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5. Calculus / 5.17 Area and volume

The big picture

This subtopic is the extension of what you learned in subtopic 5.11 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27908/). This section will investigate further possible applications of the definite integral.

- First you will see how to find areas of certain regions that are not bounded by graphs of functions.
- You will then see how regions can be approximated using rectangles and how the notation for the sum of the area of the rectangles is related to the notation used for definite integrals.
- The investigation of the relationship between sums and definite integral will lead to a formula that expresses the volume of certain solids. These volumes of revolution (sometimes called solids of revolution) will occur when the axis of revolution is the x-axis and also when the axis of revolution is the y-axis.

On the applet below you can explore how volumes of revolution are generated.

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Interactive 1. Explore How Volumes of Revolution Are Generated.

More information for interactive 1

This interactive allows users to explore volumes of revolution and understand the applications of integrals by letting users manipulate a curve and visualize its rotation around axes.

The screen is divided into two halves; on the right, there is a 3D graph with a curved blue line, and on the left side of the screen, there is an "Adjust Curve" button that allows users to drag three purple points on the curve to modify the curve's shape. Below it there are two selection buttons that enable users to choose whether to rotate the curve around the x-axis or y-axis to generate the corresponding 3D solid, illustrating how the 2D area transforms into a volume through rotation.

Users will see firsthand how calculus is used to build up volumes through integration. The visualization makes a concrete connection between the definite integral and physical volume, particularly how the integral sums countless circular slices to determine the total size of the rotated shape. By toggling between x-axis and y-axis rotation, users can compare how the orientation affects the resulting solid's form.

Concept

Throughout this subtopic, think about how **approximating** solids help you to use integrals to find exact volumes. Can you think of other areas of mathematics where integration might replace approximation? The similarity in the notation used in the **representation** of sums and definite integrals can help you to find other formulae.

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5. Calculus / 5.17 Area and volume

Area of a region revisited

In subtopic 5.5 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25533/) and subtopic 5.11 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27908/), you learned the basics of anti-differentiation, otherwise known as integration, and its use in finding the area under a curve. You initially learned how to find the area of the region between the curve and the x-axis, but you can also find the area between the curve and the y-axis.

Example 1

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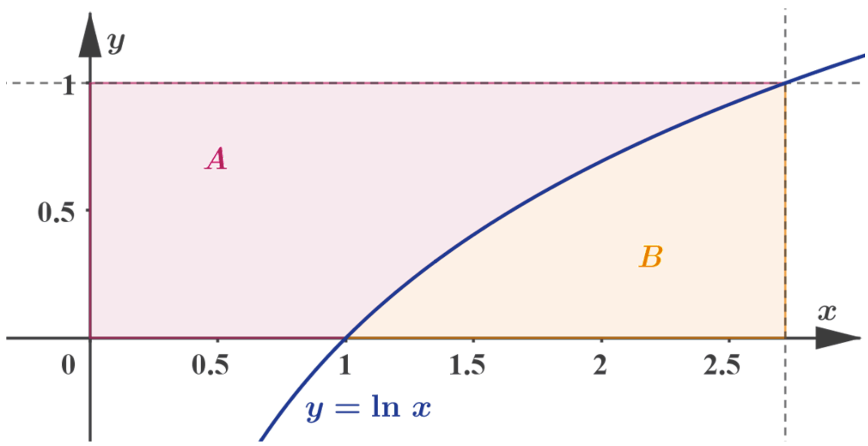
The diagram below shows part of the graph of $y = \ln x$, the line $y = 1$ and a vertical line. The two straight lines meet on the graph.



More information

The graph illustrates the function y = ln(x) alongside two straight lines: y = 1 (a horizontal line) and a vertical line that intersects the x-axis at approximately 2.718. The x-axis ranges from 0 to about 3, and the y-axis ranges from 0 to 1. Two regions, A and B, are shaded: Region A is above the curve and below y = 1, extending from x = 0 to the intersection with y = ln(x). Region B is below the curve and extends to the vertical line. The graph implies calculating the areas of these two regions.

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Find the area of the two shaded regions.

To find the position of the vertical line, find the first coordinate of the intersection of the graph of $y = \ln x$ and $y = 1$. This can be done graphically using a graphing calculator or by solving the equation $\ln x = 1$. Either way, the coordinates of this point are $(\mathrm{e}, 1)$.

Method 1

Using a calculator, you can see that the area of region B is $\int_1^{\mathrm{e}} \ln x \mathrm{d}x = 1$.

Since the area of the rectangle is $1 \times \mathrm{e} = \mathrm{e}$, the area of region A is $\mathrm{e} - 1$.

- Note, that if you want to find the area of region B first, you can either use technology as above or you can find $\int \ln x \, \mathrm{d}x$ and use this anti-derivative to find the area. The next method will follow a different approach. It will find the area of region A first.

Method 2

If $y = \ln x$, then $x = \mathrm{e}^y$.

First consider y as the independent variable and x as the dependent variable and notice that region A is bounded by the graph of $x = \mathrm{e}^y$, the y-axis and the lines $y = 0$ and $y = 1$. Hence, the area of region A is

$$\int_0^1 \mathrm{e}^y \mathrm{d}y = [\mathrm{e}^y]_0^1 = \mathrm{e}^1 - \mathrm{e}^0 = \mathrm{e} - 1$$

Since the area of the rectangle is $1 \times \mathrm{e} = \mathrm{e}$, the area of region B is $\mathrm{e} - (\mathrm{e} - 1) = 1$.

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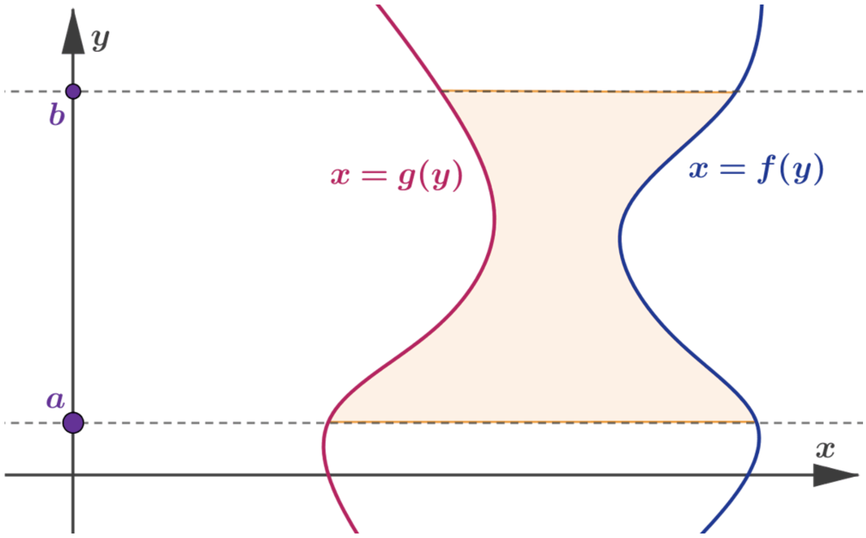
The second method of the solution in **Example 1** was based on the observation that you can use integration with respect to the variable y instead of x to find the area of certain regions.

Important

If $g(y) \leqslant f(y)$ for $a < y < b$, then the area of the region bounded by the graphs of $x = f(y)$, $x = g(y)$ and the lines $y = a$ and $y = b$ is given by

$$\int_a^b f(y) - g(y)\mathrm{d}y$$



More information

The image is a graph depicting two curves labeled (x = f(y)) and (x = g(y)), which form a shaded region between them. The curves are plotted on a coordinate plane with the horizontal axis labeled (x) and the vertical axis labeled (y). The curve (x = f(y)) is on the right, and (x = g(y)) is on the left. The shaded area between the curves extends from (y = a) to (y = b), with these points marked on the (y)-axis. The shaded region shows the integral (\int_a^b f(y) - g(y) \, dy), representing the area between the two curves over the specified interval. Both curves and the axis are marked with dashed lines to illustrate boundaries and endpoints of the interval.

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Exam tip

In the formula booklet the area of the region enclosed by a curve and y-axis is given as

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$$A = \int_a^b |x| \mathrm{d}y$$

Example 2

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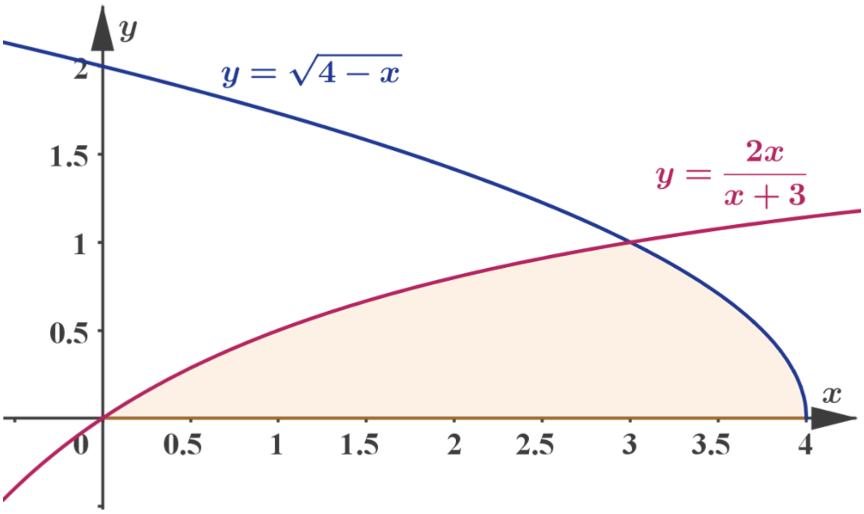
Find the area of the region (shaded on the diagram below) bounded by the graphs of $y = \sqrt{4-x}$, $y = \dfrac{2x}{x+3}$, and the x-axis.



More information

The image is a graph depicting two curves and a shaded region. The x-axis ranges from 0 to 4, and the y-axis ranges from 0 to 2. The first curve is labeled "y = sqrt(4 - x)" and is drawn in blue, starting from the y-axis at (0,2) and curving downwards towards the x-axis at (4,0). The second curve is labeled "y = 2x/(x + 3)" and is drawn in magenta, starting from the origin and curving upwards, then downwards, intersecting the first curve at approximately (3.5, 1.2). The shaded region is orange, located between these two curves and the x-axis, highlighting the area to be found. The two curves effectively bound the region from the left at x=0 to the right at x=4.

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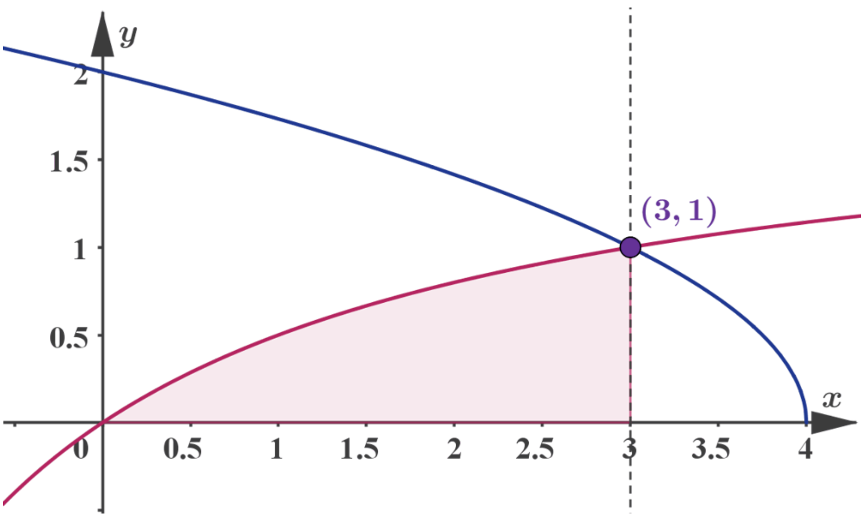
There are two ways of approaching this problem. You will need the following:

One of the solutions of $\sqrt{4-x} = \dfrac{2x}{x+3}$ gives the first coordinate of the intersection point of the graphs. The graph shows that one of the solutions is around 3. Since $\sqrt{4-3} = 1 = \dfrac{2\times 3}{3+3}$, the intersection point visible on the diagram is $(3, 1)$.

Method 1

You can split the shaded region into two parts and find each area separately.



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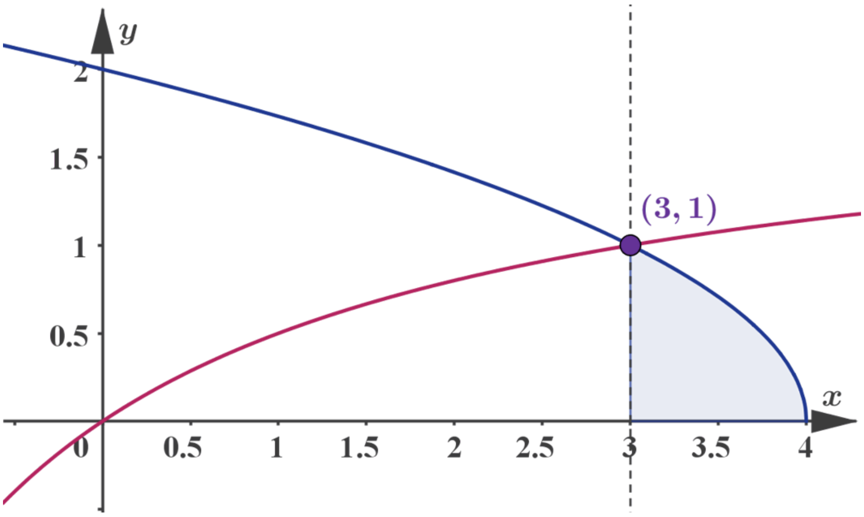

- To find the area of the region on the left, first find the anti -derivative of $y = \dfrac{2x}{x+3}$.

$$\int \frac{2x}{x+3}\mathrm{d}x = \int \frac{2(x+3)-6}{x+3}\mathrm{d}x$$
$$= \int 2 - \frac{6}{x+3}\mathrm{d}x = 2x - 6\ln(x+3) + c$$

So the area is

$$\int_0^3 \frac{2x}{x+3}\mathrm{d}x = [2x - 6\ln(x+3)]_0^3$$
$$= (6 - 6\ln 6) - (0 - 6\ln 3)$$
$$= 6 - 6(\ln 6 - \ln 3) = 6 - 6\ln\frac{6}{3} = 6 - 6\ln 2 \approx 1.8411.$$

- To find the area of the region on the right, find the anti-derivative of $y = \sqrt{4-x}$.

$$\int \sqrt{4-x}\mathrm{d}x = \int (4-x)^{1/2}\mathrm{d}x$$
$$= \frac{2}{3}(4-x)^{3/2}(-1) = -\frac{2}{3}(4-x)^{3/2} + c$$

so

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$$\int_3^4 \sqrt{4-x}\mathrm{d}x = \left[-\frac{2}{3}(4-x)^{3/2}\right]_3^4$$
$$= \left(-\frac{2}{3}(4-4)^{3/2}\right) - \left(-\frac{2}{3}(4-3)^{3/2}\right) = \frac{2}{3} \approx 0.6667.$$

Hence, the area of the shaded region is $6 - 6\ln 2 + \frac{2}{3} \approx 2.51$ units squared.

Method 2

- The blue graph on the diagram is part of the graph of $y = \sqrt{4-x}$. This can be rearranged in the form

$$\begin{aligned} y &= \sqrt{4-x} \\ y^2 &= 4-x \\ x &= 4-y^2 \end{aligned}$$

- The red graph in the diagram is part of the graph of $y = \frac{2x}{x+3}$. This can be rearranged in the form

$$\begin{aligned} y &= \frac{2x}{x+3} \\ y(x+3) &= 2x \\ yx+3y &= 2x \\ yx-2x &= -3y \\ x(y-2) &= -3y \\ x &= \frac{-3y}{y-2} \\ x &= \frac{3y}{2-y} \end{aligned}$$

You can consider y as the independent variable and x the dependent variable and think of the region as bounded by the graphs of $x = 4-y^2$ and $x = \frac{-3y}{y-2}$ over the interval $0 < y < 1$. To find the area this way, you will need

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$$\begin{aligned}\int (4-y^2) - \frac{3y}{2-y}\mathrm{d}y &= \int 4 - y^2 - \frac{-3(2-y)+6}{2-y}\mathrm{d}y \\ &= \int 4 - y^2 + 3 - \frac{6}{2-y}\mathrm{d}y \\ &= \int 7 - y^2 - \frac{6}{2-y}\mathrm{d}y \\ &= 7y - \frac{y^3}{3} - 6 \times (-1) \times \ln(2-y) + c \\ &= 7y - \frac{y^3}{3} + 6\ln(2-y) + c\end{aligned}$$

Hence, the area of the shaded region is

$$\begin{aligned}\int_0^1 (4-y^2) - \frac{3y}{2-y}\mathrm{d}y &= \left[7y - \frac{y^3}{3} + 6\ln(2-y)\right]_0^1 \\ &= \left(7 - \frac{1}{3} + 6\ln 1\right) - \left(0 - \frac{0}{3} + 6\ln 2\right) \\ &= 7 - \frac{1}{3} - 6\ln 2 \approx 2.51\end{aligned}$$

Example 3

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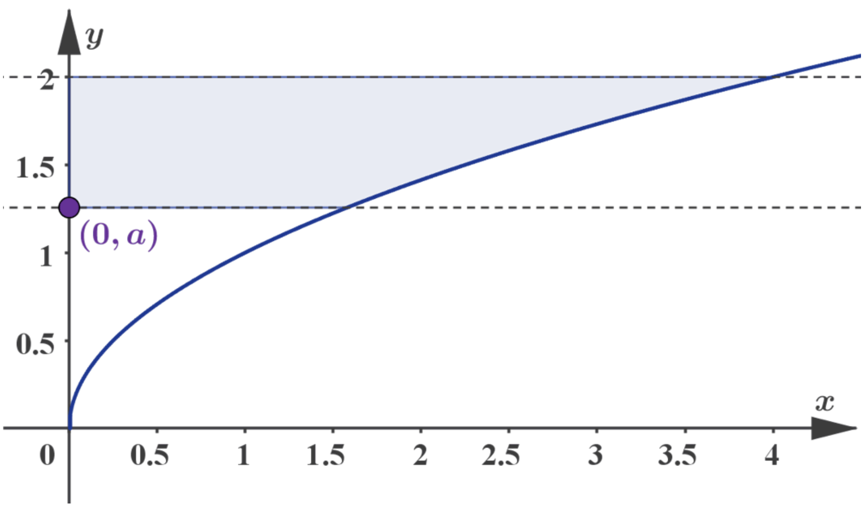
The shaded region on the diagram below is enclosed by the graph of $y = \sqrt{x}$, $y = 2$, $y = a$ and the y-axis.



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More information

The image is a graph illustrating a shaded region enclosed by four lines. **X-axis:** The x-axis ranges from 0 to 4, representing the independent variable. **Y-axis:** The y-axis ranges from 0 to 2.5, representing the dependent variable. **Equations:** The first curve is the graph of y = (\sqrt{x}), starting at (0,0) and gradually increasing as x increases. The horizontal line y = 2 is parallel to the x-axis, running approximately across the top of the graph. The line y = a is a horizontal line at an unspecified value of y, located between y = 0 and y = 2, and is marked at (0,a) on the y-axis. The shaded region is the area between y = a and y = 2 across the span of the y-axis to the point where y = (\sqrt{x}) meets y = 2. **Shading and Labeling:** The region of interest is shaded in blue, indicating the area of 2 units squared, confined to the region between y=a and y=2. The point (0,a) is marked with a dot and label on the y-axis.

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The area of the shaded region is 2 units squared. Find the value of a.

The equation of the curve, $y = \sqrt{x}$ can be written as $x = y^2$.

The area of the region is given by the integral $\int_a^2 y^2 \mathrm{d}y$, so

$$\begin{aligned} 2 &= \int_a^2 y^2 \mathrm{d}y \\ 2 &= \left[\frac{y^3}{3}\right]_a^2 \\ 2 &= \frac{2^3}{3} - \frac{a^3}{3} \\ 6 &= 8 - a^3 \\ a^3 &= 2 \\ a &= \sqrt[3]{2} \approx 1.26 \end{aligned}$$

In the examples above, the answers were found without the use of a graphing calculator. Approximate values of the answers can be calculated more quickly using a calculator.

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Example 4

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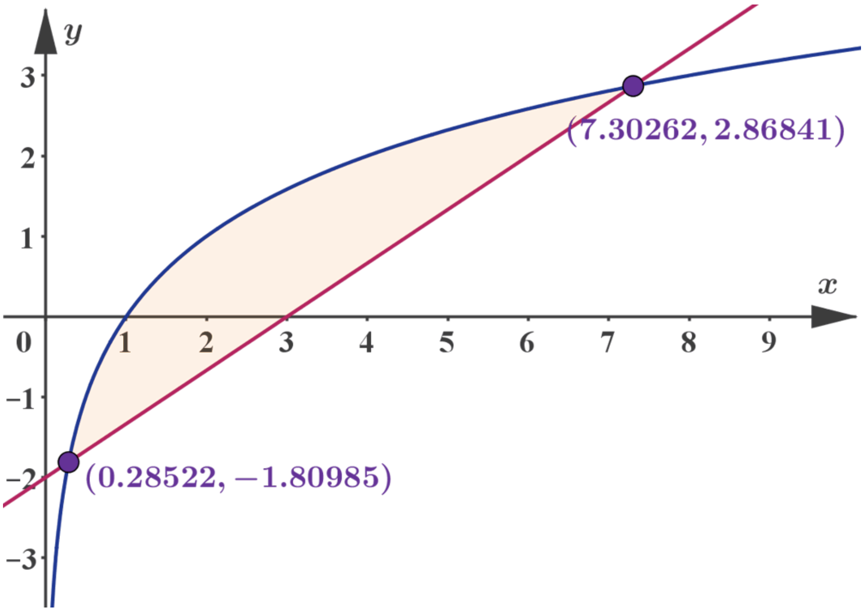
Find the area of the region enclosed by the graph of $y = \log_2 x$ and the line $2x - 3y = 6$.

First, use a calculator to plot the curve and the line and to find the intersection points. To do this, write the equation of the line in the form $y = \frac{2x-6}{3}$. The diagram below shows part of the two graphs, the coordinates of the intersection points and the region that these graphs enclose.



To find the area, use either x or y as the independent variable.

Method 1

Using $y = \log_2 x$ as the upper curve, $y = \frac{2x-6}{3}$ as the lower curve and 0.28522 and 7.30262 as bounds (of the x-values), the area is approximately

$$\int_{0.28522}^{7.30262} \log_2 x - \frac{2x-6}{3} \mathrm{d}x \approx 7.63 \text{ square units.}$$

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Method 2

You can also rearrange

- $y = \log_2 x$ as $x = 2^y$ and
- $2x - 3y = 6$ as $x = \dfrac{6+3y}{2}$.

Using these as the bounding curves and -1.80985 and 2.86841 as the bounds (of the y-values), the area is approximately

$$\int_{-1.80985}^{2.86841} \frac{6+3y}{2} - 2^y \mathrm{d}y \approx 7.63 \text{ square units.}$$

In **Example 4** it was necessary to use a calculator to find the bounds of the region. See what WolframAlpha (http://www.wolframalpha.com) gives as an answer when you type

solve {y=log_2(x), 2x-3y=6} for {x, y}

in the search line. It only gives an approximate answer, even though WolframAlpha is programmed to use all the algebraic equation solving techniques you learn about (and beyond) to find exact solutions when possible.

Exam tip

If the question does not ask for an exact value, use your calculator to find definite integrals.

Below you will find help on how to use the different calculators when the independent variable is not x . Of course, you can rename the independent variable, but it can be helpful if you are aware that calculators can find definite integrals with your choice of independent variable name.

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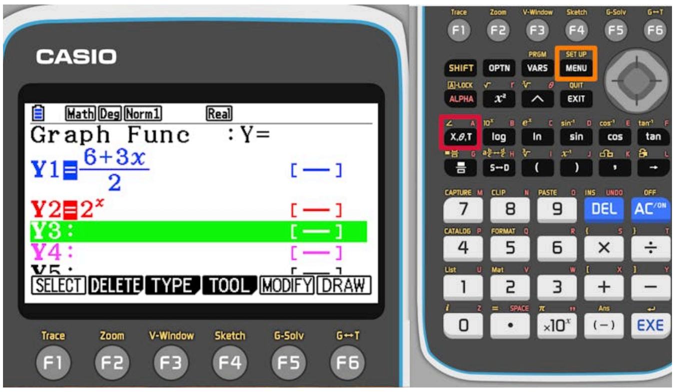
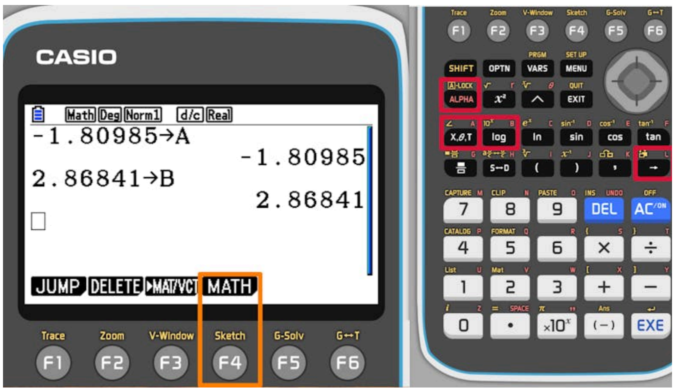
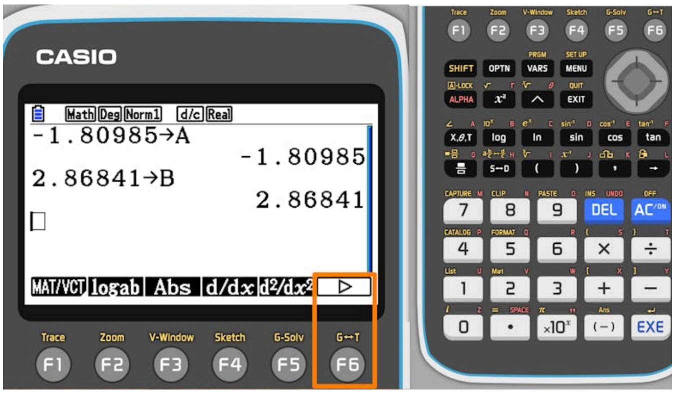
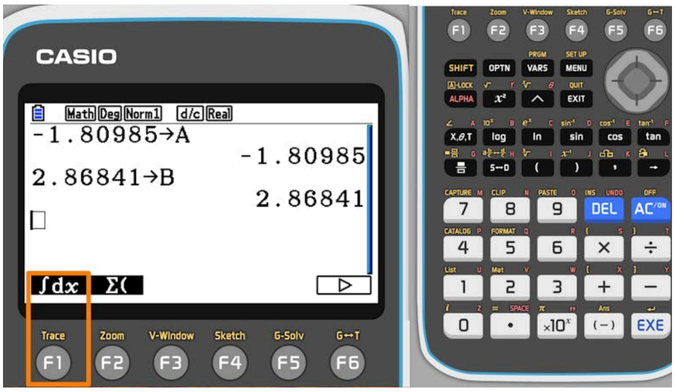
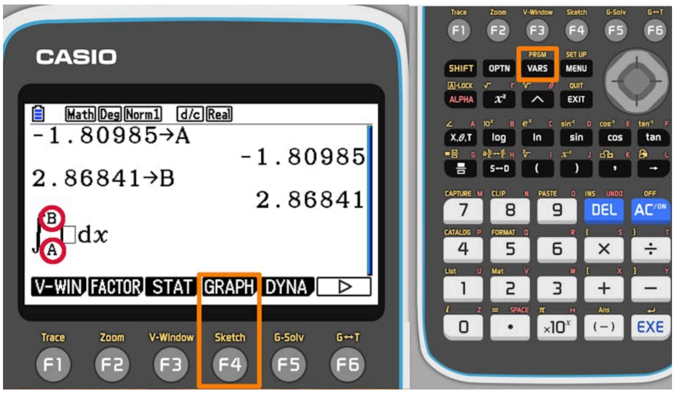
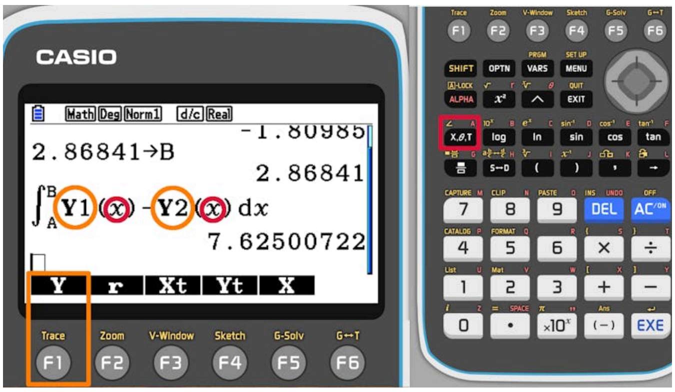
Steps	Explanation
These instructions will show you how to find the definite integral appearing in the example above. $$\int_{-1.80985}^{2.86841} \frac{6+3y}{2} - 2^y \mathrm{d}y$$ In the current operating system the integration variable needs to be x. This may change in the future, but for the moment you need to replace the variable, and find $$\int_{-1.80985}^{2.86841} \frac{6+3x}{2} - 2^x \mathrm{d}x$$ with the calculator.	
Use the variable button to enter x when you define the functions involved in the integral. This step is not necessery, but it will make the screen more easily readable, when you calculate the integral itself. Once done defining the functions, go back to the home screen.	

Steps	Explanation
Open now the calculator screen.	
Sometimes it is useful to store values in the memory of the calculator. In this case you can store the limits of integration. To ind the integral, press F4 to open the math options ...	

Steps	Explanation
... press F6 to scroll to the right ...	
... and press F1 to choose the integral template.	

Steps	Explanation
Notice, that you can use the names where you stored the limits instead of the numbers themselves. To use the function names instead of typing in the function again, press VARS and then F4 to access the variable names related to the graphical screen.	
Use the variable button to enter x and use F1 to enter the function variable names. Remember, that in the first step you stored the functions involved in Y1 and Y2.	

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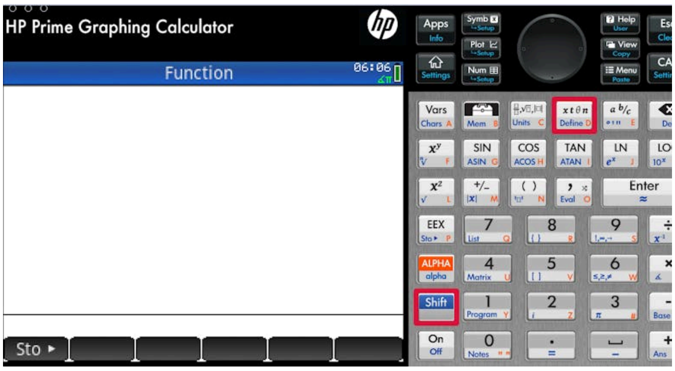
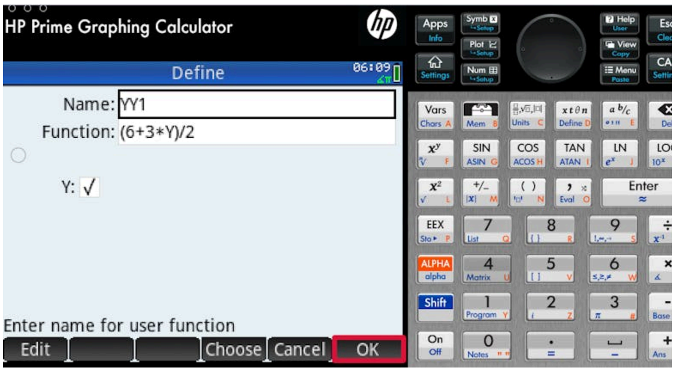
Steps	Explanation
These instructions will show you how to find the definite integral appearing in the example above. $$\int_{-1.80985}^{2.86841} \frac{6+3y}{2} - 2^y \mathrm{d}y$$ On the home screen select the option to define a function.	
You can give any name that is not used by the calculator and you can use any variable name. Press OK when you are done.	

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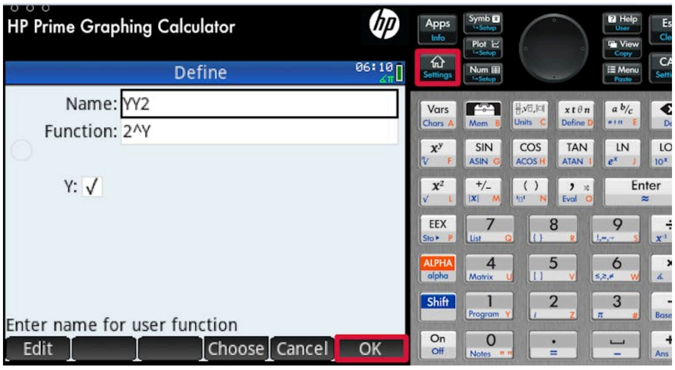
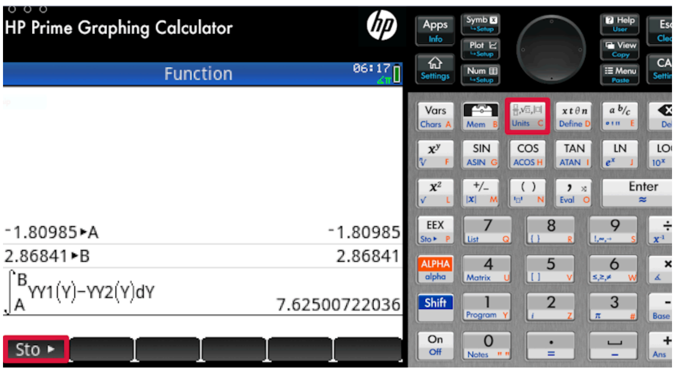
Steps	Explanation
You can define as many functions (using different names of course). When you are done, press OK and move back to the home screen.	
It is not really necessary, but sometimes it can be useful to store the values of the limits before you use the expression formatting option to enter the integral. Note that the calculator understands the functions you defined and the variable of the integration can be anything.	

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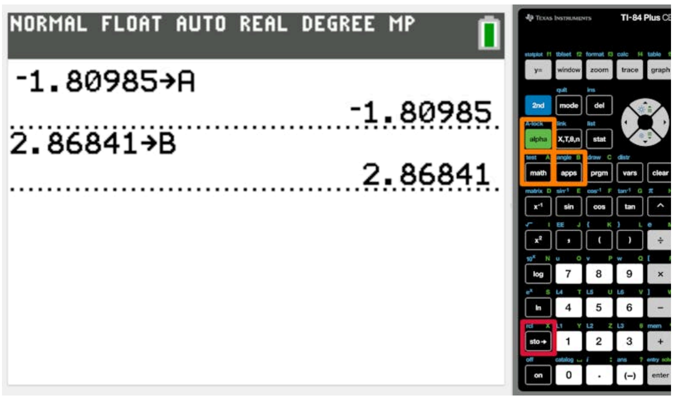
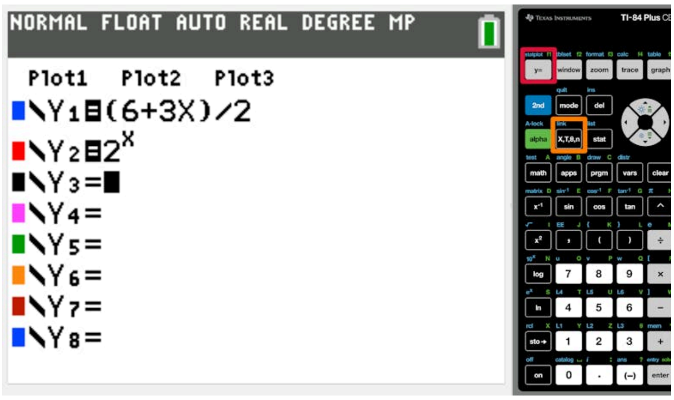
Steps	Explanation
These instructions will show you how to find the definite integral appearing in the example above. $$\int_{-1.80985}^{2.86841} \frac{6+3y}{2} - 2^y \mathrm{d}y$$ In this case it is not really necessary, but it can be useful to store values of the limits of integration in the memory of the calculator.	
You can also store the functions before starting to calculate the integral. This is again not necessary, but it will make the calculation of the integral more easily readable. Note, that to define the function you need to use x as the variable.	

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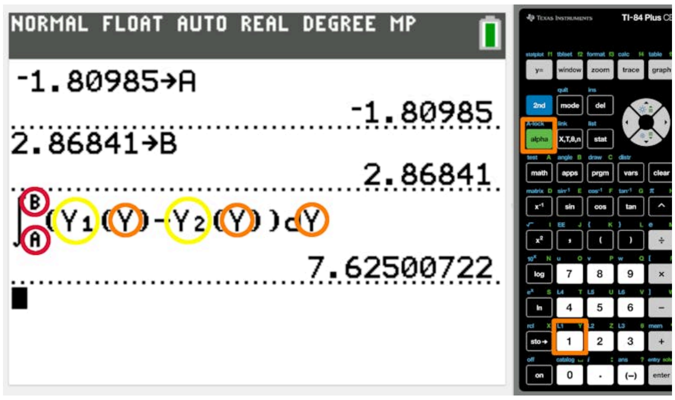
<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>Choose the numerical integral (fnInt) option in the math menu.</td><td>
</td></tr>
<tr><td>Make sure you understand every part of the integral on the screen.

You can use the variable names where you stored the limits previously (red marks).
You can use any letter as the variable of the integration (orange marks).
You can use the names of the function instead of typing in the expression again (yellow marks). See the next screen for guidance on how to find these function names.
</td><td>
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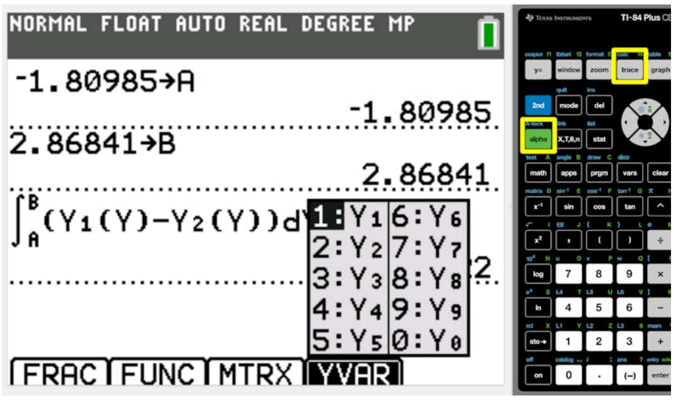
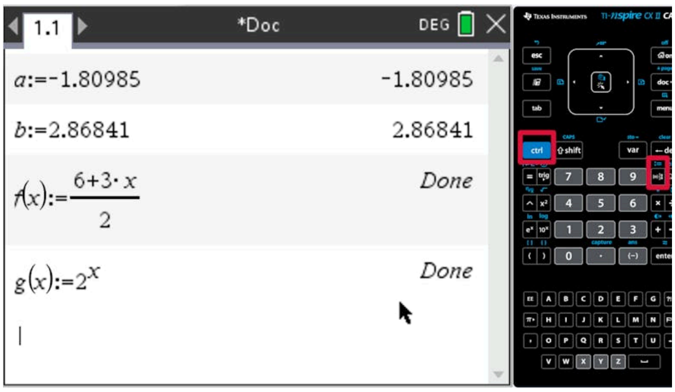
Steps	Explanation
You can access the function variable names by pressing alpha/f4.	

Steps	Explanation
These instructions will show you how to find the definite integral appearing in the example above. $$\int_{-1.80985}^{2.86841} \frac{6+3y}{2} - 2^y \mathrm{d}y$$ It is not necessary, but you can store numbers using variable names and you can also define functions that you want to use later. You can use any name and any variable name when you define a function. Make sure you use the colon equal sign in defining the functions.	

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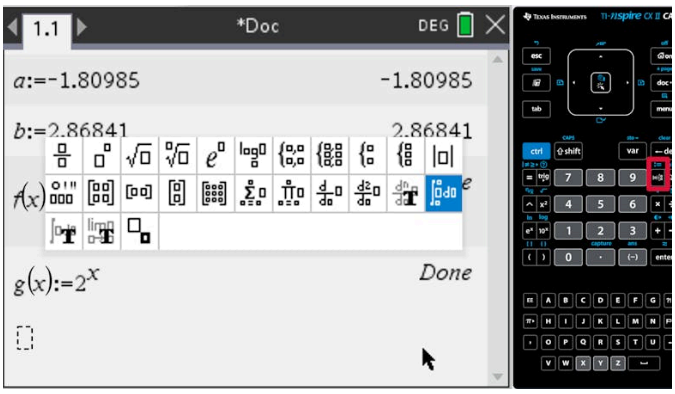
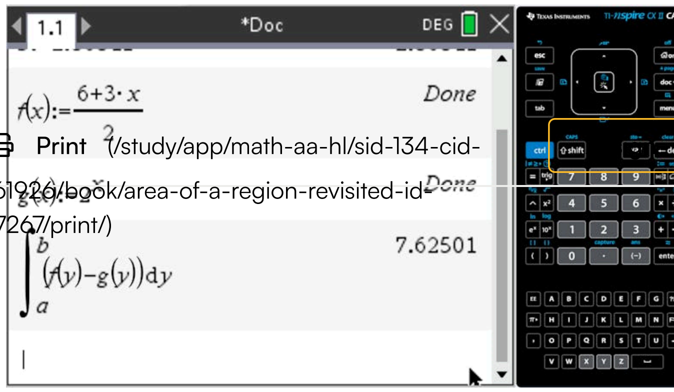
Steps	Explanation
To find a definite integral, open the template menu and choose the integral template.	
There are several things you can notice about how this integral is entered. • You can specify the limits using the names where you stored the values. • You can use the names of the functions you defined earlier. • You do not need to use the same variable name you used for defining the function. You can use any name for the integration variable.	

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/area-of-a-region-revisited-id-27267/print/)

3 section questions

Question 1

Difficulty:

★☆☆

Student view

Overview
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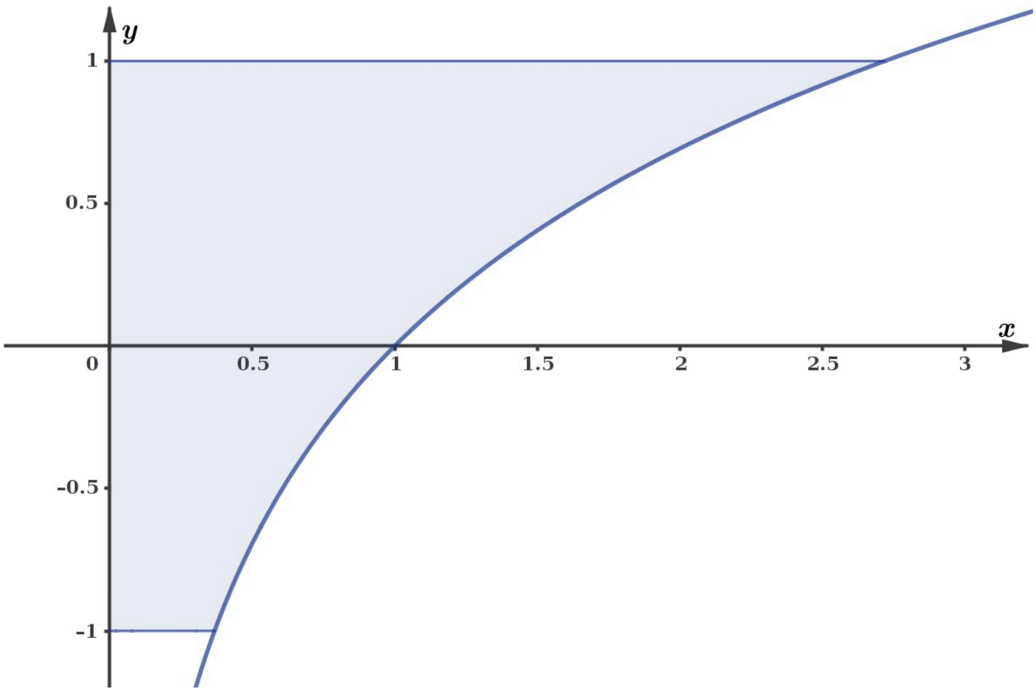
Consider the region bounded by the y-axis, the lines $y = 1$ and $y = -1$ and the graph of $y = \ln x$.



More information

Find the exact area of this region.

1 $\mathrm{e} - \frac{1}{\mathrm{e}}$ ✓

2 $\mathrm{e} + \frac{1}{\mathrm{e}}$

3 $\frac{1}{\mathrm{e}} - \mathrm{e}$

4 $\ln 2$

Explanation

In terms of y, the curve $y = \ln x$ is given by $x = e^y$.

Thus, we obtain the integral:

$$\int_{-1}^{1} \mathrm{e}^y \, \mathrm{d}y = \left[\mathrm{e}^y\right]_{-1}^{1}$$
$$= \mathrm{e}^1 - \mathrm{e}^{-1} = \mathrm{e} - \frac{1}{\mathrm{e}}$$

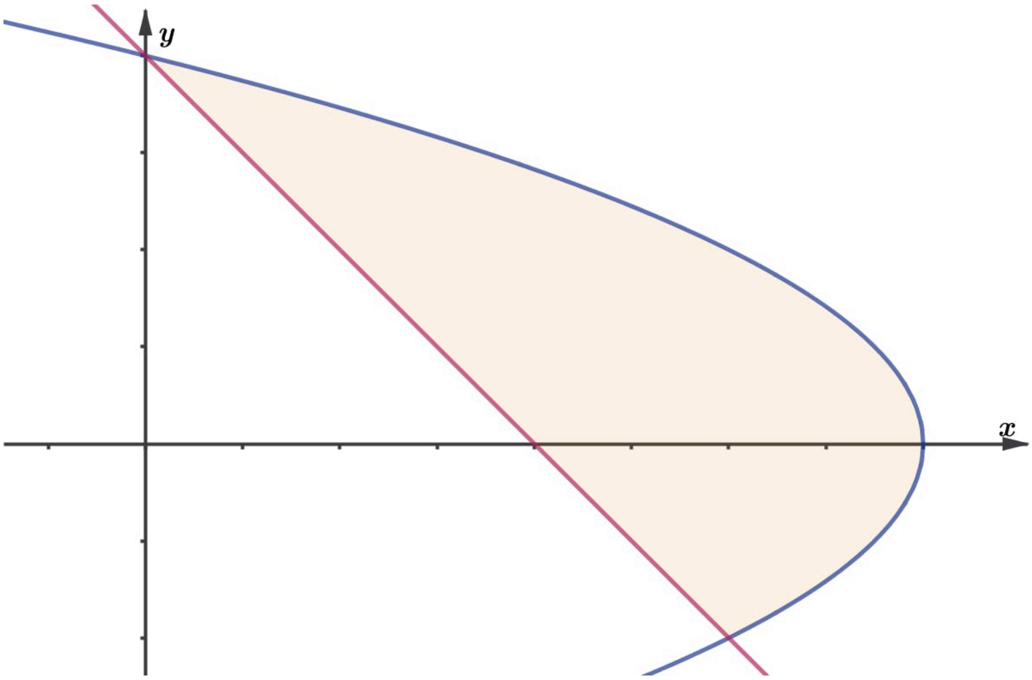
Student view

Question 2

Difficulty: ★★☆

Overview
(/study/ap aa-hl/sid-134-cid-761926/o

Consider the region bounded by the graphs of $x + y^2 = 4$ and $x + y = 2$.



More information

Find the area of the region.

Give an exact answer as a decimal or as a fraction (in the form n/m) in fully simplified form.

4.5

Accepted answers

4.5, 4,5, 9/2

Explanation

First we find the second coordinate of the intersection points of the graphs.

These are given by the solutions of the system of equation

$$\begin{cases} x + y^2 = 4 \\ x + y = 2 \end{cases}$$

Expressing x from the second equation and substituting in the first gives

$$\begin{aligned} (2-y) + y^2 &= 4 \\ y^2 - y - 2 &= 0 \\ (y-2)(y+1) &= 0 \end{aligned}$$

The solutions are $y = 2$ and $y = -1$.

Student view

Overview
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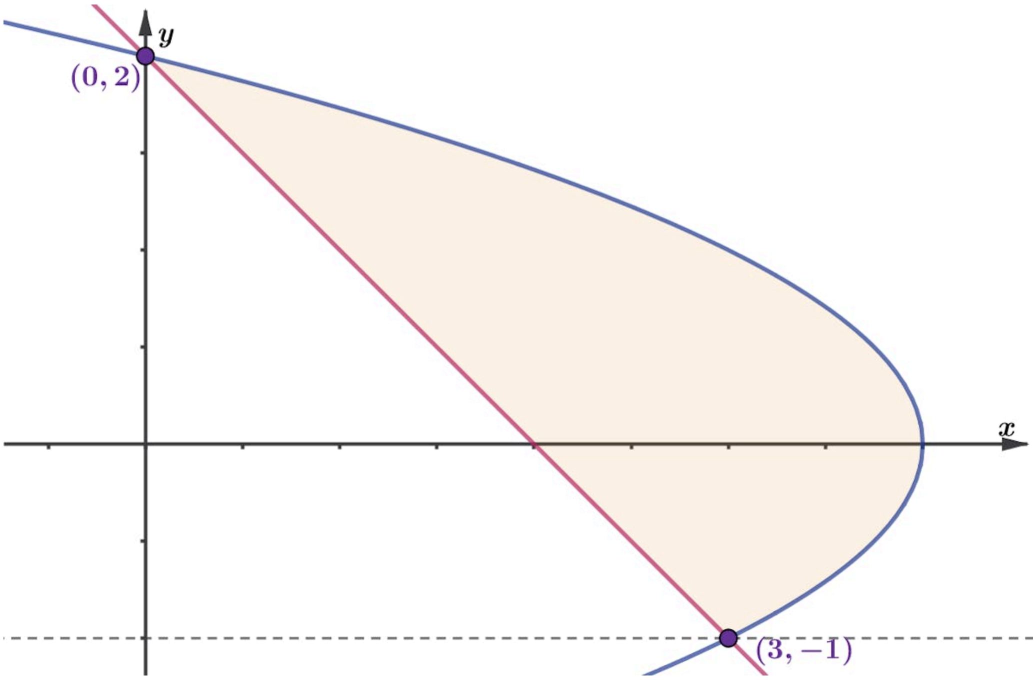


More information

We use the formula $\int_a^b f(y) - g(y)\mathrm{d}y$ to find the area, so we rearrange the defining equations of the curves.

- For the blue curve,
$$x + y^2 = 4$$
$$x = 4 - y^2$$
- For the red curve,
$$x + y = 2$$
$$x = 2 - y$$

Hence, the area of the region is

$$\begin{aligned}\int_{-1}^{2}(4-y^2)-(2-y)\mathrm{d}y &= \int_{-1}^{2} 2+y-y^2\mathrm{d}y \\ &= \left[2y+\frac{y^2}{2}-\frac{y^3}{3}\right]_{-1}^{2} \\ &= \left(4+\frac{4}{2}-\frac{8}{3}\right)-\left(-2+\frac{1}{2}-\frac{-1}{3}\right) \\ &= \frac{10}{3}-\frac{-7}{6}=\frac{9}{2}=4.5\end{aligned}$$

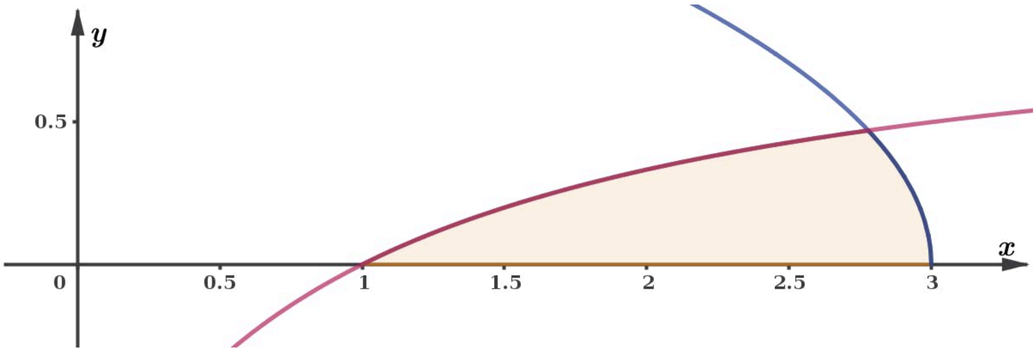
Question 3

Difficulty:

★★☆

Consider the region bounded by the x-axis and the graphs of $y = \sqrt{3-x}$ and $y = \dfrac{x-1}{x+1}$

Student
view



More information

Find the area of the region.

Give your answer as a decimal, rounded to three significant figures.

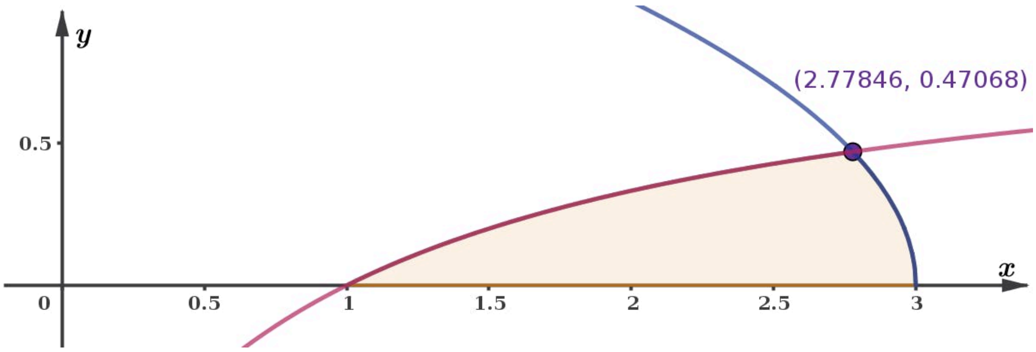
0.576

Accepted answers

0.576, 0,576, .576

Explanation

We will need the coordinates of the intersection point of the graphs.



More information

We use the formula $\int_a^b f(y) - g(y)\mathrm{d}y$ to find the area, so we rearrange the defining equations of the curves.

- For the blue curve,

$$\begin{aligned} y &= \sqrt{3-x} \\ y^2 &= 3-x \\ x &= 3-y^2 \end{aligned}$$

- For the red curve,

$$\begin{aligned} y &= \frac{x-1}{x+1} \\ y(x+1) &= x-1 \\ yx+y &= x-1 \\ yx-x &= -y-1 \\ x(y-1) &= -y-1 \\ x &= \frac{-y-1}{y-1} = \frac{y+1}{1-y} \end{aligned}$$

Overview
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Hence, the area of the region is

$$\int_0^{0.47068} (3-y^2) - \frac{y+1}{1-y} \mathrm{d}y$$

Graping calculators have applications that can find the value of this definite integral:

$$\int_0^{0.47068} (3-y^2) - \frac{y+1}{1-y} \mathrm{d}y \approx 0.576$$

Note that this area can also be found without rearranging the defining curves using integration with respect to x and splitting the region.

$$\int_1^{2.77846} \frac{x-1}{x+1} \mathrm{d}x + \int_{2.77846}^{3} \sqrt{3-x}\, \mathrm{d}x \approx 0.576$$

5. Calculus / 5.17 Area and volume

Integral as limit

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/integral-as-limit-id-27268/print/) Assign

Before turning to other applications of the definite integral, look at this approach for how to approximate the area of a region below the graph of a positive function.

Making connections

A similar approach (the trapezium rule) is part of the Applications and interpretation syllabus but not covered in the Analysis and approaches syllabus.

Exam tip

The content of this section is not part of the syllabus. For exam preparation you can safely move to the next section. Nevertheless, the discussion presented here is useful for understanding the reason why definite integral is used not only for calculating areas, but also for calculating volumes of revolution.

Student view

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On the applet below the area of a region below the graph of a positive function is approximated by the sum of the area of rectangles.

Interactive 1. The Area of a Region below the Graph of a Positive Function Is Approximated by the Sum of the Area of Rectangles.

More information for interactive 1

This interactive graph allows users to understand how definite integrals approximate the area under a curve using Riemann sums. The screen is divided in two halves, the top half shows a graph in blue line with five red dots scattered, and rectangles under the line, and the bottom half has two toggle buttons, ‘Adjust curve’ and ‘Number of rectangles’.

When ‘Adjust curve’ is selected, users can drag red points to adjust the curve's shape, modifying the region whose area users want to estimate. Behind the blue line graph is a collection of rectangles with equal base and variable heights. When ‘Number of rectangles’ is selected, a slider appears that allows the selection of 1 to 20 rectangles, with the applet instantly displaying both the rectangles and their combined area approximation.

For example, with 5 rectangles, the approximate area under the curve is 5.81, compared to the actual area of 5.82, illustrating a rough yet fairly close estimate. Increasing the number to 13 rectangles refines the estimate further, making the approximate area match the actual area of 5.82 almost exactly. This example highlights how using more, narrower rectangles results in a more accurate approximation.

As users increase the number of rectangles, users observe the approximation converging toward the true area under the curve - visually demonstrating how finer partitions yield more accurate results. The dynamic visualization makes fundamental calculus concepts tangible, showing the progression from rough estimation (few, wide rectangles) to precise calculation (many narrow rectangles) that underlies integral calculus.

Student view

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Activity

- Can you explain how the rectangles were constructed?
- Can you see that the total area of the rectangles is close to the area of the region?
- Can you suggest a way to get an even closer approximation?

In addition to using the applet above to experiment, you can also use WolframAlpha (http://www.wolframalpha.com). Type into the search line:

integral of y=x^2 for 0<x<3 10 intervals

Try to interpret the answer WolframAlpha gives. You can experiment with other functions over a different interval and different number of subintervals.

- WolframAlpha gives its interpretation of the input. This interpretation mentions the midpoint method. Try to explain what this means. Later WolframAlpha also mentions other methods (left endpoint, right endpoint) to approximate the integral. Try to explain what these other methods mean. These methods will not be discussed in detail, but it is useful to think about the similarities and differences.
- WolframAlpha also gives a visual illustration of the approximation. Is this similar to the illustration in the applet above?
- WolframAlpha also gives the symbolic form of the approximation. This symbolic form is discussed next.

Important

If f is continuous on the interval $[a, b]$ and $a = x_0 < x_1 < \ldots < x_n = b$ and $x_{i-1} \leqslant x_i^* \leqslant x_i$ for $1 \leqslant i \leqslant n$, then the sum

$$f(x_1^*)(x_1 - x_0) + \ldots + f(x_n^*)(x_n - x_{n-1}) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

is an approximation of $\int_a^b f(x)\mathrm{d}x$.

Student view

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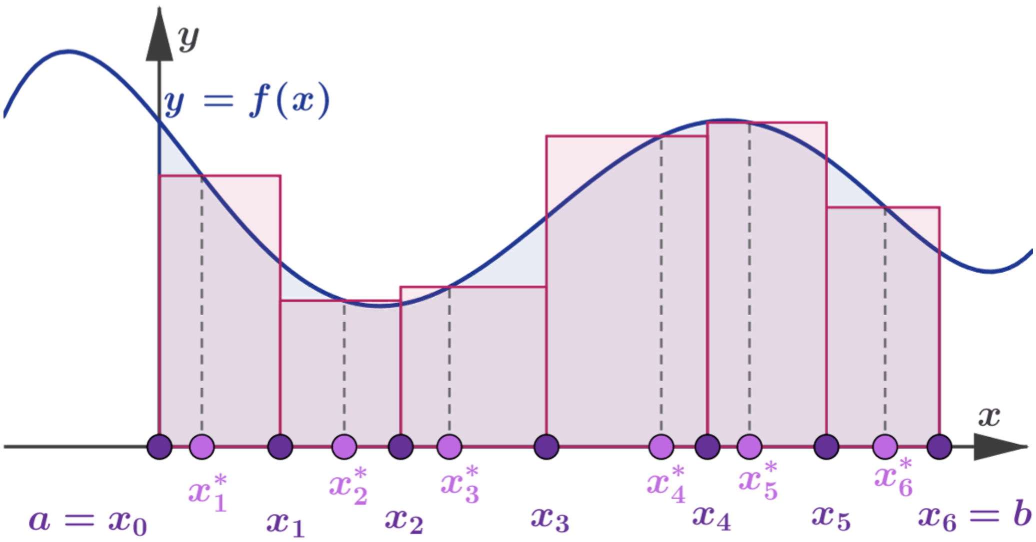
Moreover, the error of this approximation is approaching 0 as n increases without bound and the maximum length of the subintervals approaches 0.

The sum above is called the Riemann sumfor the function f corresponding to the partition $a = x_0 < x_1 < \ldots < x_n = b$ and the points $x_1^*, \ldots, x_n^*$ in the subintervals.

The diagram below illustrates the result. It shows a graph of a continuous function and rectangles that approximate the region bounded by the graph and the x-axis over a finite interval $[a, b]$.



More information

The image is a diagram illustrating a continuous function represented by a smooth curve on a graph. The X-axis is labeled with points from (a = x_0) to (x_6 = b), and rectangles are drawn between these points under the curve. The rectangles have bases corresponding to intervals ([x_0, x_1], [x_1, x_2], ..., [x_5, x_6]) and heights set to the function value at midpoints (x_1^, x_2^, ..., x_6^*). The function shown is labeled as (y = f(x)), and the graph demonstrates how the sum of the areas of the rectangles approximates the area under the curve over the interval ([a, b]).

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Student view

The first rectangle has base $x_1 - x_0$ and height $f(x_1^*)$, so the area is $f(x_1^*)(x_1 - x_0)$. Similar expressions give the area of the other rectangles, so the total area of all the rectangles is

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$$f(x_1^*)(x_1 - x_0) + \ldots + f(x_n^*)(x_n - x_{n-1}) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}).$$

To show that this area is close to the area of the region bounded by the graph of f and the x-axis over the interval $[a, b]$ (if the subintervals are short enough) is beyond the syllabus. It involves the concept of uniform continuity, which is not covered in this course. Nevertheless, the illustration is intuitively convincing and sufficient for this course without more justification.

✓ **Important**

A Riemann sum is an approximation of an integral using a finite sum of areas, typically using areas of rectangles.

Before you move on to the examples, you can try the page on Riemann sums on Wolfram MathWorld (http://mathworld.wolfram.com/RiemannSum.html) to experiment a bit more.

Example 1

★★☆

- Show that $\int_1^2 \frac{1}{x} \mathrm{d}x = \ln 2$.
- Use a Riemann sum for $y = \frac{1}{x}$ with five subintervals of $[1, 2]$ of equal length and the midpoints of these intervals to get a rational approximation of $\ln 2$.

- Using that $\int \frac{1}{x} \mathrm{d}x = \ln x + c$, you get

$$\int_1^2 \frac{1}{x} dx = (\ln x + c)\Big|_1^2 = (\ln 2 + c) - (\ln 1 + c) = \ln 2 + x - 0 - c = \ln$$

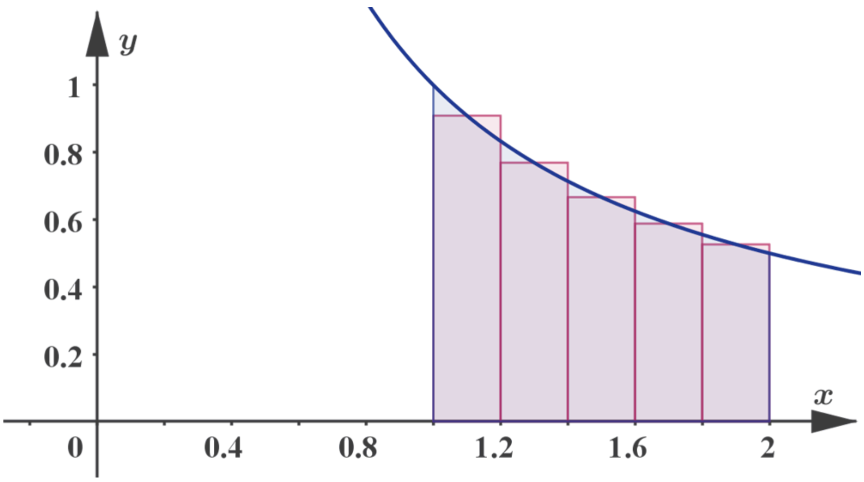
- The endpoints of the subintervals are $1, 1.2, 1.4, 1.6, 1.8, 2$.
  The midpoints of these intervals are $1.1, 1.3, 1.5, 1.7, 1.9$.
  Since the length of all subintervals is 0.2, the Riemann sum is

Student view

$$\frac{1}{1.1} \times 0.2 + \frac{1}{1.3} \times 0.2 + \frac{1}{1.5} \times 0.2 + \frac{1}{1.7} \times 0.2 + \frac{1}{1.9} \times 0.2$$
$$= 0.2\left(\frac{10}{11} + \frac{10}{13} + \frac{10}{15} + \frac{10}{17} + \frac{10}{19}\right)$$
$$= 2\left(\frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19}\right) = \frac{479378}{692835}$$

Notice that these two answers are close as $\ln 2 \approx 0.693$ and $\frac{479378}{692835} \approx 0.691$

The diagram below illustrates the region representing the definite integral and the approximating rectangles.



To get more accurate results with Riemann sums, use more rectangles with a smaller width. This will reduce the error as the area of the ‘corners’ of the rectangles will be smaller.

Example 2

★★★

- Show that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Student view

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- Use Riemann sums to find $\int_0^3 x^2 + 2\mathrm{d}x$.

- Use mathematical induction to prove $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for $n = 1, 2, 3, \ldots$

For $n = 1$, $1^2 = 1 = \frac{1 \times (1+1) \times (2 \times 1 + 1)}{6}$, so the statement is true.

If $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$, then

$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}\end{aligned}$$

The statement is true for $n = 1$ and if it is true for $n = k$ it is true for $n = k + 1$, so, by the principle of mathematical induction, it is true for all $n = 1, 2, \ldots$

- Since the question did not specify how to set up the Riemann sum, you can use n intervals of equal length and the left endpoint of each interval to determine the height of the rectangles. So,

$$\begin{aligned}x_i - x_{i-1} &= \frac{3-0}{n} = \frac{3}{n} \\ x_i &= 0 + i\frac{3}{n} = \frac{3i}{n} \\ f(x_i^*) = f(x_i) &= \frac{9i^2}{n^2} + 2 \\ f(x_i^*)(x_1 - x_{i-1}) &= \left(\frac{9i^2}{n^2} + 2\right) \times \frac{3}{n} = \frac{27i^2}{n^3} + \frac{6}{n}\end{aligned}$$

Student view

Overview
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Hence,

$$\begin{aligned}\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) &= \sum_{i=1}^{n}\left(\frac{27i^2}{n^3}+\frac{6}{n}\right)\\ &= \frac{27}{n^3}\sum_{i=1}^{n} i^2 + \frac{6}{n}\sum_{i=1}^{n} 1\\ &= \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} + n\frac{6}{n}\\ &= \frac{9(n+1)(2n+1)}{2n^2} + 6\\ &= \frac{30n^2+27n+9}{2n^2} = 15 + \frac{27}{2n} + \frac{9}{2n^2}\end{aligned}$$

The definite integral is the limit of these Riemann sums as the division of the interval $[0, 3]$ gets finer and finer. As n increases without bound, both $\frac{27}{2n}$ and $\frac{9}{2n^2}$ approach 0, so

$$\int_0^3 x^2 + 2\mathrm{d}x = \lim_{n\to\infty} 15 + \frac{27}{2n} + \frac{9}{2n^2} = 15$$

Making connections

Using the notation $\Delta x_i = x_i - x_{i-1}$ in the Riemann sum formula yields

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Notice the similarity to the notation for the definite integral it approximates,

$$\int_a^b f(x)\mathrm{d}x$$

This similarity will help you recognize how to use integration to find volumes of certain solids in the next sections.

Student view

Overview
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$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

3 section questions

Question 1

Difficulty: ★☆☆

Consider the function defined by $f(x) = \frac{4}{1+x^2}$. In this question you can use that f is decreasing on the interval $[0, 1]$ and that $\int_0^1 \frac{4}{1+x^2}\mathrm{d}x = \pi$.

Use two intervals of equal length and the Riemann sum corresponding to the left endpoints of these intervals to find a rational upper bound of π.

Give this upper bound as your answer either as a decimal or as a fraction (in the form n/m) in fully simplified form.

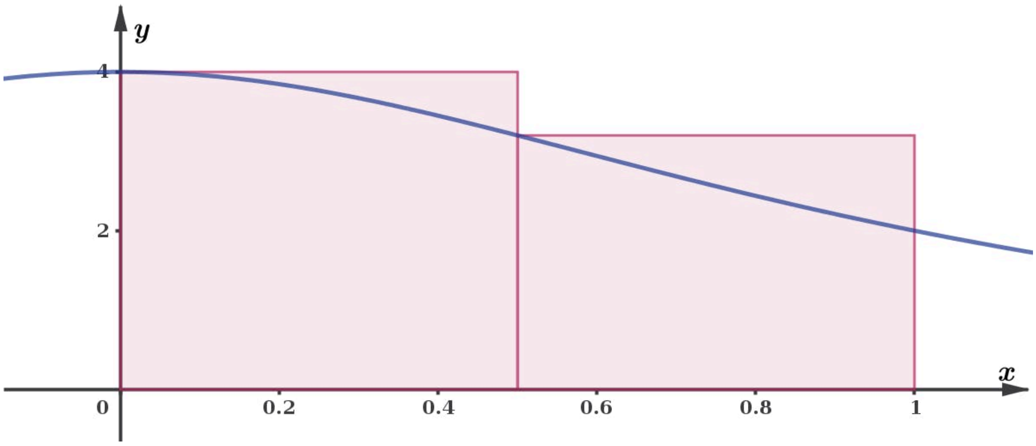
3.6 ✓

Accepted answers

3.6, 3,6, 18/5

Explanation

The diagram below shows part of the graph of f and the rectangles illustrating the Riemann sum corresponding to the left endpoints.



Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

More information

The length of the base of the rectangles is 0.5.

The Riemann sum is

$$0.5 \times \frac{4}{1+0^2} + 0.5 \times \frac{4}{1+0.5^2} = 3.6$$

According to the claims in the question, the sum of the area of the two rectangles is greater than the area of the region below the graph, which is π. Hence, 3.6 is an upper bound for π.

Question 2

Difficulty: ★☆☆

Consider the function defined by $f(x) = \frac{4}{1+x^2}$. In this question you can use that f is decreasing on the interval $[0, 1]$ and that $\int_0^1 \frac{4}{1+x^2} \mathrm{d}x = \pi$.

Use two intervals of equal length and the Riemann sum corresponding to the right endpoints of these intervals to find a rational lower bound of π.

Give this lower bound as your answer either as a decimal or as a fraction (in the form n/m) in fully simplified form.

2.6 ✓

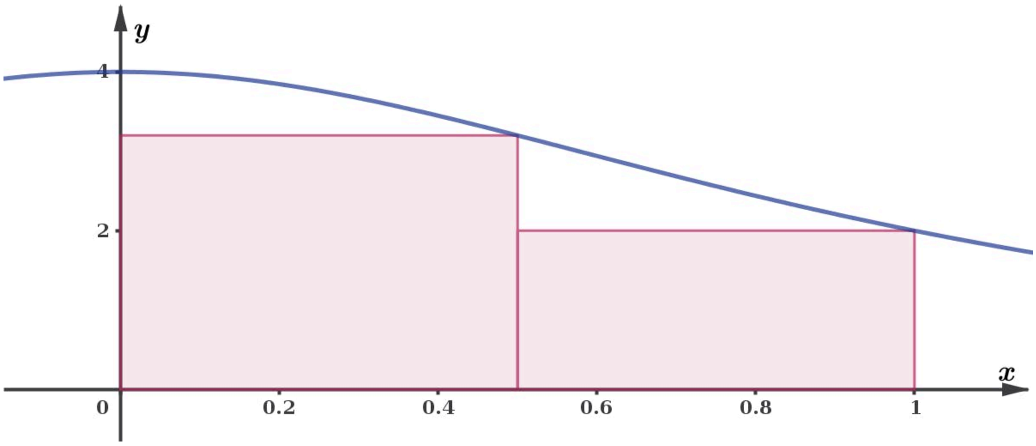
Accepted answers

2.6, 2,6, 13/5

Explanation

The diagram below shows part of the graph of f and the rectangles illustrating the Riemann sum corresponding to the right endpoints.

Student view



More information

The length of the base of the rectangles is 0.5.

The Riemann sum is

$$0.5 \times \frac{4}{1+0.5^2} + 0.5 \times \frac{4}{1+1^2} = 2.6$$

According to the claims in the question, the sum of the area of the two rectangles is smaller than the area of the region below the graph, which is π. Hence, 2.6 is a lower bound for π.

Question 3

Difficulty:

★☆☆

Consider the function defined by $f(x) = \dfrac{4}{1+x^2}$.

In this question you can use that $\displaystyle\int_0^1 \frac{4}{1+x^2}\mathrm{d}x = \pi$.

Use two intervals of equal length and the Riemann sum corresponding to the midpoints of these intervals to find a rational approximation of π.

Give this approximation as your answer either exactly as a fraction (in the form n/m) in fully simplified form or as a decimal rounded to three significant figures.

1344/425 ✓

Accepted answers

1344/425, 3.16, 3,16

Explanation

Overview
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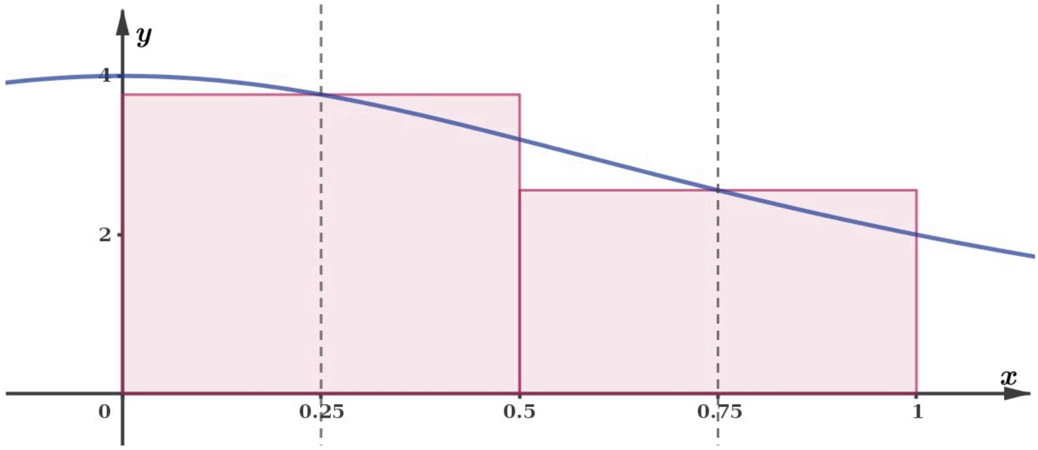
The diagram below shows part of the graph of f and the rectangles illustrating the Riemann sum corresponding to the midpoints.



More information

The length of the base of the rectangles is 0.5.

The Riemann sum is

$$0.5 \times \frac{4}{1+0.25^2} + 0.5 \times \frac{4}{1+0.75^2} = \frac{1344}{425} \approx 3.16$$

According to the claim in the question, the sum of the area of the two rectangles is an approximation area of the region below the graph, which is π. Hence, 3.16 is an approximation of π.

5. Calculus / 5.17 Area and volume

Volume of revolution about the x-axis

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On the applet below you can explore how cylinders can be used to approximate solids by rotating a region 360° about the x-axis.

Student
view

Overview
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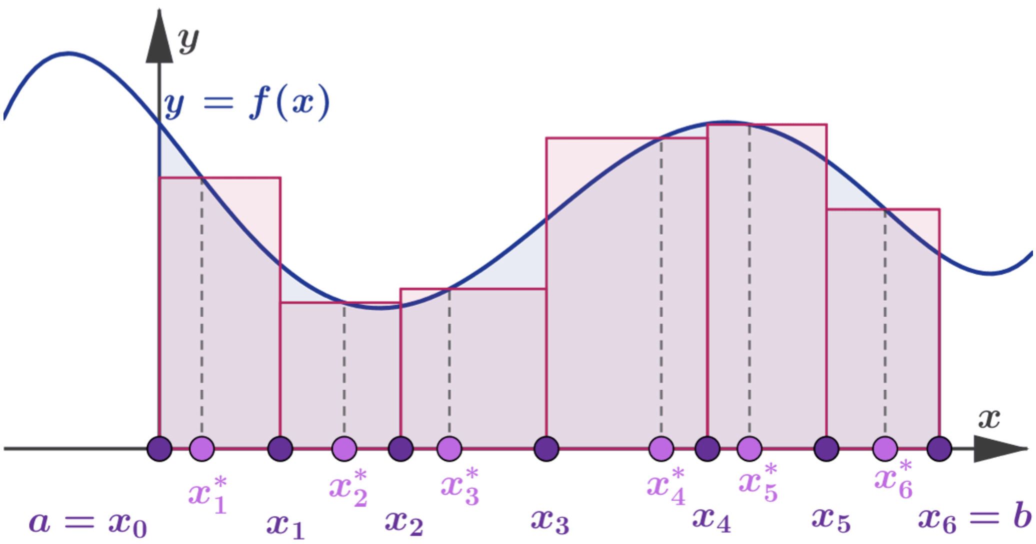
You can use these approximations to find the volume of the solid generated. The diagram below (which you have already seen when studying Riemann sums) illustrates the start of the process



More information

The diagram illustrates the process of using Riemann sums to approximate the volume of a solid. It features a curve labeled "y = f(x)" representing the function being integrated. Vertical dashed lines divide the x-axis into segments, each labeled from x_0 to x_6, with intermediary points x_1, x_2, x_3, x_4, x_5, *and* x_6 marked slightly above the x-axis. Rectangles are drawn, each with a height reaching a point on the curve at x_1, x_2, etc., indicating the approximate area under the curve. The overall shape reflects the start of an approximation process using rectangles under the curve y = f(x).

Student
view

Overview
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You get the approximating cylinders by rotating the rectangles that are used to approximate the region.

- The radius of the first cylinder (corresponding to the first rectangle) is $r = f(x_1^*)$.
- The height of the first cylinder is $h = x_1 - x_0 = \Delta x_1$.

Hence, the volume of the first cylinder is

$$\pi f(x_1^*)^2 \Delta x_1$$

The expression for the volume of the other cylinders is similar, so the total volume of all cylinders (if n cylinders are used) is

$$\sum_{i=1}^{n} \pi f(x_i^*)^2 \Delta x_i$$

- This expression is a Riemann sum for the function defined by $\pi f(x)^2$, so it is an approximation of the definite integral

  $$\int_a^b \pi f(x)^2 \mathrm{d}x$$

  Moreover, the approximation approaches 0 as the maximum length of the subintervals approaches 0.
- This expression also gives the total volume of the cylinders that approximate the solid.

As the same sum approximates both the integral and the volume, you have justified the following claim.

Another simple shape to visualize is a cone. Think of the mathematical game, Towers of Hanoi, on its side. It can be used to approximate the volume of a cone by finding the volume of each of the cylinders, or disks, and adding them up. Each disk has a volume of $V_n = \pi r^2 h = \pi f(x)^2 \Delta x$.

Student view

Overview
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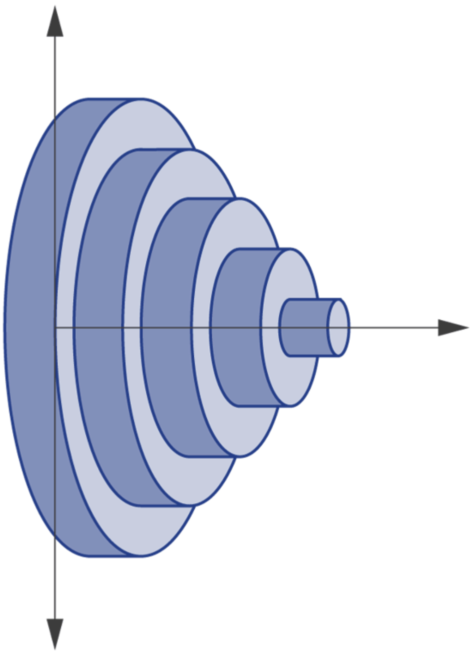

More information

The image is a diagram illustrating the disk method for approximating the volume of a cone. It shows a series of disks stacked horizontally, progressively getting smaller as they extend to the right, which collectively form a cone-like shape on its side. The left side of the diagram has larger disks, and they decrease in size towards the right, representing how volume segments contribute to the overall volume of the cone. There are arrows pointing in horizontal and vertical directions to represent axes and the directional approach of the method. The diagram visually explains the concept using the equation given: Each disk has a volume defined by V_n = πr^2h = πf(x)^2Δx, aligning with the textual description following the image about the disk method or 'disk and washer' method.

[Generated by AI]

As the number of disks approaches infinity, Δx approaches 0, and it is replaced with $\mathrm{d}x$. This technique is often referred to as the 'disk' method or 'disk and washer' method.

Important

Suppose f is a continuous function over the interval $[a, b]$ and $f(x) \geqslant 0$ for all $a < x < b$. Let R be the region bounded by the x-axis, the graph of f and the lines $x = a$ and $x = b$.

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view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Consider the solid generated by rotating R through $360°$ about the x-axis.

The **volume of this solid of revolution** is $\int_a^b \pi f(x)^2 \mathrm{d}x$.

Example 1

★☆☆

Consider the region bounded by the graph of $y = \mathrm{e}^x$, the x-axis and the lines $x = 0$ and $x = 2$.

Evaluate the volume of the solid generated by revolving the region through $360°$ about the x-axis.

Using the formula (and the rules of integration), the volume is

$$\begin{aligned} V &= \int_0^2 \pi(\mathrm{e}^x)^2 \mathrm{d}x \\ &= \pi \int_0^2 \mathrm{e}^{2x} \mathrm{d}x \\ &= \frac{\pi}{2} \int_0^2 2\mathrm{e}^{2x} \mathrm{d}x \\ &= \frac{\pi}{2} [\mathrm{e}^{2x}]_0^2 \\ &= \frac{\pi}{2} (\mathrm{e}^4 - 1) \approx 84.2 \end{aligned}$$

Example 2

★☆☆

Consider the shaded region bounded by the graph of $y = \sqrt{2x}$, the x-axis and the lines $x = 1$ and $x = 3$.

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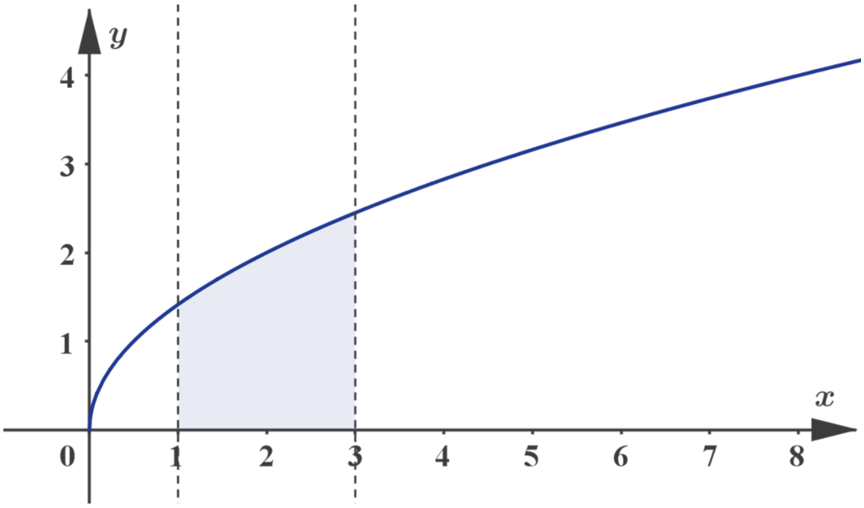

More information

The image shows a graph depicting the curve of the function y = √(2x). The x-axis is labeled from 0 to 8, with specific vertical lines marked at x = 1, x = 3, and dotted grid lines demarcating integer values. The y-axis is labeled from 0 to 4. The curve starts at the origin (0,0) and rises as it moves right, illustrating the function y = √(2x). A shaded region is present under the curve, starting at x = 1 and ending at x = 3, forming a trapezoidal shape between these x-values and the x-axis, bounded above by the curve. The task mentions evaluating the volume of the solid generated by revolving this shaded region around the x-axis.

[Generated by AI]

Evaluate the volume of the solid generated by revolving the region through 360° about the x-axis.

Using the formula (and the rules of integration), the volume is

$$\begin{aligned} V &= \int_1^3 \pi\left(\sqrt{2x}\right)^2 \mathrm{d}x \\ &= \pi \int_1^3 2x\mathrm{d}x \\ &= \pi[x^2]_1^3 \\ &= \pi(3^2 - 1^2) = 8\pi \approx 25.1 \end{aligned}$$

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Exam tip

The volume of revolution about the x-axis is given in the formula booklet as

$$V = \int_a^b \pi y^2 \mathrm{d}x$$

These examples used algebraic methods, but in an exam you can use your calculator to find definite integrals if the question does not ask for an exact value.

As with areas of regions, you can also extend the scope of the formula to the rotation of regions between two graphs.

Important

Suppose f and g are continuous functions over the interval $[a, b]$ and $f(x) > g(x) \geqslant 0$ for all $a < x < b$. Let R be the region bounded by the graphs of f and g and the lines $x = a$ and $x = b$.

Consider the solid generated by rotating R through $360°$ about the x-axis.
The **volume of this solid** is $\int_a^b \pi \left(f(x)^2 - g(x)^2\right) \mathrm{d}x$.

If you graph one of these problems with a Riemann sum, you will see 'washers with holes in the middle' instead of solid disks; hence the nickname 'disks and washers'.

The solid generated is the difference between

- the solid generated by rotating the region below the graph of f and
- the solid generated by rotating the region below the graph of g.

Hence, the volume is the difference between the volumes of these solids:

$$\int_a^b \pi f(x)^2 \mathrm{d}x - \int_a^b \pi g(x)^2 \mathrm{d}x = \int_a^b \pi \left(f(x)^2 - g(x)^2\right) \mathrm{d}x$$

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Be aware

Note that the formula above involves integrating the difference of the squares and not the square of the differences.

Example 3

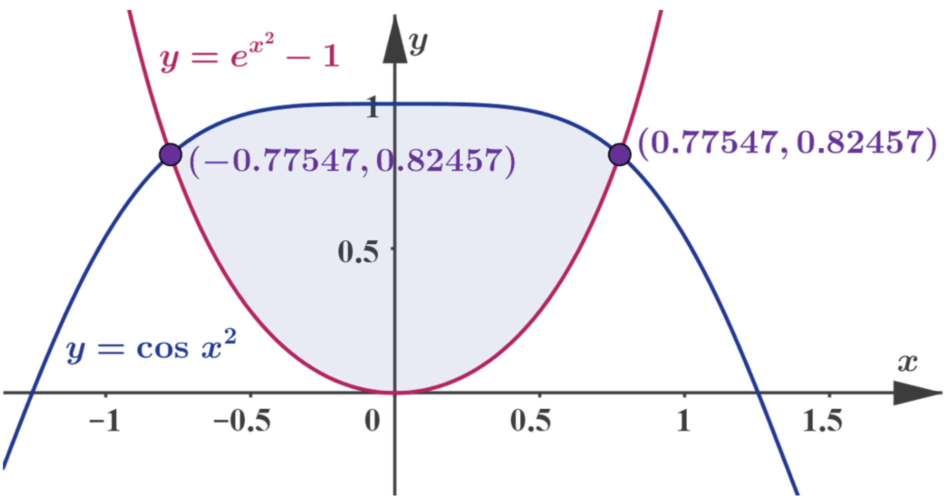
★★☆

Let R be the region enclosed by the graphs of $y = \mathrm{e}^{x^2} - 1$ and $y = \cos\left(x^2\right)$.

Find the volume of the solid generated by revolving R through $360°$ about the x-axis.

First, use a calculator to find the intersection points of the curves.

The diagram below shows the graphs, the intersection points and the shaded region R.



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Using the formula, the volume of revolution is approximately

$$\int_{-0.77547}^{0.77547} \pi\left(\left(\cos(x^2)\right)^2 - \left(\mathrm{e}^{x^2}-1\right)^2\right) \mathrm{d}x \approx 3.99$$

3 section questions

Question 1

Difficulty: ★☆☆

Consider the region bounded by the graph of $y = 2x$, the x-axis and the lines $x = 0$ and $x = 3$.

Evaluate the volume of the solid generated by revolving the region $360°$ about the x-axis.

1. 36π ✓
2. 108π
3. 18π
4. 9π

Explanation

According to the formula (and the rules of integration), the volume is

$$\begin{aligned} V &= \int_0^3 \pi\,(2x)^2\,\mathrm{d}x \\ &= 4\pi \int_0^3 x^2\,\mathrm{d}x \\ &= \frac{4\pi}{3}\left[x^3\right]_0^3 \\ &= 36\pi \end{aligned}$$

Question 2

Difficulty: ★☆☆

Consider the region bounded by the graph of $y = x^{3/2}$, the x-axis and the lines $x = 1$ and $x = 3$.

Student view

Overview
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The volume of the solid generated by revolving the region 360° about the x-axis is $n\pi$.

Find the value of n. Give your answer as an integer.

20

Accepted answers

20, n=20, n = 20, 20pi, 20π, 20 π

Explanation

According to the formula (and the rules of integration), the volume is

$$\begin{aligned} V &= \int_1^3 \pi\left(x^{3/2}\right)^2 \mathrm{d}x \\ &= \pi\int_1^3 x^3 \,\mathrm{d}x \\ &= \frac{\pi}{4}\left[x^4\right]_1^3 \\ &= 20\pi \end{aligned}$$

Question 3

Difficulty: ★★☆

Let R be the region enclosed by the graphs of $y = \mathrm{e}^{x^2} - 1$ and $y = 0.5 + \sin\left(x^2\right)$.

Find the volume of the solid generated by revolving R 360° about the x-axis.

Give your answer as a decimal, rounded to three significant figures. Do not include units in your answer.

Use your calculator in radian mode to find the answer to this question.

2.14

Accepted answers

2.14, 2,14

Explanation

First we use calculator to find the intersection points of the curves.

The diagram below shows the graphs, the intersection points and the shaded region R.

Student view

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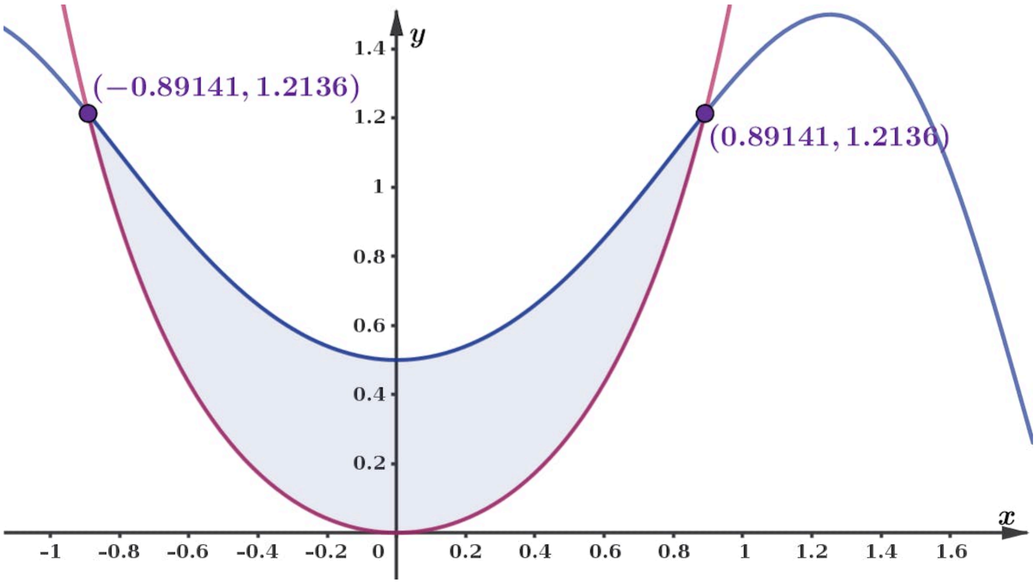

More information

Using the formula, we get that the volume of the solid of revolution is approximately

$$\int_{-0.89141}^{0.89141} \pi\left(\left(0.5+\sin(x^2)\right)^2-\left(\mathrm{e}^{x^2}-1\right)^2\right)\mathrm{d}x \approx 2.14$$

5. Calculus / 5.17 Area and volume

Volume of revolution about the y-axis

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/volume-of-revolution-about-the-yaxis-id-27270/print/) Assign

In this section, you will consider volumes of revolution where a region bounded by the y-axis and the graph of a function in y is rotated about the y-axis. You can explore solids like this using the applet below.

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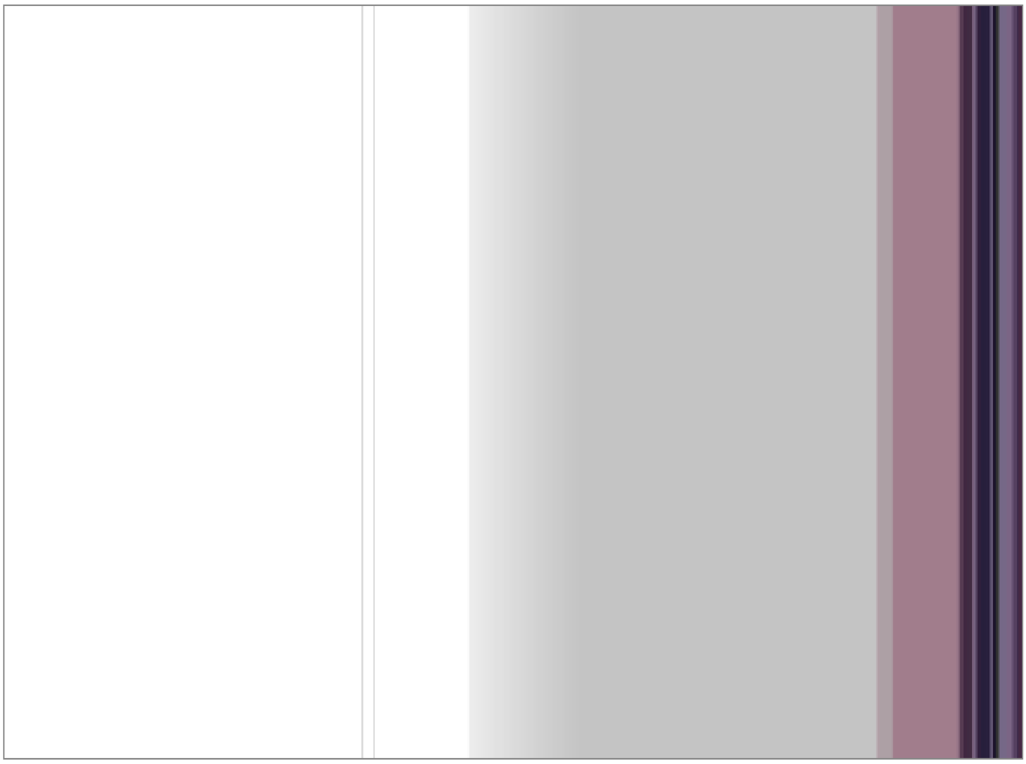

Interactive 1. Consider Volumes of Revolution Where a Region Bounded by the y-axis and the Graph of a Function in y Is Rotated about the y-axis.

More information for interactive 1

This interactive allows users to visualize volumes of revolution generated by rotating a region bounded by the y-axis and the graph of a function $f(y)$ about the y-axis.

The screen is divided in two halves, on the right there is a 3D graph with a curve in blue plotted on the XY plane with rectangles projecting on the Y-axis to this curve, and on the left there is an “Adjust curve” button that allows users to drag three red points on the curve to modify the curve's shape. Below it is a vertical slider that allows users to adjust the number of approximating rectangles, ranging from 1 to 20. There are two selection buttons that enable users to choose to show a volume figure of the curve and/or show cylinders. The number of cylinders displayed corresponds to the number of rectangles selected based on the slider’s position.

By dragging the red points, users can dynamically adjust the shape of the function graph, observing how the resulting volume and/or cylinders change in real-time and by dragging the vertical slider, the user can observe that as the number of cylinders increases, the approximation becomes more refined, closely resembling the actual volume calculated using the integral formula.

This interactive tool helps users intuitively understand the relationship between the graphical representation of a function in terms of y and the three-dimensional volume formed by its rotation, reinforcing the concept that the volume is computed via integration along the y-axis.

Student
view

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Since all you are doing is using y as the independent variable instead of x, all the methods from the previous section (/study/app/math-aa-hl/sid-134-cid-761926/book/volume-of-revolution-about-the-xaxis-id-27269/) can still be used.

Important

- Suppose f is a continuous function (of the variable y) over the interval $[a, b]$ and $f(y) \geqslant 0$ for all $a < y < b$. Let R be the region bounded by the y-axis, the graph of f and the lines $y = a$ and $y = b$.
  Consider the solid generated by rotating R 360° about the y-axis.
  The **volume of this solid** is $\int_a^b \pi f(y)^2 \mathrm{d}y$.
- Suppose f and g are continuous functions (of the variable y) over the interval $[a, b]$ and $f(y) > g(y) \geqslant 0$ for all $a < y < b$. Let R be the region bounded by the graphs of f and g and the lines $y = a$ and $y = b$.
  Consider the solid generated by rotating R 360° about the y-axis.
  The **volume of this solid** is $\int_a^b \pi \left(f(y)^2 - g(y)^2\right) \mathrm{d}y$.

Exam tip

The volume of revolution about the y-axis is given in the formula booklet as

$$V = \int_a^b \pi x^2 \mathrm{d}y$$

Example 1

★★☆

Let R be the region bounded by the y-axis, the graph of $y = \sqrt{x}$ and the line $x + 4y = 12$.

Find the volume of the solid generated by rotating R through 360° about the y-axis.

Student view

To find the intersection point of the graph and the line, solve

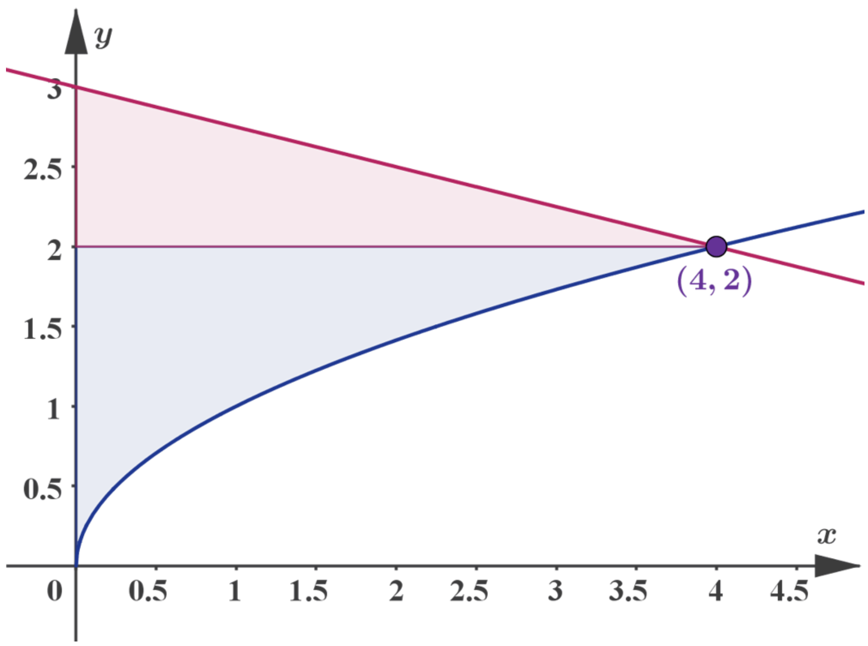
Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$$x + 4\sqrt{x} = 12$$
$$4\sqrt{x} = 12 - x$$
$$16x = (12 - x)^2$$
$$16x = 144 - 24x + x^2$$
$$x^2 - 40x + 144 = 0$$

The two solutions of this quadratic equation are $x = 4$ and $x = 36$.

- Substituting $x = 4$ into the original equations, you get $y = \sqrt{4} = 2$, and $4 + 4 \times 2$ is indeed 12. So the point $(4, 2)$ is an intersection point of the graph and the line.
- Substituting $x = 36$ into the original equations, you get $y = \sqrt{36} = 6$. However, $36 + 4 \times 6$ is not 12, so you can discard this solution.

The diagram below shows the region that is rotated and the intersection point found above.



T he diagram is split into two parts, because for $0 \leqslant y \leqslant 2$ the graph of $y = \sqrt{x}$ is rotated and for $2 \leqslant y \leqslant 3$ the graph of $x + 4y = 12$ is rotated.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

- To find the volume of the solid corresponding to the blue region, rearrange $y = \sqrt{x}$ as $x = y^2$. So the volume of this part of the solid is

$$\int_0^2 \pi\left(y^2\right)^2 \mathrm{d}y = \pi \int_0^2 y^4 \mathrm{d}y = \pi\left[\frac{y^5}{5}\right]_0^2 = \pi\left(\frac{2^5}{5} - \frac{0^5}{5}\right) = \frac{32\pi}{5}$$

- To find the volume of the solid corresponding to the red region, rearrange $x + 4y = 12$ as $x = 12 - 4y$. So the volume of this part of the solid is

$$\int_2^3 \pi(12 - 4y)^2 \mathrm{d}y = \pi\left[\frac{(12 - 4y)^3}{3 \times (-4)}\right]_2^3 = \pi\left(\frac{0^3}{-12} - \frac{4^3}{-12}\right) = \frac{16\pi}{3}$$

Hence, the volume of the solid is $\frac{32\pi}{5} + \frac{16\pi}{3} = \frac{176\pi}{15} \approx 36.9$ units cubed.

Example 2

★★☆

Let R be the region bounded by the graph of $y = \log_2 x$ and the line $2x - 3y = 6$.

Find the volume of the solid generated by rotating R through $360°$ about the y-axis.

Use a calculator to plot the curve and the line and to find the intersection points. To do this, write the equation of the line in the form $y = \frac{2x - 6}{3}$. The diagram below shows parts of the two graphs, the coordinates of the intersection points and the region these graphs enclose.

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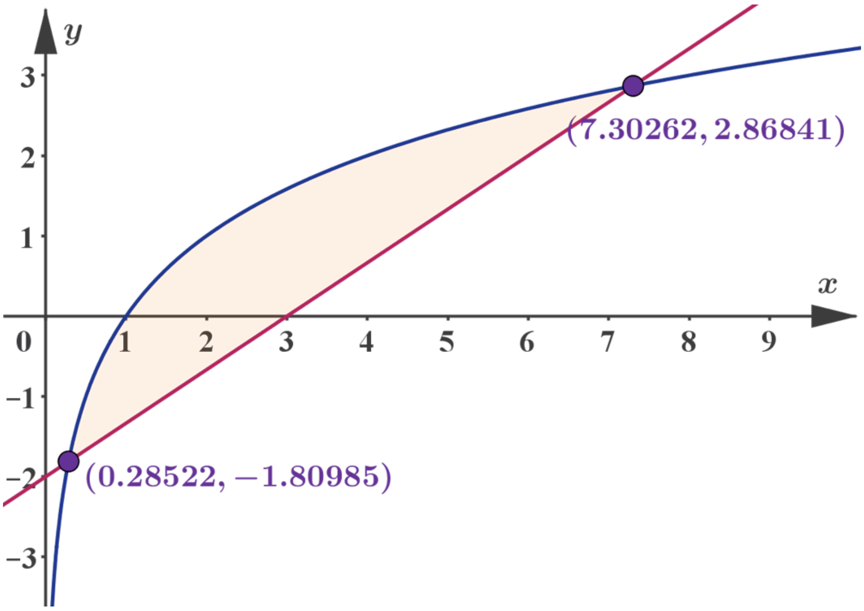

You can rearrange

- $y = \log_2 x$ as $x = 2^y$ and
- $2x - 3y = 6$ as $x = \dfrac{6+3y}{2}$.

Using these expressions and -1.80985 and 2.8684 as the bounds (of the y -values), you get that the volume is approximately

$$\int_{-1.80985}^{2.86841} \pi\left(\left(\frac{6+3y}{2}\right)^2 - \left(2^y\right)^2\right) \mathrm{d}y \approx 151 \text{ units cubed.}$$

Example 3

★★☆

The shaded region on the diagram below is enclosed by the graph of $y = \sqrt{x}$, $y = 2$, $y = a$ and the y-axis.

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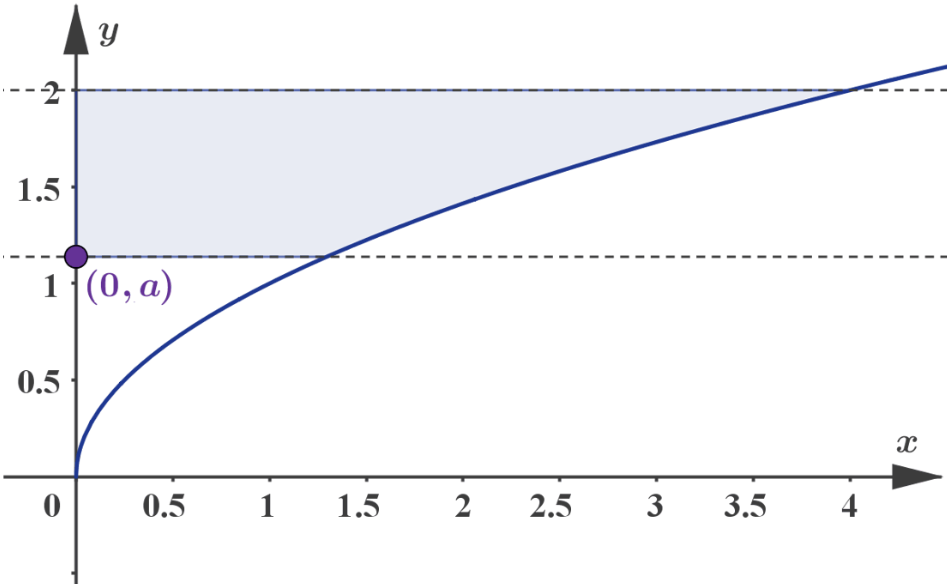

More information

The image is a diagram of a coordinate plane with the x-axis and y-axis. It features a curve representing the function y = sqrt(x), which starts at the origin (0,0) and increases to the right. A horizontal line at y=2 is depicted, creating a boundary as it intersects the y-axis and extends parallel to the x-axis. Additionally, there is a horizontal line at y=a, positioned below y=2 and above the x-axis, forming another boundary for the shaded region. The region is also confined by the y-axis to the left.

The shaded region is the area between the curve y = sqrt(x), the horizontal lines y = 2 and y = a, and the y-axis. This region is highlighted to indicate the area described in the text accompanying the image. The point (0, a) is marked to illustrate the intersection of y = a with the y-axis.

[Generated by AI]

The region is rotated $360°$ about the y-axis.

The volume of the solid generated is 6π units cubed. Find the value of a.

The equation of the curve, $y = \sqrt{x}$ can be written as $x = y^2$.

The volume of the solid is given by the integral $\int_a^2 \pi\left(y^2\right)^2 \mathrm{d}y = \pi \int_a^2 y^4 \mathrm{d}y$, so

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$$6\pi = \pi \int_a^2 y^4 \mathrm{d}y$$

$$6 = \left[\frac{y^5}{5}\right]_a^2$$

$$6 = \frac{2^5}{5} - \frac{a^5}{5}$$

$$30 = 32 - a^5$$

$$a^5 = 2$$

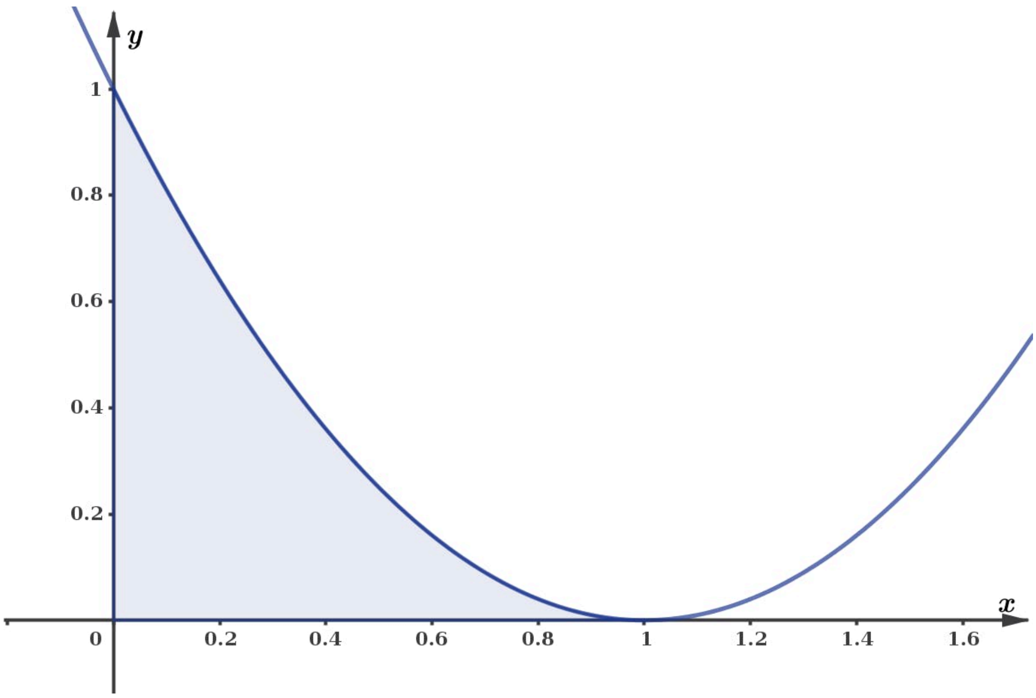
$$a = \sqrt[5]{2} \approx 1.15$$

3 section questions

Question 1

Difficulty: ★☆☆

Consider the region bounded by the graph of $y = (x-1)^2$ and the coordinate axes.



More information

The region is rotated $360°$ about the y-axis.

Find the volume of the solid generated.

1 $\frac{1}{6}\pi$ ✓

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2 $\frac{1}{6}$

3 $\frac{17}{6}\pi$

4 $\frac{17}{6}$

Explanation

Since the y-intercept of the graph is $(0, 1)$ and the graph touches the x-axis at $(1, 0)$, the y-range of the region is $0 \leqslant y \leqslant 1$.

Solving the defining equation of the graph for x in the first quadrant gives

$$\begin{aligned} y &= (x-1)^2 \\ \pm\sqrt{y} &= x - 1 \\ x &= 1 \pm \sqrt{y}. \end{aligned}$$

On the part of the curve that bounds the region the x-coordinates of the points are between 0 and 1. This means that we need to subtract $\sqrt{y}$ from 1 in the equation above, so we need the solution with the negative sign.

$$x = 1 - \sqrt{y}$$

Using the volume formula and the rules of integration gives

$$\begin{aligned} V &= \int_0^1 \pi(1-\sqrt{y})^2 \mathrm{d}y \\ &= \pi \int_0^1 y - 2\sqrt{y} + 1 \mathrm{d}y \\ &= \pi \left[\frac{y^2}{2} - \frac{4y^{3/2}}{3} + y\right]_0^1 \\ &= \pi\left[\frac{1}{2} - \frac{4}{3} + 1\right] - \pi\left[\frac{0}{2} - \frac{4 \times 0}{3} + 0\right] \\ &= \frac{1}{6}\pi. \end{aligned}$$

Question 2

Difficulty: ★★☆

Consider the region bounded by the graph of $y = 2\mathrm{e}^{-x/3} - 1$ and the coordinate axes.

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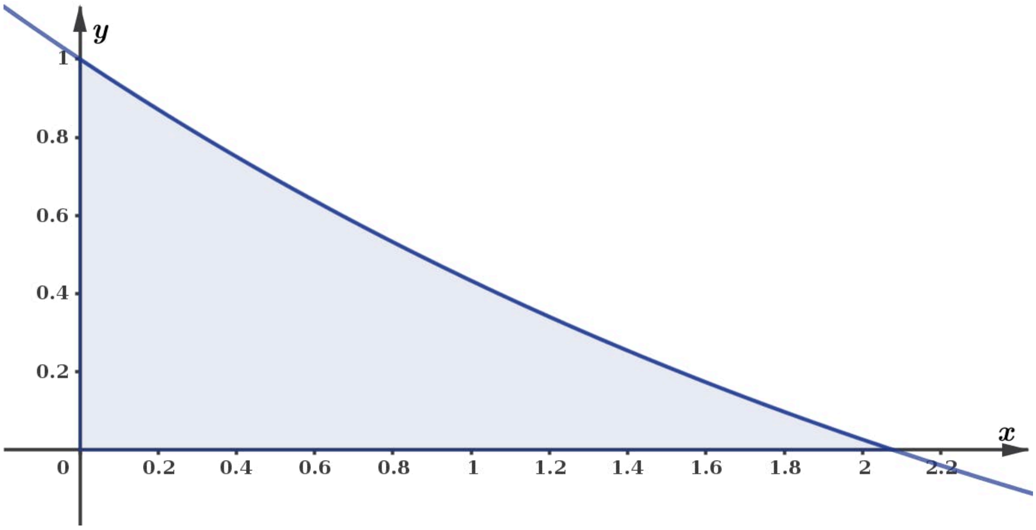

More information

The region is rotated 360° about the y-axis.

Find the volume of the solid generated. Give your answer as a decimal, rounded to three significant figures.

3.77

Accepted answers

3.77, 3,77

Explanation

Since the y-intercept of the graph is at $y = 2\mathrm{e}^{-0/3} - 1 = 2 - 1 = 1$, the y-span of the region is $0 \leqslant y \leqslant 1$.

Solving the defining equation of the graph for x gives

$$\begin{aligned} y &= 2\mathrm{e}^{-x/3} - 1 \\ \frac{y+1}{2} &= \mathrm{e}^{-x/3} \\ \ln\frac{y+1}{2} &= -x/3 \\ x &= -3\ln\frac{y+1}{2} \end{aligned}$$

Thus, the integral for the volume of revolution becomes

$$V = \int_0^1 \pi\left(-3\ln\frac{y+1}{2}\right)^2 \mathrm{d}y$$

Using our GDC, we find that

$$V = \int_0^1 \pi\left(-3\ln\frac{y+1}{2}\right)^2 \mathrm{d}y \approx 3.77$$

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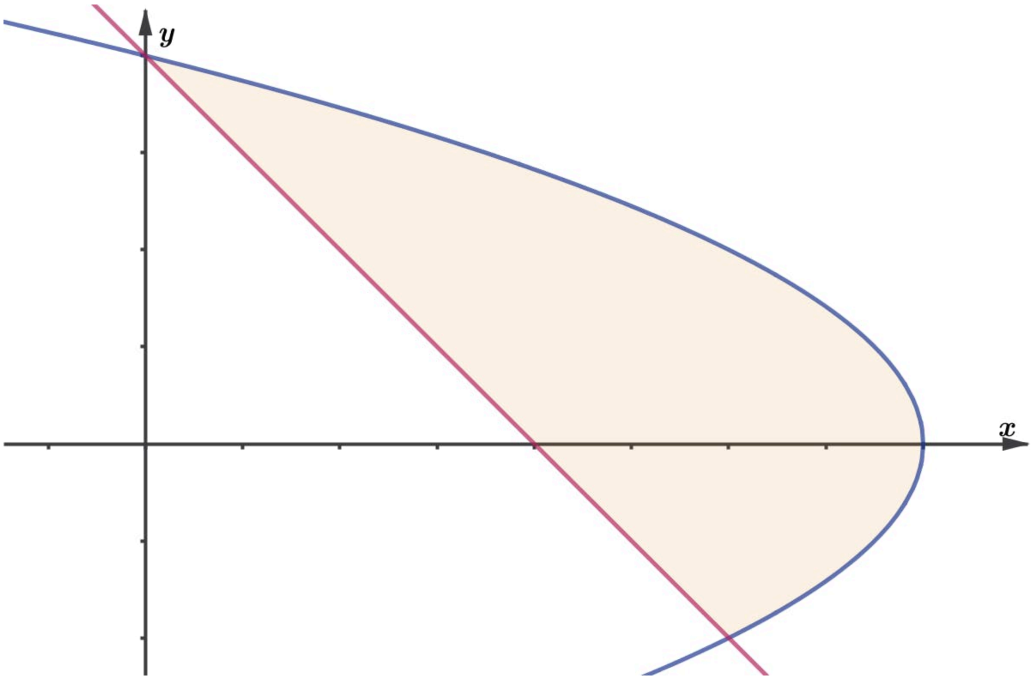
Question 3

Difficulty:

★★☆

Consider the region bounded by the graphs of $x + y^2 = 4$ and $x + y = 2$.



More information

The region is rotated 360° about the y-axis.

Find the volume of the solid generated. Give your answer as a decimal, rounded to three significant figures. Do not include units in your answer.

67.9

Accepted answers

67.9, 67,9

Explanation

First we find the second coordinate of the intersection points of the graphs.

These are given by the solutions of the system of equation

$$\begin{cases} x + y^2 = 4 \\ x + y = 2 \end{cases}$$

Expressing x from the second equation and substituting in the first gives

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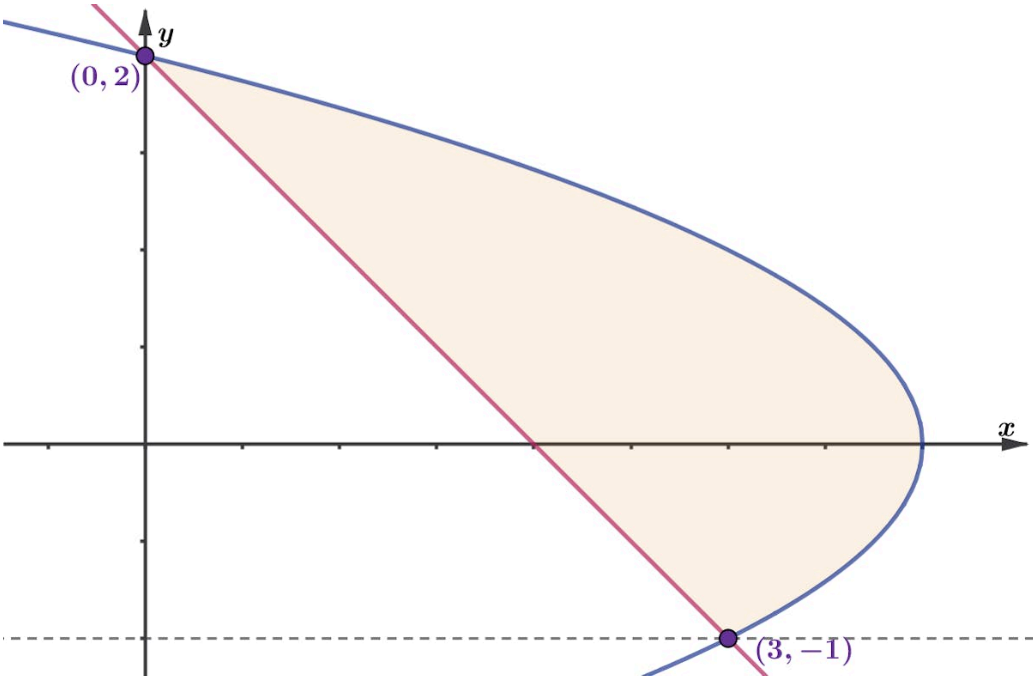
$$(2-y)+y^2=4$$
$$y^2-y-2=0$$
$$(y-2)(y+1)=0$$

The solutions are $y=2$ and $y=-1$.



More information

We use the formula $\int_a^b \pi\left(f(y)^2-g(y)^2\right)\mathrm{d}y$ to find the volume, so we rearrange the defining equations of the curves.

- For the blue curve,
  $$x+y^2=4$$
  $$x=4-y^2$$
- For the red curve,
  $$x+y=2$$
  $$x=2-y$$

Using these expressions in the formula and the graphing calculator, we get that the volume is

$$\int_{-1}^{2}\pi\left((4-y^2)^2-(2-y)^2\right)\mathrm{d}y\approx 67.9 \text{ units cubed.}$$

5. Calculus / 5.17 Area and volume

Checklist

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view

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Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-27271/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- find area of regions bounded by a curve and the y-axis
- find the volume of revolution about both the x and y-axis
- be aware that technology can also be used if an approximate value of the definite integral is enough.

5. Calculus / 5.17 Area and volume

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-27272/print/) Assign

Part 1

In this subtopic you learned a method for using approximation by cylinders to find exact volume of solids of revolution. The applet below shows another approach. You can use cylindrical shells instead of cylinders.

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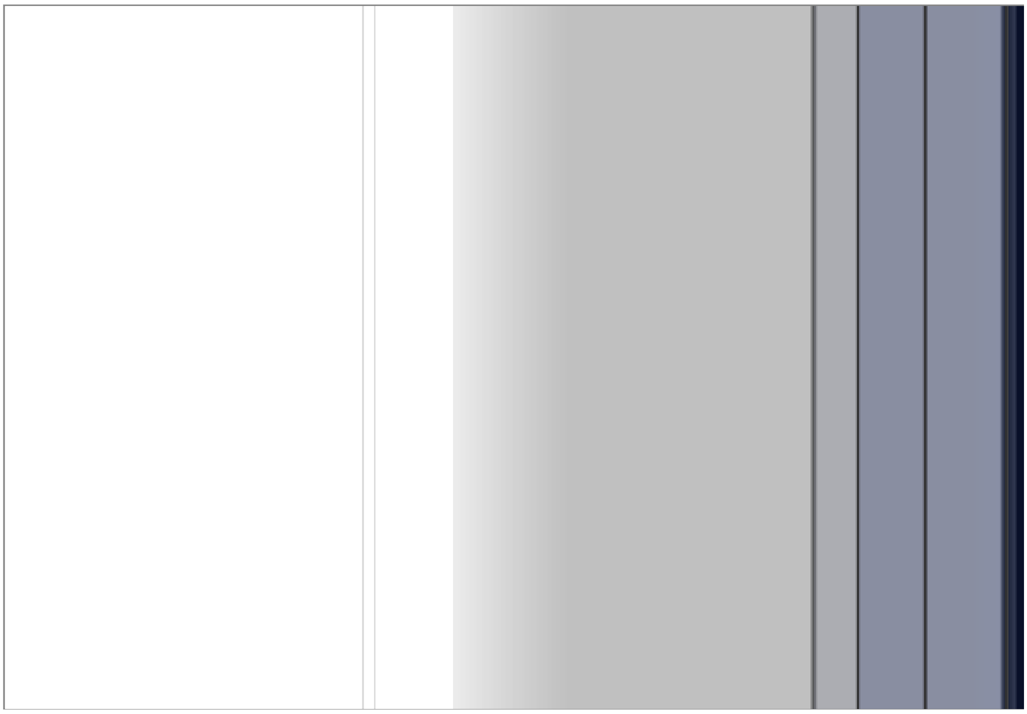

Interactive 1. Using Approximation by Cylinder Shells to Find Exact Volume of Solids of Revolution.

More information for interactive 1

This interactive allows users to visualize solids of revolution generated by rotating a region.

The screen is divided in two halves, on the right there is a graph with three axes with a curve line in blue on XY plane with rectangles projecting on X-axis, and on the left there is an “Adjust curve” button that allows users to drag three red points on the curve to modify the curve's shape. Below it a horizontal slider allows users to adjust the number of approximating rectangles, ranging from 1 to 10. There are two selection buttons that enable users to choose to show solid, which is below the “Adjust curve button”, and to show shells, which is below the horizontal slider. The number of cylindrical shells displayed corresponds to the number of rectangles selected based on the slider’s position. Below the "Show Shells" button, there is a "One by One" button that enables an additional horizontal slider, allowing users to select which individual shell is displayed in the graph. If the total number of shells is 5, the slider for the "One by One" button can be adjusted up to a maximum of 5.

By dragging the red control points, users can dynamically adjust the shape of the curve, observing how the resulting solid changes in real time. Users can select a curve and adjust the number of cylindrical shells (from 1 to 10) to approximate the solid formed by rotating the curve around an axis. The applet visually constructs each shell one by one, allowing users to observe how increasing the number of shells improves the approximation of the solid's volume.

- Use the applet to understand how this approximation works.
- Find an expression for the volume of the shells and an approximation of the volume of the solid as a sum of these shells.

Student
view

Overview
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- Can you turn this sum into an integral that expresses the volume of the solid?
- Pick a graph and find the volume of the solid generated by this graph.

Part 2

Before starting the applet below, think about how you would find the length of the river Danube (the second longest river in Europe).

Credit: Danubemap (https://commons.wikimedia.org/wiki/File:Danubemap.png) is in the public domain

More information

The image is a map illustrating the Danube River flowing through several European countries. It starts in Germany and passes southeast across Central and Eastern Europe until it reaches the Black Sea in Romania. The map includes labeled countries such as Germany, Austria, Hungary, Romania, and others along with major cities like Berlin, Vienna, Budapest, and Bucharest. The river is marked with a bold blue line running through the countries mentioned on the map, providing a clear visual path of its route. Other neighboring countries, such as Poland, Ukraine, Serbia, Italy, and France, are also shown to provide geographical context.

[Generated by AI]

Student
view

Interactive 1. Finding the line segments and approximation of the length of the curve as a sum of the lengths.

More information for interactive 1

This interactive graph allows users to visualize how to approximate the length of a curve—such as a river or a function's graph—by using straight-line segments. The screen is divided into two sections: the upper half displays a smooth blue curve marked with red control points, while the lower section provides two toggle buttons labeled 'Adjust curve' and 'Number of segments'.

When the 'Adjust curve' option is selected, users can drag the red points on the graph to reshape the curve. This feature enables dynamic editing of the path whose arc length is being approximated. A red line appears, connecting only the starting and ending points of the curve, indicating the direct distance across the curve without segment subdivisions.

When the 'Number of segments' option is selected, a slider appears beneath it, allowing users to choose a number from 1 to 50. The applet then overlays the curve with a connected series of straight-line segments—also called a polygonal path—that approximates the curve's shape. As users increase the number of segments using the slider, the polygonal path conforms more closely to the curve. This visual refinement demonstrates how increasing the number of segments leads to a better approximation of the curve's true length.

In addition, the current approximate length of the polygonal path is automatically calculated and displayed below the sliders. This value updates in real time as users adjust the curve's shape or change the number of segments, providing immediate visual and numerical feedback.

By toggling between curve adjustment and segment approximation, users can explore how geometry and calculus concepts come together. This activity builds intuition for the method of calculating arc length: by breaking a curved path into smaller linear segments and summing their lengths. It serves as a conceptual bridge to more advanced calculus ideas, such as integral formulas for arc length.

Student view

Overview
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Does the approximation on the applet match the method you suggested?

- Find an expression for the line segments and an approximation of the length of the curve as a sum of these lengths.
- Can you turn this sum into an integral that expresses the length of the curve?
- Pick a function and find the length of the graph.

You can use WolframAlpha (https://www.wolframalpha.com/) to find the length of a graph. For example, type into the search line:

length of y=x^2 for 0<x<1

Can you see that this length is expressed by an integral? Did you find this integral in the activity above?

Student view