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# The big picture

In section 5.12.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/continuity-id-26491/), you learned the basics about continuity and differentiability. In this subtopic, you will learn about some of the underlying theorems that rely on those properties.

Think about climbing a mountain and starting at sea level.

Sundial Peak, USA

Credit: Stephanie Sigafoos GettyImages

To get to the top of a 5000 foot mountain, is it necessary, at least once, to be at 4000 feet? Of course it is. You may have to go through that altitude multiple times based on the terrain, but you know you have to go through it at least once. This is an example of the intermediate value theorem.

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What if you were to climb over the mountain and go down the other side all the way to the sea (sea level)? Along your trek, would you ever be at a place where the terrain levelled off? Again, common sense says, at least at the very top of the mountain, it would be level at the point where you stopped going up and started going down. Depending on the terrain, you might encounter many level spots, but you can guarantee at least one. That is an example of Rolle's theorem.

Finally, if you were to measure the average gradient from the bottom of the mountain to the top, would there ever be a point along the trek that had an instantaneous gradient equal to that average gradient? This one is harder to picture, but as you will see in the chapter, the answer is yes, there has to be at least one such point.

Additionally, this chapter will develop a well-known power series called the Taylor series that can be used to represent a variety of complex functions with a series of polynomial terms. The accuracy of this approximation depends largely on how close the initial known point is to the point to be evaluated. The Maclaurin series is a special case of the Taylor series that uses the known point $x = 0$.

These series can be used to find approximations of complex functions, limits to functions, approximations to definite integrals, and to approximate solutions to differential equations.

### Concept

Throughout this subtopic, think back to other calculus problems you have solved in topic 5. Are there examples that you do not have the ability to solve? Are there examples that you may have the ability, but would like a more efficient method to find a 'good enough' answer more quickly? How do the techniques learned in this subtopic apply to finding **approximate** solutions to those types of problems?

5. Calculus / 5.19 Maclaurin series expansions

# Continuity theorems

In subtopic 5.12 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26489/), you learned about continuity and differentiability. As a review:

### Making connections

A function, $f(x)$, is said to be **continuous** at point $c$ if:

- $f(c)$ is defined in the range of $f(x)$
- $\lim\limits_{x \to c} f(x)$ exists

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- $\lim_{x \to c} f(x) = f(c)$.

For a function to be **differentiable** at a point, it must:

- be continuous at the point
- have local linearity at the point.

For a function to be differentiable across a domain, if must be differentiable at every point within that domain.

Based on these definitions, you can now look at a few important continuity theorems.

### Exam tip

The content of this section can be helpful for understanding the connections you will learn in the next few sections. However, the syllabus does not mention continuity theorems, so this section can safely be skipped for exam preparation.

# Intermediate value theorem

In 1817, Bernard Bolzano, a Bohemian mathematician, offered the first proof of this theorem, which is often called Bolzano's theorem. Prior to his work, the idea was considered obvious enough to require no proof.

### Important

Let $f$ be a continuous function on $[a, b]$ with $f(a) \neq f(b)$. Then, for every $y_0$ between $f(a)$ and $f(b)$, or $f(a) < y_0 < f(b)$, there is at least one $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

In simpler terms, this means that a continuous function never takes on two values without taking on all the values in between. Below is an example graph showing this theorem with a continuous function defined on a closed interval $[a, b]$. For every $y_0$, $f(a) < y_0 < f(b)$, there is at least one $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

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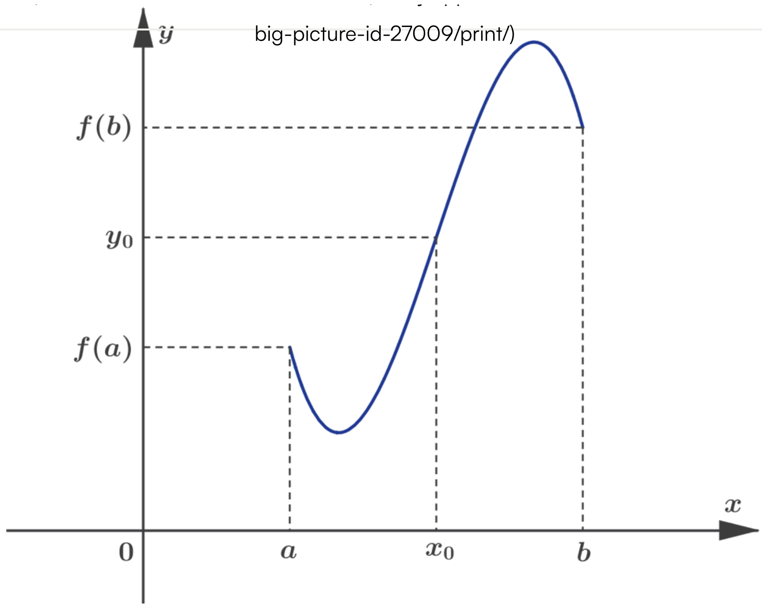


More information

This image is a graph that demonstrates a continuous function on a closed interval ([a,b]). The X-axis represents the variable (x) with the endpoints labeled (a) and (b), and the Y-axis represents the function values (f(x)). The graph includes a continuous curve which shows the progression of the function from (f(a)) to (f(b)). There is a horizontal dashed line at (y_0) which is between (f(a)) and (f(b)), indicating that somewhere between (a) and (b) (specifically at (x_0)), the function( f(x)) evaluates to (y_0). This demonstrates the intermediate value theorem, which states that for any value (y_0) between (f(a)) and (f(b)), there exists an (x_0) in the interval ((a, b)) such that (f(x_0) = y_0). The graph also shows points (f(a)), (f(b)), (x_0), and (y_0) marked with dashed lines showing their positions relative to the axes and the function curve.

[Generated by AI]

Depending on the nature of the function, there may be more than one value for $x_0$ that satisfies the condition. If the function is either increasing or decreasing throughout the interval, there will be only one solution.

## Example 1

★★☆

Show that the equation $\sin x - x = -1$ has at least one solution on the interval $[0, \pi]$.

Consider the function $f(x) = \sin x - x$ defined on the closed interval $[0, \pi]$ :

- $f$ is continuous on the closed interval $[0, \pi]$ as it is the difference between the two continuous functions $\sin x$ and $x$.
- $f(0) = \sin 0 - 0 = 0$

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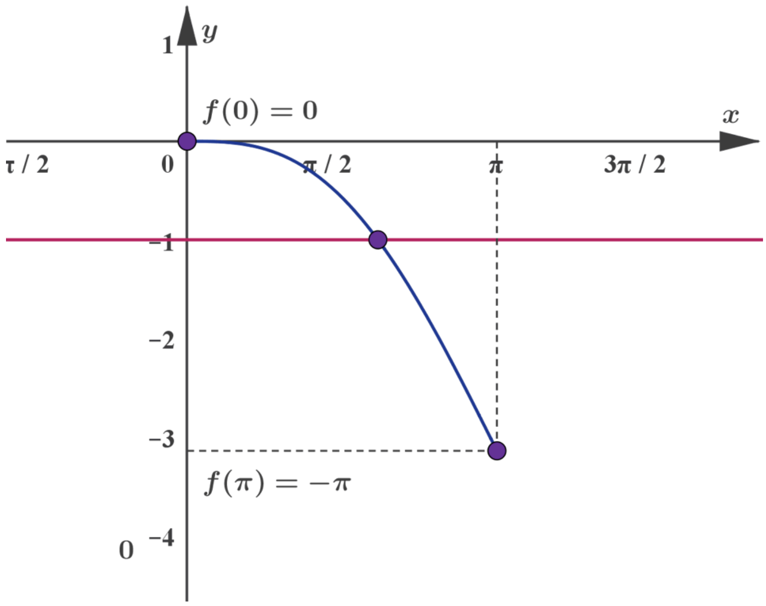
- $f(\pi) = \sin \pi - \pi = -\pi$

Thus, the two conditions of the intermediate value theorem are met and as $f(\pi) < -1 < f(0)$, by applying this theorem, you know that there is at least one $x_0 \in [0, \pi]$ such that $f(x_0) = -1$ or $\sin x_0 - x_0 = -1$. Therefore, the equation above has at least one solution on the interval $(0, \pi)$. This can be validated if you also graph the function $f(x) = \sin x - x$ on a GDC, as shown in graph below, which has one solution to $f(x) = -1$.




Also notice (although not asked for) that as the function $f(x) = \sin x - x$ is decreasing on $[0, \pi]$, the equation $\sin x - x$ has exactly one solution.

# Rolle's theorem

In 1691, the French mathematician Michel Rolle expanded on the intermediate value theorem by applying a similar idea to gradients of a function. Ironically, Rolle completed his proof without using calculus as he did not believe it was a worthwhile subject to learn in mathematics.

### Important

If the function $f$ is

- continuous on the closed interval $[a, b]$
- differentiable on the open interval $(a, b)$ and
- $f(a) = f(b)$

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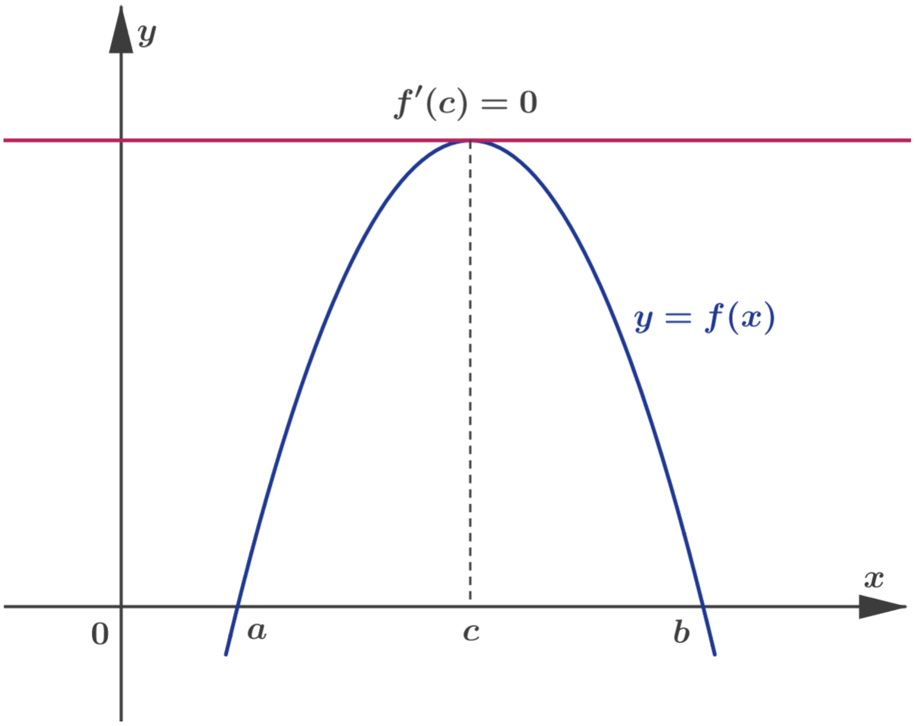
then there is a value $c$, $a < c < b$, such that $f'(c) = 0$.

In simpler terms, this means that a differentiable function  that starts and stops at the same $y$-value must somewhere along the interval have a gradient of $0$. Below is an example graph showing this theorem. A differentiable function defined on a closed interval $[a, b]$. There is a $c$, $a < c < b$, such that $f'(c) = 0$.




More information

This is a graph illustrating Rolle's Theorem. The X-axis is labeled with points at 0, a, c, and b, and the Y-axis represents the function values. The main curve on the graph is a parabola, which begins at point a, rises to a peak at point c, and descends back to point b, demonstrating a differentiable function defined over the closed interval [a, b]. At point c, the derivative of the function, f'(c), is equal to 0, indicating a horizontal tangent at this peak. There is a horizontal line at the top of the graph that signifies f'(c) = 0. This setup visually represents the conclusion of Rolle's Theorem, as somewhere along the interval, the gradient of the curve is zero.

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In the applet below, you can move the blue point on the graph. The tangent line to the curve is shown, and when $f'(c) = 0$, the colour of the tangent changes to indicate that you have found a value of $c$, $a < c < b$, illustrating the conclusion of Rolle's theorem.

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view

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**Interactive 1.** Visualizing Rolle's Theorem: Finding Critical Points

More information for interactive 1

This interactive provides a dynamic visualization of Rolle's Theorem, emphasizing not only the conditions under which the theorem holds but also the interesting possibility of multiple critical points within a given interval. The graph displays a continuous, differentiable function (shown as a blue curve) that starts and ends at the same horizontal level—points A and B —where the function values are equal, satisfying the condition $f(a) = f(b)$.
Users can interact with the graph by dragging a red point along the curve between these endpoints. At specific positions, the tangent line at the red point turns pink, visually indicating where the derivative of the function is zero—these are the critical points. This tool reveals that while Rolle's Theorem guarantees at least one such point, there can be two or even three locations where the slope is zero, showcasing more complex behavior.
The interactive experience helps learners build intuition about differentiable functions and how their slopes behave between points of equal height. By experimenting with different curve shapes that maintain ff$(a) = f(b)$, users can explore how the quantity and positions of critical points vary, thereby deepening their understanding of the geometric and analytical implications of Rolle's Theorem.

Even though Rolle did not support calculus, his theorem is quite valuable when discussing concepts such as extreme values, or maximum and minimum values.

## Example 2

★★★

Consider the equation $f(x) = x^2 - kx$ over the closed interval $[0, 2]$, where $k$ is a positive integer. What would be the value for $k$ that would allow use of Rolle's theorem? At what $x$-value would $f(x)$ have a horizontal tangent?

Student view

For Rolle's theorem to apply:

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- $f$ must be continuous on the closed interval $[0, 2]$. *It is* .
- $f$ must be differentiable on the open interval $(0, 2)$ . *It is* .
- $f(a) = f(b)$

For this last requirement to hold:

$$\begin{aligned} f(0) &= f(2) \\ 0^2 - k(0) &= 2^2 - k(2) \\ 0 &= 4 - 2k \\ k &= 2 \end{aligned}$$

Therefore, the function is $f(x) = x^2 - 2x$

To find the horizontal tangent, set

$$\begin{aligned} f'(x) &= 2x - 2 = 0 \\ x &= 1 \end{aligned}$$

# Mean value theorem

Rolle's theorem restricted the discussion of intervals to those beginning and ending at the same function value, or $f(a) = f(b)$, proving that at least one point in between was equal to $0$. In 1823, Augustin Cauchy, another French mathematician, expanded this to the more general mean value theorem.

✓ **Important**

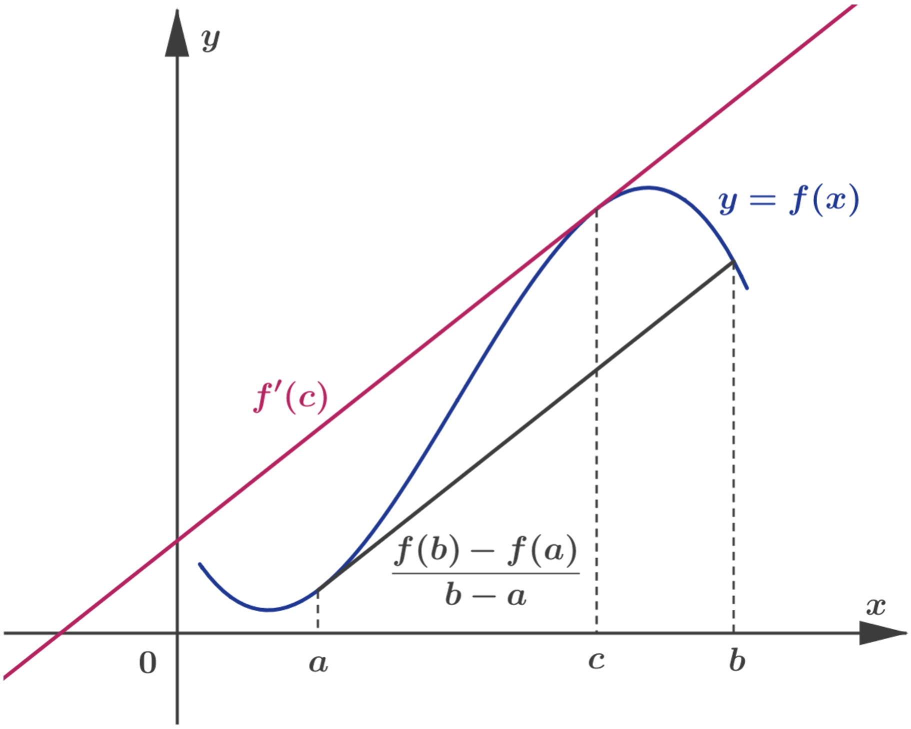
If the function $f$ is

- continuous on the closed interval $[a, b]$ and
- differentiable on the open interval $(a, b)$,

then there is a $c$, $a < c < b$, such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

In simpler terms, this means that somewhere along the segment of a differentiable function, there must be a point with a gradient equal to the gradient between the endpoints. Below is an example graph showing this theorem. A differentiable function defined on a closed interval $[a, b]$. There is a $c$, $a < c < b$, such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Student view




More information

The image is a graph illustrating the Mean Value Theorem. It features a coordinate system with labeled axes, where the x-axis is horizontal and the y-axis is vertical.

The graph shows: - A blue curve representing a function ( y = f(x) ) defined on a closed interval ([a,b]). - Point ( a ) is at the start of the interval and point ( b ) at the end. - A tangent line at point ( c ), ( a < c < b ), is shown in pink, indicating where ( f'(c) = \frac{f(b) - f(a)}{b - a} ). - A slope line from point ( a ) to ( b ) illustrating the average rate of change, labeled ( \frac{f(b) - f(a)}{b - a} ).

This visual describes the concept that there is at least one point ( c ) on the curve where the tangent line is parallel to the secant line connecting the endpoints of the interval, supporting the Mean Value Theorem in calculus.

[Generated by AI]

In the applet below, you can move the red point on the graph of the function, and once you find a point where the tangent line is parallel to the line segment connecting the endpoints of the graph, the colour of the tangent changes, indicating that you found a $c$, $a < c < b$, illustrating the conclusion of the mean value theorem.

Student
view

**Interactive 1.** Mean Value Theorem: Tangent Meets Secant

More information for interactive 1

This interactive helps users understand the Mean Value Theorem (MVT) by letting them explore a graph of a smooth curve between two points labeled “a” and “b” on the x-axis. At point “a,” the curve starts at a higher position on the y-axis, and at point “b,” it ends lower. The graph shows a curved blue line representing a continuous and differentiable function, with point “a” marked as $(a, f(a))$ and point “b” as $(b, f(b))$. A straight pink line connects these two endpoints, showing the average rate of change between them—this is called the secant line.

There is also a red movable dot on the curve. As the user drags this point along the curve between a and b, a straight orange tangent line appears, touching the curve at the red point. The goal is to find a position where the tangent line has the same slope as the secant line. When this happens, the orange tangent line becomes parallel to the pink secant line, demonstrating the Mean Value Theorem in action. The slope matching happens at a specific point “c” between a and b. The app highlights this visually, but the concept can be understood through the slope values being equal at that point.

This interaction shows that for any smooth curve between two endpoints, there is at least one point where the function’s instantaneous rate of change matches the average rate of change over the interval. It reinforces the idea that such a point must exist, even though it may not always be easy to locate by hand. By allowing users to move the point and see or hear when the slopes match, the interactive provides a strong understanding of the theorem in a hands-on, accessible way.

In the case of the applet, all three examples where the tangent is parallel to the line segment connecting the end points are very near the three local extreme points. What is the driving factor for these tangent points being so close? What situation would result in the parallel tangents occurring at the local extreme points?

## Example 3

★★★

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A rock is thrown upward with a parameterised height function of $f(t) = -5t^2 + 16t$, where $f(t)$ is in metres and $t$ is in seconds, and lands on top of a building 3 metres high (after reaching the its highest point). Use the mean value theorem to find the time, $t$, when the vertical component of the velocity is equal to the average velocity over the time of flight.

For the mean value theorem to apply:

- $f$ must be continuous on the closed interval. *It is.*
- $f$ must be differentiable on the open interval. *It is.*

First, find the time of flight:

$f(t)$ is the height at time $t$ seconds so substitute $f(t) = 3$ and solve the quadratic equation

$$\begin{aligned} f(t) &= -5t^2 + 16t = 3 \\ 5t^2 - 16t + 3 &= 0 \\ (5t-1)(t-3) &= 0 \\ t &= \frac{1}{5}, 3 \end{aligned}$$

The rock passes the 3 metre mark at $t = \frac{1}{5}$ on the way up, passes the highest point on its path, and reaches the 3 metre mark again at $t = 3$. The time of flight is 3 seconds.

From the mean value theorem, there is a time where $f'(c) = \frac{f(b) - f(a)}{b - a}$. In this case:

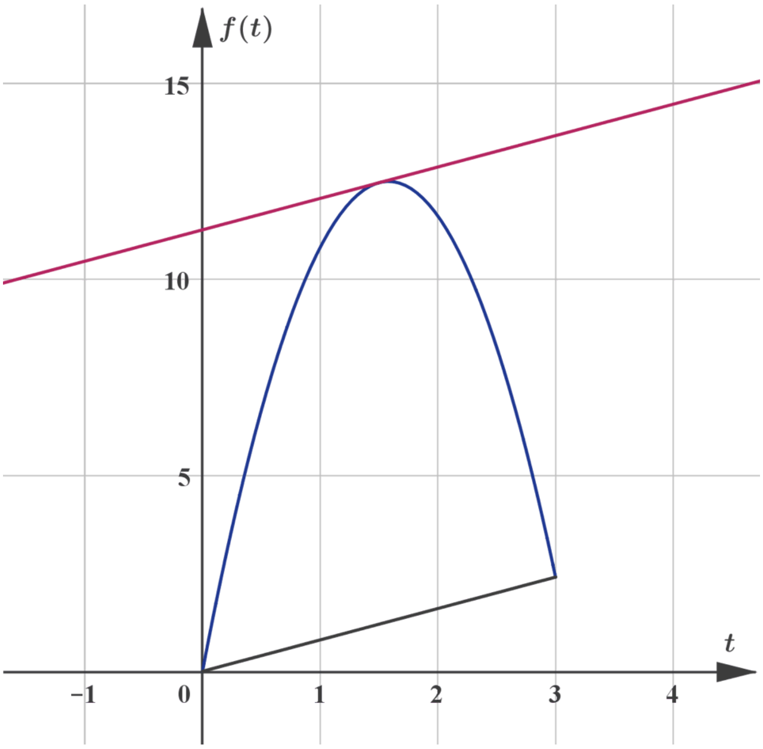
$$f'(c) = \frac{f(3) - f(0)}{3 - 0} = \frac{3 - 0}{3 - 0} = 1.$$

Therefore, you need to find the time when $f'(t) = 1$.

$$\begin{aligned} f'(t) &= -10t + 16 = 1 \\ 10t &= 15 \\ t &= \frac{3}{2} \text{ seconds} \end{aligned}$$

The graph of height against time can be seen below in blue, with the average rate of change shown in grey and the instantaneous rate of change at $t = 1.5$ shown in pink.

Student view




## 6 section questions

**Question 1**

Difficulty: ★★☆

Consider the function $f(x) = \cos x - x + 2$ defined on the interval $\left[-\frac{k\pi}{2}, \frac{k\pi}{2}\right]$, where $k$ is a positive integer. Find the minimum value of $k$ for which the intermediate value theorem cannot be applied for the function $f$ and the value $0$ as an intermediate value.

Give your answer as an integer and not in an equation (e.g. $3$ and not $3.00$ or $k = 3$).

1

### Accepted answers

1

### Explanation

The function $f(x) = \cos x - x + 2$ is a continuous function on the closed interval $\left[-\frac{k\pi}{2}, \frac{k\pi}{2}\right]$ for any positive integer $k$. The intermediate value theorem will not be applied with $0$ as the intermediate value if there is a positive integer $k$ such that $f\left(-\frac{k\pi}{2}\right)$ and $f\left(\frac{k\pi}{2}\right)$ are both positive or both negative.

If we take $k = 1$, we have $f\left(-\frac{\pi}{2}\right) = 2 + \frac{\pi}{2} > 0$ and $f\left(\frac{\pi}{2}\right) = 2 - \frac{\pi}{2} > 0$ and thus, the intermediate value theorem cannot be applied for the value $0$.

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Since $1$ is the smallest positive integer, $k = 1$ is the answer to the question.

Note, that it can also be shown that $k = 1$ is the only value when the intermediate value theorem can not be applied with $0$ as the intermediate value. This was not asked in the question, but the following is the proof:

$$f\left(-\frac{k\pi}{2}\right) = \cos\left(-\frac{k\pi}{2}\right) + \frac{k\pi}{2} + 2 \geqslant -1 + \frac{k\pi}{2} + 2 = 1 + \frac{k\pi}{2} > 0 \text{ as } k \text{ is a positive integer}$$

and

$$f\left(\frac{k\pi}{2}\right) = \cos\left(\frac{k\pi}{2}\right) - \frac{k\pi}{2} + 2 \leqslant 1 - \frac{k\pi}{2} + 2 = 3 - \frac{k\pi}{2} < 0 \text{ if } k \geqslant 2 \text{ or } > 0 \text{ if } k = 1.$$

So for $k \geqslant 2$, $f\left(-\frac{k\pi}{2}\right)$ and $f\left(\frac{k\pi}{2}\right)$ have different signs, so the intermediate value theorem can be applied with $0$ as the intermediate value.

**Question 2**

Difficulty: ★★★

Consider the function $f(x) = -(x-1)^3$. Find the minimum positive integer $a$ for which the intermediate value theorem implies that the function $f$ has a root in the interval $]-a, a[$.

Give your answer as an integer and not in an equation (for example, $1$ and not $1.00$ or $a = 1$).

2 ✓

### Accepted answers

2

### Explanation

This function $f$ is clearly continuous. We need to find a closed interval of the form $[-a, a]$ with $a$ being the minimum positive integer for which $f(-a) \times f(a) < 0$. In this case $0$ is between $f(-a)$ and $f(a)$ so the intermediate value theorem guarantees a root in $]-a, a[$.

As $f(1) = 0$, for $a = 1$ we get $f(-a) \times f(a) = 0$, so the intermediate value theorem does not guarantee a root in the open interval $]-1, 1[$.

Thus, we select the positive integer closest to $1$, i.e. we select $2$. We have $f(2) = -(2-1)^3 = -1 < 0$ and $f(-2) = -(-2-1)^3 = 27 > 0$. Thus, $a = 2$ and the interval is $]-2, 2[$.

**Question 3**

Difficulty: ★☆☆

The function defined by $f(x) = x(x-1)(x+1)(x-3)$ is continuous and differentiable everywhere, and $f(0) = f(1) = 0$. Therefore, Rolle's theorem applies on the interval $[0, 1]$.

Student view

Find the value of $0 < c < 1$ that satisfies the conclusion of Rolle's theorem. Give your answer as a decimal, rounded to three significant figures. Give the numerical value only.

0.531

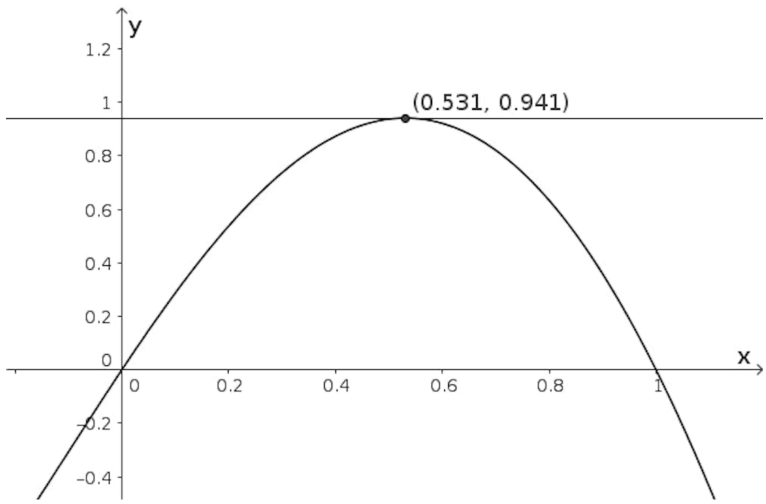
**Accepted answers**

0.531, 0,531

**Explanation**

We need to find $c$, where $f'(c) = 0$.

We can use a GDC to find the maximum or minimum of $f$ on $[0, 1]$.




More information

The answer is the $x$-coordinate of the maximum point.

**Question 4**

Difficulty:

★☆☆

The function defined by $f(x) = x(x-1)(x+1)(x-3)$ is continuous and differentiable everywhere, and $f(0) = f(3) = 0$. Therefore, Rolle's theorem applies on the interval, $[0, 3]$.

How many $c$ values satisfy the conclusion of Rolle's theorem on the interval $[0, 3]$?

2

**Accepted answers**

2, two

**Explanation**

This is easily done using a calculator by drawing the graph and counting the turning points in the interval $[0, 3]$.

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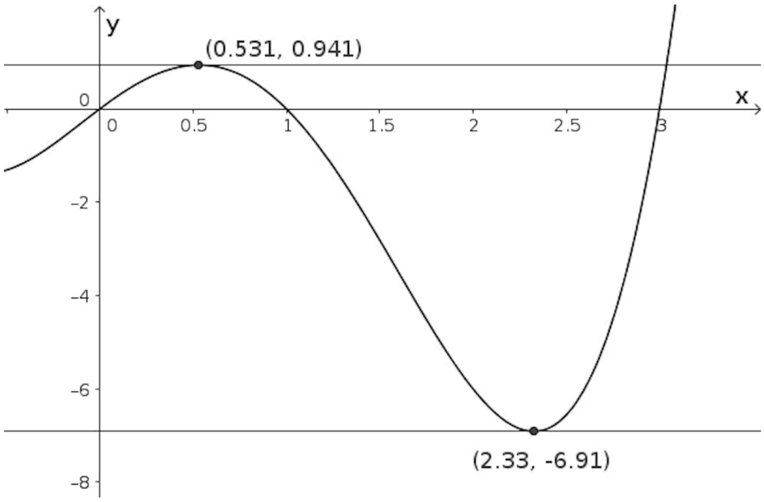



More information

There are two turning points, so there are two $c$ values between $0$ and $3$ where $f'(c) = 0$.

For the interested students, here is a way to argue without the help of a calculator.

Since $f(-1) = f(0) = f(1) = f(3) = 0$, according to Rolle's theorem, there must be $-1 < c_1 < 0$, $0 < c_2 < 1$ and $1 < c_3 < 3$ with $f'(c_1) = f'(c_2) = f'(c_3) = 0$.

Since $f'$ is a cubic polynomial function, it does not have more than three roots, so we have found all the $c$ values where $f'(c) = 0$. Since $c_2$ and $c_3$ are between $0$ and $3$ but $c_1$ is not, there are exactly two $c$ values satisfying the conlusion of Rolle's theorem in$[0, 3]$.

**Question 5**

Difficulty:

★☆☆

The function defined by $f(x) = x^4 - 3x^3 - x^2 + 6x$ is continuous and differentiable everywhere, so the mean value theorem applies to this function on any interval.

There is a unique $1 < c < 3$ that satisfies the conclusion of the mean value theorem on the interval $[1, 3]$. Find the value of $c$. Give your answer rounded to three significant figures. Give the numerical value only.

2.33 ✓

### Accepted answers

2.33, 2,33

### Explanation

We need to find $1 < c < 3$, such that $f'(c) = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{9 - 3}{2} = 3$

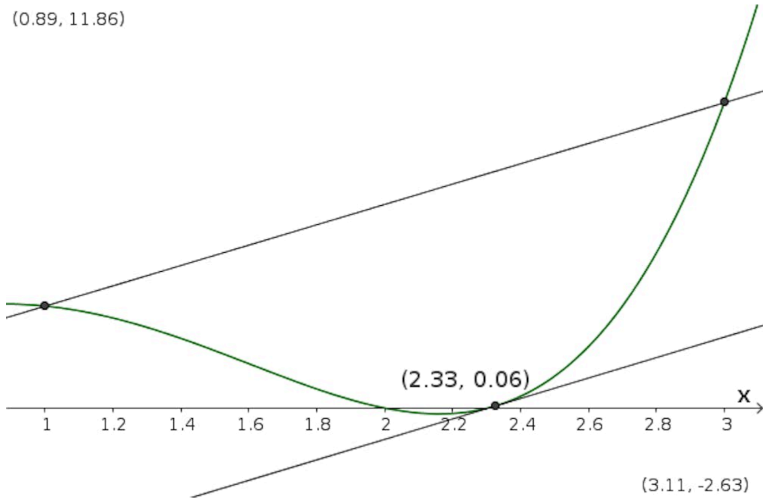
$f'(x) = 4x^3 - 9x^2 - 2x + 6$, so we need to find the solution of $4x^3 - 9x^2 - 2x + 6 = 3$.

Using our GDC, we obtain three solutions: $2.33$, $0.531$ and $-0.607$

Only $2.33$ lies between $1$ and $3$, so this is the solution we were looking for.

Student view

The diagram below illustrates the result.




More information

**Question 6**

The function defined by $f(x) = x^4 - 3x^3 - x^2 + 6x$ is continuous and differentiable everywhere, so the mean value theorem applies to this function on any interval.

Find the maximum number of $c$ values that satisfy the conclusion of the mean value theorem on any fixed interval. Give the numerical value only.

3

## Accepted answers

3, three

## Explanation

On any fixed interval $[a, b]$, $\dfrac{f(b) - f(a)}{b - a} = K$ is a fixed number.

The question asks for the maximum number of solutions for $f'(x) = K$, or after rearrangement, $f'(x) - K = 0$.

Since $f'(x) - K$ is a cubic polynomial, it cannot have more than three roots. So the maximum number of solutions is certainly not more than three.

The diagram below illustrates an example corresponding to the interval $[-1.6, 3]$ with three possible $c$ values, so the maximum number of possibilities is exactly three.

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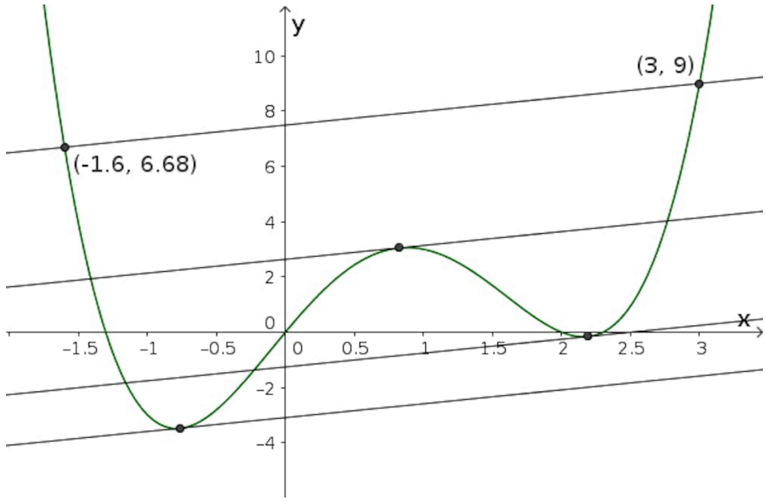


More information

5. Calculus / 5.19 Maclaurin series expansions

# Taylor polynomials

There are times when you may want to approximate complex functions with simpler functions such as polynomials. Now that you have studied quite a bit of calculus, you may have realised that polynomials are some of the easiest functions to work with, as the derivatives and integrals are very straightforward.

## Taylor polynomials

Brooke Taylor, an English mathematician, formalised one such approximation technique appropriately called a Taylor polynomial.

### Important

A Taylor polynomial, $P_k(x)$, is a polynomial of degree $k$ used for approximating $f(x)$ with the approximation centred at $a$.

$$P_k(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^k}{k!}f^{(k)}(a)$$
$$= \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notice the centre point of $a$. The value of $a$ is somewhat arbitrary. When dealing with a Taylor polynomial, choosing $a$ near your area of interest is important. As the Taylor polynomial gets farther away from the ‘centre’ point, the approximation loses its accuracy.

Student view

## Exam tip

On exams you will only meet approximating polynomials centred at $a = 0$.

Recall that, in subtopic 5.12 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26489/), you learned that higher-order derivatives are denoted by $f'(x), f''(x), f'''(x), f^{(4)}, f^{(5)}, \cdots, f^{(n)}$.

## Be aware

A Taylor polynomial of degree $k$ has $k + 1$ terms.

The first term, $f(a)$, has no derivative. It is simply the function evaluated at $a$. Every term after that has a derivative of order 1 to $k$.

## Example 1

★★☆

Approximate $f(x) = \ln x$ with a sixth-order Taylor polynomial centred at $x = 1$. Use your polynomial to find an estimate of the value of $\ln 0.4$.

To use the formula in the definition, you need $f(1)$ and the value of the first five derivatives of $f(x)$ at $x = 1$.

| | | |
|---|---|---|
| $f(x) = \ln x$ | $f(1) = \ln(1) = 0$ | $0$ |
| $f'(x) = x^{-1}$ | $f'(1) = \frac{1}{1} = 1$ | $(x-1)$ |
| $f''(x) = -x^{-2}$ | $f''(1) = -\frac{1}{1^2} = -1$ | $\frac{(x-1)^2}{2!}(-1) = -\frac{(x-1)^2}{2}$ |
| $f'''(x) = 2x^{-3}$ | $f'''(1) = \frac{2}{1^3} = 2$ | $\frac{(x-1)^3}{3!}(2) = \frac{(x-1)^3}{3}$ |
| $f^{(4)}(x) = -6x^{-4}$ | $f^{(4)}(1) = \frac{-6}{1^4} = -6$ | $\frac{(x-1)^4}{4!}(-6) = -\frac{(x-1)^4}{4}$ |
| $f^{(5)}(x) = 24x^{-5}$ | $f^{(5)}(1) = \frac{24}{1^5} = 24$ | $\frac{(x-1)^5}{5!}(24) = \frac{(x-1)^5}{5}$ |

Student view

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| | | |
|---|---|---|
| $f(x) = \ln x$ | $f(1) = \ln(1) = 0$ | $0$ |
| $f^{(6)}(x) = -120x^{-6}$ | $f^{(6)}(1) = \frac{-120}{1^6} = -120$ | $\frac{(x-1)^6}{6!}(-120) = -\frac{(x-1)^6}{6}$ |
| $P_6(x) = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6}$ | | |
| $P_6(0.4) = 0 + (-0.6) - \frac{(-0.6)^2}{2} + \frac{(-0.6)^3}{3} - \frac{(-0.6)^4}{4} + \frac{(-0.6)^5}{5} - \frac{(-0.6)^6}{6}$ $= -0.90773 \approx -0.908$ | | |

Notice the exact value is $-0.916\ldots$. This gives an error of about $0.93\%$.

## Example 2

★★☆

Approximate $f(x) = \sin x$ with a fifth-order Taylor polynomial centred at $x = 0$. Find an estimate for $\sin 1$.

To use the formula in the definition, you need $f(0)$ and the value of the first five derivatives of $f(x)$ at $x = 0$.

| | | |
|---|---|---|
| $f(x) = \sin x$ | $f(0) = \sin 0 = 0$ | $0$ |
| $f'(x) = \cos x$ | $f'(0) = \cos 0 = 1$ | $(x-0)(1) = x$ |
| $f''(x) = -\sin x$ | $f''(0) = -\sin 0 = 0$ | $\frac{(x-0)^2}{2!}(0) = 0$ |
| $f'''(x) = -\cos x$ | $f'''(0) = -\cos 0 = -1$ | $\frac{(x-0)^3}{3!}(-1) = -\frac{x^3}{6}$ |
| $f^{(4)}(x) = \sin x$ | $f^{(4)}(0) = \sin 0 = 0$ | $\frac{(x-0)^4}{4!}(0) = 0$ |
| $f^{(5)}(x) = \cos x$ | $f^{(5)}(0) = \cos 0 = 1$ | $\frac{(x-0)^5}{5!}(1) = \frac{x^5}{120}$ |
| $P_5(x) = 0 + x + 0 - \frac{x^3}{6} + 0 + \frac{x^5}{120} = x - \frac{x^3}{6} + \frac{x^5}{120}$ | | |
| $P_5(1) = 1 - \frac{1^3}{6} + \frac{1^5}{120} = \frac{101}{120} \approx 0.84167$ | | |

Student view

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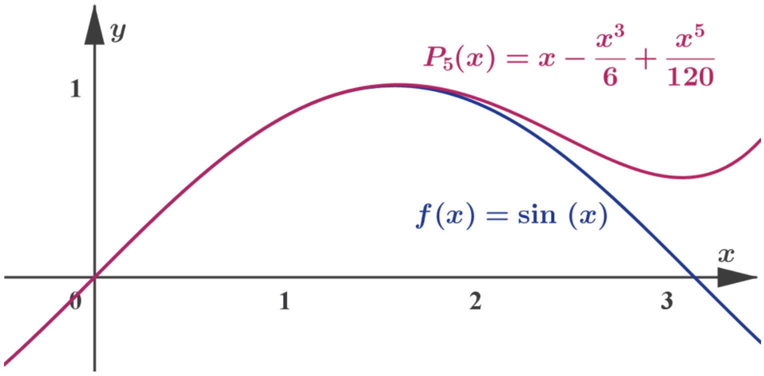


The diagram above illustrates that the assumption that $P_5(1)$ is close to sin 1. At this resolution, the Taylor polynomial of degree $5$ is indistinguishable from the original function until around $x = 2$.

Using the mean value theorem, you can further show that

$$\begin{cases} x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120} & x > 0 \\ x - \frac{x^3}{6} + \frac{x^5}{120} < \sin x < x - \frac{x^3}{6} & x < 0 \end{cases}$$

For $x = 1$, you get

$$\frac{100}{120} < \sin 1 < \frac{101}{120}$$

In fact, the exact value is $0.84147\ldots$. This gives an error of about $0.97\%$ with the third-order approximation of $\frac{100}{120}$ and about $0.023\%$ with the fifth-order approximation of $\frac{101}{120}$.

## Example 3

★★☆

Find a quadratic Taylor polynomial centred at $x = 1$ corresponding to the function defined by $f(x) = e^x \ln x$.

To use the formula in the definition, you need $f(1)$ and the value of the first two derivatives of $f(x)$ at $x = 1$.

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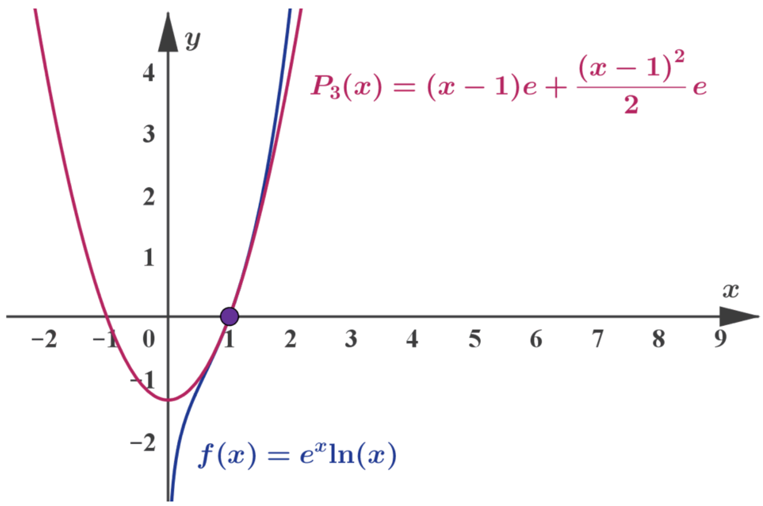
| | | |
|---|---|---|
| $f(x) = \mathrm{e}^x \ln x$ | $f(1) = \mathrm{e}^1 \ln 1 = 0$ | $0$ |
| $f'(x) = \mathrm{e}^x \ln x + \frac{\mathrm{e}^x}{x}$ | $f'(1) = \mathrm{e}^1 \ln 1 + \frac{\mathrm{e}^1}{1} = \mathrm{e}$ | $(x-1)\mathrm{e}$ |
| $f''(x) = \mathrm{e}^x \ln x + \frac{\mathrm{e}^x}{x} + \frac{x\mathrm{e}^x - \mathrm{e}^x}{x^2}$ | $f''(1) = \mathrm{e}^1 \ln 1 + \frac{\mathrm{e}^1}{1} + \frac{1\mathrm{e}^1 - \mathrm{e}^1}{1^2} = \mathrm{e}$ | $\frac{(x-1)^2}{2}\mathrm{e}$ |
| $P_2(x) = (x-1)\mathrm{e} + \frac{(x-1)^2}{2}\mathrm{e}$ | | |




The diagram above illustrates how well this quadratic Taylor polynomial approximates the original function.

## 3 section questions

**Question 1**

Difficulty: ★★☆

Let the Taylor polynomial of degree 7 for $\sin(x)$, centred at $x = 0$, be $P_7(x)$. For this polynomial, $P_7(1) = \frac{k}{5040}$.

Find the exact value of $k$. Give the numerical value only.

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Student view

**Accepted answers**

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4241

### Explanation

We use the definition of the Taylor polynomial:

$$P_7(x) = f(0) + f'(0)x + \cdots + f^{(7)}(0)\frac{x^7}{7!}$$

The first seven derivatives of $\sin x$ are $\cos x$, $-\sin x$, $-\cos x$, $\sin x$, $\cos x$, $-\sin x$ and $-\cos x$.

We need the value of $f(x) = \sin x$ and the value of these derivatives at $x = 0$: $\sin 0 = 0$, $\cos 0 = 1$, $-\sin 0 = 0$, $-\cos 0 = -1$, $\sin 0 = 0$, $\cos 0 = 1$, $-\sin 0 = 0$ and $-\cos 0 = -1$.

Hence,

$$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

thus,

$$P_7(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} = \frac{4241}{5040}$$

**Question 2**

Difficulty: ★★☆

Let the Taylor polynomial of degree 6 for $\cos(x)$, centred at $x = 0$, be $P_6(x)$. For this polynomial, $P_6(1) = \dfrac{k}{720}$.

Find the exact value of $k$. Give the numerical value only.

389 ✓

### Accepted answers

389

### Explanation

We use the definition of the Taylor polynomial:

$$P_6(x) = f(0) + f'(0)x + \cdots + f^{(6)}(0)\frac{x^6}{6!}$$

The first six derivatives of $\cos x$ are $-\sin x$, $-\cos x$, $\sin x$, $\cos x$, $-\sin x$ and $-\cos x$.

We need the value of $f(x) = \cos x$ and the value of these derivatives at $x = 0$: $\cos 0 = 1$, $-\sin 0 = 0$, $-\cos 0 = -1$, $\sin 0 = 0$, $\cos 0 = 1$, $-\sin 0 = 0$ and $-\cos 0 = -1$.

Hence,

$$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

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thus,

$$P_6(1) = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} = \frac{389}{720}$$

**Question 3**

Difficulty: ★★☆

Let the Taylor polynomial of degree $5$ for $\mathrm{e}^x$ centred at $x = 0$ be $P_5(x)$. For this polynomial, $P_5(1) = \frac{k}{60}$.

Find the exact value of $k$. Give the numerical value only.

163

**Accepted answers**

163, k=163, k = 163

**Explanation**

We use the definition of the Taylor polynomial:

$$P_5(x) = f(0) + f'(0)x + \cdots + f^{(5)}(0)\frac{x^5}{5!}$$

All derivatives of $\mathrm{e}^x$ are $\mathrm{e}^x$.

We need the value of $f(x) = \mathrm{e}^x$ and the value of these derivatives at $x = 0$: $\mathrm{e}^0 = 1$.

Hence,

$$P_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

thus,

$$\begin{aligned} P_5(1) &= 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} \\ &= \frac{163}{60} \end{aligned}$$

5. Calculus / 5.19 Maclaurin series expansions

# Error term

Student view

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In the previous section (/study/app/math-aa-hl/sid-134-cid-761926/book/taylor-polynomials-id-27283/), you learned about Taylor polynomials. For example, you saw that the fifth-order Taylor polynomial of $f(x) = \sin x$ centred at $x = 0$ is $P_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$. Furthermore, this polynomial can be used to approximate values of $f(x) = \sin x$, such as $\sin(1) \approx P_5(1) = 0.84167$. As the value from a calculator or table is $\sin(1) \approx 0.84147\ldots$, this approximation is pretty close. How would you know the accuracy if you did not have access to a calculator or table?

In general, Taylor polynomials are useful because they can be used for approximations. At the same time, an approximation is only really useful if you can estimate the error you would get from using the approximation instead of the actual value.

### Exam tip

Error analysis of approximating functions with polynomials is not part of the syllabus. However, such an analysis can be helpful to justify claims that are in the syllabus and discussed in later sections. These justifications and the content of this section can be skipped for exam preparation, but they are here for you to help deepen your understanding.

There are different ways of expressing the error term, this section will focus on the Lagrange form.

### Important

If $f$ is a function that is differentiable $n+1$ times around $a$ in the domain and the $n$th derivative is continuous on the closed interval between $a$ and $x$, then

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x)$$

where the error term can be expressed in the form:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some $c$ between $x$ and $a$.

As an application, reconsider the example involving the Taylor polynomial approximation of $\sin x$.

# Example 1

★★☆

Use the Lagrange error term to estimate the error when $\sin 1$ is approximated using a fifth-order Taylor polynomial centred at $x = 0$.

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Take the solution from **Example 2** in the previous section ( 5.19.2) and incorporate the error term,

$$\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120} + R_5(x)$$

The Lagrange form of the error term is $R_5(x) = \frac{f^{(6)}(c)}{6!}(x)^6$ for some $c$ between $0$ and $x$.

Since the approximation is taking place at $x = 1$ , the Taylor polynomial reduces to

$$\sin 1 \approx \frac{101}{120} + R_5(1)$$

The sixth derivative of $\sin x$ is $-\sin x$ , the polynomial further reduces to
$\sin 1 \approx \frac{101}{120} + \frac{(-\sin c)}{720}1^6 = \frac{101}{120} - \frac{\sin c}{720}$ for some $0 < c < 1$ .

As you consider all possible value of $c$ for $\sin c$ , you find that $|\sin c| \leqslant 1$ , so the error cannot be more than $\frac{1}{720}$ .

## 3 section questions

**Question 1**

Difficulty: ★★☆

According to the formula involving the error term, $\sin x = P_7(x) + R_7(x)$, where $P_7(x)$ is the Taylor polynomial of degree $7$ centred at $x = 0$. So, $\sin 1 = P_7(1) + R_7(1)$.

Using the Lagrange form of the error term $|R_7(1)| < \frac{1}{k}$, find the largest possible value of $k$.

There are different ways of estimating how close $P_7(1)$ is to $\sin x$. You can probably find better estimates than the one suggested above. The goal of this question is to ensure you know how to use the Lagrange form of the error term for estimations. The system will accept only the answer that follows from the process described in this section.

40320

**Accepted answers**

40320, k=40320, k = 40320

**Explanation**

The Lagrange form of the error term is $R_7(x) = \frac{f^{(8)}(c)}{8!}x^8$, for some $c$ between $0$ and $x$.

Since the eighth derivative of $\sin x$ is $\sin x$, using $x = 1$, we get that

Student view

$$R_7(1) = \frac{\sin c}{8!}1^8 = \frac{\sin c}{40320} \text{ for some } 0 < c < 1.$$

Since $|\sin c| \leqslant 1$, the error is certainly not more than $\frac{1}{40320}$.



**Question 2**

Difficulty: ★★☆

According to the formula involving the error term, $\cos x = P_6(x) + R_6(x)$, where $P_6(x)$ is the Taylor polynomial of degree $6$ centred at $x = 0$. So, $\cos 1 = P_6(1) + R_6(1)$.

Using the Lagrange form of the error term $|R_6(1)| < \frac{1}{k}$, find the largest possible value of $k$.

Note, that there are different ways of estimating how close $P_6(1)$ is to $\cos x$. You can probably find better estimates than the one suggested above. The goal of this question is to ensure you know how to use the Lagrange form of the error term for estimations. The system will accept only the answer that follows from the process described in this section.

5040 ✓

**Accepted answers**

5040, k=5040, k = 5040

**Explanation**

The Lagrange form of the error term is $R_6(x) = \frac{f^{(7)}(c)}{7!}x^7$, for some $c$ between $0$ and $x$.

Since the seventh derivative of $\cos x$ is $\sin x$, using $x = 1$, we get that

$$R_6(1) = \frac{\sin c}{7!}1^7 = \frac{\sin c}{5040}, \text{ for some } 0 < c < 1.$$

Since $|\sin c| \leqslant 1$, the error is certainly not more than $\frac{1}{5040}$.

**Question 3**

Difficulty: ★★☆

According to the formula involving the error term, $\mathrm{e}^x = P_5(x) + R_5(x)$, where $P_5(x)$ is the Taylor polynomial of degree $5$, centred at $x = 0$. So, $\mathrm{e}^1 = P_5(1) + R_5(1)$.

Using the Lagrange form of the error term $|R_5(1)| < \frac{\mathrm{e}}{k}$, find the largest possible value of $k$.

Note, that there are different ways of estimating how close $P_5(1)$ is to $\mathrm{e}^x$. You can probably find better estimates than the one suggested above. The goal of this question is to ensure you know how to use the Lagrange form of the error term for estimations. The system will accept only the answer that follows from the process described in this section.

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720

#### Accepted answers

720, k=720, k = 720

#### Explanation

The Lagrange form of the error term is $R_5(x) = \frac{f^{(6)}(c)}{6!}x^6$, for some $c$ between $0$ and $x$.

Since the sixth derivative of $\mathrm{e}^x$ is $\mathrm{e}^x$, using $x = 1$, we get that

$$R_5(1) = \frac{\mathrm{e}^c}{6!}1^6 = \frac{\mathrm{e}^c}{720} \text{ for some } 0 < c < 1.$$

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Since for $0 < c < \mathrm{e}$, $|\mathrm{e}^c| < \mathrm{e}$, the error is certainly less than $\frac{\mathrm{e}}{720}$. Hence, the result.

5. Calculus / 5.19 Maclaurin series expansions

# Taylor and Maclaurin series

## Taylor series

In section 5.19.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/taylor-polynomials-id-27283/), you learned that a Taylor polynomial can approximate a function.

$$P_k(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^k}{k!}f^{(k)}(a)$$
$$= \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If you want to make this approximation more accurate, you simply continue to add terms. If you add terms to $n \to \infty$, then you have an infinite series, or more specifically, a Taylor series. Just as the Taylor polynomial can be written using summation notation as $P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$, the Taylor series can be written as $P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

#### Important

If a function is differentiable infinitely many times at $x = a$, the Taylor series centred at $x = a$ is a power series of the form:

Student view

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \,.$$

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## Example 1

★★☆

Find the Taylor series representing $f(x) = \ln x$ centred at $x = 1$.

From **Example 1** in section 5.19.2, you found the first seven terms (sixth order) Taylor polynomial to be:

$$P_6(x) = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6}$$

You can generalise this by recognising that there is a pattern that goes on indefinitely:

$$P(x) = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \ldots$$

or by finding an algebraic term to represent the growing terms:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

When turning a long series into summation notation, a property worth reviewing is how to deal with alternating negative signs. Terms that alternate between positive and negative can be represented by terms of the form $(-1)^n$ or $(-1)^{n+1}$.

In this example, the first term is positive. Therefore, you must use $(-1)^{n+1}$ to start the progression correctly.

# Maclaurin series

The Scottish mathematician Colin Maclaurin focused his effort on the Taylor series at a centre point of $x = 0$. This series, named after him, significantly simplifies the formula.

Student view

✓ **Important**

The Maclaurin series is a Taylor series centred at $x = 0$. It is a power series of the form:

$$f(0) + (x)f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

### Exam tip

On exams you will only meet questions related to Maclaurin series. IB exam questions will not include Taylor series centred at a value other than $0$.

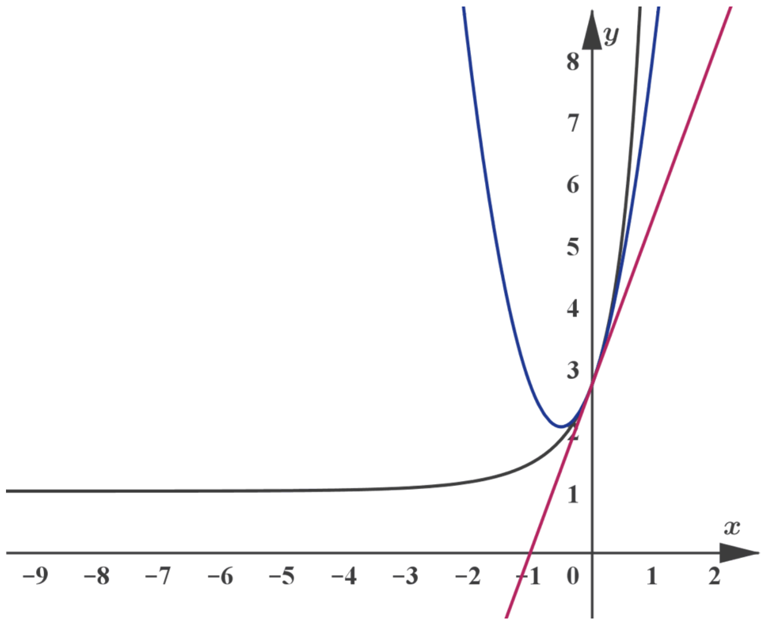
## Example 2

★★☆

Find the first three terms of the Maclaurin series expansion of $f(x) = \mathrm{e}^{\mathrm{e}^x}$.

As this is a Maclaurin series, you need $f(0)$ and the value of the first two derivatives of $f(x)$ at $x = 0$.

| | | |
|---|---|---|
| $f(x) = \mathrm{e}^{\mathrm{e}^x}$ | $f(0) = \mathrm{e}^{\mathrm{e}^0} = \mathrm{e}^1 = \mathrm{e}$ | $\mathrm{e}$ |
| $f'(x) = \mathrm{e}^{\mathrm{e}^x}\mathrm{e}^x$ | $f'(0) = \mathrm{e}^{\mathrm{e}^0}\mathrm{e}^0 = \mathrm{e}^1(1) = \mathrm{e}$ | $\mathrm{e}x$ |
| $f''(x) = \left(\mathrm{e}^{\mathrm{e}^x}\mathrm{e}^x\right)\mathrm{e}^x + \mathrm{e}^{\mathrm{e}^x}\mathrm{e}^x$ | $f''(0) = \left(\mathrm{e}^{\mathrm{e}^0}\mathrm{e}^0\right)\mathrm{e}^0 + \mathrm{e}^{\mathrm{e}^0}\mathrm{e}^0 = 2\mathrm{e}$ | $\frac{2\mathrm{e}}{2!}x^2 = \mathrm{e}x^2$ |
| $\mathrm{e}^{\mathrm{e}^x} = \mathrm{e} + x\mathrm{e} + \mathrm{e}x^2 \ldots$ | | |



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The diagram above illustrates the function $f(x) = e^{e^x}$ (black) and the Taylor approximations $P_1(x) = e + ex$ (pink) and $P_2(x) = e + ex + ex^2$ (blue).

Notice the accuracy in the positive direction versus the accuracy in the negative direction. Since this Taylor approximation was based on of a Maclaurin series, it is anchored at $x = 0$, so all three lines cross at $(0,\ \mathrm{e})$.

## Example 3

★★★

Find the first five terms of the Maclaurin series expansion of $f(x) = \mathrm{e}^x$ to estimate the value of e. How accurate is your estimate? How many terms do you need to use to be sure that your estimated value is within $10^{-3}$ of e?

As this is a Maclaurin series, you need $f(0)$ and the value of the first four derivatives of $f(x)$ at $x = 0$.

| | | |
|---|---|---|
| $f(x) = \mathrm{e}^x$ | $f(0) = \mathrm{e}^0 = 1$ | $1$ |
| $f'(x) = \mathrm{e}^x$ | $f'(0) = \mathrm{e}^0 = 1$ | $x$ |
| $f''(x) = \mathrm{e}^x$ | $f''(0) = \mathrm{e}^0 = 1$ | $\frac{x^2}{2!} = \frac{x^2}{2}$ |
| $f'''(x) = \mathrm{e}^x$ | $f'''(0) = \mathrm{e}^0 = 1$ | $\frac{x^3}{3!} = \frac{x^3}{6}$ |
| $f^{(4)}(x) = \mathrm{e}^x$ | $f^{(4)}(0) = \mathrm{e}^0 = 1$ | $\frac{x^4}{4!} = \frac{x^4}{24}$ |

$$\mathrm{e}^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

To find e , you will evaluate at $x = 1$.

$$a_6 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{65}{24} \approx 2.708$$
$$\mathrm{e} = 2.718$$

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The error is about 0.01.

To find the number of terms needed to get the error less than 0.001, use the error term

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \text{ (Note: } a = 0 \text{ as this is a Maclaurin series).}$$

Applied to this problem,

$$R_n(1) = \frac{f^{(n+1)}(c)}{(n+1)!}$$

Assume that $e$ is less than 3,

$$R_n(1) = \frac{f^{(n+1)}(\mathrm{e}^1)}{(n+1)!} \approx \frac{3^1}{(n+1)!} < 0.001$$

By trial and error:

$$\begin{aligned}
&n = 0 \;\; R_0 = 3\\
&n = 1 \;\; R_1 = 1.5\\
&n = 2 \;\; R_2 = 0.5\\
&n = 3 \;\; R_3 = 0.125\\
&n = 4 \;\; R_4 = 0.025\\
&n = 5 \;\; R_5 = 0.125\\
&n = 6 \;\; R_6 = 0.004\\
&n = 7 \;\; R_7 = 0.0006
\end{aligned}$$

The first seven terms, or a sixth-order approximation, should suffice.

- **Note** that this last part of the question is beyond what you will be asked on exams. You do not need to know the error term, so this part of the question is only here as an interesting extension.

# Maclaurin series through substitution

There are times when substitution can be used to manipulate a known simple expansion to find a more complex expansion. Many of these simple expansions will be developed in the next section (/study/app/math-aa-hl/sid-134-cid-761926/book/expansion-of-special-functions-id-27286/), but they can be demonstrated here.

## Example 4

Student view

★★☆

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Find the first three non-zero terms of the Maclaurin series expansion of $f(x) = \mathrm{e}^{x^2}$ by using derivatives.

As this is a Maclaurin series, you need $f(0)$ and the value of the first few derivatives of $f(x)$ at $x = 0$.

| | | |
|---|---|---|
| $f(x) = \mathrm{e}^{x^2}$ | $f(0) = 1$ | $1$ |
| $f'(x) = 2x\mathrm{e}^{x^2}$ | $f'(0) = 0$ | $0x = 0$ |
| $f''(x) = 2\mathrm{e}^{x^2} + 4x^2\mathrm{e}^{x^2}$ | $f''(0) = 2$ | $2\frac{x^2}{2!} = x^2$ |
| $f'''(x) = 4x\mathrm{e}^{x^2} + 8x\mathrm{e}^{x^2} + 8x^3\mathrm{e}^{x^2} = 12x\mathrm{e}^{x^2} + 8x^3\mathrm{e}^{x^2}$ | $f'''(0) = 0$ | $0\frac{x^3}{3!} = 0$ |
| $f^{(4)}(x) = 12\mathrm{e}^{x^2} + 24x^2\mathrm{e}^{x^2} + 24x^2\mathrm{e}^{x^2} + 16x^4\mathrm{e}^{x^2}$ $= 12\mathrm{e}^{x^2} + 48x^2\mathrm{e}^{x^2} + 16x^4\mathrm{e}^{x^2}$ | $f^{(4)}(0) = 12$ | $12\frac{x^4}{4!} = \frac{x^4}{2}$ |
| $\mathrm{e}^{x^2} = 1 + x^2 + \frac{x^4}{2} + \ldots$ | | |

## Example 5

★★☆

Find the first five non-zero terms of the Maclaurin series expansion of $f(x) = \mathrm{e}^{x^2}$.

From **Example 3** above, you already know $\mathrm{e}^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

Using this expansion, find:

$$\mathrm{e}^{x^2} \approx 1 + \left(x^2\right) + \frac{\left(x^2\right)^2}{2!} + \frac{\left(x^2\right)^3}{3!} + \frac{\left(x^2\right)^4}{4!} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24}$$

This example shows the benefit of using substitution to find the first few terms of a Maclaurin series.

Student
view

3 section questions ^

Overview (/study/ap aa- hl/sid- 134- cid- 761926/o

**Question 1**

Difficulty: ★☆☆

The Maclaurin expansion of $e^{-2x}$ is $a_0 + a_1x + a_2x^2 + \ldots$

Find $a_5$. Give your answer either as a rational number in a fully simplified form (for example, —2/3) or as a decimal number rounded to three significant figures.

-4/15 ✓

**Accepted answers**

-4/15, -0.267, -0,267, - 4/15, - 0.267, - 0,267, —4/15, — 4/15, —0.267, — 0.267

**Explanation**

We use the expansion given in the Formula booklet:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

and substituting

$$e^{-2x} = 1 - 2x + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + \frac{(-2x)^5}{5!} + \ldots$$

hence,

$$a_5 = \frac{(-2)^5}{5!} = \frac{-2^5}{5!} = -\frac{4}{15} \approx -0.267$$

**Question 2**

Difficulty: ★★☆

The Maclaurin expansion of $(e^x)^3$ is $a_0 + a_1x + a_2x^2 + \ldots$

Find $a_7$. Give your answer either as a rational number in a fully simplified form (for example, —2/3) or as a decimal number rounded to three significant figures.

243/560 ✓

**Accepted answers**

243/560, 0.434, 0,434, .434

**Explanation**

We use the expansion given in the Formula booklet:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots$$

and substituting

$$(e^x)^3 = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \frac{(3x)^5}{5!} + \frac{(3x)^6}{6!} + \frac{(3x)^7}{7!} + \ldots$$

hence,

$$a_7 = \frac{3^7}{7!} = \frac{243}{560} \approx 0.434$$

Student view

Overview
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**Question 3**

Difficulty: ★★★

The Maclaurin expansion of $e^{x^3}$ is $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

Find $a_9$. Give your answer either as a rational number in a fully simplified form or as a decimal number rounded to three significant figures.

1/6

**Accepted answers**

1/6, 0.167, 0,167, .167

**Explanation**

We use the expansion given in the Formula booklet:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

and substituting

$$e^{x^3} = 1 + x^3 + \frac{(x^3)^2}{2!} + \frac{(x^3)^3}{3!} + \ldots$$

hence,

$$a_9 = \frac{1}{3!} = \frac{1}{6} \approx 0.167$$

5. Calculus / 5.19 Maclaurin series expansions

# Expansion of special functions

There are many important Maclaurin series expansions. In this section, you are going to begin to learn how to manipulate these basic expansions to find expansions of more complex functions. First, you will develop a list of common expansions to refer to. There are many more, but for the scope of this course, only a selection will be considered.

## Exponential function

**Example 3** in section 5.19.4 (/study/app/math-aa-hl/sid-134-cid-761926/book/taylor-and-maclaurin-series-id-27285/) developed the expansion for the exponential function,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Student view

# Natural logarithm

To find the Maclaurin series expansion for the natural logarithm $f(x) = \ln(1+x)$, follow the process:

| Derivative | Evaluated derivative | Term |
|---|---|---|
| $f(x) = \ln(1+x)$ | $f(0) = \ln(1) = 0$ | $0$ |
| $f'(x) = \dfrac{1}{1+x}$ | $f'(0) = 1$ | $x$ |
| $f''(x) = \dfrac{-1}{(1+x)^2}$ | $f''(0) = -1$ | $\dfrac{-x^2}{2!} = \dfrac{-x^2}{2}$ |
| $f'''(x) = \dfrac{2}{(1+x)^3}$ | $f'''(0) = 2$ | $2\dfrac{x^3}{3!} = \dfrac{x^3}{3}$ |
| $f^{(4)}(x) = \dfrac{-6}{(1+x)^4}$ | $f^{(4)}(0) = -6$ | $-6\dfrac{x^4}{4!} = \dfrac{-x^4}{4}$ |

Following the pattern you can write up the full expansion.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

# Sine function

Similarly, you can find the Maclaurin series expansion for the sine function $f(x) = \sin x$:

| Derivative | Evaluated derivative | Term |
|---|---|---|
| $f(x) = \sin x$ | $f(0) = \sin 0 = 0$ | $0$ |
| $f'(x) = \cos x$ | $f'(0) = \cos 0 = 1$ | $x$ |
| $f''(x) = -\sin x$ | $f''(0) = -\sin 0 = 0$ | $0$ |
| $f'''(x) = -\cos x$ | $f'''(0) = -\cos 0 = -1$ | $-\dfrac{x^3}{3!}$ |
| $f^{(4)}(x) = \sin x$ | $f^{(4)}(0) = \sin 0 = 0$ | $0$ |
| $f^{(5)}(x) = \cos x$ | $f^{(5)}(0) = \cos 0 = 1$ | $\dfrac{x^5}{5!}$ |



$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

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The cycling of the derivatives, i.e. $f(x) = f^{(4)}(x), f'(x) = f^{(5)}(x), \ldots$, is what drives the periodic sequence $1, 0, -1, 0, 1, 0, -1, 0, \ldots$

# Cosine function

The Maclaurin series expansion for the cosine function, $f(x) = \cos x$, can be found in two different ways. You can find it formally just like the sine function:

| Derivative | Evaluated derivative | Term |
|---|---|---|
| $f(x) = \cos x$ | $f(0) = \cos 0 = 1$ | $1$ |
| $f'(x) = -\sin x$ | $f'(0) = -\sin 0 = 0$ | $0$ |
| $f''(x) = -\cos x$ | $f''(0) = -\cos 0 = -1$ | $-\frac{x^2}{2!}$ |
| $f'''(x) = \sin x$ | $f'''(0) = \sin 0 = 0$ | $0$ |
| $f^{(4)}(x) = \cos x$ | $f^{(4)}(0) = \cos 0 = 1$ | $\frac{x^4}{4!}$ |

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

A second technique would be to consider the Maclaurin series expansion for the sine function,

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, and to take the derivative of both sides:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$
$$\cos x = 1 - 3\frac{x^2}{3!} + 5\frac{x^4}{5!} - \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

## Example 1

★☆☆

Use the first three non-zero terms of the Maclaurin series to estimate the value of cos 1.

From the work above, you know that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

Student view

For $x = 1$, this gives:

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$$\cos 1 \approx 1 - \frac{1^2}{2!} + \frac{1^4}{4!} = 1 - \frac{1}{2} + \frac{1}{24} = \frac{13}{24}$$

**Note on the accuracy of this estimate:**

The Lagrange form of the error term is $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$. In this case, choose $R_5(x)$ because $R_4(x)$ would be paired with the $x^5$ term that has a coefficient of 0.

Therefore,

$$R_5(1) = \frac{f^{(6)}(c)}{6!}(1)^6 = \frac{-\cos(c)}{720}$$

Since $-1 \leqslant \cos x \leqslant 1$, the error term is $\frac{-1}{720} \leqslant R_5(1) \leqslant \frac{1}{720}$.

This gives estimates of $\frac{389}{720} \leqslant \cos 1 \leqslant \frac{391}{720}$ or $0.54028 \leqslant \cos 1 \leqslant 0.54306$.

The decimal approximation found by the calculator is $0.54030$, which does fall within this interval.

With the applet below, you can explore the accuracy of the Maclaurin polynomial approximations to the cosine and sine curves. You can control the number of non-zero non-constant terms in the expansion with the slider.

**Interactive 1.** Accuracy of the Maclaurin Polynomial Approximations to the Cosine and Sine Curves.

More information for interactive 1

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

This interactive graph allows users to explore how Maclaurin series can be used to approximate the sine and cosine functions using polynomial expressions. The graph displays the x-axis from −8 to 8 and the y-axis from −6 to 4. At the top left of the screen, there is a slider labeled “n,” which lets users choose the number of terms in the approximation from 0 to 8. Two checkboxes are available—one for cos(x) and one for sin(x)—allowing users to select which function to analyze.

Once a function is selected, the actual function is drawn in blue. As the slider value increases, a magenta curve representing the polynomial approximation appears and updates in real time. For example, if users select cos(x) and set n = 4, the interactive will display the approximation:

$1-x^2/2 + x^4/24-x^6/720$

This polynomial closely matches the true cosine curve near x = 0.

Likewise, selecting sin(x) and setting n = 3 shows the approximation:

$x-x^3/6 + x^5/120$

Users can visually observe how these polynomials improve in accuracy as more terms are added. The approximation starts off as a straight line or simple curve and gradually wraps around the shape of the true trigonometric function, especially near the origin.

The interactive also reinforces mathematical patterns: cosine uses only even powers of x with alternating signs, while sine uses odd powers of x, also alternating in sign. As more terms are added, users see that these polynomials converge to the actual function. This helps build understanding of infinite series and their practical use in approximating complex functions with simpler expressions.

# Inverse tangent function

✓ **Important**

The inverse trigonometric derivatives from subtopic 5.15 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26507/) include:

- If $f(x) = \arcsin x$, then $f'(x) = \frac{1}{\sqrt{1-x^2}}$
- If $f(x) = \arccos x$, then $f'(x) = -\frac{1}{\sqrt{1-x^2}}$
- If $f(x) = \arctan x$, then $f'(x) = \frac{1}{1+x^2}$

The Maclaurin series expansion for the inverse tangent function, $f(x) = \arctan x$ is found below:

| Derivative | Evaluated derivative | Term |
|---|---|---|
| $f(x) = \arctan x$ | $f(0) = \arctan 0 = 0$ | $0$ |
| $f'(x) = \frac{1}{1+x^2}$ | $f'(0) = \frac{1}{1+0^2} = 1$ | $x$ |
| $f''(x) = \frac{-2x}{(1+x^2)^2}$ | $f''(0) = \frac{-2(0)}{(1+0^2)^2} = 0$ | $0$ |

Student view

| Derivative | Evaluated derivative | Term |
|---|---|---|
| $f'''(x) = \frac{-2+6x^2}{(1+x^2)^3}$ | $f'''(0) = \frac{-2+6(0)^2}{\left(1+(0)^2\right)^3} = -2$ | $-2\frac{x^3}{3!} = -\frac{x^3}{3}$ |
| $f^{(4)}(x) = \frac{24x-24x^3}{(1+x^2)^4}$ | $f^{(4)}(x) = \frac{24-24(0)^3}{\left(1+(0)^2\right)^4} = 0$ | 0 |
| $f^{(5)}(x) = \frac{24-240x^2+120x^4}{(1+x^2)^5}$ | $f^{(5)}(x) = \frac{24-240(0)^2+120(0)^4}{\left(1+(0)^2\right)^5} = 24$ | $24\frac{x^5}{5!} = \frac{x^5}{5}$ |

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

The derivatives in this working are not trivial. Can you verify the result using the quotient rule?

## ✓ Important

Familiarity with the following expansions is required for this course:

- Exponential function
  $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
  Can be used for approximation for all real numbers $x$

- Natural logarithm
  $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$
  Can be used for approximation for $|x| < 1$ and for $x = 1$.

- Sine
  $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$
  Can be used for approximation for all $x$

- Cosine
  $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$
  Can be used for approximation for all $x$

- Inverse tangent
  $$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$
  Can be used for approximation for $|x| \leqslant 1$

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These formulae, without the general form or intervals of convergence, are provided in the formula booklet.

# 4 section questions

**Question 1**

Difficulty: ★★★

The Maclaurin expansion of $\cos(3x)$ is $a_0 + a_1x + a_2x^2 + \ldots$

Find $a_8$. Give your answer either as a rational number in a fully simplified form or as a decimal number rounded to three significant figures. Give the numerical value only.

729/4480 ✓

**Accepted answers**

729/4480, 0.163, 0,163

**Explanation**

We start with the expansion for $\cos x$ given in the Formula booklet and then apply it as follows:

$$\begin{array}{ll} \text{Since} & \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots \\ \text{then} & \cos 3x = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \frac{(3x)^8}{8!} - \ldots \\ \text{hence,} & a_8 = \frac{3^8}{8!} = \frac{729}{4480} \approx 0.163 \end{array}$$

**Question 2**

Difficulty: ★★☆

The Maclaurin expansion of $\ln(1 + 3x)$ is $a_0 + a_1x + a_2x^2 + \ldots$

Find the exact value of $a_6$. Give your answer either as a rational number in a fully simplified form or as a decimal number (but do not round). Give the numerical value only.

-243/2 ✓

**Accepted answers**

-243/2, -121.5, -121,5, - 243/2, - 121.5, - 121,5, —243/2, —121.5, —121,5

**Explanation**

We use the expansion for $\ln(1+x)$ given in the Formula booklet and apply it to this particular problem.

Student
view

Overview
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Since $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots$

then $\ln(1+3x) = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \frac{(3x)^5}{5} - \frac{(3x)^6}{6} + \dots$

hence, $a_6 = -\frac{3^6}{6} = \frac{243 \times 3}{3 \times 2} = \frac{243}{2} = -121.5$

**Question 3**

Difficulty: ★★☆

The Maclaurin expansion of $\ln(3+x)$ is $a_0 + a_1 x + a_2 x^2 + \dots$

Find $a_4$. Give your answer either as a rational number in a fully simplified form or as a decimal number rounded to three significant figures. Give the numerical value only.

-1/324

### Accepted answers

-1/324, -0.00309, -0,00309, - 1/324, - 0.00309, - 0,00309, —1/324, —0.00309, —0,00309

### Explanation

We use the expansion for $\ln(1+x)$ given in the Formula booklet and apply it to this particular problem.

Since $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

then
$$\ln(3+x) = \ln\left(3\left(1+\frac{x}{3}\right)\right) = \ln 3 + \ln\left(1+\frac{x}{3}\right)$$
$$= \ln 3 + \left(\frac{x}{3} - \frac{\left(\frac{x}{3}\right)^2}{2} + \frac{\left(\frac{x}{3}\right)^3}{3} - \frac{\left(\frac{x}{3}\right)^4}{4} + \dots\right)$$

hence,
$$a_4 = -\frac{1}{3^4 \times 4}$$
$$= -\frac{1}{324}$$
$$\approx -0.00309$$

**Question 4**

Difficulty: ★★☆

The Maclaurin expansion of $\arctan(2x^2)$ is $a_0 + a_1 x + a_2 x^2 + \dots$

Find $a_{14}$. Give your answer either as a rational number in a fully simplified form or as a decimal number rounded to three significant figures. Give the numerical value only.

-128/7

### Accepted answers

-128/7, -18.3, -18,3

### Explanation

We use the expansion for $\arctan x$ given in the Formula booklet and apply it to this particular problem.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Since $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

then $\arctan(2x^2) = 2x^2 - \frac{(2x^2)^3}{3} + \frac{(2x^2)^5}{5} - \frac{(2x^2)^7}{7} + \ldots$

hence,
$$\begin{aligned} a_{14} &= -\frac{2^7}{7} \\ &= -\frac{128}{7} \\ &\approx -18.3 \end{aligned}$$



# Binomial series

In subtopic 1.9 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27687/), the binomial theorem was discussed to expand expressions of the form $(a+b)^n$ for positive integer exponents. In section 1.10.4 (/study/app/math-aa-hl/sid-134-cid-761926/book/generalisation-of-the-binomial-theorem-id-26977/) this was generalized to work with exponents other than positive integers. Here, you are going to look at this extension from a different point of view by investigating the Maclaurin series expansion of $f(x) = (1+x)^p$.

| Derivative | Evaluated derivative | Term |
|---|---|---|
| $f(x) = (1+x)^p$ | $f(0) = 1$ | $1$ |
| $f'(x) = p(1+x)^{p-1}$ | $f'(0) = p$ | $px$ |
| $f''(x) = p(p-1)(1+x)^{p-2}$ | $f''(0) = p(p-1)$ | $\frac{p(p-1)}{2!}x^2$ |
| $f'''(x) = p(p-1)(p-2)(1+x)^{p-3}$ | $f'''(0) = p(p-1)(p-2)$ | $\frac{p(p-1)(p-2)}{3!}x^3$ |
| $f^{(4)}(x) = p(p-1)(p-2)(p-3)(1+x)^{p-4}$ | $f^{(4)}(x) = p(p-1)(p-2)(p-3)$ | $\frac{p(p-1)(p-2)(p-3)}{4!}x^4$ |

$$f(x) = (1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

If $p = n$ is a positive integer, then this series will have 0 coefficients after the term in $x^n$, because the $n(n-1)(n-2)\ldots$ product contains a 0. Hence, for positive integer exponents, this series simply returns the finite binomial expansion of $(1+x)^n$, as studied in the core of the course.

Student view

If $p$ is not a positive integer, then this series is infinite. It can be shown that for $-1 < x \leqslant 1$, the infinite sum is equal to $(1+x)^p$.

### Important

The formula for the **binomial series** is:

$$\begin{aligned} f(x) &= (1+x)^p \\ &= 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \frac{p(p-1)(p-2)(p-3)}{4!}x^4 + \cdots \end{aligned}$$

and can be used for approximation for $-1 < x \leqslant 1$.

The explicit equation can be written as

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n \text{ for } -1 < x \leqslant 1,$$

where the generalised binomial coefficient is defined as

$$\binom{p}{n} \equiv \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$$

### Exam tip

This formula is in the Formula booklet in the following form:

$$(a+b)^n = a^n\left(1 + n\left(\frac{b}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 + \ldots\right)$$

There are many numbers used on a regular basis where the exact value has a non-repeating infinite decimal form. Some common examples include $\pi$, $\mathrm{e}$ and $\sqrt{2}$. Maclaurin expansions can be used to calculate valid approximations to many of these values.

# Example 1

★★☆

Using the first three terms of the Maclaurin expansion of $\sqrt{1+x}$, find a rational approximation of $\sqrt{2}$.

From the work above, you know that $f(x) = (1+x)^p = 1 + px + \dfrac{p(p-1)}{2!}x^2 + \cdots$

With $\sqrt{1+x}$ rewritten as $(1+x)^{\frac{1}{2}}$, you can find the approximation as:

$$f(x) = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2!}x^2 + \cdots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

Student view

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Evaluating for $\sqrt{2}$ you get

$$\sqrt{2} = \sqrt{1+1} = f(1) \approx 1 + \frac{1}{2}(1) - \frac{1}{8}(1)^2 = 1 + \frac{1}{2} - \frac{1}{8} = 1.375$$

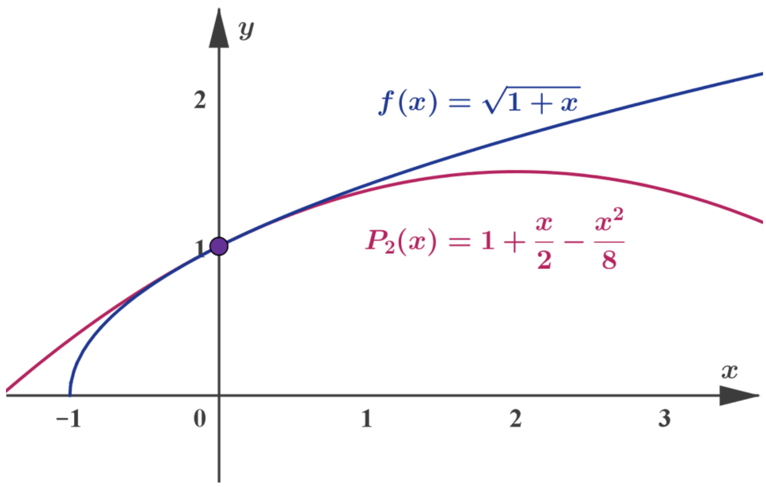
**Note on the accuracy of this approximation:**

To find the error, use $R_2(1) = \frac{f'''(c)}{3!}$ for $0 < c < 1$.

Since the third derivative is $f'''(c) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1+c)^{-\frac{5}{2}} = \frac{3}{8}(1+c)^{-\frac{5}{2}}$ (check this out for yourself), you can write the error term as $R_2(1) = \frac{\frac{3}{8}(1+c)^{-\frac{5}{2}}}{3!} = \frac{1}{16\sqrt{(1+c)^5}}$ for some $0 < c < 1$. The worst case for this is $c = 0$, so $R_2(1) = \frac{1}{16} \leqslant 0.0625$. That gives an interval of $1.3125 \leqslant \sqrt{2} \leqslant 1.4375$.

The actual value of $\sqrt{2} = 1.4142\ldots$ is in this interval.

The following diagram illustrates the original function and the approximating Taylor polynomial.




## Example 2

★★★

Using the first three terms of the Maclaurin expansion of $\sqrt{4+x}$, find a rational approximation of $\sqrt{5}$.

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From the work above, you know that $f(x) = (1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots$

With $\sqrt{4+x}$ rewritten as $2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$, you can find the approximation as:

$$f(x) = 2\left(1+\frac{x}{4}\right)^{\frac{1}{2}} = 2\left(1 + \frac{1}{2}\left(\frac{x}{4}\right) + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2!}\left(\frac{x}{4}\right)^2 + \cdots\right) = 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \cdots$$

Evaluating for $\sqrt{5}$ or $x = 1$ you get

$$\sqrt{5} = 2\sqrt{1+\frac{1}{4}} = f(1) \approx 2 + \frac{1}{4}(1) - \frac{1}{64}(1)^2 = \frac{143}{64} \approx 2.234\ldots$$

The actual value is $\sqrt{5} \approx 2.236\ldots$

The applet below illustrates the relationship of the Maclaurin expansion and the original function $f(x) = \sqrt{a+x}$ for $1 \leqslant a \leqslant 2$. The approximating polynomials do not get close to $f(x)$ for $x < -a$, because for these $x$ values, $\sqrt{a+x}$ is not defined. From the applet, you can also see that the approximating polynomials do not get close to $f(x)$ for $x > a$ either. Move the sliders to change the value of $a$ and to change the order of the approximating polynomial. The shaded region in the middle illustrates the interval where the polynomials can be used for approximation.

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/expansion-of-special-functions-id-27286/print/) Assign

**Interactive 1.** Binomial Series Visualization.

More information for interactive 1

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

This interactive visualizes the Maclaurin series approximation of the function $f(x) = \sqrt{a + x}$ within the domain $1 \leq a \leq 2$. It demonstrates how polynomial expansions of varying degrees approximate the original function, highlighting the limitations of these approximations outside specific intervals. The tool helps users understand the convergence behavior of Taylor series expansions and their dependence on both the function's properties and the chosen expansion point.

The display shows a coordinate system with x-axis ($-3$ to $3$) and y-axis ($-2$ to $2$), featuring two prominent curves: a blue curve representing the original function $f(x) = \sqrt{a + x}$ and a red curve showing the current Maclaurin polynomial approximation. There are two sliders on the top left that users can adjust - one for parameter 'a' ($1$ to $2$), which modifies the function's horizontal shift, and another for order 'n' ($0$ to $9$), which controls the degree of the approximating polynomial. A shaded region between $-a$ and $a$ indicates where the approximation remains valid, with the area outside this range clearly showing divergence between the function and its polynomial approximation.

For example, by adjusting the sliders, users observe key approximation behaviors: with $a = 1.5$ and $n = 0$, $f(x)$ is approximated by a horizontal line, while increasing to $n = 4$ yields a closer polynomial fit within $[-a, a]$. Setting $a = 2$ (with $n = 3$) expands the valid approximation interval. The visualization clearly shows the polynomial diverging outside $[-a, a]$, especially where $f(x) = \sqrt{a + x}$ becomes undefined for $x < -a$.

Through this exploration, users gain important insights about series approximations: they learn how Maclaurin polynomials approximate functions near zero, understand the concept of radius of convergence, and observe how higher-degree polynomials provide better approximations within the convergence interval. The visualization makes clear why these approximations fail outside specific bounds, particularly for functions with limited domains. These concepts are fundamental for calculus, numerical analysis, and physics applications where function approximation is essential.

5. Calculus / 5.19 Maclaurin series expansions

# Further expansions

In the previous sections, you learned about the Maclaurin expansions of $\sin x$, $\cos x$, $\ln(1 + x)$, $\arctan x$ and $(1 + x)^p$. Using these expansions, you can find expansions of combinations of these functions.

## Example 1

★★☆

Find the first three non-zero terms of the Maclaurin series of $\sin x \cos x$.

You can continue the process as before with derivatives.

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| | | |
|---|---|---|
| $f(x) = \sin x \cos x$ | $f(0) = 0$ | $0$ |
| $f'(x) = \cos^2 x - \sin^2 x$ | $f'(0) = 1$ | $x$ |
| $f''(x) = -4 \sin x \cos x$ | $f''(0) = 0$ | $0$ |
| $f'''(x) = -4\cos^2 x + 4\sin^2 x$ | $f'''(0) = -4$ | $-\frac{4x^3}{3!}$ |
| $f^{(4)}(x) = 16 \sin x \cos x$ | $f^{(4)}(0) = 0$ | $0$ |
| $f^{(5)}(x) = 16\cos^2 x - 16\sin^2 x$ | $f^{(5)}(0) = 16$ | $\frac{16x^5}{5!}$ |
| $f(x) = \sin x \cos x = x - \frac{2x^3}{3} + \frac{2x^5}{15} + \cdots$ | | |

A **second** method is through trigonometry and the common expansion for $\sin x$ **.**

Using the double angle identity $\sin 2x = 2 \sin x \cos x$ ( from subtopic 3.6.2) you get:

$$f(x) = \sin x \cos x = \frac{1}{2}\sin 2x = \frac{1}{2}\left((2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots\right) = x - \frac{2x^3}{3} + \frac{2x^5}{15} - \cdots$$

A **third** method would be to use the common expansions of $\sin x$ and $\cos x$.

You can multiply the expansions together, ensuring a complete expansion of the first three terms of each.

$$\begin{aligned} f(x) &= \sin x \cos x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \\ &= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^3}{3!} + \frac{x^5}{3!2!} - \frac{x^7}{3!4!} + \frac{x^5}{5!} - \frac{x^7}{5!2!} + \frac{x^9}{5!4!} + \cdots \\ &= x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^3}{6} + \frac{x^5}{12} - \frac{x^7}{144} + \frac{x^5}{120} - \frac{x^7}{240} + \frac{x^9}{2880} + \cdots \\ &= x - \frac{2x^3}{3} + \frac{2x^5}{15} - \cdots \end{aligned}$$

As you only went to the third term in each, you can only justify the first three non-zero terms in the expansion. Any term after that is missing some product terms from the infinite series expansions. So, keeping just the first three terms, you get:

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$$f(x)=\sin x\cos x=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)=x-\frac{2x^3}{3}+\frac{2x^5}{15}-\cdots$$

# Example 2

★★☆

Find the first three non-zero terms of the expansion of $\cos^2 x$.

All three techniques work.

Traditional:

| | | |
|---|---|---|
| $f(x)=\cos^2 x$ | $f(0)=1$ | $1$ |
| $f'(x)=-\sin 2x$ | $f'(0)=0$ | $0$ |
| $f''(x)=-\cos 2x$ | $f''(0)=-2$ | $\frac{-2x^2}{2!}=-x^2$ |
| $f'''(x)=4\sin 2x$ | $f'''(0)=0$ | $0$ |
| | | |

$$f(x)=\cos^2 x=1-x^2+\frac{x^4}{3}+\cdots$$

Trigonometry:

$$\cos 2x=2\cos^2 x-1,\text{ so }\cos^2 x=\frac{1}{2}(\cos 2x+1)$$

From the expansion for $\cos x$:

$$\begin{aligned}\cos^2 x &= \frac{1}{2}(\cos 2x+1)=\frac{1}{2}\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\cdots+1\right)\\ &=\frac{1}{2}\left(2-\frac{4x^2}{2}+\frac{16x^4}{24}-\cdots\right)=\frac{1}{2}\left(2-2x^2+\frac{2x^4}{3}-\cdots\right)=1-x^2+\frac{x^4}{3}-\end{aligned}$$

Expansion multiplication:

Student view

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$$\begin{aligned} f(x)=\cos^2 x &= \cos x\cos x=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right) \\ &= 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^4}{24}+\cdots=1-x^2+\frac{x^4}{3}+\cdots \end{aligned}$$

# Example 3

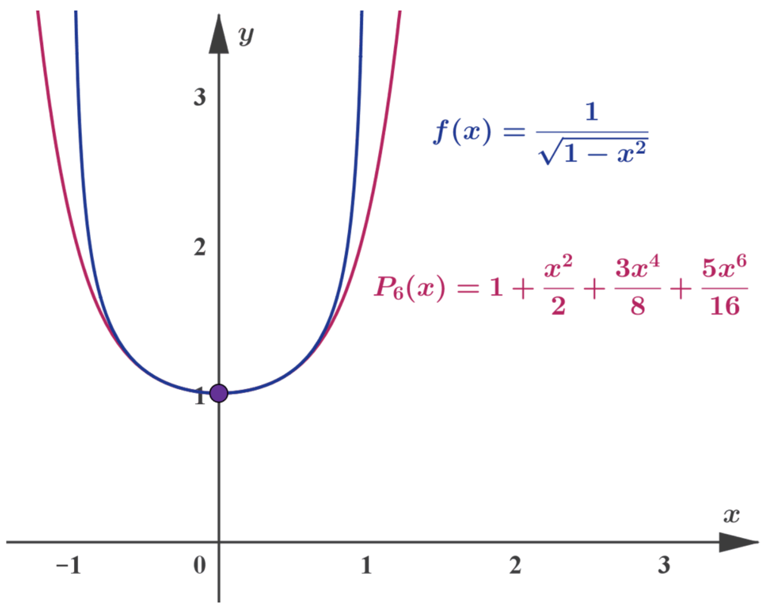
★★★

Find the first four non-zero terms of the expansion of $\frac{1}{\sqrt{1-x^2}}$. Hence, find an approximation of $\arcsin x$.

First, rewrite the function as a binomial $\frac{1}{\sqrt{1-x^2}}=\left(1+(-x^2)\right)^{-\frac{1}{2}}$. This allows you to use the binomial expansion:

$$\begin{aligned} \frac{1}{\sqrt{1-x^2}} &= \left(1+(-x^2)\right)^{-\frac{1}{2}} \\ &= 1+\left(-\frac{1}{2}\right)(-x^2)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2 \\ &= 1+\frac{x^2}{2}+\frac{3x^4}{8}+\frac{5x^6}{16}+\cdots \end{aligned}$$

The following diagram illustrates the original function and the approximating Taylor polynomial.




Student view

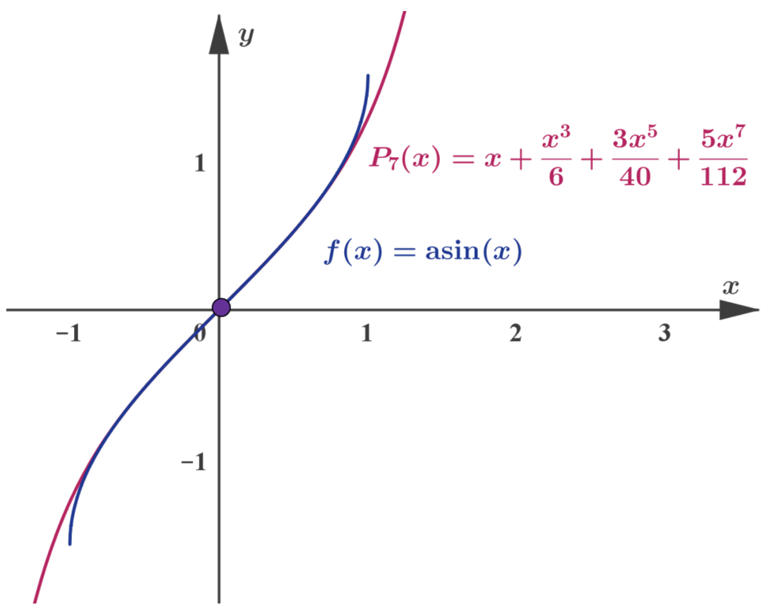
Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Since $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x$ , you can find the expansion of $\arcsin x$ by integrating the expansion of $\frac{1}{\sqrt{1-x^2}}$ term-by-term.

$$\arcsin x = \int \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} + \cdots\right)dx = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \cdots$$

Note that when integrating, a constant term also need to be added. In this case the constant term of the resulting series is 0, because $\arcsin 0 = 0$.

The following diagram illustrates the original function and the approximating Taylor polynomial.




## 3 section questions

**Question 1**

Difficulty:

★★★

The Maclaurin expansion of $\cos(2x)\sin(4x)$ is $a_0 + a_1x + a_2x^2 + \ldots$

Find $a_5$. Give your answer either as a rational number in a fully simplified form or as a decimal number rounded to three significant figures. Give the numerical value only.

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488/15

### Accepted answers

488/15, 32.5, 32,5

### Explanation

We start with the expansions for $\cos x$ and $\sin x$ given in the Formula booklet and then apply these as follows:

$$\cos(2x)\sin(4x) = \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \ldots\right)\left(4x - \frac{(4x)^3}{3!} + \frac{(4x)^5}{5!} + \ldots\right)$$

hence,

$$a_5 = \frac{4 \times 2^4}{4!} + \frac{4^3 \times 2^2}{2!3!} + \frac{4^5}{5!} = \frac{488}{15} \approx 32.5$$

## Question 2

Difficulty: ★★☆

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/binomial-function-id-28376/print/) Assign

The Maclaurin expansion of $\sin^2 x$ is $a_0 + a_1 x + a_2 x^2 + \ldots$

Find $a_6$. Give your answer either as a rational number in a fully simplified form or as a decimal number rounded to three significant figures. Give the numerical value only.

2/45

### Accepted answers

2/45, 0.0444, 0,0444

### Explanation

We use the expansion of $\sin x$ from the Formula booklet:

$$\begin{aligned}\sin^2 x &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)\\ a_6 &= \frac{1}{5!} + \frac{1}{3!3!} + \frac{1}{5!}\\ &= \frac{2}{45}\\ &\approx 0.0444\end{aligned}$$

## Question 3

Difficulty: ★★☆

The Maclaurin expansion of $\arccos x$ is $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

Find the exact value of $a_5$. Give your answer as a decimal or as a fraction in fully simplified form.

-3/40

### Accepted answers

-3/40, -0.075, -0,075, -.075

### Explanation

Student view

Overview (/study/ap aa- hl/sid- 134- cid- 761926/o

According to Example 3 in the section 5.19.7:

$$\arcsin x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \dots$$

Since $\arccos x = \frac{\pi}{2} - \arcsin x$,

$$\arccos x = \frac{\pi}{2} - \left(x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \dots\right)$$

Hence,

$$a_5 = -\frac{3}{40} = -0.075$$

5. Calculus / 5.19 Maclaurin series expansions

# Applications

## Evaluate limits

In subtopic 5.13 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26497/), L'Hôpital's rule was used to solve indeterminate forms of limits. You can also use the Maclaurin series to find these limits .

Consider for example $\lim\limits_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

Using L'Hôpital's rule , you can find the answer as follows:

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} \text{ (L'Hôpital's rule)}$$

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \frac{e^0 - 1}{2(0)} = \frac{0}{0}$$

This is still indeterminate. Apply L'Hôpital's rule again:

$$\lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$$

Now try to find it using a Maclaurin series substitution.

Student view

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First, find the Maclaurin series:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$$

Now substitute it into the function and simplify:

$$\begin{aligned}\lim_{x\to 0}\frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0}\frac{\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots\right) - 1 - x}{x^2}\\ &= \lim_{x\to 0}\frac{\frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots}{x^2}\\ &= \lim_{x\to 0}\left(\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \ldots\right) = \frac{1}{2}\end{aligned}$$

In this expansion the terms involving $x$ approach 0 as $x$ approaches 0. The limit is the value of the constant term.

$$\lim_{x\to 0}\frac{e^x - 1 - x}{x^2} = \frac{1}{2}$$

This matches the answer from L'Hôpital's rule.

## Example 1

★★★

Find $\lim\limits_{x\to 0}\dfrac{x - \ln(1+x)}{x^2}$

This could be found through multiple applications of L'Hôpital's rule.

$$\lim_{x\to 0}\left(\frac{x - \ln(1+x)}{x^2}\right) = \lim_{x\to 0}\left(\frac{1 - \frac{1}{1+x}}{2x}\right) = \lim_{x\to 0}\left(\frac{\frac{1}{(1+x)^2}}{2}\right) = \frac{1}{2}$$

But suppose the denominator was a higher-order polynomial, such as $x^4$ (as you will see in one of the section questions). Eventually, the derivatives are going to get quite challenging to complete.

Using the $\ln(1+x)$ expansion from the Formula booklet:

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$$\frac{x-\ln(1+x)}{x^2}=\frac{x-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)}{x^2}=\frac{\frac{x^2}{2}-\frac{x^3}{3}+\cdots}{x^2}=\frac{1}{2}-\frac{x}{3}+\cdots$$

In this final expansion, the constant term is $\frac{1}{2}$ and all other terms approach $0$ as $x$ approaches $0$, implying that $\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}=\frac{1}{2}$

## Example 2

★★★

Find $\lim_{n\to\infty} n\left(\sqrt{n^2+1}-n\right)$

First notice that if $n$ is large, then $\sqrt{n^2+1}$ is close to $\sqrt{n^2}=n$. In fact, the difference approaches $0$. To formally prove this, a standard trick is to multiply and divide the expression by the sum of the two expressions.

$$\begin{aligned}\sqrt{n^2+1}-n&=\frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n}\\&=\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}=\frac{1}{\sqrt{n^2+1}+n}\end{aligned}$$

In this quotient the numerator is $1$ and the denominator grows without bound as $n$ grows without bound, so the limit is indeed $0$.

In the question this difference is multiplied by $n$, so the expression is a product of a number that grows without bound and an expression that approaches $0$. This indeterminate form can be rewritten to a type $\frac{0}{0}$ by changing multiplication by $n$ to a division by $\frac{1}{n}$.

$$n\left(\sqrt{n^2+1}-n\right)=\frac{\sqrt{n^2+1}-n}{\frac{1}{n}}$$

**Method 1**

This rewriting suggests replacing $\frac{1}{n}$ with $x$, so using the substitution $n=\frac{1}{x}$:

Student view

$$
\begin{aligned}
\lim_{n\to\infty} n\left(\sqrt{n^2+1}-n\right) &= \lim_{x\to 0^+}\frac{1}{x}\left(\sqrt{\frac{1}{x^2}+1}-\frac{1}{x}\right)\\
&= \lim_{x\to 0^+}\frac{1}{x}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\\
&= \lim_{x\to 0^+}\frac{\sqrt{1+x^2}-1}{x^2}
\end{aligned}
$$

Using the binomial expansion:

$$
\sqrt{1+x}=(1+x)^{\frac{1}{2}}=1+\frac{x}{2}-\frac{x^2}{8}+\cdots
$$

By replacing $x$ with $x^2$

$$
\sqrt{1+x^2}=\left(1+x^2\right)^{\frac{1}{2}}=1+\frac{x^2}{2}-\frac{x^4}{8}+\cdots
$$

Therefore,

$$
\begin{aligned}
\lim_{n\to\infty} n\left(\sqrt{n^2+1}-n\right) &= \lim_{x\to 0^+}\frac{\sqrt{1+x^2}-1}{x^2}=\lim_{x\to 0^+}\frac{\left(1+\frac{x^2}{2}-\frac{x^4}{8}+\cdots\right)-1}{x^2}\\
&= \lim_{x\to 0^+}\frac{1}{2}-\frac{x^2}{8}+\cdots=\frac{1}{2}
\end{aligned}
$$

**Method 2**

Note that the formal proof of the limit at the beginning of this explanation is not needed for the calculation of the limit. It is just a motivation for the substitution $n=\frac{1}{x}$. This limit calculation can also lead to another calculation that uses neither L'Hôpital's rule nor the binomial series.

$$
\begin{aligned}
\lim_{n\to\infty} n\left(\sqrt{n^2+1}-n\right) &= \lim_{n\to\infty}\frac{n\left(\sqrt{n^2+1}-n\right)\left(\sqrt{n^2+1}+n\right)}{\sqrt{n^2+1}+n}\\
&= \lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}+n}\\
&= \lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}+n}\times\frac{1/n}{1/n}\\
&= \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}+1}=\frac{1}{\sqrt{1+0}+1}=\frac{1}{2}
\end{aligned}
$$

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

# Approximate definite integrals

Another application of a Maclaurin series is to find approximations to definite integrals that otherwise would be difficult to evaluate.

## Example 3

★★★

Use the first three terms of the Maclaurin expansion of $e^x$ to find an approximate value for $\int_0^1 e^{-x^2}dx$.

Using the exponential expansion $e^x = 1 + x + \frac{x^2}{2} + \ldots$, replace $x$ with $-x^2$ :

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} + R_2(-x^2)$$

Integration results in:

$$\int_0^1 e^{-x^2}dx = \int_0^1 \left(1 - x^2 + \frac{x^4}{2} + \ldots\right)dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10} + \ldots\right]_0^1$$
$$\approx 1 - \frac{1}{3} + \frac{1}{10} = \frac{23}{30}$$

**Note on the accuracy of this approximation:**

With $f(x) = e^x$, derivatives are simple: $f(x) = f'(x) = f''(x) = f'''(x) = e^x$. This leads to the Lagrange form of the error term:

$$R_2(-x^2) = \frac{f^{(3)}(c)}{3!}(x)^3 = \frac{e^c}{6}\left(-x^2\right)^3 \text{ where } 0 < c < -x^2$$

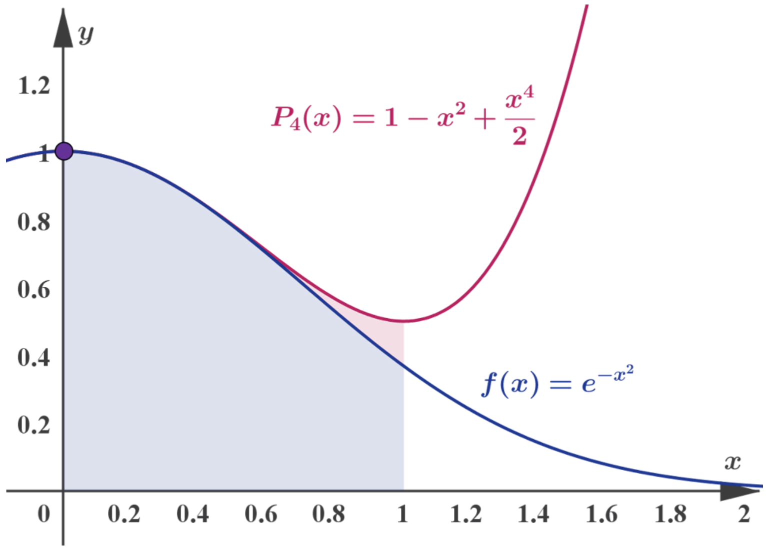
Since the integration is between 0 and 1, $0 < x < 1$, and then $-1 < c < 0$.

Therefore $\left|R_2(-x^2)\right| < \frac{e^0}{6}1^3 = \frac{1}{6}$, and the error of the approximation is less than $\int_0^1 \frac{1}{6}dx = \frac{1}{6}$

The following diagram illustrates the difference between the integral of the original function and the integral of the approximating Taylor polynomial.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o




# 3 section questions

**Question 1**

Difficulty:

★★☆

Find the exact value of $\lim_{x \to 0} \frac{x^2 - \ln(1 + x^2)}{x^4}$

0.5

**Accepted answers**

0.5, 1/2, 0,5

**Explanation**

Using the known expansion for $\ln(1 + x)$ given in the Formula booklet and simplifying:

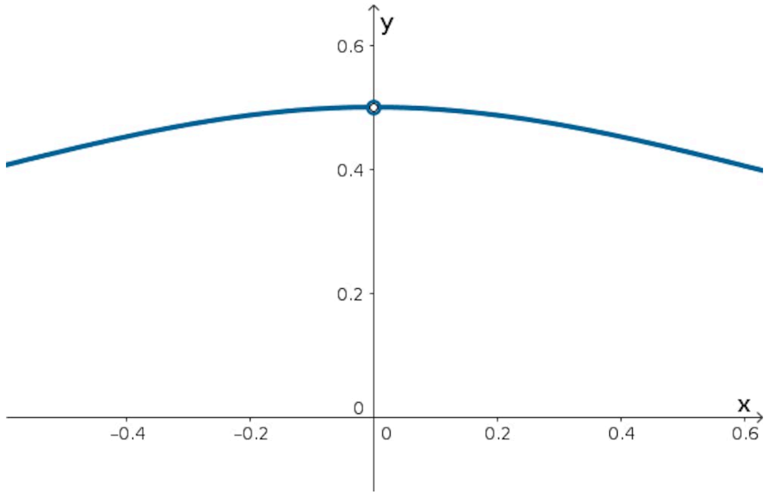
$$\begin{aligned}\frac{x^2 - \ln(1 + x^2)}{x^4} &= \frac{x^2 - \left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \dots\right)}{x^4} \\ &= \frac{\frac{x^4}{2} - \frac{x^6}{3} + \dots}{x^4} \\ &= \frac{1}{2} - \frac{x^2}{3} + \dots\end{aligned}$$

So, $\lim_{x \to 0} \frac{x^2 - \ln(1 + x^2)}{x^4} = \frac{1}{2}$

The diagram below shows the graph of $y = \frac{x^2 - \ln(1 + x^2)}{x^4}$ around $x = 0$.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o




More information

**Question 2**

Difficulty:

★★★

Find the exact value of $\lim_{n\to\infty} n\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right)$

1

### Accepted answers

1

### Explanation

We will use the substitution $n=\frac{1}{x}$ and note that if $n\to\infty$, then $x\to 0^+$:

$$\begin{aligned}\lim_{n\to\infty} n\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right) &= \lim_{x\to 0^+}\frac{1}{x}\left(\sqrt{\frac{1}{x^2}+1}-\sqrt{\frac{1}{x^2}-1}\right)\\ &= \lim_{x\to 0^+}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2}\end{aligned}$$

Using the binomial expansion:

$$\begin{aligned}(1+x)^{\frac{1}{2}} &= 1+\frac{1}{2}x+\frac{\frac{1}{2}\times\frac{-1}{2}}{2!}x^2+\frac{\frac{1}{2}\times\frac{-1}{2}\times\frac{-3}{2}}{3!}x^3+\ldots\\ &= 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+\ldots\end{aligned}$$

we can replace $x$ by $x^2$ to get

$$(1+x^2)^{\frac{1}{2}}=1+\frac{x^2}{2}-\frac{x^4}{8}+\frac{x^6}{16}+\ldots$$

and we can replace $x$ by $-x^2$ to get

$$(1-x^2)^{\frac{1}{2}}=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\ldots$$

Student view

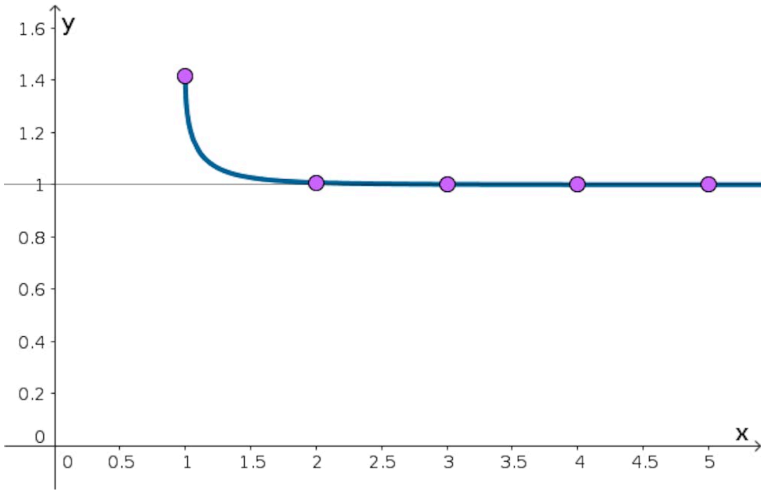
Using these expansions:

$$(1+x^2)^{\frac{1}{2}}-(1-x^2)^{\frac{1}{2}}=\left(1+\frac{x^2}{2}-\frac{x^4}{8}+\frac{x^6}{16}+\ldots\right)-\left(1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\ldots\right)$$
$$=x^2+\frac{x^6}{8}+\ldots$$

So continuing the limit calculation:

$$\begin{aligned}\lim_{n\to\infty} n\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right) &= \lim_{x\to 0^+}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2}\\ &= \lim_{x\to 0^+}\frac{x^2+\frac{x^6}{8}+\ldots}{x^2}\\ &= \lim_{x\to 0^+} 1+\frac{x^4}{8}+\ldots\\ &= 1\end{aligned}$$

The diagram below illustrates the convergence of this sequence.




More information

**Question 3**

Difficulty: ★★★

By considering the first three non-zero terms of the Maclaurin expansion of $\sin x$, find an approximation for $\int_0^1 \sin x^2 \, dx$.

Give your answer as a rational number in a fully simplified form or as a decimal rounded to three significant figures.

2867/9240 ✓

### Accepted answers

2867/9240, 0.31, 0.310, 0,310, 0.31

### Explanation

Using the known expansion of $\sin x$ given in the Formula booklet and substituting $x^2 = x$, we find

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$$\sin x^2 \approx x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120}$$

Then, integrating the series expansion term by term, we obtain

$$\begin{aligned}\int_0^1 \sin x^2 \, \mathrm{d}x &\approx \int_0^1 x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} \, \mathrm{d}x \\ &= \left[\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320}\right]_0^1 \\ &= \left(\frac{1}{3} - \frac{1}{42} + \frac{1}{1320}\right) - 0 \\ &= \frac{2867}{9240} \approx 0.310\end{aligned}$$

5. Calculus / 5.19 Maclaurin series expansions

# Expansion from a differential equation

In subtopic 5.18 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27273/) you explored various methods for solving specific types of differential equations, or finding approximate solutions when an exact solution is not available. The Taylor and Maclaurin series provide another numerical method for these approximations.

The basic idea is to use the formulae for these series from section 5.19.4 (/study/app/math-aa-hl/sid-134-cid-761926/book/taylor-and-maclaurin-series-id-27285/):

✓ **Important**

If a function can be differentiated infinitely many times at $x = a$, the Taylor series centred at $x = a$ is a power series of the form:

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Maclaurin series is a Taylor series centred at $x = 0$ . It is a power series of the form:

$$f(0) + (x)f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

When given a first-order differential equation and a starting point $(x, y)$, you can use the starting point and the evaluation of all of the necessary derivatives to approximate a solution. More terms yield a more accurate approximation.

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## Making connections

A quick review of implicit differentiation from section 5.14.1 (/study/app/math-aa-hl/sid-134-cid-761926/book/implicit-differentiation-id-26502/) may be of use in this section.

# Example 1

★★★

A solution curve of the differential equation $y' = x - y^2$ passes through the point $(0, 1)$. Find the first four terms of the Maclaurin series corresponding to this solution. Hence, using these terms, find an approximate value of $y(1)$.

The Maclaurin series can be rewritten as:

$$f(0) + (x)f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f^3(0) + \ldots = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y^3(0)}{6}x^3 + \ldots$$

This provides the first four terms (function plus three derivatives) and converts the format to match the problem statement.

Although the function $y$ is not provided, the function value is $y(0) = 1$ .

The first derivative is given and can be evaluated. $y'(0) = (0) - (1)^2 = -1$

The second derivative can be found implicitly then evaluated.

$$y''(x) = (y')' = (x - y^2)' = 1 - 2yy'$$

$$y''(0) = 1 - 2y(0)y'(0) = 1 - 2(1)(-1) = 3$$

Likewise, the third derivative can be determined and evaluated.

$$y'''(x) = (y'')' = (1 - 2yy')' = -2y'y' - 2yy''$$

$$y'''(0) = -2y(0)'y(0)' - 2y(0)y(0)'' = -2(-1)(-1) - 2(1)(3) = -8$$

Building the Maclaurin series yields:

Student view

$$y(x) \approx y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y^3(0)}{6}x^3 = 1 - x + \frac{3}{2}x^2 - \frac{8}{6}x^3$$

The approximation would be:

$$y(1) \approx 1 - (1) + \frac{3}{2}(1)^2 - \frac{8}{6}(1)^3 = \frac{1}{6}$$

# Example 2

★★★

A solution curve of the differential equation $y' = x - y^2$ passes through the point $(1, 0)$. Find the first four terms of the Maclaurin series corresponding to this solution, and using these terms, find an approximate value of $y(2)$.

Unlike **Example 1**, this Taylor series is not centred at $x = 0$, so it is not a Maclaurin series. It is a Taylor series centred at $x = 1$. The solution method, however, is quite similar.

The Taylor series can be rewritten as:

$$f(0) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f^3(a) + \ldots$$
$$= y(a) + y'(a)(x-a) + \frac{y''(a)}{2}(x-a)^2 + \frac{y^3(a)}{6}(x-a)^3 + \ldots$$

The function value is $y(1) = 0$.

The first derivative is given and can be evaluated: $y'(1) = (1) - (0)^2 = 1$

The second derivative can be determined.

$$y''(x) = (y')' = (x - y^2)' = 1 - 2yy'$$

$$y''(1) = 1 - y(1)y'(1) = 1 - 2(0)(1) = 1$$

The third derivative can be determined.

$$y'''(x) = (y'')' = (1 - 2yy')' = -2y'y' - 2yy''$$

$$y'''(1) = -2y(1)'y(1)' - 2y(1)y(1)'' = -2(1)(1) - 2(0)(1) = -2$$

Building the Taylor series yields:

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$$\begin{aligned} y(x) &\approx y(1) + y'(1)(x-1) + \frac{y''(1)}{2}(x-1)^2 + \frac{y^3(1)}{6}(x-1)^3 \\ &= (x-1) + \frac{1}{2}(x-1)^2 - \frac{2}{6}(x-1)^3 \end{aligned}$$

The approximation would be:

$$y(2) \approx (2-1) + \frac{1}{2}(2-1)^2 - \frac{1}{3}(2-1)^3 = \frac{7}{6}$$

With the following applet, you can see that for **Example 1** the estimate is very far from the solution, even if you use more terms for the expansion. For **Example 2**, the estimate is better. With the applet, you can see the actual solution curve to the differential equation $y' = x - y^2$, which is the equation considered in both examples. You can move the blue point to change the initial value and with the slider you can change the order of the approximating polynomial. In these examples, you were asked to find the first four terms, which means you were asked to find third-degree polynomials. These correspond to $n = 3$ on the applet.

**Interactive 1.** Actual Solution Curve for a Differential Equation.

More information for interactive 1

This interactive is a graph that allows the user to understand the expansion from a differential equation.

A graph is displayed on the screen with an xy axes. The graph ranges from 0 to 4 on x axis and $-1.5$ to 2 on y axis. Two curves originate from a purple movable point and are projected onto the graph: one is a blue curve, and the other is a red curve. On the top of the graph there is a horizontal slider n, which is used to change the order of the approximating polynomial from 1 to 6. The graph has the actual solution curve to the differential equation $y' = x - y2$. The blue curve represents a particular solution to the

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differential equation $y' = x - y$, while the red curve represents a different particular solution corresponding to a different initial condition. Users will move the purple point to change the initial value and will use the slider to change the order of the approximating polynomial.

For example, when the users want to find the first four terms, means to find third-degree polynomials they can slide the value of $n = 3$ on the applet.

When Users set the blue dot at 0.25 (approximately) and $n = 1$ , they will notice that the red curve is a straight line passing through 0.75 (approximately) on the x axis and blue curve goes upwards but with the blue dot at 0.25 (approximately) and $n = 3$, the red line becomes a curve passing through 0.75 (approximately) on the x axis.

By changing the position of blue dot and the order n users get a better understanding of the concept of expansion from a differential equation.

# 3 section questions

**Question 1**

Difficulty: ★★☆

A solution curve of the differential equation $y' = x + y^3$ passes through the point $(1, 0)$.

The Taylor expansion of this solution centred at $x = 1$ is $y(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + \ldots$

Find the exact value of $a_4$.

1/4

### Accepted answers

1/4, 0.25, 0,25

### Explanation

To find the Taylor expansion, we need the value and the first four derivatives at $x = 1$. Following the method outlined in the text:

$y(1) = 0$ is given in the question.

$y' = x + y^3$, so $y'(1) = 1 + 0^3 = 1$

$y'' = (x + y^3)' = 1 + 3y^2y'$, so $y''(1) = 1 + 3 \times 0^2 \times 1 = 1$

$y''' = (y'')' = 6yy'y' + 3y^2y''$, so $y'''(1) = 6 \times 0 \times 1 \times 1 + 3 \times 0^2 \times 1 = 0$

$y'''' = (y''')' = 6(y')^3 + 12yy'y'' + 6yy'y'' + 3y^2y'''$

so, $y''''(1) = 6 \times 1^3 + 18 \times 0 \times 1 \times 1 + 3 \times 0^2 \times 0 = 6$

Hence, $a_4 = \dfrac{y''''(1)}{4!} = \dfrac{6}{24} = 0.25$

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**Question 2**

Difficulty:

★★☆

A solution curve of the differential equation $y' = x + y^3$ passes through the point $(0, 1)$.

The Maclaurin expansion of this solution is $y(x) = a_0 + a_1x + a_2x^2 + \ldots$

Find the exact value of $a_3$.

3

### Accepted answers

3

### Explanation

To find the Taylor expansion, we need the value and the first three derivatives at $x = 0$. Following the method outlined in the text:

$y(0) = 1$ is given in the question.

$y' = x + y^3$, so $y'(0) = 0 + 1^3 = 1$

$y'' = (x + y^3)' = 1 + 3y^2y'$, so $y''(0) = 1 + 3 \times 1^2 \times 1 = 4$

$y''' = (y'')' = 6yy'y' + 3y^2y''$, so $y'''(0) = 6 \times 1 \times 1 \times 1 + 3 \times 1^2 \times 4 = 18$

Hence, $a_3 = \frac{y'''(0)}{3!} = \frac{18}{6} = 3$

**Question 3**

Difficulty:

★★☆

A solution curve of the differential equation $y' = x + y^3$ passes through the point $(1, 1)$.

The Taylor expansion of this solution centred at $x = 1$ is $y(x) = a_0 + a_1(x - 1) + a_2(x - 1)^2 + \ldots$

Find the exact value of $a_3$.

15/2

### Accepted answers

15/2, 7.5, 7,5

### Explanation

To find the Taylor expansion, we need the value and the first three derivatives at $x = 1$. Following the method outlined in the text:

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$y(1) = 1$ is given in the question.

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$y' = x + y^3$, so $y'(1) = 1 + 1^3 = 2$

$y'' = (x + y^3)' = 1 + 3y^2y'$, so $y''(1) = 1 + 3 \times 1^2 \times 2 = 7$

$y''' = (y'')' = 6yy'y' + 3y^2y''$, so $y'''(1) = 6 \times 1 \times 2 \times 2 + 3 \times 1^2 \times 7 = 45$

Hence, $a_3 = \frac{y'''(1)}{3!} = \frac{45}{6} = 7.5$

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# Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-27290/print/) Assign

## What you should know

By the end of this subtopic you should be able to:

- represent a function using Maclaurin series

$$f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$$

- find Maclaurin series and use these for approximation for:
  - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x$
  - $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$ for $|x| < 1$
  - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ for all $x$
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$ for all $x$
  - $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$ for $|x| \leqslant 1$

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- $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$
  for $-1 < x \leqslant 1$

- use Maclaurin series of the special functions to find other expansions using substitution, multiplication, addition, subtraction, division, composition, differentiation and integration
- use a Maclaurin series to obtain:
  - approximations of definite integrals
  - limits of the indeterminate form $\frac{0}{0}$
  - approximate solutions to differential equations.

5. Calculus / 5.19 Maclaurin series expansions

# Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-27291/print/) Assign

Use the applet below to investigate the accuracy of a Taylor polynomial approximation.

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/expansion-from-a-differential-equation-id-27289/print/) Assign

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Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

**Interactive 1.** Investigating Taylor and Maclaurin Series Approximation.

More information for interactive 1

This interactive allows the user to investigate the accuracy of a Taylor polynomial approximation.

The screen displays a graph of the xy axes, with x-axis ranging from -6 to 2 and y-axis ranging from -3 to 6. A blue curve represents the original function $f(x) = e^x$ mentioned in a box on the top left of the graph which can be altered by the user and the red line represents the Taylor polynomial $p_n(x) = e * \frac{(x-1)^0}{0!} + e * \frac{(x-1)}{1!}$ approximation at evaluation point $x$. On the left side of the graph are present 2 sliders: $a$ $(-5 \text{ to } 5)$ represents the center point and $n$ $(1 \text{ to } 10)$ represents the degree. In the given interactive users can manually enter the function $f(x)$ and find the Taylor polynomial. At the left bottom, users can manually enter the value of $x$ for which they want to calculate the error and accordingly the error term $R\_n(x)$ will be calculated and displayed. The error term $Rn(x)$ is defined as the difference between the actual function value and the Taylor polynomial approximation.

For example, at $f\,x = e^x$, $a = 1$, $n = 1$ and $x = 1.2$

$f(1.2) = e\text{^}1.2) \approx 3.32011692$

$P1_{(1.2)} = e + e(1.2 - 1)$

$= e + e(0.2)$

$= e(1 + 0.2)$

$= 1.2e$

$\approx 1.2 * 2.71828 \approx 3.261936$

$R1_{(1.2)} = f(1.2) - P1_{(1.2)}$

$\approx 3.32011692 - 3.261936$

$\approx 0.0581 8092$

Using different values of $a$, $n$ and $x$ and different functions users will get a better understanding of the concept.

- Enter $f(x) = \ln x$ as the function.
- Move the slider for $a$ to change the centre point. How does changing the centre point affect the graph of the Taylor polynomial? Why can you set a centre point outside of the domain of the function you are approximating? Is this a good idea?
- Move the slider for $n$ to change the order of the polynomial. How does adding and subtracting terms (changing $n$) affect the shape, end behaviour and accuracy of the Taylor polynomial?
- Change the $x$-value. How do this and the centre point relate to each other with respect to accuracy?
- Try other functions, for example, $\sin x$,$\cos x$ and $e^x$.

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