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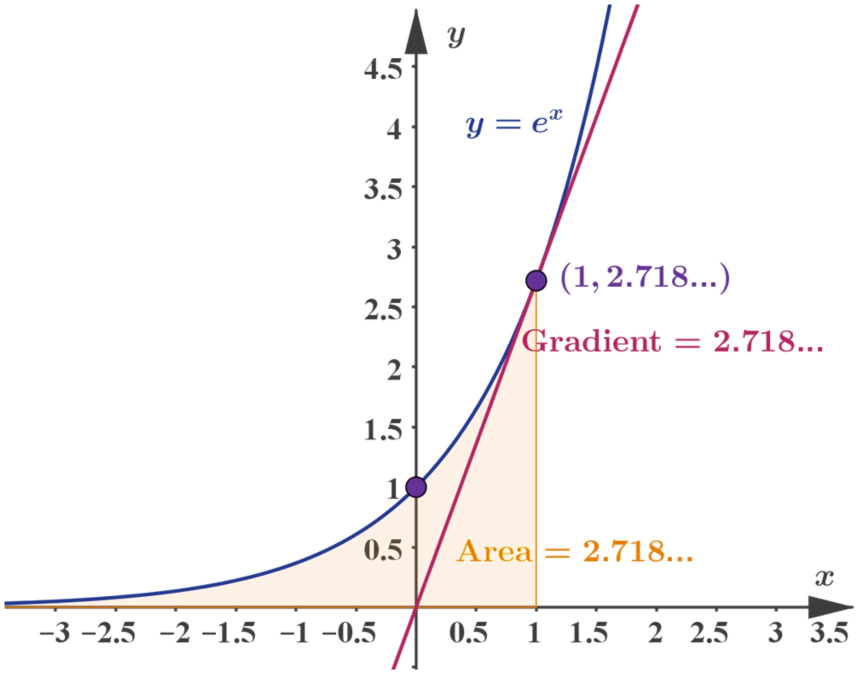
The big picture

Have you ever wondered how archaeologists or palaeontologists estimate the ages of archaeological artefacts, or what type of mathematical models are used by geologists to study the evolution of earth?

Exponential functions are a special type of function with a considerable number of modelling applications. These range from financial applications like compound interest, scientific applications such as radioactive decay curves, the spread of disease and drug concentration decay, and Newton's cooling curve, through to geography and population growth or decline.

At a slightly higher level, exponential functions can also be applied to contexts such as music, with sound decay curves; or again to geography, with methods to protect buildings from earthquakes by reducing the effect of the shock waves; and also to engineering, with a similar effect used in shock absorbers and car suspensions. There is a multitude of examples of modelling with exponential functions in many other subject areas, including physics, chemistry, biology, economics, geography and psychology. This is an area that will never run dry and it has a range of stimulating applications.



The graph of the natural exponential function $y = \mathrm{e}^x$ and some special graphical properties where the mathematical constant $\mathrm{e} = 2.718\ldots$ appears

More information

This image shows the graph of the natural exponential function, y = e^x. The graph displays the y-axis ranging from 0 to 4.5 and the x-axis from -3.5 to 3.5. Key points include:

1. A marked point at (1, 2.718...), which is the value of e, the base of natural logarithms.
2. A text annotation 'Gradient = 2.718...' is present on the graph, indicating the slope of the tangent at this point.
3. An orange shaded area under the curve, between x = 0 and x = 1, is labeled 'Area = 2.718...', representing the area under the exponential curve from x=0 to x=1.
4. The curve starts near the x-axis as x approaches -3.5 and increases steeply as x becomes positive.

This graph illustrates the key properties of the exponential function, including the slope and area under the curve at x = 1.

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In this section, you will study:

- exponential functions and their graphs
- logarithmic functions and their graphs.

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Concept

Exponential growth and decay are exhibited often in real-life scenarios when the rate of **change** — the change per instant or unit of time — of the value of a mathematical function of time is proportional to the function's current value, so its value at any time is an exponential function of time. While learning how exponential functions are used in mathematics to describe the size of anything that is growing or decreasing steadily, reflect on the techniques you learned previously in graphing functions. Which parameters in exponential **models** determine exponential growth or exponential decay?

Think about real-life phenomena such as the spread of diseases and how exponential functions can be used to model an epidemic. In many instances, people are using phrases such as 'grows or decreases exponentially'. Reflect on whether such phrases are a misleading use of the mathematical term.

2. Functions / 2.9 Exponential functions

Exponential functions

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Exponential functions and their properties

Exponential functions

Exponential functions are functions where the independent variable, say x, is the exponent of a positive number, that is, $f(x) = a^x$, $a > 0$.

Important

The **exponential function** f **with base** a is denoted by $y = a^x$, where $a > 0$, $a \neq 1$ and $x \in \mathbb{R}$.

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Note that when the base $a = 1$, the exponential function becomes the constant function and thus it is of no interest here.

Exponential function graph

In your exploration of the exponential functions of the form $y = a^x$, you will distinguish two cases:

$$0 < a < 1 \text{ and } a > 1.$$

The applet below shows the graphs of the exponential functions $y = a^x$ and $y = \left(\frac{1}{a}\right)^x$, which is the same as $y = a^{-x}$.

Interactive 1. Graphical Representation of Exponential Functions.

More information for interactive 1

This interactive enables users to explore the concept of exponential functions by visualizing the graphs of $y = a^x$ (blue graph) and its reflection $y = a^{-x}$ (red graph). By adjusting the base a from 0 to 5, users can observe how different values affect the growth or decay of the function.

When $a > 1$ (e.g., $(a = 2.4)$), the graph of $(y = a^x)$ shows exponential growth, rising rapidly as x increases. Conversely, $(y = a^{-x})$ represents exponential decay, decreasing toward zero as x increases. If $(0 < a < 1)$, the behavior flips: $(y = a^x)$ decays, while $(y = a^{-x})$ grows. The case $(a = 1)$ produces a constant function $(y = 1)$, as any number to the power of x remains 1.

The tool also highlights the symmetry between $y = a^x$ and $(y = a^{-x})$, which are reflections of each other across the *y*-axis. This interactive exploration helps users distinguish between growth and decay, understand the impact of the base a, and recognize the relationship between a function and its reciprocal exponent form.

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Activity

Use the applet to visualise the graph of $f(x) = a^x$ for different values of a.

Which graphs represent exponential growth? Which represent exponential decay? What happens when $a = 1$?

Click the box to show $y = a^{-x}$. How are the graphs of the functions $f(x) = a^x$ and $g(x) = a^{-x}$ related to each other?

Important

The exponential function $f(x) = a^x,\ a > 0$ has:

- domain $(-\infty,\ +\infty)$
- range $(0,\ \infty)$
- y -intercept $(0,\ 1)$.

If $a > 1$:

- f is a **growing exponential** function. It **continuously increases.**
- The x-axis is a horizontal asymptote: $x \to -\infty$ when $a^x \to 0^+$.

If $0 < a < 1$:

- f is a **decaying exponential** function. It **continuously decreases**.
- The x-axis is a horizontal asymptote: $x \to +\infty$ when $a^x \to 0^+$.

Notice the following:

- When $a > 1$, the graph is positive and increasing for the entire domain $(0,\ \infty)$ of the function, and as x increases, the graph gets steeper.
- The graph has no vertical asymptotes.
- $f(x) = a^x = \left(\frac{1}{a}\right)^{-x}$
- The graph of $y = a^{-x}$ is a reflection of $y = a^x$ in the y-axis.

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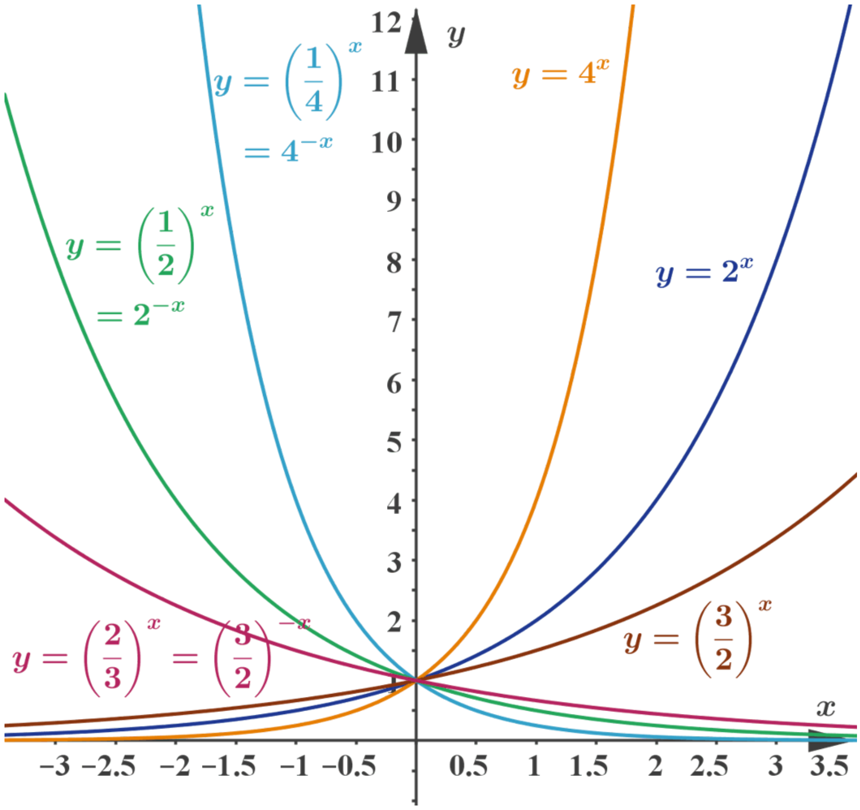
These important facts about exponential functions are shown below **.**



More information

The image is a graph displaying multiple exponential functions with different equations. The graph includes the X-axis representing the variable (x) and ranging from approximately (-3.5) to (3.5), and the Y-axis representing the variable (y) with values ranging from (0) to (12). Several curves are plotted, each labeled with their respective equations:

1. (y = \left(\frac{1}{4} \right)^x = 4^{-x}) - A decreasing curve.
2. (y = \left(\frac{1}{2} \right)^x = 2^{-x}) - Another decreasing curve, less steep.
3. (y = \left(\frac{2}{3} \right)^x = \left(\frac{3}{2} \right)^{-x}) - A decreasing curve.
4. (y = \left(\frac{3}{2} \right)^x) - An increasing curve.
5. (y = 2^x) - A steeper increasing curve.
6. (y = 4^x) - The steepest increasing curve.

The graph illustrates how exponential functions with bases greater than 1 grow while those with bases between 0 and 1 decay. The functions intersect at the origin ((0,1)).

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The range of an exponential function is always $(0, \infty)$. What does that mean regarding the value a^x for any value of x? Do exponential functions have an inverse?

The graphs of exponential functions pass the horizontal line test and thus exponential functions are one-to-one functions.

Important

For $a > 0$ and $a \neq 1$, $a^x = a^y$ if and only if $x = y$.

Making connections

Recall from subtopic 2.5 (/study/app/math-aa-sl/sid-177-cid-761925/book/the-big-picture-id-26449/)that:

- A function has an inverse if and only if the function is one-to-one. A function is one-to-one if it passes the horizontal line test.
- The graphs of two inverse functions are reflections of each other in the line $y = x$.

Example 1

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The diagram shows the graph of an exponential function of the form $y = b \cdot a^x, a > 0$. Find the value of a.

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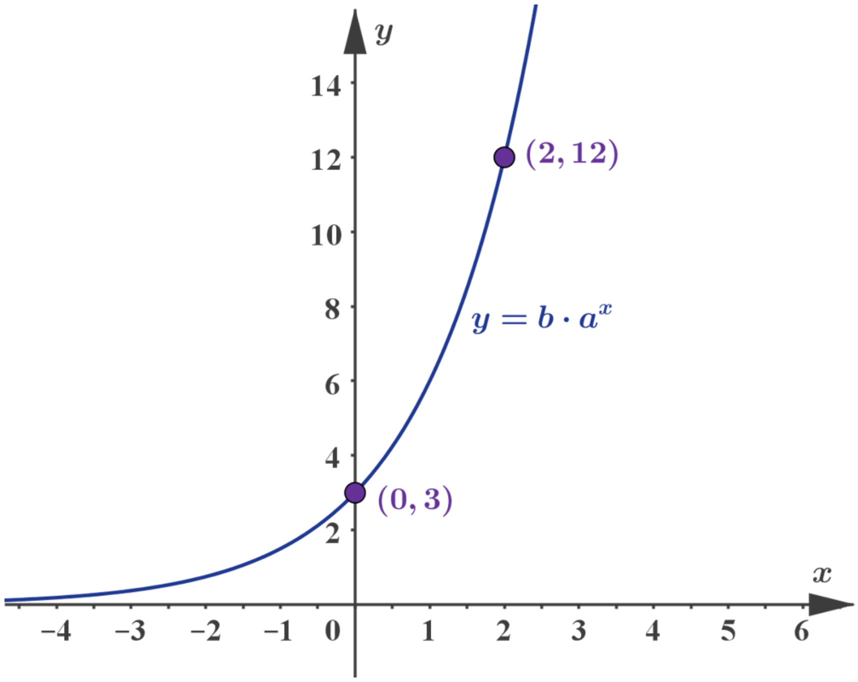

More information

The image is a graph depicting an exponential function of the form y = b * a^x. The X-axis is labeled from -5 to 6, and the Y-axis is labeled from 0 to 16. Two specific points on the graph are highlighted: (0, 3) and (2, 12). The curve starts at the point (0, 3) and rises sharply, passing through the point (2, 12). The general equation y = b * a^x is shown above the curve next to the curve's arc. The graph represents typical exponential growth.

[Generated by AI]

Steps	Explanation
$3 = ba^0$ $3 = b(1)$ $b = 3$	Substitute the coordinates of point $(0, 3)$ into the function formula and solve for b.
$12 = 3a^2$ $4 = a^2$ $a = 2$ Only the positive value a is required.	Substitute the coordinates of point $(2, 12)$ into the function formula and solve for a.

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Example 2

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Given that $f(x) = 3^x$, show that $f(x+2) = 9f(x)$. Express $f(x+a)$ in terms of $f(x)$.

Steps	Explanation
$f(x+2) = 3^{x+2} = 3^2 3^x = 9f(x)$	Substitute $x+2$ into function f and express $f(x+$ in terms of $f(x)$.
$f(x+a) = 3^{x+a} = 3^a 3^x = 3^a f(x)$	Repeat the above process for $x+a$.

Exam tip

If you want to sketch the graph of an exponential function, use the table function on your calculator to find coordinates for your graph.

3 section questions

2. Functions / 2.9 Exponential functions

Exponential modelling

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Exponential growth or decay

The natural base e

Mathematical models that follow exponential growth describe quantities where one quantity increases by the same factor over equal intervals of time, whereas exponential decay describes related quantities where one quantity decreases by the same factor over equal intervals of time.

Making connections

One of the best-known applications of exponential functions lies in the world of economics, in particular, the growth of investments earning continuously compounded interest.

Suppose a principal P is invested at an annual interest rate r, compounded once a year. If the interest is added to the principal at the end of the year, the new balance P_1 is

$$P_1 = P + Pr = P\left(1 + r\right)$$

This pattern of multiplying the previous principal by $1 + r$ is then repeated each successive year, as shown below.

Year	Balance after each compounding
0	$P = P$
1	$P_1 = P\left(1 + r\right)$
2	$P_2 = P_1\left(1 + r\right) = P\left(1 + r\right)\left(1 + r\right) = P(1 + r)^2$
3	$P_3 = P_2\left(1 + r\right) = P(1 + r)^2\left(1 + r\right) = P(1 + r)^3$
$\vdots$	$\vdots$
t	$P_t = P(1 + r)^t$

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To model more frequent (quarterly, monthly or daily) compounding periods of interest, let n be the number of compounding periods per year and let t be the number of years. Then the rate per compounding is $\frac{r}{n}$ and the account balance after t years is

Steps	Explanation
$A = P\left(1 + \frac{r}{n}\right)^{nt}$	Balance with n compounding periods per year.

Important

After t years, the balance A in an account with principal P and annual interest $r\%$ is given by the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where n is the number of times that the interest is compounded per year.

Exam tip

The annual interest rate must be written in decimal form in the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$.

If you let n increase without bound, the process approaches what is called continuous compounding. In the formula for n compounding periods per year, let $m = \frac{n}{r}$. This produces

Steps	Explanation
$A = P\left(1 + \frac{r}{mr}\right)^{mrt}$	Substitute mr for n.
$A = P\left(1 + \frac{1}{m}\right)^{mrt}$	Simplify.
$A = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$	Use property of exponents.

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As m increases without bound, the expression $\left(1+\frac{1}{m}\right)^m$ approaches the special constant number $\mathrm{e} = 2.718281692\ldots$ as you can see in the table below.

m	$\left(1+\frac{1}{m}\right)^m$
1	2
10	2.59374246
100	2.704813829
$1,000$	2.716923932
$1,000,000$	2.718280469
$10,000,000$	2.718281693
$\downarrow$	$\downarrow$
∞	e

From this you can conclude that the formula for continuous compounding is

Steps	Explanation
$A = P\mathrm{e}^{rt}$	Substitute e for $\left(1+\frac{1}{m}\right)^m$.

Important

The constant e is a famous number in mathematics and is known as the natural base e, the **natural number** e, **Euler's number** or Napier's constant .

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Euler's number e is an irrational number, meaning that it cannot be written as a fraction. It has infinite number of decimal places, so you are always approximating when working with e.

Making connections

Many everyday situations can be modelled as exponential functions. The growth of bacteria in a culture, money at a bank and the world's population are all examples of exponential growth. Examples of exponential decay include certain chemical reactions or the decay of radioactive substances.

Example 1

★★☆

Hugo deposits $\$900$ into an account that pays 4.55% interest per annum.

a) How much money will Hugo have in the account (to the nearest cent) at the end of 5 years if the interest is compounded monthly?

b) How long will it take Hugo to double his money?

	Steps	Explanation
a)	$A = P\left(1+\frac{r}{n}\right)^{nt}$	Start with the formula.
	$A = 900\left(1+\frac{0.0455}{12}\right)^{12\cdot 5}$	Substitute in the formula $n = 12$ (compounding periods per year), $r = 0.0455$ (in decimal form) and $t = 5$.
	$A = 1129.425695$	Evaluate.
	$A = 1129.43$	Round answer to the nearest cent.

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	Steps	Explanation
b)	$x = nt$	Let $x = nt$ be the number of monthly instalments until the initial capital is doubled.
	$1800 = 900\left(1 + \frac{0.0455}{12}\right)^x$	Set an equation for the balance of the account to be 1800 (double the initial account) and simplify.
	$2 = \left(1 + \frac{0.0455}{12}\right)^x$	Simplify equation.
	$\ln(2) = \ln\left(1 + \frac{0.0455}{12}\right)^x$	Take the natural logarithm on both sides of the equation.
	$\ln(2) = x \ln\left(1 + \frac{0.0455}{12}\right)$	Use logarithmic properties.
	$x = \frac{\ln(2)}{\ln\left(1 + \frac{0.0455}{12}\right)}$	Solve for x .
	$x = 183.2$	
	Hence the solution is 184 months, which is 15 years and 4 months.	

Activity

Use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ to calculate the amount in an account when $P = \$3000,\ r = 6\%,\ t = 10$ years, and compounding is done by the:

- day
- hour
- minute
- second.

Does increasing the number of compounding periods per year result in unlimited growth of the amount in the account? Report your findings in a brief report.

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International Mindedness

The 17th century saw significant advances in science and mathematics in Europe that would ultimately lead to the Age of Enlightenment in the eighteenth century. Many of the mathematical advances were driven by the needs of business, working with larger quantities of money in ever more complicated transactions. They were also driven by the scientific revolution, which required mathematics to be able to provide rigorous models that could be tested and used to help explain the universe. The development of the exponential function was a particularly significant step, and was accompanied by the development of logarithms which allowed the computation of very large numbers, as well as very small ones.

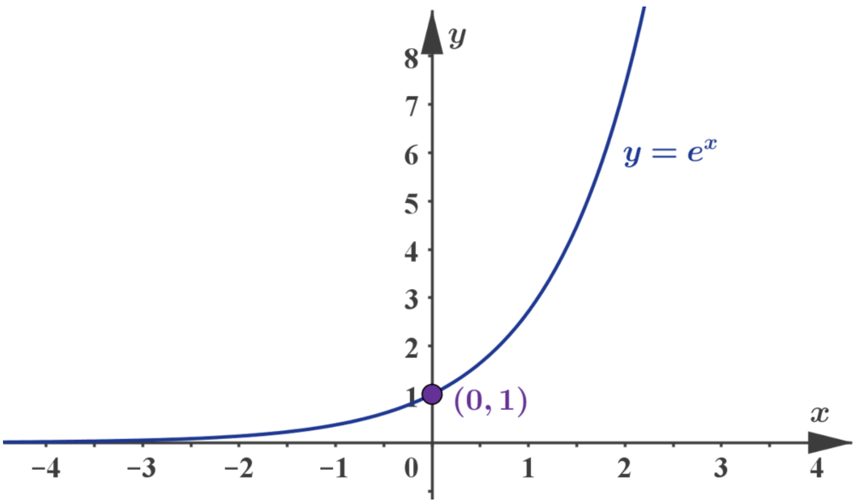
The natural exponential function

The number e comes up over and over again in different subject areas such as science, engineering, chemistry and many more. Exponential functions are used to model and explain phenomena.

Important

The function $f(x) = \mathrm{e}^x$ is called the natural exponential function.

The graph of the natural exponential function $f(x) = \mathrm{e}^x$ is shown below.



The graph of the natural exponential function $f(x) = \mathrm{e}^x$

More information

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The image features a graph illustrating the natural exponential function, y = e^x.

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X-axis: - Labeled as 'x'. - Shows values from -4 to 4. - Origin (0, 0) is marked with an arrowhead at the end of the axis.

Y-axis: - Labeled as 'y'. - Shows values from 0 to 8. - The axis extends vertically with an arrowhead at the top end.

Data curve: - The function y = e^x is depicted as a curved line starting near the x-axis for negative x-values, rising steeply as x becomes positive.

Marked Point: - The point (0, 1) is highlighted on the graph, indicating where the curve passes through the y-axis.

Trend Description: - The curve represents exponential growth, starting from near zero on the left, passing through (0, 1), and rising steeply as x increases.

[Generated by AI]

The natural exponential function is a special type of exponential function of the form $y = a^x$, where $a = \mathrm{e}$. The natural exponential function is always positive and increasing, has the x-axis as a horizontal asymptote and is one-to-one.

Important

The natural exponential function $f(x) = \mathrm{e}^x$ has the inverse function $f^{-1}(x) = \ln x$, so $\ln \mathrm{e}^x = x$.

Making connections

In subtopic 1.9 (/study/app/math-aa-sl/sid-177-cid-761925/book/the-big-picture-id-26442/)you used the binomial theorem to expand binomial expressions of the form $(a+b)^n$.

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Radioactive decay

One of the best-known applications of exponential models is in dating techniques applied by scientists in archaeology, palaeontology and geology to determine the age of an artefact.

Making connections

Radiocarbon decay, or carbon dating, is a vital tool used in archaeology when analysing plants or animals that died a long time ago, up to around $50\,000$ years. It allows archaeologists to roughly determine the age at which they died, by measuring the amount of the radioactive isotope carbon-14.

The process relies on the consumption of carbon dioxide by plants during photosynthesis, which results in the radioactive carbon-14 as a by-product, along with far greater amounts of the non-radioactive carbon-12. Since animals eat plants, they will also contain traces of carbon-14, which will continue to accumulate until they die. At this point, the carbon-14 decays at a rate determined by the 'half-life' of the radioactive isotope, and by measuring the proportion of carbon-14 to carbon-12 in the fossilised organism, archaeologists can estimate fairly accurately the age of death.

Example 2

★★☆

The amount of radioactive material, M, in grams, is modelled according to the function $M\left(t\right) = 250\mathrm{e}^{-kt}$, where $k > 0$ and t is time measured in years. It is determined that after 20 years, the amount of radioactive material present is 50 grams.

a) Show that the amount of radioactive material M decays exponentially.

b) What is the initial amount of material present?

c) Find the value of k to 3 decimal places.

d) How much material is left after 80 years? Give your answer to 3 decimal places.

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	Steps	Explanation
a)	$M(t) = 250e^{-kt} = 250\left(\frac{1}{e}\right)^{kt}$,	Use exponent rules to express th model function without negative exponent.
	$M(t)$ is decreasing and thus M models exponential decay.	Since $\frac{1}{e} < 1$.
b)	$M(0) = 250\left(\frac{1}{e}\right)^{0} = 250$ and thus the initial amount, regardless of the value of k, is 250 grams.	Substitute $t = 0$ into the functior
c)	$50 = 250e^{-20k}$ $0.2 = e^{-20k}$ $\ln(0.2) = \ln\left(e^{-20k}\right)$ $\ln(0.2) = -20k$ $k = \frac{\ln(0.2)}{-20}$ $k = 0.08047$ $k = 0.080$	Simplify equation. Take the natural logarithm of bot sides. Use properties of logarithms and solve for k. Correct to 3 decimal places.
d)	$M(t) = 250e^{-0.08047t}$	Substitute $k = -0.08047$ into th function.
	$M(80) = 250e^{-0.08047(80)} = 0.40006$	Substitute $t = 80$ into the function.
	Thus, the amount of material remaining after 80 years is 0.400 grams.	

Be aware

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All exponential models that you explored above have **time** as an independent variable. This is very often the case for exponential growth or decay.

4 section questions

2. Functions / 2.9 Exponential functions

Logarithmic functions

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Logarithmic functions and their graphs

Logarithms and exponentials go together. You have already seen that in the topic of algebra. In particular, you saw the relationship that if $2^3 = 8$ then $\log_2(8) = 3$.

Important

In general, if $a^x = b$, then $x = \log_a(b)$.

Based on this relationship, you could say that the logarithm is a ‘tool’ that helps us evaluate the **unknown exponent** of a known power. In terms of functions, you will see that the exponential and the logarithmic functions are also related in a special manner. You will explore this in the following video.

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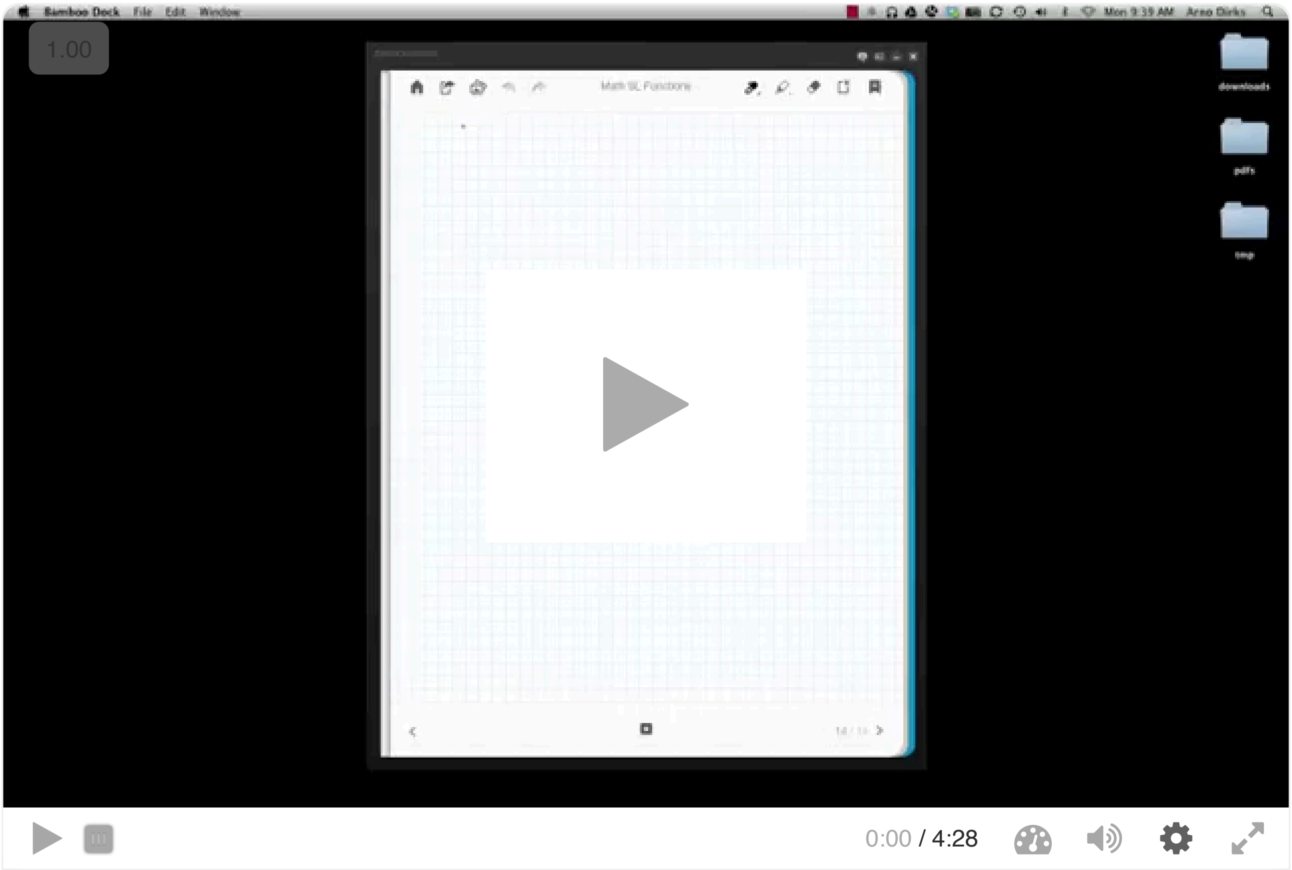

Video 1. Understanding Logarithmic Functions: The Inverse of Exponentials.

More information for video 1

1

00:00:00,467 --> 00:00:04,037

narrator: In this video we're going

to consider logarithmic functions

2

00:00:04,271 --> 00:00:06,373

and they go together

with exponential functions.

3

00:00:06,440 --> 00:00:10,177

So logarithmic function looks

like something $\log_a(x)$.

4

00:00:11,512 --> 00:00:16,717

Now let's recall that $2^3 = 8$,

5

00:00:16,783 --> 00:00:19,820

but we can relate it

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to logarithms as follow.

6

00:00:21,054 --> 00:00:23,423

So $3 = \log_2(8)$.

7

00:00:23,857 --> 00:00:26,660

So 2 is the base,

and 8 is called the argument.

8

00:00:26,960 --> 00:00:29,963

So an exponential function

then based on this would be,

9

00:00:31,098 --> 00:00:35,469

2^x as we've already seen.

10

00:00:36,036 --> 00:00:38,739

And then the logarithmic function

would then be $\log_2(x)$.

11

00:00:40,374 --> 00:00:46,079

And then the logarithmic function

would then be $f(x) = \log_2(x)$.

12

00:00:46,413 --> 00:00:47,915

So let's call it $f(x)$,

13

00:00:48,081 --> 00:00:50,651

and let's call $2^x = g(x)$.

14

00:00:50,717 --> 00:00:54,588

And let's explore.

So the dark blue function is $\log_2(x)$,

15

00:00:54,655 --> 00:00:57,791

and the light blue is 2^x,

16

00:00:57,858 --> 00:00:59,560

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the exponential function

we already explored.

17

00:00:59,793 --> 00:01:02,062

Now let's look at a few points

on those curves.

18

00:01:02,129 --> 00:01:05,098

So 0.5 gets mapped to -1

in the log curve,

19

00:01:05,265 --> 00:01:07,401

and -1 gets mapped to 0.5.

20

00:01:07,467 --> 00:01:11,872

In exponential curve, one gets mapped

to 0 in the log curve and 0 gets mapped

21

00:01:11,939 --> 00:01:13,140

to 1 on the exponential curve.

22

00:01:13,240 --> 00:01:15,909

2 gets mapped

to 1 on the logarithmic curves

23

00:01:16,143 --> 00:01:19,279

and 1 gets mapped

to 2 on this exponential curve.

24

00:01:19,513 --> 00:01:23,083

So we see that really we have a symmetry

25

00:01:23,150 --> 00:01:26,687

around $y = x$ and make it

even more poignant.

26

00:01:26,753 --> 00:01:28,322

I plot the segments between those

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27

00:01:28,388 --> 00:01:30,958

and they cut the $y = x$

line at right angles.

28

00:01:31,058 --> 00:01:34,394

In other words, we can say

this those are chart as inverse.

29

00:01:34,761 --> 00:01:39,633

The same is true for e^x

and logarithmic base e, $\ln(x)$.

30

00:01:39,800 --> 00:01:43,704

The points mirror image 1 to 0 is 0 to 1,

31

00:01:43,770 --> 00:01:46,340

and there's a reflection

of $y = x$ axis.

32

00:01:48,175 --> 00:01:51,778

And I can also take $\log_{10}(x)$

and 10^x,

33

00:01:52,012 --> 00:01:55,148

and again, they are reflected

in the $y = x$ axis.

34

00:01:55,215 --> 00:01:56,783

So they are each other's inverse.

35

00:01:58,519 --> 00:02:01,722

So if $f(x) = \log_2(x)$,

36

00:02:01,788 --> 00:02:04,725

then $f^{-1}(x) = 2^x$.

Student view

37

00:02:04,791 --> 00:02:06,260

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So let's remind ourselves what that means.

38

00:02:06,460 --> 00:02:09,429

$x = (f^{-1} \circ f)(x)$

39

00:02:09,897 --> 00:02:13,467

So for us, that means x goes

into $\log_2(x)$

40

00:02:13,734 --> 00:02:15,836

and 2 raised to that power is x.

41

00:02:16,270 --> 00:02:18,105

Similarly, of course

42

00:02:18,172 --> 00:02:21,575

they were commutative $x = (f \circ f^{-1})(x)$.

43

00:02:22,009 --> 00:02:26,747

In other words,

2^x into $\log_2$

44

00:02:26,813 --> 00:02:29,550

gives us $x \log_2(2)$,

which of course is x.

45

00:02:29,950 --> 00:02:34,655

So if $f(x) = a^x$,

where a was a positive value.

46

00:02:34,721 --> 00:02:38,926

$f^{-1}(x)$ is our logarithmic

functions $\log_a(x)$,

47

00:02:39,259 --> 00:02:41,562

such that we have the relationship

between them.

48

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00:02:41,795 --> 00:02:45,999

$\log_a(a^x) = x$,

well as $a^{\log_a(x)}$ is of also x.

49

00:02:46,133 --> 00:02:48,001

$a^{\log_a(x)}$ is also x.

50

00:02:48,969 --> 00:02:51,839

Now, just a quick reminder

about domains and ranges.

51

00:02:51,905 --> 00:02:54,141

So a^x, the domain

52

00:02:54,408 --> 00:02:58,612

x could be any real number,

but the range was only positive.

53

00:02:59,746 --> 00:03:04,985

Logarithmic functions

domain is only positive values

54

00:03:05,052 --> 00:03:07,120

and the range is any value.

55

00:03:07,354 --> 00:03:12,893

So the domain and range

of these functions match,

56

00:03:12,960 --> 00:03:15,796

which is exactly what an individual

relationship ought to be.

57

00:03:15,996 --> 00:03:18,799

So $f(x) = a^x$ and $f^{-1}(x) = \log_a(x)$.

58

Student view

00:03:19,199 --> 00:03:23,103

Now $(1, 0)$ is on the log function,

59

00:03:23,170 --> 00:03:26,373

and $(0, 1)$ is on the exponential function.

60

00:03:26,573 --> 00:03:31,011

Notice that it have no presence

in unless there is a translation,

61

00:03:31,078 --> 00:03:33,747

which we will look at later in the course.

62

00:03:33,814 --> 00:03:36,250

Now let's compare a few

logarithmic functions.

63

00:03:36,316 --> 00:03:41,054

I've put $\log_{10}(x)$

$\log_e(x) = \ln(x)$

64

00:03:41,121 --> 00:03:43,156

and $\log_2(x)$ in blue.

65

00:03:44,858 --> 00:03:49,062

Now you see that they all

intersect at the point $(1, 0)$.

66

00:03:49,329 --> 00:03:53,934

You see that the range is minus

as well as positive values,

67

00:03:54,067 --> 00:03:57,571

but the domain only

takes on positive values.

68

00:03:57,838 --> 00:04:00,707

Now let's zoom out and we see that

even though the functions

69

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00:04:00,774 --> 00:04:04,444

continue to increase,

the curve flattens out considerably

70

00:04:04,511 --> 00:04:06,413

until they're almost completely flat,

71

00:04:06,513 --> 00:04:09,049

but they'll keep on increasing

at a very slow rate.

72

00:04:09,383 --> 00:04:12,619

So that is an important feature

of the logarithmic functions.

73

00:04:13,587 --> 00:04:17,824

Similarly, if we look

at the negative range,

74

00:04:17,891 --> 00:04:22,262

then you see that they get arbitrarily

close to the y axis,

75

00:04:22,329 --> 00:04:26,200

but they continue to go

into the far negative regime.

76

00:04:26,400 --> 00:04:27,801

If you zoom in.

Important

Let $f(x) = a^x$ be an exponential function with $a > 0,\ a \neq 1$. Then, the **logarithmic function** f with **base** a is the function defined by the formula $f^{-1}(x) = \log_a(x)$ for all $x \in \mathbb{R}^+$.

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In the following applet you will visualise the graph of the exponential function $f(x) = a^x$ with $a > 0,\ a \neq 1$ and its inverse logarithmic function $f^{-1}(x) = \log_a(x)$.

Activity

Use the slider to visualise the graph of the logarithmic function when $0 < a < 1$ and $a > 1$. What do you notice? Describe the features of the graphs of logarithmic functions for each case.

Interactive 1. Graphical Representation of a Logarithmic Function.

More information for interactive 1

This interactive enables users to explore the concept of logarithmic functions and their relationship with exponential functions through a dynamic visual representation. The graph displays the x-axis ranging from −20 to 30 and the y-axis from −20 to 20, along with a dashed line representing $y = x$, which serves as the line of reflection between the exponential and logarithmic functions.

A slider in the bottom left corner allows users to adjust the value of the base a, ranging from 0 to 10. As users move the slider, two curves appear and update in real time: the exponential function $y = a^x$ shown in blue, and its inverse, the logarithmic function $y = \log ax$, shown in red. The current value of aaa and the updated function equations are displayed at the bottom of the graph. When the base $a > 1$, the exponential curve rises rapidly, while the logarithmic curve increases slowly, with a vertical asymptote at $x = 0$. The logarithmic graph always passes through the point $(1, 0)$, and as the value of a approaches 1, the logarithmic curve becomes steeper. Conversely, when $0 < a < 1$, the exponential function shows decay, and the logarithmic function becomes a decreasing curve with the same vertical asymptote at $x = 0$.

This interactive effectively demonstrates that exponential and logarithmic functions are inverses of each other, visually shown by their symmetry across the line $y = x$. It also allows users to observe how the steepness of the logarithmic function changes with the base and compare it with the

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natural logarithm l n(x), when a = e. Overall, this tool helps learners gain an intuitive understanding of inverse functions, asymptotic behavior, and the effects of varying the base on function graphs.

Important

- $y = a^x$ and $y = \log_a x$ are each other's inverse function, and hence are reflections in the line $y = x$.
- A logarithmic function is a mapping $x \mapsto \log_a(x)$ with continuous domain $x > 0$ and range $-\infty < \log_a(x) < +\infty$.
- For $\log_a(x)$, there is a **vertical asymptote** at $x = 0$ where $\log_a(x) \to -\infty$ as $x \to 0$.

Notice that the x-intercept of the function $\log_a x$ increases, but at an even slower rate, which is the opposite of the growing exponential function a^x. The steepness of the curve $y = \log_a x$ depends on the base a.

Important

There is a **natural base** for the logarithmic function equal to the natural base for the exponential function, namely Euler's number.

$$\log_e(x) \equiv \ln(x).$$

Recall that $\mathrm{e} = 2.7182818...$

The features of the logarithmic function are shown below.

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view

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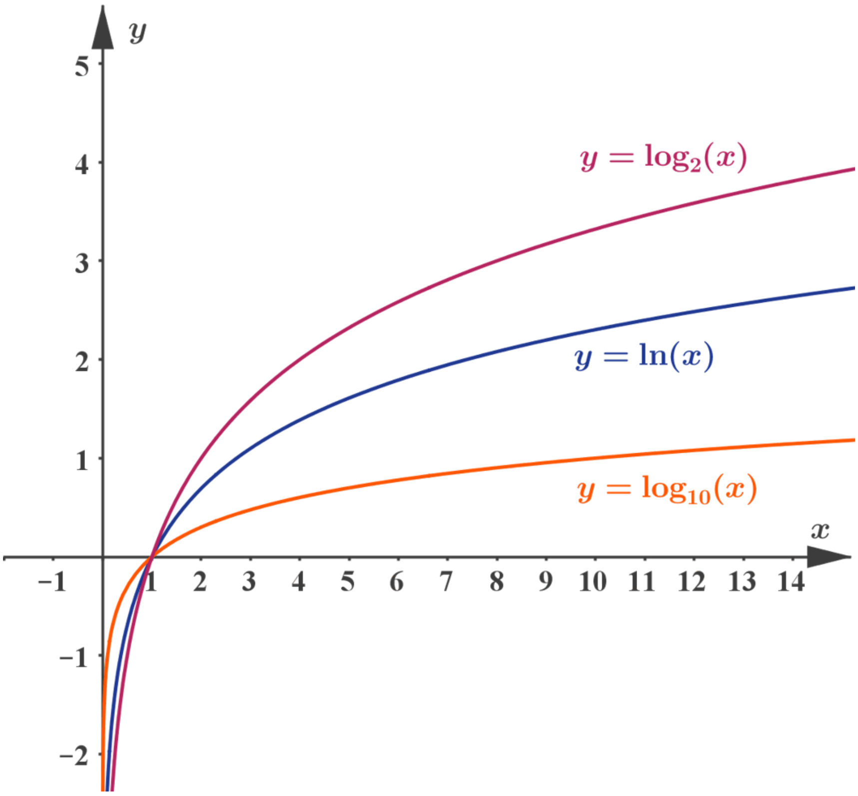

More information

The image is a graph showing three logarithmic functions. The horizontal X-axis is labeled with values ranging from -2 to 14, and the vertical Y-axis is labeled with values from -2 to 5.

Three curves are plotted: 1. y = log2(x) is represented by a purple curve, starting from negative infinity below x=1, crossing the point (1,0) and rising steeply, becoming less steep as x increases. 2. y = ln(x) is represented by a blue curve, also crossing the point (1,0), rising sharply but not as steep as log2(x), and flattening out as x increases. 3. y = log10(x) is shown in orange, crossing (1,0) and rising gradually with increasing x values, with a less steep incline.

All functions pass through the point (1,0), illustrating that regardless of the base of the logarithm, they share this feature. The functions differ in their rate of growth beyond x=1, with log2(x) growing the fastest and log10(x) growing the slowest.

[Generated by AI]

Notice that all functions $y = \log_a(x)$ go through the point $(1,\ 0)$. What is the effect of the base a in the regions $x < 1$ and $x > 1$?

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Important

As the logarithmic and exponential functions are each other's inverse functions, it is implied that:

- $\log_a a^x = x$ and $a^{\log_a x} = x,\ x > 0,\ a \neq 1$ (inverse property)
- If $\log_a x = \log_a y$, then $x = y$ (one-to-one property).

$$a^x = \mathrm{e}^{x\ln a},\quad a > 0.$$

Example 1

★★☆

Consider the function $f(x) = \log_3(x-2) + 1$

a) Find the domain and range of f.

b) Find any asymptotes and axes intercepts.

c) Sketch the graph of f showing all important features.

d) Find f^{-1}.

	Steps	Explanation
a)	$x - 2 > 0$ when $x > 2$. So, the domain is $D_f = \{x \mid x > 2\}$ and the range is $y \in \mathbb{R}$.	The logarithmic function $f(x) = \log_a x$ is defined for $x >$ and the range is always the set of real numbers $\mathbb{R}$.

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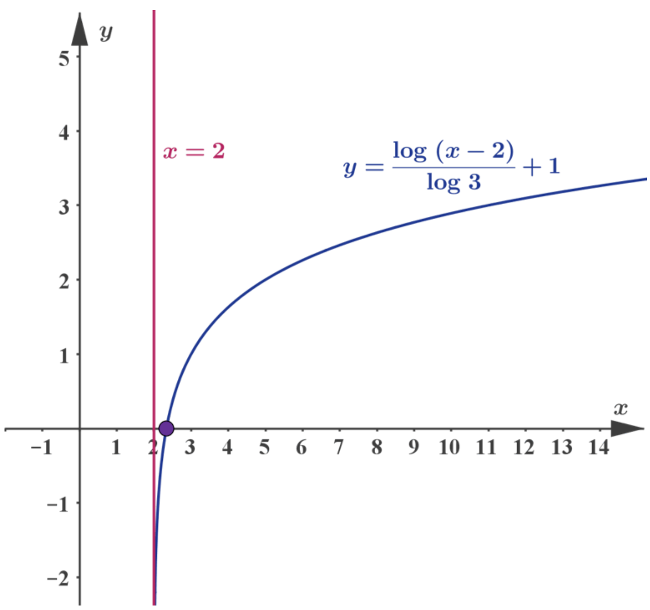
	Steps	Explanation
b)	As $x \to 2$ from the right, $y \to -\infty$, so the vertical asymptote is $x = 2$. When $x = 0$, y is undefined, so there is no y-intercept. When $y = 0$, $\log_3(x-2) = -1$ $$(x-2) = 3^{-1}$$ $$x = 2 + \frac{1}{3}$$ $$x = \frac{7}{3}.$$ So, the x-intercept is $\frac{7}{3}$.	
c)	The graph of $y = \log_3(x-2) + 1 = \frac{\log(x-2)}{\log 3} + 1$ is shown below.  	To graph the function using the GDC, it may be necessary to change the base to 10 or e.

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	Steps	Explanation
d)	f is defined by $y = \log_3(x-2)+1$. f^{-1} is defined by $x = \log_3(y-2)+1$. $x-1 = \log_3(y-2)$. $3^{x-1} = y-2$. $y = 3^{x-1}+2$. So, $f^{-1}(x) = 3^{x-1}+2$.	Find the inverse by interchanging and y in function f and solve for y

Example 2

★★☆

Consider the function $f(x) = e^{2x+1}$.

a) Find the equation defining f^{-1}.

b) Sketch the graphs of f and f^{-1} on the same set of axes.

c) State the domain and range of f and f^{-1}.

d) Find any asymptotes of f and f^{-1}.

	Steps	Explanation
a)	$y = e^{2x+1}$	Start with function f .
	$x = e^{2y+1}$	For the inverse f^{-1} interchange x ar y .
	$\ln(x) = \ln\left(e^{2y+1}\right)$	Take natural logarithm on both sides
	$\ln(x) = 2y+1$	Use logarithmic properties.

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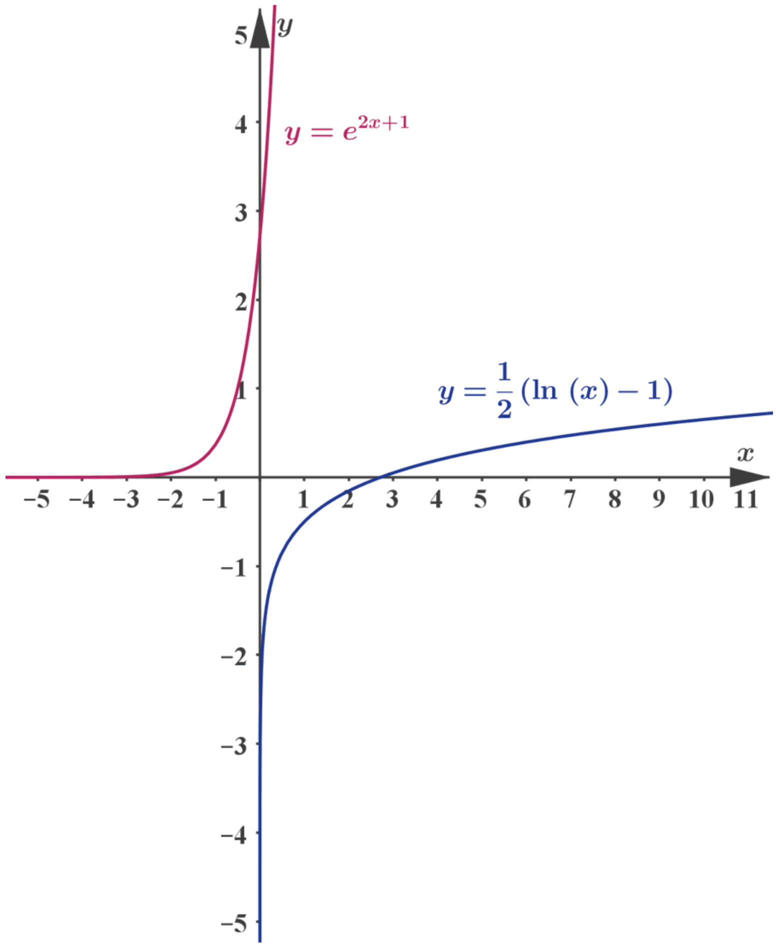
	Steps	Explanation
	$y = \frac{1}{2}(\ln(x) - 1)$	Solve for y .
	Thus, $f^{-1}(x) = \frac{1}{2}(\ln(x) - 1)$.	

b) Use the GDC to graph f and f^{-1} . The graphs are shown below.



c) The domain and range of f are $D_f = \{x | x \in \mathbb{R}\}$ and $R_f = \{y | y > 0\}$ respectively.

The domain and range of g are $D_g = \{x | x > 0\}$ and $R_g = \{y | y \in \mathbb{R}\}$.

d) Function f has a horizontal asymptote at the x -axis with equation $y = 0$.

Function f^{-1} has a vertical asymptote at the y -axis with equation $x = 0$.

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International Mindedness

With developments in banking and finance in the seventeenth century came the need to be able to easily calculate repeated percentages, especially in the context of compound interest. Jacob Bernoulli, a Swiss mathematician investigated this, with particular interest in the limit of $\left(1+\frac{1}{n}\right)^n$ as n tends towards infinity. Using methods of series expansion developed by the mathematician Blaise Pascal, Bernoulli showed that the limit lies somewhere in the range of 2 and 3. Around the same time, the exponential function was being independently discovered along with natural logarithms; it was not until the early eighteenth century that the exponential constant was first expressed by another great Swiss mathematician, Leonhard Euler, drawing together all of these ideas into a unifying concept.

3 section questions

2. Functions / 2.9 Exponential functions

Logarithmic modelling

Section Student... (0/0) Feedback Print (/study/app/math-aa-sl/sid-177-cid-761925/book/logarithmic-modelling-id-26591/print/) Assign

Various systems and phenomena, such as population growth and radioactive decay, behave in predictable ways and can be modelled by logarithmic and exponential relationships. For example, the decibel scale for the loudness of sound and the Richter scale of earthquake magnitude are both logarithmic scales. In this section, you will explore further growth and decay problems that use logarithms in their solutions.

Magnitudes of earthquakes

The magnitude of an earthquake given on the Richter scale is a measure of the energy released at the source of the earthquake. The intensity of an earthquake is a measure of the shaking produced by the earthquake's waves in a particular place.

The magnitude R of an earthquake with intensity I is given by $R = \log\left(\frac{I}{I_0}\right)$, where $I_0 = 1$ is the minimum intensity used. Find the intensities per unit of area for:

Student view

- Northern Sumatra in 2004, where $R = 9.0$

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Steps	Explanation
$R = \log\left(\frac{I}{I_0}\right)$	Start with the given equation for R.
$9.0 = \log\left(\frac{I}{1}\right)$	Substitute 1 for I_0 and 9.0 for R.
$10^{9.0} = 10^{\log(\mathrm{I})}$	Exponentiate each side.
$I = 10^{9.0}$	Inverse property.
$\mathrm{I} = 1,000,000,000$	

- Athens in 1999, where $R = 6.0$.

Steps	Explanation
$6 = \log\left(\frac{I}{1}\right)$	Substitute 1 for I_0 and 6.0 for R.
$10^{6} = 10^{\log(\mathrm{I})}$	Exponentiate each side.
$I = 10^{6}$	Inverse property.
$\mathrm{I} = 1,000,000$	

Notice that an increase of 3 units on the Richter scale (from 6 to 9) represents an increase in intensity by a factor of $\frac{1\ 000\ 000\ 000}{1\ 000\ 000} = 1000$. That is, the intensity of the earthquake in Sumatra was about 1000 times greater than that of the earthquake in Athens.

Sound waves

The loudness of sound perceived by the human ear is a function of the intensity of the sound signal. The strength of the intensity is measured by the decibel scale, which is a function of the intensity through the relation $d = 10\log\left(\frac{I}{I_0}\right)$, where d is the number of decibels, I is the sound intensity and I_0 is the threshold intensity (the weakest sound an ear can perceive). Consider the following questions, which can be answered using logarithmic functions.

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- What is the value of d if I is 10 times as great as I_0?

Steps	Explanation
$d = 10\log\left(\frac{I}{I_0}\right)$	Start with the given equation for d.
$d = 10\log\left(\frac{10I_0}{I_0}\right)$	Set $I = 10I_0$ and substitute into the equation.
$d = 10\log10$	Simplify.
$d = 10 \times 1$	Use properties of logarithms.
$d = 10$	

Setting $I = 10I_0$, you obtain the equation
$d = 10\log\left(\frac{10I_0}{I_0}\right) = 10\log10 = 10 \times 1 = 10.$

- What is the difference in d between a sound that is 100 times as intense as I_0 and 100 000 times as intense as I_0?

Steps	Explanation
$d\left(100000I_0\right) - d\left(100I_0\right) =$	Start with the difference as described in the question and write $100\ 000 = 10^5$ and $100 = 10^2$.
$d\left(10^5I_0\right) - d\left(10^2I_0\right) =$	
$10\log\left(\frac{10^5I_0}{I_0}\right) - 10\log\left(\frac{10^2I_0}{I_0}\right) =$	Simplify.
$10\log\left(10^5\right) - 10\log\left(10^2\right) =$	Use properties of logarithms.
$10 \times 5 \times \log10 - 10 \times 2 \times \log10 =$	Simplify.
$50 - 20 =$	
30	

Student view

Example 1

★★☆

A population p of 50 wolves was introduced into a forest in 2010. The population is expected to grow by the function $p = 50\mathrm{e}^{0.085t}$, where t is the time in years after 2010.

a) What will be the population in 2019?

b) Write a function for t in terms of p.

c) How many years will it take for the population to double?

	Steps	Explanation
a)	$p(t) = 50\mathrm{e}^{0.085t}$ $p(9) = 50\mathrm{e}^{0.085(9)} = 107.4$ Thus, the population in 2019 will be 107 wolves.	Substitute $t = 9$ into the population formula, as 2019 is 9 years after 2010. Round off your answer to 107 as you cannot have 0.4 of a wolf.
b)	$p = 50\mathrm{e}^{0.085t}$	Start with the exponential function for the population p .
	$\frac{p}{50} = \mathrm{e}^{0.085t}$	Simplify.
	$\ln\left(\frac{p}{50}\right) = \ln\left(\mathrm{e}^{0.085t}\right)$	Take natural logarithms on both sides.
	$\ln\left(\frac{p}{50}\right) = 0.085t$	Use properties of logarithms.
	$\ln\left(\frac{p}{50}\right) = \frac{17}{200}t$	Transform the decimal into fraction.
	$t = \frac{200}{17}\ln\left(\frac{p}{50}\right)$	Solve for t.
c)	$t = \frac{200}{17}\ln\left(\frac{p}{50}\right)$	Use the function for the time t .

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	Steps	Explanation
	$t = \frac{200}{17}\ln\left(\frac{100}{50}\right)$	Substitute $p = 100$ (double population)
	$t = \frac{200}{17}\ln(2)$	Simplify and evaluate the value for t .
	$t = 8.15$	
	Thus, it will take about 8.15 years for the population to double.	

3 section questions

2. Functions / 2.9 Exponential functions

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-sl/sid-177-cid-761925/book/checklist-id-26592/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- find the domain, range, asymptotes and axes intercepts of exponential functions
- sketch the graph of exponential functions by showing all relevant features
- find the domain, range, asymptotes and axes intercepts of logarithmic functions
- sketch the graph of logarithmic functions by showing all relevant features
- apply exponential and logarithmic models in real-life situations.

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2. Functions / 2.9 Exponential functions

Investigation

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Be aware

Notice that an exponential function $f(x) = a^x$ is a constant raised to a variable power, whereas a power function $g(x) = x^a$ is a variable raised to a constant non-negative power.

In the following activity, you will investigate the graphs of $f(x) = a^x$ and $g(x) = x^n$, where a and n are constant non-negative numbers.

Interactive 1. Graphs of Exponential and Power Functions.

More information for interactive 1

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This interactive allows users to compare the graphs of an exponential function $f(x) = a^x$ in blue curve and a polynomial function $g(x) = x^n$ in pink curve, where $a \neq 0$ and n are adjustable parameters. Users can set 'a' to values ranging from 0 to 10 in increments of 0.1 (e.g., 0.1, 0.2, ..., 9.9, 10) and adjust 'n' to integer values from 1 to 10 (e.g., 1, 2, 3, ..., 10). By changing these values, users can observe how the growth rates and shapes of the two functions evolve.

However, as 'a' and 'n' varies, it becomes clear that exponential growth (governed by a^x) eventually outpaces polynomial growth (governed by x^n), regardless of the degree n. This demonstrates the principle that exponential growth is significantly faster than polynomial growth of any degree.

By experimenting with different values of 'a' and 'n', users can draw conclusions about the conditions under which the exponential function grows faster than the polynomial function.

Compare the graphs of the functions for the same values of a and n. What do you notice when the value of a is different than n? Draw some conclusions about the values of a and n that make the exponential function grow faster than the polynomial. Reflect on the statement: 'Exponential growth is bigger and faster than polynomial growth of any degree'.

In the following applet you will investigate the graphs of $y = \dfrac{a^x}{x^a}$ and $y = \dfrac{x^a}{a^x}$.

Interactive 2. Graph of Exponential and Rational Functions.

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view

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More information for interactive 2

This interactive enables users to explore the comparative growth rates of exponential and polynomial functions through the visualization of $y = \frac{x^a}{a^x}$ in red and $y = \frac{a^x}{x^a}$ in blue. User can use **a slider** controls the parameter a (from 0 to 10), which affects both functions simultaneously. Users can observe how these ratios behave as x increases, highlighting the fundamental difference between polynomial and exponential growth.

For $y = \frac{a^x}{x^a}$, as x grows large, the function tends to infinity when $a > 1$, demonstrating how exponential functions a^x eventually dominate polynomial functions x^a. Conversely, $y = \frac{x^a}{a^x}$ approaches zero for large x, showing that polynomial growth becomes negligible compared to exponential growth over time.

The interactive clearly illustrates why exponential growth is considered "faster" than polynomial growth, regardless of the degree a. Users can experiment with different values of a to see how changing the base affects the point at which the exponential function begins to outpace the polynomial function, reinforcing key concepts about growth rates in a visual and intuitive manner.

Use the slider to adjust the value of a. Describe and compare the graphs of f and g, as $x \to \infty$.

How would you explain the behaviour of the functions as x tends to really large numbers?

Rate subtopic 2.9 Exponential functions

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