Index

The big picture

In this subtopic, you will learn about the application of differentiation in context.

Have you ever wondered about what type of function models the shape of hanging chains or cables?

Credit: Creative-TouchGetty Images

Credit: bauhaus1000 Getty Images

Credit: picturist Getty Images

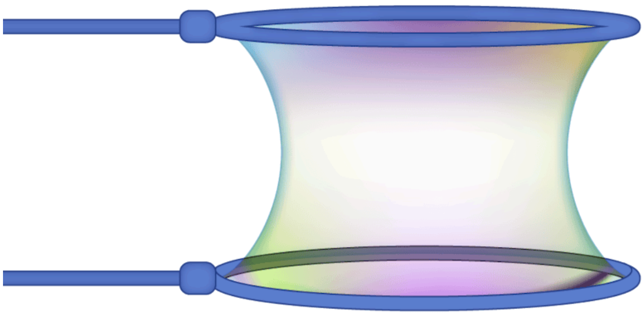
Hanging curves

These curves look like parabolas, but in fact they are not. In your physics studies you may have heard about this curve (or you may in your future studies). It is called a catenary. The name is derived from the Latin word for chain. It is the shape with minimum potential energy. Interestingly, this shape is often used in architecture. For example, the cross section of the roof structure at Keleti railway station in Budapest, Hungary, can be modelled by such a curve.

Washington Dulles International Airport

Credit: joeravi Getty Images:

Yet another interesting 3D connection is the shape of the soap film between two circular wire frames. The cross-section is a catenary.

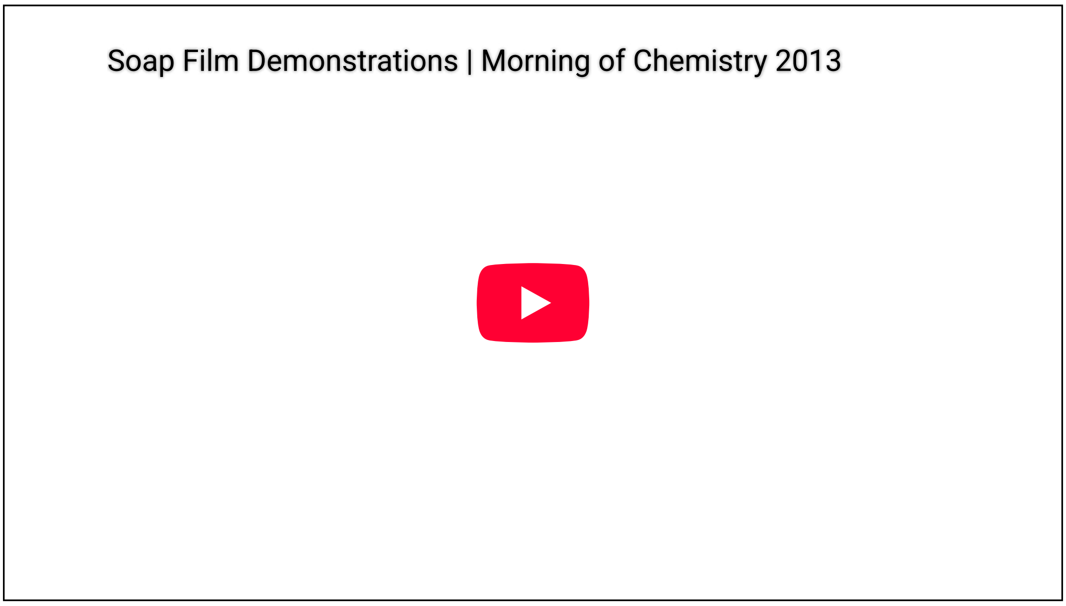


More information

The image shows a soap film forming a surface between two circular wire frames spaced apart. The film spans between the frames, creating a 3D shape that resembles a catenary, characterized by its minimal surface area property. The film appears to have a smooth, curved surface, bending inward between the two circles, and displaying a slight iridescent sheen, which is typical of thin films.

[Generated by AI]

The shape of the soap film has the property that it minimises the surface area connecting the boundary. Watch this video to see some more examples of 3D shapes formed by soap film.



Finally, take a look at the video below. It is an illustration of how nature finds the 2D shape that has the maximum area with a fixed perimeter.

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Video 2. Soap Film Loops.

More information for video 2

The video shows a circular plastic loop with a handle being dipped in a dark liquid. When the loop is lifted from the liquid, a soap film stretched across the circular frame of the loop is formed. The handle of the loop with the soap film is then placed in a wooden holder.

A thin thread with a small slip knot at its end is placed in the soapy film. The thread is carefully manipulated within the soap film, demonstrating the precision required to interact with such a fragile medium. When the thread is poked gently, the thread gets pulled and a circular hole is created in the soap film.

Another shorter thread with a small slip knot at the end is placed into the soap film. When the thread is poked gently, the thread gets pulled and a smaller circular hole is created in the soap film. The two circular holes created by the threads move when disturbed and get attracted to one another.

The threads are removed gently one after the other without breaking the soap film.

In all of the examples above the connecting theme is optimisation, that is, finding minimum or maximum values. However, the mathematical tools needed to work out these optimum shapes are beyond the scope of high school mathematics. In this subtopic, you will see how to use the tools that you have already learned to solve simpler problems. These will serve as a foundation for your future studies, where you may learn to tackle these more advanced questions.

Concept

Calculus is an important tool for investigating relationships in real life. In this subtopic, you will see some examples of this, but, of course, there are plenty more that are not discussed here. While studying the examples in this book, look out for possible similarities with other applications where you could use calculus as a modelling tool to

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investigate a problem and find an optimum solution.

Theory of Knowledge

Stationary points and points of inflection provide the knower with knowledge regarding change. Mathematics as an area of knowledge is quite unique in the degree to which it can make claims to the precision of change. Think about other areas of knowledge and consider the methodologies through which they identify change.

Knowledge Question: To what extent does methodology create knowledge, versus knowledge creating methodology?

5. Calculus / 5.8 Optimisation

Maximum and minimum points

Take another look at the activity you carried out in section 5.7.4 (/study/app/math-aa-hl/sid-134-cid-761926/book/relationship-between-graphs-and-derivative-graphs-id-27792/). The purpose for repeating it here is to highlight a relationship that was not mentioned before.

Activity

On the applet below you can see the graph of the derivative of a certain function. By moving the red point, the applet will show you the graph and give some information about its shape.

- Look for a relationship between the shape of the curve around the turning points and the nature of the turning points.

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Interactive 1. Graph of the Derivative of a Certain Function.

More information for interactive 1

This interactive allows users to visualize the graph of the derivative of a given function. By moving a red dot along the $x - axis$, users can observe the graph of $y = f(x)$. As the red dot moves to the right, the behavior of the derivative, $f'(x)$, is revealed. Also as we move the red point the necessary information about the behavior of the function and its derivative appears below the graph.

When $f'(x)$ is positive, the function $f(x)$ is increasing. If the graph of $f'(x)$ is increasing, the second derivative, $f''(x)$, is positive, and the graph of $f(x)$ is concave up. If the graph of $f'(x)$ reaches a local maximum, this indicates a point of inflection for $f(x)$. Moving the red dot further to the right, if $f(x)$ continues increasing but the graph of $f'(x)$ starts decreasing, $f''(x)$ becomes negative, and the graph of $f(x)$ is concave down. When the graph of $f'(x)$ reaches zero without changing the sign, it marks an inflection point where $f(x)$ pauses its increase or decrease.

In another scenario, when the red dot moves forward and the graph of $f'(x)$ reaches zero while changing from positive to negative, this indicates a local maximum for $f(x)$. As the dot moves further, if $f'(x)$ is negative, $f(x)$ is decreasing, and if $f'(x)$ is decreasing, $f''(x)$ is negative, making the graph of $f(x)$ concave down. When $f'(x)$ reaches a local minimum, it marks an inflection point for $f(x)$. Moving the dot further, if $f(x)$ continues decreasing but $f'(x)$ starts increasing, $f''(x)$ becomes positive, making the graph of $f(x)$ concave up. When $f'(x)$ changes from negative to positive, this indicates a local minimum for $f(x)$.

This interactive tool helps users visualize how the graph of the derivative changes as the function changes.

Did you notice that if a graph is concave down at a turning point, then this turning point is a local maximum point? Similarly, if the graph is concave up at a turning point, then this turning point must be a local minimum point. Since you already know a connection between the

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second derivative and the shape of a graph, this can be translated to the following claim.

Important

Second derivative test

- If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum point on the graph of f.
- If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum point on the graph of f.

Note that this does not reveal anything about the nature of the stationary point if both the first and second derivative are 0. In this case the second derivative test is inconclusive. It could be that the point is a minimum point (for example, the point $(0, 0)$ on the graph of $y = x^4$) and it could be that the point is a maximum point (for example, the point $(0, 0)$ on the graph of $y = -x^4$). It could also be that the point is neither a minimum nor a maximum point (for example, the point $(0, 0)$ on the graph of $y = x^3$).

Example 1

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Identify and classify the stationary points on the graph of $y = 4x^5 - 45x^4 + 120x^3 + 40x^2 - 480x + 1200$.

Steps	Explanation
$y' = 20x^4 - 180x^3 + 360x^2 + 80x - 480 = 0$	At the stationary points the derivative is 0.
The solutions are $x = -1$, $x = 2$ and $x = 6$.	Graphic display calculators have applications that can solve polynomial equations such as this one.
The stationary points are $(-1, 1551)$, $(2, 768)$ and $(6, -1536)$.	You can find the y-coordinate of the stationary points by substituting these values into the equation of the curve.

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Steps	Explanation
$y'' = 80x^3 - 540x^2 + 720x + 80$ At $x = -1$ $y''(-1) = -1260 < 0$, so $(-1, 1551)$ is a local maximum point. At $x = 6$ $y''(6) = 2240 > 0$, so $(6, -1536)$ is a local minimum point. At $x = 2$ $y''(2) = 0$, so the second derivative test is inconclusive.	To classify the stationary points, you can use the second derivative test.
Checking the derivative at $x = 1$ and $x = 3$ $y'(1) = -200 < 0$ $y'(3) = -240 < 0$ Since the derivative does not change sign at $(2, 768)$, this is a horizontal point of inflexion.	To identify the nature of the stationary point where the second derivative test is inconclusive, you can see whether or not the derivative changes sign at that point.
	Note that this last conclusion can also be justified differently, for example by examining the values at $x = -1$, $x = 2$ and $x = 6$.

You might ask why all this calculation is needed when calculators can show you the graph and you can visually identify the nature of the stationary points. This of course is a valid point and the numbers in the question make it unreasonable to ask it on an exam. However, the explanation above is still a good illustration of what can be done without using the graphing ability of the calculator.

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27795/print/) Assign

In **Example 1**, an algebraic method was used to classify the nature of the stationary points. Below you will find instructions on how to use your graphic display calculator to find the *values* of the first and second derivative at a given point, without the need to find the derivatives.

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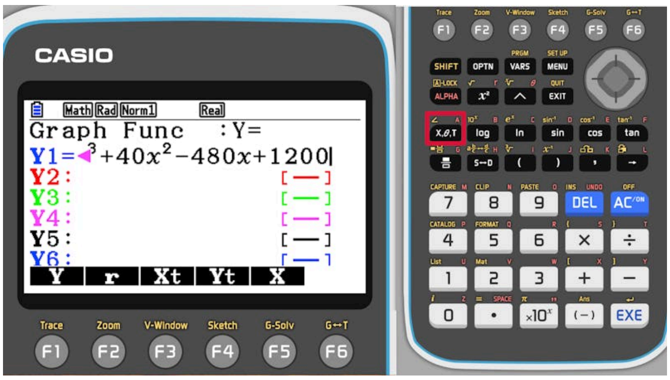
Steps	Explanation
In these instructions you will see how to find $f'(6)$ and $f''(6)$ for the polynomial in Example 1. To define the polynomial, enter the graph mode.	
Enter the formula, $$4x^5 - 45x^4 + 120x^3 \\ +40x^2 - 480x + 1200.$$ You could also define the derivative and second derivative functions here, but let's now look at a different approach. Go back to the home screen ...	

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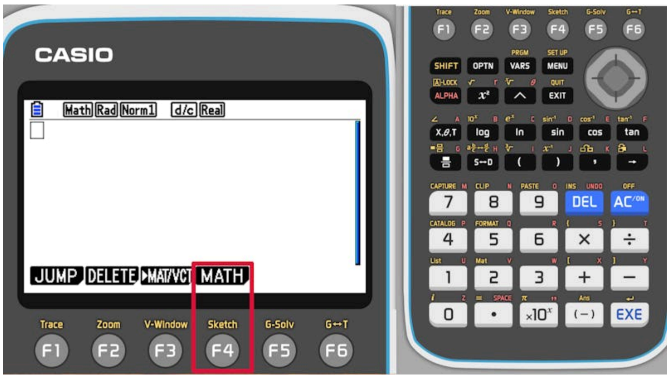
<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>... and choose the calculator mode.</td><td>


</td></tr>
<tr><td>Press F4 to select the math options ...</td><td>


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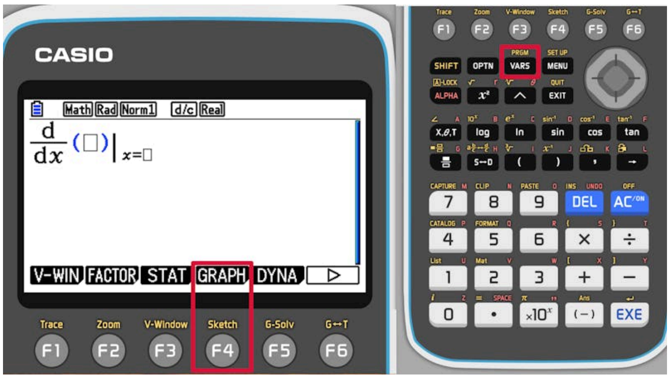
Steps	Explanation
... and press F4 again to bring up a template for the first derivative.	
You do not need to type in the expression again, you already stored it in Y1. To access this name, press VARS and F4 to find variable names related to graphs.	

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/maximum-and-minimum-points-id-27796/print/) Assign

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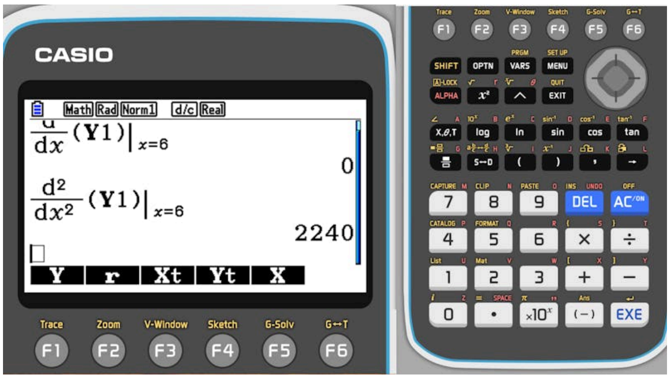
Steps	Explanation
Use F1 to print the name for the function and type 6 to indicate, that you would like to find the value of the derivative at $x = 6$. Press EXE to find the value of the derivative.	
You can use similar steps to find the value of the second derivative. The difference is, that you need to choose the template for the second derivative instead of the first derivative.	

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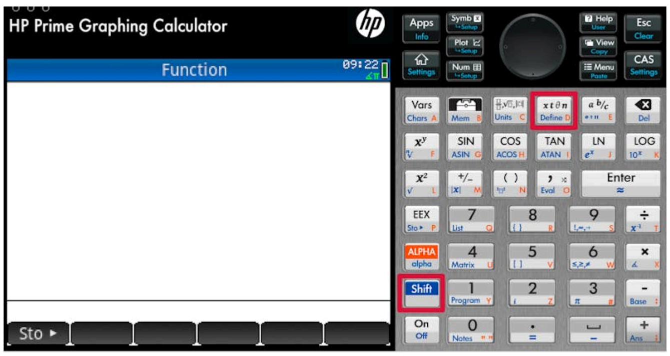
Steps	Explanation
In these instructions you will see how to find $f'(6)$ and $f''(6)$ for the polynomial in Example 1. You can use functions and define the derivatives in the functions application, but let's look at a different approach here. Enter the home screen of any application.	
You can define functions. To access this feature, press 2nd/Define.	

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<tr><th>Steps</th><th>Explanation</th></tr>
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Give a name to your function and enter the formula,

$$4x^5 - 45x^4 + 120x^3$$
$$+40x^2 - 480x + 1200.$$

Notice, that the calculator recognized, that x is the variable of the function. We do not need it here, but you can have more than one variables for user defined functions.

Once done, press OK.

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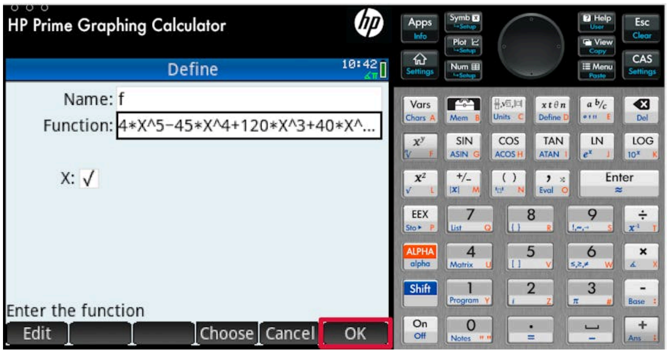

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You can use the derivative template to look for the value of the derivative.

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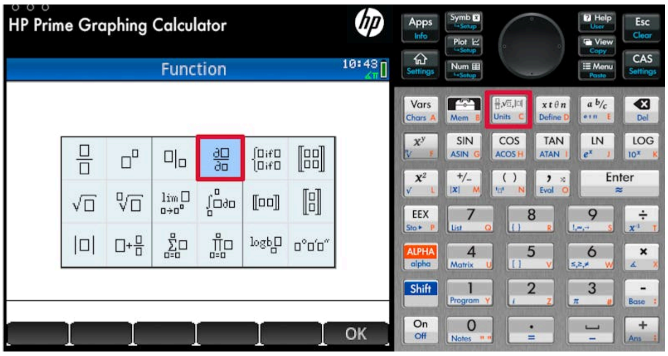

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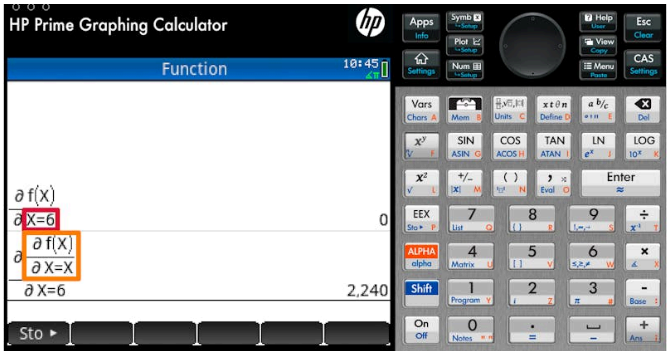
Steps	Explanation
Use the name of the function you just defined instead of typing in the expression again. Use the syntax $x = 6$ to indicate that you would like to find the value at $x = 6$. There is no template for second derivative, but you can use the derivative template twice to find the derivative of the derivative. Note, that in the inner derivative you are defining the derivative function, so use the syntax $x = x$.	

Steps	Explanation
In these instructions you will see how to find $f'(6)$ and $f''(6)$ for the polynomial in Example 1. To define the polynomial, enter the screen, where you can define functions.	

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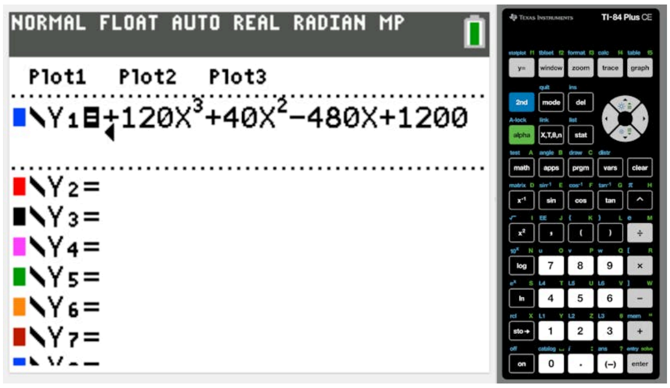
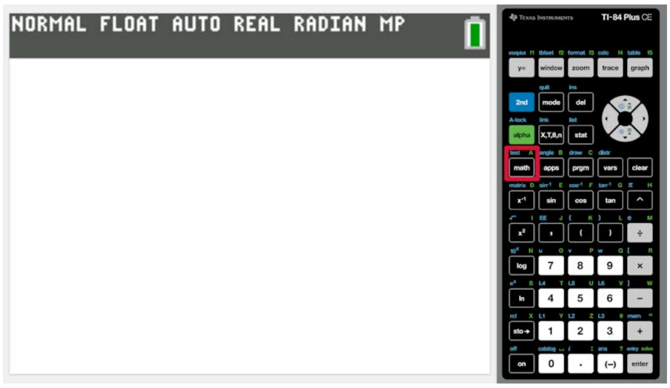
Steps	Explanation
Enter the formula, $$4x^5 - 45x^4 + 120x^3$$ $$+40x^2 - 480x + 1200.$$ You could also define the derivative and second derivative functions here, but let's now look at a different approach. Go back to the home screen ...	
... press math ...	

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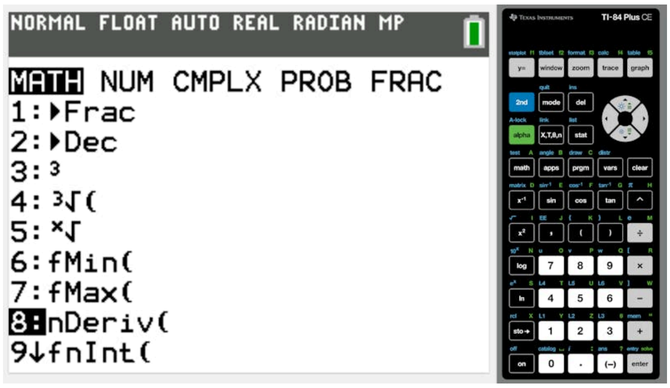
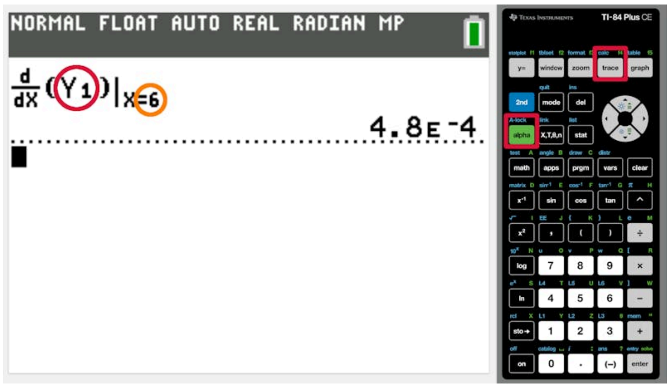
Steps	Explanation
... and choose the option (nDeriv) to find numerical derivatives.	
When filling the template, use the name of the function you just defined instead of typing in the expression again. Use the syntax $x = 6$ to indicate that you would like to find the value at $x = 6$.	

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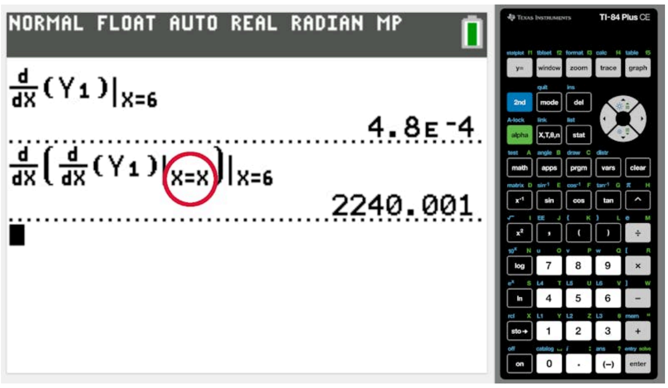
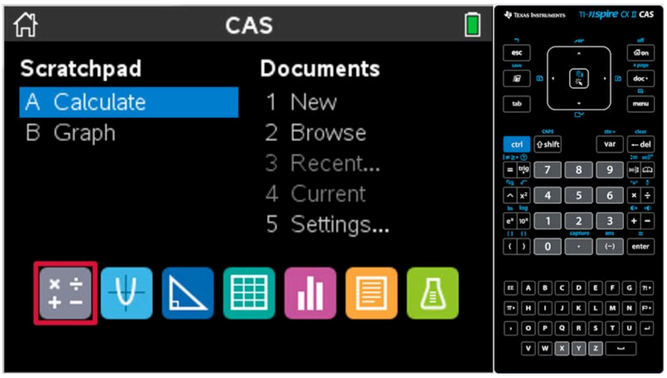
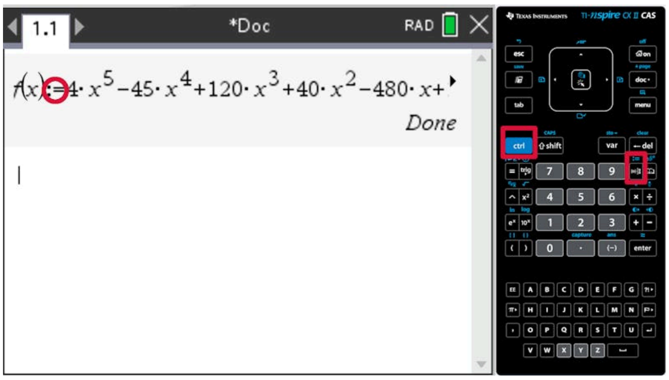
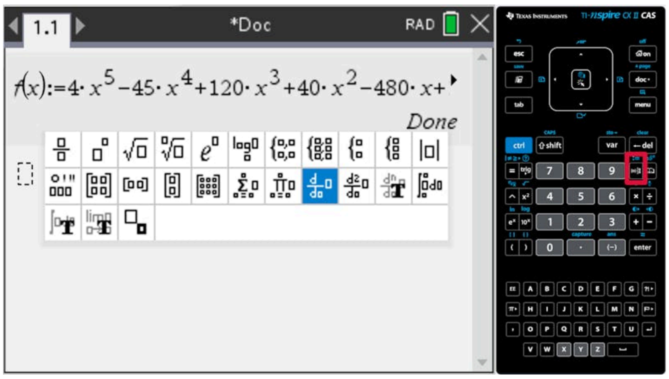
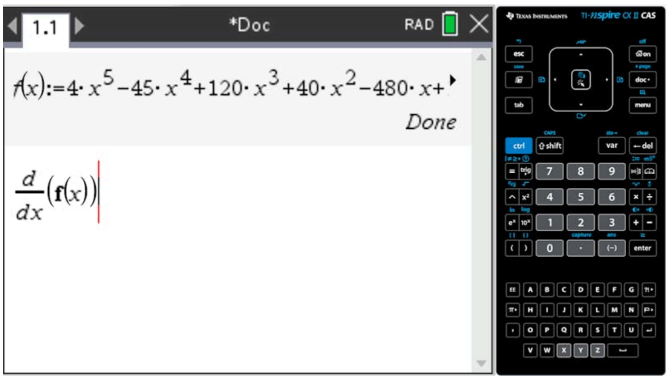
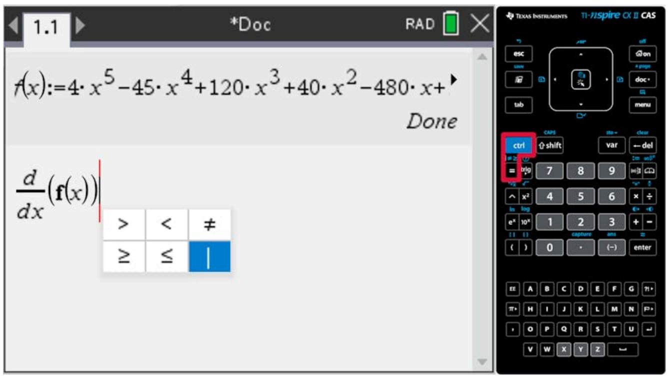
Steps	Explanation
There is no template for second derivative, but you can use the derivative template twice to find the derivative of the derivative. Note, that in the inner derivative you are defining the derivative function, so use the syntax $x = x$. Notice, that the calculator does not give the exact value. It uses numerical algorithms, so the result is only approximate.	

Steps	Explanation
In these instructions you will see how to find $f'(6)$ and $f''(6)$ for the polynomial in Example 1. There are several ways of doing this, let's see now how to work this out on a calculator page (without graphing any functions).	

Steps	Explanation
Give any name to your function and use colon equal to indicate, that this is a definition. Enter the formula, $$4x^5 - 45x^4 + 120x^3 + 40x^2 - 480x + 1200.$$	
Use the template menu to find a template for the derivative.	

Steps	Explanation
When filling the template, use the name of the function you just defined instead of typing in the expression again. Don't press enter just yet, you still need to tell the calculator that you would like to evaluate the derivative at $x = 6$.	
Choose the vertical line (that stands for "evaluated at")	

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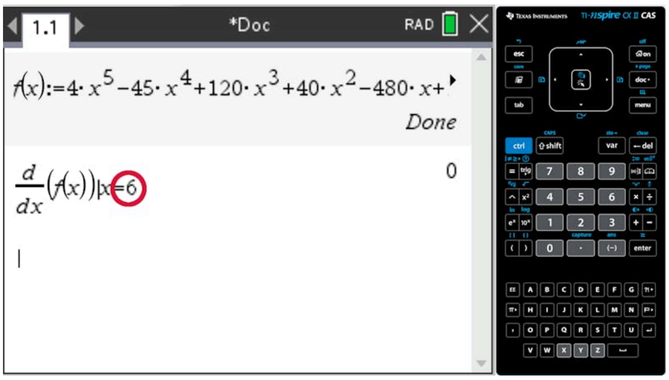
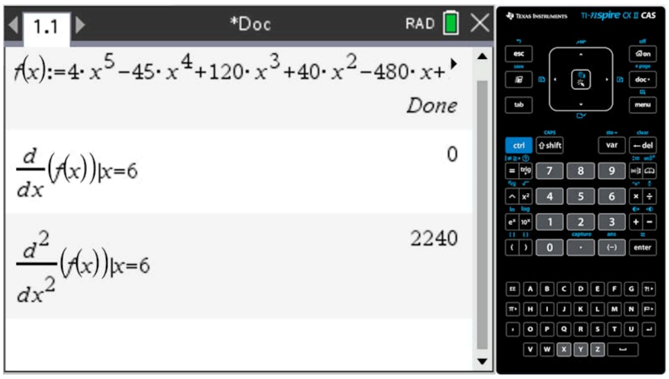
Steps	Explanation
Use the syntax $x = 6$ to indicate that you would like to find the value of the derivative evaluated at $x = 6$.	
You can use similar steps to find the value of the second derivative. The difference is, that you need to choose the template for the second derivative instead of the first derivative.	

Example 2

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Identify and classify the stationary points on the graph of $y = x + 2\cos x$ in the interval $[0, 2\pi]$.

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Steps	Explanation
$y' = 1 - 2\sin x = 0$ $\sin x = \frac{1}{2}$	At the stationary points the derivative is 0.
$x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$	There are two solutions of this equation between 0 and 2π.
The stationary points are $\left(\frac{\pi}{6}, \frac{\pi}{6} + \sqrt{3}\right)$ and $\left(\frac{5\pi}{6}, \frac{5\pi}{6} - \sqrt{3}\right)$	You can find the y-coordinate of the stationary points by substituting these values into the equation of the curve.
$y'' = -2\cos x$ • At $x = \frac{\pi}{6}$ $y''\left(\frac{\pi}{6}\right) = -\sqrt{3} < 0$, so $\left(\frac{\pi}{6}, \frac{\pi}{6} + \sqrt{3}\right)$ is a local maximum point. • At $x = \frac{5\pi}{6}$ $y''\left(\frac{5\pi}{6}\right) = \sqrt{3} > 0$, so $\left(\frac{5\pi}{6}, \frac{5\pi}{6} - \sqrt{3}\right)$ is a local minimum point.	To classify the stationary points, you can use the second derivative test.

Example 3

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Identify and classify the stationary points on the graph of $y = x^2 \mathrm{e}^{-x}$.

Steps	Explanation
$y' = 2x\mathrm{e}^{-x} + x^2(-\mathrm{e}^{-x}) = 0$ $(2x - x^2)\mathrm{e}^{-x} = 0$ $x(2 - x)\mathrm{e}^{-x} = 0$	At the stationary points the derivative is 0.
$x = 0$ or $x = 2$	$\mathrm{e}^{-x} \neq 0$

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Steps	Explanation
The stationary points are $(0, 0)$ and $(2, 4e^{-2})$.	You can find the y-coordinate of the stationary points by substituting these values into the equation of the curve.
$y'' = (2-2x)e^{-x} + (2x - x^2)(-e^{-x})$ $= (x^2 - 4x + 2)e^{-x}$ • At $x = 0$ $y''(0) = 2 > 0$, so $(0, 0)$ is a local minimum point. • At $x = 2$ $y''(2) = -2e^{-2} < 0$, so $(2, 4e^{-2})$ is a local maximum point.	To classify the stationary points, you can use the second derivative test.

Example 4

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Find the smallest and largest value of $y = \ln(4-x) + 2\ln x$ for $1 \leq x \leq 3$.

Steps	Explanation
$y' = -\frac{1}{4-x} + \frac{2}{x} = 0$ $\frac{-x + 2(4-x)}{x(4-x)} = 0$ $8 - 3x = 0$ $x = \frac{8}{3}$	To find the turning points, you can look for the values where the derivative is 0.
$y = \ln\left(4 - \frac{8}{3}\right) + 2\ln\left(\frac{8}{3}\right)$ $= \ln\left(\frac{4}{3} \times \left(\frac{8}{3}\right)^2\right)$ $= \ln\left(\frac{256}{27}\right)$	You can find the y-coordinate of the stationary point by substituting this value into the equation of the curve.

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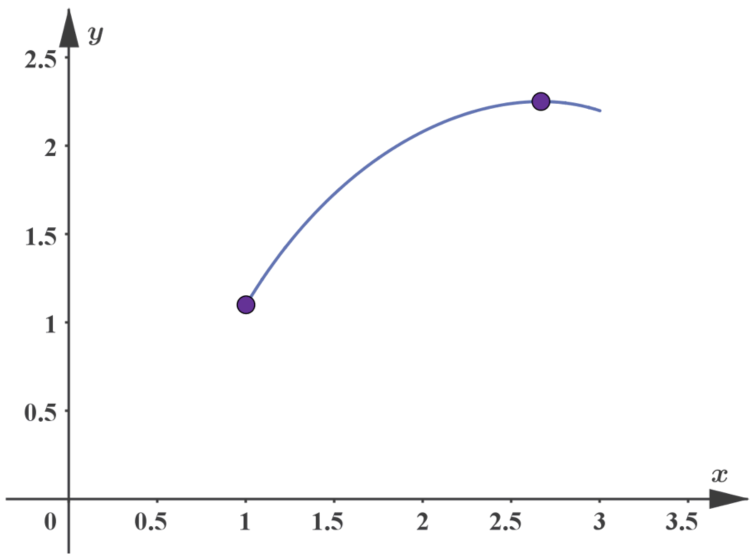
Steps	Explanation
$y(1) = \ln 3 + 2\ln 1 = \ln 3 < \ln\left(\frac{256}{27}\right)$ $y(3) = \ln 1 + 2\ln 3 = \ln 9 < \ln\left(\frac{256}{27}\right)$	To find the minimum and maximum value, you can investigate the value at the stationary point compared to the values at the endpoints of the domain.
Hence, the curve is increasing until it reaches the stationary point and from there it is decreasing.	The curve can only change direction at a stationary point.
The largest value of $y = \ln(4 - x) + 2\ln x$ for $1 \le x \le 3$ is $\ln\left(\frac{256}{27}\right)$ and the smallest value is $\ln 3$.	$\ln 3 < \ln 9$

The diagram below shows the graph with the maximum and minimum points indicated.



While thinking about the previous problem, you may notice the following.

Important

If f is a differentiable function defined on the interval $[a, b]$, then the largest and smallest values of the function are $f(a)$, $f(b)$ or the y-coordinate of one of the

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turning points.

You can investigate the claim above on the following applet. Move the red points around to change the shape of the graph. The applet shows you the domain and the range.

Interactive 1. Graph Showing Domain and Range.

More information for interactive 1

This interactive allows users to explore the concepts of global maximum and minimum values of a differentiable function defined on a closed interval $[a, b]$. It is designed to help users understand how critical points, endpoints, and turning points affect the largest and smallest function values.

There is a graph shown in the interactive with maximum and minimum defined on the x and y axes. The curve has 4 interactive red dots, one in the starting and ending each, and another 2 in between. By moving the red points, users can change the shape of the graph and observe how the function's maximum and minimum values are determined within the given domain.

The applet visually displays the domain and highlights the global maximum and minimum values, which are either the function values at the endpoints $f(a)$ and $f(b)$ or the y-coordinates of the function's turning points.

The applet provides a clear representation of how the domain (set of x-values) and range (set of y-values) are affected by modifications to the function. Users can observe how changing the function's curvature impacts the possible maximum and minimum values within the given interval.

3 section questions

Question 1

Difficulty:

★★☆

The point $(a, a - \sqrt{b})$ is a local minimum point on the graph of $y = x + 2\sin x$ between 0 and 2π.

Find the value of b. Give an exact answer as an integer.

3

Accepted answers

3, b=3

Explanation

Steps	Explanation
$y' = 1 + 2\cos x = 0$ $\cos x = -\dfrac{1}{2}$	At the stationary points the derivative is 0.
$x = \dfrac{2\pi}{3}$ or $x = \dfrac{4\pi}{3}$	There are two solutions of this equation between 0 and 2π.
The stationary points are $\left(\dfrac{2\pi}{3}, \dfrac{2\pi}{3} + \sqrt{3}\right)$ and $\left(\dfrac{4\pi}{3}, \dfrac{4\pi}{3} - \sqrt{3}\right)$.	You can find the y-coordinate of the stationary points by substituting these values into the equation of the curve.

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
$y'' = -2\sin x$ • At $x = \frac{2\pi}{3}$ $y''\left(\frac{2\pi}{3}\right) = -\sqrt{3} < 0$, so $\left(\frac{2\pi}{3}, \frac{2\pi}{3} + \sqrt{3}\right)$ is a local maximum point. • At $x = \frac{4\pi}{3}$ $y''\left(\frac{4\pi}{3}\right) = \sqrt{3} > 0$, so $\left(\frac{4\pi}{3}, \frac{4\pi}{3} - \sqrt{3}\right)$ is a local minimum point.	To classify the stationary points, you can use the second derivative test.
Hence, $b = 3$.	

Question 2

Difficulty:

★☆☆

Find the smallest value of $f(x) = \frac{x}{2} + \frac{2}{x}$ for $1 \leq x \leq 5$.

Give an exact answer as an integer.

2

Accepted answers

2, y=2

Explanation

Steps	Explanation
$f(1) = \frac{1}{2} + \frac{2}{1} = 2.5$ $f(5) = \frac{5}{2} + \frac{2}{5} = 2.9$	The smallest value can be one of the values at the endpoints of the domain.
$f'(x) = \frac{1}{2} - \frac{2}{x^2} = 0$ $\frac{x^2 - 4}{2x^2} = 0$ $x^2 = 4$	The smallest value can also be the y-coordinate of a stationary point. To find the x-coordinate of the stationary points, you can solve $f'(x) = 0$.
$x = 2$	You need the solution in $[1, 5]$.

Student view

Steps	Explanation
The only stationary point in $[1,5]$ is $(2, f(2)) = (2,2)$.	$\frac{2}{2} + \frac{2}{2} = 2$
The smallest value of $f(x) = \frac{x}{2} + \frac{2}{x}$ for $1 \leq x \leq 5$ is 2.	The minimum of 2.5, 2.9 and 2 is 2.

Question 3

Difficulty:

★★☆

Find the largest value of $f(x) = \frac{x}{2} + \frac{2}{x}$ for $1 \leq x \leq 5$.

Give an exact answer as a decimal.

2.9

Accepted answers

2.9, y=2.9

Explanation

Steps	Explanation
$f(1) = \frac{1}{2} + \frac{2}{1} = 2.5$ $f(5) = \frac{5}{2} + \frac{2}{5} = 2.9$	The largest value can be one of the values at the endpoints of the domain.
$f'(x) = \frac{1}{2} - \frac{2}{x^2} = 0$ $\frac{x^2-4}{2x^2} = 0$ $x^2 = 4$	The largest value can also be the y-coordinate of a stationary point. To find the x-coordinate of the stationary points, you can solve $f'(x) = 0$.
$x = 2$	You need the solution in $[1,5]$.
The only stationary point in $[1,5]$ is $(2, f(2)) = (2,2)$.	$\frac{2}{2} + \frac{2}{2} = 2$
The largest value of $f(x) = \frac{x}{2} + \frac{2}{x}$ for $1 \leq x \leq 5$ is 2.9.	The maximum of 2.5, 2.9 and 2 is 2.9.

Optimisation in context

In this section you will see some problems that can be solved using the techniques you learned in the previous sections.

Example 1

★☆☆

There are 40 apple trees in an orchard. Each tree produces 600 apples in a year. For each additional tree planted in the orchard, the output of every tree drops by 10 apples.

How many trees should be added to the existing trees in the orchard to maximise the total production of apples?

Steps	Explanation
Let P be the production of the orchard (total number of apples). Let T be the number of trees added to the orchard. Let A be the number of apples on a tree.	Introducing variables is useful for translation of the problem into the language of mathematics. The question gives the relationship between these variables.
$P = (40 + T)A$	The production is the number of trees multiplied by the number of apples on a tree.
$A = 600 - 10T$	The number of apples on a tree decreases when new trees are planted.

Student view

Overview
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Steps	Explanation
$P = (40+T)(600-10T) = 24\,000 + 200T - 10T^2$	The combination of these two relationships give the production c the orchard in terms of the numbe of new trees.
$\frac{\mathrm{d}P}{\mathrm{d}T} = 200 - 20T = 0$ $T = 10$	When production is at an optimum the rate of change, $\frac{\mathrm{d}P}{\mathrm{d}T}$, is 0.
Since $\frac{\mathrm{d}^2P}{\mathrm{d}T^2} = -20 < 0$, the addition of 10 trees would correspond to the maximum production.	To see if this stationary point corresponds to a maximum or a minimum production, you can use the second derivative test.
The maximum production of the orchard is $25\,000$ apples in a year and it is achieved by adding 10 new trees to the existing 40.	$P(10) = (40+10)(600 - 10 \times 1$ $= 25\,000$

Example 2

★☆☆

A farmer wants to enclose a rectangular part of his field. He has 800 metres of fencing available to use. What is the largest possible area that he can enclose?

Steps	Explanation
Let x and y be the length of the two sides of the rectangle.	Introducing variables is useful for translation of the problem into the language of mathematics.
$2x + 2y = 800$ $x + y = 400$ $y = 400 - x$	The given perimeter can be used to express y in terms of x. Since both x and y are positive, the meaningful x-values are $0 < x < 400$.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
$A = xy$ $= x(400 - x)$ $= 400x - x^2$	The area can also be expressed in terms of x.
$\frac{\mathrm{d}A}{\mathrm{d}x} = 400 - 2x = 0$ $x = 200$ For $x = 200$, the area is $A = 200 \times 200 = 40\,000$. Since $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -2 < 0$, the point $(200, 40\,000)$ is a local maximum point.	You can use the first and second derivative to search for and classify the turning points.
Hence, the maximum rectangular area that the farmer can enclose with 800 metres of fencing is $40\,000$ square metres.	If there is exactly one turning point on the graph, then it gives the global maximum or minimum.

Note that instead of using the derivative, you can get the same result by finding the vertex of the quadratic expression that gives the area in terms of x.

Example 3

★★☆

Petra is preparing for a craft fair. She needs to decide on the quantity of her product to take to the fair. To make the product, she needs to invest USD 30. After this initial investment, the materials to make each item cost USD 10. From past experience, she estimates that if she charges \$ p for each item, then she will sell $100 - 2p$ items at the fair.

- How many items should she make, and how much should she charge for each item, to maximise her profit?
- What is this maximum profit?

Steps	Explanation
Let n be the number of items she takes to the fair and let p be the price of each item.	Introducing variables is useful for translation of the problem into the language of mathematics.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
$n = 100 - 2p$ $2p = 100 - n$ $p = 50 - \frac{n}{2}$	If she wants to sell all the items, she needs to price them according to her past experience.
$I(n) = np$ $= n\left(50 - \frac{n}{2}\right)$ $= 50n - \frac{n^2}{2}$	Her income at the fair is the product of the number of items sold and the price of each item.
$C(n) = 30 + 10n$	Her cost is the sum of the initial investment and the cost of the materials to make the items for the fair.
$P(n) = I(n) - C(n)$ $= \left(50n - \frac{n^2}{2}\right) - (30 + 10n)$ $= -\frac{n^2}{2} + 40n - 30$	The profit is the cost subtracted from the income.
The maximum profit corresponds to $n = -\frac{40}{2 \times \left(-\frac{1}{2}\right)} = 40$	The turning point of the parabola $y = ax^2 + bx + c$ is on the axis of symmetry, $x = -\frac{b}{2a}$.
So Petra should take 40 items to the fair and sell each item for $p = 50 - \frac{40}{2} = 30$ USD.	
The maximum profit she can achieve is $P(40) = -\frac{40^2}{2} + 40 \times 40 - 30 = 770$ USD.	

Example 4

★★☆

Student view

Overview
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The applet below illustrates a method of making a cone from a paper disk. When a sector is cut from the disk and the cut sides are pulled together, the paper forms a cone.

Interactive 1. Method of Making a Cone From a Paper Disk.

More information for interactive 1

The interactive allows users to explore the formation of a cone from a paper disc by cutting out a sector from a circular disc and joining the remaining edges.

As users change the cutout angle, the applet dynamically displays the updated cone shape. The curved boundary of the remaining disc becomes the circumference of the cone's base.

By adjusting the cutout angle, users can observe how changes in the sector size affect the cone's dimensions and ultimately its volume.

The interactive tool continuously displays the calculated volume of the cone at the bottom, enabling real-time observation of how changes in the sector angle impact the volume. Users can find a pattern in the volume with respect to the angle. If the angle of cutout increases, the volume of the cone decreases. The volume of the cone displays when the angle of cutout is 66° is $0.40304R^3$. Here, R is the radius of the circular sheet.

This applet provides an engaging way to visualize 3D shape formation from a 2D perspective, making it a valuable tool for students studying geometry, mathematics, and engineering concepts.

What is the maximum volume of the cone that can be formed from a disk of radius 10 cm?

Student view

Overview
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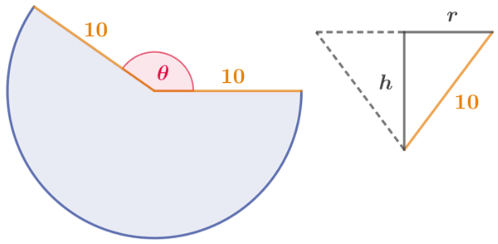
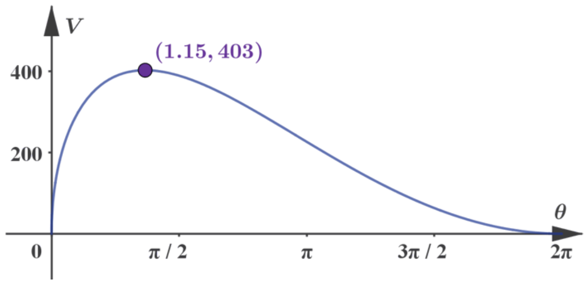
Steps	Explanation
 Let the cut-out angle be θ (measured in radians). Let the height of the corresponding cone be h and the radius of the base of the cone is r.	Drawing a sketch and introducing variables are useful tools for translation of the problem into the language of mathematics.
$$\begin{aligned} 10(2\pi - \theta) &= 2r\pi \\ r &= \frac{5(2\pi - \theta)}{\pi} \end{aligned}$$	The length of the shaded arc and the circumference of the base circle of the cone is the same.
$$\begin{aligned} h^2 + r^2 &= 10^2 \\ h^2 &= 100 - r^2 \\ h &= \sqrt{100 - r^2} \end{aligned}$$	The slant height of the cone is the same as the radius of the paper disk, so you can use Pythagoras' theorem to express the height of the cone in terms of the radius of the base.
$$V = \frac{1}{3}r^2\pi h$$	The volume of the cone can be expressed in terms of the height and the radius of the base.
$$\begin{aligned} V &= \frac{1}{3}r^2\pi h \\ &= \frac{1}{3}r^2\pi\sqrt{100 - r^2} \\ &= \frac{1}{3}\left(\frac{5(2\pi - \theta)}{\pi}\right)^2\pi\sqrt{100 - \left(\frac{5(2\pi - \theta)}{\pi}\right)^2} \end{aligned}$$	Putting all this together gives the volume in terms of the size of the cut-out angle.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
	Graphic display calculators have applications that can find the maximum of this expression for $0 \leq \theta \leq 2\pi$ (these are the meaningful central angles of the cut).
So the maximum volume of the cone that can be formed starting with a disk of radius 10 cm is 403 cm^3.	

Although it was not asked for in **Example 4**, the graph also indicates that this optimum cone can be formed by cutting out a sector with central angle approximately 1.15 radians (or 66°).

If you are interested, you can work out the exact solution (by solving $V' = 0$).

The exact cut-out angle that gives the maximum volume is $\theta = \dfrac{2}{3}(3 - \sqrt{6})\pi$

(or $360 - 120\sqrt{6}$ degrees), and the maximum volume is $V = \dfrac{2000\pi}{9\sqrt{3}}$.

Example 5

★★☆

A company manufactures robots that can travel both on land and in water. Different models have different speeds on land and in water. The robots are tested on a circular lake with diameter 300 m. The aim is to reach the opposite point at the end of the diameter, starting from a point on the shore. On the applet below, you can investigate the total travel times using different speeds and different ways of crossing the lake.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Interactive 2. Total Travel Times at Various Speeds and Methods for Lake Crossing.

More information for interactive 2

This interactive lets users explore the optimal path for the shortest time for travelling across a circular lake. There are two windows. On the left side of the interface, there is a speed selector, in which the x-axis depicts speed in water, ranging from 0.38 m/s to 2.38 m/s and the y-axis represents the speed on land from 1 m/s to 3 m/s. Placing the red dot gives a combination of land and water speed.

On the right side of the interface, there is a circular lake in which the object has to move from point A to B via C. A represents the starting point on the shore. O is the center of the lake. B is the destination on the opposite shore.

The straight line represents movement in water and the curved line represents the movement on land. The objective of this interactive is to find the shortest time to travel from A to B using a perfect combination of AC and CB and land and water speed. The total time is displayed in real time on the bottom left, enabling users to simulate various crossing strategies, such as traveling directly across the lake or combining land and water routes to find the shortest time to travel.

For example, if we keep the speed selector on the extreme top right corner, making the fastest speed on land 2.38 m/s and water 3 m/s and the line from A to B completely straight. That is, point C on point B, using only the waterway, we will get the total time as 126.32 seconds. However, if we keep point C on point A , using only land, we will get a time of 157.08 seconds. Now the user needs to find a combination of land and water, giving the fastest way of traveling from point A to C

The purpose of this interactive applet is to explore the fastest possible route for amphibious robots to cross a circular lake while considering their different speeds on land and in water.

The company is testing two robots, both of which can travel on land with a speed of 3 metres per second.

Student view

- The first robot can travel in water with a speed of 1 metre per second.

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

- The second robot can travel in water with a speed of 2 metres per second.

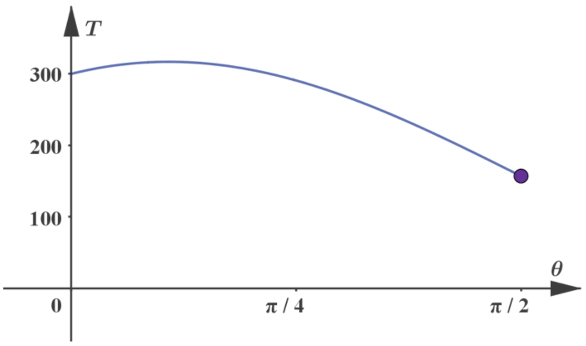
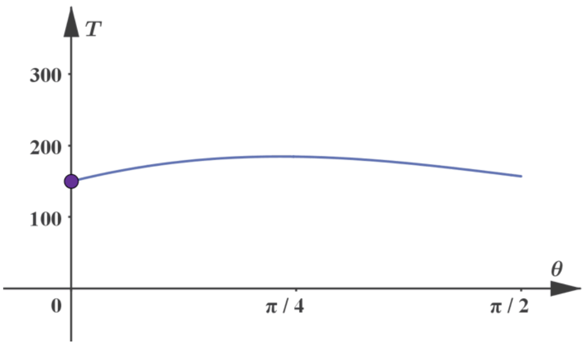
What is the shortest time in which these robots can cross the lake? Which way should they go?

Steps	Explanation
Let A be the starting point, let B the target, let O the centre of the circle and let C be the point the robot is aiming for across the lake. Let θ be the angle $\mathrm{C\hat{A}B}$ (measured in radians). The radius of the circle is $\mathrm{OA} = \mathrm{OB} = \mathrm{OC} = 150$.	Introducing variables is useful for translation of the problem into the language of mathematics.
$\mathrm{C\hat{A}O} = \mathrm{A\hat{C}O}$, so $\mathrm{A\hat{O}C} = \pi - 2\theta$. $\mathrm{AC}^2 = 150^2 + 150^2 - 2 \times 150 \times 150 \cos(\pi - 2\theta)$ $= 45\,000 - 45\,000 \cos(\pi - 2\theta)$ $\mathrm{AC} = \sqrt{45\,000 - 45\,000 \cos(\pi - 2\theta)}$	You can use the cosine rule in the isosceles triangle AOC to express AC in terms of θ.
The arc length formula gives the length of arc CB: $150 \times 2\theta = 300\theta$.	$\mathrm{C\hat{O}B} = \pi - (\pi - 2\theta) = 2\theta$

To calculate the time needed for the robot to cross the lake, you can use the knowledge that time is the distance divided by the speed.

Steps	Explanation
The crossing time of the first robot is $T(\theta) = \dfrac{\sqrt{45\,000 - 45\,000 \cos(\pi - 2\theta)}}{1} + \dfrac{300\theta}{3}$	The first robot has a speed of 3 metres per second on land and 1 metre per second in water.

Student view

Steps	Explanation
	Investigating the graph gives the shortest time.
The shortest time corresponds to $\theta = \frac{\pi}{2}$, i.e. when the robot does not go in the water at all. The shortest crossing time for the first robot is $\frac{300\pi/2}{3} \approx 157$ seconds.	Note that the shortest time does not correspond to a turning point on the graph.
The crossing time of the second robot is $T(\theta) = \frac{\sqrt{45\,000 - 45\,000\cos(\pi - 2\theta)}}{2} + \frac{300\theta}{3}$	The second robot has a speed of 3 metres per second on land and 2 metres per second in water.
	Investigating the graph gives the shortest time.
The shortest time corresponds to $\theta = 0$, so when the robot is in the water all the time. The shortest crossing time for the second robot is $\frac{300}{2} = 150$ seconds.	Note that the shortest time does not correspond to a turning point on the graph.

Student view

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Note that in the explanation the cosine rule was used to find the length of AC. Triangle AOC can also be split to two right triangles, and since triangle AOC is isosceles, the length of AC can also be found without using the cosine rule. This expression, $\text{AC} = 300\cos(\theta)$, can also be used to find the answer to the question.

If you experiment with the applet from **Example 5** you will see that for this particular shape of lake the shortest time always corresponds to either moving all the way around the lake on land or swimming straight across. For this question there is no way of getting a shorter time by swimming to some point first and then moving on land from there.

3 section questions

Question 1

Difficulty: ★☆☆

A farmer wants to enclose a rectangular part of her field next to a river. She has 800 metres of fencing available to use and no fence is needed along the river.

What is the largest possible area (in square metres) that she can enclose?

Give an exact answer as an integer, without units.

80000 ✓

Accepted answers

80000, 80000m^2

Explanation

Steps	Explanation
Let x and y be the length and width of the rectangle (y is along the river).	Introducing variables is useful for translation of the problem into the language of mathematics.
$2x + y = 800$ $y = 800 - 2x$	The given length of the fence can be used to express y in terms of x. Since both x and y are positive, the meaningful x-values are $0 < x < 400$.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
$A = xy$ $= x(800 - 2x)$ $= 800x - 2x^2$	The area can also be expressed in terms of x.
$\frac{dA}{dx} = 800 - 4x = 0$ $x = 200$ For $x = 200$, the area is $A = 200 \times 400 = 80\,000$. Since $\frac{d^2A}{dx^2} = -4 < 0$, the point $(200, 80\,000)$ is a local maximum point.	You can use the first and second derivative to search for and classify the turning points.
Hence, the maximum rectangular area the farmer can enclose along the river with 800 metres of fencing is $80\,000$ square metres.	If there is exactly one turning point on the graph, then it gives the global maximum or minimum.

Question 2

Difficulty: ★★☆

A company is producing fruit juice. They decide to distribute the juice in small cuboidal cartons of volume 243 cubic centimetres. To place their preferred image on the face of the box, one edge of the box should be twice as long as another edge.

Find the length of the longest edge of the box that uses the smallest amount of packaging material.

Give an exact answer as an integer, without units.

9 ✓

Accepted answers

9, 9cm

Explanation

Steps	Explanation
Let x, $2x$ and y be the length of the edges.	Introducing variables is useful for translation of the problem into the language of mathematics.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
$243 = x \times 2x \times y$ $y = \frac{243}{2x^2}$	The given volume can be used to express y in terms of x. Since both x and y are positive, the meaningful x-values are $x > 0$.
$A = 2(2x^2 + xy + 2xy)$ $= 4x^2 + 6xy$ $= 4x^2 + 6x \times \frac{243}{2x^2}$ $= 4x^2 + \frac{729}{x}$	The surface area can also be expressed in terms of x.
$\frac{\mathrm{d}A}{\mathrm{d}x} = 8x - \frac{729}{x^2} = 0$ $x^3 = \frac{729}{8}$ $x = \frac{9}{2}$ For $x = \frac{9}{2}$, the surface area is $A = 4 \times \left(\frac{9}{2}\right)^2 + \frac{729}{9/2} = 243.$ For $x > 0$, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 8 + \frac{1458}{x^3} > 0$, so $\left(\frac{9}{2}, 243\right)$ is a local minimum point.	You can use the first and second derivative to search for and classify the turning points.
For $x = \frac{9}{2}$, the other sides are $2x = 9$ and $y = \frac{243}{4.5 \times 9} = 6$. So the longest edge of the box that uses the smallest amount of packaging material is 9 centimetres.	If there is exactly one turning point on the graph, it gives the global maximum or minimum.

Question 3

Difficulty: ★★☆

Alex is preparing for a craft fair. He needs to decide on the quantity of his product to take to the fair. To make the product, he needs to invest US$40. After this initial investment the materials to make each item cost US$8.

Student view

Overview (/study/ap aa-hl/sid-134-cid-761926/o

From past experience, he estimates that if he charges US$$p$ for each item, then he can sell $200 - 4p$ items at the fair.

What is the maximum profit Alex can achieve at the fair?

Give an exact answer as an integer, without the dollar sign.

1724

Accepted answers

1724, 1724$, $1724, 1724dollar, 1724dollars, 1724 dollar, 1724 dollars

Explanation

Steps	Explanation
Let n be the number of items Alex takes to the fair and let p be the price of each item.	Introducing variables is useful for translation of the problem into the language of mathematics.
$$\begin{aligned} n &= 200 - 4p \\ 4p &= 200 - n \\ p &= 50 - \frac{n}{4} \end{aligned}$$	If he wants to sell all the items, he needs to price them according to his past experience.
$$\begin{aligned} I(n) &= np \\ &= n\left(50 - \frac{n}{4}\right) \\ &= 50n - \frac{n^2}{4} \end{aligned}$$	His income on the fair is the product of the number of pieces and the price of each piece.
$C(n) = 40 + 8n$	His cost is the sum of the initial investment and the cost of the materials to make the items for the fair.
$$\begin{aligned} P(n) &= I(n) - C(n) \\ &= \left(50n - \frac{n^2}{4}\right) - (40 + 8n) \\ &= -\frac{n^2}{4} + 42n - 40 \end{aligned}$$	The profit is the cost subtracted from the income.
The maximum profit corresponds to $$n = -\frac{42}{2 \times (-1/4)} = 84.$$	The turning point of the parabola $y = ax^2 + bx + c$ is on the axis of symmetry, $x = -\frac{b}{2a}$.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
The maximum profit Alex can achieve is $$P(84) = -\frac{84^2}{4} + 42 \times 84 - 40 = 1724$$ dollars.	

5. Calculus / 5.8 Optimisation

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-27798/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- identify and classify stationary points on a graph:
  - use the first derivative to find stationary points as possible turning points
  - use the second derivative test to see whether a stationary point is a minimum or a maximum
  - recognise when the second derivative test is not conclusive and use alternative methods to see the nature of the stationary point
- understand the difference between local and global minimum and maximum points:
  - recognise when a global minimum or maximum does not correspond to a turning point.
- translate optimisation questions in context into mathematical language and use appropriate methods to find optimum solutions:
  - either use a graphic display calculator and visual inspection of the graph or use the derivative to identify turning points.

Student view

5. Calculus / 5.8 Optimisation

Investigation

In The big picture (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27795/) you saw an example of how nature chooses the shape with maximum area when the perimeter is fixed. In section 5.8.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/optimisation-in-context-id-27797/) you found the rectangle with maximum area when the perimeter is fixed. In this investigation you will explore triangles.

On the applet below move the vertices of the triangle and try to find a triangle that has maximum area. You can either move the points around freely or you can tick the box and restrict movement so that the perimeter does not change.

- What do you notice?
- Can you prove what you noticed?

Interactive 3. Maximum Area of a Triangle With Adjustable Vertices and Fixed Perimeter.

More information for interactive 3

The interactive allows users to explore the properties of triangles by moving the vertices to maximize the area while optionally fixing the perimeter. Users can freely adjust the positions of the triangle's vertices by moving the red dots or choose to restrict movement to maintain a constant perimeter by clicking the checkbox ‘Fix perimeter.’ The applet dynamically updates the area of the triangle as users adjust the vertices.

The applet displays the current perimeter and area, enabling users to observe how changes in the triangle's shape affect these measurements. The users will also notice that two triangles with the same perimeter can have different areas. For example, fixing the perimeter at 12.83 can give an area of 7 and 7.2 in two different triangles. The primary goal is to find the triangle configuration that yields the maximal area for a given perimeter.

Student view

When the checkbox is enabled, the perimeter remains constant while users move the vertices. Despite maintaining the same perimeter, different shapes yield different areas. The maximum area configuration appears to be a more symmetric triangle, often approaching an equilateral shape.

Through this interactive, users can understand that the maximum area of a triangle given a fixed perimeter occurs when the triangle is equilateral. Users are encouraged to notice patterns or properties that emerge during this exploration and to consider how they might prove their observations.