Further differentiation

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The big picture

Earlier in your calculus studies, you learned the basics of finding the slope of a tangent and applied this process to polynomial functions. This technique is valuable in finding optimum solutions to simple problems.

In this subtopic, you will learn about some tools that will help you to find the derivatives of the types of functions you encountered in topic 2 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26012/) and topic 3 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26035/), including combinations of these. In the following sections, you will apply these tools to solving problems.

The process of finding derivatives is based on a set of rules. The derivatives of complex functions can be built up using basic derivatives. The formula booklet contains these basic derivatives and the rules for combining them; you will learn how to apply these in this subtopic.

In some sense, this process is similar to any complicated system. If you understand the building blocks and the connections, you understand the whole system. The mechanical watch in the following video is a good example.

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Video: How a Mechanical Watch Works | Explained in 5 Minutes.

More information

The title "HOW A MECHANICAL WATCH WORKS" is displayed in over a blurred image of a partially disassembled mechanical watch. The watch's intricate components, including black plates, silver and gold gears, and small ruby-colored jewels, are visible. In the background, watchmaking tools such as tweezers and a movement holder can be seen, setting the stage for a detailed exploration of the watch's inner workings.

The camera zooms in on the watch movement, focusing on the gear train. The interplay of gears is highlighted, with the words “MADE SWISS” etched on a prominent silver gear. The focus then shifts to the balance wheel, and the text “EXPLAINED IN 5 MINUTES” appears on the right side of the frame. The screen transitions to black, introducing the first chapter with the text: “CHAPTER 1: THE POWERSOURCE.”

A quartz watch movement is shown, secured in a holder. A gloved hand uses tweezers to hold a small silver button cell battery, labeled "BATTERY." Below the movement, text explains that this is the power source of a quartz watch. The scene transitions to a coiled mainspring, gleaming in its relaxed state, held by tweezers. The text "MAINSPRING" identifies it as the power source of a mechanical watch. The mainspring is wound into a cylindrical mainspring barrel by gloved hands using a specialized tool. Social media handles for “/reddeadrestoration” on Facebook and “@reddeadrestoration” on Instagram appear in the upper left corner, along with a note about a Patreon page. The camera focuses on the barrel as the mainspring is inserted and secured with a flat metal disc. A transparent barrel is used to demonstrate the process, offering a clear view of the mainspring inside. The text “MAINSPRING BARREL” identifies the component.

The screen fades to black, introducing “CHAPTER 2: THE GEAR TRAIN.” Tweezers place gold-colored gears onto the mainplate, each labeled with text: “ESCAPE WHEEL,” “SECOND WHEEL,” and “THIRD WHEEL.” A black plate is secured over the gear train with screws, and the “CENTER WHEEL” is positioned and labeled. Additional plates and components are added, with small ruby jewels identified as “BEARINGS.” A toothpick gently touches the mainspring barrel, causing it and the gear train to move. Text explains that the movement of the mainspring barrel drives the gear train.

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The screen transitions to “CHAPTER 3: WINDING THE MAINSPRING.” A winding stem is inserted into the movement, and the watch is gently wound, engaging the gears. The “CROWN WHEEL” is labeled as it interacts with the winding stem. The “RATCHET WHEEL” is shown rotating during the winding process, while the “CLICK SPRING” and “CLICK LEVER” are added with precision. The movement is wound further, and text explains that winding the mainspring drives the gear train. A gold watch face with hour and minute hands is displayed, but the hands move rapidly, demonstrating uncontrolled motion. Text notes that the gears and hands are still turning without regulation.

The screen fades to black, introducing “CHAPTER 4: THE ESCAPEMENT.” A small, intricate part with ruby jewels, the pallet fork, is held by tweezers and labeled. The pallet fork is carefully inserted into the movement and secured. A balance cock is added and screwed into place. Text explains that the mainspring is no longer driving the gear train, allowing it to save energy. The pallet fork is manually moved with tweezers, demonstrating its interaction with the escape wheel. The watch face is shown again, with the second hand advancing in distinct steps, illustrating the controlled release of energy.

The screen transitions to “CHAPTER 5: THE BALANCE.” The balance wheel and hairspring assembly are shown spinning. Key components are labeled: “IMPACT JEWEL,” “BALANCE WHEEL,” and “HAIRSPRING.” The balance wheel assembly is inserted into the movement and secured. A close-up reveals the impact jewel resting between the forks of the pallet fork. Slow-motion footage highlights the interaction between the escape wheel, pallet fork, and balance wheel. The balance wheel oscillates, driving the gear train and regulating the watch's motion. The gold watch face is shown again, with the second hand ticking smoothly. Text notes that the balance wheel rotates at 21,600 vibrations per hour, or six steps per second.

The fully assembled watch movement is displayed, and the screen fades to black before showing the movement again. The text “SUBSCRIBE” appears with an arrow pointing down, followed by a message thanking patrons and subscribers, with special mentions for Diego Macias and Jim Janson. The balance wheel continues to oscillate, and the gears turn, bringing the demonstration to a close.

Theory of Knowledge

Differentiation rules are used to find derivatives (rates of change) in mathematics. This is done so we have knowledge about a function and can thus make predictions.

The fact that maths has predictive validity is a strength of mathematics. However, not all areas of knowledge have high degrees of predictive validity. History, for example, has zero predictive validity and religious knowledge systems have very little predictive validity.

Knowledge Question: To what extent does predictive validity demonstrate quality of knowledge?

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Concept

Don't be put off by what seems to be a complicated list of rules. The best way to learn to apply them is with a lot of repetitive practice, rather than simply trying to memorise them. This will help you to see a common pattern in seemingly different examples. Look for the inner structure of a complicated looking function, and look for the relationship between the simpler functions that are used as the building blocks.

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Power rule revisited

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/power-rule-revisited-id-28326/print/) Assign

In subtopic 5.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26147/) you encountered the sum rule and the constant factor rule for differentiation. These rules will be used, without quotation, from now on. Here is a combination of these rules to refresh your memory.

Important

If $h(x) = af(x) \pm bg(x)$, then the derivative is $h'(x) = af'(x) \pm bg'(x)$.

You have also seen the formula for the derivatives of power functions.

Important

- The derivative of $f(x) = c$ is $f'(x) = 0$.
- The derivative of $f(x) = x$ is $f'(x) = 1$.
- The derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$.

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In subtopic 5.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26147/) you only used this formula for integer (positive or negative) exponents, but the formula is true for any real number exponent. This is what you will practise now.

Example 1

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Find the derivatives of $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$.

Steps	Explanation
$f(x) = \sqrt{x} = x^{\frac{1}{2}}$	You can use the formula if $f(x)$ is in the form x^n.
$f'(x) = \frac{1}{2}x^{\frac{1}{2-1}}$ $= \frac{1}{2}x^{\frac{-1}{2}}$ $= \frac{1}{2\sqrt{x}}$	The formula can be used for $n = \frac{1}{2}$.
$g(x) = \frac{1}{\sqrt{x}} = x^{\frac{-1}{2}}$	You can use the formula if $g(x)$ is in the form x^n.
$g'(x) = -\frac{1}{2}x^{-1/2-1}$ $= -\frac{1}{2}x^{-3/2}$ $= -\frac{1}{2\sqrt{x^3}}$ $= -\frac{1}{2x\sqrt{x}}$	The formula can be used for $n = -\frac{1}{2}$.

Example 2

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Student view

Find the derivative of $y = \frac{3x^2 - 7}{\sqrt[3]{x}}$.

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Steps	Explanation
$y = \frac{3x^2-7}{\sqrt[3]{x}}$ $= \frac{3x^2}{\sqrt[3]{x}} - \frac{7}{\sqrt[3]{x}}$ $= 3x^{2-\frac{1}{3}} - 7x^{-\frac{1}{3}}$ $= 3x^{\frac{5}{3}} - 7x^{-\frac{1}{3}}$	To use the formula, the expression needs to b rewritten as a combination of expressions of the form ax^n.
$\text{dyd}x = 3 \times \frac{5}{3}x^{\frac{5}{3}-1} - 7 \times -\frac{1}{3}x^{-\frac{1}{3}-1}$ $= 5x^{\frac{2}{3}} + \frac{7}{3}x^{-\frac{4}{3}}$ $= 5\sqrt[3]{x^2} + \frac{7}{3\sqrt[3]{x^4}}$	

Example 3

★★☆

Find the derivatives of the following expressions.

a) $y = 5\sqrt[3]{x^4}$

b) $y = \frac{3}{7\sqrt{x}}$

c) $y = \sqrt{x}\left(3x^2 - 9\right)$

d) $y = \frac{2x^3}{\sqrt[4]{x}} - \frac{\sqrt[4]{x}}{2x^3}$

e) $y = \frac{\left(\sqrt{x} - 3\right)^2}{\sqrt{x}}$

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	Steps	Explanation
a)	$y=5\sqrt[3]{x^4}=5x^{\frac{4}{3}}$	$\frac{\mathrm{d}y}{\mathrm{d}x}=5\times\frac{4}{3}x^{\frac{4}{3}-1}$ $=\frac{20}{3}x^{\frac{1}{3}}$ $=\frac{20\sqrt[3]{x}}{3}$
b)	$y=\frac{3}{7\sqrt{x}}=\frac{3}{7}x^{-\frac{1}{2}}$	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{7}\times-\frac{1}{2}x^{-\frac{1}{2}-1}$ $=-\frac{3}{14}x^{-\frac{3}{2}}$ $=-\frac{3}{14x\sqrt{x}}$
c)	$y=\sqrt{x}\left(3x^2-9\right)$ $=3x^{\frac{1}{2}+2}-9x^{\frac{1}{2}}$ $=3x^{\frac{5}{2}}-9x^{\frac{1}{2}}$	$\frac{\mathrm{d}y}{\mathrm{d}x}=3\times\frac{5}{2}x^{\frac{5}{2}-1}-9\times\frac{1}{2}x^{\frac{1}{2}-1}$ $=\frac{15}{2}x^{\frac{3}{2}}-\frac{9}{2}x^{-\frac{1}{2}}$ $=\frac{15x\sqrt{x}}{2}-\frac{9}{2\sqrt{x}}$
d)	$y=\frac{2x^3}{\sqrt[4]{x}}-\frac{\sqrt[4]{x}}{2x^3}$ $=2x^{3-\frac{1}{4}}-\frac{1}{2}x^{\frac{1}{4}-3}$ $=2x^{\frac{11}{4}}-\frac{1}{2}x^{-\frac{11}{4}}$	$\frac{\mathrm{d}y}{\mathrm{d}x}=2\times\frac{11}{4}x^{\frac{11}{4}-1}-\frac{1}{2}\times\frac{-11}{4}x^{\frac{-11}{4}-1}$ $=\frac{11}{2}x^{\frac{7}{4}}+\frac{11}{8}x^{\frac{-15}{4}}$ $=\frac{11x\sqrt[4]{x^3}}{2}+\frac{11}{8x^3\sqrt[4]{x^3}}$
e)	$y=\frac{\left(\sqrt{x}-3\right)^2}{\sqrt{x}}$ $=\frac{\left(x-6\sqrt{x}+9\right)}{\sqrt{x}}$ $=x^{\frac{1}{2}}-6+9x^{-\frac{1}{2}}$	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}x^{\frac{1}{2}-1}-0+9\times\frac{-1}{2}x^{-\frac{1}{2}-1}$ $=\frac{1}{2}x^{-\frac{1}{2}}-\frac{9}{2}x^{-\frac{3}{2}}$ $=\frac{1}{2\sqrt{x}}-\frac{9}{2x\sqrt{x}}$

3 section questions

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5. Calculus / 5.9 Further differentiation

Derivative of the exponential function

The next goal is to learn about derivatives of functions other than power functions. This section concentrates on the exponential function which you learned about in subtopic 2.9 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27495/).

Activity

On the applet below you can see the graph of $y = a^x$ and the derivative for different values of the base a. Move the slider to change the base. What do you notice?

Interactive 1. Derivative of Exponential Functions.

More information for interactive 1

This interactive allows users to practice finding the derivatives of functions of the form for $y = k^{ax+b}$ randomly generated parameters a, b, and k. To differentiate such functions, users will first apply the chain rule, where $u = ax + b$, and then differentiate accordingly:

On applying chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

$\frac{dy}{dx} = ak^{(ax+b)}$

For example, find the derivative of $y = (4x + 3)^2$

Here, $\frac{dy}{dx} = 2(4x + 3)4$

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$\frac{dy}{dx} = 8(4x + 3)$

Users can generate new questions by clicking on **New Question**, allowing them to practice different levels of problems based on the same concept. They can also verify their solutions by selecting the **Show Answers** option. By using this interactive, users will develop a strong grasp of the chain rule in differentiation and improve their problem-solving skills.

From the applet you may have noticed that the derivative of $y = a^x$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = ka^x$ for some value of k, which depends on the base a. The applet also shows you that for a specific value of the base the derivative is the same as the function. To find the value of this special base, open WolframAlpha (http://www.wolframalpha.com/) and type the following into the command line:

solve y'=y

This will search for functions with the property that the derivative is the same as the function itself.

In the answer WolframAlpha gives you can see that the functions with this property are constant multiples of the exponential function $y = \mathrm{e}^x$, where the base is Euler's number, e≈ 2.71828. You learned about this number in subtopic 1.5 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26006/) and subtopic 2.9 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27495/).

The formula booklet contains this information.

Important

If $f(x) = \mathrm{e}^x$, then the derivative is $f'(x) = \mathrm{e}^x$.

You can investigate a generalisation of this claim with the following applet.

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Activity

The applet below gives the derivatives of functions of the form $y = k\mathrm{e}^{ax+b}$ for randomly generated parameters a, b and k.

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Keep generating examples until you can predict the derivative without looking at the answer.

Interactive 2. Finding the Derivative of Exponential Function.

More information for interactive 2

This interactive allows users to practice differentiating functions of the form $y = c \cdot e^{(ax+b)}$ for randomly generated parameters a, b, and c. Users can click on Click here for new questions to generate different functions of this type and apply differentiation rules to find their derivatives. To verify their solutions, they can use the Show answers option.

To differentiate $y = c \cdot e^{(ax+b)}$, users will apply the chain rule:

$$\frac{d}{dx}(c \cdot e^{(ax+b)}) = cae^{(ax+b)}$$

For example, for the function $y = 7 \cdot e^{(-3x+9)}$ the derivative is:

$$\frac{dy}{dx} = 7 \cdot (-3)e^{(ax+b)} = -21e^{(ax+b)}$$

Through this interactive, users will develop a strong understanding of differentiating exponential functions and applying the chain rule effectively.

Did you notice the following relationship?

Important

If $f(x) = ke^{ax+b}$, then the derivative is $f'(x) = kae^{ax+b}$.

In the following examples you can see how this formula is used to find derivatives of certain types of functions.

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Example 1

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Find the derivative of $f(x) = \frac{2e^{3x}}{5e^4}$.

Steps	Explanation
$f(x) = \frac{2e^{3x}}{5e^4} = \frac{2}{5}e^{3x-4}$	You can apply the formula if the expression is in the form ke^{ax+b}.
$f'(x) = \frac{2}{5} \times 3e^{3x-4} = \frac{6}{5}e^{3x-4}$	

Example 2

★★☆

Find the derivative of $y = \frac{7}{6e^{2x}}$.

Steps	Explanation
$y = \frac{7}{6e^{2x}} = \frac{7}{6}e^{-2x}$	You can apply the formula if the expression is in the form ke^{ax+b}.
$\frac{dy}{dx} = \frac{7}{6} \times (-2)e^{-2x} = -\frac{7}{3}e^{-2x} = -\frac{7}{3e^{2x}}$	

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Derivative of the natural logarithm function

In this section, you will see how to differentiate functions of the form $y = k\ln(ax + b)$, where k, a and b are real parameters.

Activity

To start, take a look at the following argument.

$$\begin{aligned} y &= \ln x \\ x &= \mathrm{e}^y \\ \frac{\mathrm{d}x}{\mathrm{d}y} &= \mathrm{e}^y \\ \frac{\mathrm{d}x}{\mathrm{d}y} &= x \end{aligned}$$

- Do you agree that each line is the consequence of the line before it?
- Did you notice that in the differentiation step x is the dependent and y is the independent variable?
- Suggest a formula for $\frac{\mathrm{d}y}{\mathrm{d}x}$, the derivative of $y = \ln x$.

The activity above gives an informal justification of the formula for the derivative of $y = \ln x$, which you can find in the formula booklet.

Important

If $f(x) = \ln x$, then the derivative is $f'(x) = \frac{1}{x}$.

You can investigate a generalisation of this claim with the following applet.

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Activity

The applet below gives the derivatives of functions of the form $y = k\ln(ax + b)$ for randomly generated parameters a, b and k.

Keep generating examples until you can predict the derivative without looking at the answer.

Interactive 1. Derivatives of Natural Logarithm Functions.

More information for interactive 1

This interactive allows the user to find the derivatives of natural logarithm functions of the form $y = k\ln(ax + b)$ for randomly generated parameters a, b and k. Users can generate new differentiation problems by clicking on ‘Click for a new question’. After attempting the solution, they can click "Show Answer" to reveal the correct answer. To find the derivative users will first apply the formula $\frac{d}{dx}\ln(x) = \frac{1}{x}$

For example, find the derivative of $y = [11\ln(5x - 5)]$.

$$\Rightarrow \frac{dy}{dx} = \frac{11}{(5x-5)}\frac{d(5x-5)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{55}{5x-5}$$

The users will develop a better understanding of the concept of natural logarithm. This technique will also help them in finding derivatives of complicated functions. Eventually, helping users to understand how it has applications in various fields like physics, economics and finance.

Did you notice the following relationship?

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✓ **Important**

If $f(x) = k\ln(ax+b)$, then the derivative is $f'(x) = \dfrac{ka}{ax+b}$.

In the following examples you can see how this formula is used to find derivatives of certain types of functions.

Example 1

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Find the derivative of $f(x) = 2\ln(3x-4)^5$.

Steps	Explanation
$f(x) = 2\ln(3x-4)^5$ $= 2\times 5\ln(3x-4)$ $= 10\ln(3x-4)$	You can apply the formula if the expression is in the form $f(x) = k\ln(ax+b)$.
$f'(x) = \dfrac{10\times 3}{3x-4} = \dfrac{30}{3x-4}$	

Example 2

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Find the derivative of $y = \dfrac{\ln\sqrt{7-12x}}{3}$.

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Steps	Explanation
$y = \frac{\ln \sqrt{7-12x}}{3} = \frac{1}{3}\ln(7-12x)^{\frac{1}{2}} = \frac{1}{3} \times \frac{1}{2}\ln(7-12x) = \frac{1}{6}\ln(7-12x)$	You can apply the formula if the expression is in the form $y = k\ln(ax+b)$.
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{1}{6} \times (-12)}{7-12x} = \frac{2}{12x-7}$	

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5. Calculus / 5.9 Further differentiation

Derivative of trigonometric functions

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/derivative-of-trigonometric-functions-id-28206/print/) Assign

The next goal is to find the formulae for the derivatives of $y = \sin x$ and $y = \cos x$.

Activity

In the applet below move the red point on the graph. The applet shows the tangent line at the given point and also shows a trace of the gradient function.

- Can you formulate a conjecture for the derivatives of $y = \sin x$ and $y = \cos x$?
- Does it matter whether the angles are measured in degrees or radians?

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Interactive 1. Exploring the Derivatives of Sine and Cosine Functions.

More information for interactive 1

This interactive allows the users to find the formula for the derivative of trigonometric functions $y = \sin x$ and $y = \cos x$. Users can drag the red point along the curve to observe how the slope of the tangent line changes at different points. The slope value is displayed in real time. Users can also choose whether the angle is measured in degrees or radians using the appropriate checkbox. The “sine” and “cosine” buttons allow users to switch between viewing the sine and cosine curves.

The interactive also shows a trace of the gradient function. The users will gain the knowledge that when x is measured in radians:

the derivative of $f(x) = \sin x$ is $f'(x) = \cos x$

the derivative of $f(x) = \cos x$ is $f'(x) = -\sin x$

For instance, if $f(x) = \cos x$ then $f'(x) = -\sin x$ and gradient $m = f'(x)$.

At $x = 0$, the gradient will be $m = f'(0) = -\sin(0) = 0$.

You may have noticed the following information.

Important

When x is measured in radians:

- the derivative of $f(x) = \sin x$ is $f'(x) = \cos x$
- the derivative of $f(x) = \cos x$ is $f'(x) = -\sin x$.

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You can investigate a generalisation of this claim with the following applet.

Activity

The applet below gives the derivatives of functions of the form $y = k\sin(ax + b)$ and $y = k\cos(ax + b)$ for randomly generated parameters a, b and k.

Keep generating examples until you can predict the derivative without looking at the answer.

Interactive 2. Derivative of Inverse Trigonometric Functions.

More information for interactive 2

This interactive allows the users to apply the formula for the derivative of trigonometric functions $y = \sin x$ and $y = \cos x$. The interactive helps in understanding the derivatives of functions of the form $y = k\sin(ax + b)$ and $y = k\cos(ax + b)$ for randomly generated parameters a, b and k. Users can click the " Click here of a new question" to solve different problems, allowing them to practice and analyze the results. By clicking on the 'Show answers' users can verify their result as well. The derivative of $f(x) = k\sin(ax + b)$ *is* $f'(x) = k\,a\cos(ax + b)$

the derivative of $f(x) = k\cos(ax + b)$ *is* $f'(x) = -k\,a\sin(ax + b)$

For example, if $f(x) = 4\sin(2x + 3)$

Therefore, $f'(x) = 8\cos(2x + 3)$

The users will deepen their knowledge of the concept through practice. The concept is crucial for understanding the rate of change (derivative) of these functions and how it is useful in the field of physics, engineering and modelling periodic formulas.

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Did you notice the following relationship?

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Important

When x is measured in radians:

- the derivative of $f(x) = k\sin(ax+b)$ is $f'(x) = ka\cos(ax+b)$
- the derivative of $f(x) = k\cos(ax+b)$ is $f'(x) = -ka\sin(ax+b)$

In the following examples you can see how this formula is used to find derivatives of certain types of functions.

Example 1

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Find the derivative of $f(x) = 5\sin(6x+4)$.

Steps	Explanation
$\begin{aligned} f(x) &= 5\sin(6x+4) \\ f'(x) &= 5\times 6\cos(6x+4) \\ &= 30\cos(6x+4) \end{aligned}$	You can use $f'(x) = ka\cos(ax+b)$, where $k=5$, $a=$ and $b=4$.

To differentiate trigonometric functions in the next examples other tools are needed.

- You will see in later sections how to use the product and chain rules to find the derivatives.
- The solutions you see in this section use the following double angle identities. These are not part of the syllabus, so you do not need to remember them for your exam. Nevertheless, it is interesting to see how these identities can be used to find the derivatives.

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \end{aligned}$$

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Example 2

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Find the derivative of $y = \sin^2 x$.

Steps	Explanation
$y = \sin^2 x = \frac{1}{2} \times 2\sin^2 x$ $= \frac{1}{2}(1 - \cos 2x)$ $= \frac{1}{2} - \frac{1}{2}\cos 2x$	You can use the double angle formula $\cos 2x = 1 - 2\sin^2 x$ to express $\sin^2 x$ in terms of $\cos 2x$.
$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 - \frac{1}{2} \times 2 \times (-\sin 2x)$ $= \sin 2x$	You can use $f'(x) = -ka\sin(ax + b)$, where $k =$ $a = 2$ and $b = 0$.

Example 3

★★★

Find the derivative of $y = \sin^3 x \cos x$.

Steps	Explanation
$y = \sin^3 x \cos x$ $= \sin^2 x(\sin x \cos x)$ $= \frac{1}{4} \times 2\sin^2 x(2\sin x \cos x)$ $= \frac{1}{4}(1 - \cos 2x)\sin 2x$ $= \frac{1}{4}\sin 2x - \frac{1}{8} \times 2\cos 2x \sin 2x$ $= \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x$	You can start by writing the product differently and using the double angle formulae to find an equivalent form as a sum of trigonometric expressions.

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Steps	Explanation
$\frac{dy}{dx} = \frac{1}{4} \times 2\cos 2x - \frac{1}{8} \times 4\cos 4x$ $= \frac{\cos 2x - \cos 4x}{2}$	

3 section questions

5. Calculus / 5.9 Further differentiation

Chain rule

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/chain-rule-id-28327/print/) Assign

In this section, you will learn about differentiating composite functions.

Activity

The applet below gives the derivatives of functions of the form $y = g(f(x))$.

- Identify f and g in the examples.
- State the derivatives, f' and g'.
- Can you see how $\frac{dy}{dx}$ is formed using f, g, f' and g'?

Keep generating examples until you can predict the derivative without looking at the answer.

Interactive 1. Application of the Chain Rule in Differentiation.

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More information for interactive 1

This interactive allows users to explore the differentiation of simple and composite functions, focusing on the application of the chain rule in finding derivatives of functions of the form $y = g(f(x))$. Users can click on **Click here for a new question** to generate different problems, enabling them to practice and analyze their results. They can also verify their answers by selecting the **Show Answers** option, reinforcing their understanding of the differentiation process.

The chain rule states that $(f(x) = g(h(x))$, then the derivative is given by $f'(x) = g'(h(x))f'(x)$.

For example, if For example, $y = \sin(5x)$ Applying the chain rule helps users understand how the derivative is computed step by step.

$\Rightarrow y' = \cos(5x)\frac{d(5x)}{dx}$

⇒ \(y'=5\cos(5x)\)

Through this interactive, users will strengthen their knowledge of the chain rule and its application in differentiating composite functions. They will develop a deeper understanding of how the rate of change of one function influences the rate of change of another, enhancing their problem-solving skills in calculus.

You may have noticed the following rule for the derivative of composite functions. This is called the chain rule.

Important

- If $h(x) = g(f(x))$, then $h'(x) = g'(f(x)) \times f'(x)$.

The formula booklet gives this rule in the following form:

- If $y = g(u)$, where $u = f(x)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$.

Example 1

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- Identify the two functions that are used to form the composite functions in the first column of the table below.
- Find the derivatives missing from the table.

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$h(x) = g(f(x))$	$g(u)$	$u = f(x)$	$g'(u)$	$f'(x)$	$h'(x)$
$(x^2 - 2x)^5$					
$\dfrac{1}{x^3 - 4x^2}$					
e^{-x^2}					
$\ln(x^4 + 2x - 3)$					
$\sin x^2$					
$\sin^2 x$					
$\cos(2x^3 - 3x)$					
$e^{\cos x}$					
$\cos e^x$					
$\ln \sin x$					
$\sin \ln x$					

- For $h(x) = (x^2 - 2x)^5$
  - $g(u) = u^5$
  - $u = f(x) = x^2 - 2x$
  - $g'(u) = 5u^4$
  - $f'(x) = 2x - 2$
  - $h'(x) = g'(f(x))f'(x) = 5(x^2 - 2x)^4(2x - 2)$
- For $h(x) = \dfrac{1}{x^3 - 4x^2}$
  - $g(u) = \dfrac{1}{u} = u^{-1}$
  - $u = f(x) = x^3 - 4x^2$
  - $g'(u) = -u^{-2} = \dfrac{-1}{u^2}$
  - $f'(x) = 3x^2 - 8x$

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- $h'(x) = g'(f(x))f'(x) = \dfrac{-(3x^2 - 8x)}{(x^3 - 4x^2)^2}$

- For $h(x) = \mathrm{e}^{-x^2}$
  - $g(u) = \mathrm{e}^u$
  - $u = f(x) = -x^2$
  - $g'(u) = \mathrm{e}^u$
  - $f'(x) = -2x$
  - $h'(x) = g'(f(x))f'(x) = -2x\mathrm{e}^{-x^2}$
- For $h(x) = \ln(x^4 + 2x - 3)$
  - $g(u) = \ln u$
  - $u = f(x) = x^4 + 2x - 3$
  - $g'(u) = \dfrac{1}{u}$
  - $f'(x) = 4x^3 + 2$
  - $h'(x) = g'(f(x))f'(x) = \dfrac{4x^3 + 2}{x^4 + 2x - 3}$
- For $h(x) = \sin x^2$
  - $g(u) = \sin u$
  - $u = f(x) = x^2$
  - $g'(u) = \cos u$
  - $f'(x) = 2x$
  - $h'(x) = g'(f(x))f'(x) = 2x \cos x^2$
- For $h(x) = \sin^2 x$
  - $g(u) = u^2$
  - $u = f(x) = \sin x$
  - $g'(u) = 2u$
  - $f'(x) = \cos x$
  - $h'(x) = g'(f(x))f'(x) = 2 \sin x \cos x$
- For $h(x) = \cos(2x^3 - 3x)$
  - $g(u) = \cos u$
  - $u = f(x) = 2x^3 - 3x$
  - $g'(u) = -\sin u$
  - $f'(x) = 6x^2 - 3$
  - $h'(x) = g'(f(x))f'(x) = -(6x^2 - 3) \sin(2x^3 - 3x)$
- For $h(x) = \mathrm{e}^{\cos x}$
  - $g(u) = \mathrm{e}^u$
  - $u = f(x) = \cos x$
  - $g'(u) = \mathrm{e}^u$
  - $f'(x) = -\sin x$

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- $h'(x) = g'(f(x))f'(x) = -\sin x \mathrm{e}^{\cos x}$
- For $h(x) = \cos \mathrm{e}^x$
  - $g(u) = \cos u$
  - $u = f(x) = \mathrm{e}^x$
  - $g'(u) = -\sin u$
  - $f'(x) = \mathrm{e}^x$
  - $h'(x) = g'(f(x))f'(x) = -\mathrm{e}^x \sin \mathrm{e}^x$
- For $h(x) = \ln \sin x$
  - $g(u) = \ln u$
  - $u = f(x) = \sin x$
  - $g'(u) = \dfrac{1}{u}$
  - $f'(x) = \cos x$
  - $h'(x) = g'(f(x))f'(x) = \dfrac{\cos x}{\sin x}$
- For $h(x) = \sin \ln x$
  - $g(u) = \sin u$
  - $u = f(x) = \ln x$
  - $g'(u) = \cos u$
  - $f'(x) = \dfrac{1}{x}$
  - $h'(x) = g'(f(x))f'(x) = \dfrac{\cos \ln x}{x}$

Example 2

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Find the derivative of $f(x) = 2\ln(3x-4)^5$.

You encountered this function in section 5.6.3. This solution shows you how to find the derivative using the chain rule, rather than the laws of logarithms.

Steps	Explanation
$f(x) = g(u(x))$, where $g(u) = 2\ln u$ and $u(x) = (3x-4)^5$	You can write $f(x)$ as a composition.

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Steps	Explanation
$g'(u) = \dfrac{2}{u}$ $u'(x) = 3 \times 5(3x-4)^4 = 15(3x-4)^4$	To use the chain rule, you need the derivatives of g and u.
$f'(x) = \dfrac{2}{(3x-4)^5} \times 15(3x-4)^4$ $= \dfrac{30}{3x-4}$	According to the chain rule, $f'(x) = g'(u(x)) \times u'(x)$.

In the examples above you found derivatives of compositions of two functions. The function in the next example is built as a composition of more than two functions.

Example 3

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Consider the functions defined by $f(x) = \mathrm{e}^x$, $g(x) = \sqrt{1+x^2}$ and $h(x) = \sin x$.

- Find $g'(x)$.
- Find $l(x) = (g \circ h)(x)$ and $l'(x)$.
- Find $m(x) = (f \circ g \circ h)(x)$ and $m'(x)$.

Recall, that $p \circ q$ is the composition of the functions p and q, so $(p \circ q)(x) = p(q(x))$.

Steps	Explanation
$g(x) = s(u(x))$, where $s(u) = \sqrt{u} = u^{\frac{1}{2}}$, and $u(x) = 1 + x^2$	You can write $g(x)$ as a composition.
$s'(u) = \dfrac{1}{2}u^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{u}}$ $u'(x) = 2x$	To use the chain rule, you need the derivatives of s and u.
$g'(x) = \dfrac{1}{2\sqrt{1+x^2}} \times 2x = \dfrac{x}{\sqrt{1+x^2}}$	According to the chain rule, $g'(x) = s'(u(x)) \times u'(x)$.

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Steps	Explanation
$l(x) = g(h(x)) = g(\sin x)$ $= \sqrt{1+\sin^2 x}$	
$l'(x) = \frac{\sin x}{\sqrt{1+\sin^2 x}} \times \cos x$ $= \frac{\sin x \cos x}{\sqrt{1+\sin^2 x}}$	According to the chain rule, $l'(x) = g'(h(x)) \times h'(x)$.
$m(x) = f(g(h(x))) = f(l(x))$ $= f(\sqrt{1+\sin^2 x})$ $= \mathrm{e}^{\sqrt{1+\sin^2 x}}$	
$m'(x) = \mathrm{e}^{\sqrt{1+\sin^2 x}} \times \frac{\sin x \cos x}{\sqrt{1+\sin^2 x}}$	According to the chain rule, $m'(x) = f'(l(x)) \times l'(x)$.

Making connections

It was not emphasised in the previous example, but if you put together the pieces you can notice that the derivative of $m(x) = f(g(h(x)))$ is in fact

$$m'(x) = f'(g(h(x))) \times g'(h(x)) \times h'(x).$$

This type of pattern is the reason why this rule is called the chain rule.

Can you find the similar formula for the derivative of the composition of four functions?

The following example illustrates a situation where, instead of being given a formula, you are given only partial information about a function.

Example 4

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For some function, f, you are given that $f'(10) = 7$.

Let $g(x) = f(x^2 - 15)$.

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Find $g'(5)$.

Steps	Explanation
$u'(x) = 2x$	$g(x)$ is the composition of $f(u)$ and $u(x) = x^2 - 15$. To use the chain rule, you need the derivative of u.
$g'(x) = f'(u(x)) \times u'(x) = 2xf'(x^2 - 15)$	
$g'(5) = 2 \times 5 \times f'(5^2 - 15)$ $= 10 \times f'(10)$ $= 10 \times 7 = 70$	You can get $g'(5)$ by substituting $x = 5$ and using the information given in the question.

The table below summarises information about some basic derivatives and also shows the chain rule applied to these functions.

The information in the first two columns is given in the formula booklet.

$f(x)$	$f'(x)$	$g(x) = g(ax + b)$	$g'(x)$	$h(x) = f(u(x))$	$h'(x)$
x^n	nx^{n-1}	$(ax + b)^n$	$an(ax + b)^{n-1}$	$(u(x))^n$	$n(u(x))^{n-1}u$
$\sin x$	$\cos x$	$\sin(ax + b)$	$a\cos(ax + b)$	$\sin u(x)$	$u'(x)\cos x$
$\cos x$	$-\sin x$	$\cos(ax + b)$	$-a\sin(ax + b)$	$\cos u(x)$	$-u'(x)\sin x$
e^x	e^x	e^{ax+b}	$a\mathrm{e}^{ax+b}$	$\mathrm{e}^{u(x)}$	$\mathrm{e}^{u(x)}u'(x)$
$\ln x$	$\frac{1}{x}$	$\ln(ax + b)$	$\frac{a}{ax + b}$	$\ln u(x)$	$\frac{u'(x)}{u(x)}$

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5 section questions

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Product rule

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/product-rule-id-28328/print/) Assign

In this section, you will learn about differentiating products of functions.

Activity

The applet below gives the derivatives of functions of the form $y = u(x)v(x)$.

- Identify u and v in the examples.
- State the derivatives, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$.
- Can you see how $\frac{\mathrm{d}y}{\mathrm{d}x}$ is formed using u, v, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$?

Keep generating examples until you can predict the derivative without looking at the answer.

Interactive 1. Application of the Product Rule in Differentiation.

More information for interactive 1

This interactive tool helps users practice differentiating product functions of the form $y = u(x)v(x)$. Each time users generate a new question using the “click here for a new question”, they'll see a different product of functions to work with. The tool guides users through a clear process: first identifying the two

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functions u(x) and v(x), then finding their separate derivatives $\frac{du}{dx}$ and $\frac{dv}{dx}$, and finally combining them using the product rule formula $\frac{dy}{dx} = u. \frac{dv}{dx} + v. \frac{du}{dx}$

Example: The given question is $y = \sin(x).x^3$

The Answer is - $\frac{dy}{dx} = \sin(x) \cdot 3x^2 + x^3 \cdot \cos(x)$

Users can work with various function combinations, from simpler products like polynomials and trigonometric functions to more complex pairings. The "Show answer" feature lets users check their work against the correct solution, helping them identify any mistakes and learn from them. By repeatedly solving different examples, users will naturally recognize the pattern of the product rule and gain confidence in applying it.

Through consistent practice with these interactive exercises, users will develop the ability to quickly and accurately differentiate any product functions they encounter in their calculus studies.

You may have noticed the following rule for the derivative of the product of two functions. This is called the product rule.

Important

- If $h(x) = u(x)v(x)$, then $h'(x) = u(x)v'(x) + v(x)u'(x)$.

The formula booklet gives this rule in the following form:

- If $y = uv$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$.

Example 1

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- The expressions in the first column of the table below are products of two simpler expressions. Identify these two expressions.
- Find the derivatives missing from the table.

$h(x) = u(x)v(x)$	$u(x)$	$v(x)$	$u'(x)$	$v'(x)$	$h'(x)$
$(x^3 + 7)\mathrm{e}^{-x}$					

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$h(x)=u(x)v(x)$	$u(x)$	$v(x)$	$u'(x)$	$v'(x)$	$h'(x)$
$(x^2+1)^3\ln(2x)$					
$e^{x^2}\cos x$					
$\sin(5-2x)\ln x$					

- For $h(x)=(x^3+7)e^{-x}$
  - $u(x)=x^3+7$
  - $v(x)=e^{-x}$
  - $u'(x)=3x^2$
  - $v'(x)=-e^{-x}$
  - $$\begin{aligned}h'(x)&=u(x)v'(x)+v(x)u'(x)\\&=(x^3+7)\times(-e^{-x})+e^{-x}\times 3x^2\\&=-e^{-x}(x^3-3x^2+7)\end{aligned}$$
- For $h(x)=(x^2+1)^3\ln(2x)$
  - $u(x)=(x^2+1)^3$
  - $v(x)=\ln(2x)$
  - $u'(x)=3(x^2+1)^2\times 2x=6x(x^2+1)^2$
  - $v'(x)=\frac{1}{2x}\times 2=\frac{1}{x}$
  - $$\begin{aligned}h'(x)&=u(x)v'(x)+v(x)u'(x)\\&=(x^2+1)^3\times\frac{1}{x}+\ln(2x)\times 6x(x^2+1)^2\\&=(x^2+1)^2\left(\frac{x^2+1}{x}+6x\ln(2x)\right)\end{aligned}$$
- For $h(x)=e^{x^2}\cos x$
  - $u(x)=e^{x^2}$
  - $v(x)=\cos x$
  - $u'(x)=2xe^{x^2}$
  - $v'(x)=-\sin x$
  - $$\begin{aligned}h'(x)&=u(x)v'(x)+v(x)u'(x)\\&=e^{x^2}\times(-\sin x)+\cos x\times 2xe^{x^2}\\&=e^{x^2}(2x\cos x-\sin x)\end{aligned}$$
- For $h(x)=\sin(5-2x)\ln x$
  - $u(x)=\sin(5-2x)$
  - $v(x)=\ln x$
  - $u'(x)=\cos(5-2x)\times(-2)=-2\cos(5-2x)$

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- $v'(x) = \dfrac{1}{x}$
- $$\begin{aligned} h'(x) &= u(x)v'(x) + v(x)u'(x) \\ &= \sin(5-2x) \times \frac{1}{x} + \ln x \times (-2\cos(5-2x)) \\ &= \frac{\sin(5-2x)}{x} - 2\ln x \cos(5-2x) \end{aligned}$$

Example 2

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Find the derivative of $y = \sin^3 x \cos x$.

Steps	Explanation
$y = uv$ for $u = \sin^3 x$ and $v = \cos x$	You can write the expression as a product.
$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 3\sin^2 x \cos x$ $\dfrac{\mathrm{d}v}{\mathrm{d}x} = -\sin x$	To use the product rule, you need the derivatives of u and v.
$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= \sin^3 x \times (-\sin x) + \cos x \times 3\sin^2 x \cos x \\ &= 3\sin^2 x \cos^2 x - \sin^4 x \end{aligned}$	

Note that you have already seen the derivative of the function from **Example 2** in the previous section (/study/app/math-ai-hl/sid-132-cid-761618/book/chain-rule-id-28327/). There, the answer for the derivative was $\dfrac{\cos 2x - \cos 4x}{2}$. Can you prove (without using differentiation) that these two expressions are equivalent?

In the examples above, you found derivatives of products of two functions. The function in the next example is built as a product of more than two functions.

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Example 3

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Find the derivative of $f(x) = \sqrt{x}\mathrm{e}^x \sin x$.

Steps	Explanation
$f(x) = u(x)v(x)$ with $u(x) = \sqrt{x}$ and $v(x) = \mathrm{e}^x \sin x$	You can write the expression as a product.
$v(x) = g(x)h(x)$ with $g(x) = \mathrm{e}^x$ and $h(x) = \sin x$	To find the derivative of v, it can also be written as a product.
$g'(x) = \mathrm{e}^x$ $h'(x) = \cos x$	To find v' using the product rule, you need the derivatives of g and h.
$\begin{aligned} v'(x) &= g(x)h'(x) + h(x)g'(x) \\ &= \mathrm{e}^x \times \cos x + \sin x \times \mathrm{e}^x \\ &= \mathrm{e}^x(\sin x + \cos x) \end{aligned}$	
$u'(x) = \dfrac{1}{2\sqrt{x}}$	To find f' using the product rule, in addition to v', you also need u
$\begin{aligned} f'(x) &= u(x)v'(x) + v(x)u'(x) \\ &= \sqrt{x} \times \mathrm{e}^x(\sin x + \cos x) + \mathrm{e}^x \sin x \times \frac{1}{2\sqrt{x}} \\ &= \frac{\mathrm{e}^x(2x \sin x + 2x \cos x + \sin x)}{2\sqrt{x}} \end{aligned}$	

Making connections

It was not emphasised in the previous example, but if you put together the pieces you may notice that the derivative of $f = ugh$ is

$$f' = u'gh + ug'h + ugh'.$$

Do you see the pattern?

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Can you find the similar formula for the derivative of the product of four functions?

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The following example illustrates a situation where, instead of being given a formula, you are given only partial information about a function.

Example 4

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For some function, f, you are given that $f(5) = -3$ and $f'(5) = 2$.

Let $g(x) = (x^2 - 15)f(x)$.

Find $g'(5)$.

Steps	Explanation
$u'(x) = 2x$	$g(x)$ is the product of $f(x)$ and $u(x) = x^2 - 15$. To use the product rule, you need the derivative of u.
$g'(x) = u(x)f'(x) + f(x)u'(x)$ $= (x^2 - 15)f'(x) + 2xf(x)$	
$g'(5) = (5^2 - 15) \times f'(5) + 2 \times 5 \times f(5)$ $= 10 \times 2 + 10 \times (-3) = -10$	You can get $g'(5)$ by substituting $x = 5$ and using the information given in the question.

3 section questions

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Quotient rule

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/quotient-rule-id-28329/print/) Assign

In this section, you will learn about differentiating quotients of functions.

Activity

The applet below gives the derivatives of functions of the form $y = \frac{u(x)}{v(x)}$.

- Identify u and v in the examples.
- State the derivatives, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$.
- Can you see how $\frac{\mathrm{d}y}{\mathrm{d}x}$ is formed using u, v, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$?

Keep generating examples until you can predict the derivative without looking at the answer.

Interactive 1. Application of the Quotient Rule in Differentiation.

More information for interactive 1

This interactive helps the user master derivatives of functions in the form $y = \frac{u(x)}{v(x)}$.

Each problem presents different differentiable functions for $u(x)$ and $v(x)$ and the user can generate new questions using the button “Click here for a new question” challenging the user to: identify components, compute $\frac{du}{dx}$ and $\frac{dv}{dx}$, and predict $\frac{dy}{dx}$. Through repeated practice, user can notice a consistent pattern behind the quotient rule:

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$\frac{dy}{dx} = (\frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2})$

Users can check the answer by clicking the “Show the answer” checkbox.

Example: The given question is $y = \frac{\sin(x)}{x^3}$.

The Answer is - $\frac{dy}{dx} = \frac{x^3 \cos(x) - \sin(x).3x^2}{(x^3)^2}$

The tool generates diverse examples - from simple polynomials to trigonometric and exponential functions - helping you recognize this pattern across various cases. Users are able to develop intuition for how numerator and denominator derivatives interact, moving beyond memorization to genuine understanding. Immediate feedback lets you verify predictions and refine your approach.

You may have noticed the following rule for the derivative of the quotient of two functions. This is called the quotient rule.

✓ **Important**

- If $h(x) = \frac{u(x)}{v(x)}$, then $h'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$.

The formula booklet gives this rule in the following form:

- If $y = \frac{u}{v}$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$.

Example 1

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- The expressions in the first column of the table below are quotients of two simpler expressions. Identify these two expressions.
- Find the derivatives missing from the table.

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$h(x) = \dfrac{u(x)}{v(x)}$	$u(x)$	$v(x)$	$u'(x)$	$v'(x)$	$h'(x)$
$\dfrac{x^3+7}{\mathrm{e}^{-x}}$					
$\dfrac{(x^2+1)^3}{\ln(2x)}$					
$\dfrac{\mathrm{e}^{x^2}}{\cos x}$					
$\dfrac{\sin(5-2x)}{\ln x}$					

- For $h(x) = \dfrac{x^3+7}{\mathrm{e}^{-x}}$
  - $u(x) = x^3 + 7$
  - $v(x) = \mathrm{e}^{-x}$
  - $u'(x) = 3x^2$
  - $v'(x) = -\mathrm{e}^{-x}$
  - $$\begin{aligned} h'(x) &= \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2} \\ &= \frac{\mathrm{e}^{-x} \times 3x^2 - (x^3+7) \times (-\mathrm{e}^{-x})}{(\mathrm{e}^{-x})^2} \\ &= \frac{\mathrm{e}^{-x}(x^3+3x^2+7)}{(\mathrm{e}^{-x})^2} \\ &= \mathrm{e}^{x}(x^3+3x^2+7) \end{aligned}$$
- For $h(x) = \dfrac{(x^2+1)^3}{\ln(2x)}$
  - $u(x) = (x^2+1)^3$
  - $v(x) = \ln(2x)$
  - $u'(x) = 3(x^2+1)^2 \times 2x = 6x(x^2+1)^2$
  - $v'(x) = \dfrac{1}{2x} \times 2 = \dfrac{1}{x}$

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$$h'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$$
- $$= \frac{\ln(2x) \times 6x(x^2+1)^2 - (x^2+1)^3 \times \frac{1}{x}}{(\ln(2x))^2}$$
$$= \frac{(x^2+1)^2 \left(6x\ln(2x) - \frac{x^2+1}{x}\right)}{(\ln(2x))^2}$$

- For $h(x) = \frac{e^{x^2}}{\cos x}$
  - $u(x) = e^{x^2}$
  - $v(x) = \cos x$
  - $u'(x) = 2xe^{x^2}$
  - $v'(x) = -\sin x$

  $$h'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$$
  - $$= \frac{\cos x \times 2xe^{x^2} - e^{x^2} \times (-\sin x)}{\cos^2 x}$$
  $$= \frac{e^{x^2}(2x + \tan x)}{\cos x}$$
- For $h(x) = \frac{\sin(5-2x)}{\ln x}$
  - $u(x) = \sin(5-2x)$
  - $v(x) = \ln x$
  - $u'(x) = \cos(5-2x) \times (-2) = -2\cos(5-2x)$
  - $v'(x) = \frac{1}{x}$

  $$h'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$$
  - $$= \frac{\ln x \times (-2\cos(5-2x)) - \sin(5-2x) \times \frac{1}{x}}{(\ln x)^2}$$
  $$= -\frac{2x\ln x\cos(5-2x) + \sin(5-2x)}{x(\ln x)^2}$$

Example 2

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Express the derivative of $g = \frac{1}{f}$ in terms of f and f'.

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Method 1 (using the quotient rule)

Steps	Explanation
$g = \frac{u}{v}$ with $u = 1$ and $v = f$.	You can think of g as a quotient.
$u' = 0$ $v' = f'$	To find the derivative using the quotient rule, you need the derivatives of u and v.
$g' = \frac{vu' - uv'}{v^2}$ $= \frac{f \times 0 - 1 \times f'}{f^2}$ $= -\frac{f'}{f^2}$	

Method 2 (using the chain rule)

Steps	Explanation
For $h(u) = \frac{1}{u} = u^{-1}$, $g(x) = (h \circ f)(x) = h(f(x))$	You can think of g as a composition.
$h'(u) = -u^{-2} = -\frac{1}{u^2}$	To find the derivative using the chain rule, you need the derivative of h.
$g'(x) = h'(f(x)) \times f'(x)$ $= -\frac{1}{(f(x))^2} \times f'(x)$ $= -\frac{f'(x)}{(f(x))^2}$	

The following example illustrates a situation where, instead of being given a formula, you are given only partial information about a function.

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Example 3

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For some function, f, you are given that $f(5) = -3$ and $f'(x) = 2$.

let $g(x) = \dfrac{x^2 - 15}{f(x)}$ and $h(x) = \dfrac{f(x)}{x^2 - 15}$.

Find $g'(5)$ and $h'(5)$.

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Steps	Explanation
$u'(x) = 2x$	Both $g(x)$ and $h(x)$ are quotients of $f(x)$ an $u(x) = x^2 - 15$. To use the quotient rule, yo need the derivative of u.
$g'(x) = \frac{f(x)u'(x) - u(x)f'(x)}{(f(x))^2} = \frac{2xf(x) - (x^2 - 15)f'(x)}{(f(x))^2}$	
$g'(5) = \frac{2 \times 5 \times f(5) - (5^2 - 15) \times f'(5)}{(f(5))^2} = \frac{2 \times 5 \times (-3) - (5^2 - 15) \times 2}{(-3)^2} = -\frac{50}{9}$	You can get $g'(5)$ by substituting $x = 5$ and using the information given in the question.
$h'(x) = \frac{u(x)f'(x) - f(x)u'(x)}{(u(x))^2} = \frac{(x^2 - 15)f'(x) - 2xf(x)}{(x^2 - 15)^2}$	
$h'(5) = \frac{(5^2 - 15) \times f'(5) - 2 \times 5 \times f(5)}{(5^2 - 15)^2} = \frac{(5^2 - 15) \times 2 - 2 \times 5 \times (-3)}{(5^2 - 15)^2} = \frac{20 + 30}{100} = \frac{1}{2}$	You can get $h'(5)$ by substituting $x = 5$ and using the information given in the question.

3 section questions

5. Calculus / 5.9 Further differentiation

Related rates of change

Student view

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Consider filling a glass with water from a slowly dripping tap.

Over time, the rate at which the water enters the glass remains constant. Will the height of the water rise at a constant rate?

To answer this question you need to find a connection between the volume of the water in the cup and the height of the water. Depending of the shape of the cup this can be expressed as $V = f(h)$ for some function, f.

Since the rate of the flow of water is given, you have information about $\frac{dV}{dt}$, the rate of change of the volume with respect to time. You are asked to figure out how the height of the water is changing, so the question asks about $\frac{dh}{dt}$.

You can find this relationship by differentiating the relationship between V and h with respect to time. The key point is that both V and h are changing, so they are both functions of time, $V(t)$ and $h(t)$. To simplify the expressions, the t is usually not added, but this dependence on time is important to remember. You can use the chain rule to find the derivative.

$$V = f(h)$$
$$\frac{dV}{dt} = \frac{df}{dh}(h) \times \frac{dh}{dt}$$

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Notice that in this relationship you know $\frac{dV}{dt}$ and $f(h)$, so you can solve this equation for $\frac{dh}{dt}$.

Problems about related rates of change involve you using known values and rates to find the rate at which some other variable is changing.

You can use the following method.

✓ **Important**

Strategy for solving related rates of change problems.

1. **Understand the problem.** In particular, identify:
   a) What rate of change do you *want*?
   b) At what *time* do you want that rate of change?
   c) What rate of change have you been *given*?
2. **Develop a model.** Often, drawing or studying a diagram is the best way to start.
3. **Write an equation relating what you want and what you know.** This is often a geometric fact, and you may have extra variables that you need to eliminate.
4. **Differentiate both sides with respect to time.** All the basic rules still apply, especially the chain rule.
5. **Substitute any known values.**
6. **Answer the question.** Don't forget to include units.

Example 1

★★☆

An open cone of height $10\,\text{cm}$ and base of radius $5\,\text{cm}$ is being filled with water at a rate of $1.5\,\text{cm}^3\,\text{s}^{-1}$. When the height of the water is $5\,\text{cm}$, what is the rate at which the water level is rising?

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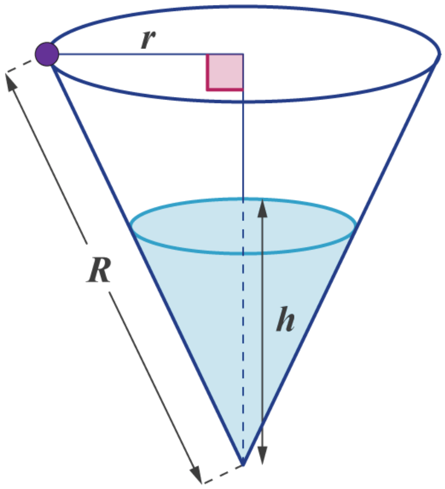

More information

The image shows a diagram of an open cone, filled partially with water. The cone is oriented upright with the circular base at the top and tapering downwards. The labels provided in the image denote specific measurements: "r" represents the radius of the base, "R" is the slant height, and "h" is the height of the water in the cone. There is a red square at the top indicating a specific area, possibly the cross-section relevant to the radius. The water within the cone is depicted using a blue shade. Dotted lines indicate the measurements of the height and the possible radius within the cone. The diagram helps visualize a geometric scenario possibly involving calculations related to volume and rate of change.

[Generated by AI]

For this problem, you would like to find the rate at which the height is increasing, $\frac{dh}{dt}$, when the height of the water is 5 cm. You know that the cone is being filled at a rate of $1.5\text{ cm}^3\text{ s}^{-1}$, so $\frac{dV}{dt} = 1.5$. You also know that $h = 5$ at the point in time you are interested in. Finally, you know the dimensions of the cone; specifically, the height is 10 cm and the radius is 5 cm. All of this is shown in the diagram.

Since you are working with volume and height, you need a geometric formula that includes both of these, i.e. the formula for the volume of a cone. Then you need to replace r using the ratio given for the cone.

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Steps	Explanation
$V = \frac{1}{3}\pi r^2 h$	Volume of a cone
$\frac{r}{h} = \frac{5}{10}$ $r = \frac{5h}{10} = \frac{h}{2}$ $V = \frac{1}{12}\pi h^3$	From the dimensions of the cone
$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$	Differentiate both sides
$1.5 = \frac{\pi}{4}5^2\frac{dh}{dt}$	Substitute the known values: $\frac{dV}{dt} = 1.5$ and $h = 5$
$\frac{dh}{dt} = \frac{6}{25\pi} \approx 0.0764$	Rearrange and solve for $\frac{dh}{dt}$.

Exam tip

such practical applications, it is always worth considering whether your result makes sense.

Thus, for the example of a cone being filled at a constant rate, and considering the rate of change of the water level, it is clear that the water level changes at an ever-slower rate as it increases because the surface area increases with increasing water level. You can see that the equation satisfies this because $\frac{dh}{dt}$ is inversely proportional to the square of the height. Therefore, as h increases, $\frac{dh}{dt}$ decreases.

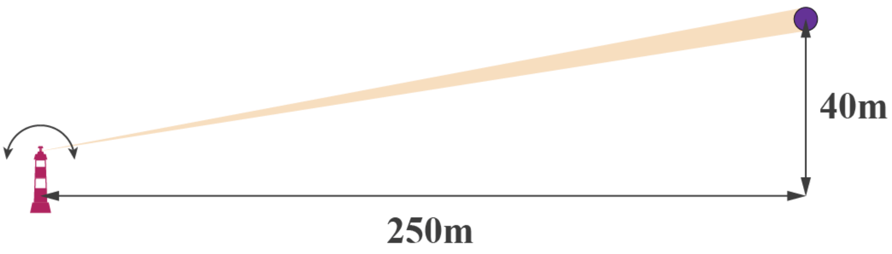
Example 2

★★★

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

A lighthouse is located $250\,\text{m}$ off the coast. Its light rotates at a rate of 2 revolutions per minute. How fast does the light beam pass a point on the coastline that is $40\,\text{m}$ from the point directly opposite the lighthouse?



More information

The diagram illustrates a lighthouse positioned 250 meters off the coast. A light beam rotates around the lighthouse. The diagram shows lines and measurements indicating various distances. The lighthouse projects a light beam that sweeps across a point on the coastline 40 meters from a point directly opposite the lighthouse. There is a horizontal line measuring 250 meters from the lighthouse to the coast, and a vertical line measuring 40 meters from that point on the coast to the position where the light beam reaches.

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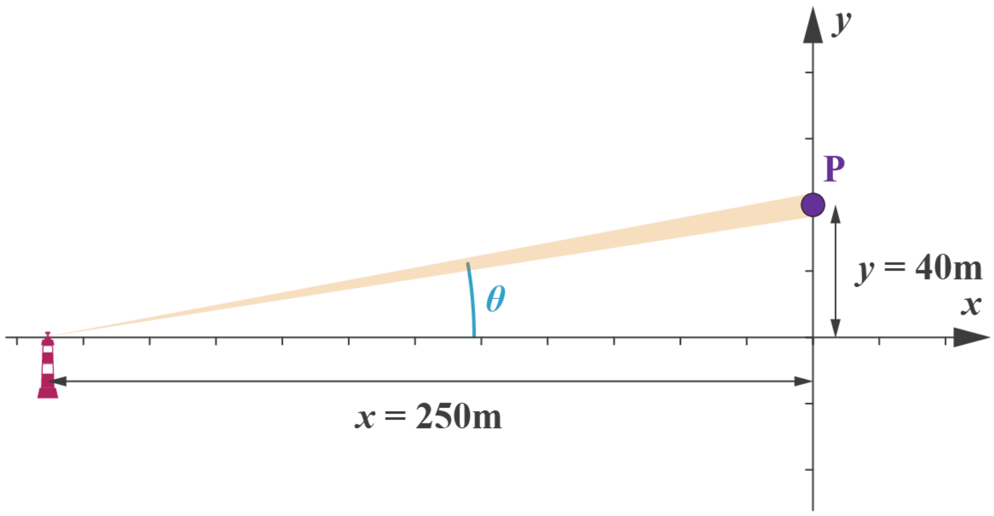
On the diagram below the coastline is the y-axis and the lighthouse is positioned on the x-axis. Angle θ represents the angle between the x-axis and the light beam.

Student view



For this problem, you would like to find the rate at which the light beam is moving along the shore. You know that the lighthouse is 250 m from the shore and the light beam is rotating at 2 revolutions per minute. One revolution is 2π radians, so the beam is rotating $2 \times 2\pi$ radians in 60 seconds. Therefore,

$$\frac{d\theta}{dt} = ()\text{mfrac}\frac{\pi \text{ radians}}{15 \text{ seconds}}, \text{ or } \frac{\pi}{15}\text{rad s}^{-1}.$$

You also know that $y = 40$ at the point of time you are interested in, as the beam is 40 m along the shoreline. From basic trigonometry, you can find the angle at this time:

$$\tan\theta = \frac{y}{x} = \frac{40}{250}$$
$$\theta = 0.1587$$

Finally, you know you are working with a right triangle as you are in Cartesian space. Using basic trigonometry you can relate θ and y with $y = 250\tan\theta$.

$$\frac{dy}{dt} = 250\sec^2\theta\frac{d\theta}{dt} \qquad \text{Find the derivative of both sic}$$
$$\frac{dy}{dt} = \frac{250}{\cos^2\theta}\frac{d\theta}{dt}$$
$$\frac{dy}{dt} = \frac{250}{\cos^2 0.1587}\frac{\pi}{15} = 53.7 \text{ m s}^{-1} \qquad \text{Substitute the values you kno}$$

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Be aware

You will inevitably apply the chain rule when working with related rates, so this gives you a hint for how to relate the information you have been given to what you want to find.

3 section questions

5. Calculus / 5.9 Further differentiation

Checklist

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What you should know

By the end of this subtopic you should be able to:

- find derivatives of:
  - $f(x) = \sin x$
  - $f(x) = \cos x$
  - $f(x) = \tan x$
  - $f(x) = e^x$
  - $f(x) = \ln x$
- find derivatives of complex functions using the:
  - power rule $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$
  - chain rule $y = g(u)$, where $u = f(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$
  - product rule $y = uv \times \frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$
  - quotient rule $y = \frac{u}{v} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$
- solve related rates of change using the steps:
  - understand the problem
  - develop a model

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- write an equation relating what you want and what you have been given
- differentiate both sides with respect to time
- substitute any known values
- answer the question.

5. Calculus / 5.9 Further differentiation

Investigation

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In this subtopic, you were shown some tools that will help you to find the derivatives of a wide range of functions.

You learned how to work with

- linear combinations $(af(x) + bg(x))$
- compositions $(f(g(x)))$
- products $(f(x)g(x))$
- quotients $\left(\frac{f(x)}{g(x)}\right)$.

Activity

In the applet below you can investigate the derivatives of functions of the form $y = f(x)^{g(x)}$.

- Change the base and the exponent and take a look at the derivative.
  The default question of the applet asks about the derivative of $y = x^x$.
  - Can you find this without looking at the answer?
  - If you got the correct derivative, well done.
  - If the formula you suggested does not match the actual derivative, can you explain why?

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- Can you find a general formula for the derivative of $y = f(x)^{g(x)}$?

Interactive 1. Exploring Derivatives of Exponential Functions.

More information for interactive 1

Rate subtopic 5.9 Further differentiation

Help us improve the content and user experience.

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