TOPIC 3
GEOMETRY AND TRIGONOMETRY

SUBTOPIC 3.1
THREE-DIMENSIONAL SPACE

Teacher view

Index

The big picture

Geometric figures and their transformations are widely used in design and architecture. Apart from engineering calculations of buildings, architects use geometry to define the spatial form of a building, to create harmonious forms and also to create environmentally conscious buildings. The Petronas Twin Towers in Kuala Lumpur, Malaysia, are among the many skyscrapers built in recent history. The design of the towers starts with a square where the vertices of the square represent unity, harmony, stability and rationality. This square is rotated through 90° and small circular infills complete the floor plate of the towers, as seen in the photograph. The next stage of the design moves from 2D to 3D using the base plate, which changes in size as the tower is built upwards. From a distance, the towers give the illusion of cylinders at the bottom and cones or pyramids on the upper floors.

How many square metres of glass do you think are used in these towers? What is the total volume of the structure, and how many air conditioning units are needed to cool the building in the hot tropical weather?

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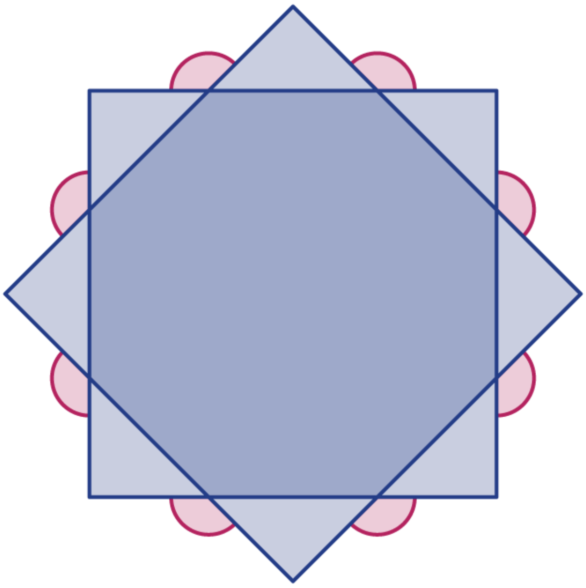
To help answer these types of question, you will be learning about 3D shapes and forms, and their relation to 2D objects.

The Petronas Twin Towers, Kuala Lumpur, Malaysia

Credit: Nikada Getty Images



Concept

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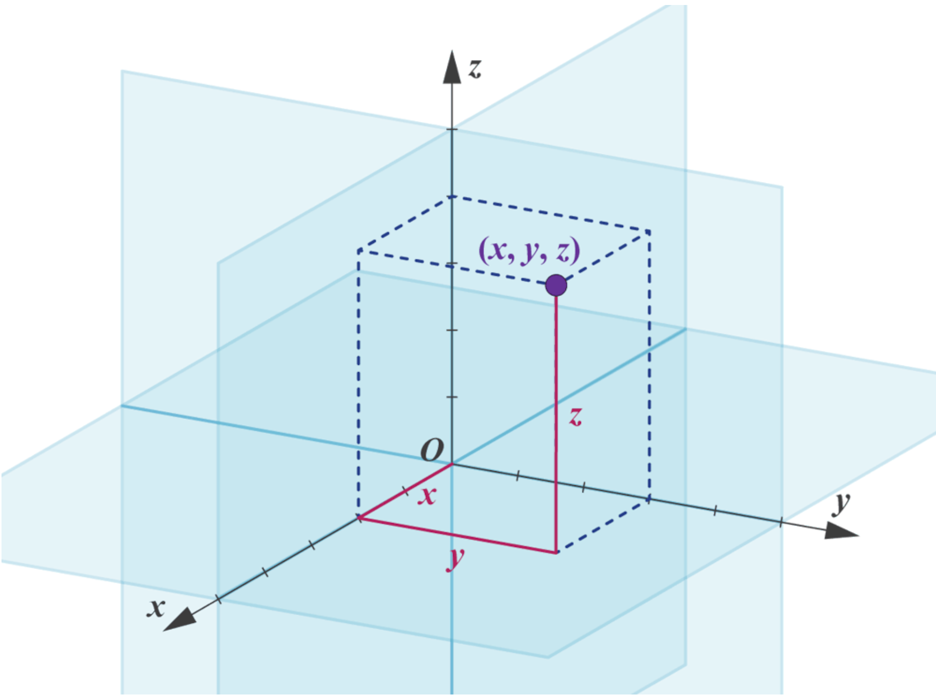
3D shapes such as the right pyramid, cone and sphere are widely used in design and architecture. Identifying elements such as faces, edges and vertices helps you to calculate surface area and volume which define the space occupied, and you can use modelling to create new designs.

The 3D coordinate system

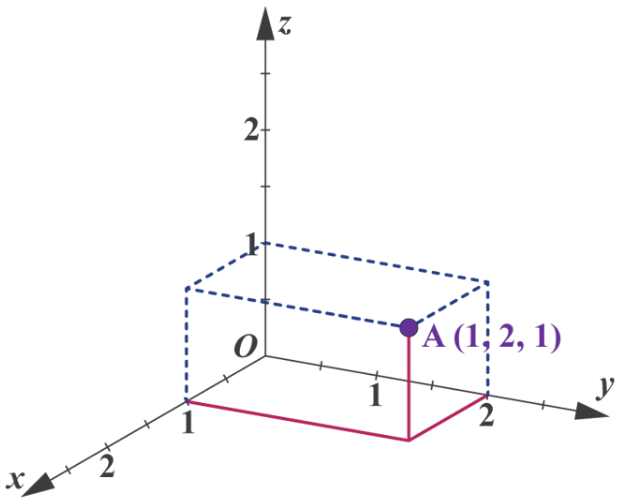
Plotting points in 3D

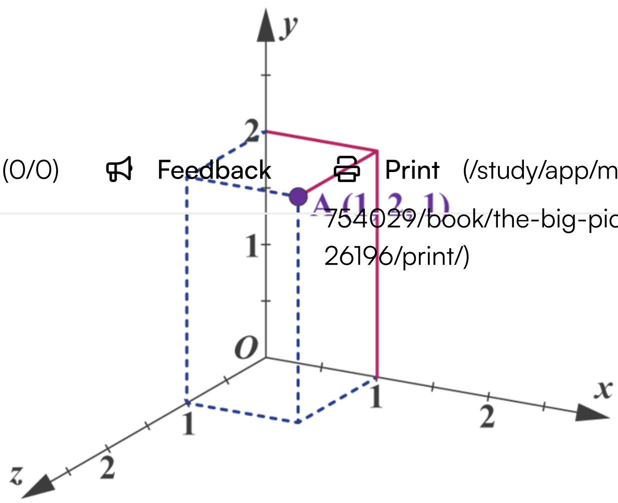
The Cartesian coordinate system consists of three perpendicular planes intersecting at a point called 'the origin'. Similar to the 2D coordinate plane, each point is defined by its perpendicular distance from each plane, as shown in the diagram below. By convention, you use (x, y, z), always in the same order, to represent the coordinates of a point in 3D.



More information

As it is a perpendicular system, you can think of the coordinates of a point as a vertex of a cuboid, where one of the vertices is at the origin, and three of the edges are on each axis. In the diagram below, point A (1,2,1) is the vertex of a cuboid with edges of lengths 1, 2 and 1 drawn on the x, y and z axes, respectively.





More information

Important

You use (x, y, z) to represent the coordinates of a point in 3D space.

Example 1

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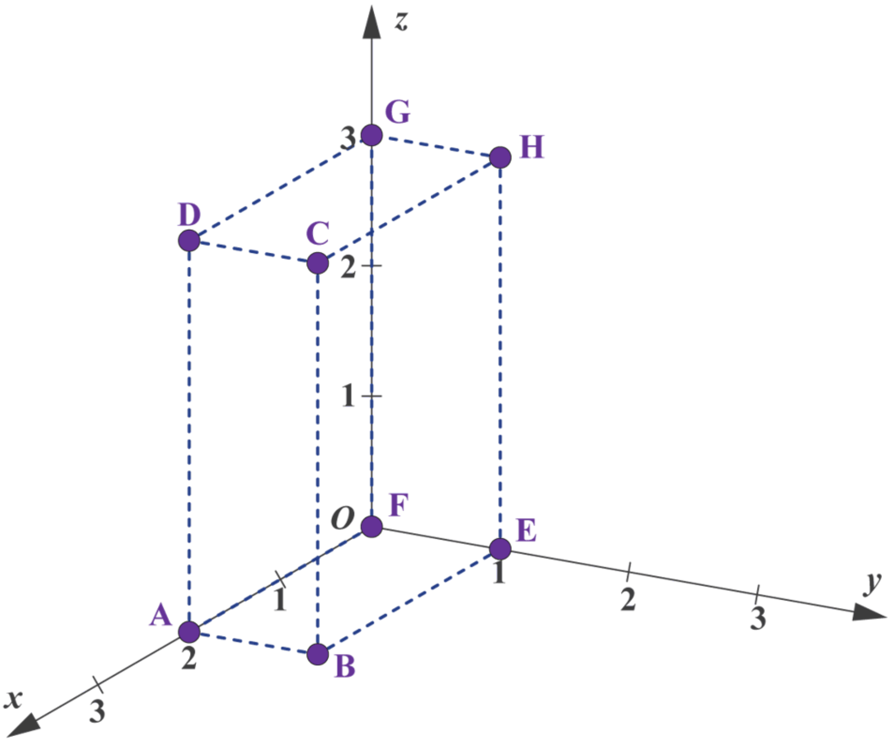

More information

The image depicts a cuboid drawn in a 3D coordinate system. The cuboid is displayed with vertices labeled as A, B, C, D, E, F, G, and H. The coordinate axes are labeled x, y, and z. The axis lines have scale markings: the x-axis extends from 0 to 3, the y-axis from 0 to 3, and the z-axis from 0 to 3, each marked at unit intervals. The vertices are connected by dotted lines, forming a rectangular prism. The cuboid sits on the base formed by points A, B, E, and F with its top formed by points C, D, G, and H. Each axis appears to define the space in which the cuboid is centered, with the point O at the origin.

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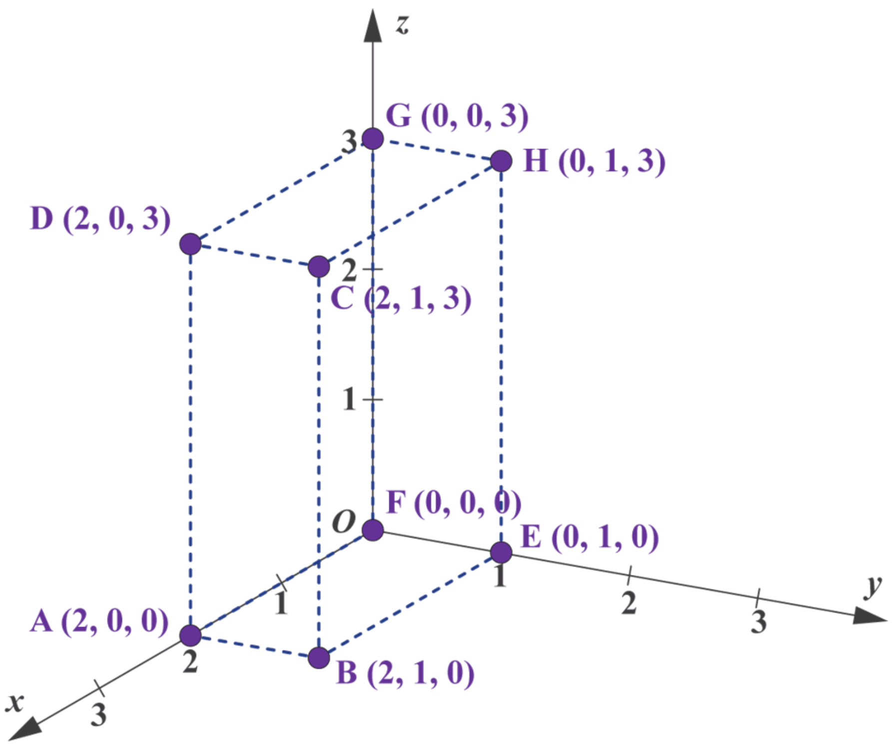
The figure above shows a cuboid drawn in the 3D coordinate system.

a) Write down the letters with coordinates (0,0,0) and (2,0,0) respectively.

b) Write down the coordinates of each of the vertices of the cuboid.

a) F (0,0,0) and A (2,0,0).

b) Using (x, y, z) as $(\text{length}, \text{ width}, \text{ height})$ gives B (2,1,0), C (2,1,3), D (2,0,3), E (0,1,0), G (0,0,3) and H (0,1,3).



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Midpoint of a line segment

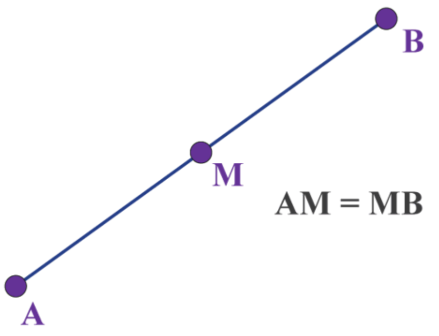
Similar to the midpoint formula in the 2D Cartesian plane, the midpoint of a line segment in 3D is the average of the coordinates of the endpoints.

Exam tip

The formula booklet gives you the following formula:

The coordinates of the midpoint of a line segment with endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by

$$\mathrm{M}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$



More information

The image is a diagram showing a line segment with three labeled points: A, M, and B, displaying a collinear relationship. Each point is marked by a dot on the line. The text AM = MB indicates that point M is equidistant from points A and B, making M the midpoint of the line segment AB. The diagram visually represents the concept of a midpoint in geometry.

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Example 2

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If the endpoints of a line segment are $\text{A}\ (1, -1, 2)$ and $\text{B}\ (2, -2, 3)$ find the coordinates of the midpoint of the segment AB.

Steps	Explanation
Use	$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$
which gives	$M\left(\frac{1+2}{2}, \frac{-1+(-2)}{2}, \frac{2+3}{2}\right)$ $M\left(\frac{3}{2}, -\frac{3}{2}, \frac{5}{2}\right)$
So the midpoint has coordinates	$\text{M}\left(\frac{3}{2}, -\frac{3}{2}, \frac{5}{2}\right)$

Example 3

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The coordinates of the two of vertices of a parallelogram ABCD are $\text{A}\ (2, 0, 0)$ and $\text{B}\ (0, 0, 2)$.

If the intersection of diagonals of the parallelogram is $\text{M}\ (0, 2, 0)$, find the coordinates of the other vertices.

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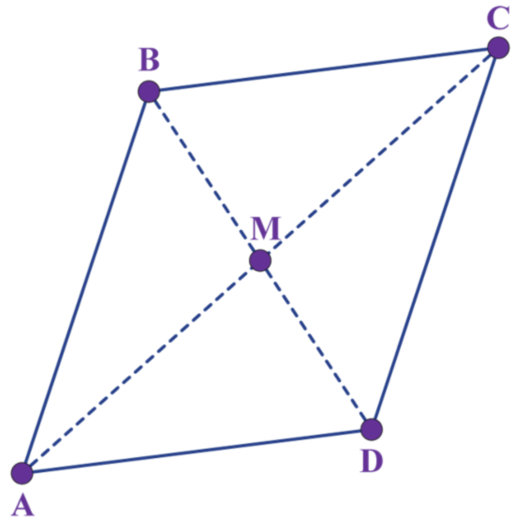

More information

This is a diagram of a parallelogram. It has four vertices labeled as A, B, C, and D. The point M, labeled at the center, represents the intersection of the diagonals of the parallelogram. The diagonals are shown as dashed lines, connecting vertex A to C, and B to D. The solid lines connect the vertices to form the parallelogram shape. The diagram seems to use a coordinate system with midpoint M at (0, 2, 0), although specific coordinates for other points are not shown in the image.

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M $(0, 2, 0)$ is the midpoint of AC. A has coordinates $(2, 0, 0)$, and C has coordinates (x_1, y_1, z_1), so,

$$0 = \frac{2 + x_1}{2} \Rightarrow x_1 = -2$$

$$2 = \frac{0 + y_1}{2} \Rightarrow y_1 = 4$$

$$0 = \frac{0 + z_1}{2} \Rightarrow z_1 = 0.$$

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Therefore C has coordinates $(-2, 4, 0)$.

As $\text{M}(0, 2, 0)$ is also the midpoint of BD, where the endpoints are $\text{B}(0, 0, 2)$ and $\text{D}(x_2, y_2, z_2)$,

$$0 = \frac{0 + x_2}{2} \Rightarrow x_2 = 0$$

$$2 = \frac{0 + y_2}{2} \Rightarrow y_2 = 4$$

$$0 = \frac{2 + z_2}{2} \Rightarrow z_2 = -2$$

So D has coordinates $(0, 4, -2)$.

Distance between two points

The distance between $\text{A}(x_1, y_1, z_1)$ and $\text{B}(x_2, y_2, z_2)$ could be found using Pythagoras' theorem, as shown in the diagram below. First, using right-angled triangle AKC, you can find the length of AC, and then using triangle ACB you can find the length of AB.

$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

or

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

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Exam tip

The formula booklet gives you the following formula:

The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

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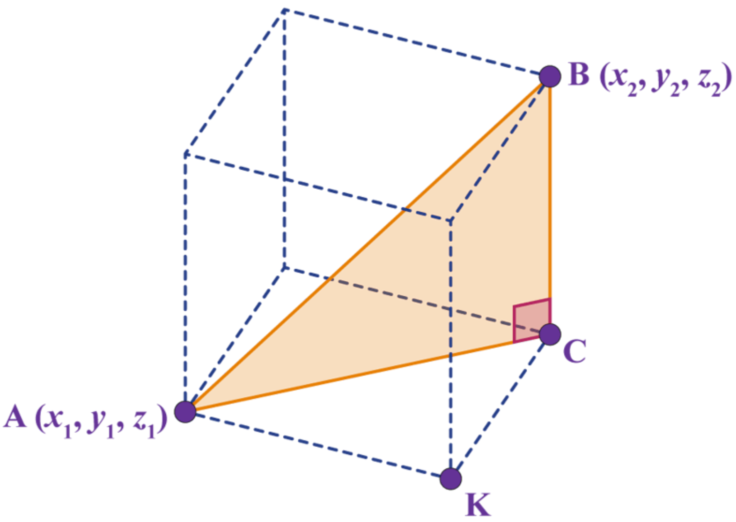
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$



More information

The image is a 3D geometric diagram showing a cube with a triangular plane inside, connecting points A(x1, y1, z1), B(x2, y2, z2), C, and K. Point A is at the front bottom left and Point B is at the top right corner of the cube. Lines connecting these points form an orange triangular plane, marked as ABC. Point K is on the base of the cube under point B, and Point C is located on the bottom right corner of the face near the observer. The lines forming the cube and triangle are dashed, while the edges of the triangle are solid, highlighting its shape inside the cube. A right angle is indicated at point C, suggesting a perpendicular relationship with the base of the cube.

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Example 4

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a) Find the exact distance between points A (−1,2,3) and B (−1,−2,4).

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b) Round your answer in (a) to 3 significant figures.

$$AB = \sqrt{(-1-(-1))^2 + (2-(-2))^2 + (3-4)^2}$$

a) $AB = \sqrt{17}$ (exact)

b) $AB = 4.12$ (3 significant figures)

Exam tip

Pay attention to the accuracy that the question asks for. An exact answer should be left in surd form or as a multiple of π .

For example, $\frac{\pi}{2}$ is an exact value but when rounded to 3 significant figures it is 1.57.

Example 5

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A remote-controlled drone is 500 m north and 200 m east of its operator, and it is 20 m high. It is travelling due north at a constant height at a speed of 2 km h $^{-1}$.

a) What will be its coordinates to nearest metre after 30 minutes?

b) At this time, how far will the drone be from the operator? Give your answer to the nearest metre.

a) Initial coordinates of the drone (north, east, height) are (500,200,20) 30 min = 0.5 hour so the distance it will travel is given by

$$\text{distance} = \text{speed} \times \text{time} = 0.5 \times 2 = 1 \text{ km} = 1000 \text{ m.}$$

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As it is travelling due north the new coordinates will be (1500,200,20).

b) The operator is at the origin, at coordinates (0,0,0) and the new coordinates of the drone are (1500,200,20).

Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$$d = \sqrt{1500^2 + 200^2 + 20^2} = 1513 \text{ m to nearest metre.}$$

4 section questions

3. Geometry and trigonometry / 3.1 Three-dimensional space

The right pyramid

Definition

A pyramid is a polyhedron formed by connecting a polygonal base to its apex . Each triangle formed by the edge of the base and the apex is called a lateral face .

If the apex of the pyramid is directly above the centre of the base, then it is called a right pyramid .

The diagram shows a right pyramid with a rectangular base ABCD. Point E is its apex and G is the centre of the base. Line segment EG is perpendicular to the base of the pyramid.

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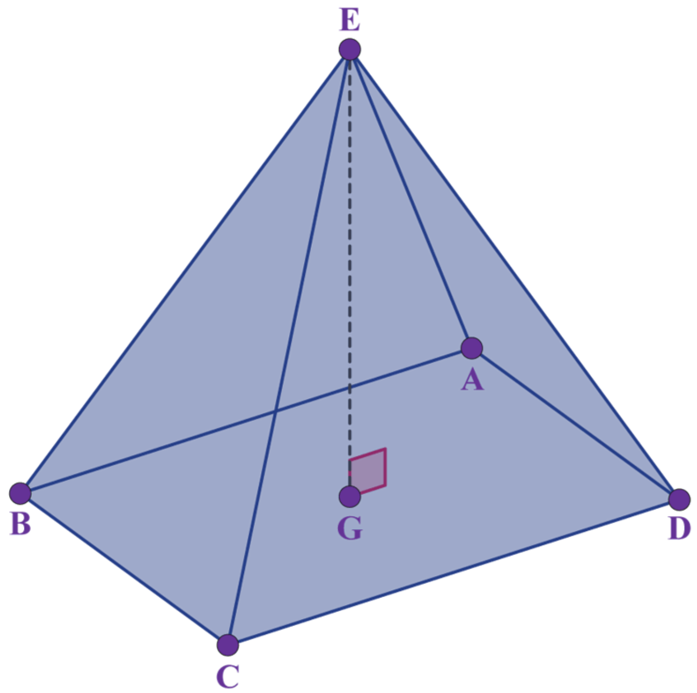

More information

The diagram features a right pyramid with a rectangular base labeled ABCD. Point E is the apex of the pyramid, and point G is the center of the rectangular base. The line segment EG is perpendicular to the base, indicating that E is directly above G. The four corners of the rectangular base are labeled B, C, A, and D in a clockwise order. The pyramid's apex and base are connected by triangular faces.

The base, ABCD, is shown as a parallelogram to represent the three-dimensional shape of the base when viewed from an angle. The vertical segment EG represents the height of the pyramid, and a right angle is marked at point G to highlight the perpendicular relationship between the line segment EG and the base ABCD.

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The base of a pyramid can have three or more sides. On the next diagram, you can see the triangular based pyramid which is also known as a tetrahedron.

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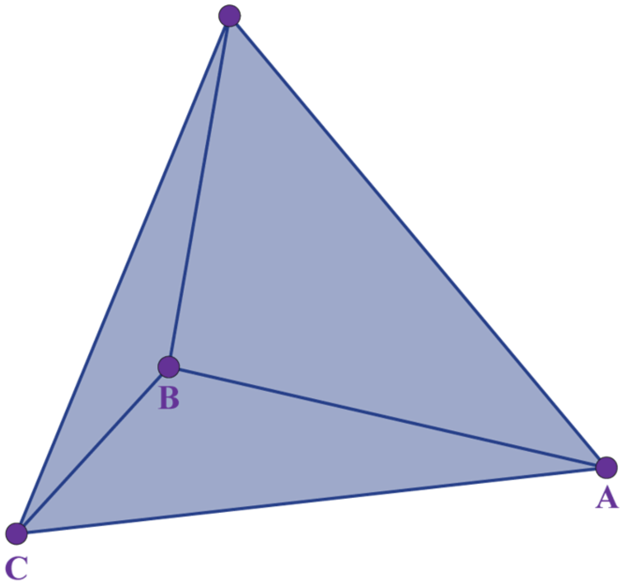

More information

The image is a diagram representing a triangular based pyramid also known as a tetrahedron. The tetrahedron is composed of a triangle with three vertices labeled A, B, and C. Lines connect these three points, forming the triangular base of the pyramid. There are no additional labels or text within the triangle. The diagram highlights the structure and shape of the tetrahedron as described in the accompanying text, emphasizing the triangular nature of the base and the overall pyramid-like structure of the shape.

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The diagram above shows a right pyramid with a triangle as the base (tetrahedron).

In right pyramids, the apex is exactly above the centre of the base. The illustration in the **Important** box shows a regular hexagonal based pyramid.

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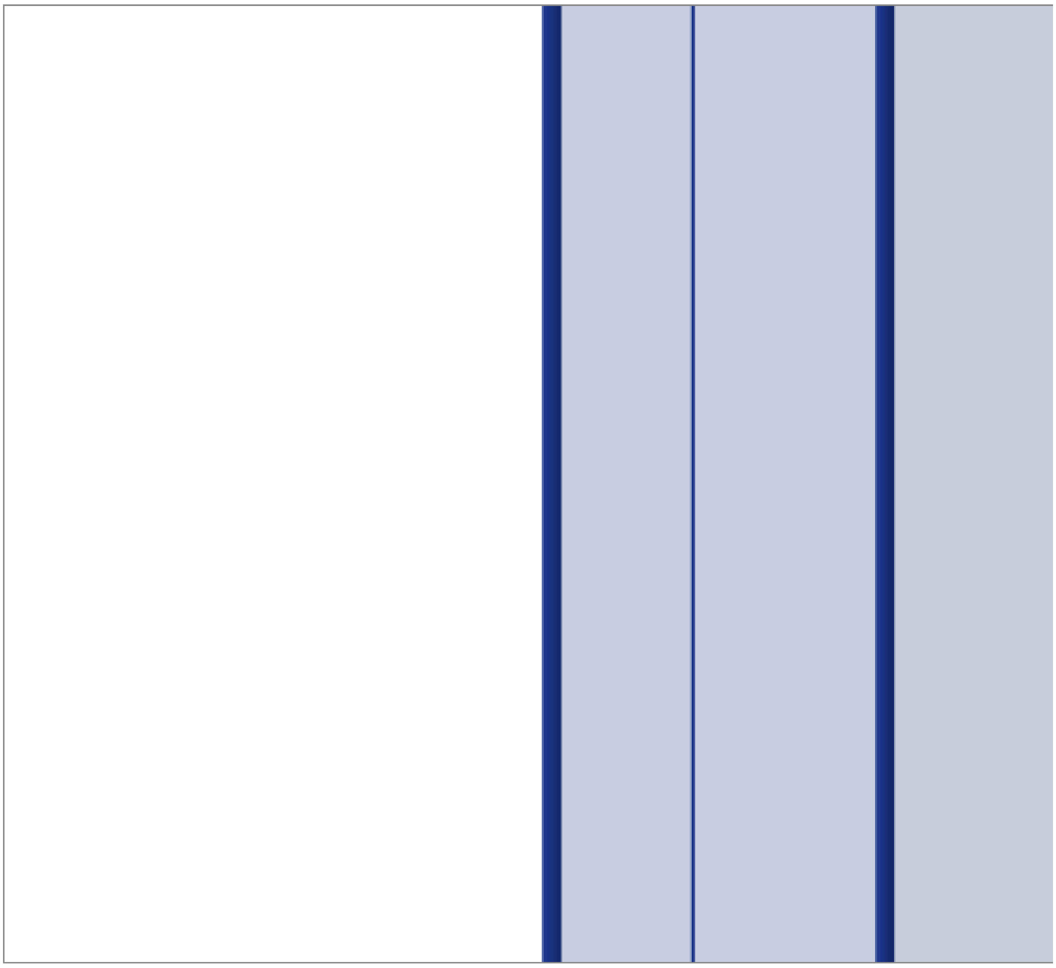
Important

- The centre of the base is the intersection of diagonals, G .
- Length GK is the perpendicular height of the pyramid, h.

- Point H is the midpoint of edge ED.
- Connecting K to H gives the height of triangle KDE.
- The height of face KDE is the slant height of the pyramid, l.
- Connecting centre G to point H gives the height of the triangle formed by points GED; let $\mathrm{GH} = b$.
- Triangle KGH is a right-angled triangle.

You need to be able to spot right-angled triangles to be able to calculate information about the pyramid.



Interactive 3. Net of a Square Based Pyramid.

Credit: GeoGebra (https://www.geogebra.org/m/Aur2JZd5) Lee W Fisher

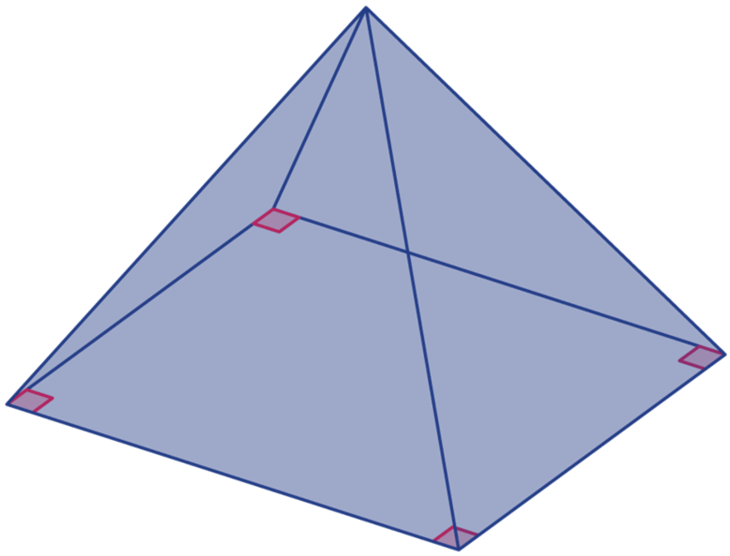
More information for interactive 3

The diagram illustrates a right pyramid with a regular hexagonal base. In the right pyramid the apex K is positioned directly above the center G of the base. The perpendicular height h of the pyramid is represented by the line segment GK, which is perpendicular to the base. Point H is the midpoint of edge ED, and connecting K to H gives the slant height ‘l‘of the pyramid, which is the height of the triangular face KDE.

Connecting the center G to point H provides the height b of the triangle formed by points GED . The right-angled triangle KGH is essential for calculating various properties of the pyramid. By identifying and analyzing right-angled triangles within the pyramid, such as KGH, one can determine important measurements like the perpendicular height, slant height, and other geometric properties. This visualization helps in understanding the structural relationships and performing calculations related to the pyramid's dimensions.

Surface area of a right pyramid

The net of a pyramid includes a base of n sides and n triangular faces, one on each edge of the base.



More information

The image shows the net of a square-based pyramid. It consists of a square in the center, representing the base, with a triangle attached to each of the square's sides. The net forms a symmetric arrangement, where the triangles extend outward from each side of the square. This structure can be folded along the edges to form a three-dimensional square pyramid, where the triangles become the pyramid's lateral faces.

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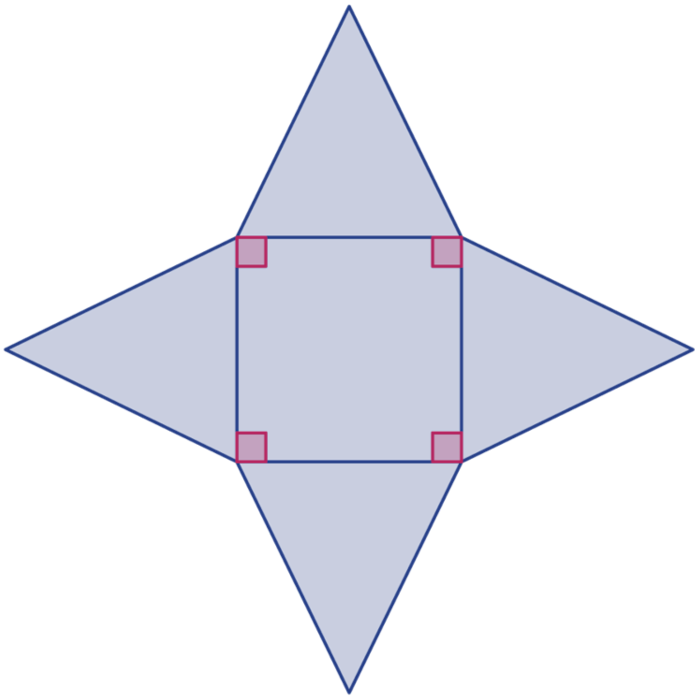
The square-based pyramid above has a base with four sides. Therefore, the net is made up of four triangles and a square base.

The square-based pyramid has five faces in total, shown below.



More information

The image is a diagram showing the net of a square-based pyramid. It appears as a diamond shape with four congruent triangles forming the sides, arranged around a central square. Each triangle is connected to one side of the square, creating a cross-like pattern. The four triangular faces are meant to fold up from the center square to form the pyramid's shape, with the square acting as the base. Small squares are marked near each connecting edge, possibly indicating tabs or connection points for assembly. This diagram illustrates how the two-dimensional net of a pyramid can be folded to form a three-dimensional shape.

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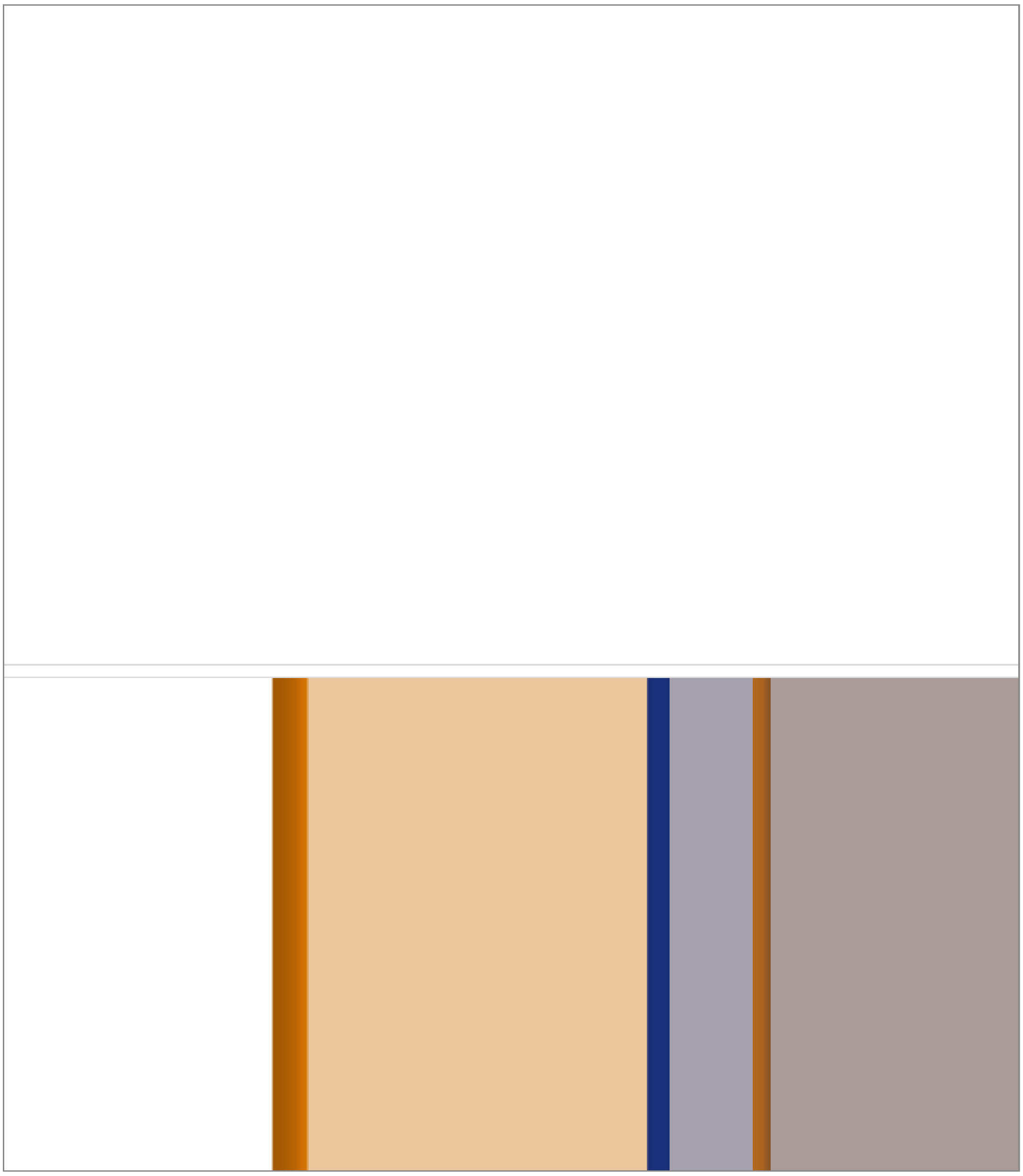
An applet for this net can be found below.



interactive 1. Surface Area of a Right Pyramid.

More information for interactive 1

This interactive visualization demonstrates how to find the surface area of a right pyramid. By adjusting the slider ‘c’, users can observe the transition between the 2D net and the 3D pyramid. As c moves from 0 to 1, the pyramid's faces gradually unfold. At $c = 0$, the pyramid is fully constructed. As c increases, the triangular faces lift outward, revealing the 2D net. At $c = 1$, the net is completely unfolded, showing the square base and four separate triangular faces.

This helps in understanding how the total surface area is composed. As the slider moves to lower values, the triangular faces gradually fold upward, forming the pyramid. At the lowest position of the slider, the

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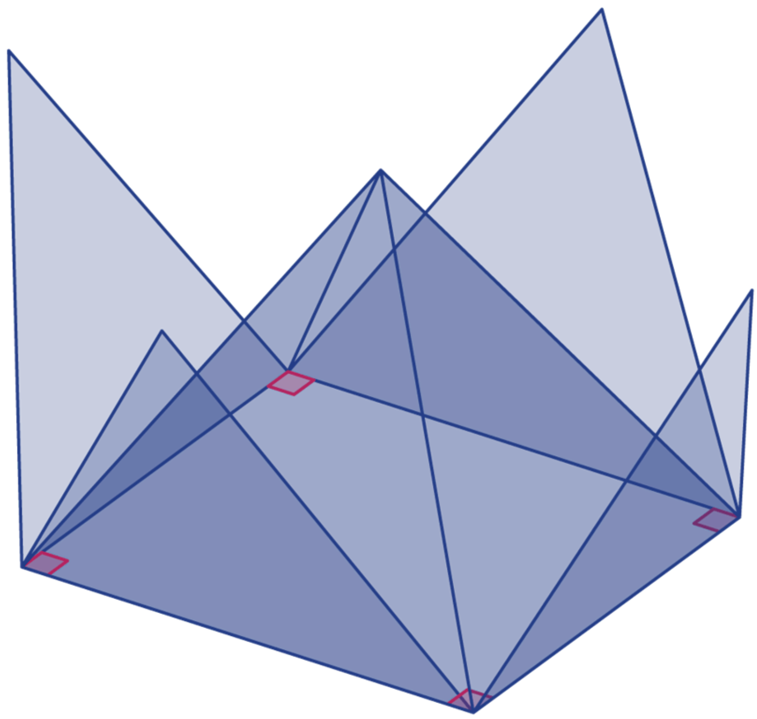
pyramid is fully constructed, showing how the lateral faces meet at the apex. This interactive approach allows users to explore how the surface area is determined by adding the base area and the combined area of the triangular faces.

The final diagram shows the net partially folded.



More information

A partially folded net diagram is shown. The diagram features a series of diamond shapes, each outlined in red, connected at intervals along the interior. The net folds upwards, creating a series of angular peaks and valleys, suggesting a three-dimensional structure at a midway folding stage. The diamonds likely denote connection points or folds critical for understanding the folding process. Additional text or labeling is not present in the image, focusing on the structural representation.

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Important

Total surface area of a right pyramid is the sum of the areas of the triangular faces and the base area.

Example 1

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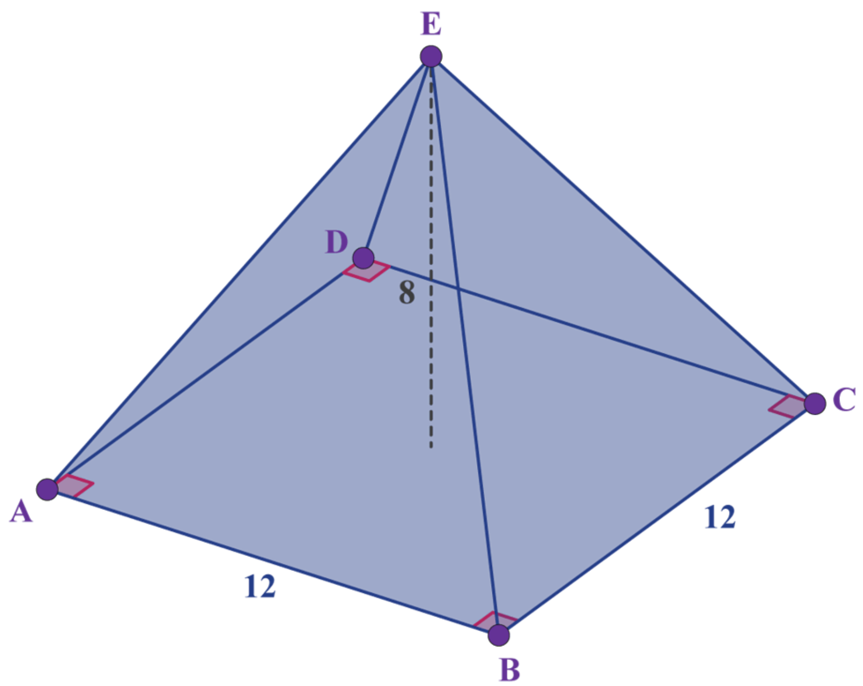
Pyramid ABCDE is square-based, as shown below. The length of the side of the square is 12 cm. The perpendicular height of the pyramid is 8 cm. Find the surface area of the pyramid.



More information

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The image displays a square-based pyramid labeled as pyramid ABCDE. The base of the pyramid, ABCD, is a square with each side measuring 12 cm. The apex of the pyramid is point E, which is vertically above the center of the square base. The perpendicular height from the apex E to the base is indicated at point D and measures 8 cm. The vertices of the square are labeled as A, B, C, and D, each marked with a small circle at these points. Angles at the base are marked as right angles, emphasizing the geometric structure. The image is titled "Pyramid ABCDE" and illustrated in a purple and blue gradient tone.

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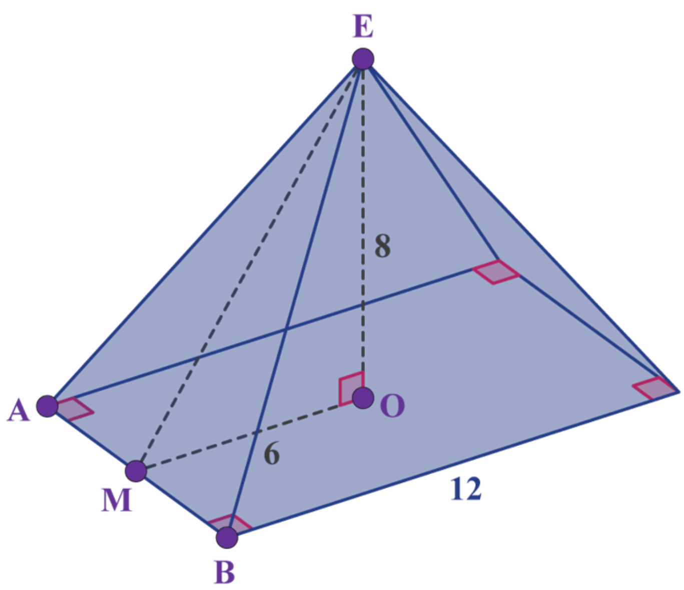
Examine the figure to see what information you have been given for triangle ABE.



You know that the length $AB = 12$ cm since the base of the pyramid is a square.

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The dimension 8 cm is not on triangle ABE since 8 is the distance OE – the perpendicular height of the pyramid.

Introduce points O (the centre of the base) and M (the midpoint of the edge AB).

The length EM is the height of triangle ABE. It is also the hypotenuse of triangle MOE.

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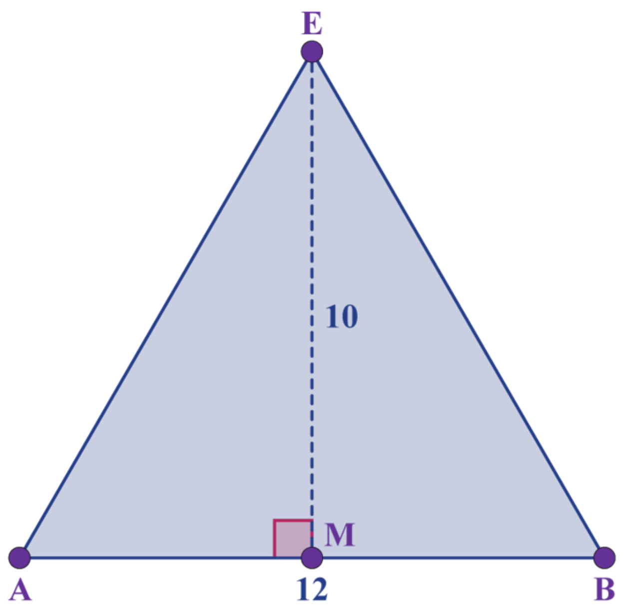
$$AM = \frac{1}{2} \text{ of } AB \text{, so } AM = 6 \text{ cm.}$$

Therefore, using Pythagoras' theorem,

$$EM = \sqrt{6^2 + 8^2} = 10.$$

The diagram shows triangle AEB.



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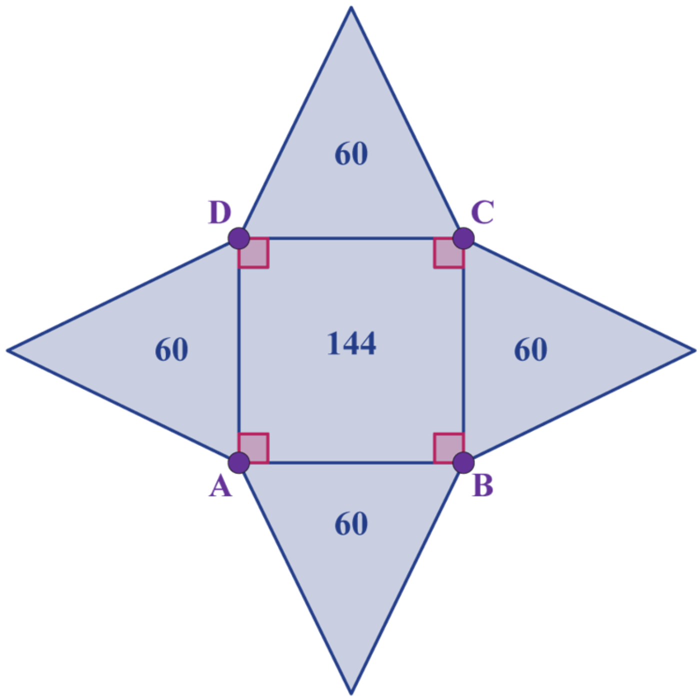
The area of the face ABE is

$$\text{Area}_{\Delta\text{ABE}} = \frac{1}{2}bh = \frac{1}{2} \times 12 \times 10 = 60\,\text{cm}^2.$$

The area of the base of the pyramid = $12 \times 12 = 144\,\text{cm}^2$.



Therefore, the total surface area is $4 \times 60 + 144 = 384\,\text{cm}^2$.

Volume of a right pyramid

Important

The volume of a pyramid is $\frac{1}{3}$ of the volume of the prism with the same height and base. Therefore, the formula for the volume of the pyramid is

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$$V = \frac{1}{3}\text{ (base area)} \times \text{height}$$

You can watch a demonstration of finding the volume of a square-based right pyramid in the following video.

Exam tip

The formula booklet gives the following formula:

Volume of a right pyramid $V = \frac{1}{3}A \times h$, where A is the area of the base, h is the height.

Example 2

★☆☆

Pyramid ABCDE is square based, as shown in the diagram below. The length of the side of the square is 12 cm. The perpendicular height of the pyramid is 8 cm.

Find the volume of the pyramid.

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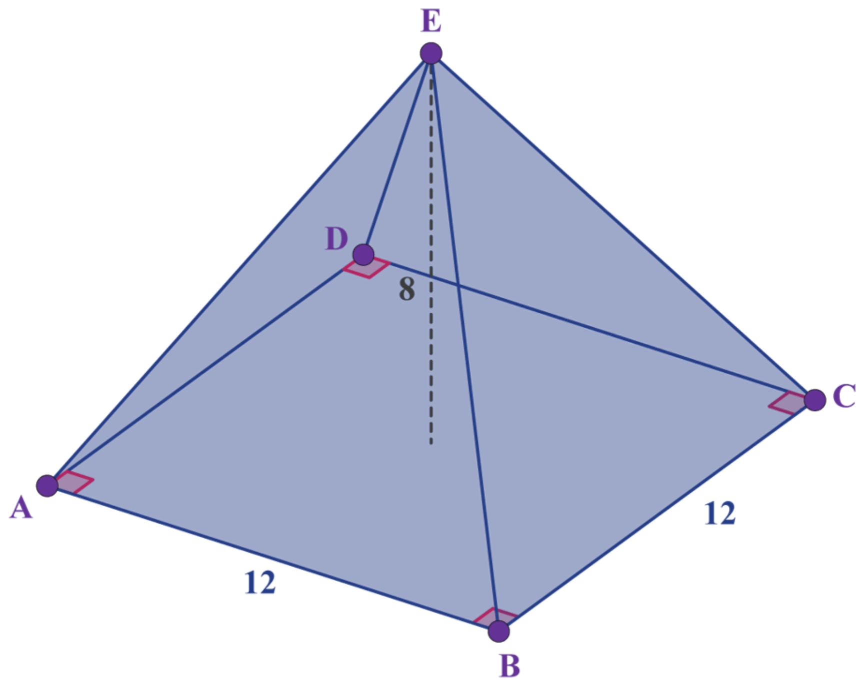

More information

The image depicts a geometric diagram of a pyramid with a square base labeled ABCD. The vertices are labeled A, B, C, D, and the apex E. The base ABCD is a square with sides measuring 12 units each, as indicated by labels on the side AB and side BC. The height of the pyramid is represented by a dotted line from the apex E to a point D on the base ABCD, marked as 8 units. Angles at vertices A, B, and C appear to be right angles, indicated by small squares.

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To find the volume of the pyramid, use the formula

$$V = \frac{1}{3}Ah,$$

where A is the area of the base and h is the perpendicular height from the base to the apex (vertex E).

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The base is a square. Therefore, $A = 12 \times 12 = 144.$

$$V = \frac{1}{3} \times 144 \times 8 = 384$$

So the volume of the pyramid is 384 cm ^{3}.

3 section questions

3. Geometry and trigonometry / 3.1 Three-dimensional space

The cone

Definition

A cone is a solid with a circular base and an apex. It has two faces, the circular base and a sector of a circle which has the circumference of the base as its arc.

The diagram shows a cone with base radius *r* and perpendicular height *h* along with its net. The length $\mathrm{AB} = l$ is called the slant height of the cone. AGB is a right triangle. The circumference of the base of the circle is equal to the length of the arc BE.

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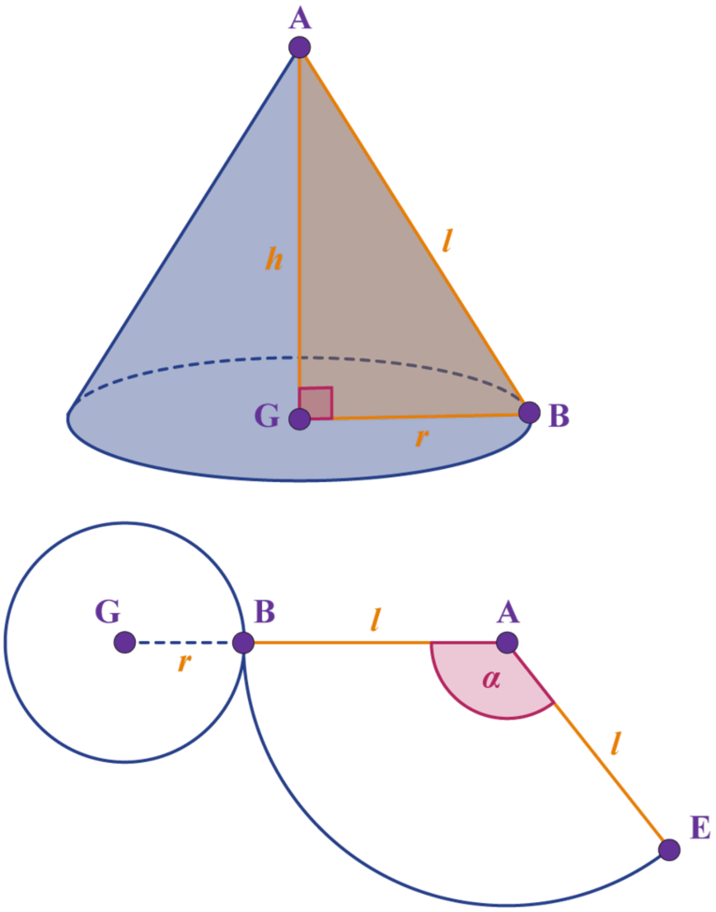

More information

The diagram illustrates a three-dimensional cone with a circular base. The cone is shown alongside its net, which is used to represent the cone as a two-dimensional shape. The cone itself has a base with radius 'r' and a perpendicular height 'h', with its apex labeled as point A, the base center as point G, and the base perimeter intersecting at point B. The slant height of the cone, denoted by 'l', is the hypotenuse of the right triangle AGB.

The net of the cone is displayed below the cone. It consists of a circle with radius 'r', centered at point G, representing the base of the cone, and an adjoining circular sector that forms the lateral surface of the cone when folded. The sector has a radius equal to the slant height 'l' of the cone. The arc BE of this sector is equal in length to the circumference of the cone's base. The labeling on the diagram includes the points A, B, G, and E, and the values for radius 'r', slant height 'l', and perpendicular height 'h'. This illustrates the geometric relationship between the physical cone and its geometric representation as a net.

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Area of the curved surface of a cone

Exam tip

The formula booklet gives you the following formula:

Area of the curved surface of a cone $A = \pi r l$, where r is the radius, l is the slant height.

Example 1

★★☆

A piping nozzle

turbojet Getty Images

The photograph shows a piping nozzle, which is used to apply icing onto cakes in a decorative manner. It is the curved surface of a cone. There is no base.

A piping nozzle has the following dimensions. The radius of the base is 15 mm and the radius of the top is 3 mm. It is formed by cutting a small cone from the top of a larger cone. The perpendicular height of the large cone is 36 mm. The

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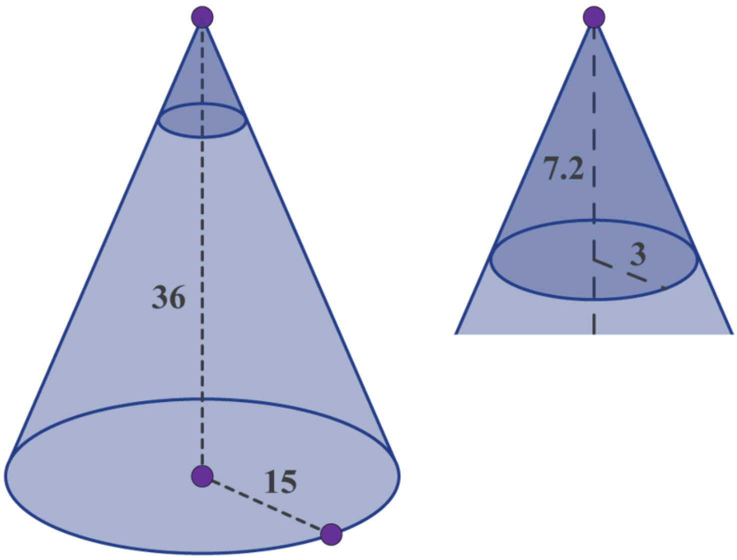
perpendicular height of the cone that is removed from the top is 7.2 mm.

Find the surface area of the piping nozzle, correct to 3 significant figures.



Steps	Explanation
Call the curved surface area of the larger cone A_1, the curved surface area of the smaller cone A_2 and the area required A_{nozzle}.	You are only interested in surface area, as the pipin does not have a base. You need to find the curv area of the larger cone, th the curved surface area o cone.
Using Pythagoras' theorem $l = \sqrt{15^2 + 36^2} = 39$.	From the formula booklet surface has area $A = \pi rl$ the value of h but not l. U Pythagoras' theorem to fi

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Steps	Explanation
$A_1 = \pi \times 15 \times 39 = 1837.83\ldots \text{mm}^2$	Use the formula $A = \pi rl$
The slant height of the smaller cone is $l = \sqrt{3^2 + 7.2^2} = 7.8$.	Use Pythagoras' theorem
So $A_2 = \pi \times 3 \times 7.8 = 73.51\ldots \text{mm}^2$.	Use $A = \pi rl$.
$A_{\text{nozzle}} = 1837.83 - 73.51\ldots$ $= 1764.31\ldots$ $= 1760$ (3 significant figures)	Use $A_{\text{nozzle}} = A_1 - A_2$.
So the surface area of the piping nozzle is $1760\,\text{mm}^2$ to 3 significant figures.	Remember to give the un answer.

Volume of a right cone

Finding the volume of a cone is similar to the volume of a pyramid.

$$V = \frac{1}{3}\,(\text{base area}) \times \text{height}.$$

In this case, the base area is a circle $A = \pi r^2$. Therefore, $V = \dfrac{1}{3}\pi r^2 h$.

Exam tip

The formula booklet gives you the following formula:

Volume of a right cone $V = \dfrac{1}{3}\pi r^2 h$, where *r* is the radius and *h* is the height.

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Example 2

★☆☆

Find the exact volume of the cone with base radius 3 cm and perpendicular height 7 cm .

Using the volume formula $V = \frac{1}{3}\pi r^2 h$

$$\begin{aligned} V &= \frac{1}{3}\pi(3)^2 7 \\ &= 21\pi \end{aligned}$$

So the volume is $V = 21\pi$ cm ^{3}.

Exam tip

When the question is asking for an exact value, and π is involved in the calculation, leave your answer in terms of π. Do not round your answer, and make sure to include units.

2 section questions

3. Geometry and trigonometry / 3.1 Three-dimensional space

The sphere and hemisphere

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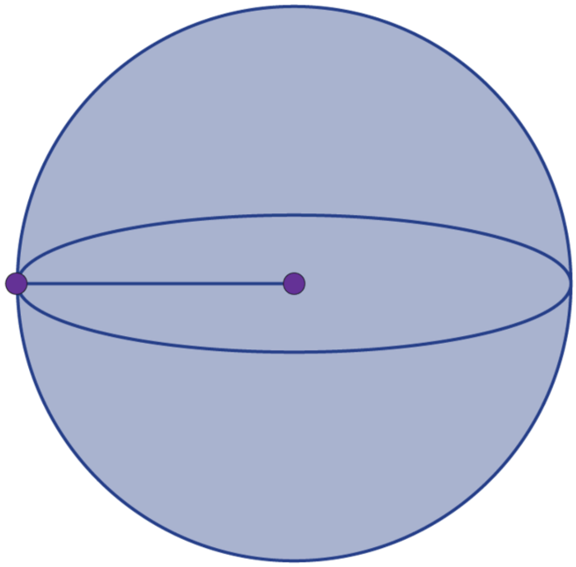

More information

The image is a Venn diagram consisting of two overlapping circles. The larger circle on the right is blue, representing the superset of items labeled as "universal set." Inside this circle, near the center, is a smaller purple circle labeled "B," likely representing a subset. On the left, partially overlapping the blue circle, is another smaller purple circle labeled "A," representing another subset. The overlap area between "A" and the blue section indicates the common elements shared by sets "A" and "B." This Venn diagram visually depicts the relationships between different sets or groups, facilitating understanding of how the subsets intersect with the universal set.

[Generated by AI]

International Mindedness

‘When a semicircle with fixed diameter is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a sphere.’

This is how Euclid defined a sphere in his *Elements, Book 11 Definition 14*. It was written c. 300 BCE. *The Elements* was one of the very first text books. People used it to study mathematics for more than 2000 years!

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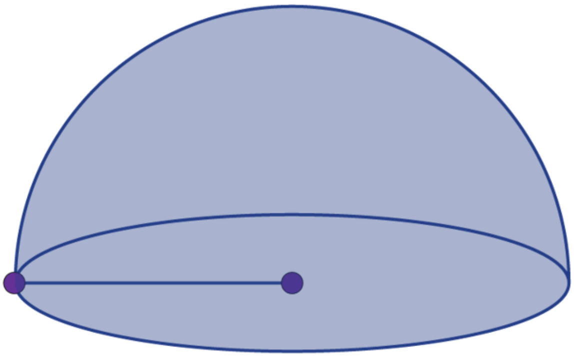
Hemisphere

A hemisphere is one half of a sphere. In world affairs, you often refer to the northern hemisphere or the southern hemisphere.

If you rotate an object, say the Earth, clockwise, then as seen from above the plane the rotation is clockwise, but if you look underneath the plane ...



More information

The image is a diagram illustrating the perspective of rotation. It shows a semicircular shape that represents an object, such as the Earth, viewed from a side angle. On the left side of the semicircle, a dot is marked, indicating the viewpoint from above, where the rotation appears clockwise. On the opposite side, another dot represents the viewpoint from below, where the rotation appears counterclockwise. The diagram effectively visualizes how the direction of rotation can seem different depending on the observer's perspective, as described in the surrounding text.

[Generated by AI]

... down here the rotation is counterclockwise! Here is a simple experiment you might like to try. Take a plate. Hold it up between you and a friend. Rotate it clockwise. They will tell you that it is rotating counterclockwise. It's a matter of perspective!

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Surface area of a sphere

Exam tip

The formula booklet gives you the following formula:

Surface area of a sphere $A = 4\pi r^2$, where *r* is the radius.

Example 1

★☆☆

Find the surface area of a sphere with radius 3.4 cm. Round your answer to the nearest cm^2.

Using the area formula $A = 4\pi r^2$

$$\begin{aligned} A &= 4\pi(3.4)^2 \\ &= 145.267\ldots \end{aligned}$$

So the area is $145\ \text{cm}^2$ to the nearest cm ^{2}.

Volume of a sphere

Exam tip

The formula booklet gives you the following formula:

Volume of a sphere $V = \dfrac{4}{3}\pi r^3$, where *r* is the radius.

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Example 2

★☆☆

Find the volume of a sphere with radius 3.4 cm. Round your answer to nearest cm^3.

Using the volume formula $V = \frac{4}{3}\pi r^3$

$$V = \frac{4}{3}\pi(3.4)^3$$
$$= 164.636\ldots$$

So the volume is $165\ \text{cm}^3$ to the nearest cm^3.

2 section questions

3. Geometry and trigonometry / 3.1 Three-dimensional space

Combined solids

Combining shapes in design and architecture

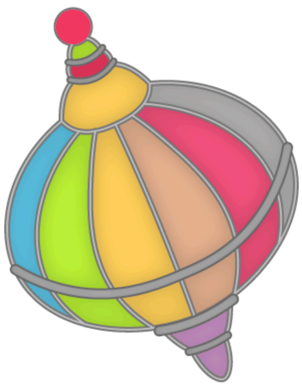
Many everyday objects and many buildings are made up of a combination of several 3D geometrical shapes. The photograph shows a row of historical town houses. Can you identify some of the 3D shapes you have studied so far in these buildings? How are they combined?

Student view

Town houses

Credit: amedved Getty Images

The diagram below shows a spinning top made up of several different sized spheres, and the bottom part is a cone. Why do you think the centre is a sphere and the bottom part is a cone?



Example 1

★★☆

A plastic model of an ice-cream cone comprises a cone with height 8 cm and a hemisphere with radius 4 cm.

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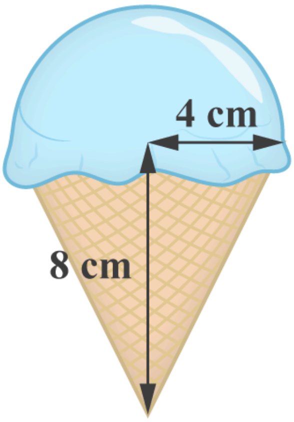
Calculate the total volume and the total surface area of the model.

Give your answer correct to 3 significant figures.



$$\text{volume of the combined solids} = \text{volume of the cone} + \text{volume of the h}$$

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) \\ &= \frac{1}{3}\pi(4)^2(8) + \frac{1}{2}\left(\frac{4}{3}\pi(4)^3\right) \\ &= 268.082\ldots \end{aligned}$$

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So the volume of the combined solid is 268 cm^3 correct to 3 significant figures.

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You can use the surface area formulas to find the combined surface area.

- The area of the curved surface of a hemisphere is $2\pi r^2$.
- The area of the curved surface of a cone is πrl, where the slant height can be calculated using Pythagoras' theorem, $l = \sqrt{h^2 + r^2}$.

You can find the total surface area combining these formulae.

$$\begin{aligned} A &= 2\pi r^2 + \pi r\sqrt{h^2 + r^2} \\ &= 2\pi(4^2) + \pi(4)\sqrt{8^2 + 4^2} \\ &= 212.929\ldots \end{aligned}$$

So the surface area of the combined solid is $213\ \text{cm}^2$ correct to 3 significant figures.

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Theory of Knowledge

As seen in the video below, there are many people who sincerely believe that the earth is flat. Is it possible to use trigonometry or geometry to 'prove' that the earth is not flat? Consider the knowledge question, 'How much knowledge must one have in regard to something before we can say for "certain" that it is true?'

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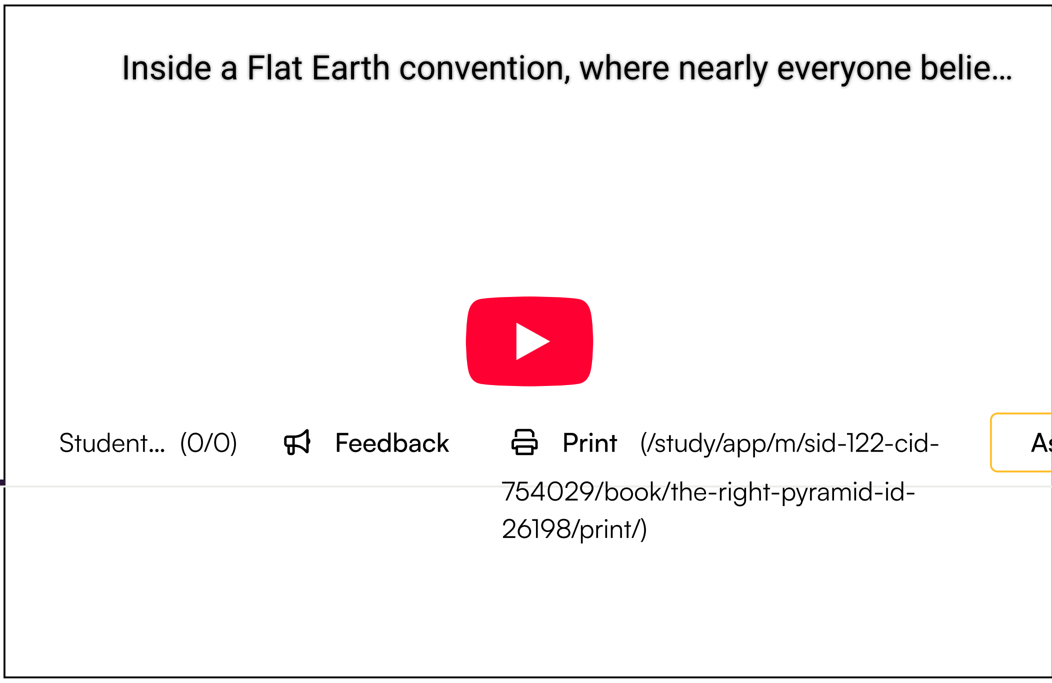

2 section questions

Angles between lines

Right triangles in 3D shapes

As you have seen in previous sections, you can draw right-angled triangles within 3D shapes. In the diagram below, you can see a right triangle formed by the diagonal of the base DB, DL and BL. Using these right triangles, you can calculate the angles between lines, such as angle LBD $= \beta$ in the diagram.

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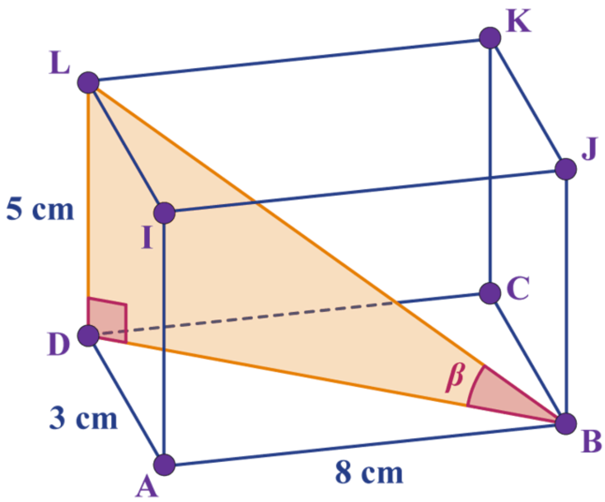

More information

The image is a 3D diagram of a cuboid with points labeled A, B, C, D, I, J, K, and L at each corner. A right-angled triangle is highlighted within the cuboid, with points D, B, and L forming the vertices of the triangle. The base of the triangle is the diagonal DB of the cuboid's bottom face, with a length of 8 cm marked. The perpendicular side DL (vertical leg) of the triangle is 5 cm long, and the horizontal leg DA is 3 cm. The angle at B, labeled as β, is the angle of interest and forms part of the triangle with a right angle marked at D. The cuboid's vertical faces and additional dimensions are depicted but not labeled. The diagram illustrates how a 3D shape can be analyzed through its 2D triangular cross-section to calculate angles.

[Generated by AI]

Suppose three points are drawn on the edges of a cuboid and joined to form a triangle. Sometimes the triangle will be right-angled; sometimes it will not be right-angled.

Use the applet below to move the points on the edges of the cuboid to get a sense of when a triangle will be right-angled.

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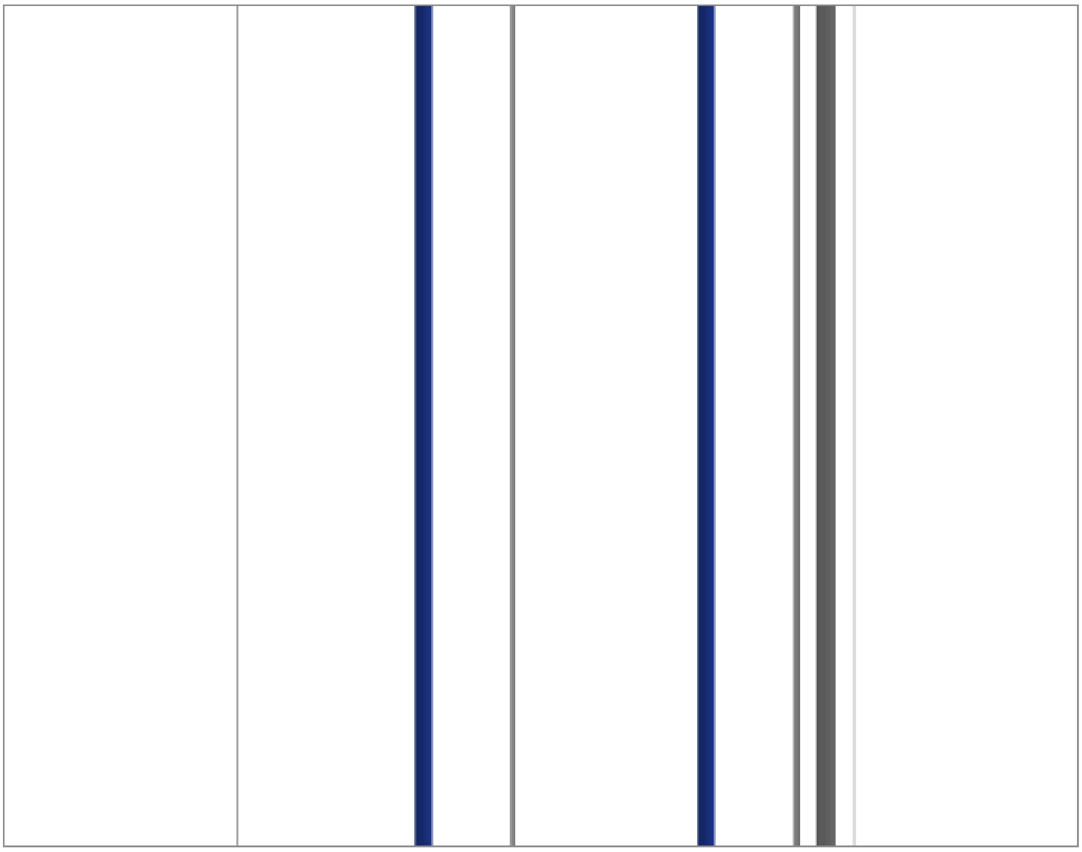

Interactive 2. Right Triangles in 3D Shapes.

Credit: GeoGebra (https://www.geogebra.org/m/jMXT6f1P) Lee W Fisher

More information for interactive 2

The image shows a 3D cuboid with labeled vertices: A, B, C, D, E, F, G, and H. The edges of the cuboid are highlighted in dark blue, and purple points are placed along its edges. These points can be moved along the edges to form different triangles.

This interactive allows users to explore the properties of triangles formed within a cuboid by moving its eight vertices. By adjusting the positions of these points along the edges, users can observe how different triangles are created and identify when a triangle becomes right-angled. As the points are moved, the dimensions of the cuboid change, altering the relationships between its edges and diagonals. Users can experiment by aligning points to form right-angled triangles along faces, space diagonals, and edge connections.

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Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/combined-solids-id-26201/print/) Assign

Interactive 3. Formation of Right Triangles in 3D Shapes.

Credit: GeoGebra (https://www.geogebra.org/m/zagcpBoZ) Lee W Fisher

More information for interactive 3

This interactive allows users to explore the formation of right-angled triangles within a cuboid by moving three points along the edges.

As the user moves the points, a continuous angle measurement is shown, updating in real-time to indicate whether the triangle remains right-angled or not.

As users adjust the points, the triangle dynamically updates, and a 90° angle is displayed when the conditions for a right-angled triangle are met. A right-angled triangle appears when one side is parallel to an edge of the cuboid, and the second shorter side lies within one of the cuboid’s faces. No matter where the third point is placed along an edge, as long as these conditions are satisfied, the triangle remains right-angled.Hence, the interactive clears the understanding of the conditions required to get a right angled triangle.

Notice that when one edge of the triangle is parallel to an edge of the cuboid, (forming the first shorter side) and the second shorter side lies on a face of the cuboid, the resulting triangle is right-angled, regardless of where on an edge the third point is.

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Example 1

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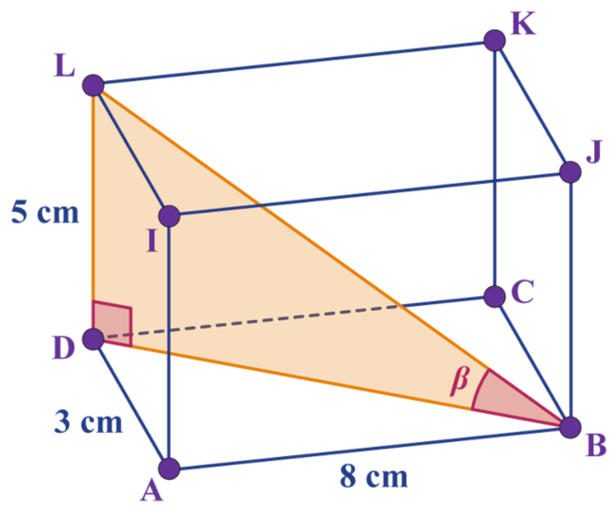

More information

The diagram shows a cuboid with vertices labeled A, B, C, D, I, J, K, and L. The edges of the cuboid have the following lengths: AB = 8 cm, AD = 3 cm, and AI = 5 cm. The diagram appears to illustrate spatial relationships and angles. Particular emphasis is placed on the triangle formed by vertices D, I, and B, which is highlighted in orange. Vertex D has a perpendicular angle marked, and the angle at vertex B is labeled as beta (β). The diagram helps in understanding the geometrical representation of the cuboid's dimensions and the relationships between points and angles.

[Generated by AI]

In the diagram, the cuboid has lengths $AB = 8$ cm, $AD = 3$ cm and $AI = 5$ cm.

Calculate the angle between the diagonal DB and LB.

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Consider the right-angled triangle drawn in the cuboid.

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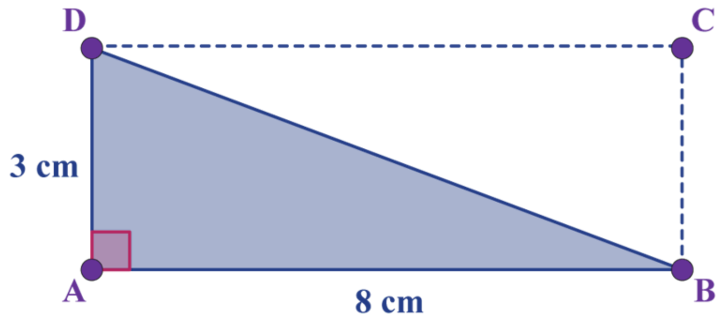
First, you need to find the length of the diagonal of the base ABCD.



You need to find the length of the diagonal of the base ABCD using Pythagoras' theorem:

$$\begin{aligned} DB^2 &= 3^2 + 8^2 \\ &= \sqrt{73} \end{aligned}$$

So the length of the diagonal is $\sqrt{73}$cm.

Exam tip

Do not round your answer at this stage. Leave it in surd form.

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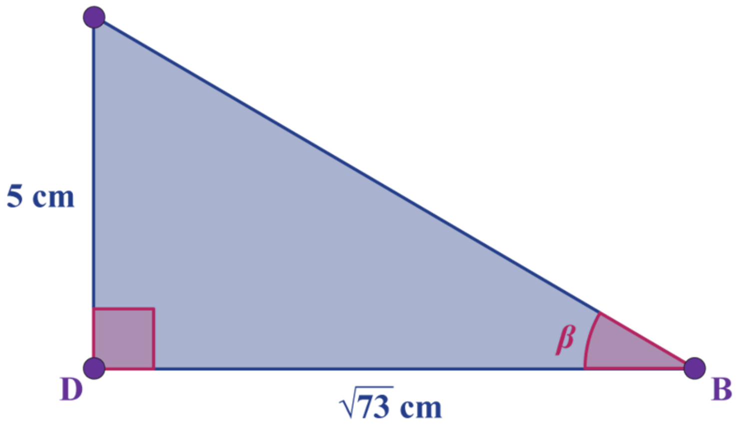

Now that you have the two of the sides of the right-angled triangle, you can use trigonometric ratios. You know the opposite and adjacent sides to the angle you need to find, so use the tangent ratio:

$$\tan\beta = \left(\frac{5}{\sqrt{73}}\right)$$

$$\begin{aligned}\beta &= \tan^{-1}\left(\frac{5}{\sqrt{73}}\right)\\ &= 30.336\ldots^\circ\end{aligned}$$

So $\beta = 30.3^\circ$ to 3 significant figures.

Exam tip

Only round the final answer.

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Example 2

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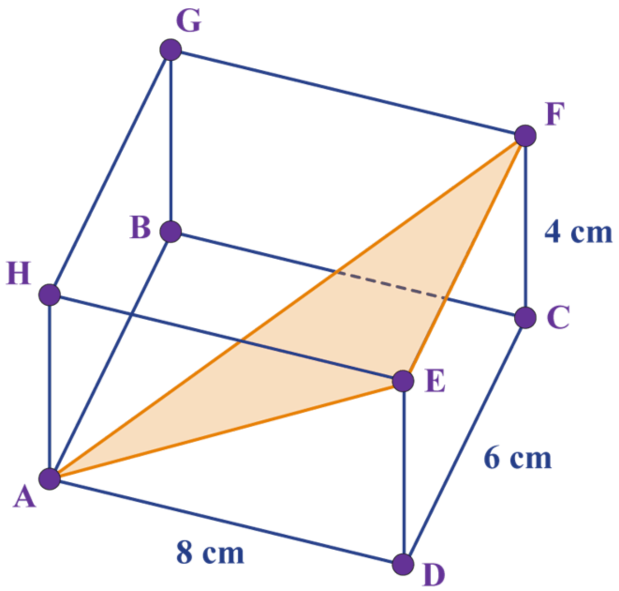
Cuboid ABCDEFGH has dimensions 8 cm, 6 cm and 4 cm, as shown in the diagram below.

Calculate the angle between line segments FA and AE .



More information

This is a diagram of a 3D geometric shape with eight points labeled A, B, C, D, E, F, G, and H. The shape appears to resemble a cuboid with additional lines connecting the labeled points. The face AEFC is highlighted in orange, suggesting it's the area of interest. The lengths of some of the line segments are given: segment AE is 8 cm, and segments AD and EF are 6 cm and 4 cm, respectively. The task is to calculate the angle between the line segments FA and AE. Each segment represents an edge of the shape and is part of the overall geometric configuration.

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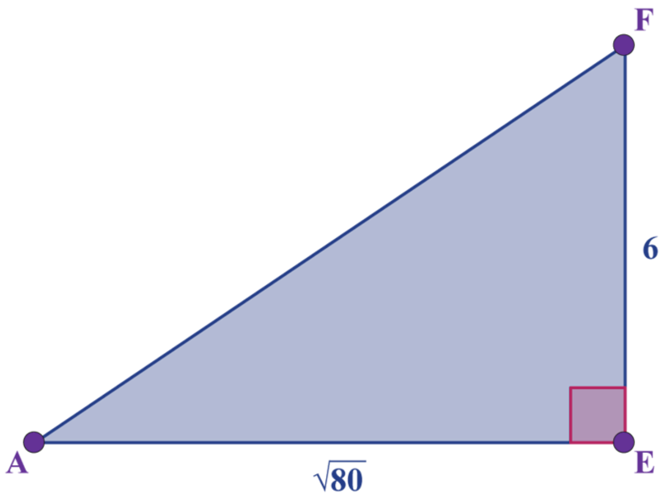
Steps	Explana
Calculate the length of AE: $$(AE)^2 = 8^2 + 4^2 = 80$$ Therefore, $AE = \sqrt{80}$. 	Before you isolat triangle, make su know which side hypotenuse. The is at E; therefore hypotenuse is th

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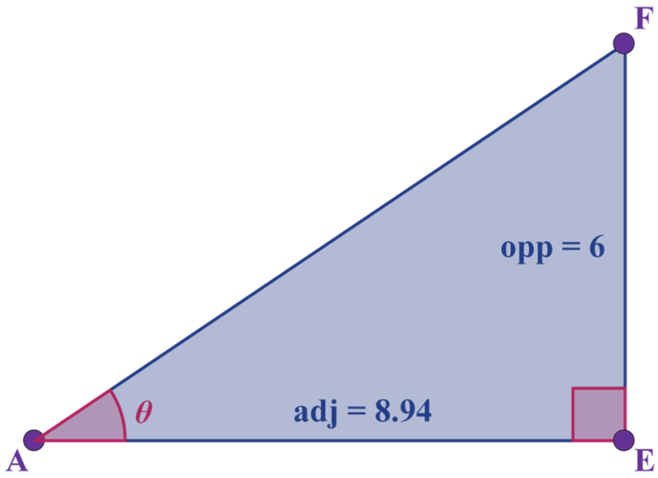
Steps	Explana
 $\tan\theta = \left(\frac{6}{\sqrt{80}}\right)$ $\theta = \tan^{-1}\left(\frac{6}{\sqrt{80}}\right)$ $= 33.854\ldots^\circ$ So $\theta = 33.9^\circ$ to 3 significant figures.	The final step is $\angle\mathrm{FAE}$.

Exam tip

Make sure your calculator mode is set to degrees when you are solving this type of problem.

3 section questions

Student view

3. Geometry and trigonometry / 3.1 Three-dimensional space

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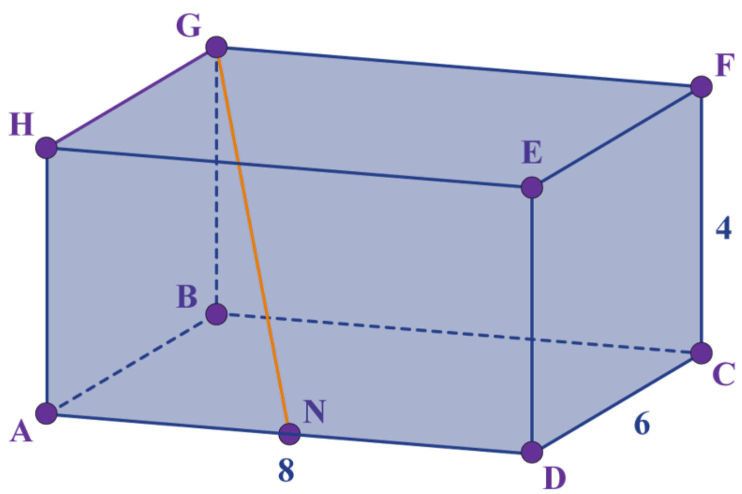
Angle between a line and a plane

To calculate the angle between a line and a plane, or the face of a solid, you use a three-step calculation. The first step is simply to determine the correct line to use on the face of the solid. The next two steps are the same as in the previous example.

This time, you need to calculate the angle between the line GN and the base ABCD. N is the midpoint of AD.



More information

The image shows a 3D diagram of a cuboid labeled with various points: A, B, C, D, E, F, G, H, and N. The points form the vertices of a rectangular cuboid. The line GN connects point G at the top with point N on the base, given that N is the midpoint of the line segment AD on the cuboid's base ABCD. The base ABCD is marked with side lengths, with side AB equaling 8 and side AD equaling 6. Additionally, the height from the base to the top of the cuboid, represented by points such as G and F, is 4. Line GN needs to be analyzed to determine the angle between it and the base of the cuboid.

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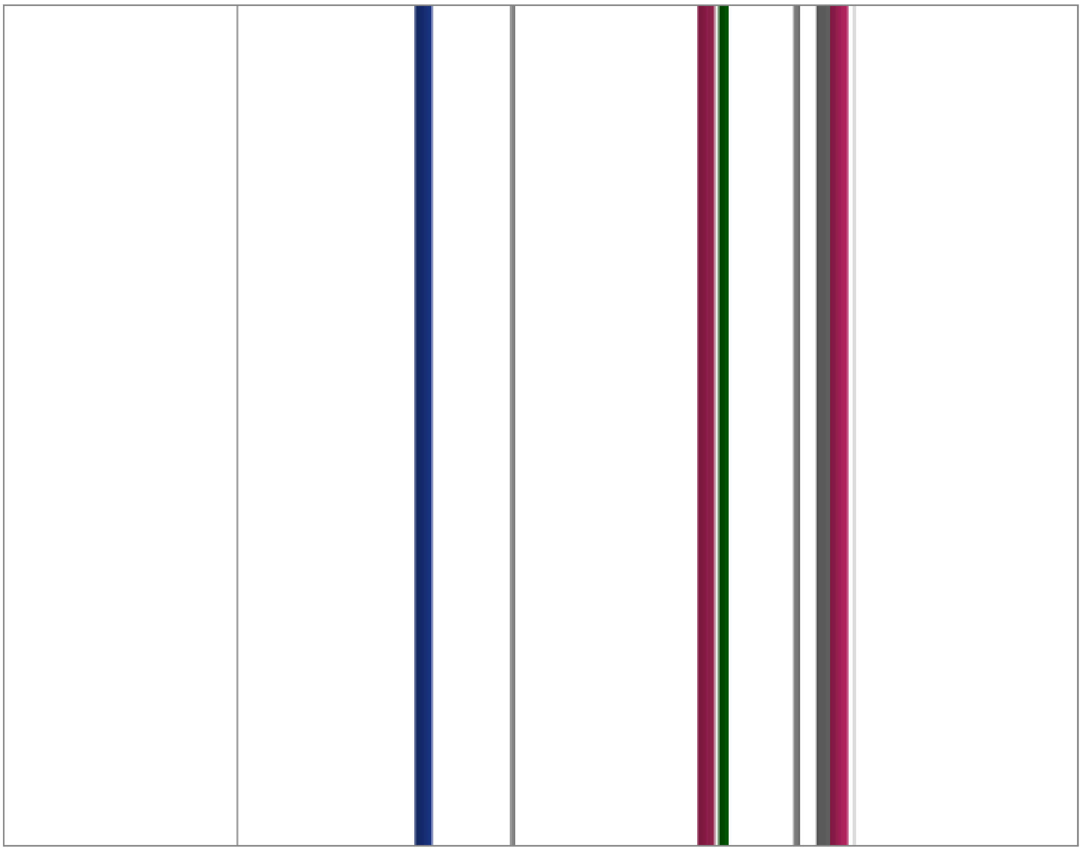
[Generated by AI]

Step 1 Project the upper point to the base. In other words, drop a line from G to the base. That takes you to the point B. Your triangle of study is GBN. Because GB is perpendicular to the base ABCD, the triangle GBN is right-angled at B. Click on the link below to enable you to rotate the cuboid so that you can see it better.



Interactive 1. Angle Between a Line and a Plane.

More information for interactive 1

This interactive applet demonstrates a step-by-step method to calculate the angle between a line and a plane, specifically between the line GN and the base ABCD of a cuboid. Here, N is the midpoint of AD. Users can manipulate the cuboid's vertices to explore different configurations, allowing them to practice and gain a deeper understanding of this concept. The process involves projecting a point onto the base plane to form a right-angled triangle, calculating the necessary side lengths using geometric principles such as the Pythagorean theorem, and then applying trigonometric functions, such as tangent, to determine the angle between the line and the plane.

They learn how to apply the Pythagorean theorem to determine unknown side lengths and use

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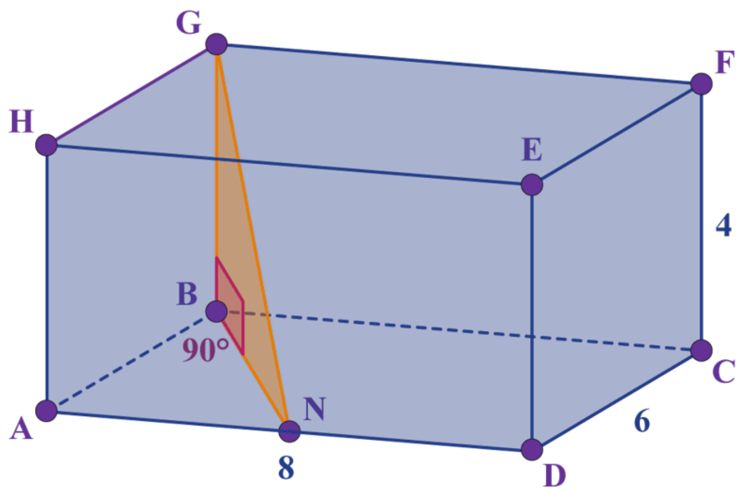
trigonometric ratios to calculate the desired angle.

For example, to find the angle between GN and the base ABCD, the user must first determine at least two sides of the right-angled triangle. Given that $BG = 4$ cm, they need to calculate BN. With $AB = 6$ cm and $AN = 4$ cm, applying the Pythagorean theorem gives $BN^2 = 6^2 + 4^2 = 36 + 16 = 52$, leading to $BN = \sqrt{52}$.

Using trigonometric functions, such as $\tan\theta = \frac{\text{opposite}}{\text{Adjacent}}$, they can then compute the angle.

Since the cuboid can be rotated and viewed from different angles in 3D, users can interactively explore how the angle between the line and the plane changes with different perspectives. This hands-on exploration enhances spatial reasoning skills and provides a clearer understanding of geometric transformations in three-dimensional space, making abstract mathematical concepts more tangible and engaging.



More information

The image is a 3D geometric diagram representing a rectangular prism. The diagram is labeled with points A to H on the prism’s vertices. Lines are drawn between these points to form rectangles and triangles, showing different two-dimensional shapes on the prism’s surfaces. There are specific labels such as BG = 4 cm, AB = 6 cm, AN = 4 cm, and BN is calculated as the square root of 52. The image

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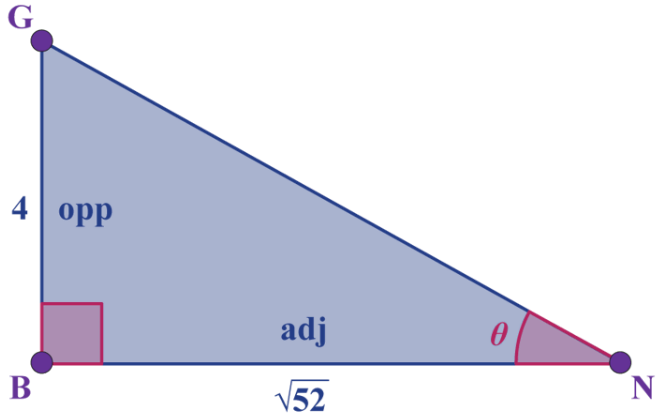
includes a highlighted orange triangle within the blue prism, demonstrating a right angle of 90° at point B. The labels 8, 6, and 4 along the edges signify the dimensions of the prism. This structure is useful for solving geometric problems related to angles and sides within 3D shapes.

[Generated by AI]

Step 2 To calculate the required angle, you need to know at least two sides of the triangle. You know that $BG = 4$ cm. Next you need to calculate BN. Now $AB = 6$ and $AN = 4$, so $(BN)^2 = 6^2 + 4^2 = 52$; therefore, $BN = \sqrt{52}$.

Step 3 Now you are ready to calculate $\angle \text{GNB}$, which is the angle between GN and the base ABCD. This angle is marked as θ in the diagram below.



More information

The image depicts a right triangle with vertices labeled G, N, and B. The side GN is the hypotenuse. The side GB is labeled with a length of 4 units, and BN with a length of (\sqrt{52}). There is a right angle marked at vertex B. The angle (\theta), formed between GN and BN, is highlighted as well. This diagram

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is used to illustrate the calculation of the angle (\angle {\rm{GNB}}), based on trigonometric relationships, specifically using the tangent function (\tan \theta = \frac{4}{\sqrt{52}}). An angle (\theta) is marked near vertex N.

[Generated by AI]

$$\tan\theta = \left(\frac{4}{\sqrt{52}}\right)$$

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{4}{\sqrt{52}}\right)\\ &= 29.017\ldots^\circ\end{aligned}$$

So $\theta = 29.0^\circ$ (to 3 significant figures).

Example 1

★★☆

The diagram below shows a square-based right pyramid ABCDE. The height of the pyramid is 6 cm and each of the sides of the base measures 10 cm. M is the midpoint of the line segment CD.

Find the following angles. Give your answers correct to 3 significant figures.

a) $\angle\text{EMG}$

b) $\angle\text{ECM}$

c) $\angle\text{ECG}$

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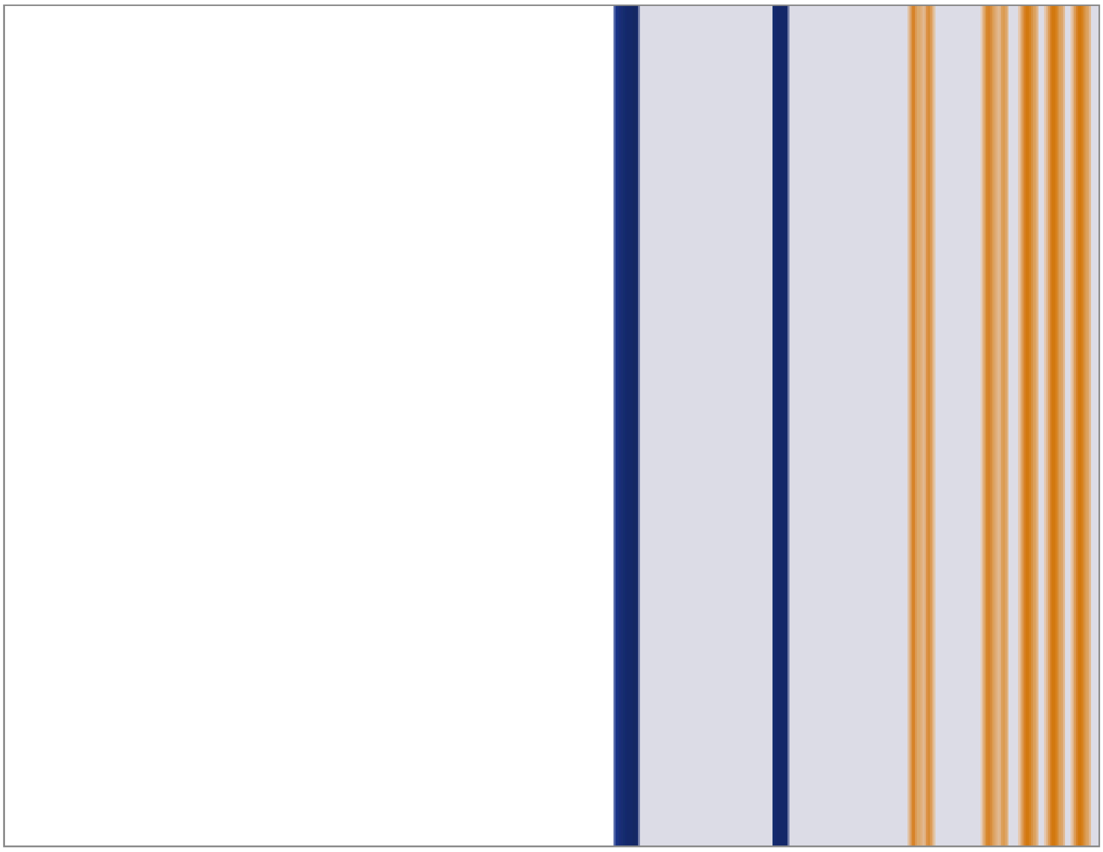

Interactive 2. Square-Based Right Pyramid ABCDE.

More information for interactive 2

This interactive tool provides an engaging and dynamic way to explore the geometric properties of a **square-based right pyramid ABCDE.** The pyramid has a **height of 6 cm**, and each side of its square base measures 10 cm. The midpoint of the base edge **CD** is labeled as **M**, serving as a key reference point for angle calculations.

The interactivity allows for **rotation of the 3D pyramid**, offering multiple perspectives to better understand its structure. Additionally, **points A and B can be moved,** enabling adjustments to the base dimensions. As the base expands or compresses, the internal angles of the pyramid change, demonstrating how different elements of the shape interact.

This tool focuses on guiding through the calculation of three important angles:

- $\angle EMG$**:** This angle is part of the right-angled triangle **EGM**, where **G is the foot of the perpendicular from E to the base.** Since this triangle is right-angled at G, trigonometry can be used to determine the other angles.
- $\angle ECM$ **:** This angle lies within the triangular face **ECM**, which is formed by the slant height of the pyramid and a segment of the base.
- $\angle ECG$**:** This angle illustrates the relationship between the apex and different sections of the base.

By allowing direct manipulation of the pyramid's structure, this tool provides an interactive experience

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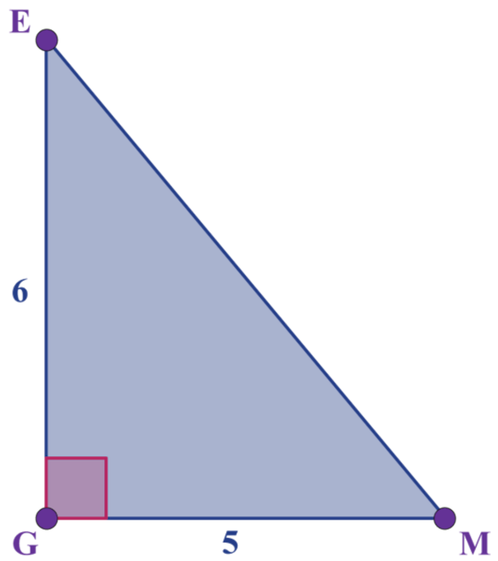
that enhances the understanding of **3D geometry, trigonometry, and spatial relationships.** The ability to rotate the shape and modify its dimensions makes it easier to explore geometric concepts in a more intuitive and accessible way.

To calculate $\angle\text{EMG}$, isolate triangle EGM, which is right-angled at G. This angle may also be described as the angle between EM and the base ABCD.

To calculate $\angle\text{ECM}$, isolate triangle EMC, which is right-angled at M.

To calculate $\angle\text{ECG}$, isolate triangle EGC, which is right-angled at G. This angle may also be described as the angle between EC and the base.



a) Let $\angle\text{EMG} = \theta$.

So $\tan\theta = \dfrac{6}{5}$.

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Therefore, $\theta = \tan^{-1}\dfrac{6}{5} = 50.2°$ (to 3 significant figures).

In triangle EMG you can also find the the length of $[\text{EM}]$, which will be used later.

$$\text{EM} = \sqrt{5^2 + 6^2} = \sqrt{61}$$

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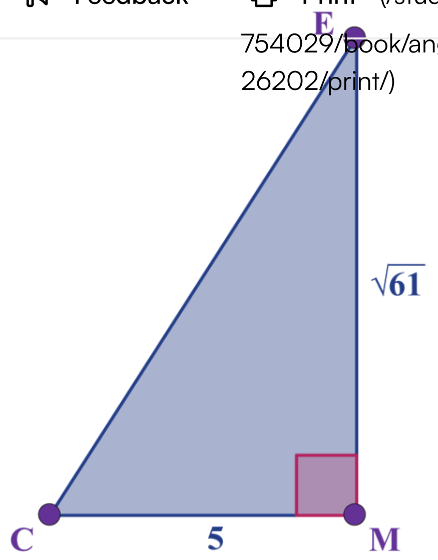

b) Let $\angle\text{ECM} = \alpha$.

So, $\tan\alpha = \dfrac{\sqrt{61}}{5}$.

Therefore, $\alpha = \tan^{-1}\dfrac{\sqrt{61}}{5} = 57.4°$ (to 3 significant figures).

In triangle EMC you can also find the the length of $[\text{EC}]$, which will be used later.

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$$\text{EC} = \sqrt{5^2 + \sqrt{61}^2} = \sqrt{86}$$

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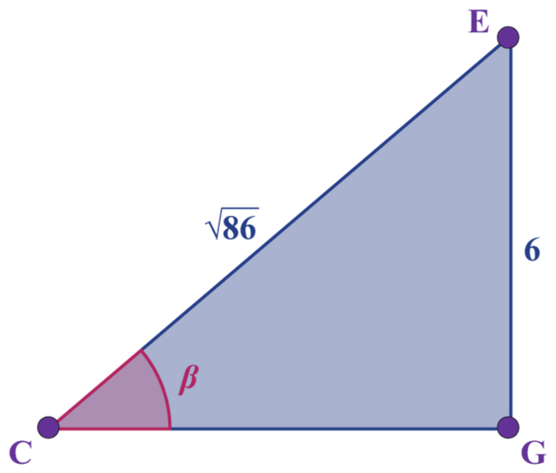

c) Relative to the angle marked β (which is $\angle\text{ECG}$), you know the opposite side and the hypotenuse of the triangle. So you can use the sine ratio:

$$\sin\beta = \frac{6}{\sqrt{86}}$$

Therefore, $\beta = \sin^{-1}\frac{6}{\sqrt{86}} = 40.3°$ (to 3 significant figures).

You can also form right-angled triangles in cuboids and use these to find the angles between lines.

3 section questions

3. Geometry and trigonometry / 3.1 Three-dimensional space

Checklist

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What you should know

By the end of this subtopic you should be able to:

- find the midpoint and the distance of any two points in 3D
- select the correct formulae for finding areas and volumes of 3D shapes from the formula booklet
- count the faces on a pyramid or prism
- calculate the total surface area of a solid by adding together the area of each of its faces
- calculate the volume of pyramids, cones, spheres and combinations of these solids.

Investigation

As you saw at the start of this subtopic, many geometric shapes form the bases of architectural designs. Here you have two examples from two different cities in the world.

The first one is the Petronas twin towers, Kuala Lumpur, Malaysia and the second one is the Kuwait towers of Kuwait City.

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Petronas twin towers, Kuala Lumpur, Malaysia

Credit: fotoVoyager GettyImages

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Kuwait towers, Kuwait City, Kuwait

Credit: Anson Fernandez Dionisio GettyImages

Your task is to model one of these two pairs of towers to estimate the height, surface area and volume of the building.

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Consider the following questions to complete this task.

- What are the relevant elements? What information do you need to solve this problem?
- What mathematical strategies will you use to solve the problem? What mathematics do you need to do? What technology will you use to help you?
- Do your solutions make sense in real-life?
- Consider the accuracy of your solutions – justify the accuracy and explain any reasons why inaccuracy exists.
- Write a report showing how you solved the problem. Include reasoning and justification for all your decisions and the final solutions, as well as reflection.

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