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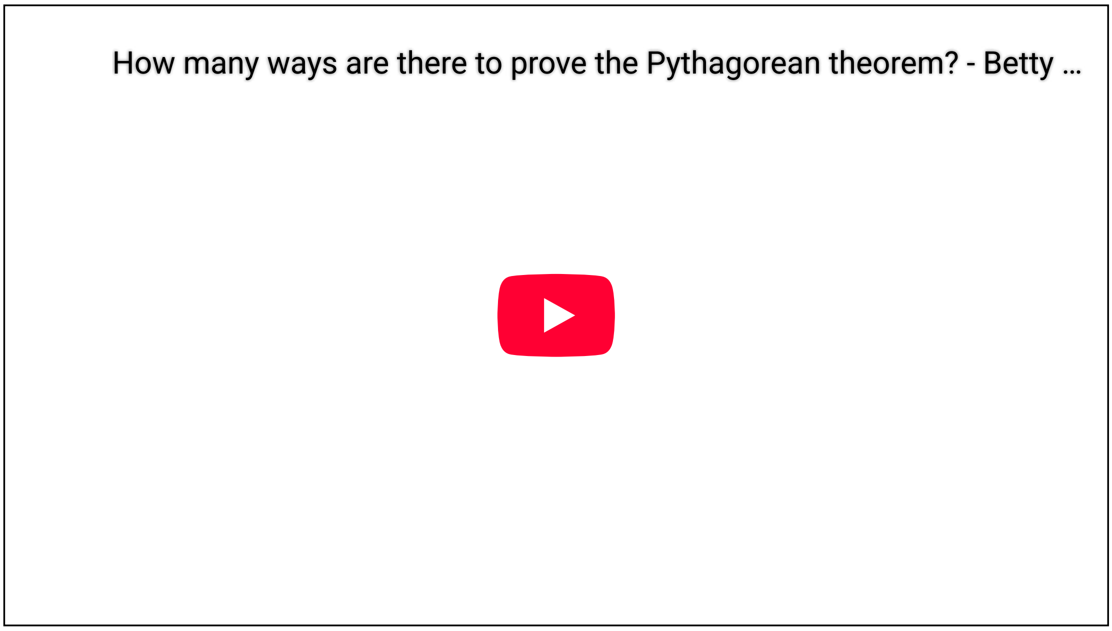
The big picture

In mathematics you use many equations, such as Pythagoras' theorem, to solve problems or answer questions.

While it is helpful to use $a^2 + b^2 = c^2$ when you want to find information about a right-angled triangle, how do you know that this equation is always true?

Showing that $a^2 + b^2 = c^2$ is always true is a proof. Watch this video to see the proof for Pythagoras' theorem. Note that proofs can be elegant, and the approaches can be varied and creative.



Proofs are very important in mathematics; you can see this if you do some research on Millennium Prize Problems.

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

In this subtopic you will study how to use deductive reasoning and algebraic manipulation in proofs.

Concept

In this subtopic you will learn how to prove equivalence by LHS to RHS arguments where you can't work across the equals sign. As you work on these proofs, consider why it is so important to work on the two sides of the equals sign separately.

Theory of Knowledge

Mathematics has strict standards for 'proving' something true. Consider how other areas of knowledge (AOKs) 'prove' knowledge as true while you contemplate the following Knowledge Question: What factors within an area of knowledge contribute to standards of proof within that AOK?

1. Number and algebra / 1.6 Deduction

Identities

Mathematicians distinguish between identities and equations by using the $\equiv$sign for identities and the more familiar $=$ sign for equations.

An identity is a special kind of equation. **Example 1** will help you see what makes an identity special.

Example 1

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Student view

a) Solve $3x + 1 = -2\left(x + 1\right) + 5x + 3.$

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b) Hence, decide whether $3x + 1 = -2(x + 1) + 5x + 3$ is an identity.

	Steps	Explanation
a)	$3x + 1 = -2(x + 1) + 5x + 3$ $3x + 1 = -2x - 2 + 5x + 3$ $3x + 1 = 3x + 1$ $0 = 0$	Solve for x.
	Any value of x is a solution to this question, and no particular value can be found.	
b)	Hence, $3x + 1 = -2(x + 1) + 5x + 3$ is an identity because it is true for all values of x. It should be written as $3x + 1 \equiv -2(x + 1) + 5x + 3.$	

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Important

An identity is an equation that is always true. If it contains a variable, it is true for all values of that variable. Identities are presented using the $\equiv$ sign.

Exam tip

There are slight variations in how mathematicians use the term identity. Some mathematicians require the two sides to be defined on the same set of values, or specify a domain for the identity. Others consider an equation to be an identity if the two sides evaluate to the same value when both are defined. In the guide IB uses the term identity in this less restrictive sense.

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Example 2

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a) Solve $2x + 7 = 10 + x$.

b) Hence, decide whether $2x + 7 = 10 + x$ is an identity.

	Steps	Explanation
a)	$2x + 7 = 10 + x \quad \Leftrightarrow \quad x = 3$	Solve for x.
b)	$2x + 7 = 10 + x$ is true only when $x = 3$, so it is not an identity.	

Example 3

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Identify which of the following are identities and justify your answer:

a) $x^2 + 20 = -9x$

b) $a = \frac{1}{a}$

c) $a \times \left(-\frac{1}{a}\right) = -1$

d) $\frac{1}{2} + \frac{1}{7} = \frac{9}{14}$

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	Steps	Explanation
a)	$x^2 + 20 = -9x$ $\Leftrightarrow\ x^2 + 9x + 20 = 0$ $\Leftrightarrow\ (x+5)(x+4) = 0$ $\Leftrightarrow\ x = -5 \text{ or } x = -4$	Solve for x.
	This equation is true only for $x = -5$ and $x = -4$, so it is not an identity.	
b)	$a = \frac{1}{a} \quad \Leftrightarrow \quad a^2 = 1 \quad \Leftrightarrow \quad a = \pm 1$	Solve for a.
	This equation is true only for $a = \pm 1$, so it is not an identity.	
c)	$a \times \left(-\frac{1}{a}\right) = -1 \quad \Leftrightarrow \quad -\frac{a}{a} = -1 \quad \Leftrightarrow \quad -1 = -1$	Solve for a.
	This equation is true for all values of a, so it is an identity.	
d)	$\frac{1}{2} + \frac{1}{7} = \frac{7}{2 \times 7} + \frac{2}{2 \times 7} = \frac{2+7}{14} = \frac{9}{14}$, so it is an identity.	Find the common denominator and ac the fractions.

3 section questions

1. Number and algebra / 1.6 Deduction

LHS to RHS proof

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In the previous section (/study/app/math-aa-hl/sid-134-cid-761926/book/identities-id-27667/) you learned how to distinguish between identities and equalities. Now you will learn how to prove that one side of an identity is equal to the other.

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To do this you need to algebraically manipulate the left-hand side (LHS) of the equation separately from the right-hand side (RHS) until they look the same.

Example 1

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Show that $\frac{1}{n}+\frac{1}{n^2}=\frac{n+1}{n^2}$.

Steps	Explanation
$\text{LHS}=\frac{1}{n}+\frac{1}{n^2}=\frac{n}{n}\times\frac{1}{n}+\frac{1}{n^2}$ $=\frac{n+1}{n^2}$	Only work on one side at a time.
$\text{LHS}=\text{RHS}$ Therefore $\frac{1}{n}+\frac{1}{n^2}=\frac{n+1}{n^2}$	Conclude your work with the statement LHS = RHS.

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Example 2

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Show that $\frac{1}{2}+\frac{1}{3}=\frac{2}{3}+\frac{1}{6}$.

$$\text{LHS}=\frac{1}{2}+\frac{1}{3}=\frac{3}{3}\times\frac{1}{2}+\frac{2}{2}\times\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

$$\text{RHS}=\frac{2}{3}+\frac{1}{6}=\frac{2}{2}\times\frac{2}{3}+\frac{1}{6}=\frac{4}{6}+\frac{1}{6}=\frac{5}{6}$$

$$\text{LHS}=\text{RHS}$$

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Therefore

$$\frac{1}{2}+\frac{1}{3}=\frac{2}{3}+\frac{1}{6}$$

Be aware

Work on the LHS separately from working on the RHS. Never work across the equals sign when you are trying to prove that two sides of an equation are equal.

Any algebraic manipulation that uses the fact that both sides are equal is working across the equals sign. Some examples are:

- adding/subtracting the same value to both sides
- moving a term from one side to another
- dividing/multiplying both sides by the same value
- taking a root of both sides
- raising both sides to an equal power.

Can you explain why there is a restriction about working across the equals sign?

International Mindedness

You can conclude a proof by saying LHS = RHS. Another way to conclude a proof is to say QED, which is an abbreviation for the Latin phrase 'quod erat demonstrandum' meaning 'that which was to be demonstrated.' If you do maths in another language, you may find that instead of the Latin version there is an equivalent abbreviation for the phrase in your language.

Example 3

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Show that $\frac{(x+y)^2+(x-y)^2}{2}=x^2+y^2$.

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Steps	Explanation
$\text{LHS} = \frac{x^2+2xy+y^2+x^2-2xy+y^2}{2}$ $= \frac{2x^2+2y^2}{2} = \frac{2\left(x^2+y^2\right)}{2} = x^2+y^2$	
QED	You can use QED or 'LHS = RHS and therefor the original equation is true' to conclude your proof.

Be aware

The method of LHS to RHS proof can also be used to check that your solution to a question is correct. This would be done by showing that one side (your solution) is equivalent to the other side (the original conditions you were given).

2 section questions

1. Number and algebra / 1.6 Deduction

Deductive proof

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A deductive proof, which is also known as direct proof, is used to show that a conditional statement is true by using established facts and a series of logical arguments to arrive at the required conclusion.

A conditional statement is of the form:

'If P is true, then Q is true.'

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This is written as $P \Rightarrow Q$. The first part of the deductive proof is to assume that P is true. Then you try to arrive at Q being true from there.

Making connections

Deductive reasoning is used in the scientific method, where you start with a theory and a hypothesis and you use observations to try to confirm the hypothesis.

Example 1

★☆☆

Prove that the sum of two consecutive integers is odd.

Steps	Explanation
'The sum of two consecutive integers is odd' is a $P \Rightarrow Q$ statement. You can see this if you rephrase it to: 'If two numbers are consecutive integers, then their sum is odd.'	
Let a and b be the two consecutive integers.	This is where you assume tha is true. The two numbers **are** consecutive integers.
Let $b = a + 1$ where $a \in \mathbb{Z}$	
$\Rightarrow a + b = a + a + 1 = 2a + 1$	
$\Rightarrow 2a + 1$ is odd since $2n + 1$ where $n \in \mathbb{Z}$ is an odd integer.	

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Be aware

In these proofs you will often work with specific kinds of numbers. You need to know that:

- $\mathbb{N}$ represents the set of natural numbers $\{0,\ 1,\ 2,\ 3,\ 4,\ \ldots\}$
- $\mathbb{Z}$ represents the set of integers $\{0,\ \pm 1,\ \pm 2,\ \pm 3, \ldots\}$
- $\mathbb{Q}$ represents the set of rational numbers
- $\mathbb{R}$ represents the set of real numbers

Notation such as $\mathbb{Z}^+$ means that you are only dealing with the positive numbers in this set.

Notation such as $a \in \mathbb{Z}$ means that a is in the set of integers (a is an integer).

Exam tip

Note that there is no universal agreement among mathematicians whether to include 0 in the set of natural numbers. The IBDP guide considers 0 to be a natural number, this is why it is included in the set $\mathbb{N}$ in the definition above.

Exam tip

An even integer can be written as $2n$ where $n \in \mathbb{Z}$.

An odd integer can be written as $2n + 1$ where $n \in \mathbb{Z}$.

Example 2

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Prove that the difference of a 3-digit number and that number written in reverse is always divisible by 3.

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Steps	Explanation
Let a be a 3-digit number written as $a = x_1x_2x_3$ where $0 \leq x_1, x_2, x_3 \leq 9 \text{ and} x_1 \neq 0.$	
a written in reverse is $b = x_3x_2x_1$	
$\Rightarrow a = 100x_1 + 10x_2 + x_3$ $b = 100x_3 + 10x_2 + x_1$	Rewriting a as a sum is helpful in this kind of question.
$\Rightarrow a - b = 100x_1 + 10x_2 + x_3 - 100x_3 - 10x_2 - x_1 = 99x_1 - 99x_3$.	
$3 \times 33\left(x_1 - x_3\right)$ is divisible by 3, so $a - b$ is divisible by 3.	

Exam tip

An IB exam question will often be structured so that the deductive proof is in the part of the question with the command term ‘Hence, or otherwise’. This command term tells you that you can do this question using the previous result or in some other way. So, while you are not required to use the previous result in the proof, it is probably the easiest place to start.

Example 3

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a) Show that $(2n - 1)(2n + 1) = 4n^2 - 1$, where $n \in \mathbb{Z}$.

b) Hence, or otherwise, prove that the product of two consecutive odd numbers is always odd.

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	Steps	Explanation
a)	$\text{LHS} = (2n-1)(2n+1) = 4n^2 - 2n + 2n - 1 = 4n^2 - 1$	To show equality always use LHS to RHS proof.
b)	Let $2n-1$ and $2n+1$ where $n \in \mathbb{Z}$ be two consecutive odd numbers.	Using the result fro part a.
	$\Rightarrow \quad (2n-1)(2n+1) = 4n^2 - 1$	
	$4n^2 - 1$ is odd since $4n^2 = 2\left(2n^2\right)$ and is therefore even.	

Example 4

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a) Show that $\frac{1}{7} + \frac{1}{9} = \frac{16}{63}$.

b) Show that $\frac{1}{2n-1} + \frac{1}{2n+1} = \frac{4n}{4n^2-1}$.

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	Steps	Explanation
a)	$\text{LHS} = \frac{9}{9} \times \frac{1}{7} + \frac{7}{7} \times \frac{1}{9} = \frac{9+7}{9 \times 7} = \frac{16}{63}$ $\text{LHS} = \text{RHS}$	To show equality alway use LHS to RHS proo
b)	$\text{LHS} = \frac{1}{2n-1} + \frac{1}{2n+1}$ $= \frac{2n+1}{2n+1} \times \frac{1}{2n-1} + \frac{2n-1}{2n-1} \times \frac{1}{2n+1}$ $= \frac{2n+1+2n-1}{(2n+1)(2n-1)} = \frac{4n}{4n^2-1}$ $\text{LHS} = \text{RHS}$	To show equality alway use LHS to RHS proo

3 section questions

1. Number and algebra / 1.6 Deduction

Checklist

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What you should know

By the end of this subtopic, you should be able to

- distinguish between an equation and an identity
- use algebraic manipulations to construct a LHS to RHS proof
- recognise a conditional statement
- construct a deductive proof
- recognise and interpret set notation for number systems:
  - $\mathbb{N}$ represents the set of natural numbers $\{0,\ 1,\ 2,\ 3,\ 4,\ \ldots\}$
  - $\mathbb{Z}$ represents the set of integers $\{0,\ \pm 1,\ \pm 2,\ \pm 3, \ldots\}$
  - $\mathbb{Q}$ represents the set of rational numbers

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view

- $\mathbb{R}$ represents the set of real numbers.

Investigation

Investigation 1

Part 1

Use the applet below to generate a variety of sums of fractions. Note down your observations, paying attention to any patterns. If you don’t see any patterns right away, generate more examples.

Deduce a general rule for any pattern you observe. Decide if this rule is an identity.

Use an appropriate strategy to prove any identities that you formulate.

Interactive 1. Sum of Fractions.

More information for interactive 1

This interactive enables users to explore the concept of fraction addition by generating random sums and observing patterns. Users can click the "New Sum" button to create different fraction addition problems, allowing them to practice and analyze the results. By recording their observations, they can identify recurring patterns and deduce general rules. If a pattern holds true for all cases, it may represent a mathematical identity, which can then be proven using algebraic techniques or other appropriate strategies.

For example, consider the sum $\left(-\frac{1}{3}+\frac{1}{12}\right) = -\frac{1}{4}$. Here, the denominators (3 and 12) share a relationship, as 12 is a multiple of 3. Simplifying the fractions to a common denominator (12) yields $\left(-\frac{4}{12}+\frac{1}{12}\right) = -\frac{3}{12}$, which reduces to $-\frac{1}{4}$. This suggests a pattern where the sum of certain fractions with related denominators simplifies neatly.

This tool provides a hands-on way to explore fraction arithmetic, recognize patterns, and verify mathematical rules through repeated practice and analysis.

Part 2

Generate your own identities for sums and differences of fractions. How many identities are possible? Do you think that all of them can be found by looking at patterns like you did in the first activity? Justify your thinking.

Investigation 2

Watch the video below to see a deductive proof that all triangles are equilateral. You know from experience that this is not true. Try to find the incorrect step in the deductive logic. Consider using software to draw a diagram that corresponds to the situation presented in the proof to visualise what is happening.

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All Triangles are Equilateral - Numberphile

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