Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Table of contents

Notebook

Glossary

Reading assistance

TOPIC 1
NUMBER AND ALGEBRA

0 (https://intercom.help/kognity)

SUBTOPIC 1.12
INTRODUCTION TO COMPLEX NUMBERS

Student view

Show all topics

Index

The big picture

Heron of Alexandria came across square roots of negative numbers in the 1st century while calculating volumes of pyramids.

These numbers make their next big appearance in mathematics in the 16th century with the Italian mathematician, Girolamo Cardano, who was working on finding solutions to cubic equations.

Roots of negative numbers made mathematicians uncomfortable and seemed to present no real-world application and were therefore called ‘imaginary numbers’. Interestingly, Cardano was able to use imaginary numbers to find real solutions to cubic equations. You can learn more about this by watching the video below.

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Imaginary numbers were developed further in the 18th and 19th centuries by Leonard Euler, Carl Fredrich Gauss, and William Rowan Hamilton.

Now complex numbers are used in many applied fields, for example electrical engineering, and make electricity, cell phones and computers possible in your daily life.

Concept

Initially, mathematicians had two problems with complex numbers: the inability to see their connection to the real world and lack of structures to represent these kinds numbers. As you work through this section consider how the various forms of representation of complex numbers enhance your understanding of their nature.

Theory of Knowledge

If imaginary numbers don't actually 'exist' in the real world, and complex numbers are formed using imaginary numbers, can complex numbers be said to exist?

On the flip side, if complex numbers do indeed exist in the real world (as demonstrated by their ability to be expressed as a Cartesian number), do they, by proxy, prove the existence of imaginary numbers?

This calls into question reason, the accuracy of deduction and role of perception in existence. A knowledge question that emerges is: 'Is the application of deductive logic sufficient to establish truth?'

1. Number and algebra / 1.12 Introduction to complex numbers

Complex solutions to quadratic equations

Important

A quadratic function is defined as $y = ax^2 + bx + c$, where a, b and c are called the coefficients. In this course, you will only work with quadratic equations that have real coefficients.

Student view

Overview (/study/ap ai- hl/sid- 132- cid- 761618/ov

There are several ways to solve quadratic equations, including factorising, completing the square, graphing and using the quadratic formula. In this subtopic we will focus on the quadratic formula and graphed solutions.

Important

If $P(x) = ax^2 + bx + c$ then the values of x such that $P(x) = 0$ are the solutions to the quadratic equation $P(x) = 0$.

A zero of a quadratic function $P(x)$ is a value of the variable that makes the function equal to zero. The roots of a quadratic equation are the solutions to $P(x) = 0$. The roots of the quadratic equation $P(x) = 0$ are the zeros of $P(x)$ and the x-intercepts of the graph of $y = P(x)$.

Solutions to a quadratic equation can be found using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula is given in the Prior Learning section of the IB formula booklet.

Example 1

★☆☆

Find the solutions to $y = x^2 - 5x + 6$ using the quadratic formula.

Steps	Explanation
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \frac{5 \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$ $= \frac{5 \pm \sqrt{25 - 24}}{2}$ $= \frac{5 \pm 1}{2}$ $x = 3$ or $x = 2$	In $y = x^2 - 5x + 6,\ a = 1,\ b = -5$ and $c = 6$.

Student view

Activity

Try to solve each of the following quadratic equations using the quadratic formula.

$$y = x^2 + \frac{1}{2}x - \frac{3}{2} \qquad y = x^2 - 2x + 1 \qquad y = x^2 + x + 2$$

$$y = x^2 - 4x + 13 \qquad y = x^2 + 6x + 5 \qquad y = x^2 - 6x + 9$$

Comment on any patterns that you notice.

Solutions to quadratic equations can also be found by graphing and locating the \(x\) -intercepts on the graph. Use graphing to find solutions to the quadratic above. How do your graphical results compare with the ones you found algebraically? Comment on any patterns that you notice.

Your findings in the activity should have alerted you to the fact that the nature of the roots of a quadratic equation depend on the value of $b^2 - 4ac$ in $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Important

There are three possible outcomes for the roots of a quadratic equation. Each can be predicted by looking at the value of the discriminant , $\Delta = b^2 - 4ac$, of the quadratic formula.

- If the discriminant is positive $(\Delta > 0)$ there are two real distinct roots. The graph has two different x-intercepts.
- If the discriminant is zero $(\Delta = 0)$ there is one repeated real root. The graph just touches the x-axis at one point.
- If the discriminant is negative $(\Delta < 0)$ there are no real roots. There are two complex roots. The graph does not have any x-intercepts.

Exam tip

The formula for the discriminant is in the IB formula booklet.

In order to solve quadratic equations where $\Delta < 0$, a new type of number needs to be defined.

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

✓ **Important**

The imaginary unit i is defined as $\mathrm{i} = \sqrt{-1}$.

A complex number is written in the form $z = a + b\mathrm{i}$ where $a, b \in \mathbb{R}$, and $\mathrm{i} = \sqrt{-1}$
.

You will learn more about the properties of complex numbers in the following sections. In this section, you will focus on writing out complex solutions for quadratic equations in $a + b\mathrm{i}$ form.

Example 2

★★☆

Write the solutions to $y = x^2 + 6x + 25$ in terms of i.

Steps	Explanation
$x = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(25)}}{2(1)}$ $= \frac{-6 \pm \sqrt{-64}}{2}$ $= \frac{-6 \pm \sqrt{64} \times \sqrt{-1}}{2}$ $= \frac{-6 \pm 8i}{2}$ $x = -3 + 4\mathrm{i}$ or $x = -3 - 4\mathrm{i}$	Use the quadratic formula. Use $\mathrm{i} = \sqrt{-1}$.

Example 3

★★☆

Write the solutions to $x^2 + 14x = -50$ in terms of i.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$x^2 + 14x + 50 = 0$	Before you can solve a quadratic equation you need to write it in the form $ax^2 + bx + c = 0$.
$x = \frac{-14 \pm \sqrt{(14)^2 - 4(1)(50)}}{2(1)}$ $= \frac{-14 \pm \sqrt{-4}}{2}$ $= \frac{-14 \pm \sqrt{4} \times \sqrt{-1}}{2}$ $x = -7 + \mathrm{i}$ or $x = -7 - \mathrm{i}$	

Activity

Explain why the complex solutions to quadratic equations always come in $a + b\mathrm{i}$ and $a - b\mathrm{i}$ pairs. Do you think that this is still the case if the coefficients of the quadratic are also complex numbers?

Important

The complex solutions to quadratic equations with real coefficients are always in the form $a + b\mathrm{i}$ and $a - b\mathrm{i}$. If $z = a + b\mathrm{i}$ then $a - b\mathrm{i}$ is denoted by z^* and called the complex conjugate of z.

This result is called the conjugate root theorem.

3 section questions

1. Number and algebra / 1.12 Introduction to complex numbers

Cartesian form of complex numbers

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Defining complex numbers allows you to solve equations that involve square roots of negative numbers.

✓ **Important**

The number i is defined as $\mathrm{i} = \sqrt{-1}$ or as the solution to $\mathrm{i}^2 = -1$.

Leonhard Euler, a Swiss mathematician who lived in the 1700s, was the first mathematician to use the notation $\mathrm{i} = \sqrt{-1}$.

Example 1

★☆☆

Simplify each of the following:

a) i^2

b) i^3

c) i^4

Question Part	Explanation
a)	$\mathrm{i}^2 = -1$
b)	$\mathrm{i}^3 = \mathrm{i}^2 \times \mathrm{i} = -1 \times \mathrm{i} = -\mathrm{i}$
c)	$\mathrm{i}^4 = \mathrm{i}^3 \times \mathrm{i} = -\mathrm{i} \times \mathrm{i} = -\mathrm{i}^2 = -(-1) = 1$

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Making connections

While surds are not defined for negative values in the square root such as $\mathrm{i} = \sqrt{-1}$, it is useful to remember that $\sqrt{-1}$ follows the same rules as surds.

Some useful rules for surds are:

$$\sqrt{a} \times \sqrt{a} = a$$

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$

$$b\sqrt{a} + c\sqrt{a} = (b + c)\sqrt{a}$$

where $a \geq 0$ and $b \geq 0$.

Important

The set of complex numbers is represented by $\mathbb{C}$ and in Cartesian form defined as $z = a + b\mathrm{i}$ where $a, b \in \mathbb{R}$ and $\mathrm{i} = \sqrt{-1}$.

The complex number, z, is made up of two parts:

1. The real part which is represented by: $\mathrm{Re}(z) = a$.
2. The imaginary part which is represented by: $\mathrm{Im}(z) = b$.

If z is purely real number , $\mathrm{Im}(z) = 0$ and $z = a$.

If z is purely imaginary number , $\mathrm{Re}(z) = 0$ and $z = b\mathrm{i}$.

Example 2

★☆☆

Given that $z = 3 - 2\mathrm{i}$, find $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$\mathrm{Re}\,(z) = 3$ $\mathrm{Im}\,(z) = -2$	Using: $z = a + b\mathrm{i}$ $\mathrm{Re}\,(z) = a$ $\mathrm{Im}\,(z) = b$

Example 3

★★☆

Consider the complex number $z = 3x\mathrm{i} + y^2 + 1$, where x and y are real.

Find the values of x and y if $\mathrm{Re}\,(z) = 2$ and $\mathrm{Im}\,(z) = 9$.

Steps	Explanation
$\mathrm{Re}\,(z) = y^2 + 1 = 2$ $\mathrm{Im}\,(z) = 3x = 9$	$z = 3x\mathrm{i} + y^2 + 1 = \underbrace{y^2 + 1}_{a} + \underbrace{3x}_{b}\,\mathrm{i}$
$3x = 9 \quad \Leftrightarrow \quad x = 3$	
$y^2 + 1 = 2 \quad \Leftrightarrow \quad y^2 = 1 \quad \Leftrightarrow \quad y = \pm\sqrt{1}$ $\Leftrightarrow \quad y = \pm 1$	

Activity

Use what you know about surds to completely simplify each of the following (the first one has been done for you):

$$\left(3 + 2\sqrt{5}\right) + \left(-1 + 3\sqrt{5}\right) = 2 + 5\sqrt{5}$$

Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

$$\left(4-\sqrt{2}\right)-\left(7+3\sqrt{2}\right)=$$

$$\left(2+3\sqrt{3}\right)\times\left(5-\sqrt{3}\right)=$$

$$\frac{-3+2\sqrt{6}}{1-\sqrt{6}}=$$

Hence, deduce the rules for addition, subtraction, multiplication, and division of complex numbers.

Important

Let $z_1 = a + b\mathrm{i}$ and $z_2 = c + d\mathrm{i}$, then the following rules apply to addition, subtraction, multiplication, division and equality.

Addition:

$$z_1 + z_2 = (a + c) + (b + d)\,\mathrm{i}$$

Subtraction:

$$z_1 - z_2 = (a - c) + (b - d)\,\mathrm{i}$$

Multiplication:

$$z_1 \times z_2 = (a + b\mathrm{i})\,(c + d\mathrm{i}) = ac + ad\mathrm{i} + bc\mathrm{i} + bd\mathrm{i}^2 = (ac - bd) + (ad + bc)\,\mathrm{i}$$

Division:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{a+b\mathrm{i}}{c+d\mathrm{i}} \times \frac{c-d\mathrm{i}}{c-d\mathrm{i}} = \frac{ac - ad\mathrm{i} + bc\mathrm{i} - bd\mathrm{i}^2}{c^2 - cd\mathrm{i} + cd\mathrm{i} - d^2\mathrm{i}^2}\\ &= \frac{(ac+bd) + (bc-ad)\,\mathrm{i}}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\mathrm{i}\end{aligned}$$

Equality:

$$z_1 = z_2 \text{ if and only if } a = c \text{ and } b = d$$

Exam tip

The rules for operations with complex numbers are not in the IB formula booklet but you do not need to memorise them for the exam since you work with complex numbers in the same way as surds.

Student
view

Section Student (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid- Assign

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Example 4

★★☆

Find the real numbers a and b if $(2-3\mathrm{i})(a+b\mathrm{i}) = 4+7\mathrm{i}$.

Steps	Explanation
$\begin{aligned}(2-3\mathrm{i})(a+b\mathrm{i}) &= 2a+2b\mathrm{i}-3a\mathrm{i}-3b\mathrm{i}^2\\ &= 2a+3b+(2b-3a)\mathrm{i}\end{aligned}$	Use the multiplication rule.
$2a+3b+(2b-3a)\mathrm{i} = 4+7\mathrm{i}$ $\begin{cases}2a+3b=4\\ -3a+2b=7\end{cases}$ $a = \frac{4-3b}{2}$ $-3\left(\frac{4-3b}{2}\right)+2b=7$ $\Leftrightarrow\ -6+\frac{13}{2}b=7$ $\Leftrightarrow\ \frac{13}{2}b=13$ $\Leftrightarrow\ b=2$ $a=\frac{4-3b}{2}=\frac{4-3(2)}{2}=-1$	Use equality. Solve the system of equations by using elimination or substitution.

Example 5

★★☆

Write $\frac{1-2\mathrm{i}}{1+3\mathrm{i}}$ in $a+b\mathrm{i}$ form.

Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

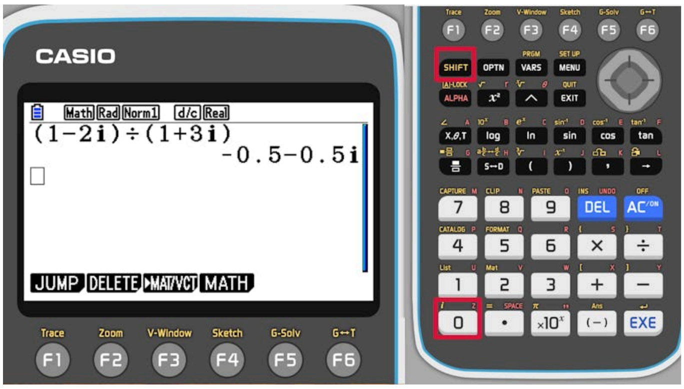
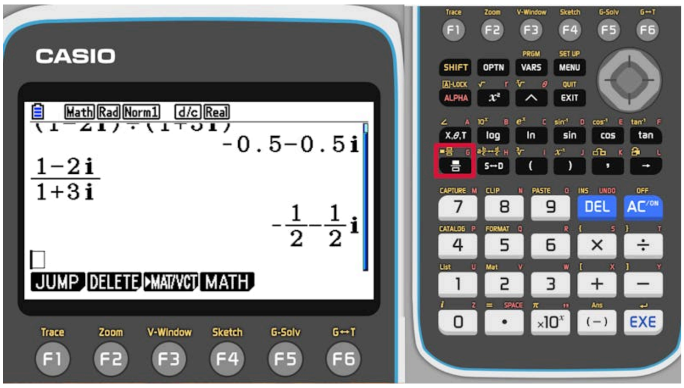
Steps	Explanation
$\frac{1-2\text{i}}{1+3\text{i}} = \frac{1-2\text{i}}{1+3\text{i}} \times \frac{1-3\text{i}}{1-3\text{i}}$ $= \frac{1-3\text{i}-2\text{i}+6\text{i}^2}{1+9}$ $= \frac{-5-5\text{i}}{10}$ $= -\frac{1}{2} - \frac{1}{2}\text{i}$	Multiply the numerator and denominator by the complex conjugate. (Compare this with rationalising the denominator when working with surds.)

In addition to performing these calculations analytically, you should also be able to use your calculator to add, subtract, multiply and divide complex numbers.

Steps	Explanation
To work with complex numbers, open the calculator option.	

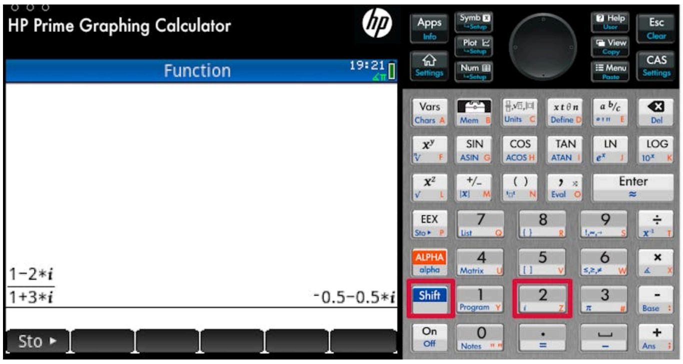
Student
view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

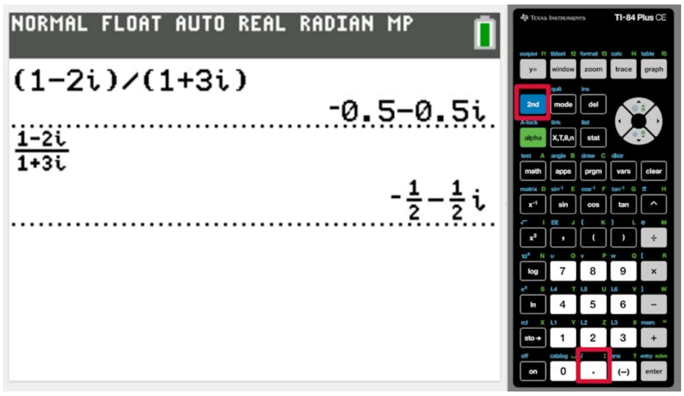
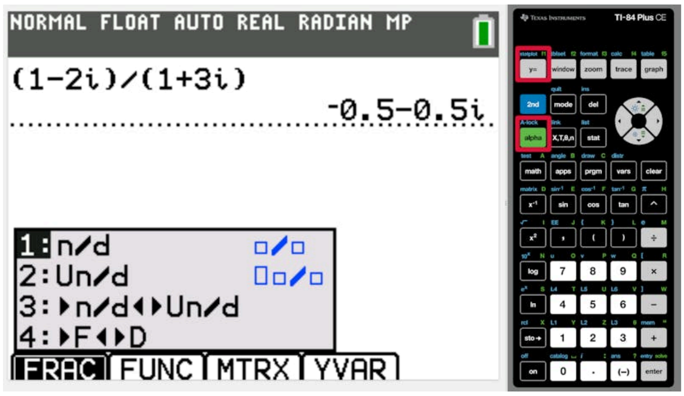
Steps	Explanation
Use the imaginary unit in your expressions, otherwise you can use the operations that you use for real numbers.	
If you use fractions, you will get the answer in different form.	

Student
view

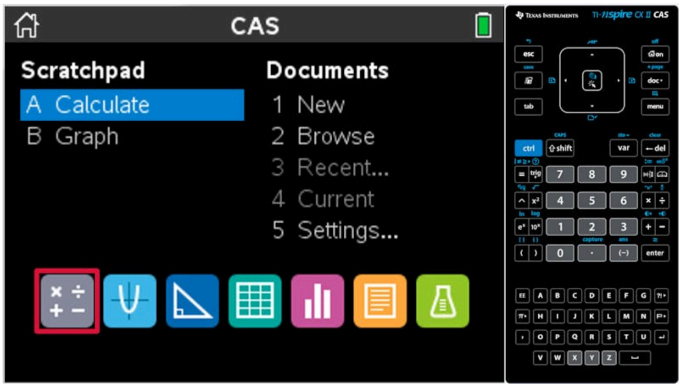
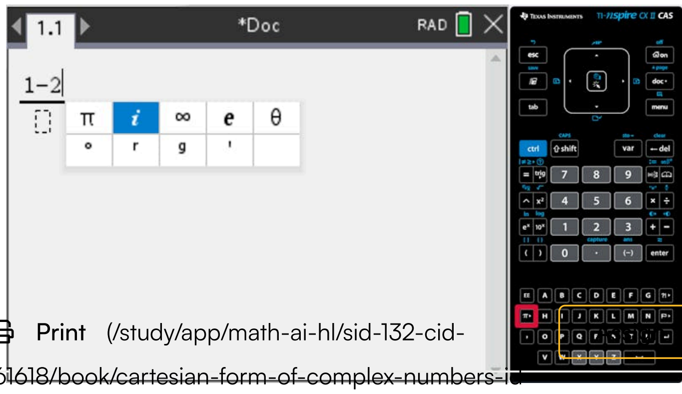
Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Steps	Explanation
To work with complex numbers, enter the home screen of any application.	
Use the imaginary unit in your expressions, otherwise you can use the operations that you use for real numbers.	

Student
view

Steps	Explanation
Use the imaginary unit in your expressions, otherwise you can use the operations that you use for real numbers. Note, that you get your answer in different format when you enter your expression in different format.	
You can access the fraction form (the second form on the previous screen) using alpha/f1.	

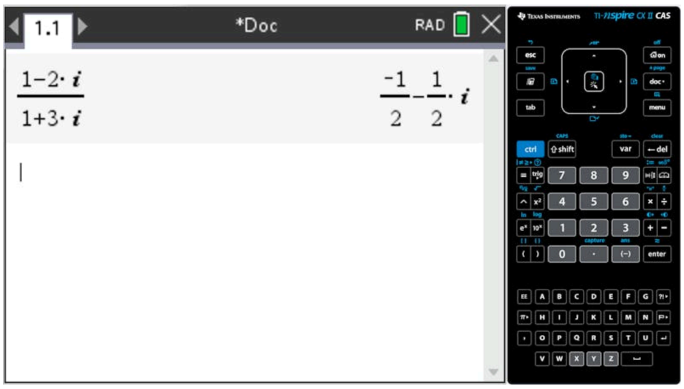
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
To work with complex numbers, open a calculator page.	
Use the imaginary unit in your expressions, otherwise you can use the operations that you use for real numbers.	

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/cartesian-form-of-complex-numbers-id-27423/print/)

Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Steps	Explanation
	

Important

The numbers $c + d\mathrm{i}$ and $c - d\mathrm{i}$ that are used in division of complex numbers such that $\frac{z_1}{z_2} = \frac{a + b\mathrm{i}}{c + d\mathrm{i}} \times \frac{c - d\mathrm{i}}{c - d\mathrm{i}}$ are called complex conjugates.

If $z = a + b\mathrm{i}$, then its complex conjugate is $z^* = a - b\mathrm{i}$.

Example 6

★★☆

For two complex numbers z and w, show that $(z + w)^* = z^* + w^*$.

Let $z = a + b\mathrm{i}$ and $w = c + d\mathrm{i}$

$$\text{LHS} = (z + w)^* = ((a + b\mathrm{i}) + (c + d\mathrm{i}))^* = ((a + c) + (b + d)\,\mathrm{i})^* = (a + c) - (b +$$

$$\text{RHS} = (a - b\mathrm{i}) + (c - d\mathrm{i}) = (a + c) + (-b - d)\,\mathrm{i} = (a + c) - (b + d)\,\mathrm{i}$$

Student
view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

$$\text{LHS} = \text{RHS}$$

Example 7

★★★

Solve the system of equations

$$\begin{aligned} z + 2\text{i} &= \text{i}\left(3 - w\right) \\ 7\text{i}\left(z - 1\right) &= 2\left(z + 2w\right) \end{aligned}$$

where $z, w \in \mathbb{C}$.

Steps	Explanation
From the first equation: $$z = 3\text{i} - w\text{i} - 2\text{i} = \text{i} - w\text{i}$$	Solve the system of equations by substitution.
Substitute $z = \text{i} - w\text{i}$ into the second equation: $$\begin{aligned} 7\text{i}\left(\text{i} - wi - 1\right) &= 2\left(\text{i} - w\text{i} + 2w\right) & \Leftrightarrow \\ -7 + 7w - 7\text{i} &= 2\text{i} - 2w\text{i} + 4w & \Leftrightarrow \\ 3w + 2w\text{i} &= 7 + 9\text{i} & \Leftrightarrow \\ w\left(3 + 2\text{i}\right) &= 7 + \text{i} & \Leftrightarrow \\ w &= \frac{7 + 9\text{i}}{3 + 2\text{i}} = 3 + \text{i} \end{aligned}$$	Use your calculator to carry ou the division.
$z = \text{i} - \left(3 + \text{i}\right)\text{i} = 1 - 2\text{i}$	Substitute $w = 3 + \text{i}$ into $z = \text{i} - w\text{i}$ to find z .

6 section questions ⌄

1. Number and algebra / 1.12 Introduction to complex numbers

Powers of complex numbers

Student
view

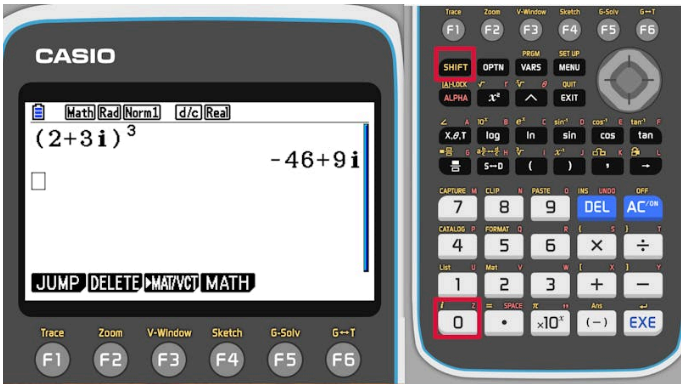
Example 1

★★☆

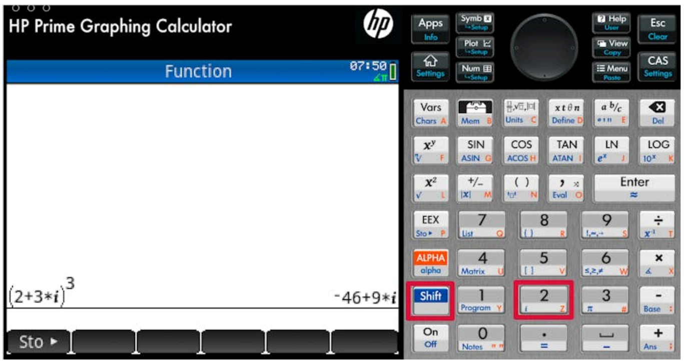
Write $(2+3\mathrm{i})^3$ in $a+b\mathrm{i}$ form.

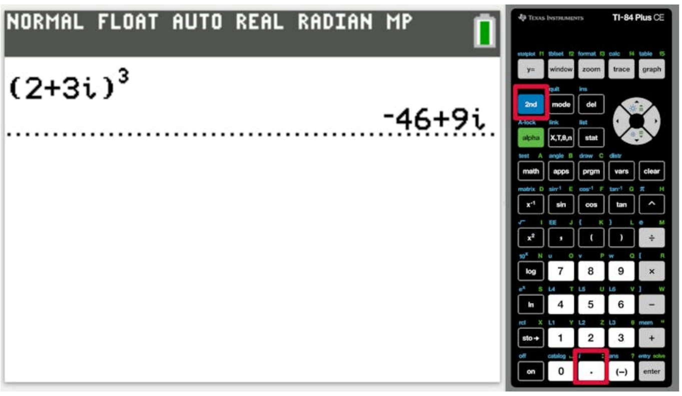
$$\begin{aligned}(2+3\mathrm{i})(2+3\mathrm{i})(2+3\mathrm{i}) &= \left(4+12\mathrm{i}+9\mathrm{i}^2\right)(2+3\mathrm{i})\\ &= (-5+12\mathrm{i})(2+3\mathrm{i})\\ &= -10-15\mathrm{i}+24\mathrm{i}+36\mathrm{i}^2\\ &= -46+9\mathrm{i}\end{aligned}$$

As you can see from **Example 1**, finding powers of complex numbers analytically in Cartesian form is tedious and time consuming. In this course you will be able to use your calculator to find powers of complex numbers.

Steps	Explanation
Your calculator understands complex numbers. Make sure you know where to find the symbol for the imaginary unit.	

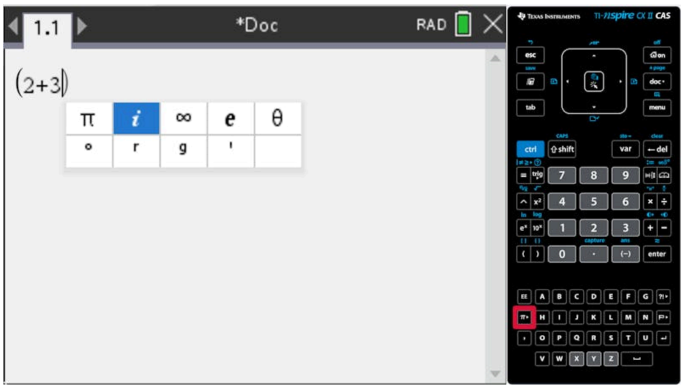
Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Steps	Explanation
Your calculator understands complex numbers. Make sure you know where to find the symbol for the imaginary unit.	

Steps	Explanation
Your calculator understands complex numbers. Make sure you know where to find the symbol for the imaginary unit.	

Student
view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Steps	Explanation
Your calculator understands complex numbers. Make sure you know where to find the symbol for the imaginary unit.	
	

Use your calculator to find the answer to **Example 1.**

The applet below gives you more practice questions for finding powers of complex numbers.

Student
view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Interactive 1. Practice Questions for Finding Powers of Complex Numbers.

More information for interactive 1

This interactive activity allows users to practice calculating powers of complex numbers and expressing the result in the standard a + bi form. By clicking the “new question” button, users are presented with a fresh expression involving a complex number raised to a power. They are encouraged to use a calculator to compute the result and then write it in the correct form. After solving, users can check their answer by selecting the “Show the answer” checkbox, which provides immediate feedback and reinforces the learning process.

For example

To write $(4+5i)^2$ in $a+bi$ form, we expand the expression:

$$(4+5i)^2 = (4+5i)(4+5i)$$

Using the distributive property

$$(4+5i)(4+5i) = 4\cdot 4 + 4\cdot(5i) + (5i)\cdot 4 + (5i)\cdot(5i)$$

$$= 16 + 20i + 20i + 25i^2$$

Combine the real and imaginary terms:

$$= 16 + (20+20)i + 25i^2$$
$$= 16 + 40i + 25i^2$$

Recall that $i^2 = -1$:

Student view

$= 16 + 40i + 25(-1)$

$= 16 + 40i - 25$

$= (16 - 25) + 40i$

$= -9 + 40i$

The final answer is $-9 + 40i$

3 section questions

1. Number and algebra / 1.12 Introduction to complex numbers

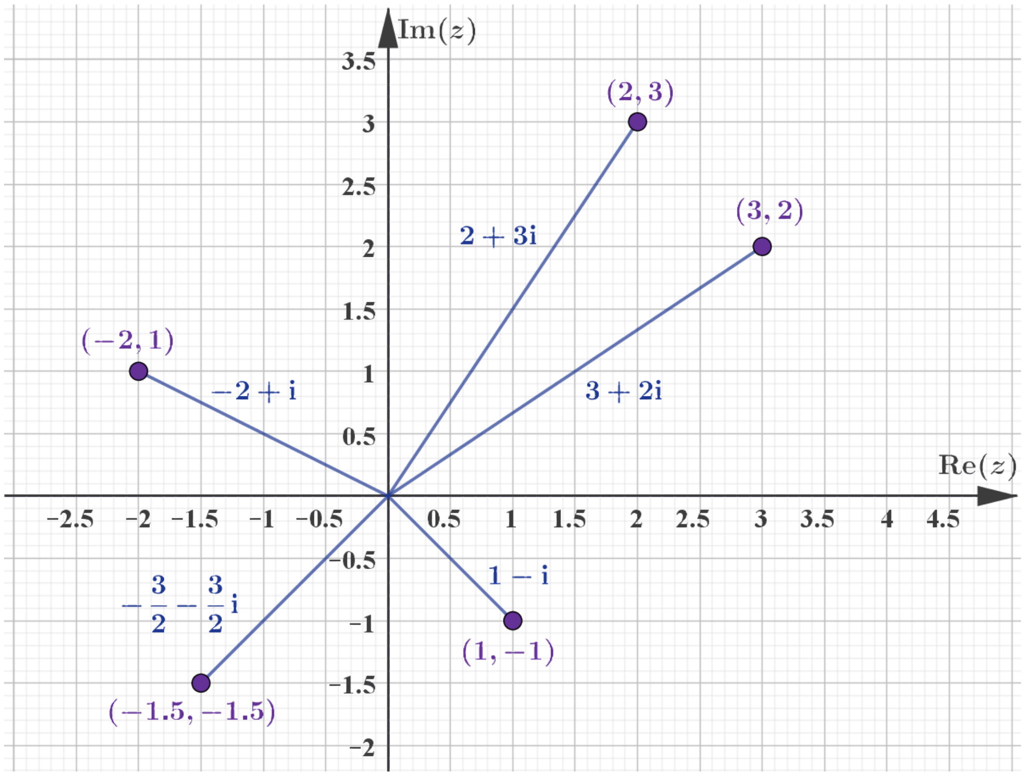
Graphing in the complex plane

Complex numbers can be represented on a two-dimensional plane called the Argand diagram. Similar to the Cartesian plane, which you are familiar with, the Argand diagram has a vertical and a horizontal axis.

The numbers on the horizontal axis represent the real part of the complex number and the numbers on the vertical axis represent the imaginary part.

Complex numbers are represented by vectors on the Argand plane. You will learn more about vectors in topic 3 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26035/), but for now you just need to know how to plot complex numbers on the Argand plane. Examples are shown below.

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov



More information

The image is a diagram showing complex numbers on an Argand plane, which is a grid with real and imaginary axes. The horizontal axis represents the real part of the complex numbers, while the vertical axis represents the imaginary part. Several complex numbers are plotted as points and labeled with coordinates:

1. (-2, 1)
2. (1, 2)
3. (2, 3)
4. (-1.5, -1.5)

These points are connected to the origin (0,0) with vectors, illustrating how complex numbers can be visualized as vectors on this plane. The grid is marked with numbers to assist in identifying exact positions on both axes. This visual representation helps demonstrate the relationship of complex numbers as vectors.

[Generated by AI]

Student
view

International Mindedness

John-Robert Argand and Carl Friedrich Gauss developed the concept of the complex plane during roughly the same time period. Although one was working in France and the other was working in Germany, they came up with similar ideas independently. Hence, the Argand diagram is also referred to as the Gaussian plane.

As you can see in the graph above, plotting $z = a + b\mathrm{i}$ on an Argand diagram requires you to plot a point corresponding to a units on the real axis and b units on the imaginary axis. Connecting this point to the origin with a line allows you to visualise the complex number as a vector. Vectors are always drawn with an arrow indicating direction, but this is sometimes omitted when graphing complex numbers. You can see this notation in the solution to **Example 1**.

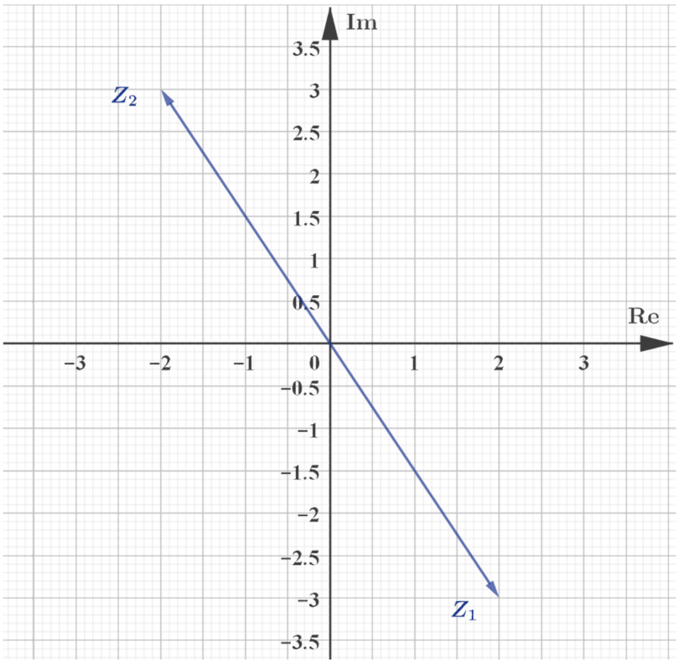
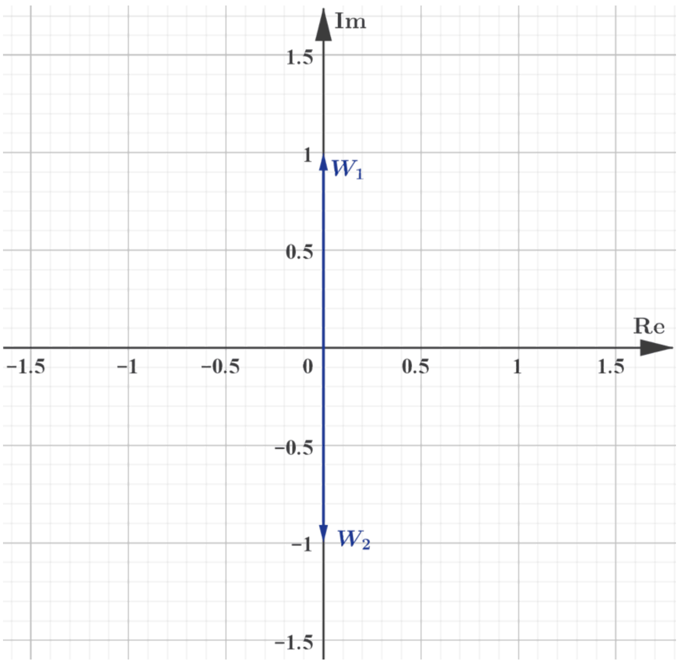
Example 1

★☆☆

a) Plot $z_1 = 2 - 3\mathrm{i}$ and $z_2 = -2 + 3\mathrm{i}$ on an Argand diagram.

b) Plot $w_1 = \mathrm{i}$ and $w_2 = -\mathrm{i}$ on an Argand diagram.

c) Hence, describe the graphical relationship between z and $-z$.

	Steps	Explanation
a)		z_1 is plotted at 2 on the real axi and -3 on the imaginary axis. z_2 is plotted at -2 on the real axis and 3 on the imaginary axis.
b)		w_1 is plotted at 0 on the real ax and 1 on the imaginary axis. w_2 is plotted at 0 on the real axis and -1 on the imaginary axis.

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

	Steps	Explanation
c)	$-z$ is the image of z rotated by $180°$ $-z$ is the image of z after a reflection in the horizontal and vertical axes.	There are two ways to describe the relationship.

Modulus-argument form of a complex number

In addition to describing $z = a + b\text{i}$ on the Argand plane by giving the coordinates (a, b), you can also describe the same complex number by defining its distance from the origin and the angle that the line representing the complex number makes with the positive horizontal axis.

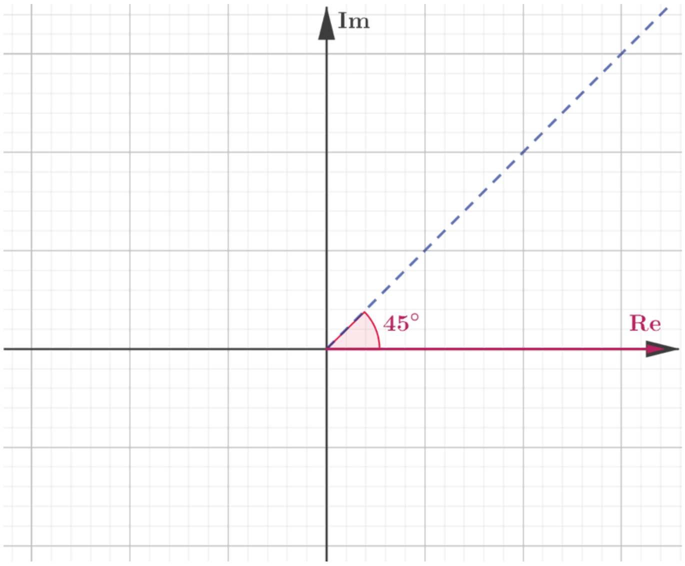
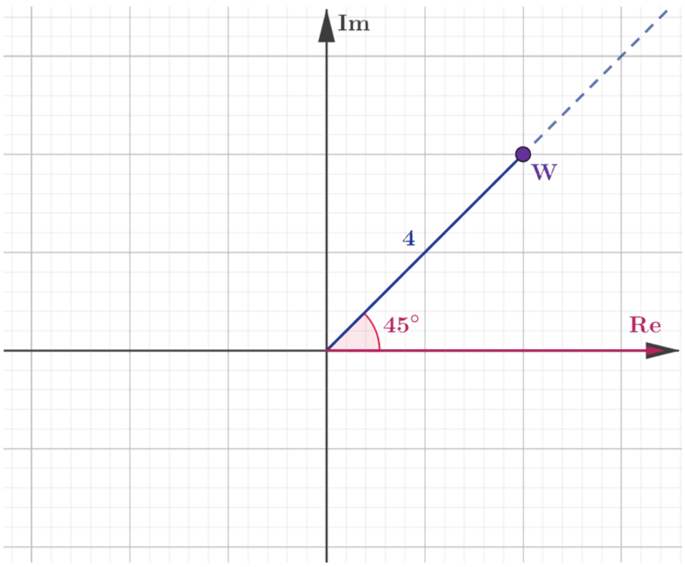
Example 2

★☆☆

Sketch the graph for the complex number w on the Argand plane, given that it is 4 units away from the origin and makes an angle of $45°$ with the positive horizontal axis.

Student
view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Steps	Explanation
	The positive horizontal axis is shown in red. An angle of approximately $45°$ is drawn in using a dashed line.
	Approximately 4 units are measured out along the line making a $45°$ angle with the positive horizontal axis.

Student
view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Exam tip

A question that uses the command term **sketch** is asking you to create a graph or a diagram that gives a general idea of the shape and proportions.

Important information such as length, angles, coordinates of points of intersection, equations of curves, and so on should be labelled. A sketch does not need to be drawn on graph paper with a precise scale.

Important

The distance from the origin to the point representing a complex number on the Argand plane is called the modulus of a complex number . It is denoted by $\text{mod}\,(\text{z}) = |z|$.

The angle the vector representing a complex number makes with the positive horizontal axis is called the argument of a complex number . It is denoted by $\arg\,(\text{z})$
.

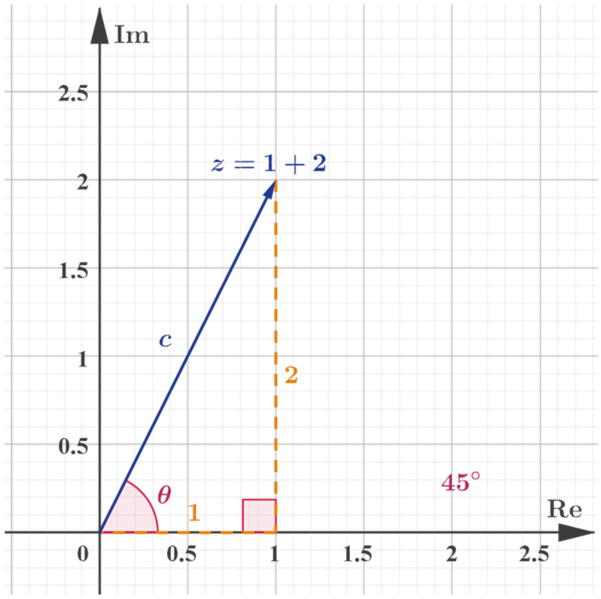
Example 3

★☆☆

Given that $z = 1 + 2\text{i}$, find $|z|$ and $\arg\,(z)$.

Student
view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
Using Pythagoras' theorem: $$\lvert z\rvert=\sqrt{1^2+2^2}=\sqrt{5}$$ Using $\tan\theta=\dfrac{\text{opposite}}{\text{adjacent}}$: $$\begin{aligned}\arg(z)&=\theta\\ \tan\theta&=\frac{2}{1}=2\\ (\theta&=\tan^{-1}2\\ &=63.4^\circ(3\text{ significant figures})\\ \arg(z)&=63.4^\circ\end{aligned}$$	You should plot z on an Argand plane to visualise what you are working with.  A right-angled triangle can be drawn with sides of length 1 and 2 units to represent the complex number.

Important

Given that $z=x+y\text{i}$, $\text{mod}(z)=\lvert z\rvert=\sqrt{x^2+y^2}$.

Activity

Find the argument of each of the following, giving your answers to 1 decimal place, where appropriate.

$$2+3\text{i} \qquad -1+\sqrt{2}\text{i} \quad -3-5\text{i} \qquad 3-2\text{i}$$

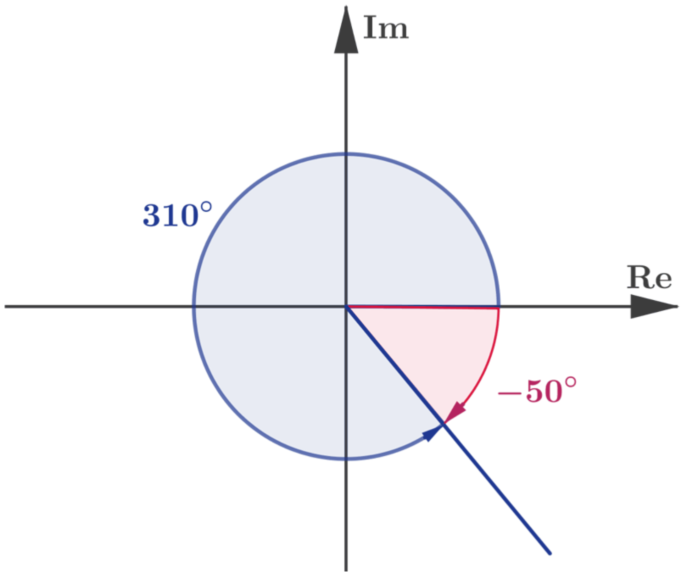
What difficulties do you encounter in your calculations when the complex number is not in the first quadrant?

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Suggest some solutions to the problems you describe.

When you state the value of an angle there are a few equivalent possibilities. For instance, the angle of $310°$ is equivalent to $-50°$, as shown in below. You can learn more about this in subtopic 3.8 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27441/).



More information

The image is a diagram of a complex plane with both real and imaginary axes labeled as 'Re' and 'Im,' respectively. A circle is centered at the origin, and two angles, 310° and -50°, are marked on the circle. The angle of 310° is measured in the counterclockwise direction and is in blue, while the angle of -50° is measured in the clockwise direction and is in red. This demonstrates the equivalence of the angles in terms of their positions on the circle. The diagram illustrates how angles can be represented using both positive and negative measurements based on direction.

[Generated by AI]

Activity

The accepted convention for working with the argument of complex numbers is shown in the applet below. Drag the purple point through the quadrants and note

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

the relationship between $\arg(\mathrm{z})$ and $\tan^{-1}\left(\frac{y}{x}\right)$. Describe what you notice.

Interactive 1. Graphing Complex Numbers in the Argand Plane.

More information for interactive 1

This interactive demonstrates the argument "*arg z*" of a complex number $z = x + iy$ and its relation to $\arctan(\frac{y}{x})$. The purple point represents the complex number and can be dragged across quadrants to observe how the argument changes. In the first quadrant, $\arg z = \arctan(\frac{y}{x})$. However, in the second and third quadrants, where x is negative, the argument is adjusted by adding $180°$ and subtracting $180°$, respectively. In the fourth quadrant, the argument is negative and follows standard conventions. As the point moves, the argument smoothly transitions between $-180°$ and $180°$, illustrating how angle adjustments are necessary to correctly represent complex numbers in polar form.

For example, in the first quadrant, if we have a complex number $z = 2.82 + 2.26i$, then both x and y are positive. The argument is calculated directly using the inverse tangent function: $\arg z = \arctan(\frac{y}{x}) = \arctan\frac{(2.26}{2.82}) \approx 38.71°$. This shows how the argument of z is simply the angle between the positive real axis and the line connecting the origin to the point $(2.82, 2.26)$ in the complex plane.

Important

Given that $z = x + yi$, then the principal value of the argument is

- in the first and fourth quadrants when $Re(z) > 0$, $\arg(\mathrm{z}) = \tan^{-1}\left(\frac{y}{x}\right)$

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

- in the second quadrant when $Re\,(z) < 0$ and $Im\,(z) > 0$, $\arg\,(\mathrm{z}) = 180 + \tan^{-1}\left(\frac{y}{x}\right)$
- in the third quadrant when $Re\,(z) < 0$ and $Im\,(z) < 0$, $\arg\,(\mathrm{z}) = -180 + \tan^{-1}\left(\frac{y}{x}\right)$.

Be aware

The argument of a complex number can be given in degrees or in radians. In this subtopic all questions will be in degrees, which you are already familiar with. You will study radians later in the course.

Example 4

★★☆

Find the modulus and argument for each of the following:

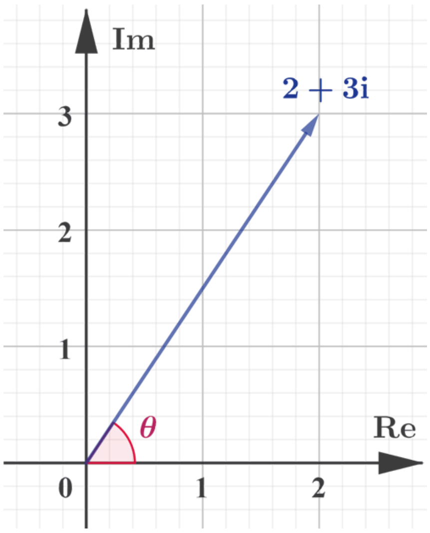
a) $2 + 3\mathrm{i}$

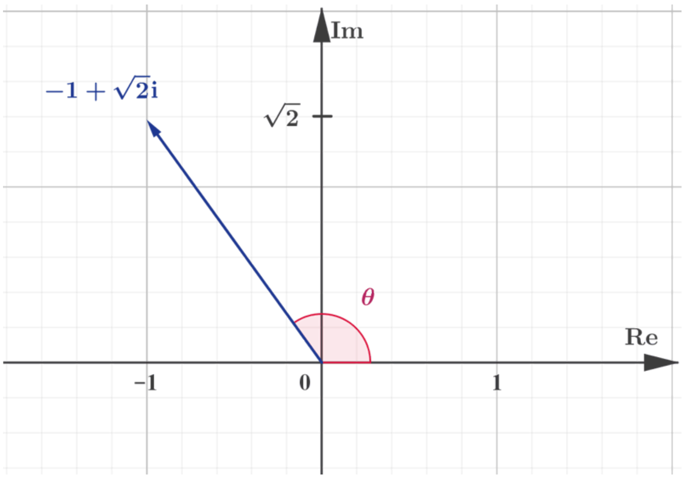
b) $-1 + \sqrt{2}\mathrm{i}$

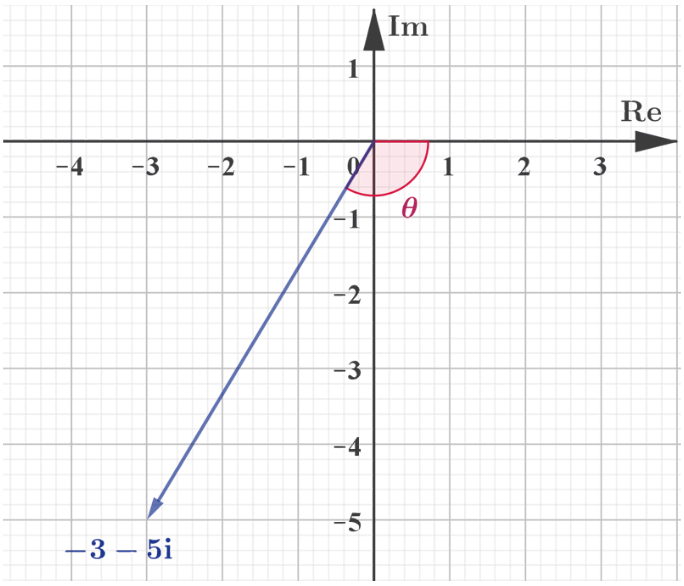
c) $-3 - 5\mathrm{i}$

d) $3 - 2\mathrm{i}$

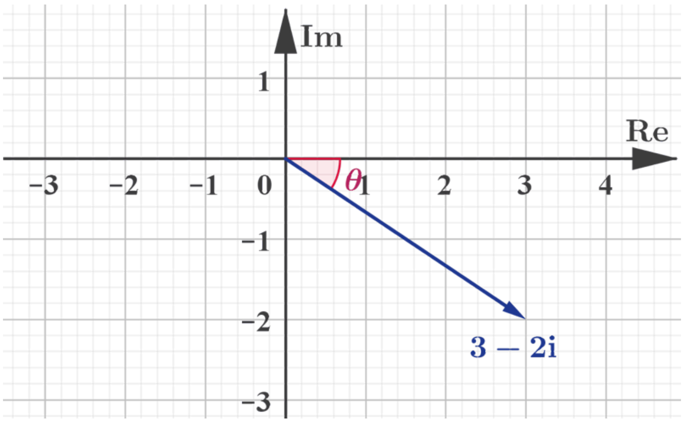
Student view

	Steps	Explanation
a)	 $\lvert z\rvert = \sqrt{2^2+3^2} = \sqrt{13}$ $\arg(z) = \tan^{-1}\frac{3}{2} = 56.3°$ (3 significant figures)	You should always make a sketch of the Argand diagram for the complex number if you are working with the modulus and argument. Check that your final answer for the modulus matches the diagram you have drawn.

	Steps	Explanation
b)	 $\lvert z\rvert = \sqrt{(-1)^2 + \sqrt{2}^2} = \sqrt{3}$ $\arg(z) = 180 + \tan^{-1}\frac{\sqrt{2}}{-1}$ $= 180 + (-54.7356)$ $= 125^\circ$ (3 significant figures)	

	Steps	Explanation
c)	 $\lvert z\rvert = \sqrt{(-3)^2 + (-5)^2} = \sqrt{34}$ $\arg(\text{z}) = -180 + \tan^{-1}\frac{-5}{-3}$ $= -180 + 59.0362$ $= -121^{\circ}\ (3 \text{ significant figures})$	

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

	Steps	Explanation
d)	 $|z| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$ $\arg(z) = \tan^{-1}\frac{-2}{3} = -33.7^\circ$ (3 significant figures)	

Be aware

You should always make a sketch of the Argand diagram for the complex number if you are working with the modulus and argument.

At this point it is hard to see the benefits of working with complex numbers using the modulus and argument as it is clearly easier to graph them in $a + b$i form. You will learn of the benefits of the modulus–argument form in the next two subtopics.

4 section questions

1. Number and algebra / 1.12 Introduction to complex numbers

Checklist

Student view

What you should know

By the end of this subtopic you should be able to:

- find complex roots of quadratic equations
- recognise that complex solutions to quadratic equations with real coefficients come in complex conjugate pairs
- given $z = a + b\mathrm{i}$, identify that $\mathrm{Re}\,(z) = a$ and $\mathrm{Im}\,(z) = b$
- add, subtract, multiply and divide complex numbers analytically and using a calculator
- write the complex conjugate of a complex number and use it to perform division
- use the equality property of complex numbers to solve for variables
- solve systems of equations with complex number coefficients and solutions
- find powers of complex numbers using the calculator
- represent complex numbers on the Argand plane
- find the modulus and argument of a complex number and use these to graph complex numbers.

Investigation

Part 1

Simplify each of the following:

$$\mathrm{i}^2 \quad \mathrm{i}^3 \quad \mathrm{i}^4 \quad \mathrm{i}^5 \quad \mathrm{i}^6 \quad \mathrm{i}^7$$

Summarise any patterns that you find. You may need to complete further examples (e.g. i^8 and so on) if you have trouble finding the pattern in the ones given.

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Use the patterns that you found to simplify the following. (You can use a calculator to check your answers.)

$$\mathrm{i}^{375} \qquad \mathrm{i}^{2702} \qquad \mathrm{i}^{3756}$$

Write a general rule for i^n where $n \in \mathbb{N}$.

Part 2

Let $z_1 = 2 + \mathrm{i}$ and $z_2 = -1 - \mathrm{i}$.

Find:

$$(z_1 + z_2)^* \qquad (z_1 - z_2)^* \qquad (z_1 \times z_2)^* \qquad \left(\frac{z_1}{z_2}\right)^*$$

$$z_1^* \times z_2^* \qquad z_1^* - z_2^* \qquad \frac{z_1^*}{z_2^*} \qquad z_1^* + z_2^*$$

Comment on any patterns that you notice. Show that your observations hold true for any two complex numbers

Rate subtopic 1.12 Introduction to complex numbers

Help us improve the content and user experience.

Student view