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The big picture

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-28203/print/) Assign

In subtopic 5.5 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26177/) you were introduced the concept of both the indefinite integral and definite integral. You also learned how to use technology to find approximate values of definite integrals and how the concept of the definite integral is connected to the area of certain regions. In subtopic 5.8 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-5-id-27890/) you also learned how to approximate regions with trapezoids.

Using the fundamental theorem of calculus introduced in section 5.11.7 (/study/app/math-ai-hl/sid-132-cid-761618/book/newtonleibniz-formula-id-28200/), our work in this subtopic will depend on the techniques to find indefinite integrals introduced in section 5.5.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/antiderivatives-of-power-functions-id-26180/) and further developed in subtopic 5.11 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27512/).

- First you will learn how to find the exact area of certain regions.
- You will then learn how regions can be approximated using rectangles (instead of trapezoids, which you have already seen) and how the notation for the sum of the area of the rectangles is related to the notation used for definite integrals.
- The investigation of the relationship between sums and definite integral will lead us to a formula that expresses the volume of certain solids. We will discuss solids of revolution when the axis of revolution is the x-axis and also when the axis of revolution is the y-axis.

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On the applet below you can explore how solids of revolution are generated.

Interactive 1. Exploring Solids of Revolution: Area and Volume.

More information for interactive 1

This interactive visualization helps users understand how rotating a 2D curve around an axis can create a 3D solid. On the right side of the screen, a 3D graph with labeled X, Y, and Z axes displays a curve drawn in the XY-plane. The curve starts at the origin, rises to a peak, then dips downward, creating an arc-like shape. This curve serves as the base for generating a solid of revolution. On the left side, users can choose whether to rotate the curve around the x-axis or y-axis using two checkboxes. When the "adjust curve" option is selected, three red dots appear on the curve. These dots can be dragged vertically to reshape the curve, allowing users to explore how different curve profiles create different 3D shapes. A draggable red point on a circular guide allows users to manually spin the curve around the chosen axis. As users make adjustments, the 3D view updates in real time to show how the rotated curve forms various solids—such as cones, bowls, or hourglass shapes—depending on the curve's shape and the selected axis of rotation.

For example, a user could simulate designing a wine glass by adjusting the curve to have a narrow base and wide top, then rotating it about the y-axis.

This demonstration connects visual geometry with calculus principles by showing how solids of revolution are built from thin circular slices—called disks or washers—whose areas are integrated to calculate volume. The dynamic, hands-on manipulation offers a concrete way to understand an abstract concept, especially for students studying calculus or 3D modeling.

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Concept

Throughout this subtopic, think about how the **relationship** between indefinite and definite integrals helps you to find exact areas of specific regions bounded by curves.

Think also about how **approximating** solids helps us use integrals to find exact volumes. Can you think of other areas of mathematics where integration might replace approximation?

5. Calculus / 5.12 Area and volume

Definite integral revisited

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/definite-integral-revisited-id-28209/print/) Assign

In section 5.5.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/definite-integrals-id-26179/) you were introduced to the notation $\int_a^b f(x)\mathrm{d}x$ to represent the area of the region bounded by the graph of the positive function f above the interval $[a, b]$. In this section, you will see how to extend the meaning of this notation for not necessarily positive functions.

Activity

- Use your graphic display calculator to find some definite integrals.
- Sketch the graphs of the functions you used and find the area below the graph.
- Compare the results you get.
- Experiment with functions that are not always positive. For example, find $\int_{-1}^{1} x^3\mathrm{d}x$. What do you notice?

In the previous activity, you should have found that $\int_{-1}^{1} x^3\mathrm{d}x = 0$. This is clearly not the area of the region bounded by the graph and the x-axis. Although there is a connection between area and definite integral, these two concepts are not the same.

Student view

The applet below shows a graph of a function f. It also shows a region and lets you modify the lines that bound the region from the left and from the right. The applet gives the area of the region and also the value of the integral of the function between the given limits. Can you find a relationship between the area of the region and the value of the definite integral?

Interactive 1. Adjusting Bounds for Definite Integrals.

More information for interactive 1

This interactive tool allows users to visually explore the concept of definite integrals and their relationship to geometric area under a curve.

At the top of the screen is a graph with labeled x- and y-axes, where the x-axis ranges from 0 to 5 and the y-axis ranges from -1 to 1. A blue curve representing the function $f(x)$ is plotted on the graph. The area between this curve and the x-axis, over a selected interval, is shaded in light blue. This shaded region represents the definite integral of the function over that interval. Two red dots on the x-axis define the bounds of integration, and vertical lines rise from these points to the curve, clearly showing the interval in question.

Below the graph are two toggle options: “Adjust curve” and “Adjust the bounds of the region.” When “Adjust curve” is selected, users can drag red points on the curve up or down to change its shape. When “Adjust the bounds” is selected, users can move the red points along the x-axis to change the interval of integration.

As users adjust either the curve or the integration bounds, the tool updates the shaded region and displays both the signed integral and the total geometric area. This distinction is key: the definite integral gives a net value, where area below the x-axis subtracts from the total, while geometric area always adds up the absolute values regardless of position.

For instance, a user might observe that when the curve dips below the x-axis, the definite integral decreases while

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the geometric area increases. This side-by-side display helps users understand how integrals can represent both net change and total accumulation, and builds a deeper conceptual understanding of integral calculus through direct, interactive feedback.

Activity

- Move both points to make the function positive between them. What do you notice?
- Move both points to make the function negative between them. What do you notice?
- What happens if you change the order of the two points?
- How is the value of the integral related to the area of the parts of the region?

The observations above justify the following definition.

Important

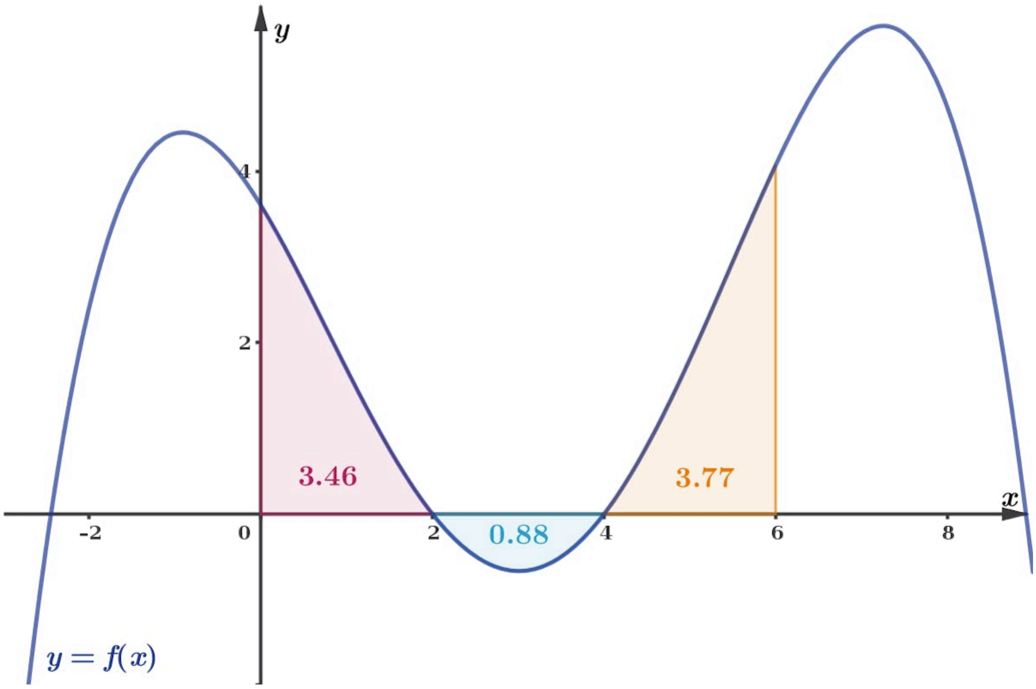
- For $a < b$, the **definite integral** (of the continuous function f)
$$\int_a^b f(x)\mathrm{d}x$$
is the signed area of the region bounded by the lines $x = a$, $x = b$, the x-axis and the graph of $y = f(x)$.
In the signed area, the part of the region above the x-axis counts as positive, the part of the region below the x-axis counts as negative.
- For $a = b$, the value of the definite integral is 0.
- For $a > b$,
$$\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$$

Example 1

★☆☆

Student view

The diagram below shows part of the graph of $y = f(x)$. There are also three shaded regions on the diagram. The area of these regions are 3.46, 0.88 and 3.47 units squared.



More information

The image is a graph depicting the curve of the function y=f(x). The graph has an x-axis and a y-axis with labels at regular intervals. The y-axis ranges from -2 to 6, and the x-axis ranges from -2 to 8. The curve has a peak before descending again, forming hills and valleys.

Three regions are shaded under the curve indicating areas between the curve and the x-axis. The first shaded region lies to the left of the y-axis, between x=-2 and x=0, with an area labeled as 3.46 units squared. The middle region is between x=2 and x=4, with an area labeled as 0.88 units squared. The final region is to the right, between x=4 and x=6, with a labeled area of 3.77 units squared.

The curve exhibits a peak between x=0 and x=2, a trough between x=2 and x=4, and rises again to end at another peak extending beyond x=6.

[Generated by AI]

- Find $\int_0^4 f(x)\mathrm{d}x$.
- Find $\int_6^2 f(x)\mathrm{d}x$.

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- Since the region above the x axis counts as positive and the region below the x-axis counts as negative,
$$\int_0^4 f(x)\mathrm{d}x = 3.46 - 0.88 = 2.58.$$
- Similarly to the previous part,
$$\int_2^6 f(x)\mathrm{d}x = -0.88 + 3.77 = 2.89.$$
Hence,
$$\int_6^2 f(x)\mathrm{d}x = -\int_2^6 f(x)\mathrm{d}x = -2.89.$$

In the example above, the areas of some regions were used to find the value of definite integrals. It is more common to use the definite integral to find the area of a region.

Example 2

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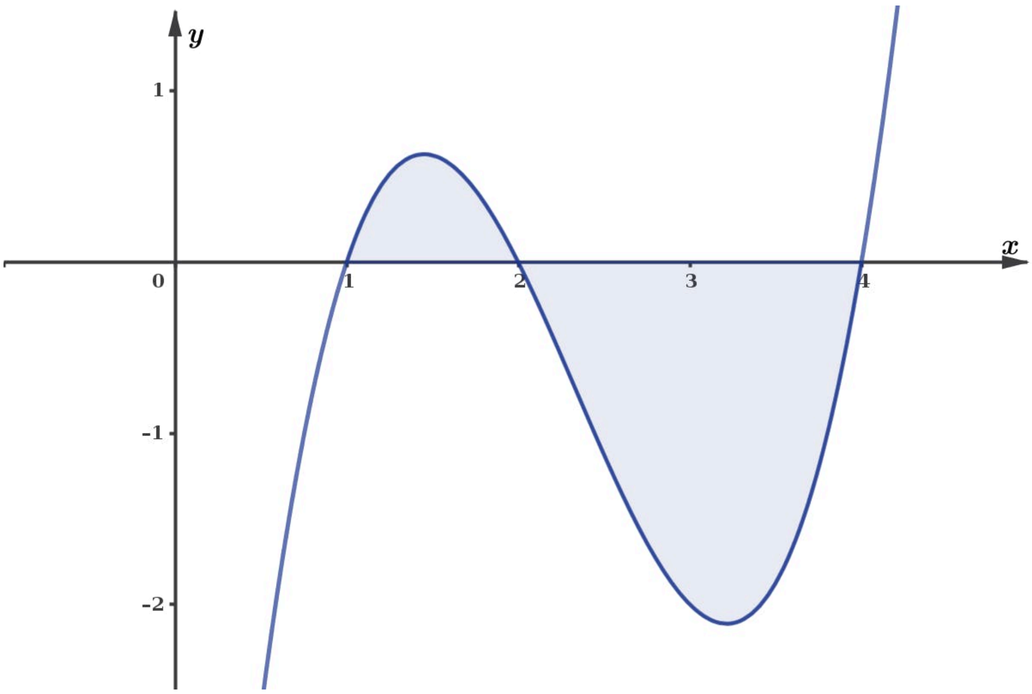
The diagram below shows part of the graph of $y = x^3 - 7x^2 + 14x - 8$.



More information

Student view

The image is a graph illustrating the equation y = x^3 - 7x^2 + 14x - 8. The X-axis and Y-axis intersect at the origin (0,0). The graph has a cubic curve that begins slightly above the X-axis, dips below zero between x=1 and x=2, rises to a local maximum around x=2.5, and then dips again steeply before rising towards positive infinity as x increases beyond 4. The Y-axis scale ranges from approximately -2 to 2. Key inflection points are visible, with roots near x=1 and x=4, and local extrema between these. The curve highlights the behavior of the polynomial function in both negative and positive domains of x.

[Generated by AI]

- Find $\int_1^2 x^3 - 7x^2 + 14x - 8\mathrm{d}x$.
- Find $\int_2^4 x^3 - 7x^2 + 14x - 8\mathrm{d}x$.
- Find $\int_1^4 x^3 - 7x^2 + 14x - 8\mathrm{d}x$.
- Find the area of the shaded region.

Graphic display calculators have applications that can find the definite integrals. The values we get are

- $\int_1^2 x^3 - 7x^2 + 14x - 8\mathrm{d}x \approx 0.41667 \approx 0.417$.
- $\int_2^4 x^3 - 7x^2 + 14x - 8\mathrm{d}x \approx -2.66667 \approx -2.67$.
- $\int_1^4 x^3 - 7x^2 + 14x - 8\mathrm{d}x \approx -2.25000 = -2.25$.

The area of the shaded region is the sum of the area of the regions above and below the x-axis. The first integral gives the area of the region above the x-axis. The second integral without the negative sign gives the area of the region below the x-axis. So the total area is

$$0.41667 + 2.66667 = 3.08334 \approx 3.08 \text{ units squared.}$$

Example 3

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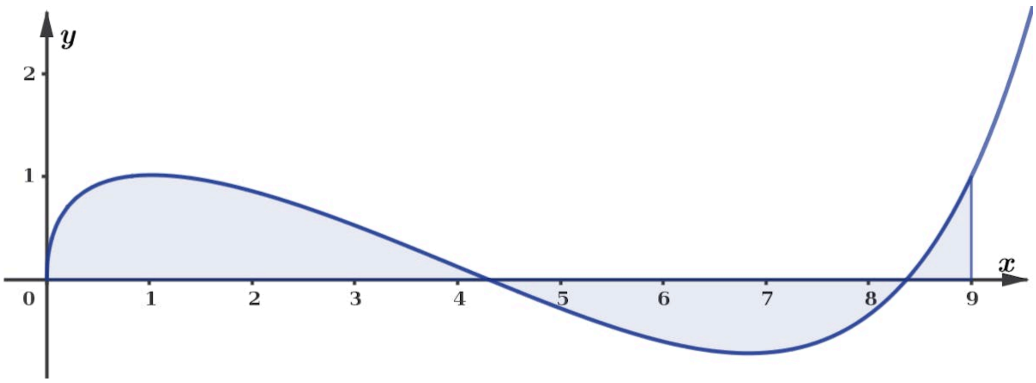
Find the area of the region between x-axis, the graph of $y = 2\sqrt{x} - x + 2^{x-7}$ and the vertical line $x = 9$.



More information

The image is a graph depicting the function (y = 2\sqrt{x} - x + 2^{x-7}). The x-axis is labeled from 0 to 9, with markers at each integer. The y-axis starts at 0 and extends upwards, with markers at each integer up to 3.

The curve starts slightly above the x-axis around x=0, reaches a peak slightly above y=2 between x=1 and x=2, then dips below the x-axis around x=5. It rises again, crossing back above the x-axis before x=9. The area between the curve and the x-axis is shaded, illustrating the region in question.

The graph identifies both the axis labels with 'x' for the horizontal axis and 'y' for the vertical axis. The curve's shape features an initial rise, a peak, and subsequent descent followed by a rise towards the endpoint at x=9.

[Generated by AI]

First of all, note that there is no left bound given for the region. This means, that we need to use $x = 0$ as the left bound, since the graph starts at the origin.

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Method 1

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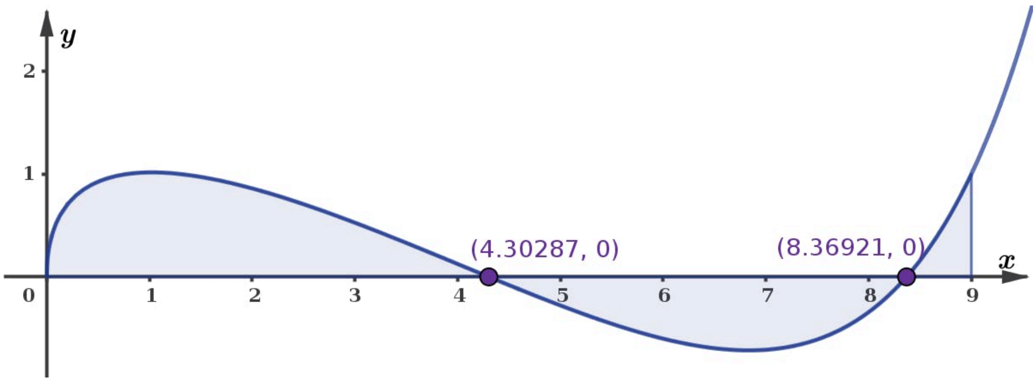
Since there are parts below and above the x-axis, you need to find the values where the graph intersects the x-axis and use these values as bounds to find the areas of the regions separately.

The diagram below shows the intersection points.



- The area of the first part of the region is approximately
$$\int_0^{4.30287} 2\sqrt{x} - x + 2^{x-7}\mathrm{d}x \approx 2.85464.$$
- For the second part, find
$$\int_{4.30287}^{8.36921} 2\sqrt{x} - x + 2^{x-7}\mathrm{d}x \approx -1.87848.$$
So the area of the second part is approximately 1.87848 units squared.
- The area of the third part of the region is approximately
$$\int_{8.36921}^{9} 2\sqrt{x} - x + 2^{x-7}\mathrm{d}x \approx 0.28335.$$

Hence, the area of the shaded region is approximately
$$2.85464 + 1.87848 + 0.28335 = 5.01647 \approx 5.02 \text{ units squared.}$$

Method 2

When using technology to find the area, you can also use a shortcut. The diagram below also shows the graph of

$$y = \left|2\sqrt{x} - x + 2^{x-7}\right|.$$

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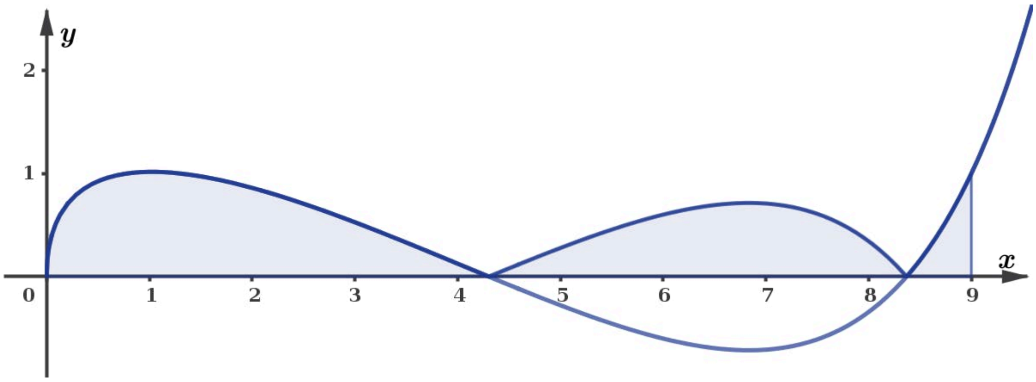

Note that this graph is above the x-axis, and the area bounded by this graph and the x-axis is the same as the area of the original regions. Hence you can find this area without the need to find the x-intercepts.

$$\int_0^9 \left|2\sqrt{x} - x + 2^{x-7}\right| \mathrm{d}x \approx 5.02$$

Exam tip

In the formula booklet, the formula for the area of a region enclosed by a curve and the x-axis is given as

$$A = \int_a^b |y| \mathrm{d}x.$$

This of course represents the area of the region bounded by the x-axis and the graph of $y = f(x)$ over the interval $[a, b]$. Because of the absolute value, this formula gives the correct area even if the graph of $y = f(x)$ has parts below the x-axis.

Note, that in **Example 3** above, we only found an approximate area. To find the exact area is beyond the capabilities of WolframAlpha (http://www.wolframalpha.com). If you are interested in the WolframAlpha gives as an answer, type

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area below 2sqrt(x)-x+2^(x-7) for 0<x<9

in the search line. However, it is beyond even the HL syllabus to fully understand the answer you get.

Using the applet below, you can check your understanding.

Interactive 2. Area between a Curve and Vertical Lines.

More information for interactive 2

This interactive tool provides a dynamic way to practice calculating areas between curves and the x-axis using definite integrals.

The screen has a graph of the xy axis where the x-axis ranges from -6 to 8. A parabola is projected on the graph with two vertical dotted lines $x = a$ and $x = b$. By clicking the “Click here for a new question” button on the top left corner, it updates the graph and generates random problems where users find the area bounded by a function $y = f(x)$, the x-axis, and two vertical lines $x = a$ and $x = b$.

With the “show answer” checkbox on the bottom left corner, users can reveal the correct answer to check their work, reinforcing understanding through immediate feedback. The applet helps distinguish between signed integrals (which may be negative) and actual areas (always positive), particularly when functions cross the x-axis. By solving these problems, users develop crucial skills for working with regions above and below the x-axis and learn when to divide integrals at roots.

For example, Find the area of the region between the x-axis, the graph of $y = x^2 - x - 44.89$ and the vertical lines $x = -4$ and $x = 6.2$.

$= \int_{-4}^{6.2} (x^2 - x - 44.89)\, dx$

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$= F(x) = \frac{x^3}{3} - \frac{x^2}{2} - 44.89x$

We need to evaluate $F(6.2) - F(-4)$

$F(6.2) = \frac{(6.2)^3}{3} - \frac{(6.2)^2}{2} - 44.89(6.2)$

$= \frac{238.328}{3} - \frac{38.44}{2} - 278.318$

$= 79.442666... - 19.22 - 278.318$

$\Rightarrow F(6.2) = -218.095333...$

$F(-4) = \frac{(-4)^3}{3} - \frac{(-4)^2}{2} - 44.89(-4)$

$= -\frac{64}{3} - \frac{16}{2} + 179.56$

$= -21.33333... - 8 + 179.56$

$\Rightarrow F(-4) = 150.226666...$

$\text{Area} = F(6.2) - F(-4) = -218.095333... - 150.226666...$

$\text{Area} = -368.322$

Since area cannot be negative, we take the absolute value of the result:

$\text{Area} = |-368.322| = 368.322$ units squared.

The random problem generator and instant feedback create an effective environment for self-paced learning, strengthening both your conceptual understanding and practical ability to apply integrals to area calculations.

4 section questions

5. Calculus / 5.12 Area and volume

Area and integral

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In section 5.12.1 (/study/app/math-ai-hl/sid-132-cid-761618/book/definite-integral-revisited-id-28209/) you learned about the connection between area of regions bounded by a graph and the x-axis and the definite integral. You have also learned how to find areas using a graphic display calculator (GDC). In this section we will use the Newton-Leibniz formula to find exact values of areas. First we start with an abstract example.

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Be aware

Sometimes the form $\int_a^b f'(x)\mathrm{d}x = f(b) - f(a)$ is used to present the Newton-Leibniz formula.

Example 1

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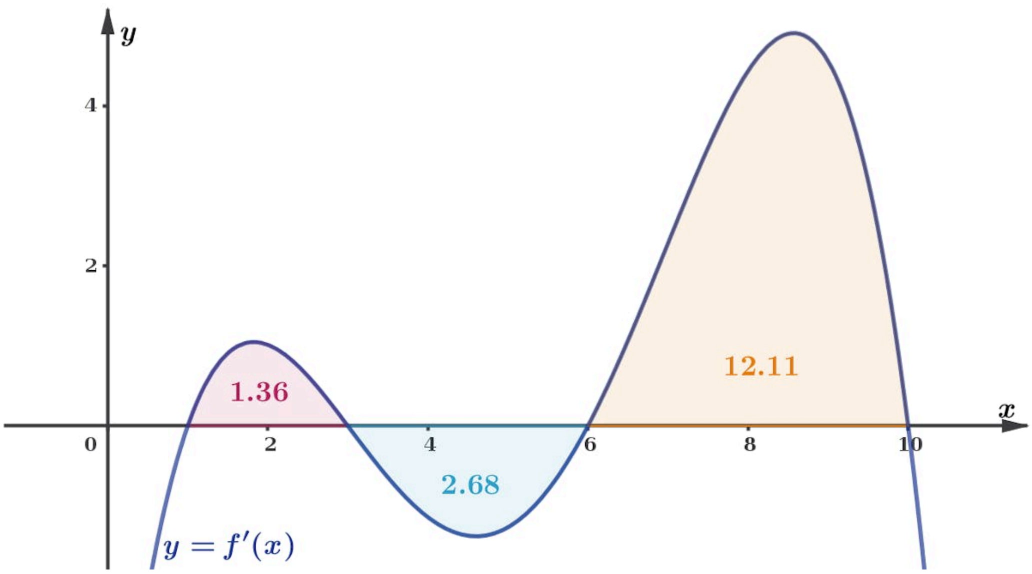
The diagram below shows the graph of $y = f'(x)$ and three shaded regions between the graph and the x-axis. The x-intercepts of the graph are $(1, 0)$, $(3, 0)$, $(6, 0)$ and $(10, 0)$. The area of the three regions are 1.36, 2.68 and 12.11 units squared.



More information

The image is a graph depicting the function (y = f'(x)) and highlighting three distinct regions between the graph's curve and the x-axis.

1. **X-axis:** Represents the variable (x) ranging from 0 to 10. The axis includes key x-intercepts at points (1,0), (3,0), (6,0), and (10,0).

2. **Y-axis:** Represents (f'(x)) with a scale up to 5 and down to -2.

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3. **Graph**: Features a curve that crosses the x-axis at the intercepts mentioned above.

4. **Shaded Regions:**

- Between x=1 and x=3, area: 1.36 (above x-axis)
- Between x=3 and x=6, area: 2.68 (below x-axis)
- Between x=6 and x=10, area: 12.11 (above x-axis)

Each region is clearly labeled with its respective area value. The graph overall demonstrates the changes in (f'(x)) across the given interval, with noticeable peaks and troughs at the turning points between the shaded areas.

[Generated by AI]

Given that $f(1) = 5$, find $f(10)$.

Use $\int_a^b f'(x)\mathrm{d}x = f(b) - f(a)$ for $a = 1$ and $b = 10$.

$$\begin{aligned} \int_1^{10} f'(x)\mathrm{d}x &= f(10) - f(1) \\ 1.36 - 2.68 + 12.11 &= f(10) - 5 \\ f(10) &= 15.79 \end{aligned}$$

The next examples show how to use antiderivatives to find the exact area of specific regions.

Example 2

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Find the exact area of the shaded region between the graph of $y = \cos x$ and the x-axis.

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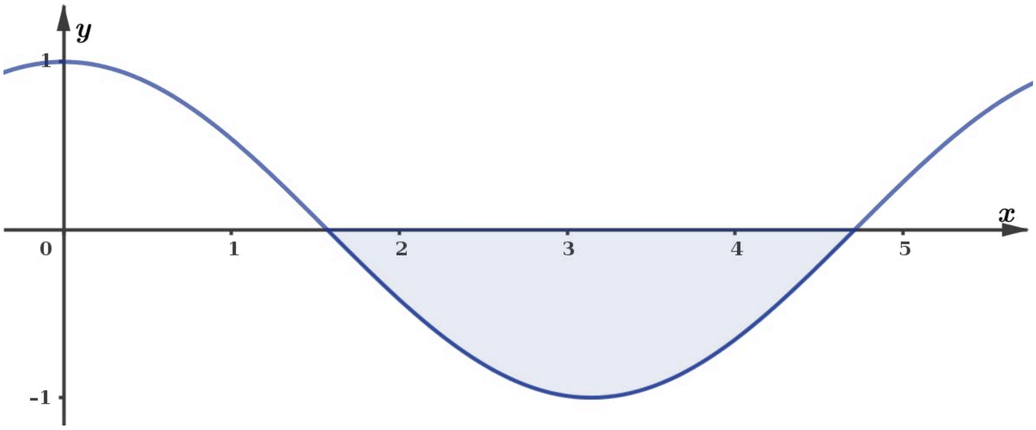

More information

The image depicts a graph of the function y = cos x. The x-axis is labeled with tick marks at 0, 1, 2, 3, 4, 5, and 6, while the y-axis features values at 1 and -1. The curve of the cosine function starts at (0, 1), moves downward to intersect the x-axis, reaches a minimum point at (π, -1), and continues upwards, intersecting the x-axis again at (2π, 1). The shaded area is visible between the curve from approximately x = 0 to x = 2π and remains beneath the x-axis, illustrating the region described in the problem statement as the exact area to find between the curve and the x-axis.

[Generated by AI]

The bounds of the region are the x-intercepts of the graph, so $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

The region is below the x-axis, so the area is the absolute value of the definite integral.

Since $\int \cos x \mathrm{d}x = \sin x + c$, the area is

$$\left|\int_{\pi/2}^{3\pi/2} \cos x \mathrm{d}x\right| = \left|\Big[\sin x\Big]_{\pi/2}^{3\pi/2}\right| = \left|\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right| = |-1-1| = 2.$$

Example 3

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Student view

Find the exact area of the shaded region enclosed by the graph of $y = (x+2)(x-1)(x-5)$ and the x-axis.

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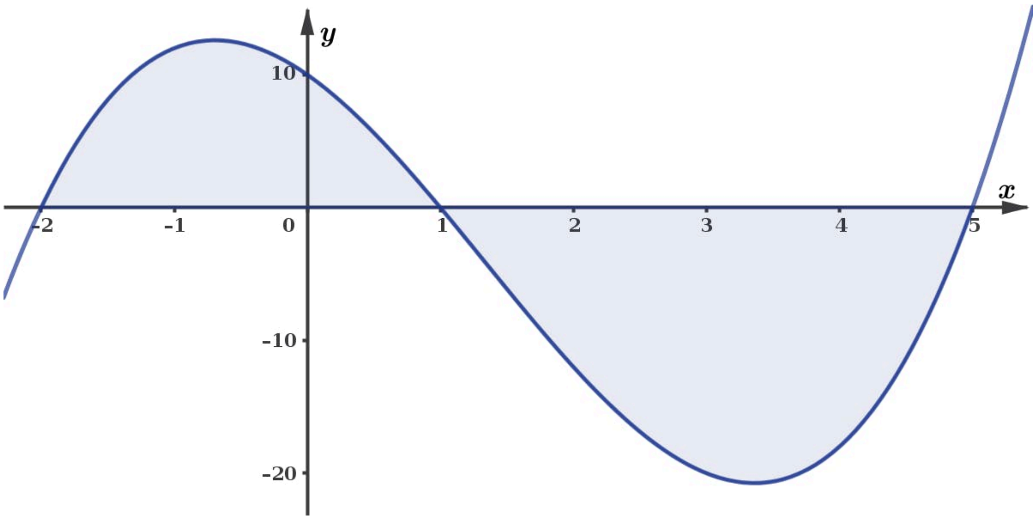

More information

The image displays a graph of a cubic function (y = (x+2)(x-1)(x-5)). The x-axis is labeled with values from -2 to 5, and the y-axis is labeled with values from -20 to 10. The curve intersects the x-axis at points x = -2, x = 1, and x = 5. The graph has a shaded region which represents the area bounded by the curve and the x-axis, appearing above the x-axis between x = -2 and x = 1, and below the x-axis between x = 1 and x = 5. The curve has a local maximum above the x-axis between x = -2 and x = 1 and a local minimum below the x-axis between x = 1 and x = 5.

[Generated by AI]

First, expand the given expression,

$$(x+2)(x-1)(x-5) = x^3 - 4x^2 - 7x + 10.$$

To find the area of the shaded region, we use the integral

$$\int x^3 - 4x^2 - 7x + 10 \mathrm{d}x = \frac{x^4}{4} - \frac{4x^3}{3} - \frac{7x^2}{2} + 10x + c.$$

Since the shaded region has parts above and below the x-axis, we set up and find two integrals to find the two areas separately.

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$$\int_{-2}^{1} x^3 - 4x^2 - 7x + 10\mathrm{d}x$$
$$= \left[\frac{x^4}{4} - \frac{4x^3}{3} - \frac{7x^2}{2} + 10x\right]_{-2}^{1}$$
$$= \left(\frac{1}{4} - \frac{4}{3} - \frac{7}{2} + 10\right) - \left(\frac{16}{4} - \frac{-32}{3} - \frac{28}{2} - 20\right)$$
$$= \frac{65}{12} - \frac{-58}{3} = \frac{99}{4}$$

$$\int_{1}^{5} x^3 - 4x^2 - 7x + 10\mathrm{d}x$$
$$= \left[\frac{x^4}{4} - \frac{4x^3}{3} - \frac{7x^2}{2} + 10x\right]_{1}^{5}$$
$$= \left(\frac{625}{4} - \frac{500}{3} - \frac{175}{2} + 50\right) - \left(\frac{1}{4} - \frac{4}{3} - \frac{7}{2} + 10\right)$$
$$= \frac{-575}{12} - \frac{65}{12} = \frac{-160}{3}$$

Hence, the total area of the two shaded parts is $\frac{99}{4} + \frac{160}{3} = \frac{937}{12} \approx 78.1$ units squared.

- Note, that this calculation is not reasonable to expect on an exam without a calculator. However, since the question asked for the exact value, we need to use the Newton-Leibniz formula to find this area. A GDC would only give an approximate answer. Nevertheless, you can use the calculator to check your answer:

$$\int_{-2}^{5} \left|(x+2)(x-1)(x-5)\right| \mathrm{d}x \approx 78.083333.$$

Note the absolute value around the expression within the integral. With the use of the absolute value, there is no need to find two integrals to find the area of the region. However, this is only helpful if you use the calculator, because with the absolute value you do not know how to find the antiderivative.

⚠ Be aware

- If the area of a region bounded by the graph of $y = f(x)$ and the x-axis is calculated using the Newton-Leibniz formula, then the parts above and

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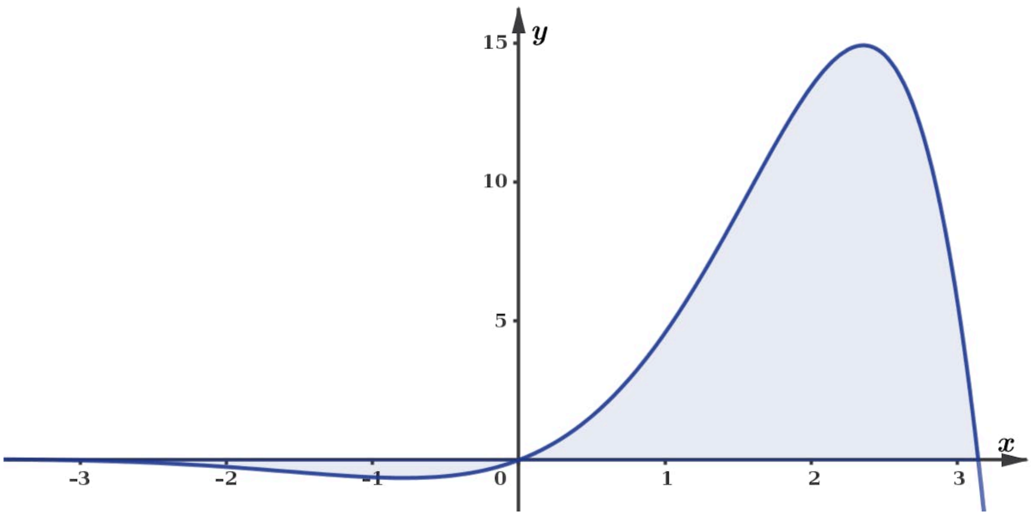
below the x-axis need to be considered separately.

- If the area is calculated by calculator, then the formula $\int_a^b \left|f(x)\right| \mathrm{d}x$ can be used.
  The similar formula without the absolute value only gives the correct answer if the graph of f is above the x-axis.

Example 4

★★☆

- Show that if $f(x) = \mathrm{e}^x(\sin x - \cos x)$, then $f'(x) = 2\mathrm{e}^x \sin x$.
- Find the exact area of the shaded region enclosed by the graph of $y = 2\mathrm{e}^x \sin x$ and the x-axis for $-\pi \leq x \leq \pi$.



More information

The image is a graph with x and y axes. The x-axis ranges from -4 to +4 and the y-axis ranges from 0 to 15. A smooth curve rises from about (0, 0) to peak around (2.5, 15) and then descends back toward the x-axis after 3. The area under the curve is shaded. The y-axis is labeled from 0 to 15 in increments of 5 units, and tick marks are indicated along the negative and positive x-axis to mark -3, -2, -1, 0, 1, 2, 3.

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To find the derivative, use the product rule.

$$\begin{aligned} f(x) &= \mathrm{e}^x(\sin x - \cos x) \\ f'(x) &= \mathrm{e}^x(\sin x - \cos x) + \mathrm{e}^x(\cos x - (-\sin x)) \\ &= \mathrm{e}^x(\sin x - \cos x + \cos x + \sin x) \\ &= 2\mathrm{e}^x \sin x \end{aligned}$$

Since the shaded region has parts above and below the x-axis, we set up and find two integrals to find the two areas separately.

The x-intercepts of the graph are the points, where $\sin x = 0$, so $(-\pi, 0)$, $(0, 0)$ and $(\pi, 0)$.

Using the previous calculation we get

$$\begin{aligned} &\int_{-\pi}^{0} 2\mathrm{e}^x \sin x \mathrm{d}x \\ &= \left[\mathrm{e}^x(\sin x - \cos x)\right]_{-\pi}^{0} \\ &= \left(\mathrm{e}^0(\sin 0 - \cos 0)\right) - \left(\mathrm{e}^{-\pi}(\sin(-\pi) - \cos(-\pi))\right) \\ &= 1 \times (0 - 1) - \frac{1}{\mathrm{e}^\pi} \times (0 - (-1)) = -1 - \frac{1}{\mathrm{e}^\pi} \end{aligned}$$

$$\begin{aligned} &\int_{0}^{\pi} 2\mathrm{e}^x \sin x \mathrm{d}x \\ &= \left[\mathrm{e}^x(\sin x - \cos x)\right]_{0}^{\pi} \\ &= \left(\mathrm{e}^\pi(\sin(\pi) - \cos(\pi))\right) - \left(\mathrm{e}^0(\sin 0 - \cos 0)\right) \\ &= \mathrm{e}^\pi \times (0 - (-1)) - 1 \times (0 - 1) = 1 + \mathrm{e}^\pi \end{aligned}$$

Hence, the total area of the two shaded parts is $1 + \mathrm{e}^\pi + 1 + \frac{1}{\mathrm{e}^\pi} = 2 + \mathrm{e}^\pi + \frac{1}{\mathrm{e}^\pi}$ units squared.

- Note that we did not use a calculator to find the answer. Nevertheless, if available, it is always a good idea to use a calculator to check the result.
  - $2 + \mathrm{e}^\pi + \frac{1}{\mathrm{e}^\pi} \approx 25.1839$
  - $\int_{-\pi}^{\pi} \left|2\mathrm{e}^x \sin x\right| \mathrm{d}x \approx 25.1839$

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Looking ahead

Note that in **Example 4** it was given that $\int 2e^x \sin x dx = e^x(\sin x - \cos x) + c$.

- $\int 2e^x \cos x dx$ is similar. Can you find it?
- How about $\int xe^x dx$ and $\int x \sin x dx$?

You have not learned the tools yet, but you can still make a conjecture, check it using differentiation and modify your conjecture if it does not work. You can also take a look at the answer given by WolframAlpha (http://www.wolframalpha.com) gives as an answer. Type:

integral of 2*e^x*cos(x)

(or the other expressions) in the search line.

Be aware

As you have already seen, not all functions have antiderivatives that can be expressed using the functions we know. Areas of regions bounded by such curves can only be found using technology. Typical examples of functions like this are the ones defined by $f(x) = e^{x^2}$ and $g(x) = \cos(x^2)$. Take a look at the answer given by WolframAlpha when you ask it to find the area bounded by these curves over the interval $[0, 1]$.

3 section questions

5. Calculus / 5.12 Area and volume

Regions bounded by two curves

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/regions-bounded-by-two-curves-id-28211/print/) Assign

It is possible to calculate the area of a region that is enclosed by two different functions by integration.

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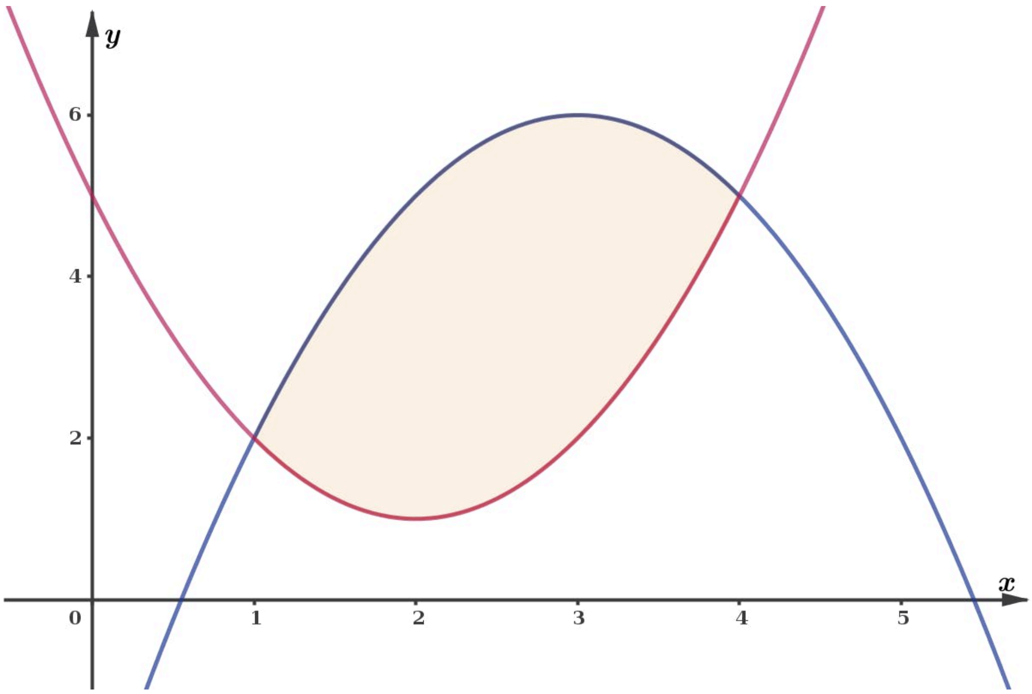
Example 1

★☆☆

Find the area of the region enclosed by the graphs of $y = x^2 - 4x + 5$ and $y = 6x - x^2 - 3$.



More information

The image is a graph displaying two curves and the region enclosed between them. The X-axis represents the variable x, ranging from 0 to 6, and the Y-axis represents the variable y, ranging from 0 to 8. Two functions are plotted: (y = x^2 - 4x + 5) (a downward opening parabola) and (y = 6x - x^2 - 3) (an upward opening parabola). The curves intersect at two points, forming a closed region between them, which is shaded. The area between these curves needs to be calculated as per the given task.

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To find the bounds of the region from left and right we find the intersection points of the two curves. This can either be done using a GDC or by solving

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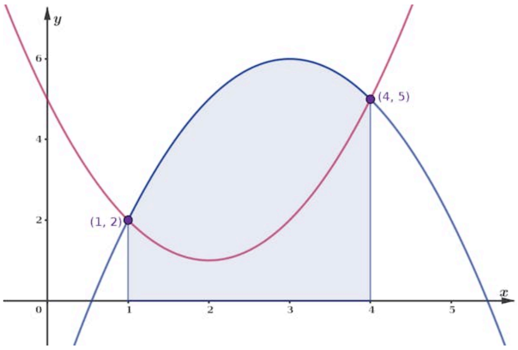
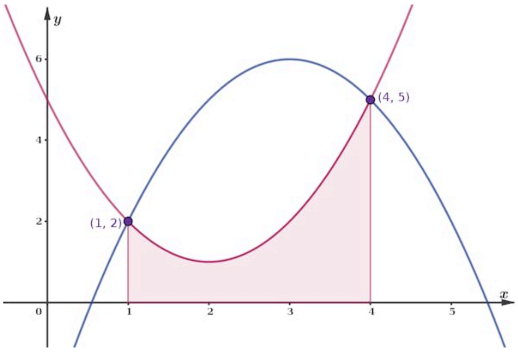
$$x^2 - 4x + 5 = 6x - x^2 - 3$$
$$2x^2 - 10x + 8 = 0$$
$$x^2 - 5x + 4 = 0$$
$$(x-1)(x-4) = 0$$

The solutions are $x = 1$ and $x = 4$.

On the diagrams below we see two regions. The difference of the area of these regions gives the area of the region in the question.





We can find these areas either by calculator or using integration.

- The area of the region below the graph of $y = 6x - x^2 - 3$ (the one on the left) is

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$$\int_1^4 6x - x^2 - 3\mathrm{d}x = \left[\frac{6x^2}{2} - \frac{x^3}{3} - 3x\right]_1^4$$
$$= \left(48 - \frac{64}{3} - 12\right) - \left(3 - \frac{1}{3} - 3\right) = 15$$

- The area of the region below the graph of $y = x^2 - 4x + 5$ (the one on the right) is

$$\int_1^4 x^2 - 4x + 5\mathrm{d}x = \left[\frac{x^3}{3} - \frac{4x^2}{2} + 5x\right]_1^4$$
$$= \left(\frac{64}{3} - 32 + 20\right) - \left(\frac{1}{3} - 2 + 5\right) = 6$$

Hence, the area of the shaded region in the question is $15 - 6 = 9$ units squared.

Generalising the solution above, we can make the following claim.

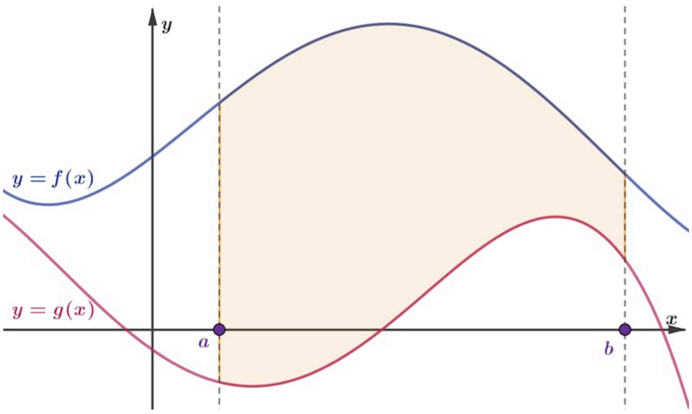
Important

If $g(x) \leqslant f(x)$ for $a < x < b$, then the area of the region bounded by the graphs of $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is given as

$$\int_a^b f(x)\mathrm{d}x - \int_a^b g(x)\mathrm{d}x$$

The same area is also given as

$$\int_a^b f(x) - g(x)\mathrm{d}x$$



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More information

The image shows a graph representing the functions y = f(x) and y = g(x) on a Cartesian plane. The x-axis is labeled 'x,' and the y-axis is labeled 'y.' The graph highlights the region between the two curves, y = f(x) and y = g(x), by shading the area between them with a different color. The region of interest is between points 'a' and 'b' on the x-axis, denoting the limits of integration for the function. The upper curve y = f(x) is shown in blue, and the lower curve y = g(x) is depicted in red. The points where the curves intersect the x-axis are labeled as 'a' and 'b,' and are marked with a purple dot. Vertical dashed lines highlight the bounds a and b.

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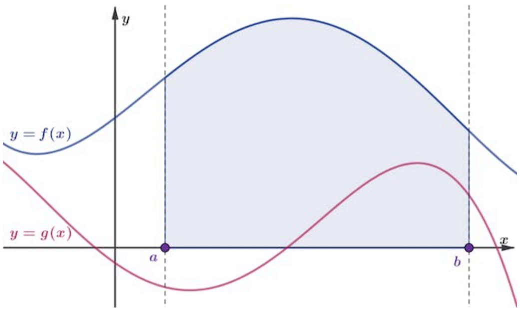
Instead of a formal proof we illustrate the method with the graphs illustrating the claim. As in **Example 1**, we look at two regions.



More information

The image is a graph displaying two functions, y = f(x) and y = g(x). The X-axis is labeled with 'x', and the Y-axis is labeled with 'y'. Two points on the X-axis are marked as 'a' and 'b'. The curve y = f(x) is positioned above y = g(x) between these points, forming a shaded area. The difference between the two curves in this interval indicates the area under the curve y = f(x) and above y = g(x). The intersection points with the X-axis and the vertical lines projecting from 'a' and 'b' highlight the region of interest. Both curves are smooth, showing periodic wave-like behaviors.

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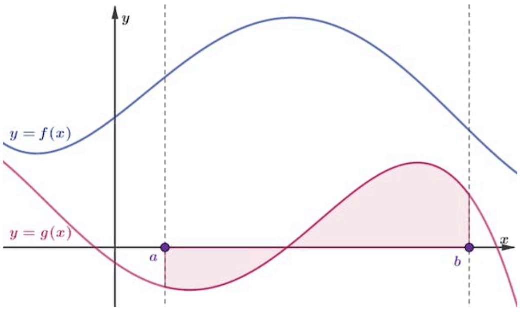

More information

The graph displays two curves, y = f(x) and y = g(x). The x-axis is labeled with points 'a' and 'b' where vertical dashed lines extend upwards. The y-axis is labeled as 'y'. Between the points 'a' and 'b', there's a shaded area under the curve y = g(x) and above the x-axis. The curve y = f(x) is shown in blue and moves upward then downward, while the curve y = g(x) is shown in red, rising and then falling. Points 'a' and 'b' are marked with small circles on the x-axis.

[Generated by AI]

The region on the left illustrates $\int_a^b f(x)\mathrm{d}x$ and the region on the right illustrates $\int_a^b g(x)\mathrm{d}x$.

- Since the area of the part of the region below the x-axis counts as negative in the second integral, when we subtract this integral, this area is actually added to the area of the region on the diagram on the left.
- The area of the part of the second region that is above the x-axis counts as positive in the integral, so when we subtract the second integral, this area is subtracted from the area of the region on the diagram on the left.

This means that $\int_a^b f(x)\mathrm{d}x - \int_a^b g(x)\mathrm{d}x$ gives the correct area even when part of the second curve is below the x-axis. There are other cases (for example part of the graph of f can also be below the x-axis), but we will not cover all the cases here.

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The proof of the second form of the area is the consequence of the Newton-Leibniz formula.

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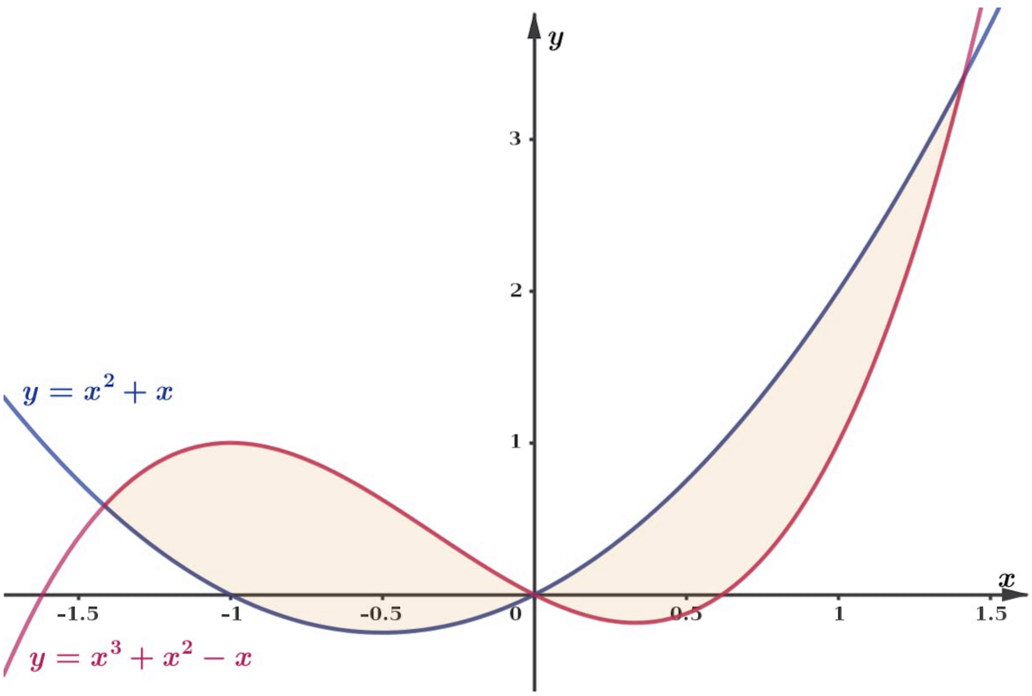
If $F'(x) = f(x)$ and $G'(x) = g(x)$, then

$$\begin{aligned}\int_a^b f(x)\mathrm{d}x - \int_a^b g(x)\mathrm{d}x &= \Big[F(x)\Big]_a^b - \Big[G(x)\Big]_a^b \\ &= \Big(F(b) - F(a)\Big) - \Big(G(b) - G(a)\Big) \\ &= \Big(F(b) - G(b)\Big) - \Big(F(a) - G(a)\Big) \\ &= \Big[F(x) - G(x)\Big]_a^b \\ &= \int_a^b f(x) - g(x)\mathrm{d}x\end{aligned}$$

Example 2

★★☆

Find the area of the region enclosed by the graphs of $y = x^2 + x$ and $y = x^3 + x^2 - x$.



More information

The graph displays two curves, y = x^2 + x and y = x^3 + x^2 - x, overlaid on a Cartesian plane with x and y axes. The x-axis ranges from -1.5 to 1.5, and the y-axis ranges from -1 to 3. The curves are plotted in distinct colors with y = x^2 + x shown in blue and y = x^3 + x^2 - x in red. These curves intersect at two points. The regions enclosed between the curves are shaded, indicating the area to be calculated. The curves appear to intersect at

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approximately x = -1 and x = 1, enclosing a symmetric region between them. The blue curve has a parabolic shape opening upwards, while the red curve is a cubic polynomial with a more complex trajectory, starting below the x-axis, curving above it, and crossing back down.

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Method 1 (without calculator)

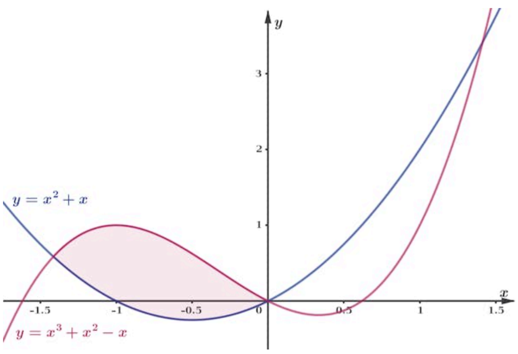
To apply the claim, you need to know the bounds of the region and to know which is the upper and which is the lower curve.

To find the bounds, solve

$$\begin{aligned} x^3 + x^2 - x &= x^2 + x \\ x^3 - 2x &= 0 \\ x(x^2 - 2) &= 0 \end{aligned}$$

The solutions are $x = 0$ and $x = \pm\sqrt{2}$.

Since the curves cross each other at $x = 0$, we use two integrals to find the area of the two parts of the region separately.



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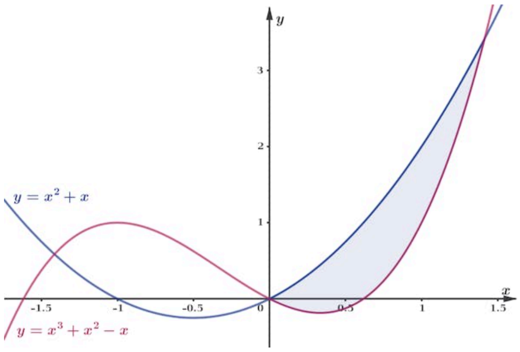

For the region on the left, the cubic curve is above the quadratic, so the area is

$$\begin{aligned}\int_{-\sqrt{2}}^{0}(x^3+x^2-x)-(x^2+x)\mathrm{d}x &= \int_{-\sqrt{2}}^{0} x^3-2x\mathrm{d}x\\ &= \left[\frac{x^4}{4}-x^2\right]_{-\sqrt{2}}^{0}\\ &= \left(\frac{0^4}{4}-0^2\right)-\left(\frac{\left(-\sqrt{2}\right)^4}{4}-\left(-\sqrt{2}\right)^2\right)\\ &= 0-(1-2)=1\end{aligned}$$

For the region on the right, the cubic curve is below the quadratic, so the area is

$$\begin{aligned}\int_{0}^{\sqrt{2}}(x^2+x)-(x^3+x^2-x)\mathrm{d}x &= \int_{0}^{\sqrt{2}} 2x-x^3\mathrm{d}x\\ &= \left[x^2-\frac{x^4}{4}\right]_{0}^{\sqrt{2}}\\ &= \left(\left(\sqrt{2}\right)^2-\frac{\left(\sqrt{2}\right)^4}{4}\right)-\left(0^2-\frac{0^4}{4}\right)\\ &= (2-1)-0=1\end{aligned}$$

Hence, the area of the region enclosed by the two curves is $1+1=2$ units squared.

- Note the interesting fact, that the two parts of the region has the same area even though they have different shapes. Can you explain why this is true?

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Method 2 (using GDC)

Using a GDC the task is easier, although this way only gives an approximate value for the area.

Calculators can give you the intersection points of the curves. These are

$$(-1.41421, 0.58579), (0,0) \text{ and } (1.41421, 3.41421).$$

To eliminate the need to check which is the upper and which is the lower curve, integrate the absolute value of the difference to work out that the area of the shaded region is approximately

$$\int_{-1.41421}^{1.41421} \left|(x^3+x^2-x)-(x^2+x)\right| \mathrm{d}x \approx 2 \text{ units squared.}$$

- Note that the absolute value is important. Without the absolute value,
  $$\int_{-1.41421}^{1.41421} (x^3+x^2-x)-(x^2+x)\mathrm{d}x \approx 0$$
  which is clearly not the correct value for the area. Can you explain the reason?

Be aware

The condition $f(x) \geqslant g(x)$ is important. If this is not satisfied, then the integral

$$\int_a^b f(x)-g(x)\mathrm{d}x$$

does not give the area bounded by the curves $y=f(x)$ and $y=g(x)$.

Exam tip

Even if the condition $f(x) \geqslant g(x)$ is not satisfied everywhere, on a calculator paper, you can use

$$\int_a^b \left|f(x)-g(x)\right|\mathrm{d}x$$

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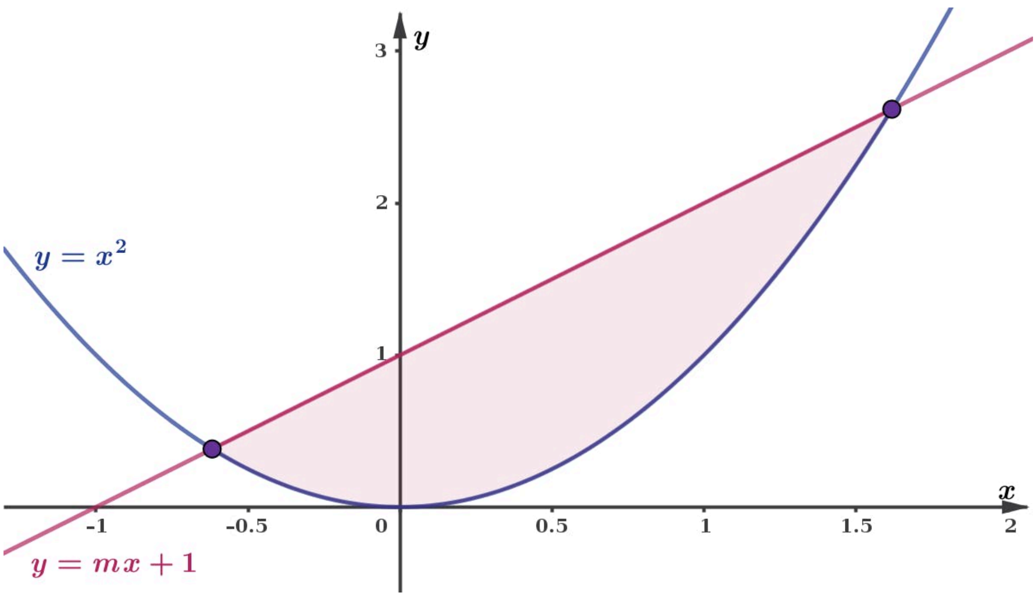
to find the area bounded by the curves $y = f(x)$ and $y = g(x)$ above the interval $[a, b]$.

Example 3

★★★

The area of the region bounded by the graph of $y = x^2$ and the line $y = mx + 1$ is 2 units squared. There are two such values of m. Find the positive one.

The diagram below illustrates the region.



To find the bounds of the region, solve

$$x^2 = mx + 1$$
$$x^2 - mx - 1 = 0$$

The solutions are $x_1 = \dfrac{m - \sqrt{m^2 + 4}}{2}$ and $x_2 = \dfrac{m + \sqrt{m^2 + 4}}{2}$.

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Hence, an expression for the area of the region is

$$\int_{x_1}^{x_2} mx + 1 - x^2 \mathrm{d}x$$

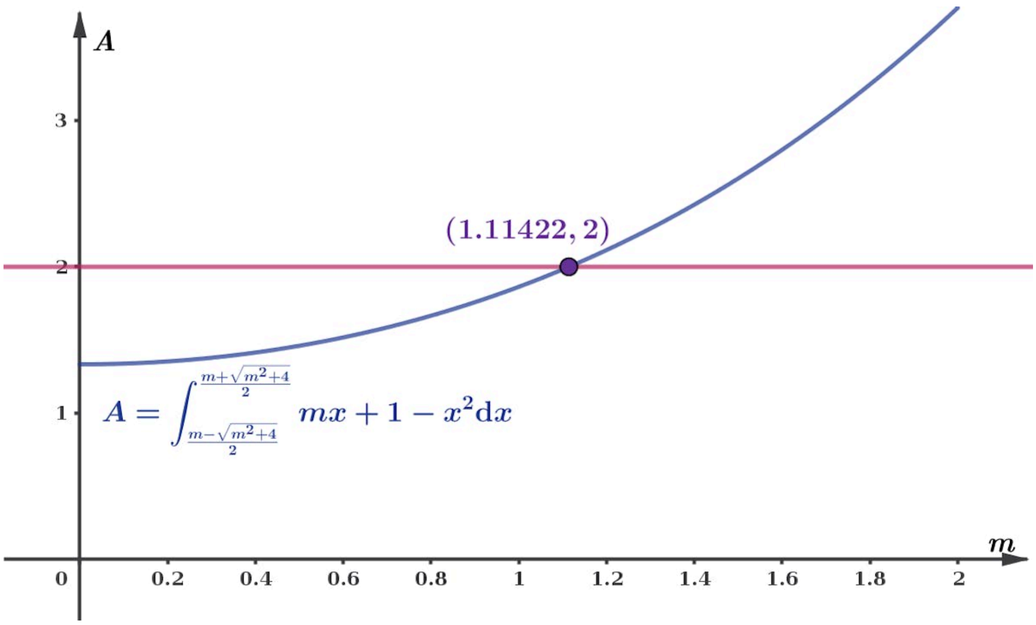
On a GDC you can define the function

$$A : m \mapsto \int_{x_1}^{x_2} mx + 1 - x^2 \mathrm{d}x$$

so in other words the function, where $A(m)$ is the value of the integral, where the uknown m is both in the expression and in the limits of integration. To see how to do this on some of the calculators, see the instructions after the solution.

You can then plot the graph and see at which value of m it crosses the line $y = 2$. This value will be the solution to the question.

The diagram below shows the graphs and the intersection point.



Hence, the value you are looking for is $m \approx 1.11$

Here is some help on how to work out the solution to **Example 3** using different calculators.

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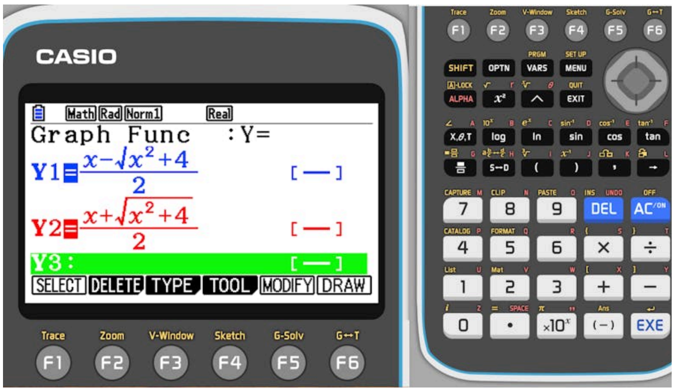
Steps	Explanation
These instructions show you how to find the value of m, such that $$\int_{\frac{m-\sqrt{m^2+4}}{2}}^{\frac{m+\sqrt{m^2+4}}{2}} mx+1-x^2\mathrm{d}x=2$$ From the main menu select the graph option.	
Define two functions, one for each root of the quadratic equation. These will be used in the limit of integration.	

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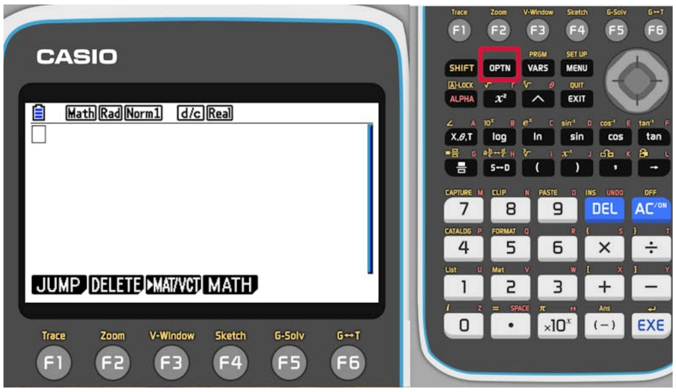
<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>Go back to the main menu and choose the calculator option.</td><td>
</td></tr>
<tr><td>You will need to find the numerical equation solver (SolveN) option.

Press OPTN to start your search ...</td><td>
</td></tr>
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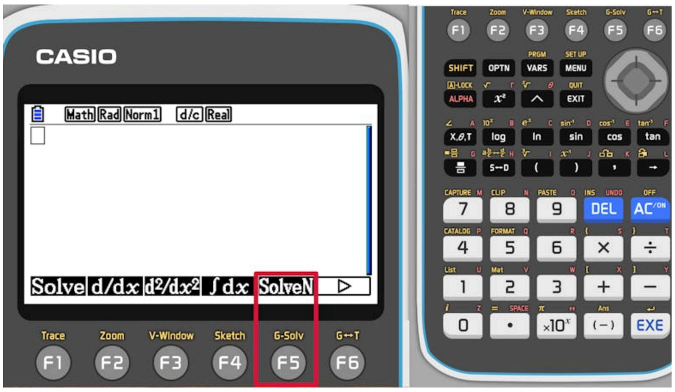
Steps	Explanation
... press F4 for the calculus related options ...	
... and F5 to open the numerical equation solver (SolveN).	

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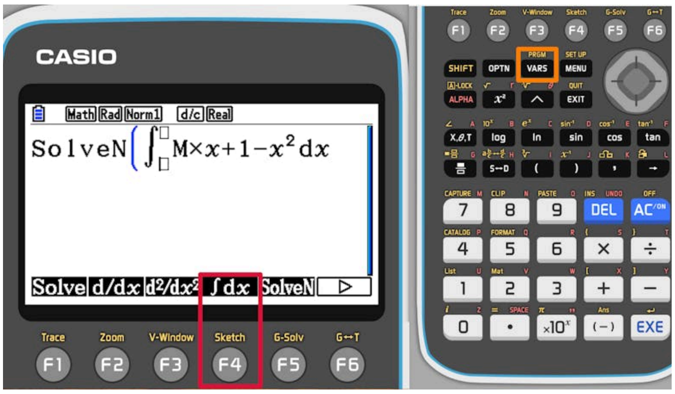
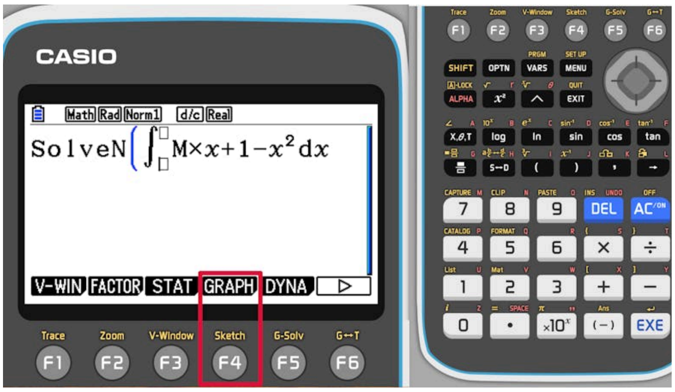
Steps	Explanation
Press F4 to bring up the integral template and enter the function. Note, that you will need a variable name for m. You can enter any letter using the Alpha key. To use the predefined expressions for the limit, you need to find the function variable names. Press VARS to search for these ...	
... and press F4 to access the graph related variable names.	

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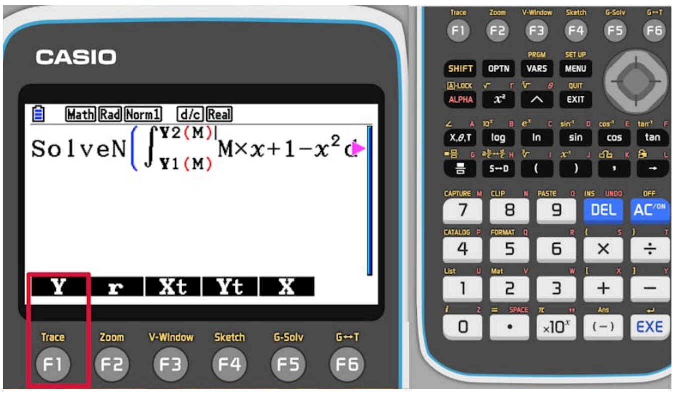
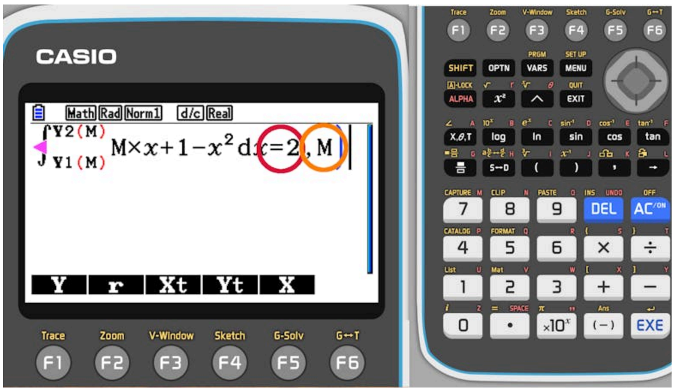
Steps	Explanation
Use F1 for the function variable name. Note, that you need these functions evaluated at m, so use the format on the screenshot. Once done, move to the end of the line to finish the expression.	
You need to tell the calculator that you are looking for the value, where the integral is 2, so set the integral equal to 2. You also need to tell the calculator, that the variable you want the equation to be solved is m, so type a comma and the variable name after the equation. Once done, close the bracket and press enter.	

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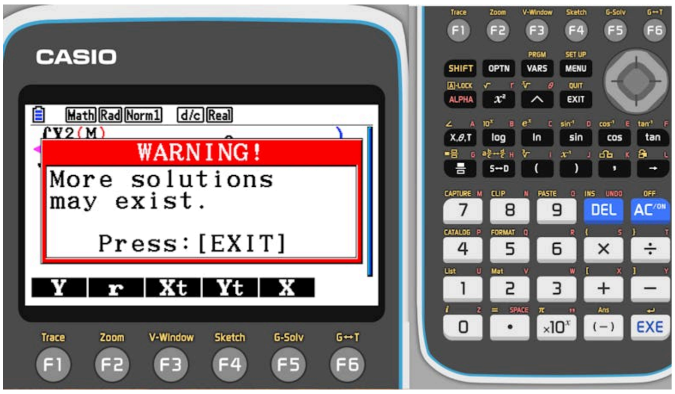
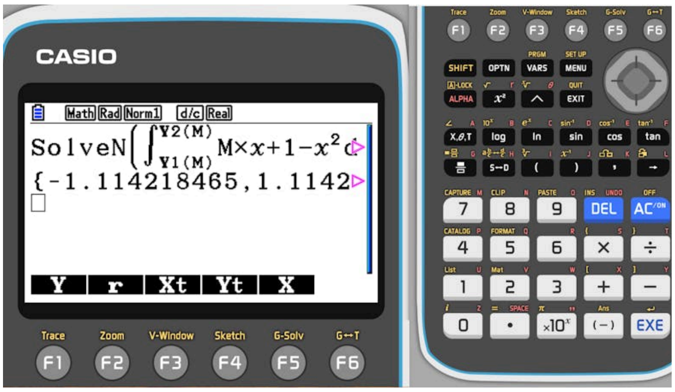
Steps	Explanation
You will see the usual warning that the calculator may not show you all the solutions.	
In this case the calculator shows both the negative and the positive solutions.	

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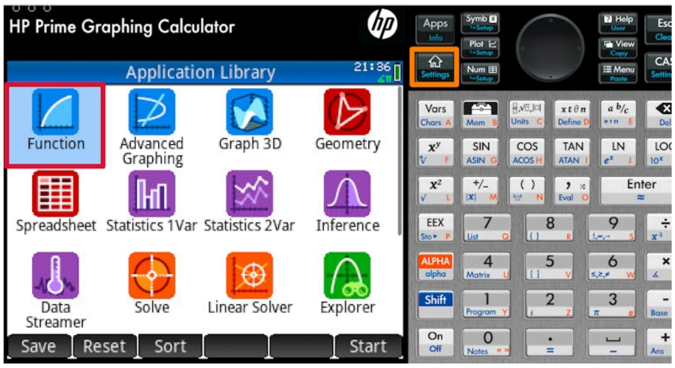
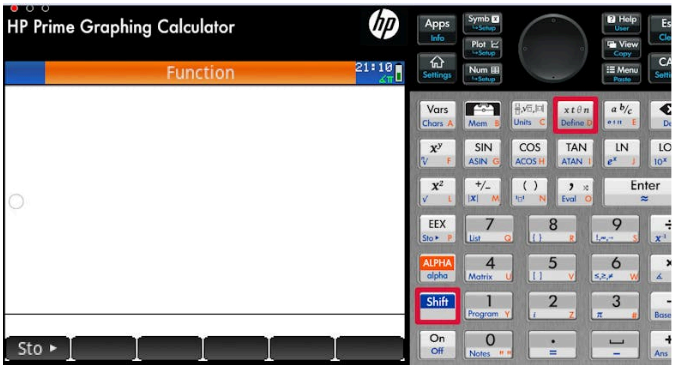
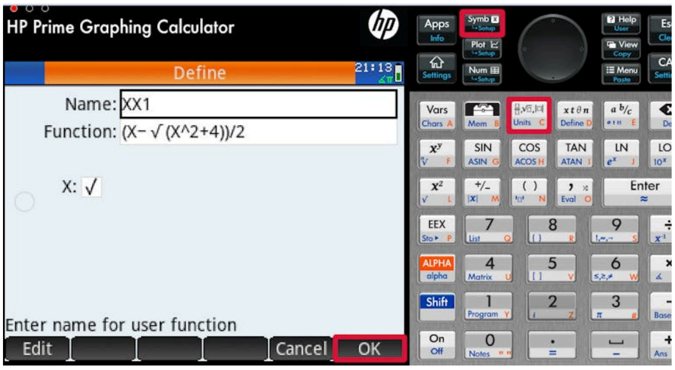
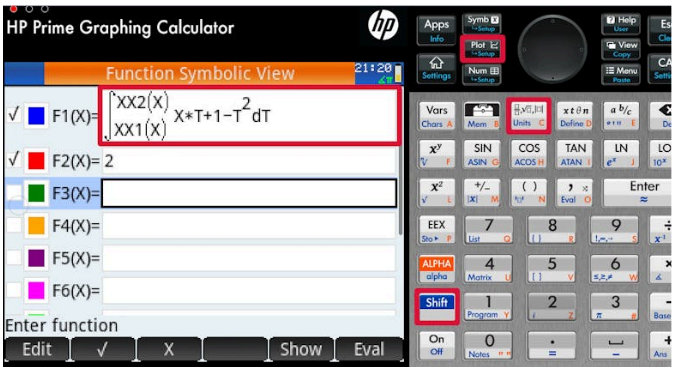
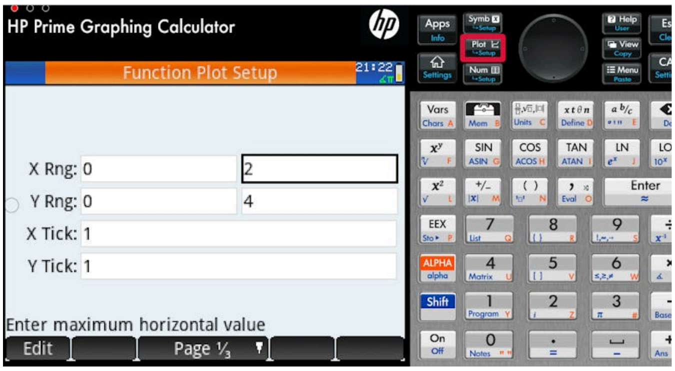
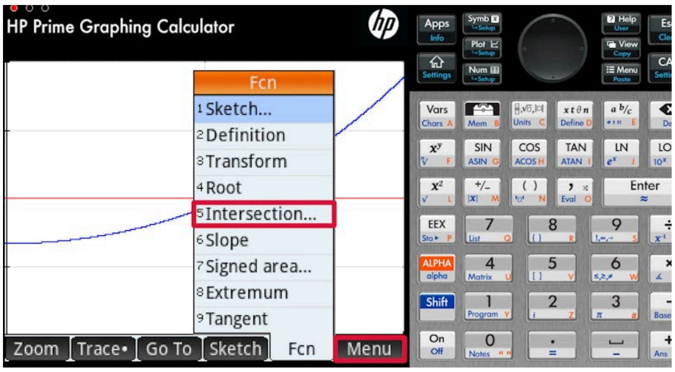
Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
These instructions show you how to find the value of m, such that $$\int_{\frac{m-\sqrt{m^2+4}}{2}}^{\frac{m+\sqrt{m^2+4}}{2}} mx+1-x^2\mathrm{d}x=2$$ Find the function application and enter the home screen.	
On the home screen select the option to define a user defined function.	

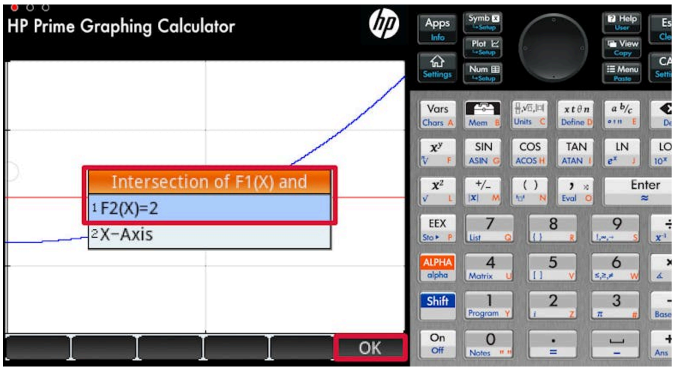
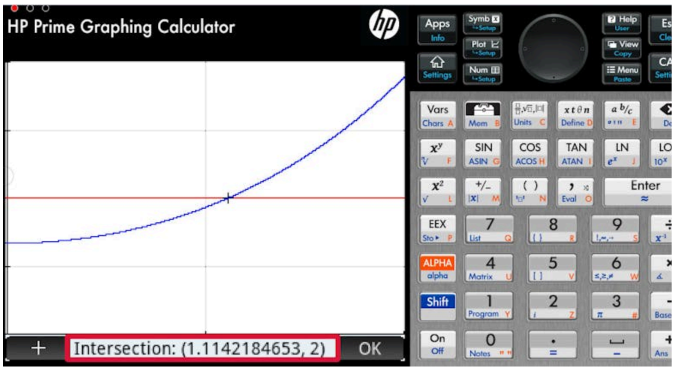
Student view

Steps	Explanation
Write the expression for one of the roots. Note that X is used as the variable instead of m in the example above, because this is the name of the variable the calculator uses in graphing functions. Define also an other function for the other roots of the quadratic equation. Once done, enter the symbolic view.	
In the symbolic view you can define the area of the region between the parabola and the line as a function of the slope. Note that in the example you used m for the slope and x as the integration variable. Here you need to use X for the slope and any other letter as the integration variable (since for the calculator X is the independent variable for the graphs). Define also the constant 2 function, because you would like to find the slope when the area is exactly 2. Once done, enter the setup for the plot.	

Student view

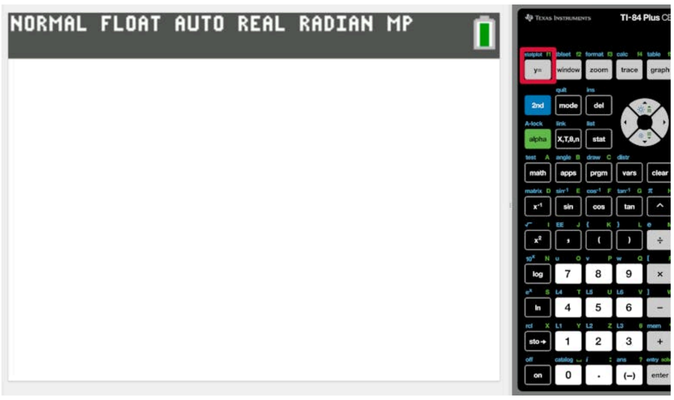
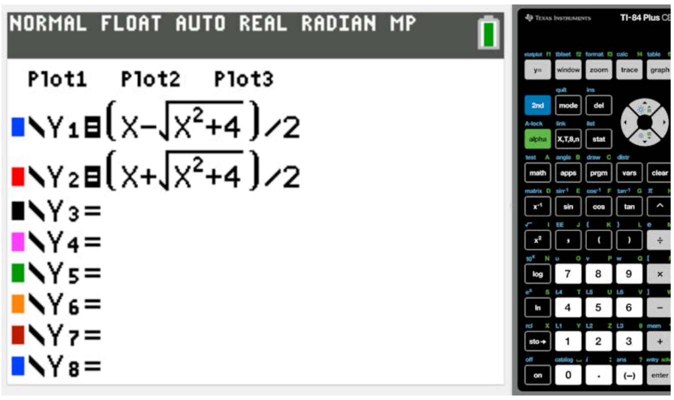
Steps	Explanation
Enter the range for the plot. Since you want to see the intersection point, 2 should be in the y-range. You may need to experiment with the horizontal range. Once done, enter the plot view.	
In the plot view select to find the intersection in the function menu.	

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
Select the two curves and press OK.	
The calculator moves the cursor to the intersection point and displays the coordinates.	

Student view

Overview (/study/ap ai- hl/sid- 132- cid- 761618/ov

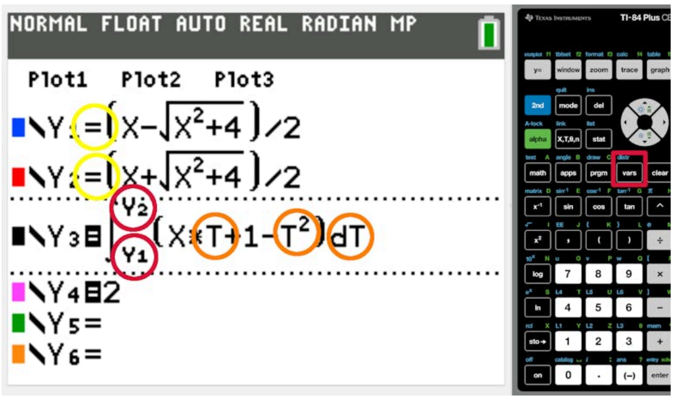
Steps	Explanation
These instructions show you how to find the value of m, such that $$\int_{\frac{m-\sqrt{m^2+4}}{2}}^{\frac{m+\sqrt{m^2+4}}{2}} mx+1-x^2\mathrm{d}x=2$$ From the main screen choose the option to enter functions.	
Define two functions, one for each root of the quadratic equation. These will be used in the limit of integration. Once done, move down to the third line.	

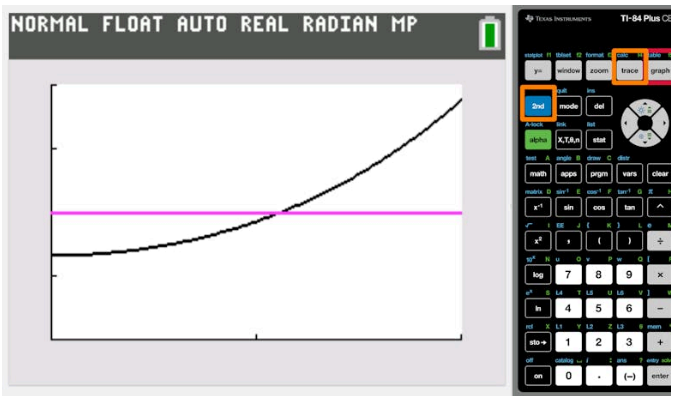
Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

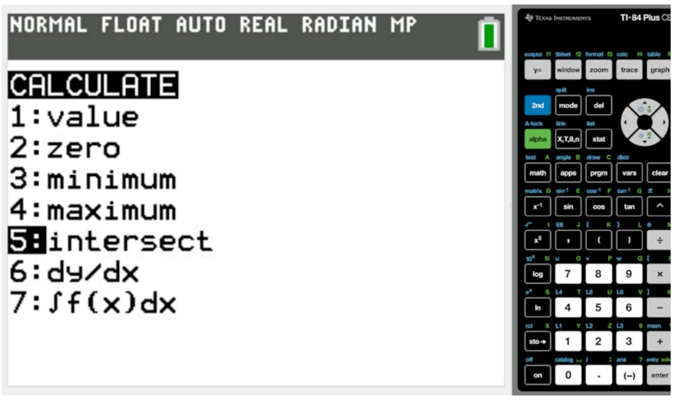
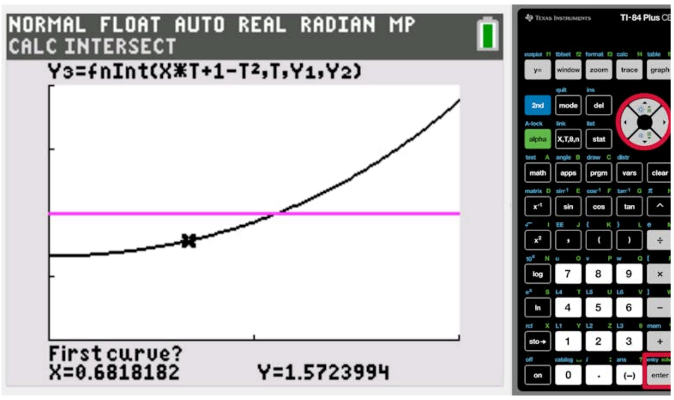
Steps	Explanation
You will need to write an integral expression for the third function, so press math and choose the numerical integral (fnInt) option.	

Student
view

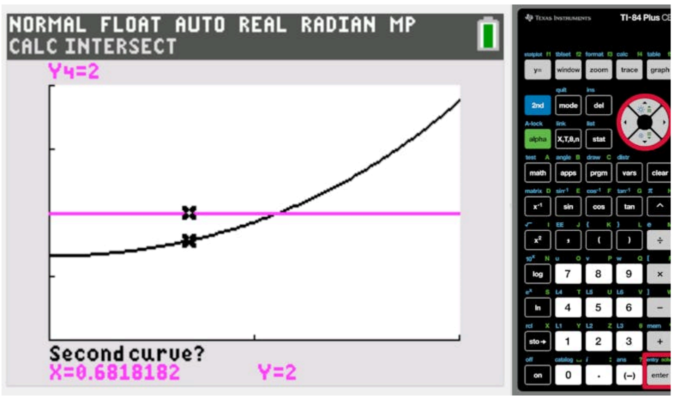
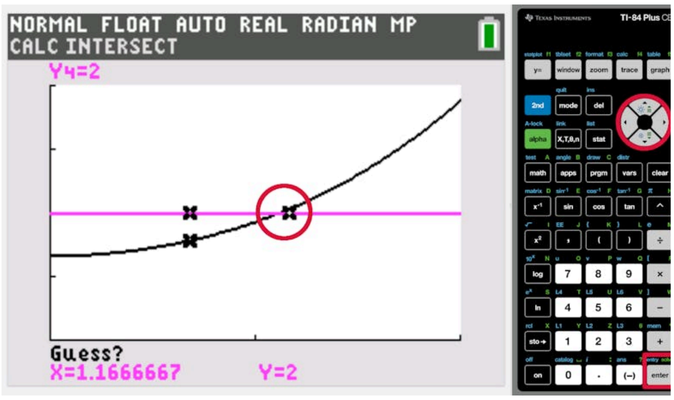
Steps	Explanation
There are several things you need to be careful when you enter the integral expression. • Use the functions you defined previously instead of typing the expressions in the limits of the integration. You can access the names for example by pressing the variable button (and looking for the function variables). • You need to change the variaable names from the one in the expression you have on paper. the calculator has x as the independent variable, so change m to x. Since this way x is already used, you also need to change the integration variable to any letter. Enter also the constant function $y = 2$ (since you are interested in the value of m where the integral is 2). Once all this is done, unselect the first two functions, because you do not need to see the graphs of those.	

Steps	Explanation
To set the viewing window, press window. Enter the range for the plot. Since you want to see the intersection point, 2 should be in the y-range. You may need to experiment with the horizontal range.	
Press graph to see the graphs and press 2nd/calc to start the search for the intersection point.	

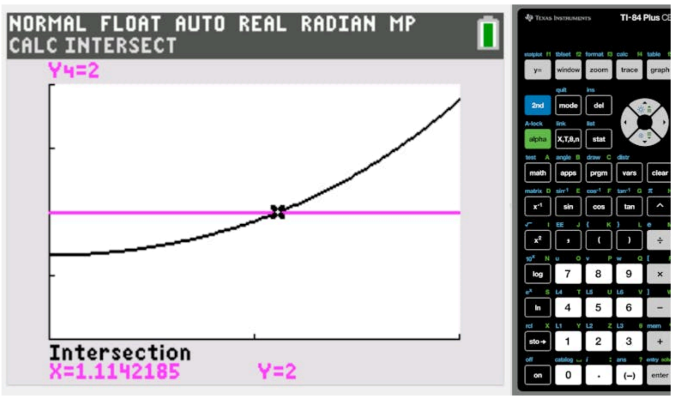
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

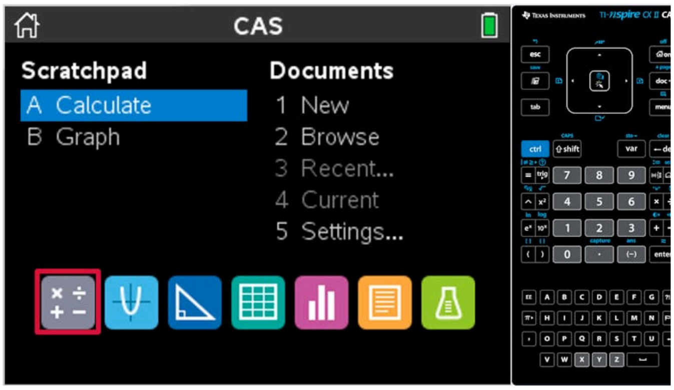
Steps	Explanation
Select the option to find the intersection points ...	
... select the first curve and press enter to confirm ...	

Student view

Steps	Explanation
... select the second curve and press enter to confirm ...	
... move the cursor close to the intersection point and press enter to confirm your guess.	

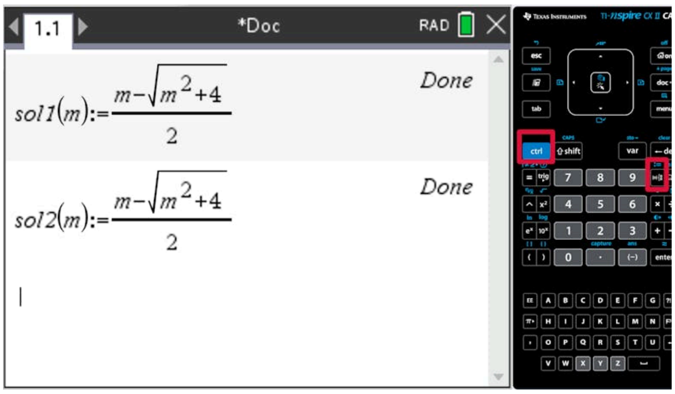
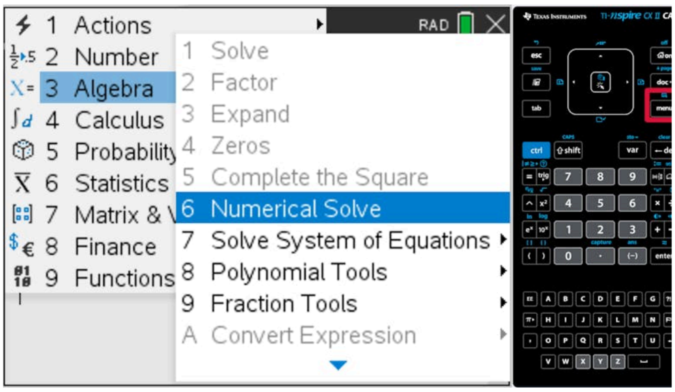
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
The calculator moves the cursor to the intersection point and displays the coordinates.	

Steps	Explanation
These instructions show you how to find the value of m, such that $$\int_{\frac{m-\sqrt{m^2+4}}{2}}^{\frac{m+\sqrt{m^2+4}}{2}} mx+1-x^2\mathrm{d}x=2$$ Open a calculator page.	

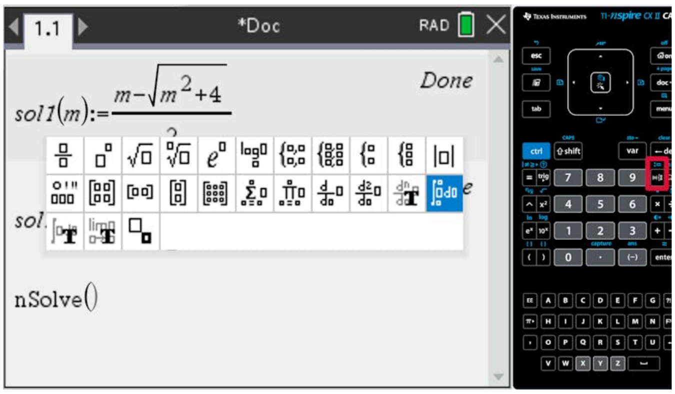
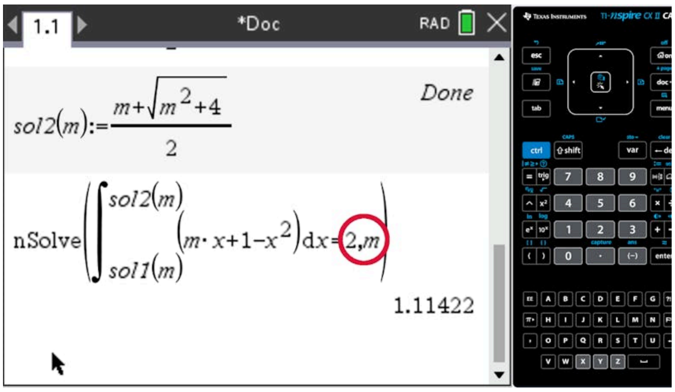
Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
Define expressions for the roots. These will be useful to enter as the limits of the integral. Note, that the colon equal sign is used to indicate a definition. You can give any name to the functions you are defining.	
To find the solution to the question, open the menu and search for the numerical equation solver option.	

Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Steps	Explanation
You want to solve an equation involving an integral, so open the template menu and choose the integral teplate.	
Type in the expressions. Remember to use the predefined expressions in the limits, this will improve readability. Set the integral equal to 2 (since that is the equation you would like to solve) and add a comma and m to indicate, that you would like the calculator to solve this equation for m. Press enter to get the solution.	

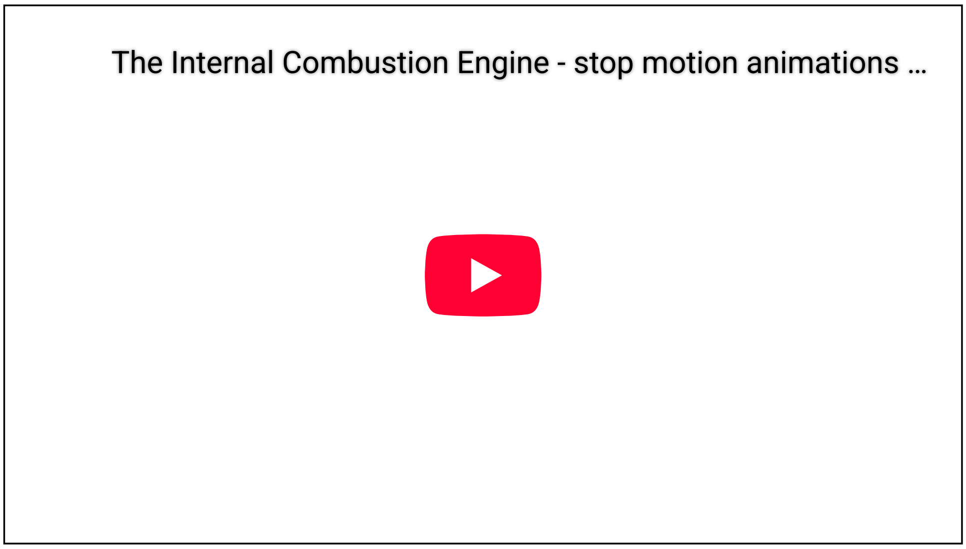
Example 4

★★☆

In a car engine, the pressure and volume of the gas changes periodically. This cycle can be illustrated on a so-called PV diagram. Take a look at the video that explains what such a diagram means.

Student
view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov



Video 1. The Internal Combustion Engine - stop motion animations and the PV cycle (Otto cycle)

More information for video 1

1

00:00:01,300 --> 00:00:04,333

man: My class and I had a go

at making short stop motion animations

2

00:00:04,400 --> 00:00:06,900

to show how the internal

combustion engine works.

3

00:00:07,733 --> 00:00:10,667

This rather nice engine

is from a Ferrari Testarossa.

4

00:00:12,900 --> 00:00:15,433

The white particles

on the left hand side here

5

00:00:15,500 --> 00:00:18,600

represent the mixture of particles

of air and fuel.

6

00:00:19,333 --> 00:00:22,600

The inlet valve opens

as the piston moves down.

7

Student
view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

00:00:23,333 --> 00:00:25,467

As the volume of the cylinder increases,

8

00:00:25,700 --> 00:00:27,667

this draws air and fuel mixture

9

00:00:27,733 --> 00:00:30,567

into the chamber

at constant atmospheric pressure.

10

00:00:31,733 --> 00:00:34,567

The piston then continues to move upwards,

11

00:00:34,933 --> 00:00:38,567

either forced upwards by the momentum

of the crank or the force generated

12

00:00:38,633 --> 00:00:40,433

by the other cylinders in the engine.

13

00:00:41,233 --> 00:00:43,633

This compresses the fuel

and the air mixture

14

00:00:43,700 --> 00:00:45,733

in a roughly adiabatic compression.

15

00:00:47,667 --> 00:00:51,900

In a petrol engine, the spark plug

at the top of the cylinder ignites

16

00:00:51,967 --> 00:00:53,433

the fuel at this point here.

17

00:00:56,000 --> 00:00:58,433

This is at the point

of maximum compression.

18

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

00:00:59,267 --> 00:01:02,333

This ignition causes a rapid

increase in pressure.

19

00:01:03,167 --> 00:01:04,333

After combustion,

20

00:01:04,400 --> 00:01:07,233

I've changed the particles

in the animation from whites to blue

21

00:01:07,400 --> 00:01:09,500

to represent the combustion products.

22

00:01:10,733 --> 00:01:14,100

The combustion products are now at

a very hot temperature and pressure

23

00:01:14,500 --> 00:01:15,367

and a very high pressure,

24

00:01:15,467 --> 00:01:18,033

and they force the piston

back down the cylinder.

25

00:01:18,667 --> 00:01:20,300

This is called the power stroke.

26

00:01:21,500 --> 00:01:24,967

In the final phase of the cycle,

the exhaust valve opens

27

00:01:25,700 --> 00:01:28,200

as the exhaust products

are at a pressure much higher

28

00:01:28,267 --> 00:01:29,867

than atmospheric pressure,

Student view

29

00:01:29,933 --> 00:01:32,600

they tend to rush out

of the open exhaust port.

30

00:01:34,233 --> 00:01:36,833

Finally, the momentum

of the piston continues

31

00:01:36,900 --> 00:01:39,033

to push it back to the starting position

32

00:01:39,100 --> 00:01:41,000

ready for the next inlet stroke.

33

00:01:45,500 --> 00:01:47,933

Some people like to remember

the four stages

34

00:01:48,000 --> 00:01:49,367

of the engine cycle.

35

00:01:49,433 --> 00:01:55,100

With these four words suck, squeeze,

bang, and blow.

36

00:01:59,067 --> 00:02:00,467

Let's have a look at some animations

37

00:02:00,533 --> 00:02:03,200

that my students created

during their recent physics lesson.

38

00:02:06,967 --> 00:02:10,067

Although this first animation

is rather simplistic

39

00:02:10,133 --> 00:02:12,767

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

and doesn't include much detail

of the cylinder and valves,

40

00:02:13,133 --> 00:02:15,367

I rather like it

for its scientific accuracy.

41

00:02:16,067 --> 00:02:17,533

The students have represented fuel

42

00:02:17,600 --> 00:02:20,400

and air particles

with blue and white dots,

43

00:02:20,600 --> 00:02:24,033

and they change to orange

to indicate combustion has taken place.

44

00:02:27,000 --> 00:02:29,933

This second animation is rather

neat and nicely drawn.

45

00:02:30,267 --> 00:02:31,867

The students have used an orange color

46

00:02:31,933 --> 00:02:34,033

to indicate the high pressure

in the cylinder,

47

00:02:34,267 --> 00:02:37,367

and they have changed the color

of the combustion products to black.

48

00:02:37,933 --> 00:02:39,700

One slight problem with this animation

49

00:02:39,767 --> 00:02:41,567

is that the number

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

of particles in the cylinder

50

00:02:41,633 --> 00:02:44,733

doesn't seem to stay constant even

when the valves are shut.

51

00:02:47,633 --> 00:02:49,733

This animation has been excellently drawn

52

00:02:49,800 --> 00:02:52,567

and also cleverly engineered

using some split pins

53

00:02:52,633 --> 00:02:54,333

to show the rotation of the crank.

54

00:02:55,267 --> 00:02:59,133

This nicely illustrates how the momentum

of the crank carries the piston upwards

55

00:02:59,200 --> 00:03:02,067

during the compression

and exhaust strokes.

56

00:03:05,500 --> 00:03:08,167

I like how this final animation

uses paper confetti

57

00:03:08,233 --> 00:03:11,133

from a hole punch to represent

the different particles,

58

00:03:11,300 --> 00:03:16,233

white particles for air, yellow for fuel,

and orange for the combustion products.

59

00:03:16,867 --> 00:03:19,400

The orange combustion products

Student
view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

cleverly spread out

60

00:03:19,467 --> 00:03:21,000

from the point of the spark plug.

61

00:03:21,733 --> 00:03:23,900

The other students in the group

affectionately christened

62

00:03:23,967 --> 00:03:26,433

this animation, the rubbish compressor.

63

00:03:29,767 --> 00:03:32,467

I'm now going to explain the stages

of the engine cycle

64

00:03:32,533 --> 00:03:33,567

in a little more detail.

65

00:03:34,333 --> 00:03:37,667

This diagram on the right

is a pressure volume diagram.

66

00:03:38,767 --> 00:03:41,800

P here for the pressure

of the gas on the y-axis

67

00:03:42,167 --> 00:03:44,367

and V here for the volume of the cylinder.

68

00:03:44,500 --> 00:03:45,767

on the x-axis.

69

00:03:46,767 --> 00:03:49,167

The cycle shown is the idealized cycle

70

00:03:49,233 --> 00:03:51,867

for a petrol engine or auto cycle.

Student
view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

71

00:03:52,567 --> 00:03:54,933

The green dot makes

its way around the cycle

72

00:03:55,000 --> 00:03:57,067

roughly in time with the cylinder.

73

00:03:57,667 --> 00:04:00,467

You might be able

to spot the different stages of the cycle

74

00:04:00,600 --> 00:04:01,733

on the diagram.

75

00:04:03,233 --> 00:04:06,567

I'm now going to look at each

of those stages in a little more detail.

76

00:04:09,100 --> 00:04:11,100

In the inlet or suck stroke,

77

00:04:11,167 --> 00:04:14,067

the volume of the cylinder increases

as the valve is opened

78

00:04:14,133 --> 00:04:16,567

to the atmosphere,

drawing in air and fuel.

79

00:04:17,367 --> 00:04:21,567

Because the valve is open, the pressure

remains at atmospheric pressure.

80

00:04:23,833 --> 00:04:28,300

In the compression stroke,

the valves are closed.

81

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

00:04:28,600 --> 00:04:31,100

As the piston makes a tight

fit with the cylinder,

82

00:04:31,167 --> 00:04:32,633

the gas is compressed.

83

00:04:33,500 --> 00:04:37,167

This curved line on the graph

is an adiabatic compression,

84

00:04:37,533 --> 00:04:40,433

one in which no heat is exchanged

with the surroundings.

85

00:04:41,100 --> 00:04:44,800

This means that the pressure

and temperature of the gas both rise.

86

00:04:47,067 --> 00:04:49,967

The ignition happens very quickly

in a petrol engine.

87

00:04:51,833 --> 00:04:56,233

In the idealized auto cycle, the volume

of the cylinder remains constant,

88

00:04:56,800 --> 00:05:00,833

while the pressure and temperature

of the gas increase during combustion.

89

00:05:01,733 --> 00:05:05,267

In reality, of course, the line can't be

completely vertical

90

00:05:05,333 --> 00:05:07,733

as the piston never stops moving.

91

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

00:05:10,233 --> 00:05:13,500

The power stroke is represented

in the auto cycle

92

00:05:13,567 --> 00:05:15,933

by an adiabatic expansion,

93

00:05:16,200 --> 00:05:17,867

this second curved line here.

94

00:05:18,900 --> 00:05:21,900

During this expansion, the volume

of the cylinder increases

95

00:05:21,967 --> 00:05:24,667

and the pressure and temperature

of the gas decrease.

96

00:05:25,633 --> 00:05:28,700

This does work on the piston

pushing it down.

97

00:05:30,700 --> 00:05:32,567

The exhaust phase is represented

98

00:05:32,733 --> 00:05:35,300

by another vertical line

on the auto cycle.

99

00:05:35,933 --> 00:05:39,033

The pressure reduces at constant volume.

100

00:05:40,067 --> 00:05:43,500

When the exhaust valve opens,

the gas is at high pressure

101

00:05:43,567 --> 00:05:45,733

and so it rushes out of the open valve.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

102

00:05:50,000 --> 00:05:53,600

The final exhaust stroke pushes

out any remaining gas

103

00:05:53,667 --> 00:05:54,600

at low pressure

104

00:05:54,667 --> 00:05:57,833

and brings the green dot back

to the starting position.

105

00:05:59,900 --> 00:06:02,967

Hopefully, you can identify

all of those processes

106

00:06:03,067 --> 00:06:06,000

as the green dot makes its

way around the auto cycle.

107

00:06:07,100 --> 00:06:09,600

The area enclosed by the graph

108

00:06:09,767 --> 00:06:14,200

represents the amount

of work done by the engine per cycle.

109

00:06:17,400 --> 00:06:18,767

I hope you've enjoyed this video

and the animations

110

00:06:18,833 --> 00:06:21,833

that my students and I created.

111

00:06:22,367 --> 00:06:25,367

Why not have a look at some of the other

stop motion animation projects

112

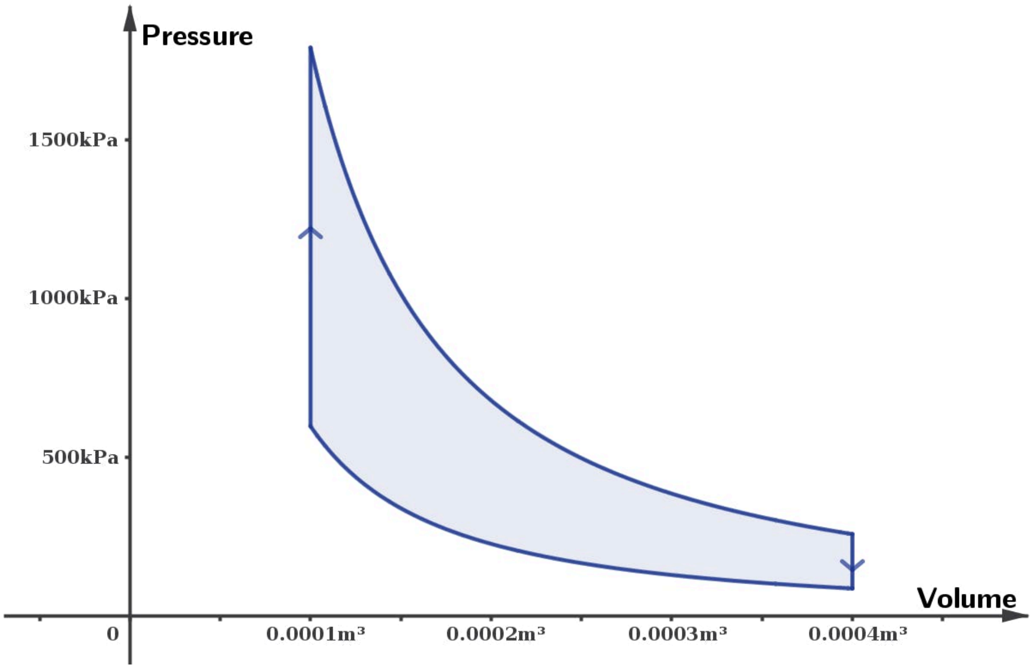
Student view

00:06:25,433 --> 00:06:28,833

that I've created with some classes

by clicking one of the links below?

The diagram below shows multiple steps of a thermodynamic cycle, similar to the one you have seen in the video. The area inside the curve represents the work done during the cycle.



More information

This is a diagram illustrating a thermodynamic cycle characterized by a pressure-volume curve. The graph has a vertical Y-axis labeled 'Pressure' with values at intervals (0, 500, 1000, 1500 kPa), and a horizontal X-axis labeled 'Volume' with values at intervals (0, 0.0001, 0.0002, 0.0003, 0.0004 m^3). The curve forms a closed loop, beginning at a higher pressure and lower volume, expanding to a higher volume, and then contracting back to the starting point. This area inside the curve represents the work done during the cycle. The bounding equations of the curves are given as PV^{1.4}=0.0015 and PV^{1.4}=0.0045.

[Generated by AI]

Student view

The equation of the bounding curves are $PV^{1.4} = 0.0015$ and $PV^{1.4} = 0.0045$.

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

The vertical boundaries are at $V = 0.0001$ and $V = 0.0004$.

Volume is measured in cubic metres, pressure is measured in kilopascals, work is measured in joules.

Find how much work is done during the cycle.

The equations of the curves are $P = \dfrac{0.0015}{V^{1.4}}$ and $P = \dfrac{0.0045}{V^{1.4}}$

The area between the two curves is

$$\int_{0.0001}^{0.0004} \frac{0.0045}{V^{1.4}} - \frac{0.0015}{V^{1.4}} \mathrm{d}V \approx 0.127$$

The work done during the cycle is $0.127 \text{ kJ} = 127 \text{ J}$

3 section questions

5. Calculus / 5.12 Area and volume

Area of a region revisited

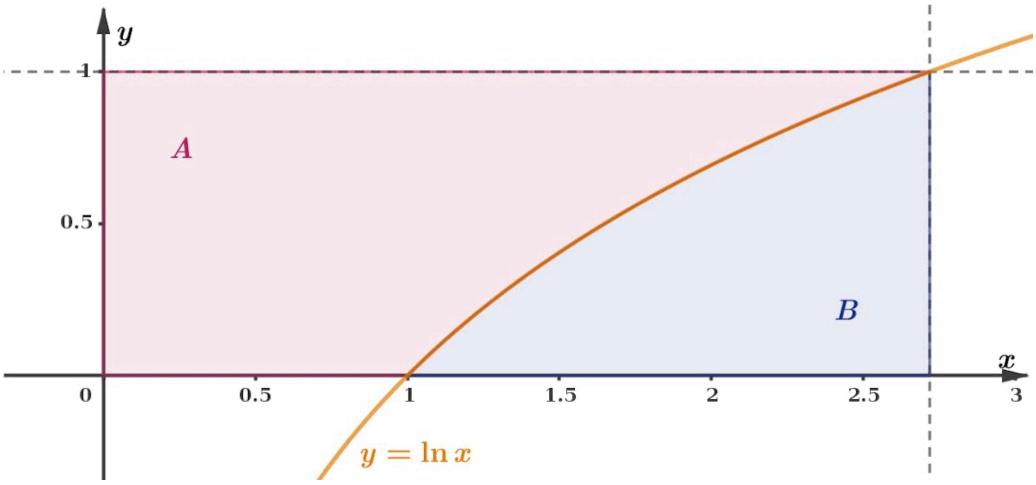
Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/area-of-a-region-revisited-id-28212/print/) Assign

Example 1

★☆☆

The diagram below shows part of the graph of $y = \ln x$, the line $y = 1$ and a vertical line. The two straight lines meet on the graph.

Student view



More information

The image is a diagram featuring the graph of the logarithmic function y = ln x. The graph is plotted with x ranging from 0 to 3 on the horizontal axis (x-axis) and y ranging from 0 to a little over 1 on the vertical axis (y-axis).

Two key straight lines are present: a horizontal line representing y = 1 and a vertical line where x is approximately 2.7. These lines intersect with the curve of y = ln x, creating a right boundary as x approaches 3.

The diagram highlights two shaded areas: - Area A is colored fainter in a pinkish tone, positioned below the line y = 1, to the left of the intersection point between y = ln x and the vertical line. - Area B is in a bluish hue, positioned to the right, between the logarithmic curve and the horizontal line y = 1, extending vertically down to the x-axis limits.

The diagram's axes are both numbered, allowing for deducing numerical relationships between the lines and the curve, illustrating changes concerning the given logarithmic function.

[Generated by AI]

Find the area of the two shaded regions.

To find the position of the vertical line, find the first coordinate of the intersection of the graph of $y = \ln x$ and $y = 0$.

$$\ln x = 1$$
$$x = \mathrm{e}^1 = \mathrm{e}$$

Method 1

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Using a calculator you can find the area of region B is $\int_1^{\mathrm{e}} \ln x \mathrm{d}x = 1$.

Since the area of the rectangle is $1 \times \mathrm{e} = \mathrm{e}$, the area of region A is $\mathrm{e} - 1$.

- Note, that if you want to find the area of region B first, you can either use technology, as you did above, or you can find $\int \ln x \mathrm{d}x$ and use this antiderivative to find the area. The next method shows a different approach by finding the area of region A first.

Method 2

If $y = \ln x$, then $x = \mathrm{e}^y$.

You can consider y as the independent variable and x as the dependent variable and notice that region A is bounded by the graph of $x = \mathrm{e}^y$, the y-axis and the lines $y = 0$ and $y = 1$. Hence, the area of region A is

$$\int_0^1 \mathrm{e}^y \mathrm{d}y = \left[\mathrm{e}^y\right]_0^1 = \mathrm{e}^1 - \mathrm{e}^0 = \mathrm{e} - 1.$$

Since the area of the rectangle is $1 \times \mathrm{e} = \mathrm{e}$, the area of region B is $\mathrm{e} - (\mathrm{e} - 1) = 1$.

The second method of the solution in **Example 1** was based on the observation that we can use integration with respect to the variable y instead of x to find the area of certain regions.

✓ Important

If $g(y) \leq f(y)$ for $a < y < b$, then the area of the region bounded by the graphs of $x = f(y)$, $x = g(y)$ and the lines $y = a$ and $y = b$ is given by

$$\int_a^b f(y) - g(y) \mathrm{d}y.$$

Student view

Overview
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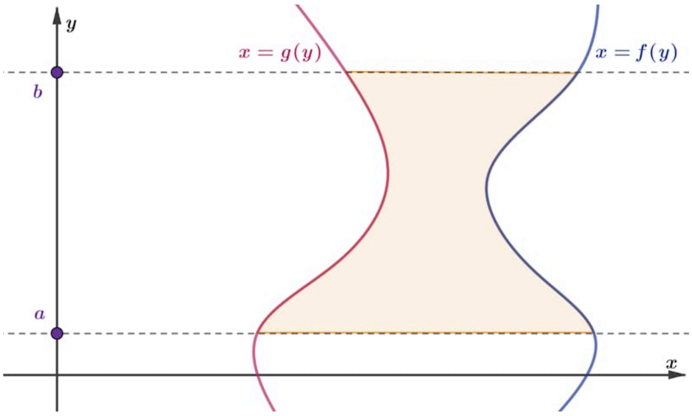

More information

The image is a graph illustrating the concept of finding the area between two curves. The graph displays two curves, labeled x = g(y) in red and x = f(y) in blue, bounded by two horizontal dashed lines, y = a and y = b in purple. These lines represent the limits of integration. The region between the two curves is shaded to indicate the area of interest. The y-axis is oriented vertically, while the x-axis is horizontal, providing a typical Cartesian coordinate system setup. The curves x = g(y) and x = f(y) show different shapes, indicating variability in the functions represented, with a visibly enclosed shaded area between them.

[Generated by AI]

Exam tip

In the formula booklet, the area of the region enclosed by a curve and y-axis is given as

$$A = \int_a^b |x|\mathrm{d}y.$$

Example 2

★★☆

Find the area of the region (shaded on the diagram below) bounded by the graphs of $y = \sqrt{4-x}$ and $y = \dfrac{2x}{x+3}$.

Student view

Overview
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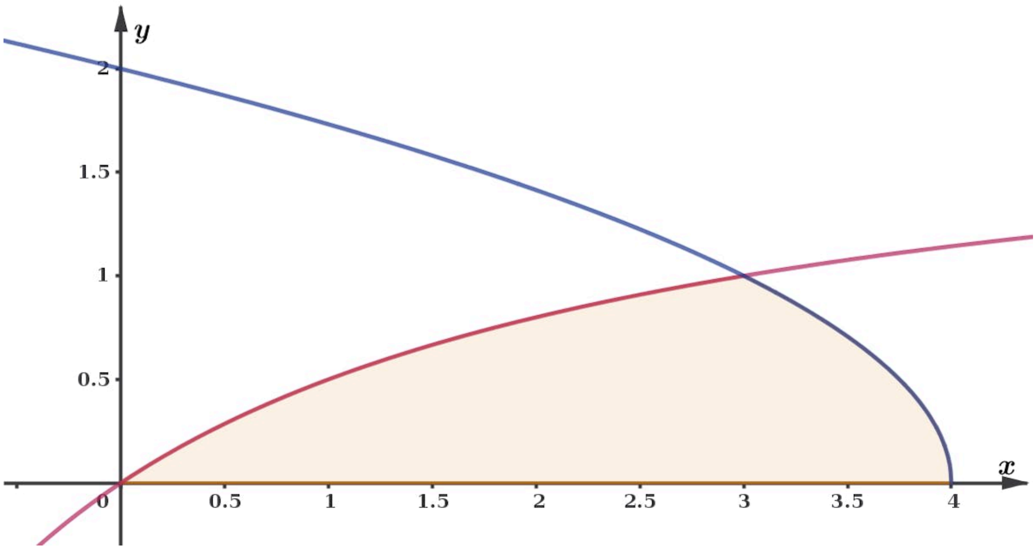

More information

The image is a graph depicting the shaded area between two curves. The horizontal axis is labeled 'x' and ranges from 0 to 4. The vertical axis is labeled 'y' and goes from 0 to just above 2. The first curve is y=√(4-x), which starts at the point (0,2) and curves downward to intersect the x-axis at (4,0). The second curve is y=2x/(x+3), which starts from the origin, increases and then approaches a horizontal asymptote near y=0.67 as x increases. The shaded area, representing the region between these curves, starts at the origin and extends to approximately x=4.

[Generated by AI]

There are two ways of approaching this problem. You will need the following:

- The blue graph on the diagram is part of the graph of $y = \sqrt{4-x}$. In the second approach, you will need to understand that this equality can be rearranged in the form

$$\begin{aligned} y &= \sqrt{4-x} \\ y^2 &= 4-x \\ x &= 4-y^2. \end{aligned}$$

- The red graph on the diagram is part of the graph of $y = \dfrac{2x}{x+3}$. In the second approach, you will need to understand that this equality can be rearranged in

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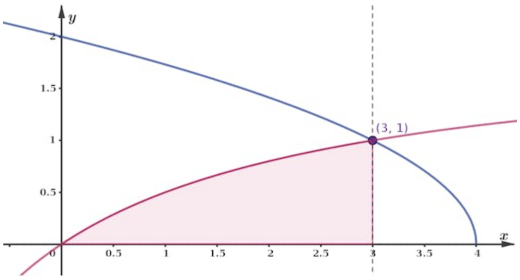
the form

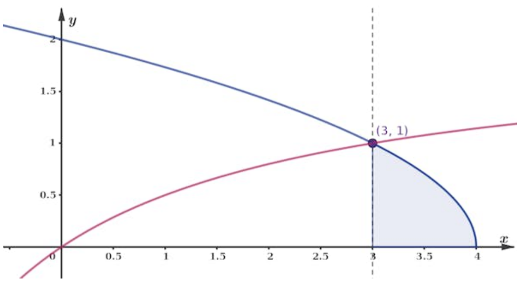
$$\begin{aligned} y &= \frac{2x}{x+3} \\ y(x+3) &= 2x \\ yx + 3y &= 2x \\ yx - 2x &= -3y \\ x(y-2) &= -3y \\ x &= \frac{-3y}{y-2} \\ x &= \frac{3y}{2-y} \end{aligned}$$

- One of the solutions of $\sqrt{4-x} = \frac{2x}{x+3}$ gives the first coordinate of the intersection point of the graphs. From the graph given in the question, you can see that one of the solutions is near $x = 3$. Since $\sqrt{4-3} = 1 = \frac{2 \times 3}{3+3}$, you can deduce that the intersection point visible on the diagram is $(3, 1)$.

Method 1

Split the shaded region into two and find the areas separately.





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- To find the area of the region on the left, first find the antiderivative of $y = \frac{2x}{x+3}$.

$$\int \frac{2x}{x+3}\mathrm{d}x = \int \frac{2(x+3)-6}{x+3}\mathrm{d}x$$
$$= \int 2 - \frac{6}{x+3}\mathrm{d}x = 2x - 6\ln(x+3) + c$$

So the area is

$$\int_0^3 \frac{2x}{x+3}\mathrm{d}x = \Big[2x - 6\ln(x+3)\Big]_0^3$$
$$= (6 - 6\ln 6) - (0 - 6\ln 3)$$
$$= 6 - 6(\ln 6 - \ln 3) = 6 - 6\ln\frac{6}{3} = 6 - 6\ln 2 \approx 1.8411.$$

- To find the area of the region on the right, first find the antiderivative of $y = \sqrt{4-x}$.

$$\int \sqrt{4-x}\mathrm{d}x = \int (4-x)^{1/2}\mathrm{d}x$$
$$= \frac{2}{3}(4-x)^{3/2}(-1) = -\frac{2}{3}(4-x)^{3/2} + c$$

so

$$\int_3^4 \sqrt{4-x}\mathrm{d}x = \left[-\frac{2}{3}(4-x)^{3/2}\right]_3^4$$
$$= \left(-\frac{2}{3}(4-4)^{3/2}\right) - \left(-\frac{2}{3}(4-3)^{3/2}\right) = \frac{2}{3} \approx 0.6667.$$

Hence, the area of the shaded region is $6 - 6\ln 2 + \frac{2}{3} \approx 2.51$ units squared.

Method 2

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Consider y as the independent variable and x as the dependent variable and think of the region as bounded by the graphs of $x = 4 - y^2$ and $x = \dfrac{-3y}{y-2}$ over the interval $0 < y < 1$. To find the area this way, you will need

$$\begin{aligned}\int (4-y^2) - \frac{3y}{2-y}\mathrm{d}y &= \int 4 - y^2 - \frac{-3(2-y)+6}{2-y}\mathrm{d}y \\ &= \int 4 - y^2 + 3 - \frac{6}{2-y}\mathrm{d}y \\ &= \int 7 - y^2 - \frac{6}{2-y}\mathrm{d}y \\ &= 7y - \frac{y^3}{3} - 6 \times (-1) \times \ln(2-y) + c \\ &= 7y - \frac{y^3}{3} + 6\ln(2-y) + c.\end{aligned}$$

Hence, the area of the shaded region is

$$\begin{aligned}&\int_0^1 (4-y^2) - \frac{3y}{2-y}\mathrm{d}y \\ &= \left[7y - \frac{y^3}{3} + 6\ln(2-y)\right]_0^1 \\ &= \left(7 - \frac{1}{3} + 6\ln 1\right) - \left(0 - \frac{0}{3} + 6\ln 2\right) \\ &= 7 - \frac{1}{3} + 6\ln 2 \approx 2.51.\end{aligned}$$

Example 3

★★☆

The shaded region on the diagram below is enclosed by the graph of $y = \sqrt{x}$, $y = 2$, $y = a$ and the y-axis.

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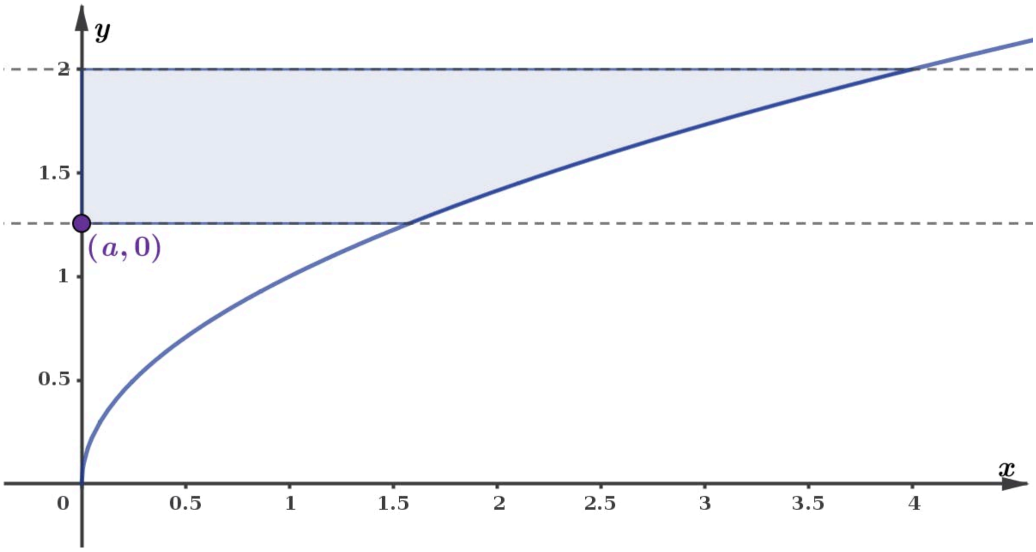

More information

The image depicts a graph with a shaded region enclosed by the curves and lines. The x-axis is marked from 0 to approximately 4.5, and the y-axis is marked from 0 to 2.5. The curve of (y=\sqrt{x}) begins at the origin (0,0) and ascends, moving to the right. A horizontal line at (y=2) extends from the y-axis and intersects (y=\sqrt{x}). Another horizontal line, labeled (y=a), is drawn parallel to the x-axis, intersecting the left point of the shaded region at approximately (a, 0). The shaded region lies above this horizontal line (y=a), below (y=2), and to the left of (y=\sqrt{x}). The area of the shaded region is stated to be 2 square units in the accompanying text.

[Generated by AI]

The area of the shaded region is 2 units squared. Find the value of a.

The equation of the curve, $y = \sqrt{x}$ can be written as $x = y^2$.

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The area of the region is given by the integral $\int_a^2 y^2 \mathrm{d}y$, so

$$
\begin{aligned}
2 &= \int_a^2 y^2 \mathrm{d}y \\
2 &= \left[\frac{y^3}{3}\right]_a^2 \\
2 &= \frac{2^3}{3} - \frac{a^3}{3} \\
6 &= 8 - a^3 \\
a^3 &= 2 \\
a &= \sqrt[3]{2} \approx 1.26.
\end{aligned}
$$

In the examples above, you found the answers without the use of a GDC. Approximate values of the answers can of course be calculated more quickly using a GDC.

Example 4

★★☆

Find the area of the region enclosed by the graph of $y = \log_2 x$ and the line $2x - 3y = 6$.

Use a GDC to plot the curve and the line, and to find the intersection points. To do this, write the equation of the line in the form $y = \dfrac{2x-6}{3}$. The diagram below shows parts of the two graphs, the coordinates of the intersection points and the region that these graphs enclose.

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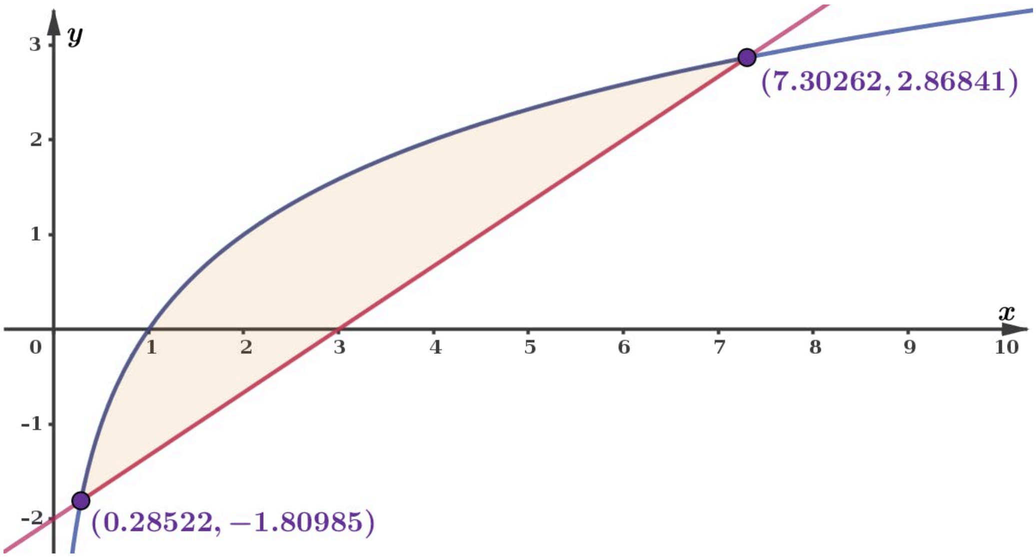

To find the area, use either x or y as the independent variable.

Method 1

Using $y = \log_2 x$ as the upper curve, $y = \dfrac{2x-6}{3}$ as the lower curve and 0.28522 and 7.30262 as bounds (of the x-values), the area is approximately

$$\int_{0.28522}^{7.30262} \log_2 x - \frac{2x-6}{3}\mathrm{d}x \approx 7.63 \text{ units squared.}$$

Method 2

You can also rearrange

- $y = \log_2 x$ as $x = 2^y$ and
- $2x - 3y = 6$ as $x = \dfrac{6+3y}{2}$.

Using these as the bounding curves and -1.80985 and 2.86841 as the bounds (of the y-values), the area is approximately

$$\int_{-1.80985}^{2.86841} \frac{6+3y}{2} - 2^y\mathrm{d}y \approx 7.63 \text{ units squared.}$$

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In **Example 4** it was necessary to use a GDC to find the bounds of the region. See the answer given by WolframAlpha (http://www.wolframalpha.com) when you type

solve {y=log_2(x), 2x-3y=6} for {x,y}

in the search line. It only gives an approximate answer even though WolframAlpha is programmed to use all the algebraic equation solving techniques you have learned about (and beyond) to find exact solutions when possible.

Exam tip

If the question does not ask for an exact value, use your GDC to find definite integrals.

Below you will find help on how to use the different calculators when the independent variable is not x. Of course, you can rename the independent variable, but it can be helpful if you are aware that calculators can find definite integrals with your choice of independent variable name.

Steps	Explanation
These instructions will show you how to find the definite integral appearing in the example above. $$\int_{-1.80985}^{2.86841} \frac{6+3y}{2} - 2^y \mathrm{d}y$$ In the current operating system the integration variable needs to be x. This may change in the future, but for the moment you need to replace the variable, and find $$\int_{-1.80985}^{2.86841} \frac{6+3x}{2} - 2^x \mathrm{d}x$$ with the calculator.	

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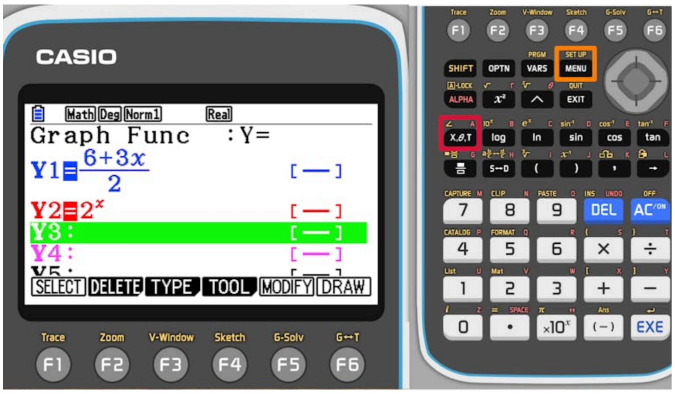
Steps	Explanation
Use the variable button to enter x when you define the functions involved in the integral. This step is not necessery, but it will make the screen more easily readable, when you calculate the integral itself. Once done defining the functions, go back to the home screen.	
Open now the calculator screen.	

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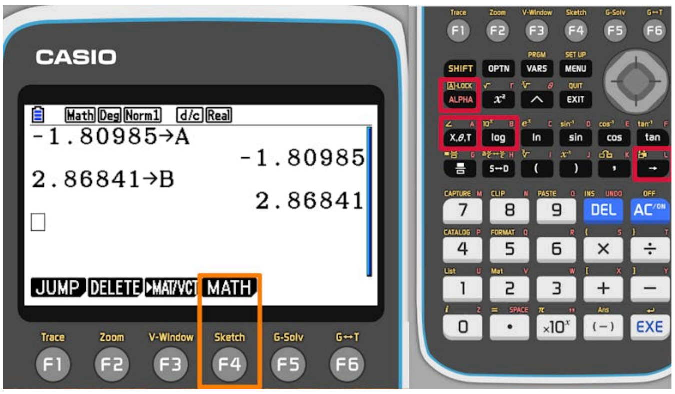
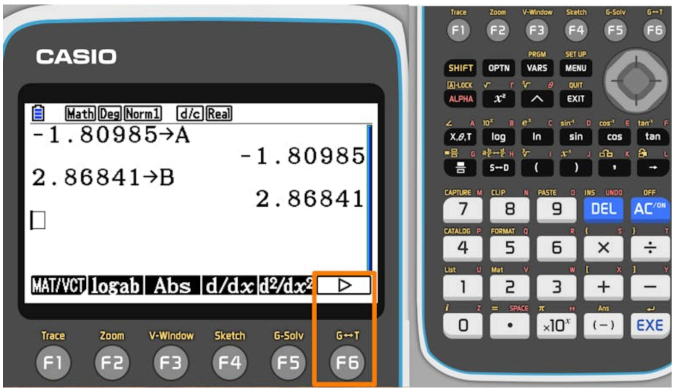
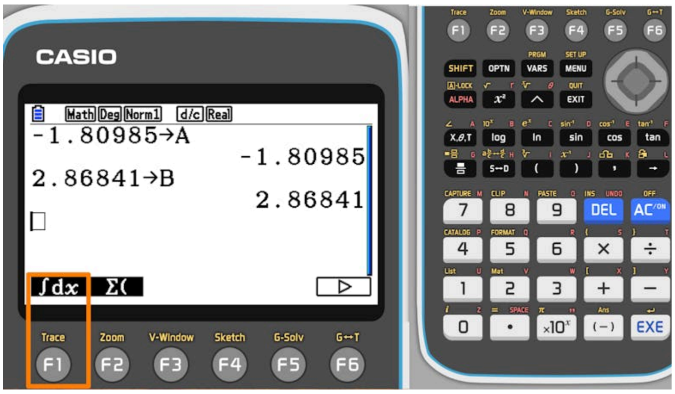
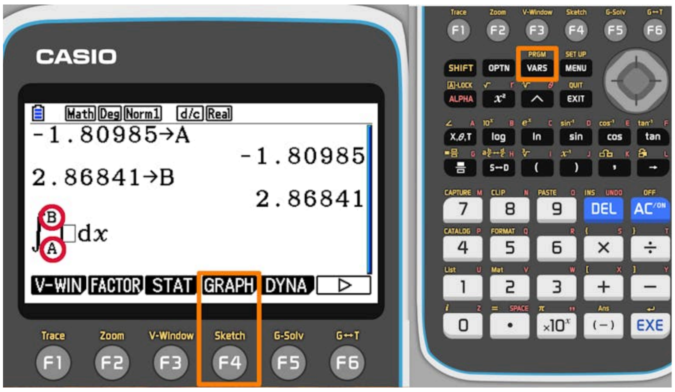
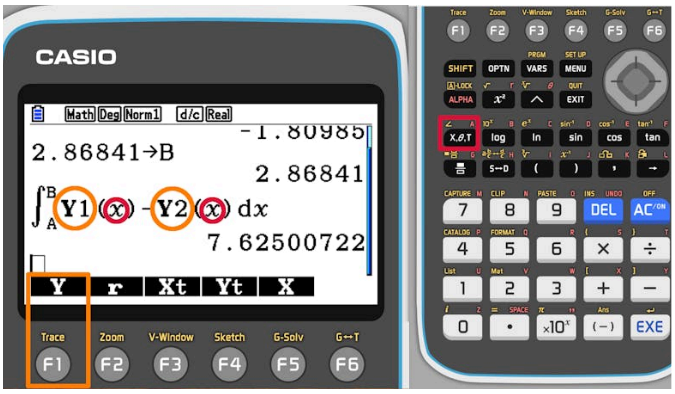
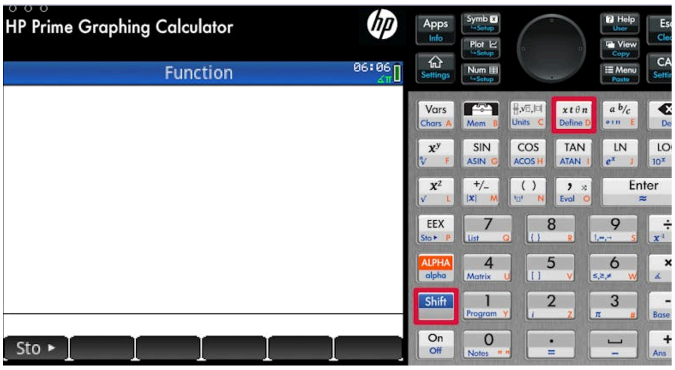
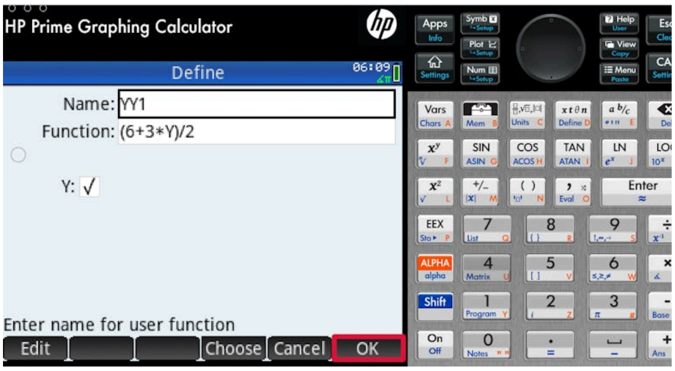
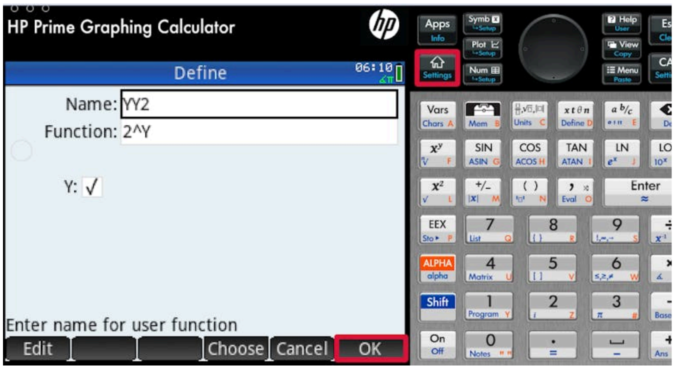
view

Steps	Explanation
Sometimes it is useful to store values in the memory of the calculator. In this case you can store the limits of integration. To ind the integral, press F4 to open the math options ...	
... press F6 to scroll to the right ...	

Steps	Explanation
... and press F1 to choose the integral template.	
Notice, that you can use the names where you stored the limits instead of the numbers themselves. To use the function names instead of typing in the function again, press VARS and then F4 to access the variable names related to the graphical screen.	

Steps	Explanation
Use the variable button to enter x and use F1 to enter the function variable names. Remember, that in the first step you stored the functions involved in Y1 and Y2.	

Steps	Explanation
These instructions will show you how to find the definite integral appearing in the example above. $$\int_{-1.80985}^{2.86841} \frac{6+3y}{2} - 2^y \mathrm{d}y$$ On the home screen select the option to define a function.	

Steps	Explanation
You can give any name that is not used by the calculator and you can use any variable name. Press OK when you are done.	
You can define as many functions (using different names of course). When you are done, press OK and move back to the home screen.	

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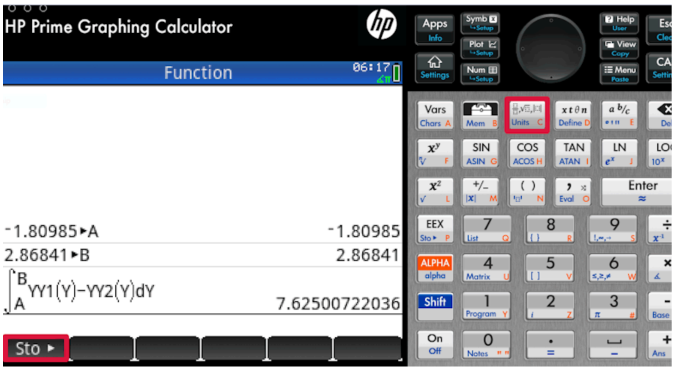
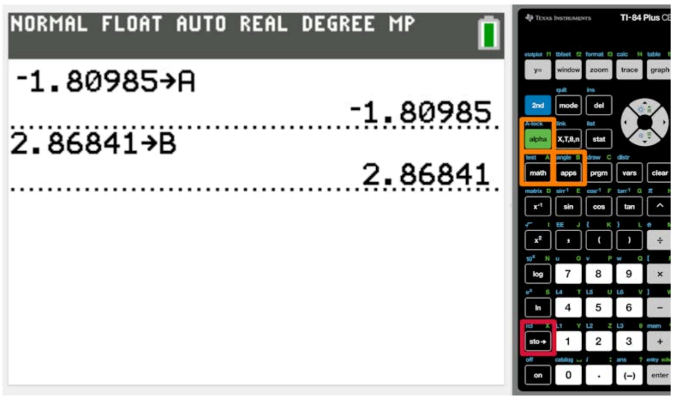
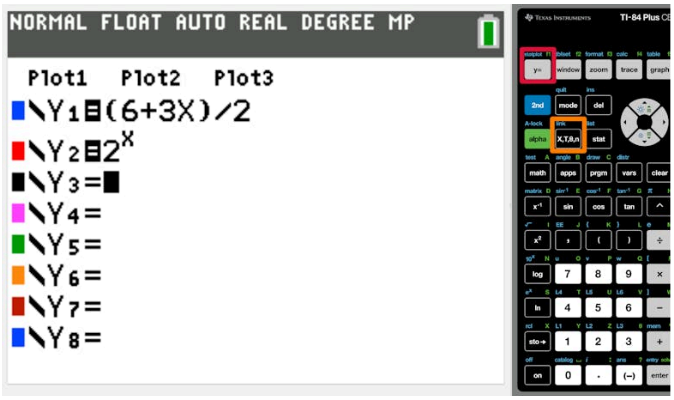
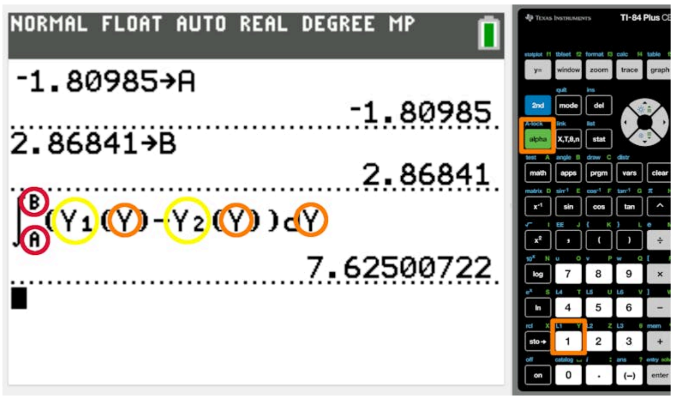
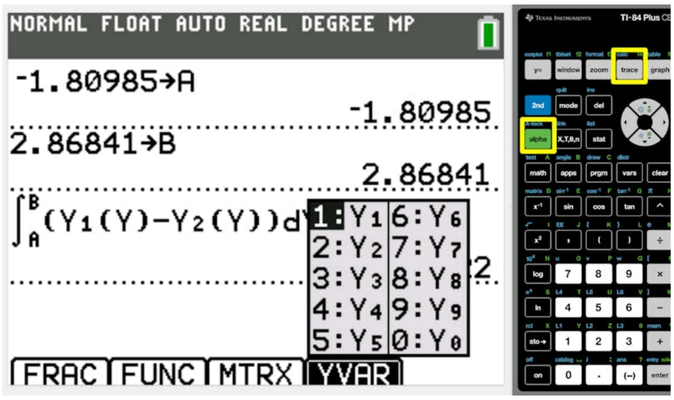
Steps	Explanation
It is not really necessary, but sometimes it can be useful to store the values of the limits before you use the expression formatting option to enter the integral. Note that the calculator understands the functions you defined and the variable of the integration can be anything.	

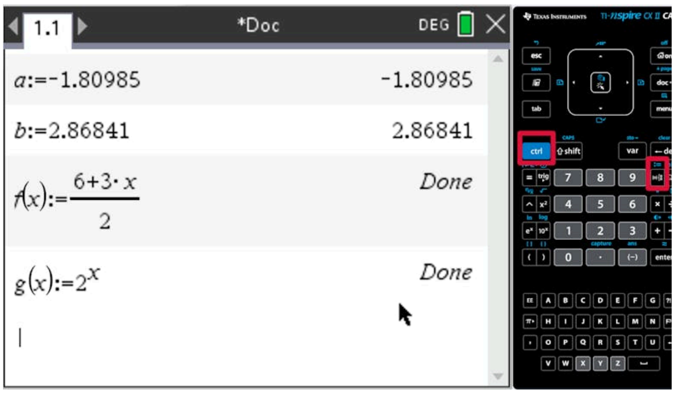
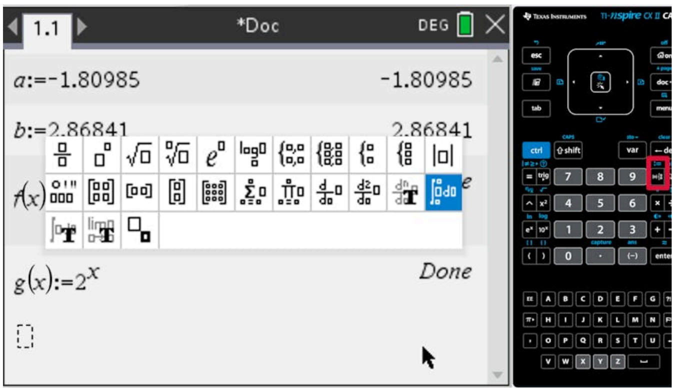
Steps	Explanation
These instructions will show you how to find the definite integral appearing in the example above. $$\int_{-1.80985}^{2.86841} \frac{6+3y}{2} - 2^y \mathrm{d}y$$ In this case it is not really necessary, but it can be useful to store values of the limits of integration in the memory of the calculator.	

Student view

<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>You can also store the functions before starting to calculate the integral. This is again not necessary, but it will make the calculation of the integral more easily readable.

Note, that to define the function you need to use x as the variable.</td><td>
</td></tr>
<tr><td>Choose the numerical integral (fnInt) option in the math menu.</td><td>
</td></tr>
</table>

Steps	Explanation
Make sure you understand every part of the integral on the screen. • You can use the variable names where you stored the limits previously (red marks). • You can use any letter as the variable of the integration (orange marks). • You can use the names of the function instead of typing in the expression again (yellow marks). See the next screen for guidance on how to find these function names.	
You can access the function variable names by pressing alpha/f4.	

Steps	Explanation
These instructions will show you how to find the definite integral appearing in the example above. $$\int_{-1.80985}^{2.86841} \frac{6+3y}{2} - 2^y \mathrm{d}y$$ It is not necessary, but you can store numbers using variable names and you can also define functions that you want to use later. You can use any name and any variable name when you define a function. Make sure you use the colon equal sign in defining the functions.	
To find a definite integral, open the template menu and choose the integral template.	

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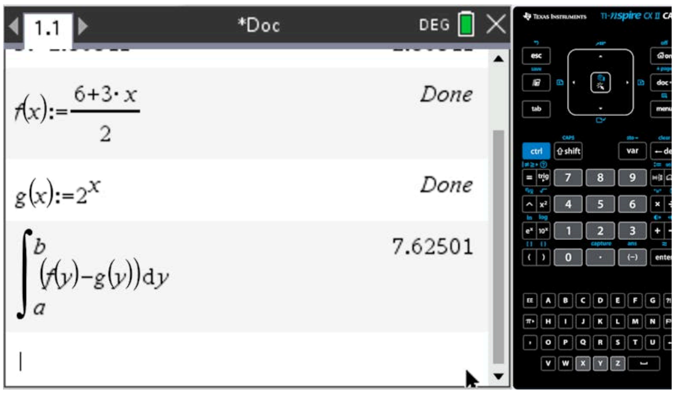
Steps	Explanation
Theer are several things you can notice about how this integral is entered. • You can specify the limits using the names where you stored the values. • You can use the names of the functions you defined earlier. • You do not need to use the same variable name you used for defining the function. You can use any name for the integration variable.	

3 section questions

5. Calculus / 5.12 Area and volume

Integral as limit

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/integral-as-limit-id-28213/print/) Assign

Before turning to other applications of the definite integral, consider an approach on how to approximate the area of a region below the graph of a positive function. The method considered here will be slightly different from the trapezoidal rule that you already saw in subtopic 5.8 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-5-id-27890/).

Exam tip

The content of this section is not part of the syllabus. For exam preparation you can safely move to the next section. Nevertheless, the discussion presented here is useful for understanding the reason why definite integral is used not only for

Student view

calculating areas, but also for calculating volumes of revolution.

On the applet below, the area of a region below the graph of a positive function is approximated by the sum of the area of rectangles.

Interactive 1. Riemann Sum Approximation.

More information for interactive 1

This interactive line graph allows users to understand how definite integrals approximate the area under a curve using Riemann sums.The screen is divided in two halves, the top half shows a graph in blue line with red dots line and the bottom half has two toggle buttons, ‘Adjust curve’ and ‘Number of rectangles’.

When ‘Adjust curve’ is selected users can drag red points to adjust the curve's shape, modifying the region whose area users want to estimate. Behind the blue line graph is a collection of rectangles with equal base and variable heights. When ‘Number of rectangles’ is selected a slider appears, that allows selection of 1 to 20 rectangles, with the applet instantly displaying both the rectangles and their combined area approximation.

As users increase the number of rectangles, users observe the approximation converging toward the true area under the curve - visually demonstrating how finer partitions yield more accurate results. The dynamic visualization makes fundamental calculus concepts tangible, showing the progression from rough estimation (few, wide rectangles) to precise calculation (many narrow rectangles) that underlies integral calculus.

For example, a user sets the number of rectangles to 4 using the slider. The graph immediately displays 4 wide rectangles under the curve, with their total area shown as 5.40 (approximate area) and Area of the region 5.82.

When the user increases the number to 10 rectangles, the visualization updates to show narrower rectangles fitting more precisely under the curve, and the approximate area becomes 5.80 - closer to the true area of 5.82. Finally,

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when set to 20 rectangles, the approximation improves further to 5.82, demonstrating how increasing partitions leads to more accurate area calculations. The interactive display clearly shows the rectangles becoming thinner and the approximation converging toward the exact area as the number increases.

Activity

- Can you explain how the rectangles were constructed?
- Can you see that the total area of the rectangles is close to the area of the region?
- Can you suggest a way to get an even closer approximation?

Besides using the applet above to experiment, you can also use, for example, WolframAlpha (http://www.wolframalpha.com). Type

integral of y=x^2 for 0<x<3 using 10 intervals

in the search line and try to interpret the answer WolframAlpha gives. You can of course experiment with other functions over a different interval and different number of subintervals.

- WolframAlpha gives its interpretation of the input. This interpretation mentions the midpoint method. Try to explain what this means. Later WolframAlpha also mentions other methods (left endpoint, right endpoint) to approximate the integral. Try to explain what these other methods mean. We will not talk about these methods in detail, but it is useful to think about the similarities and differences.
- WolframAlpha also gives a visual illustration of the approximation. Is this similar to the illustration in the applet above?
- WolframAlpha also gives the symbolic form of the approximation. This symbolic form is what we discuss next.

Important

If f is continuous on the interval $[a, b]$ and $a = x_0 < x_1 < \cdots < x_n = b$ and $x_{i-1} \leqslant x_i^* \leqslant x_i$ for $1 \leqslant i \leqslant n$, then the sum

Student view

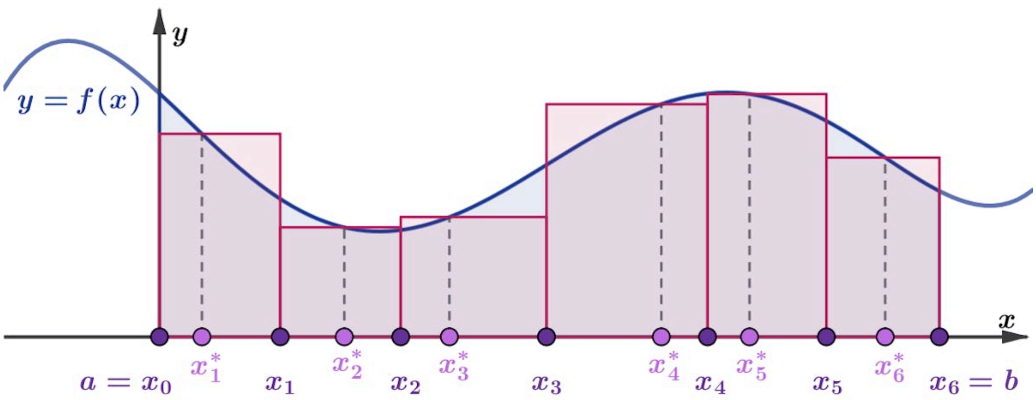
$$f(x_1^*)(x_1 - x_0) + \cdots + f(x_n^*)(x_n - x_{n-1}) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

is an approximation of $\int_a^b f(x)\mathrm{d}x$.

Moreover, the error of this approximation is approaching 0 as n increases without bound and the maximum length of the subintervals approaches 0.

The sum above is called the **Riemann sum** for the function f corresponding to the partition $a = x_0 < x_1 < \cdots < x_n = b$ and the points $x_1^*, \ldots, x_n^*$ in the subintervals.

The diagram below illustrates the claim. It shows a graph of a continuous function and rectangles that approximate the region bounded by the graph and the x-axis over a finite interval $[a, b]$.



More information

The image displays a graph of a continuous function y = f(x) depicted by a smooth curve. The graph shows the x-axis labeled from a = x_0 to x_6 = b and the y-axis, though no specific values are given. Several vertical rectangles are drawn from the x-axis to the curve, all bounded between points on the x-axis, indicating an approximation of the area under the curve over a finite interval [a, b]. Each rectangle has its base extending between consecutive x marks such as x_0 to x_1, x_1 to x_2, and so on, with heights determined by the function's value at specific points, labeled as x_*1*, x_2, etc. These points where the height of rectangles is evaluated are marked on the x-axis. The function's curve indicates it is oscillating, with both peaks and troughs visible, crossing the x-axis at various points. The rectangles demonstrate how their heights approximate the function's value at specific x* points, illustrating the concept of Riemann sums.

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The first rectangle has base $x_1 - x_0$ and height $f(x_1^*)$, so the area is $f(x_1^*)(x_1 - x_0)$. Similar expressions give the area of the other rectangles, so the total area of all the rectangles is

$$f(x_1^*)(x_1 - x_0) + \cdots + f(x_n^*)(x_n - x_{n-1}) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

To show that this area is close to the area of the region bounded by the graph of f and the x-axis over the interval $[a, b]$ (if the subintervals are short enough) is beyond the syllabus. It involves the concept of uniform continuity, which we do not learn about. Nevertheless, the illustration is intuitively convincing, so we will use this claim without more justification.

✓ **Important**

A Riemann sum is an approximation of an integral using a finite sum of areas, typically using areas of rectangles.

Before moving on to the examples, you can try the page on Riemann sums (http://mathworld.wolfram.com/RiemannSum.html) on Wolfram MathWorld to experiment a bit more.

Example 1

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- Show that $\int_1^2 \frac{1}{x} \mathrm{d}x = \ln 2$
- Use a Riemann sum for $y = \frac{1}{x}$ with five subintervals of $[1, 2]$ of equal length and the midpoints of these intervals to get a rational approximation of $\ln 2$.

- Using $\int \frac{1}{x} \mathrm{d}x = \ln x + c$, gives

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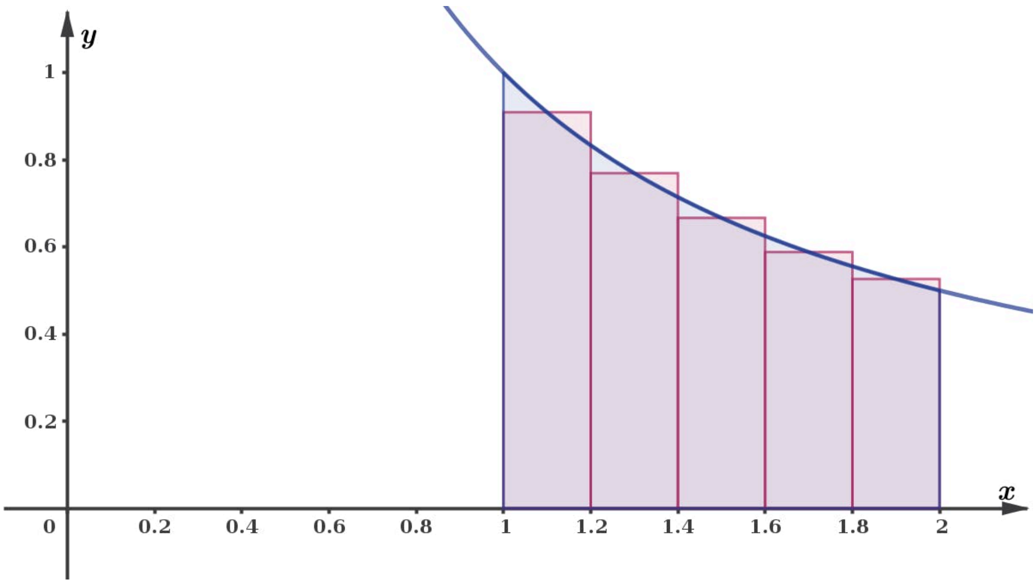
$$\int_1^2 \frac{1}{x}\mathrm{d}x = \Big[\ln x\Big]_1^2 = \ln 2 - \ln 1 = \ln 2 - 0 = \ln 2$$

- The endpoints of the subintervals are $1, 1.2, 1.4, 1.6, 1.8, 2$.
  The midpoints of these intervals are $1.1, 1.3, 1.5, 1.7, 1.9$.
  Since the length of all subintervals is 0.2, the Riemann sum is

$$\begin{aligned}&\frac{1}{1.1}\times 0.2 + \frac{1}{1.3}\times 0.2 + \frac{1}{1.5}\times 0.2 + \frac{1}{1.7}\times 0.2 + \frac{1}{1.9}\times 0.2\\ &= 0.2\left(\frac{10}{11} + \frac{10}{13} + \frac{10}{15} + \frac{10}{17} + \frac{10}{19}\right)\\ &= 2\left(\frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19}\right) = \frac{479\,378}{692\,835}\end{aligned}$$

The diagram below illustrates the region representing the definite integral and the approximating rectangles.



Example 2

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Use Riemann sums to find $\int_0^3 x^2 + 2\mathrm{d}x$

In you solution you can use the following formula that gives the sum of the squares of the first n integers.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Since the question did not specify how to set up the Riemann sum, use n intervals of equal length and the left endpoint of each interval to determine the height of the rectangles. So,

$$\begin{aligned} x_i - x_{i-1} &= \frac{3-0}{n} = \frac{3}{n} \\ x_i &= 0 + i\frac{3}{n} = \frac{3i}{n} \\ f(x_i^*) = f(x_i) &= \frac{9i^2}{n^2} + 2 \\ f(x_i^*)(x_1 - x_{i-1}) &= \left(\frac{9i^2}{n^2} + 2\right) \times \frac{3}{n} = \frac{27i^2}{n^3} + \frac{6}{n} \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) &= \sum_{i=1}^{n} \left(\frac{27i^2}{n^3} + \frac{6}{n}\right) \\ &= \frac{27}{n^3}\sum_{i=1}^{n} i^2 + \frac{6}{n}\sum_{i=1}^{n} 1 \\ &= \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} + n\frac{6}{n} \\ &= \frac{9(n+1)(2n+1)}{2n^2} + 6 \\ &= \frac{30n^2 + 27n + 9}{2n^2} = 15 + \frac{27}{2n} + \frac{9}{2n^2} \end{aligned}$$

The definite integral is the limit of these Riemann sums as the division of the interval $[0, 3]$ gets finer and finer. As n increases without bound, both $\frac{27}{2n}$ and $\frac{9}{2n^2}$ approach

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0, so

$$\int_0^3 x^2 + 2\mathrm{d}x = \lim_{n\to\infty} 15 + \frac{27}{2n} + \frac{9}{2n^2} = 15$$

Making connections

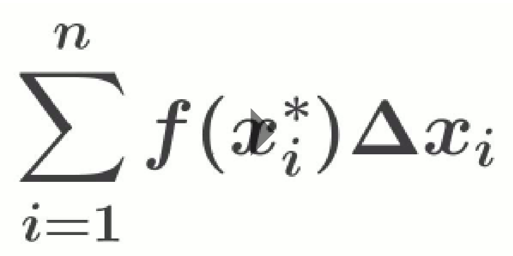
If in the formula of the Riemann sum you use the notation $\Delta x_i = x_i - x_{i-1}$, you get

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Notice the similarity of this notation to the notation for the definite integral it approximates,

$$\int_a^b f(x)\mathrm{d}x.$$

This similarity will help you recognise how to use integration to find the volumes of certain solids in the next sections.



More information

The image depicts a mathematical expression known as a Riemann sum, used in calculus to approximate the integral of a function. The formula is represented using sigma notation: the summation symbol (Σ) from i equals 1 to n, followed by the function f of x sub i star, multiplied by delta x sub i. This notation is a fundamental concept in understanding integration as it symbolizes the addition of areas of rectangles under a curve, as the number of rectangles (n) approaches infinity, their width (Δx) shrinks, providing an approximation of areas under curves.

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3 section questions

Solid of revolution about the x-axis

On the applet below, you can explore how cylinders can be used to approximate solids that you get by rotating a region 360° about the x-axis.

Interactive 1. Solid of Revolution Visualization.

More information for interactive 1

This interactive allows users to visualize and explore solids of revolution through cylindrical approximation.

The screen is divided into two halves. On the right side a 3D graph is displayed with the XYZ axis. On the left side of the screen, an Adjust curve checkbox is there which allows to modify the 2D curve $y = f(x)$ using 3 red points which change the shape of the curve. The number of rectangles projecting from the x-axis to the curve is corresponding to the number of cylinders set by the users by moving the vertical slider between 1 to 20. Two checkboxes, Show solids and show cylinders, allows users to toggle the visibility of them.

By adjusting the curve $y = f(x)$, users can see how rotating a function around the x-axis generates a three-dimensional shape. The tool features two display modes: one showing the exact solid of revolution, and another revealing the approximating cylinders used in the calculation.

Users can modify the number of cylindrical disks from 1 to 20, observing how increasing this number improves the volume approximation. As more cylinders are added, their combined volume converges to the exact value given by the integral $\pi\int_a^b f(x)^2 dx$, demonstrating the relationship between Riemann sums and definite integrals in volume

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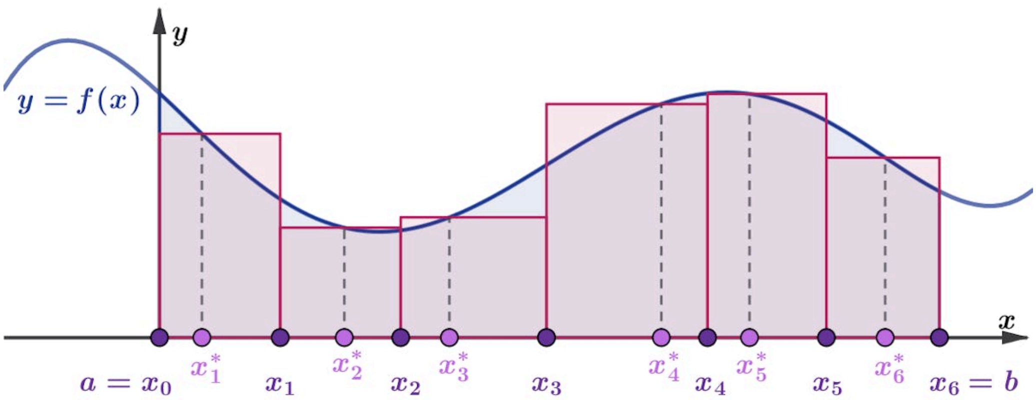
calculations.

For example, if a user checks both "Show solid" and "Show cylinders" boxes and sets the number of cylinders to 5, they can see how the 3D shape is approximated. The curve (set by default or adjusted by the user) is divided into 5 sections along the x-axis, with each section creating a cylindrical disk when rotated. The visualization clearly shows the "stair-step" approximation of the smooth solid, where the cylinders are either slightly larger or smaller than the actual solid. As the user moves the slider to increase the number of cylinders to 15 or 20, they can observe how the approximation becomes smoother and more accurate, with the cylindrical disks fitting much closer to the true shape of the solid of revolution. This demonstrates the fundamental concept that more cylinders (thinner disks) create better approximations of the exact volume.

You can use these approximations to find the volume of the solid generated. The diagram below (which we have already seen when we discussed Riemann sums) illustrates the start of the process.



More information

The image is a diagram illustrating the process of using Riemann sums to approximate the volume of a solid of revolution. The diagram consists of a curve labeled 'y = f(x)' plotted on a two-dimensional graph with axes labeled 'x' and 'y'. Below the curve, a series of rectangles are drawn, representing the Riemann sums. The base of each rectangle is aligned with segments on the x-axis marked from a = x_0 to x_6 = b, with additional points x_1, x_2, x_3, x_4, x_5, *and* x_6 highlighted. Each rectangle's height corresponds to the value at these points on the x-axis. The rectangles show steps of the approximation process, which involves rotating these rectangles to form cylindrical shells, as mentioned in the accompanying text. This graphical method aims to provide a visual understanding of integration and volume approximation techniques.

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You get the approximating cylinders by rotating the rectangles that are used to approximate the region.

- The radius of the first cylinder (corresponding to the first rectangle) is $r = f(x_1^*)$.
- The height of the first cylinder is $h = x_1 - x_0 = \Delta x_1$.

Hence, the volume of the first cylinder is $\pi r^2 h = \pi f(x_1^*)^2 \Delta x_1$

The expression for the volume of the other cylinders is similar, so the total volume of all cylinders (if n cylinders are used) is

$$\sum_{i=1}^{n} \pi f(x_i^*)^2 \Delta x_i$$

- This expression is a Riemann sum for the function defined by $\pi f(x)^2$, so it is an approximation of the definite integral

  $$\int_a^b \pi f(x)^2 \mathrm{d}x$$

  Morover, the approximation approaches 0 as the maximum length of the subintervals approaches 0.
- This expression also gives the total volume of the cylinders that approximate the solid.

As the same sum approximates both the integral and the volume, the following statement is justified.

Important

Suppose f is a continuous function over the interval $[a, b]$ and $f(x) \geqslant 0$ for all $a < x < b$. Let R be the region bounded by the x-axis, the graph of f and the lines $x = a$ and $x = b$.

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Consider the solid generated by rotating R 360° about the x-axis.

The **volume of this solid** is $\int_a^b \pi f(x)^2 \mathrm{d}x$.

Example 1

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Consider the region bounded by the graph of $y = \mathrm{e}^x$, the x-axis and the lines $x = 0$ and $x = 2$.

Evaluate the volume of the solid generated by revolving the region 360° about the x-axis.

According to the formula (and the rules of integration), the volume is

$$\begin{aligned} V &= \int_0^2 \pi(\mathrm{e}^x)^2 \mathrm{d}x \\ &= \pi \int_0^2 \mathrm{e}^{2x} \mathrm{d}x \\ &= \frac{\pi}{2} \int_0^2 2\mathrm{e}^{2x} \mathrm{d}x \\ &= \frac{\pi}{2} \left[\mathrm{e}^{2x}\right]_0^2 \\ &= \frac{\pi}{2}(\mathrm{e}^4 - 1) \approx 84.2 \end{aligned}$$

Example 2

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Consider the shaded region enclosed by the graph of $y = \sin x$ and the x-axis.

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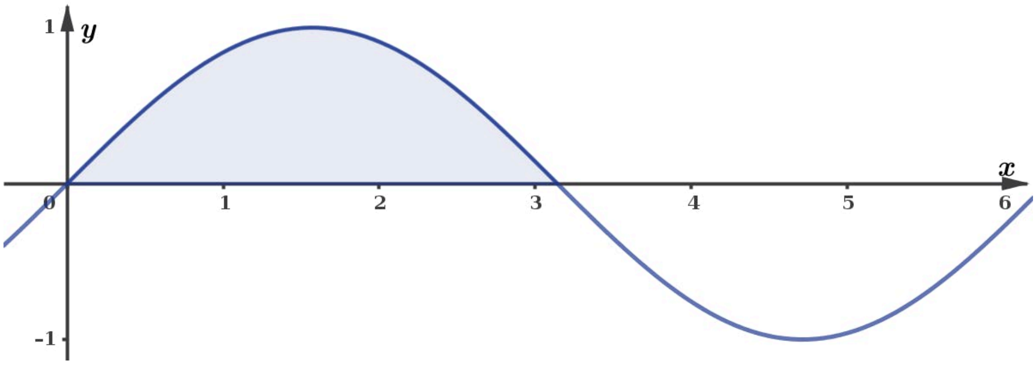

More information

The image shows a graph depicting the function y = sin(x). The X-axis is labeled with values ranging from 0 to 6. The Y-axis ranges from -1 to 1. The curve starts at the origin (0,0), reaches its peak at (π/2, 1), then descends back through (π, 0), continues to the minimum at (3π/2, -1), and finally ascends back to (2π, 0). The region enclosed by the curve and the X-axis between 0 and π is shaded, indicating the area of interest for calculating the volume of revolution around the X-axis.

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Evaluate the volume of the solid generated by revolving the region 360° about the x-axis.

The question does not specify the bounds of the region. These are given by the x-intercepts, so these bounds are $x = 0$ and $x = \pi$.

Use the formula to set up the integral and then use calculator to find the integral.

$$V = \int_0^\pi \pi(\sin x)^2 \mathrm{d}x \approx 4.93$$

Example 3

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Consider the shaded region bounded by the graph of $y = \sqrt{2x}$, the x-axis and the lines $x = 1$ and $x = 3$.

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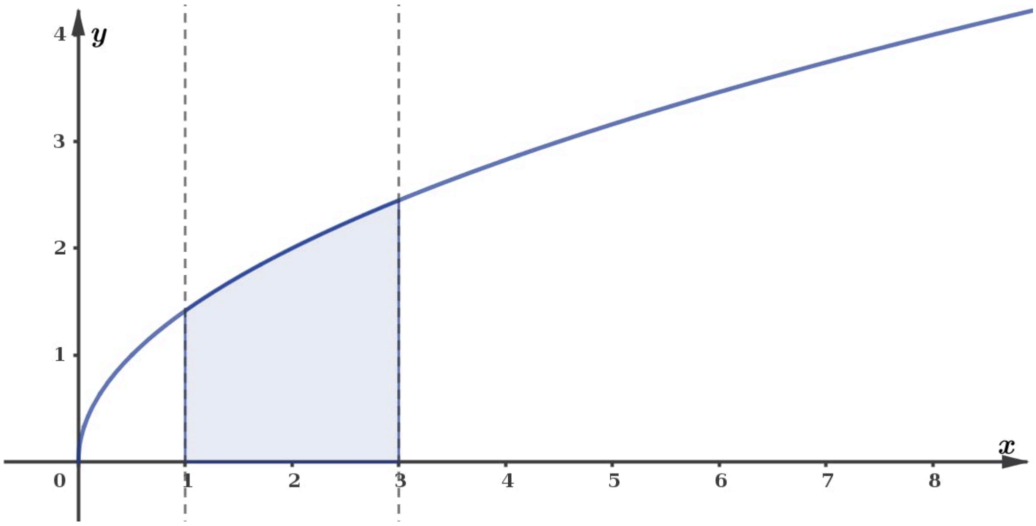

More information

The image shows a graph of the function (y = \sqrt{2x}) on a Cartesian plane. The X-axis is labeled from 0 to 8, and the Y-axis is labeled from 0 to 4. The curve (y = \sqrt{2x}) rises from the origin and curves upward to the right. A shaded region is bounded by this curve, the X-axis, and the vertical lines at (x = 1) and (x = 3).

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Evaluate the volume of the solid generated by revolving the region 360° about the x-axis.

According to the formula (and the rules of integration), the volume is

$$\begin{aligned} V &= \int_1^3 \pi\left(\sqrt{2x}\right)^2 \mathrm{d}x \\ &= \pi \int_1^3 2x \mathrm{d}x \\ &= \pi\left[x^2\right]_1^3 \\ &= \pi(3^2 - 1^2) = 8\pi \approx 25.1 \end{aligned}$$

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view

Exam tip

The volume of revolution about the x-axis is given in the formula booklet as

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$$V = \int_a^b \pi y^2 \mathrm{d}x$$

In the examples, you used algebraic methods, but use your calculator in an exam to find definite integrals if the question does not ask for an exact value.

As with areas of regions, we can also extend the scope of the formula to the rotation of regions between two graphs.

Important

Suppose f and g are continuous functions over the interval $[a, b]$ and $f(x) > g(x) \geqslant 0$ for all $a < x < b$. Let R be the region bounded by the graphs of f and g and the lines $x = a$ and $x = b$.

Consider the solid generated by rotating R $360°$ about the x-axis.
The **volume of this solid** is $\int_a^b \pi \left(f(x)^2 - g(x)^2\right) \mathrm{d}x$.

The solid generated is the difference of

- the solid generated by rotating the region below the graph of f and
- the solid generated by rotating the region below the graph of g.

Hence, the volume is the difference of the volumes of these solids,

$$\int_a^b \pi f(x)^2 \mathrm{d}x - \int_a^b \pi g(x)^2 \mathrm{d}x = \int_a^b \pi \left(f(x)^2 - g(x)^2\right) \mathrm{d}x$$

Be aware

Note that in the formula above you are integrating the difference of the squares and not the square of the differences.

Example 4

Student view

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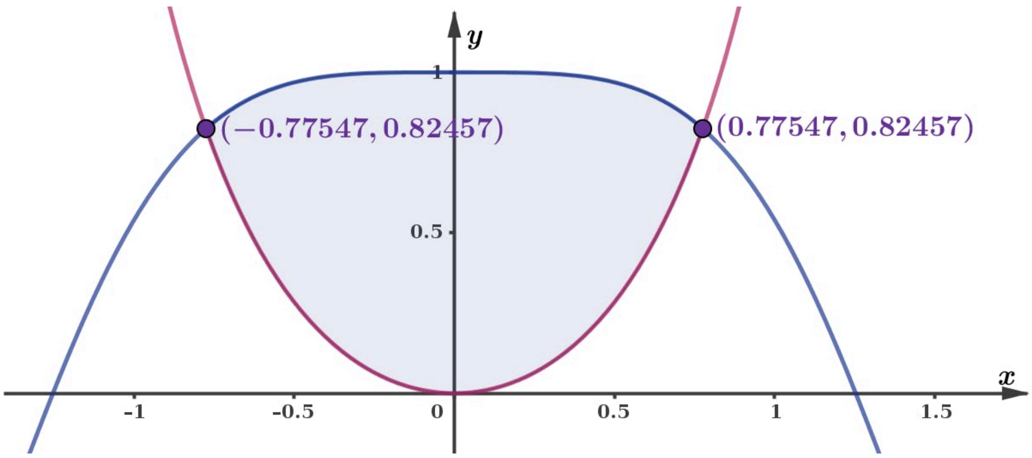
Let R be the region enclosed by the graphs of $y = \mathrm{e}^{x^2} - 1$ and $y = \cos\left(x^2\right)$.

Find the volume of the solid generated by revolving R 360° about the x-axis.

First, use a calculator to find the intersection points of the curves.

The diagram below shows the graphs, the intersection points and the shaded region R.



Using the formula, you will see that the volume of the solid of revolution is approximately

$$\int_{-0.77547}^{0.77547} \pi\left(\left(\cos(x^2)\right)^2 - \left(\mathrm{e}^{x^2} - 1\right)^2\right) \mathrm{d}x \approx 3.99$$

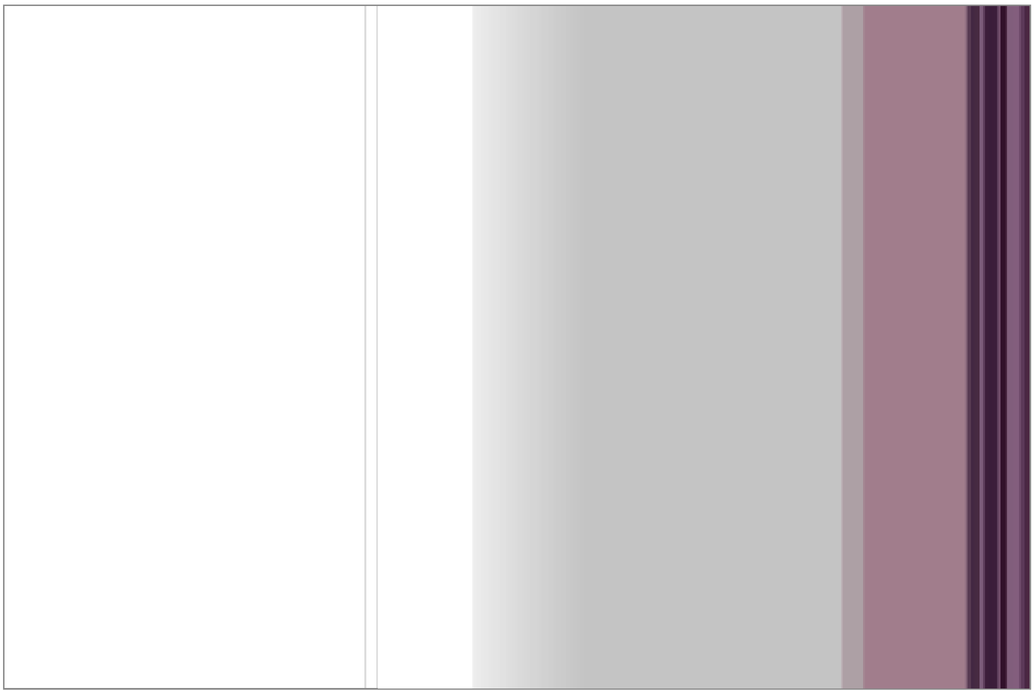
3 section questions

5. Calculus / 5.12 Area and volume

Solid of revolution about the y-axis

Student view

This section considers solids of revolution where a region bounded by the y-axis and the graph of a function in y is rotated about the y-axis. You can explore solids like this on the applet below.



Interactive 1. Solid of Revolution Around the Y-Axis.

More information for interactive 1

This interactive allows users to visualize solids of revolution generated by rotating a region bounded by the y-axis and the graph of a function f (y) about the y-axis.

The screen is divided into two panels.

On the right, there is a 3D graph with a curve in blue plotted on the XY plane with rectangles projecting on the Y-axis to this curve. On the left, there is an “Adjust curve” button that allows users to drag three red points on the curve to modify the curve's shape. By dragging the red points, users can dynamically adjust the shape of the function graph. Below it is a vertical slider that allows users to adjust the number of approximating rectangles, ranging from 1 to 20.

There are two selection buttons that enable users to choose. 1. To show a solid figure of the curve with a wide top and a narrow bottom with smooth tapering in between. 2. To show multiple cylinders along the solid figure. The number of cylinders displayed corresponds to the number of rectangles selected based on the slider’s position. By observing how the resulting solid and/or cylinders change in real-time and by dragging the vertical slider, the user can observe that as the number of cylinders increases, the approximation becomes more refined, closely resembling the actual volume calculated using the integral formula.

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This interactive tool helps users intuitively understand the relationship between the graphical representation of a function in terms of y and the three-dimensional solid formed by its rotation, reinforcing the concept that the volume is computed via integration along the y-axis.

Since the only change is using y as the independent variable instead of x, all the claims from the previous section translate directly.

Important

- Suppose f is a continuous function (of the variable y) over the interval $[a, b]$ and $f(y) \geq 0$ for all $a < y < b$. Let R be the region bounded by the y-axis, the graph of f and the lines $y = a$ and $y = b$.
  Consider the solid generated by rotating R $360°$ about the y-axis.
  The **volume of this solid** is $\int_a^b \pi f(y)^2 \mathrm{d}y$.
- Suppose f and g are continuous functions (of the variable y) over the interval $[a, b]$ and $f(y) > g(y) \geq 0$ for all $a < y < b$. Let R be the region bounded by the graphs of f and g and the lines $y = a$ and $y = b$.
  Consider the solid generated by rotating R $360°$ about the y-axis.
  The **volume of this solid** is $\int_a^b \pi \left(f(y)^2 - g(y)^2\right) \mathrm{d}y$.

Exam tip

The volume of revolution about the y-axis is given in the formula booklet as

$$V = \int_a^b \pi x^2 \mathrm{d}y.$$

Example 1

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Let R be the region bounded by the y-axis, the graph of $y = \sqrt{x}$ and the line $x + 4y = 12$.

Find the volume of the solid generated by rotating R $360°$ about the y-axis.

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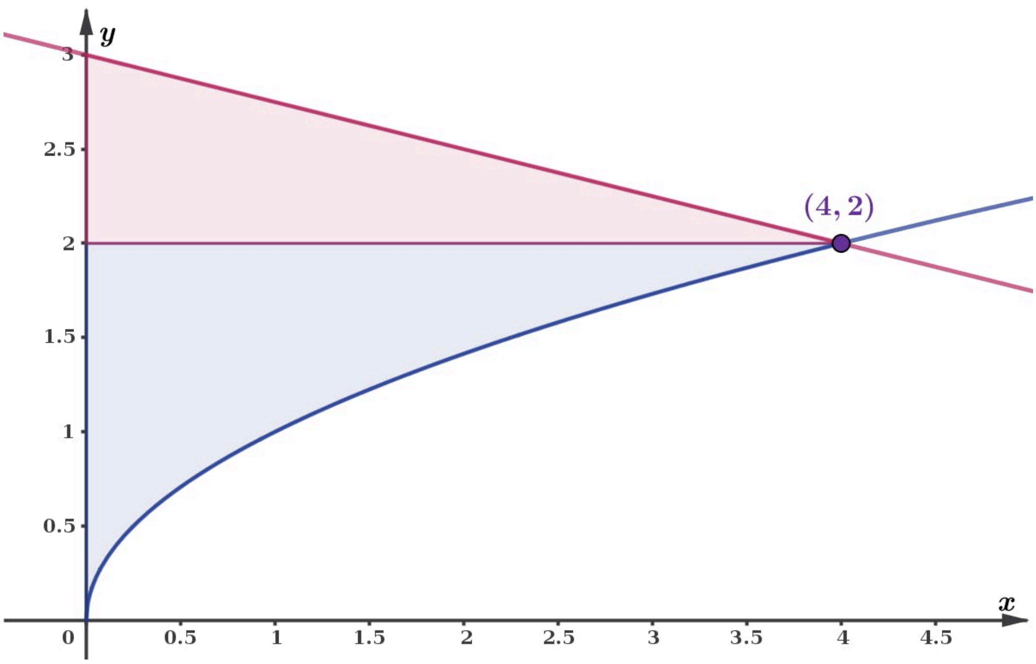
To find the intersection point of the graph and the line, solve

$$\begin{aligned} x + 4\sqrt{x} &= 12 \\ 4\sqrt{x} &= 12 - x \\ 16x &= (12 - x)^2 \\ 16x &= 144 - 24x + x^2 \\ x^2 - 40x + 144 &= 0 \end{aligned}$$

The two solutions of this quadratic equation are $x = 4$ and $x = 36$.

- Substituting $x = 4$ in the original equations, we get $y = \sqrt{4} = 2$, and $4 + 4 \times 2$ is indeed 12. So the point $(4, 2)$ is an intersection point of the graph and the line.
- Substituting $x = 36$ in the original equations, we get $y = \sqrt{36} = 6$. However, $36 + 4 \times 6$ is not 12, so discard this solution.

The diagram below shows the region that is rotated and the intersection point found above.



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The diagram is split into two parts, because for $0 \leq y \leq 2$ the graph of $y = \sqrt{x}$ is rotated and for $2 \leq y \leq 3$ the graph of $x + 4y = 12$.

- To find the volume of the solid corresponding to the blue region, rearrange $y = \sqrt{x}$ as $x = y^2$. So the volume of this part of the solid is

$$\int_0^2 \pi\left(y^2\right)^2 \mathrm{d}y = \pi \int_0^2 y^4 \mathrm{d}y = \pi\left[\frac{y^5}{5}\right]_0^2 = \pi\left(\frac{2^5}{5} - \frac{0^5}{5}\right) = \frac{32\pi}{5}.$$

- To find the volume of the solid corresponding to the red region, rearrange $x + 4y = 12$ as $x = 12 - 4y$. So the volume of this part of the solid is

$$\int_2^3 \pi(12 - 4y)^2 \mathrm{d}y = \pi\left[\frac{(12 - 4y)^3}{3 \times (-4)}\right]_2^3 = \pi\left(\frac{0^3}{-12} - \frac{4^3}{-12}\right) = \frac{16\pi}{3}.$$

Hence, the volume of the solid is $\frac{32\pi}{5} + \frac{16\pi}{3} = \frac{176\pi}{15} \approx 36.9$ units cubed.

Example 2

★★☆

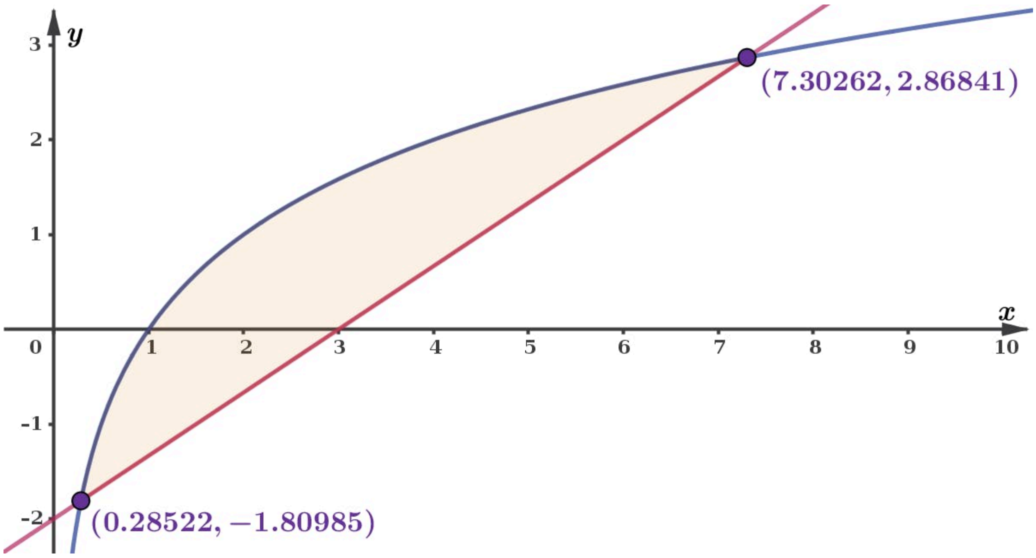
Let R be the region bounded by the graph of $y = \log_2 x$ and the line $2x - 3y = 6$.

Find the volume of the solid generated by rotating R 360° about the y-axis.

Use a GDC to plot the curve and the line and to find the intersection points. To do this, write the equation of the line in the form $y = \frac{2x - 6}{3}$. The diagram below shows parts of the two graphs, the coordinates of the intersection points and the region these graphs enclose.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov



This can be rearranged

- $y = \log_2 x$ as $x = 2^y$ and
- $2x - 3y = 6$ as $x = \dfrac{6+3y}{2}$.

Using these expressions and -1.80985 and 2.86841 as the bounds (of the y-values), gives the volume as approximately

$$\int_{-1.80985}^{2.86841} \pi\left(\left(\frac{6+3y}{2}\right)^2 - (2^y)^2\right) \mathrm{d}y \approx 151 \text{ units cubed.}$$

Example 3

★★☆

The shaded region on the diagram below is enclosed by the graph of $y = \sqrt{x}$, $y = 2$, $y = a$ and the y-axis.

Student view

Overview
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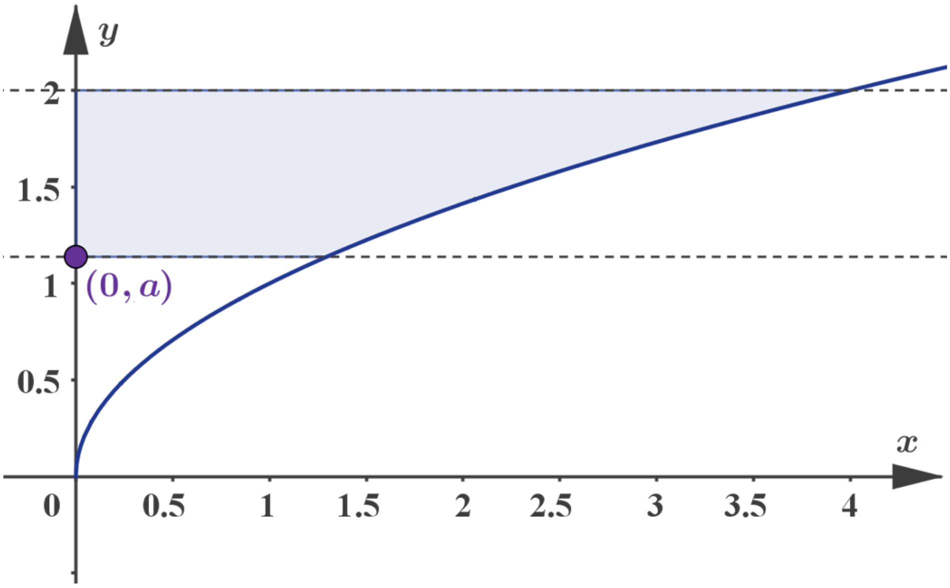

More information

The diagram shows a graph with a shaded region. The function (y=\sqrt{x}) is plotted on a coordinate plane with the x-axis labeled from 0 to 4 and the y-axis labeled from 0 to 2. The shaded region is enclosed by the curve (y=\sqrt{x}), horizontal lines at (y=2) and (y=a), and the y-axis. A point is marked at ((0, a)), which is the intersection of the line (y=a) with the y-axis. The region is shaded above the curve (y=\sqrt{x}) up to (y=2) and between the y-axis and (y=a). This shaded area is rotated 360 degrees about the y-axis.

[Generated by AI]

The region is rotated 360° about the y-axis.

The volume of the solid generated is 6π units cubed. Find the value of a.

The equation of the curve, $y = \sqrt{x}$ can be written as $x = y^2$.

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Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

The volume of the solid is given by the integral $\int_a^2 \pi\left(y^2\right)^2 \mathrm{d}y = \pi \int_a^2 y^4 \mathrm{d}y$, so

$$\begin{aligned} 6\pi &= \pi \int_a^2 y^4 \mathrm{d}y \\ 6 &= \left[\frac{y^5}{5}\right]_a^2 \\ 6 &= \frac{2^5}{5} - \frac{a^5}{5} \\ 30 &= 32 - a^5 \\ a^5 &= 2 \\ a &= \sqrt[5]{2} \approx 1.15. \end{aligned}$$

3 section questions

5. Calculus / 5.12 Area and volume

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/checklist-id-28216/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- find areas of regions bounded by the graph of $y = f(x)$ and the x-axis
  - be aware that if exact values are needed, parts of the region above and below the x-axis need to be dealt with separately
- find area of regions between the graph of two functions
  - be aware that several integrals might be needed in case the graphs intersect
- find area of regions bounded by a curve and the y-axis
- find volume of solids of revolution about both the x and y-axis
- be aware that technology can also be used if approximate values of the definite integral are enough
  - be aware that in certain cases the tools we learn about are not enough to find exact values, so the use of technology is necessary.

Student view

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5. Calculus / 5.12 Area and volume

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/investigation-id-28217/print/) Assign

Investigation 1

In the 3rd century BCE Archimedes proved a certain relationship between the area of a parabolic segment and a related triangle. He used this relationship to find the area of parabolic segments without the use of calculus. The applet below shows you the graph of $y = x^2$ and a chord. It also shows the triangle corresponding to this chord, where the third vertex is below the midpoint of the chord. The applet also shows the area of the parabolic segment and the area of the triangle. Note that the scale on the two axes is not $1 : 1$, the values of the areas are the actual values, not the areas of the shapes on the diagram.

Interactive 1. Parabolic Segment and Triangle Area Comparison.

More information for interactive 1

Student view

This interactive allows users to explore the result by Archimedes regarding the area of a parabolic segment, which is the region bounded by a **parabola**, and a chord of the parabola.

The interface displays the parabola of $y = x2$ along with a chord connecting two points on it. Users can manipulate the two red dots, which represent the endpoints of the chord, to observe how the areas of the enclosed regions change dynamically.

A **triangle** is formed by connecting the ends of the chord to a third point directly below the midpoint of the chord—this point lies on the parabola and its vertical projection intersects the x-axis at the midpoint of the chord.

As users move the endpoints of the chord, they will observe that both the area of the parabolic segment and the area of the triangle change accordingly. The relationship between these two areas becomes evident.

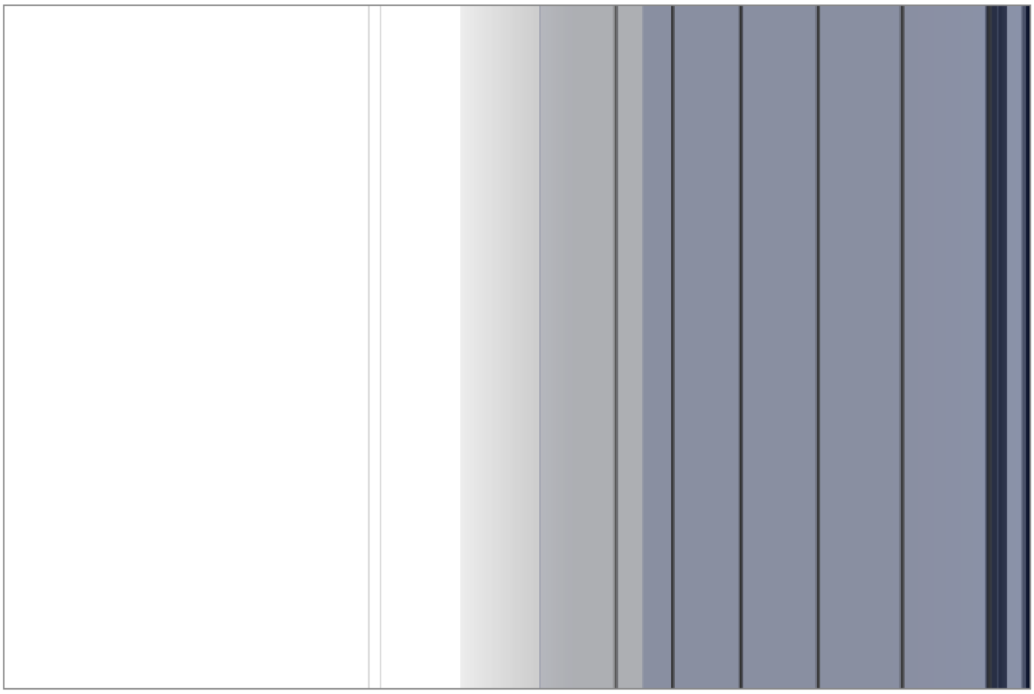
For example, when the area of the triangle is **11.49**, the area of the parabolic segment is **15.32**, and when the area of the triangle is 13.35, the area of the parabolic segment is 17.80, which implies as the area of the triangle increases, the area of the parabolic segment follows a proportional increase.

This visualization helps users investigate and understand the relationship between these two areas, reinforcing key mathematical concepts about parabolic segments, integration, and geometric properties.

- Move the endpoints of the chord and investigate the relationship between the two area values.
- Creating a scatter plot can be useful. Can you find the relationship?

Investigation 2

In the previous subtopic you learned about a method that used approximation by cylinders to find exact volume of solids of revolution. The applet below shows another approach. You can use cylindrical shells instead of cylinders.



Interactive 2. Exploring Volume with Shells.

More information for interactive 2

This interactive allows users to visualize solids of revolution generated by rotating a region.

The screen is divided into two panels.

On the right, there is a graph with three axes with a blue downward curve on the XY plane, starting at the origin and ending on the axis with rectangles projecting on X-axis.

On the left there is an “Adjust curve” button that allows users to drag three red points on the curve to modify the curve's shape. Below it is a horizontal slider that allows users to adjust the number of approximating rectangles, ranging from 1 to 10.

There are two selection buttons that enable users to choose to show a solid figure, which is below the “Adjust curve button”, and to show shells, which is below the horizontal slider. The number of cylindrical shells displayed corresponds to the number of rectangles selected based on the slider’s position. Below the "Show Shells" button, there is a "One by One" button that enables an additional horizontal slider, allowing users to select which individual shell is displayed in the graph. If the total number of shells is 5, the slider for the "One by One" button can be adjusted up to a maximum of 5.

By dragging the red control points, users can dynamically adjust the shape of the curve, observing how the resulting solid changes in real time. Users can select a curve and adjust the number of cylindrical shells (from 1 to 10) to approximate the solid formed by rotating the curve around an axis. The applet visually constructs each shell one by one, allowing users to observe how increasing the number of shells improves the approximation of the solid's volume.

Student
view

- Use the applet to understand how this approximation works.

Overview
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- Find an expression for the volume of the shells and an approximation of the volume of the solid as a sum of these shells.
- Can you turn this sum to an integral that expresses the volume of the solid?
- Pick a graph and find the volume of the solid generated by this graph.

Investigation 3

Before starting the applet below, think about how you would find the length of the river Danube (the second longest river in Europe).

Map showing the flow of the Danube River

Source: https://commons.wikimedia.org/wiki/File:Danubemap.png

(https://commons.wikimedia.org/wiki/File:Danubemap.png)

More information

The map depicts the course of the Danube River as it flows through several European countries. The river starts in Germany and passes through or borders Austria, Slovakia, Hungary, Croatia, Serbia, Bulgaria, Moldova, Ukraine, and Romania before emptying into the Black Sea. The major cities along its route include Vienna, Bratislava, Budapest, and Belgrade. The countries and geographical areas are labeled, and international borders are highlighted. The overall layout demonstrates the river's path across Central and Eastern Europe, showing the connection between the countries through this major waterway.

[Generated by AI]

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Overview
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Interactive 3. Curve Approximation Exploration.

More information for interactive 3

This interactive graph allows users to visualize how to approximate the length of a curve— a function's graph—by using straight-line segments. The screen is divided into two sections: the upper section displays a smooth blue curve marked with red control points, while the lower section provides two toggle buttons labeled 'Adjust curve' and 'Number of segments'.

When the 'Adjust curve' option is selected, users can drag the red points on the graph to reshape the curve. This feature enables dynamic editing of the path whose arc length is being approximated. A red line appears, connecting only the starting and ending points of the curve, indicating the direct distance across the curve without segment subdivisions.

When the 'Number of segments' option is selected, a slider appears beneath it, allowing users to choose a number from 1 to 50. The applet then overlays the curve with a connected series of straight-line segments—also called a polygonal path—that approximates the curve's shape. As users increase the number of segments using the slider, the polygonal path conforms more closely to the curve. This visual refinement demonstrates how increasing the number of segments leads to a better approximation of the curve's true length.

In addition, the current approximate length of the polygonal path is automatically calculated and displayed below the sliders. This value updates in real-time as users adjust the curve's shape or change the number of segments, providing immediate visual and numerical feedback. For example, the length of the curve with one segment is 5.16 units, while changing the segments to 50, the length is 6.32 units.

By toggling between curve adjustment and segment approximation, users can explore how geometry and calculus concepts come together. This activity builds intuition for the method of calculating arc length: by breaking a curved path into smaller linear segments and summing their lengths. It serves as a conceptual bridge to more advanced calculus ideas, such as integral formulas for arc length.

Student view

Overview
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- Does the approximation on the applet matches the method you suggested?
- Find an expression for the line segments and an approximation of the length of the curve as a sum of these lengths.
- Can you turn this sum into an integral that expresses the length of the curve?
- Pick a function and find the length of the graph.

You can use WolframAlpha (https://www.wolframalpha.com/) to find the length of a graph. Type, for example,

length of y=x^2 for 0<x<1

into the search line. Do you see that this length is expressed by an integral? Did you find this integral in the activity above?

Rate subtopic 5.12 Area and volume

Help us improve the content and user experience.

Student view