The big picture

It may have been a while since you discovered that:

- you cannot divide by zero
- any number divided by infinity tends to zero
- zero times anything is zero
- infinity multiplied by or added to any number is still infinity.

You have, however, used these facts several times in this course – for example, when investigating asymptotes of rational functions.

You may now ask the question, 'what happens when you obtain an expression that combines two functions that tend to either zero or infinity?' Some combinations are obvious.

For example, $0 \times 0 = 0$ and $\infty + \infty = \infty$. Other combinations, however, are not as obvious.

So, what can you conclude when you obtain an expression that comes to $0 \times \infty$ or $\frac{0}{0}$?

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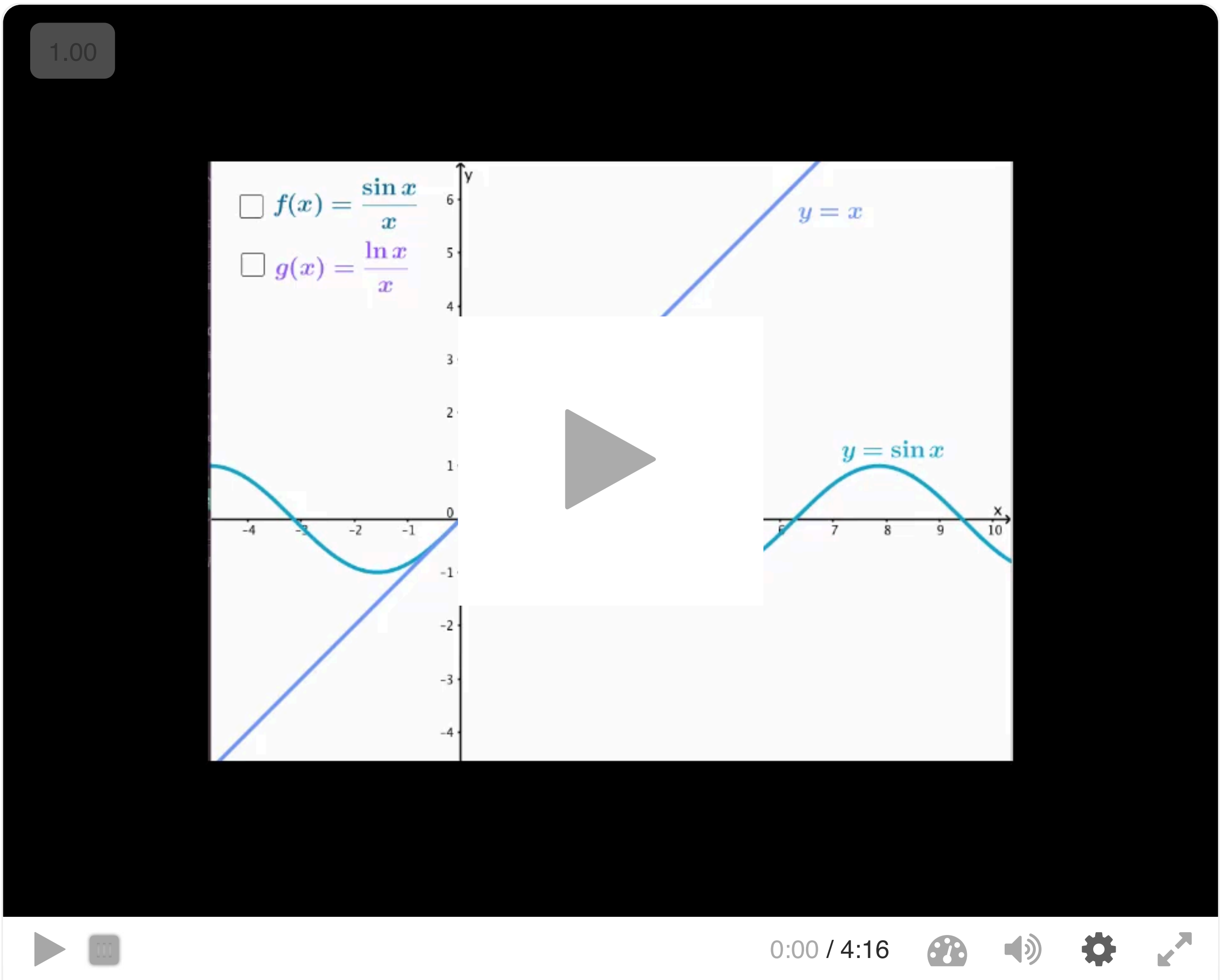
Some of the combinations, such as the latter two, are said to be of an indeterminate form and they do not allow immediate evaluation. This does not necessarily mean that their evaluation cannot yield a well-defined answer. Some expressions of indeterminate form do have well-defined, usually finite, answers and you need to learn how to obtain them.

The fact that some indeterminate forms are well defined is explored in the following video, which considers limits of the following two expressions:

- $\lim\limits_{x\to 0}\left[\dfrac{\sin x}{x}\right]$ of indeterminate form $\dfrac{0}{0}$
- $\lim\limits_{x\to \infty}\left[\dfrac{\ln x}{x}\right]$ of indeterminate form $\dfrac{\infty}{\infty}$



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Video 1. Exploring Indeterminate Forms.

More information for video 1

1

00:00:00,100 --> 00:00:02,100

narrator: In this video,

we're going to investigate functions

2

00:00:02,167 --> 00:00:04,433

with such the ratio

of composite functions,

3

00:00:04,500 --> 00:00:07,467

and particularly in the limit

when those individual functions

4

00:00:07,533 --> 00:00:12,000

become zero and or infinity,

giving us a ratio of $\frac{0}{0}$

5

00:00:12,133 --> 00:00:13,667

or $\frac{\infty}{\infty}$.

6

00:00:13,733 --> 00:00:15,633

And we're gonna do that

with a well-known example,

7

00:00:15,700 --> 00:00:17,833

$\frac{\sin(x)}{x}$ and $\frac{\ln(x)}{x}$.

8

00:00:17,900 --> 00:00:19,933

Let's start with $\frac{\sin(x)}{x}$

9

Student view

00:00:20,000 --> 00:00:23,900

and see that of course it's made up of

the ratio of $\sin(x)$ and x.

10

00:00:24,000 --> 00:00:25,533

So let's plot those two functions

11

00:00:25,667 --> 00:00:28,500

and put a point on each

of these and follow the points.

12

00:00:28,600 --> 00:00:31,667

Now, $\sin(x)$ of course,

oscillates between 1 and minus 1

13

00:00:31,733 --> 00:00:34,100

and $y = x$ of course, is a straight line

14

00:00:34,167 --> 00:00:36,567

that increases

if you go from left to right.

15

00:00:36,633 --> 00:00:39,533

Now of course you see that

when it approaches $x = 0$,

16

00:00:39,667 --> 00:00:41,933

both of those curves

go through the origin,

17

00:00:42,000 --> 00:00:44,567

that is both of them evaluate to zero.

18

00:00:44,667 --> 00:00:49,200

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So what happens when you take

the function $\frac{\sin(x)}{x}$

19

00:00:49,267 --> 00:00:50,900

and what happens when x becomes 0?

20

00:00:51,000 --> 00:00:52,900

Now this function

is $\frac{\sin(x)}{x}$,

21

00:00:52,967 --> 00:00:55,133

which is plotted here in dark blue line.

22

00:00:55,200 --> 00:00:57,433

And let's also follow

the point over there.

23

00:00:57,533 --> 00:01:00,633

Now of course you see that when $\sin(x)$

24

00:01:00,700 --> 00:01:03,867

and x have opposite sin,

then $\frac{\sin(x)}{x}$ is negative.

25

00:01:03,933 --> 00:01:05,433

$\frac{\sin(x)}{x}$ is negative.

26

00:01:05,933 --> 00:01:08,700

So here when x gets close to 0

27

00:01:08,767 --> 00:01:14,233

and $\sin(x)$ and x each turn to 0,

$\frac{\sin(x)}{x}$

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28

00:01:14,500 --> 00:01:16,167

evaluates to 1.

29

00:01:16,367 --> 00:01:22,500

So it is well defined even though

its form at that point is $\frac{0}{0}$,

30

00:01:22,567 --> 00:01:24,933

which is a indeterminant form

31

00:01:25,000 --> 00:01:27,300

and you cannot necessarily

conclude anything.

32

00:01:27,800 --> 00:01:31,000

Now for large x behavior,

maybe it's easy to think of

33

00:01:31,267 --> 00:01:34,567

$\frac{\sin(x)}{x}$'s $\frac{1}{x} \times \sin(x)$.

34

00:01:34,700 --> 00:01:37,933

So here we've plotted

$\frac{1}{x}$ instead of $y = x$,

35

00:01:38,200 --> 00:01:40,333

and you can see that $\sin(x)$ oscillates

36

00:01:40,400 --> 00:01:42,700

with exactly the same

periodicity as $\sin(x)$

37

00:01:42,767 --> 00:01:46,300

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that is 2π,

but that is amplitude is modulated

38

00:01:46,467 --> 00:01:50,800

by the function $\frac{1}{x}$,

so that it amplitude decreases

39

00:01:50,900 --> 00:01:52,367

as you go to higher x.

40

00:01:52,467 --> 00:01:54,867

And of course you can also

see that it's an even function

41

00:01:54,933 --> 00:01:56,900

as it is symmetric around the y axis.

42

00:01:57,633 --> 00:02:01,367

So when $\sin(x)$ and x go to 0

43

00:02:01,467 --> 00:02:02,500

and x goes to 0,

44

00:02:02,567 --> 00:02:06,233

$\frac{\sin(x)}{x}$ is a well-defined function.

45

00:02:06,433 --> 00:02:07,833

Now let's look at $\frac{\ln(x)}{x}$.

46

00:02:07,900 --> 00:02:11,133

And again, let's spot

$y = x$ and $y = \ln(x)$.

Student
view

47

00:02:11,200 --> 00:02:13,633

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We see that both of these functions

are increasing functions.

48

00:02:13,700 --> 00:02:15,367

Indeed, when we follow them,

49

00:02:15,700 --> 00:02:18,700

of course, x's to be positive

when we follow them as x increases,

50

00:02:18,967 --> 00:02:20,267

both of them increase.

51

00:02:20,333 --> 00:02:25,467

Although of course $y = x$ increases

faster than $y = \ln(x)$.

52

00:02:25,600 --> 00:02:27,733

Now the question is what happens

53

00:02:27,800 --> 00:02:31,900

when we consider the ratio

of $\frac{\ln(x)}{x}$,

54

00:02:31,967 --> 00:02:35,433

when x goes infinity,

because both of those turn to infinity.

55

00:02:35,500 --> 00:02:38,300

Again, you can think of this

as $\frac{1}{x} \times \ln(x)$ instead

56

00:02:38,367 --> 00:02:39,733

of $\frac{\ln(x)}{x}$.

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57

00:02:40,067 --> 00:02:43,500

But let's stick

to the $\frac{\ln(x)}{x}$

58

00:02:43,600 --> 00:02:47,467

and investigate what happens to the ratio

of the functions as x grows.

59

00:02:47,533 --> 00:02:49,267

Because of course we are gonna end up

60

00:02:49,333 --> 00:02:51,333

with the form $\frac{\infty}{\infty}$.

61

00:02:51,500 --> 00:02:54,700

And we here we have plotted $\frac{\ln(x)}{x}$.

62

00:02:54,933 --> 00:02:58,300

And you can see that as x grows

63

00:02:58,367 --> 00:03:00,433

$\frac{\ln(x)}{x}$ decreases

64

00:03:00,500 --> 00:03:04,467

and gets asymptotically

close to the x axis,

65

00:03:04,533 --> 00:03:06,267

that is y goes to 0.

66

00:03:07,000 --> 00:03:10,200

Now, perhaps we can zoom

Student view

in a little bit here

67

00:03:10,267 --> 00:03:15,100

to see that indeed the denominator

of $\frac{\ln(x)}{x}$

68

00:03:15,167 --> 00:03:17,300

in some sense wins this fight.

69

00:03:17,433 --> 00:03:20,700

So you can see that while both $y = x$

70

00:03:20,800 --> 00:03:23,133

and $\ln(x)$ are increasing functions,

71

00:03:23,200 --> 00:03:28,067

$\frac{\ln(x)}{x}$ is a function

that increases until it hits a maximum

72

00:03:28,133 --> 00:03:29,233

and then slides down.

73

00:03:29,300 --> 00:03:31,433

It's a decreasing function all the way

74

00:03:31,533 --> 00:03:35,267

and gets asymptotically

close to the x axis.

75

00:03:35,667 --> 00:03:37,167

So these are the two functions.

76

00:03:37,233 --> 00:03:40,567

$\frac{\sin(x)}{x}$,

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$\frac{\ln(x)}{x}$.

77

00:03:40,633 --> 00:03:43,300

The first one be takes

on an indeterminant form,

78

00:03:43,400 --> 00:03:47,533

$\frac{0}{0}$ when x goes to 0.

79

00:03:47,833 --> 00:03:50,300

And the second one takes

on an indeterminant form

80

00:03:50,500 --> 00:03:55,333

of $\frac{\infty}{\infty}$

when x goes to infinity.

81

00:03:55,833 --> 00:03:58,200

And we've seen that even though

82

00:03:59,067 --> 00:04:01,500

these forms being indeterminant

83

00:04:01,567 --> 00:04:05,033

will not allow us

in principle to conclude what happens.

84

00:04:05,100 --> 00:04:08,133

It seems that in those particular

cases for those functions,

00:04:08,267 --> 00:04:10,867

they are well-defined

and we need to determine

86

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00:04:10,933 --> 00:04:15,200

in which cases we have a well-defined

limit and in which cases we do not.

87

00:04:15,267 --> 00:04:16,200

And that was that.

In a previous subtopic (see section 5.12.1 (/study/app/math-aa-hl/sid-134-cid-761926/book/limits-and-convergence-id-26490/)), you studied the algebra of continuous functions and the limits of functions. There you discovered that for the limit of a continuous function, you can evaluate the function at that limit, i.e. $\lim_{x \to a} f(x) = f(a)$. However, if an expression is formed by using algebraic operations on functions, you may no longer be able to find the limit. Such expressions have an indeterminate form. Although there are many indeterminate forms, you will only study $\frac{0}{0}$ and $\frac{\infty}{\infty}$ using l'Hôpital's rule. In subtopic 5.19 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27009/), you will revisit this topic and apply the Maclaurin series to solve the same type of problem.

Concept

Throughout this subtopic, think about how your maths from previous courses and topics might help you to evaluate sophisticated functions. For example, in economics you might encounter the compound interest formula, $A = P \times \left(1 + \frac{r}{n}\right)^{nt}$, where A is the amount, P is the principal, r is the interest rate, n is the number of times interest is compounded per year, and t is the number of years. As you increase the number of times the interest is compounded per year towards infinity, what will be the result?

This simplifies to $A = P \times \left(1 + \frac{r}{\infty}\right)^{\infty t} = P \times 1^{\infty}$, but what **quantity** does that **represent** ?

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Can you think of other examples in science and engineering where you might need to find the limit of an expression that appears to be indeterminate?

5. Calculus / 5.13 Limits of indeterminate forms

L'Hôpital's rule

An indeterminate form is a function consisting of two or more smaller functions whose limit cannot be determined from the limits of the smaller functions. There are a variety of indeterminate forms that functions can take:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty-\infty,\ 0\times\infty,\ 0^0,\ 1^\infty \text{ and } \infty^0$$

In this course, you will study only those indeterminate forms that can be written as $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Making connections

Although you did not call them indeterminate forms, when you analysed holes and asymptotes of rational functions in subtopic 2.8 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27718/), you were investigating functions that had values of $\frac{0}{0}$.

There are times when methods from subtopic 2.8 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27718/) on rational functions are sufficient to deal with such indeterminate forms. At other times, you need a new method.

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Example 1

★☆☆

Find $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

It is clear that, given the denominator, the domain for this function does not include $x = 3$. Does the limit $x \to 3$ make sense?

You can factorise and simplify.

$$\frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3$$

Thus, you have the graph of the function $y = \frac{x^2 - 9}{x - 3}$, as shown below, in which $x = 3$ is taken out of the domain.

From your study of rational functions, you know that this is classified as a removable discontinuity, or a hole. This can be seen by substituting $x = 3$ into the original expression, which gives a denominator of 0. By substituting $x = 3$ into the simplified expression, $y = (3) + 3 = 6$, you can algebraically determine the coordinates of the hole as $(3, 6)$. Graphically, it looks like this:

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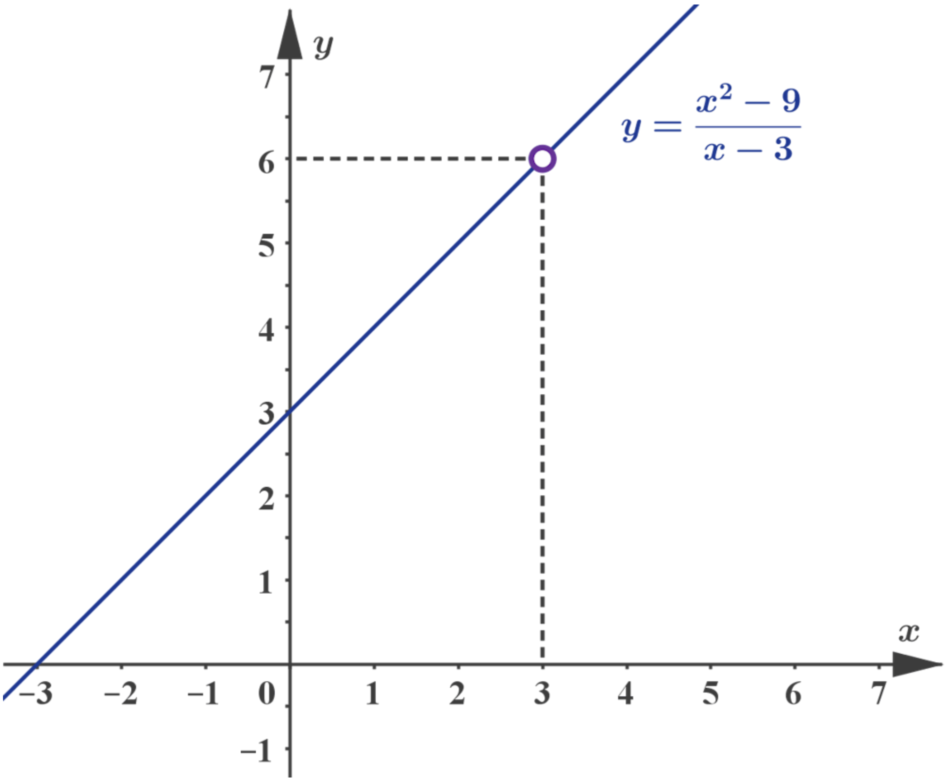

Hence, you see that the limit exists and that $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$.

In the last subtopic (see section 5.12.4 (/study/app/math-aa-hl/sid-134-cid-761926/book/first-principles-id-26493/)) you learned the definition of the derivative by first principles, i.e., $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. In the same way as you factorised the expression in **Example 1** and cancelled out the denominator, when finding the derivative, you rearrange and factorise the numerator of the derivative equation and cancel out the denominator h.

However, there are times when you cannot simplify a function to circumvent an indeterminate form and then you have to resort to alternative methods.

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L'Hôpital's rule

Important

L'Hôpital's rule is stated as follows:

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$, and if $g'(x) \neq 0$, then

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, provided the latter limit exists.

You should note that l'Hôpital's rule does not work for all indeterminate forms. It is limited to the forms $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$.

Exam tip

On exams it is not an expectation to check the condition that $g'(x) \neq 0$, but you do need to check the limits of the numerator and denominator before applying the rule.

International Mindedness

L'Hôpital's rule is named after Guillaume de l'Hôpital, a French mathematician who published a treatise entitled *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* . The rule was, in fact, discovered by his mentor, the Swiss mathematician Johann Bernoulli, who was acknowledged in the preface of his treatise.

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Example 2

★★☆

Find $\lim_{x \to -2} \frac{x^3 - 3x + 2}{x^3 + x^2 + 4}$ by

a) simplifying the quotient

b) using l'Hôpital's rule.

Let $h(x) = \frac{x^3 - 3x + 2}{x^3 + x^2 + 4}$

Then $h(-2) = \frac{(-2)^3 - 3(-2) + 2}{(-2)^3 + (-2)^2 + 4} = \frac{0}{0}$, which is an indeterminate form.

From subtopic 2.12 , you know that you can use the factor theorem to factorise the numerator and denominator:

$$x^3 - 3x + 2 = (x+2)(x-1)^2$$
$$x^3 + x^2 + 4 = (x+2)\left(x^2 - x + 2\right)$$

a) You can simplify the quotient and evaluate the limit:

$$\lim_{x \to -2} \frac{x^3 - 3x + 2}{x^3 + x^2 + 4} = \lim_{x \to -2} \frac{(x+2)(x-1)^2}{(x+2)(x^2 - x + 2)}$$
$$= \lim_{x \to -2} \frac{(x-1)^2}{(x^2 - x + 2)} = \lim_{x \to -2} \frac{((-2) -}{\left((-2)^2 - (-\right.}$$

b) As the limit is of the indeterminate form $\frac{0}{0}$, you may apply l'Hôpital's rule:

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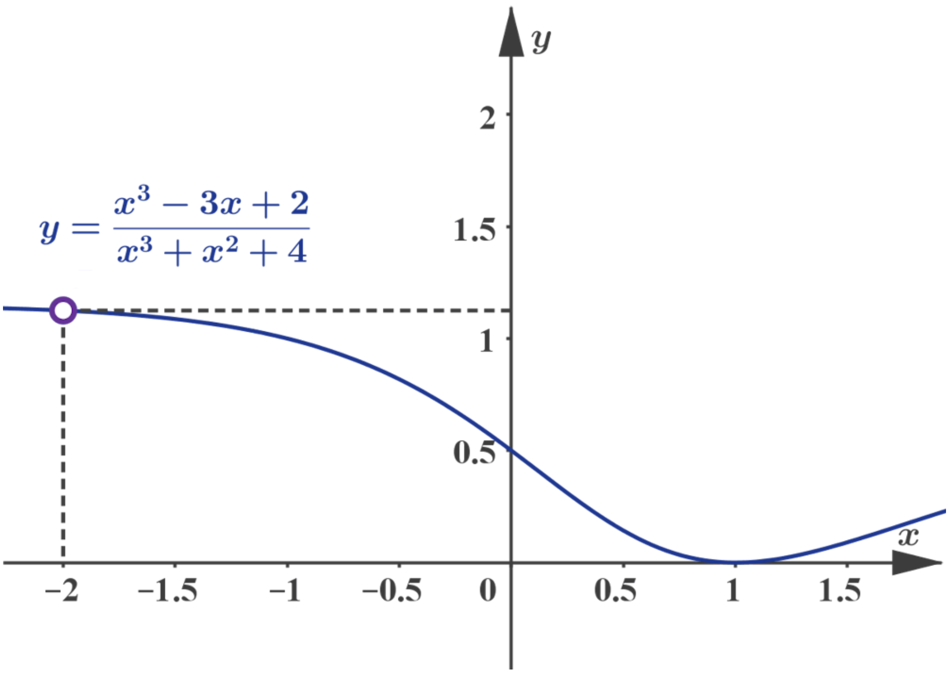
$$\lim_{x \to -2} \frac{f(x)}{g(x)} = \lim_{x \to -2} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to -2} \frac{x^3 - 3x + 2}{x^3 + x^2 + 4} = \lim_{x \to -2} \frac{3x^2 - 3}{3x^2 + 2x} = \frac{3(-2)^2 - 3}{3(-2)^2 + 2(-2)} = \frac{9}{8}$$

Of course, both approaches give the same result, which can also be seen in the figure below.



In The big picture (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26497/), you investigated the classic example of $\lim_{x \to 0} \frac{\sin x}{x}$.

Example 3

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Use l'Hôpital's rule to find $\lim_{x \to 0} \frac{\sin x}{x}$.

Let $h(x) = \frac{\sin x}{x}$

Then $h(0) = \frac{\sin 0}{0} = \frac{0}{0}$, which is an indeterminate form .

This allows you to use l'Hôpital's rule.

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = \frac{1}{1} = 1$$

This was shown in the introductory video to this subtopic and is also illustrated in the figure below, where both $\frac{\sin x}{x}$, which is not defined at $x = 0$, and its derivative, namely, $\frac{\cos x}{1} = \cos x$, which is defined at $x = 0$, are shown.

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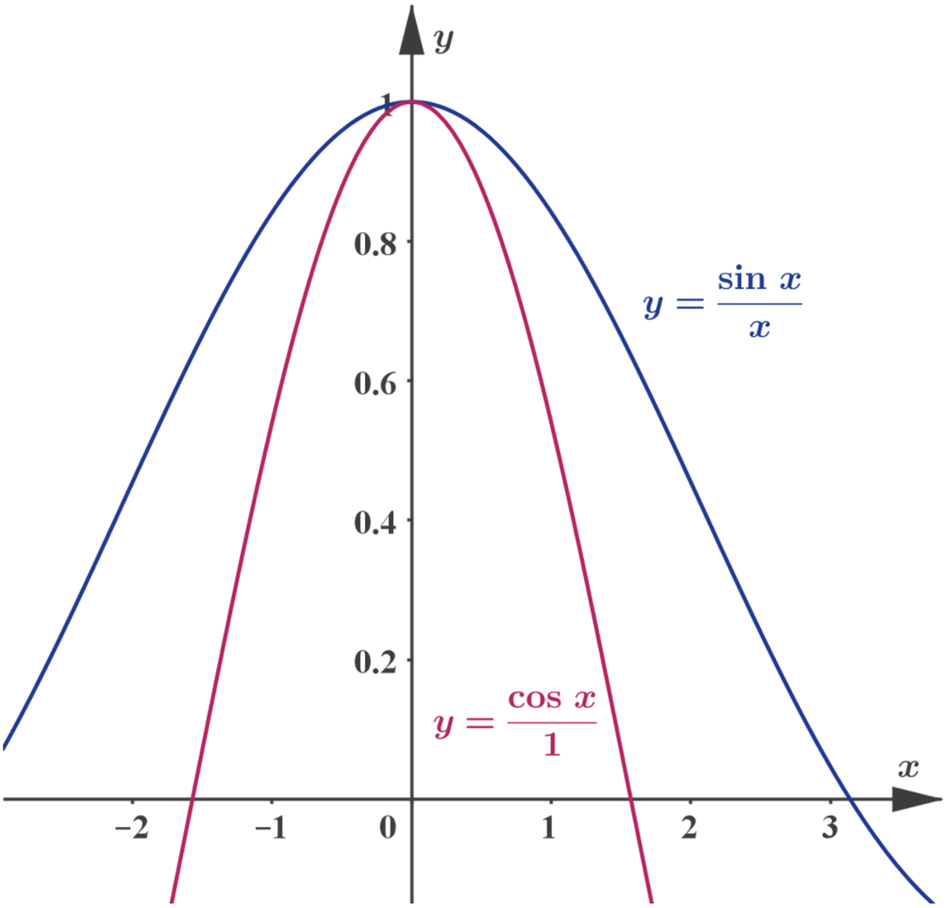

Example 4

★★☆

Evaluate $\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}}$

Let $h(x) = \frac{\ln x}{2\sqrt{x}}$

Then $h(\infty) = \frac{\ln(\infty)}{2\sqrt{\infty}} = \frac{\infty}{\infty}$, which is an indeterminate form.

This allows you to use L'Hôpital's rule.

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$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

$$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

Activity

Open up a graphing utility, such as Geogebra. Investigate $\lim_{x\to\pi} \frac{\sin x}{\tan x}$

- Plot the function $h(x) = \frac{\sin x}{\tan x}$
- Plot the numerator $f(x) = \sin x$
- Plot the denominator $f(x) = \tan x$
- Zoom in on the point $(\pi, 0)$. As the curves become linear, what does the relationship appear to be?
- Confirm your answer by solving using l'Hôpital's rule.

Repeated use of l'Hôpital's rule

An interesting advantage of l'Hôpital's rule is that there is no technical requirement for $g'(x) \neq 0$. If the result of l'Hôpital's rule yields another function of indeterminate form, there is the possibility that you can still find a limit. L'Hôpital's rule may be repeatedly applied if the limit of the quotient after differentiation continues to be of the indeterminate forms $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. Typically, this occurs when $f(x)$ or $g(x)$ is a polynomial and slowly reduces to a constant through repeated derivatives.

Example 5

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Evaluate $\lim_{x\to 0} \frac{2x - \sin 2x}{4x^3}$

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Let $h(x) = \dfrac{2x - \sin 2x}{4x^3}$

Then $h\left(0\right) = \dfrac{2(0) - \sin 2(0)}{4{(0)}^3} = \dfrac{0}{0}$, which is an indeterminate form.

This allows you to use L'Hôpital's rule.

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

$$\lim_{x\to 0} \frac{2x - \sin 2x}{4x^3} = \lim_{x\to 0} \frac{2 - 2\cos 2x}{12x^2} = \frac{2 - 2\cos 0}{12{(0)}^2} = \frac{0}{0}$$

This is still indeterminate. Apply L'Hôpital's rule again:

$$\lim_{x\to 0} \frac{2 - 2\cos 2x}{12x^2} = \lim_{x\to 0} \frac{4\sin 2x}{24x} = \frac{4\sin 2(0)}{24(0)} = \frac{0}{0}$$

This is still indeterminate. Apply L'Hôpital's rule again:

$$\lim_{x\to 0} \frac{4\sin 2x}{24x} = \lim_{x\to 0} \frac{8\cos 2x}{24} = \frac{8\cos 2(0)}{24} = \frac{8}{24} = \frac{1}{3}$$

Thus, after applying L'Hôpital's rule three times you find that

$$\lim_{x\to 0} \frac{2x - \sin 2x}{4x^3} = \frac{1}{3}$$

This result is far from obvious, considering the original function.

The result is also illustrated in the figure below, where both $\dfrac{2x - \sin 2x}{4x^3}$, which is not defined at $x = 0$, and its third derivative, namely, $\dfrac{8\cos 2x}{24}$, which is defined at $x = 0$, are shown.

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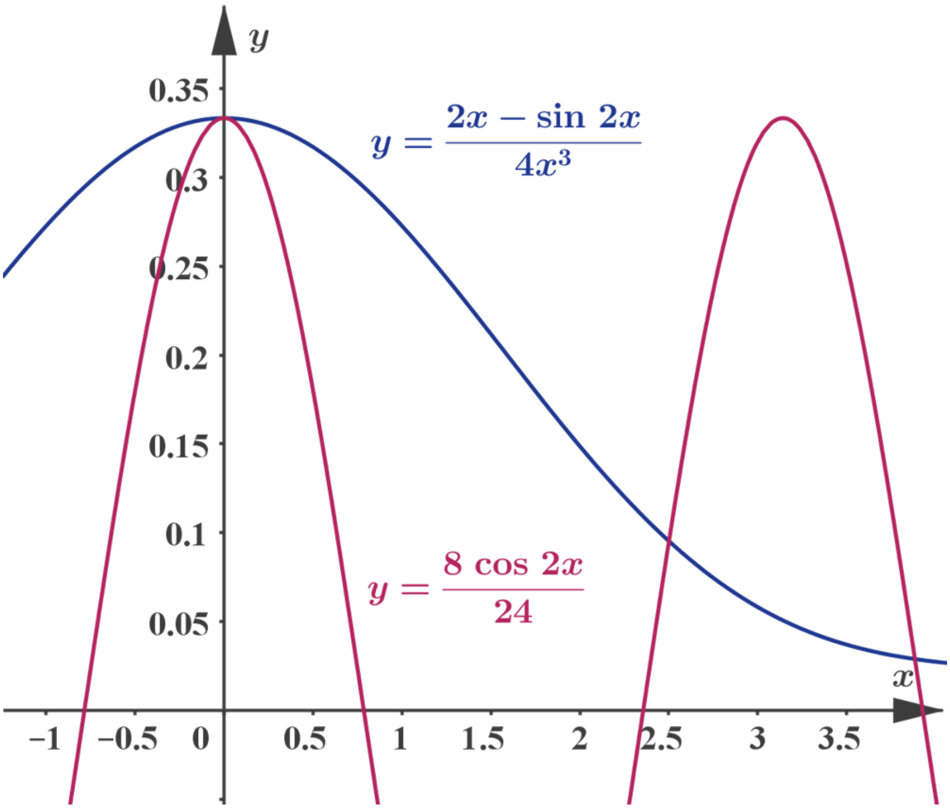

Use of l'Hôpital's rule on other indeterminate forms

You can sometimes handle the other indeterminate forms by using algebra to get $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ instead.

Example 6

★★★

Evaluate $\lim_{x \to 0} x \ln x$.

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Let $h(x) = x \ln x$

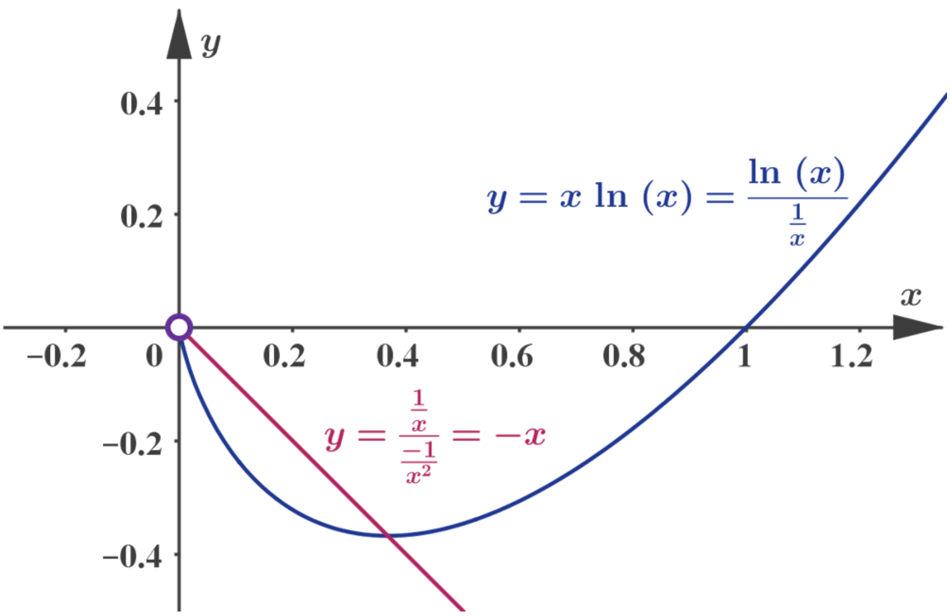
Then $h(0) = 0 \ln 0 = 0 \cdot \infty$, which is an indeterminate form, but not one covered by l'Hôpital's rule. However, you may rearrange:

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \frac{\ln 0}{\frac{1}{0}} = \frac{-\infty}{\infty}$$, which is also an indeterminate form, but one that allows you to use l'Hôpital's rule.

Using $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$,

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0}(-x) = 0$$

Thus, $\lim_{x\to 0} x \ln x = 0$. This result is also illustrated in the figure below.



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Example 7

★★★

Evaluate $\lim_{x \to 0} x^x$.

Let $h(x) = x^x$

Then $h(0) = 0^0$, which is an indeterminate form, but not one covered by L'Hôpital's rule. However, you may rearrange

$$\begin{aligned} y &= x^x \\ \ln y &= \ln x^x \\ &= x \ln x \\ &= \frac{\ln x}{1/x} \end{aligned}$$

This is also an indeterminate form, but one that allows you to use l'Hôpital's rule. You solved this in the last example, so you know that

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln x}{1/x} = \lim_{x \to 0}(-x) = 0$$

If $\lim_{x \to 0} \ln y = 0$, then $\lim_{x \to 0} y = \mathrm{e}^0 = 1$

Since e^x is continuous, and as you have set $y = x^x$, you conclude that $\lim_{x \to 0} x^x = 1$.

This limit is also illustrated in the figure below.

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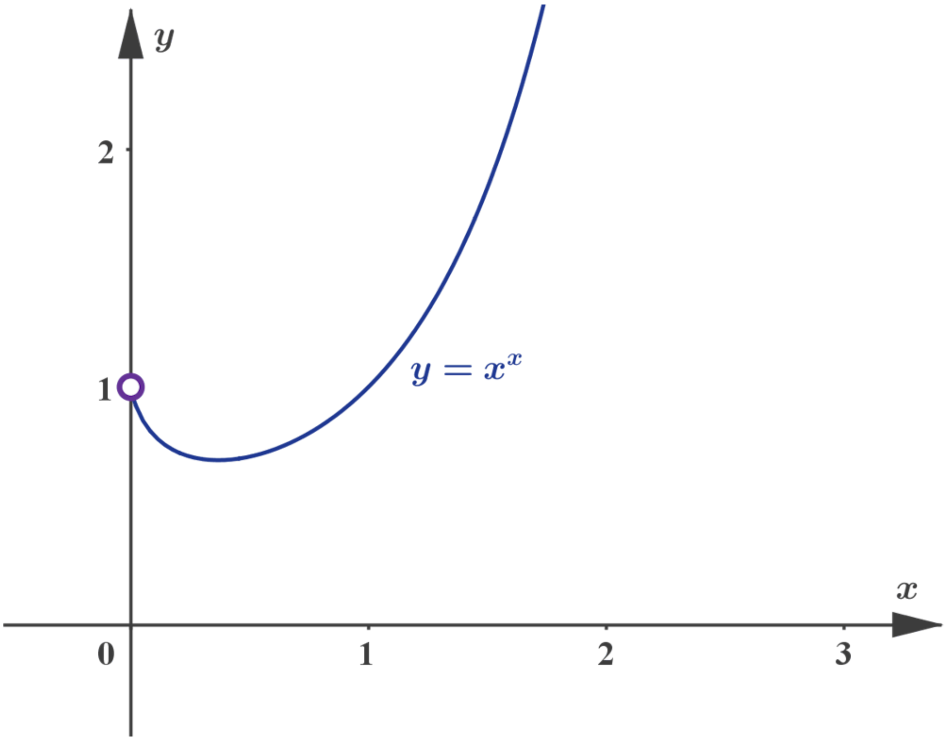

Exam tip

You may only apply l'Hôpital's rule to indeterminate forms $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$
.

However, you may use the result that if $f(x) > 0$ and $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} \frac{1}{f(x)} = \infty$ and vice versa to rearrange the expression into an appropriate quotient form.

Making connections

You will revisit indeterminate forms later when you study the Maclaurin series in subtopic 5.19 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27009/). The Maclaurin series will provide a second way of finding these limits.

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6 section questions

Question 1

Difficulty:

★★☆

Evaluate $\lim_{x\to\infty} \dfrac{x}{x+\ln x}$

1

Accepted answers

1

Explanation

Let $h(x) = \dfrac{x}{x+\ln x}$

Then $h(\infty) = \dfrac{\infty}{\infty + \ln\infty} = \dfrac{\infty}{\infty}$, which is an indeterminate form.

This allows you to use l'Hôpital's rule.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

$$\lim_{x\to\infty} \frac{x}{x+\ln x} = \lim_{x\to\infty} \frac{1}{1+1/x} = \frac{1}{1+0} = 1$$

Question 2

Difficulty:

★★☆

Evaluate $\lim_{x\to\infty} \dfrac{\ln x}{x}$

0

Accepted answers

0

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Explanation

Let $h(x) = \dfrac{\ln x}{x}$

Then $h(\infty) = \dfrac{\ln \infty}{\infty} = \dfrac{\infty}{\infty}$, which is an indeterminate form.

This allows you to use l'Hôpital's rule.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0$$

Question 3

Difficulty:

★★☆

Evaluate $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{x-\frac{\pi}{2}}{\cos x}$

-1

Accepted answers

-1

Explanation

Let $h(x) = \dfrac{x-\frac{\pi}{2}}{\cos x}$

Then $h\left(\frac{\pi}{2}\right) = \dfrac{\frac{\pi}{2}-\frac{\pi}{2}}{\cos\frac{\pi}{2}} = \dfrac{0}{0}$, which is an indeterminate form.

This allows you to use l'Hôpital's rule.

$$\lim_{x\to\frac{\pi}{2}} \frac{f(x)}{g(x)} = \lim_{x\to\frac{\pi}{2}} \frac{f'(x)}{g'(x)}$$

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$$\lim_{x \to \frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{\cos x} = \lim_{x \to \frac{\pi}{2}} \frac{1}{-\sin x} = \frac{1}{-1} = -1$$

Question 4

Difficulty:

★★☆

Evaluate $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos x}{\sin 2x}$.

Give an exact answer either as a decimal or as a fraction in fully simplified form.

0.5 ✓

Accepted answers

0.5, 0,5, 1/2, .5

Explanation

Method 1

Let $h(x) = \dfrac{\cos x}{\sin 2x}$

Then $h\left(\dfrac{\pi}{2}\right) = \dfrac{\cos \frac{\pi}{2}}{\sin 2\left(\frac{\pi}{2}\right)} = \dfrac{0}{0}$, which is an indeterminate form.

This allows you to use L'Hôpital's rule.

$$\lim_{x \to \frac{\pi}{2}} \frac{f(x)}{g(x)} = \lim_{x \to \frac{\pi}{2}} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\sin 2x} = \lim_{x \to \frac{\pi}{2}} \frac{-\sin x}{2\cos 2x} = \frac{-1}{-2} = \frac{1}{2}$$

Method 2

Using the double angle formula gives $\dfrac{\cos x}{\sin 2x} = \dfrac{\cos x}{2\sin x \cos x} = \dfrac{1}{2\sin x}$, when x is not $\dfrac{\pi}{2}$. Hence,

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$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\sin 2x} = \lim_{x \to \frac{\pi}{2}} \frac{1}{2 \sin x} = \frac{1}{2 \sin \frac{\pi}{2}} = \frac{1}{2}$$

Question 5

Difficulty:

★★☆

Evaluate $\lim\limits_{x \to 1} \dfrac{\ln x}{x^2 - x}$

1

Accepted answers

1

Explanation

Let $h(x) = \dfrac{\ln x}{x^2 - x}$

Then $h\left(0\right) = \dfrac{\ln 1}{1^2 - 1} = \dfrac{0}{0}$, which is an indeterminate form.

This allows you to use l'Hôpital's rule.

$$\lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to 1} \frac{\ln x}{x^2 - x} = \lim_{x \to 1} \frac{1/x}{2x - 1} = \frac{1}{1} = 1$$

Question 6

Difficulty:

★★★

Evaluate $\lim\limits_{x \to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x} \right)$

0

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Accepted answers

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0

Explanation

Let $h(x) = \frac{1}{\sin x} - \frac{1}{x}$

Then $h(0) = \frac{1}{0} - \frac{1}{0} = \infty - \infty$, which is an indeterminate form, but not one covered by l'Hôpital's rule. However, you may rearrange:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} = \frac{0-0}{0(0)} = \frac{0}{0}$$

This is also an indeterminate form, but one that allows you to use l'Hôpital's rule.

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} = \frac{1-1}{0+0(1)} = \frac{0}{0}$$

This is still indeterminate. Apply l'Hôpital's rule again:

$$\begin{aligned} \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} &= \lim_{x \to 0} \frac{\sin x}{\cos x + \cos x - x \sin x} \\ &= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2-0} = 0 \end{aligned}$$

Thus, $\lim_{x \to 0} \frac{1}{\sin x} - \frac{1}{x} = 0$

5. Calculus / 5.13 Limits of indeterminate forms

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-26499/print/) Assign

What you should know

By the end of this subtopic you should be able to:

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- use l'Hôpital's rule to find limits of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$

if $\lim_{x\to a} f(a) = \lim_{x\to a} g(a) = 0$ or $\lim_{x\to a} f(a) = \lim_{x\to a} g(a) = \pm\infty$, and if $g'(x) \neq 0$,

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, provided the latter limit exists

- rearrange expressions into the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

5. Calculus / 5.13 Limits of indeterminate forms

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-26500/print/) Assign

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Interactive 1. Finding the Limit of an Indeterminate Equation.

Credit: GeoGebra (https://www.geogebra.org/m/Ay848ERd) Chris Mizell

More information for interactive 1

This interactive visualization allows users to explore the behavior of two functions, their ratios, and their derivatives. This provides valuable insight into the application of L'Hôpital's Rule, limits, and the concept of removable discontinuities (holes) in rational functions.

A graph of two functions $f(x) = x^2 - 6x + 8$ and $g(x) = x^3 - 6x^2 + 8x$ is displayed on an XY-axis, where the x-axis ranges from 0 to 5 and the y-axis ranges from -8 to 8. The function $f(x)$ is represented by a blue parabolic curve intersecting the Y-axis at 8 and the X-axis at 2 and 4. The function $g(x)$ is represented by a pink cubic function curve passing through the origin and intersecting the X-axis at 2 and 4. A horizontal slider on the top of the graph allows the users to choose the value of *a* from 0 *to* 8. At the left bottom of the graph a play button can move the value of a from 8 to 0 automatically and let users observe the changes in real time.

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For example in the given interactive when $f(x) = x^2 - 6x + 8$ and $g(x) = x^3 - 6x^2 + 8x$ and $a = 3.24$, the tangent of $f(x)$ is given in blue line with $\text{gradient} = 0.48$ and tangent of $g(x)$ is given in pink line with $\text{gradient} = 0.6128$. When the value of the functions at $a = 3.24$ are:

$f(3.24) = -0.9424$

$g(3.24) = -3.0534$

The ratio of the functions at $a = 3.24$ is displayed as:

$$\frac{f(3.24)}{g(3.24)} = -\frac{0.9424}{-3.0534} = 0.3086$$

The ratio of their first derivative is:

$$\frac{f(3.24)}{g(3.24)} = -\frac{0.48}{0.6128} = 0.3086$$

Also, if the users select the value of $a = 2$, then

f2g2=0

In this case, the interactive applies L'Hôpital's Rule, evaluating the ratio of the derivatives instead:

$$\frac{f(2)}{g(2)} = \frac{-2}{-4} = 0.5.$$

The gradient of the function $f(x)$ is -2 while the gradient of $g(x)$ is -4.

The interactive will help users in studying the relation between different functions and their derivatives at a specific given point.

Finding the limit of an equation of indeterminate form can seem like magic. One of the examples occurs when a rational function of the form evaluated at a given x-value reduces to. Although you studied this case in topic 2.8 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27718/), it may be useful at this point to further study the graphs of each component and see how l'Hôpital's rule applies.

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Look at the applet above and study the function. As you drag the a-value left and right, what happens to the values of the numerator and denominator? What about the first derivatives of the numerator and denominator? As you approach $x = 2$, what happens to the function? Is it defined? What about the value provided by l'Hôpital's rule? Does it provide a limit? Does this fit with what you learned about holes and asymptotes in topic 2.8 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27718/) ?

Now try some rational functions for yourself, varying the equations for $f(x)$ and $g(x)$. What happens as you approach a vertical asymptote? What happens when you approach a hole? Do the derivatives of the numerator and denominator, and the resulting rational equation from l'Hôpital's rule, adequately predict the outcome?

Now for a real challenge. Can you open up a blank Geogebra window at https://www.geogebra.org/graphing (https://www.geogebra.org/graphing)and investigate some of the other indeterminate forms?

Rate subtopic 5.13 Limits of indeterminate forms

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