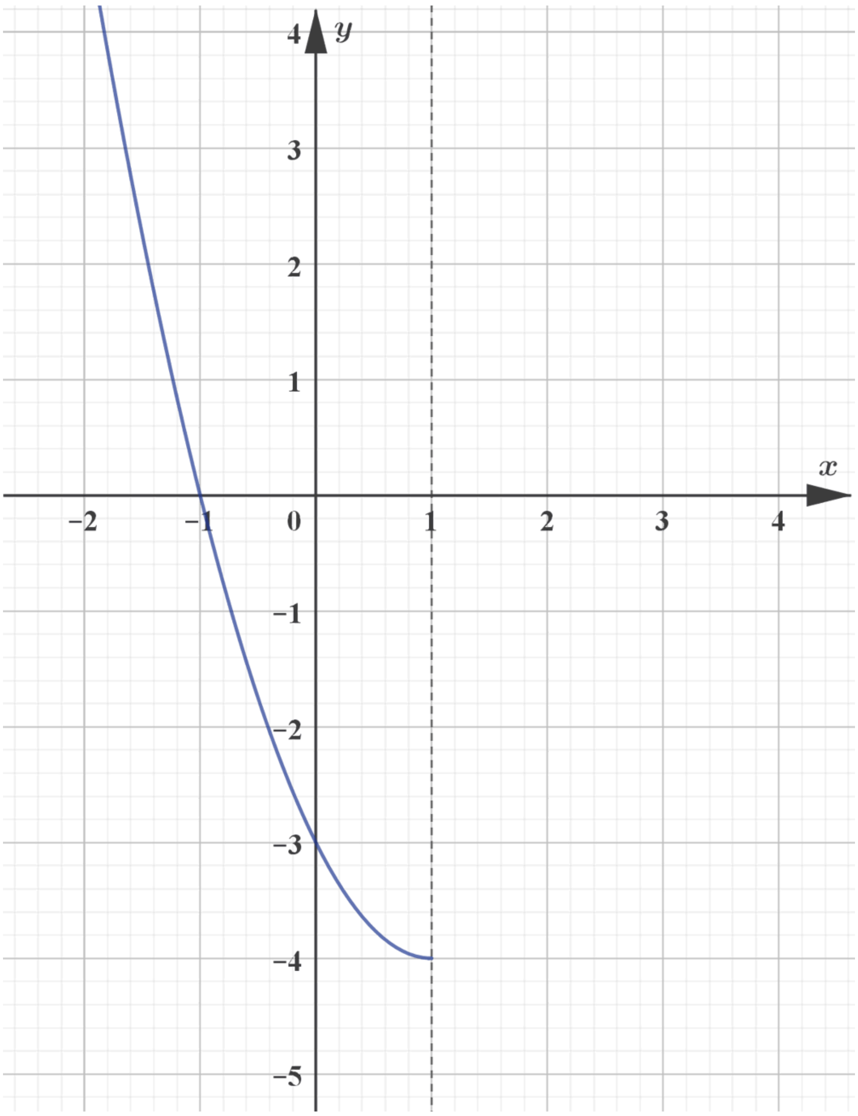
For example, the part which is to the left of the axis of symmetry (red dotted line) will have only one x-value mapped to one y-value. In order to make this partition, you restrict the domain of the function to $(-\infty, x_{min}]$ or $[x_{min}, \infty)$, so that the graph would look as shown below.

The function considered here is $g(x) = x^2 - 2x - 3$.



More information

The image is a graph of the function g(x) = x^2 - 2x - 3 plotted on a Cartesian coordinate system. The X-axis represents the range of x-values and is labeled with values from -2 to 5. The Y-axis represents the range of g(x) values and is labeled with values from -6 to 5. The graph is a downward-opening parabola, crossing the Y-axis at the point (0,-3) and having a vertex at the minimum point (1,-4). The parabolic curve also intersects the X-axis at points (-1, 0) and (3, 0). The overall shape of the curve is symmetric with respect to the line x=1, which is the axis of symmetry.

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The minimum point is $(1, -4)$ and the domain can be restricted to either $(-\infty, 1]$ or $[1, \infty)$.

For simplicity, take the domain to be $(-\infty, 1]$

The range is $[-4, \infty)$. You may recall that the range of a function is equal to the domain of its inverse. (Subtopic 2.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25390/)). Therefore, the domain of the inverse, in this case, is $[-4, \infty)$.

Finding the inverse of the above function using any of the methods stated above, you get:

$$g^{-1}(x) = 1 \pm \sqrt{x+4}$$

In order to identify which of the two options is reasonable, you need to apply the restricted domain $(-\infty, 1]$ of $g(x)$, because this is the range of $g^{-1}(x)$.

Be aware

Always keep in mind both the domain and range of the original function and be careful to apply them to the inverse function as required.

Hence, the above inverse function $g^{-1}(x) = 1 \pm \sqrt{x+4}$, should have its y-values less than or equal to 1.

This is possible only when you choose, $g^{-1}(x) = 1 - \sqrt{x+4}$, since $1 - \sqrt{x+4} \leqslant 1$.

Exam tip

Always use the range of the inverse function to eliminate the unwanted part before you give the final answer. This process will have a ‘reason’ mark and only one right answer will be considered for the final answer mark in the examination.

Example 3

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1. $f(x) = (x-1)^2 - 5 \implies f^{-1}(x) = 1 \pm \sqrt{x+5}$
2. $f(x) = x^2 + 2x - 1 \implies f^{-1}(x) = -1 \pm \sqrt{2+x}$

Give the correct inverse functions for the above two functions by restricting their domains.

1. $f(x) = (x-1)^2 - 5 \Rightarrow f^{-1}(x) = 1 \pm \sqrt{x+5}$
   Minimum point $(1,\ -5)$
   Restricted domain: $(-\infty,\ 1]$
   All y values of $f^{-1}(x) = 1 \pm \sqrt{x+5} \leq 1 \Rightarrow f^{-1}(x) = 1 - \sqrt{x+5}$.
2. $f(x) = x^2 + 2x - 1 \Rightarrow f^{-1}(x) = -1 \pm \sqrt{2+x}$
   Minimum point $(-1,\ -2)$
   Restricted domain: $(-\infty,\ -1]$
   All y values of $f^{-1}(x) = -1 \pm \sqrt{2+x} \leq -1 \Rightarrow f^{-1}(x) = -1 - \sqrt{2+x}$

Note: If the restricted domains in the above were given as $[1,\ \infty)$ and $[-1,\ \infty)$, your answer should be $1 + \sqrt{x+5}$ and $-1 + \sqrt{2+x}$ respectively.

Making connections

Think about trigonometric functions and their inverses (subtopic 3.9 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26855/)). Do you need to restrict the domain to find the inverse trigonometric functions? Why?

Activity

Consider $f(x) = 3x + 2$ and $g(x) = x^2 + 1$.

Find

(1) $f^{-1}(x)$

(2) $g^{-1}(x)$

(3) $f \circ g(x)$

(4) $(f \circ g)^{-1}(x)$

(5) $\left(f^{-1} \circ g^{-1}\right)(x)$

(6) $\left(g^{-1} \circ f^{-1}\right)(x)$

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What do you infer from the answers to questions (4), (5) and (6)? Explain your answer with more examples.

International Mindedness

Imagine studying graphs and inverses of a function without the notation that we are using now ($f(x)$).

How complicated it would have been to find the domain and range, or to find out whether the function is odd or even? When was this notation developed? Who discovered it? How did it become internationally accepted?

Theory of Knowledge

Is there any connection between the notation used in mathematics and culture or history? Can mathematics exist without the effect of culture? Justify your answers with examples.

3 section questions

Question 1

Difficulty:

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The function $f(x) = x^2 - 2x + 1$ with a restricted domain $x \leqslant a$ has an inverse . Find the largest possible value of a.

1

Accepted answers

1, a=1

Explanation

Minimum point of the function is $(1, 0)$

Hence the domain should be restricted either as $x \leqslant 1$ or $x \geqslant 1$

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Question 2

Difficulty:

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If $f(x) = \ln(1+x)$ find the value of $f^{-1}(0)$.

0

Accepted answers

0

Explanation

Finding inverse:

$$\begin{aligned} f(x) &= \ln(1+x) \\ y &= \ln(1+x) \\ \mathrm{e}^y &= 1+x \\ x &= \mathrm{e}^y - 1 \\ f^{-1}(x) &= \mathrm{e}^x - 1 \\ f^{-1}(0) &= \mathrm{e}^0 - 1 \\ f^{-1}(0) &= 1-1 \\ &= 0 \end{aligned}$$

Question 3

Difficulty:

★☆☆

Which of the following can be the domain of $f(x) = x^2 + 4x$ if f has an inverse?

1 $x \leq -2$

2 $x \leq -1$

3 $x \geq -3$

4 $x \leq 4$

Explanation

The function has minimum at $(-2, -4)$. Hence the domain is $x \leqslant -2$ or $x \geqslant -2$.

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Checklist

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What you should know

By the end of this subtopic you should be able to:

- identify odd and even functions
- find the inverse of a function
- restrict the domain of a function so that it is possible for the inverse to exist
- identify self-inverse functions.

2. Functions / 2.14 Odd and even functions

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-26761/print/) Assign

Investigate the restrictions that should be applied to the domain of a cubic function so that it has an inverse.

- Sketch a cubic function

$$g(x) = x^3 + 4x^2 + 2$$

- Is this a one-to-one function?
  Explain your answer?
- If the domain is restricted between $x = a$ and $x = b$, where both a and b are finite real numbers, the function becomes one-to-one. Find the largest possible domain $[a, b]$.

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Use the idea from the cubic function and investigate the domain and range of all the inverse trigonometric functions. Use the following as a guideline for investigating other trigonometric functions too:

- Sketch the graph of $y = \sin(x)$.
- Find the largest possible domain for which the inverse can exist.
- Find the domain and range of $\sin^{-1}(x)$.
- Sketch both $y = \sin(x)$ and $y = \sin^{-1}(x)$ within the restricted domain.

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