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Teacher view
0 (https://intercom.help/kognity)

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1. Number and algebra / 1.15 Proof

The big picture

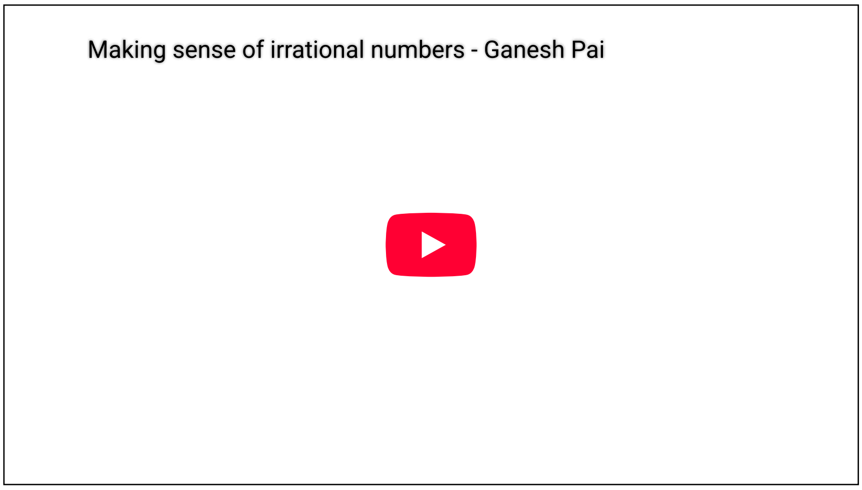
Generally, there are three types of ideas in mathematics: axioms, theorems and conjectures.

Axioms are the basic assumption that do not require proof. They are used along with logical reasoning to prove conjectures which then become theorems.

Exploration of new areas of mathematical thinking requires the use of conjectures to push the boundaries of mathematical research. Mathematicians want to be able to prove new conjectures so that they will become theorems that will not be proven false at a later date.

In this subtopic you will learn two formal methods of proof – proof by contradiction and proof by induction.

One of the earliest examples of the use of a proof by contradiction was very controversial. Watch the video to learn more.



Concept

Formal proofs , such as proof by induction, require a specific sequence of steps to create a logical argument and to prove a mathematical statement. How do you know that a proof is valid?

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Theory of Knowledge

Mathematics has strict standards for ‘proving’ something true. Consider how other areas of knowledge (AOKs) ‘prove’ knowledge as true while you contemplate the following Knowledge Question: What factors within an area of knowledge contribute to standards of proof within that AOK?

1. Number and algebra / 1.15 Proof

Proof by contradiction

A proof by contradiction is mostly used for ‘If… then…’ statements. It starts by identifying what is being implied and assuming that the implication is false.

This allows you to use theorems and axioms to show that the assumption that the implication is false leads to a logical contradiction or an absurd result. This proves that the original implication must be true.

International Mindedness

The idea of working to show that a result is absurd is called *reductio ad absurdum.* This Latin name is given to a concept that originated with Greek philosophers.

Example 1

★☆☆

Prove by contradiction that if n^2 is even then n is also even.

Steps	Explanation
Assume that it is not true that if n^2 is even then n is also even. This would mean that there is an n which is odd, but n^2 even. $\therefore n = 2k+1$ where $k \in \mathbb{Z}$ $n^2 = (2k+1)^2 = 4k^2+4k+1 = 2\left(2k^2+2k\right)+1$ $\therefore n^2$ is odd, since $2k^2+2k$ is an integer number This is a contradiction.	Start by assuming that the opposite is true.
Therefore if n^2 is even then n is also even is proven by contradiction.	

Student view

Example 2

★★☆

Steps	Explanation
Assume $$\sqrt{2}=\frac{p}{q}$$ where $p, q \in \mathbb{Z}$ and p and q have no common factors.	Start by assuming that the opposite is true. Assume that $\sqrt{2}$ is a rational number. Use the definition of a rational number to write your assumption.
$\sqrt{2}=\frac{p}{q} \quad \Leftrightarrow \quad 2=\frac{p^2}{q^2} \quad \Leftrightarrow \quad 2q^2=p^2$ $\therefore p^2$ is an even integer and p is an even integer. $\therefore p=2a$ where $a \in \mathbb{Z}$ $$2q^2=(2a)^2 \quad \Leftrightarrow \quad 2q^2=4a^2 \quad \Leftrightarrow \quad q^2=2a^2$$ $\therefore q^2$ is an even integer and q is an even integer.	If n^2 is even then n is even as proven in **Example 1.**
If p and q are both even, then they have a common factor of 2 which contradicts the initial statement. Therefore, no p and q can be found such that $\sqrt{2}=\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and p and q have no common factors. The statement that $\sqrt{2}$ is irrational is proven by contradiction.	

Example 3

★★★

Prove by contradiction that $\sqrt{3}$ is irrational.

You can show this claim using a method similar to the one used in Example 2. You can use divisibility by 3 instead of 2. Try it! Let's look at a different approach now.

Student view

Steps	Explanation
Assume $\sqrt{3}=\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and p and q have no common factors.	Start by assuming that the opposite is true. Assume that $\sqrt{3}$ is a rational number. Use the definition of a rational number to write your assumption.
$\sqrt{3}=\frac{p}{q} \quad \Leftrightarrow \quad 3=\frac{p^2}{q^2} \quad \Leftrightarrow \quad 3q^2=p^2$	Here you must consider two cases: q is even and q is odd.
In the case where q is even: $q=2a$ where $a \in \mathbb{Z}$ $3(2a)^2=p^2 \quad \Leftrightarrow \quad 2\left(6a^2\right)=p^2$ $2\left(6a^2\right)$ is even $\therefore$ p^2 as well as p are even. If p and q are both even, then they have a common factor of 2 which contradicts the initial statement.	
In the case that q is odd: $q=2a+1$ where $a \in \mathbb{Z}$. $3(2a+1)^2=p^2 \Leftrightarrow 3\left(4a^2+4a+1\right)=p^2 \Leftrightarrow 2\left(6a^2+6a\right)+3=p^2$ $2\left(6a^2+6a\right)$ is even $\therefore$ $2\left(6a^2+6a\right)+3$ is odd. $\therefore$ p^2 and p are odd. This means that $p=2b+1$ where $b \in \mathbb{Z}$. $3q^2=p^2 \quad \Leftrightarrow \quad 3(2a+1)^2=(2b+1)^2$ $\Leftrightarrow \quad 12a^2+12a+3=4b^2+4b+1$ $\Leftrightarrow \quad 6a^2+6a+\frac{3}{2}=2b^2+2b+\frac{1}{2}$ $\Leftrightarrow \quad 2\left(3a^2+3a\right)+1=2\left(b^2+b\right)$ One side of the equation is even, and the other side is odd. This is a contradiction.	
Therefore, no p and q can be found such that: $\sqrt{3}=\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and p and q have no common factors. The statement that $\sqrt{3}$ is irrational is proven by contradiction.	

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You can also use proof by contradiction to prove that there are infinitely many prime numbers. This was originally proved by Euclid who lived in the 4th and 3rd centuries BC. Watch the video below to see how proof by contradiction works in this case and how it compares with Euclid's original proof.

2 section questions

1. Number and algebra / 1.15 Proof

Using counterexamples

Goldbach's conjecture states that every even integer greater than 2 can be expressed as the sum of two primes.

Activity

Show that Goldbach's conjecture holds true for $4, 6, 8, 10, 12, 14$ and 16.

Do these examples prove the conjecture for all even numbers greater than 2?

In the activity you showed that Goldbach's conjecture is true for 7 specific examples but you cannot use examples to prove the conjecture for all cases. In fact, Goldbach's conjecture, which was made in 1742, remains one of the big unproven problems in mathematics even though it has been shown to hold for approximately 10^{18} cases. You can learn more about this conjecture by watching the video below.

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Goldbach Conjecture - Numberphile

While a million examples cannot be used to prove that a conjecture is true, a single example can be used to prove that a conjecture is false. In this case you are using a counterexample that shows that the statement is not always true and is therefore false.

Example 1

★☆☆

Prove that the following conjecture is false:

For $n \in \mathbb{Z}$, if n^2 is divisible by 4, then n is divisible by 4.

Steps	Explanation
Let $n = 2$. Then $2^2 = 4$;, which is divisible by 4. But $n = 2$ is not divisible by 4. The statement is proven false by counterexample.	You can think of the squares that are divisible by 4 and work backwards. $1^2 = 1 \ \text{(not divisible)}$ $2^2 = 4 \ \text{(divisible)}$

Example 2

★☆☆

Show that the following statement is not always true: there are no positive integer solutions to $x^2 + y^2 = 8$.

Steps	Explanation
Let $x = 2$ and $y = 2$ then $2^2 + 2^2 = 8$. Therefore $x = 2$ and $y = 2$ is a positive integer solution to $x^2 + y^2 = 8$. The counterexample shows that the statement is not always true.	

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Example 3

★★☆

Prove that the following conjecture is false:

$$n^2 + 41n + 41 \text{ is prime for } n \in \mathbb{N}$$

Steps	E:
<table><tr><td>n</td><td>$n^2 + 41n + 41$</td></tr><tr><td>1</td><td>83</td></tr><tr><td>2</td><td>127</td></tr><tr><td>3</td><td>173</td></tr><tr><td>4</td><td>221</td></tr><tr><td>5</td><td>271</td></tr><tr><td>...</td><td></td></tr></table> When $n = 4$, $4^2 + 41 \times 4 + 41 = 221$. This is not a prime number since $221 = 13 \times 17$. The statement is proven false by counterexample.	You need to finc conjecture is no One way to do t In this case you various numbers the output. This by using a sprea your GDC. You can see that get is $221 = 13$ prime.
When $n = 41$, $41^2 + 41 \times 41 + 41 = 41\,(41 + 41 + 1) = 41 \times 83$, which is not prime. The statement is proven false by counterexample.	The trial and err consuming and distinguishing bi primes. Another way to think about it alg If $n^2 + 41n + 4$ can be written a $n^2 + 41n + 41$ Since $41n + 41$ sense to use $n =$ $41^2 + 41 \times 41$ - .

Example 4

★★☆

Write a counterexample to the statement: if n is prime then $2^n - 1$ is prime.

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view

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Steps	Explanation
n / $2^n - 1$: 2 / 3; 5 / 31; 7 / 127; 11 / 2047	Using trial and error.
Let $n = 11$, $2^{11} - 1 = 2047 = 23 \times 89$, which is not prime. Therefore, $n = 11$ is a counterexample to this claim.	

n	$2^n - 1$
2	3
5	31
7	127
11	2047

Be aware

The most efficient way to check that a number is prime is to divide it by prime numbers starting with the smallest primes that you know and going up sequentially.

2 section questions

1. Number and algebra / 1.15 Proof

Proof by induction

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/proof-by-induction-id-27003/print/) Assign

A proof by induction is a formal proof that has four specific steps. These steps are outlined in the example below.

Example 1

★★☆

Prove that the sum of the first n square numbers is

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/proof-by-contradiction-id-27001/print/) Assign

$$\frac{1}{6}n(n+1)(2n+1) \text{ for } n \in \mathbb{Z}^+ .$$

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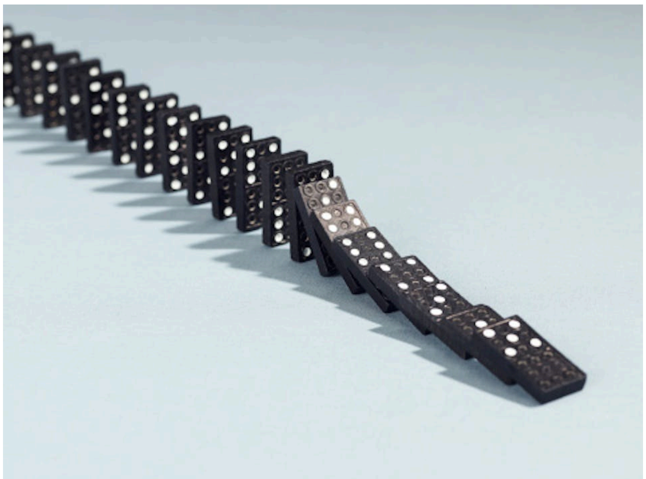
Step	Action
Proposition	Let P_n be the proposition that $1^2+2^2+3^2+\ldots+n^2=\frac{1}{6}n(n+1)(2n+1)$.
Step 1	**Show** that the proposition is true for a specific case, usually $n=1$, i.e. show that P_1 holds.
Step 2	**Assume** (make sure that you include the word 'assume' when writing your solution in the exams) that the proposition is true for a general case, i.e. assume $1^2+2^2+3^2+\ldots+k^2=\frac{1}{6}k(k+1)(2k+1)$ is true for some $k\in\mathbb{Z}^+$.
Step 3	**Prove** that if the proposition is true for the general case, it is true for the next, i.e. prove that it is true for $n=k+1$.
Step 4	**Induce** that since it is true for a particular case (step 1), and given any general case (step 2), it is true for the next (step 3), the proposition is true for all cases.

You will not need to formally prove the validity of this method, but you can understand why it works by considering the following:

Since whenever the statement is true for $n=k$ it is also true for $n=k+1$ and because it is true for $n=1$ it is thus true for $n=1+1=2$. Equally, since it is true for $n=2$ it is also true for $n=2+1=3$, etc. Hence, it is natural to accept that the statements are true for all $n\in\mathbb{Z}^+$.

Another way to think about a proof by induction is to consider an analogy. You can think of the terms $P_1, P_2, P_3, \ldots$ for the proposition for P_n as dominoes placed in row, as seen in the image below. If you can show that the $(k+1)$th domino falls if the kth one falls and you can show that the starting domino falls, then you know that all the dominoes have fallen.

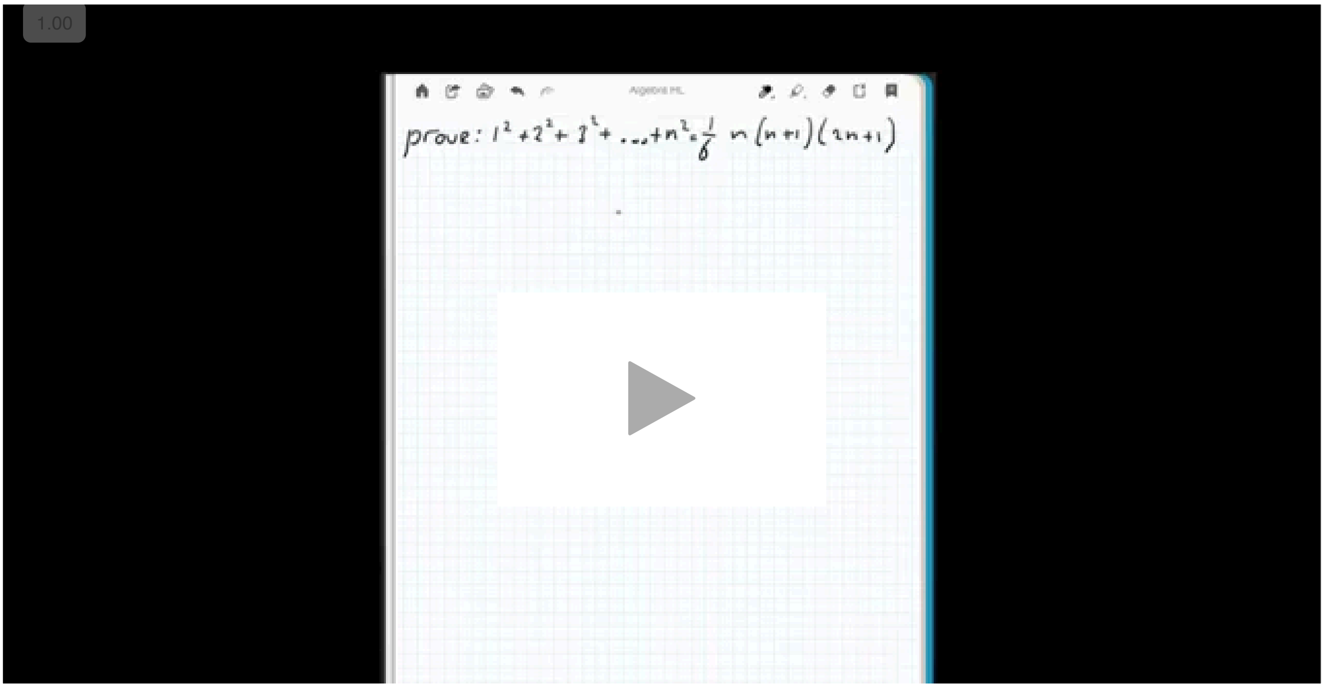


A row of dominoes

Credit: Getty Images Walker and Walker

The full proof for $1^2+2^2+3^2+\ldots n^2=\frac{1}{6}n\left(n+1\right)\left(2n+1\right)$ for $n\in\mathbb{Z}^+$ is shown in the video below.

Student view



Video 1. Proof by Induction: Example.

More information for video 1

1

00:00:00,267 --> 00:00:02,102

narrator: In this video,

we're going to take a look

2

00:00:02,169 --> 00:00:05,272

at proof by induction

and we're gonna take as an example,

3

00:00:05,472 --> 00:00:08,008

$1^2 + 2^2 + 3^2 + \quad + n^2 = \frac{1}{6}n(n+1)(2n+1)$

the statement that 1^2

plus 2^2 plus 3^2

4

00:00:08,075 --> 00:00:10,844

plus, et cetera, et cetera, all the way up

to n^2

5

00:00:10,911 --> 00:00:15,482

is equal to a sixth n into $n + 1$

into $2n + 1$.

6

00:00:15,849 --> 00:00:18,785

So we start by making a proposition.

7

00:00:19,119 --> 00:00:23,056

So let $P(n)$ be the proposition that indeed

8

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00:00:23,457 --> 00:00:26,994

this sum 1^2 plus 2^2

plus 3^2, et cetera,

9

00:00:27,060 --> 00:00:28,395

all the way up to n^2

10

00:00:28,462 --> 00:00:32,199

is equal to one sixth n into $n+1$

into $2n+1$.

11

00:00:32,633 --> 00:00:34,067

So we follow the four step procedure.

12

00:00:34,134 --> 00:00:37,070

So step number one is the following,

13

00:00:37,137 --> 00:00:42,376

show that $P(n)$ holds for in this case n

equals 1, but you can take any.

14

00:00:42,809 --> 00:00:44,745

$\text{LHS} = 1^2 = 1$

So the left hand side

is n^2, which is $1^2 = 1$.

15

00:00:44,811 --> 00:00:47,247

$\text{RHS}: \frac{1}{6} \times 1 \times (1+1)(2\times 1+1) = \frac{1}{6} \times 2 \times 3 = 1$

The right hand side is a 6th times 1,

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16

00:00:47,614 --> 00:00:52,486

times $1+1$ in $2\times 1+1$,

which is a 6 times 2 times 3,

17

00:00:52,553 --> 00:00:53,921

which of course is equal to 1.

18

00:00:54,121 --> 00:00:56,223

So the left hand side

is equal to right hand side

19

00:00:56,290 --> 00:00:59,660

and therefore we conclude that $P(1)$ is true.

20

00:01:00,427 --> 00:01:02,996

Now, step number two is the assumption

21

00:01:03,263 --> 00:01:06,033

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that $P(n)$ is true for a general case.

22

00:01:06,099 --> 00:01:07,301

In this case, $n = k$.

23

00:01:07,701 --> 00:01:09,770

Now it's good to write down

what that actually entails.

24

00:01:09,837 --> 00:01:12,072

So 1^2 plus 2^2 plus 3^2

25

00:01:12,139 --> 00:01:15,909

plus all the way up to k^2 is equal

to one sixth k into $k + 1$

26

00:01:15,976 --> 00:01:17,044

into $2k + 1$.

27

00:01:17,778 --> 00:01:19,513

So now we're gonna

get to step number three

28

00:01:20,314 --> 00:01:23,550

and in step number three,

we are going to test or show

29

00:01:23,650 --> 00:01:28,455

or investigate or prove that the next step

and equals $k + 1$ is true.

30

00:01:28,622 --> 00:01:31,825

Now it is good practice

to write down what this is again.

31

00:01:31,892 --> 00:01:33,894

So 1^2 plus 2^2 plus 3^2

32

00:01:33,961 --> 00:01:37,397

plus all the way up to k^2

plus the next one, $(k + 1)^2$.

33

00:01:37,497 --> 00:01:41,602

Now you see that the first

part is the kth,

34

00:01:41,668 --> 00:01:43,604

the surgeon that is $P(k)$.

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35

00:01:43,837 --> 00:01:45,672

Now it is always the case that

36

00:01:45,739 --> 00:01:47,774

you're gonna need step number

two and step number three.

37

00:01:47,841 --> 00:01:49,443

So write it down explicitly.

38

00:01:49,710 --> 00:01:52,446

So we replace the first part

with the step number two.

39

00:01:52,613 --> 00:01:55,849

So we left with one sixth

of k into $k + 1$ into $2k + 1$

40

00:01:55,916 --> 00:01:57,117

plus $(k + 1)^2$.

41

00:01:57,618 --> 00:01:59,486

Now you see there's a common effect

of $k + 1$.

42

00:01:59,553 --> 00:02:02,923

So I'm gonna write it rather

interestingly like this.

43

00:02:03,457 --> 00:02:06,927

And then I'm gonna take out

the one sixth $k + 1$ leaving

44

00:02:06,994 --> 00:02:11,098

inside k into $2k + 1$

plus 6 into $k + 1$.

45

00:02:11,298 --> 00:02:15,636

So let's multiply the brackets

$2k^2 + k+$

46

00:02:15,769 --> 00:02:17,070

$6k + 6$.

47

00:02:17,504 --> 00:02:19,239

And now we see that we've got a quadratic

48

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00:02:19,306 --> 00:02:22,843

and therefore we can actually

factorize this $(k + 2)$,

49

00:02:22,910 --> 00:02:24,411

$(2k + 3)$.

50

00:02:24,945 --> 00:02:27,314

And then we're going to write as explicitly

51

00:02:27,381 --> 00:02:30,083

as $k + 2$ is the same as $k + 1 + 1$

52

00:02:30,184 --> 00:02:34,221

and $2k + 3$ is the same

as $2(k + 1) + 1$.

53

00:02:34,521 --> 00:02:37,224

And now you can see that

if we make the substitution

54

00:02:37,291 --> 00:02:39,993

$a = k + 1$,

we have a sixth a into $a + 1$

55

00:02:40,394 --> 00:02:41,428

into $2a + 1$.

56

00:02:41,495 --> 00:02:44,164

So we have shown that we get

57

00:02:44,231 --> 00:02:47,334

to the $k + 1$ assertion

of the proposition.

58

00:02:47,901 --> 00:02:50,537

Now this is not the end of it.

You must do step number four.

59

00:02:50,604 --> 00:02:52,773

Otherwise you have not shown

60

00:02:53,073 --> 00:02:55,475

or have not fulfilled

the proof by induction.

61

00:02:55,709 --> 00:02:58,278

So if the proposition

is true for $n = k$,

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62

00:02:58,679 --> 00:03:00,714

then it is true for $n = k + 1$.

63

00:03:00,781 --> 00:03:01,982

We've shown that.

64

00:03:02,850 --> 00:03:07,054

Now since it is true for $n = 1$,

65

00:03:07,120 --> 00:03:11,325

which we've shown that it therefore

must be true for $n = 1 + 1$ is 2.

66

00:03:12,593 --> 00:03:15,629

Furthermore, as it is true for $n = 2$,

67

00:03:16,296 --> 00:03:20,901

it must then also be true

for $n = 2 + 1$,

68

00:03:20,968 --> 00:03:22,302

which is $n = 3$.

69

00:03:22,870 --> 00:03:26,573

And therefore we can conclude

that indeed this is true.

70

00:03:26,640 --> 00:03:31,678

That is the proposition

is true for all positive integers.

71

00:03:31,979 --> 00:03:33,180

Now this step is crucial

72

00:03:33,247 --> 00:03:36,316

even though you're not doing any

mathematics really in terms of equation.

73

00:03:36,450 --> 00:03:38,785

So notice that we include step number two,

74

00:03:39,119 --> 00:03:41,455

we include step number three,

75

00:03:41,555 --> 00:03:45,292

and we include step number one

in step number four,

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76

00:03:45,492 --> 00:03:46,994

and that is proof by induction.

Exam tip

Step 3 is the most algebraically difficult part of a proof by induction. You should always use the expressions from Step 2 in your work for Step 3. Doing so will help you to find patterns and simplify algebraic expressions in your work for Step 3 and will also make your proof valid.

Why must you use the expressions from Step 2 in Step 3 of a proof by induction to make it valid?

Important

Principle of mathematical induction

For a list of infinitely many propositions, P_n, $n = 1,\ 2,\ 3,\ \ldots$

if P_1 is true, and whenever P_k is true then P_{k+1} is also true, then P_n is true for all positive integers, n.

Be aware

In this section you will learn proofs by induction for:

- Divisibility
- Sequences
- Inequalities
- Complex numbers

Be aware

To show that a number, a, is divisible by an integer b , you can write $a = bm$, where $m \in \mathbb{Z}$.

Example 2

★★☆

Prove that $9^n - 1$ is divisible by 8 for all $n \in \mathbb{Z}^+$.

Step	Action
Proposition	Let P_n be the proposition that $9^n - 1$ is divisible by 8.

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Step	Action
Step 1	**Show** that P_n holds for $n = 1$. That is, $9^1 - 1 = 8 = 8 \times 1$, thus $\frac{9^1 - 1}{8} = 1$, which shows that $9^1 - 1$ is divisible by 8. Hence, the proposition holds for $n = 1$.
Step 2	**Assume** P_n to be true for $n = k$, i.e. $9^k - 1 = 8m,\ m \in \mathbb{Z}$.
Step 3	**Investigate** P_{k+1}: $$\begin{aligned} 9^{k+1} - 1 &= 9 \times 9^k - 1 \\ &= 9 \times (8m + 1) - 1 \quad \text{[using step 2]} \\ &= 72m + 8 \\ &= 8(9m + 1) \\ &= 8 \times l,\ l \in \mathbb{Z}. \end{aligned}$$ Hence $\frac{9^{k+1} - 1}{8} = l$, and $9^{k+1} - 1$ is thus divisible by 8, which is the $(k+1)$th assertion of P_n.
Step 4	If the proposition holds for $n = k$, it holds for $n = k + 1$. Since it holds for $n = 1$, by the principle of mathematical induction, the proposition P_n holds for all positive integers.

Exam tip

It may be tempting to skip Step 4 in a proof by induction because it is repetitive, but you need to remember that a proof by induction is not complete without this step. It is important that your written work includes all parts of this step.

Example 3

★★★

Prove that $2^n > n$ for all $n \in \mathbb{Z}^+$.

Step	Action
Proposition	Let P_n be the proposition that $2^n > n$.
Step 1	For $n = 1$, $2^1 = 2 > 1$, thus P_1 is true.
Step 2	Assume that P_n holds for $n = k$, i.e. $2^k > k$.

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Step	Action
Step 3	Investigate the proposition for $n = k + 1$. However, we start this time, from step 2. $$\begin{aligned} 2^k &> k \\ 2 \times 2^k &> 2 \times k \\ 2^{k+1} &> 2k \\ 2^{k+1} &> k + 1 \qquad [\text{since } k \geqslant 1,\, 2k = k + k \geqslant k + 1]. \end{aligned}$$ We can also solve as in previous examples starting from 2^{k+1} as follows: $$2^{k+1} = 2^k \times 2 > k \times 2 \qquad (1) \qquad [\text{from step 2 since } 2^k > k]$$ And as, $k \times 2 = 2k = k + k > k + 1$ (since $k > 1$), the (1) above becomes $$2^{k+1} > k + 1$$ Hence, P_{k+1} holds.
Step 4	If the proposition holds for $n = k$, it holds for $n = k + 1$. Since it holds for $n = 1$, by the principle of mathematical induction, the proposition P_n holds for all positive integers.

Example 4

★★★

Prove that $(1 + i)^{4n} = (-4)^n$, for all $n \in \mathbb{Z}^+$.

Step	Action
Proposition	Let P_n be the proposition that $(1 + i)^{4n} = (-4)^n$, for all $n \in \mathbb{Z}^+$.
Step 1	Let $n = 1$ and evaluate: $$\begin{aligned} (1 + i)^4 &= [(1 + i)(1 + i)][(1 + i)(1 + i)] \\ &= [1 + 2i - 1][1 + 2i - 1] \\ &= (2i)(2i) \\ &= 4i^2 \\ &= -4 = (-4)^1 \end{aligned}$$ So P_1 is true.
Step 2	Assume that P_n holds for $n = k$, i.e. $(1 + i)^{4k} = (-4)^k$.
Step 3	Let $n = k + 1$ and investigate: $$\begin{aligned} (1 + i)^{4(k+1)} &= (1 + i)^{4k}(1 + i)^4 \\ &= (-4)^k(-4)^1 \qquad [\text{using Step 2 and Step 1}] \\ &= (-4)^{k+1} \end{aligned}$$ Hence, if P_k is true, then P_{k+1} is also true.
Step 4	If the proposition holds for $n = k$, it holds for $n = k + 1$. Since it holds for $n = 1$, by the principle of mathematical induction, the proposition P_n holds for all positive integers.

Student view

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Exam tip

Note that the phrasing for each step of the proof is slightly different in each Example. You do not need to memorise an exact phrase for each of the four steps in a proof by induction as long as you communicate the information necessary at each step. The key words for steps 1—3 are show, assume and prove. These must be included in your written work.

Be aware

The symbol $\forall$ means 'for all'. For example, if you see $\forall\ n > 1$ it means 'for all n greater than 1'.

4 section questions

1. Number and algebra / 1.15 Proof

Proof by induction continued

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/proof-by-induction-continued-id-27004/print/) Assign

In this section, you will learn how to use proof by induction for questions involving trigonometry and calculus which are studied in Topic 3 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25407/) and Topic 5 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25542/). You may choose to come back to these proofs after completing these topics.

Trigonometry

Example 1

★★☆

Prove that $\sin(x + n\pi) = (-1)^n \sin x$, for all $n \in \mathbb{Z}^+$.

Step	Action
Proposition	Let P_n be the proposition that $\sin(x + n\pi) = (-1)^n \sin x$, for all $n \in \mathbb{Z}^+$.
Step 1	Let $n = 1$, then $\begin{aligned}\sin(x + 1 \times \pi) &= \sin(x + \pi)\\ &= \sin x \cos \pi + \sin \pi \cos x \\ &= \sin x \times (-1) + 0 \times \cos x \quad \text{[compound angle identities]}\\ &= -\sin x = (-1)^1 \sin x\end{aligned}$ Hence, it is true for $n = 1$.

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Step	Action
Step 2	Assume that P_n holds for $n = k$, i.e. $\sin(x + k \times \pi) = (-1)^k \sin x$.
Step 3	Let $n = k + 1$ and investigate: $$\begin{aligned}\sin[x + (k+1) \times \pi] &= \sin[(x + k \times \pi) + \pi] \\ &= \sin(x + k \times \pi)\cos\pi + \sin\pi\cos(x + k \times \pi) \\ &= -1 \times \sin(x + k \times \pi) \\ &= -1 \times (-1)^k \sin x \quad \text{[by step 2]} \\ &= (-1)^{k+1} \sin x\end{aligned}$$ Hence, if P_k is true, then P_{k+1} is also true.
Step 4	If the proposition holds for $n = k$, it holds for $n = k + 1$. Since it holds for $n = 1$, by the principle of mathematical induction, the proposition P_n holds for all positive integers.

Example 2

★★☆

Prove that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for all $n \in \mathbb{Z}^+$.

Step	Action
Proposition	Let P_n be the proposition that $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$, for all $n \in \mathbb{Z}^+$.
Step 1	Let $n = 1$, for which $$(\cos\theta + \mathrm{i}\sin\theta)^1 = \cos\theta + \mathrm{i}\sin\theta.$$ Hence, it is true for $n = 1$.
Step 2	Assume that P_n holds for $n = k$, i.e. $(\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + \mathrm{i}\sin k\theta$.
Step 3	Let $n = k + 1$ and investigate: $$\begin{aligned}(\cos\theta + \mathrm{i}\sin\theta)^{k+1} &= (\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + \mathrm{i}\sin\theta) \\ &= (\cos k\theta + \mathrm{i}\sin k\theta)(\cos\theta + \mathrm{i}\sin\theta) \quad \text{[using step 2]} \\ &= \cos k\theta\cos\theta + \mathrm{i}\cos k\theta\sin\theta + \mathrm{i}\cos\theta\sin k\theta - \sin\theta\sin k\theta \\ &= \cos k\theta\cos\theta - \sin k\theta\sin\theta + \mathrm{i}(\sin k\theta\cos\theta + \cos k\theta\sin\theta) \\ &= \cos(k\theta + \theta) + \mathrm{i}\sin(k\theta + \theta) \quad \text{[compound angle identities]} \\ &= \cos((k+1)\theta) + \mathrm{i}\sin((k+1)\theta)\end{aligned}$$ So P_{k+1} is true if P_k is true.
Step 4	If the proposition holds for $n = k$, it holds for $n = k + 1$. Since it holds for $n = 1$, by the principle of mathematical induction, the proposition P_n holds for all positive integers.

Making connections

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

The proof in Example 2 is for De Moivre's theorem which you studied in section 1.14.1 (/study/app/math-aa-hl/sid-134-cid-761926/book/de-moivres-theorem-id-26995/).

Calculus

Example 3

★★★

Prove that $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$, for all $n \in \mathbb{Z}^+$.

Step	Action
Proposition	Let P_n be the proposition that $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$ for all $n \in \mathbb{Z}^+$.
Step 1	Let $n = 1$. Then using differentiation from first principles, $$\begin{aligned}\mathrm{d}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^1\right] &= \lim_{h\to 0}\left(\frac{(x+h)-x}{h}\right)\\ &= \lim_{h\to 0}\left(\frac{h}{h}\right) = 1 = 1x^0 = 1x^{1-1}\end{aligned}$$ So true for $n = 1$.
Step 2	Assume that P_n holds for $n = k$, i.e. $\frac{\mathrm{d}}{\mathrm{d}x}x^k = kx^{k-1}$.
Step 3	Let $n = k+1$. Then $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{k+1}\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[x \times x^k\right]\\ &= \frac{\mathrm{d}}{\mathrm{d}x}[x] \times x^k + x \times \frac{\mathrm{d}}{\mathrm{d}x}\left[x^k\right] && [\text{using the product rule}]\\ &= 1 \times x^k + x \times kx^{k-1} && [\text{using step 1 and 2}]\\ &= x^k + kx^k = (k+1)\,x^k\end{aligned}$$ So P_{k+1} is true if P_k is true.
Step 4	If the proposition holds for $n = k$, it holds for $n = k+1$. Since it holds for $n = 1$, by the principle of mathematical induction, the proposition P_n holds for all positive integers.

Example 4

★★★

Prove that $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x\mathrm{e}^{-x}\right] = (-1)^{n+1}\mathrm{e}^{-x}\,(n-x)$, for all $n \in \mathbb{Z}^+$.

Student view

Overview
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Step	Action
Proposition	Let P_n be the proposition that $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x\mathrm{e}^{-x}\right]=(-1)^{n+1}\mathrm{e}^{-x}(n-x)$, for all $n\in\mathbb{Z}^+$.
Step 1	Let $n=1$. Then $\frac{\mathrm{d}}{\mathrm{d}x}\left[x\mathrm{e}^{-x}\right]=\frac{\mathrm{d}}{\mathrm{d}x}[x]\times\mathrm{e}^{-x}+x\times\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-x}\right]\quad[\text{using product rule}]$ $=1\times\mathrm{e}^{-x}+x\times\left(-\mathrm{e}^{-x}\right)$ $=\mathrm{e}^{-x}(1-x)=(-1)^2\mathrm{e}^{-x}(1-x)$ $=(-1)^{1+1}\mathrm{e}^{-x}(1-x)\,.$ So P_1 is true.
Step 2	Assume that P_n holds for $n=k$, i.e. $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[x\mathrm{e}^{-x}\right]=(-1)^{k+1}\mathrm{e}^{-x}(k-x)$.
Step 3	Let $n=k+1$. Then $\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left[x\mathrm{e}^{-x}\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}x\mathrm{e}^{-x}\right)$ $=\frac{\mathrm{d}}{\mathrm{d}x}\left[(-1)^{k+1}\mathrm{e}^{-x}(k-x)\right]\quad[\text{using step 2}]$ $=(-1)^{(k+1)}\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-x}(k-x)\right]$ $=(-1)^{k+1}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-x}\right]\times(k-x)+\mathrm{e}^{-x}\frac{\mathrm{d}}{\mathrm{d}x}[(k-x)]\right\}\quad[\text{product rule}]$ $=(-1)^{k+1}\left\{-\mathrm{e}^{-x}(k-x)+\mathrm{e}^{-x}\times-1\right\}$ $=(-1)^{k+1}\left\{-\mathrm{e}^{-x}(k-x)-\mathrm{e}^{-x}\right\}$ $=(-1)^{k+1}\left\{(-1)\,\mathrm{e}^{-x}((k-x)+1)\right\}$ $=(-1)^{(k+1)+1}\mathrm{e}^{-x}((k+1)-x)\,.$ So P_{k+1} is true if P_k is true.
Step 4	If the proposition holds for $n=k$, it holds for $n=k+1$. Since it holds for $n=1$, by the principle of mathematical induction, the proposition P_n holds for all positive integers.

2 section questions

1. Number and algebra / 1.15 Proof

Checklist

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What you should know

By the end of this subtopic you should be able to:

- write appropriate assumptions for a proof by contradiction
- prove conjectures to be true by contradiction and by induction
- prove conjectures to be false by counterexample

Student view

Overview
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- understand and use the four steps in a proof by induction
- recognise when it is appropriate to prove conjectures using proof by induction.

1. Number and algebra / 1.15 Proof

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-27006/print/) Assign

Part 1

Describe the characteristics that all statements that can be proved by induction have in common.

Hence, create your own example of a statement that can be proved by induction. Explain how it fits with the characteristics that you described.

Share your work with a peer and see if they agree that your statement can be proved by induction.

Part 2

Read the following proof by induction:

Prove that 9^n is a multiple of 5 if $n \in \mathbb{Z}^+$.

Let P_n be the proposition that $9^n = 5a$ where $n, a \in \mathbb{Z}^+$.

Assume that P_n holds true for $n = k$; $9^k = 5a$.

Let $n = k + 1$ and investigate:

$$\text{LHS} = 9^{k+1} = 9 \times 9^k = 9 \times 5a = 5 \times (9a)$$

Hence, if P_k is true, then P_{k+1} is also true.

If the proposition holds for $n = k$, it holds for $n = k + 1$. By the principle of mathematical induction, the proposition P_n holds for all positive integers.

Is proof by induction suitable for the proposition given in this proof?

Is this proof valid? Explain your reasoning and justify your answer.

Student view