1.11 Teacher view

# Partial fractions







# The big picture



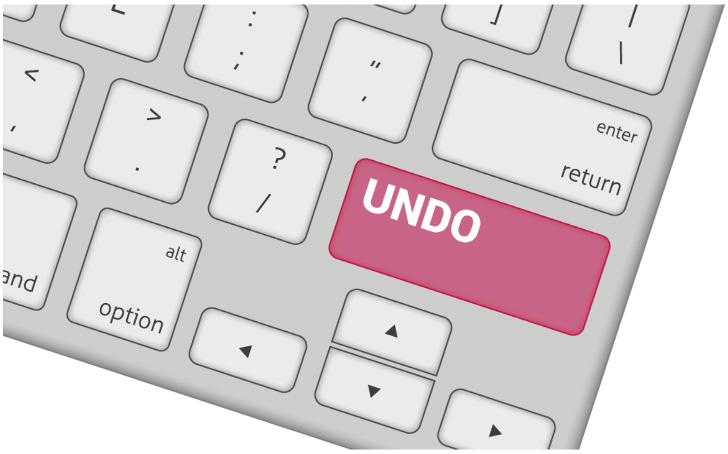
There may be instances where you would find it useful to ‘undo’ some mathematical operations. Since there is no ‘Undo’ button in mathematics, you need to learn specific algebraic techniques for reversing mathematical operations.

You already learned the algebra skills needed for solving a simple equation such as

$$2x + 3 = 5.$$

To solve this equation, you undo all the operations that were done to $x$ (multiplication by $2$, then addition of $3$), by doing them in the reverse order. That is, you subtract $3$ and then divide by $2$ on both sides of the equation, to get the answer $x = 1$.

In this subtopic you will learn a technique for undoing the addition of algebraic fractions. This skill will be useful later in the course for solving certain equations and for calculating integrals when you study calculus.




### Concept

Mathematical operations are often easier to do in one direction than in the other. For example, it is easier to expand $(2x+1)(3x-2)$ into $6x^2 - x - 2$ than it is to factorise $6x^2 - x - 2$ into $(2x+1)(3x-2)$ . Patterns observed when working in the easier direction are often helpful for figuring out how to work in the harder direction.

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The two forms — ‘combined’ and ‘separated’ — are equivalent , though in different contexts one form may be more useful than the other. As you study how to express algebraic fractions in the equivalent form of partial fractions , pay attention to which patterns are used as shortcuts in the process, and consider in what contexts it would be more convenient to use the ‘separated’ versus the ‘combined’ form.

1. Number and algebra / 1.11 Partial fractions

# Partial fractions

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## Activity

Write each expression as a single fraction.

$$\frac{3}{x+1}+\frac{1}{x-2}$$

$$\frac{5}{x+4}-\frac{2}{x-5}$$

$$\frac{2x}{x-6}+\frac{4}{x}$$

Describe how you can use your observations from the examples above to write $\frac{1}{x^2-3x+2}$ as a sum or difference of two fractions.

Consider:

$$\frac{2}{x+1}+\frac{3}{x-3}=\frac{2(x-3)}{(x+1)(x-3)}+\frac{3(x+1)}{(x+1)(x-3)}=\frac{2x-6+3x+3}{x^2-2x-3}=\frac{5x-3}{x^2-2x-3}.$$

This process can be done in reverse, starting with $\frac{5x-3}{x^2-2x-3}$ to obtain the partial fractions $\frac{2}{x+1}$ and $\frac{3}{x-3}$.

This splitting of an algebraic expression into two (or more) parts is called decomposition.

What do you notice about the factors of $x^2-2x-3$ and the denominators of the partial fractions of $\frac{5x-3}{x^2-2x-3}$?

## Making connections

You already know that a numerical fraction such as $\frac{7}{18}$ whose numerator is less than the denominator is called a proper fraction. This terminology can be generalised to algebraic fractions.

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**Important**

An algebraic fraction where the degreeof the numerator is less than the degree of the denominator is called a proper fraction .

# Example 1

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Express $\frac{2x+1}{x^2+x-2}$ as a sum of partial fractions.

| Steps | Explanation |
|---|---|
| $x^2+x-2=(x+2)(x-1)$ | The first step is to factorise the denominator. Each factor will become the denominator of one of the partial fractions. |
| $\frac{2x+1}{x^2+x-2} \equiv \frac{A}{x+2}+\frac{B}{x-1}$ | Now write the given expression as a sum of partial fractions, using the letters $A$ and $B$ to represent the unknown numerators.<br><br>Note the use of the identity symbol $\equiv$ here. It means that the two sides are the same for all values of the variable $x$ . |
| $\frac{2x+1}{x^2+x-2} \equiv \frac{A(x-1)}{(x+2)(x-1)}+\frac{B(x+2)}{(x+2)(x-1)}$<br>$\equiv \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$ | Make a common denominator and add the partial fractions. |
| $A(x-1)+B(x+2) \equiv 2x+1$ | Since the denominators are the same on both sides of the identity (one is just the factorised form of the other), the numerators must be equal.<br><br>The next few steps of the solution can be done using one of two methods. |

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| Steps | Explanation |
|---|---|
| **Method 1**<br>$Ax - A + Bx + 2B \equiv 2x + 1$<br>$(A+B)\,x + (-A+2B) \equiv 2x + 1$<br>$\therefore A + B = 2$<br>$-A + 2B = 1$ | Equate the coefficients by using the following property:<br>$ax + b \equiv cx + d$<br>$\Leftrightarrow a = c$ and $b = d$. |
| $\begin{cases} A + B = 2 \\ -A + 2B = 1 \end{cases} \quad \Leftrightarrow \quad 3B = 3 \quad \Leftrightarrow \quad B = 1$<br>$A + 1 = 2 \quad \Leftrightarrow \quad A = 1$ | Solve the system of equations for $A$ and $B$ using substitution or elimination. |
| **Method 2**<br>$A\,(x-1) + B\,(x+2) \equiv 2x + 1$<br>Let $x = 1$:<br>$A\,(1-1) + B\,(1+2) = 2\,(1) + 1 \quad \Leftrightarrow \quad 3B = 3 \quad \Leftrightarrow \quad B = 1$<br>Let $x = -2$:<br>$A\,(-2-1) + B\,(-2+2) = 2\,(-2) + 1$<br>$\Leftrightarrow \quad -3A = -3 \quad \Leftrightarrow \quad A = 1$ | Pick values of $x$ that would make $(x-1) = 0$ and $(x+2) = 0$ . |
| $\dfrac{2x+1}{x^2+x-2} \equiv \dfrac{1}{x+2} + \dfrac{1}{x-1}$ | Substitute the values for $A$ and $B$ and write the final result. |

Why do you think that $\dfrac{2x+1}{x^2+x-2} \equiv \dfrac{A}{x+2} + \dfrac{B}{x-1}$ in **Example 1** is written with the identity symbol ($\equiv$) rather than the equals sign ($=$) ?

Why are you allowed to substitute $x = 1$ and $x = -2$ into $A\,(x-1) + B\,(x+2) \equiv 2x + 1$ for Method 2 in **Example 1** ?

## Making connections

In subtopic 2.7 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27705/) you will learn how factorisation can be used to solve quadratic equations. If you need to review factorising skills, you can do so in this subtopic.

## International Mindedness

Mathematicians have been developing techniques for factorising polynomials since the 16th century.

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Contributions to this topic were made by mathematicians from all over the world.

# Example 2

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Write $\dfrac{3x+5}{2x^2-5x-3}$ as a sum of partial fractions.

| Steps | Explanation |
|---|---|
| $2x^2-5x-3=(2x+1)(x-3)$ | Factorise the denominator. |
| $\dfrac{3x+5}{2x^2-5x-3}\equiv\dfrac{A}{2x+1}+\dfrac{B}{x-3}$ | Write the partial fractions with $A$ and $B$ standing for the unknown numerators. |
| $3x+5\equiv A(x-3)+B(2x+1)$ | Combine the partial fractions over a common denominator and then set the numerators of both sides to be equal. |
| **Method 1**<br>$3x+5\equiv Ax-3A+2Bx+B\equiv(A+2B)x+(-3A+B)$<br>$\begin{cases}3=A+2B\\5=-3A+B\end{cases}\Leftrightarrow\ 5=-3(3-2B)+B$<br>$\Leftrightarrow\ 5=-9+7B\ \Leftrightarrow\ B=2$<br>$A=3-2(2)=-1$ | Equate the coefficients by using the following property:<br>$ax+b\equiv cx+d$<br>$\Leftrightarrow a=c$ and $b=d$. |
| $\dfrac{3x+5}{2x^2-5x-3}\equiv-\dfrac{1}{2x+1}+\dfrac{2}{x-3}$ | Write the final result. |
| **Method 2**<br>$3x+5\equiv A(x-3)+B(2x+1)$<br>Let $x=3$:<br>$3(3)+5=A(3-3)+B(2(3)+1)$<br>$\Leftrightarrow\ 14=7B\ \Leftrightarrow\ B=2$<br>Let $x=-\dfrac{1}{2}$:<br>$3\left(-\dfrac{1}{2}\right)+5=A\left(-\dfrac{1}{2}-3\right)+B\left(2\left(-\dfrac{1}{2}\right)+1\right)$<br>$\Leftrightarrow 3.5=-3.5A\ \Leftrightarrow\ A=-1$ | Pick values of $x$ that would make $(x-3)=0$ and $(2x+1)=0$ . |

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| Steps | Explanation |
|---|---|
| $\dfrac{3x+5}{2x^2-5x-3} \equiv -\dfrac{1}{2x+1} + \dfrac{2}{x-3}$ | Write the final result. |

### Making connections

Partial fractions are useful when integrating, which you will study in subtopic 5.15 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26507/).

You can see another example of finding partial fractions in the video below.

❖ Partial Fraction Decomposition - Example 1 ❖

### Exam tip

In the examination, you will only be asked to find partial fractions for proper algebraic fractions with a maximum of two distinct linear factors of the form $(ax+b)$ in the denominator.

## Example 3

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Write $\dfrac{3x+1}{(x+2)\left(x^2-2x+1\right)}$ as a sum of partial fractions.

| Steps | Explanation |
|---|---|
| $(x+2)\left(x^2-2x+1\right) = (x+2)(x-1)(x-1)$ | Factorise the denominator. |

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| Steps | Explanation |
|---|---|
| $\frac{3x+1}{(x+2)(x^2-2x+1)} \equiv \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ | In this case there is a repeated linear factor $(x-1)$ in the denominator.<br><br>There are three possibilities for the denominators of the partial fractions.<br><br>(You can check for yourself that using just $\frac{A}{x+2} + \frac{B}{x-1}$ or $\frac{A}{x+2} + \frac{C}{(x-1)^2}$ won't work.) |
| $3x+1 \equiv A(x-1)^2 + B(x+2)(x-1) + C(x+2)$ | Put the partial fractions over a common denominator and equate the numerators on both sides of the identity. |
| Let $x=1$:<br><br>$3(1)+1 = A(1-1)^2 + B(1+2)(1-1) + C(1+2)$<br><br>$\Leftrightarrow \quad 4 = 3C \quad \Leftrightarrow \quad C = \frac{4}{3}$<br><br>Let $x=-2$:<br><br>$3(-2)+1 = A(-2-1)^2 + B(-2+2)(-2-1) + C(-2+2)$<br><br>$\Leftrightarrow \quad -5 = 9A \quad \Leftrightarrow \quad A = -\frac{5}{9}$ | Pick values of $x$ that would make $(x-1)=0$ and $(x+2)=0$.<br><br>Note that the value of $B$ can't be found using this method. |
| $3x+1 \equiv A(x-1)^2 + B(x+2)(x-1) + C(x+2)$<br><br>$3x+1 \equiv Ax^2 - 2Ax + A + Bx^2 + Bx - 2B + Cx + 2C$<br><br>$\Leftrightarrow \begin{cases} 0x^2 = Ax^2 + Bx^2 \\ 3x = -2Ax + Bx + Cx \\ 1 = A - 2B + 2C \end{cases}$ | So we have to go back to the method of equating coefficients. |
| $0 = A + B \quad \Leftrightarrow \quad 0 = -\frac{5}{9} + B \quad \Leftrightarrow \quad B = \frac{5}{9}$ | You already know $A = -\frac{5}{9}$ and $C = \frac{4}{3}$ , so you can use any of the three equations to find the value of $B$ . |
| $\frac{3x+1}{(x+2)(x^2-2x+1)} \equiv -\frac{5}{9(x+2)} + \frac{5}{9(x-1)} + \frac{4}{3(x-1)^2}$ | Write the final result. |

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# 4 section questions

1. Number and algebra / 1.11 Partial fractions

# Checklist

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### What you should know

By the end of this subtopic you should be able to:

- Rewrite proper algebraic fractions that have a maximum of two distinct linear factors in the denominator as sums of partial fractions.

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# Investigation

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### Exam tip

In the examination:

- You will be expected to decompose only proper algebraic fractions.
- The denominator of the given algebraic fraction can be factorised into, at most, two distinct linear terms.

Investigate one or more of the following cases that are not covered by the exam.

Try to create your own examples and discuss your findings with reference to them.

Explore whether the decomposition technique you learned in section 1.11.1 (/study/app/math-aa-hl/sid-134-cid-761926/book/partial-fractions-id-26981/) can be applied to improper fractions such as $\frac{2x^2+7}{x^2-x-6}$ or $\frac{6x^3-39x^2-20x+68}{(2x+1)(x-7)}$, where the degree of the numerator is greater than or equal to the degree of the denominator.

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Investigate how you might extend the technique of section 1.11.1 (/study/app/math-aa-hl/sid-134-cid-761926/book/partial-fractions-id-26981/) to algebraic fractions whose denominator can be factorised into three (or more) distinct linear factors, such as $\frac{4x^2-35x-71}{(x+2)(x+1)(x-3)}$.

Think about how you might find the partial fraction decomposition for an algebraic fraction whose denominator has a quadratic factor that cannot be further factorised, such as $\frac{x^2+3x-8}{(x^2+1)(x-3)}$.

Consider how you could split an algebraic fraction like $\frac{3x^2+12x-20}{(x-1)^3}$ into partial fractions, where the linear factors of the denominator are not distinct.

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