TOPIC 5
CALCULUS

SUBTOPIC 5.1
INTRODUCTION TO DIFFERENTIATION

Overview
(/study/ap
122-
cid-
754029/

Teacher view

Index

5. Calculus / 5.1 Introduction to differentiation

The big picture

In topic 5 you will learn some of the concepts of calculus, building on your previous work on functions, and extending your knowledge towards investigating changes. Calculus has many applications in a wide range of areas. To see some examples, take a look at the following video.

Activity

While watching the video and listening to what the narrator says

- look for expressions that are related to change
- think about how you would analyse the data collected by scientists.

Greenland Ice Sheet Changing

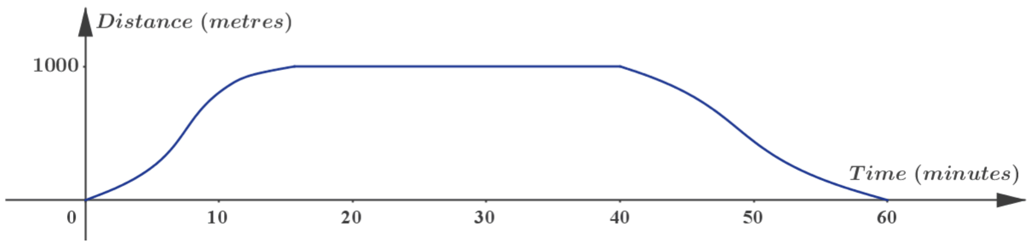
Student
view

Overview
(/study/ap
122-
cid-
754029/

To begin your investigation, consider the following situation:

Ali walks from his home to the grocery store and then he walks back home.

This diagram shows information about his journey.



More information

The graph represents a journey plotted with distance in meters on the Y-axis and time in minutes on the X-axis. The X-axis ranges from 0 to 60 minutes, marked at 10-minute intervals. The Y-axis ranges up to 1000 meters. Starting at 0, the graph shows a sharp increase reaching approximately 1000 meters at around 10 minutes, indicating rapid movement. From 10 to 40 minutes, the graph forms a plateau at 1000 meters, suggesting a pause or constant speed. Subsequently, there is a gradual decline from 40 to 60 minutes, where the distance returns to 0 meters, indicating a return to the starting point or end of the journey.

[Generated by AI]

Activity

Think of some questions about Ali’s journey that could be answered by interpreting the graph.

Student
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Did you ask questions related to distance and time?

Overview
(/study/ap
122-
cid-
754029/

Did you ask questions related to speed?

How is speed calculated? From your previous studies, you may know that the average speed of a journey is the distance travelled divided by the time taken for the journey.

Example 1

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Use the graph above to answer the following questions.

- How far is the store from Ali's home?
- How long did it take for Ali to get to the store?
- How much time did Ali spend in the store?
- What was Ali's average speed on the way back home?
- What was Ali's maximum speed along this journey?

The graph shows that in the first 15 minutes Ali was walking towards the store, which is 1000 metres away from his home.

Then Ali spent 25 minutes in the store before walking back home. This took him 20 minutes.

His average speed on the way back was

$$\frac{1000 \text{ metres}}{20 \text{ minutes}} = \frac{1\text{km}}{\frac{1}{3}\text{hour}} = 3 \text{ kmh}^{-1}$$

The answer to the last question is not covered here – it will be considered in section 5.1.4.

In this subtopic, you will investigate how to move from the concept of average rate of change to instantaneous rate of change. You will do this using the concept of a limit.

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view

Overview
(/study/ap
122-
cid-
754029/

Concept

While studying this subtopic, think about the relationship between average and instantaneous rate of change . Think of an average rate of change over shorter and shorter intervals. This gives an approximation of the instantaneous rate of change.

5. Calculus / 5.1 Introduction to differentiation

Concept of a limit

Let's begin to explore the concept of a limit by looking at an example.

Example 1

★☆☆

Consider the sequence of numbers,

$$a_n = \frac{6n-4}{3n+2} \text{ for } n = 1, 2, 3, \ldots.$$

- Find $a_1, a_2, a_3, a_4, a_{10}$ and a_{100}.
- Without calculating the exact value, give an estimate of a_{1000}.

$$a_1 = \frac{6 \times 1 - 4}{3 \times 1 + 2} = \frac{2}{5} = 0.4$$

$$a_2 = \frac{6 \times 2 - 4}{3 \times 2 + 2} = \frac{8}{8} = 1$$

$$a_3 = \frac{6 \times 3 - 4}{3 \times 3 + 2} = \frac{14}{11} \approx 1.27$$

$$a_4 = \frac{6 \times 4 - 4}{3 \times 4 + 2} = \frac{20}{14} \approx 1.43$$

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view

Overview
(/study/ap
122-
cid-
754029/

$$a_{10} = \frac{6 \times 10 - 4}{3 \times 10 + 2} = \frac{56}{32} = 1.75$$

$$a_{100} = \frac{6 \times 100 - 4}{3 \times 100 + 2} = \frac{596}{302} \approx 1.97$$

Since 1000 is large compared with 4 and 2,

$$a_{1000} = \frac{6 \times 1000 - 4}{3 \times 1000 + 2} \approx \frac{6 \times 1000}{3 \times 1000} = 2.$$

Activity

Use the following applet to investigate the behaviour of a similar sequences of numbers.

Based on your investigation, find an approximation for $\frac{an+b}{cn+d}$ if n is large and $c \neq 0$. Express this approximation in terms of a, b, c and d.

Interactive 1. Concept of Limit.

More information for interactive 1

The interactive allows users to explore the behavior of sequence defined by the general form $\frac{a_n+b}{c_n+d}$, where a, b, c, and d are constants, $n = 1, 2, \ldots, 50$ and $c \neq 0$. By plotting terms up to a_{50}, users can observe how the sequence behaves as n becomes large. The concept of a limit describes the value that the sequence approaches as n

Student view

Overview
(/study/ap
122-
cid-
754029/

increases. When n is large, the terms b and d become insignificant compared to an and cn, causing the sequence to approximate $\frac{a}{c}$. This is because the dominant terms an and cn dictate the sequence's behavior, leading to the ratio $\frac{a}{c}$ as n approaches infinity. For example, in the sequence $\frac{2n+5}{3n+2}$, the terms stabilize around $\frac{2}{3}$ for large n. Users can generate new sequences of type $\frac{a_n+b}{c_n+d}$ by clicking on the "New Sequence" tab for more practice.

On the graph, the limit is visualized as a horizontal line that the sequence approaches. Initially, the terms may vary, but as n increases, they cluster closer to this line, indicating convergence. By plotting terms up to a_{50}, users can observe this stabilization, gaining insight into how the sequence's behavior changes with larger n.

For large values of n, $\frac{an+b}{cn+d} \approx \frac{a}{c}$.

Did you find this formula? Can you explain how you obtained it?

✓ **Important**

The number A is called the limit of the sequence $\{a_n\}$ if a_n is close to A for large values of n.

Example 2

★☆☆

Consider the sequence defined by $a_n = \frac{3n-2}{2n+4}$.

Write down the limit of this sequence.

For large values of n, $\frac{3n-2}{2n+4} \approx \frac{3}{2} = 1.5$, so the limit of this sequence is 1.5.

Similarly to sequences, you can talk about limits of functions.

Student
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Important

The number A is called the limit (at infinity) of the function f if $f(x)$ is close to A for large values of x.

Note that this is closely related to the concept of a horizontal asymptote . If A is the limit (at infinity) of f, then the line $y = A$ is a horizontal asymptote of the graph of $y = f(x)$.

Example 3

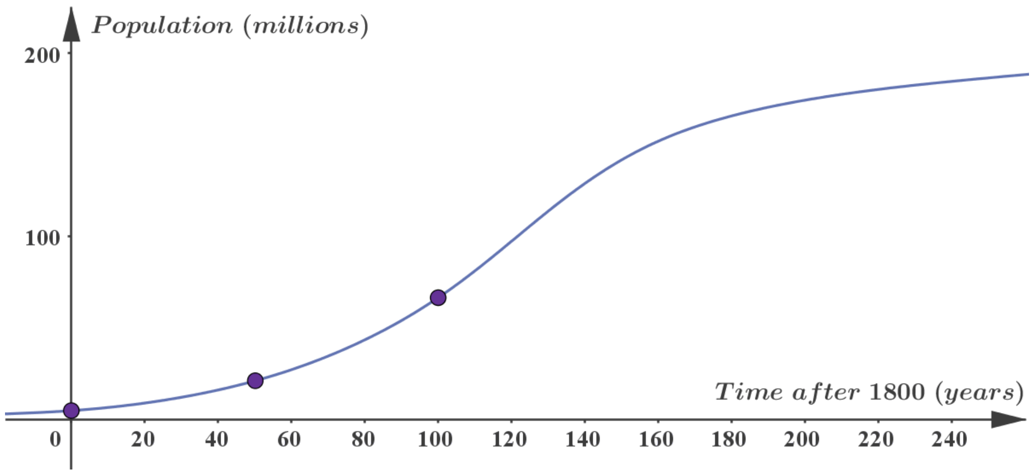
★★☆

The population of the United States in 1800, 1850 and 1900 was 5.3, 23.1 and 76 million people respectively. This population growth can be modelled by

$$P(t) = \frac{189.4}{1 + 34.74\mathrm{e}^{-0.031476t}},$$

where t is the time in years after 1800.

The diagram below illustrates this growth.



More information

The graph illustrates the growth of population measured in millions over time, in years after 1800. The X-axis represents the time in years after 1800, ranging from 0 to 240. The Y-axis represents the population in millions, ranging from 0 to 200. The curve on the graph shows an initial slow increase until around 100 years after 1800, where the rate of population growth begins to accelerate. The acceleration continues sharply upwards and seems to plateau slightly toward the end, suggesting a slowing of growth after the steep increase. Specific data points are marked on the curve, indicating notable positions at 0, 100, and 240 years where the population is significantly changing.

[Generated by AI]

Find the limit of this function as t increases without bound and interpret what this limit means.

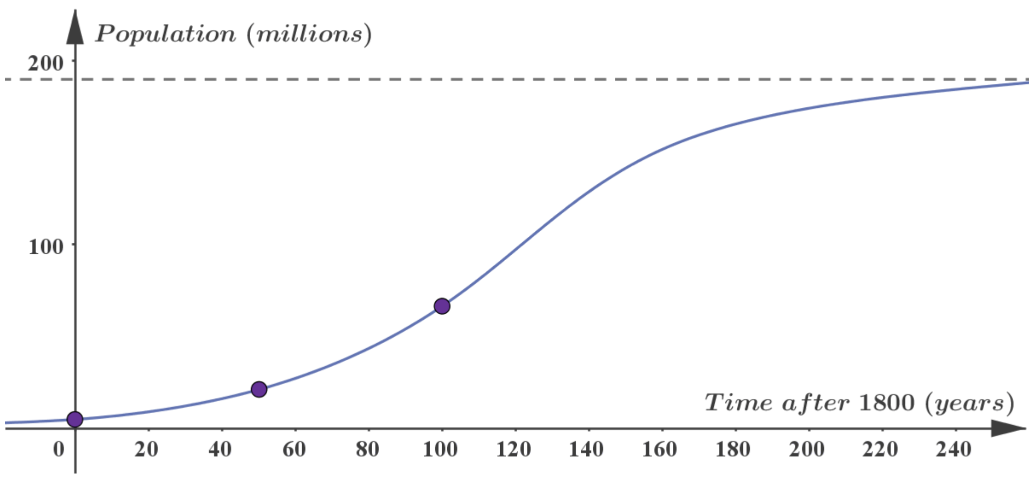
For large values of t

- $\mathrm{e}^{-0.031476t}$ is close to 0,
- so $1 + 34.74\mathrm{e}^{-0.031476t}$ is close to 1,
- so $\dfrac{189.4}{1 + 34.74\mathrm{e}^{-0.031476t}}$ is close to 189.4.

Hence, the limit of P (at infinity) is 189.4.

This means, that this model predicts a population increase in the US that approaches, but does not go beyond 189.4 million people.

The diagram below shows the horizontal asymptote to the graph of P.



Overview
(/study/ap
122-
cid-
754029/

Example 4

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The table below contains information on the temperature of a cup of tea.

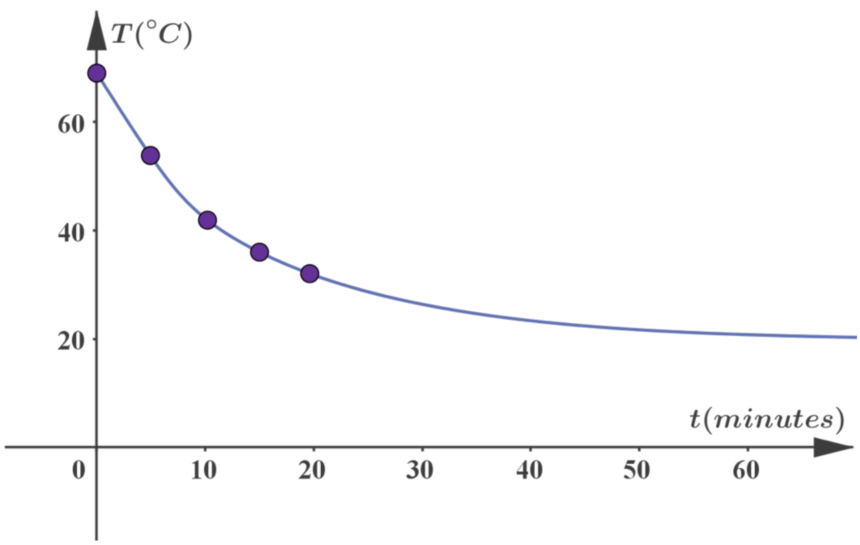
Time (minutes)	0	5	10	15	20
Temperature (°C)	69	54	43	36	31

The cooling can be approximately modelled by the function

$$T(t) = 19.8 + 49.2\mathrm{e}^{-0.0744t},$$

where $T(t)$ is the temperature in degrees Celsius t minutes after the measurement is started.

The diagram below illustrates this function and the data points.



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More information

Overview
(/study/ap
122-
cid-
754029/

The graph plots temperature (T) in degrees Celsius on the vertical Y-axis against time (t) in minutes on the horizontal X-axis. The X-axis is labeled from 0 to 60 in increments of 10, and the Y-axis is labeled from 0 to 60 in increments of 20. A curve that slopes downward from left to right is shown on the graph. The curve starts at 60°C at time 0, moving through several plotted points, and gradually approaches but never quite reaches 0 as time increases. The general trend is a decrease in temperature over time, indicating a cooling process.

[Generated by AI]

Find the limit of this function as t increases without bound and interpret what this limit means.

For large values of t

- $\mathrm{e}^{-0.0744t}$ is close to 0,
- so $19.8 + 49.2\mathrm{e}^{-0.0744t}$ is close to 19.8.

Hence, the limit of T (at infinity) is 19.8.

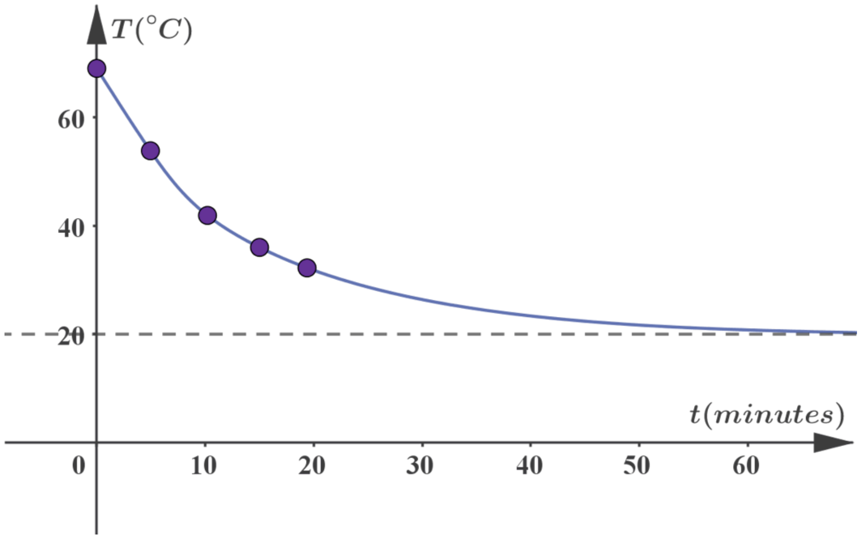
So, this model predicts that the temperature of the cup of tea is decreasing to 19.8 °C.

What do you think this tells you about the temperature of the room?

The diagram below shows the horizontal asymptote to the graph of T.

Student
view

Overview
(/study/app/m/sid-122-cid-754029/



So far you have only looked at limits at infinity. You will now turn your attention to a different kind of limit, which will become important as the subtopic progresses.

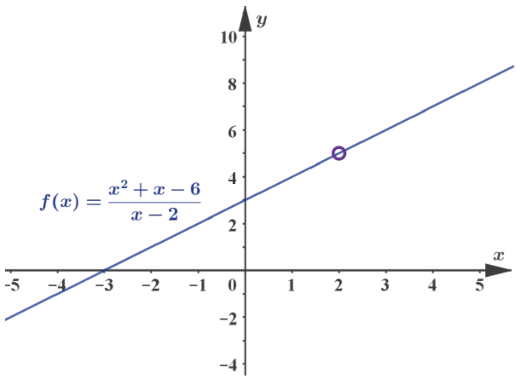
Activity

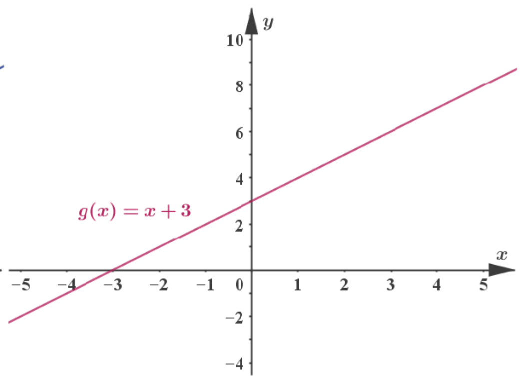
Consider the functions defined by $f(x) = \dfrac{x^2 + x - 6}{x - 2}$ and $g(x) = x + 3$.

- Find $f(n)$ and $g(n)$ where *n* is an integer. What do you notice?
- Use your calculator to sketch the graph of both functions for $-4 \leq x \leq 5$.

What do you notice? Did the calculator draw the correct graph?

The two diagrams below show part of the graphs of the functions from the activity above.





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More information

The image contains two graphs of functions displayed on separate coordinate planes. The left graph represents the function f(x) = (x² + x - 6) / (x - 2), displayed as a blue line. The X-axis ranges from -5 to 5 and the Y-axis ranges from -4 to 10. The line crosses the X-axis approximately at x = -3, and includes a small circular break at x = 2, indicating a discontinuity. The right graph represents the function g(x) = x + 3 as a red line. This graph also has X-axis values ranging from -5 to 5 and Y-axis values from -4 to 10. The line crosses the Y-axis at y = 3. Both graphs indicate distinct linear relationships with slightly different slopes and intercepts. The visual difference between the graphs lies in the discontinuity of f(x) and the starting intercept of g(x).

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The difference between the two graphs is not big, but important.

- For $x = 2$, $f(2)$ is undefined, because you have $2 - 2 = 0$ in the denominator. This is indicated by the empty circle on the graph of f at the point with x-coordinate 2. At the same time, $g(x)$ is defined everywhere, no empty circle on the graph.
- For $x \neq 2$,
$$f(x) = \frac{x^2 + x - 6}{x - 2} = \frac{(x+3)\cancel{(x-2)}}{\cancel{x-2}} = x + 3 = g(x),$$
so for $x \neq 2$ the two graphs match.

Since $g(2) = 5$, the comparison of the two graphs tell us that the empty circle on the graph of f is at $(2, 5)$. Even though f is not defined at $x = 2$, you can still see that if x is close to 2, then $f(x)$ is close to 5. The mathematical way of expressing this relationship is saying that the limit of the function f at $x = 2$ is 5.

Important

The number A is called the limit of the function f at $x = a$ if $f(x)$ is close to A when x is close to a.

Note that the function does not have to be defined at $x = a$ in order to have a limit. If the function happens to be defined at $x = a$ and the limit is the same as $f(a)$, we say that the function is continuous at $x = a$.

Overview
(/study/ap
122-
cid-
754029/

Example 5

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Consider the function defined by

$$f(x) = \frac{x^4 + x^3 - 7x^2 - x + 6}{x^2 + 2x - 3}.$$

Find the limit of f at $x = 1$.

Note that since $1^2 + 2 \times 1 - 3 = 0$, $f(1)$ is undefined, so we cannot find the limit as the quotient of the numerator and the denominator at $x = 1$.

Method 1
You can use the calculator to find this limit in several ways.

- You can, for example, find $f(1.1) = -1.89$, $f(1.01) = -1.9899$, $f(1.001) = -1.998999$, etc. and make a conjecture. It looks like that these values are getting closer and closer to -2, so the limit is -2.
- You can also sketch the graph with a graphic display calculator tracing the graph close to the point where $x = 1$. Investigating the y -coordinates of the points on the graph close to x =1 will give you the same limit.

Method 2
You can search for the factor form of the numerator and the denominator. This is not easy, but if you happen to notice that

$$x^4 + x^3 - 7x^2 - x + 6 = (x-1)(x+1)(x-2)(x+3)$$

and

$$x^2 + 2x - 3 = (x-1)(x+3),$$

then, after cancellation, you get that for $x \neq 1$ and $x \neq -3$

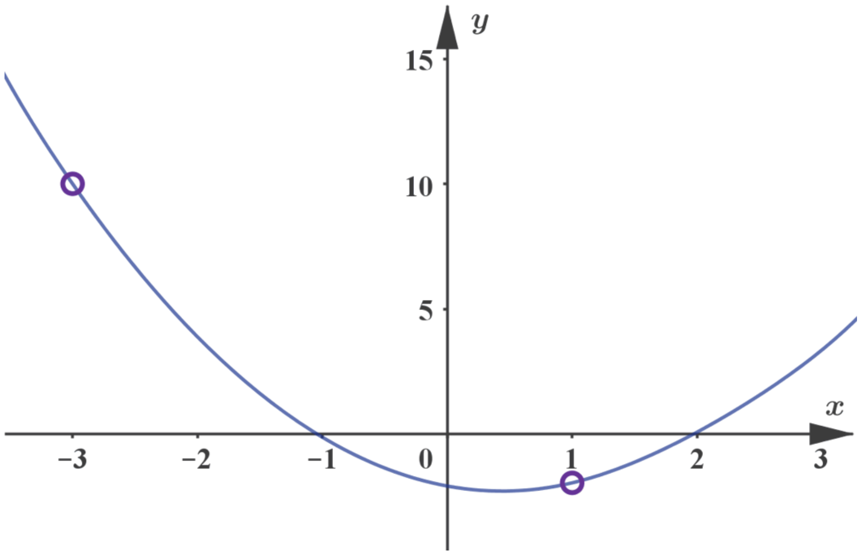
$$f(x) = (x+1)(x-2).$$

Student
view

Overview
(/study/ap
122-
cid-
754029/

So the limit is $(1+1)(1-2)=-2$.

The diagram below shows the relevant part of the graph of f.



3 section questions

5. Calculus / 5.1 Introduction to differentiation

Gradient at a point

In the previous section, you investigated limits of number sequences and functions. In this section, you will extend the concept of a limit to a more general case.

Activity

- The applet below shows a curve, a point and the line called the tangent to the graph. Move the point along the curve. What do you think a tangent is?
- When you tick the "adjust secant" box, a second point appears and a secant line is drawn. Move the red point and explore what happens to the secant line

Student
view

as the two points on the curve get closer and closer to each other.

- How is the tangent related to the secant lines?
- A triangle is also shown on the applet. Can you give a reason why it is there?

Interactive 1. Gradient at a Point.

More information for interactive 1

This interactive allows users to explore the concepts of tangents and secants in relation to a curve, helping them understand the fundamental idea of limits in calculus. Users can adjust the curve by ticking the “Adjust curve” box and moving the red points to shape it as desired. Once the curve is set, selecting the "Move Point" option enables users to drag a point along the curve and observe how the tangent slope updates dynamically in real time. This slope is represented by the tangent line at that point, which touches the curve at exactly one location.

When users check the "Adjust Secant" box, a second point appears, and a secant line is drawn between the two points on the curve. The slope of this secant line represents the average rate of change between the two points. As the second point moves closer to the first, the secant line gradually aligns with the tangent line. This demonstrates how the tangent line is the limiting position of secant lines as the distance between the two points approaches zero, reinforcing the definition of the derivative. The interactive visually conveys how the slope of the secant line converges to the slope of the tangent line, highlighting their relationship in the context of differentiation.

A right triangle is also displayed in the applet, representing the difference in x-values (horizontal leg) and y-values (vertical leg) between the two points forming the secant line. The ratio of these differences, $\Delta y/\Delta x$, gives the slope of the secant line. As the second point moves closer, this ratio approaches the derivative, which is the instantaneous rate of change at the given point. This triangle provides a geometric interpretation of the difference quotient, helping

Overview (/study/app/m/sid-122-cid-754029/o

users understand how the derivative is computed from secant slopes.

They visualize derivatives as instantaneous rates of change, reinforcing core calculus concepts through interactive exploration.

Important

For a function f, the slope of the tangent at the point $(a, f(a))$ is called the gradient of the graph of $y = f(x)$ at $x = a$.

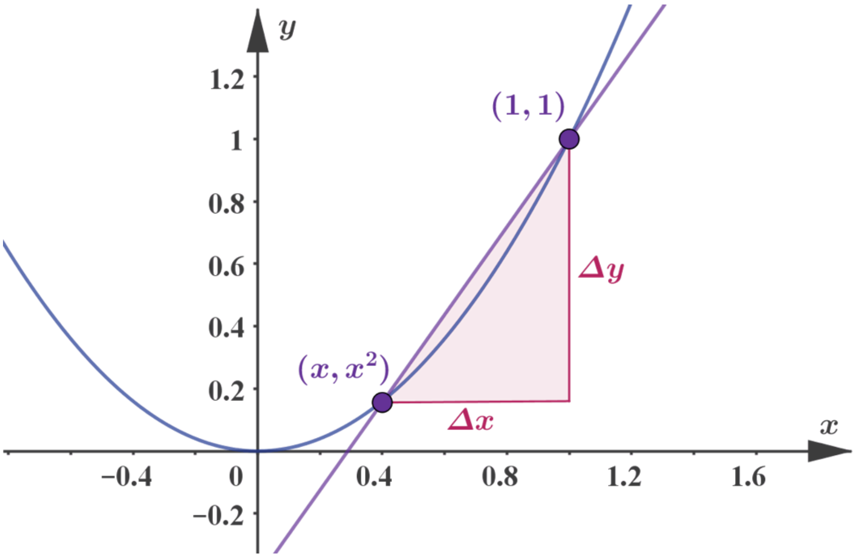
Example 1

★★★

Find the gradient of the graph of $y = x^2$ at $x = 1$.

The applet above shows that the secant line approaches the tangent as the two points move closer and closer to each other. This also means that the limit of the gradient of the secant line is the gradient of the tangent.

The diagram below shows part of the graph of $y = x^2$ and a secant line through the points (x, x^2) and $(1, 1)$ on the graph.



Student view

The gradient of this secant line is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{x^2 - 1}{x - 1}.$$

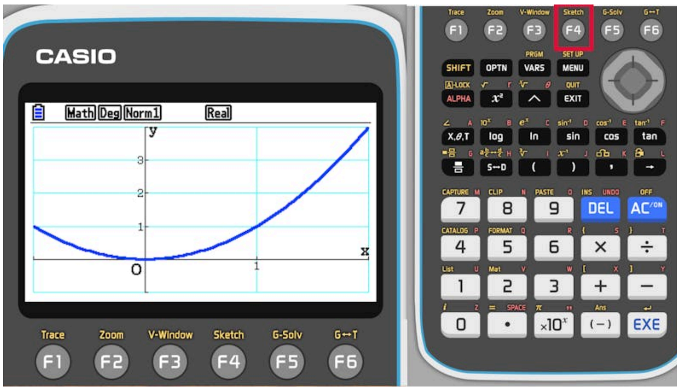
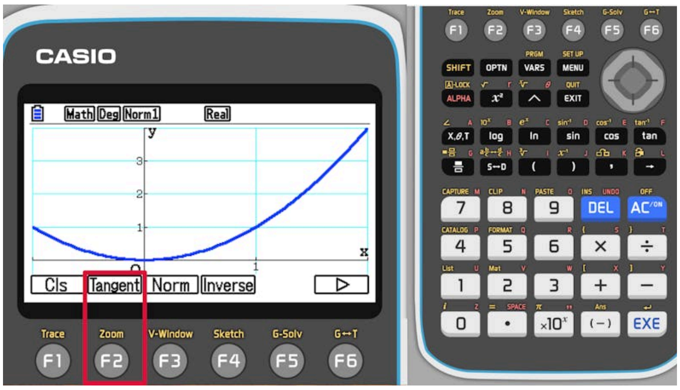
You need to find the limit of this expression as the point (x, x^2) gets closer and closer to the point $(1, 1)$.

- You can use a calculator to find this limit by, for example, investigating the values of the expression for $x = 0.9$, $x = 0.99$, $x = 0.999$ etc.
- You may also notice that $x^2 - 1 = (x + 1)(x - 1)$, so the fraction simplifies to
$$\frac{x^2 - 1}{x - 1} = \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = x + 1.$$

Using either method, you get that the limit is 2, so the slope of the tangent at $(1, 1)$, and hence the gradient of the graph at $x = 1$, is 2.

Even for this simple function, this was a lengthy process. In later sections, you will learn methods that will help you to find the exact value of the gradient without calculating limits. Graphing calculators have built in applications that can find approximate values of gradients of curves. Here are the instructions on how to access these applications on some common models.

Overview
(/study/app/m/sid-122-cid-754029/

Step	Explanation
These instructions assume that you have the graph of $y = x^2$ on the screen in the window $-1 < x < 2$ and $-1 < y < 4$. We will find the gradient at $x = 1$ to confirm the calculation of Example 1. Press F4 to bring up the options of "sketch".	
Press F2 to draw a tangent line.	

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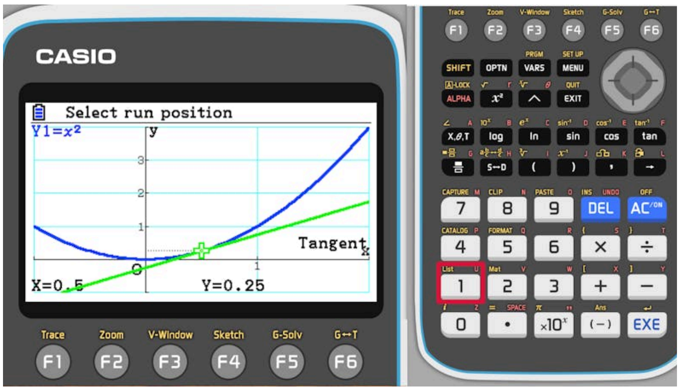
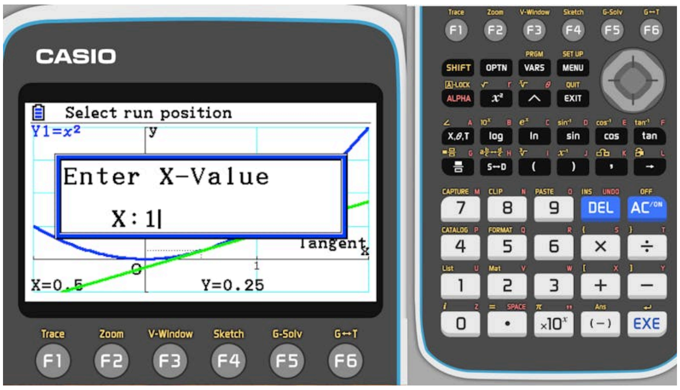
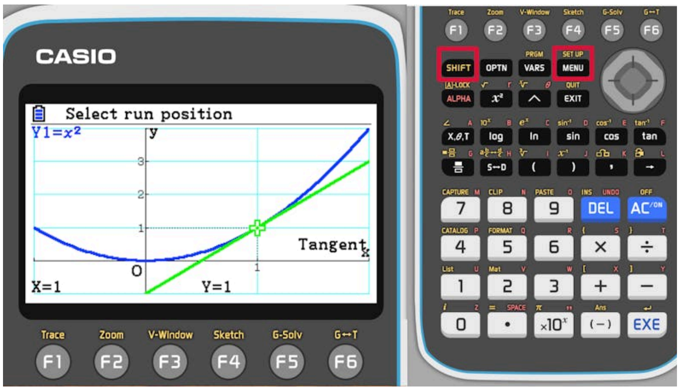
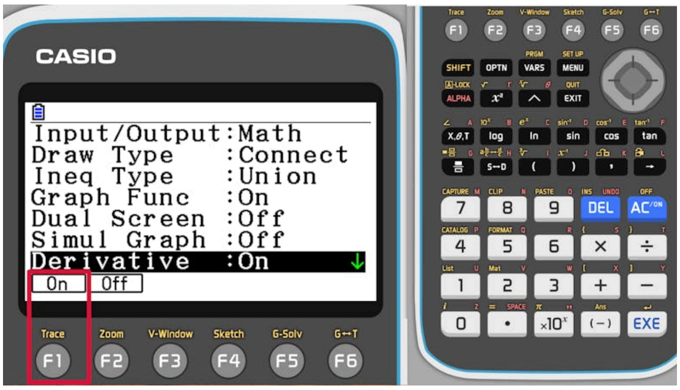
Overview
(/study/ap
122-
cid-
754029/

Step	Explanation
The tangent line will be drawn at the position of the cursor. You can change this position by entering the x-coordinate, so in this example press 1.	
Press EXE to confirm the value.	

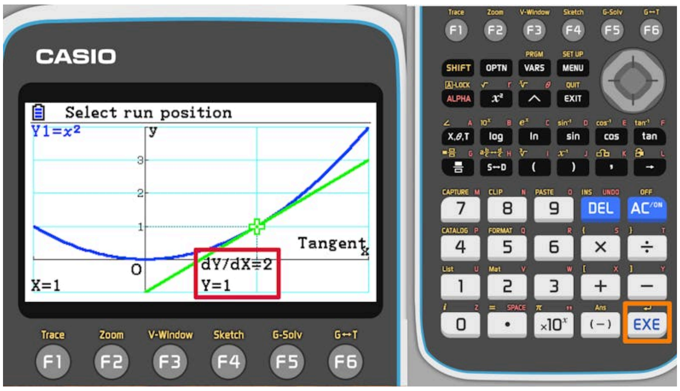
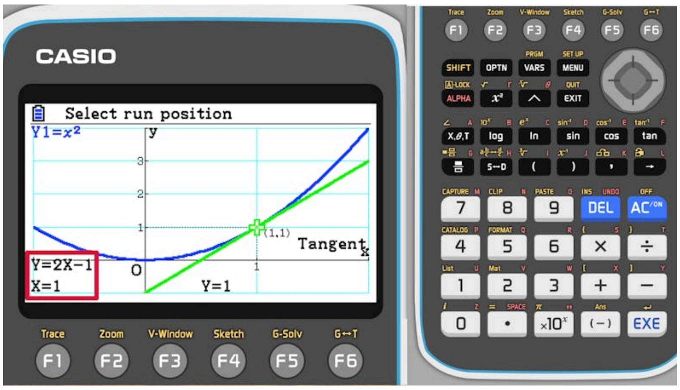
Student
view

Overview
(/study/ap
122-
cid-
754029/

Step	Explanation
The tangent is now at the correct position. You may alredy see the gradient, but if not, enter setup.	
Press F1 to turn on the derivative.	

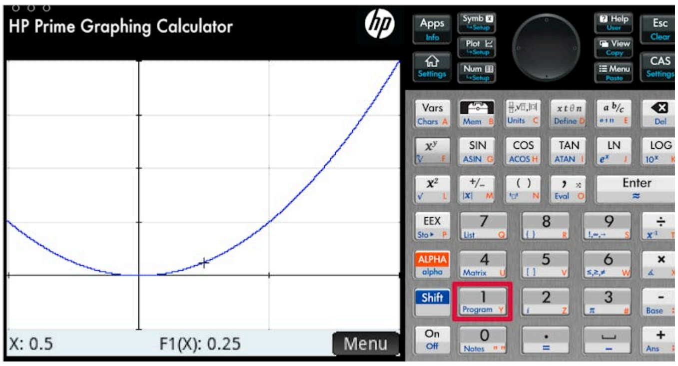
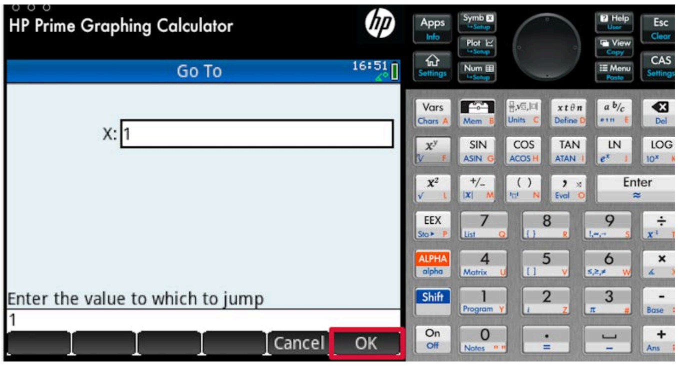
Student
view

Overview (/study/ap 122-cid-754029/

Step	Explanation
The gradient is now displayed. The notation used for the gradient will be explained later in this subtopic. Press EXE.	
In this view the equation of the tangent line is displayed.	

Student view

Overview
(/study/ap
122-
cid-
754029/

Step	Explanation
These instructions assume that you have the graph of $y = x^2$ on the screen in the window $-1 < x < 2$ and $-1 < y < 4$. We will find the gradient at $x = 1$ to confirm the calculation of Example 1. First, move the cursor to the position where you are interested in the gradient. Press a number (in this case 1) to enter the x-coordinate.	
Enter the x-coordinate of the point and press OK to confirm your choice.	

Student
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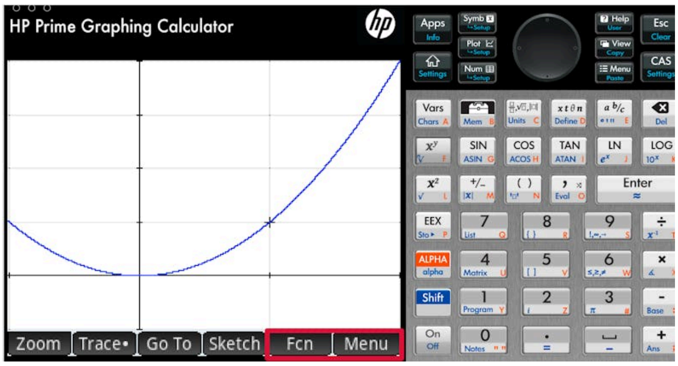
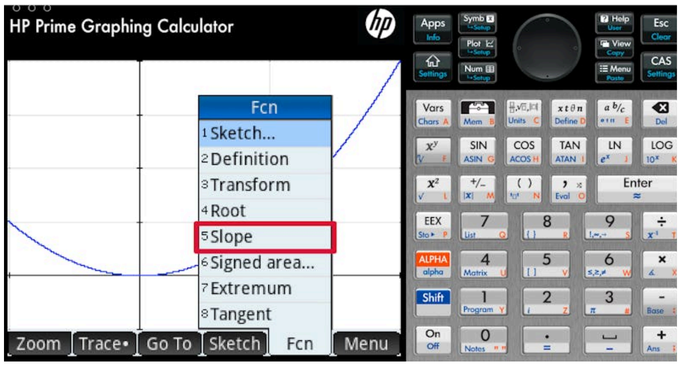
Step	Explanation
Now the cursor is at the correct position. To see the gradient, pull up the menu options related to functions.	
Choose the option to see the slope (this calculator uses this term instead of gradient).	

Overview
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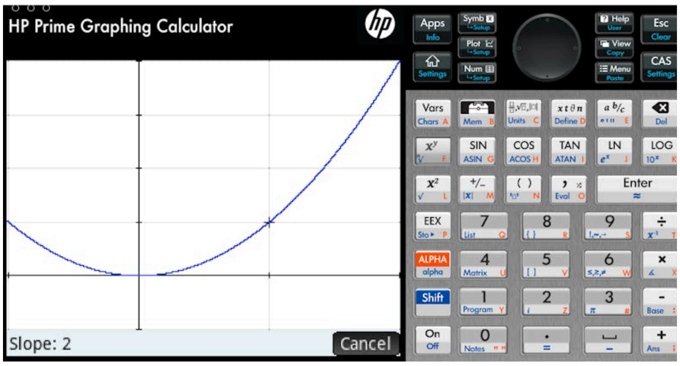
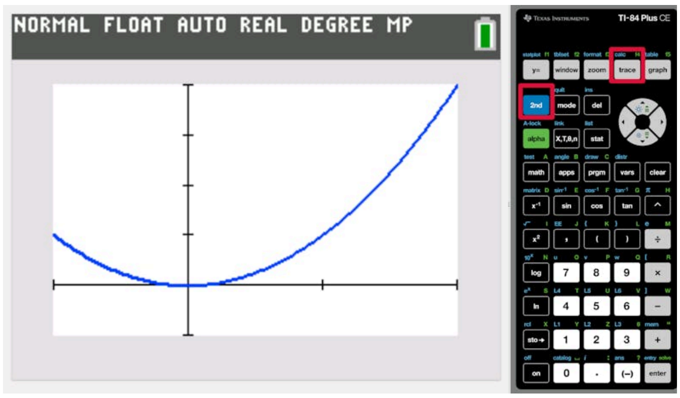
122-
cid-
754029/

Step	Explanation
The slope (gradient) is displayed on the screen.	

Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/concept-of-a-limit-id-26271/print/) Assign

Step	Explanation
These instructions assume that you have the graph of $y = x^2$ on the screen in the window $-1 < x < 2$ and $-1 < y < 4$. We will find the gradient at $x = 1$ to confirm the calculation of Example 1. Open the options to analyse the graph (calc).	

Student view

Overview
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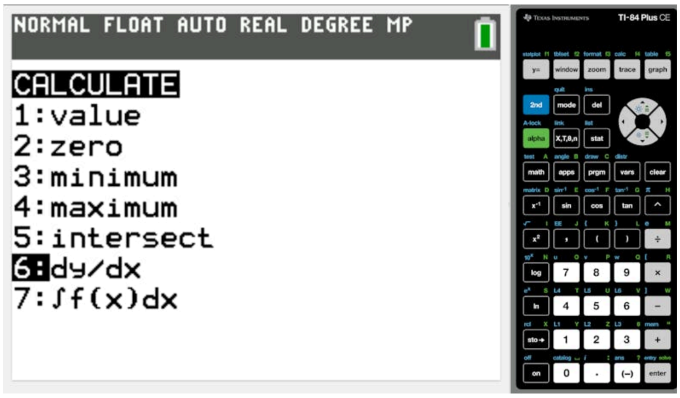
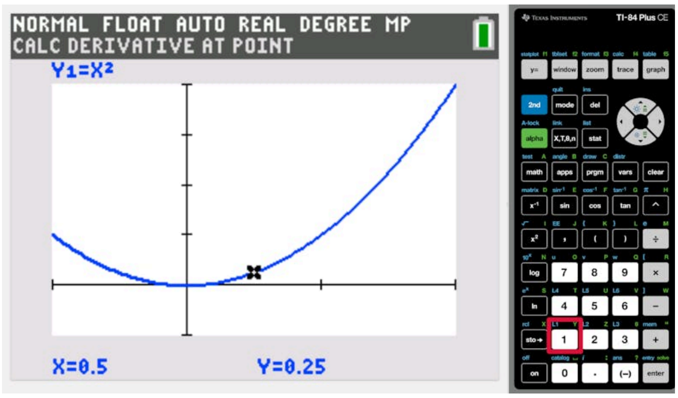
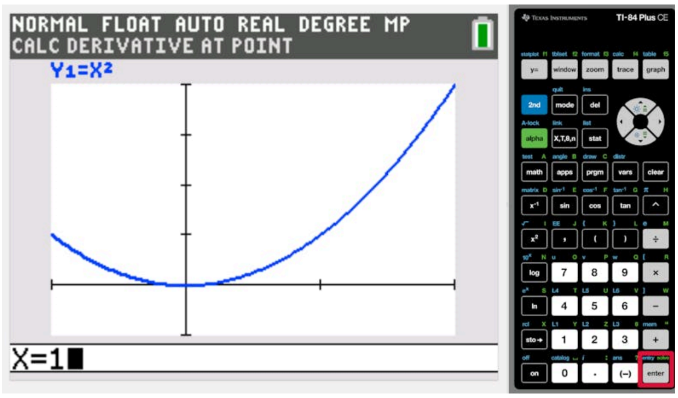
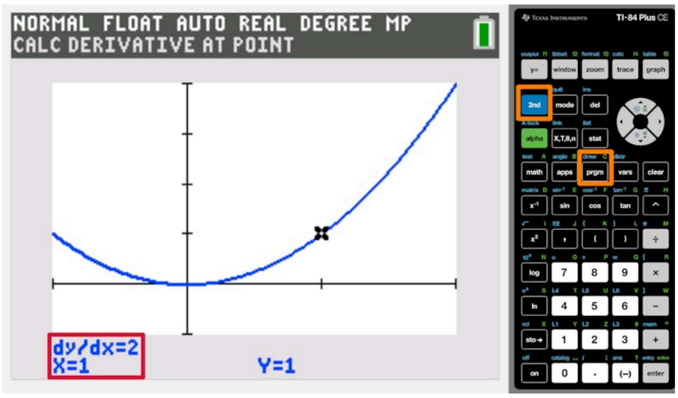
Step	Explanation
Choose the option to find the gradient. The notation used for the gradient will be explained later in this subtopic.	
The calculator is now waiting for the position, where you are interested in the gradient. It offers a position, but you can change it by entering the x-coordinate, so in this example press 1.	

Student
view

Step	Explanation
Enter the position and confirm the value by pressing enter.	
The gradient is now displayed. If you are also interested in the tangent line, enter the draw menu.	

Overview
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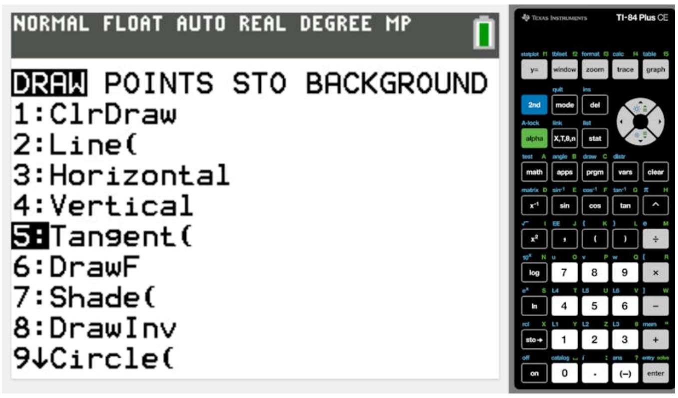
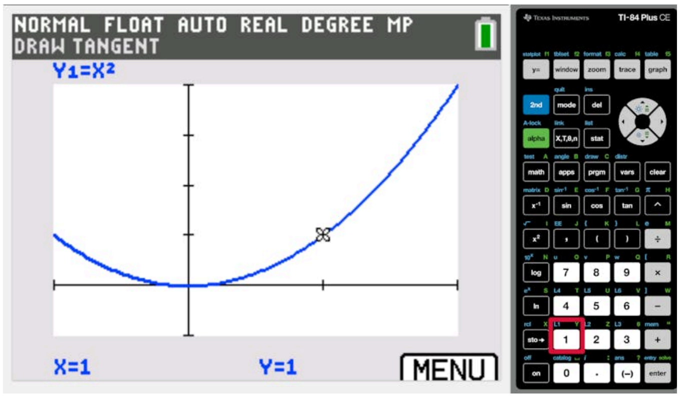
Step	Explanation
Choose the option to draw the tangent line.	
The calculator is now waiting for the position, where you are interested in the gradient. It offers a position, but you can change it by entering the x-coordinate. In this case the position is the same as the one where you found the gradient, so you can simply accept it, or press any number to change it.	

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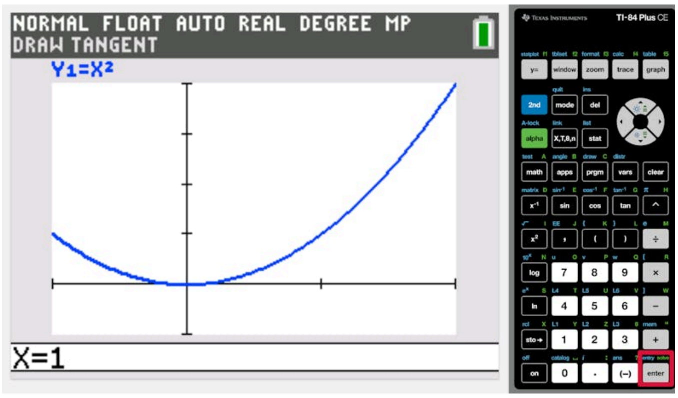
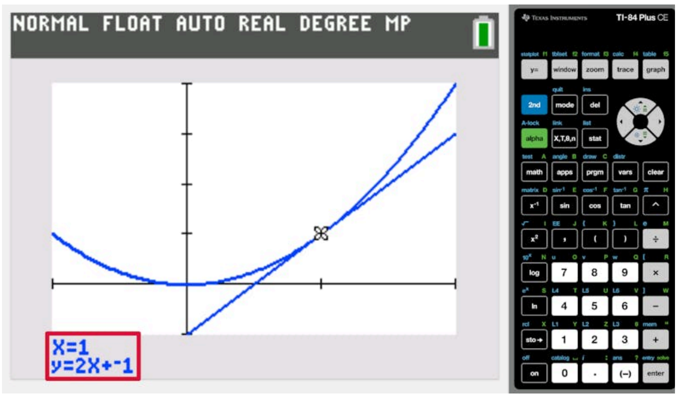
Step	Explanation
Enter the position and confirm the value by pressing enter.	
The tangent line is drawn and the equation is displayed.	

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Overview
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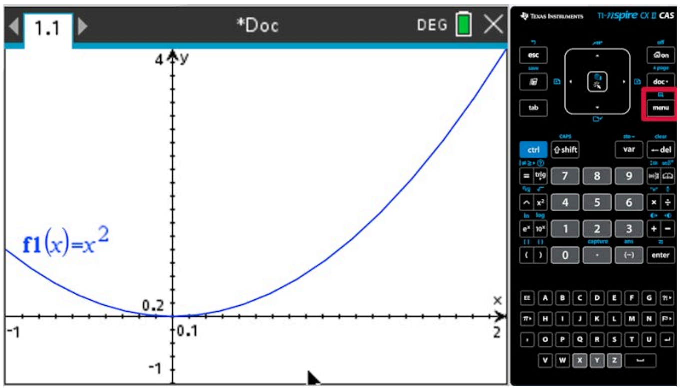
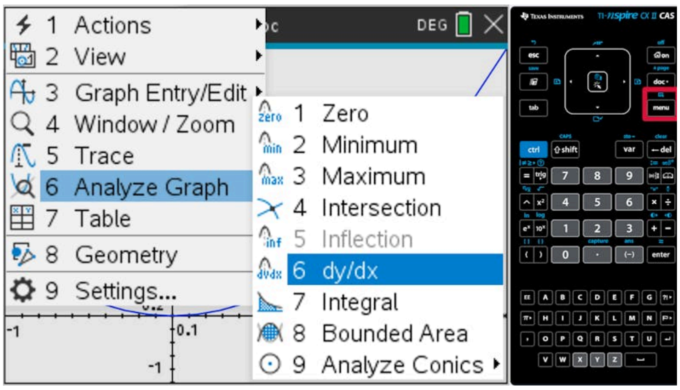
Step	Explanation
These instructions assume that you have the graph of $y = x^2$ on the screen in the window $-1 < x < 2$ and $-1 < y < 4$. We will find the gradient at $x = 1$ to confirm the calculation of Example 1. Start with opening the menu system.	
Choose the option to find the gradient. The notation used for the gradient will be explained later in this subtopic.	

Student
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Overview
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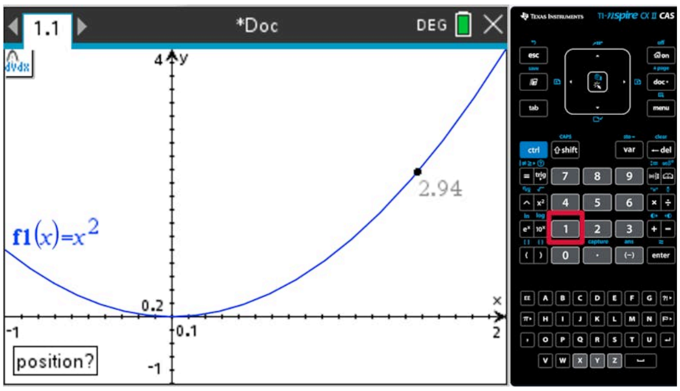
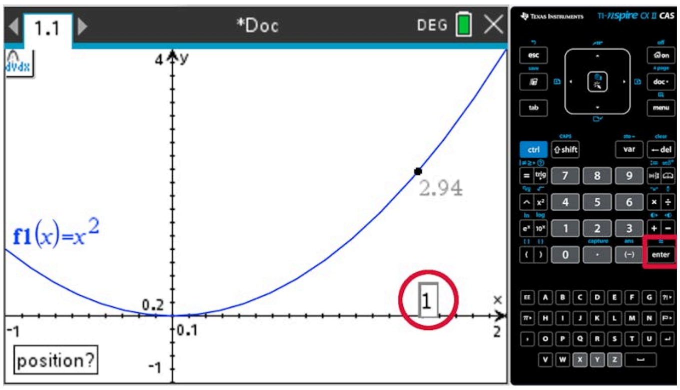
Step	Explanation
The calculator is now waiting for the position, where you are interested in the gradient. It offers a position, but you can change it by dragging the point on the graph or entering the x-coordinate. In this example press 1.	
Enter the position and confirm the value by pressing enter.	

Student view

Overview
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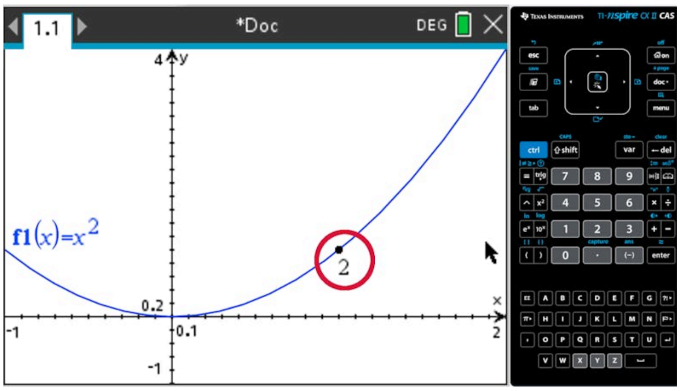
Step	Explanation
The gradient is now displayed.	

Example 2

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Use your calculator to find the gradient of the given graphs at the given value of x.

Curve	x-value	Gradient
$y = \mathrm{e}^x$	0	
$y = \dfrac{5 + x^2}{2x + 3}$	2	
$y = \sqrt{x + 3}$	-1	
$y = \dfrac{\sqrt{x}}{1 + \ln x}$	e	

Student view

Sketch the graphs and use your calculator to find the gradient at the given value. It is good practice to include the tangent in your sketch.

Overview
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122-
cid-
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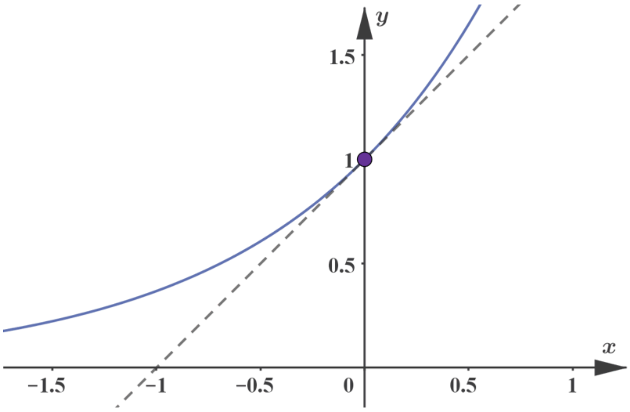
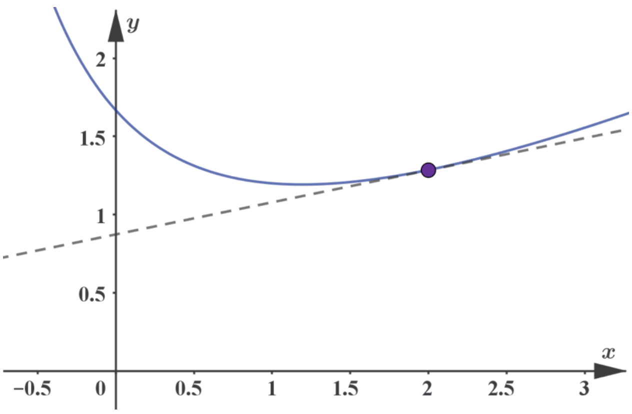
Curve	x-value	Gradient
$y = \mathrm{e}^x$ 	0	1.00
$y = \frac{5+x^2}{2x+3}$ 	2	0.204

Student view

Overview
(/study/ap
122-
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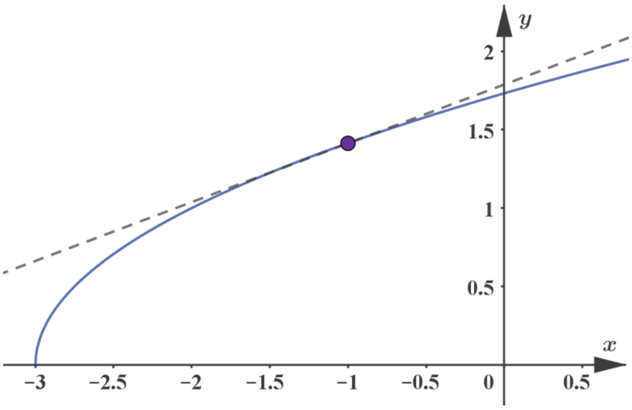
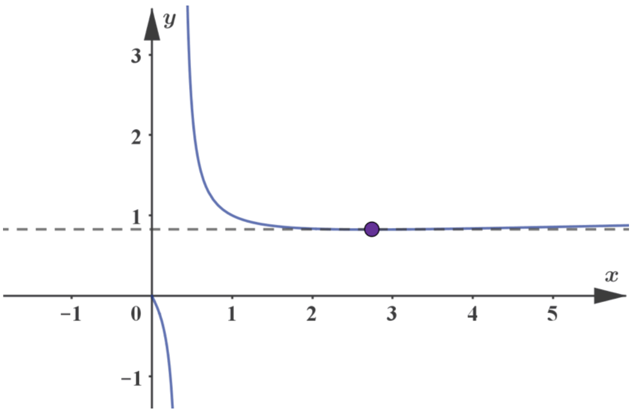
Curve	x-value	Gradient
$y = \sqrt{x+3}$ 	-1	0.354
$y = \dfrac{\sqrt{x}}{1 + \ln x}$ 	e	0.00

3 section questions

Student view

Gradient function

In the previous section, you saw what it means when you talk about the gradient of a curve. You also saw how to use the calculator to find the gradient of a graph of a function.

Activity

Move the red point on the applet below. The applet shows you the tangent to the graph and the gradient of this tangent. It also shows the trace of a second point.

- Describe the relationship between this second point and the point you are moving around.
- Describe the relationship between the trace of this second point and the original curve.

Interactive 1. Gradient Function.

More information for interactive 1

Overview
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122-
cid-
754029/

This interactive provides an interactive way to explore the relationship between a curve, its tangent, and the gradient of the tangent. Users can adjust the shape of the curve by ticking “Adjust curve” and moving the red points. Once the curve is set, by ticking” Move Point,” they can move a point along the curve to observe how the gradient of the tangent changes at different points.

As the user moves the point along the curve, the interactive displays the tangent line at that point and its gradient. Additionally, a second point is shown, and its trace is displayed as the primary point moves. The relationship between the second point and the moving point is that the second point's position is determined by the gradient of the tangent at the primary point. Essentially, the second point's trace represents the gradient function of the original curve.

The trace of the second point forms a new graph that illustrates how the gradient of the tangent changes along the original curve.

This new graph provides a visual representation of the derivative of the original curve, showing how the slope of the tangent varies at different points, with the slope value displayed numerically.

Your observations in this Activity form an introduction to the concept of the gradient function.

Important

The function g, for which $g(a)$ is the gradient of the graph of $y = f(x)$ at the point $(a, f(a))$ is called the derivative (or gradient function) of the function f.

Common forms of notation for the derivative are f', $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}}{\mathrm{d}x}f$.

International Mindedness

The forms of notation mentioned above were introduced by Joseph-Louis Lagrange and Gottfried Leibniz. Other, less common, notations were introduced by Leonhard Euler and Isaac Newton. You could do some research to find out more about the contributions of these famous mathematicians to the development of mathematics and, in particular, to calculus.

Student
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Example 1

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Overview
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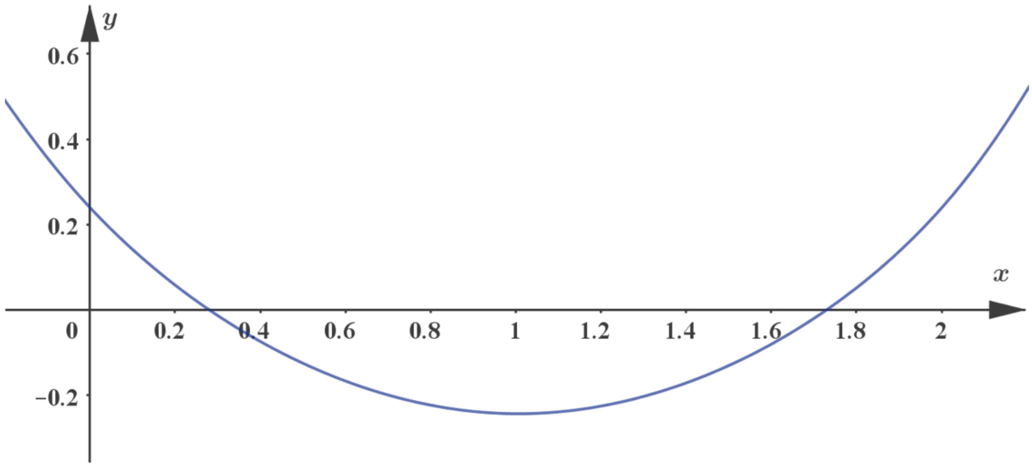
The diagram below shows the graph of a function, f.



More information

The image depicts the graph of a mathematical function, specifically a parabola. The X-axis, labeled 'x', represents the independent variable and ranges from -0.4 to 2.2. The Y-axis, labeled 'y', represents the dependent variable and ranges from -0.2 to 0.6. The curve starts at the top right, descends to a minimum point, and ascends again, illustrating a parabolic shape that opens upwards. This shape suggests the likely presence of a quadratic equation. The graph includes grid lines for more precise reading of data points along the axes.

[Generated by AI]

For some integers a, b and c, $f'(a) = 0$, $f'(b) = 1$ and $f'(c) = -1$.

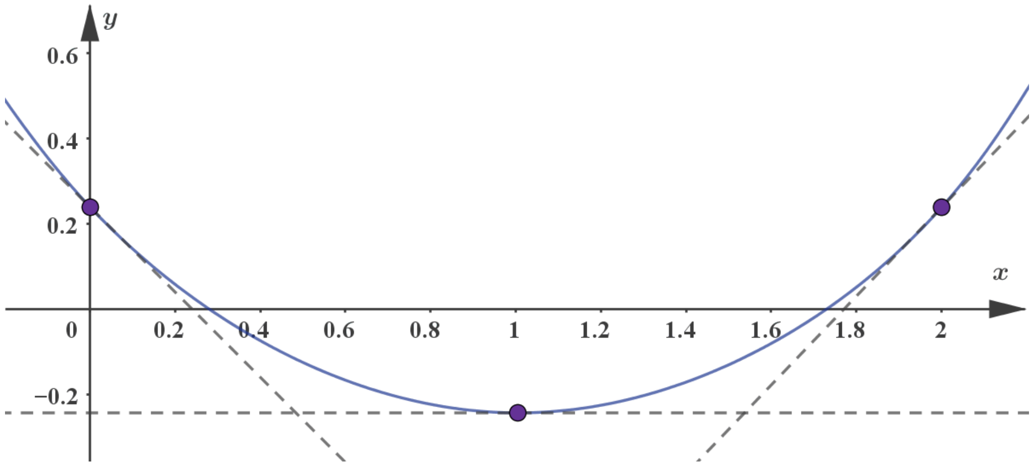
Find the value of a, of b and of c.

There are three integers in the domain of the graph you see on the diagram: 0, 1 and 2.

The diagram below shows the corresponding points on the graph and the tangents to the graph at these points.

Student
view

Overview (/study/ap 122- cid- 754029/



- For $x = 0$, the corresponding tangent has a negative gradient.

Hence, $f'(0) < 0$, which means that $c = 0$.

- For $x = 1$, the corresponding tangent is horizontal.

Hence, $f'(1) = 0$, which means that $a = 1$.

- For $x = 2$, the corresponding tangent has a positive gradient.

Hence, $f'(2) > 0$, which means that $b = 2$.

Example 2

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The diagram below shows the graph of f', the derivative of f.

Student view

Overview
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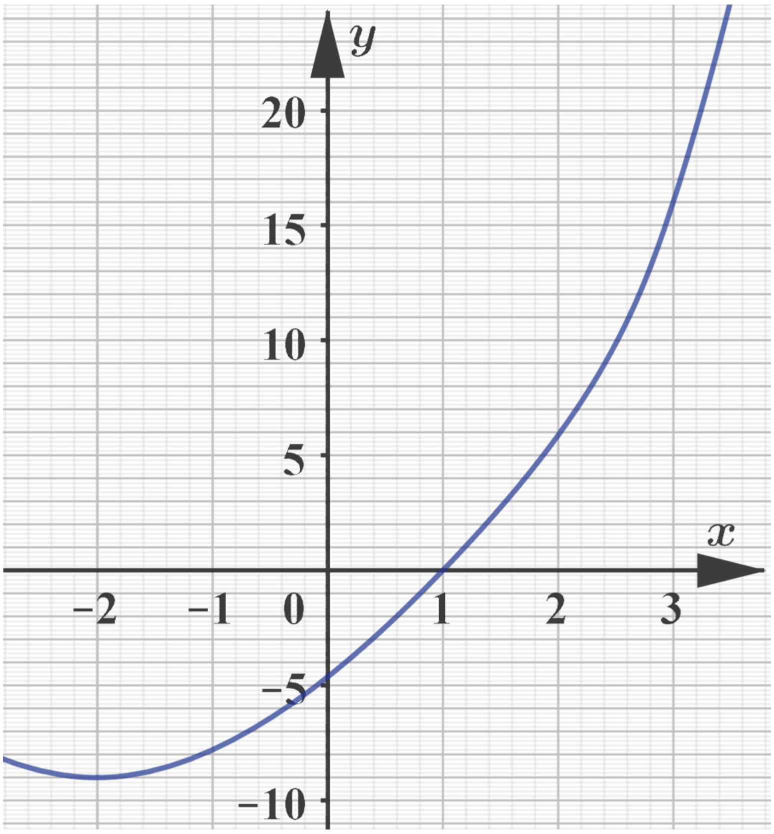

More information

The image is a graph illustrating the derivative of a function f'. The x-axis represents the variable x, with marks at intervals of 1 ranging from -3 to 3. The y-axis denotes the derivative value, marked at intervals of 5, ranging from -10 to 20. The curve begins below the x-axis, indicating negative derivative values, then crosses the x-axis around x = 0, marking a zero point, and steeply rises toward y = 20 as x increases beyond 2. This suggests that the function f' starts with a negative slope, transitions through zero, and increases significantly for positive x values, indicating increasing values of the original function f as x progresses.

[Generated by AI]

Find the gradient of the tangent to the graph of f at the points where $x = -2$, $x = 1$ and $x = 3$.

The diagram below shows the points on the graph of f' corresponding to the given x-values.

Student
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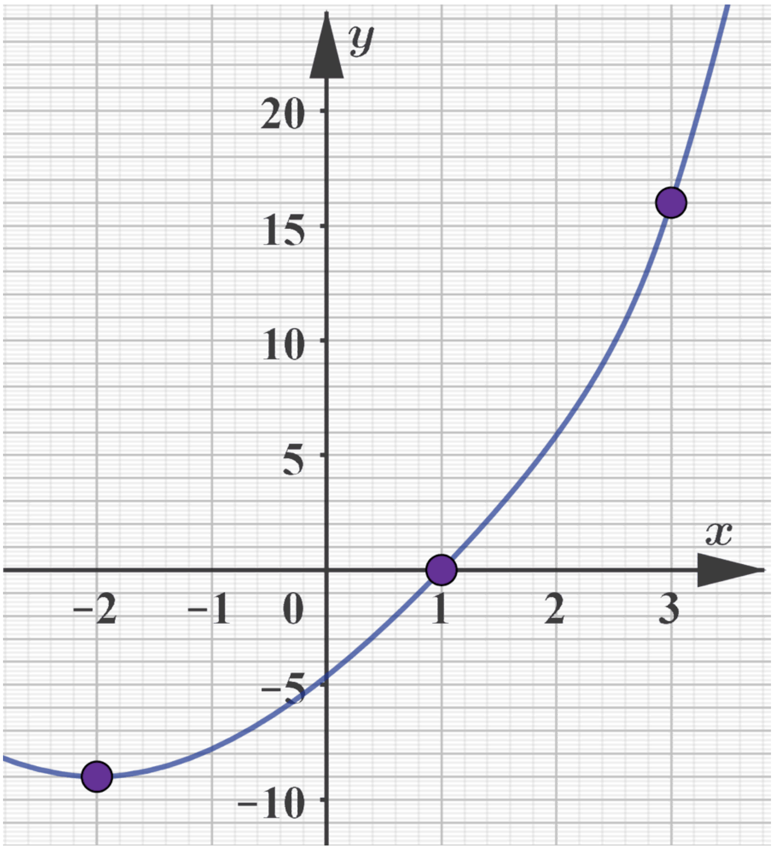

- According to the diagram, $f'(-2) = -9$. This means that the gradient of the tangent (which is the same as the gradient of f) at the point where $x = -2$ is -9.
- Similarly, since $f'(1) = 0$, the gradient of the tangent to the graph of f at the point where $x = 1$ is 0. The graph of f has a horizontal tangent at $x = 1$.
- Similarly, since $f'(3) = 16$, the gradient of the tangent to the graph of f at the point where $x = 3$ is 16.

Example 3

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The derivative of the function f is given by $f'(x) = 2x - 6$.

- Find the gradient of the tangent to the graph of f at the point where $x = 5$.
- Find the value of x at the point where the tangent to the graph of f is horizontal.

Overview
(/study/app/m/sid-122-cid-754029/

- The gradient of the tangent to the graph of f at the point where $x = 5$ is
$$f'(5) = 2 \times 5 - 6 = 4.$$
- The graph of f has a horizontal tangent when $f'(x) = 0$, that is, when
$$\begin{aligned} 2x - 6 &= 0 \\ 2x &= 6 \\ x &= 3. \end{aligned}$$

Making connections

You should now be familiar with the concept of the derivative function. In the next section, you will see how to find the derivative function and understand why it is a useful tool.

3 section questions

5. Calculus / 5.1 Introduction to differentiation

Rate of change

In the example in section 5.1.0 (/study/app/m/sid-122-cid-754029/book/the-big-picture-id-26270/), the last question was left unanswered. The question is repeated here. This time you will be able to give an answer using the concept of differentiation.

This is the scenario you were given:

Ali walks from his home to the grocery store and then he walks back home.

This diagram shows information about his journey.

Student view

Overview
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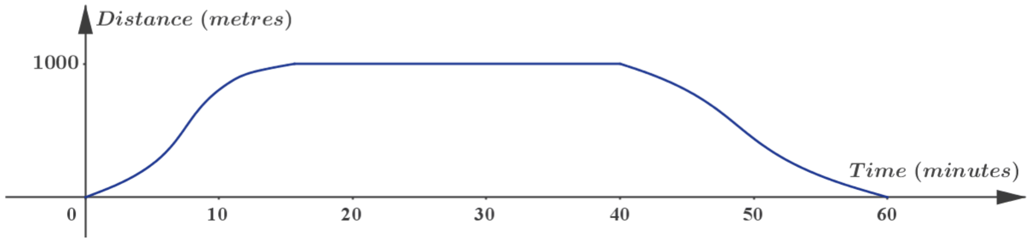

More information

The graph illustrates the journey of a person, depicting the distance in meters from their home over time in minutes. The X-axis represents time, labeled "Time (minutes)", ranging from 0 to 60 minutes. The Y-axis represents distance, labeled "Distance (metres)", with a range from 0 to 1000 meters.

Key observations: 1. From 0 to 10 minutes, the curve rises steeply, indicating an increase in distance up to 1000 meters. 2. Between 10 and 40 minutes, the curve is flat, suggesting that the distance remains constant at 1000 meters. 3. From 40 to 60 minutes, the curve descends back to 0 meters, indicating a return to the starting point.

This graph shows a journey that starts, maintains a constant distance, and then returns over a time span of 60 minutes.

[Generated by AI]

His distance, s (measured in metres) from home t minutes after he started his trip is given by

$$s(t) = \begin{cases} -\frac{16}{27}t^3 + \frac{40}{3}t^2 & \text{for } 0 \leqslant t < 15 \\ 1000 & \text{for } 15 \leqslant t < 40 \\ \frac{1}{4}t^3 - \frac{75}{2}t^2 + 1800t - 27000 & \text{for } 40 \leqslant t \leqslant 60 \end{cases}$$

Find Ali's maximum speed during this shopping trip.

You can now answer the question using differentiation.

The average speed on a part of the journey is the distance covered, divided by the time taken. This can be represented by the gradient of the line connecting the points on the graph corresponding to the starting and finishing time of that section. If you consider shorter and shorter time intervals, then these average speed values get closer and closer to a specific

Student
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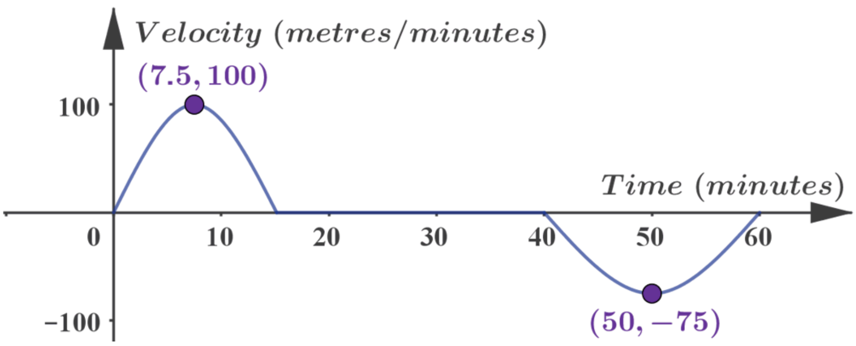
Overview (/study/app/m/sid-122-cid-754029/

value, which you call the (instantaneous) speed of Ali. This is similar to how you found the gradient of a curve as a limit of slopes of secant lines in section 5.1.2 (/study/app/m/sid-122-cid-754029/book/gradient-at-a-point-id-26272/).

Hence, Ali's speed, at any given time, is the value of the derivative of the position function:

$$v(t) = s'(t)$$

In section 5.1.1 (/study/app/m/sid-122-cid-754029/book/concept-of-a-limit-id-26271/), you saw how to use a calculator to find the value of the derivative at a particular value of t . Graphing calculators have applications that can put these derivative values together and draw the derivative function. The diagram below shows the derivative graph for Ali's journey.



More information

The graph displays velocity over time. The X-axis represents time in minutes, extending from 0 to 60, and the Y-axis represents velocity in metres per minute, with a range from -100 to 100. The graph shows a curve that peaks at 7.5 minutes with a velocity of 100 metres/minute and dips to a low point at 50 minutes with a velocity of -75 metres/minute. There's also a note that for time values between 40 and 60 minutes, the derivative is below the horizontal axis, indicating a change in direction. Two key points are highlighted: (7.5, 100) and (50, -75).

[Generated by AI]

Note that for time values between 40 and 60 minutes the derivative graph is below the horizontal axis. This indicates the direction of Ali's movement from the store on the way back home.

Student view

Overview
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122-
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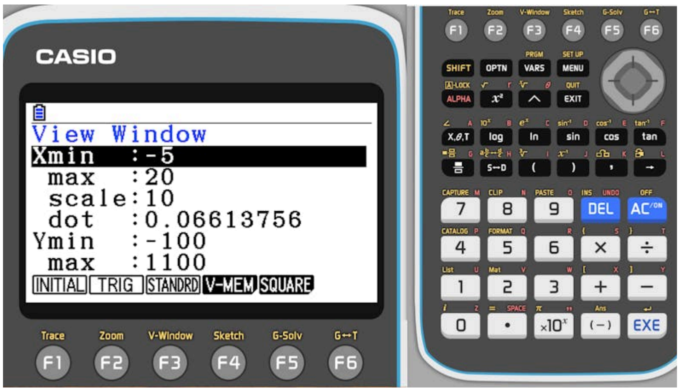
Graphing calculators also have applications to find maximum and minimum points on graphs. The diagram also shows these points. From the graph, you can read that, on the way from his home to the store, Ali’s speed increases in the first 7.5 minutes, reaches a maximum speed of 100 metres per minute, then decreases to 0 when he reaches the store. On the way back home, Ali’s maximum speed is 75 metres per minute.

Hence, during this trip, Ali’s maximum speed is 100 metres per minute.

Below you can find help on how to draw the derivative graph on different calculators.

Step	Explanation
This instruction shows you two features. It explains how to graph the derivative without algebraically finding it. It also illustrates how to graph a function on a restricted domain. On the last slide you will see the derivative graph of the first part of Ali's journey from the example above. This screen shows you the viewing window settings used.	

Student
view

Overview
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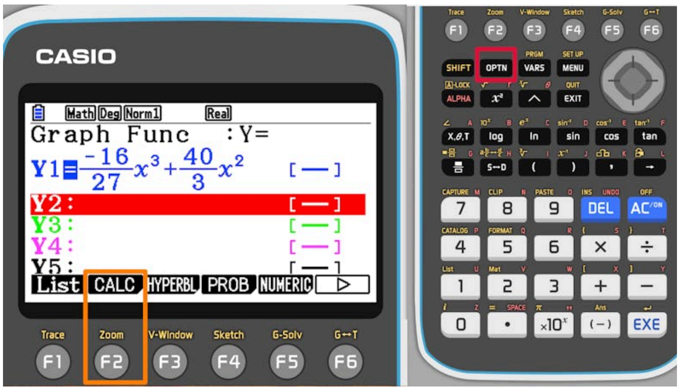
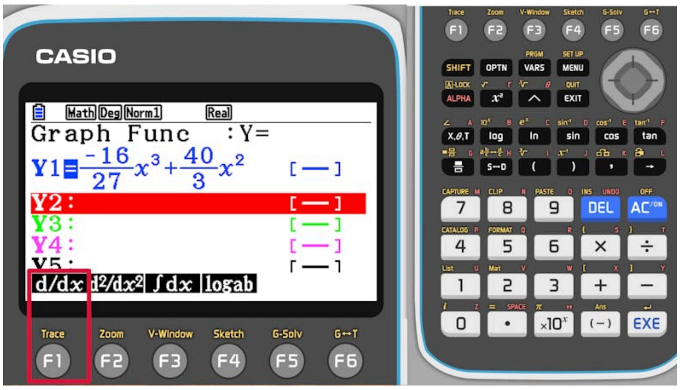
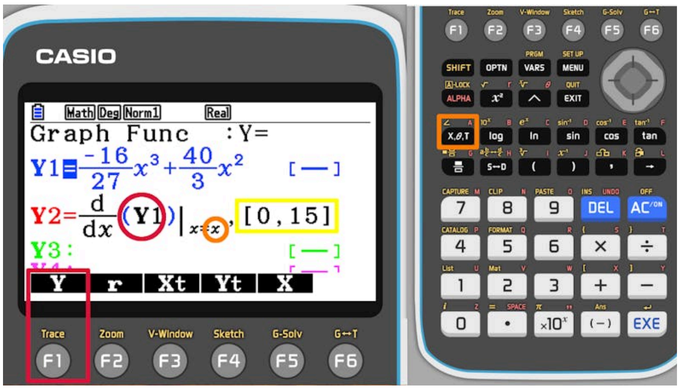
Step	Explanation
Enter the function in Y1 and move to Y2 to start entering the expression for the derivative. Press option and the press F2 to bring up the calculus options.	
Choose the derivative option.	

Student
view

Step	Explanation
An alternative way of accessing the derivative operation is through the catalog of all available operations.	
It is important to understand all parts of the definition of Y2. • Instead of typing in the expression of the function to differentiate, you can use the name (red markers). • Since you want to define a function, you need to use the variable, where the derivative is evaluated (orange markers). • You can use interval notation to indicate, that you only want to see the function for $0 \leqslant x \leqslant 15$ (yellow marker). One done, press EXE.	

Overview
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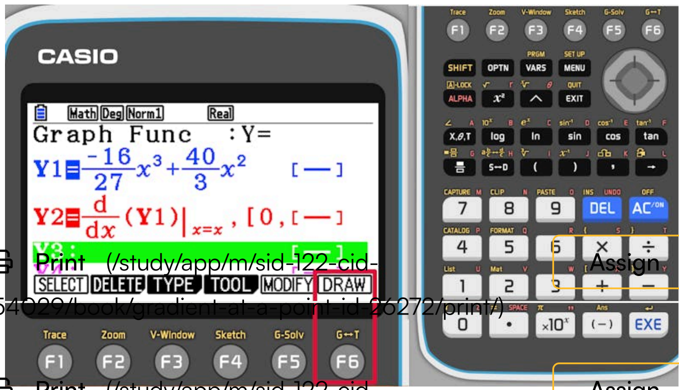
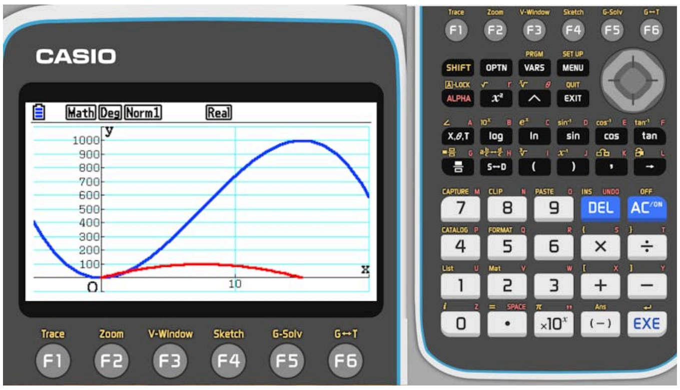
Step	Explanation
Once both the function (Y1) and its derivative on a restricted domain (Y2) is defined, press F6 to see the graphs.	
Notice, that the derivative is only displayed for $0 \leqslant x \leqslant 15$.	

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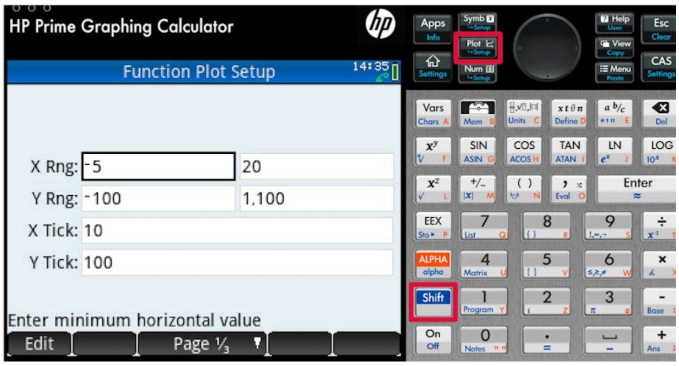
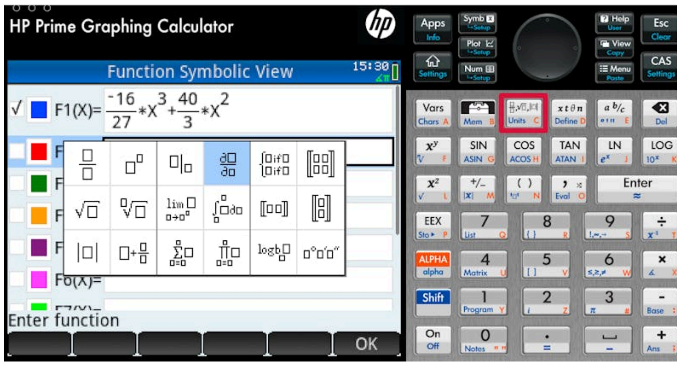
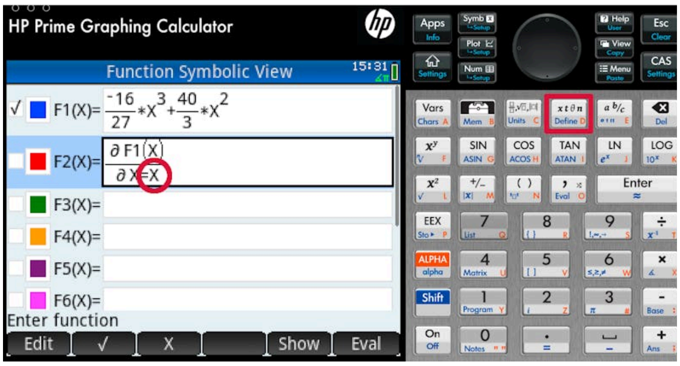
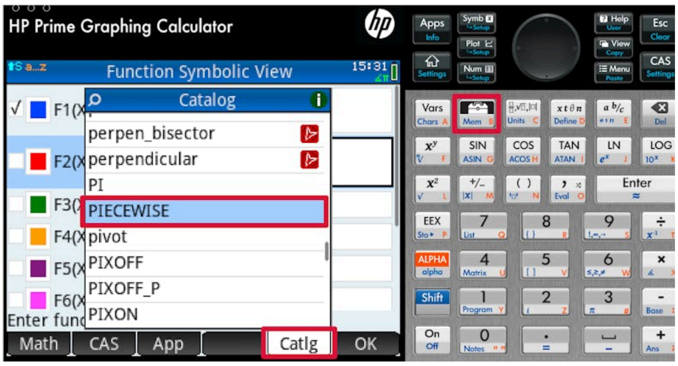
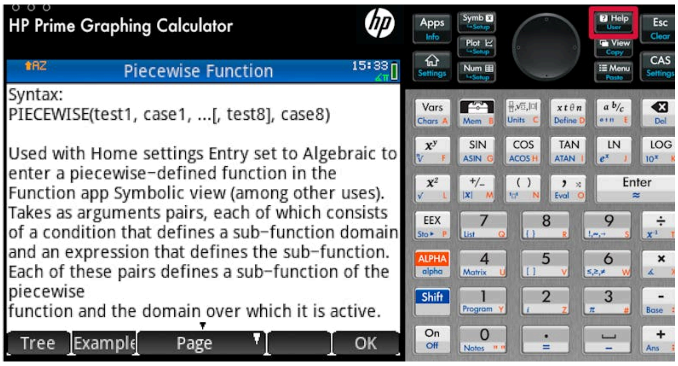
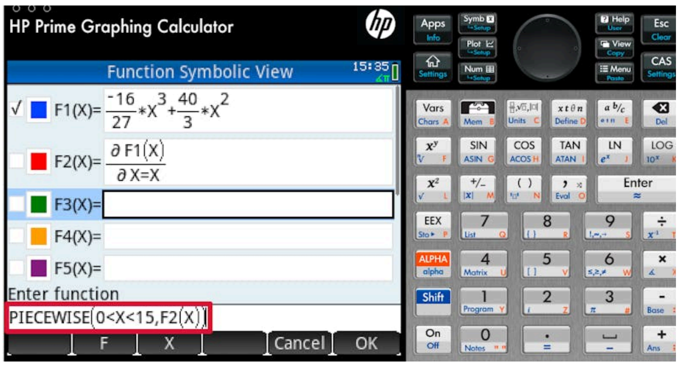
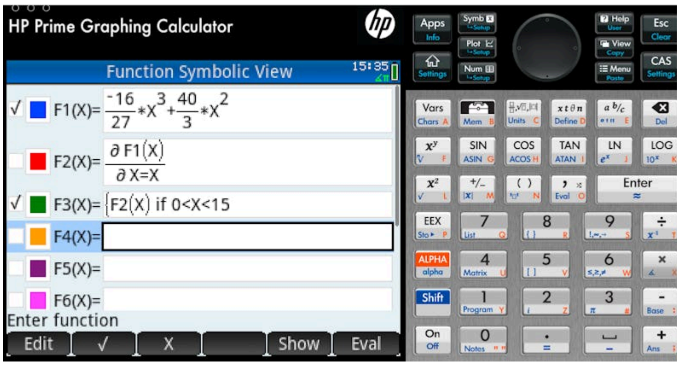
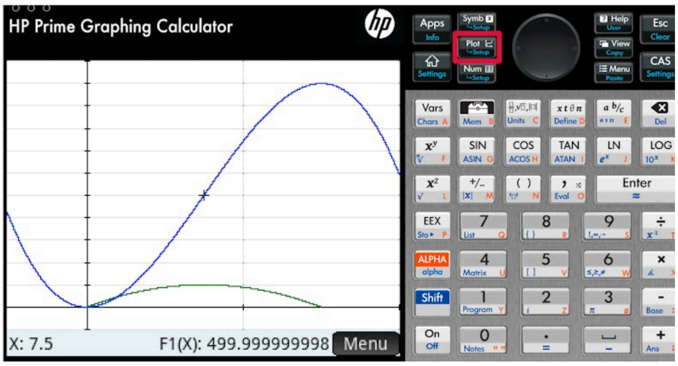
Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/gradient-function-id-26273/print/) Assign

Student
view

Step	Explanation
This instruction shows you two features. It explains how to graph the derivative without algebraically finding it. It also illustrates how to graph a function on a restricted domain. On the last slide you will see the derivative graph of the first part of Ali's journey from the example above. This screen shows you the viewing window settings used.	
Enter the function and move to the next line. To enter the derivative, open the template menu and choose the template for a derivative.	

Step	Explanation
Note the format of the derivative. You need to have an equality in the denominator to indicate the place where the derivative is evaluated at. Since you want to define a function, you need to use the variable instead of a number here. Do not mark this function with a tick. The goal is to display only the part for $0 \leqslant x \leqslant 15$.	
You can find the tool (PIECEWISE) to restrict the domain of a function in the catalog part of the toolbox.	

Step	Explanation
As with all tools, you can take a look at the help screen. With this tool you can define functions piece by piece (you can find an example on the second page of the help screen, which is not shown here). If you choose only one piece, it will simply restrict the domain of the function.	
Take a look at carefully the definition of F3. The values are the same as the values of F2, but F3 is only defined for x-values between 0 and 15.	

Step	Explanation
There are now the three functions defined. F1 is the original, F2 is the derivative (with no tickmark, so it will not be drawn), and F3 is the derivative with the domain restricted to $0 < x < 15$.	
You can see the graphs in the plot view. Notice, that the derivative is only displayed for $0 < x < 15$.	

Overview
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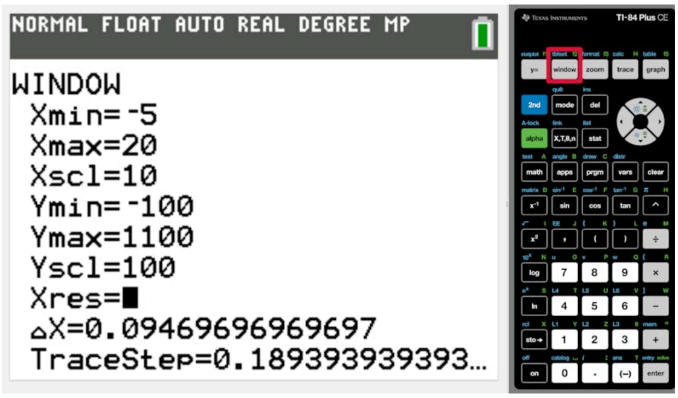
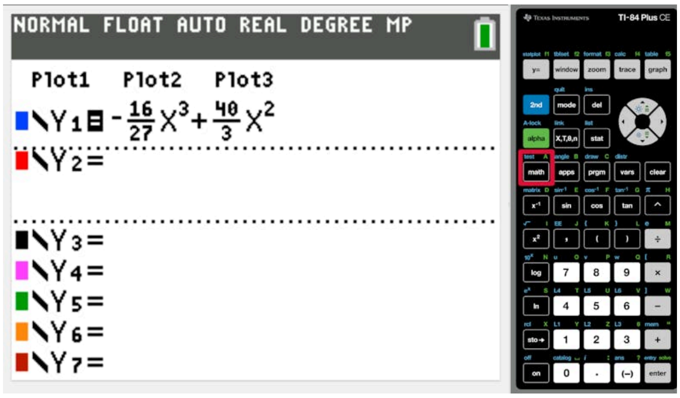
Step	Explanation
This instruction shows you two features. It explains how to graph the derivative without algebraically finding it. It also illustrates how to graph a function on a restricted domain. On the last slide you will see the derivative graph of the first part of Ali's journey from the example above. This screen shows you the viewing window settings used.	
To enter the derivative of Y1 in Y2, open the math menu ...	

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Overview
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122-
cid-
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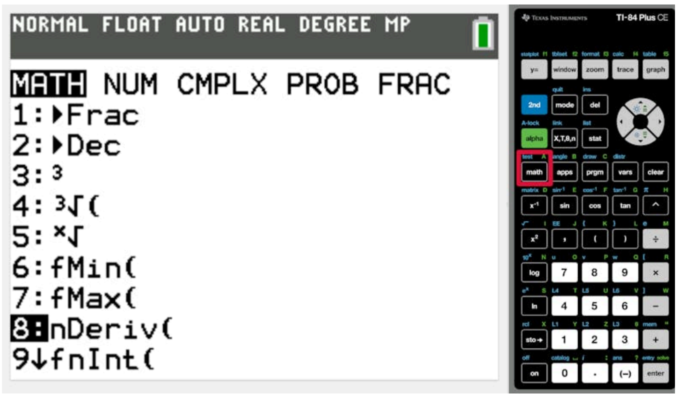
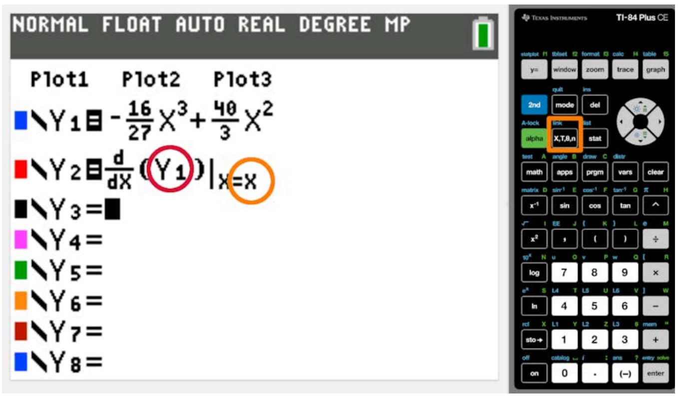
<table>
<tr><th>Step</th><th>Explanation</th></tr>
<tr><td>... and choose the numerical derivative (nDeriv) option.</td><td>
</td></tr>
<tr><td>It is important to understand all parts of the definition of Y2.

Instead of typing in the expression of the function to differentiate, you can use the name (red marker, see a bit further down in these instructions on how to access this name).
Since you want to define a function, you need to use the variable, where the derivative is evaluated (orange markers).
</td><td>
</td></tr>
</table>

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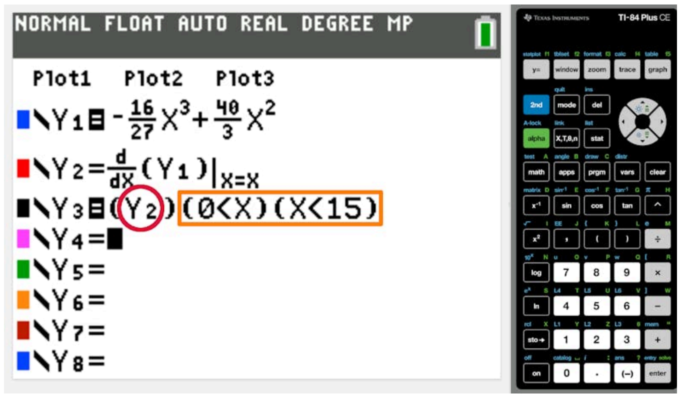
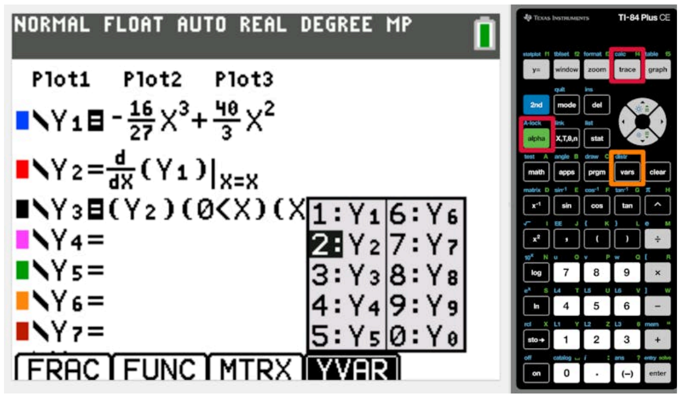
Overview (/study/app/m/sid-122-cid-754029/...)

Step	Explanation
To display the derivative only for $0 < x < 15$, unhighlight the equality sign at Y2 (by moving over it and pressing enter). This means, that Y2 will not be shown on the graph. Define a new function, Y3, which is the same as Y2 (red marker) but the domain is restricted (orange marker). See the next two images for guidance on how to access the function name Y1 and the inequality symbol.	
The names of the functions can be accessed through alpha/f4 (shown on the screen) or alternatively through the variable button (not shown).	

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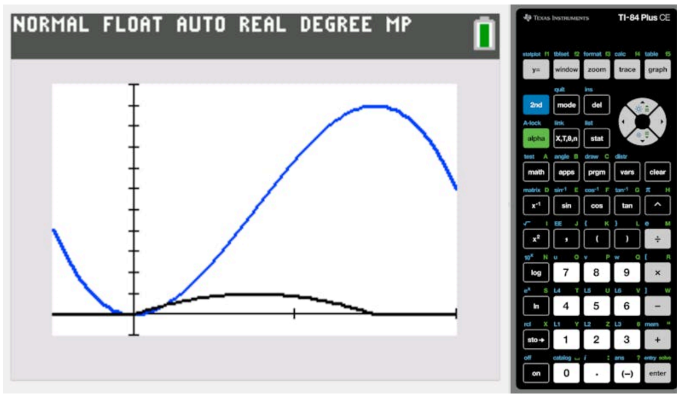
Step	Explanation
The inequality symbols can be accessed using the test menu. Once you entered all three functions (and made sure Y2 is not displayed), press graph to see the graphs.	
You can see two graphs. Notice, that the derivative is only displayed for $0 < x < 15$. For some reason, the calculator displays that the second function is 0 outside the specified domain. This is of course not true (it is not defined outside the domain), but you should be aware, that the calculator displays the value 0 instead of undefined. This may change in future versions of the operating system.	

Overview
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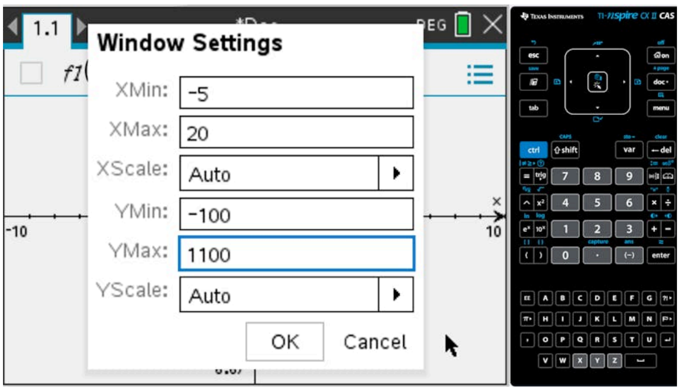
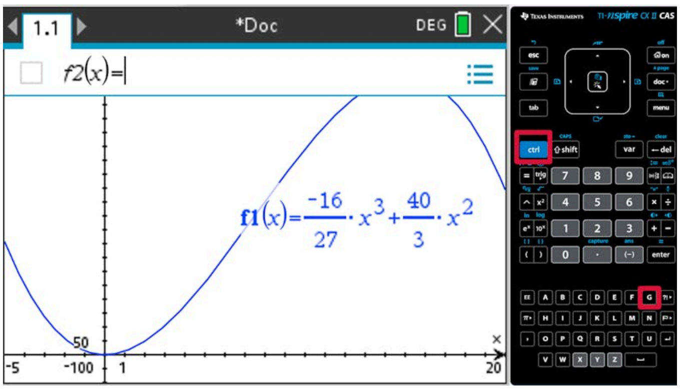
122-
cid-
754029/

Step	Explanation
This instruction shows you two features. It explains how to graph the derivative without algebraically finding it. It also illustrates how to graph a function on a restricted domain. On the last slide you will see the derivative graph of the first part of Ali's journey from the example above. This screen shows you the viewing window settings used.	
Once the main function is defined, press ctrl/G to start defining a new function (the derivative).	

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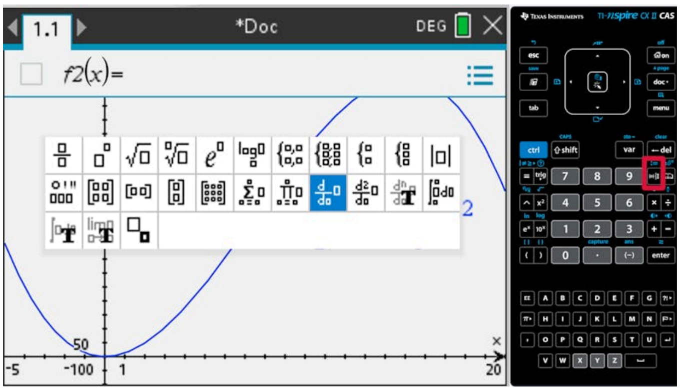
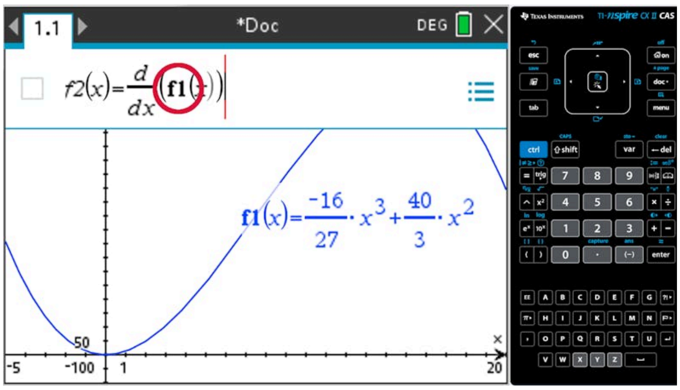
Overview (/study/ap 122- cid- 754029/

Step	Explanation
Open the collection of templates and choose the one corresponding to the derivative.	
Instead of typing in the expression again, you can use the name of the main function. Don't press enter just yet, this is the place you can tell the calculator about domain restriction.	

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Overview
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cid-
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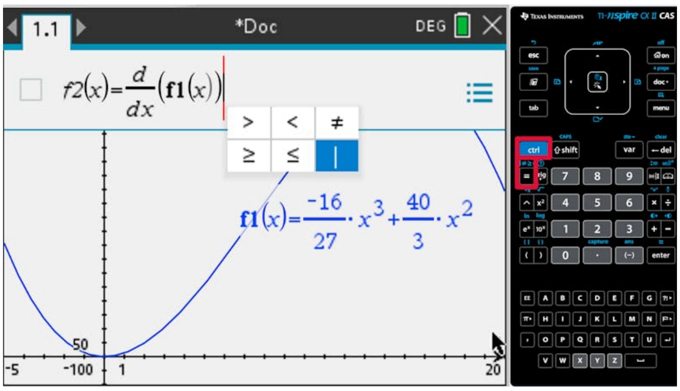
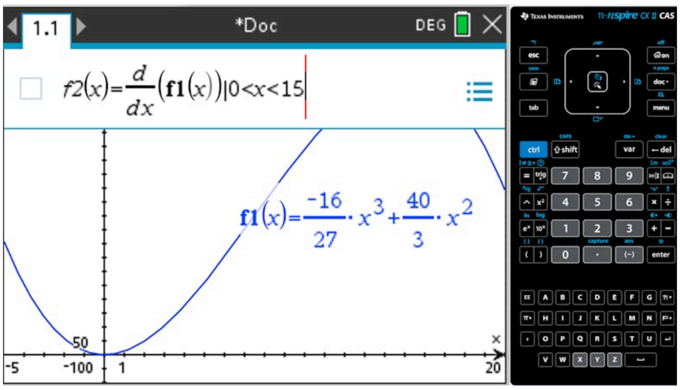
Step	Explanation
Use the vertical line to separate the definition from the condition and use the inequality symbols to set the condition.	
The second function is the derivative of the first for $0 < x < 15$.	

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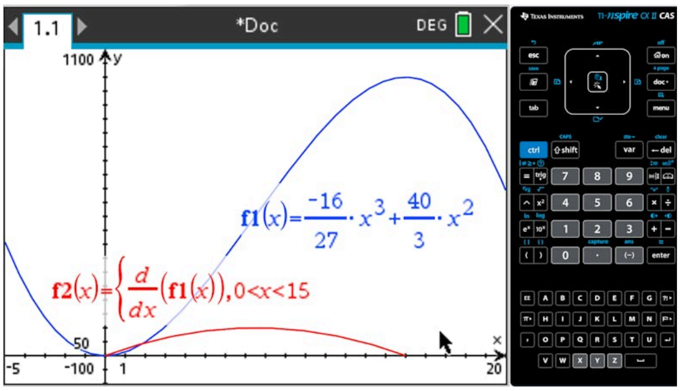
Overview (/study/app/m/sid-122-cid-754029/

Step	Explanation
You can see two graphs. Notice, that the derivative is only displayed for $0 < x < 15$.	

Important

The velocity of an object is the rate of change of its position.

The velocity is given by the derivative of the position function. This can be formalised as

$$v = s' = \frac{\mathrm{d}s}{\mathrm{d}t}$$

where *v* is the velocity, *s* is the position and s' and $\frac{\mathrm{d}s}{\mathrm{d}t}$ can both be used to represent the derivative of *s* with respect to time.

In general, if some quantity, V, depends on some other quantity, r, then the derivative

$$V' = \frac{\mathrm{d}V}{\mathrm{d}r}$$

gives the rate of change of V with respect to r.

You will now revisit two examples from section 5.1.2 (/study/app/m/sid-122-cid-754029/book/gradient-at-a-point-id-26272/).

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Overview
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122-
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Example 1

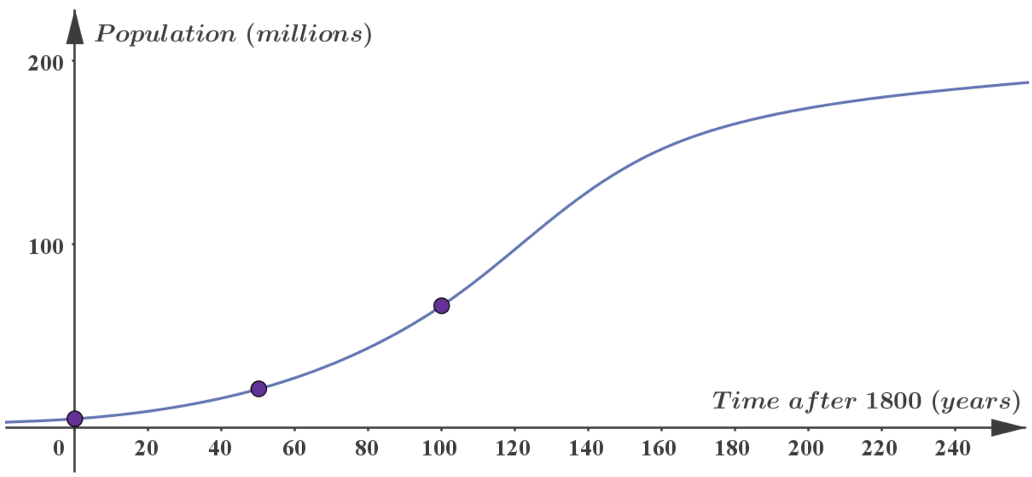
★★☆

The population of the United States in 1800, 1850 and 1900 was 5.3, 23.1 and 76 million people respectively. This population growth can be modelled by

$$P(t) = \frac{189.4}{1 + 34.74\mathrm{e}^{-0.031476t}}$$

where t is the time in years after 1800.

The diagram below illustrates this growth.



More information

The graph depicts the growth of population in millions over the years following 1800. The x-axis represents the time in years after 1800, ranging from 0 to 240. The y-axis represents the population in millions, with values ranging from 0 to 200.

The curve begins near the origin, showing a slow initial increase in population. Around 60 years after 1800, there is a noticeable upward trend, indicating a significant increase. By 100 years after 1800, the population reaches around 100 million. The upward slope continues sharply until around 160 years after 1800, where the curve begins to flatten, suggesting a slowing in growth rate as it approaches 200 million.

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122-
cid-
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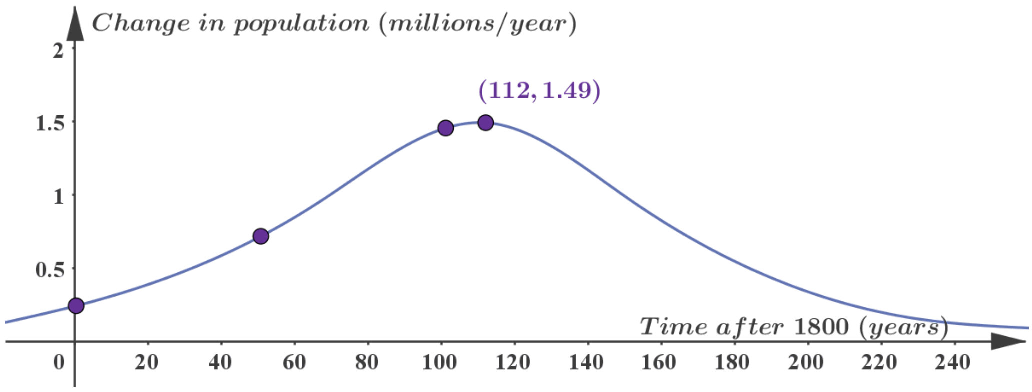
- Use the model to find at what rate the population was changing in 1800, 1850 and 1900.
- Explain how this model predicts the rate of change of the population.

The rate of population change at any time is given by the gradient of the graph of this model. Graphing calculators have applications that can find the value of the derivative and also to draw the derivative graph. This is shown on the diagram below.



- Since time is measured in years after 1800, the rate of change of population in 1800 corresponds to the value *t* = 0. $P'(0) \approx 0.162$, so the population is growing at a rate of 0.162 million people per year.
- The rate of population change in 1850 corresponds to the value *t* = 50. $P'(50) \approx 0.638$, so the population is growing at a rate of 0.638 million people per year.
- The rate of the population change in 1900 corresponds to the value *t* = 100. $P'(100) \approx 1.43$, so the population is growing at a rate of 1.43 million people per year.

The derivative graph shows an increase in population at an increasing rate until $t = 112$. This corresponds to 1912, when the maximum rate of population increase was 1.49 million people per year. Since 1912, the model still shows an increase in population (as the derivative is positive), but at a decreasing rate.

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Example 2

★★☆

The table below contains information on the temperature of a cup of tea.

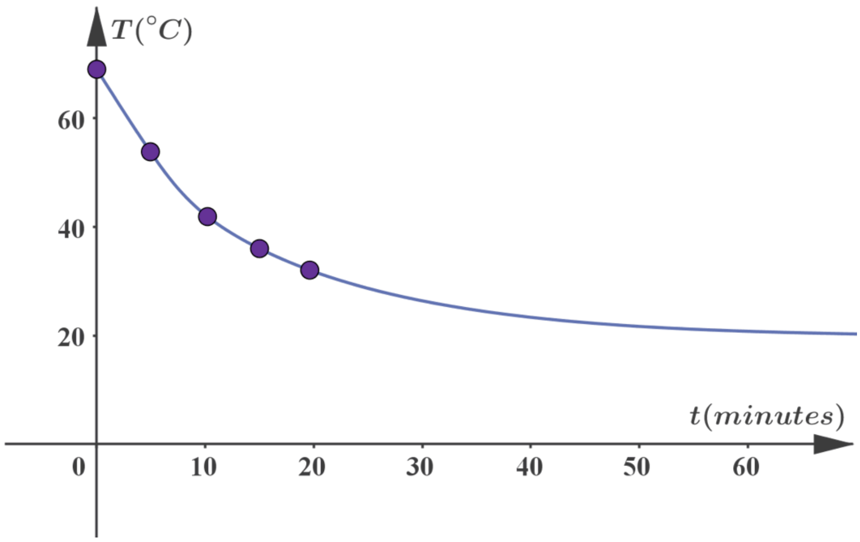
Time (minutes)	0	5	10	15	20
Temperature (°C)	69	54	43	36	31

The cooling can be approximately modelled by the function

$$T(t) = 19.8 + 49.2\mathrm{e}^{-0.0744t}$$

where $T(t)$ is the temperature in Celsius degrees t minutes after the measurement is started.

The diagram below illustrates this function and the data points.



More information

The diagram displays a graph with time in minutes on the X-axis ranging from 0 to 60, and temperature in degrees Celsius on the Y-axis ranging from 0 to 60. The curve shows a decreasing trend from the top left to the bottom right of the graph, illustrating how the temperature decreases over time. Data points along the curve illustrate measurements at specific intervals. Initially, the temperature is at 60°C at time 0, decreasing as time progresses. The decrease becomes less steep as time increases, forming a curve that flattens towards the right end of the graph.

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122-
cid-
754029/

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- Find the average rate of change of temperature in the first 20 minutes.
- Find the average rate of change of temperature in the first 5 minutes.
- Find the initial rate of change of temperature.

- The temperature change in the first twenty minutes is $31 - 69 = -38$ (the negative sign indicates that the temperature was decreasing).

Hence, the average rate of change of temperature in the first 20 minutes is $\frac{-38}{20} = -1.9$ degrees per minute.

- Similarly, the average rate of change in the first 5 minutes is $\frac{54 - 69}{5} = -3$ degrees per minute.
- The initial rate of change is the value of the derivative of T at $t = 0$.

Graphing calculators have applications to find the derivative. The value you get is $T'(0) \approx -3.66$

Hence, the temperature is initially decreasing at a rate of 3.66 degrees per minute.

Theory of Knowledge

As you've progressed through your studies, you have likely noticed the complexity of the questions you're being asked to answer increasing. You have perhaps even come across some questions that you found so difficult that you wondered whether or not a solution even exists! That is a very epistemological thought. A related knowledge question is 'Do all valid questions have valid answers?'

3 section questions

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Checklist

What you should know

By the end of this subtopic you should be able to:

- understand the concept of the
  - limit of a sequence
  - limit of a function both at infinity and at a finite point
- estimate the value of a limit
  - using table of values
  - using graphs
- be aware that tangents to curves are limits of secant lines
- understand the concept of the gradient of a curve
- understand the concept of the derivative as the gradient function
- understand that a gradient expresses a rate of change
- understand the connection and difference between average rate of change and instantaneous rate of change.

Investigation

Investigation 1

Watch this video (https://www.youtube.com/watch?v=A6shAlaU_SA) of the women's 50 m freestyle final at the European Swimming Championship in 2016. The winning time of Pernille Blume was 24.07 seconds.

The red markers on the lane dividers are at 15 m, 25 m and 35 m.

Overview
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122-
cid-
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Did the speed of the swimmers change during the race?

Investigation 2

In the video below you can watch grass grow.

Does grass grow at a constant rate or does the growth rate change?

Grass Seed Germination and Grass Growing Time Lapse

Investigation 3

In this video you can see the temperature of a cup of tea change as it cools down.

Can you estimate the room temperature?

Can you estimate the room temperature by looking at only the first half of the video?

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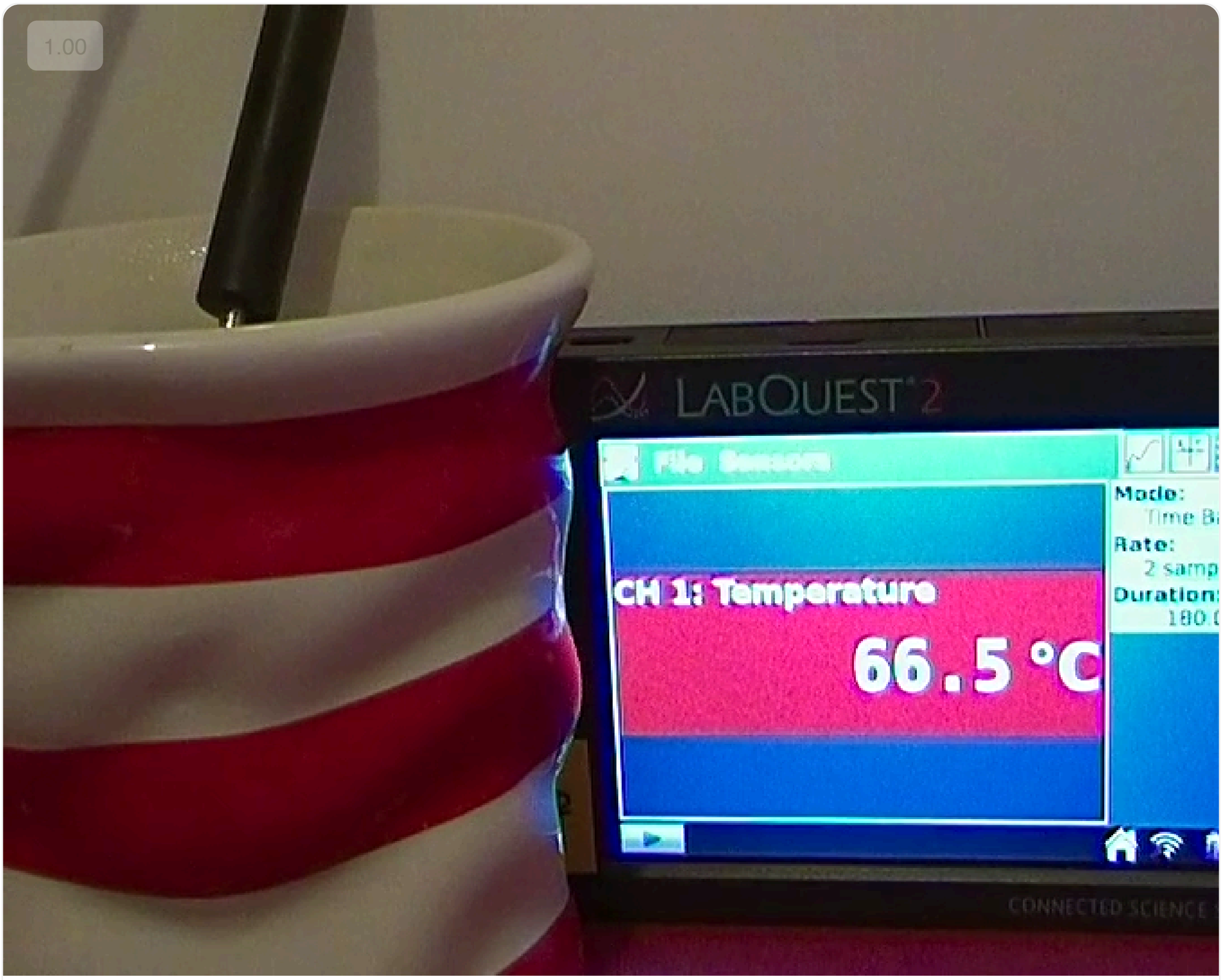
Overview
(/study/ap
122-
cid-
754029/



0:03 / 49:58

Video 1. Analyzing Temperature Changes in a Cooling Drink.

More information for video 1

This interactive video demonstrates the cooling process of a cup of tea, allowing viewers to observe and analyze how temperature of a cup of tea changes as it cools down. The video has a total duration of 49:58 minutes. The tea starts at 66.6° C and gradually cools down, reaching 39.3° C by the end of the video.

By watching this process, users can explore key thermodynamic concepts, such as **Newton's Law of Cooling**, which describes how an object's temperature approaches that of its surroundings. The video provides an opportunity to estimate the room temperature by analyzing the cooling trend and understanding the factors that influence heat transfer.

Rate subtopic 5.1 Introduction to differentiation

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