2. Functions / 2.8 Rational functions

The big picture

The world's largest pendulum is in the Oregon Convention Centre in the USA. It has a ball that weighs over 400 kg, swinging from a 21 metre cable.

The motion of pendulums has fascinated mathematicians and physicists for hundreds of years – it provides a mechanism by which the passing of time can be measured, due to its striking regularity. If you ever used to play on a swing when you were a child, reflect back on whether the time it took to perform a full swing depended on how high up you were.

A very simple rule governing cyclical motion, as shown in pendulums, is given by the formula:

$$f = \frac{1}{T}$$

Here, f is the frequency of swings, or the number of swings per second, and T is the period, or the time in seconds to make a complete swing. So if it takes 10 seconds for a full swing, the frequency will be 0.1 s $^{-1}$, or 0.1 hertz. The variables f and T are reciprocal – the larger one variable is, the smaller the other will be. You could say that f is a reciprocal function of T.

In this subtopic, you will learn about:

- the reciprocal function
- rational functions.

Watch this video, which shows the synchronous motion of a number of pendulums.

Video 1. Pendulum Waves.

More information for video 1

The video begins with a title card against a black background, introducing the demonstration as "Pendulum Waves" in bold white text. The scene transitions to display a wooden platform supporting a metal A-frame structure with a wave pendulum with 15 balls hanging from the frame with different lengths in ascending order. The camera zooms in to provide a closer view of the wave pendulums, emphasizing the precision and arrangement of the setup.

The presenter aligns the balls with a wooden plank.

The balls rest against the plank, ensuring they are positioned uniformly before the demonstration begins.

This meticulous preparation highlights the importance of accuracy in the experiment.

The presenter slowly removes the wooden plank, releasing the balls simultaneously.

As the balls begin to swing, the strings create a visually captivating wave-like motion.

The pendulums move in a synchronized pattern initially, but as time progresses, they gradually desynchronize, creating a mesmerizing and dynamic display of motion. Eventually, the pendulums realign into a similar wave pattern, demonstrating the cyclical nature of the phenomenon.

This sequence showcases the principles of physics, particularly the relationship between pendulum length, frequency, and synchronization.

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Concept

While learning about the algebraic and graphical **representation** of the reciprocal and rational functions, reflect on how horizontal and vertical asymptotes help you to sketch the graph of these functions. What happens when one polynomial is divided by another? How can you rewrite rational functions? Which techniques can be used to sketch functions?

Think of real-life examples that can be **modelled** with the reciprocal function and rational functions.

Theory of Knowledge

The key knowledge issue of formalism vs. Platonism in regard to mathematics is discussed in other TOK boxes throughout the course; however, it seems apropos to contemplate mathematics' rational origins in the context of rational functions.

Knower bias is a key factor in knowledge production and reception; however, at first glance, it seems that mathematics is immune to such biases because it is built on reason and has a very high level of real-world predictive validity.

Knowledge Question: To what extent can knowledge be free from bias?

2. Functions / 2.8 Rational functions

The reciprocal function

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Properties of the reciprocal function

You know that for every real number x other than zero, its **reciprocal** number is defined as $\frac{1}{x}$. There is a function that maps any $x \neq 0$ to $\frac{1}{x}$ and it is called the reciprocal function.

Important

The **reciprocal function** is the function that maps any $x \neq 0$ to its reciprocal number $\frac{1}{x}$.

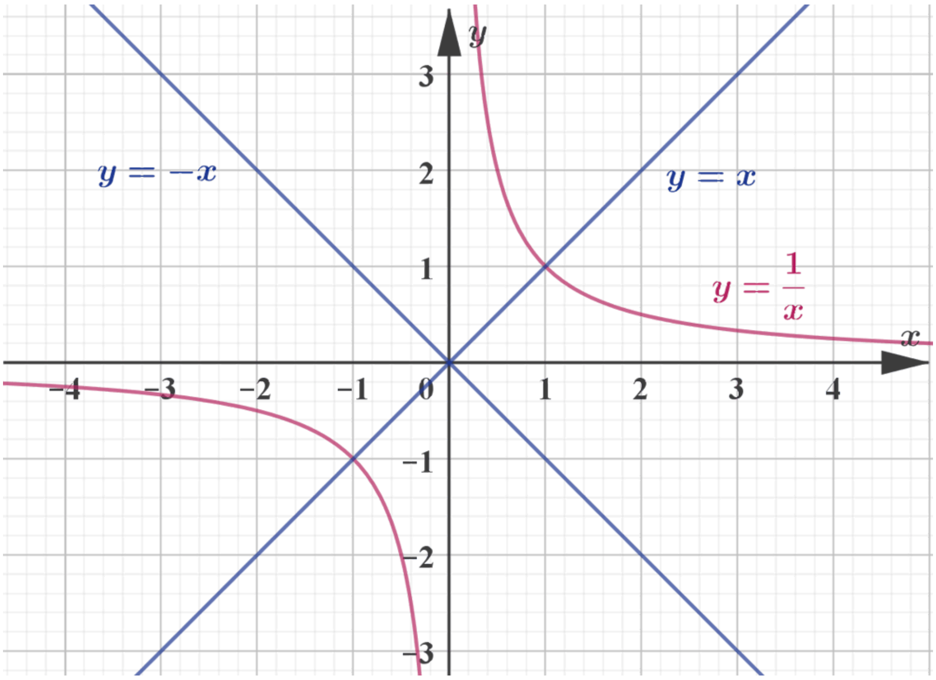
It has the form $f(x) = \frac{1}{x}$.

Activity

The statement $x \neq 0$ means an input value of zero is not permitted. Divide one by zero on your calculator: what happens? Now divide one by a very small number on your GDC, and then by an even smaller one. What happens? Why do you think you cannot divide by zero?

The reciprocal function has several important properties and they are worth remembering. The reciprocal function $f(x) = \frac{1}{x}$ is shown below.

Student view



More information

The image displays a graph of the reciprocal function (f(x) = \frac{1}{x}). The graph features two asymptotic branches existing in the first and third quadrants of the Cartesian plane. These branches curve closer to the x-axis and y-axis but never touch them, illustrating asymptotic behavior.

The X-axis and Y-axis are numbered from -4 to 4. The function (y = \frac{1}{x}) is shown as a red curve, decreasing in both the positive and negative direction of the axis.

Two diagonal blue lines are present in the graph: one with a positive slope labeled (y = x) and another with a negative slope labeled (y = -x). The graph possesses symmetry about the origin, shown by the intersecting blue diagonal lines. These lines form a cross through the origin and divide the graph into four quadrants, emphasizing the symmetry of the reciprocal function.

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The graph of the reciprocal function $f(x) = \frac{1}{x}$ has two branches, which exist in the first and third quadrants only. The function is decreasing on the intervals $(-\infty,\ 0)$ and $(0,\ +\infty)$, and the graph is symmetric with respect to the origin, which implies symmetry about the lines $y = x$ and $y = -x$.

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Also, as $x \to 0^{+}$, the function f increases without bound, $f(x) \to +\infty$, and as $x \to 0^{-}$, the function f decreases without bound, $f(x) \to -\infty$. Therefore, the line $x = 0$ (y-axis) is a vertical asymptote of the graph, and the domain of the reciprocal function is$(-\infty,\ 0) \cup (0,\ +\infty)$.

In the figure above , observe that the graph of the reciprocal function f has a horizontal asymptote at the line $y = 0$ (x-axis). The values of $f(x) = \frac{1}{x}$ approach zero as x increases or decreases without bound. Using mathematical notation: as $x \to +\infty,\ f(x) = \frac{1}{x} \to 0^{+}$, and as $x \to -\infty,\ f(x) = \frac{1}{x} \to 0^{-}$. The range of the reciprocal function is $(-\infty,\ 0) \cup (0,\ +\infty)$.

- Reflect on why the reciprocal function $f(x) = \frac{1}{x}$ never equals zero. Does the reciprocal function have axes intercepts ?
- Use the graph to explain why the reciprocal function has an inverse and find the inverse function analytically. Can you explain why the reciprocal function is called a self-inverse function ?

Be aware

A **self-inverse function** is a function that is the inverse of itself, which means that the inverse of a self-inverse function turns out to be the function itself.

- Use the graph of the reciprocal function to explain why the reciprocal function is a **self-inverse function.**

Making connections

Recall that the graph of function f and its inverse f^{-1} are symmetric about the line $y = x$.

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Important

A function f is a self-inverse function if, and only if, f has an inverse f^{-1} and $f(f(x)) = x$, or written another way, $f(x) = f^{-1}(x)$.

Activity

Prove that any function of the form $f(x) = \frac{a}{x}$ is self-inverse.

International Mindedness

Gottfried Wilhelm von Leibniz (1646—1716) was a German philosopher and mathematician who made many contributions to the mathematical notation that is used today. Leibniz proved that the gradients of the graphs of two functions that are the inverse of each other are reciprocals.

General context of the reciprocal function

The reciprocal of a function f is defined as $y = \frac{1}{f(x)} = [f(x)]^{-1}$.

Important

The **reciprocal of a function** f is a function g such that $g(x) = \frac{1}{f(x)} = [f(x)]^{-1}$, for all x where $f(x) \neq 0$.

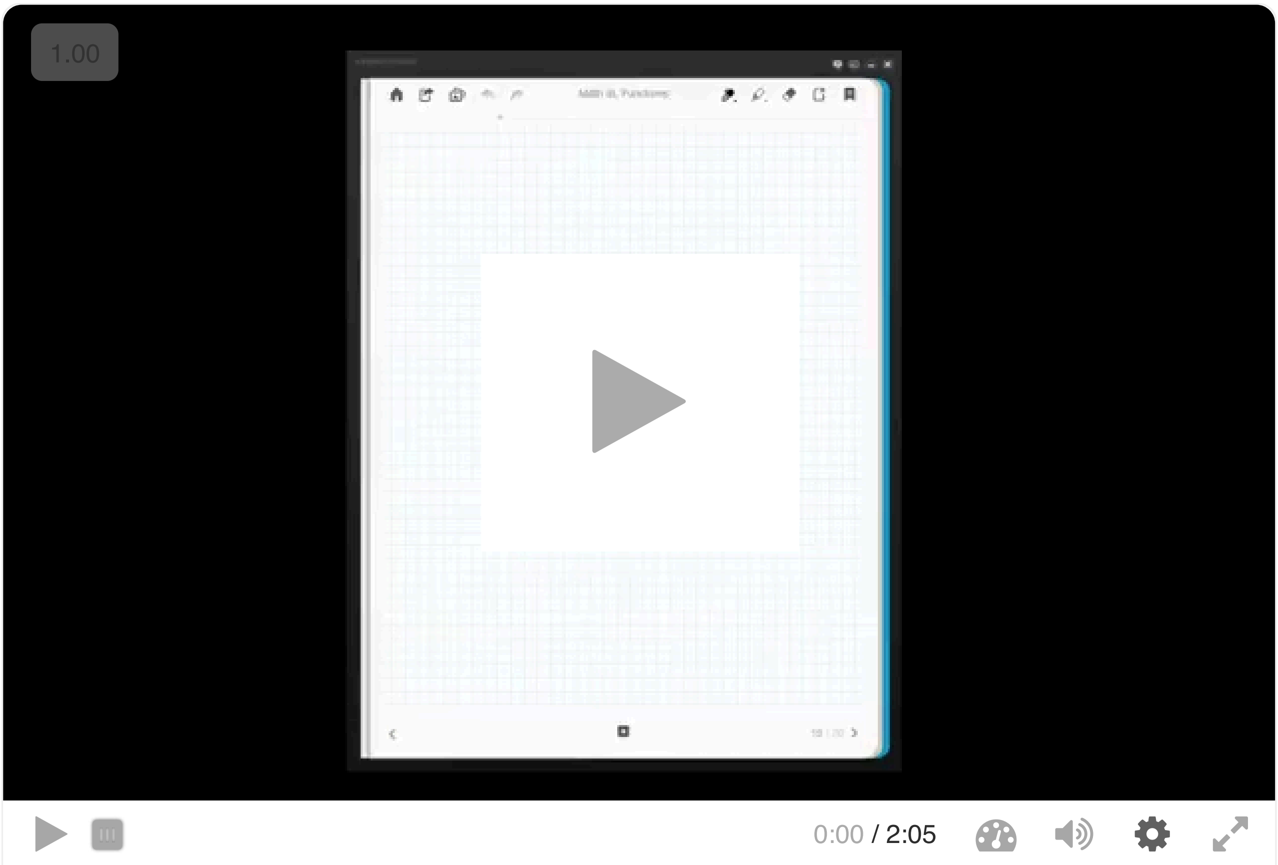
Practically, the reciprocal of a function f is the composition of the function f and the reciprocal function $\frac{1}{x}$, where f applies first and $\frac{1}{x}$ applies second.

Student view

Be aware

Be extremely careful to not mistake the reciprocal of a function f, $[f(x)]^{-1}$, for its inverse function, f^{-1}.

Consider the function $f(x) = 2x - 1$. In the following video you can explore the relationship between the graph of $y = f(x)$ and $y = \dfrac{1}{f(x)}$.



Video 1. Understanding the Reciprocal Function: Properties and Graphs.

More information for video 1

1

00:00:00,434 --> 00:00:03,370

narrator: In this video we're going

to look at reciprocal functions

2

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00:00:03,437 --> 00:00:06,039

and particularly from the point of view

3

00:00:06,106 --> 00:00:09,676

of comparing the graph of one

with the graph of the other one.

4

00:00:09,810 --> 00:00:13,413

As an example, we're going to

take $f(x) = 2x - 1$,

5

00:00:13,580 --> 00:00:16,350

and we're going to compare

that graphically with $1/f(x)$

6

00:00:16,416 --> 00:00:19,653

which of course is equal

to $1/(2x - 1)$.

7

00:00:19,753 --> 00:00:21,054

But the focus is gonna be graph.

8

00:00:21,121 --> 00:00:25,492

So let's look at the graph here.

y versus x, let's first plot $f(x)$.

9

00:00:25,592 --> 00:00:27,794

Now clearly it's a line

10

00:00:27,995 --> 00:00:31,164

and it's obvious

that it's y intercept is minus 1

11

00:00:31,231 --> 00:00:33,734

and it's x intercept is 0.5.

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view

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12

00:00:33,800 --> 00:00:37,204

So that is $y = 2f(x)$.

13

00:00:38,105 --> 00:00:41,475

Now let's start by indicating the zero,

14

00:00:41,708 --> 00:00:45,212

which of course is at 0.5, x is 0.5.

15

00:00:45,546 --> 00:00:48,815

And let's look at the right hand side,

which in this case is positive.

16

00:00:48,982 --> 00:00:53,153

And then $f(x)$ is negative

on the left hand side of that dotted line.

17

00:00:55,055 --> 00:00:58,559

Now we also see that this line increases

18

00:00:59,193 --> 00:01:03,664

without bounds towards large x,

meaning that the value

19

00:01:03,730 --> 00:01:05,799

of $f(x)$ tends to positive infinity

20

00:01:06,300 --> 00:01:08,101

and it decreases in the other direction,

21

00:01:08,168 --> 00:01:10,771

meaning that if you track down

x to minus infinity,

22

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00:01:10,838 --> 00:01:13,140

the function itself

tends to minus infinity.

23

00:01:13,307 --> 00:01:17,878

Now let's plot on the same set of axis

to graph of one of f (x) in blue.

24

00:01:18,111 --> 00:01:21,882

So where f (x) is equal to zero one

of f (x) is equal to vertical asymptote,

25

00:01:21,949 --> 00:01:23,617

which is indicated by the blue line.

26

00:01:23,750 --> 00:01:27,588

Remember, when f (x) is positive,

then one of f (x) is positive.

27

00:01:27,688 --> 00:01:31,692

So we know that the asymptote

gets reached on the left hand side

28

00:01:31,758 --> 00:01:34,194

negatively and right hand side positively.

29

00:01:34,895 --> 00:01:38,365

f (x) increases without bounds

towards positive infinity

30

00:01:38,432 --> 00:01:40,634

means that one of f (x) decreases

31

00:01:40,968 --> 00:01:45,239

about the x axis, and the same happens

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but an opposite on the minus sign.

32

00:01:45,305 --> 00:01:49,243

The y intercept is minus 1, well,

one over minus 1 is still minus 1.

33

00:01:49,309 --> 00:01:53,413

So the one of f (x) also

has a y intercept minus 1.

34

00:01:53,680 --> 00:01:57,885

So now we connect the little pieces

that we can with a continuous curve,

35

00:01:58,318 --> 00:02:02,256

and we found ourselves

y = 1/f (x)

36

00:02:02,589 --> 00:02:04,958

from the curve y = f (x).

- The graph of $y = 2x - 1$ has an x-intercept at $2x - 1 = 0 \Leftrightarrow x = \frac{1}{2}$. The graph of $y = \frac{1}{2x-1}$ has a vertical asymptote at $x = \frac{1}{2}$.
- The graph of $y = 2x - 1$ is positive for $x > \frac{1}{2}$ and negative for $x < \frac{1}{2}$. Thus, the graph of $y = \frac{1}{2x-1}$ is positive for $x > \frac{1}{2}$ and negative for $x < \frac{1}{2}$.
- The y-intercept of $y = 2x - 1$ is at $y = -1$, so the y-intercept of $y = \frac{1}{2x-1}$ is at $y = \frac{1}{-1} = -1$.
- The graph of $y = 2x - 1 \to +\infty$ as $x \to +\infty$. Thus, the graph of $y = \frac{1}{2x-1} \to 0$ as $x \to +\infty$ from above the x-axis.

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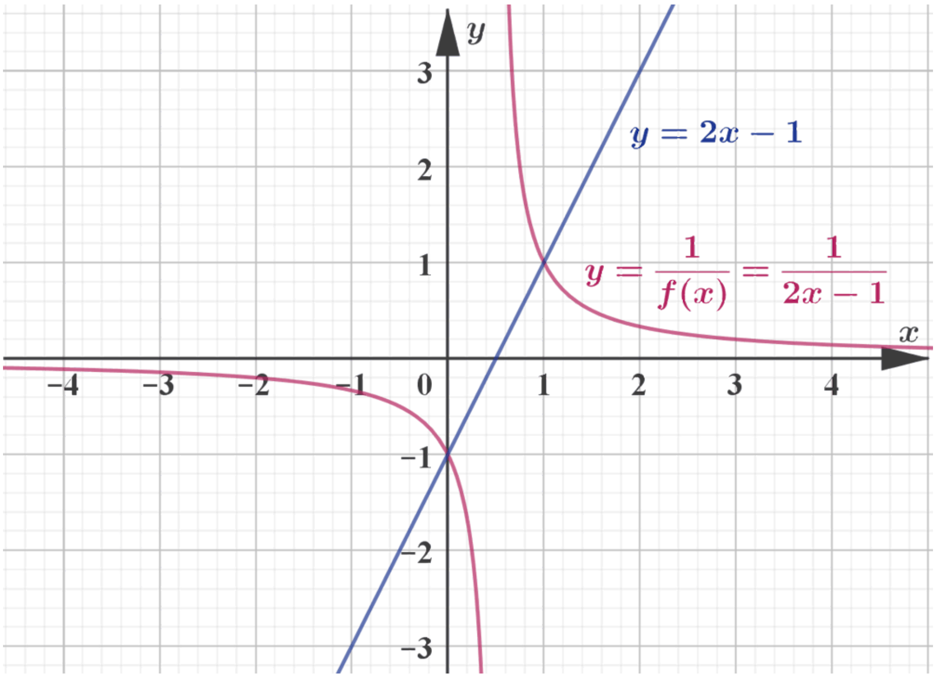
- Similarly, the graph of $y = 2x - 1 \to -\infty$ as $x \to -\infty$. Thus, the graph of $y = \dfrac{1}{2x-1} \to 0$ as $x \to -\infty$ from below the x-axis.

The graphs of $y = f(x) = 2x - 1$ and $y = \dfrac{1}{f(x)} = \dfrac{1}{2x-1}$ are shown below.



More information

The image is a graph displaying two functions on a Cartesian coordinate plane. The first function is a blue line representing the equation (y = 2x - 1). It is a straight line with a positive slope that passes through the y-axis at -1. The second function is a pink curve representing the equation (y = \frac{1}{{2x - 1}}). This curve has two distinct branches: one approaches zero from the positive y-axis as x approaches positive infinity, and another branch moves downwards crossing near the y-axis and approaches negative infinity as x approaches zero from the left.

The x-axis and y-axis intersect at the origin and both have markings at regular intervals indicating units. The graph also shows horizontal and vertical grid lines for reference. These graphs illustrate the relationship and intersections between a linear function and its reciprocal function.

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3 section questions

Rational functions

Rational functions of the form $f\left(x\right)=\frac{ax+b}{cx+d}$

You may consider rational functionsto be a generalisation of the reciprocal function, certainly in the way in which we are going to approach graphs of rational functions.

Important

Rational functions are functions that are the ratio of two polynomial functions.

In this section, you will explore rational functions involving linear functions, that is, functions of the form $f\left(x\right)=\frac{ax+b}{cx+d}$, where a, b, c and d are real numbers and $c\neq 0$. You can visualise the graph of various rational functions of this form in the applet below.

Activity

Use the sliders to visualise various rational functions of the form $f(x) = \dfrac{ax+b}{cx+d}$.

Describe some of the graphical features of rational functions.

Form rules about:

- the intercepts of the rational functions of the form $f(x) = \dfrac{ax+b}{cx+d}$.
- the vertical asymptote and horizontal asymptote of rational functions of the form $f(x) = \dfrac{ax+b}{cx+d}, c \neq 0$.

Interactive 1. Graphical Representation of Rational Functions.

More information for interactive 1

This interactive tool allows users to visually explore rational functions of the form $f(x) = \dfrac{ax-b}{cx-d}$ By adjusting the coefficients a, b, c, and d using sliders, the equation and graphs changes in real time. a can be adjusted between −5 to 2, b from −5 to 5, c from 1 to 5 and d from −5 to 5. The tool automatically calculates and displays the

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vertical asymptote ($x = -\frac{d}{c}$), horizontal asymptote ($y = \frac{a}{c}$), x-intercept ($\frac{-b}{a}, 0$), and y-intercept ($0, \frac{b}{a}$).

For example, with $f(x) = \frac{(-x+4)}{(2x+3)}$, the graph shows a vertical asymptote at $x = -1.5$, horizontal asymptote at $y = -0.5$, x-intercept at $(4, 0)$, and y-intercept at $(0, 1.33)$. Users can manipulate the coefficients to see how the graph transforms - watching the asymptotes shift positions, observing intercepts move, and noting how the curve's shape changes between different configurations.

The interactive provides an intuitive way to connect the algebraic form of rational functions with their graphical behavior, helping users develop a deeper understanding of these important mathematical functions.

Important

The graph $y = f(x)$ has:

- a **vertical asymptote** when the denominator, $cx + d$, is equal to zero. Thus, the equation of the vertical asymptote is $x = -\frac{d}{c}$.
- an x **-intercept** when $f(x) = 0 \Leftrightarrow ax + b = 0 \Leftrightarrow x = -\frac{b}{a}$. Hence, the zero is a consequence of the numerator.
- a **horizontal asymptote** $y = \frac{a}{c}$ as $x \to \pm\infty$.
- a y **-intercept** at $y = f(0) = \frac{b}{d}$.

For example, consider the function $f(x) = \frac{x+2}{3x-2}$, $x \neq \frac{2}{3}$. The graph of function f is shown below.

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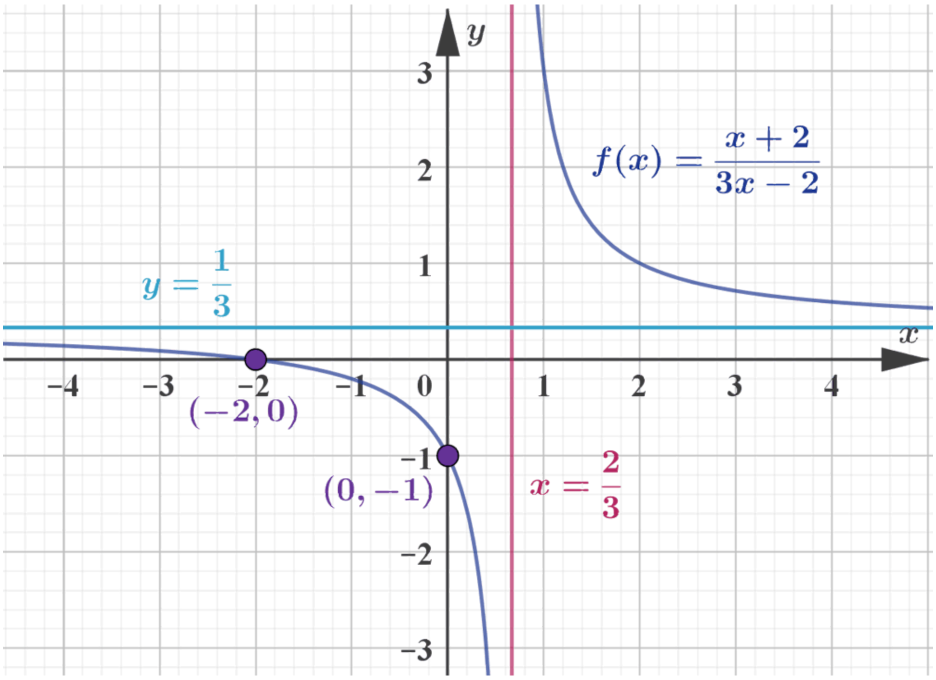

More information

The image is a graph of the function (f(x) = \frac{x + 2}{3x - 2}) with asymptotes and specific points highlighted. The X-axis represents the variable (x) and is marked with integers including -3, -2, -1, 0, 1, 2, and 3. The Y-axis represents (f(x)), with integers similarly marked from -3 to 3.

Visible on the graph is a curve representing the function. There is a vertical asymptote at (x = \frac{2}{3}), highlighted by a vertical line, indicating that as (x) approaches (\frac{2}{3}), the function increases to positive infinity on one side and decreases to negative infinity on the other.

A horizontal asymptote is seen along the line (y = \frac{1}{3}), showing that as (x) moves towards positive or negative infinity, the function approaches this value.

Points marked on the curve include ((-2, 0)) and ((0, -1)), indicated by purple dots. The curve shows a smooth transition from one asymptote to the other while passing through these specific points.

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As $x \to \frac{2}{3}^+$, function f increases without bound, $f(x) \to +\infty$ and as $x \to \frac{2}{3}^-$, function f decreases without bound, $f(x) \to -\infty$. Therefore, the graph has a vertical asymptote at the line $x = \frac{2}{3}$.

Further, as x increases without bound, the graph of function f approaches the line $y = \frac{1}{3}$ from above, and as x decreases without bound, the graph of function f approaches the line $y = \frac{1}{3}$ from below. Using mathematical notation:

- as $x \to +\infty$, $f(x) \to \frac{1}{3}^+$
- as $x \to -\infty$, $f(x) \to \frac{1}{3}^-$

and thus, the line $y = \frac{1}{3}$ is a horizontal asymptote of the graph.

The x-intercept of the graph of the function f occurs when $f(x) = 0 \Leftrightarrow \frac{x+2}{3x-2} = 0 \Leftrightarrow x + 2 = 0 \Leftrightarrow x = -2$. Finally, the y-intercept of the graph is -1, as $f(0) = \frac{2}{-2} = -1$.

Notice that for the horizontal asymptote, you consider the behaviour of the function as $|x| \to \pm\infty$. Although the concept of the limithas not been covered yet, it is useful to consider how this rule is derived analytically. A standard method is to multiply the rational function by a factor $\frac{1/x}{1/x}$, which is equal to 1.

$$f(x) = \left(\frac{ax+b}{cx+d}\right)\frac{\frac{1}{x}}{\frac{1}{x}} = \frac{a + \frac{b}{x}}{c + \frac{d}{x}}.$$

Consider what happens at the extremes of the domain. As $x \to \pm\infty$, the two terms divided by x become vanishingly small, i.e. $\frac{b}{x} \to 0$ and $\frac{d}{x} \to 0$, and thus $f(x) \to \frac{a}{c}$, which is the value of the horizontal asymptote.

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Example 1

★★☆

Consider the function $f\left(x\right)=\dfrac{4x+5}{3x-1}$.

a) Find the asymptotes of the function.

b) State the domain and range of the function.

c) Find the axes intercepts.

d) Discuss the behaviour of the function as it approaches its asymptotes.

e) Sketch the graph of the function.

	Steps	Explanati
a)	The function is not defined for $x=\frac{1}{3}$ and thus the graph has a vertical asymptote at the line $x=\frac{1}{3}$. The function has a horizontal asymptote at the line $y=\frac{4}{3}$.	A function has a v asymptote at the p the function is not A rational function form $f\left(x\right)=\frac{ax}{cx}$ horizontal asympt line $y=\frac{a}{c}$.
b)	The domain of the function is $$Df=\left\{x\vert x\in\mathbb{R},x\neq\tfrac{1}{3}\right\}.$$ The range of the function is $$Rf=\left\{y\vert y\in\mathbb{R},y\neq\tfrac{4}{3}\right\}.$$	The graph of the f does not intersect asymptotes.

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	Steps	Explanati
c)	The y-intercept of the function is -5, as $f(0) = \frac{5}{-1} = -5$.	Substitute $x = 0$ in function.
	The x-intercept of the function is $-\frac{5}{4}$, as $f(x) = 0 \Leftrightarrow \frac{4x+5}{3x-1} = 0 \Leftrightarrow 4x + 5 = 0 \Leftrightarrow x = -\frac{5}{4}$.	Solve the equation .
d)	As $x \to +\infty$, $f(x) \to \frac{4}{3}^{+}$.	For large values of function is positive
	As $x \to -\infty$, $f(x) \to \frac{4}{3}^{-}$.	There are values o such that the func smaller than $\frac{4}{3}$.

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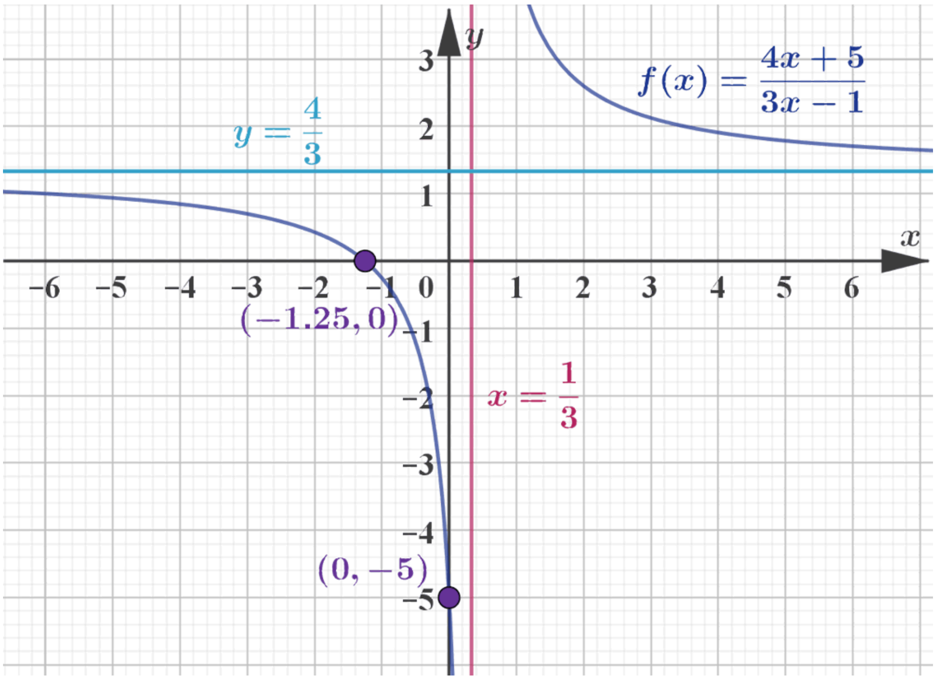
	Steps	Explanati
e)	The graph of the function is sketched below, showing all relevant features. 	

Remember that when **sketching** a graph, you must indicate the special features and the general shape of the graph, but it does not have to be an exact drawing.

Exam tip

When sketching rational functions, look for the following four different graphical features, even if they are not explicitly requested in the question:

- x **-intercept** .
- **vertical asymptote** , in the form $x = k$, where k is a constant.
- **horizontal asymptote** , in the form $y = k$, where k is a constant.

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- **y -intercept .**

Be aware

The graph of a rational function has two branches that appear either in quadrants 1 and 3 or in quadrants 2 and 4. Check the value of the function for a couple of x values to determine where the two branches lie.

5 section questions

2. Functions / 2.8 Rational functions

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-sl/sid-177-cid-761925/book/checklist-id-26586/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- find the domain and range of reciprocal functions
- sketch the graph of reciprocal functions, by showing features such as axes intercepts and asymptotes
- determine whether functions are self-inverse functions
- find the domain and range of rational functions
- sketch the graph of rational functions, by showing features such as axes intercepts and asymptotes.

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2. Functions / 2.8 Rational functions

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-sl/sid-177-cid-761925/book/investigation-id-26587/print/) Assign

This applet shows the graph of the function $y = \frac{a}{x^n}$, where you can choose the values of a and n.

Interactive 1. Graphical Representation of a Function.

More information for interactive 1

This interactive enables users to explore the concept of reciprocal power functions through the graph of $y = \frac{a}{x^n}$. By adjusting the parameters "a" (ranging from -300 to 300) and "n" (ranging from 1 to 9), users can observe how these changes affect the shape and behavior of the graph.

If a is negative, the graph flips: the right branch rises from left to right, and the left branch appears above the $x - \text{axis}$. This inversion occurs because the negative numerator reverses the sign of the output.

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Varying n introduces further complexity. For odd n (e.g., 3, 5), the graph retains symmetry about the origin but becomes steeper near the y-axis and flatter away from it. For even n (e.g., 2, 4), the graph appears only in the first and third quadrants (if $a > 0$) or second and fourth quadrants (if $a < 0$), with each branch becoming more compressed as n increases.

Through experimentation, users can discover how a controls vertical scaling and reflection, while n determines the steepness and symmetry of the curve, deepening their understanding of reciprocal power functions.

Set $n = 1$, so the function has the form $y = \frac{a}{x}$, and choose a positive value of a. Explain, by thinking about what the function does, why the part of the graph to the right of the y-axis falls from the left to right.

Explain why the part of the graph to the left of the y-axis is entirely below the x-axis.

Choose a negative value of a. How does this change the graph?

Explore the effect of changing n to different values. Describe your observations and try to explain them.

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