Index

The big picture

In subtopic 5.1 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26130/) you learned about the derivative of a function. This subtopic moves a step forward; you will be working with the derivative of the derivative. Remember that the derivative describes a rate of change; it is sometimes useful to know how the change is changing. This is why you learn about the second derivative. A typical application is the description of the movement of an object. Speed describes the rate of change of position. Acceleration describes the rate of change of speed, so is the rate of change of the rate of change of the position.

Another application is the description of the shape of curves. You can investigate this using the applet below.

Activity

The applet below shows the graph of a function and a point on the graph with both the tangent and the normal drawn at this point. The applet also tells the value of the first and second derivative at the point. In addition to this information, the applet also shows either a parabola or a circle as an approximation to the curve around the fixed point.

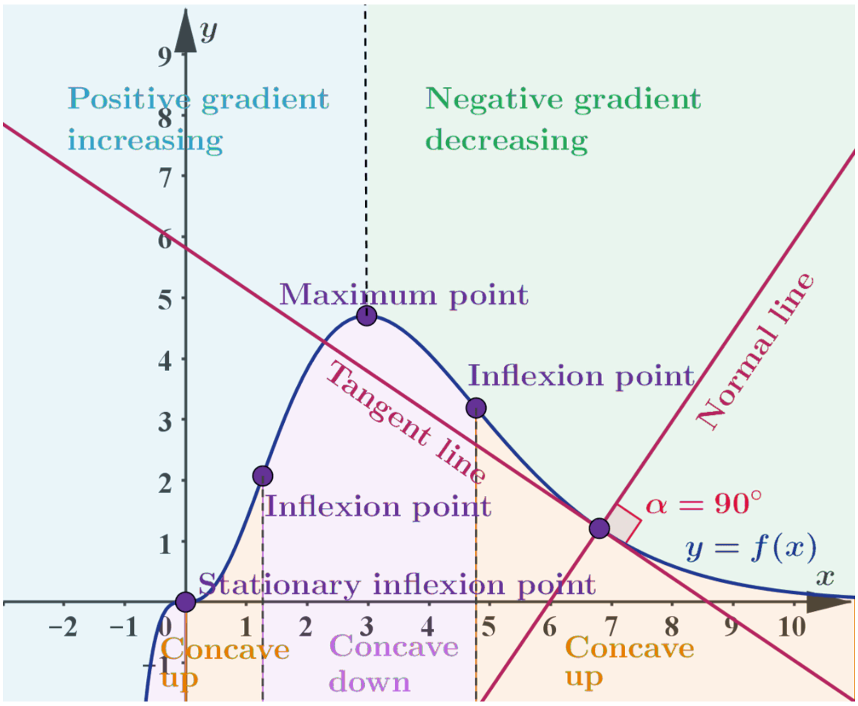
- First move the points on the left and right of the fixed point to create a parabola through the three points. Note that this is similar to how you created the tangent as a limit of secants in section 5.1.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/gradient-at-a-point-id-26133/). What do you notice?

- Next, move the points to create a circle through the three points. What do you notice?

Take a look at the video below. It shows an interesting application of investigating the curvature of curves drawn on two-dimensional surfaces.

The figure below provides a quick preview of a few of the new concepts covered in this chapter.

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov



More information

The graph illustrates a two-dimensional Cartesian plane with an X-axis and Y-axis. The X-axis ranges from -2 to 10, while the Y-axis ranges from 0 to 10. Several key features are labeled on the graph:

- The curve shown passes through points labeled as "Maximum point," "Inflexion point," "Stationary inflexion point," and another "Inflexion point."
- The areas under and around the curve are shaded with labels "Concave up" and "Concave down."
- There are lines labeled as "Tangent line" and "Normal line" intersecting the curve.
- The angle marked at one intersection is labeled as α = 90°.
- Text labels indicate gradient changes such as "Positive gradient increasing" and "Negative gradient decreasing."
- The mathematical expression y = f(x) is shown near the curve, emphasizing that the curve represents a function.

The graph visually indicates key mathematical properties and inflection points of a function, illustrating the changes in curvature direction and the nature of the gradient at various points.

[Generated by AI]

Student view

Concept

When graphing functions, think about the relationship between the sign of the first and second derivatives and the shape of the graph of the function.

Increasing/decreasing behaviour revisited

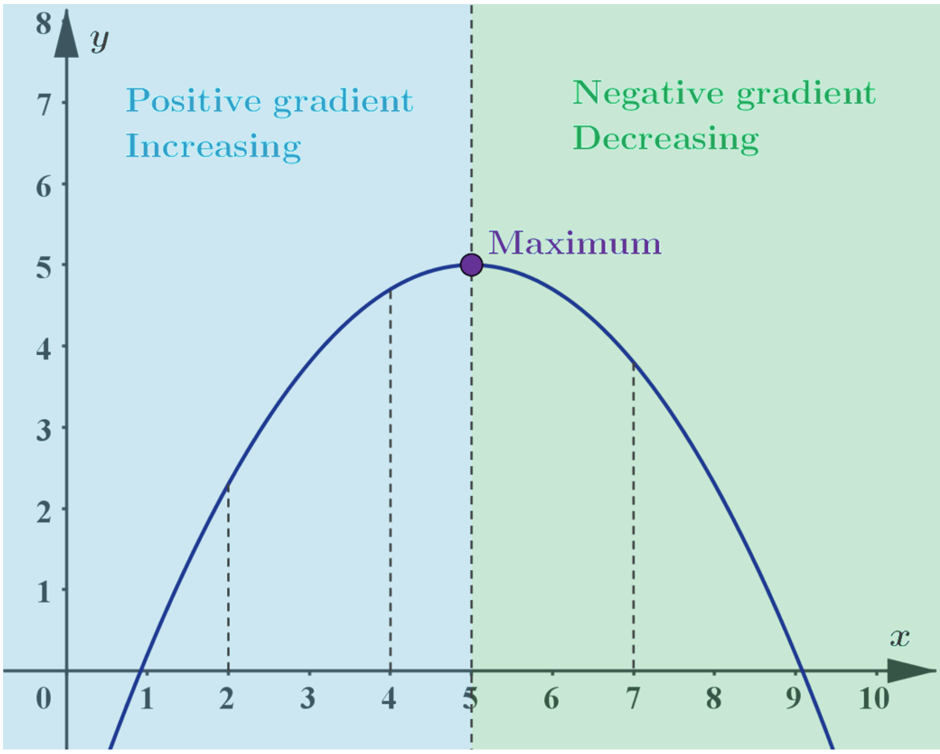
In subtopic 5.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26143/) you saw the relationship between the sign of the derivative and the shape of the graph. Here is a reminder of what you have observed there.

Important

- If $f'(x) > 0$ on an interval $]a, b[$, then f is increasing on $[a, b]$
- If $f'(x) < 0$ on an interval $]a, b[$ then f is decreasing on $[a, b]$
- If the function f is differentiable and $(a, f(a))$ is a turning point of the graph of f, then $f'(a) = 0$.

As a reminder, consider the following quadratic function:

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov



More information

The image depicts a graph of a quadratic function. The X-axis is labeled (x) and ranges from 0 to 10. The Y-axis is labeled (y) and ranges from 0 to 8. A parabola opens downwards, with its vertex at the point (5,5), labeled as 'Maximum'. To the left of the vertex (from x=0 to x=5), the parabola is increasing, which aligns with the label 'Positive gradient Increasing' on a blue background. To the right of the vertex (from x=5 to x=10), the parabola is decreasing, aligning with the label 'Negative gradient Decreasing' on a green background. The graph visually conveys the concept of increasing and decreasing intervals for quadratic functions.

[Generated by AI]

To define a function as increasing or decreasing, an interval must be identified. In the graph above, this interval is increasing along $[2, 4]$ as the slope between any two points within that interval is positive. The function decreasing in the interval $[7, 9]$. More interestingly, the function is also increasing in $[2, 5]$ and decreasing in $[5, 9]$. As $(5, 5)$is a turning point, it can be included in either the increasing and decreasing interval depending on what direction is being analysed.

In subtopic 5.1 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-26130/), you were given the derivative functions, but since you are now able to differentiate, in this subsection you will need to find them for yourself. You may be given a function and be asked

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

to sketch its graph. You should be able to draw these sketches without the use of graphic display calculators.

Example 1

★★☆

Consider the function defined by $f(x) = x + \frac{1}{x}$ for $0 < x$.

a) Identify the intervals where f is increasing and decreasing.

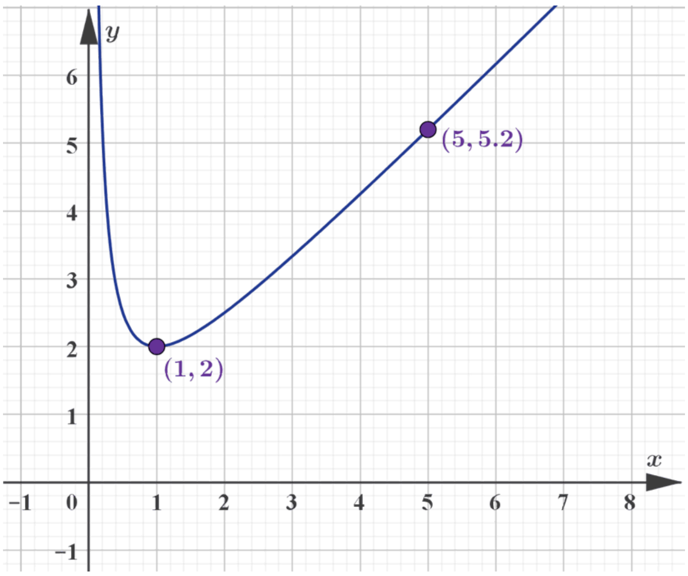
b) Identify the turning point on the graph of f .

c) Sketch the graph of f .

	Steps	Explanation
a)	$f(x) = x + x^{-1}$ $f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$ $= \frac{x^2-1}{x^2}$	To use the observations above, you need the derivative of f .
	Since the denominator is positive, $f'(x) > 0$ if $x^2 > 1$ so for $1 < x$ $f'(x) < 0$ if $x^2 < 1$ so for $0 < x < 1$.	You need to identify the intervals within the domain $0 < x$, where the derivative negative and positive.
	Hence, f is increasing on $[1, \infty[$ and decreasing on $]0, 1]$.	The turning point can be added to both intervals.
b)	The graph arrives at $(1, f(1)) = (1, 2)$ as a decreasing curve and leaves this point as an increasing curve, so the point $(1, 2)$ is a minimum point of the graph.	At the turning point the derivative is 0 and changes sign.

Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

	Steps	Explanation
c)	$f(0)$ is not defined and the $x = 0$ line is a vertical asymptote.	To sketch the graph, you ne to investigate the behaviour the endpoint of the domain.
		The sketch should match th observations above. On this sketch a second poi is drawn besides the turning point as a guide for an accurate sketch.

Example 2

★★☆

Consider the function defined by $f(x) = \mathrm{e}^x \sin x$ for $-1 \leqslant x \leqslant 3$.

a) Identify the values of x where $f'(x) = 0$.

b) Identify the intervals where f is increasing and decreasing.

c) Identify the turning points on the graph of f .

d) Find the axes intercepts of the graph of f .

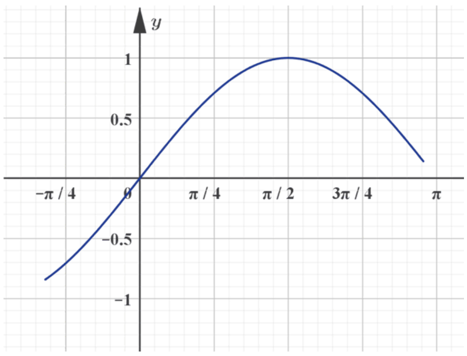
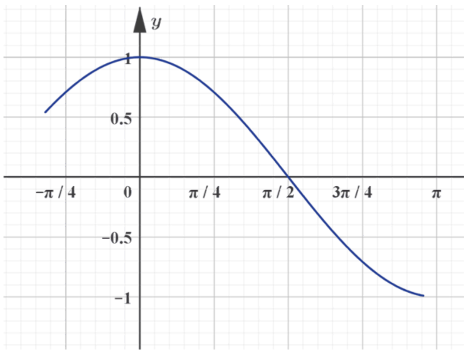
Student
view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

e) Given that $f(-1) \approx -0.3$ and $f(3) \approx 2.8$ sketch the graph of f .

	Steps	Explanation
a)	$f'(x) = e^x \cos x + e^x \sin x$ $= e^x(\cos x + \sin x)$	You can use the product rule to finc the derivative.
	$f'(x) = 0$ $e^x(\cos x + \sin x) = 0$ $\cos x + \sin x = 0$ $\cos x = -\sin x$	$e^x \neq 0$
	$x = -\frac{\pi}{4}$ $x = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$	This equation has two solutions between -1 and 3.

Student view

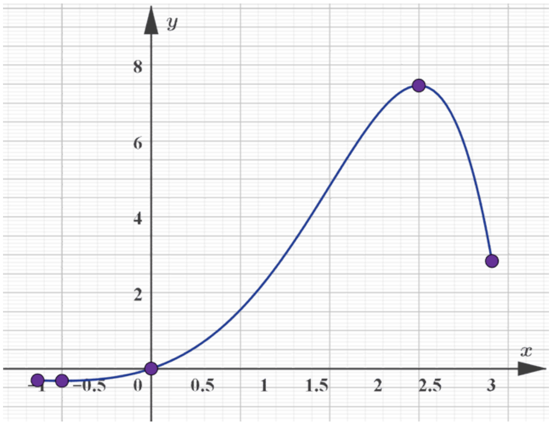
	Steps	Explanation
b)	For $-1 \leqslant x < -\frac{\pi}{4}$, $\sin x < -\frac{1}{\sqrt{2}}$ and $\cos x < \frac{1}{\sqrt{2}}$ so $\cos x + \sin x < 0$ For $-\frac{\pi}{4} < x < \frac{\pi}{4}$, $\sin x > -\frac{1}{\sqrt{2}}$ and $\cos x > \frac{1}{\sqrt{2}}$, so $\cos x + \sin x > 0$ For $\frac{\pi}{4} < x < \frac{3\pi}{4}$, $\sin x > \frac{1}{\sqrt{2}}$ and $\cos x > \frac{1}{\sqrt{2}}$ so $\cos x + \sin x > 0$ For $\frac{3\pi}{4} < x \leqslant 3$, $\sin x < \frac{1}{\sqrt{2}}$ and $\cos x < -\frac{1}{\sqrt{2}}$ so $\cos x + \sin x < 0$	You need to identify the intervals within the domain $[-1, 3]$ where th derivative is negative and where it i positive. Since $e^x > 0$ for any x , only the sig of $\cos x + \sin x$ needs to be checked. A sketch of the sine and cosine cur on the given domain can be helpful  

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

	Steps	Explanation
	Hence, f is increasing on $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ and decreasing on $\left[-1, -\frac{\pi}{4}\right]$ and $\left[\frac{3\pi}{4}, 3\right]$.	
c)	The point $\left(-\frac{\pi}{4}, f\left(-\frac{\pi}{4}\right)\right)$ is a local minimum point.	At $x = -\frac{\pi}{4}$ the derivative is changing from negative to positive, so at $\left(-\frac{\pi}{4}, f\left(-\frac{\pi}{4}\right)\right)$ the curve is changing from decreasing to increasing.
	The point $\left(\frac{3\pi}{4}, f\left(\frac{3\pi}{4}\right)\right)$ is a local maximum point.	At $x = \frac{3\pi}{4}$ the derivative is changi from positive to negative, so at $\left(\frac{3\pi}{4}, f\left(\frac{3\pi}{4}\right)\right)$ the curve is changing from increasing to decreasing.
d)	$f(0) = \mathrm{e}^0 \sin 0 = 1 \times 0 = 0$ so the y-intercept is $(0, 0)$.	The y-intercept is the point with x-coordinate 0 .
	$f(x) = 0$ $\mathrm{e}^x \sin x = 0$	The x-intercepts are the points with -coordinate 0.
	$\sin x = 0$ The only solution of this equation in $[-1, 3]$ is $x = 0$. Hence, the only point the graph intersects the coordinate axes is the origin of the coordinate system.	$\mathrm{e}^x \neq 0$

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

	Steps	Explanation
e)		The sketch should match the observations above.

3 section questions

5. Calculus / 5.10 Second derivative

Second derivative

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/second-derivative-id-28187/print/) Assign

In this section you will see some examples of how the derivative of a function is used. You will also go one step further and explore examples where the derivative of the derivative is also needed.

Example 1

★★★

Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Calculate the velocity of a skydiver

Credit: vuk8691 Getty Images

According to a simplified model, the velocity, $v(t)$, (measured in metres per second) of a skydiver t seconds after jumping out of a plane satisfies the equation $100\frac{\mathrm{d}v}{\mathrm{d}t} = 1000 - 40v^2$.

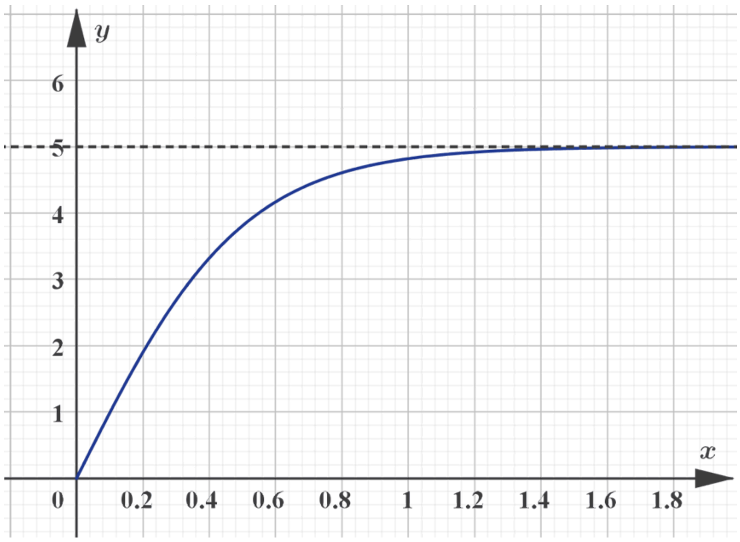
a) Show that $v(t) = \frac{5(\mathrm{e}^{4t} - 1)}{\mathrm{e}^{4t} + 1}$ satisfies this equation.

b) Draw the graph of v and comment on the shape in the context of the question.

	Steps	Explanation
a)	$\frac{dv}{dt} = \frac{5(4\mathrm{e}^{4t} - 0)(\mathrm{e}^{4t} + 1) - 5(\mathrm{e}^{4t} - 1)(4\mathrm{e}^{4t} + 0)}{(\mathrm{e}^{4t} + 1)^2}$ $= \frac{20\mathrm{e}^{8t} + 20\mathrm{e}^{4t} - 20\mathrm{e}^{8t} + 20\mathrm{e}^{4t}}{(\mathrm{e}^{4t} + 1)^2}$ $= \frac{40\mathrm{e}^{4t}}{(\mathrm{e}^{4t} + 1)^2}$	The quotient rule can b used to find the derivative.
	$100\frac{dv}{dt} = \frac{4000\mathrm{e}^{4t}}{(\mathrm{e}^{4t} + 1)^2}$	Left-hand side.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

	Steps	Explanation
	$$\begin{aligned}1000-40v^2&=1000-40\times\left(\frac{5(\mathrm{e}^{4t}-1)}{\mathrm{e}^{4t}+1}\right)^2\\&=\frac{1000(\mathrm{e}^{4t}-1)^2-40\times25(\mathrm{e}^{4t}-1)^2}{(\mathrm{e}^{4t}+1)^2}\\&=\frac{1000\left((\mathrm{e}^{8t}+2\mathrm{e}^{4t}+1)-(\mathrm{e}^{8t}-2\mathrm{e}^{4t}+1)\right)}{(\mathrm{e}^{4t}+1)^2}\\&=\frac{4000\mathrm{e}^{4t}}{(\mathrm{e}^{4t}+1)^2}\end{aligned}$$	Right-hand side.
	Since the left and right hand side simplify to the same expression, $v(t)=\frac{5(\mathrm{e}^{4t}-1)}{\mathrm{e}^{4t}+1}$ indeed satisfies the equation.	
b)		
	According to this model, the velocity of the skydiver approaches, but does not reach 5 metres per second.	The graph has a horizontal asymptote, $v=5$.

As you have already learned, a derivative represents the rate of change at a given point, or the gradient of a tangent touching the graph at a specified value. The derivative itself is a function. Does that mean that you can find the derivative of the derivative?

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Consider the function $f(x) = \sin 2x$.

You know from earlier sections that $f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(\sin 2x) = 2\cos 2x$.

What is the derivative of the derivative, or $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$?

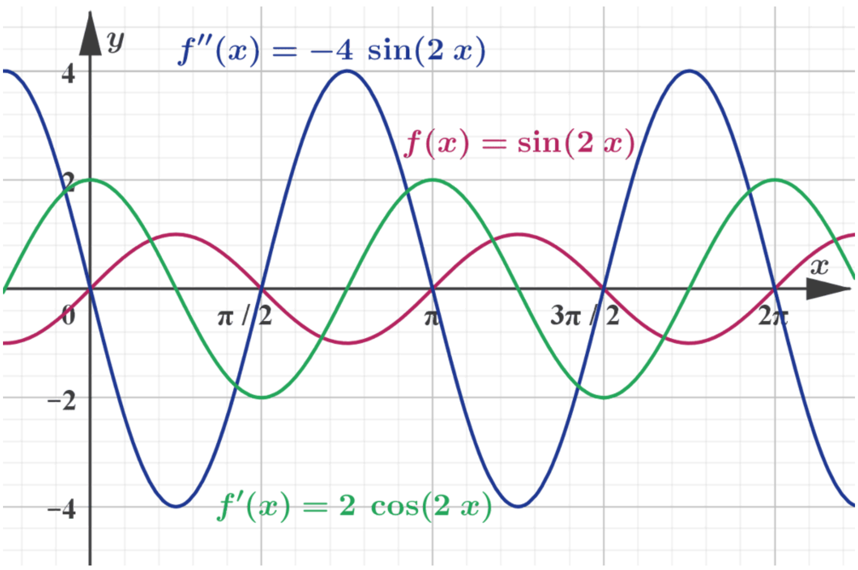
By treating the first derivative as a function in its own right, you can then take the derivative of that function to be $-4\sin 2x$.

The first derivative can be represented using the notations $\frac{\mathrm{d}y}{\mathrm{d}x}$, y', $f'(x)$, or in Newtonian mechanics, $\dot{x}$.

Similarly, the second derivative can be represented by $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, y'', $f''(x)$ or $\mathrm{d}\dot{x}$.

So, for the example above, $f(x) = \sin 2x$, $f'(x) = 2\cos 2x$, and $f''(x) = -4\sin 2x$.

The graph of all three functions looks like this:



More information

The image is a graph illustrating three trigonometric functions labeled as follows: (f(x) = \sin(2x)), (f'(x) = 2\cos(2x)), and (f''(x) = -4\sin(2x)). The X-axis represents the angle (x) and is marked with intervals such as (0), (\frac{\pi}{2}), (\pi), (\frac{3\pi}{2}), and (2\pi). The Y-axis represents the function values, with marked values from (-4) to (4) at regular

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

intervals.

The graph includes: - Three curves representing the functions. The primary function (f(x) = \sin(2x)), shown in magenta, illustrates two complete sine waves within the given range. - Its first derivative, (f'(x) = 2\cos(2x)), in green, displays cosine waves. - The second derivative, (f''(x) = -4\sin(2x)), in blue, shows sine waves scaled by a factor of (-4).

Each curve shows periodic behavior, reflecting the characteristics of sine and cosine functions, and can be used to analyze points of inflection, maxima, and minima. The relationships among these curves are especially visible at key points like (x = \pi) and (x = \frac{3\pi}{2}), indicating changes in increase or decrease in their slopes and inflection points.

[Generated by AI]

Note the relationships among the graphs. At $x = \pi$, what does the original function, $f(x)$, appear to be doing? Is it increasing or decreasing? What about the first and second derivatives? What about at $x = \frac{3\pi}{2}$? Can you determine a relationship that identifies a stationary point as a maximum or minimum based on the second derivative?

You can continue to take more derivatives, such as the third, fourth, and fifth derivative, in a similar manner, but this course will only require differentiation up to the second derivative.

In **Example 1**, the equation you needed to check involved the velocity function and its derivative. Equations like this are called differential equations. You were asked only to verify the given solution. In the higher level extension of this course you will learn methods of solving certain differential equations. For the moment, if you are interested in other solutions, you can type

solve 100v'=1000-40v^2

into the search line of WolframAlpha (http://www.wolframalpha.com)and interpret the result.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

The example above lets you investigate the velocity, $v(t)$ of the skydiver. It is a natural to ask if there is a way to investigate the position. Since the velocity is the change of position, $s(t)$ we can replace $v(t)$ with $s'(t)$ in the equation above. But what can you do with $v'(t)$?

✓ **Important**

The derivative of the derivative of $y = f(x)$ is called the second derivative.

Different forms of notation used for the second derivative are, for example, y'', f'' and $\frac{d^2y}{dx^2}$.

WolframAlpha (http://www.wolframalpha.com)understands these notations. If you are interested in the function describing the motion of the skydiver, type

solve {100s''=1000-40(s')^2, s'(0)=0, s(0)=-3000}

into the search line . The solution will tell you the position if the skydiver jumps from the plane at the altitude of 3000 metres with 0 metres per second initial speed.

Before taking a look at some further applications, here is an exercise to practise finding the second derivative.

Example 2

★☆☆

Find the first and second derivatives.

$f(x)$	$f'(x)$	$f''(x)$
$mx + c$		
$ax^2 + bx + c$		
$ax^3 + bx^2 + cx + d$		
$\sqrt{x}$		

Student view

$f(x)$	$f'(x)$	$f''(x)$
$\ln x$		
e^{-x}		
$\sin x$		

$f(x)$	$f'(x)$	$f''(x)$
$mx + c$	m	0
$ax^2 + bx + c$	$2ax + b$	$2a$
$ax^3 + bx^2 + cx + d$	$3ax^2 + 2bx + c$	$6ax + 2b$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$-\frac{1}{4x\sqrt{x}}$
$\ln x$	$\frac{1}{x}$	$-\frac{1}{x^2}$
e^{-x}	$-e^{-x}$	e^{-x}
$\sin x$	$\cos x$	$-\sin x$

Example 3

★★★

The radius of the best approximating circle of a curve at a given point is called the radius of curvature. For the graph of $y = f(x)$ this radius is given by

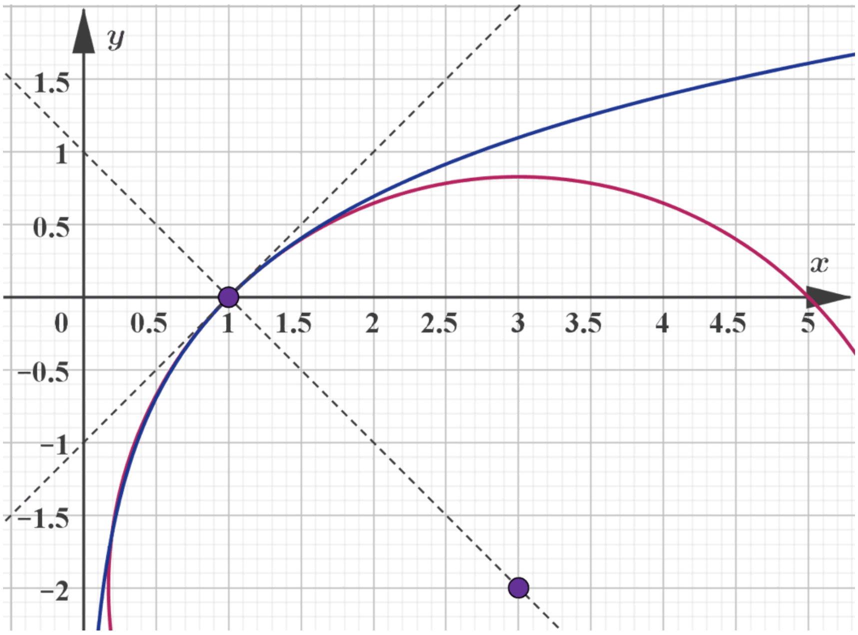
$$R = \left| \frac{(1 + y'^2)^{\frac{3}{2}}}{y''} \right|$$

Student view

The centre of this best approximating circle is on the normal to the curve.

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Find the radius and the centre of the best approximating circle to the graph of $y = \ln x$ at the x-intercept.



More information

The image is a graph with a grid layout illustrating the curve of the logarithmic function (y = \ln x) and an approximating circle. The X-axis is marked with numbers at intervals (e.g., 0.5, 1, 1.5, 2, etc.) extending from negative to positive values, while the Y-axis is similarly labeled at regular intervals (e.g., -2, -1, 0, 1, 2). The curve begins at the point of the intersection at the X-axis and rises logarithmically. A circle, approximating the curve, is centered along the X-axis to represent a point of interest. The circle intersects the logarithmic curve near the point where (y = \ln x) crosses the plane. There are two distinguishable points emphasized on the graph, possibly representing points of tangency or intersection. These features illustrate the concept of locally approximating the logarithmic curve with a circle at a specific point.

[Generated by AI]

Student view

Steps	Explanation
The x-intercept is $(1, 0)$.	$\ln 1 = 0$

Steps	Explanation
$y'(x) = \frac{1}{x}$ $y'(1) = \frac{1}{1} = 1$ $y''(x) = -\frac{1}{x^2}$ $y''(1) = -\frac{1}{1^2} = -1$	To find the radius, you need the first and second derivative at $x = 1$.
$R = \left\lvert \frac{\left(1+y'^2\right)^{\frac{3}{2}}}{y''} \right\rvert$ $= \left\lvert \frac{\left(1+1^2\right)^{\frac{3}{2}}}{-1} \right\rvert = 2\sqrt{2}$	
The equation of the normal is $y = 1 - x$.	The gradient of the tangent is $y'(1) = 1$, so the gradie of the normal is $\frac{-1}{1} = -1$.
The centre of the best approximating circle is $(3, -2)$.	Since the angle between the normal and the x-axis is $45°$ and since the radius is $2\sqrt{2}$ the centre is 2 units to the right and 2 units down from the x-intercept.

Activity

Before the next example take a look at the applet below. It illustrates the movement of an object when it is hung from a spring. The applet can illustrate the movement in an ideal situation when there is no friction (so the object would move up and down forever) or in a more realistic situation, when the amplitude of the movement is decreasing.

- Can you suggest a model that might describe the position of the object in the ideal situation?
- Can you modify this model so that it might be applicable in the realistic scenario?

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

Example 4

★★☆

According to the laws of physics, in the ideal situation (with no friction) the position, $y(t)$, of an object placed at the end of a vertical spring can be described by the differential equation $my'' + ky = 0$, where m is the mass of the object and is a constant determined by the properties of the spring.

For certain values of the constants, this equation becomes $8y'' + 16y = 0$.

A solution of this equation is $y(t) = \sin ct$, where $c > 0$.

Find the value of c.

Student
view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$y(t) = \sin ct$ $y'(t) = c\cos ct$ $y''(t) = -c^2 \sin ct$	To use the equation, you need the second derivative.
$8 \times (-c^2 \sin ct) + 16\sin ct = 0$ $(16 - 8c^2)\sin ct = 0$	Substituting in the equation.
$16 - 8c^2 = 0$ $c^2 = 2$	Since $c > 0$ and the equality needs to be true for any valu of t .
$c = \sqrt{2}$	The question asked for the positive solution.

In the example above you were asked only to verify the given solution. If you are interested in other solutions, you can type

solve 8y''+16y=0

or

solve m*y''+k*y=0

into the search line of WolframAlpha (http://www.wolframalpha.com)and interpret the result.

Example 5

★★★

In a more realistic model, the position, $y(t)$ of an object placed at the end of a vertical spring can be described by the differential equation $my'' + cy' + ky = 0$, where m is the mass of the object and c and k are constants determined by the properties of the spring and the surroundings.

Student view

For certain values of the parameters, this equation becomes $2y'' + 3y' + y = 0$.

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Show that $y(t) = a\mathrm{e}^{-t/2} + b\mathrm{e}^{-t}$ is a solution of this differential equation for any value of a and b.

Steps	Explanation
$y'(t) = -\frac{a}{2}\mathrm{e}^{\frac{-t}{2}} - b\mathrm{e}^{-t}$ $y''(t) = \frac{a}{4}\mathrm{e}^{\frac{-t}{2}} + b\mathrm{e}^{-t}$	To check the equality, you need the first and second derivatives.
$\begin{aligned} 2y'' + 3y' + y &= 2\left(\frac{a}{4}\mathrm{e}^{\frac{-t}{2}} + b\mathrm{e}^{-t}\right) \\ &\quad + 3\left(-\frac{a}{2}\mathrm{e}^{\frac{-t}{2}} - b\mathrm{e}^{-t}\right) \\ &\quad + \left(a\mathrm{e}^{\frac{-t}{2}} + b\mathrm{e}^{-t}\right) \\ &= \left(\frac{2a}{4} - \frac{3a}{2} + a\right)\mathrm{e}^{\frac{-t}{2}} \\ &\quad + (2b - 3b + b)\mathrm{e}^{-t} \\ &= 0 \times \mathrm{e}^{\frac{-t}{2}} + 0 \times \mathrm{e}^{-t} = 0 \end{aligned}$	Substitute the first and second derivatives into th equation and check that the right-hand side is zero.

If you are interested in the solution of the general equation, you can type

solve m*y''+c*y'+k*y=0

into the search line of WolframAlpha (http://www.wolframalpha.com)and interpret the result.

Making connections

In this section, you learned about the second derivative and some of its applications. You might ask, why stop here; why not take the derivative of the derivative? Higher-order derivatives are indeed useful, for example, for finding more and more accurate approximations to functions. You have already seen the tangent (a linear approximation) and in the The big picture (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27511/) section a quadratic approximation. To find these more and more accurate approximations, you will use higher-order derivatives, which you will learn about in subtopic 5.12 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-28203/).

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

3 section questions

5. Calculus / 5.10 Second derivative

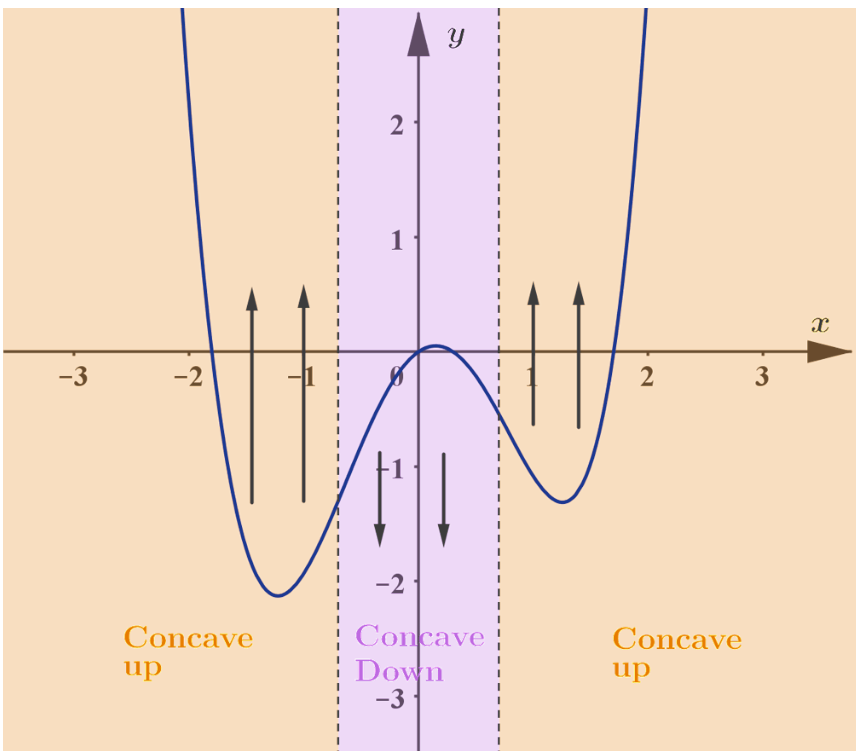
Concavity

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/concavity-id-28188/print/) Assign

In previous subtopics, you have seen how you can use the sign of the derivative to identify intervals where a graph is increasing and decreasing. This was also helpful in identifying turning points on the graph. In this section, you will see how to use the second derivative to get more information about the shape of a curve.

In a very simple sense, the concavity of a function can be thought of as how the function opens up. If it looks like a bowl, or part of a bowl, opening up, then it is concave up. If it looks like an upside down bowl, then it is concave down.

The diagram below is a representation of a fourth-order polynomial and the concavity of the function.



Student view

More information

The diagram illustrates a fourth-order polynomial graph. The X-axis ranges from -3 to 3 with integer intervals, while the Y-axis ranges from -3 to 3 with integer intervals as well. The polynomial curve shows two distinct concave up regions and a concave down region between them. The concave up areas occur approximately when x < -1 and x > 1. The concave down region is between -1 and 1. Vertical arrows indicate these sections, with 'Concave Up' labeled under the sections outside -1 to 1, and 'Concave Down' labeled between -1 and 1. The graph visually emphasizes these transitions of concavity with colorful background sections. Labels are placed at the relevant sections to denote the change in concavity.

[Generated by AI]

The following is a summary of the terminology related to the second derivative and the shape of the graph of a function.

Important

- The graph of a differentiable function is concave up on an interval $]a, b[$ if the graph is above all the tangents.
- The graph of a differentiable function is concave down on an interval $]a, b[$ if the graph is below all the tangents.
- A point on a graph where the concavity is changing is called a point of inflexion.

The following is a summary of how you can use the second derivative to investigate concavity.

Important

- If $f''(x) > 0$ for every $a < x < b$ then the graph of f is concave up on $]a, b[$.
- If $f''(x) < 0$ for every $a < x < b$ then the graph of f is concave down on $]a, b[$.

Student view

- If $f''(x) = 0$ and $f''(x)$ changes sign at $x = a$, then the point $(a, f(a))$ is a point of inflexion of the graph.

Example 1

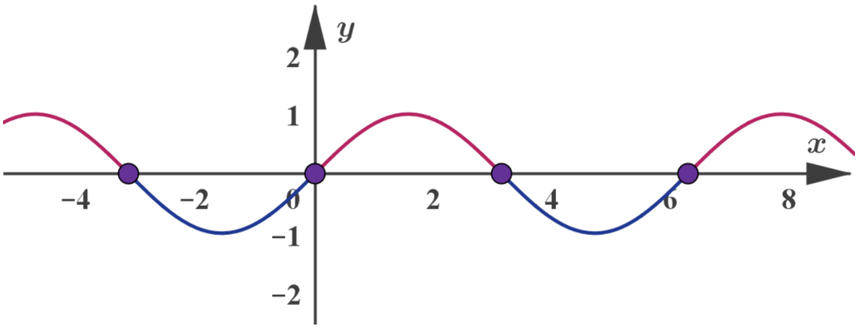
★☆☆

a) Find the intervals where the graph of $y = \sin x$ is concave up and the intervals where the graph is concave down.

b) Find the points of inflexion on the graph of $y = \sin x$.

	Steps	Explanation
a)	$y' = \cos x$ $y'' = -\sin x$	You can investigate concavity using the second derivative.
	$y''(x) > 0$ in $](2k-1)\pi, 2k\pi[$ for $k = 0, \pm 1, \pm 2, \ldots$	$\sin x$ is negative in the third and fourth quadrant of the unit circle.
	The graph of $y = \sin x$ is concave up in the intervals $](2k-1)\pi, 2k\pi[$ for $k = 0, \pm 1, \neq 2, \ldots$	
	$y''(x) < 0$ in $]2k\pi, (2k+1)\pi[$ for $k = 0, \pm 1, \neq 2, \ldots$	$\sin x$ is positive in the first and second quadrant of the unit circle.
	The graph of $y = \sin x$ is concave down in the intervals $]2k\pi, (2k+1)\pi[$for $k = 0, \pm 1, \neq 2, \ldots$	
b)	The points of inflexion are where $y''(x)$ is 0 and changes sign, so $(0, 0), (\neq \pi, 0), (\pm 2\pi, 0), \ldots$	$\sin x = 0$ for $x = 0, \pm\pi, \pm 2\pi, \ldots$

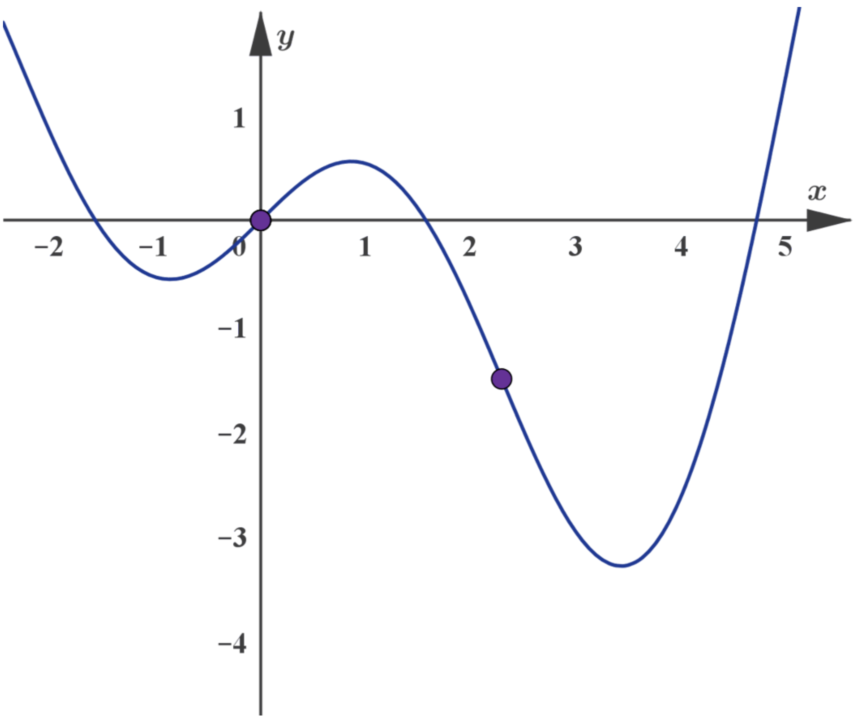
Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov



Example 2

★☆☆

Find the two points of inflexion on the graph of $y = x\cos x$ that are shown on the diagram below.



More information

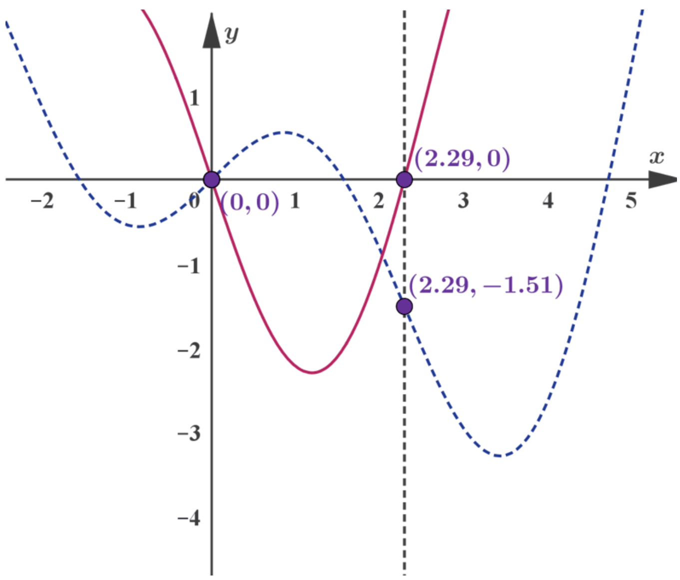
Student
view

The diagram shows the graph of the function (y = x\cos x) plotted on a Cartesian coordinate system. The X-axis is labeled with values from -2 to 5, and the Y-axis is labeled from -4 to 1. The graph features a wavy curve that represents the equation (y = x\cos x). Two points of inflection are highlighted on the curve: one is around x = 0 and

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

the other near x = 2.5. At x = 0, the curve has a positive slope, indicating a change from concave up to concave down. At x = 2.5, the curve shifts from concave down to concave up, showing an inflection point.

[Generated by AI]

Steps	Explanation
$y' = \cos x + x(-\sin x)$ $= \cos x - x\sin x$ $y'' = -\sin x - (\sin x + x\cos x)$ $= -2\sin x - x\cos x$	You can identify points of inflexion by investigating the second derivative.
$-2\sin x - x\cos x = 0$	At points of inflexion, the second derivative is 0.
The solutions within the domain on the diagram in the question are $x = 0$ and $x = 2.29$. The corresponding points of inflexion are $(0, 0)$ and $(2.29, -1.51)$.	Graphic display calculators have applications to solve equations like this. 

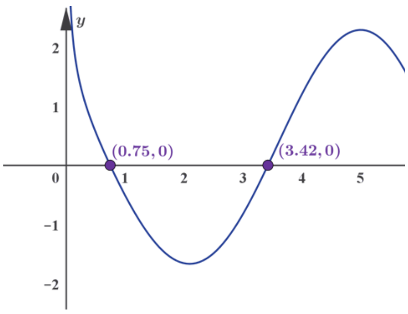
Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Example 3

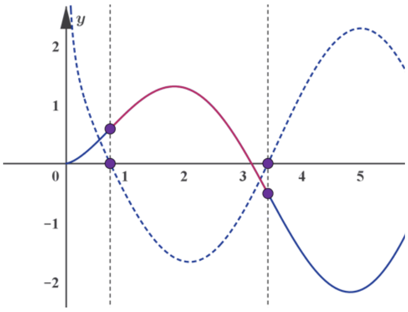
★★★

Find the interval where the graph of $y = \sqrt{x}\sin x$ is concave down in the interval $[0, 2\pi]$.

Steps	Explanation
$y' = \frac{1}{2\sqrt{x}}\sin x + \sqrt{x}\cos x$ $= \frac{\sin x + 2x\cos x}{2\sqrt{x}}$ $y'' = \frac{(3\cos x - 2x\sin x)2\sqrt{x} - (\sin x + 2x\cos x)\frac{1}{\sqrt{x}}}{4x}$ $= \frac{6x\cos x - 4x^2\sin x - \sin x - 2x\cos x}{4x\sqrt{x}}$ $= \frac{4x\cos x - \left(4x^2+1\right)\sin x}{4x\sqrt{x}}$	You can investigate concavity using the second derivative.
$\frac{4x\cos x - \left(4x^2+1\right)\sin x}{4x\sqrt{x}} = 0$	The second derivative is 0 whe concavity changes.
The solutions of this equation between 0 and 2π are $x = 0.746$ and $x = 3.42$.	Graphic display calculators hav applications to solve equations like this. 

Student view

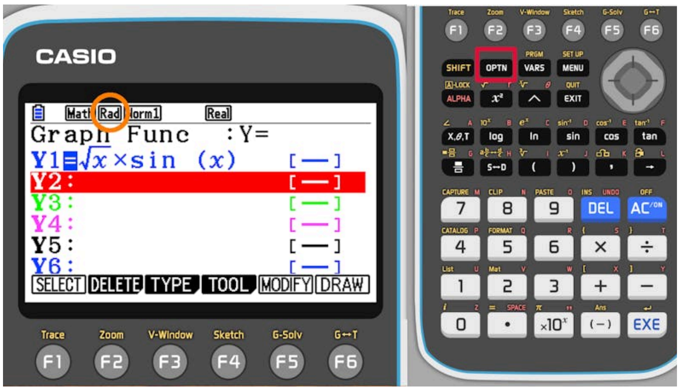
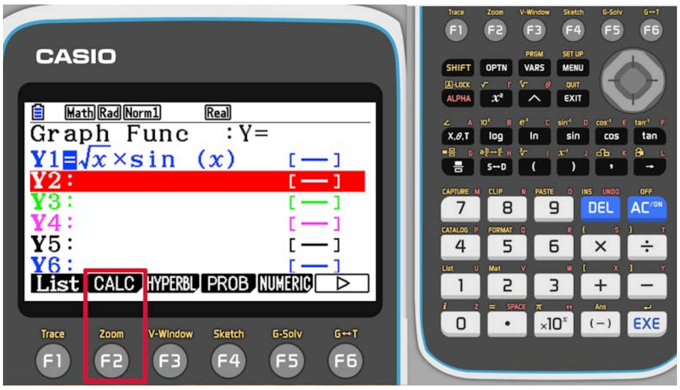
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
The graph of $y = \sqrt{x}\sin x$ is concave down on $]0.746, 3.42[$.	The graph is concave down on the interval where the second derivative is negative. 

Steps	Explanation
In these instructions you will see how to find the solution of $f''(x) = 0$ for $f(x) = \sqrt{x}\sin x$ on the interval $0 \leq x \leq 2\pi$. From the main menu, choose the graph option.	

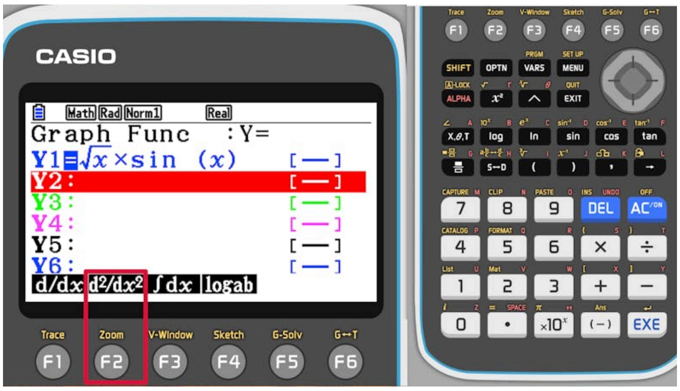
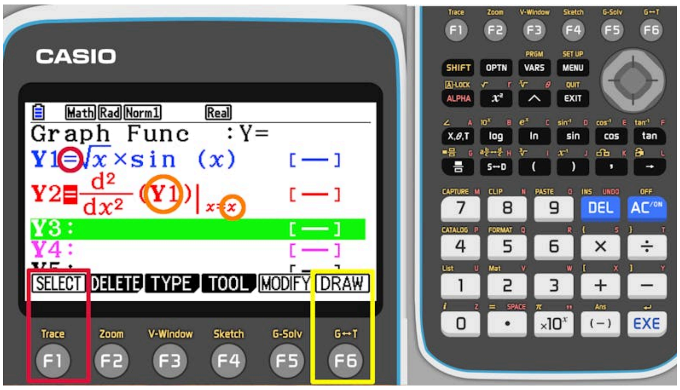
Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
Enter the definition of the function and make sure your calculator is in radian mode. The next step is telling the calculator that the second function is the second derivative of Y1. Press OPTN to find the template for the second derivative ...	
... press F2 to bring up the calculus related options ...	

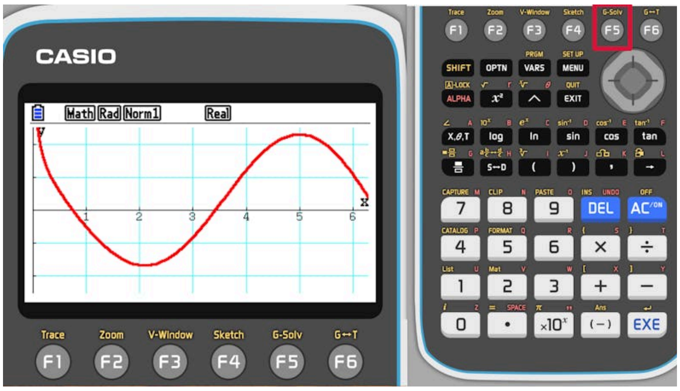
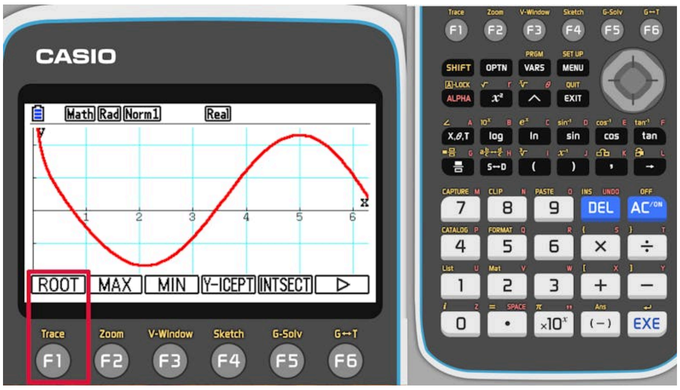
Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

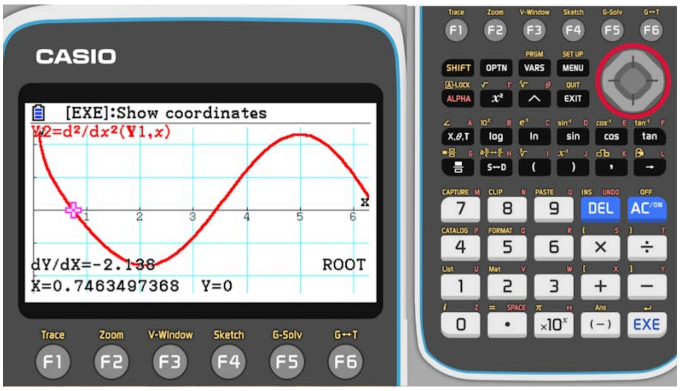
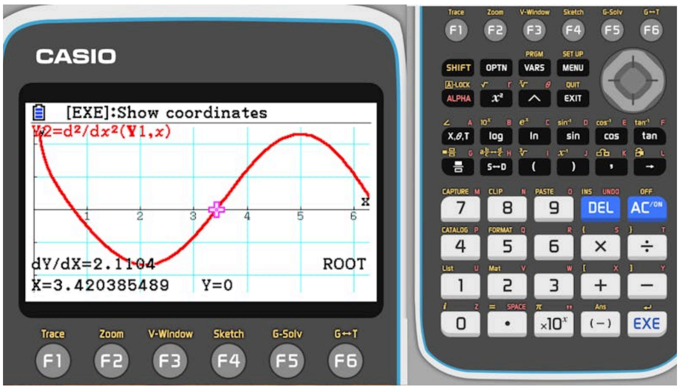
Steps	Explanation
... and press F2 again to insert the second derivative template.	
Use the function name instead of retyping the expression. Use the variable x to indicate, that you are defining a function, not evaluating the second derivative at a given point. Make sure you unselect the function itself (Y1) so that you do not see it on the graph view. Once done, press F6 to draw the graph.	

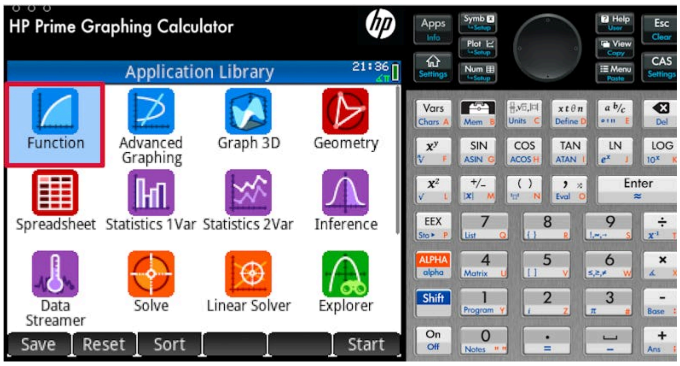
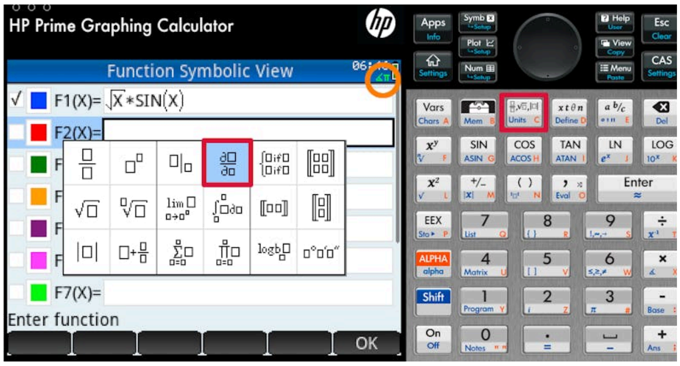
Student
view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

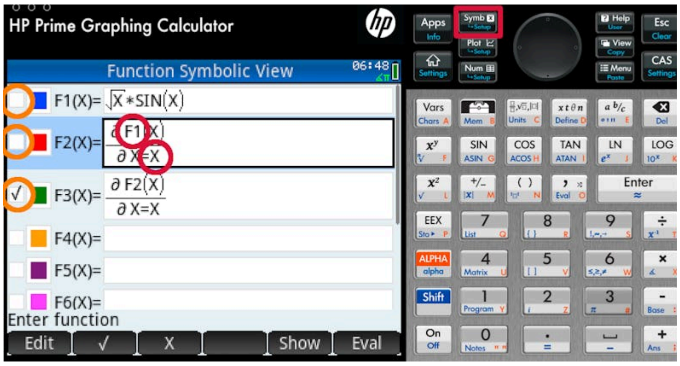
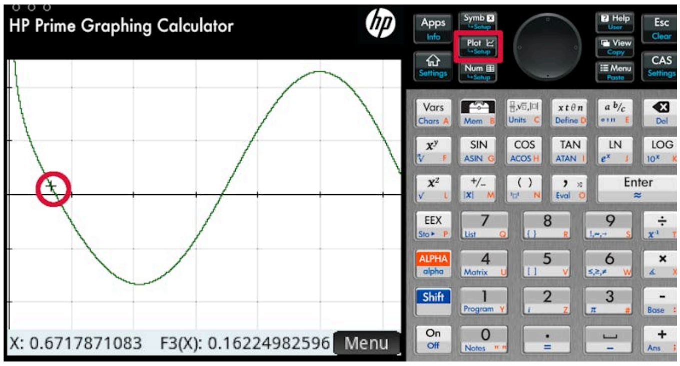
Steps	Explanation
You probably will need to adjust the window so that you only see the graph of this second derivative for $0 \leq x \leq 2\pi$. Once done, press F5 (G-Solve) to bring up options to analyze the graph.	
You are interested in the x-intercepts, so press F1 to find the roots.	

Student
view

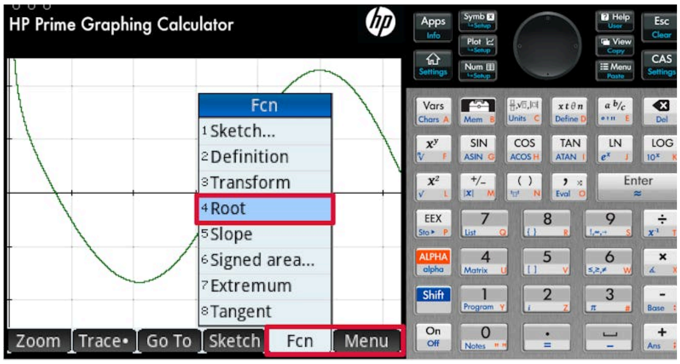
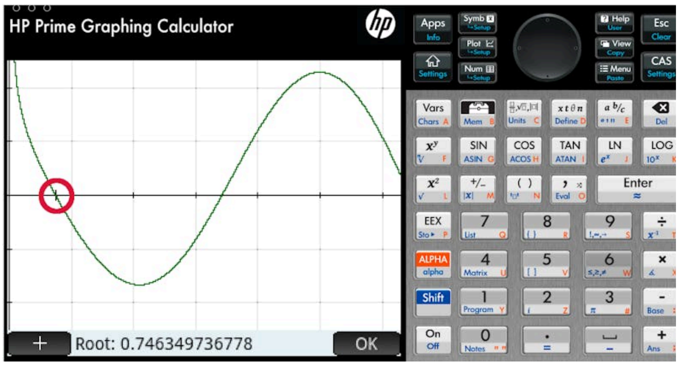
Steps	Explanation
The calculator moves the cursor to one of the x-intercepts, and displays the coordinates. You can move left and right to move between the roots visible on the screen.	
On this screenshot you can see the coordinates of the second x-intercept.	

Steps	Explanation
In these instructions you will see how to find the solution of $f''(x) = 0$ for $f(x) = \sqrt{x}\sin x$ on the interval $0 \leq x \leq 2\pi$. Choose the function application.	
In symbolic view, enter the definition of the function and make sure your calculator is in radian mode. The next step is telling the calculator that you would like to graph the second derivative of F1. Bring up the templates and choose the template for the derivative.	

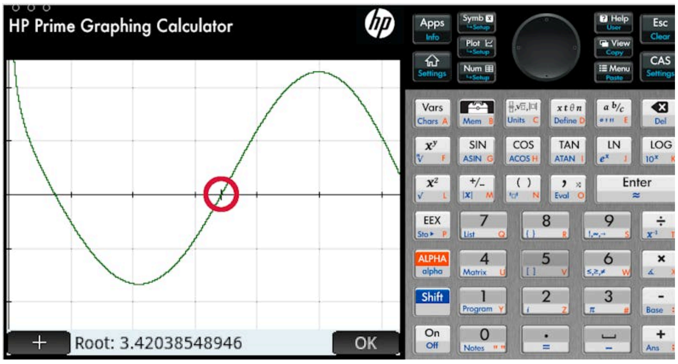
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

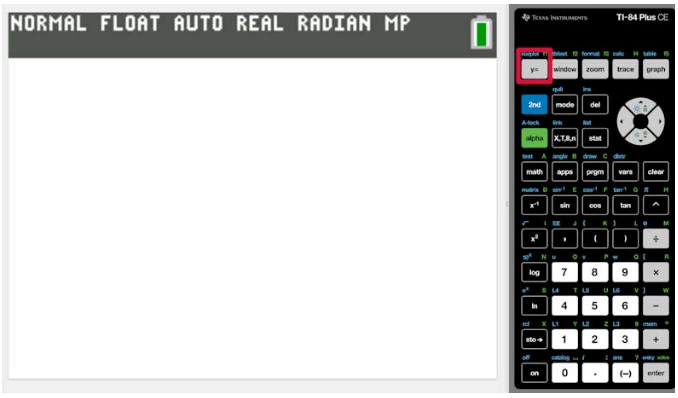
Steps	Explanation
Use the function name instead of retyping the expression. Use the syntax $x = x$ to indicate, that you are defining a function, not evaluating the derivative at a given point. You will need two steps, the derivative of F1 and the derivative of this derivative. Make sure you unselect the function and the derivative (F1 and F2) so that you do not see these in the plot view.	
Change now to plot view. You probably will need to adjust the window so that you only see the graph of this second derivative for $0 \leq x \leq 2\pi$. You are interested in the x-intercepts, so move the cursor close to one of these.	

Student view

Steps	Explanation
Tap on Menu and select to find the roots among the options to analyze a function.	
The calculator moves the cursor to the x-intercept and displays the first coordinate.	

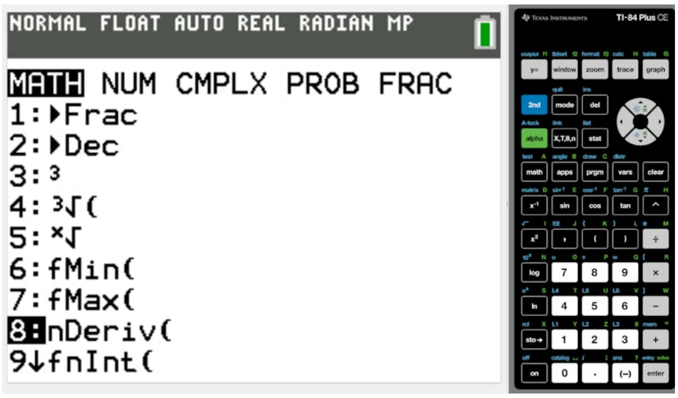
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
If you search for the roots with the cursor close to the other x-intercept, the calculator will find that one.	

Steps	Explanation
In these instructions you will see how to find the solution of $f''(x) = 0$ for $f(x) = \sqrt{x}\sin x$ on the interval $0 \leq x \leq 2\pi$. Enter the function definition screen.	

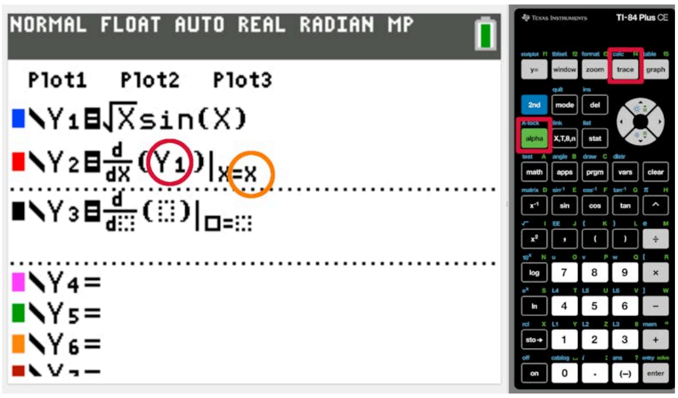
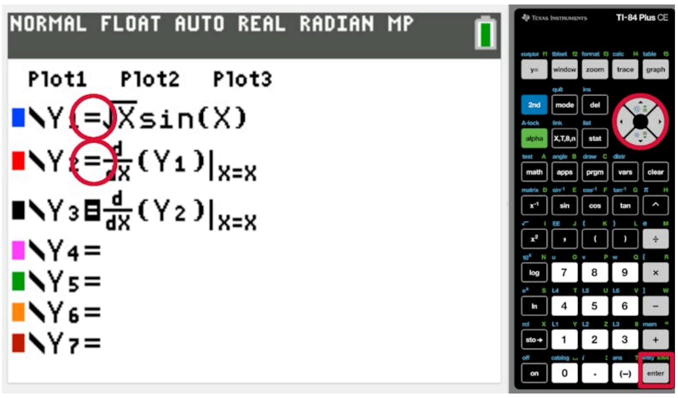
Student view

<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>
Enter the definition of the function and make sure your calculator is in radian mode.
The next step is telling the calculator that you would like to graph the second derivative of Y1. Press math ...
</td><td>


</td></tr>
<tr><td>
... and choose the option (nDeriv) to find the numerical derivative.
</td><td>


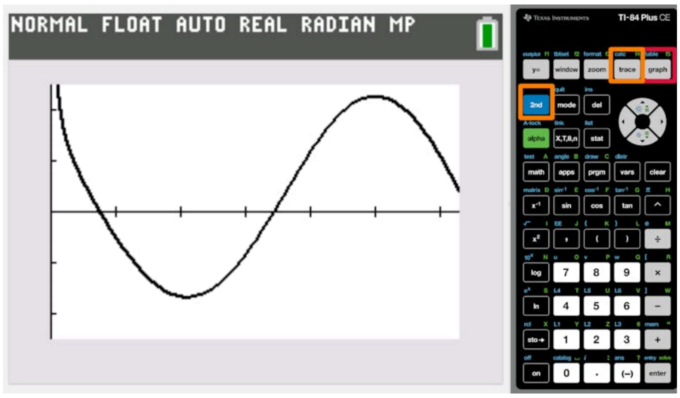
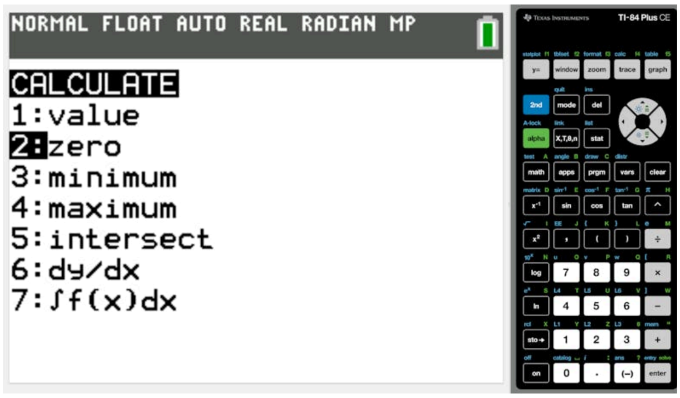
</td></tr>
</table>

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
Use the function name instead of retyping the expression (you can access it either by pressing the vars button or through alpha/f4). Use the syntax $x = x$ to indicate, that you are defining a function, not evaluating the derivative at a given point. You will need two steps, the derivative of Y1 and the derivative of this derivative.	
Make sure you unselect the function and the derivative (Y1 and Y2) so that you do not see these in the plot view. You can unselect these by moving over the equality sign and pressing enter.	

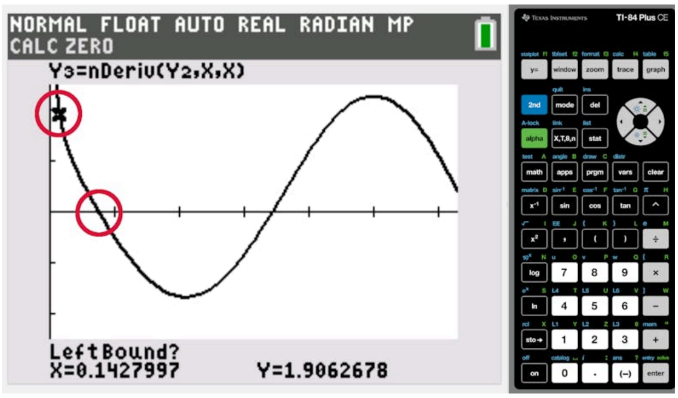
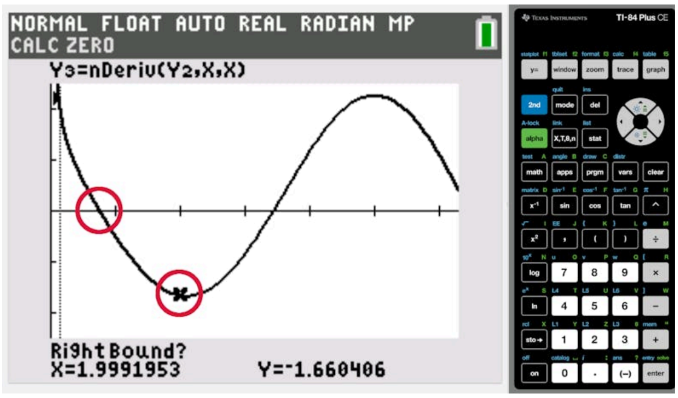
Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

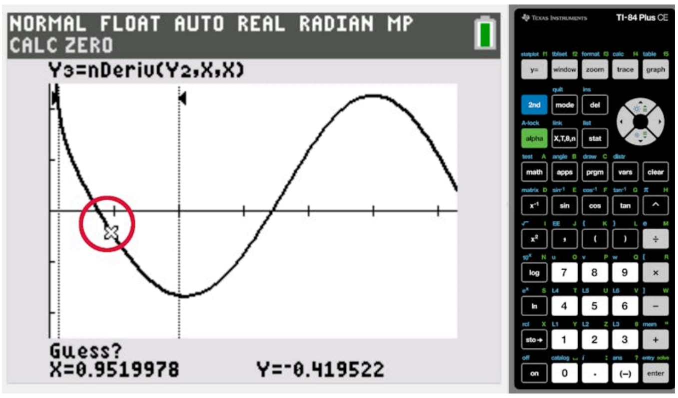
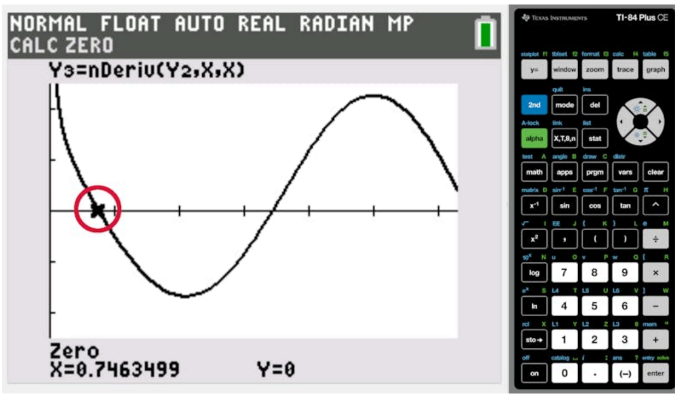
Steps	Explanation
Press graph to bring up the plot. You probably will need to adjust the window so that you only see the graph of the second derivative for $0 \leq x \leq 2\pi$. Once done, bring up options to analyze the graph (2nd calc).	
You are interested in the x-intercepts, so choose the option to find the zeros.	

Student
view

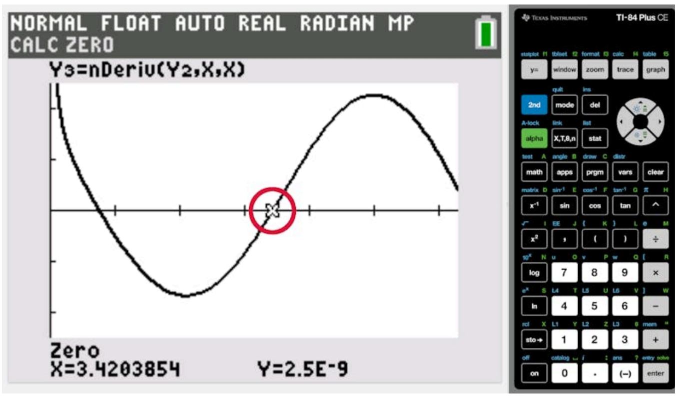
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

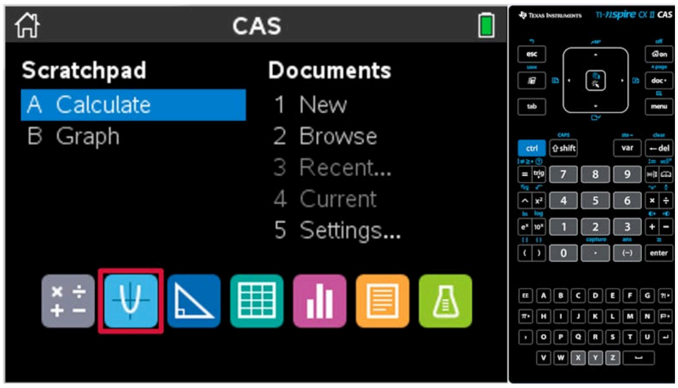
Steps	Explanation
The calculator needs more information, so it asks questions. First it asks for a left bound, so move the cursor to the left of the x-intercept you woudl like to find and press enter.	
The next step is to give an upper bound for the zero.	

Student view

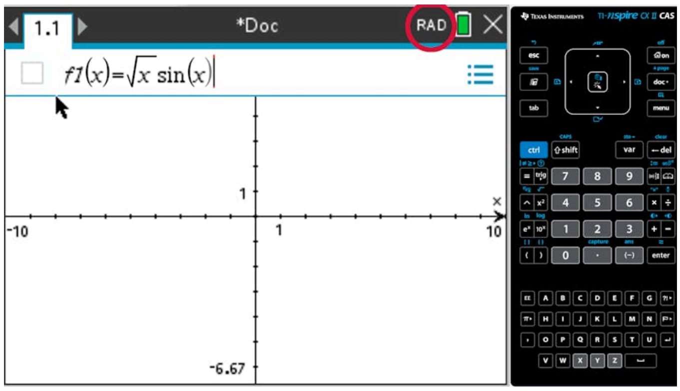
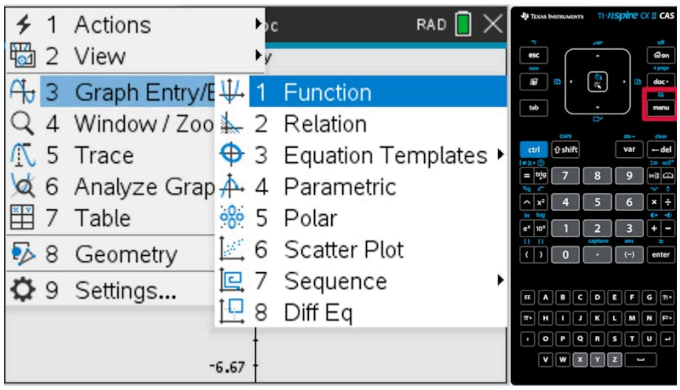
Steps	Explanation
Finally, move the cursor close to the x-intercept to tell the calculator your guess (it needs this as a starting point for the numerical algorithm) and press enter.	
The calculator moves the cursor the x-intercept, and displays the coordinates.	

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

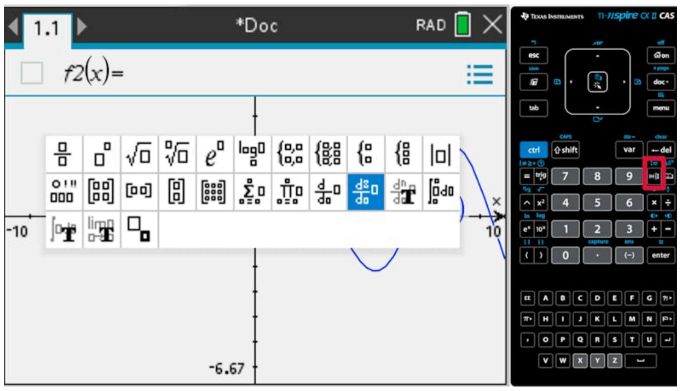
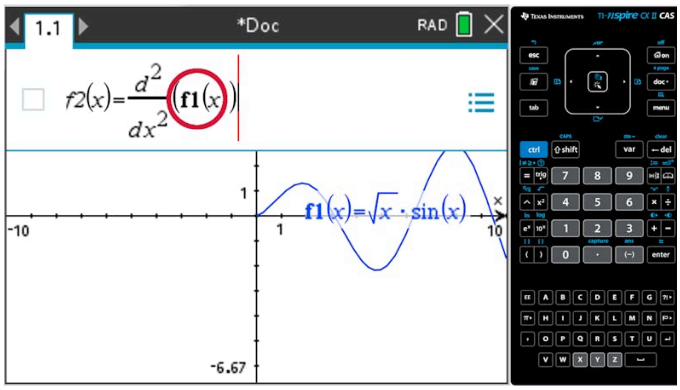
Steps	Explanation
Follow the same steps (but specify different left and right bounds) to find the other x-intercept.	

Steps	Explanation
In these instructions you will see how to find the solution of $f''(x) = 0$ for $f(x) = \sqrt{x}\sin x$ on the interval $0 \leq x \leq 2\pi$. From the home screen, open a graphing page.	

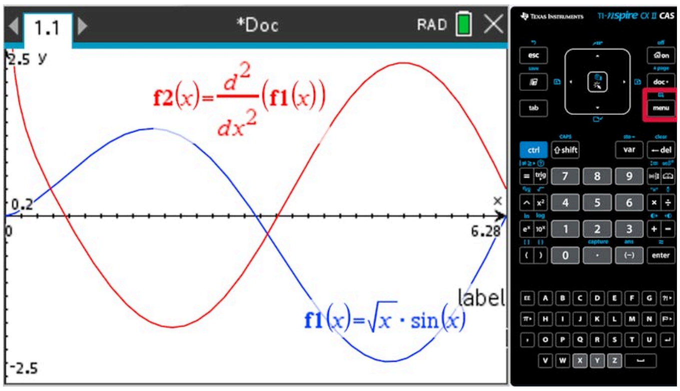
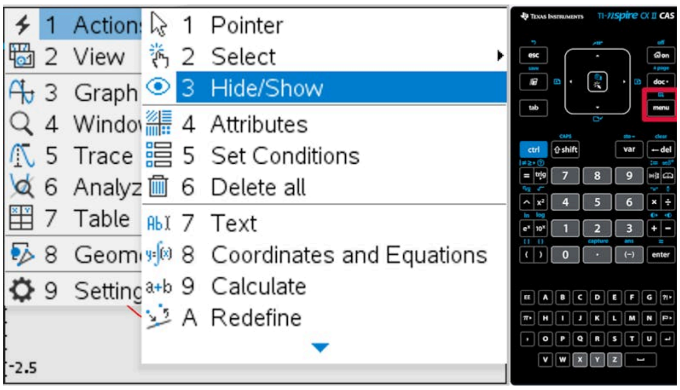
Student
view

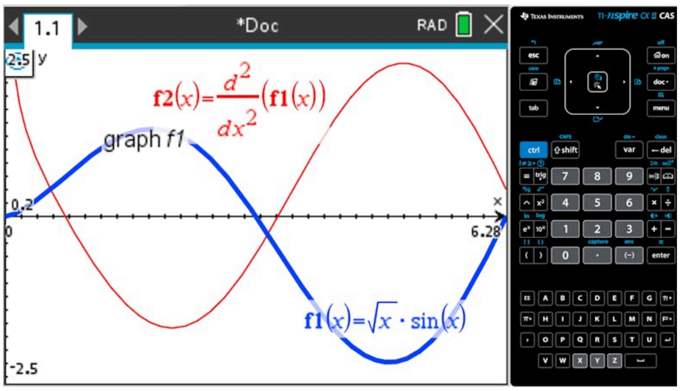
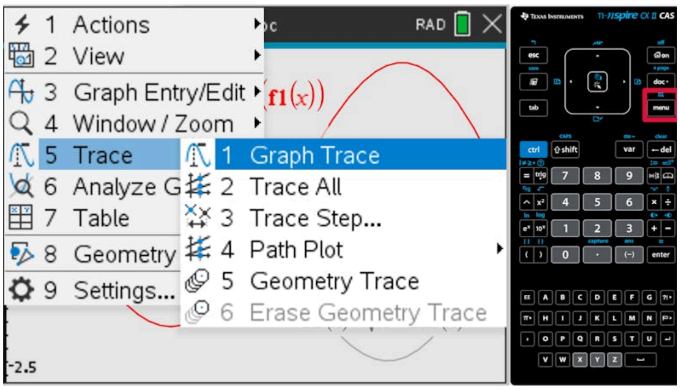
Steps	Explanation
Enter the definition of the function and make sure your calculator is in radian mode. In the newest operating system you can change between degree and radian mode by simply clicking on the word.	
The next step is telling the calculator that you would like to see the graph of the second derivative of the function you just defined. Add a new function to the document. One way of doing this is through the menu.	

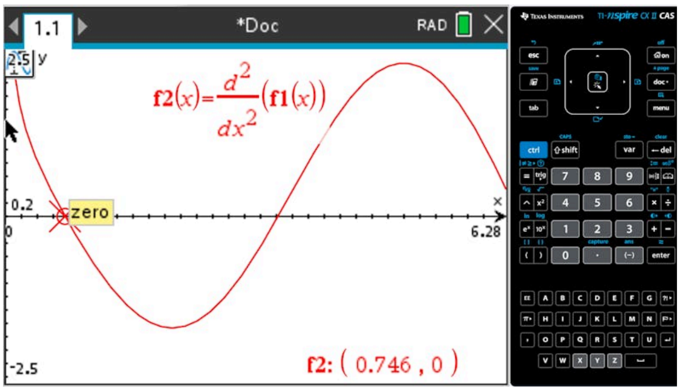
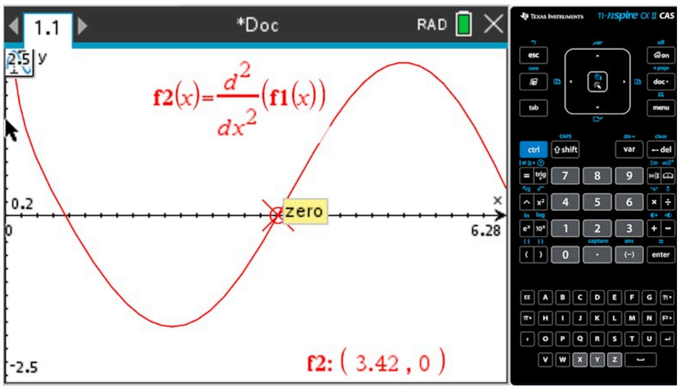
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
Press the button to bring up the template options and find the template for the second derivative.	
In filling the blanks in the template, use the function name instead of retyping the expression.	

Student view

Steps	Explanation
You can now see the graph of the function and the second derivative. You probably will need to adjust the window so that you only see the graph of this second derivative for $0 \leq x \leq 2\pi$. If you want to hide the graph of the function (which is not needed for finding the solutions of $f''(x) = 0$), press Menu ...	
... and find the option to hide/show objects.	

Steps	Explanation
If you move over the graph of the function and press enter, it will remove it from the view.	
You are interested in the x-intercepts. There are several ways of finding these. Probably the quickest is through tracing the graph.	

Steps	Explanation
Tracing the graph has the feature, that when you move close to an important point (like the x-intercept), the calculator will jump to that point, shows the type and displays the coordinates.	
To find the other x-intercept, all you need to do is to move the cursor close to it.	

Example 4

★★☆

Find the x-coordinate of the point of inflexion on the graph of $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$y' = 3ax^2 + 2bx + c$ $y'' = 6ax + 2b$	You can identify points of inflexion by investigating the second derivative.
$6ax + 2b = 0$ $6ax = -2b$ $x = -\frac{2b}{6a} = -\frac{b}{3a}$	At the point of inflexion, $y'' = 0$
So the x-coordinate of the point of inflexion on the graph of $y = ax^3 + bx^2 + cx + d$ is $-\frac{b}{3a}$.	

3 section questions

5. Calculus / 5.10 Second derivative

Stationary points

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/stationary-points-id-28189/print/) Assign

Arguably the most important concept in differential calculus is that of stationary points.

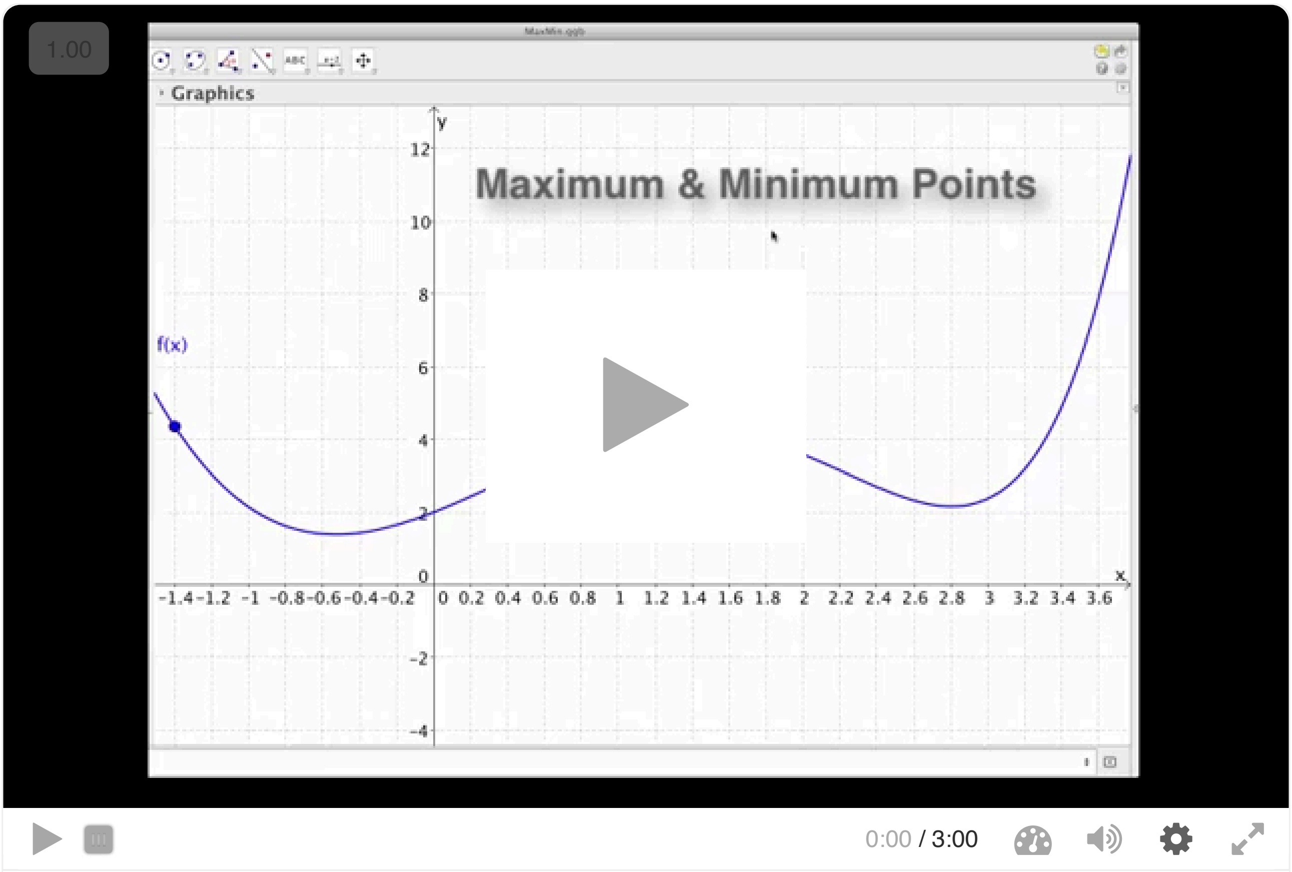
Stationary points, as the name implies, are points along a curve where that curve is momentarily stationary, meaning that it neither goes up nor down. In other words, the tangent to the curve at stationary points is horizontal. Horizontal lines have a zero gradient. Hence, at a stationary point, the gradient of a curve is zero and its gradient function evaluates to zero. Note that the value of the curve itself can be positive, negative or zero. It is the behaviour of its gradient that determines where the stationary points occur.

There are three types of stationary points: maxima, minima and stationary points of inflexion. In this section, you will learn about maxima and minima; points of inflexion are covered in the next section. In general, maxima and minima are also referred as extreme points, or extrema. In the video below, you can explore stationary points, distinguish

Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

between local maxima and local minima, and learn how the first derivative and second derivative functions can help you to determine where these points occur and their nature, i.e. whether they are maxima or minima.



Video 1. Maximum and Minimum Points.

More information for video 1

1

00:00:00,400 --> 00:00:03,604

narrator: In this video, we're going to

look at maximum minimum points

2

00:00:03,670 --> 00:00:07,808

along our curve

as a first application of differentiation

3

00:00:07,875 --> 00:00:08,775

to curve sketching.

4

00:00:09,209 --> 00:00:14,481

So when we trace our curve,

Student
view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

then you can see that over here

5

00:00:14,815 --> 00:00:17,584

it seems to go through a flat portion.

6

00:00:18,252 --> 00:00:21,421

And here again it seems

to go through a flat portion

7

00:00:21,555 --> 00:00:23,223

as well as turning around.

8

00:00:23,657 --> 00:00:27,794

And one more point over here,

it's going from down to up.

9

00:00:28,228 --> 00:00:32,533

And here I've indicated those

three special points with vertical lines.

10

00:00:32,900 --> 00:00:35,169

And we already know from talking about

11

00:00:35,235 --> 00:00:37,070

increasing and decreasing functions,

12

00:00:37,137 --> 00:00:41,308

that on either side

of those dotted red lines,

13

00:00:41,375 --> 00:00:44,278

the functions

are increasing or decreasing,

14

00:00:44,344 --> 00:00:48,782

which has a bearing on the sign

of the gradient function.

15

Student
view

00:00:48,849 --> 00:00:50,817

So let's have heat of gradient function.

16

00:00:51,618 --> 00:00:53,353

And here we're stepping

along the gradient function,

17

00:00:53,420 --> 00:00:56,990

along you see that

at this turning around point,

18

00:00:57,090 --> 00:00:59,092

the gradient seems to be zero.

19

00:00:59,226 --> 00:01:00,928

The tangent line is horizontal.

20

00:01:00,994 --> 00:01:05,232

And again, here too, the gradient is zero

21

00:01:05,866 --> 00:01:07,901

as we turn around on apparent function.

22

00:01:08,035 --> 00:01:09,436

And the last points equally.

23

00:01:09,636 --> 00:01:12,773

So the gradient function goes through zero

24

00:01:13,240 --> 00:01:16,877

as the blue curve

goes through the special point.

25

00:01:18,011 --> 00:01:21,615

Now those points are called

either minimum or maximum point,

26

00:01:21,682 --> 00:01:23,183

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

depending on their behavior.

27

00:01:23,250 --> 00:01:27,621

So the minimum point and the maximum point

are very different.

28

00:01:27,688 --> 00:01:29,923

At the minimum point is like a valley,

29

00:01:29,990 --> 00:01:32,192

I'm walking through it,

going from down to up.

30

00:01:32,292 --> 00:01:34,561

A maximum point is like a summit,

31

00:01:34,628 --> 00:01:39,499

I'm going up to down

if I look at increasing x direction.

32

00:01:40,801 --> 00:01:44,771

Now, we've also already learned

that we can take the second derivative

33

00:01:44,838 --> 00:01:50,444

and we're gonna see what kind of bearing

that has on our maxima and minima.

34

00:01:51,044 --> 00:01:53,514

So as we step along our parent function

35

00:01:53,947 --> 00:01:59,152

and we go get to our first minimum,

you see that a gradient function is zero,

36

00:01:59,753 --> 00:02:03,323

but our second derivative

f double prime is positive.

Student
view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

37

00:02:04,057 --> 00:02:05,859

On the other end,

when we come through our maximum,

38

00:02:05,959 --> 00:02:07,728

the gradient function again is zero,

39

00:02:07,895 --> 00:02:10,531

but a second derivative is negative.

40

00:02:12,232 --> 00:02:15,369

And in our last minimum point

along this curve,

41

00:02:15,435 --> 00:02:17,838

again, we find that

a gradient function is zero,

42

00:02:17,938 --> 00:02:21,408

but a second derivative

evaluates to a positive value.

43

00:02:23,110 --> 00:02:24,878

Now there are special regions

44

00:02:24,945 --> 00:02:30,017

called concave up and concave down,

and they are indicated by the sign

45

00:02:30,083 --> 00:02:32,352

of the second derivative function.

46

00:02:32,452 --> 00:02:35,322

So where the second

derivative function is positive,

47

00:02:35,389 --> 00:02:39,226

Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

we have concave up

and concave down when it's negative.

48

00:02:39,293 --> 00:02:40,994

So the function kind of looks downward.

49

00:02:41,061 --> 00:02:43,297

This is the blue function

of parent function.

50

00:02:43,363 --> 00:02:47,234

And concave up

is when the function really looks upwards

51

00:02:47,401 --> 00:02:51,205

and the sign of the second derivative

is positive.

52

00:02:52,973 --> 00:02:55,876

So here we have looked

at the parent function

53

00:02:55,943 --> 00:02:58,545

in more detail using

both the first derivative

54

00:02:58,612 --> 00:03:00,347

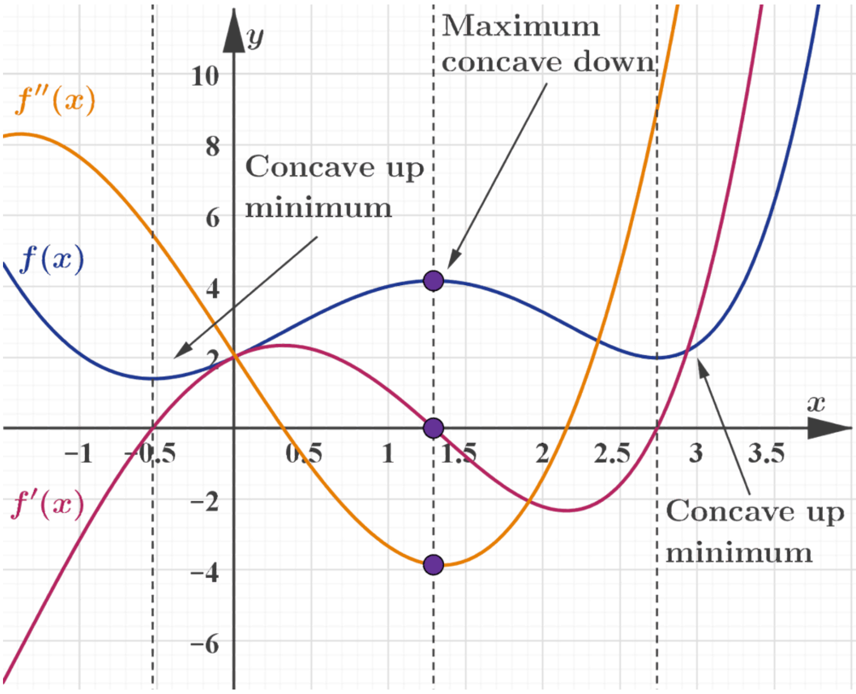
as well as the second derivative.

The video investigates the graph of $f(x)$, and the relationship between its first and second derivative functions, $f'(x)$ and $f''(x)$, respectively, and its maximum and minimum points .

The diagram below shows a comparison of how the first and second derivative functions, $f'(x)$ and $f''(x)$, respectively, behave at and around maximum and minimum points along a curve $f(x)$. Note also that in a concave up region the curve opens upwards, whereas in a concave down region the curve opens downwards.

Student
view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov



More information

The image is a diagram displaying the behavior of the first derivative (f'(x)) and second derivative (f''(x)) at various points along a function (f(x)) curve. The diagram is set on a grid with a vertical (Y-axis) running from -8 to 9 labeled 'Contributions to amplitude', and a horizontal (X-axis) running from -3 to 5 labeled 'Contributions to frequency'. The curve (f(x)) (in blue) goes from left to right, showing maximum and minimum points as peaks.

- The positive first derivative (f'(x)) (in magenta) is shown as a separate curve which intersects (f(x)) at the maximum and minimum points, changing from increasing to decreasing.
- The second derivative (f''(x)) (in orange) emphasizes concavity; in the regions of the diagram where this line is above zero, the curve (f(x)) is concave upwards, and where it's below zero, the curve is concave downwards.

Key points marked with purple dots illustrate where (f'(x) = 0), as these are turning points where the slope of (f(x)) changes from positive to negative or vice versa, highlighting maximal or minimal points.

[Generated by AI]

Do you see any relationships between the values of the first and second derivatives, the type of extreme points, and concavity? In general:

Student
view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

- For all stationary points, $f'(c) = 0$.
- Additionally, for maxima:
  - $f'(c)$ transitions from positive to negative
  - $f''(c) < 0$
  - $f(x)$ is concave down at $x = c$.
- Similarly, for minima:
  - $f'(c)$ transitions from negative to positive
  - $f''(c) > 0$
  - $f(x)$ is concave up at $x = c$.

Example 1

★☆☆

Find the coordinates of the stationary point of the curve $f(x) = -3 + 8x - 2x^2$ and determine its nature.

Find the first derivative

$$f'(x) = 8 - 4x$$

Set the first derivative equal to 0

$$f'(x) = 8 - 4x = 0$$

Solve for x

$$x = 2$$

Solve for y

$$y = f(2) = -3 + 8(2) - 2(2)^2 = 5$$

Find the second derivative

$$f''(x) = -4$$

Student
view

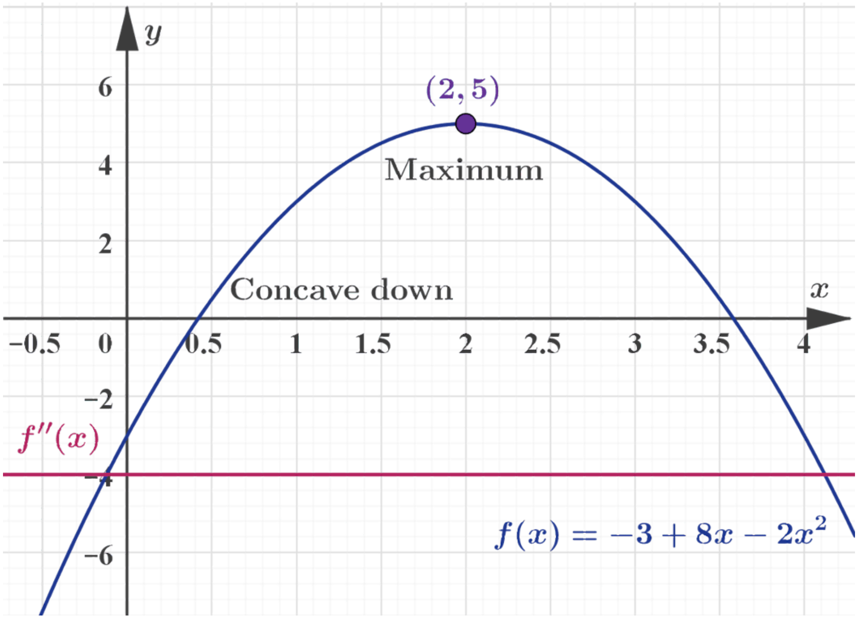
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Evaluate the second derivative

$$f''(2) = -4 < 0$$

There is a stationary point at $(2, 5)$ and it is a maximum as the function is concave down at that point (as it is everywhere along the curve).

The diagram below shows the maximum point of the curve $f(x) = -3 + 8x - 2x^2$, and that the second derivative function is negative everywhere, meaning that the curve is concave down.



Example 2

★★☆

Find the stationary point of the curve $f(x) = \dfrac{e^x}{x}, x > 0$ and determine its nature.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Find first derivative (quotient rule)

$$f'(x) = \frac{x\mathrm{e}^x - \mathrm{e}^x}{x^2}$$

Set first derivative equal to 0

$$f'(x) = \frac{x\mathrm{e}^x - \mathrm{e}^x}{x^2} = 0$$

Solve for x

$$\mathrm{e}^x(x-1) = 0$$

$$x = 1$$

Solve for y

$$y = f(1) = \frac{1\mathrm{e}^1 - \mathrm{e}^1}{1^2} = 0$$

Find second derivative

$$\begin{aligned} f''(x) &= \frac{x^2(\mathrm{e}^x + x\mathrm{e}^x - \mathrm{e}^x) - (x\mathrm{e}^x - \mathrm{e}^x)(2x)}{(x^2)^2} \\ &= \frac{\mathrm{e}^x\left(x^2 - 2x + 2\right)}{x^3} \end{aligned}$$

Evaluate second derivative

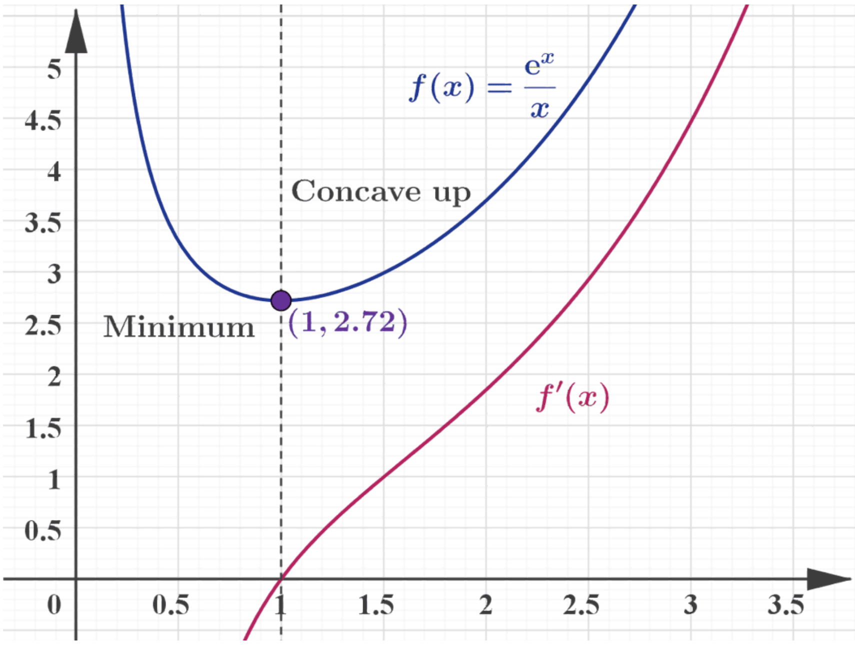
$$f''(1) = \frac{\mathrm{e}^1\left(1^2 - 2(1) + 2\right)}{1^3} = \mathrm{e} > 0$$

Note that, in this case, the first derivative could be rewritten as $f'(x) = \dfrac{\mathrm{e}^x(x-1)}{x^2}$. Since e^x and x^2 both have to be non-negative, the sign of the first derivative is driven by $(x-1)$. Since the sign of that term changes from negative to positive as the function passes $x = 1$, you can classify the point as a minimum without going through the work of the second derivative.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

The diagram below shows the curve $f(x) = \frac{e^x}{x}, x > 0$ in blue with its minimum point. By looking at the sign of the gradient function (red) on either side of the stationary point, you can deduce that it is indeed a minimum. The curve in the region shown is concave up.



3 section questions

5. Calculus / 5.10 Second derivative

Points of inflexion

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/points-of-inflexion-id-28190/print/) Assign

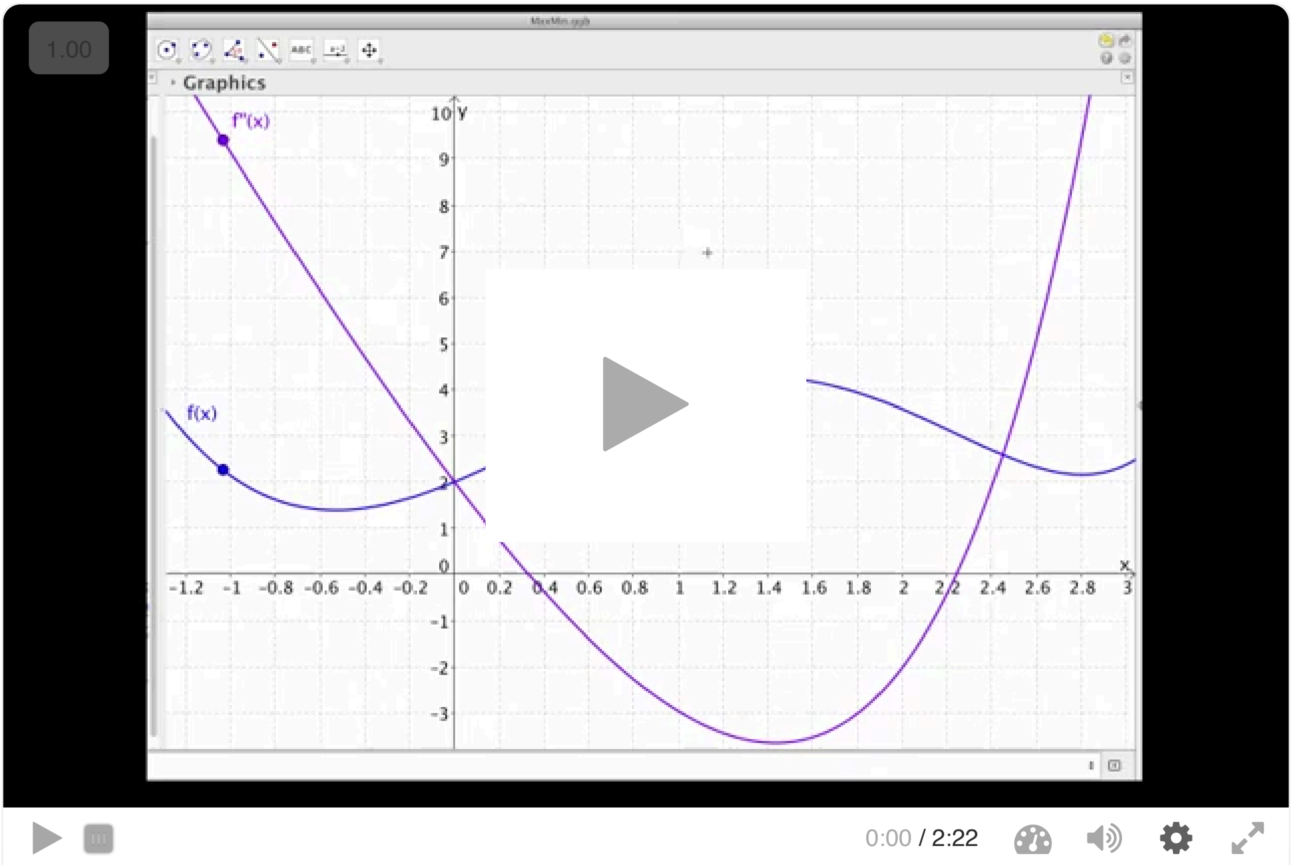
As you learned in section 5.10.4 (/study/app/math-ai-hl/sid-132-cid-761618/book/stationary-points-id-28189/), stationary points are points along a curve where the gradient is equal to 0. What happens when the gradient of the first derivative (or the second derivative) is equal to 0?

Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

When you analysed a function earlier, you found concavity by looking at the sign of the second derivative. If $f''(x) > 0$, the function is concave up; if $f''(x) < 0$, the function is concave down. Somewhere in between, there has to be a point where $f''(x) = 0$. What does this represent?

This is investigated in the video below.



Video 1. Points of Inflexion.

More information for video 1

1

00:00:00,467 --> 00:00:02,169

narrator:

In this video, we're going to investigate

2

00:00:02,236 --> 00:00:04,805

what the second derivative

function f '' (x)

3

00:00:04,872 --> 00:00:08,208

can tell us about the parent function

Student
view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

f (x) the blue function here.

4

00:00:08,675 --> 00:00:11,712

Now, we already seen that

when I hit the stationary point,

5

00:00:11,945 --> 00:00:13,247

in this case, a minimum,

6

00:00:13,313 --> 00:00:16,617

the second derivative evaluation

to positive value.

7

00:00:16,683 --> 00:00:17,818

But we're gonna look at some detail

8

00:00:17,885 --> 00:00:21,455

in what are the implication

when the second derivative function

9

00:00:21,522 --> 00:00:22,589

goes through zero.

10

00:00:24,024 --> 00:00:25,893

What does it tell

us about the parent function?

11

00:00:25,959 --> 00:00:28,195

So here we're gonna maximum

the second derivative

12

00:00:28,262 --> 00:00:30,063

evaluates to a negative value.

13

00:00:30,130 --> 00:00:31,431

We already seen that.

14

00:00:31,498 --> 00:00:33,433

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

And here we're

gonna go again through zero.

15

00:00:33,500 --> 00:00:36,003

What does it tell us

about the blue function?

16

00:00:36,770 --> 00:00:39,006

And here at the minimum

of the blue function,

17

00:00:39,239 --> 00:00:41,241

the second derivative is a positive value.

18

00:00:42,376 --> 00:00:44,545

Now let's add the first

derivative function to it

19

00:00:44,611 --> 00:00:47,281

and see if we can get

a little bit more understanding.

20

00:00:47,347 --> 00:00:50,484

So here, of course, we're stepping down,

the blue dot is stepping down,

21

00:00:50,551 --> 00:00:53,720

the gradient is negative

when it hits the stationary point,

22

00:00:53,820 --> 00:00:57,558

a minimum,

the first derivative is equal to zero.

23

00:00:57,724 --> 00:01:00,994

And here we're gonna look at some detail

at what the second derivative is zero.

24

Student view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

00:01:01,061 --> 00:01:03,630

So we see that the gradient is positive,

25

00:01:03,897 --> 00:01:06,533

but the gradient function hits a maximum,

26

00:01:06,600 --> 00:01:08,969

meaning that on that point,

27

00:01:09,036 --> 00:01:11,705

it's at the steepest point

along the curve.

28

00:01:11,772 --> 00:01:14,842

After that, it's still positive,

the gradient,

29

00:01:14,942 --> 00:01:16,910

but it gets less steep.

30

00:01:18,812 --> 00:01:20,747

And then we're gonna see this point again,

31

00:01:20,814 --> 00:01:22,950

that even though the gradient is negative

32

00:01:23,016 --> 00:01:26,119

completely around this region,

where the second derivative is zero,

33

00:01:26,386 --> 00:01:28,255

it is at its minimum point,

34

00:01:28,322 --> 00:01:31,124

so it's most negative

that the gradient gets,

Student
view

35

00:01:34,928 --> 00:01:37,731

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

after that becomes less negative.

36

00:01:38,465 --> 00:01:41,468

Now those places

where the second derivative is zero

37

00:01:42,503 --> 00:01:45,706

demarcates a difference

between a concave up and concave down.

38

00:01:45,772 --> 00:01:50,277

So the concavity changes

yet changes from concave down

39

00:01:50,511 --> 00:01:51,879

to concave up.

40

00:01:53,747 --> 00:01:55,983

And those points

then are very special points,

41

00:01:56,049 --> 00:01:57,584

which we call inflexion points.

42

00:01:57,651 --> 00:02:00,153

Inflexion points

is where the concavity changes.

43

00:02:00,354 --> 00:02:03,156

So that means that

the second derivative goes through zero

44

00:02:03,223 --> 00:02:05,826

and the first derivative

45

00:02:05,893 --> 00:02:09,162

has the same sign on either

side is very important

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

46

00:02:09,229 --> 00:02:14,635

that you always check that both those

situations are true.

47

00:02:15,502 --> 00:02:17,638

So those are inflexion points.

48

00:02:17,704 --> 00:02:21,108

It's where the concavity

of the function changes.

As outlined in section 5.10.4 (/study/app/math-ai-hl/sid-132-cid-761618/book/stationary-points-id-28189/), a point of inflexion is where the concavity of the function changes from up to down or down to up. This necessarily means that the sign of the second derivative changes. A point of inflexion is where the gradient function has reached its local maximum or minimum, meaning that, for that interval, the function has reached its greatest or smallest value for a gradient. For a point to be classified as a point of inflexion, it must satisfy the following two conditions:

1. the second derivative at that point evaluates to zero; and
2. the sign of the second derivative changes on either side of the point of inflexion.

Exam tip

You must always check whether a point where $f''(x) = 0$ is definitely a point of inflexion, and not an extremum, by checking the second condition.

This second condition can be difficult to check with confidence without the aid of a graphical tool. The idea is to check the second derivative close to the candidate point, but determining how close is close enough may prove challenging. For the purpose of this course, the functions will be relatively simple, for example, basic polynomials.

Student view

Just as there are stationary points that are not points of inflexion (i.e. extrema), there are also points of inflexion that are not stationary points. Remember that a stationary point is simply a point where $f'(x) = 0$ and a point of inflexion is a point where the concavity is

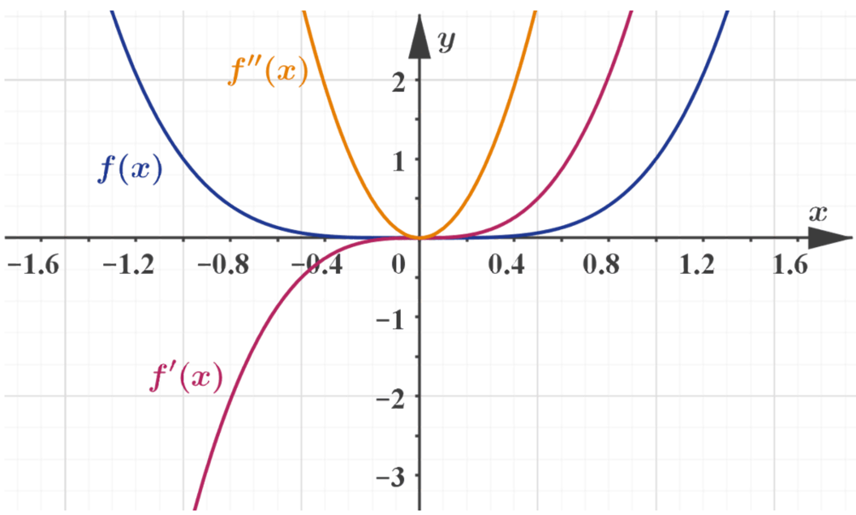
Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

changing sign. It is possible, and quite common, for the concavity to change at a point where $f'(x) \neq 0$.

As an example, examine the following functions at $x = 0$:

$f(x)$	$f'(x)$	$f''(x)$	**Classification**
$f(x) = x^4$ $f(0) = 0$	$f'(x) = 4x^3$ $f'(0) = 0$	$f''(x) = 12x^2$ $f''(0) = 0$	Minimum
$f(x) = x^3$ $f(0) = 0$	$f'(x) = 3x^2$ $f'(0) = 0$	$f''(x) = 6x$ $f''(0) = 0$	Stationary inflexion point
$f(x) = \sin x$ $f(0) = 0$	$f'(x) = \cos x$ $f'(0) = 1$	$f''(x) = -\sin x$ $f''(0) = 0$	Non-stationary inflexion point

For all three functions, the graphs go through the origin and the second derivative is equal to 0. For $f(x) = x^4$ and $f(x) = x^3$, the first derivative is also equal to zero, but by looking just to the left and right of the critical point, you can see that the sign of the second derivative of $f(x) = x^4$ does not change; it goes from positive to zero to positive. For $f(x) = x^3$, the second derivative changes from negative to zero to positive. For $f(x) = \sin x$, the first derivative is not equal to zero, but the second derivative is equal to zero and changes sign from positive to zero to negative. The following three diagrams show these relationships graphically.



More information

Student view

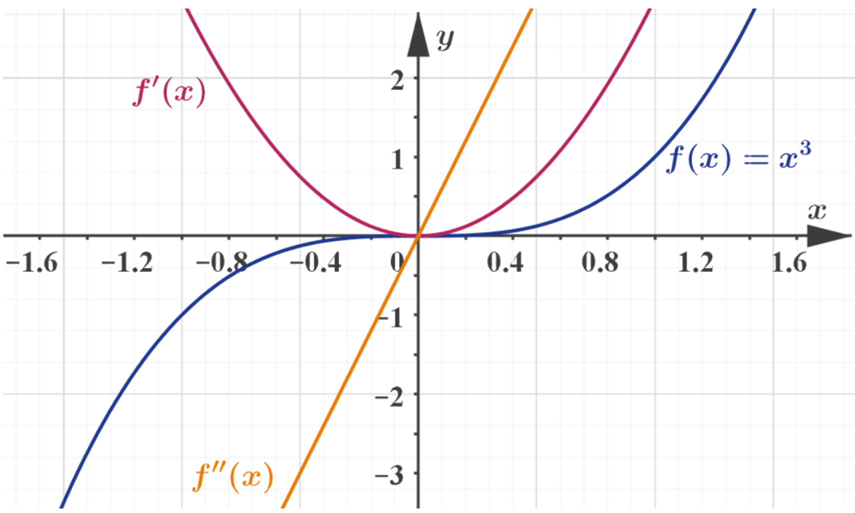
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

The image shows a graph plotting the function (f(x) = x^4) along with its first and second derivatives, (f'(x)) and (f''(x)).

- The X-axis represents the variable (x), ranging approximately from -1.6 to 1.6.
- The Y-axis represents the function values, ranging approximately from -3 to 3.
- The graph has three curves:
- The blue curve represents (f(x) = x^4), which is relatively flat around the origin and concave upwards.
- The pink curve represents the first derivative (f'(x)), which crosses the origin and exhibits a minimum point here.
- The orange curve represents the second derivative (f''(x)), which is a parabola curving upwards, starting above 0 when (x) is close to 0, indicating a minimum at this point.
- The diagram visualizes how the second derivative remains positive except at the minimum point, demonstrating the curvature behavior of (f(x)).

[Generated by AI]

The diagram above shows the curve $f(x) = x^4$ and its two derivative functions, $f'(x)$ and $f''(x)$. This is an example of a minimum.



More information

Student view

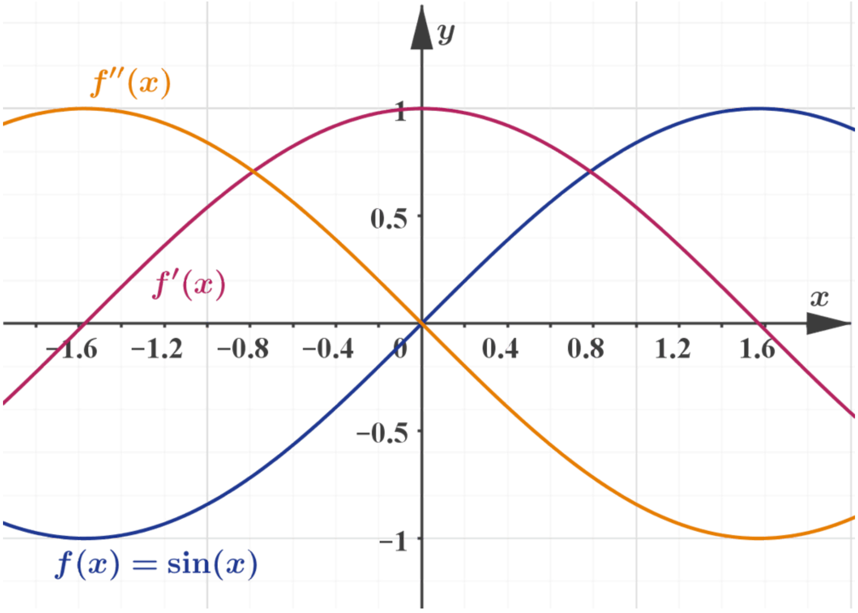
This is a graph showing three curves: f(x) = x^3, f'(x), and f''(x) on a grid. The x-axis is labeled 'x' and ranges from -1.6 to 1.6, with intervals labeled every 0.8 units. The y-axis is labeled 'y' and ranges from -3 to 3, with intervals labeled every 1 unit. The blue curve represents f(x) = x^3, showing a gradual increase with an inflection point at the origin.

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

The red curve represents f'(x), starting from negative values, intersecting the x-axis at the origin, and increasing thereafter. The orange curve represents f''(x), a straight line passing through the origin, indicating changes in concavity. The diagram illustrates the concept of a non-stationary point of inflexion, where the first derivative f'(x) changes from negative to positive and the second derivative f''(x) is non-zero at the inflection point.

[Generated by AI]

The diagram above shows the curve $f(x) = x^3$ and its two derivative functions, $f'(x)$ and $f''(x)$. This is an example of a non-stationary point of inflexion.



More information

The image is a graph depicting the function f(x) = sin(x) and its derivatives, plotted on a grid with labeled axes. The x-axis ranges from -16 to at least 1.5, and the y-axis ranges approximately from -1 to 1. The graph shows three curves: the original sinusoidal curve in blue labeled f(x) = sin(x), the first derivative in pink labeled f'(x), and the second derivative in orange labeled f''(x). The curves illustrate how the original function and its derivatives intersect and diverge over this domain, demonstrating a stationary point of inflection where the curves intersect at x = 0.

[Generated by AI]

Student
view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

The diagram above shows the curve $f(x) = \sin x$ and its two derivative functions, $f'(x)$ and $f''(x)$. This is an example of a stationary point of inflexion.

2 section questions

5. Calculus / 5.10 Second derivative

First and second derivative tests

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/first-and-second-derivative-tests-id-28191/print/) Assign

First derivative

As you learned in subtopic 5.9 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-28208/), the first derivative can help determine the shape of a graph. In addition to determining where a function is increasing or decreasing, the way in which $f'(x)$ changes around a critical point helps to determine whether the critical point is a maximum, a minimum or neither.

Important

First derivative test. Given that $f'(x) = 0$ or $f'(c)$ does not exist:

- If $f'(x)$ changes from positive to negative at c, then $f(x)$ has a local maximum at $x = c$.
- If $f'(x)$ changes from negative to positive at c, then $f(x)$ has a local minimum at $x = c$.
- If $f'(x)$ does not change sign at c, then $f(x)$ has neither a local minimum nor maximum at $x = c$.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Second derivative

In sections 5.10.4 (/study/app/math-ai-hl/sid-132-cid-761618/book/stationary-points-id-28189/) and 5.10.5 (/study/app/math-ai-hl/sid-132-cid-761618/book/points-of-inflexion-id-28190/), you learned that the second derivative can also help determine the shape of the graph. In addition to determining the concavity of a function, $f''(x)$ may also help determine whether the critical point is a maximum, a minimum or neither.

✓ **Important**

Second derivative test. Given that $f(x)$ is continuous near c:

- If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at $x = c$.
- If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local maximum at $x = c$.
- If $f'(c) = 0$ and $f''(c) = 0$ or is undefined, then the second derivative test is inconclusive. $f(x)$ may have a maximum, minimum or neither at $x = c$.

Summary of behaviour

	$f'(x)$	$f'(x)$ **behaviour**	$f''(x)$	$f''(x)$ **behaviour**
Local minimum	$f'(x) = 0$	negative $\rightarrow 0 \rightarrow$ positive	$f''(x) > 0$	positive concave up
Local maximum	$f'(x) = 0$	positive $\rightarrow 0 \rightarrow$ negative	$f''(x) < 0$	negative concave down
Stationary inflexion point	$f'(x) = 0$	negative $\rightarrow 0 \rightarrow$ negative positive $\rightarrow 0 \rightarrow$ positive	$f''(x) = 0$	negative $\rightarrow 0 \rightarrow$ positive positive $\rightarrow 0 \rightarrow$ negative
Non-stationary inflexion point	$f'(x) \neq 0$	negative positive	$f''(x) = 0$	negative $\rightarrow 0 \rightarrow$ positive positive $\rightarrow 0 \rightarrow$ negative

3 section questions ⌄

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

5. Calculus / 5.10 Second derivative

Optimisation

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/optimisation-id-28192/print/) Assign

Optimisation involves finding the best or most effective use of a resource. It might mean minimising cost, maximis ing revenue, minimising man-hours, maximising attendees, or any of a number of situations. The key to optimisation problems is to identify the independent variable (the resource you control), the dependent variable (the objective you are interested in optimising), the mathematical relationship between the two, and the absolute extremum over the domain of all possible values.

Often, when evaluating a problem for optimisation, you will find a number of stationary points. These are sometimes called candidate points. They can include points with a gradient of zero, end points if there is a restricted domain, and any points of discontinuity. Once you have the list of candidate points, you can evaluate the objective for each of these points and select the best result.

A good procedure to follow is:

1. Define the variables and choose symbols for them.
2. Find the relation, i.e. the function, between them.
3. Differentiate the function and find each stationary point, i.e. set the function for the first derivative to zero and solve the resulting equation. Identify any other candidate points.
4. Determine whether each stationary point is a local maximum or local minimum.
5. Evaluate all remaining candidate points and select the best value.

Remember, there are a variety of conditions that result in a gradient of zero. A point of this nature could be a local or absolute maximum, a local or absolute minimum, or a horizontal point of inflexion. That is why Step 4 of the process is so important.

Consider the following problem.

Student view

Example 1

★★☆

Overview (/study/ap ai- hl/sid- 132- cid- 761618/ov

There are 40 apple trees in an orchard. Each tree produces 600 apples in a year. For each additional tree planted in the orchard, the output of every tree drops by 10 apples. How many trees should be added to the existing trees in the orchard to maximise the total production of apples?

Steps	Explanation
P — Production of orchard. T — Trees added.	Definition of variables.
Let P be the total number of trees and x be the production per tree. $P = (40 + T)(600 - 10T) = -10T^2 + 200T + 24000$	Define relation.
$\frac{\mathrm{d}P}{\mathrm{d}T} = -20T + 200 = 0$ $T = 10$	Differentiate, set to zero an solve.
$T = 0$	Natural end point — add n trees.
$\frac{\mathrm{d}^2P}{\mathrm{d}T^2} = -20 < 0$	Differentiate, $T = 10$ is maximum.
$P(0) = 40(600) = 24\,000$ $P(10) = (40 + 10)(600 - 10(10)) = 25\,000$	Evaluate candidate points.
Add 10 trees.	

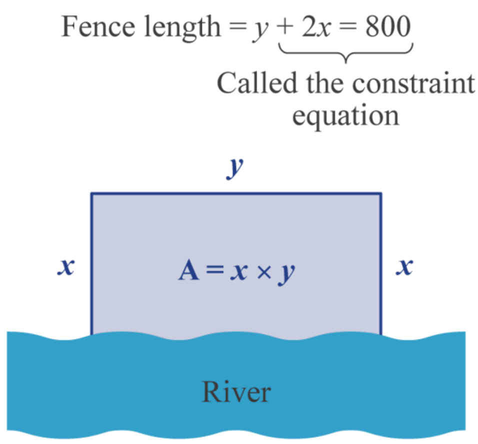
Example 2

★★☆

A farmer has $800\,\mathrm{m}$ of fencing to fence off a rectangular field that borders a river. No fence is needed along the river. What are the dimensions of the field that gives it its largest area?

Student view

Although not necessarily one of the steps, drawing a diagram may help with any geometric typ
of problem.



x — width y — length	Definition of variables.
$A = xy$, area $y + 2x = 800$, constraint (fencing) $y = -2x + 800$ $A = x(-2x + 800) = -2x^2 + 800x$	Define relations.
$\frac{\mathrm{d}A}{\mathrm{d}x} = -4x + 800 = 0$ $x = 200, y = 400$	Differentiate, set to zero ar solve.
$x = 0, y = 800$ $x = 400, y = 0$	Natural end points.
$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -4 < 0$	Differentiate; $x = 200$ is maximum.
$A(200, 400) = 80000$ $A(0, 800) = 0$ $A(400, 0) = 0$	Evaluate candidate points.
$x = 200, y = 400$ provides the maximum area of $80\,000\ \mathrm{m}^2$.	

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

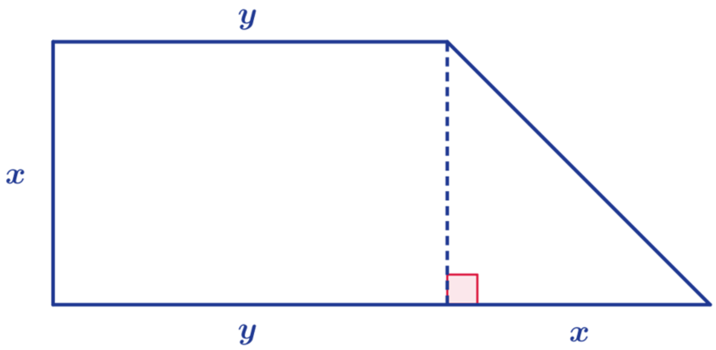
Be aware

1. Include a sketch whenever the problem involves anything geometrical and define the variables explicitly.
2. The functions involved can only be relations between two variables. If there are more than two variables in the problem, say three, then look for information that relates two of them in a constraint equation.
3. In these problems, you normally have to use all the information that is given such as numbers and relationships. If you are stuck, see whether you have overlooked some information in the question.
4. Usually the resulting functions are relatively easy to differentiate, either by hand or with your calculator.

Example 3

★★★

A farmer has 400 m of fencing to fence off the field shown. The field comprises two smaller pieces, one rectangle and one right-angled triangle, which are attached as shown in the figure. The fence will be placed along the solid blue perimeter of the field. Find the length x that gives the largest area of the field.



More information

Student
view

Overview
(/study/ap
ai-
hl/sid-
132-
cid-
761618/ov

The image shows a geometric diagram of a field composed of two parts: a rectangle and a right-angled triangle. Both shapes are joined together by one of the shorter sides of the triangle, forming a larger complex shape. The rectangle and triangle share a common side labeled as "y," representing their shared width.

The rectangle has a width of "y" and a length of "x." To the right of this rectangle is a right-angled triangle, where the base is labeled as "x" and the height is marked as "y," making the hypotenuse form the longest side.

This setup is encapsulated by a solid blue outline representing the fencing placed along the perimeter of both the rectangle and triangle combined. A dashed blue line divides the rectangle and the triangle to emphasize their separate areas. The task is to maximize the area of this combined shape with a total fencing length of 400 meters.

[Generated by AI]

Steps	Explanation
x and y are defined in the diagram.	Definition of variables.
$A = xy + \frac{x^2}{2}$, area $2x + 2y + x\sqrt{2} = 400$, constraint (fencing) $y = 200 - x - \frac{x\sqrt{2}}{2}$ $A = \left(200 - x - \frac{x\sqrt{2}}{2}\right)x + \frac{x^2}{2} = 200x - x^2\frac{\left(1+\sqrt{2}\right)}{2}$	Define the relationships.
$\frac{\mathrm{d}A}{\mathrm{d}x} = 200 - x\left(1+\sqrt{2}\right) = 0$ $x = \frac{200}{1+\sqrt{2}}, y = \frac{100\sqrt{2}}{1+\sqrt{2}}$	Differentiate, set to zero and solve.
$x = 0, y = 200$ $x = \frac{400}{2+\sqrt{2}}, y = 0$	Natural end points.
$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -\left(1+\sqrt{2}\right) < 0$	Differentiate; $x = \frac{200}{1+\sqrt{}}$ is maximum.

Student
view

Steps	Explanation
$A\left(\dfrac{200}{1+\sqrt{2}}\right) \approx 8284$ $A(0) = 0$ $A\left(\dfrac{400}{2+\sqrt{2}}\right) \approx 6863$	Evaluate candidate point:
$x = \dfrac{200}{1+\sqrt{2}}$ provides the maximum area of $8284\ \mathrm{m}^2$.	

3 section questions

5. Calculus / 5.10 Second derivative

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/checklist-id-28193/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- find the second derivative of a function and apply it in context
- understand the different notation used for the second derivative
- sketch the first and second derivative without algebraically finding these derivatives
- understand the relationship between the sign of the derivative and the increasing/decreasing behaviour of the graph of a function
- understand the relationship between the sign of the second derivative and the concavity of the graph of a function
- identify stationary points and classify these as maxima, minima and points of inflexion
- identify points of inflexion and classify these as stationary and non-stationary
- draw sketches based on information about the first and second derivative
- translate word problems into functions relating variables, and use constraint equations to ensure that the functions are dependent on one variable only;

Student
view

then apply differentiation to find the optimum solution (maximum or minimum).

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

5. Calculus / 5.10 Second derivative

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/investigation-id-28194/print/) Assign

Investigation 1

Using the applet below you can investigate the derivatives of $y = x\mathrm{e}^x$, $y = x^2\mathrm{e}^x$ and $y = x^3\mathrm{e}^x$.

Before you move the slider, find the first and second derivatives. You can even go further to differentiate the second derivative to find the third derivative.

- Finding further derivatives by hand is good practice, but is a very repetitive process. Move the slider to see the first 50 derivatives.
  - The applet uses the notation you will learn in subtopic 5.12 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-28203/). Can you see the pattern in the notation?
- Can you find a pattern in the derivatives? Can you find the hundredth derivative?

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Interactive 1. Graph of First, Second, and Third Derivatives.

More information for interactive 1

This interactive allows the users to investigate the derivatives of $y = xe^x$, $y = x^2e^x$ and $y = x^3e^x$. The screen is divided in two halves vertically. On the right there is an equation $y = x^1 * e^x$; on the left it is mentioned "Exponent" with three options to select, namely, 1, 2, and 3. A slider is also provided to control which derivative is being displayed, ranging from the first derivative to the 50th derivative.

The applet uses differential notation commonly seen in calculus, such as $\frac{d^n y}{dx^n}$, which helps users become familiar with mathematical notation. By observing successive derivatives, users can identify patterns in differentiation.

As users move the slider, the function updates to show the corresponding derivative.

$$\frac{d^1 y}{dx^1} = (x + 1)e^x$$

(first derivative)

$$\frac{d^2 y}{dx^2} = (x + 2)e^x$$

(second derivative)

$$\frac{d^{50} y}{dx^{50}} = (x + 50)e^x$$

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

(50th derivative)

When 2 is selected, the equation changes to $y = x^2 * e^x$, and the derivatives change accordingly.

$$\frac{d^1y}{dx^1} = (x^2 + 2x)e^x$$

(first derivative)

$$\frac{d^2y}{dx^2} = (x^2 + 4x + 2)e^x$$

(second derivative)

$$\frac{d^{50}y}{dx^{50}} = (x^2 + 100x + 2450)e^x$$

(50th derivative)

Lastly, when 3 is selected, the derivative appears accordingly.
That is, for the function $y = x^3e^x$

$$\frac{d^1y}{dx^1} = (x^3 + 3x^2)e^x$$

(first derivative)

$$\frac{d^2y}{dx^2} = (x^3 + 6x^2 + 6x + 0)e^x$$

(second derivative)

$$\frac{d^{50}y}{dx^{50}} = (x^3 + 150x^2 + 7350x + 117600)e^x$$

(50th derivative)

Student view

The users will find a pattern when they find the derivatives in order and hence will understand how the pattern can be given a general notation. Additionally helping them in finding the derivatives easily.

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

You can use WolframAlpha (https://www.wolframalpha.com/) to check the hundredth derivative you found. Type for example:

100th derivative of x^3*e^x

into the search line .

Investigation 2

In this course, a lot of time is spent focusing on solving problems by hand or with the use of a graphic display calculator. Once you have finished college, you may find that there are easier ways of solving problems with other forms of technology, such as Microsoft Excel or Google Sheets.

Consider the first example in section 5.10.7 (/study/app/math-ai-hl/sid-132-cid-761618/book/optimisation-id-28192/):

There are 40 apple trees in an orchard. Each tree produces 600 apples in a year. For each additional tree planted in the orchard, the output of every tree drops by 10 apples. How many trees should be added to the existing trees in the orchard to maximise the total production of apples?

Open a spreadsheet program of your choice. In the first column, label the first four columns 'Trees Added', 'Trees in Orchard', 'Tree Output', and 'Orchard Production'.

In the 'Trees Added' column, enter numbers 0 and 1 in cells A2 and A3. You can highlight these two cells and copy down to add more trees.

In the 'Trees in Orchard' column (cell B2), type the formula '=40+A2'. Highlight and copy that cell down.

In the 'Tree Output' column (cell C2), type the formula '=600-10*C2'. Highlight and copy that cell down.

In the 'Orchard Production' column (cell D2), type the formula '=B2*C2' . Highlight and copy that cell down.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

In column D, what is the maximum production? How many trees were added?

Your spreadsheet should look something like this:

Trees Added	Trees in Orchard	Tree Output	Orchard Production
0	40	600	24 000
1	41	590	24 190
2	42	580	24 360
3	43	570	24 510
4	44	560	24 640
5	45	550	24 750
6	46	540	24 840
7	47	530	24 910
8	48	520	24 960
9	49	510	24 990
10	50	500	25 000
11	51	490	24 990
12	52	480	24 960
13	53	470	24 910
14	54	460	24 840
15	55	450	24 750
16	56	440	24 640
17	57	430	24 510

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Trees Added	Trees in Orchard	Tree Output	Orchard Production
18	58	420	24 360
19	59	410	24 190
20	60	400	24 000

This was a discrete case. Trees are added in increments of 1. For a continuous case, you decide on the increments yourself, and can vary them when you get close.

Select another example or section question from section 5.10.7 (/study/app/math-ai-hl/sid-132-cid-761618/book/optimisation-id-28192/) and try to use a spreadsheet to solve it.

Student view