TOPIC 1
NUMBER AND ALGEBRA

SUBTOPIC 1.13
FURTHER COMPLEX NUMBERS

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The big picture

Complex numbers can be written in a variety of forms. These include the Cartesian form, $z = a + bi$ that you studied in the previous subtopic, as well as polar and Euler forms that you will study now.

Defining complex numbers in Euler and polar forms leads to Euler's identity which is the famous equation $e^{i\pi} + 1 = 0$.

Many mathematicians call this equation the most beautiful of all mathematical equations. What do you think makes this equation beautiful? Do you find it beautiful?

International Mindedness

The idea of beauty and what is considered beautiful differs for different cultures. Think about what is considered beautiful in your culture and do some research to find what is considered beautiful in one other culture.

How well does the beauty found in Euler's identity correspond to the ideas of beauty in these two cultures?

Watch the video below to see one mathematician explain his thinking about this equation.

The Most Beautiful Equation in Math

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Concept

Complex numbers span a number of mathematical fields. The representationof complex numbers and understanding of their properties involves the use of trigonometry, algebra, and geometry. How are connections between these fields made in the case of complex numbers? What do these connections tell you about the nature of complex numbers and the study of mathematics?

1. Number and algebra / 1.13 Further complex numbers

Polar and Euler forms

Polar form

In the previous subtopic you learned how to find the modulus and argument of a complex number. Use what you know about the modulus and argument to complete the following activity.

Activity

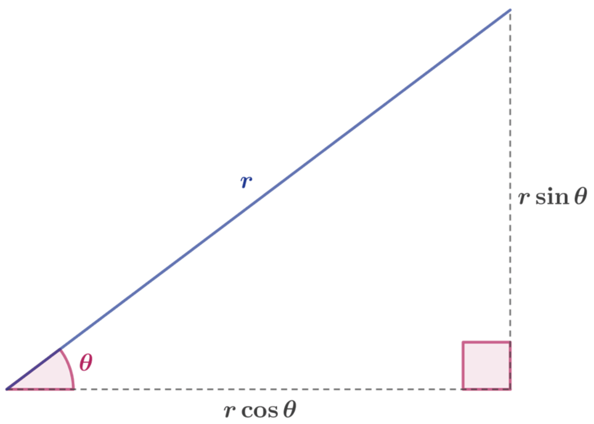
You are given the modulus and argument for the following complex numbers.

Write each in Cartesian form.

$$\arg(z) = 27^\circ\ ,\ |z| = 3 \quad \arg(w) = 65^\circ, |w| = 10 \quad \arg(a) = 127^\circ, |a| = 5$$

Hence, write an expression for a and b in $z = a + b\mathrm{i}$ in terms of $\arg(z) = \theta$ and $|z| = r$.

You can use a right-angled triangle to express a and b in $z = a + b\mathrm{i}$ in terms of the modulus (r) and the argument (θ) of the complex number. The diagram below shows this relationship.



More information

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The image is a diagram of a right-angled triangle used to represent the components of a complex number in polar form. The hypotenuse of the triangle is labeled 'r', which represents the modulus of the complex number. One of the legs of the triangle is labeled 'r cos θ', and the other leg is labeled 'r sin θ'. These correspond to the real and imaginary components of the complex number, respectively, when expressed in terms of the modulus and argument. The angle θ is located at the junction of 'r cos θ' and the hypotenuse 'r', representing the argument of the complex number. The right angle is marked at the intersection between 'r sin θ' and 'r cos θ'. This diagram visually illustrates the relationships between the modulus, argument, and the real and imaginary components of a complex number in polar form.

[Generated by AI]

Since, $z = a + b\mathrm{i}$, $a = r\cos\theta$ and $b = r\sin\theta$, a complex number can be written as $z = r\cos\theta + \mathrm{i}r\sin\theta$. This is called the polar form of a complex number and is also referred to as the modulus–argument form .

Important

The polar (modulus—argument) form of a complex number is $z = r(\cos\theta + \mathrm{i}\sin\theta)$, where r is the modulus and θ is the argument of the complex number.

Be aware

You will often see $z = r\mathrm{cis}\theta$ which is shorthand for $z = r(\cos\theta + \mathrm{isin}\theta)$. Both of these forms of writing a complex numbers are given in the IB formula booklet.

Why are the letters cis used in $z = r\,\mathrm{cis}\,\theta$ notation?

Making connections

Generally, the argument of a complex number in polar form is given in radians rather than degrees unless specified in the question. If you are unfamiliar with radians you can learn about them by visiting subtopic 3.4 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27860/).

Example 1

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Write $z = 3 - 2\mathrm{i}$ in polar form.

Steps	Explanation
$\lvert z\rvert = \sqrt{(3)^2 + (-2)^2} = \sqrt{13}$	Leave as an exact answer.

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Steps	Explanation
$\tan\theta = -\frac{2}{3} \Leftrightarrow \tan^{-1}\left(-\frac{2}{3}\right) = \theta \Leftrightarrow \theta = -0.588$	Use your calculator for the inverse of the tangent. Round to 3 significant figures. Angles measured in radians are usually not denoted with a unit symbol.
$z = \sqrt{13}(\cos(-0.588) + \mathrm{i}\sin(-0.588)) = \sqrt{13}\mathrm{cis}(-0.588)$	Either notation is acceptable for the answer. Note that sine and cosine are both periodic with period 2π, so any multiple of 2π can be added to the argument to get the same complex number.

Be aware

When you are using your calculator to evaluate trigonometric ratios and find the inverses make sure that the mode of the calculator (radians or degrees) matches the unit in which the angle is measured in the question.

Euler form

You can use the modulus and argument to write a complex number in Euler form. This is also referred to as exponential form.

Important

If $\arg z = \theta$ and $|z| = r$, then the complex number z written in Euler (exponential) form is $z = re^{\mathrm{i}\theta}$.

Making connections

There are a number of ways to derive Euler's formula, $e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$, and therefore to show that a complex number can be written in Euler form but they require knowledge of calculus, which you will study in topic 5 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26130/).

Example 2

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Write $w = -5 + \mathrm{i}$ in Euler form.

Steps	Explanation
$\|w\| = \sqrt{(-5)^2 + (1)^2} = \sqrt{26}$	Leave as an exact answer.

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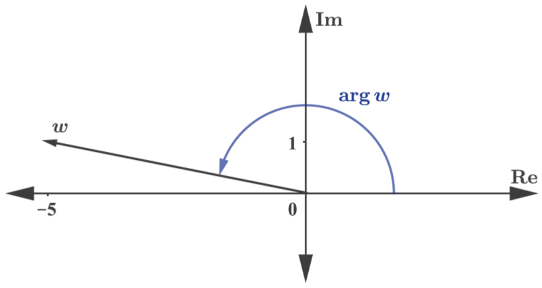
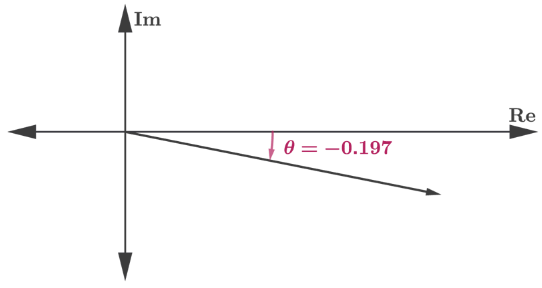
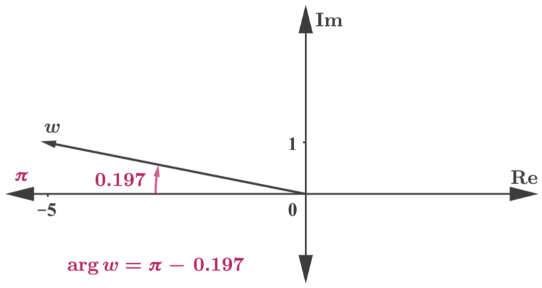
Steps	Explanation
$\tan\theta = -\frac{1}{5}$ $\theta = \tan^{-1}\left(-\frac{1}{5}\right) = -0.197$	The calculator can only give you values for θ in the first and fourth quadrants.
Argand diagram for w :  Diagram for $\theta = -0.197$:  Finding $\arg w$:  $\arg w = \pi - 0.197 = 2.94$	You should always make a sketch of a complex number on the Argand plane when finding the argument. Remember that positive angles are measured anticlockwise and negative angles are measured clockwise starting from the positive x-axis. You can see from the diagrams for w and $\theta = -0.197$ that the argument of $w \neq -0.197$. You should use the reference angle of 0.197 in the second quadrant to find the $\arg w$.
$w = \sqrt{26}e^{2.94\mathrm{i}}$	

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Be aware

The calculator gives you values for tangent inverses in the first and fourth quadrants. You should always sketch an Argand diagram for your complex number when finding its argument. You can review this skill by visiting section 1.12.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/cartesian-form-of-complex-numbers-id-27423/).

Example 3

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Write $z = 2e^{\frac{\pi}{2}\mathrm{i}}$ in Cartesian form.

Steps	Explanation
$z = 2e^{\frac{\pi}{2}\mathrm{i}} = 2\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right) = 2\left(0 + \mathrm{i}\times(1)\right) = 2\mathrm{i}$	Rewrite in polar form and evaluate sine and cosine.

The table below shows more examples of converting between Cartesian, polar and Euler forms.

Cartesian $z = x + iy$	$r = \|z\|$	$\theta = \mathrm{Arg}(z)$	Polar form $z = r\,\mathrm{cis}\,\theta$	Euler form $z = r\,\mathrm{e}^{\mathrm{i}\theta}$
$2 + \mathrm{i}3$	$\sqrt{13}$	0.983 rad	$\sqrt{13}\,\mathrm{cis}\,(0.983)$	$\sqrt{13}\,\mathrm{e}^{\mathrm{i}0.983}$
$3 + 2\mathrm{i}$	$\sqrt{13}$	0.588 rad	$\sqrt{13}\,\mathrm{cis}\,(0.588)$	$\sqrt{13}\,\mathrm{e}^{\mathrm{i}0.588}$
$-2 + \mathrm{i}$	$\sqrt{5}$	2.68 rad	$\sqrt{5}\,\mathrm{cis}\,(2.68)$	$\sqrt{5}\,\mathrm{e}^{\mathrm{i}2.68}$
$1 - \mathrm{i}$	$\sqrt{2}$	$-\frac{\pi}{4}$	$\sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$	$\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}$
$-\frac{3}{2} - \frac{3}{2}\mathrm{i}$	$\frac{3}{\sqrt{2}}$	$-\frac{3\pi}{4}$	$\frac{3}{\sqrt{2}}\mathrm{cis}\left(-\frac{3\pi}{4}\right)$	$\frac{3}{\sqrt{2}}\,\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}$

You can use your calculator to convert complex numbers from Euler to Cartesian form and from Cartesian to Euler form. If you need to practise more conversion questions, you can create your own examples and check your work using the calculator.

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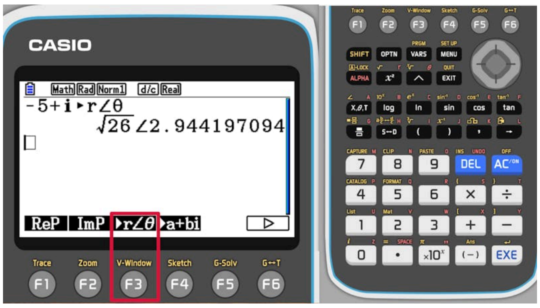
Steps	Explanation
To convert a complex number from Cartesian form to polar form, choose the options (OPTN) and press F3 to see the options related to complex numbers ...	
... press F6 to scroll to see more options ...	
... and press F3 for the conversion. Note, that the calculator shows the modulus and the argument separated by an angle symbol $\angle$. If you want to convert from polar form to Cartesian form, you need to press F4 at this point.	

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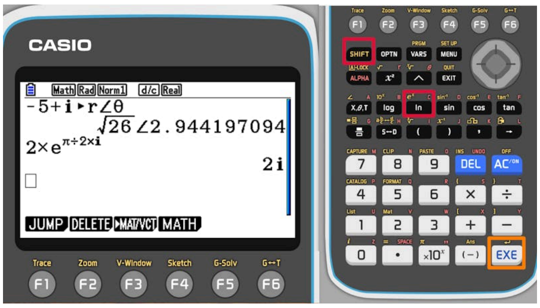
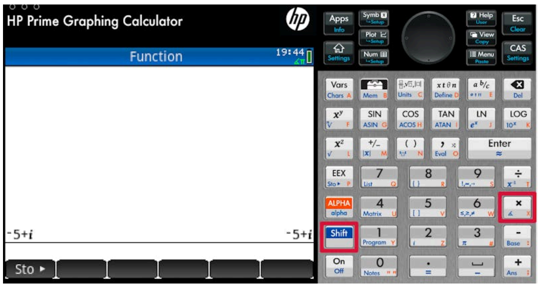
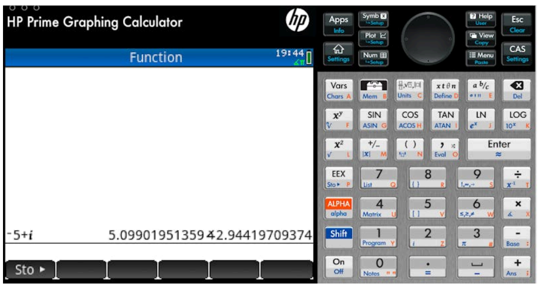
Steps	Explanation
The calculator also understands the Euler form of a complex number. Enter the expression and press EXE. The Cartesian form will be displayed.	

Steps	Explanation
To convert a complex number between Cartesian and polar form, press the angle symbol (Shift ∡). This will do the conversion both ways.	
Note, that the calculator shows the modulus and the argument separated by an angle symbol ∡.	

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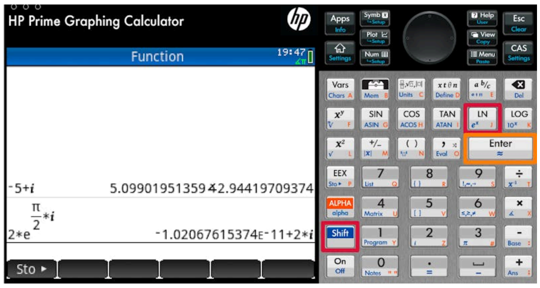
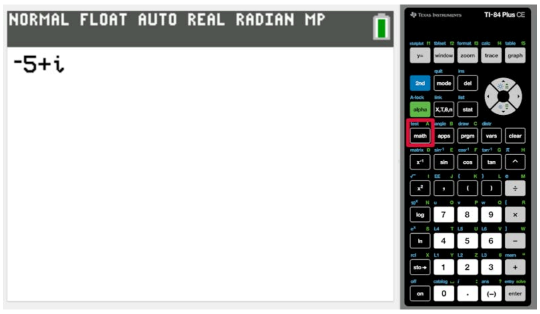
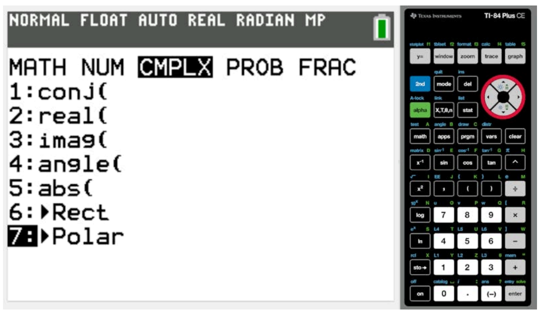
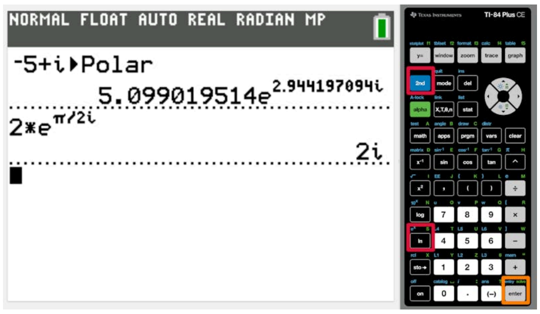
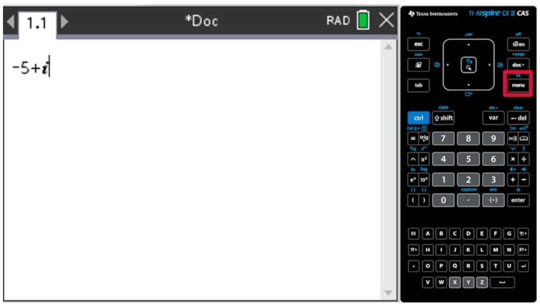
Steps	Explanation
The calculator also understands the Euler form of a complex number. Enter the expression and press enter. The Cartesian form will be displayed. Note, that this conversion is not exact. In this case the real part is exactly 0, but instead of the exact value the calculator gives a small number.	

Steps	Explanation
To convert a complex number from Cartesian form to polar form, choose the math options ...	
... and find the option to convert to polar form. Note that if you want to convert from polar to Cartesian form, you need to choose option 6 here.	

Steps	Explanation
You will get the modulus and the argument written using the Euler notation.	
If you enter the expression in Euler form and press enter, the Cartesian form will be displayed.	

Steps	Explanation
To convert a complex number from Cartesian form to polar form, open the menu …	

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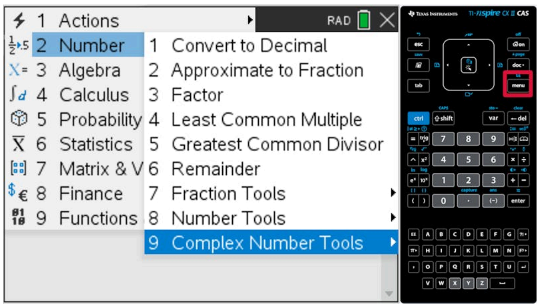
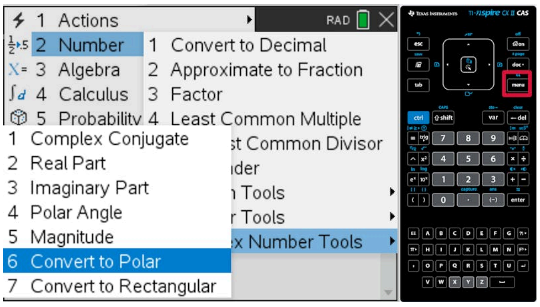
Steps	Explanation

... navigate to the complex number tools, ...

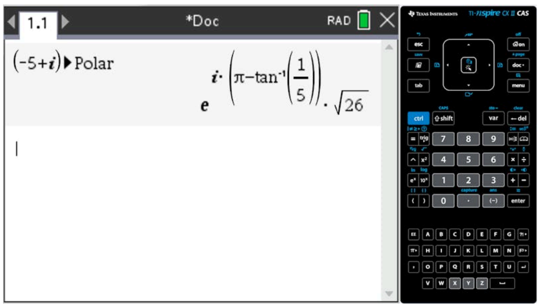


... and choose the option to convert to polar.

Note, that if you need to convert from polar form to Cartesian form, you need to choose option 7 here.



If you press enter, you get an exact expression. This may be useful in some ocasions, but more often it is not very helpful.



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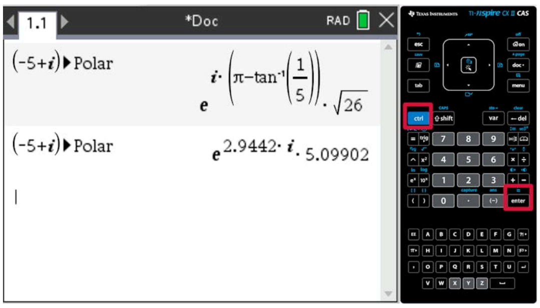
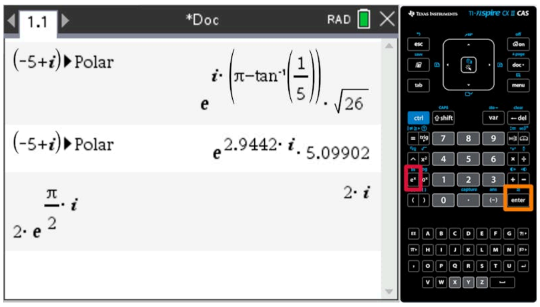
Steps	Explanation
To get the approximate value, you need to press ctrl/ ≈. You will get the modulus and the argument written using the Euler notation.	
If you enter the expression in Euler form and press enter, the Cartesian form will be displayed.	

Exam tip

In topic 3 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26035/) you will learn how to find exact ratios and inverses for a set of special angles using a table such as the one shown below.

In an exam, you will be expected to apply this skill to your work with complex numbers in polar and Euler forms.

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Degrees	0	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

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Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Degrees	0	$30°$	$45°$	$60°$	$90°$
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined

4 section questions

1. Number and algebra / 1.13 Further complex numbers

Multiplication and division

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/multiplication-and-division-id-27570/print/) Assign

Working with complex numbers in Euler and polar forms simplifies calculations for multiplication and division.

Multiplication

Activity

Complete the following table:

z_1	z_2	$z_1 \times z_2$	Polar form of z_1	Polar form of z_2	Polar form of $z_1 \times z_2$
$2-\mathrm{i}$	$3+2\mathrm{i}$	$8+\mathrm{i}$			
$-3-4\mathrm{i}$	$7-\mathrm{i}$				
$1+\mathrm{i}$	$1-2\mathrm{i}$				

Comment on any patterns that you notice.

Important

Given $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$, then

$$z_1 \times z_2 = r_1 \times r_2(\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2)),$$

and given $z_1 = r_1 e^{\mathrm{i}\theta_1}$ and $z_2 = r_2 e^{\mathrm{i}\theta_2}$ then

$$z_1 \times z_2 = r_1 \times r_2 e^{\mathrm{i}(\theta_1+\theta_2)}.$$

Making connections

The result for multiplication of two complex numbers in polar form can be derived using the compound angle identities . These identities are not in the syllabus, you do not need to know the proof presented below.

Compound angle identities are:

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$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Multiplication in polar form:

$$\begin{aligned} z_1 \times z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 \times r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2) \end{aligned}$$

Group real terms together. Group imaginary terms together.

$$= r_1 \times r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2))$$

Use compound angle identities.

$$\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \cos(\theta_1 + \theta_2)$$

$$\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 = \sin(\theta_1 + \theta_2)$$

$$z_1 \times z_2 = r_1 \times r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Example 1

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Consider the complex numbers $z_1 = -1 + \mathrm{i}$ and $z_2 = 2\sqrt{3} - 2\mathrm{i}$.

Convert to polar form and multiply.

Steps	Explanation
$\lvert z_1\rvert = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$ $\theta = \tan^{-1}\left(\frac{1}{-1}\right) = -\frac{\pi}{4} \approx -0.785$ Since z_1 is in the second quadrant: $\arg z_1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \approx 2.356$ $z_1 = \sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4}\right)$	Convert z_1 .
$\lvert z_2\rvert = \sqrt{(2\sqrt{3})^2 + (-2)^2} = 4$ $\theta = \tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right) = -\frac{\pi}{6} \approx -0.524$ Since z_2 is in the fourth quadrant: $\arg z_2 = -\frac{\pi}{6} \approx -0.524$ $z_2 = 4\,\mathrm{cis}\left(-\frac{\pi}{6}\right)$	Convert z_2 .
$z_1 \times z_2 = 4\sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4} + \left(-\frac{\pi}{6}\right)\right)$ $= 4\sqrt{2}\,\mathrm{cis}\left(\frac{7\pi}{12}\right)$	Multiply.

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Steps	Explanation
$z_1 \times z_2 = 4\sqrt{2}\,\text{cis}\,(2.356 + (-0.524))$ $= 4\sqrt{2}\,\text{cis}\,(1.83)$	The answer if you are doing this question without using the exact values.

Division

Rules for division of complex numbers in Euler and polar forms can be derived using the same methods as multiplication.

✓ **Important**

Given $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos\ (\theta_1 - \theta_2) + \mathrm{i}\sin\ (\theta_1 - \theta_2)\right).$$

And given $z_1 = r_1 e^{\mathrm{i}\theta_1}$ and $z_2 = r_2 e^{\mathrm{i}\theta_2}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{\mathrm{i}(\theta_1 - \theta_2)}.$$

Example 2

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Given that $z = 3e^{2.17\mathrm{i}}$ and $w = 4\,\text{cis}\,(-0.172)$, write $\frac{w}{z}$ in polar form.

$$\frac{w}{z} = \frac{4}{3}\text{cis}\,(-0.172 - 2.17) = \frac{4}{3}\text{cis}\,(-2.34)$$

Example 3

★★★

Consider the complex numbers $z_1 = 1 - \mathrm{i}$ and $z_2 = \sqrt{3} + \mathrm{i}$.

Convert to polar form and find $\frac{z_1}{z_2}$.

Then use these results to find the **exact** values of $\cos\left(-\frac{5\pi}{12}\right)$ and $\sin\left(-\frac{5\pi}{12}\right)$.

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Steps	Explanation
$\|z_1\| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$ $\theta = \tan^{-1}\left(-\frac{1}{1}\right) = -\frac{\pi}{4}$ Since z_1 is in the fourth quadrant $\arg z_1 = -\frac{\pi}{4}$. $z_1 = \sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$	For this question you should use exact values of the inverse trigonometric ratios because the second part of the question asks you for values of $-\frac{5\pi}{12}$. You may need to refer to the table of exact values given in **section 1.13.1** .
$\|z_2\| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$ $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ Since z_2 is in the first quadrant $\arg z_2 = \frac{\pi}{6}$. $z_2 = 2\text{cis}\left(\frac{\pi}{6}\right)$	
$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\text{cis}\left(-\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}\text{cis}\left(-\frac{5\pi}{12}\right)$	
$\frac{z_1}{z_2} = \frac{1-\text{i}}{\sqrt{3}+\text{i}}$ $= \left(\frac{1-\text{i}}{\sqrt{3}+\text{i}}\right) \times \left(\frac{\sqrt{3}-\text{i}}{\sqrt{3}-\text{i}}\right)$ $= \frac{\sqrt{3} - \text{i} - \sqrt{3}\text{i} + \text{i}^2}{3+1}$ $= \frac{\sqrt{3}-1}{4} + \frac{-\sqrt{3}-1}{4}\text{i}$	Divide in Cartesian form.
$\text{Re}\left(\frac{z_1}{z_2}\right) = \frac{\sqrt{2}}{2}\cos\left(-\frac{5\pi}{12}\right) = \frac{\sqrt{3}-1}{4} \Leftrightarrow$ $\cos\left(-\frac{5\pi}{12}\right) = \frac{\sqrt{3}-1}{4} \times \frac{2}{\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$ $\text{Im}\left(\frac{z_1}{z_2}\right) = \frac{\sqrt{2}}{2}\sin\left(-\frac{5\pi}{12}\right) = \frac{-\sqrt{3}-1}{4} \Leftrightarrow$ $\sin\left(-\frac{5\pi}{12}\right) = \frac{-\sqrt{3}-1}{4} \times \frac{2}{\sqrt{2}} = \frac{-\sqrt{6}-\sqrt{2}}{4}$	Equate real and imaginary parts in polar and Cartesian forms to find exact values.

6 section questions

1. Number and algebra / 1.13 Further complex numbers

Powers of complex numbers

Student view

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/powers-of-complex-numbers-id-27571/print/) Assign

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Activity

Let $z = r \operatorname{cis} \theta$. Use your knowledge of multiplication in polar form to find:

z^2 z^3 z^4 z^5

Propose a conjecture for the result of z^n .

Important

De Moivre's theorem

For $z = r(\cos \theta + \mathrm{i} \sin \theta)$, $z^n = r^n(\cos n\theta + i \sin n\theta)$, where $n \in \mathrm{Z}$.

This can also be written as $z^n = r^n \operatorname{cis} n\theta$ or $z^n = r^n e^{\mathrm{i}n\theta}$.

De Moivre's theorem can be shown by repeated application of multiplication or division. It can also be illustrated using the exponential form of complex numbers.

International Mindedness

Abraham De Moivre (1667—1754), for whom the theorem is named, was a French mathematician who worked in London. In addition to his work with complex numbers, he also made advances in the field of probability and normal distributions. In 1710, he was appointed to a Royal Society commission tasked with deciding whether Isaac Newton or Gottfried Liebniz should be credited with the invention of calculus.

Example 1

★☆☆

Show that if n is a positive integer and $z = re^{\mathrm{i}\theta}$, then $z^n = r^n e^{\mathrm{i}n\theta}$.

Steps	Explanation
$z^n = (re^{\mathrm{i}\theta})^n = r^n(e^{\mathrm{i}\theta})^n = r^n e^{\mathrm{i}n\theta}$	The properties of exponents are true even when the base is a complex number and the exponent is a positive integer .

De Moivre's theorem is very useful when working with powers of complex numbers. Consider the following examples.

Example 2

★★☆

Given that $z = 2 + 2\sqrt{3}\mathrm{i}$, find z^3 in $a + b\mathrm{i}$ form.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Method 1

Steps	Explanation
$z^3 = (2)^3 + 3(2)^2(2\sqrt{3}\mathrm{i}) + 3(2)(2\sqrt{3}\mathrm{i})^2 + (2\sqrt{3}\mathrm{i})^3$ $= 8 + 24\sqrt{3}\mathrm{i} - 72 - 24\sqrt{3}\mathrm{i}$ $= -64$	Using the binomial theorem in Cartesian form.

Method 2

Steps	Explanation
$\lvert z\rvert = \sqrt{(2)^2 + (2\sqrt{3})^2} = 4$ $\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$ Since z is in the first quadrant, $\arg z = \frac{\pi}{3}$. $z = 4\,\mathrm{cis}\frac{\pi}{3}$	Convert to polar form.
$z^3 = 4^3\,\mathrm{cis}\,\pi = 64(\cos\pi + \mathrm{i}\sin\pi) = -64$	Use De Moivre's theorem.

Would you want to use the binomial theorem method for a question asking you to find z^{10} ?

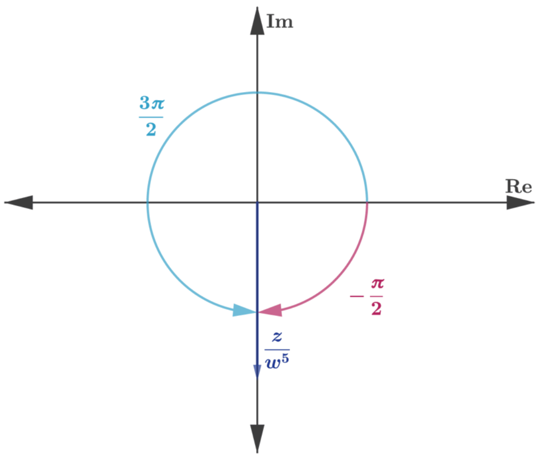
Example 3

★★☆

Write $\frac{1+\mathrm{i}}{(1-\mathrm{i})^5}$ in polar form.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
Let $z = 1 + \mathrm{i}$: $$\lvert z\rvert = \sqrt{2}$$ $$\theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$$ Since z is in the first quadrant, $\arg z = \frac{\pi}{4}$. $$z = \sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}$$ Let $w = 1 - \mathrm{i}$: $$\lvert w\rvert = \sqrt{2}$$ $$\theta = \tan^{-1} - \frac{1}{1} = -\frac{\pi}{4}$$ Since w is in the fourth quadrant, $\arg w = -\frac{\pi}{4}$ $$w = \sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$$	Convert to polar form.
$w^5 = (\sqrt{2})^5\,\mathrm{cis}\left(-\frac{5\pi}{4}\right)$	Use De Moivre's theorem.
$$\frac{z}{w^5} = \frac{\sqrt{2}}{(\sqrt{2})^5}\,\mathrm{cis}\left(\frac{\pi}{4} - \left(-\frac{5\pi}{4}\right)\right)$$ $$= \left(\sqrt{2}\right)^{-4}\mathrm{cis}\,\frac{6\pi}{4}$$ $$= \frac{1}{4}\,\mathrm{cis}\,\frac{3\pi}{2}$$	Divide.
 $$\frac{z}{w^5} = \frac{1}{4}\,\mathrm{cis}\left(-\frac{\pi}{2}\right)$$	The argument of the result is not in the range for the principle argument which is $-\pi$ to π . You need to find an angle that is equivalent to $\frac{3\pi}{2}$ in this range.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Exam tip

Unless a question specifies otherwise, you are expected to write the principle argument in your final answers. Always check that your final answer gives θ in the range $-\pi < \theta \leq \pi$. If this is not the case find the equivalent angle in the range of the principle argument.

Example 4

★★☆

If $w = \sqrt{3} - \mathrm{i}$, find $\dfrac{1}{w^2}$.

Method 1

Steps	Explanation
$\lvert w\rvert = \sqrt{4} = 2$ $\theta = \tan^{-1} - \dfrac{1}{\sqrt{3}} = -\dfrac{\pi}{6}$ Since w is in the fourth quadrant, $\arg w = -\dfrac{\pi}{6}$. $w = 2\,\mathrm{cis}\left(-\dfrac{\pi}{6}\right)$	Convert to polar form.
$\dfrac{1}{w^2} = w^{-2}$ $= 2^{-2}\,\mathrm{cis}\left(\dfrac{2\pi}{6}\right)$ $= \dfrac{1}{4}\,\mathrm{cis}\left(\dfrac{\pi}{3}\right)$	

Method 2

Steps	Explanation
$\lvert w\rvert = \sqrt{4} = 2$ $\theta = \tan^{-1} - \dfrac{1}{\sqrt{3}} = -\dfrac{\pi}{6}$ Since w is in the fourth quadrant, $\arg w = -\dfrac{\pi}{6}$. $w = 2\ \mathrm{cis}\left(-\dfrac{\pi}{6}\right)$	Convert to polar form.
$w^2 = 2^2\ \mathrm{cis}\left(-\dfrac{2\pi}{6}\right)$ $= 4\ \mathrm{cis}\left(-\dfrac{\pi}{3}\right)$	

Student view

Steps	Explanation
$1 = 1\,\text{cis}\,0$ $\frac{1}{w^2} = \frac{1}{4}\text{cis}\left(0 - \left(-\frac{\pi}{3}\right)\right)$ $= \frac{1}{4}\text{cis}\,\frac{\pi}{3}$	Divide.

4 section questions

1. Number and algebra / 1.13 Further complex numbers

Geometric interpretations

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/geometric-interpretations-id-27572/print/) Assign

Conjugates and negatives

Activity

Let $z = 2 + \text{i}$ and let z^* be its complex conjugate. Sketch $z, -z, z^*$ and $-z^*$ on an Argand plane. Comment on any patterns that you notice.

Write $z, -z, z^*$ and $-z^*$ in polar form and closely examine the arguments of these numbers.

Given that $z = r\,\text{cis}\,\theta$, write an expression for $-z, z^*$ and $-z^*$ in terms of $r.\ \theta$ and π .

Important

If $z = r\,\text{cis}\,\theta$, then: $-z = r\,\text{cis}\,(\theta + \pi)$ $z^* = r\,\text{cis}\,(2\pi - \theta)$ $-z^* = r\,\text{cis}\,(\pi - \theta)$

Why can't you write $-z = -r\,\text{cis}\,\theta$ instead of $-z = r\,\text{cis}\,(\theta + \pi)$?

Exam tip

The IB formula booklet does not give you any information about the geometric relationships for complex numbers. However, you do not need to memorise them so long as you can work them out them using Argand diagrams.

Addition and subtraction

Activity

In the first applet below, add z_1 and z_2 algebraically and compare your result with the diagram shown. Move the purple points to change z_1 and z_2 and add them again.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Describe the geometrical properties of the sum of z_1 and z_2 .

In the second applet below, find $z_1 - z_2$ algebraically and compare your result with the diagram shown. Move the purple points to change z_1 and z_2 and repeat the calculations.

Describe the geometrical properties of $-z_2$ and $z_1 - z_2$.

Interactive 1. Geometric Interpretation of Complex Number Addition.

Credit: GeoGebra (https://www.geogebra.org/m/FEqW6uZs) GeoGebra Materials Team

More information for interactive 1

The interactive applet allows users to explore the addition of complex numbers both visually and algebraically. It features an Argand plane with an x-axis ranging from -10 to 12 and a y-axis from -8 to 8. Two movable points, $Z1$ and $Z2$, are shown on the plane with vector arrows originating from the origin $(0, 0)$, representing the complex numbers. As users drag the purple points to change the values of $Z1$ and $Z2$, the applet dynamically updates to show the sum $Z1 + Z2$ as a new point on the plane. The numeric values of $Z1$, $Z2$, and their sum are also displayed, helping users connect algebraic addition to its geometric interpretation.

For example, $Z1$ is $(1.27, 4.15)$ and $Z2$ is $(4.86, -1.18)$ then $Z1 + Z2$ will be $(6.12, 2.96)$.

Through this visual, learners can understand that adding complex numbers corresponds to vector addition — where placing the tail of $Z2$ at the head of $Z1$ (or vice versa) gives a diagonal representing the sum. This reinforces the concept that complex number addition can be viewed as the addition of directed line segments in the Argand diagram.

Student view

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(/study/app/math-ai-hl/sid-132-cid-761618/ov

Interactive 2. Geometric Interpretation of Complex Numbers.

Credit: GeoGebra (https://www.geogebra.org/m/VZAzYFHh) GeoGebra Materials Team

More information for interactive 2

This interactive applet allows users to explore the subtraction of complex numbers both visually and algebraically. The display features an Argand plane with the x-axis ranging from -10 to 12 and the y-axis from -8 to 8. Two movable points, $Z1$ and $Z2$, are shown in blue on the plane with directional vector arrows starting from the origin $(0, 0)$, representing complex numbers.

As users drag the purple points to change the values of $Z1$ and $Z2$, the applet dynamically calculates and displays the difference $Z2 - Z1$ as a green vector. Additionally, the negative of $Z1$, represented as $-Z1$, is shown as a red vector pointing in the opposite direction of $Z1$. These vectors provide a visual representation of how subtraction works geometrically in the complex plane. The diagram also uses dashed lines to connect the tip of $Z1$ to $Z2$, illustrating the resultant vector $Z2 - Z1$ as the diagonal of the triangle formed by the two original vectors.

For example, when $Z1 = (1, 3)$ and $Z2 = (4, 2)$, then $Z2 - Z1 = (3, -1)$, which is shown in green. The applet highlights how subtracting complex numbers is equivalent to vector subtraction: drawing a vector from the tip $Z1$ of to the tip of $Z2$ yields the result. This visualization reinforces the geometric interpretation of complex number subtraction as adding the inverse vector $-Z1$ to $Z2$.

Making connections

Complex numbers can be represented as vectors on the Argand plane. The addition of complex numbers follows the parallelogram rule for addition of vectors. Subtraction can be done by adding a negative vector.

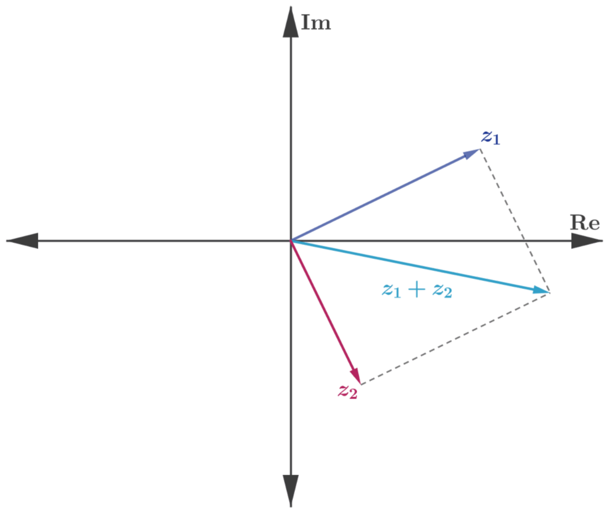
Important

The sum of two complex numbers can be represented in the Argand diagram as the diagonal of the parallelogram formed by the two complex numbers, as shown in the first diagram below.

The difference of two complex numbers can be represented in the Argand diagram as the sum of z_1 and $-z_2$, as seen in the second diagram below.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov



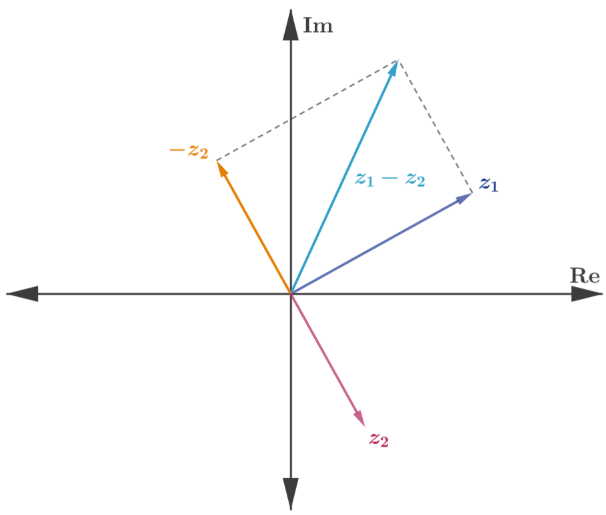
More information

The image is a diagram depicting vector addition in the complex plane. It features two axes: the horizontal axis labeled 'Re' for the real part and the vertical axis labeled 'Im' for the imaginary part. Three vectors are shown originating from the origin:

1. Vector z1, represented by a blue line, extends to the point labeled 'z1'.
2. Vector z2, depicted as a magenta line, extends to the point labeled 'z2'.
3. A cyan line represents the resultant vector z1 + z2, extending to the point labeled 'z1 + z2'.

Dashed lines indicate the components and resultant paths of the vectors. The arrows convey direction and are situated within a Cartesian coordinate system representing complex numbers.

[Generated by AI]



More information

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

The image shows a diagram on a complex plane with real (Re) and imaginary (Im) axes. The following vectors are displayed: z1 extending from the origin to its endpoint, z2 extending in another direction, and z1 - z2 connecting the endpoints of z2 to z1, forming a parallelogram. The vector -z2 is also shown, extending from the origin in the direction opposite to z2. Arrows indicate the vectors, and the diagram illustrates vector addition and subtraction on the complex plane.

[Generated by AI]

Example 1

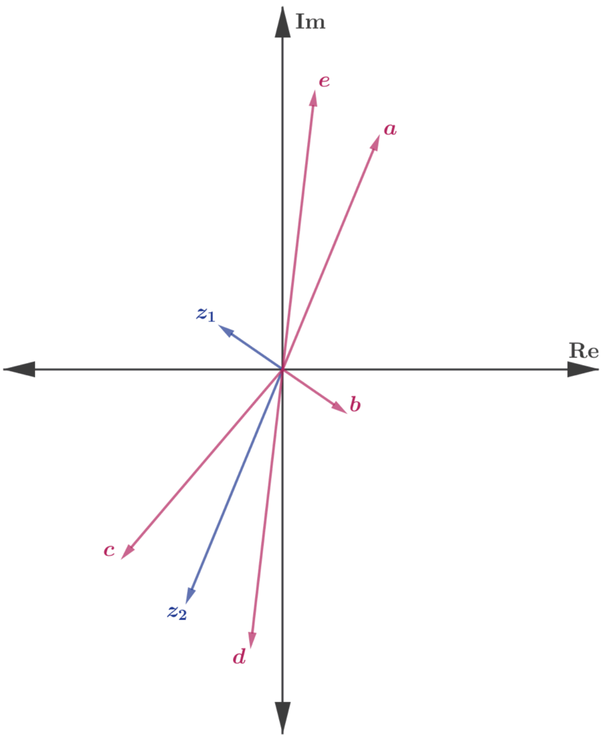
★☆☆

Determine which vector in the diagram shown represents:

a) $z_1 + z_2$

b) $-z_1$

c) $z_2 - z_1$.



More information

The image is a diagram of the complex plane with a vertical axis labeled 'Im' for imaginary numbers and a horizontal axis labeled 'Re' for real numbers. At the center, where the axes meet, several vectors originate and extend outward in different directions. The vectors are labeled 'a', 'b', 'c', 'd', and 'e' in pink, with varying lengths and angles on the plane.

Student view

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Two vectors labeled 'z1' and 'z2' are shown in blue, adding to the complexity of the diagram. Additionally, the image text associate with the image, expressed as c) (z_2 - z_1), implies a focus on the vector or mathematical representation of the difference between two points 'z2' and 'z1' on the complex plane. Each vector indicates a particular direction and magnitude relative to the origin of the axis.

[Generated by AI]

	Steps	Explanation
a)	$z_1 + z_2$ is represented by c.	The sum should look like the diagonal of a parallelogram formed by z_1 and z_2 .
b)	$-z_1$ is represented by b.	$-z_1$ should be the same length as z_1 and rotated 180° about the origin.
c)	$z_2 - z_1$ is represented by d.	$z_2 - z_1$ is the sum of z_2 and $-z_1$ or z_2 and b .

Multiplication by a real number

Activity

Let $z = 2\,\text{cis}\,\frac{\pi}{4}$. Sketch $2z, 3z, 5z, \frac{1}{2},$ and $\frac{1}{4}z$ on an Argand diagram.

Describe the geometric significance of multiplying a complex number by a real number.

Important

If $z = r\,\text{cis}\,\theta$ and a is a positive real number, then $a \times z = a \times r\text{cis}\theta$.

It can be said that the modulus is stretched by a factor of a and the argument remains unchanged.

Example 2

★☆☆

Explain why it is unnecessary to describe the geometric significance of division by a real number separately to multiplication.

Division is the same as multiplication by a reciprocal which is already described under multiplication.

Student view

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Multiplication and division of complex numbers

Example 3

★★☆

Given that $w = r \operatorname{cis} \theta$ and $z = m \operatorname{cis} \varphi$, find:

a) $w \times z$

b) $\frac{w}{z}$.

Hence, describe geometrically how w is related to:

c) $w \times z$

d) $\frac{w}{z}$.

a) $w \times z = (r \times m) \operatorname{cis}(\theta + \varphi)$

b) $\frac{w}{z} = \frac{r}{m} \operatorname{cis}(\theta - \varphi)$

c) When w was multiplied by z it was stretched by a factor of m and rotated anticlockwise through an angle φ .

d) When w was divided by z it was stretched by a factor of $\frac{1}{m}$ and rotated clockwise through an angle φ .

✓ **Important**

When a complex number is multiplied or divided by another complex number, it is stretched and rotated.

Example 4

★☆☆

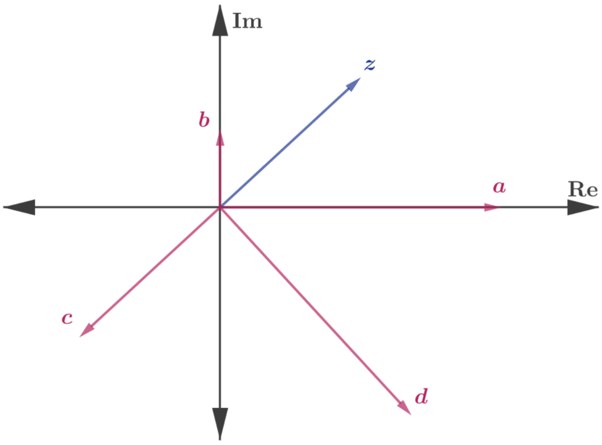
Determine which of the vectors, in the diagram shown, is the most appropriate representation of

a) $z \times \left(\frac{1}{2} \operatorname{cis} \frac{\pi}{4}\right)$

b) $\frac{z}{\frac{2}{3} \operatorname{cis} \frac{\pi}{2}}$

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov



More information

The image is a diagram of the complex plane featuring axes labeled Im (imaginary) and Re (real). There are four vectors extending from the origin, labeled a, b, c, and d, and one vector labeled z. The vector a points to the right along the real axis, vector b points upward along the imaginary axis, vector c points to the bottom left, and vector d points downward to the right. The vector z is angled upwards between the real and imaginary axes, suggesting a complex number's representation with both real and imaginary components. The diagram illustrates the geometric representation of complex numbers and their operations in a two-dimensional space.

[Generated by AI]

a) Multiplying z by $\frac{1}{2}$ cis $\frac{\pi}{4}$ will stretch z by a factor of $\frac{1}{2}$ and rotate it anticlockwise by $\frac{\pi}{4}$. This is most closely represented by vector b .

b) Dividing z by $\frac{2}{3}$ cis $\frac{\pi}{2}$ will stretch z by a factor of $\frac{3}{2}$ and rotate it clockwise by $\frac{\pi}{2}$.

This is most closely represented by vector d .

3 section questions

1. Number and algebra / 1.13 Further complex numbers

Addition of sinusoidal functions

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/addition-of-sinusoidal-functions-id-27573/print/) Assign

Properties of the exponential and polar forms of complex numbers are used for addition of periodic models, namely, sine and cosine.

Addition of sine and cosine functions has a very important real-world application in the field of electric circuits, where voltages from AC (alternating current) sources can be modelled as sine or cosine functions.

Student view

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Be aware

Sine and cosine models are studied in topic 2 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26012/). If you have not yet studied trigonometric models, you should come back to this subtopic once you have.

Activity

Let $z_1 = 5\ \text{cis}\theta$ and $z_2 = 2\text{cis}\left(\theta + \frac{\pi}{2}\right)$.

Write down an expression for $\text{Re}(z_1)$ and $\text{Re}(z_2)$. What would $\text{Re}(z_1)$ and $\text{Re}(z_2)$ look like if you graphed them?

Write down an expression for $\text{Im}(z_1)$ and $\text{Im}(z_2)$. What would $\text{Im}(z_1)$ and $\text{Im}(z_2)$ look like if you graphed them?

Let $w = r\ \text{cis}(\theta + a)$, where a is a constant.

Describe the graph of $\text{Re}(w)$ and the graph of $\text{Im}(w)$.

As you can see from the activity, the real part of a complex number corresponds to a cosine function and the imaginary part to a sine function. This information is generalised as follows.

Important

Cosine and sine functions can be expressed as real and imaginary parts of complex numbers.

If $f(x) = r\cos(ax+b)$, then $f(x) = \text{Re}(r\ \text{cis}(ax+b)) = \text{Re}(re^{(ax+b)\text{i}})$.

If $g(x) = r\sin(ax+b)$, then $g(x) = \text{Im}(r\ \text{cis}(ax+b)) = \text{Im}(re^{(ax+b)\text{i}})$.

Example 1

★☆☆

Write $f(x) = 2\sin(3x+1)$ in terms of a complex number in polar form.

Steps	Explanation
$f(x) = 2\sin(3x+1)$ $= \text{Im}(2\,\text{cis}(3x+1))$	Use $g(x) = r\sin(ax+b)$ $= \text{Im}(r\,\text{cis}(ax+b))$.

The ability to write cosine and sine models in terms of complex numbers allows you to add these models together.

Example 2

★★☆

Student view

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a) Let $z_1 = a + b\mathrm{i}$ and $z_2 = c + d\mathrm{i}$.

Show that $\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$ and that $\mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$.

b) Hence, show that if $f(x) = 2\cos(x+1)$ and $g(x) = 3\cos(x+\pi)$, then $f(x) + g(x) = \mathrm{Re}\left(e^{x\mathrm{i}}\left(2\mathrm{e}^{\mathrm{i}} + 3\mathrm{e}^{\pi\mathrm{i}}\right)\right)$.

c) Hence, write $f(x) + g(x)$ in $A\cos(Bx + C)$ form.

	Steps	Explanation
a)	$z_1 + z_2 = a + c + (b+d)\mathrm{i}$ $\mathrm{Re}(z_1) = a, \mathrm{Re}(z_2) = c$ $\mathrm{Re}(z_1 + z_2) = a + c = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$ $\mathrm{Im}(z_1) = b, \mathrm{Im}(z_2) = d$ $\mathrm{Im}(z_1 + z_2) = b + d = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$	
b)	$f(x) = \mathrm{Re}\left(2\mathrm{e}^{(x+1)\mathrm{i}}\right)$ $g(x) = \mathrm{Re}\left(3\mathrm{e}^{(x+\pi)\mathrm{i}}\right)$	Rewrite the functions in terms of complex numbers in exponent form.
	$f(x) + g(x) = \mathrm{Re}\left(2\mathrm{e}^{(x+1)\mathrm{i}} + 3\mathrm{e}^{(x+\pi)\mathrm{i}}\right)$ $= \mathrm{Re}\left(2\mathrm{e}^{x\mathrm{i}}\mathrm{e}^{\mathrm{i}} + 3\mathrm{e}^{x\mathrm{i}}\mathrm{e}^{\pi\mathrm{i}}\right)$ $= \mathrm{Re}\left(\mathrm{e}^{x\mathrm{i}}\left(2\mathrm{e}^{\mathrm{i}} + 3\mathrm{e}^{\pi\mathrm{i}}\right)\right)$	Using $\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$, add the functions. Use exponent rules to rewrite in the required form.
c)	$2\mathrm{e}^{\mathrm{i}} + 3\mathrm{e}^{\pi\mathrm{i}} \approx 2.55\mathrm{e}^{2.42\mathrm{i}}$	Use your calculator to add and convert to exponential form. Calculator instructions for converting between Cartesian and polar/Euler forms are found in **section 1.13.1** .
	$f(x) + g(x) = \mathrm{Re}\left(\mathrm{e}^{x\mathrm{i}}\left(2\mathrm{e}^{\mathrm{i}} + 3\mathrm{e}^{\pi\mathrm{i}}\right)\right)$ $= \mathrm{Re}\left(\mathrm{e}^{x\mathrm{i}}\left(2.55\mathrm{e}^{2.42\mathrm{i}}\right)\right)$ $= \mathrm{Re}\left(2.55\left(\mathrm{e}^{(x+2.42)\mathrm{i}}\right)\right)$ $= 2.55\cos(x + 2.42)$	Using $r\cos(ax+b) = \mathrm{Re}\left(r\mathrm{e}^{(ax+b)\mathrm{i}}\right)$.

Think of how you can use your calculator to check the result from **part c** in **Example 2**, i.e. that $2\cos(x+1) + 3\cos(x+\pi) = 2.55\cos(x+2.42)$.

Example 3

★★★

Given that $f(x) = r_1\sin(ax + b_1)$ and $g(x) = r_2\sin(ax + b_2)$, show that $f(x) + g(x) = \left|r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}}\right|\sin(ax + \arg(r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}}))$.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$f(x) = \mathrm{Im}(r_1\mathrm{e}^{(ax+b_1)\mathrm{i}})$ $g(x) = \mathrm{Im}(r_2\mathrm{e}^{(ax+b_2)\mathrm{i}})$	Rewrite the functions in terms of complex numbers in exponential form.
$\begin{aligned} f(x)+g(x) &= \mathrm{Im}\left(r_1\mathrm{e}^{(ax+b_1)\mathrm{i}} + r_2\mathrm{e}^{(ax+b_2)\mathrm{i}}\right) \\ &= \mathrm{Im}\left(r_1\mathrm{e}^{ax\mathrm{i}}\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{ax\mathrm{i}}\mathrm{e}^{b_2\mathrm{i}}\right) \\ &= \mathrm{Im}\left(\mathrm{e}^{ax\mathrm{i}}(r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}})\right) \end{aligned}$ Let $r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}} = z$. Then: $r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}} = \lvert z\rvert\,\mathrm{e}^{(\arg z)\mathrm{i}}$ It follows that: $\begin{aligned} f(x)+g(x) &= \mathrm{Im}\left(\mathrm{e}^{ax\mathrm{i}}(r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}})\right) \\ &= \mathrm{Im}\left(\mathrm{e}^{ax\mathrm{i}}(\lvert z\rvert\mathrm{e}^{(\arg z)\mathrm{i}})\right) \\ &= \mathrm{Im}\left(\lvert z\rvert\mathrm{e}^{(ax+\arg z)\mathrm{i}}\right) \end{aligned}$	Add using properties of exponents and complex numbers.
$\begin{aligned} f(x)+g(x) &= \mathrm{Im}(\lvert z\rvert\mathrm{e}^{(ax+\arg z)\mathrm{i}}) \\ &= \lvert z\rvert\sin(ax+\arg z) \end{aligned}$ Since, $\begin{aligned} r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}} &= z, \\ \lvert z\rvert &= \lvert r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}}\rvert \text{ and} \\ \arg z &= \arg(r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}}). \end{aligned}$ Therefore, $f(x)+g(x) = \lvert r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}}\rvert\sin(ax+\arg(r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}})).$	Rewrite in sine form.

The results from **Examples 2** and **3** can be written as follows.

✓ **Important**

If $f(x) = r_1\sin(ax+b_1)$ and $g(x) = r_2\sin(ax+b_2)$, then

$$f(x)+g(x) = \lvert r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}}\rvert\sin(ax+\arg(r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}})).$$

If $f(x) = r_1\cos(ax+b_1)$ and $g(x) = r_2\cos(ax+b_2)$, then

$$f(x)+g(x) = \lvert r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}}\rvert\cos(ax+\arg(r_1\mathrm{e}^{b_1\mathrm{i}} + r_2\mathrm{e}^{b_2\mathrm{i}})).$$

Note that these results only apply when adding two functions with the same period.

These relationships are not given in the exam formula booklet. You should be able to derive them in the exam.

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Be aware

When you work with real-world models, instead of the period of a sine or cosine model sometimes the frequency is mentioned. For example, $f(x) = 2\sin(3x+1)$ and $g(x) = 4\sin\left(3x + \frac{\pi}{4}\right)$ are described as having the same frequency rather than equal period.

The value of b in $f(x) = r\sin(x-b)$ is called phase shift . For example, in $f(x) = 4\sin\left(x - \frac{\pi}{4}\right)$ the phase shift is $\frac{\pi}{4}$.

Example 4

★★★

a) Two AC sources provide voltage at the same frequency. One has a maximum voltage of 110V and the other has a maximum of 80V . The models are of the form

$$V_1 = 110\sin(Bt+\pi) \text{ and } V_2 = 80\sin\left(Bt + \frac{\pi}{4}\right)$$

Find a model for the sum of these sources in the form $V = A\sin(Bt + C)$, where t represents time.

b) Hence, state the maximum voltage of the sum of these two sources.

	Steps	Explanation
a)	$V_1 = \text{Im}\left(110e^{(Bt+\pi)\text{i}}\right)$ $V_2 = \text{Im}\left(80e^{\left(Bt+\frac{\pi}{4}\right)\text{i}}\right)$	Rewrite each model in terms of complex numbers in exponential form.
	$V = V_1 + V_2$ $= \text{Im}\left(\text{e}^{Bt\text{i}}\left(110\text{e}^{\pi\text{i}} + 80\text{e}^{\frac{\pi}{4}\text{i}}\right)\right)$ $= \text{Im}\left(\text{e}^{Bt\text{i}}\left(77.8\text{e}^{2.33\text{i}}\right)\right)$ $= 77.8\sin(Bt + 2.33)$	
b)	The maximum voltage is 77.8V .	Maximum voltage is the amplitude of the sine model.

Be aware

Voltage is measured in units of volts. The symbol for volts is V . Sine and cosine models are only applicable to AC (alternating current) sources because the voltage from these sources oscillates in a periodic fashion. DC (direct current) sources supply a steady voltage.

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Example 5

★★★

Two electrical sources have models of the form $V = A\cos(x + C)$. The difference of the phase shifts is $30°$.

One of the sources has a maximum output of 20V and the other has a maximum of 35V.

Find the maximum output voltage when these two sources are combined.

Steps	Explanation
$V_1 = 20\cos(t)$ $V_2 = 35\cos(t + 30°)$	Create a model for each source. A possible pair with the given difference in phase shift is to use a phase shift of $0°$ for source 1 and $30°$ for source 2.
$V_1 = \text{Re}\left(20\text{e}^{t\text{i}}\right)$ $V_2 = \text{Re}\left(35\text{e}^{(t+30°)\text{i}}\right)$	Rewrite in terms of complex numbers in exponential form.
$V_1 + V_2 = \text{Re}\left(\text{e}^{t\text{i}}(20 + 35\text{e}^{30°\text{i}})\right)$ $= \text{Re}\left(\text{e}^{t\text{i}}(53.3\text{e}^{19.2°\text{i}})\right)$ $= 53.3\cos(t + 19.2°)$ The maximum voltage for the two sources is 53.3 V (3 significant figures).	

Exam tip

Be careful to use your calculator in the appropriate mode, radians or degrees, when working with complex numbers. Some calculators only perform complex number operations in radians.

In that case, a question given in degrees should be converted to radians for the calculation step and then back to degrees for the final answer.

International Mindedness

The frequency of an AC voltage source differs in different countries.

Some countries use a 50 Hz frequency while others use 60 Hz. The maximum output of an AC voltage source also differs, with most countries either using 110V or 220V.

Electronic devices such as laptops and cell phones use DC power while the electricity that comes from the mains is AC. This is why there is a converter in the plug or attached to the laptop power cord. It converts an AC voltage source to DC and does this for a range of $100–240\text{V}$ and $50–60\text{Hz}$. This enables you to simply plug in your electronic device to the mains when you travel to a different country.

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Appliances, such as washing machines use an AC power source. This means that companies have to manufacture different models for $110\mathrm{V}$ and $220\mathrm{V}$ power sources.

2 section questions

1. Number and algebra / 1.13 Further complex numbers

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/checklist-id-27574/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- convert between Cartesian, polar and Euler forms of complex numbers
- multiply and divide complex numbers in polar and Euler forms
- understand and describe the geometric significance of operations with complex numbers as represented in the Argand diagram
- add sine and cosine functions using properties of complex numbers
- understand how voltage sources can be modelled using sine and cosine functions and added using properties of complex numbers.

1. Number and algebra / 1.13 Further complex numbers

Investigations

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/investigations-id-27575/print/) Assign

Part 1

Create your own examples for values of a and b in $z = a + b\mathrm{i}$ to investigate the relationships between the modulus and arguments of z and $\frac{1}{z}$.

Hence, write an expression for $\frac{1}{z}$ in terms of r and θ given that $z = r\,\mathrm{cis}\theta$.

Part 2

Let $z = 2 + 3\mathrm{i}$ and $w = -4 + \mathrm{i}$.

Show that the following properties are true for z and w:

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$$|z \times w| = |z| \times |w| \text{ and } \left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

Generalise to all complex numbers.

Part 3

Sunrise times and sunset times can be modelled with sine or cosine functions. Find data for sunrise and sunset times for a specific region and create a model for sunrise times and another model for sunset times.

Explain what information can be gained from the addition of these two functions.

Determine whether the functions you generated can be added using the technique that you learned in this subtopic. Explain your reasoning.

Rate subtopic 1.13 Further complex numbers

Help us improve the content and user experience.

Student view