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Teacher view



(https://intercom.help/kognity)



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Reading
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2. Functions / 2.13 Further rational functions

The big picture

Some people define infinity as the value when a number is divided by 0. Is this correct? There are many ways of interpreting infinity theory. How would you define infinity?

The video below explains what it means by dividing a number by 0.

Why can't you divide by zero? - TED-Ed



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In the above video, you saw that $\frac{1}{0}$ is undefined. But what happens to $\frac{1}{x}$ as x is very close to 0? Or when the value of x is large?

In this subtopic, you will learn about rational functions and their behaviour, including vertical, horizontal and oblique asymptotes.



Concept

Representation

Representing a rational function as a graph helps us to better understand its behaviour in terms of its curvature, turning points, asymptotes and axes intercepts.

- Is it only rational functions that have asymptotes? Give reasons for your answer.



Theory of Knowledge

The key knowledge issue of formalism vs. Platonism in regard to mathematics is discussed in other TOK boxes throughout the course; however, it seems apropos to contemplate mathematics' rational origins in the context of rational functions.

Knower bias is a key factor in knowledge production and reception; however, at first glance, it seems that mathematics is immune to such biases because it is built on reason and has a very high level of real-world predictive validity.

Knowledge Question: To what extent can knowledge be free from bias?

2. Functions / 2.13 Further rational functions



Types of rational functions

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A rational function can be expressed in the form

$$\frac{f(x)}{g(x)}, \text{ where } f(x) \text{ and } g(x) \text{ are polynomial functions and } g(x) \neq 0.$$

There are three different types of rational functions:

1. The degree of $f(x)$ is greater than the degree of $g(x)$,
for example, $\frac{(2x^3 + 3x^2 - 4x + 8)}{3x + 2}$
2. The degree of $f(x)$ is less than the degree of $g(x)$, for example,
 $\frac{3x^2 - 2x + 8}{4x^3 - 5x^2 + 7x - 1}$
3. The degrees of $f(x)$ and $g(x)$ are equal, for example, $\frac{3x + 1}{2x - 5}$.

Making connections

Recall how you change an improper fraction into a mixed number (see [section 2.12.2 \(/study/app/math-aa-hl/sid-134-cid-761926/book/algebra-of-polynomials-id-26605/\)](/study/app/math-aa-hl/sid-134-cid-761926/book/algebra-of-polynomials-id-26605/)).

If 15 is divided by 2, the quotient is 7 and the remainder is 1; hence using the division algorithm:

$$15 = 2 \times 7 + 1 \text{ or } \frac{15}{2} = 2 + \frac{1}{2}$$

But, when 14 is divided by 2, you factorise 14 as 2×7 and cancel the twos in the numerator and the denominator to get an answer 7 as shown below:

$$\frac{14}{2} = \frac{7 \times 2}{2} = 7$$

You will do the same process when you have a polynomial in the numerator with degree higher than that in the denominator.

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Section Example 1

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Feedback



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Simplify the rational function $\frac{x^2 + 5x + 6}{x + 2}$ for $x \neq -2$.

$$\frac{x^2 + 5x + 6}{x + 2} = \frac{(x + 2)(x + 3)}{(x + 2)} \quad [\text{Factorising the numerator}]$$

$$= x + 3 \quad [\text{Cancelling } (x + 2)]$$

Example 2



Express $\frac{f(x)}{g(x)}$ in the form:

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)},$$

where $f(x) = 2x^2 + 3x + 4$ and $g(x) = 2x + 1$.

You are asked to write $\frac{f(x)}{g(x)} = \frac{2x^2 + 3x + 4}{2x + 1}$ in the form

$$q(x) + \frac{r(x)}{2x + 1}.$$

In this form $r(x)$ is the remainder when $f(x) = 2x^2 + 3x + 4$ is divided by $2x + 1$, so according to the remainder theorem it is the constant $f(-0.5)$.

$$r = f(-0.5) = 2(-0.5)^2 + 3(-0.5) + 4 = 3$$

Let's substitute this value to find $q(x)$.



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$$\frac{2x^2 + 3x + 4}{2x + 1} = q(x) + \frac{3}{2x + 1}$$

$$2x^2 + 3x + 4 = q(x)(2x + 1) + 3$$

$$2x^2 + 3x + 1 = q(x)(2x + 1)$$

$$q(x) = \frac{2x^2 + 3x + 1}{2x + 1} = \frac{(x + 1)(2x + 1)}{2x + 1}$$

$$q(x) = x + 1$$

Hence, $\frac{2x^2 + 3x + 4}{2x + 1} = (x + 1) + \frac{3}{2x + 1}$

Alternatively, you can also use polynomial division to find the quotient and the remainder.

$$\begin{array}{r}
 \begin{array}{c} g(x) \leftarrow 2x + 1 \end{array} \overline{) \begin{array}{c} \begin{array}{c} x + 1 \rightarrow \text{Quotient } q(x) \\ 2x^2 + 3x + 4 \rightarrow f(x) \\ 2x^2 + x \\ \hline 2x + 4 \\ 2x + 1 \\ \hline 3 \rightarrow \text{Remainder } r(x) \end{array} \end{array}
 \end{array}$$



✓ Important

Before proceeding to the next subsection, remember the following :

$$\frac{1}{x} \rightarrow 0 \text{ as } x \rightarrow \infty$$

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$$\frac{1}{x} \rightarrow \infty \text{ as } x \rightarrow 0 \text{ and } x > 0$$

$$\frac{x}{n} \rightarrow \infty, \text{ when } x \rightarrow \infty. \text{ Here } n \text{ is any positive number.}$$

$$\frac{0}{n} = 0, \text{ where } n \neq 0 \text{ is any number.}$$

3 section questions

2. Functions / 2.13 Further rational functions

Vertical asymptotes



Activity

Consider the following rational functions and find their values when $x = 0$

1. $\frac{3x + 5}{2x + 3}$

2. $\frac{1}{x + 3}$

3. $\frac{x + 3}{5}$

4. $\frac{x^2 - 3x + 5}{2x + 1}$

5. $\frac{2}{x}$

6. $\frac{x + 4}{x}$

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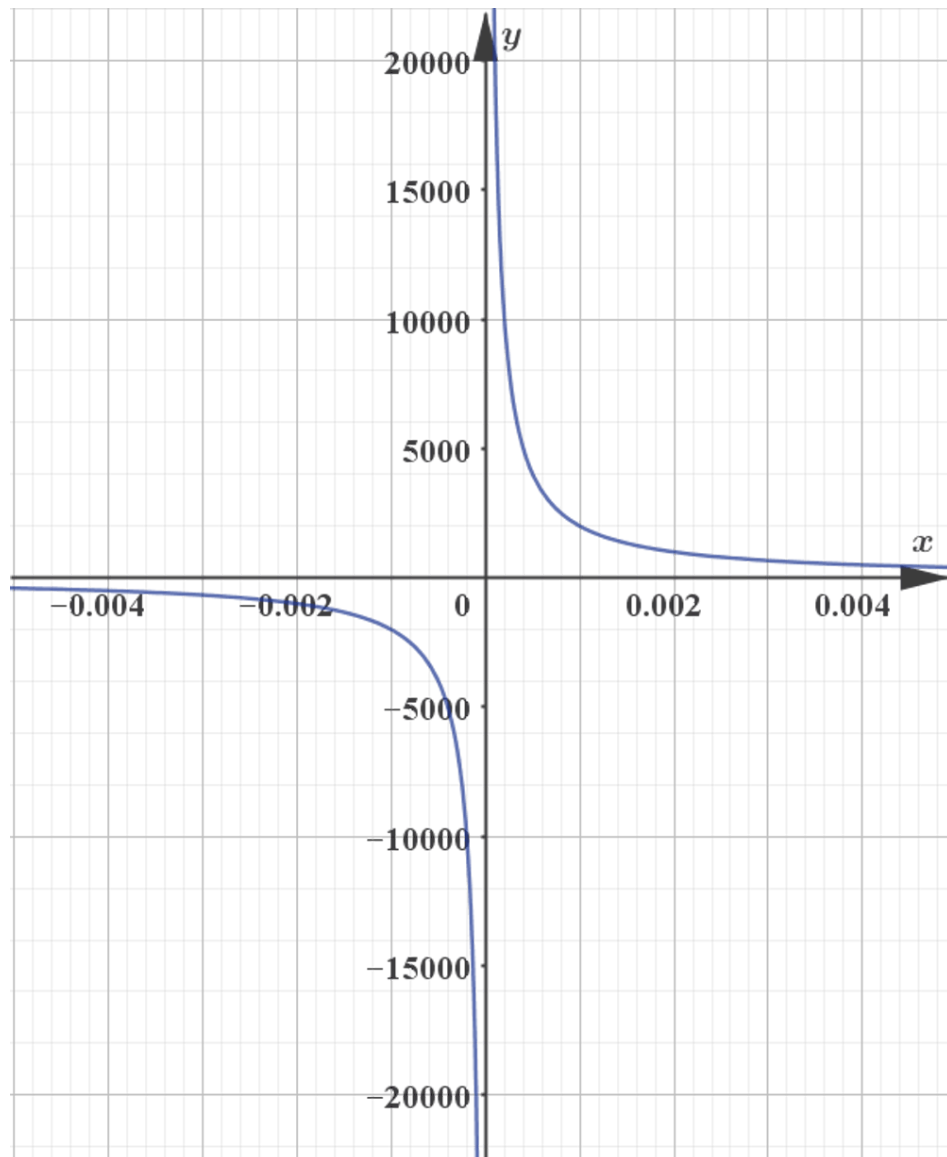
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$$7. x + \frac{1}{x}$$

For examples 1 to 4, the values of the functions at $x = 0$ are finite numbers, but in examples 5 to 7, the value for $x = 0$ would be undefined. For these functions, the denominator becomes 0 when $x = 0$.

When you examine the graphs of the last three examples, you can see a specific pattern around $x = 0$. For example, the graph of $y = \frac{2}{x}$ near $x = 0$ would look like this:



Graph of $f(x) = \frac{2}{x}$

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[More information](#)

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The graph represents the function $f(x) = 2/x$. The X-axis shows values in increments around zero, highlighting the behavior of the function near this point. The Y-axis represents the output of the function, also in a range of values centered around zero. This graph displays two symmetrical branches, one in the first quadrant and one in the third quadrant, approaching the Y-axis asymptotically. The curve does not intersect the axis, illustrating the vertical asymptote at $x = 0$. As x approaches zero from the positive side, the value of $f(x)$ increases sharply, moving towards positive infinity. Conversely, as x approaches zero from the negative side, the curve decreases sharply towards negative infinity. These behaviors indicate a vertical asymptote along the line $x = 0$, illustrating the curve's approach to but never reaching the line, both from the positive and negative sides.

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From the graph you can see that the curve never touches the y -axis, which is given by the equation $x = 0$. Such lines (that the curve is getting close to) are called asymptotes for a function. In this case, $x = 0$ is a vertical asymptote. Vertical asymptotes will be of the form $x = k$.

Look at the graphs below of examples 6 and 7 along with their vertical asymptotes and observe how the graph behaves near $x = 0$.



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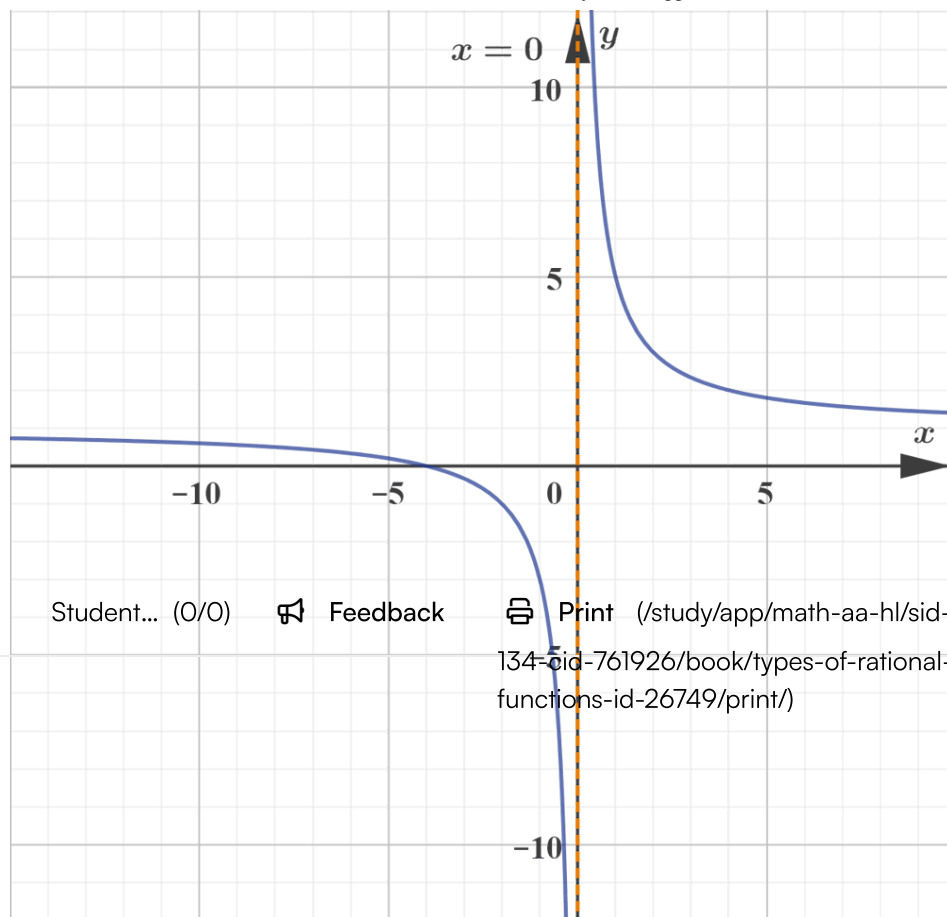
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Graph of $f(x) = \frac{x+4}{x}$



More information

The graph represents the function $(f(x) = \frac{x+4}{x})$. It appears on a grid with x and y axes. The x-axis ranges from -10 to 10, and the y-axis ranges from -10 to 10 as well. There is a vertical asymptote at $(x = 0)$, shown as a dashed line, where the function is undefined. To the left of the vertical asymptote, as (x) approaches 0 from the negative side, the curve runs from negative values and approaches negative infinity. To the right of the vertical asymptote, as (x) approaches 0 from the positive side, the curve ascends towards positive infinity. The curve forms a decreasing branch in the second quadrant tending towards the x-axis as (x) moves away from 0, and an increasing branch in the first quadrant moving away from the x-axis.

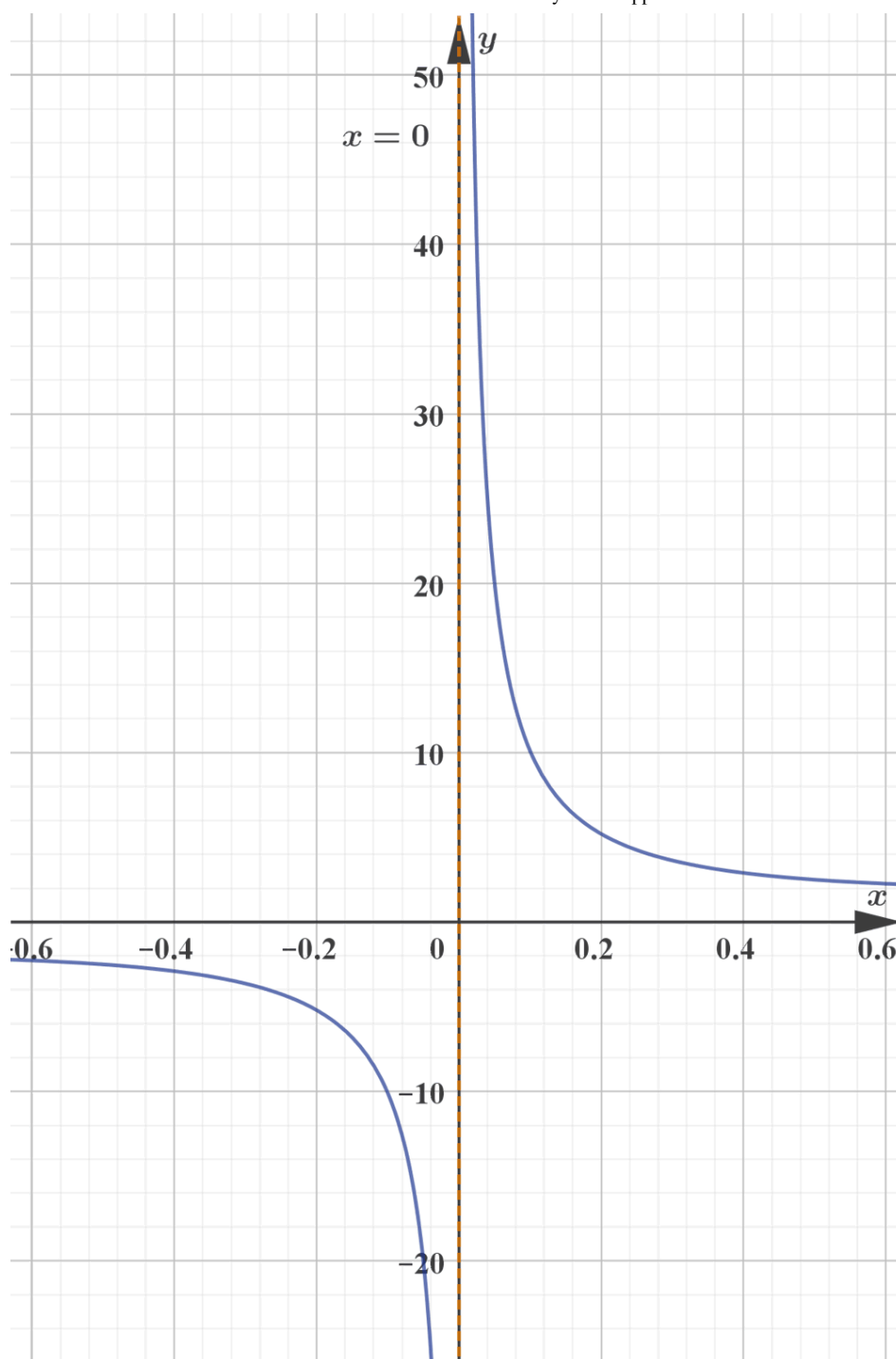
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Graph of $f(x) = \frac{x+1}{x}$

More information

The image shows a graph of the function $f(x) = \frac{x+1}{x}$, depicted on a coordinate plane with a grid. The X-axis ranges from -0.6 to 0.6, and the Y-axis ranges from -50 to 50. The graph features a vertical asymptote at $(x = 0)$, where the function is undefined. This is shown as a dashed vertical line.



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To the left of the asymptote, the curve descends from the top left, crossing the X-axis near ($x = -1$), and continues downward sharply to approach negative infinity as (x) nears 0. To the right of the asymptote, the curve ascends sharply from negative infinity, crosses the X-axis near ($x = -1$), and continues upward towards the right edge of the graph, stabilizing towards a value between 0 and 1 as (x) increases.

Overall, the graph illustrates the nature of rational functions with a vertical asymptote and how such a function behaves on either side of the asymptote.

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In the three examples above the denominator was 0 for $x = 0$. Think about what happens for other rational functions. For example, what would be the vertical asymptote for the function $f(x) = \frac{1}{x-1}$?

✓ Important

When looking for vertical asymptotes, investigate the function around the places where the denominator is 0.

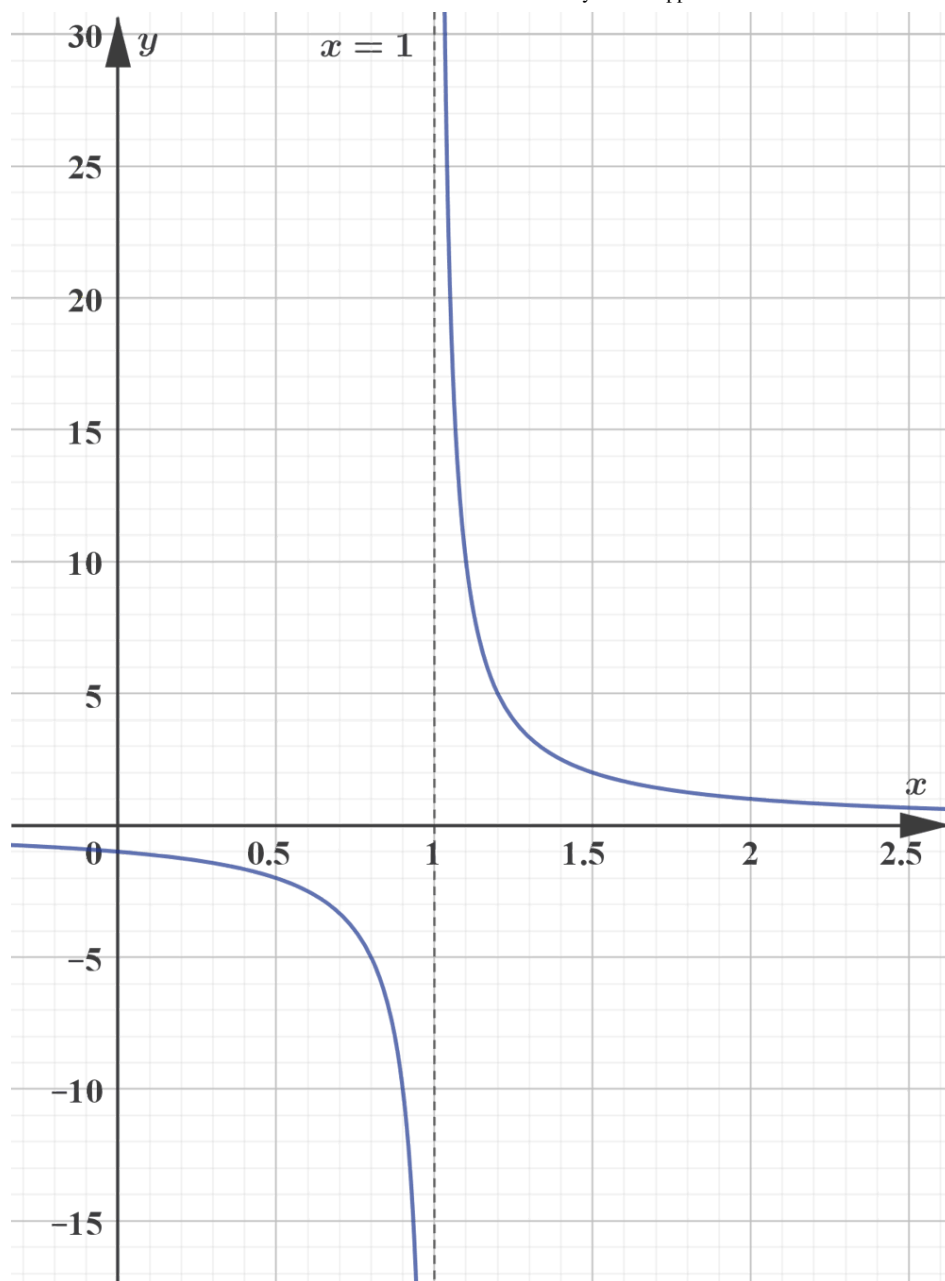
In this function $x = 1$ gives a 0 denominator, the line $x = 1$ is a vertical asymptote. Check this on the graph shown below.



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Graph of $f(x) = \frac{1}{x-1}$

More information

The graph depicts the function ($f(x) = \frac{1}{x-1}$). The X-axis represents the values of (x) ranging from -3 to 3. The Y-axis displays the function values, ranging from -30 to 30.

The graph has a vertical asymptote at ($x = 1$), where the function is undefined. To the left of ($x = 1$), the graph descends sharply as it approaches ($x = 1$) from its negative infinity on the Y-axis, rising back from negative infinity as (x) continues to decrease. To the right of ($x = 1$), the graph ascends sharply from its negative infinity back towards the X-axis, approaching it asymptotically as (x) increases. The curve is symmetrical around ($x = 1$) for a reciprocal function, showing a classic hyperbolic shape indicative of rational functions with vertical asymptotes.



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⚠ Exam tip

Do not forget to use a GDC if access to it is allowed. Be careful to use an appropriate window in order to see all places where the denominator of a rational function is 0.

✓ Important

The graph of a rational function has vertical asymptote $x = k$ if the magnitude of $f(x)$ tends to infinity as x approaches k . This can only happen when the denominator approaches 0, so if k is the root of the polynomial in the denominator.

Example 1



Find the vertical asymptote of the function defined by $f(x) = \frac{3x - 5}{2x + 7}$.

The denominator is 0 when $x = -3.5$.

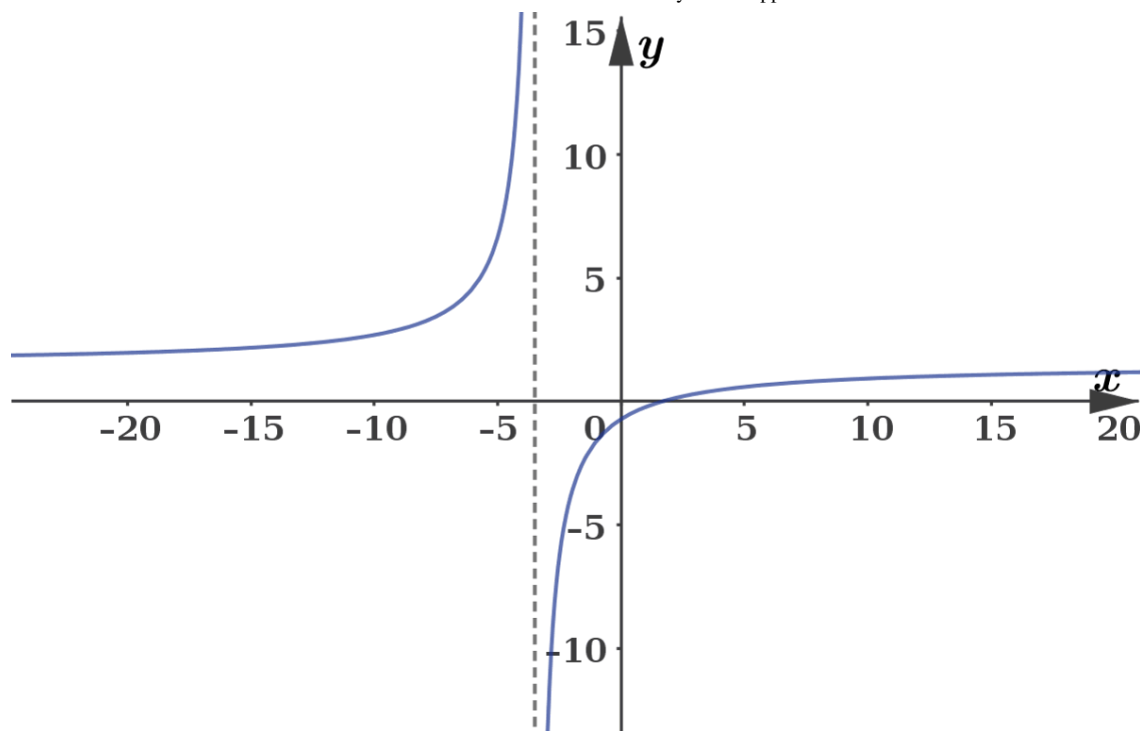
The vertical asymptote of the function is $x = -3.5$. This can be confirmed by checking the graph.



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Example 2



Find the vertical asymptotes of the function defined by $f(x) = \frac{3x - 5}{x^2 - 2x - 15}$.

You can use the factor form of the denominator to find the roots.

$$x^2 - 2x - 15 = (x + 3)(x - 5)$$

The denominator is 0 when $x = -3$ and when $x = 5$.

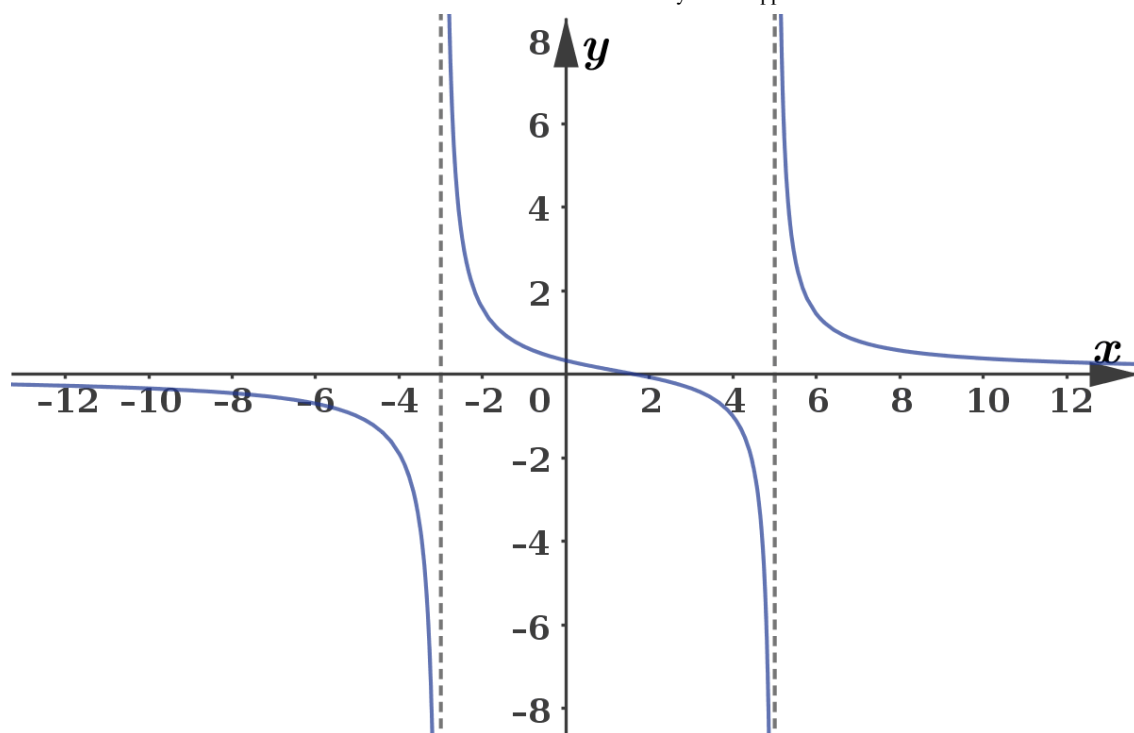
There are two vertical asymptotes, $x = -3$ and $x = 5$. This can be confirmed by checking the graph.



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Example 3



Find the vertical asymptote of the function defined by $f(x) = \frac{3x - 15}{x^2 - 2x - 15}$.

You can use the factor form of the denominator to find the roots.

$$x^2 - 2x - 15 = (x + 3)(x - 5)$$

The denominator is 0 when $x = -3$ and when $x = 5$.

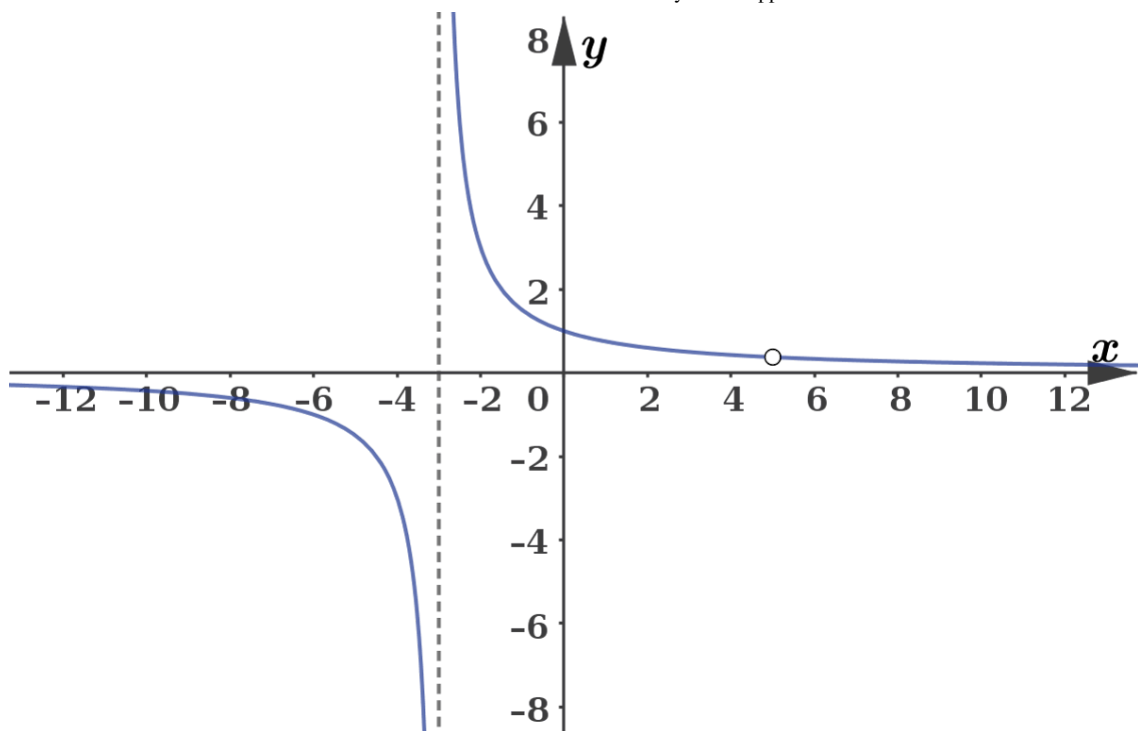
Checking the graph, you can see that only one of these roots corresponds to a vertical asymptote.



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The reason is that for $x = 5$ the numerator is also 0, not just the denominator. The fraction in the function definition can be simplified for $x \neq 5$.

$$\frac{3x - 15}{x^2 - 2x - 15} = \frac{3 \cancel{(x - 5)}}{(x + 3) \cancel{(x - 5)}} = \frac{3}{x + 3}$$

The function in the question is not defined when x is 5, but it has only one vertical asymptote. The equation of this asymptote is $x = -3$.

Example 4



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Find the vertical asymptotes of the function defined by $f(x) = \frac{3x - 15}{x^2 - 2x + 15}$.



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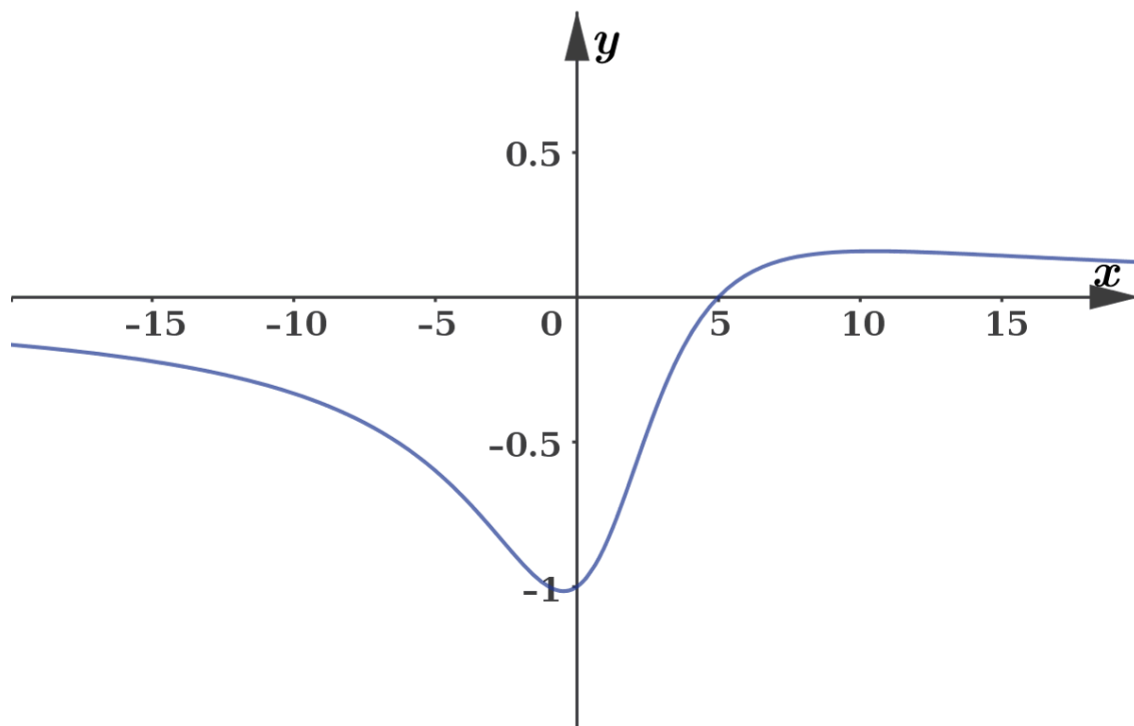
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The discriminant of the denominator is $(-2)^2 - 4 \times 1 \times 15 = -56$. This is negative, so the quadratic polynomial in the denominator has no roots. This means that the rational function is defined everywhere, it has no vertical asymptotes. This can be confirmed by checking the graph.



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2. Functions / 2.13 Further rational functions

Horizontal asymptotes



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Look at the graph of the function $f(x) = \frac{x+1}{x^2+4x+5}$

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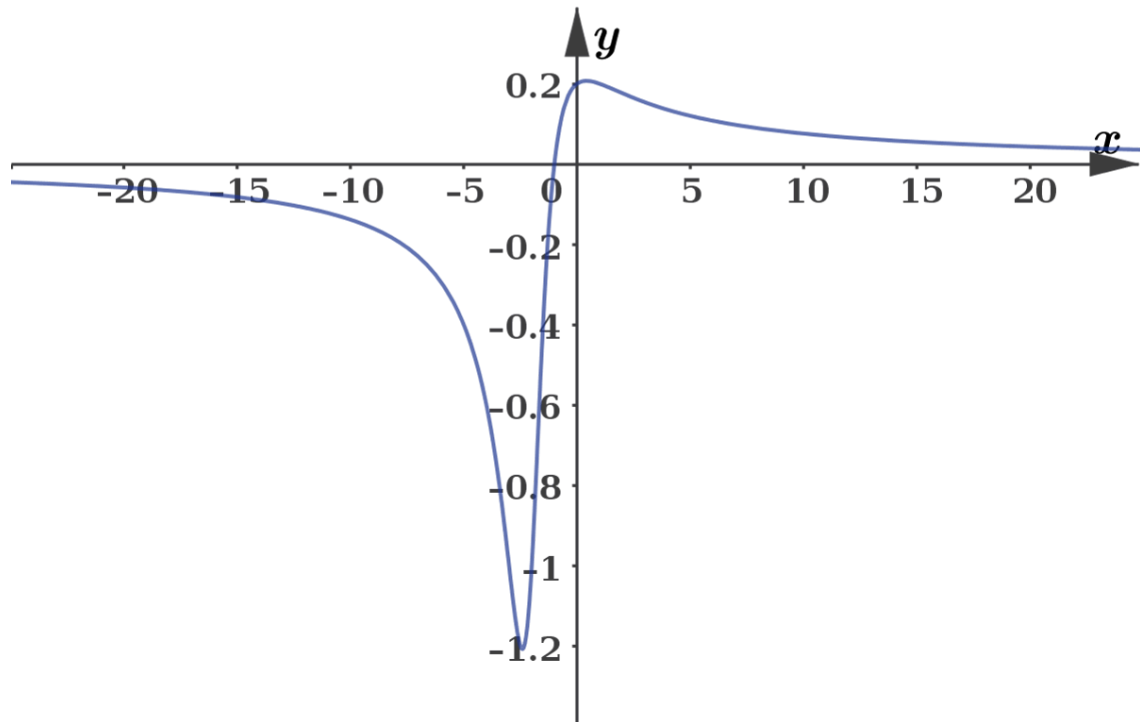
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Graph of $f(x) = \frac{x+1}{x^2+4x+5}$

More information

The image shows a graph of the function $f(x) = \frac{x+1}{x^2+4x+5}$. The X-axis ranges from -25 to 25, while the Y-axis ranges from -1.2 to 0.2. The curve has a notable dip below the x-axis before $x = 0$ and rises sharply up reaching a peak shortly after $x = 0$, then gradually approaches 0 as x moves towards positive infinity. Similarly, moving towards negative infinity, the graph approaches 0 from below the x-axis. This creates a horizontal asymptote at $y = 0$ on both sides of the graph. At $x = 0$, the graph meets the y-axis at approximately -1.

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What do you think will be value of the function as x approaches infinity or minus infinity? It looks like it is going to be a number very close to 0, but never 0, which means that the x -axis is an asymptote for the function. In this case, it is a horizontal asymptote.

✓ Important

A horizontal asymptote is the line of the form $y = k$, where the value of the function approaches k as x approaches infinity or minus infinity.

You can find the horizontal asymptote of rational functions analytically. The method is different depending on the degree of the polynomials in the numerator and the denominator.

- If the numerator is a constant polynomial and the denominator has a degree greater than 1, then the x -axis is a horizontal asymptote. This is true, since $\frac{n}{y}$ approaches 0 as y approaches infinity.
- If neither the numerator nor the denominator is a constant polynomial, then divide both the numerator and the denominator by x^n , where n is the degree of the denominator.



Activity

There are some rational functions for which horizontal asymptotes will not occur. What does this mean? Investigate and find out what should be the connection between the degrees of the polynomials in the numerator and the denominator so that there will be no horizontal asymptote.

Example 1



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Find the horizontal asymptote of the function defined by $f(x) = \frac{3x - 5}{2x + 7}$.

Both the numerator and the denominator are linear, so let's multiply both the numerator and denominator by $\frac{1}{x}$. This does not change the function values for $x \neq 0$.

$$\frac{3x - 5}{2x + 7} = \frac{3x - 5}{2x + 7} \times \frac{\frac{1}{x}}{\frac{1}{x}} = \frac{3 - \frac{5}{x}}{2 + \frac{7}{x}}$$

In this new form both $\frac{5}{x}$ and $\frac{7}{x}$ approach 0 as x approaches either infinity or minus infinity. Using this you can make the following observations.

- The numerator approaches $3 - 0 = 3$ as x approaches either infinity or minus infinity.
- The denominator approaches $2 + 0 = 2$ as x approaches either infinity or minus infinity.

Taking the quotient gives that the function values approach $\frac{3}{2}$ as x approaches either infinity or minus infinity.

This means that the equation of the horizontal asymptote is $y = \frac{3}{2}$. You can confirm this by checking the graph.



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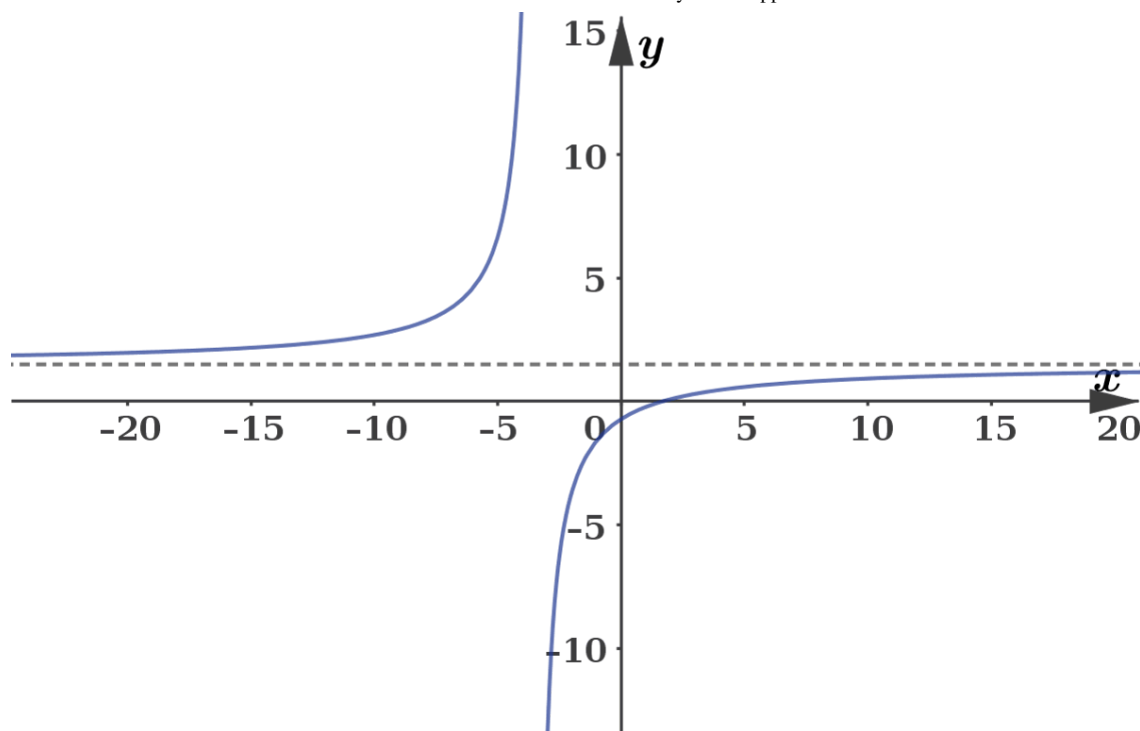
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Example 2



Find the horizontal asymptote of the function defined by $f(x) = \frac{3x - 5}{x^2 - 2x - 15}$.

The denominator is quadratic, so let's multiply both the numerator and denominator by $\frac{1}{x^2}$. This does not change the function values for $x \neq 0$.

$$\frac{3x - 5}{x^2 - 2x - 15} = \frac{3x - 5}{x^2 - 2x - 15} \times \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{\frac{3}{x} - \frac{5}{x^2}}{1 - \frac{2}{x} - \frac{15}{x^2}}$$

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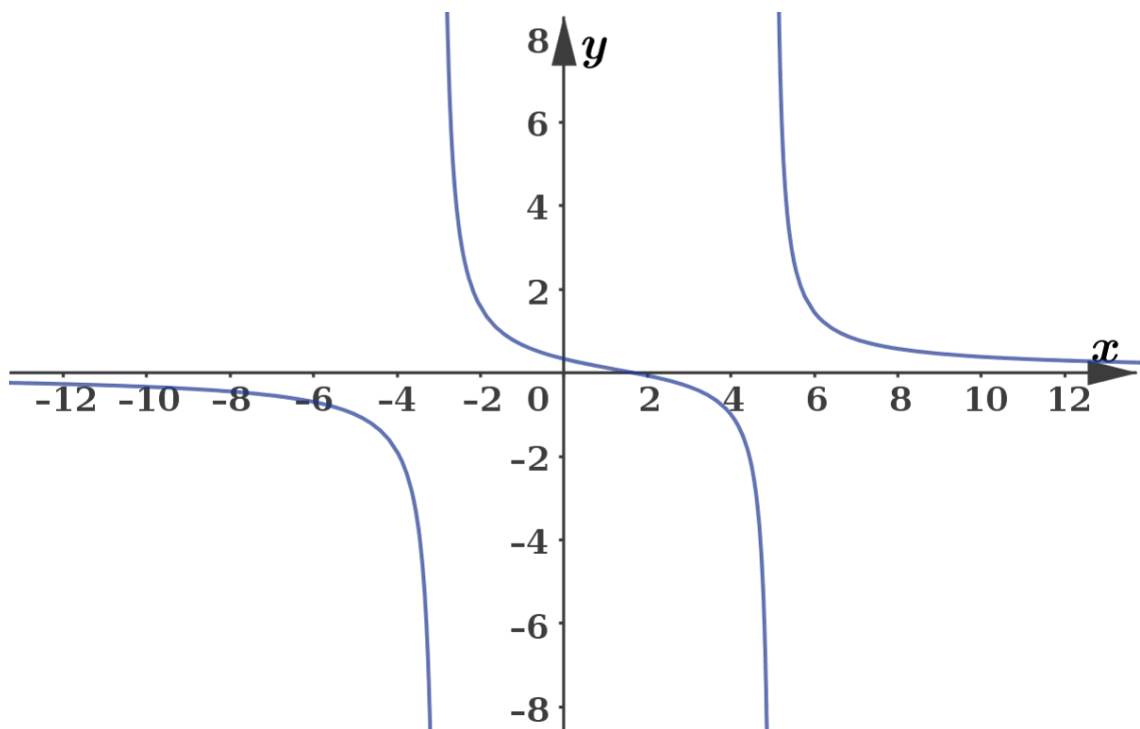
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In this new form $\frac{3}{x}$, $\frac{5}{x^2}$, $\frac{2}{x}$ and $\frac{15}{x^2}$ all approach 0 as x approaches either infinity or minus infinity. Using this you can make the following observations.

- The numerator approaches $0 - 0 = 0$ as x approaches either infinity or minus infinity.
- The denominator approaches $1 - 0 - 0 = 1$ as x approaches either infinity or minus infinity.

Taking the quotient gives that the function values approach $\frac{0}{1} = 0$ as x approaches either infinity or minus infinity.

This means that the equation of the horizontal asymptote is $y = 0$. The horizontal asymptote is the x -axis. You can confirm this by checking the graph.



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Example 3



Find the horizontal asymptote of the function defined by $f(x) = \frac{3x^2 - 5}{x^2 - 2x - 15}$.

The denominator is quadratic, so let's multiply both the numerator and denominator by $\frac{1}{x^2}$. This does not change the function values for $x \neq 0$.

$$\frac{3x^2 - 5}{x^2 - 2x - 15} = \frac{3x^2 - 5}{x^2 - 2x - 15} \times \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{3 - \frac{5}{x^2}}{1 - \frac{2}{x} - \frac{15}{x^2}}$$

In this new form $\frac{5}{x^2}$, $\frac{2}{x}$ and $\frac{15}{x^2}$ all approach 0 as x approaches either infinity or minus infinity. Using this you can make the following observations.

- The numerator approaches $3 - 0 = 3$ as x approaches either infinity or minus infinity.
- The denominator approaches $1 - 0 - 0 = 1$ as x approaches either infinity or minus infinity.

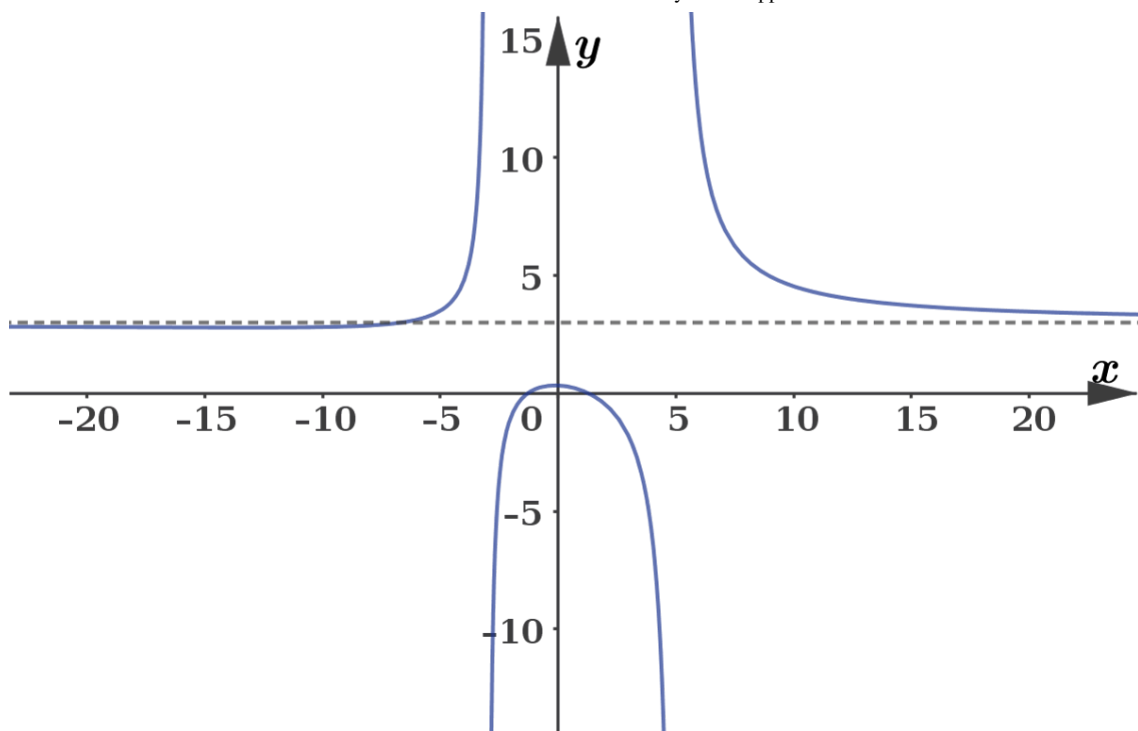
Taking the quotient gives that the function values approach $\frac{3}{1} = 3$ as x approaches either infinity or minus infinity.

This means that the equation of the horizontal asymptote is $y = 3$. You can confirm this by checking the graph.

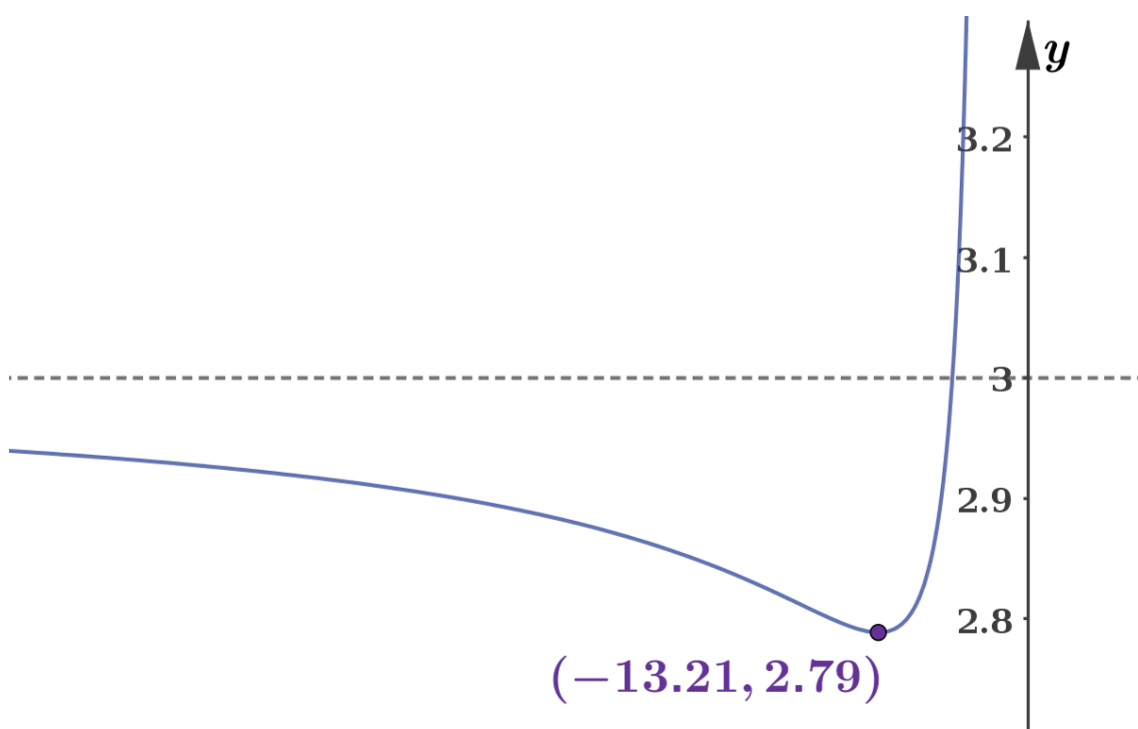
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Note that on the left the graph crosses the line $y = 3$ and then approaches it from below. You can confirm this by zooming in vertically around the asymptote.



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2. Functions / 2.13 Further rational functions

Oblique asymptotes

Oblique lines are those which are neither vertical nor horizontal. The equation of such lines can be written in the form $y = mx + c$ for some $m \neq 0$.

Some rational functions have oblique asymptotes. This means that the points on the graph of the function get closer and closer to the line as x approaches infinity or negative infinity.

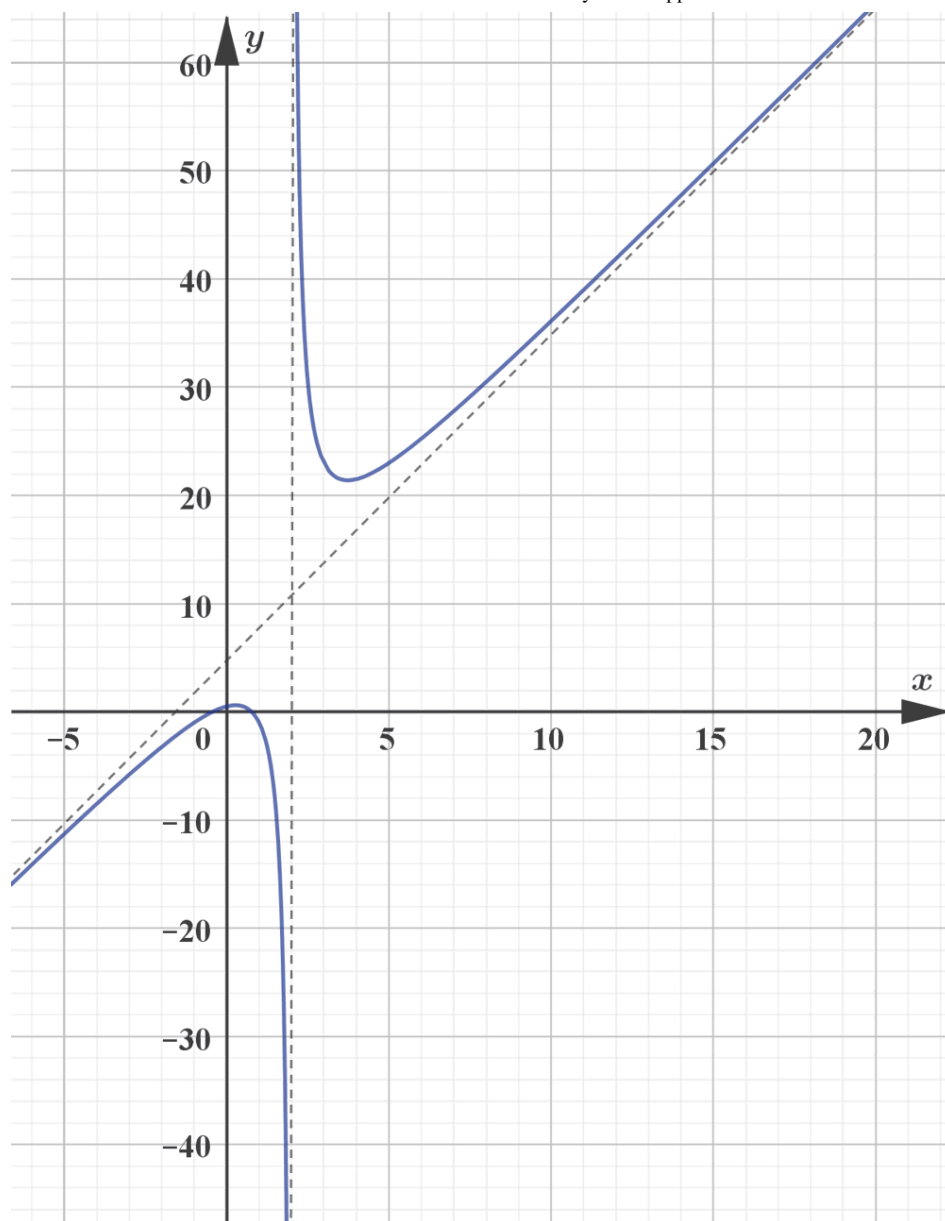
Some examples of rational functions that have oblique asymptotes are given below.



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Graph of $f(x) = \frac{3x^2 - x - 1}{x - 2}$

More information

The image is a graph of the rational function $f(x) = \frac{3x^2 - x - 1}{x - 2}$. The X-axis represents the variable x , with points marked at regular intervals, ranging from -15 to 20. The Y-axis measures the output values of the function, ranging from -50 to 50 at intervals of 10.

The graph features an oblique asymptote, indicating that as x approaches infinity or negative infinity, the curve approaches, but never touches, a slanted line. The curve has a vertical asymptote at $x = 2$, where the function is undefined.



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The graph displays two branches: one in the second and third quadrants that dips sharply from the left before leveling out as it approaches $x = 2$ from the left, and another in the first quadrant rising as it moves further to the right, following the slope of the oblique asymptote.

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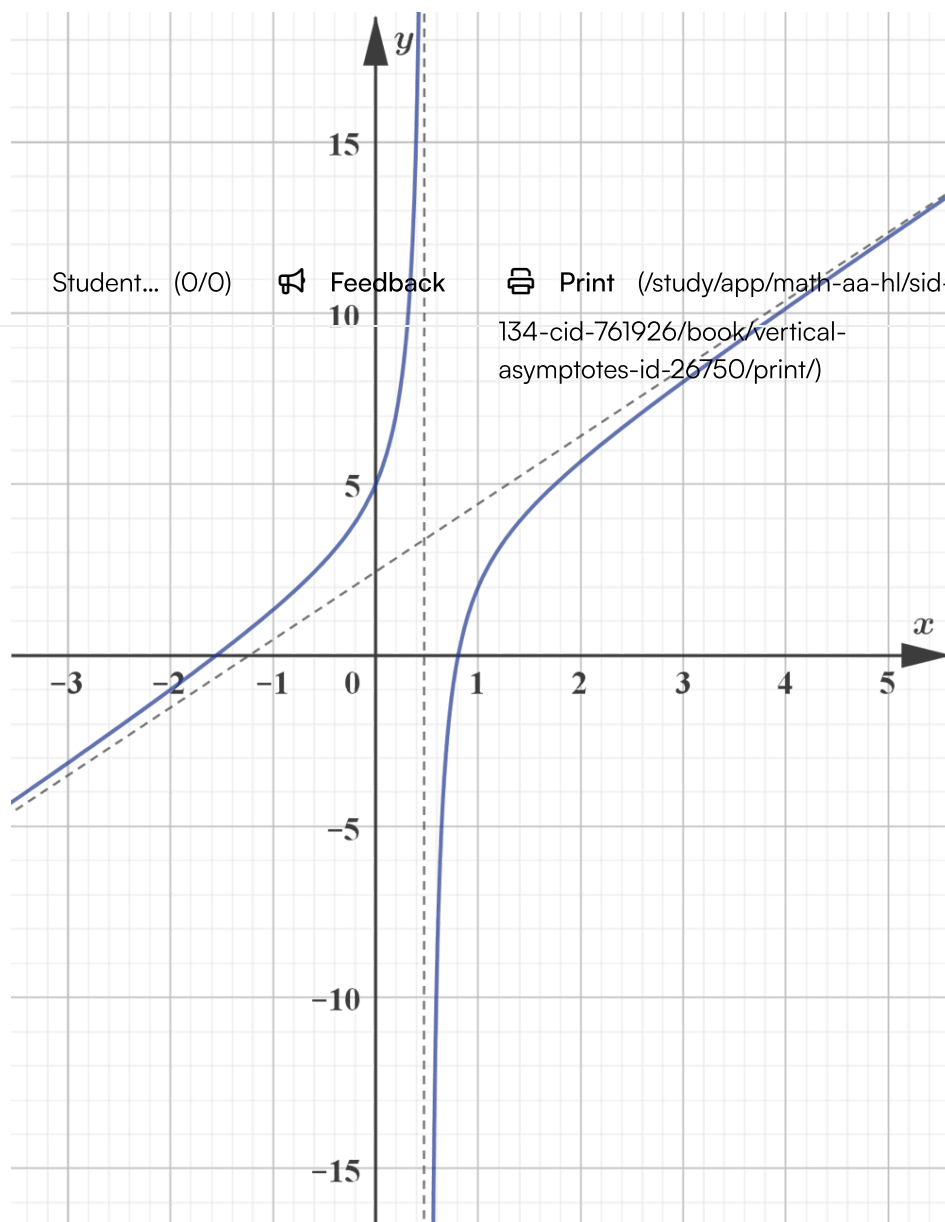
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Assign



$$\text{Graph of } f(x) = \frac{4x^2 + 3x - 5}{2x - 1}$$

More information



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The graph depicts a rational function ($f(x) = \frac{4x^2 + 3x - 5}{2x - 1}$). The X-axis ranges from -4 to 4 and the Y-axis ranges approximately from -15 to 15. There are visible vertical and horizontal asymptotes. The vertical asymptote is at ($x = 0.5$), where the function approaches infinity as it nears this x-value from



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either direction. The horizontal asymptote is on ($y = 2$), indicating that as (x) moves towards infinity on either side, the function approaches this line. The curve shows that the function dips into the negative as (x) is less than 0.5 and rises towards the positive as (x) increases above 0.5, illustrating a significant transformation around the asymptotes.

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These asymptotes are seen only in some rational functions with certain conditions relating to the degrees of the polynomials in the numerator and the denominator.



Activity

Use the GeoGebra app and graph the following functions by entering the numerator and denominator in $f(x)$ and $g(x)$ boxes.

$$1. \frac{2x - 5}{3x + 1}$$

$$2. \frac{4x - 1}{x^2 + 3x + 2}$$

$$3. \frac{x^2 + x - 8}{x^2 + 3x - 1}$$

$$4. \frac{x^2 - 3x + 1}{3x - 1}$$

$$5. \frac{2x^2 - 3x - 5}{8x + 3}$$

$$6. \frac{x^3 - 3x^2 + 4x - 1}{x^2 + 4x + 3}$$

$$7. \frac{3x^3 - 2x^2 + 7x - 1}{x - 3}$$



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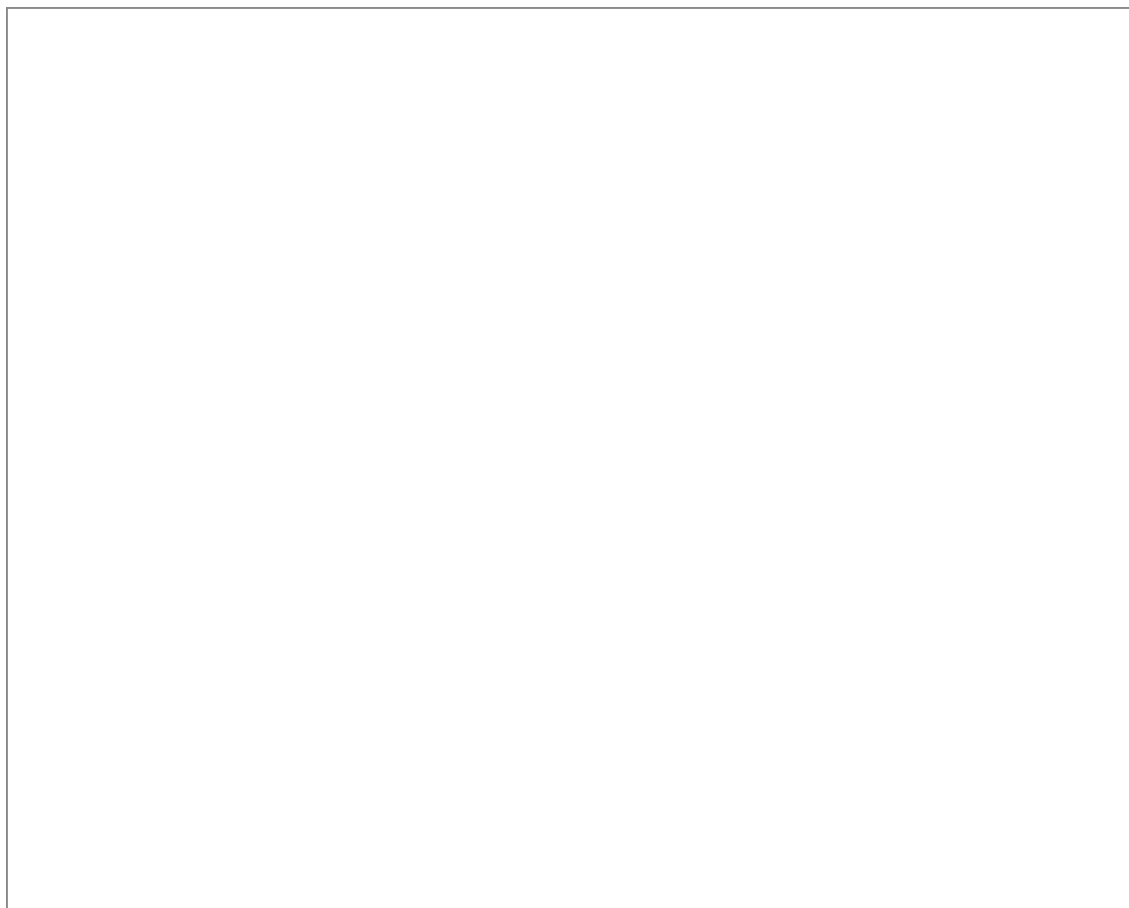
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Interactive 1. Graph the given Functions by Entering the Numerator and Denominator in $f(x)$ and $g(x)$ Boxes.

Credit: [GeoGebra](https://www.geogebra.org/m/dFP5VgR7)  (<https://www.geogebra.org/m/dFP5VgR7>) Mathguru, Mäkiö

 More information for interactive 1

This interactive allows the users to understand asymptotes of the rational function.

A graph is displayed on the xy -axis with the x -axis ranging from -40 to 20 and y -axis ranging from -100 to 250 . The users can fill the values of $f(x)$ and $g(x)$ in their respective textboxes on the bottom

left corner of the graph, and their rational form will be $h(x) = \frac{f(x)}{g(x)}$.

If users choose $f(x) = 5x^5 + 2x^2$ and $g(x) = (x^2 + 1)(x - 3)(x + 3)$, then $h(x) = \frac{5x^5 + 2x^2}{x^4 - 8x^2 - 9}$.

The graph projects asymptotes after the users input the values.

On the top left corner, users can select the three checkboxes: 'show zero of denominator', 'show vertical asymptote', and 'show asymptote as x tends to plus or minus infinity'.



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For the above function, on clicking the check box 'show zero of the denominator' two purple dots appear on both the negative and positive sides of the x-axis at $x = -3$ and $x = 3$, indicating the zeros of the denominator. If the users select 'show vertical asymptote', it plots two vertical asymptotes at $x = -3$ and $x = 3$. On clicking the checkbox 'show asymptote as x tends to plus or minus infinity', a pink dotted line is projected on the graph representing an oblique asymptote with a function at infinity, $P(x) = 5x$.

By practicing more questions users will understand that some of the functions have oblique asymptotes and some do not. Users understand those functions which have oblique asymptotes and, for these functions, users can compare the degree of the numerator and the denominator.

Some of these functions have oblique asymptotes and some do not. Identify those functions which have oblique asymptotes and, for these functions, compare the degree of the numerator and the denominator. Did you see the horizontal and oblique asymptotes in the examples?

- Examples 1, 2 and 3 have horizontal asymptotes.
- Examples 4, 5 and 6 have oblique asymptotes.
- Example 7 has neither horizontal nor oblique asymptote.

✓ Important

For a rational function to have an oblique asymptote, the degree of the numerator should be **exactly one more** than the degree of the denominator.

In order to find the equation of an oblique asymptote, you can divide the numerator by the denominator and express the function using the division algorithm ([section 2.13.1 \(/study/app/math-aa-hl/sid-134-cid-761926/book/types-of-rational-functions-id-26749/\)\)](/study/app/math-aa-hl/sid-134-cid-761926/book/types-of-rational-functions-id-26749/)).



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In example 4 given above, $f(x) = \frac{x^2 - 3x + 1}{3x - 1} = \left(\frac{1}{3}x - \frac{8}{9}\right) + \frac{\frac{1}{9}}{3x - 1}$

As x approaches infinity, the fractional part, $\frac{\frac{1}{9}}{3x - 1}$ approaches 0. This means that $f(x)$ gets closer and closer to $\frac{1}{3}x - \frac{8}{9}$.

The line $y = \frac{1}{3}x - \frac{8}{9}$ is the oblique asymptote of $f(x)$.

In the following example you can see a method that does not use polynomial division to find the oblique asymptote.

Example 1



Find the oblique asymptote of the function defined by $f(x) = \frac{x^2 + 3x}{2x + 2}$.

Assign

Step 1.

If you consider only the terms with the highest degree in the numerator and the denominator, the quotient is $\frac{x^2}{2x} = \frac{1}{2}x$. Consider the difference between this expression and the function value.

$$\frac{x^2 + 3x}{2x + 2} - \frac{1}{2}x = \frac{(x^2 + 3x) - x(x + 1)}{2x + 2} = \frac{2x}{2x + 2}$$

Step 2.



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This difference is a quotient of two linear terms. You can multiply both the numerator and the denominator by $\frac{1}{x}$ to investigate the values as x approaches infinity.

$$\frac{2x}{2x+2} = \frac{2x}{2x+2} \times \frac{\frac{1}{x}}{\frac{1}{x}} = \frac{2}{2 + \frac{2}{x}}$$

Since $\frac{2}{x}$ approaches 0 as x approaches infinity, this quotient approaches $\frac{2}{2-0} = 1$ as x approaches infinity. Let's subtract this value, too.

$$\frac{2x}{2x+2} - 1 = \frac{2x - (2x+2)}{2x+2} = -\frac{2}{2x+2}$$

Step 3.

You got this expression by subtracting first $\frac{1}{2}x$, then 1 from the function value.

$$\frac{x^2 + 3x}{2x+2} - \left(\frac{1}{2}x + 1\right) = -\frac{2}{2x+2}$$

Since this difference approaches 0 as x approaches infinity, you found the equation of the oblique asymptote:

$$y = \frac{1}{2}x + 1$$

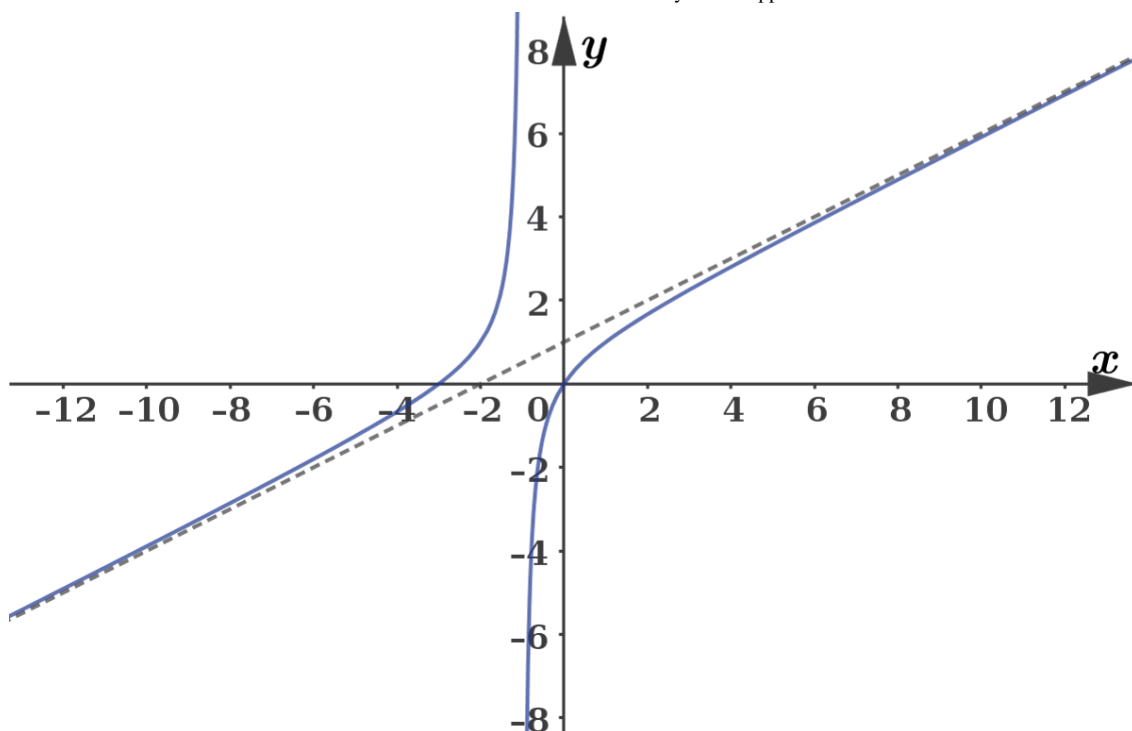
This can be confirmed by checking the graph.



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2. Functions / 2.13 Further rational functions

Sketching the graph of rational functions

If you need to sketch a rational function without a GDC, what do you think you need to show?



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You know that to graph any function, you need to know the axes intercepts.



How will you find the axes intercepts for a rational function?

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 x -intercept: when $y = 0$ y -intercept: when $x = 0$

For a function $f(x) = \frac{3x - 5}{2x + 1}$,

- the x -intercept would be found by setting $f(x) = 0$

$$\frac{3x - 5}{2x + 1} = 0$$

$$3x - 5 = 0 \Rightarrow x = \frac{5}{3}$$

From the above, you should notice that x -intercept can be found by simply equating the numerator to 0.

- the y -intercept is found by setting $x = 0$.

Substituting $x = 0$ in $\frac{3x - 5}{2x + 1}$, you would get $y = -5$, which gives $(0, -5)$ as the y -intercept.

Be aware

This might be tricky if the function has the y -axis as its vertical asymptote. For example, $f(x) = \frac{1}{x}$ has a vertical asymptote at $x = 0$. Hence, you say that there is no y -intercept.

In order to sketch the graph of a rational function, you need to know:

1. asymptotes (vertical, horizontal and oblique)
2. axes intercepts.

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Consider this example.



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Sketch the function $f(x) = \frac{x+1}{3x-2}$ without using a GDC or any other graphing software.

You can deduce the following:

- vertical asymptote: denominator = 0

$$3x - 2 = 0$$

$$x = \frac{2}{3}$$

- horizontal asymptote: Dividing numerator and denominator by x , you get,

$$f(x) = \frac{1 + \frac{1}{x}}{3 - \frac{2}{x}}$$

considering $x \rightarrow \infty$ gives,

$$y = \frac{1}{3} \text{ is the horizontal asymptote.}$$

- There is no oblique asymptote since the degrees of numerator and denominator are equal.
- x -intercept: numerator = 0

$$x + 1 = 0$$

$$x = -1. \text{ Hence the } x\text{-intercept is } (-1, 0).$$

- y -intercept:

$$x = 0$$

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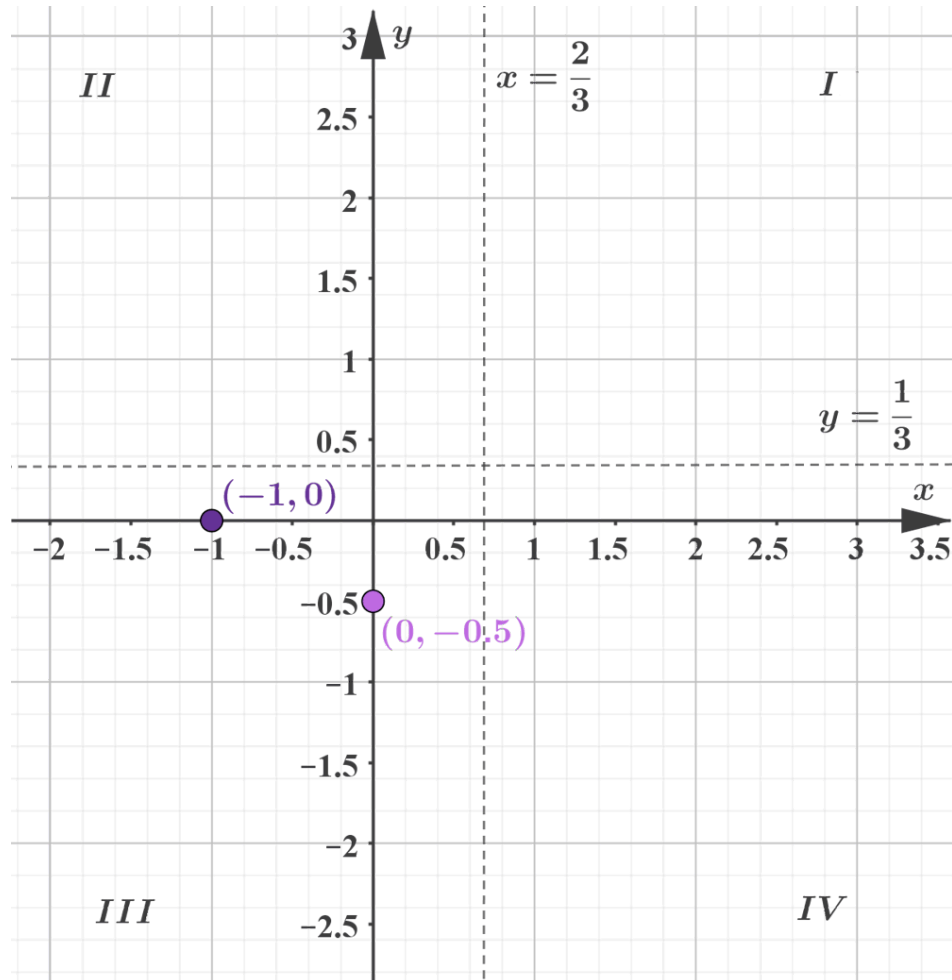
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$y = \frac{1}{-2}$. Hence the y -intercept is $\left(0, -\frac{1}{2}\right)$.

Putting all the information into a coordinate system, you will get:



 More information

This is a graph with a coordinate system containing two labeled points. The first point is labeled as $(-1, 0)$ indicating it is located at the x-coordinate of -1 and y-coordinate of 0 . The second point is labeled as $(0, -0.5)$ showing it is at the x-coordinate of 0 and y-coordinate of -0.5 . The x-axis ranges from approximately -3.5 to 4.5 , and the y-axis ranges from -3 to 3 . Above the graph, some text reads ' $t = 2$ ' and ' n ', while on the right side, there seems to be a division or a marking represented as ' $y = 1/3$ '.

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Can you complete the graph with this information?

There are 4 parts around the two asymptotes (marked as I, II, III, and IV) where the graph could be drawn. How do you find the part(s) in which the function exists?

If you consider the vertical asymptote, you can see the four options given below:

1. $y \rightarrow \infty, x \rightarrow +\frac{2}{3}$ from the left of the asymptote

2. $y \rightarrow \infty, x \rightarrow +\frac{2}{3}$ from the right of the asymptote

3. $y \rightarrow -\infty, x \rightarrow +\frac{2}{3}$ from the left of the asymptote

4. $y \rightarrow -\infty, x \rightarrow +\frac{2}{3}$ from the right of the asymptote

Start by considering each option and finding which suits the given function.

What is meant by 'from the left/right of the asymptote'?

It simply means at points whose x values are slightly less than and slightly more than $\frac{2}{3}$.

So substitute a value for x that is very close to $\frac{2}{3}$ on either side.

For example, choose $\frac{2}{3} - \frac{1}{10} = \frac{17}{30} = 0.57$ and $\frac{2}{3} + \frac{1}{10} = \frac{23}{30} = 0.77$ for x values and substitute them into the given function:

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$$f(0.57) = \frac{0.57 + 1}{3(0.57) - 2} = \frac{1.57}{-0.29} < 0 \quad (\text{you do not need to find the actual value})$$

$$f(0.77) = \frac{0.77 + 1}{3(0.77) - 2} = \frac{3.3}{0.31} > 0$$

The above value shows that near $x = \frac{2}{3}$ from the left side, the value of y is negative and from the right side the value is positive.

Hence the options are:

$$y \rightarrow -\infty, x \rightarrow +\frac{2}{3} \text{ from the left of the asymptote}$$

and

$$y \rightarrow \infty, x \rightarrow +\frac{2}{3} \text{ from the right of the asymptote}$$

You can make a small sketch of this information on the graph now. The red marks shown on either side of the graph below indicate the position of the graph on either side of the vertical asymptote.

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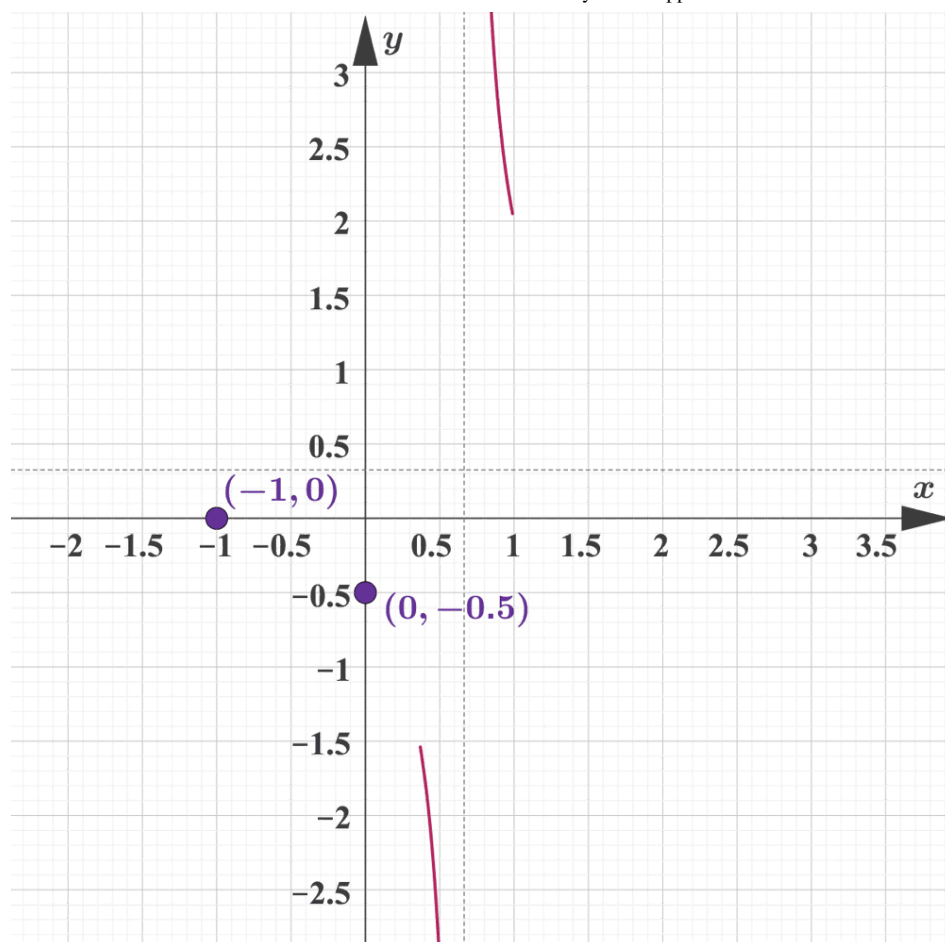
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$$f(x) = \frac{x + 1}{3x - 2}$$

More information

The image is a graph depicting the function $f(x) = \frac{x + 1}{3x - 2}$. The graph includes a vertical asymptote located at $x = \frac{2}{3}$ as indicated by the red marks extending vertically through the graph. The horizontal axis is labeled with x values ranging from -2 to 5, while the vertical axis has markings for y -values. Two significant points are highlighted on the graph: $(0, -0.5)$ on the y -axis and one at approximately $(-1, 0)$. The curve of the graph approaches the vertical asymptote on both sides but does not cross it, suggesting the behavior of the function near $x = \frac{2}{3}$. Intercepts and the overall trend of the function are indicated by the curve, which shows increasing and decreasing segments as it approaches and recedes from the asymptote.

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Using the intercepts, you can complete the graph as shown below:



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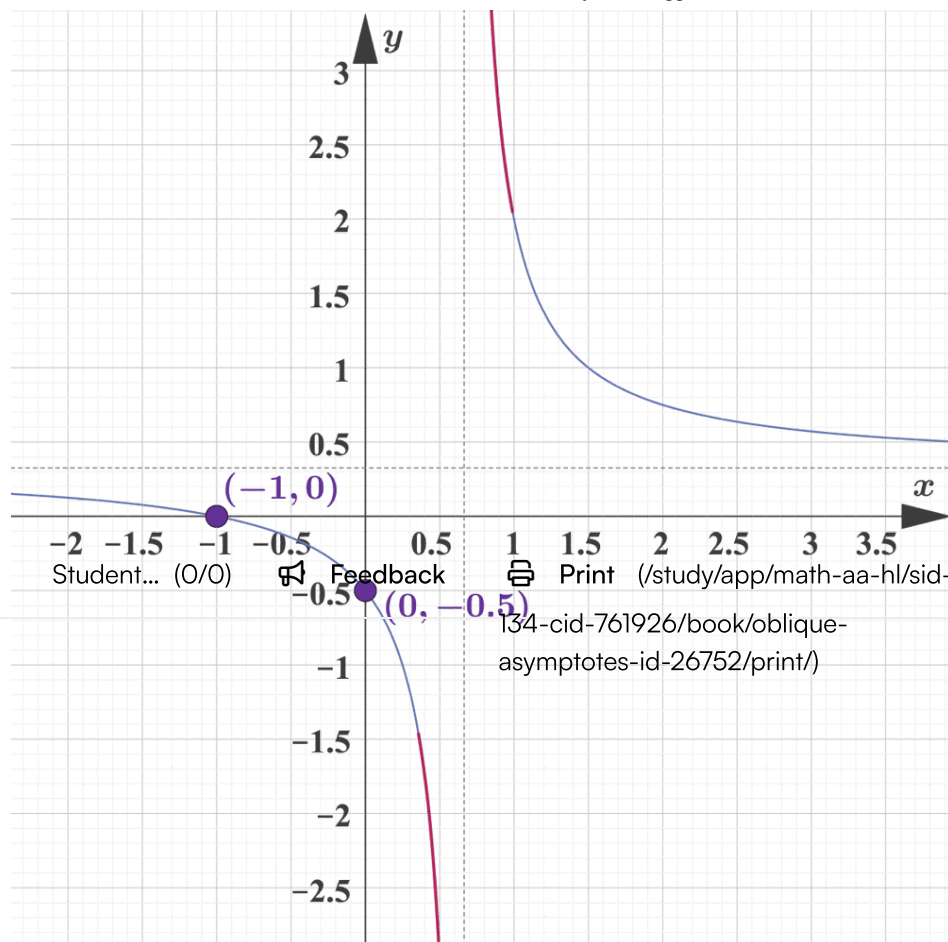
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Section



$$f(x) = \frac{x + 1}{3x - 2}$$

More information

The image is a graph displaying the function $f(x) = \frac{x + 1}{3x - 2}$. The X-axis ranges from -5 to 5, while the Y-axis also ranges from -5 to 5. The graph shows a hyperbolic curve with labeled points $(-1, 0)$ and $(0, -0.5)$, indicating intercepts. Asymptotes are visible, with a vertical asymptote at $x = 2/3$ and a horizontal asymptote at $y = 1/3$. The graph is depicted on a grid to illustrate these features clearly. There is a noticeable curve that approaches the asymptotes but does not intersect them.

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Example 1

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Sketch the graph of the function:

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$$f(x) = \frac{2 - 3x}{x^2 + 3x - 3}$$

Intercepts: $\left(0, -\frac{2}{3}\right)$ and $\left(\frac{2}{3}, 0\right)$ (1)

Horizontal asymptote:

$$f(x) = \frac{\frac{2}{x^2} - \frac{3}{x}}{1 + \frac{3}{x} - \frac{3}{x^2}} \quad \text{[Dividing the numerator and denominator by } x^2]$$

As $x \rightarrow \infty$, $f(x) \rightarrow 0$. Therefore, $y = 0$, the x -axis, will be the horizontal asymptote. (2)

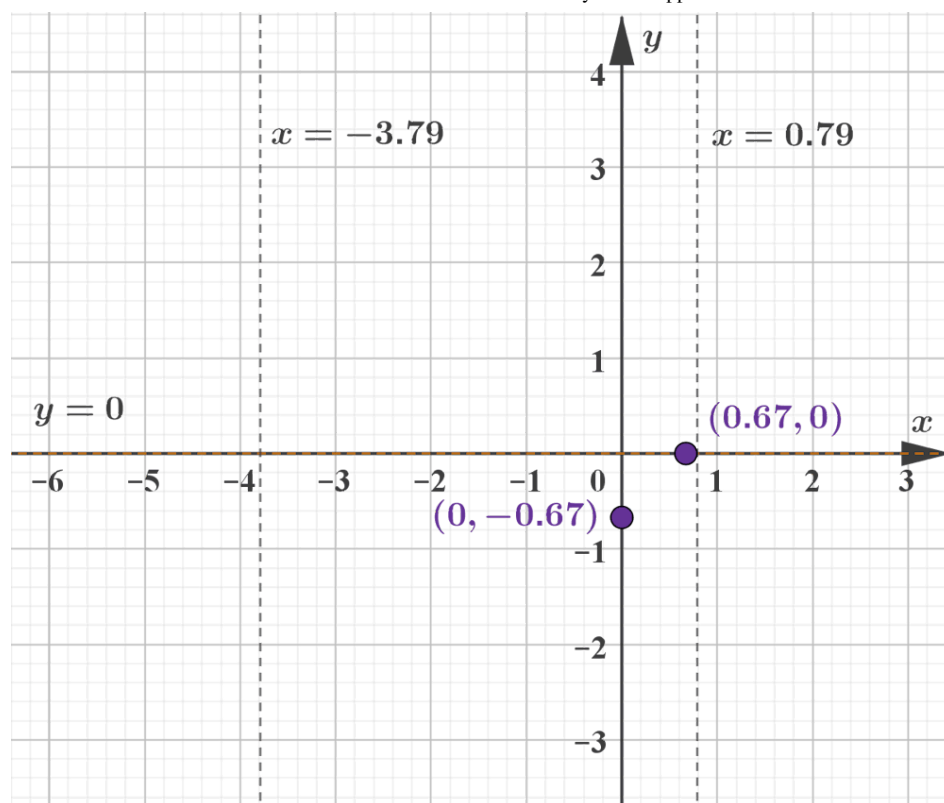
Vertical asymptote: Solving for x : $x^2 + 3x - 3 = 0$ gives $x = -3.79$ and $x = 0.79$. (3)

Putting all the above information on a graph:

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Graph of $f(x) = \frac{2 - 3x}{x^2 + 3x - 3}$



Finding the behaviour of the graph on either side of the vertical asymptotes:

$$f(-4) = \frac{2 - 3(-4)}{16 - 12 - 3} = \frac{14}{1} = 14 > 0$$

$$f(-3) = \frac{2 - 3(-3)}{9 - 9 - 3} = \frac{11}{-3} < 0$$

$$f(1) : \frac{2 - 3(1)}{1 + 3 - 3} = -\frac{1}{1} < 0$$

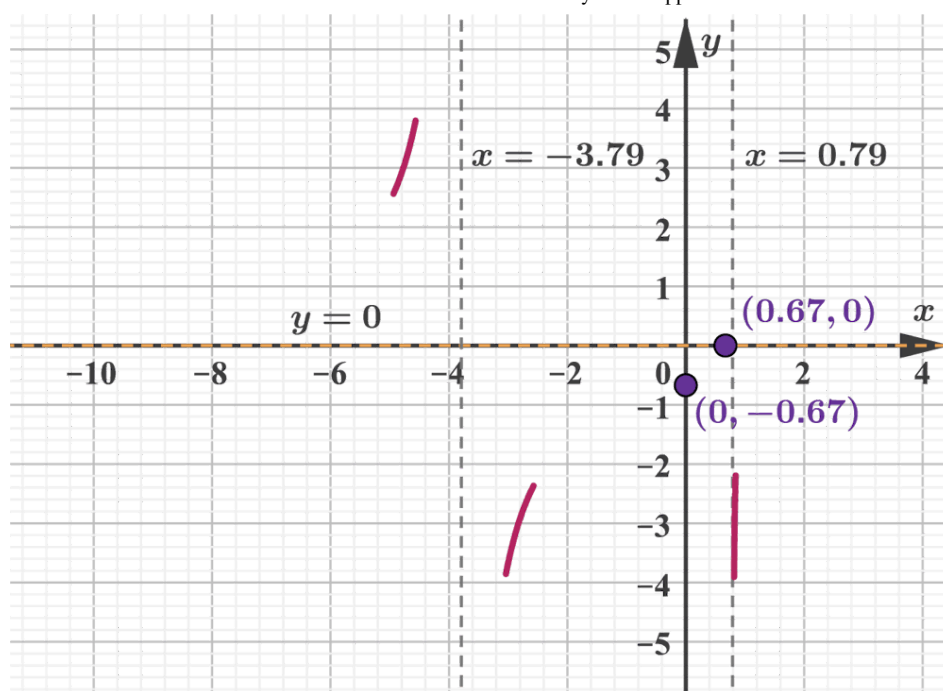
Combining all the above onto the graph:



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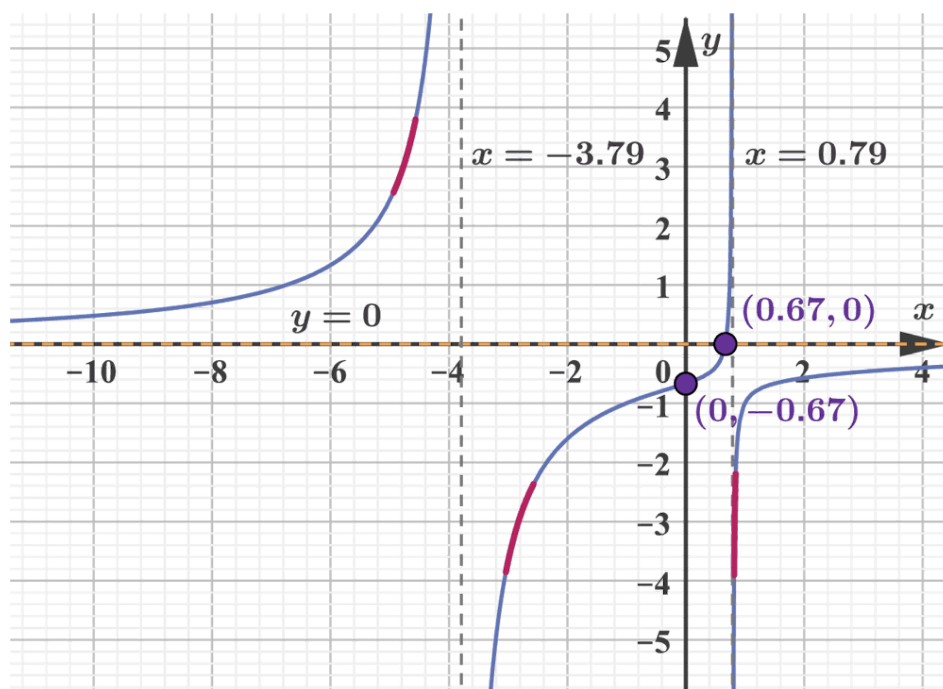
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Graph of $f(x) = \frac{2 - 3x}{x^2 + 3x - 3}$



After joining the intercepts and completing the other parts, the completed graph looks like this:



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$$\text{Graph of } f(x) = \frac{2 - 3x}{x^2 + 3x - 3}$$



International Mindedness

The concept of rational numbers could be seen in ancient Greek, Egyptian and Indian mathematics. Soon after the term 'function' was introduced in the 17th century by Leibnitz, rational functions gained importance and became one of the major concepts that play an important role in higher mathematics, for example in ring theory, Mandelbrot's fractals and the Padé approximation. Find out how mathematicians used rational functions to invent/discover new concepts.



Theory of Knowledge

Does graphing a rational function help you to analyse the function better than algebraic methods? What are the advantages and disadvantages of the two methods? For example, how reliable are your graphs in graphical method and how logical are your assumptions in algebraic method?

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2. Functions / 2.13 Further rational functions

Checklist



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What you should know

By the end of this subtopic you should be able to:

- simplify rational functions
- find the vertical, horizontal and oblique asymptotes of rational functions
- sketch the graph of a rational function with or without using a GDC.

2. Functions / 2.13 Further rational functions

Investigation

Section

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Investigate more rational functions of the forms:

$$f(x) = \frac{ax + b}{cx^2 + dx + e} \text{ and } h(x) = \frac{ax^2 + bx + c}{dx + e}$$

Use the applet below to carry out your investigation. Enter different numbers for a , b , c , d and e to understand the behaviour of the two types of functions given above. Click on the check boxes to show either $f(x)$ or $h(x)$ and observe the graphs. To see the asymptotes, click on the check box 'show asymptotes'



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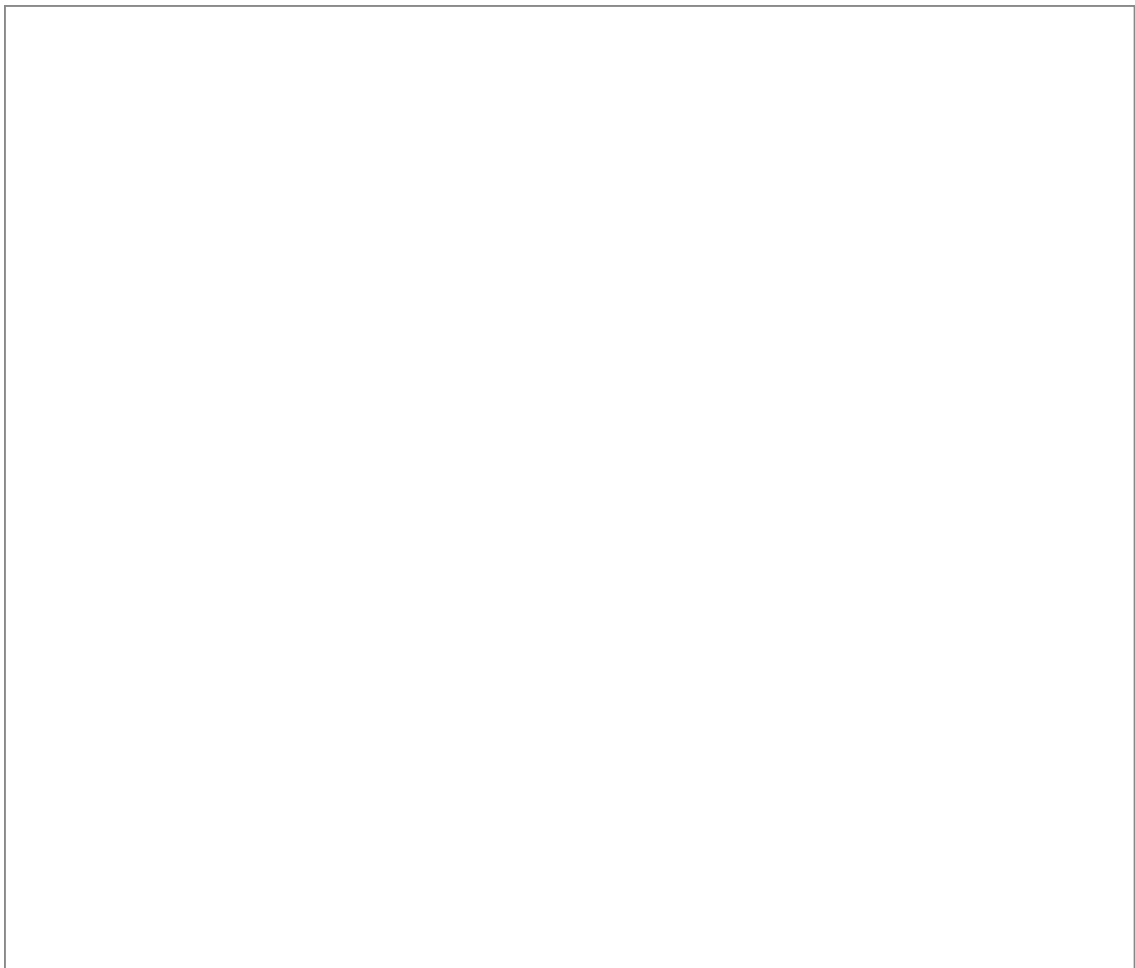
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Interactive 1. Investigate given Rational Functions.

More information for interactive 1

This interactive allows the users to Investigate the rational functions of the forms:

$$f(x) = \frac{ax+b}{cx^2+dx+e} \text{ and } h(x) = \frac{ax^2+bx+c}{dx+e}$$

A graph is displayed with xy axes, where x-axis ranges from -2 to 12 and y-axis ranges from -5 to 15 .

On the top of the graph, Users can enter different numbers for a, b, c, d and e in their respective textboxes, to understand the behavior of the two types of functions. On Clicking on the checkboxes show $f(x)$ with a blue curve and/or $h(x)$ with a red curve and observe the graphs. To see the asymptotes, click on the check box 'show asymptotes'.

If users input values as $a = 1$, $b = 3$, $c = 1$, $d = 1$ and $e = 1$, $h(x) = \frac{x^2+3x+1}{x+1}$

On clicking 'show $h(x)$ ', a pink line displays on the graph representing $h(x)$. On clicking 'show $f(x)$ ', a blue line displays on the graph representing $f(x)$. By clicking on 'show asymptote', dotted lines (in black colour) will appear representing the asymptotes of the given function.

Users can figure out what type of rational functions have one asymptote, horizontal or no horizontal asymptote, cases where there can be more than one horizontal asymptote, and can be more asymptotes on only one side of the graph.



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Different values of a , b , c , d and e give you different forms of functions, in addition to those given above. How is the type of rational function related to these values? Are there values that might not give any rational function at all?

Check points for this investigation:

- What types of rational functions have one vertical asymptote? Two vertical asymptotes?
- What type of rational functions would have a horizontal asymptote/no horizontal asymptote?
- Will it be possible to have more than one horizontal asymptote? Give reasons.
- How can you sketch the graph of a rational function that has only horizontal asymptotes but no vertical asymptotes? What more information do you need to sketch such functions?
- Can there be asymptotes on only one side of the graph, say, only on the negative x -axis?
- Can there be more types of asymptotes than stated in this section (vertical, horizontal or oblique)? If yes, what are they? If not, why not? Explain with examples

Rate subtopic 2.13 Further rational functions

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