The big picture

As more and more powerful computers become available, artists have access to increasingly advanced tools to create animations. Take a look at the following video.

Video 1. Lenard-Jones Potential

More information for video 1

On the left side of the screen, a 3D modeling program presents an overview of the scene, displaying white spheres arranged within a gray, gridded environment. This static view serves as a reference for the overall setup, with a 3D axis manipulator in the corner indicating spatial orientation.

On the right side, a dynamic viewport provides a closer look at how the spheres interact. As the camera zooms in, the spheres appear to move in response to each other—when they are far apart, they attract, but as they get closer, this attraction turns into repulsion.

To the far right, a statistics panel updates in real-time, displaying key performance metrics such as draw calls, vertices, and frame rate. Despite the continuous motion in the right viewport, certain metrics remain stable, indicating that the scene's complexity remains consistent.

As the zoom progresses, the spheres fill more of the screen, revealing their shifting spatial relationships. This interplay of attraction and repulsion creates a mesmerizing pattern

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Do you think the creator specified the position of all balls for each frame of the animation? This would be quite time-consuming. It is easier to use a rule that determines the movement of the balls relative to all the others. Did you notice that when the balls are far away, they attract each other, but when they get close to each other this attraction changes to repulsion? This model is built in the animator software Blender. You can find the manual page here (https://docs.blender.org/manual/en/2.80/physics/forces/force_fields/types/lennard_jones.html).

The important property of the model is that it has a minimum point. In this section, you will learn about methods to find minimum and maximum points on graphs of functions.

Concept

When looking for maximum and minimum points on a graph, think of these as points where the instantaneous rate of **change** of the function value is zero.

Theory of Knowledge

Stationary points and points of inflexion provide the knower with knowledge regarding change. Mathematics as an area of knowledge is quite unique in the degree to which it can make claims to the precision of change. Think about other areas of knowledge and consider the methodologies through which they identify change.

Knowledge Question: To what extent does methodology create knowledge, versus knowledge creating methodology?

5. Calculus / 5.6 Stationary points

Stationary points

Take another look at the activity you carried out in subtopic 5.2.1 (/study/app/m/sid-122-cid-754029/book/graph-properties-id-26278/). The purpose of repeating it here is to highlight a relationship that was not mentioned before.

Activity

On the applet, you can specify the sign of the derivative and a possible graph is then shown. The applet also highlights two points on the graph and the corresponding tangent lines.

Student view

Experiment with changing the sign of the derivative on the different intervals and explain what you notice about the highlighted points.

Interactive 1. Graph Properties.

More information for interactive 1

This interactive graph helps users explore the relationship between a function and its derivative by allowing them to analyze how the function's behavior determines the sign of $f'(x)$. The tool visually represents a function $f(x)$ and enables users to investigate where it is increasing or decreasing by adjusting red points along the curve. The graph is divided into colored regions, representing intervals where the function changes its behavior.

Users can move the red point along the curve to observe how the derivative behaves at different positions. By selecting the appropriate checkboxes—either $f'(x) > 0$ or $f'(x) < 0$ they can classify each region based on whether the function is increasing or decreasing.

When $f'(x) > 0$, the function is increasing, meaning its graph slopes upward, indicating that as x increases, $f'(x)$ also increases. This happens because the tangent lines at those points have positive slopes.

Conversely, when $f'(x) < 0$, the function is decreasing, meaning its graph slopes downward, showing that as x increases, $f(x)$ decreases. This is because the tangent lines have negative slopes.

This interactive helps users visually understand how the function's behavior determines the sign of the derivative, reinforcing the fundamental relationship between differentiation and the shape of a graph.

The following is a summary of some important features you might have noticed.

Important

A point where the tangent line to a graph is horizontal is called a stationary point. For a differentiable function, f, you can find these points by solving $f'(x) = 0$.

The table below gives the different type of stationary points you will meet in this course.

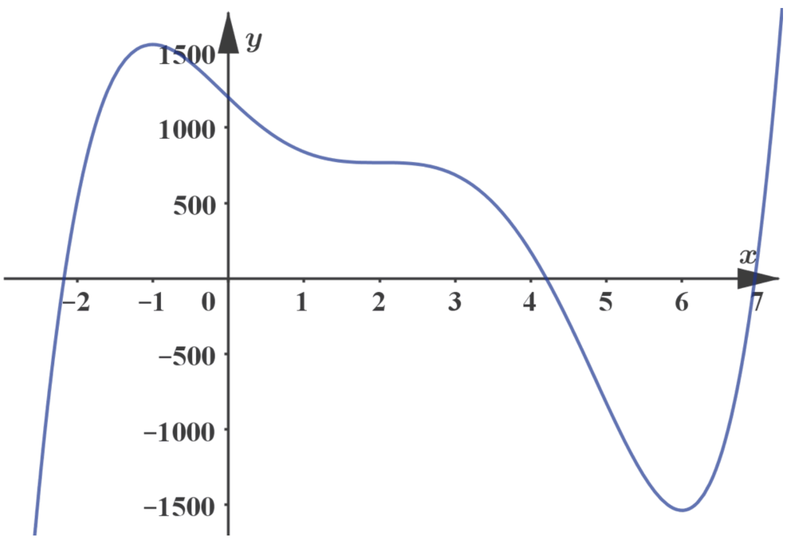
Steps	Explanation
Local maximum point	The derivative is 0, and it changed sign from positive to negative as you pass through the point in the direction of increasing x.
Local minimum point	The derivative is 0, and it changes sign from negative to positive as you pass through the point in the direction of increasing x.
Horizontal point of inflexion	The derivative is 0, but it does not change sign as you pass through the point.

The following examples show you how to find and identify stationary points.

Example 1

★★☆

Identify and classify the stationary points on the graph of
$y = 4x^5 - 45x^4 + 120x^3 + 40x^2 - 480x + 1200.$



More information

The image is a graph of the polynomial function y = 4x^5 - 45x^4 + 120x^3 + 40x^2 - 480x + 1200. The X-axis ranges approximately from -3 to 8 and the Y-axis ranges from -1500 to 1500, showing the value of 'y' depending on 'x'. The graph depicts several stationary points and curves.

Starting from the left, the graph rises to a peak around x = -2 where it reaches a local maximum above y = 1500. Moving towards the right, it descends through the origin (x = 0) and reaches a minimum around x = 1 where y is approximately -500.

After this minimum, the graph rises to another local maximum near x = 3, where y is approximately 800.Continuing rightwards, it falls steeply to another local minimum around x = 6, where y is about -1500. Beyond this, the graph rises sharply again with an upward trend.

[Generated by AI]

Steps	Explanation
$y' = 20x^4 - 180x^3 + 360x^2 + 80x - 480 = 0$	At the stationary points the derivative is 0.
The solutions are $x = -1$, $x = 2$ and $x = 6$.	Graphic display calculators have applications that can solve a polynomial equation like this. To see how these applications work, take a look at the instructions in **section 1.8.1**.
The stationary points are $(-1, 1551)$, $(2, 768)$ and $(6, -1536)$.	You can find the y-coordinate of the stationary points by substituting these values into the equation of the curve.
• The point $(-1, 1551)$ is a local maximum point. • The point $(2, 768)$ is a horizontal point of inflexion. • The point $(6, -1536)$ is a local minimum point.	Inspecting the graph gives the nature of the stationary points.

Example 2

★★☆

Find the minimum point on the graph of $y = \dfrac{1}{x^{12}} - \dfrac{2}{x^6}$ for $x > 0$.

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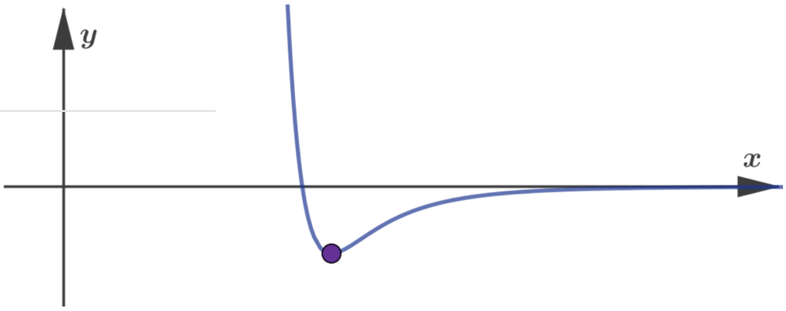


More information

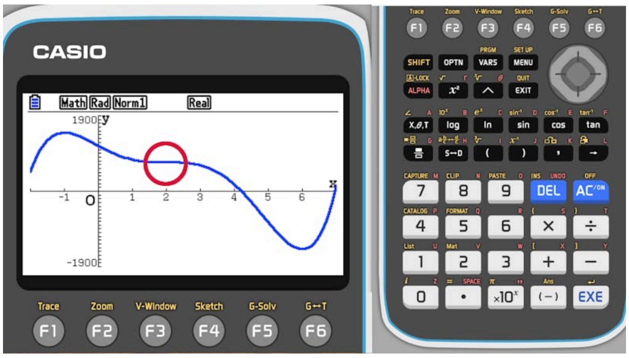
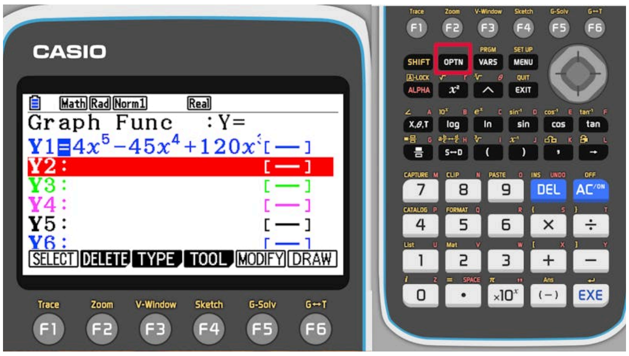
The graph represents the function (y=\dfrac{1}{x^{12}}-\dfrac{2}{x^6}) for (x>0). The X-axis represents the values of (x), and the Y-axis represents the calculated values of (y). The graph includes a noticeable minimum point at the vertex of the curve. Initially, as (x) approaches zero, the curve rises sharply. The curve reaches a minimum point and then ascends slowly as (x) increases, approaching the horizontal axis but remaining above it. These observations suggest an initial steep decrease followed by a gradual ascent.

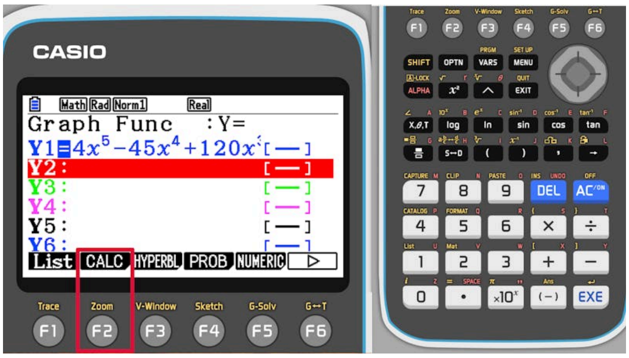
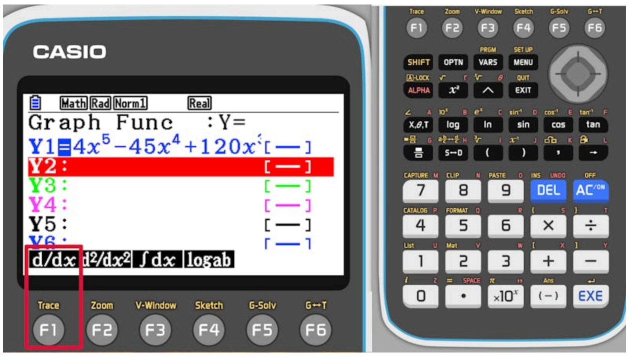
[Generated by AI]

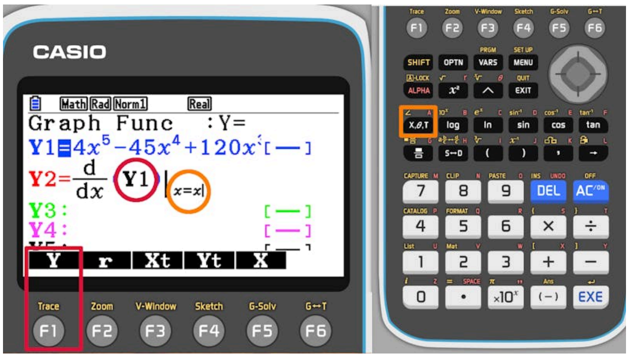
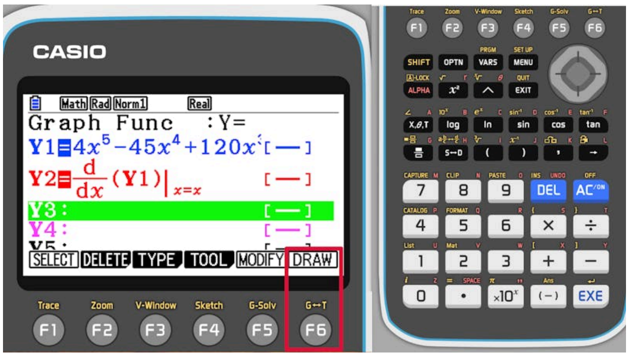
Student
view

Steps	Explanation
$y = \frac{1}{x^{12}} - \frac{2}{x^6}$ $= x^{-12} - 2x^{-6}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = -12x^{-13} - 2 \times (-6)x^{-7}$ $= -12x^{-7}(x^{-6} - 1)$	You can use the derivative to find the minimum point.
$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $-12x^{-7}(x^{-6} - 1) = 0$ $x^{-6} - 1 = 0$ $\frac{1}{x^6} = 1$ $1 = x^6$	At the minimum point the derivative is 0.
The only positive solution of this equation is $x = 1$.	The x-coordinate of the minimum point is the solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.
$y = \frac{1}{1^{12}} - \frac{2}{1^6} = 1 - 2 = -1$	The y-coordinate of the minimum point is the value of the expression at $x = 1$.
Hence, the minimum point on the graph is $(1, -1)$.	

In the examples above, the solutions used algebraic methods to find the stationary points on a graph. A graphic display calculator (GDC) has applications that can find minimum and maximum points directly, so the algebraic method is often not needed. However, in **Example 1** the graph also has a stationary point that is not a turning point. Below, you can see the steps on how to use a GDC to find this point without algebraically finding the derivative first.

Steps	Explanation
In these instructions you will see how to use the calculator to find the stationary point, which is not a maximum or minimum point. The example used here is the graph of $$y = 4x^5 - 45x^4 + 120x^3 + 40x^2 - 480x + 1200$$ It is assumed, that you have the graph on the screen. The viewing window used in this exmaple is $-2 < x < 7$ and $-2000 < y < 2000$. The goal is to find the point somewhere in the highlighted circle, where the tangent line is horizontal	
Besides the function, you will need the graph of the derivative. You can ask the calculator to draw the derivative graph without finding it algebraically. To access this option, press OPTN ...	

Steps	Explanation
... press F2 to access the options related to calculus, ...	
... and press F1 to open the template for entering the derivative.	

Steps	Explanation
Use the name of the function instead of typing it in again. Use the format $x = x$ to indicate, that you are entering a function, not a value.	
Once the derivative function is defined, press F6 to see the graphs.	

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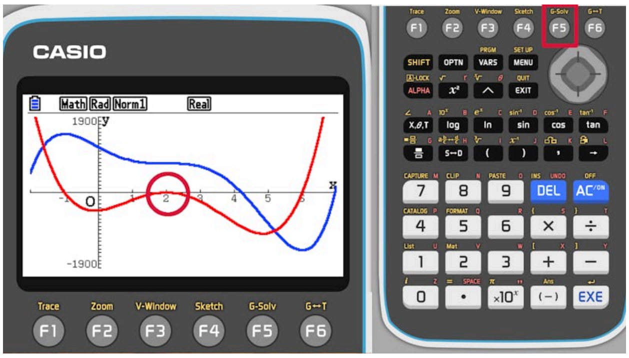
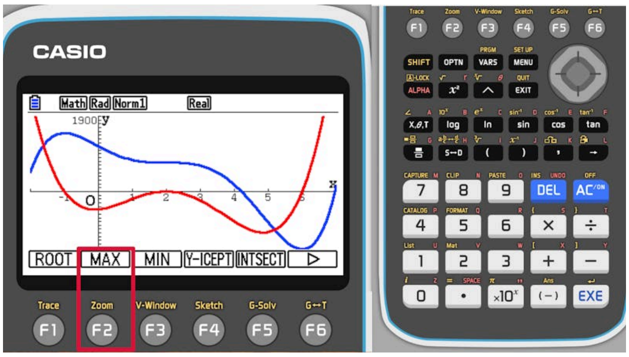
Steps	Explanation
You will need to find the point, where the derivative graph touches the x-axis. Press F5 (G_Solve) to see the options to analyze the graph.	

Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/stationary-points-id-27480/print/) Assign

You can either look for the root, or find the local maximum of the derivative. Since the graph does not cross the x-axis, it is numerically more stable to look for the maximum, so press F2.	

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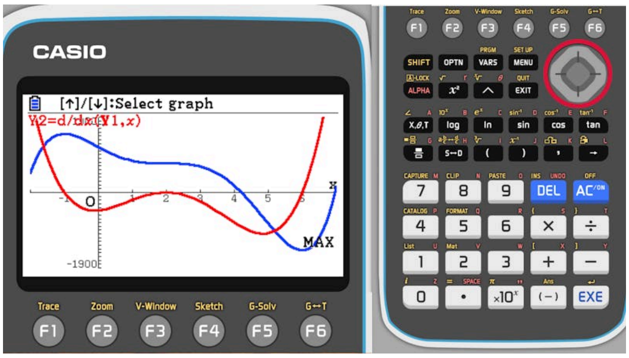
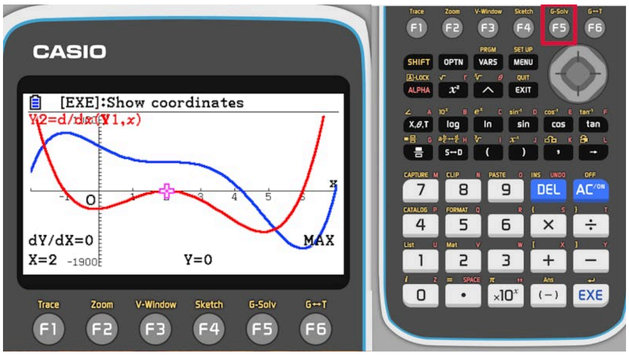
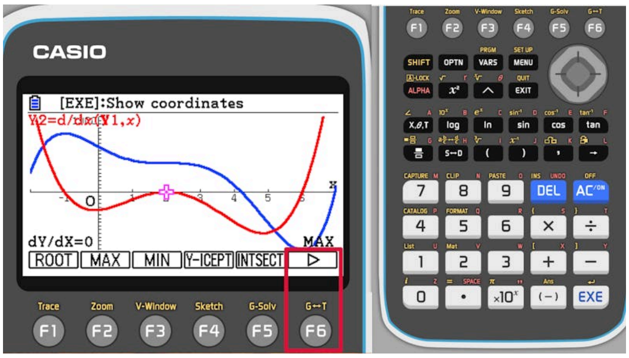
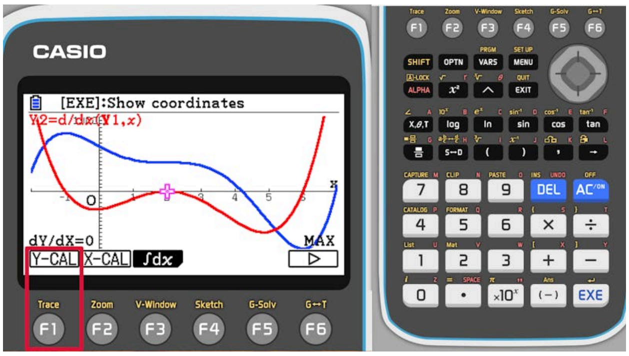
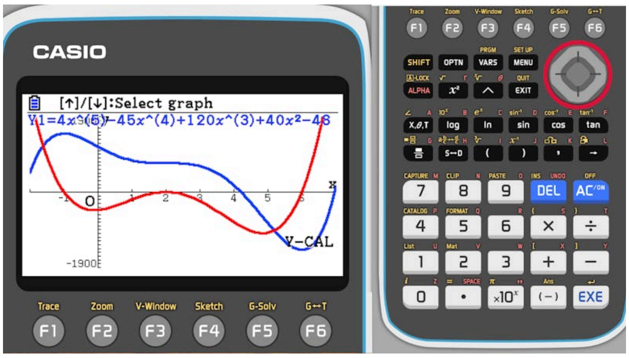
Steps	Explanation
Since there are two graphs on the screen, the calculator needs to know which one you are interested in. Select the derivative graph by moving up/down.	
The calculator moves the cursor to the point, where the derivative graph touches the x-axis, and displays its coordinates. You will need the corresponding point on the original graph, so make a note of this x-value (or store it in the memory of the calculators) and press F5 again to see the options to analyze graphs.	

Steps	Explanation
The option to find a value of a function is not among the ones you see on this screen, so press F6 to see the other options ...	
... and press F1 to calculate a y-value.	
Choose the original function, ...	

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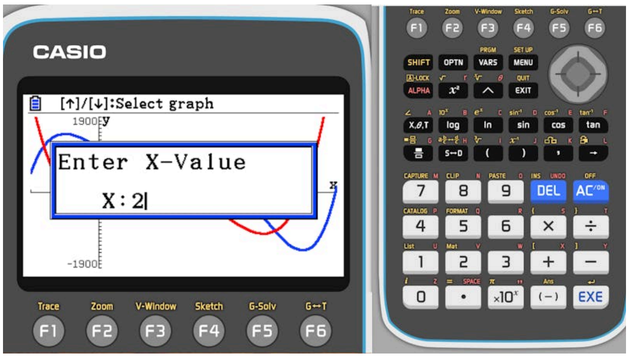
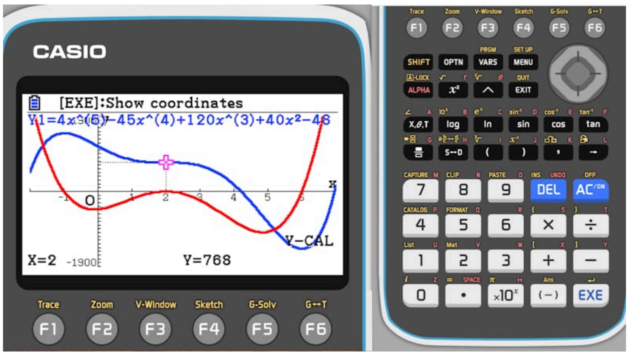
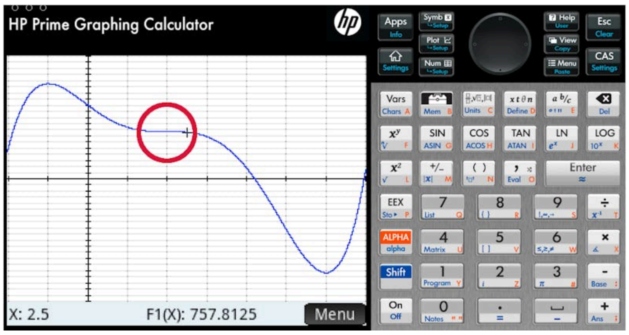
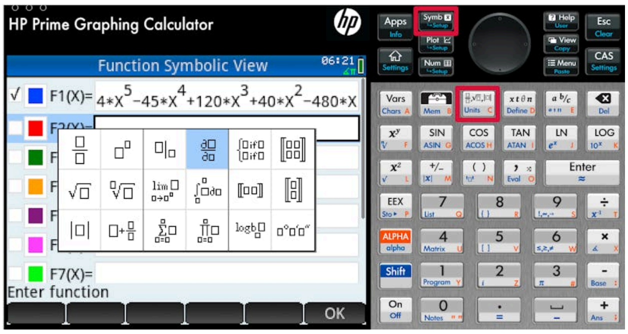
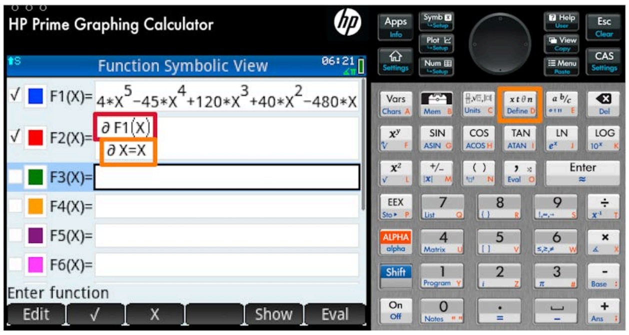
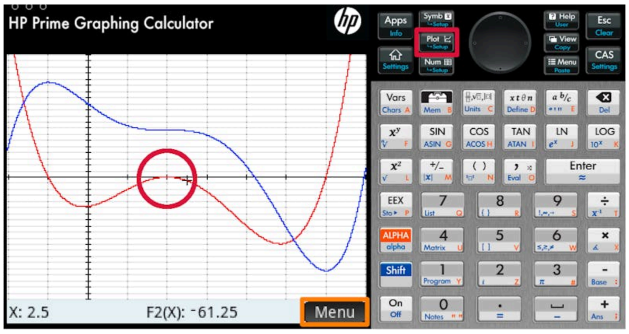
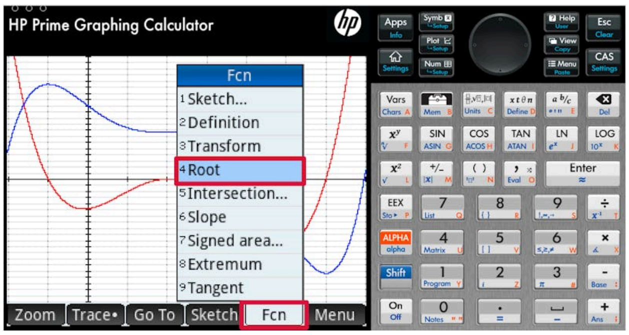
Steps	Explanation
and enter the x-value you found earlier.	
The calculator moves the cursor to the stationary point on the original graph and displays its coordinates.	

Student view

Steps	Explanation
In these instructions you will see how to use the calculator to find the stationary point, which is not a maximum or minimum point. The example used here is the graph of $y = 4x^5 - 45x^4 + 120x^3 + 40x^2 - 480x + 1200$ It is assumed, that you have the graph on the screen. The viewing window used in this exmaple is $-2 < x < 7$ and $-2000 < y < 2000$. The goal is to find the point somewhere in the highlighted circle, where the tangent line is horizontal	
Besides the function, you will need the graph of the derivative. You can ask the calculator to draw the derivative graph without finding it algebraically. You can access this option using the tamplates in symbolic view.	

Steps	Explanation
Use the name of the function instead of typing it in again. Use the format $x = x$ to indicate, that you are entering a function, not a value.	
Change to plot view to see the graphs. You will need to find the point, where the derivative graph touches the x-axis. Tap on menu to see the options to analyze the graph.	
You can either look for the root, or find the local maximum of the derivative.	

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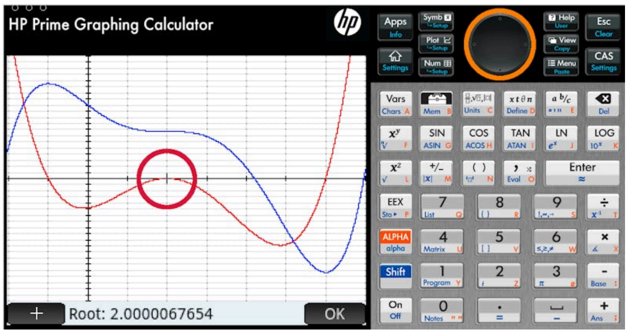
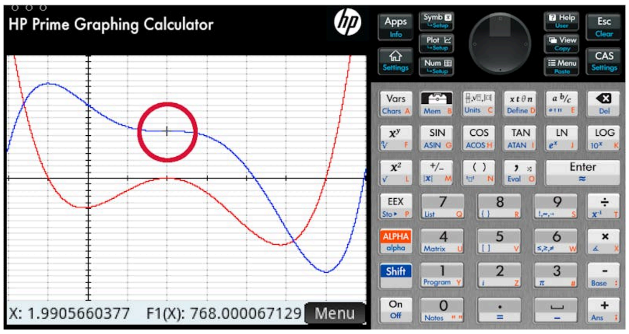
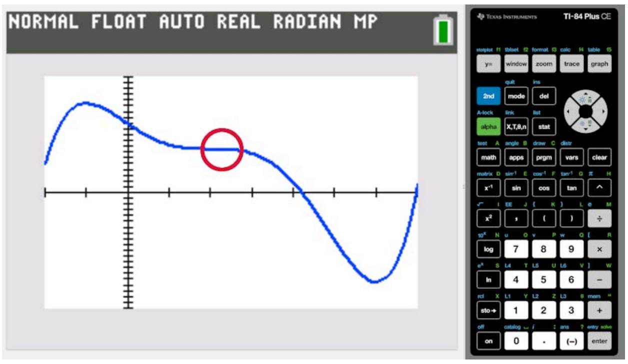
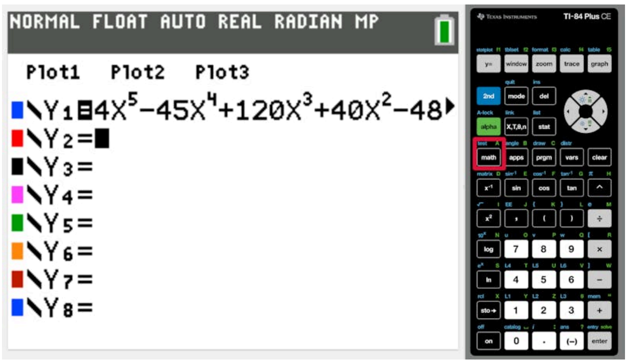
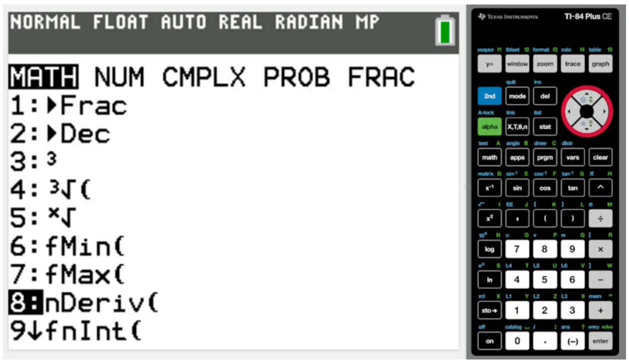
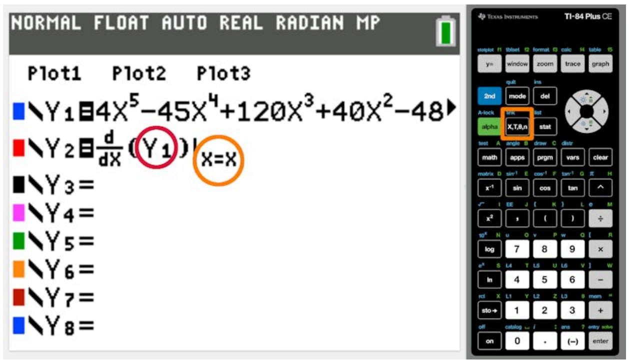
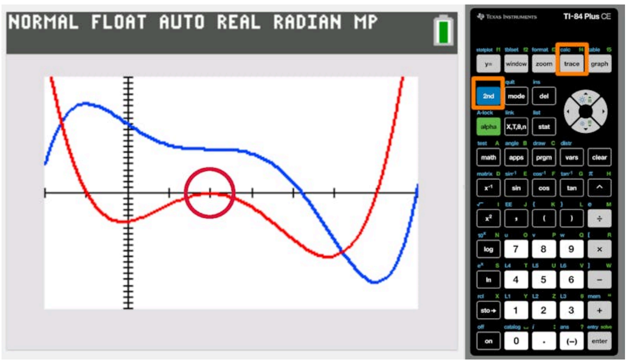
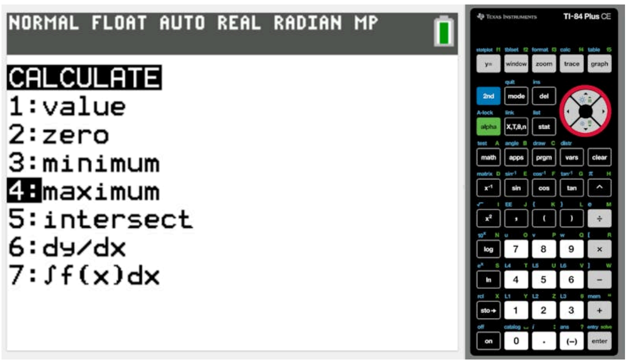
Steps	Explanation
The calculator moves the cursor to the point, where the derivative graph touches the x-axis, and displays the x-coordinate. To see the corresponding point on the original graph, move up to change between the graphs.	
The calculator moves the cursor to the stationary point on the original graph and displays its coordinates. Note, that the coordinates are only approximate, because the calculator uses numerical algorithms, not computer algebra systems to find these values.	

Student
view

Steps	Explanation
In these instructions you will see how to use the calculator to find the stationary point, which is not a maximum or minimum point. The example used here is the graph of $y = 4x^5 - 45x^4 + 120x^3 + 40x^2 - 480x + 1200$ It is assumed, that you have the graph on the screen. The viewing window used in this exmaple is $-2 < x < 7$ and $-2000 < y < 2000.$ The goal is to find the point somewhere in the highlighted circle, where the tangent line is horizontal	
Besides the function, you will need the graph of the derivative. You can ask the calculator to draw the derivative graph without finding it algebraically. To access this option, press math ...	

Steps	Explanation
and choose the option (nDerive) to open the template for entering the derivative.	
Use the name of the function instead of typing it in again. Use the format $x = x$ to indicate, that you are entering a function, not a value.	

Steps	Explanation
Change to the screen to see the graphs. You will need to find the point, where the derivative graph touches the x-axis. Press 2nd/calc to see the options to analyze the graph.	
You can either look for the zero, or find the local maximum of the derivative. Since the graph does not cross the x-axis, it is numerically more stable to look for the maximum, so select this option.	

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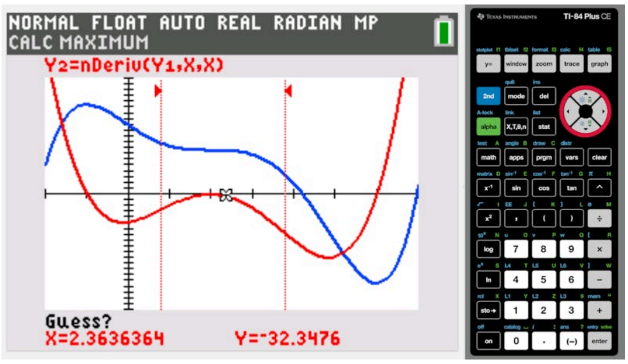
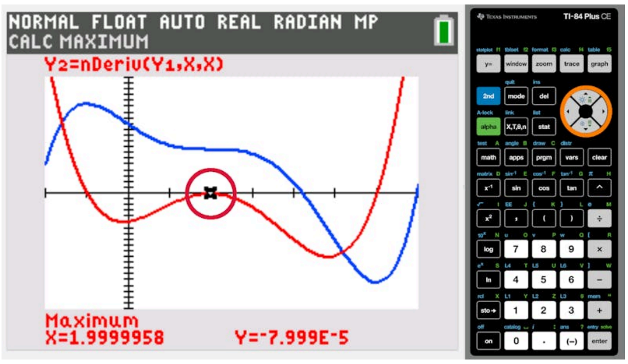
Steps	Explanation
Move the cursor to the derivative graph, set a lower and upper bound for the maximum point and finally move the cursor close to the point you want to find. Confirm your selection by pressing enter.	
The calculator moves the cursor to the point, where the derivative graph touches the x-axis, and displays the coordinates. To see the corresponding point on the original graph, move up to change between the graphs.	

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<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>

The calculator moves the cursor to the stationary point on the original graph and displays its coordinates.

Note, that the coordinates are only approximate, because the calculator uses numerical algorithms, not computer algebra systems to find these values.

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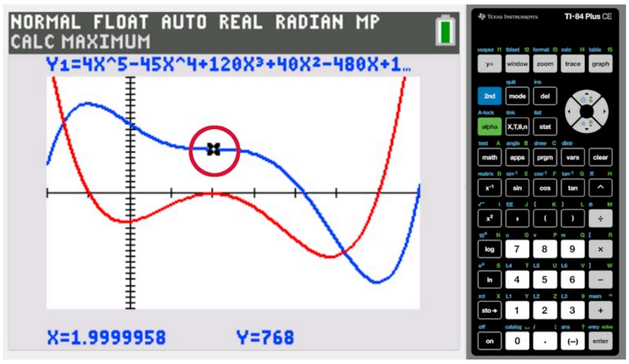

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$$y = 4x^5 - 45x^4 + 120x^3 + 40x^2 - 480x + 1200$$

It is assumed, that you have the graph on the screen. The viewing window used in this exmaple is $-2 < x < 7$ and $-2000 < y < 2000$.

The goal is to find the point somewhere in the highlighted circle, where the tangent line is horizontal

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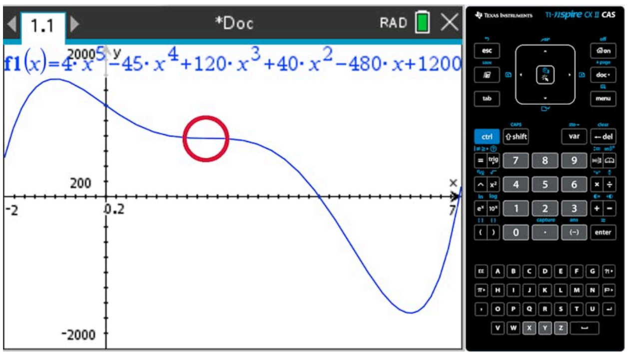

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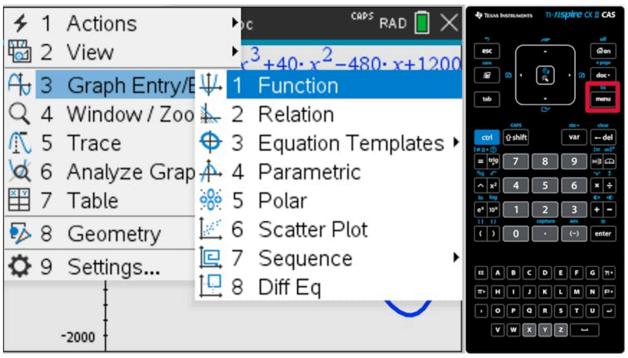
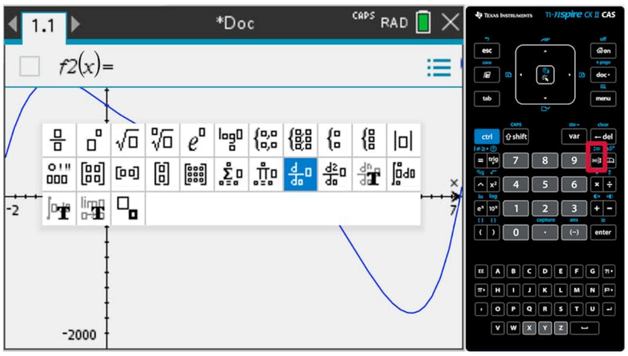
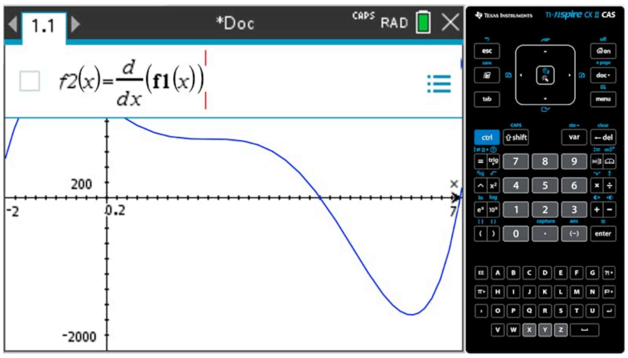
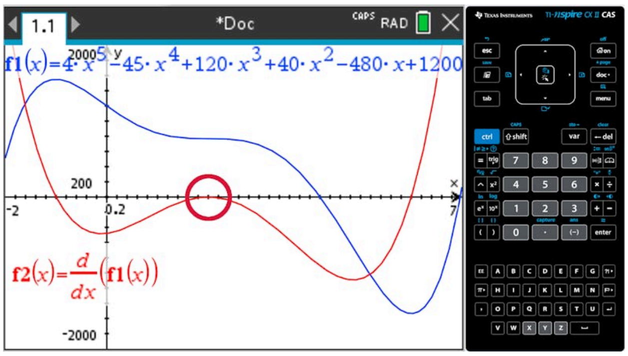
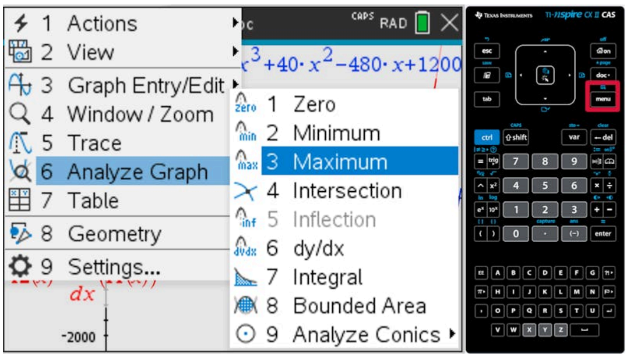
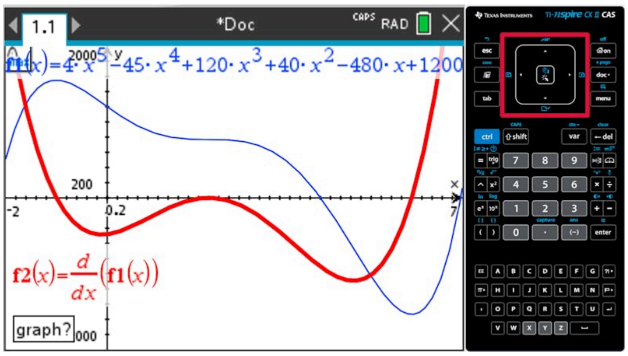
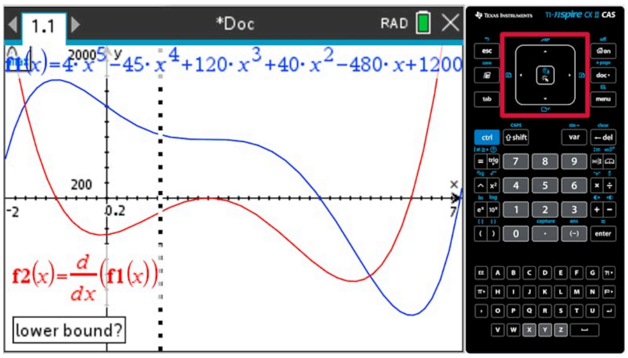
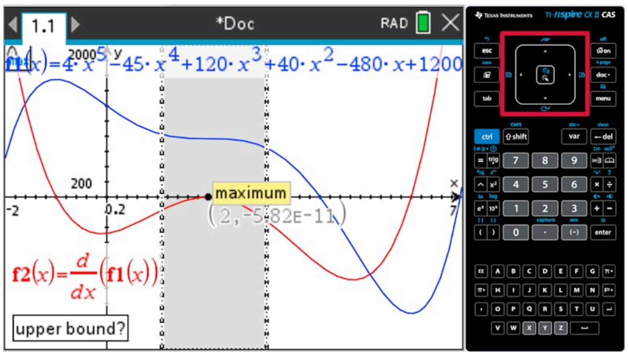
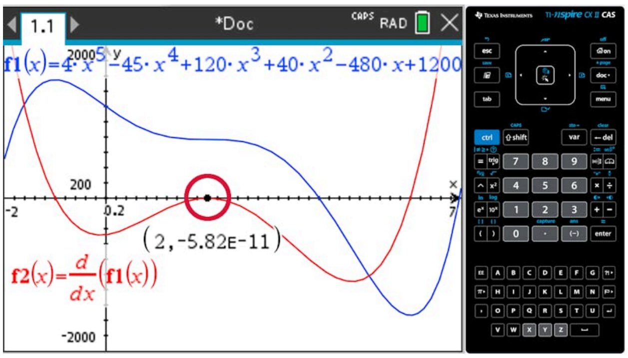
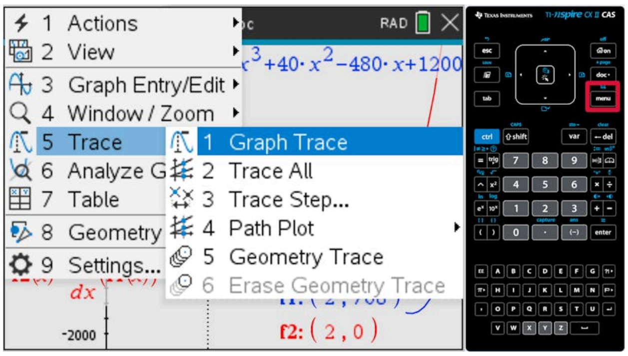
Steps	Explanation
Besides the function, you will need the graph of the derivative. You can add a second function using the menu.	
You can ask the calculator to draw the derivative graph without finding it algebraically. To access this option, use the template menu.	

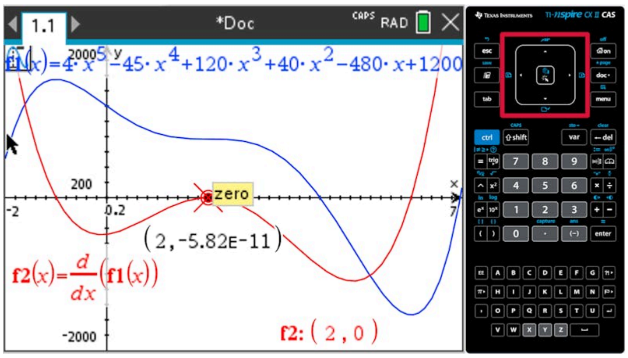
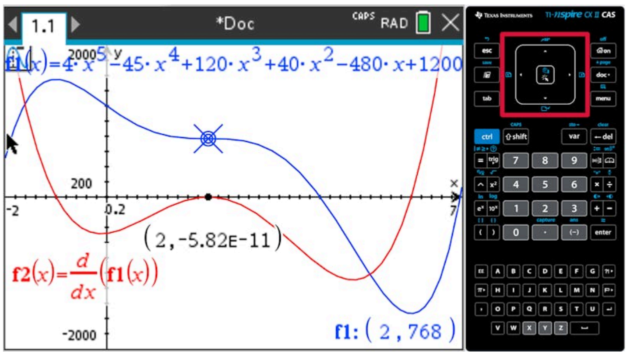
Student view

Steps	Explanation
Use the name of the function instead of typing it in again.	
You will need to find the point, where the derivative graph touches the x-axis.	

Steps	Explanation
You can either look for the zero, or find the local maximum of the derivative. Since the graph does not cross the x-axis, it is numerically more stable to look for the maximum, so select this option.	
Since there are two graphs on the screen, the calculator needs to know which one you are interested in. Select the derivative graph ...	

Steps	Explanation
choose a lower bound ...	
... and an upper bound.	

Steps	Explanation
The calculator shows the point, where the derivative graph touches the x-axis, and displays the coordinates.	
To see the corresponding point on the other graph, choose to trace the graphs.	

Steps	Explanation
Move left/right until the calculator locks on the point you have found …	
… then move up to jump to the original graph. The calculator moves the cursor to the stationary point on the original graph and displays its coordinates.	

Example 3

★★☆

Find the stationary point on the graph of $y = \sqrt{x}\left(x - 3\sqrt{x} + 3\right)$.

Overview
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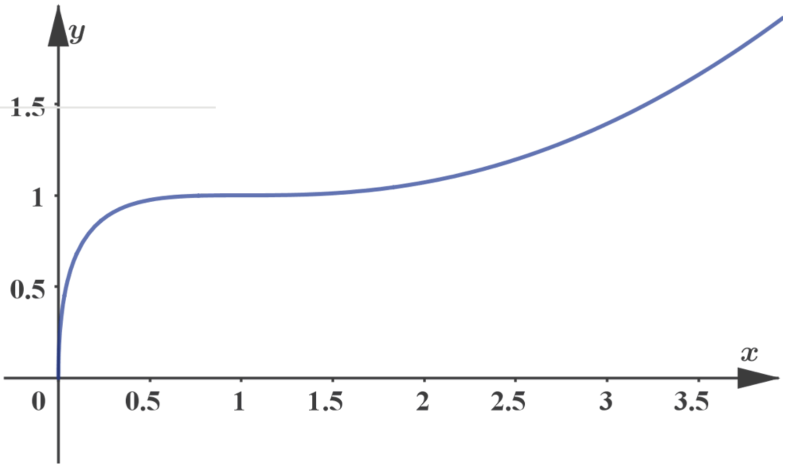
cid-
754029/
TOPIC 5
CALCULUS
SUBTOPIC 5.6
STATIONARY POINTS

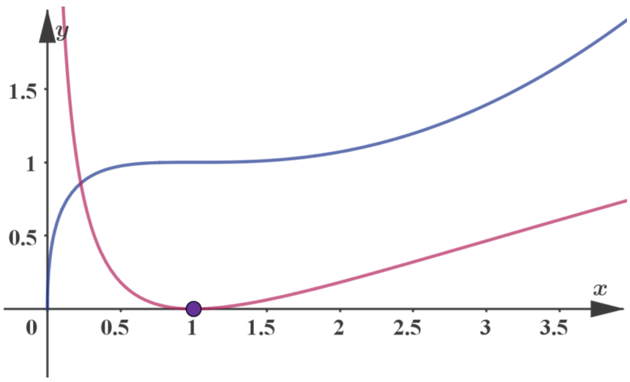


More information

The image is a graph showing the function y = sqrt(x)(x−3sqrt(x)+3). The X-axis, labeled as 'x', ranges from 0 to approximately 3.5, with intervals marked at 0.5, 1.5, 2, 2.5, 3, and 3.5. The Y-axis, labeled as 'y', ranges from 0 to 2, with intervals at 0.5, 1, 1.5, and 2.

The curve starts at the origin (0,0) and initially rises steeply. It approaches a nearly horizontal trend as x increases towards 1, with a slight dip before increasing again. As x continues to increase beyond 1, the curve rises gradually, showing an overall increasing trend. There is an inflection point where the curve transitions from concave upwards to concave downwards.

[Generated by AI]

Steps	Explanation
	A GDC can graph the derivative and find the solution of $y' = 0$. To see how this works on your calculator, follow the steps in the instructions above this example.

Student
view

Overview
(/study/ap
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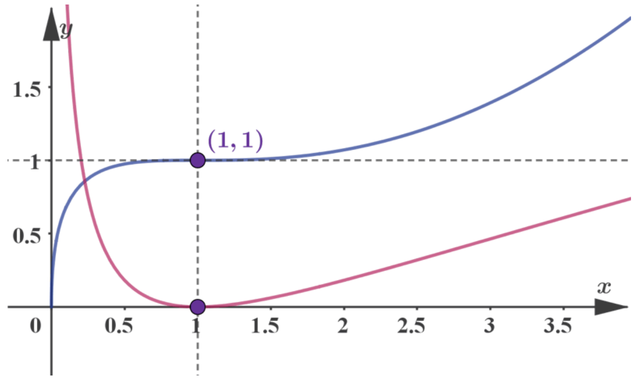
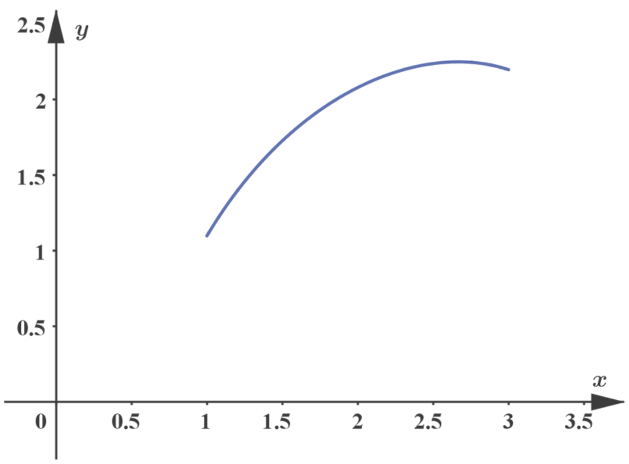
TOPIC 5
CALCULUS
SUBTOPIC 5.6
STATIONARY POINTS

Steps	Explanation
	A GDC can also find the value of the original expression for the x-value found in the previous step.

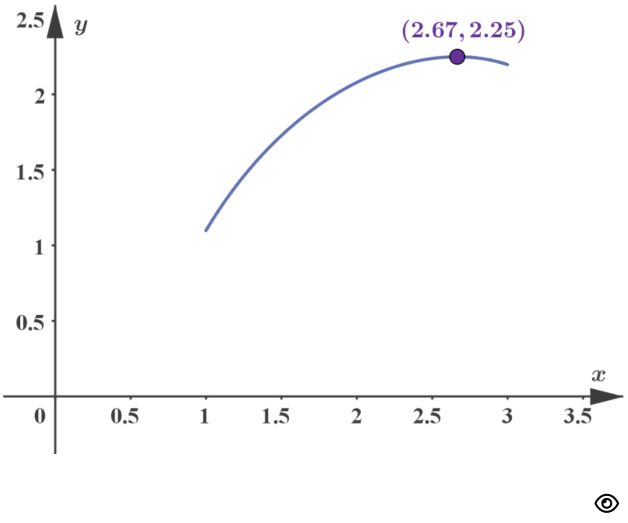
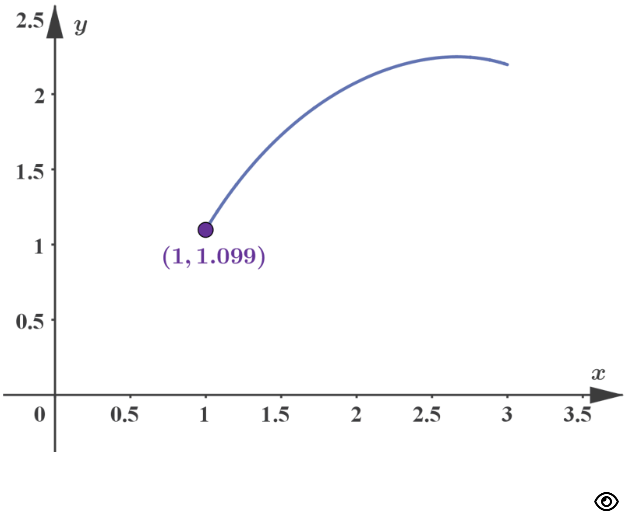
Example 4

★★☆

Find the smallest and largest value of $y = \ln(4 - x) + 2\ln x$ for $1 \leqslant x \leqslant 3$.

Steps	Explanation
	A GDC can plot the graph on a limited domain. To see how these applications work on certain calculators, take a look at the instructions in **section 5.1.4**.

Student
view

Steps	Explanation
	You can see that the maximum is at a turning point. A GDC can find this point on the graph.
The smallest value is 1.10. 	The minimum is not at a turning point. Instead it is the left endpoint of the domain.

While thinking about the previous problem, you may have noticed the following.

Important

If f is a differentiable function defined on the interval $[a, b]$, then the largest and smallest values of the function are either $f(a)$, $f(b)$ or the y-coordinate of one of the turning points.

The largest value is the global maximum and the smallest value is the global minimum of the function on the given domain.

You can investigate this on the following applet. Move the red points around to change the shape of the graph. The applet shows you the domain and the maximum and minimum values.

Interactive 1. Graph Showing Domain and Range.

More information for interactive 1

This interactive allows users to explore the concepts of global maximum and minimum values of a differentiable function defined on a closed interval $[a, b]$. It is designed to help users understand how critical points, endpoints, and turning points affect the largest and smallest function values.

There is a graph shown in the interactive with maximum and minimum defined on the x and y axes. The curve has 4 interactive red dots, one in the starting and ending each, and another 2 in between. By moving the red points, users can change the shape of the graph and observe how the function's maximum and minimum values are determined within the given domain.

The applet visually displays the domain and highlights the global maximum and minimum values, which are either the function values at the endpoints $f(a)$ and $f(b)$ or the y-coordinates of the function's turning points.

The applet provides a clear representation of how the domain (set of x-values) and range (set of y-values) are affected by modifications to the function. Users can observe how changing the function's curvature impacts the possible maximum and minimum values within the given interval.

Student
view

3 section questions

Checklist

What you should know

By the end of this subtopic, you should be able to:

- find the values of x where the gradient of a curve is zero
  - solving $f'(x) = 0$ algebraically when a polynomial equation is involved
  - using a calculator to graph the derivative (without algebraically finding it first) and use this graph to solve $f'(x) = 0$.
- identify stationary points on a curve
  - recognising whether a stationary point is a local maximum, a local minimum or a horizontal point of inflexion
- find global maximum and minimum values
  - being aware that on a restricted domain the global maximum and minimum values may not correspond to a stationary point.

Investigation

In this investigation you will explore functions of the form $E = \frac{A}{r^{12}} - \frac{B}{r^6}$. This model was proposed by Sir John Edward Lennard-Jones to express the potential energy of interaction (E) between two neutral atoms or molecules in terms of the distance of the particles (r). The parameters A and B depend on the types of particles, temperature, pressure and so on. You have already seen a short explanation of this model in the big picture section.

This equation makes it possible to model the movement of particles relative to each other and to create animations like the one in this interactive simulation (https://phet.colorado.edu/en/simulation/states-of-matter).

On the applet below, you can change the shape of the curve using two parameters:

1. σ, the horizontal axis intercept (the distance, where the potential is zero) and
2. ε, the depth of the potential well.

Interactive 1. Exploring the Lennard-Jones Potentia Function.

More information for interactive 1

The interactive applet allows users to explore the Lennard-Jones potential, a model proposed by Sir John Edward Lennard-Jones to describe the potential energy of interaction E between two neutral atoms or molecules as a function of the distance r between them. The potential energy is given by the equation $E = \frac{A}{r^{12}} - \frac{B}{r^6}$, where *A* and *B* are parameters that depend on factors such as the types of particles, temperature, and pressure. Within the applet, users can dynamically manipulate the Lennard-Jones potential curve by directly interacting with two key points. Dragging the first distinct point along the horizontal axis directly controls the value of σ, which is the distance at which the potential energy is zero and is related to the approximate size of the interacting particles. As you drag this point, you will observe how the position where the curve crosses the horizontal axis shifts, and consequently, the displayed equation for E will update to reflect the new σ value (which influences A and B). Dragging the second distinct point, typically located near the minimum of the potential well, directly controls the value of ϵ, which represents the depth of the potential well and the strength of the attractive interaction. Adjusting this point will change the vertical position of the minimum of the curve, and the displayed equation for E will update to reflect the new ϵ value (again, influencing A and B). By directly manipulating these points and observing the corresponding changes in the potential energy curve and the equation, users can gain an intuitive understanding of how the parameters σ and ϵ influence the interaction between atoms or molecules.

1. Find the value of A and B in $E = \frac{A}{r^{12}} - \frac{B}{r^6}$ in terms of σ and ε.
2. Find the horizontal coordinate of the minimum point in terms of σ and ε.

Note that the values of σ and ε on the applet are not realistic. This applet is just an illustration to show the shape of the curve. If you are interested, you can do some research on realistic values and also on how scientists find the parameters of the Lennard-Jones model for specific materials.