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Teacher view

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1. Number and algebra / 1.16 Systems of linear equations

The big picture

Mathematicians have been interested in solving equations at least as far back as the Babylonian civilization which started around 2000 BC. Applications included estimation of volumes of grain storage containers and calculations of land areas.

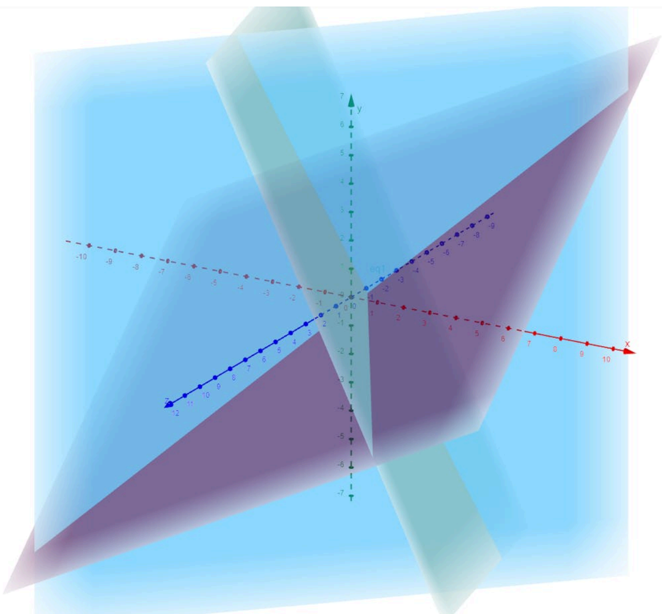
Thousands of years of work on this topic have produced many different ways to solve equations and systems of equations.

Solution techniques include the use of approximation, algebraic manipulations and graphing.

Real-life situations that are modelled using equations contain restrictions and dependencies between multiple variables that require the use of a systems with more than one equation. In this subtopic, you will focus on solving systems of linear equations such as:

$$2x + y - z = 2 \qquad -x + 2y + z = 1 \qquad x + z = 2$$

Computers and graphic display calculators provide many tools for solving systems of equations. One way in which a system of three linear equations can be solved is by creating a 3D graph as shown in below.



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The image is a 3D graph showing three intersecting planes, representing a system of three linear equations. The graph is set in a 3D coordinate system with X, Y, and Z axes. The X-axis is marked from -10 to 10, the Y-axis from -7 to 7, and the Z-axis from -8 to 8. The planes each have a unique orientation and meet at a single point where all equations are satisfied. The X-axis is colored red, the Y-axis is green, and the Z-axis is blue. The intersection point of the three planes likely signifies the solution of the system of equations. The spatial relationship and intersection are clearly important for understanding how these equations interact.

[Generated by AI]

Concept

In this subtopic you will use your algebraic methods and a graphic display calculator to generate solutions to some systems of equations. Consider the following questions:

- How will you know that the solutions generated by the graphic display calculator are valid?
- Does the validity of the solution depend on the context of the question?

Theory of Knowledge

Solving a linear equation can be very satisfying. When you solve for y or x you have confidence in your solution because it 'works'. Your solution elegantly completes the puzzle so to speak. It is this certainty that creates the level of confidence many people have in the area of knowledge of mathematics.

This leads to the Knowledge Question: 'Is the level of certainty within an area of knowledge positively correlated with the value of that area of knowledge?'

1. Number and algebra / 1.16 Systems of linear equations

Systems with two equations

By now, you will have solved simultaneous equations numerous times. For example, you will have solved pairs of equations such as:

$$\begin{aligned}2x+y&=3\\x+5y&=-3\end{aligned}$$

You may have solved these either by substitution:

$$\begin{cases}2x+y=3\\x+5y=-3\end{cases}\rightarrow\begin{cases}2x+y=3\\x=-3-5y\end{cases}\rightarrow\begin{cases}-6-10y+y=3\\x=-3-5y\end{cases}\rightarrow\begin{cases}y=-1\\x=-3-5y\end{cases}\rightarrow\begin{cases}y=-1\\x=2\end{cases}$$

or by elimination:

$$\begin{cases}2x+y=3\\x+5y=-3\end{cases}\rightarrow\begin{cases}2x+y=3\\2x+10y=-6\end{cases}\rightarrow\begin{cases}2x+y=3\\y-10y=3-(-6)\end{cases}\rightarrow\begin{cases}2x+y=3\\-9y=9\end{cases}\rightarrow\begin{cases}2x-1=3\\y=-1\end{cases}\rightarrow\begin{cases}x=2\\y=-1\end{cases}$$

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Both methods, of course, yield the same result.

Example 1

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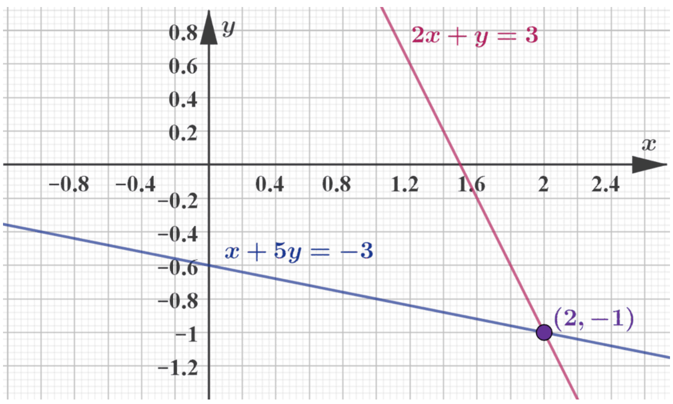
Solve $\begin{cases} 2x - y = 3 \\ x + y = 5 \end{cases}$

Steps	Explanation
$\begin{cases} 2x - y = 3 \\ x + y = 5 \end{cases} \rightarrow \begin{cases} 2x - y = 3 \\ y = 5 - x \end{cases}$ $\rightarrow \begin{cases} 2x - (5 - x) = 3 \\ y = 5 - x \end{cases}$ $\rightarrow \begin{cases} 3x - 5 = 3 \\ y = 5 - x \end{cases}$ $\rightarrow \begin{cases} 3x = 8 \\ y = 5 - x \end{cases}$ $\rightarrow \begin{cases} x = \frac{8}{3} \\ y = 5 - \frac{8}{3} \end{cases}$ $\rightarrow \begin{cases} x = \frac{8}{3} \\ y = \frac{7}{3} \end{cases}$	The solution shows the substitution method. The elimination method can also be used and will produce the same result.

The geometric significance of the solution yields important general properties of systems of equations.

Geometrically, a solution to a system of equations is a set of all points that satisfy all of the equations in the system. In the case of a system of two linear equations, you are looking at the intersection between two lines as shown below.



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This is a graph depicting two lines forming a system of equations. The graph has a grid background with labeled axes. The x-axis ranges from -0.5 to 3 and is labeled with 0, 0.5, 1, 1.5, 2, and 2.5. The y-axis ranges from -1.5 to 1.5 and is labeled similarly.

One line is marked in blue, representing the equation "x - 5y = -5." This line slopes downward.

The other line is red, representing the equation "2x - y = 3," and it slopes upward.

The point where these lines intersect is labeled (2, -1), shown with a purple dot at the intersection. This point satisfies both equations, marking the solution to the system of equations as described in the text before the image.

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Activity

Solve each of the following systems of equations by elimination or substitution and examine the graphs for each system using a calculator.

$$\begin{cases} x+5y=7 \\ -2x+y=-3 \end{cases} \qquad \begin{cases} 2x+3y=5 \\ -4x-6y=10 \end{cases} \qquad \begin{cases} 3x-2y=4 \\ 6x-4y=8 \end{cases}$$

Explain your algebraic results in the context of the graphs of each system. Generalise your findings to describe the possible solutions to any system of two linear equations in two variables.

Important

There are three possible outcomes of solving a system of two linear equations in two variables:

1. A unique solution — the lines intersect at a point. There is a unique ordered pair, (x, y), for the variables.
2. An infinite number of solution — the lines intersect in a line producing an infinite set of intersection points. There are infinitely many ordered pairs, (x, y), for the variables.
3. No solution — the lines do not have any intersection points. There are no values for the variables that satisfy both equations in the system.

A system that has no solutions is called inconsistent. A system that has a unique or infinitely many solutions is called consistent .

Consider what you will find in the algebra when you try to solve each type of system of equations using elimination or substitution.

Example 2

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Find the value of k where k is a real number, such that $\begin{cases} 2x+y=3 \\ x-ky=2 \end{cases}$ has no solutions.

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Steps	Explanation
$\begin{cases} 2x+y=3 \\ x-ky=2 \end{cases} \rightarrow \begin{cases} 2x+y=3 \\ 2x-2ky=4 \end{cases}$ $\rightarrow \begin{cases} y-(-2ky)=3-4 \\ 2x-2ky=4 \end{cases}$ $\rightarrow \begin{cases} y(1+2k)=-1 \\ 2x-2ky=4 \end{cases}$ $\rightarrow \begin{cases} y=-\dfrac{1}{1+2k} \\ 2x-2ky=4 \end{cases}$ If $1+2k=0$ there is no solution because you cannot divide by 0. $1+2k=0 \;\Leftrightarrow\; k=-\dfrac{1}{2}$ $k=-\dfrac{1}{2}$ gives no solutions.	The solution shows the elimination method. The substitution method can also be used and will produce the same result.

Example 3

★★★

The system $\begin{cases} x+my=2 \\ (m-1)x+2y=m \end{cases}$ has infinitely many solutions.

Find a value of m.

Steps	Explanation
$\begin{cases} x+my=2 \\ (m-1)x+2y=m \end{cases} \rightarrow \begin{cases} x=2-my \\ (m-1)(2-my)+2y=m \end{cases}$ $\rightarrow \begin{cases} x=2-my \\ 2m-m^2y-2+my+2y=m \end{cases}$ $\rightarrow \begin{cases} x=2-my \\ y\left(-m^2+m+2\right)=-m+2 \end{cases}$	
Consider the left-hand side: $y\left(-m^2+m+2\right)=0$ $\Leftrightarrow y(-m+2)(m+1)=0$ $\Leftrightarrow m=2$ or $m=-1$ Consider the right-hand side: $-m+2=0 \Leftrightarrow m=2$	$0=0$ for the second equation will be a case with infinitely many solutions.

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Steps	Explanation
There are infinitely many solutions for $m = 2$.	You can check your answer by examining the system generated when $m = 2$. $\begin{cases} x + 2y = 2 \\ x + 2y = 2 \end{cases}$ Both equations represent the same line so there are infinitely many solutions to the system.

Example 4

★★★

a) Find a general formula for the unique solution to $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$, where $a,\ b,\ c,\ d \neq 0$.

b) Hence, solve $\begin{cases} 2\mathrm{i}x + (2+3\mathrm{i})\,y = 3 \\ (1+\mathrm{i})\,x + 2y = 5 \end{cases}$

	Steps	Explanation
a)	$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \to \begin{cases} acx + bcy = ce \\ acx + ady = af \end{cases}$ $\to \begin{cases} bcy - ady = ce - af \\ acx + ady = af \end{cases}$ $\to \begin{cases} y\,(bc - ad) = ce - af \\ acx + ady = af \end{cases}$ $\to \begin{cases} y = \dfrac{ce - af}{bc - ad} \\ acx + ady = af \end{cases}$ $\therefore y = \dfrac{ce - af}{bc - ad}$	Use substitution or elimination to solve for one of the variables. In this case, the solution shows the use of elimination to solve for y.

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	Steps	Explanation
	$cx + dy = f \quad \Leftrightarrow$ $cx + d\left(\frac{ce - af}{bc - ad}\right) = f \quad \Leftrightarrow$ $cx = f - d\left(\frac{ce - af}{bc - ad}\right) \quad \Leftrightarrow$ $cx = f\left(\frac{bc - ad}{bc - ad}\right) - d\left(\frac{ce - af}{bc - ad}\right) \quad \Leftrightarrow$ $cx = \frac{bcf - adf - cde + adf}{bc - ad} \quad \Leftrightarrow$ $cx = \frac{bcf - cde}{bc - ad} \quad \Leftrightarrow$ $cx = \frac{c\left(bf - de\right)}{bc - ad} \quad \Leftrightarrow$ $x = \frac{bf - de}{bc - ad}$	Now substitute the expression for y into the second equation and solve for x.
	The general solution is: $$x = \frac{bf - de}{bc - ad}, \quad y = \frac{ce - af}{bc - ad}$$ where $bc - ad \neq 0$.	

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	Steps	Explanation
b)	In $\begin{cases} 2\mathrm{i}x + (2+3\mathrm{i})\,y = 3 \\ (1+\mathrm{i})\,x + 2y = 5 \end{cases}$, $a = 2\mathrm{i},\ b = 2+3\mathrm{i},\ c = 1+\mathrm{i},\ d = 2,\ e = 3,\ f = 5.$ $x = \frac{bf - de}{bc - ad}$ $= \frac{(2+3\mathrm{i})(5) - (2)(3)}{(2+3\mathrm{i})(1+\mathrm{i}) - (2\mathrm{i})(2)}$ $= \frac{10 + 15\mathrm{i} - 6}{-1 + 5\mathrm{i} - 4\mathrm{i}} = \frac{4 + 15\mathrm{i}}{-1+\mathrm{i}}$ $= \frac{4+15\mathrm{i}}{-1+\mathrm{i}} \times \frac{-1-\mathrm{i}}{-1-\mathrm{i}}$ $= \frac{11 - 19\mathrm{i}}{2}$ $= 5.5 - 9.5\mathrm{i}$ $y = \frac{ce - af}{bc - ad}$ $= \frac{(1+\mathrm{i})(3) - (2\mathrm{i})(5)}{(2+3\mathrm{i})(1+\mathrm{i}) - (2\mathrm{i})(2)}$ $= \frac{3 + 3\mathrm{i} - 10\mathrm{i}}{-1+\mathrm{i}}$ $= \frac{3 - 7\mathrm{i}}{-1+\mathrm{i}} \times \frac{-1-\mathrm{i}}{-1-\mathrm{i}}$ $= \frac{-10 + 4\mathrm{i}}{2}$ $= -5 + 2\mathrm{i}$	Use the result from part a) to solve.

Exam tip

Systems of linear equations with complex coefficients can be solved by elimination or substitution. However, it is easier to solve these questions using the result that the general solution to

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \text{ is } x = \frac{bf - de}{bc - ad},\ y = \frac{ce - af}{bc - ad}, \text{ where } bc - ad \neq 0.$$

This result is useful for solving systems of linear equations with complex coefficients, but it is not given in the IB formula booklet. You should be prepared to derive it in the exam instead of memorising it.

3 section questions

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1. Number and algebra / 1.16 Systems of linear equations

Systems with three equations

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You can use elimination and substitution to solve systems with three equations and three variables, but there is a more efficient way to approach these questions called row reduction. To do this you will write the system of equation using a matrix. A matrix is a rectangular array of terms or elements that follows specific algebraic rules. In this section you will learn how to perform row operations with a matrix to solve systems of linear equations.

The first step in solving a system of linear equations is to rewrite them into a different form that allows for easy and quick manipulation, a form that is called the augmented matrix form. Let us do that for an example of the most general systems of linear equations we will come across:

$$\begin{array}{rcrcrl} x & + & 2y & + & 3z & = 1 \\ -x & + & 3y & + & 2z & = 2 \\ 2x & + & y & - & 3z & = 3 \end{array} \quad \rightarrow \quad \begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 3 & 2 & 2 \\ 2 & 1 & -3 & 3 \end{pmatrix}$$

Remember, the augmented matrix is just a different representation of the same information, which is encoded in the coefficients in front of the variables.

You can now manipulate the rows of an augmented matrix in a way that is similar to the process of elimination to get to a matrix that looks like:

$$\begin{pmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{pmatrix}$$

where you can see the triangle of 0s in the bottom left.

This form of the matrix is called row echelon form and it allows you to easily find the value of z since the last row reads $hz = \mathrm{i}$ and therefore $z = \frac{\mathrm{i}}{h}$. From here you can use the second row which reads $ey + fz = g$ to find y, and so on for x.

The process of finding the row echelon form is called Gaussian elimination.

✓ **Important**

Gaussian elimination allows you to go from an augmented matrix to row echelon form.

To do this you can use any of the following operations as needed:

- multiply any row by a non-zero number
- replace a row with the sum of the same row and a non-zero multiple of another row
- swap position of rows.

How are the operations in Gaussian elimination related to the process of solving a system of equations by elimination?

Example 1

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Solve the system of equations

$$\begin{aligned} 2x + y - z &= 4 \\ x + y + 2z &= 2 \\ 2x - 2y - 2z &= 10 \end{aligned}$$

Steps	Explanation
$\begin{aligned} 2x + y - z &= 4 \\ x + y + 2z &= 2 \\ 2x - 2y - 2z &= 10 \end{aligned} \rightarrow \begin{pmatrix} 2 & 1 & -1 & 4 \\ 1 & 1 & 2 & 2 \\ 2 & -2 & -2 & 10 \end{pmatrix}$	Start by forming an augmented matrix.
$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & -1 & 4 \\ 2 & -2 & -2 & 10 \end{pmatrix}$	Next apply row reduction in steps. First, choose, if you can, a row that starts with a 1 to be the top row. To do this you are using the fact that you can swap rows.
$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & -1 & 4 \\ 2 & -2 & -2 & 10 \end{pmatrix} \xrightarrow[R_3 - 2\times R_1]{R_2 - 2\times R_1} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -5 & 0 \\ 0 & -4 & -6 & 6 \end{pmatrix}$	To make the first entry in the second and third rows 0 you can subtract twice row 1 from row 2, so new $R_2 - 2 \times R_1$, and from row 3, so new $R_3 - 2 \times R_1$. Here you are using the fact that you can replace a row with the sum of the same row and a non-zero multiple of another row.
$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -5 & 0 \\ 0 & -4 & -6 & 6 \end{pmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -5 & 0 \\ 0 & -2 & -3 & 3 \end{pmatrix}$	Notice that all the numbers in the new row 3 are multiples of 2, and since it is easier to work with small numbers, you can divide through by 2.
$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -5 & 0 \\ 0 & -2 & -3 & 3 \end{pmatrix} \xrightarrow{R_3 - 2\times R_2} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -5 & 0 \\ 0 & 0 & 7 & 3 \end{pmatrix}$	Then subtract twice row 2 from row 3, $R_3 - 2 \times R_2$. You have arrived at a matrix that is in the form: $\begin{pmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{pmatrix}$
From the bottom row: $7z = 3 \Leftrightarrow z = \dfrac{3}{7}$	You can read the value of z from the bottom row and use it in the rows above to find the values of y and then of x.

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Steps	Explanation
$-y-5z=0 \Leftrightarrow y=-5z=-5\times\frac{3}{7}=-\frac{15}{7}$.	Now substitute the result for z in the second row to obtain y.
$x+y+2z=2 \Leftrightarrow x=-y-2z+2=\frac{15}{7}-2\times\frac{3}{7}+2=\frac{23}{7}$.	Substitute z and y into row 1 to get x.
The solution is $$x=\frac{23}{7},\ y=-\frac{15}{7},\ z=\frac{3}{7}.$$	

Does row reduction have any connection to solving systems of equations by substitution or by elimination or is it a completely new method? Why is it possble to swap the order of the rows in the augmented matrix without changing the problem? Why is it easier to perform Gaussian elimination when the first row starts with 1?

International Mindedness

Row reduction is also called Gaussian elimination in honour of Carl Friedrich Gauss (1777—1855). However, this method for solving systems of equations was described by Chinese mathematicians as early as 150 BCE.

Activity

Solutions to a system of three linear equations can be interpreted graphically as the intersection of the three planes represented by the three equations. Sketch a situation where the three planes

- do not intersect with each other at all
- intersect in a point
- intersect in a line.

Can you think of another way that they three planes can intersect that you were not asked to sketch?

Important

The entries in a matrix in the form, $\begin{pmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{pmatrix}$ where $a, e \neq 0$, can tell you about the nature of the solutions to a system of three linear equations with three variables. The three possibilities are:

- If $h \neq 0$, there is one unique solution. There is one point (x, y, z) where all three planes intersect.
- If $h = 0$ and $i \neq 0$, there is no solution. The planes do not intersect.
- If $h = 0$ and $i = 0$, there are infinitely many solutions. The planes intersect in a line.

A system that has no solutions is called inconsistent. A system that has a unique or infinitely many solutions is called consistent.

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Be aware

The operations that are used in Gaussian elimination can be done in a variety of order. When you are checking your work against the given solutions or comparing it with the work of your peers you may find that the steps are not the same. This is fine, so long as the final solution is the same.

Example 2

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Consider the system of equations given by

$$\begin{aligned} 2x &- y + z = 5 \\ x &+ y - z = 2 \\ 3x &- 3y + 3z = k \end{aligned}$$

a) Find the value of k for which this system is inconsistent.

b) Find the value of k for which this system is consistent.

c) For the consistent system, find the solutions.

	Steps	Explanation
a)	$\begin{pmatrix} 2 & -1 & 1 & 5 \\ 1 & 1 & -1 & 2 \\ 3 & -3 & 3 & k \end{pmatrix}$	Write as an augmented matrix
	$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & -1 & 1 & 5 \\ 3 & -3 & 3 & k \end{pmatrix} \xrightarrow[R_3-3\times R_1]{R_2-2\times R_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -3 & 3 & 1 \\ 0 & -6 & 6 & k-6 \end{pmatrix} \xrightarrow{R_3-2\times R_2} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -3 & 3 & 1 \\ 0 & 0 & 0 & k-8 \end{pmatrix}$	Swap row 2 with row 1. Perform the row elimination.
	The system is inconsistent if $k-8 \neq 0$. The system is inconsistent for $k > 8$ or $k < 8$.	
b)	If $k = 8$, the last row gives $0 = 0$, there are infinitely many solutions and the system is consistent.	
c)	For $k = 8$, the reduced matrix is $\begin{aligned} x+y-z &= 2 \\ -3y+3z &= 1 \\ 0 &= 0 \end{aligned}$ Let, the second equation gives $y = t - \frac{1}{3}$.	Since the system has infinitely many solutions and you do not know specific value so of z, you can use a parameter to represent z. Use $z = t$ in the second equation to find an expression for y.

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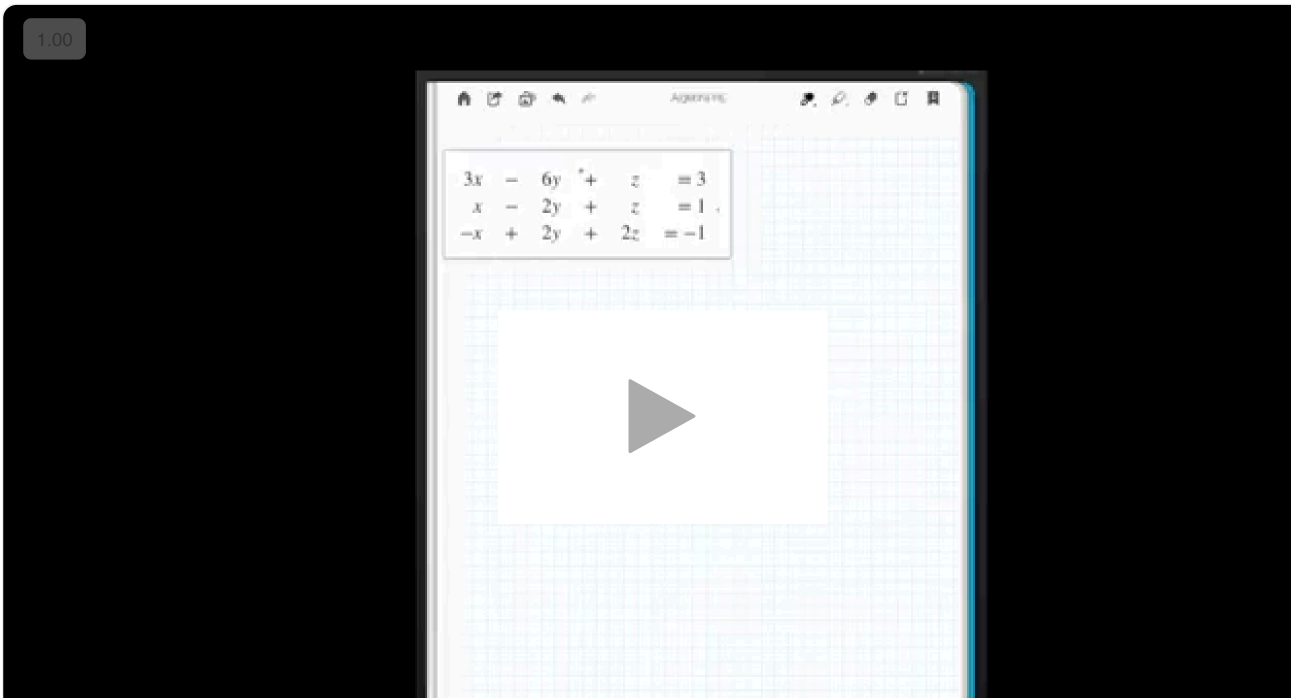
	Steps	Explanation
	$x = \frac{7}{3}$	Use $z = t$ and $y = t - \frac{1}{3}$ in the first equation to get x.
	The solution is $x = \frac{7}{3},\ y = t - \frac{1}{3},\ z = t.$	Since $t \in \mathbb{R}$, you can find a solution point for this system by picking a value of t. r example: if $t = 1$, the solution is $x = \frac{7}{3},\ y = \frac{2}{3},\ z = 1$. Since there are infinitely many solutions there are infinitely many values of t that can be used.

You will usually see examples where $\begin{pmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{pmatrix}$ where $a, e \neq 0$, and it is unlikely that you will encounter an exam question where this is not the case so it is not presented in the Examples. You can see how to work with a system where $e = 0$ in the video below, which shows the solution to:

$$\begin{cases} 3x - 6y + z = 3 \\ x - 2y + z = 1 \\ -x + 2y + 2z = -1 \end{cases}$$



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Video 1. Systems with Three Equations: Example.

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More information for video 1

1

00:00:00,200 --> 00:00:02,703

narrator: In this video, we're going

to use Gaussian elimination

2

00:00:02,769 --> 00:00:05,539

to solve the following

system of linear equations.

3

00:00:05,906 --> 00:00:07,074

Now, to start off, we're going

4

00:00:07,140 --> 00:00:09,409

to write the information in the equations

5

00:00:09,476 --> 00:00:11,311

in augmented matrix forms.

6

00:00:11,378 --> 00:00:14,848

So we use just the coefficients

because that's what the information lies,

7

00:00:15,182 --> 00:00:17,117

and it has the augmented matrix.

8

00:00:17,384 --> 00:00:20,354

Normally, I would like

the first row to start

9

00:00:20,420 --> 00:00:22,089

with the one if at all possible.

10

00:00:22,389 --> 00:00:25,025

So that is basically

just swapping two rows

11

00:00:25,092 --> 00:00:28,328

around in this case,

which is interchanging two equations,

12

00:00:28,395 --> 00:00:29,730

which of course one can always do.

13

00:00:30,030 --> 00:00:33,767

So this is the augmented matrix

from with which I'm going to work.

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00:00:34,234 --> 00:00:38,805

Now you see that if I take the second row

and subtract three times the first row,

15

00:00:38,872 --> 00:00:39,940

I'm gonna create some zeros.

16

00:00:40,007 --> 00:00:43,043

So first I'm gonna rewrite the first row

because I'm gonna leave that alone.

17

00:00:43,110 --> 00:00:44,444

So $(1 \quad -2 \quad 1 \quad | \quad 1)$,

18

00:00:44,545 --> 00:00:47,181

and then I'm gonna subtract

three times the first row

19

00:00:47,247 --> 00:00:49,383

from the second row

to give me this number.

20

00:00:49,650 --> 00:00:52,553

Then I want to create

some more zeros in the bottom,

21

00:00:52,619 --> 00:00:55,322

and that is basically

adding row one to row three.

22

00:00:55,455 --> 00:00:59,393

And here is what I get

of the row operations.

23

00:00:59,459 --> 00:01:01,662

And what I conclude

from here is from the last row,

24

00:01:01,728 --> 00:01:04,031

the $3z = 0$.

25

00:01:04,231 --> 00:01:06,934

Now, that is only true

if z itself, of course, is 0.

26

00:01:07,234 --> 00:01:10,204

So we found one solution

from the last row,

27

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00:01:10,304 --> 00:01:11,505

and that is $z = 0$.

28

00:01:11,572 --> 00:01:15,108

Now if I look at the second row,

I've got $0 \times y$

29

00:01:15,242 --> 00:01:18,078

minus $2 \times z = 0$, but z is already 0.

30

00:01:18,145 --> 00:01:19,580

So zero times y is 0.

31

00:01:19,947 --> 00:01:22,549

And of course, this is true for any y,

32

00:01:22,616 --> 00:01:25,252

any y value will satisfy this,

33

00:01:25,319 --> 00:01:28,121

and therefore I'm gonna let $y = t$ some parameter

34

00:01:28,188 --> 00:01:33,260

then from the first row,

$x - 2y + z = 1$,

35

00:01:33,594 --> 00:01:36,563

I get $x = 1 + 2t$.

36

00:01:36,930 --> 00:01:41,034

So here are my three solutions

for the unknowns x, y,

37

00:01:41,101 --> 00:01:44,271

and z for the system of linear equations.

Example 3

★★☆

Determine the value of k for which $\begin{cases} kx + y + z = 1 \\ x + y + z = 1 \\ x + 2y - z = 2 \end{cases}$ has infinitely many solutions.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
$\begin{pmatrix}1 & 1 & 1 & 1\\ k & 1 & 1 & 1\\ 1 & 2 & -1 & 2\end{pmatrix}$	Write the augmented matrix swapping row 2 for row 1. Proceed with Gaussian elimination.
$\xrightarrow{R_2 - k\times R_3} \begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1-2k & 1+k & 1-2k\\ 1 & 2 & -1 & 2\end{pmatrix}$	To get a zero in the first entry of row 2 you can subtract kR_3.
$\xrightarrow{R_3 - R_1} \begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1-2k & 1+k & 1-2k\\ 0 & 1 & -2 & 1\end{pmatrix}$	To get a zero in the first entry of row 3 you can subtract R_1.
$\xrightarrow{\text{swap } R_2 \text{ and } R_3} \begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1 & -2 & 1\\ 0 & 1-2k & 1+k & 1-2k\end{pmatrix}$	It is easier to work with expressions with k in row 3.
$\xrightarrow{R_3 + 2k\times R_2} \begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1 & -2 & 1\\ 0 & 1 & 1-3k & 1\end{pmatrix}$ $\xrightarrow{R_3 - R_2} \begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1 & -2 & 1\\ 0 & 0 & 3-3k & 0\end{pmatrix}$	
$3-3k=0 \quad \Leftrightarrow \quad k=1$	There are infinitely many solutions when row 3 gives you $0=0$.

Be aware

Gaussian elimination can also be applied to systems of four or more equations with four or more unknowns. Using matrices and the operations of Gaussian elimination helps you to keep your work organised in a way that would not be possible if you tried to solve a system of four or more equations using elimination or substitution.

4 section questions

1. Number and algebra / 1.16 Systems of linear equations

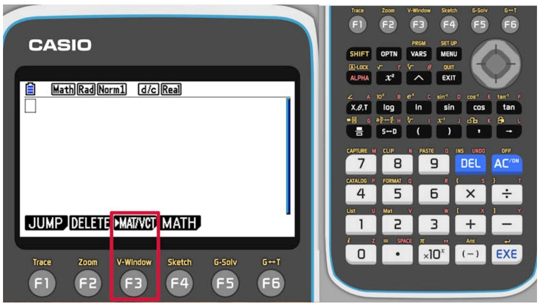
Using technology

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/using-technology-id-27380/print/) Assign

Gaussian elimination is a useful algorithm, but it is time-consuming and tedious. Your calculator can perform this operation for you. In fact, it can take the work further to create a row-reduced echelon form of the matrix from which you can read the solutions directly.

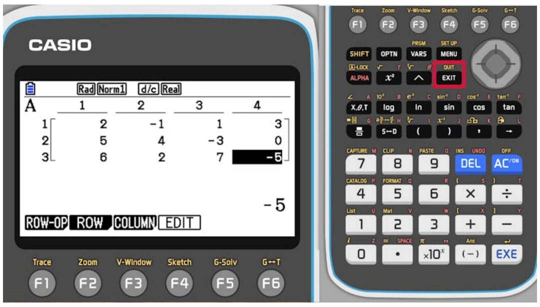
Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
To find the reduced row echelon form of the matrix $$\begin{pmatrix} 2 & -1 & 1 & 3 \\ 5 & 4 & -3 & 0 \\ 6 & 2 & 7 & -5 \end{pmatrix},$$ open the calculator mode, ...	
... and press F3 to open the page to work with matrices.	
The calculator needs to know the matrix. You can store it in any of the available matrix memory. First press F3 to tell the dimensions.	

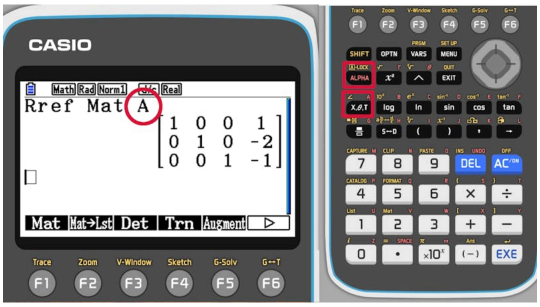
Student view

Overview (/study/ap aa-hl/sid-134-cid-761926/o

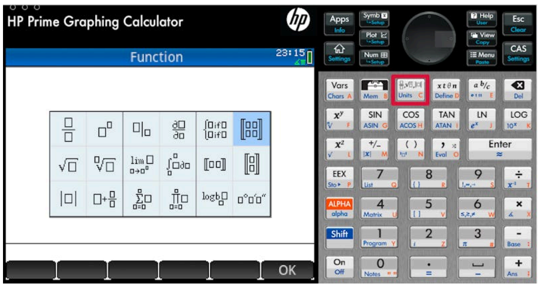
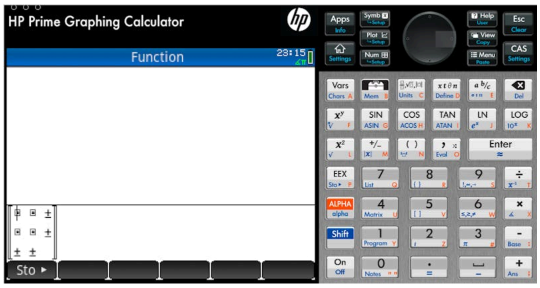
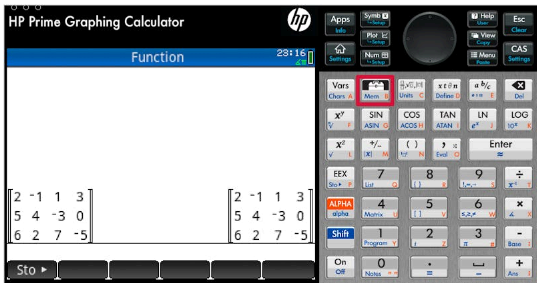
Steps	Explanation
The matrix has three rows and four columns.	
Enter the entries of the matrix. When you are done, press EXIT until you get back to the main calulation screen. The matrix is now stored in matrix memory A.	
You now need to find the option that calculates the reduced row echelon form. Press OPTN, ...	

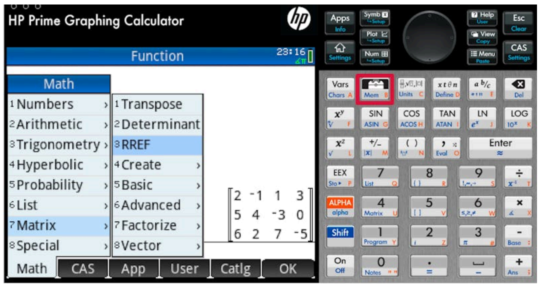
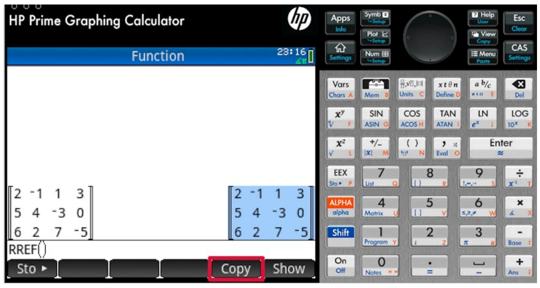
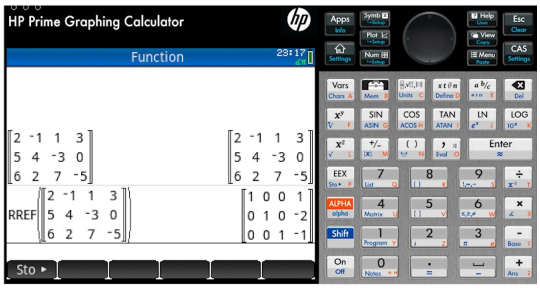
Student view

Steps	Explanation
... press F2 for the matrix related options, ...	
... press F6 to scroll to see the next options, ...	
... and finally F5 to choose the tool that calculates the reduced row echelon form (Rref).	

Steps	Explanation
Scroll back to the previous options and press F1 to choose a matrix ...	
... and choose the letter corresponding to the memory where you stored the matrix. After pressing EXE, the reduced matrix is displayed.	

Steps	Explanation
To find the reduced row echelon form of the matrix $\begin{pmatrix} 2 & -1 & 1 & 3 \\ 5 & 4 & -3 & 0 \\ 6 & 2 & 7 & -5 \end{pmatrix},$ enter the home screen of any application.	

Steps	Explanation
The calculator needs to know the matrix, so open the template menu and choose the matrix template.	
Originally you are offered to fill a 2×2 matrix template. Start entering the entries, the template will automatically expand.	
When the matrix is entered, you need to find the option that calculates the reduced row echelon form. Open the toolbox ...	

Steps	Explanation
... and find the tool that calculates the reduced row echelon form (RREF).	
One way of copying the matrix inside the brackets after RREF is to tap on the matrix and then choose the copy option.	
After pressing enter, the reduced matrix is displayed.	

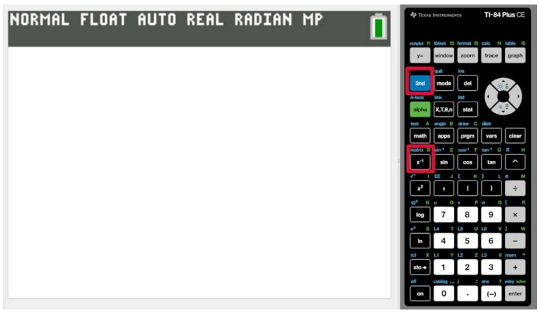
<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>

To find the reduced row echelon form of the matrix

$$\begin{pmatrix} 2 & -1 & 1 & 3 \\ 5 & 4 & -3 & 0 \\ 6 & 2 & 7 & -5 \end{pmatrix},$$

open the options to work with matrices ...

</td><td>



</td></tr>
<tr><td>

... and choose to edit any of the matrices.

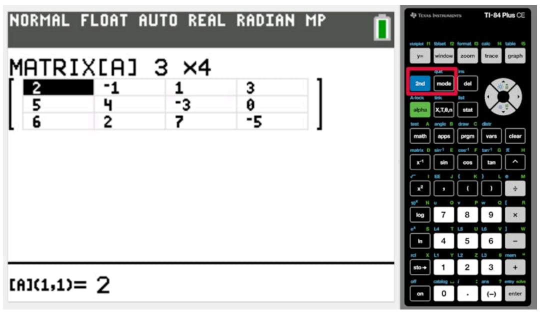
</td><td>

</td></tr>
<tr><td>

Enter the entries of the matrix.

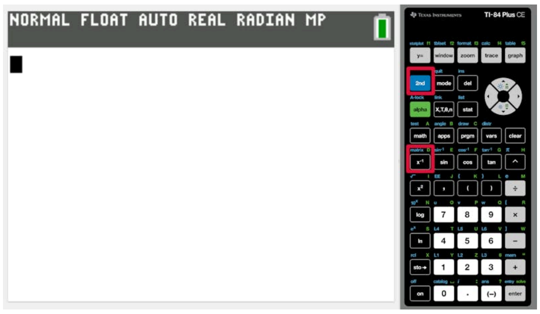
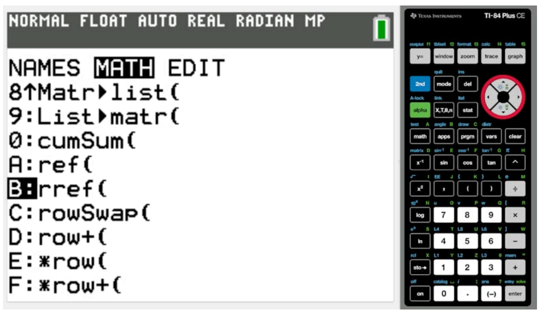
When you are done, quite the editing mode to get back to the main calulation screen. The matrix is now stored in matrix memory A.

</td><td>

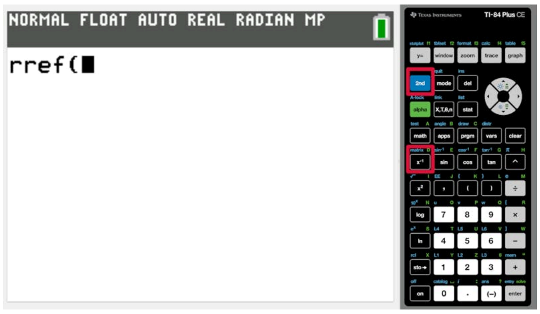
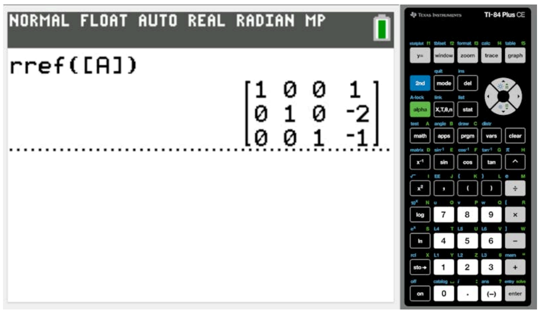


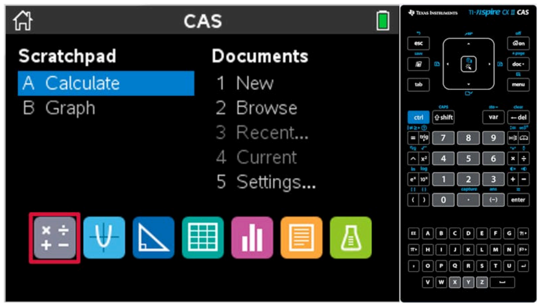
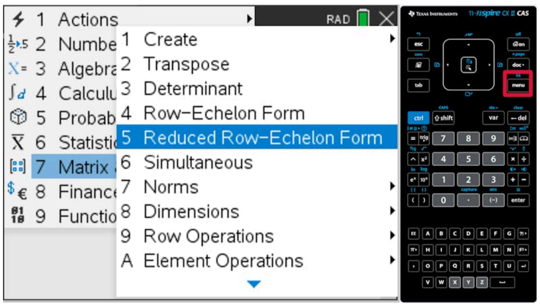
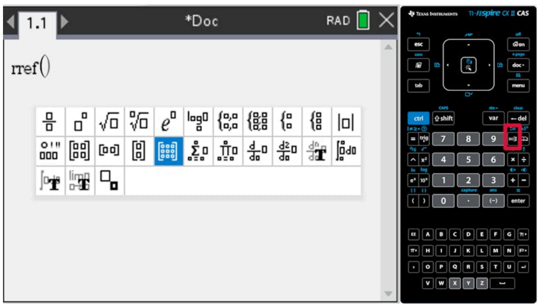
</td></tr>
</table>

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
You now need to find the option that calculates the reduced row echelon form. Open the options to work with matrices again ...	
... choose the math options and scroll down ...	
... until you find the tool that calculates the reduced row echelon form (rref).	

Student view

Steps	Explanation
You need to tell the calculator which matrix you want to work with, so open the matrix menu yet again...	
... and choose the name of the matrix where you stored the entries.	
After pressing enter, the reduced matrix is displayed.	

Steps	Explanation
To find the reduced row echelon form of the matrix $$\begin{pmatrix} 2 & -1 & 1 & 3 \\ 5 & 4 & -3 & 0 \\ 6 & 2 & 7 & -5 \end{pmatrix},$$ open a calculator page.	
Open the menu and find the tool that calculates the reduced row echelon form of a matrix.	
To enter the matrix, open the template menu and choose the matrix template.	

Overview
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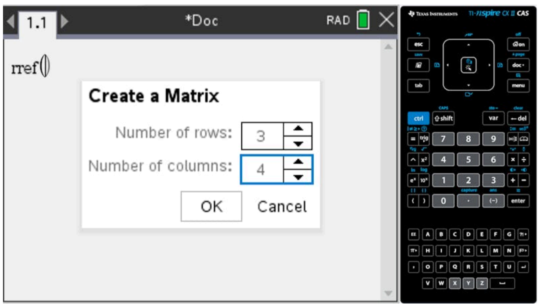
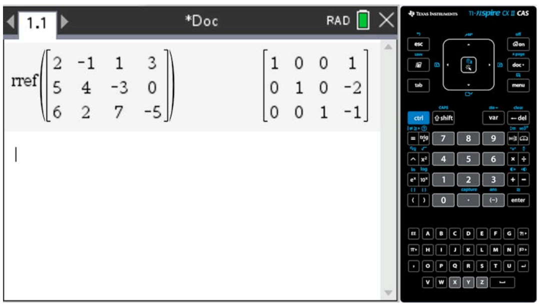
Steps	Explanation
The matrix has three rows and four columns.	
Enter the entries of the matrix. After pressing enter, the reduced matrix is displayed.	

Example 1

★☆☆

$$\text{Solve}\begin{cases} 2x - y + z = 3 \\ 5x + 4y - 3z = 0 \\ 6x + 2y + 7z = -5 \end{cases}$$

Student view

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Steps	Explanation
The row-reduced echelon form given by the calculator is: $$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$ The solution is: $$x = 1,\ y = -2,\ z = -1.$$	If you used row echelon form, you would get: $$\begin{pmatrix} 1 & 0.3333\ldots & 1.1666\ldots & -0.8333\ldots \\ 0 & 1 & -3.7857\ldots & 1.7857\ldots \\ 0 & 0 & 1 & -1 \end{pmatrix}$$ You would then need to use $z = -1$ and find values of y and x from the first two rows.

You can also use a specifically dedicated systems of equations solver on your calculator rather than use matrices.

Exam tip

If you are using your calculator to solve a system of equations, you need to write down the system being solved and state the solution. You do not need to show the steps used.

Steps	Explanation
These instructions show you how the equation solver application presents the infinitely many solutions of the following system of equations. $$\begin{aligned} 2x - y + z &= 5 \\ x + y - z &= 2 \\ 3x - 3y + 3z &= 8 \end{aligned}$$ To access the equation solver, open the equation mode ...	

Student
view

Overview
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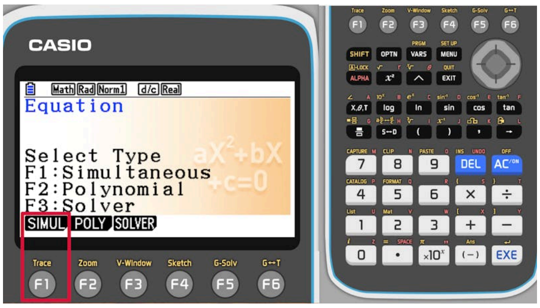
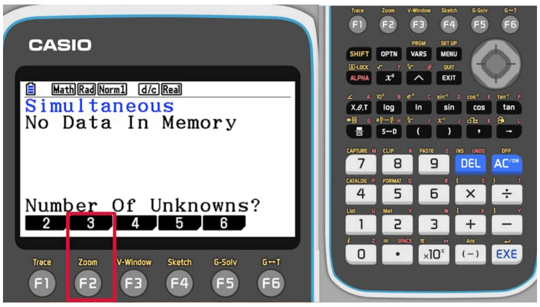
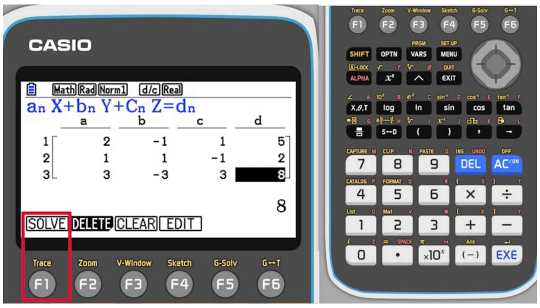
Steps	Explanation
... and press F1 to solve a system of linear equations.	
First you will be asked to set the number of unknowns. In this example the equation has three unknowns, so press F2.	
Enter the coefficients of the equations and press F1 to find the solution.	

Student
view

Overview
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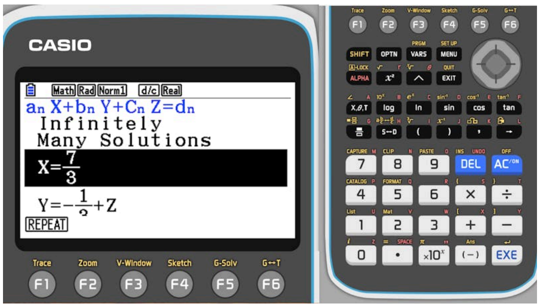
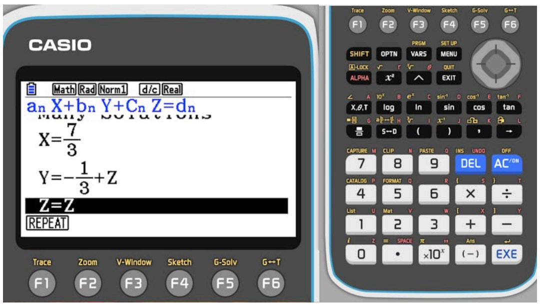
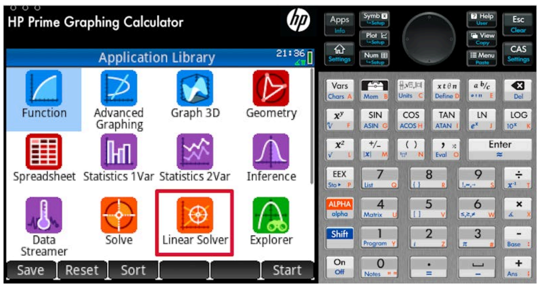
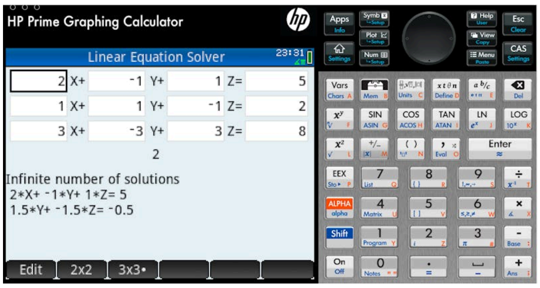
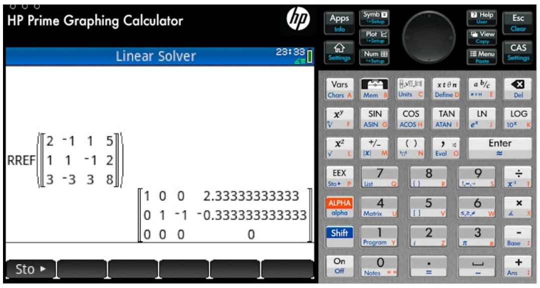
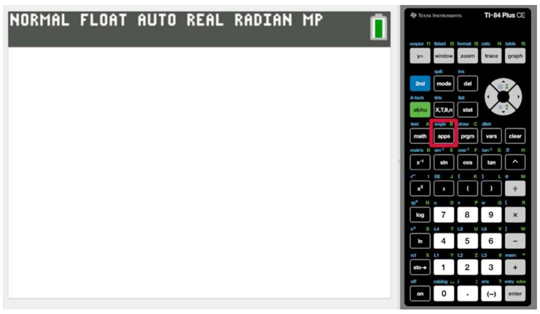
Steps	Explanation
The solution does not fit on the screen, so you will need to scroll down.	
Make sure you understand how the infinitely many solutions are presented. In this case the variable z is used as a parameter.	

Steps	Explanation
These instructions show you how the equation solver application presents the infinitely many solutions of the following system of equations. $$\begin{aligned} 2x - y + z &= 5 \\ x + y - z &= 2 \\ 3x - 3y + 3z &= 8 \end{aligned}$$ To access the equation solver, open the linear solver application.	

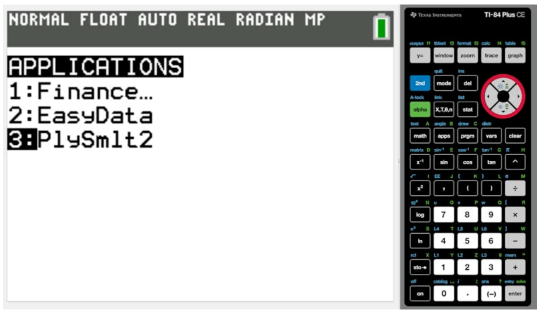
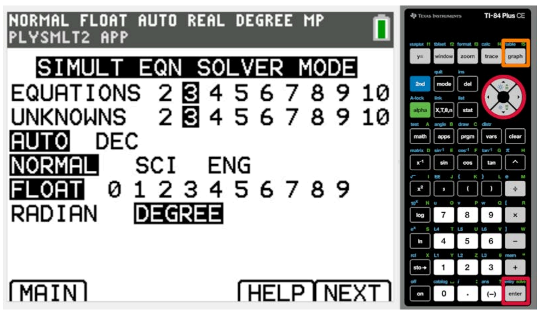
Student
view

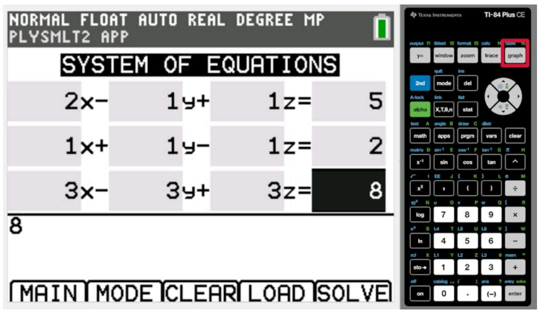
Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

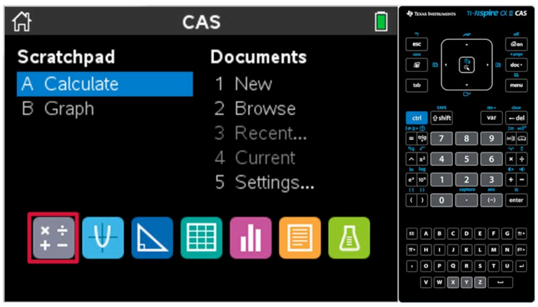
Steps	Explanation
This calculator can solve linear equation systems in two or three unknowns. The solution is updated real time as you enter the coefficients. Note, that you do not get a parametric description of the infinitely many solutions, only a confirmation that the solution is not unique and a reduced, equivalent equation system. You will need to work out the parametric description using this reduced system. This may change though in future operating systems.	
In the current operating system the reduced row echelon form of the coefficient matrix gives a result that can give the parametric form more easily. The instructions on how to find the reduced row echelon form can be found a bit earlier in this section.	

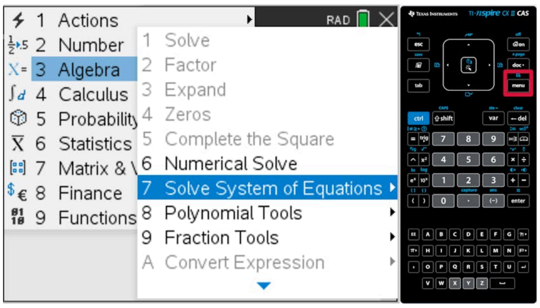
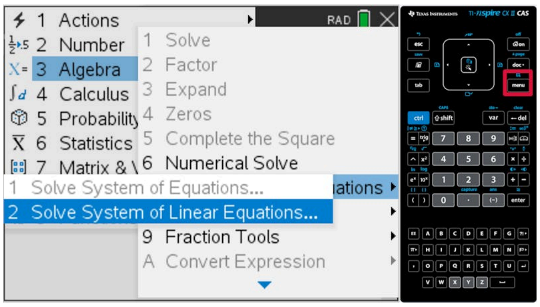
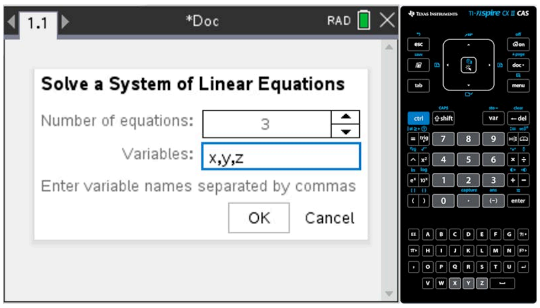
Steps	Explanation
These instructions show you how the equation solver application presents the infinitely many solutions of the following system of equations. $$\begin{aligned} 2x - y + z &= 5 \\ x + y - z &= 2 \\ 3x - 3y + 3z &= 8 \end{aligned}$$ To access the equation solver, open the application menu ...	

Student view

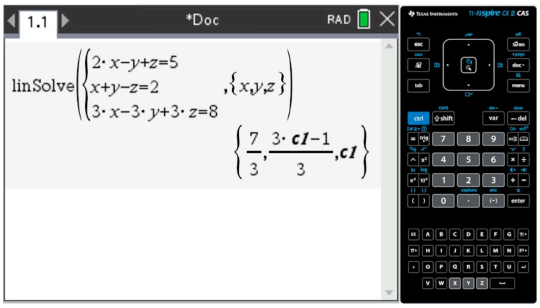
Steps	Explanation
... and choose the equation solver application (PlySmlt2).	
Choose the option to solve a simultaneous equation system.	
Use the arrow keys to select the number of equations and unknowns. When you set all options, press the graph button to move to the next screen.	

Steps	Explanation
Enter the coefficients of the equations and press the graph button to find the solution.	
Make sure you understand how the infinitely many solutions are presented. In this case the variable z is used as a parameter.	

Steps	Explanation
These instructions show you how the equation solver application presents the infinitely many solutions of the following system of equations. $$\begin{aligned}2x-y+z&=5\\x+y-z&=2\\3x-3y+3z&=8\end{aligned}$$ You can access the equation solver from a calculator page.	

<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>Open the menu, find the option to solve system of equations, ...</td><td>
</td></tr>
<tr><td>... and choose the option to solve a system of linear equations.</td><td>
</td></tr>
<tr><td>Enter the number of equations and the names of the variables.</td><td>
</td></tr>
</table>

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
Enter the equations and press enter to find the solution. Make sure you understand how the infinitely many solutions are presented. In this case the variable $c1$ is used as a parameter.	

Example 2

★★☆

$$\text{Solve} \begin{cases} 2x - y + z = 5 \\ x + y - z = 2 \\ 3x - 3y + 3z = 8 \end{cases}$$

Steps	Explanation
The solution given by the calculator is: $z = t,\ y = t - \dfrac{1}{3},\ x = \dfrac{7}{3}.$	You can use row reduced echelon form to get: $\begin{pmatrix} 1 & 0 & 0 & 2.333\ldots \\ 0 & 1 & -1 & -0.333\ldots \\ 0 & 0 & 0 & 0 \end{pmatrix}$ or you can use the systems of equations solver.

3 section questions

1. Number and algebra / 1.16 Systems of linear equations

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-27381/print/) Assign

What you should know

By the end of this subtopic you should be able to:

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

- solve systems of two linear equations with two unknowns using elimination and substitution
- solve systems of two equations and two unknowns with complex coefficients
- use Gaussian elimination to solve systems of three linear equations with three unknowns
- determine the nature of solutions when solving a system of two equations by substitution or elimination and when solving a system of three equations using Gaussian elimination
- determine values of coefficients which would produce a specified kind of solution to a system of equations
- solve systems of equations using a calculator.

1. Number and algebra / 1.16 Systems of linear equations

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-27382/print/) Assign

Interactive 1. Investigating the Intersection of Linear Equations.

More information for interactive 1

This interactive allows the users to explore linear programming questions such as the one shown in the applet. In the interactive users can see three lines in blue, purple and orange color. Each line has two red dots on it. Users can move these dots to change the linear inequalities and can select the options of $>$, $\geq$, $<$, $\leq$ from the dropdown menu of each equation to find and understand how the shaded region changes. The graph will be a solid line when the inequality $\geq$ The selected and shaded part will represent the side of the boundary line where the inequality is true. When the users select $<$ the boundary line will be a dashed line. When the users drag the points of blue line such that $y \geq 3.75x + 16.5$, the shaded region will be on the left of this line. The users will have a better understanding of how the graph changes as they move the sliders and what connection the graph and shaded area have to a solution to a system of equations.

For example taking three Inequalities as:

$y \leq 1.2x + 7.6$

$y < -0.67 + 7.67$

$y < 0.5x - 4$

In the graph on the right the blue solid line represents the boundary $y = 1.2x + 7.6$. Since the inequality is $y \leq$, the line is solid, indicating that points on the line are included in the solution. The solution to this inequality includes all points on or below the solid blue line. The pink dashed line represents the boundary $y = -0.67 + 7.67$. Since the inequality is $y <$, the line is dashed, indicating that points on the line are not included in the solution. The solution to this inequality includes all points strictly above the dashed pink line. The orange dashed line represents the boundary $y = 0.5x - 4$. Since the inequality

Student view

is y <, the line is dashed, indicating that points on the line are not included in the solution. The solution to this inequality includes all points strictly above the dashed orange line. The light grey shaded region represents the solution set to the system of all three inequalities. It includes all the points that satisfy all three conditions simultaneously. Users will notice that the boundaries formed by the dashed pink and orange lines are not included in the solution set. Through this interactive users will gain the knowledge of the linear programming models in real-life situations involving optimization, such as maximizing profit or minimizing costs, to find the best solution within constraints in areas like manufacturing, transportation, and resource allocation.

Part 1

Explore linear programming questions such as the one shown in the applet. What do you think is represented by the shaded region? How does the graph change as you move the sliders? What connection does the graph and shaded area have to a solution to a system of equations? What kind of real-life situations are modelled with linear programming? How are these questions different from ones involving a solution to a system of linear equations?

Part 2

Research the simplex method for solving linear programming questions. Compare this method with Gaussian elimination.