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5. Calculus / 5.17 Qualitative and analytical techniques for coupled systems

The big picture

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27938/print/) Assign

In subtopic 5.16 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27933/), Euler's method was expanded to numerically approximate a solution to coupled systems. In this section, analytical and qualitative techniques will be applied to a special case of coupled differential equations, that is, a linear system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$$

Consider the system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5x + 12y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -15x$$

The figure below shows the phase portrait of the system with two proposed initial conditions,

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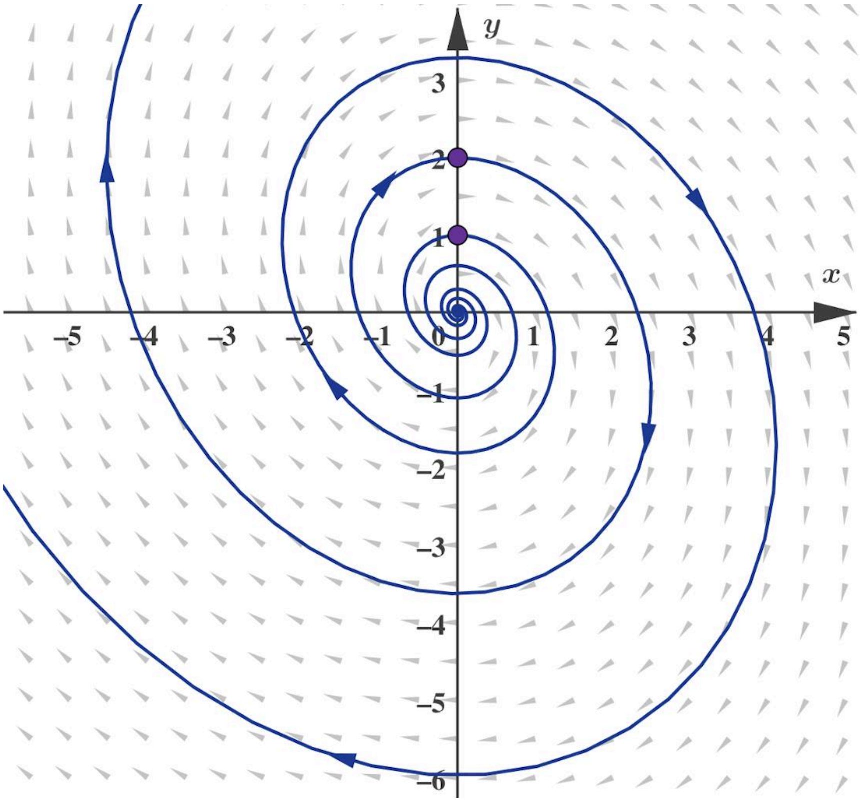

More information

The image is a graph displaying a phase portrait of a dynamical system with two initial conditions marked as purple points along the vertical y-axis. The system is visualized with paths that spiral outward from the origin without crossing each other, reflecting the behavior described in the accompanying text. The spiral paths are blue, while the underlying vector field is represented by gray arrows. The x-axis ranges from -5 to 5, and the y-axis ranges from -6 to 3, both labeled with numerical markings. The paths demonstrate outward spiraling patterns.

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Notice how the solutions spiral out from the origin, yet never cross.

Applying techniques from subtopic 1.15 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27436/), eigenvaluesand eigenvectorswill help you find analytic solutions as well as classify the stability and geometric nature of equilibrium points. Phase portraits, like slope fields from subtopic 5.15 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27928/), will allow you to sketch proposed trajectories.

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Concept

Throughout this subtopic, think about **relationships** between variables in real life and how they can be **modelled**. The previous subtopic discussed predator—prey relationships. Here are some further examples to consider:

- How would this apply to two competing businesses in the same region? Could an increase in business by a competitor hurt your business? What about complementary businesses?
- What is the relationship between a strong or weak economy and student applications for college?
- Can you find relationships in history that could be modelled in this manner?
- Does the decline in one nation affect the future of another?

To paraphrase George Box, 'All models are wrong, but some are useful.' How can you use these **models** to your benefit?

5. Calculus / 5.17 Qualitative and analytical techniques for coupled systems

Analytical techniques

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/analytical-techniques-id-27939/print/) Assign

In the study of derivatives, you learned a variety of rules that allow you to find the derivative of nearly any function. In many cases, there are multiple ways of finding an answer. Although numerical techniques are available, analytical solutions are plentiful. In transitioning to integration, the maths gets a little harder. In many cases, the integral is fairly straightforward to find, especially using rules such as the chain rule, product rule and quotient rule, but there are many integrals that are beyond the abilities of analytical techniques and require numerical approximations. With differential equations, it is even worse. A first-order differential equation can sometimes be solved analytically, such as through separation of variables as covered in subtopic 5.14.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/exact-solution-separable-equations-id-27924/), but systems of differential equations can be solved analytically in special cases only. This subsection will cover one of these special cases and the technique for solution.

Of particular interest are systems of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$$

Student view

where a, b, c, and d are constants.

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The analytical technique for finding the general solution builds on eigenvalues and eigenvectors learned in subtopic 1.15 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27436/). The basic steps are listed below.

Important

To find the general solution to a system of coupled differential equations of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$$

- Determine the characteristic polynomial of the system and solve for the eigenvalues (λ_1, λ_2) .
- Determine appropriate eigenvectors for the eigenvalues $\left(\overrightarrow{p_1}, \overrightarrow{p_2}\right)$.
- Substitute the eigenvalues and eigenvectors into the general solution equation $\overrightarrow{x} = Ae^{\lambda_1 t}\overrightarrow{p_1} + Be^{\lambda_2 t}\overrightarrow{p_2}$.
- If given, apply initial conditions to solve for A and B.
- If required, evaluate for given t.

Exam tip

The exact solution for coupled linear differential equations is in the formula booklet in the following form.

$$\boldsymbol{x} = A\mathrm{e}^{\lambda_1 t}\boldsymbol{p}_1 + B\mathrm{e}^{\lambda_2 t}\boldsymbol{p}_2$$

The first two steps should be reviewed from subtopic 1.15 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27436/). Substitution into the formula is also fairly straightforward. Solving for constants A and B involves evaluating each equation for the initial time and setting them equal to the appropriate initial values. This results in a system with two equations and two unknowns. These results should be very close to the results from Euler's method in subtopic 5.16 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27933/).

Exam tip

Calculation of exact solutions is only required for the case of real distinct eigenvalues.

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Example 1

★★☆

a) Find the solution to the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + 2y$$

with initial condition of $x_0 = 1$, $y_0 = 2$. Evaluate at $t = 1.0$.

$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x$ $\frac{\mathrm{d}y}{\mathrm{d}t} = -x + 2y$	$\begin{pmatrix} -3 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$	**Rewrite system as a matrix.**
Following the steps outlined above: Determine the eigenvalues $\det(B - \lambda I) = (-3 - \lambda)(2 - \lambda) = 0$ $\lambda = -3, 2$		Find determinant of $(B - \lambda I)$.
Determine the eigenvectors $\lambda_1 = -3$ $-3x = -3x$ $-x + 2y = -3y$ $x = 5y$ $\overrightarrow{p_1} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$	$\lambda_2 = 2$ $-3x = 2x$ $-x + 2y = 2y$ $x = 0$ $\overrightarrow{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	Substitute λ to find eigenvectors.
Determine the general solution $\overrightarrow{x} = A\mathrm{e}^{-3t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + B\mathrm{e}^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$		Apply eigenvalues and eigenvectors.

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$\frac{dx}{dt} = -3x$ $\frac{dy}{dt} = -x + 2y$	$\begin{pmatrix} -3 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$	**Rewrite system as a matrix.**
Apply initial conditions $5A = 1$ $A + B = 2$ $A = \frac{1}{5}, B = \frac{9}{5}$ $\vec{x} = \frac{1}{5}e^{-3t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} + \frac{9}{5}e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$		Apply initial conditions $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
Rewrite without vector notation $x(t) = e^{-3t}$ $y(t) = \frac{1}{5}e^{-3t} + \frac{9}{5}e^{2t}$		
Evaluate for t $\vec{x}(1) = \frac{1}{5}e^{-3(1)}\begin{pmatrix} 5 \\ 1 \end{pmatrix} + \frac{9}{5}e^{2(1)}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} e^{-3} \\ e^{-3}/5 + 9e^{2}/5 \end{pmatrix} \approx \begin{pmatrix} 0.0498 \\ 13.310 \end{pmatrix}$		Solve for $t = 1.0$.

b) Compute the approximate solution numerically using a step size of $\Delta t = 0.002$.

The table below shows the calculation using a spreadsheet. Note that you can also get this approximation with your calculator, although it will take a long time calculating 500 iterations. On exams you will not get a long calculation like this.

k	t	x_k	y_k	$f_1(x_k, y_k, t_k)$	$f_2(x_k, y_k, t_k)$
0	0	1	2	-3	3
1	0.002	0.994	2.006	-2.982	3.018
2	0.004	0.988036	2.012036	-2.96411	3.036036
3	0.006	0.982108	2.018108	-2.94632	3.054108
4	0.008	0.976215	2.024216	-2.92865	3.072217

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k	t	x_k	y_k	$f_1(x_k, y_k, t_k)$	$f_2(x_k, y_k, t_k)$
5	0.01	0.970358	2.030361	−2.91107	3.090364
⋮	⋮	⋮	⋮	⋮	⋮
499	0.998	0.049637	13.2045	−0.14891	26.35936
500	1	0.049339	13.25721	−0.14802	26.46509

At $t = 1.0$, $\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} 0.0493 \\ 13.257 \end{pmatrix}$

Comparing the analytical results with those from Euler's method, $\begin{pmatrix} 0.0498 \\ 13.310 \end{pmatrix} \approx \begin{pmatrix} 0.0493 \\ 13.257 \end{pmatrix}$. It is comforting to know that the numerical approximation is roughly the same as the analytical answer, with the x-value being within 0.6% and the y-value being within 0.4% of the actual values. Typically, all answers are given to 3 significant figures. The y-components in this example went to 5 significant figures to better determine the error.

Example 2

★★☆

Find the general solution to the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x + y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x - 3y$$

with initial condition of $x_0 = 1, y_0 = 2$. Evaluate at $t = 2.0$. Then compute the approximate solution numerically using a step size of $\Delta t = 0.004$.

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$\frac{dx}{dt} = -4x + y$ $\frac{dy}{dt} = 2x - 3y$	$\begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$	**Rewrite system as a matrix.**
Following the steps outlined above: Determine the eigenvalues $\det(B - \lambda I) = (-4-\lambda)(-3-\lambda) - 2 = 0$ $\lambda^2 + 7\lambda + 10 = 0$ $\lambda = -5, -2$		Find determinant of $(B - \lambda I)$.
Determine the eigenvectors $\lambda_1 = -5$ $-4x + y = -5x$ $y = -x$ $2x - 3y = -5y$ $2x = -2y$ $\vec{p_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\lambda_2 = -2$ $-4x + y = -2x$ $y = 2x$ $2x - 3y = -2y$ $2x = y$ $\vec{p_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$	Substitute (λI) to find eigenvectors.
Determine the general solution $\vec{x} = A\mathrm{e}^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + B\mathrm{e}^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$		Apply eigenvalues and eigenvectors.
Apply initial conditions $A + B = 1$ $-A + 2B = 2$ $A = 0, B = 1$ $\vec{x} = \mathrm{e}^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$		Apply initial conditions $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
Rewrite without vector notation $x(t) = \mathrm{e}^{-2t}$ $y(t) = 2\mathrm{e}^{-2t}$		
Evaluate for t $\vec{x}(2) = \mathrm{e}^{-2(2)}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} \mathrm{e}^{-4} \\ 2\mathrm{e}^{-4} \end{pmatrix} \approx \begin{pmatrix} 0.0183 \\ 0.0366 \end{pmatrix}$		Solve for $t = 2.0$.

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Approximation

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The table below shows the calculation using a spreadsheet. Note that you can also get this approximation with your calculator, although it will take a long time calculating 500 iterations. On exams you will not get a long calculation like this.

k	t	x_k	y_k	x'_k	y'_k
0	0	1	2	-2	-4
1	0.004	0.992	1.984	-1.984	-3.968
2	0.008	0.984064	1.968128	-1.96813	-3.93626
3	0.012	0.976191	1.952383	-1.95238	-3.90477
4	0.016	0.968382	1.936764	-1.93676	-3.87353
5	0.02	0.960635	1.92127	-1.92127	-3.84254
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
499	1.996	0.018169	0.036337	-0.03634	-0.07267
500	2	0.018024	0.036047	-0.03605	-0.07209

At $t = 2.0$, $\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} 0.0180 \\ 0.0360 \end{pmatrix}$

Example 3

★★☆

Find the general solution to the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x + y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3x + 2y$$

with initial condition of $x_0 = 0, y_0 = 1$. Evaluate at $t = 0.5$. Then compute the approximate solution numerically using a step size of $\Delta t = 0.001$.

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$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x + y$ $\frac{\mathrm{d}y}{\mathrm{d}t} = 3x + 2y$	$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$	**Rewrite system as a matrix.**
Following the steps outlined above: Determine the eigenvalues $\det(B - \lambda I) = (4-\lambda)(2-\lambda) - 3 = 0$ $\lambda^2 - 6\lambda + 5 = 0$ $\lambda = 5, 1$		Find determinant of $(B - \lambda I)$.
Determine the eigenvectors $\lambda_1 = 5$ $4x + y = 5x$ $y = x$ $3x + 2y = 5y$ $3x = 3y$ $\overrightarrow{p_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\lambda_2 = 1$ $4x + y = x$ $y = -3x$ $3x + 2y = y$ $3x = -1y$ $\overrightarrow{p_2} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$	Substitute (λI) to find eigenvectors.
Determine the general solution $\overrightarrow{x} = A\mathrm{e}^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\mathrm{e}^{t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$		Apply eigenvalues and eigenvectors.
Apply initial conditions $A + B = 0$ $A - 3B = 1$ $A = \frac{1}{4}, B = -\frac{1}{4}$ $\overrightarrow{x} = \frac{1}{4}\mathrm{e}^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{4}\mathrm{e}^{t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$		Apply initial conditions $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
Rewrite without vector notation $x(t) = \frac{1}{4}\mathrm{e}^{5t} - \frac{1}{4}\mathrm{e}^{t}$ $y(t) = \frac{1}{4}\mathrm{e}^{5t} + \frac{3}{4}\mathrm{e}^{t}$		
Evaluate for t $\overrightarrow{x}(0.5) = \frac{1}{4}\mathrm{e}^{5(0.5)}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{4}\mathrm{e}^{0.5}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ $\approx \begin{pmatrix} 2.63 \\ 4.28 \end{pmatrix}$		Solve for $t = 0.5$.

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Approximation

The table below shows the calculation using a spreadsheet. Note that you can also get this approximation with your calculator, although it will take a long time calculating 500 iterations. On exams you will not get a long calculation like this.

k	t	x_k	y_k	x'_k	y'_k
0	0	1	1	1	2
1	0.001	0.001	1.002	1.006	2.007
2	0.002	0.002006	1.004007	1.012031	2.014032
3	0.003	0.003018	1.006021	1.018093	2.021096
4	0.004	0.004036	1.008042	1.024187	2.028193
5	0.005	0.00506	1.01007	1.030312	2.035322
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
499	0.499	2.599987	4.246649	14.6466	16.29326
500	0.5	2.614633	4.262943	14.72148	16.36978

At $t = 0.5$, $\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} 2.61 \\ 4.26 \end{pmatrix}$

2 section questions

5. Calculus / 5.17 Qualitative and analytical techniques for coupled systems

Qualitative techniques

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In subtopic 5.14 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27922/), you learned how to analytically solve differential equations through separation of variables. In subtopic 5.15 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27928/), you saw how slope

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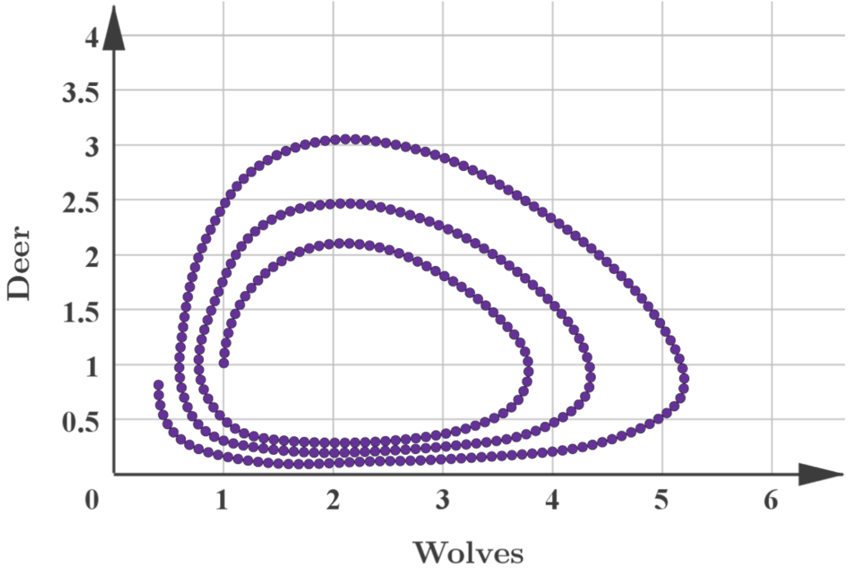
fields and isoclines could help you to visualise the equations and see how the solutions evolve over time.

With coupled systems, graphs are just as powerful. With three variables (x, y and t), **phase portraits** depict the path of the x and y variables over time, t.

Consider a system similar to the predator–prey model from **Example 3** in section 5.16.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/numerical-solutions-of-coupled-systems-id-27935/):

$$\frac{\mathrm{d}W}{\mathrm{d}t} = WD - W$$
$$\frac{\mathrm{d}D}{\mathrm{d}t} = 2D - WD$$
$$W(0) = 1,\ D(0) = 1$$

Using Euler's method with a step size of 0.05 produces the following scatter plot.



More information

This scatter plot depicts a phase portrait illustrating the population dynamics between deer and wolves. The X-axis represents the population of wolves, ranging from 0 to 6, while the Y-axis represents the population of deer, ranging from 0 to 4. Data points form concentric elliptical patterns, suggesting cyclic fluctuations in the populations of the two species. As the wolf population increases, the deer population exhibits a periodic decrease and increase, indicative of predator-prey interactions. The plot helps visualize how changes in each population can affect the other over time.

[Generated by AI]

Student view

An application on GeoGebra found at https://www.geogebra.org/m/utcMvuUy (https://www.geogebra.org/m/utcMvuUy)produces a phase portrait of:

Interactive 1. Phase Portraits and Euler's Method.

More information for interactive 1

This interactive graph displays a phase portrait representing a system of two differential equations, often used to model predator—prey dynamics.

The horizontal axis is labeled W, representing the wolf population, and the vertical axis is labeled D, representing the deer population. The graph ranges from 0 to 6 on the W-axis and 0 to 4 on the D-axis, defining the phase plane, where each point corresponds to a state of the system. A field of small gray arrows shows the direction field (vector field), indicating how the populations of wolves and deer are expected to change at each point in the plane.

A blue closed curve represents a solution trajectory of the system, showing how the populations evolve over time from a specific initial condition. This initial point is marked by a red dot and a surrounding red circle, and a small arrowhead on the curve indicates the direction of motion as time increases. The closed-loop nature of the trajectory suggests periodic behavior, where the populations oscillate in a repeating cycle. This is typical of predator—prey systems, where increases in one population lead to changes in the other in a feedback loop.

This graph also illustrates how solution curves behave globally and how systems can settle into stable limit cycles—cyclical patterns that repeat over time. Through interacting with this graph, users can explore how different numerical methods, such as Euler's method, approximate these curves. While the app shows a perfect cycle, methods like Euler's may produce spirals due to small numerical errors. Reducing the step size in Euler's method can improve accuracy and bring the approximation closer to the true trajectory.

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In both cases, the solution over time starts off in the same direction (up) and rotates clockwise around an apparent centre. Why does Euler's method produce a spiral while the application produces a closed circuit? The answer has to do with the small error introduced by the approximation used in Euler's method. If you were to reduce the step size, you would find the spiral gets tighter, eventually closing in on the circuit as the step size approaches 0.

As in section 5.17.1 (/study/app/math-ai-hl/sid-132-cid-761618/book/analytical-techniques-id-27939/), systems of particular interest are of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$$

where a, b, c and d are constants.

In subtopic 1.15 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27436/) you learned how to find eigenvalues and eigenvectors. Besides providing a method for finding an analytical solution, the eigenvectors also have graphical meaning. Consider the linear system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + 3y$$

The eigenvalues of this system are $\lambda_1 = 4$ and $\lambda_2 = 1$. The associated eigenvectors are $\overrightarrow{p_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\overrightarrow{p_2} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Can you verify using the techniques from the last section? Plotting the direction field for this system with the eigenvectors overlaid produces:

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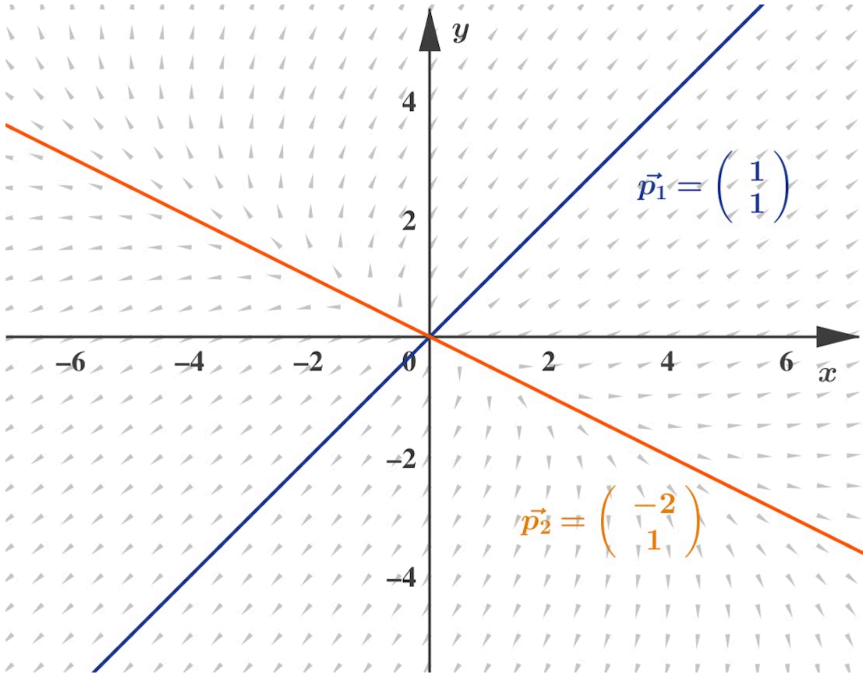

More information

The image is a graph displaying a direction field with eigenvectors overlaid. The X-axis and Y-axis intersect at the origin and range from -6 to 6 and -4 to 4, respectively. On this graph, two lines are drawn representing the eigenvectors.

The blue line represents the eigenvector (\overrightarrow{p_1} = \left(

1 1

\right)), pointing diagonally toward the top right quadrant.

The orange line represents the eigenvector (\overrightarrow{p_2} = \left(

−2 1

\right)), pointing diagonally toward the bottom right quadrant.

The direction field itself is depicted by small arrows spread across the graph, illustrating the vector field's behavior. The field suggests that trajectories starting on the eigenvector (\overrightarrow{p_1}) will continue in that direction, while trajectories not initially on an eigenvector will migrate towards (\overrightarrow{p_1}), associated with the larger eigenvalue (\lambda_1 = 4).

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Notice that any initial value along an eigenvector will continue along the eigenvector. Any initial value not on an eigenvector will over time migrate towards the eigenvector associated with the larger eigenvalue, in this case $\overrightarrow{p_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as it is associated with $\lambda_1 = 4$. This eigenvector is often referred to as the primary eigenvector. By evaluating the eigenvalues and eigenvectors, you can predict the solution curve and sketch a phase portrait for the coupled system.

Exam tip

Although some graphing calculators can draw direction fields, these applications are not available on exams. The direction fields on diagrams in this section are drawn to help understanding, but do not count on your calculator drawing it for you on exams.

The trajectories of the solution curves look differently for different systems. The behaviour depends on the eigenvalues. The following examples show these different situations.

Example 1

★☆☆

Consider the following system from **Example 1** in section 5.17.1 (/study/app/math-ai-hl/sid-132-cid-761618/book/analytical-techniques-id-27939/):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + 2y$$

with initial condition $x_0 = 1,\ y_0 = 2$.

Sketch a proposed trajectory.

From **Example 1** in **section 5.17.1**:

$$\lambda_1 = -3 \qquad \lambda_2 = 2$$
$$\overrightarrow{p_1} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \qquad \overrightarrow{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Direction field with proposed trajectory:

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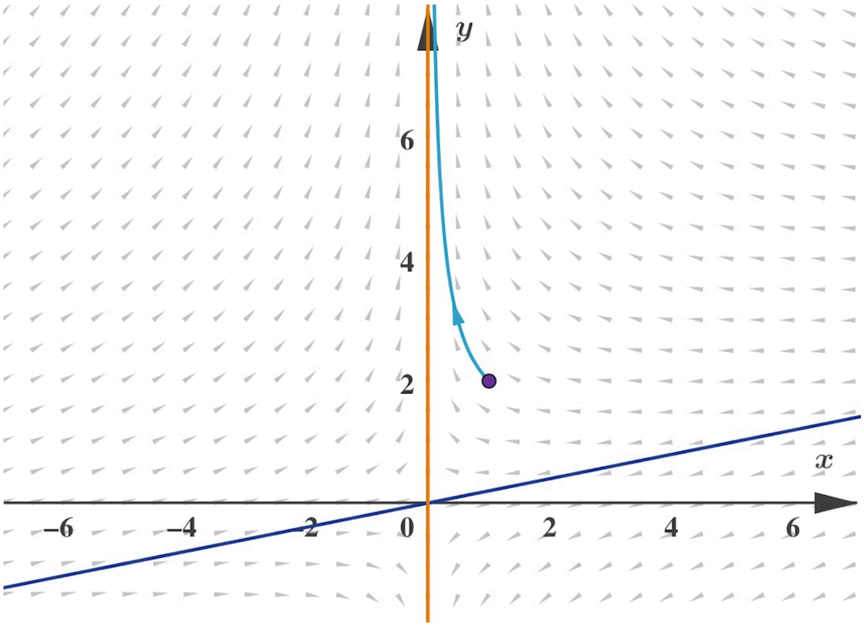


The proposed trajectory (pale blue) follows the direction arrows and approaches the eigenvector assocoated with the positive eigenvalue (orange) moving away from the origin. Also notice that if the initial value had been on the other eigenvector (dark blue), the trajectory would have tracked directly to the origin.

The origin constitutes an unstable equilibrium point . If the particle begins at the origin, it will not be driven off the point. If it starts just a little bit away from the point and not on the eigenvector associated with the negative eigenvalue, then it will travel away from the origin approaching the eigenvector associated with the positive eigenvalue. This is an example of a saddle.

The key indicator is that both eigenvalues are real and have opposite signs.

Note that on exams you will not see the direction field. However, using Euler method you can produce a sequence of points and you can use these points as guide to sketch a trajectory. You can find help on how to ask your calculator to show you these points in section 5.16.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/numerical-solutions-of-coupled-systems-id-27935/).

Example 2

★★☆

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Consider the following system from **Example 2** in section 5.17.1 (/study/app/math-ai-hl/sid-132-cid-761618/book/analytical-techniques-id-27939/):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x + y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x - 3y$$

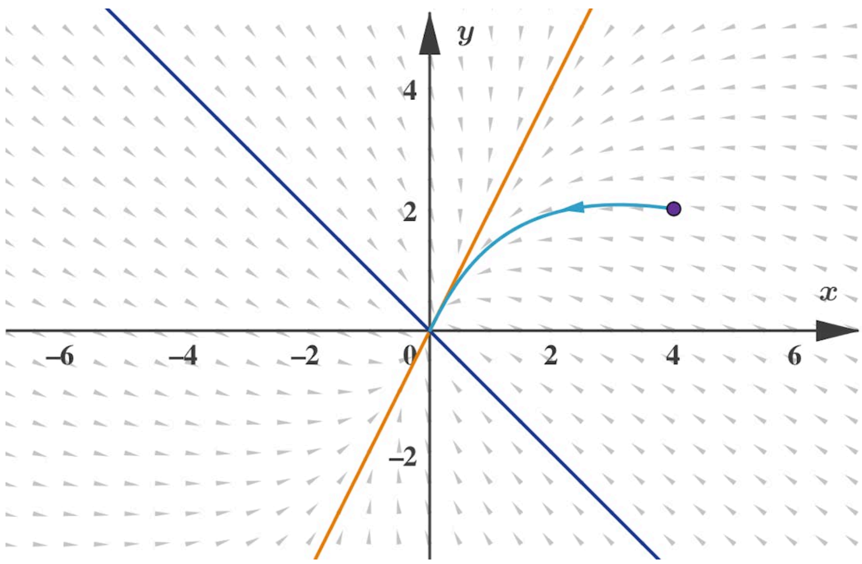
with initial condition $x_0 = 4, y_0 = 2$.

Sketch a proposed trajectory.

From **Example 2** in section 5.17.1 (/study/app/math-ai-hl/sid-132-cid-761618/book/analytical-techniques-id-27939/):

$$\lambda_1 = -5 \qquad \lambda_2 = -2$$
$$\overrightarrow{p_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \overrightarrow{p_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Direction field with proposed trajectory:



The proposed trajectory (pale blue) follows the direction arrows and approaches the eigenvector associated with the numerically larger eigenvalue (orange) moving towards the origin. Also notice that if the initial value had been on the other eigenvector (dark blue), the trajectory would have tracked directly to the origin.

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The origin constitutes a stable equilibrium point . If the particle begins at the origin, it will not be driven off the point. If it starts away from the origin, it will still travel towards the origin approaching the eigenvector associated with the eigenvalue closer to 0. This is an example of a sink.

The key indicator is that both eigenvalues are real and negative.

Note that on exams you will not see the direction field. However, using Euler method you can produce a sequence of points and you can use these points as guide to sketch a trajectory. You can find help on how to ask your calculator to show you these points in section 5.16.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/numerical-solutions-of-coupled-systems-id-27935/).

Example 3

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Consider the following system from **Example 3** in section 5.17.1 (/study/app/math-ai-hl/sid-132-cid-761618/book/analytical-techniques-id-27939/):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x + y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3x + 2y$$

with initial condition $x_0 = 0, y_0 = 1$.

Sketch a proposed trajectory.

From **Example 3** in section 5.17.1 (/study/app/math-ai-hl/sid-132-cid-761618/book/analytical-techniques-id-27939/):

$$\lambda_1 = 5 \qquad \lambda_2 = 1$$
$$\overrightarrow{p_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \overrightarrow{p_2} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Direction field with proposed trajectory:

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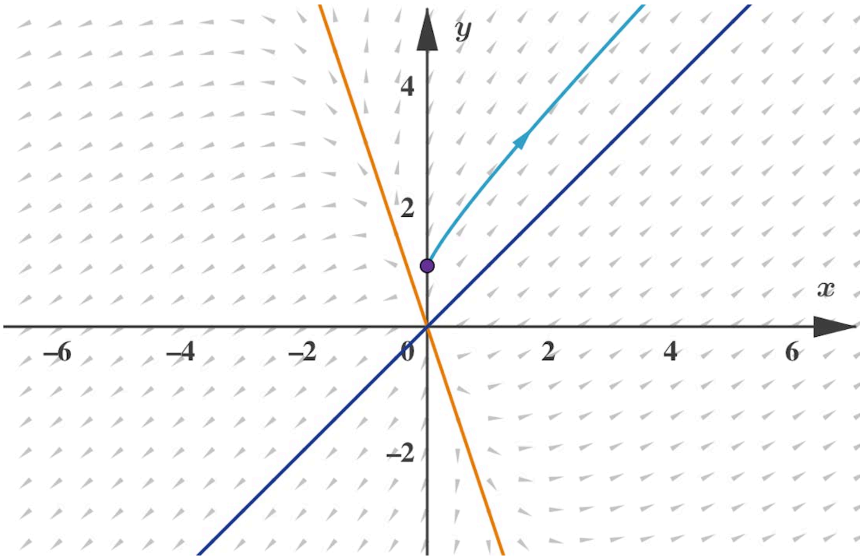

The proposed trajectory (pale blue) follows the direction arrows and approaches the direction of the eigenvector associated with the larger eigenvalue (dark blue) moving away from the origin. Also notice that if the initial value had been on the other eigenvector (orange), the trajectory would have tracked directly away from the origin.

The origin constitutes an unstable equilibrium point. If the particle begins at the origin, it will not be driven off the point. If it starts just a little bit away from the origin, it will travel away from the origin approaching the direction of the eigenvector associated with the larger eigenvalue. This is an example of a source .

The key indicator is that both eigenvalues are real and positive.

Note that on exams you will not see the direction field. However, using Euler method you can produce a sequence of points and you can use these points as guide to sketch a trajectory. You can find help on how to ask your calculator to show you these points in section 5.16.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/numerical-solutions-of-coupled-systems-id-27935/).

Example 4

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Consider the following system:

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$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x + y$$

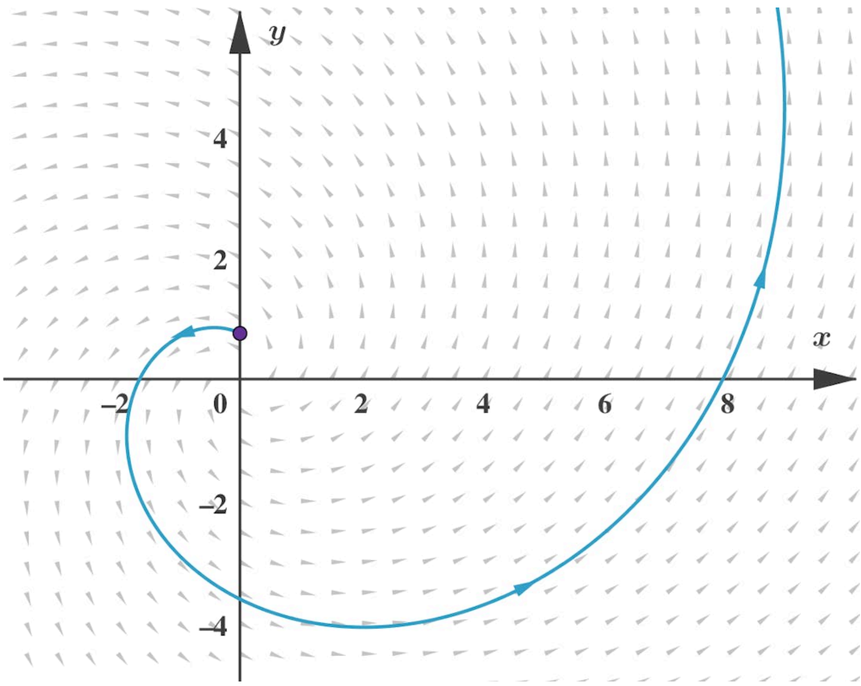
with initial condition $x_0 = 0, y_0 = 0.75.$

Sketch a proposed trajectory.

Eigenvalues

$$\det(B - \lambda I) = (1 - \lambda)(1 - \lambda) + 4 = 0$$
$$\lambda^2 - 2\lambda + 5 = 0$$
$$\lambda = 1 \pm 2i$$

Direction field with proposed trajectory:



The proposed trajectory follows the direction arrows and spirals outwards away from the origin.

The origin constitutes an unstable equilibrium point. If the particle begins at the origin, it will not be driven off the point. If it starts just a little bit away from the origin, it will spiral away from the origin. This is an example of a spiral source .

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The key indicator is that both eigenvalues are complex with positive real components.

Note that on exams you will not see the direction field. However, using Euler method you can produce a sequence of points and you can use these points as guide to sketch a trajectory. You can find help on how to ask your calculator to show you these points in section 5.16.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/numerical-solutions-of-coupled-systems-id-27935/).

Example 5

★★☆

Consider the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x - 4y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x + y$$

with initial condition $x_0 = 6, y_0 = 2$.

Sketch a proposed trajectory.

Eigenvalues

$$\det(B - \lambda I) = (-3 - \lambda)(1 - \lambda) + 8 = 0$$
$$\lambda^2 + 2\lambda + 5 = 0$$
$$\lambda = -1 \pm 2i$$

Direction field with proposed trajectory:

Student view

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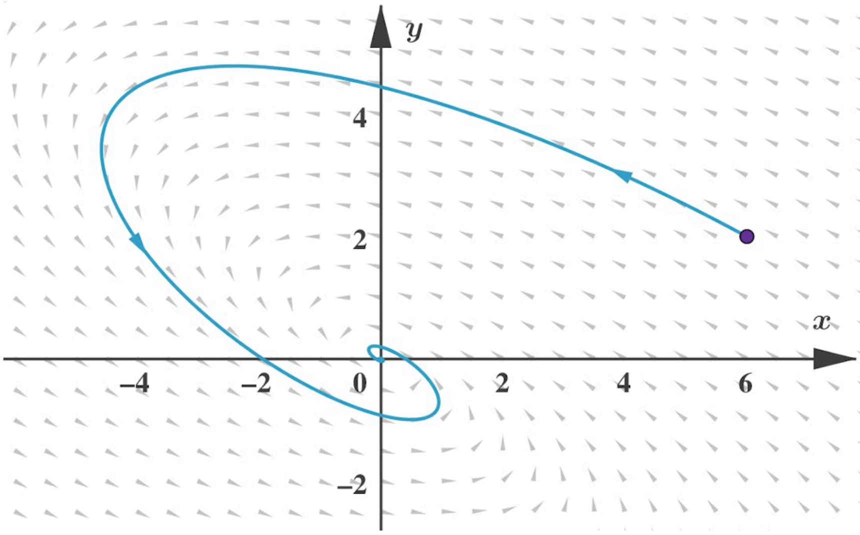

The proposed trajectory follows the direction arrows and spirals inward towards the origin.

The origin constitutes a stable equilibrium point. If the particle begins at the origin, it will not be driven off the point. If it starts away from the origin, it will spiral towards the origin. This is an example of a spiral sink .

The key indicator is that both eigenvalues are complex with negative real components.

Note that on exams you will not see the direction field. However, using Euler method you can produce a sequence of points and you can use these points as guide to sketch a trajectory. You can find help on how to ask your calculator to show you these points in section 5.16.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/numerical-solutions-of-coupled-systems-id-27935/).

Example 6

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Consider the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x - 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + y$$

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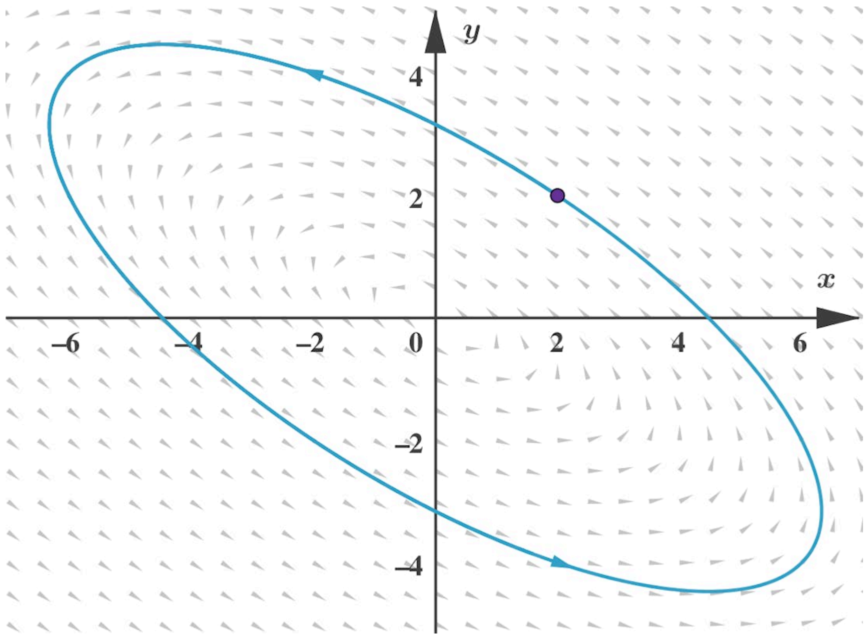
with initial condition $x_0 = 2, y_0 = 2$.

Sketch a proposed trajectory.

Eigenvalues

$$\begin{aligned}\det(B - \lambda I) = (-1-\lambda)(1-\lambda) + 2 &= 0 \\ \lambda^2 + 1 &= 0 \\ \lambda &= \pm i\end{aligned}$$

Direction field with proposed trajectory:



The proposed trajectory follows the direction arrows and follows an ellipse around the origin.

The origin constitutes a neutrally stable equilibrium point . If the particle begins at the origin, it will not be driven off the point. If it starts away from the origin, it will rotate around the origin without getting closer or farther away over the long-term. This is an example of a **centre,** and is the limit between a spiral sink and a spiral source.

The key indicator is that both eigenvalues are pure imaginary; they are complex roots with real components of zero.

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Note that on exams you will not see the direction field. However, using Euler method you can produce a sequence of points and you can use these points as guide to sketch a trajectory. You can find help on how to ask your calculator to show you these points in section 5.16.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/numerical-solutions-of-coupled-systems-id-27935/).

In this case you will need to be a bit careful, because the sequence of points you get from your calculator will not be perfectly on an ellipse. By evaluating the eigenvalues you will need to recognise that the trajectory is a cycle, even though the sequence of points you get from your calculator will have a slight spiralling trend either in or out.

Notice that in examples 4, 5 and 6 the trajectories spiraled anticlockwise. All of these examples were of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$$

with negative b. In general, if the system has distinct non-real eigenvalues, then the trajectories spiral clockwise if $b > 0$ and anticlockwise if $b < 0$. You can check this by considering the direction of the trajectory on the y-axis, for example at the point, where $x = 0$ and $y = 1$.

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=0,y=1} = a \times 0 + b \times 1 = b$$
$$\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{x=0,y=1} = c \times 0 + d \times 1 = d$$

If $b > 0$, then the direction points towards the first quadrant, indicating a clockwise spiral. If $b < 0$, then the direction points towards the third quadrant, indicating an anticlockwise spiral.

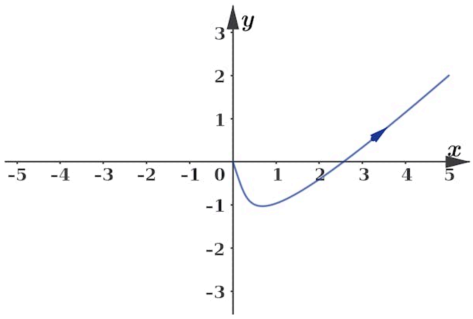
✓ **Important**

On the exam you will need to be able to analyse trajectories for coupled systems of the following form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$$

On exams, the system will have distinct non-zero eigenvalues. If the eigenvalues are:

Student view

Eigenvalues	Description
real positive	All solutions will move away from the origin. The origin is a source and is unstable.  More information The image is a graph with a curve displaying a trend in a Cartesian coordinate system. The horizontal axis, labeled 'x', ranges from -5 to 5, while the vertical axis, labeled 'y', ranges from -3 to 3. At the origin (0,0), the curve begins and moves downward to a minimum near (1, -1), then upward beyond (3, 2). An arrow along the curve at (2, 1) indicates the direction, illustrating that all solutions move away from the origin, which is a source and unstable. [Generated by AI]

Overview
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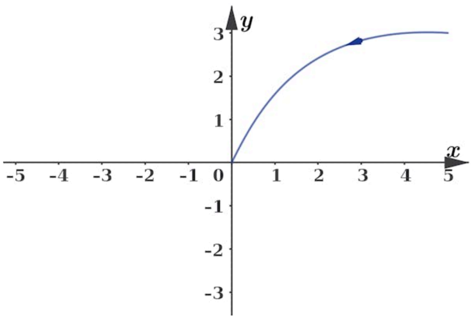
Eigenvalues	Description
real negative	All solutions will move towards the origin. The origin is a sink and is stable.  More information The image is a graph on an x-y coordinate plane. The x-axis ranges from -5 to 5 and the y-axis ranges from -3 to 3. There is a curve that starts near the point (1, 1) and approaches the point (0, 0) from the positive x-axis, indicating a trajectory that is moving towards the origin. The y-axis is labeled 'y' and the x-axis is labeled 'x'. The text before the image indicates that "All solutions will move towards the origin. The origin is a sink and is stable," suggesting that the origin is a point of attraction in this system. [Generated by AI]

Student view

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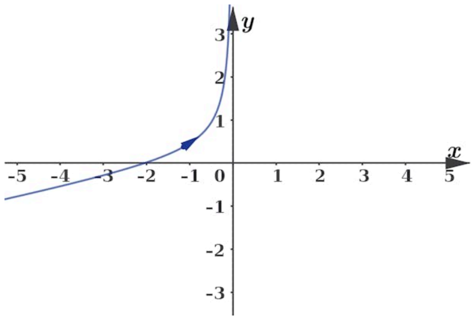
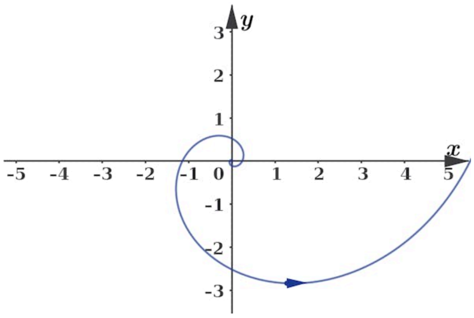
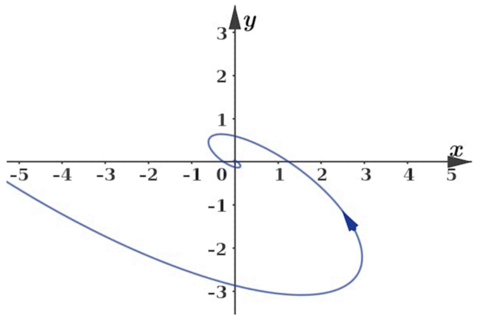
Eigenvalues	Description
real with different signs (one positive, one negative)	The origin is a saddle and is unstable.  More information The image is a graph featuring a two-dimensional coordinate system with labeled axes. The X-axis is marked with numbers ranging from -5 to 5, incrementing by 1. The Y-axis is marked from -3 to 3, also incrementing by 1. The origin is labeled "0," the X-axis is noted with an "x," and the Y-axis is noted with a "y." A curved arrow is drawn on the graph, starting from the left side of the negative X-axis at around -5 and progressing upward to the positive side of the Y-axis, just above "3." The arrow indicates a function behavior, showing an upward trend as it moves from left to right across the graph. [Generated by AI]

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Eigenvalues	Description
complex with positive real part	All solutions move away from the origin in a spiral. The origin is a spiral source and is unstable.  More information The image is a graph on a Cartesian plane with labeled axes. The x-axis ranges from -5 to 5, and the y-axis ranges from -3 to 3. The graph depicts a spiral path starting from the origin, moving counter-clockwise, and expanding outward. The starting point at the origin suggests instability, termed as a spiral source. The path exits the origin, loops around once in the second quadrant, decreases to around -1.5 on the y-axis, and crosses into the fourth quadrant before finishing in the positive x-direction past 5 in the fourth quadrant. The spiral's behavior illustrates solutions moving away from the origin in a spiral. [Generated by AI]

Student view

Eigenvalues	Description
complex with negative real part	All solutions move towards the origin in a spiral. The origin is a spiral sink and is stable.  More information The graph features an X-axis ranging from -5 to 5 and a Y-axis ranging from -3 to 3. The graph displays a spiral trajectory converging towards the origin (0,0), indicating it as a spiral sink. Starting from the point around (3, 1), the spiral loops inwards, approaching the origin in decreasing amplitude spirals along both axes. The Y-axis is labeled as 'y', and the X-axis as 'x', with an arrow on each axis indicating positive direction. The behavior illustrates stability at the origin, as all plotted solutions move inwards. [Generated by AI]

Overview
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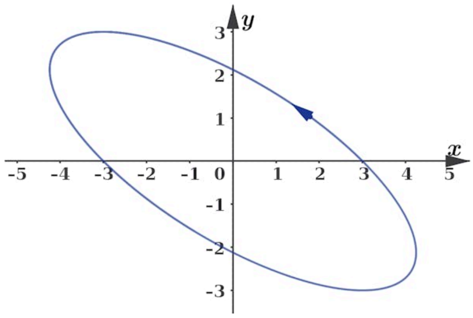
Eigenvalues	Description
complex with zero real part (pure imaginary)	All solutions will circle the origin in an elliptical orbit. The origin is a centre and is neutrally stable.  More information The image is a graph illustrating an elliptical orbit centered at the origin on an xy-plane. The x-axis ranges from -5 to 5, and the y-axis ranges from -3 to 3. The ellipse is tilted, with the longer axis oriented roughly diagonally from the bottom left to the top right. There's a blue arrow along the ellipse, indicating the direction of travel in a clockwise manner. The labeling of the x-axis and y-axis with 'x' and 'y' respectively is visible, and the origin is marked at the intersection of these axes. [Generated by AI]

If the eigenvalues are non-real, then

- the trajectories spiral clockwise if $b > 0$.
- the trajectories spiral anticlockwise if $b < 0$.

3 section questions

5. Calculus / 5.17 Qualitative and analytical techniques for coupled systems

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/checklist-id-27941/print/) Assign

What you should know

By the end of this subtopic given a coupled linear system of the form:

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$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

you should be able to:

- analytically determine the general solution through the use of eigenvalues and eigenvectors
- apply initial conditions to solve for the constants
- evaluate the solution at a given point in time
- qualitatively determine the future trajectory through the use of phase portraits
- through the use of phase portraits or eigenvalues, classify a solution as
  - stable
  - unstable
  - neutrally stable
  - sink
  - source
  - saddle
  - spiral sink
  - spiral source
  - centre
- for spiral trajectories, determine if they spiral clockwise or anticlockwise.

5. Calculus / 5.17 Qualitative and analytical techniques for coupled systems

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/investigation-id-27942/print/) Assign

Consider the system:

$$\frac{dx}{dt} = 5x + 9y$$
$$\frac{dy}{dt} = 2x + 2y$$

Confirm through the techniques you learned in subtopic 1.15 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27436/) and subtopic 5.17 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27938/) that the eigenvalues are $\lambda_1 = 8, \lambda_2 = -1$ and the

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eigenvectors are $\overrightarrow{p_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \overrightarrow{p_2} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. This is a saddle.

Interactive 1. Investigating Saddle Points in Phase Portraits.

More information for interactive 1

This interactive graph allows users to explore saddle points in phase portraits of systems of differential equations. The top half of the screen displays a vector field in the xy-plane, where the horizontal axis represents x and the vertical axis represents y, both ranging approximately from -5 to 5. Small gray arrows across the plane show the direction field, indicating how the system evolves at each point. A blue trajectory represents the solution curve for a particular initial condition, with an arrowhead showing the direction of motion as time t increases. A red dot marks the initial condition, which users can move to explore different solution paths. The system appears to evolve from the red dot, tracking through the direction field.

Below the graph, users can manually enter a system of differential equations. An optional slider labeled Integration Length controls how long the solution is traced out over time. For example, when you enter the equations $x'(t) = 5x + 9y$ and $y'(t) = 2x + 2y$, the system exhibits saddle point behavior at the origin. The blue trajectory changes direction sharply depending on its starting point:

- When placed near $(3, 1)$, the solution curves outward, tracking along the primary eigenvector direction.
- Near $(-3, 2)$, the trajectory initially follows the secondary eigenvector and heads toward the origin, which serves as an unstable equilibrium.

As users move the red dot around the plane, they can observe how different initial conditions yield different behaviors—either approaching or moving away from the saddle point. This helps illustrate the concept of eigenvectors and eigenvalues in dynamical systems and how they influence the system's local and global behavior.

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Open up the applet found above.

Enter in the system of equations. What do you see? Where does the trajectory go (look for the arrow)? Where did it come from before $t = 0$?

Move the green dots around. These indicate the initial conditions.Can you see the pattern in the direction field going diagonally from the upper-left to the lower-right? This equates to the secondary eigenvector $\overrightarrow{p_2} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

As you move the green dots around, where do the trajectories travel to? Are there 'lines' where they change drastically? You should see these lines identify with the eigenvectors.

Try placing a green dot near the point $(3, 1)$. You should see that the line makes a sharper turn as it approaches the point, and once it is on the point, the line tracks back towards the origin in negative time (it tracks directly out along the primary eigenvector).

Now try to place the other dot on point $(-3, 2)$. As you get really close, the trajectory appears to go straight down the secondary eigenvector. If you could place it exactly on the point, it would track all the way to the origin, an unstable equilibrium point for a saddle.

Try some of the other systems you saw in this chapter or make some up yourself. Can you find spiral sinks and sources? Can you find a centre?

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