Index

The big picture

The second big concept in calculus is integration. There are two important aspects to our study of integration:

1. Integration as anti-differentiation
2. Integration to find the area under curves.

Although you have already covered rudimentary integration of polynomial functions in the SL sections, it is now time to more formally discuss integration and the various techniques to solve more complex problems.

Integration as differentiation

In earlier algebra topics, you have studied operations and their inverses, for example,

Function	Inverse
x^2	$\sqrt{x}$
$\sin x$	$\arcsin x$
e^x	$\ln x$

By using an operation and its inverse in series, you mathematically complete a circle. For example, $\ln\left(e^x\right) = x$.

Integration and differentiation are inverse operations of each other. This means that if you start with a function, say $f(x)$, and if you integrate $f(x)$ and follow this by differentiating the result of the integration, you obtain again $f(x)$. Similarly, if you start with a function, say $g(x)$, and then you differentiate $g(x)$ and follow this by integrating the result of the differentiation, you obtain a function that belongs to the same family of functions as the function you started with, namely $g(x)$, possibly with a vertical translation. It is clear, therefore, that as differentiating a function gives another function (the derivative function), integrating a function also gives a function. You can show this inverse operation relationship as

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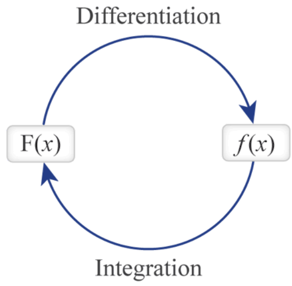

More information

The diagram illustrates the inverse relationship between differentiation and integration. It features a circular flow with two main components. On the right side, there is a box labeled "f(x)," and on the left side, a box labeled "F(x)." Arrows connect these components in a loop. The arrow from "f(x)" to "F(x)" is labeled "Integration," indicating that integrating "f(x)" results in "F(x)." Conversely, the arrow from "F(x)" to "f(x)" is labeled "Differentiation," indicating that differentiating "F(x)" returns "f(x)." This illustrates how integrating a function results in its antiderivative, while differentiating an antiderivative yields the original function.

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This concept is important. But why? Integration is considerably more difficult than differentiation. Hence, after integrating a function, you can differentiate the result to see whether you get the original function back. If you do, then you probably got the right result. If not, you most likely made a mistake when integrating rather than in the process of checking your answer by differentiating. In fact, without seeing any of the rules for integrating functions, you can probably predict the result when integrating $f(x) = x^2$.

Integration to find areas under curves

When you studied differentiation, you learned that evaluating the derivative function at a particular point along a curve gives the value of the gradient of the tangent line to the curve at that point. In other words, values of the derivative function give the steepness of the curve at any point. Similarly, you can evaluate the result of integration. If you use two different x-values along the curve, the value from the process of integration gives the area under the curve between these two x-values and the x-axis. Although the process will be developed in this chapter, this concept will be covered in depth in subtopic 5.12 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-28203/).

Concept

In the most basic sense, the **relationship** between differentiation and integration is that integration is nothing more than an anti-derivative. Integration ‘unwraps’ the work of differentiation, just as differentiation ‘unwraps’ the work of integration. With that in mind, recall your knowledge of differentiation. How many functions do you think you can integrate already?

Theory of Knowledge

You have been studying various elements of calculus that are designed to provide you with knowledge of unknowns. In essence, you've been learning how to use mathematics to create knowledge. However, did you ever consider your work logically flawed because of circularity?

One of the most famous mathematicians who ever lived, Bertrand Russell, was very troubled by what he saw as a flaw in the knowledge production process of mathematics. He believed that if one could not prove that 2 + 2 = 4, all the maths that flowed from arithmetic was flawed and invalidated. Russell and other colleges such as Alfred

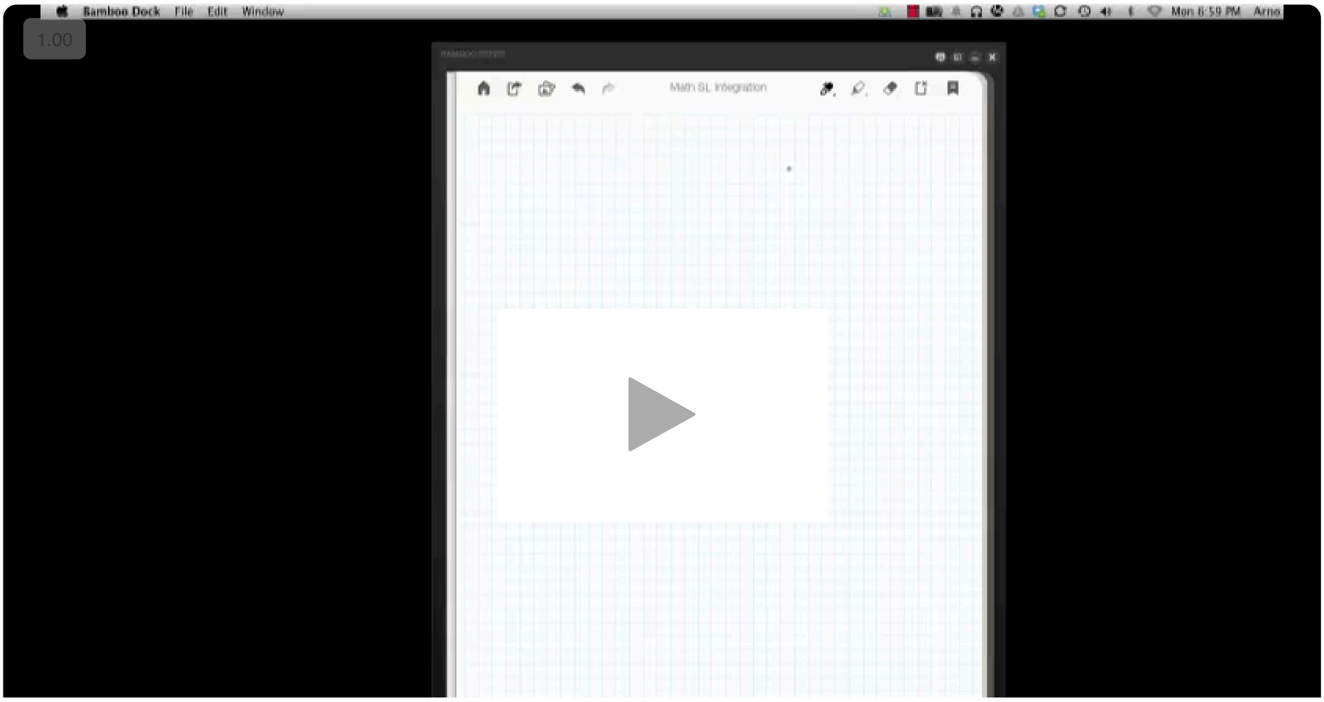
Student view

Whitehead spent their entire lives trying to work out a proof for basic arithmetic that avoided what he saw as the fatal flaw of circularity.

Knowledge Question: Can knowledge be valid without validation?

Integrals of power functions

Finding the integral, or anti-derivative is like working backwards when compared with finding the derivative. As you found out in subtopic 5.5 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26177/), the derivatives of $f(x) = x^2, g(x) = x^2 + 1, h(x) = x^2 + 2$ and $j(x) = x^2 + \pi$ are the same, namely, $2x$. For that reason, indefinite integrals include a $+C$ on the end. The C added on the end represents the fact that any value is possible in the original function, such as the 1, 2 and π in the prior examples. The following video explores indefinite integration of power functions.



Video 1. Integrating Power Functions.

More information for video 1

1

00:00:00,567 --> 00:00:03,867

narrator: In this video

we're going to look at integration

2

00:00:04,100 --> 00:00:09,167

of power laws and in particularly

the indefinite integration of power laws.

3

00:00:09,800 --> 00:00:14,167

And the driving idea

is really the idea that integration

4

00:00:14,733 --> 00:00:18,933

is anti differentiation, and we're gonna

use it over and over again to guide us

5

00:00:19,000 --> 00:00:20,433

and to check our answers.

6

00:00:20,867 --> 00:00:24,300

So let's start with a typical

power law x squared,

7

00:00:24,633 --> 00:00:27,100

and we already noted

if we differentiate x squared,

8

00:00:27,167 --> 00:00:30,900

then the outcome

will be 2 times x to power 1.

9

00:00:31,100 --> 00:00:32,600

So really this idea that means

10

00:00:32,667 --> 00:00:35,200

that if we integrate 2 times x to power 1,

11

00:00:35,267 --> 00:00:37,100

we ought to get x squared back.

12

00:00:37,767 --> 00:00:42,333

Integration then should start

with 2x to the power 1

13

00:00:42,467 --> 00:00:44,800

and take us to x squared.

14

00:00:44,867 --> 00:00:48,133

Alright, so we started

with 2x to the power 1,

15

00:00:48,300 --> 00:00:49,600

if we balance the powers first,

16

00:00:49,667 --> 00:00:52,833

need to add 1 convert it

to multiplying 2 and divide by 2.

17

00:00:53,667 --> 00:00:56,033

Well, let's see,

again, let's take x to power 3.

18

00:00:56,100 --> 00:00:58,900

Differentiate 3x squared

and the idea of anti differentiation,

19

00:00:59,000 --> 00:01:01,400

that should take us back

to x to the power of 3.

20

00:01:01,700 --> 00:01:04,400

So 3x squared

should go to x to the power of 3

21

00:01:04,867 --> 00:01:06,267

3x squared,

22

00:01:06,600 --> 00:01:11,400

you need to add 1 to the power

and and divide again by 3.

23

00:01:11,900 --> 00:01:13,967

Now let's check this for x to the power 5.

24

00:01:14,033 --> 00:01:15,433

So if we integrate that,

25

00:01:15,700 --> 00:01:20,167

it goes to x to the power

of 5 plus 1 divided by 5 plus 1.

26

00:01:20,933 --> 00:01:22,367

Now let's check with differentiation.

27

00:01:22,500 --> 00:01:24,133

So differentiation of power law,

28

00:01:24,433 --> 00:01:28,233

and you subtract 1

from the power, all power comes down

29

00:01:28,333 --> 00:01:29,833

and divide by what ahead.

30

00:01:29,933 --> 00:01:32,400

So it seems indeed that this is the case.

31

00:01:32,500 --> 00:01:35,333

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So we seem to have found

that if I integrate x

32

00:01:35,433 --> 00:01:36,600

to the power n,

33

00:01:36,867 --> 00:01:40,733

it goes to x to the n plus 1 times

1 over n plus 1.

34

00:01:42,367 --> 00:01:44,533

But really, let's wait a minute here.

35

00:01:45,433 --> 00:01:48,500

If you let y equals x

squared plus any number,

36

00:01:49,000 --> 00:01:53,800

then we also know that the y by the x

is 2x regardless of the number,

37

00:01:53,900 --> 00:01:55,100

so for all c.

38

00:01:55,567 --> 00:01:59,367

And here we see that I've got an x squared

and a derivative is 2x

39

00:01:59,500 --> 00:02:01,733

and I've created

a tangent line at a point.

40

00:02:02,067 --> 00:02:04,067

Now if I translate x squared

41

00:02:04,333 --> 00:02:06,633

up and down by arbitrary numbers,

42

00:02:06,700 --> 00:02:09,367

then you see that the steepness

of the tangent line doesn't change,

43

00:02:09,433 --> 00:02:11,467

the gradient function

doesn't change either.

44

00:02:12,267 --> 00:02:16,967

So we need to take this into consideration

when we do integration of power laws

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45

00:02:17,033 --> 00:02:18,633

and indeed integration of anything.

46

00:02:20,100 --> 00:02:23,900

So really then x to the power of n,

47

00:02:23,967 --> 00:02:28,033

if I integrate that, it seems to go to

x n plus 1

48

00:02:28,100 --> 00:02:31,233

times 1 over n plus 1 plus a constant.

49

00:02:31,300 --> 00:02:33,900

And this constant is called

the constant of integration.

50

00:02:34,133 --> 00:02:37,800

It is, if you recall,

equal to a vertical translation

51

00:02:37,900 --> 00:02:39,233

of a function.

52

00:02:39,633 --> 00:02:42,000

Now this is a good time

to establish some notation.

53

00:02:42,233 --> 00:02:43,700

So if I have a function f of x,

54

00:02:43,767 --> 00:02:46,933

and I integrate that respect to dx,

55

00:02:47,000 --> 00:02:49,367

again, another function plus a constant.

56

00:02:49,433 --> 00:02:52,833

So the signage, the squarely sign

57

00:02:52,967 --> 00:02:56,700

and dx is the way to write

the indefinite integrand,

58

00:02:56,767 --> 00:02:58,633

you need to write it all the time,

59

00:03:00,267 --> 00:03:03,367

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Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Nota bene, be very careful

that the guiding idea

60

00:03:03,433 --> 00:03:05,833

has been that if you differentiate

what comes out of integration,

61

00:03:05,933 --> 00:03:08,600

you get back to function

that went into the integration.

62

00:03:08,667 --> 00:03:11,100

Little f of x is sometimes

called the integrand.

63

00:03:11,667 --> 00:03:16,500

So we conclude that if a integrated

power law x to the n,

64

00:03:16,767 --> 00:03:22,500

then what I get x to the n

plus 1 times 1 over n plus 1 plus c.

65

00:03:22,933 --> 00:03:25,800

And that is our first standard integral.

Here is the notation used for an indefinite integral.

✓ **Important**

The indefinite integral of $f(x)$ is given by $\int f(x)\mathrm{d}x = F(x) + C, C \in \mathbb{R}$, where $\frac{d(F(x))}{\mathrm{d}x} = f(x)$ and $F(x)$ is known as the anti-derivative of $f(x)$.

As discussed in section 5.5.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/antiderivatives-of-power-functions-id-26180/), an indefinite integral is expressed without limits, while a definite integral is expressed with upper and lower limits of integration .

In the video, you saw the following important standard integrals for power functions.

In subtopic 5.5 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26177/), you learned that $\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, n \neq -1$. In that subtopic, this formula was applied almost exclusively to polynomials, but this is not a requirement. The process will work for any real powers.

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Important

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

In addition to differentiation of basic polynomials in standard form, you will also study the following special case of the chain rule:

- If the derivative of $y = f(x)$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$, then if the function $f(x)$ is transformed by a linear transformation, i.e. $y = f(ax+b)$, you get $\frac{\mathrm{d}y}{\mathrm{d}x} = af'(ax+b)$.

For example, you can verify that $\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$.

Using either this special case or the chain rule, $\frac{\mathrm{d}}{\mathrm{d}x}\left((5x+3)^2\right) = 10\,(5x+3)$, $\frac{\mathrm{d}}{\mathrm{d}x}\left((-7x+3)^2\right) = -14\,(-7x+3)$, and $\frac{\mathrm{d}}{\mathrm{d}x}\left((ax+b)^2\right) = 2a\,(ax+b)$.

Working backwards with the idea of finding the anti-derivative, you can see that $\int 10\,(5x+3)\,dx = (5x+3)^2 + C$, $\int -14\,(-7x+3)\,\mathrm{d}x = (-7x+3)^2 + C$, and $\int 2a\,(ax+b)\,dx = (ax+b)^2 + C$.

More importantly, you can predict $\int (3x+4)\,\mathrm{d}x = \frac{1}{6}(3x+4)^2 + C$, $\int 7\,(5x+8)\,\mathrm{d}x = \frac{7}{10}(5x+8)^2 + C$, and $\int (ax+b)\,\mathrm{d}x = \frac{1}{2a}(ax+b)^2 + C$.

The same philosophy works for higher-order binomials as well as the sum rule for integration.

Be aware

The indefinite integral of x^n is given by $\frac{x^{n+1}}{n+1} + C, n \neq -1$. This equation is in the IB formula booklet.

Under a linear transformation, the indefinite integral of $(ax+b)^n$ is given by $\frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$. This equation is **not** given in the IB formula booklet.

Example 1

★☆☆

Find the indefinite integral of $f(x) = 5x^7$.

$$\int 5x^7 \mathrm{d}x = \frac{5}{8}x^8 + C$$

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Example 2

★☆☆

Find the indefinite integral of $g(x) = 3(4x+9)^3$.

$$\int 3(4x+9)^3 \mathrm{d}x = \frac{3}{4\times 4}(4x+9)^4 + C = \frac{3}{16}(4x+9)^4 + C$$

Example 3

★☆☆

Find the indefinite integral of $h(x) = 8x^{\frac{3}{2}} - 2 + \frac{1}{x^2}$.

$$\int \left(8x^{\frac{3}{2}} - 2 + \frac{1}{x^2}\right) dx = \int \left(8x^{\frac{3}{2}} - 2 + x^{-2}\right) dx = 8\left(\frac{2}{5}\right) x^{\frac{5}{2}} - 2x - x^{-1} = \frac{16}{5}x^{\frac{5}{2}} - 2x - \frac{1}{x} + C$$

4 section questions ⌄

5. Calculus / 5.11 Further integration

Integrals of reciprocal functions

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/integrals-of-reciprocal-functions-id-27514/print/) Assign

In the last section, you saw that $\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, n \neq -1$. Why is there a restriction that $n \neq -1$? Think through the power rule. Applying the power rule does not work well here as the denominator would be 0. Recall that, in section 5.9.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/derivative-of-the-natural-logarithm-function-id-28204/), the derivative of the natural logarithm was defined as $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$. Therefore, it makes sense that the integral, or the anti-derivative, is defined as $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C$.

✓ **Important**

$\int \frac{1}{x} dx = \ln|x| + C$

There is one small caveat to this definition. The natural logarithm, as with any logarithmic power function, has a limited domain of $x > 0$. This integral does not account for the presence of negative numbers. The easiest solution is to force them to be positive by using the absolute value, i.e., $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C$. Since integration is merely anti-differentiation, comparing domains and ranges of the functions still works. Both functions are undefined at $x = 0$ and defined for $x > 0$. By using the absolute value, negative values that are defined in the reciprocal function will now be treated as positive values by the logarithmic function.

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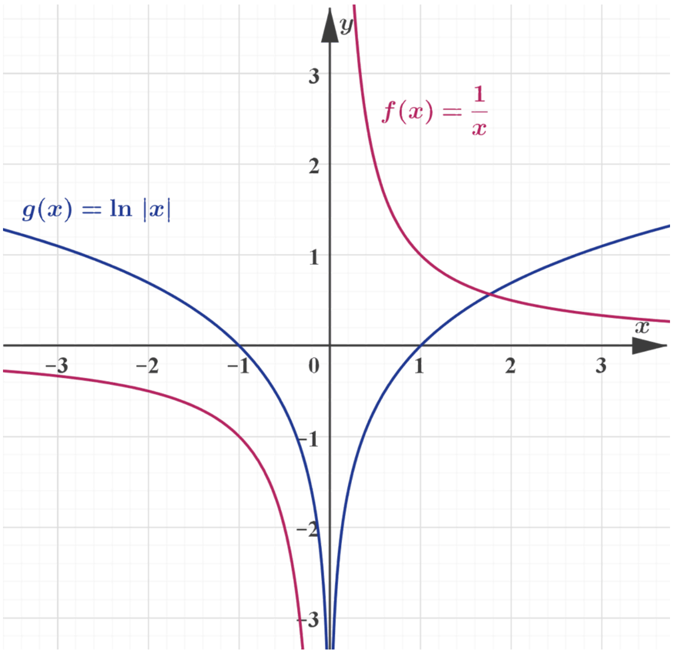
Does this make sense? Can you just put the absolute value sign in there and trust everything works out? The following figure (where $f(x) = \dfrac{1}{x}$ is presented in green and $\int f(x)\mathrm{d}x = g(x) = \ln|x|$ is presented in blue) shows why this works.



More information

The image is a graph showing two curves on a grid. The X-axis represents values from -4 to 4. The Y-axis represents values from -4 to 4. There are two curves: (f(x) = \dfrac{1}{x}) in pink and (g(x) = \ln |x|) in blue.

Curve Description: - The curve (f(x) = \dfrac{1}{x}) has branches in the first and third quadrants. In the first quadrant, it approaches the Y-axis asymptotically from above as (x) approaches zero and decreases as (x) increases. In the third quadrant, it approaches the Y-axis asymptotically from below. - The curve (g(x) = \ln |x|) in blue lies below the X-axis in the first quadrant, starting from negative infinity at (x=0) and increasing slowly as (x) increases. In the second quadrant, it mirrors this behavior with symmetry around the Y-axis.

Observations: The slope of (g(x)) is noted to be negative for (x < 0) and positive for (x > 0), demonstrating how the anti-differentiation of (f(x)) aligns with (g(x)).

[Generated by AI]

As you can see, the slope of $g(x)$ is negative when $x < 0$ and positive when $x < 0$. For $x < 0$, the relationship can be justified through the concept of anti-differentiation. For $x < 0$, the relationship can be justified based on the symmetry of the graph. Furthermore, the domain restriction of $f(x)$ lines up well with the domain restriction of $g(x)$, with the one-sided limits of $f'(x)$ at 0 approaching negative infinity from both sides.

Be aware

The indefinite integral of $\dfrac{1}{x}$ is given by $\ln|x| + C$. This equation is in the IB formula booklet.

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Under a linear transformation, the indefinite integral of $\frac{1}{ax+b}$ is given by $\frac{1}{a}\ln|ax+b|+C$. This equation is **not** given in the IB formula booklet.

Example 1

★☆☆

Find the indefinite integral of $f(x)=\frac{1}{x^3}+\frac{1}{x}$.

$$\int\left(\frac{1}{x^3}+\frac{1}{x}\right)\mathrm{d}x=\int\left(x^{-3}+\frac{1}{x}\right)\mathrm{d}x=-\frac{1}{2}x^{-2}+\ln|x|+C=-\frac{1}{2x^2}+\ln|x|+C$$

Example 2

★☆☆

Find the indefinite integral of $g(x)=\frac{1}{3x+7}$.

$$\int\frac{1}{3x+7}\mathrm{d}x=\frac{1}{3}\ln|3x+7|+C$$

4 section questions ⌄

5. Calculus / 5.11 Further integration

Integrals of exponential functions

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/integrals-of-exponential-functions-id-28196/print/) Assign

As you learned in section 5.9.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/derivative-of-the-exponential-function-id-28205/), finding the derivative of an exponential function with Euler's number, {\text{e}}, as a base is relatively simple, i.e. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x)=\mathrm{e}^x$. Therefore, the anti-derivative should be just as simple, $\int\mathrm{e}^x\mathrm{d}x=\mathrm{e}^x+C$. If you analyse the related special case from earlier sections, it should make sense that $\int\mathrm{e}^{ax+b}\mathrm{d}x=\frac{1}{a}\mathrm{e}^{ax+b}+C$. Can you confirm this formula by using the chain rule to find the derivative?

Be aware

The indefinite integral of e^x is given by e^x+C. This equation is in the IB formula booklet.

Under a linear transformation, the indefinite integral of e^{ax+b} is given by $\frac{1}{a}\mathrm{e}^{ax+b}+C$. This equation is not given in the IB formula booklet.

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view

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Example 1

★☆☆

Find the indefinite integral of $f(x) = \mathrm{e}^{5x}$.

$$\int \mathrm{e}^{5x}\mathrm{d}x = \frac{1}{5}\mathrm{e}^{5x} + C$$

Example 2

★☆☆

Find the indefinite integral of $f(x) = \mathrm{e}^{-\frac{x}{\pi}+\frac{1}{3}}$.

$$\int \mathrm{e}^{-\frac{x}{\pi}+\frac{1}{3}}\mathrm{d}x = -\pi\mathrm{e}^{-\frac{x}{\pi}+\frac{1}{3}} + C$$

4 section questions

5. Calculus / 5.11 Further integration

Integrals of trigonometric functions

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/integrals-of-trigonometric-functions-id-28197/print/) Assign

Once again, you will see that for trigonometric functions integration is nothing more than anti-differentiation. In subtopic 5.9.4 (/study/app/math-ai-hl/sid-132-cid-761618/book/derivative-of-trigonometric-functions-id-28206/), you found that $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$. You can therefore conclude that $\int \sin x \,\mathrm{d}x = -\cos x + C$ and $\int \cos x \,\mathrm{d}x = \sin x + C$.

Be aware

The indefinite integrals $\int \sin x \,\mathrm{d}x = -\cos x + C$ and $\int \cos x \,\mathrm{d}x = \sin x + C$ are in the IB formula booklet.

Under a linear transformation, the indefinite integrals $\int \sin(ax+b)\mathrm{d}x = \frac{-\cos(ax+b)}{a} + C$ and $\int \cos(ax+b)\mathrm{d}x = \frac{\sin(ax+b)}{a} + C$ are not given in the IB formula booklet.

Example 1

★☆☆

Find the indefinite integral of $f(x) = 2\cos(3x+1)$.

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$$\int 2\cos(3x+1)\mathrm{d}x = \frac{1}{3}\times 2\sin(3x+1)+C = \frac{2}{3}\sin(3x+1)+C$$

Example 2

★☆☆

Find the indefinite integral of $f(x)=\frac{1}{2}\sin\left(\frac{1}{3}x\right)-3\cos\left(\frac{\pi}{2}x+1.8\right)$.

$$\begin{aligned}\int\left(\frac{1}{2}\sin\left(\frac{1}{3}x\right)-3\cos\left(\frac{\pi}{2}x+1.8\right)\right)\mathrm{d}x &= -\frac{1}{\frac{1}{3}}\times\frac{1}{2}\cos\left(\frac{1}{3}x\right)-\frac{1}{\frac{\pi}{2}}\times 3\sin\left(\frac{\pi}{2}x+1.8\right)+C\\ &= -\frac{3}{2}\cos\left(\frac{1}{3}x\right)-\frac{6}{\pi}\sin\left(\frac{\pi}{2}x+1.8\right)+C\end{aligned}$$

Exam tip

- It is worth remembering the integrals of $\sin(ax+b)$ and $\cos(ax+b)$.
- Be careful where the minus sign goes: $\int\sin x\,\mathrm{d}x=-\cos x+C$.
- Do a quick check that your answer is correct by differentiating your integration result to see whether you get back to the function you integrated.

4 section questions

5. Calculus / 5.11 Further integration

Integration by substitution

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/integration-by-substitution-id-28198/print/) Assign

Integration by inspection

Subtopic 5.9 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-28208/) described some useful rules for differentiation:

- Chain rule: If $y=g(u)$, where $u=f(x)$, then $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}$
- Product rule: If $y=uv$, then $\frac{\mathrm{d}y}{\mathrm{d}x}=u\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}u}{\mathrm{d}x}$
- Quotient rule: If $y=\frac{u}{v}$, then $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{v\frac{\mathrm{d}u}{\mathrm{d}x}-u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

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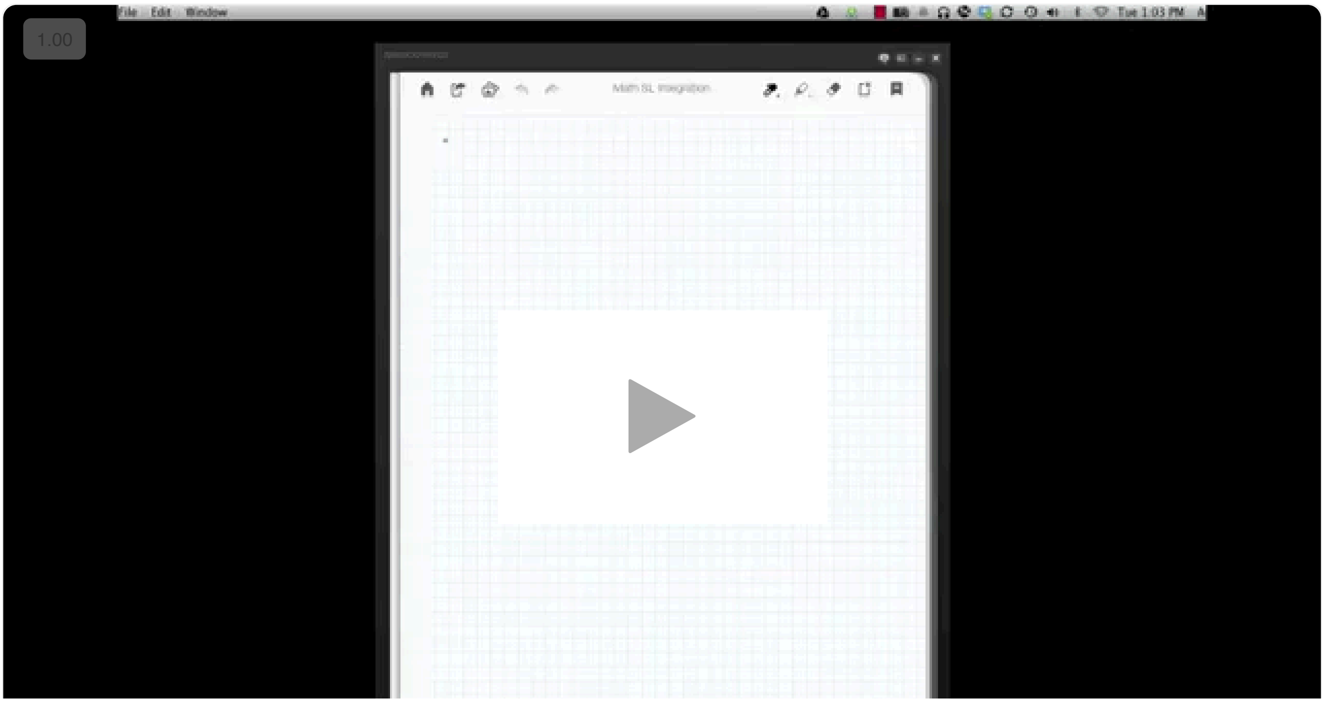
Unfortunately, there are no similar rules for integration that allow you to integrate any combination of functions.

However, using the integration as anti-differentiation insight as well as the chain rule for differentiation, there are some integrands that involve products and quotients that can be integrated. The following video explores this and shows you a form of integrand involving a product that can be integrated.



Video 1. Substitution Method in Integration.

More information for video 1

1

00:00:00,467 --> 00:00:01,401

narrator: In this short video,

2

00:00:01,468 --> 00:00:05,038

we're going to have a first look

at integration by substitution,

3

00:00:05,506 --> 00:00:07,741

which can be quite involving.

4

00:00:08,008 --> 00:00:10,377

The first part involves

doing it by inspection,

5

00:00:10,444 --> 00:00:14,147

which once again hinges

upon the idea that integration

6

00:00:14,448 --> 00:00:17,918

can be considered as anti differentiation.

7

00:00:18,418 --> 00:00:24,124

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view

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Let's consider function sin of 5x
and let's differentiate that respect to x.
8
00:00:24,424 --> 00:00:27,895
Now, we already know what we get
is 5 cosine of 5x.
9
00:00:28,595 --> 00:00:31,031
Hence using the idea
of entity differentiation.
10
00:00:31,098 --> 00:00:34,902
We know that if we integrate
5 cosine of 5x,
11
00:00:35,636 --> 00:00:39,239
then we should get sin 5x plus this
integration constant C.
12
00:00:39,573 --> 00:00:43,443
Similarly, if I look at x square plus
2 raised to the power of 6,
13
00:00:43,510 --> 00:00:46,180
and if I differentiate that respect to x,
14
00:00:46,246 --> 00:00:50,884
then what I get is 6 times x square plus
2 to the power of 5
15
00:00:50,951 --> 00:00:53,554
times 2x using the chain rule.
16
00:00:53,620 --> 00:00:57,191
Therefore, if I integrate x square plus 2
17
00:00:57,257 --> 00:00:59,359
to the power of 5 times 2x,
18
00:00:59,860 --> 00:01:02,329
I see that they're using portion
of the right hand side
19
00:01:02,396 --> 00:01:05,666
and therefore I took the left
hand side x square plus 2
20
00:01:05,732 --> 00:01:08,035
to the power of 6 except

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view

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that I didn't have 6,

21

00:01:08,101 --> 00:01:10,871

so I need to divide by 6.

22

00:01:11,471 --> 00:01:13,407

So what we have, if the integrand,

23

00:01:13,473 --> 00:01:19,913

what I'm integrating is f as a function

of u of x times u prime of x dx,

24

00:01:20,581 --> 00:01:23,317

then the result of that

will be capital F of u

25

00:01:23,717 --> 00:01:25,018

of u of x plus C,

26

00:01:25,252 --> 00:01:30,591

if capital F of u

is equal to the integration

27

00:01:31,491 --> 00:01:34,895

of f of u du, which looks

perhaps a little bit strange

28

00:01:34,995 --> 00:01:36,964

but of course used just a variable.

29

00:01:37,331 --> 00:01:42,336

So I can say capital F of x is equal

to the integral little f of x dx.

30

00:01:42,669 --> 00:01:45,539

And this allows you

to use the standard integrals,

31

00:01:45,606 --> 00:01:50,444

which you find in the IB formula booklet.

32

00:01:51,712 --> 00:01:53,747

Let us return the original question.

33

00:01:54,014 --> 00:01:58,218

5 cosine of 5x can be written as

cosine 5 of x times 5.

34

00:01:58,352 --> 00:02:01,855

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If I integrate that, then I can make the

35

00:02:02,523 --> 00:02:05,993

identification, cos of 5x is f of u,

36

00:02:06,059 --> 00:02:09,763

which is cos of u,

but only if I let u of x equals 5 of x.

37

00:02:10,163 --> 00:02:12,766

Now if I differentiate 5 of x to get 5,

38

00:02:12,866 --> 00:02:15,936

and now I see that

the 5 appears over there.

39

00:02:16,870 --> 00:02:20,240

Now since the integral of cos of u du

40

00:02:20,307 --> 00:02:24,978

equals sin of u plus C

, then I can conclude that

41

00:02:25,045 --> 00:02:31,118

cosine 5 of x times 5 integrated

is sin 5x plus C.

42

00:02:33,654 --> 00:02:37,591

And of course I can check

this using differentiation

43

00:02:37,658 --> 00:02:40,427

to make sure that I get the integrand back

44

00:02:41,862 --> 00:02:43,163

always worth remembering.

45

00:02:43,230 --> 00:02:47,534

Similarly, here,

if I identify x square plus 2

46

00:02:47,601 --> 00:02:49,770

to the power 5 as u to the 5,

47

00:02:49,837 --> 00:02:53,073

I can do that only if I let u equals

x squared plus 2,

48

00:02:53,140 --> 00:02:55,275

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which of course differentiate becomes 2x.

49

00:02:55,409 --> 00:02:58,879

And again, I noticed

that I have an f of u times

50

00:02:58,946 --> 00:03:00,581

u prime times dx.

51

00:03:00,747 --> 00:03:06,019

Now if I remember that the integral

of u to the 5 is one sixth u6,

52

00:03:06,286 --> 00:03:10,591

then I know that x square plus 2

all raise to the power 5 times 2x

53

00:03:11,091 --> 00:03:16,630

integrated is one sixth x square plus 2

to the power 6 plus C.

54

00:03:17,231 --> 00:03:20,400

Now remember that this helps us very much

55

00:03:20,467 --> 00:03:23,904

by using an integrand

which has a product rule

56

00:03:23,971 --> 00:03:27,674

in that because unlike differentiation,

there is no such thing

57

00:03:27,741 --> 00:03:30,577

as a straight product rule in integration.

The video shows that:

$$\int f(u(x))u'(x)\mathrm{d}x = F(u(x)) + C, \text{ where } F(u) = \int f(u)\mathrm{d}u.$$

It investigates the following two integrals and evaluates them using this result:

$$\int 5\cos 5x\mathrm{d}x = \sin 5x + C, \text{ and}$$

$$\int \left(x^2+2\right)^5 2x\mathrm{d}x = \frac{1}{6}\left(x^2+2\right)^6 + C.$$

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These integrals were found by inspection, using an understanding of anti-differentiation and working backwards from the derivative. They answered the question, 'What function would allow me to take a derivative that results in this function?'

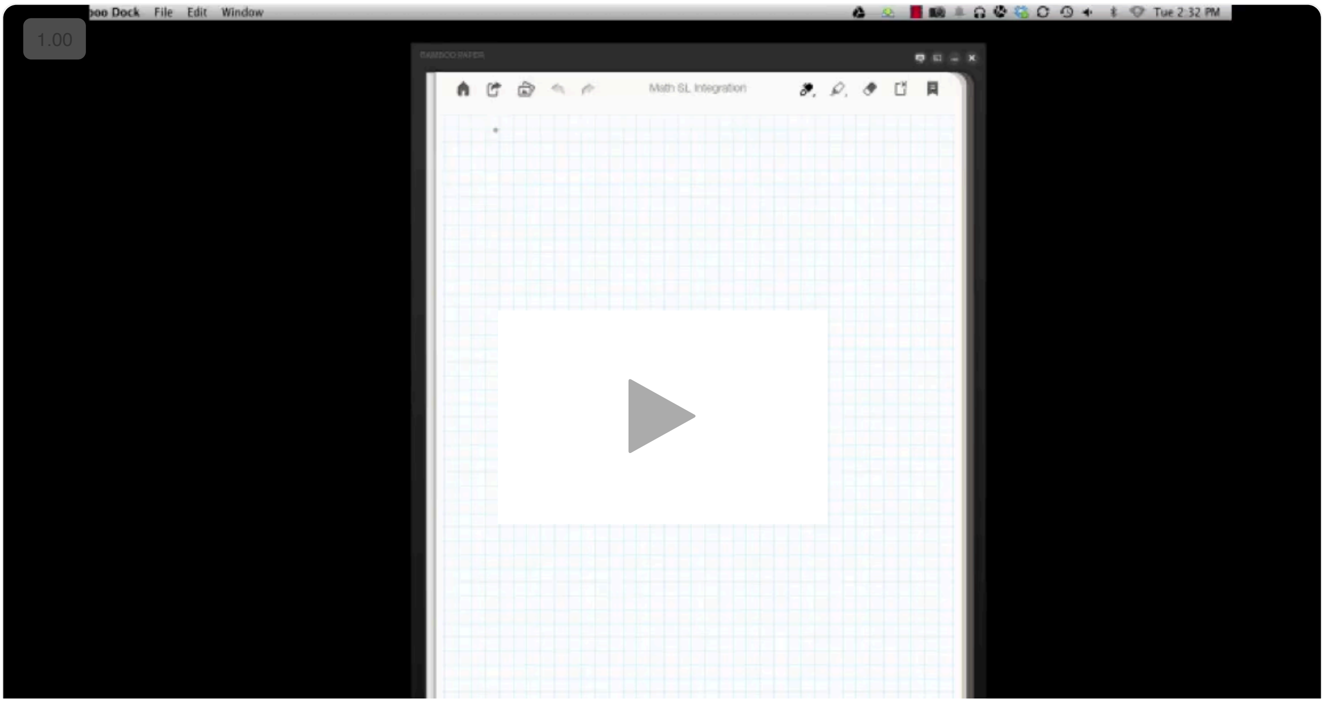
Notice that, as integration is working backwards from differentiation, this is really an example of anti-differentiation of the chain rule.

As always, upon differentiating the outcome of the integration, the result should be the original integrand. This gives a way of checking your answer.

This is commonly referred to as integration by inspection. As you gain more experience, you will get better at noticing patterns and be able to integrate a wider range of functions by inspection.

Integration by substitution

When integration by inspection does not seem to work, it is time to use a more formal approach, which is integration by substitution . This involves using a formal change of variable. The process is explored in the following video.



Video 2. Integration Using Substitution and the Chain Rule.

More information for video 2

1

00:00:00,433 --> 00:00:03,900

narrator: Now integration by inspection

required you to see the relationship

2

00:00:03,967 --> 00:00:06,200

between the two components

in the integrand

3

00:00:06,600 --> 00:00:07,900

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and that is not always very easy.

4

00:00:08,000 --> 00:00:10,933

So we're gonna do integration

by substitution proper

5

00:00:11,133 --> 00:00:13,167

by actually changing the variable.

6

00:00:13,633 --> 00:00:15,100

And let's look at an example first.

7

00:00:16,733 --> 00:00:20,633

So we're gonna look

at the integration of x times sin

8

00:00:20,833 --> 00:00:22,033

of x squared dx,

9

00:00:23,000 --> 00:00:24,967

and we're gonna make

a substitution formally,

10

00:00:25,133 --> 00:00:27,067

and I'm gonna say let u be the variable

11

00:00:27,133 --> 00:00:29,067

that makes this complicated,

which is x squared.

12

00:00:29,133 --> 00:00:31,467

And then we're gonna

differentiate u prime,

13

00:00:31,533 --> 00:00:33,400

which is really du by dx,

14

00:00:33,600 --> 00:00:35,633

which is simple enough, it's 2x.

15

00:00:35,733 --> 00:00:40,733

Now first of all, we are going to

rewrite du is 2x times dx,

16

00:00:40,933 --> 00:00:45,767

so that actually we get

1 over 2x times du is equal to dx

17

00:00:45,833 --> 00:00:47,100

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because what we need to do,

18

00:00:47,167 --> 00:00:49,733

we are actually going

to change all the x's into integration

19

00:00:49,800 --> 00:00:53,000

and that includes the measure dx.

20

00:00:53,467 --> 00:00:55,567

And then we're gonna make

all the substitutions.

21

00:00:55,633 --> 00:00:59,867

So we had x times sin x squared dx,

22

00:01:00,067 --> 00:01:04,167

which now becomes integral of x times sin

23

00:01:04,667 --> 00:01:10,167

of u times du by 2x

and you see that the x is cancel out.

24

00:01:10,233 --> 00:01:14,600

So we left with a half times

integral of sin of u du,

25

00:01:14,800 --> 00:01:16,500

which using our standard integrals

26

00:01:16,567 --> 00:01:20,233

becomes minus a half cosine of u plus C.

27

00:01:20,300 --> 00:01:23,667

Of course, now we need to go back to axis.

28

00:01:23,733 --> 00:01:27,100

So a half cosine of x squared plus C.

29

00:01:27,800 --> 00:01:29,200

Don't forget always to check.

30

00:01:29,667 --> 00:01:33,733

So if we differentiate minus half

cosine of x squared,

31

00:01:35,100 --> 00:01:38,400

if we differentiate that,

then you should get

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32

00:01:38,700 --> 00:01:41,100

the integrand so minus a half times

33

00:01:41,167 --> 00:01:46,933

minus sin of x squared times

2x using our chain rule.

34

00:01:47,000 --> 00:01:50,033

And you see that we left

with sin x square times x,

35

00:01:50,100 --> 00:01:51,567

which was the integrand.

36

00:01:52,333 --> 00:01:55,633

The other example

is the integral of x divided

37

00:01:55,867 --> 00:01:59,800

by the square root of x squared minus 4.

38

00:02:03,267 --> 00:02:07,400

So let's use as our substitution u equals

39

00:02:07,467 --> 00:02:09,167

x squared minus 4 here,

40

00:02:09,233 --> 00:02:14,467

and then we're gonna differentiate

du by dx is equal to 2x

41

00:02:14,833 --> 00:02:19,033

and that again we are going to

write dx as a function of du,

42

00:02:19,100 --> 00:02:20,367

which is the same as before.

43

00:02:20,433 --> 00:02:21,867

Then we're gonna make a substitution.

44

00:02:21,933 --> 00:02:24,300

So we add the integral

of x divided a square root

45

00:02:24,367 --> 00:02:26,167

of x squared minus 4 dx,

46

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00:02:26,233 --> 00:02:30,000

which now becomes the integral

of x times u to the minus half.

47

00:02:30,067 --> 00:02:32,767

We write explicitly as a power law times

48

00:02:33,000 --> 00:02:34,933

du divided by 2x.

49

00:02:35,167 --> 00:02:36,300

So the x's cancel

50

00:02:36,400 --> 00:02:38,900

and we left with a half times the integral

51

00:02:38,967 --> 00:02:40,767

of u to the minus a half du,

52

00:02:41,067 --> 00:02:45,933

which we can use as standard integrals

for becomes a half times 2

53

00:02:46,000 --> 00:02:48,600

times u to the plus a half plus C.

54

00:02:48,667 --> 00:02:50,800

And now we substitute back in favor of x,

55

00:02:50,867 --> 00:02:52,833

so it becomes x square minus 4

56

00:02:52,900 --> 00:02:54,833

to the power of one half plus C.

57

00:02:55,067 --> 00:02:56,300

Now of course we are going to check

58

00:02:56,367 --> 00:02:59,200

after all this work,

so we are gonna differentiate

59

00:02:59,400 --> 00:03:01,533

x square minus 4

to the power of one a half,

60

00:03:01,700 --> 00:03:04,567

and then that of course becomes one a half

61

00:03:04,900 --> 00:03:08,300

x squared minus 4 to the power

of minus a half

62

00:03:08,367 --> 00:03:10,733

times 2x using the chain rule.

63

00:03:10,800 --> 00:03:13,433

And we find that we recover d integrant.

64

00:03:13,500 --> 00:03:14,667

And that was that.

The procedure laid out in the video is as follows:

1. Make the substitution for a part of the integrand, $u = f(x)$.
2. Differentiate: $\frac{\mathrm{d}u}{\mathrm{d}x} = f'(x)$.
3. Substitute u into the integrand, and also substitute $\frac{\mathrm{d}u}{f'(x)} = \mathrm{d}x$.
4. Make any cancellations in the integrand that involve the variable x (there should be no x remaining) and now integrate with respect to the variable u.
5. In the final expression, change back to x by making use of $f(x) = u$.

Using this procedure, the following results are obtained in the video:

1. $\int x \sin x^2 \mathrm{d}x = -\frac{1}{2}\cos x^2 + C$, and
2. $\int \frac{x}{\sqrt{x^2-4}}\mathrm{d}x = \left(x^2-4\right)^{\frac{1}{2}} + C$.

The most difficult part of the process is choosing u . This comes with experience, but in the most straightforward cases, u is the inside of the most complex part of the function. Often, u will be the argument of a trigonometric, logarithmic, exponential, rational or radical function. In more complex cases, you may have to do some algebra first to make it work.

Just because one mathematician integrates by inspection and another integrates by substitution does not make either of them wrong. Both methods, if done correctly, will yield the same result. It is quite normal for you to start out using substitution frequently and then to use inspection more as you gain experience.

⚠ Be aware

- You must remove of all the x after the substitution, and end up with an integral in one variable only, such as u . This includes the substitution $\frac{\mathrm{d}u}{f'(x)} = \mathrm{d}x$.
- Do not forget to substitute back to obtain an expression in the original variable, usually x , after you have integrated using the substituted variable.

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Example 1

★★☆

Find the indefinite integral of $f(x) = 12x\cos 3x^2$.

Steps	Explanation
$\int 12\cos 3x^2 \mathrm{d}x$	
$u = 3x^2$	$\frac{\mathrm{d}u}{\mathrm{d}x} = 6x$ or $\mathrm{d}x = \frac{\mathrm{d}u}{6x}$
$\int 12x\cos 3x^2 \mathrm{d}x$	Substitute for u
$= \int 2\cos u \mathrm{d}u$	Cancel
$= 2\sin u$	Integrate
$= 2\sin 3x^2 + C$	Substitute for x

Example 2

★★☆

Find the indefinite integral of $f(x) = \frac{x}{x^2+1}$.

Steps	Explanation
$\int \frac{x}{x^2+1}\mathrm{d}x$	
$u = x^2 + 1$	$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ or $\mathrm{d}x = \frac{\mathrm{d}u}{2x}$
$\int \frac{x}{x^2+1}\mathrm{d}x$	
$= \int \frac{x}{u}\frac{\mathrm{d}u}{2x}$	Substitute for u
$= \int \frac{1}{2u}\mathrm{d}u$	Cancel
$= \frac{1}{2}\ln\lvert u\rvert$	I ntegrate
$= \frac{1}{2}\ln\lvert x^2+1\rvert + C$	Substitute for x

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This example leads to an interesting result:

1. If $f(x) > 0$, then $\int \frac{f'(x)}{f(x)} \mathrm{d}x = \ln f(x) + C$.
2. In general, $\int \frac{f'(x)}{f(x)} \mathrm{d}x = \ln |f(x)| + C$.

Note that the absolute value is not required as $x^2 + 1$ is always positive. Therefore, the answer could have been given as $= \frac{1}{2} \ln(x^2 + 1) + C$.

There will be times when the substitution is too complex to recognise, especially when dealing with trigonometric functions. Often, a trigonometric identity is useful in identifying the substitution.

Example 3

★★☆

Find the indefinite integral of $f(x) = \sin^3 x$.

Steps	Explanation
$\sin^3 x = \sin^2 x \cos x = (1 - \cos^2 x) \sin x = \sin x - \cos^2 x \sin x$ $\int \sin^3 x \mathrm{d}x = \int (\sin x - \cos^2 x \sin x)\, \mathrm{d}x = \int \sin x \mathrm{d}x - \int \cos^2 x \sin x \mathrm{d}x$	
$u = \cos x$	$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$ or $\mathrm{d}x = \frac{\mathrm{d}u}{-\sin x}$
$\int \sin x \mathrm{d}x = -\cos x$	Integrate first term
$\int \cos^2 x \sin x \mathrm{d}x = \int u^2 \sin x \frac{\mathrm{d}u}{-\sin x}$	Substitute for u in the second term
$= -\int u^2 \mathrm{d}u$	Cancel
$= -\frac{1}{3} u^3$	Integrate
$= -\frac{1}{3} \cos^3 x$	Substitute for x
$\int \sin^3 x \mathrm{d}x = -\cos x + \frac{1}{3} \cos^3 x + C$	

4 section questions ⌄

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5. Calculus / 5.11 Further integration

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Area below the derivative graph

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/area-below-the-derivative-graph-id-28199/print/) Assign

Subtopic 5.5 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26177/) introduced the concept of both the indefinite integral and definite integral. You learned how to find approximate values of definite integrals by technology and how the concept of the definite integral is connected to the area of certain regions. Within the current subtopic you have seen how to find certain indefinite integrals. The next section connects the concepts of indefinite and definite integrals. As an illustration of this, look again at the applet (below) first introduced in section 5.5.7 (/study/app/math-ai-hl/sid-132-cid-761618/book/investigation-id-26184/).

The applet shows the graph of a function f and the graph of its derivative, f'. It also lets you move a point on the x-axis and calculates the definite integral of $f'(x)$ over the interval from the origin to this moving point. If you use the notation p for the x-coordinate of this moving point, the applet shows the point $\mathrm{P}\left(p, \int_0^p f'(x)\mathrm{d}x\right)$, and also shows the trace of this point.

Interactive 1. Area Under the Derivative Curve.

More information for interactive 1

In this interactive user can investigate the relationship between the geometrical concept of the area of a region and the process of differentiation/anti-differentiation. The interactive shows the graph of a function f and the graph of its derivative, f'. It also lets you move a point on the x-axis and calculates the definite integral of $f'(x)$ over the interval from the origin to this moving point. If you use the notation p for the x-coordinate of this moving point, the applet displays the point $P\left(p, \int_0^p f'(x)dx\right)$ and shows the trace of this point as you move it. By adjusting the curve and observing the trace of *P*, you can explore how the integral of the derivative relates to the original function.

The interactive applet visually explores the relationship between the geometrical concept of the area of a region and the process of differentiation/anti-differentiation. The interactive allows users to explore the relationship between a function $f(x)$ and its derivative $f'(x)$. The applet displays the graph of $f(x)$ in blue and the graph of its derivative $f'(x)$ in pink. There are two options for understanding the interactive. 'Trace point P' and 'Adjust Curve'
When 'Trace point P' is selected, users can move a red point P along the x-axis, starting from the origin, and the applet calculates the definite integral of $f'(x)$ over the interval from 0 to p, where p is the x-coordinate of P. The point $P\left(p, \int_0^p f'(x)dx\right)$ traces its path as P moves, highlighted in light blue on the graph.

The area under the derivative curve f' over an interval $[0, p]$ represents the net change in the original function f over that interval. This is because the definite integral of f' from 0 to p gives $f(p) - f(0)$. For example, in the provided diagram, the applet shows $\int_0^{2.49} f'(x)dx = 0.46$, meaning that $f(2.49) = f(0) + 0.46$. If $f(0)$ is known, this allows you to determine the value of f at any point p by adding the area under f' to the initial value. This connection highlights how integration and differentiation are inverse processes, with the area under the derivative curve directly contributing to the

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change in the original function.

By observing the trace of point P, the user discovers that the computed integral reconstructs f (x), demonstrating the Fundamental Theorem of Calculus.

The final challenge asks the user to formalise this observation mathematically.

The applet provides an engaging, hands-on way to understand how differentiation/anti-differentiation can be determined by the area of a region. Through this interactive applet, users gain an intuitive grasp of the Fundamental Theorem of Calculus.

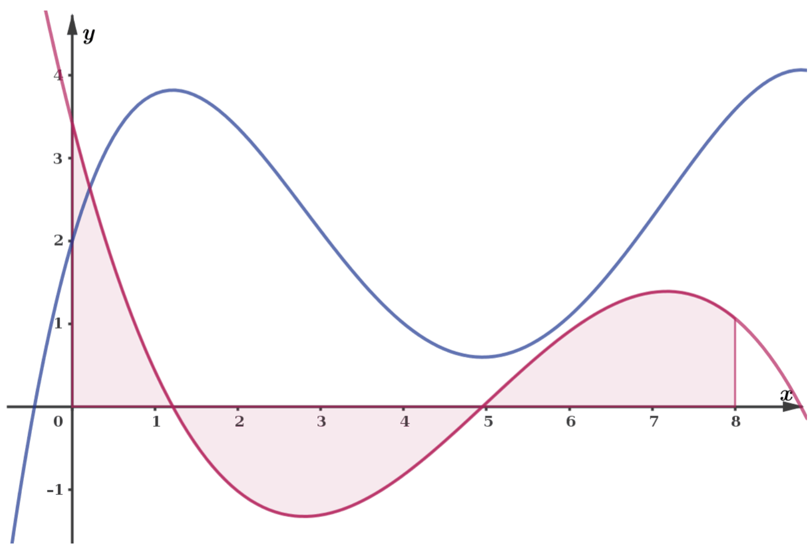
Activity

- Move the red point on the x-axis.
- Is there something you notice about the trace of point P?
- Change the shape of the curve and repeat the process. Do you have to modify your observation?

Example 1

★★☆

The diagram below shows the graph of a function f and the derivative, f'. The diagram also shows the region illustrating $\int_0^8 f'(x)\mathrm{d}x$.



More information

The image is a diagram depicting the graph of a function (f) and its derivative (f'). The X-axis represents the interval from 0 to 8. The Y-axis indicates the values of function (f) and its derivative. The graph shows a blue curve representing (f) and a red curve illustrating (f'). Additionally, there is a shaded red region between the X-axis and the curve representing the area under the derivative from X=0 to X=8. This shaded area corresponds to the integral (\int_0^8 f'(x)\mathrm{d}x). The calculated value for this integral is shown as 1.59 after the image. The diagram demonstrates how the integral of (f') over this interval results in the given area value.

[Generated by AI]

Student view

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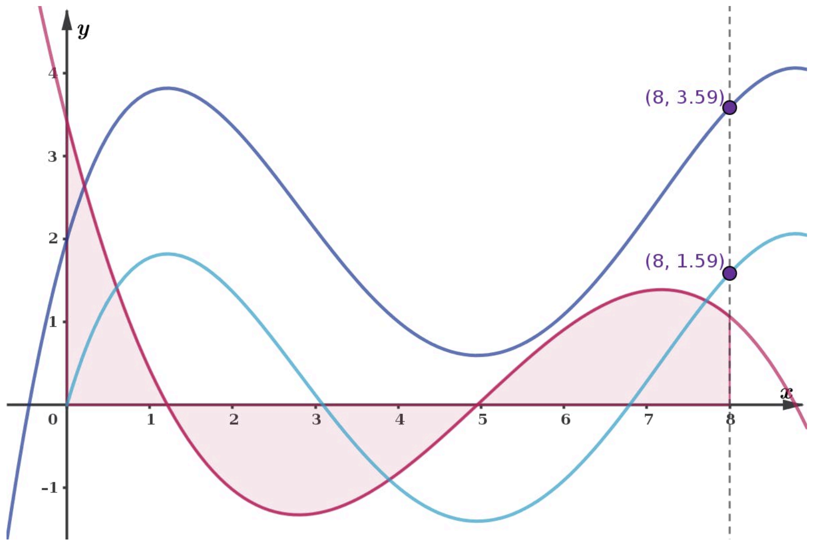
The value of the integral is $\int_0^8 f'(x)\mathrm{d}x = 1.59$ and $f(0) = 2$.

Find $f(8)$.

You can observe in the activity above that the graph of the area function $A(p) = \int_0^p f'(x)\mathrm{d}x$ is a translation of the graph of f.



Since $A(0) = \int_0^0 f'(x)\mathrm{d}x = 0$ and $f(0) = 2$, this is a shift by 2 units, so

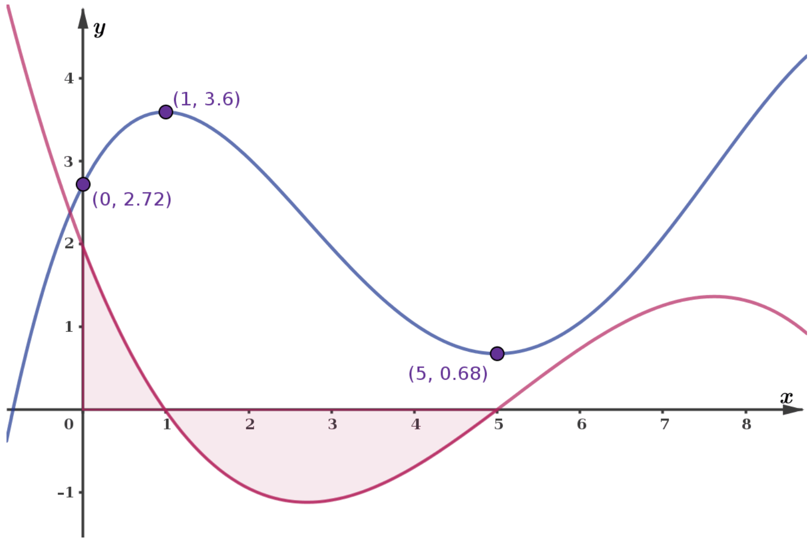
$$f(8) = 2 + A(8) = 2 + \int_0^8 f'(x)\mathrm{d}x = 2 + 1.59 = 3.59.$$

Example 2

★★☆

The diagram below shows the graph of a function f and the derivative, f'. The diagram also shows the y-intercept, $(0, 2.72)$, the local maximum point, $(1, 3.6)$ and the local minimum point, $(5, 0.68)$ of the the graph of f.

Student view



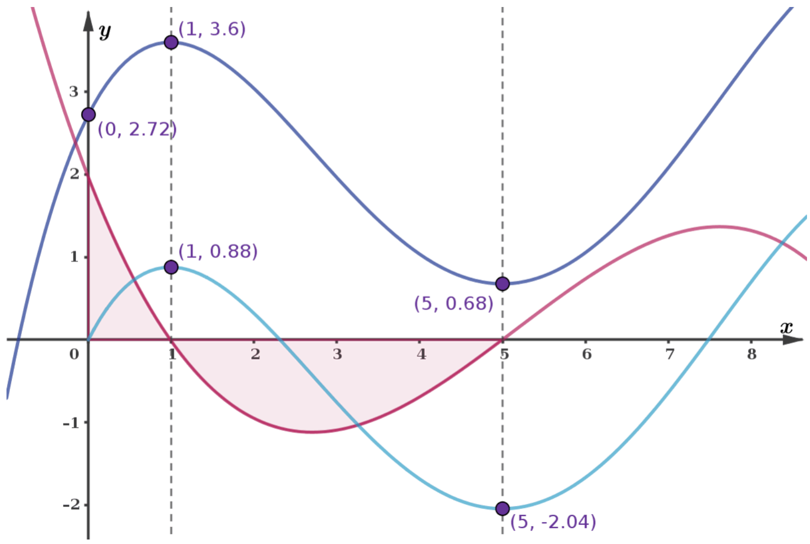
More information

The image displays a graph with two curves representing a function (f) and its derivative (f'). The x-axis is labeled from 0 to 8, and the y-axis is labeled from -1 to 4. Key points on the graph of (f) include the y-intercept (0, 2.72), a local maximum at (1, 3.6), and a local minimum at (5, 0.68). The areas where the derivative (f') is below the x-axis are shaded to indicate regions of negative slope, creating a visual contrast between positive and negative gradients along the curve. The task is to find the area of the shaded region bounded by the x-axis and the derivative (f'). This visualization is useful for understanding the relationship between the function (f) and its rate of change.

[Generated by AI]

Find the area of the shaded region bounded by the x axis and the graph of f'.

Besides the information given in the question, the diagram below also shows the graph of the area function $A(p) = \int_0^p f'(x)\mathrm{d}x$. You will use the observation that the graph of f and the graph of A are translations of each other. More precisely, $f(p) = A(p) + f(0)$.



Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

- Using the coordinates of the y-intercept of f, you get $f(0) = 2.72$.
- Using the coordinates of the local maximum of f, you get

$$\begin{aligned} f(1) &= 3.6 \\ A(1) + f(0) &= 3.6 \\ \int_0^1 f'(x)\mathrm{d}x &= 3.6 - f(0) = 3.6 - 2.72 = 0.88 \end{aligned}$$

  Since the local maximum point on the graph of f corresponds to an x-intercept of the graph of f', the area of the shaded region above the x-axis is 0.88.
- Using the coordinates of the local minimum of f and the rules of the definite integral, you get

$$\begin{aligned} f(5) &= 0.68 \\ A(5) + f(0) &= 0.68 \\ \int_0^5 f'(x)\mathrm{d}x &= 0.68 - f(0) = 0.68 - 2.72 = -2.04 \\ \int_1^5 f'(x)\mathrm{d}x &= \int_0^5 f'(x)\mathrm{d}x - \int_0^1 f'(x)\mathrm{d}x \\ \int_1^5 f'(x)\mathrm{d}x &= -2.04 - 0.88 = -2.92 \end{aligned}$$

  Since the local minimum point on the graph of f also corresponds to an x-intercept of the graph of f', this means that the area of the shaded region below the x-axis is 2.92.

Adding these two values, the area of the shaded region you get $0.88 + 2.92 = 3.8$ units squared.

In the next section you will formalise the observations of the activity, and apply the claim stated to find exact values of definite integrals.

3 section questions

5. Calculus / 5.11 Further integration

Newton–Leibniz formula

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/newtonleibniz-formula-id-28200/print/) Assign

The following claim connects the concepts of definite and indefinite integrals.

Important

If f is a continuous function defined on the interval $[a, b]$ and F is an anti-derivative of f (so $F'(x) = f(x)$), then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Student view

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The proof of this claim goes beyond the syllabus. If you are interested in the claim in full generality and a formal proof, search for **Newton-Leibniz formula** or the **fundamental theorem of calculus** on the internet.

To fully understand the connection between definite and indefinite integrals and the similarity in the notation, let us look at how to apply the claim above. Note that finding the anti-derivative mentioned in the claim involves finding the indefinite integral.

✓ **Important**

You can find the definite integral $\int_a^b f(x)\,\mathrm{d}x$ in two steps:

1. Find F, an anti-derivative of f by finding the indefinite integral $\int f(x)\,\mathrm{d}x$.
2. Evaluate this anti-derivative at $x = b$ and $x = a$ and find the difference of these values: $\int_a^b f(x)\,\mathrm{d}x = \left[\int f(x)\,\mathrm{d}x\right]_a^b = \left[F(x)\right]_a^b = F(b) - F(a).$

Consider a simple polynomial function, $f(x) = 12x^3$. Finding the indefinite integral using the power rule results in

$$F(x) = \int f(x)\mathrm{d}x = \int 12x^3\mathrm{d}x = 12\left(\frac{1}{4}\right)x^4 = 3x^4 + C.$$

To make this a definite integral requires limits. To find the definite integral of $f(x) = 12x^3$ across $[1, 2]$:

$$\int_a^b f(x)\mathrm{d}x = \left[\int f(x)\mathrm{d}x\right]_a^b = \left[F(x)\right]_a^b = F(b) - F(a)$$
$$\int_1^2 12x^3\mathrm{d}x = \left[\int 12x^3\mathrm{d}x\right]_1^2 = \left[3x^4\right]_1^2 = 3(2)^4 - 3(1)^4 = 48 - 3 = 45.$$

Note that the integration constant was not used in this last line. It does not matter which antiderivative you use, the definite integral is the same. You can use any constant, it will cancel out when taking the difference $F(b) - F(a)$.

Example 1

★☆☆

- Find the exact value of $\int_0^\pi \sin x\,\mathrm{d}x$.
- Find the exact value of $\int_0^1 x^2\,\mathrm{d}x$.
- Find the exact value of $\int_1^2 \frac{1}{x}\,\mathrm{d}x$.
- Find the exact value of $\int_{-2}^1 \mathrm{e}^x\,\mathrm{d}x$.

Student view

- Since $\int \sin x\,\mathrm{d}x = -\cos x$,
  $$\int_0^\pi \sin x\,\mathrm{d}x = \left[-\cos x\right]_0^\pi = (-\cos\pi) - (-\cos 0) = 1 + 1 = 2.$$

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- Since $\int x^2 \, \mathrm{d}x = \frac{x^3}{3}$,
  $$\int_0^1 x^2 \, \mathrm{d}x = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$
- Since $\int \frac{1}{x} \, \mathrm{d}x = \ln x$,
  $$\int_1^2 \frac{1}{x} \, \mathrm{d}x = \left[\ln x\right]_1^2 = \ln 2 - \ln 1 = \ln 2.$$
- Since $\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x$,
  $$\int_{-2}^1 \mathrm{e}^x \, \mathrm{d}x = \left[\mathrm{e}^x\right]_{-2}^1 = \mathrm{e}^1 - \mathrm{e}^{-2} = \mathrm{e} - \frac{1}{\mathrm{e}^2}.$$

Example 2

★★☆

Find the indefinite and definite integrals in the table below.

Steps	Explanation
$\int \cos(2x - \pi) \, \mathrm{d}x =$	$\int_{\frac{\pi}{3}}^{3\pi} \cos(2x - \pi) \, \mathrm{d}x =$
$\int \frac{2x}{x^2+1} \, \mathrm{d}x =$	$\int_1^2 \frac{2x}{x^2+1} \, \mathrm{d}x =$
$\int x(x-3)^2 \, \mathrm{d}x =$	$\int_0^3 x(x-3)^2 \, \mathrm{d}x =$
$\int x^2 \mathrm{e}^{x^3} \, \mathrm{d}x =$	$\int_{-1}^1 x^2 \mathrm{e}^{x^3} \, \mathrm{d}x =$

Steps	Explanation
$\int \cos(2x - \pi) \, \mathrm{d}x = \frac{1}{2}\sin(2x - \pi) + c$	$\int_{\frac{\pi}{3}}^{3\pi} \cos(2x - \pi) \, \mathrm{d}x = \frac{\sqrt{3}}{4}$
$\int \frac{2x}{x^2+1} \, \mathrm{d}x = \ln(x^2+1) + c$	$\int_1^2 \frac{2x}{x^2+1} \, \mathrm{d}x = \ln \frac{5}{2}$
$\int x(x-3)^2 \, \mathrm{d}x = \frac{x^4}{4} - 2x^3 + \frac{9x^2}{2} + c$	$\int_0^3 x(x-3)^2 \, \mathrm{d}x = \frac{27}{4}$
$\int x^2 \mathrm{e}^{x^3} \, \mathrm{d}x = \frac{\mathrm{e}^{x^3}}{3} + c$	$\int_{-1}^1 x^2 \mathrm{e}^{x^3} \, \mathrm{d}x = \frac{\mathrm{e}^2 - 1}{3\mathrm{e}}$

With the following applets you can check your understanding.

Student view

Interactive 1. Definite and Indefinite Integrals.

More information for interactive 1

This interactive allows the users to understand and practice **Newton-Leibniz rule**, it establishes a fundamental connection between differentiation and integration under definite integrals, demonstrating how they are the inverse operations. Users can apply this rule to fully understand the connection between definite and indefinite integrals and the similarity in the notation. To find the area under the curve $f(x)$ from a to b, the users can find the antiderivative $F(x)$, evaluate it at a and b and then subtract the result. For example, $f(x) = x^2$ from $x = 1$ to $x = 3$.

The antiderivative of x^2 is $F(x) = \frac{1}{3}(x^3)$.

On applying Newton Leibniz rule,

$$F(x) = \int_1^3 x^2 dx = F(3) - F(1)$$

$$= \left(\frac{1}{3}\right) \cdot (3^3) - \left(\frac{1}{3}\right) \cdot (1)^3$$
$$= 9 - \frac{1}{3}$$
$$= \frac{26}{3}$$

Users can practice with new questions and check their answers by clicking on the button 'Click here for a new question' on top of the screen. This interactive helps users to understand calculus under curves and solve many other problems in calculus and related fields.

Interactive 2. Definite and Indefinite Integrals.

More information for interactive 2

This interactive allows users to understand and practice the Newton—Leibniz rule, which establishes a fundamental connection between differentiation and integration in the context of definite integrals. It demonstrates how integration and differentiation are inverse operations and highlights the relationship between definite and indefinite integrals, including their notational similarities.

To calculate the area under a curve $f(x)$ from a to b, users are guided to find the antiderivative $F(x)$, evaluate it at the endpoints a and b, and subtract

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the results: $F(b) - F(a)$.

For example, if $f(x) = x^2$ from $x = 1$ to $x = 3$.

The antiderivative of x^2 is $F(x) = \frac{1}{3}(x^3)$.

On applying Newton Leibniz rule, $F(x) = \int_1^3 x^2 dx = F(3) - F(1)$

$$= \left(\frac{1}{3}\right)(3^3) - \left(\frac{1}{3}\right)(1^3)$$

$$= 9 - \frac{1}{3}$$

$$= \frac{26}{3}$$

Users can generate multiple problems using the “Click here for new question” option and check their answers with “Show answers.” This hands-on approach helps solidify users' conceptual understanding of integration as accumulation and provides practical experience in solving calculus problems involving areas under curves.

Example 3

★★☆

What is the value of $m > 0$ if $\int_m^{m^2} \frac{1}{x}\, \mathrm{d}x = 1$?

For $x > 0$, $\int \frac{1}{x}\, \mathrm{d}x = \ln x + c$. So,

$$\int_m^{m^2} \frac{1}{x}\, \mathrm{d}x = \big[\ln x\big]_m^{m^2} = \ln m^2 - \ln m = \ln \frac{m^2}{m} = \ln m.$$

Hence,

$$\begin{aligned} \ln m &= 1 \\ m &= \mathrm{e}. \end{aligned}$$

Below is a list of some properties of definite integrals. These properties are sometimes useful in evaluating integrals.

✓ **Important**

If f and g are continuous functions (defined on the given intervals), then

- $\int_a^b f(x) \pm g(x)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x \pm \int_a^b g(x)\, \mathrm{d}x$
- $\int_a^b cf(x)\, \mathrm{d}x = c\int_a^b f(x)\, \mathrm{d}x$
- $\int_a^b f(x)\, \mathrm{d}x + \int_b^c f(x)\, \mathrm{d}x = \int_a^c f(x)\, \mathrm{d}x$

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All of these are consequences of the Newton-Leibniz formula.

If $F'(x) = f(x)$ and $G'(x) = g(x)$, then

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x &= \Big[F(x)\Big]_a^b \pm \Big[G(x)\Big]_a^b \\ &= \Big(F(b) - F(a)\Big) \pm \Big(G(b) - G(a)\Big) \\ &= \Big(F(b) \pm G(b)\Big) - \Big(F(a) \pm G(a)\Big) \\ &= \Big[F(x) \pm G(x)\Big]_a^b \\ &= \int_a^b f(x) \pm g(x)\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}c\int_a^b f(x)\,\mathrm{d}x &= c\Big[F(x)\Big]_a^b \\ &= c\Big(F(b) - F(a)\Big) \\ &= cF(b) - cF(a) \\ &= \Big[cF(x)\Big]_a^b \\ &= \int_a^b cf(x)\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x &= \Big[F(x)\Big]_a^b + \Big[F(x)\Big]_b^c \\ &= \Big(F(b) - F(a)\Big) + \Big(F(c) - F(b)\Big) \\ &= F(c) - F(a) \\ &= \Big[F(x)\Big]_a^c \\ &= \int_a^c f(x)\,\mathrm{d}x\end{aligned}$$

Example 4

★☆☆

For the functions f and g,

$$\int_0^5 f(x)\,\mathrm{d}x = 4, \int_0^2 f(x)\,\mathrm{d}x = 1 \text{ and } \int_5^2 g(x)\,\mathrm{d}x = 6.$$

Find $\int_2^5 2f(x) - 7g(x)\,\mathrm{d}x$.

According to the property

$$\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x,$$

Student view

the following is true:

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$$\begin{aligned}\int_0^2 f(x)\,\mathrm{d}x + \int_2^5 f(x)\,\mathrm{d}x &= \int_0^5 f(x)\,\mathrm{d}x \\ 1 + \int_2^5 f(x)\,\mathrm{d}x &= 4 \\ \int_2^5 f(x)\,\mathrm{d}x &= 4 - 1 = 3\end{aligned}$$

According to the property

$$\int_a^b g(x)\,\mathrm{d} = -\int_b^a g(x)\,\mathrm{d}x,$$

the following is true:

$$\int_2^5 g(x)\,\mathrm{d} = -\int_5^2 g(x)\,\mathrm{d}x = -6$$

Using

$$\int_a^b kf(x)\,\mathrm{d} = k\int_a^b f(x)\,\mathrm{d}x,$$

the following is also true:

$$\begin{aligned}\int_2^5 2f(x) - 7g(x)\,\mathrm{d}x &= 2\int_2^5 f(x)\,\mathrm{d}x - 7\int_2^5 g(x)\,\mathrm{d}x \\ &= 2 \times 3 - 7 \times (-6) = 48\end{aligned}$$

5 section questions

5. Calculus / 5.11 Further integration

Checklist

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What you should know

By the end of this subtopic you should be able to:

- find the integral of a function using the following standard integrals:
  - $\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, n \neq -1$
  - $\int (ax+b)^n \mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$
  - $\int \frac{1}{x} \mathrm{d}x = \ln x + C$
  - $\int \frac{1}{ax+b} \mathrm{d}x = \frac{1}{a} \ln |ax+b| + C$
  - $\int \mathrm{e}^x \mathrm{d}x = \mathrm{e}^x + C$

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- $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$
- $\int \sin x dx = -\cos x + C$
- $\int \sin(ax+b) dx = \frac{-\cos(ax+b)}{a} + C$
- $\int \cos x dx = \sin x + C$
- $\int \cos(ax+b) dx = \frac{\sin(ax+b)}{a} + C$

- find indefinite and definite integrals of functions by inspection and substitution.

5. Calculus / 5.11 Further integration

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/investigation-id-28202/print/) Assign

Throughout this subtopic, you studied the relationships between functions, their derivatives and their anti-derivatives (or integrals). Basically, a derivative of a function describes the slope of a tangent line to the curve at a specific x-value, and the integral describes the area between the curve and the x-axis along an interval. Explore the applet and consider these questions:

- Can you see the relationships between the slope of $f(x)$ and the value of $f'(x)$?
- How about the slope of $\int_0^x f(x')\, dx'$ and the value of $f(x)$? What about the other direction?
- Does the area under $f(x)$ relate to the value of $\int_0^x f(x')\, dx'$?
- Does the area under $f'(x)$ relate to the value of $f(x)$?

Interactive 1. Relationships Between Functions, Their Derivatives and Their Anti-Derivatives.

More information for interactive 1

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This interactive tool allows users to explore the relationships between a function, its derivative, and its antiderivative (integral) through dynamic visualizations. The graph of $y = f(x)$ represents the function, $y = f'(x)$ represents its derivative, and $\int_0^x f(x')\,dx'$ represents its antiderivative. By interacting with the graph, users can observe how changes in the function affect its derivative and integral, reinforcing key calculus concepts.

A derivative represents the **rate of change** of a function. As users move the red dot along the function $f(x)$, they will see how the slope at that point determines the corresponding value of $f'(x)$. When $f(x)$ is increasing, its derivative is positive, and when $f(x)$ is decreasing, its derivative is negative. At points where $f(x)$ has a local maximum or minimum, the slope is zero, meaning $f'(x)$ also equals zero at those points. This visually confirms that the derivative measures how steeply a function is changing at any given moment.

The integral, on the other hand, represents the **accumulated area** under the function's curve. As users move the red dot, they will notice that the integral increases when $f(x)$ is positive and decreases when $f(x)$ is negative. This means the integral continuously sums up the values of $f(x)$ over an interval, reflecting the net accumulation of the function's values over time. The integral graph also shows how an area under the x-axis contributes negatively to the accumulation, whereas an area above the x-axis contributes positively.

By observing these interactions, users will develop an intuitive understanding of how differentiation and integration are inverse operations. The derivative graph confirms how the slope of $f(x)$ translates into rate of change, while the integral graph reveals how accumulated areas relate to the original function. This interactive serves as a powerful visual aid, reinforcing the fundamental principles of calculus by linking function behavior, instantaneous rates of change, and accumulated quantities in a hands-on manner.

Rate subtopic 5.11 Further integration

Help us improve the content and user experience.

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