TOPIC 2
FUNCTIONS

SUBTOPIC 2.6
MODELLING SKILLS

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Teacher view

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The big picture

Mathematical modelling is the process by which mathematics is used to help us to understand the world around us. The topics that you are learning on this course, such as algebra, calculus and vectors, are all used in these models, as well as many other topics that you may go on to discover. They all have one thing in common – the reduction of real-life, which is often chaotic and unpredictable, into an abstract simplification based on initial assumptions.

The figure below shows the basic process in a flowchart. You start with a real-life scenario, which you can call a ‘problem’ as there is typically something to be solved. You then reduce it in scope by ignoring all of those surrounding factors that make the real-world problems so complex, and devise a set of starting assumptions around the problem. An example might be stating that a massive object is a particle with zero volume. You then test your model, which may be a formula, by calculating with it and seeing it the results conform to what is really happening. If it works, you can then apply the model to understanding the scenario and perhaps predicting its past and future states. You might then want to refine the model (going back to our particle, you could allow it to have volume). If the model does not work, then go back and refine it, then try again. And that’s basically what many mathematicians do in their working lives to help mankind learn, predict and understand our surroundings.

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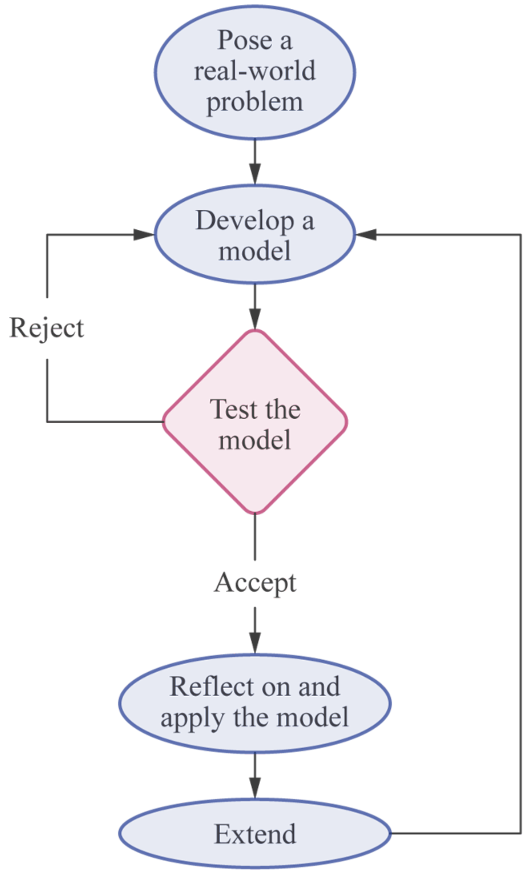

More information

The flowchart represents a problem-solving process. It begins with "Pose a real-world problem" at the top, indicating the start of the process. An arrow directs to "Develop a model," showing the next step in the process. From "Develop a model," an arrow leads to "Test the model," which is depicted in a diamond shape, representing a decision point.

From "Test the model," there are two potential pathways. If the model is rejected, indicated by a left arrow labeled "Reject," the process loops back to "Develop a model" to iterate the process. If the model is accepted, indicated by a downward arrow labeled "Accept," the process continues to "Reflect on and apply the model." From there, an arrow leads to "Extend," at the bottom, indicating further development or application of the model.

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Concept

Modelling real-life phenomena requires an in-depth understanding of functions as represented by formulae and graphs. While learning how to develop a **model** — either by using algebraic techniques or by fitting data using functions — reflect on whether mathematical models are completely accurate **representations** of physical situations. Think about the limitations of mathematical models when applied to real-life scenarios.

Student view

Solving equations

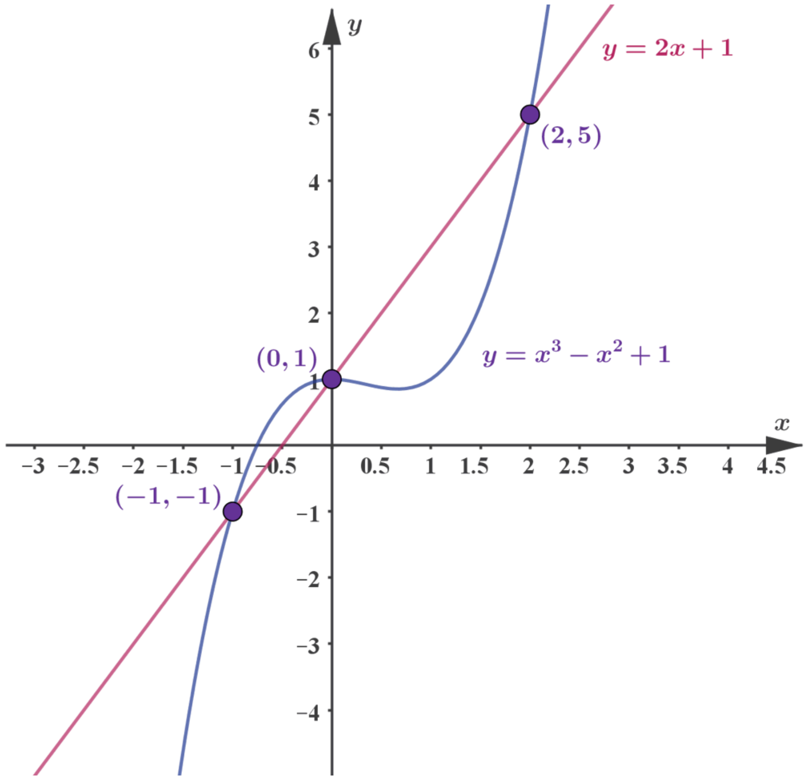
Graphical approach

Solving equations using technology

When modelling real-life phenomena, it is natural to end up with an equation. Solving an equation means finding all the values that make the statement true. In general, there are two methods for solving an equation: analytic methods and graphical methods. In this section, you will focus on how to solve equations using a graphical method.

To solve an equation graphically, you draw the graph for each side of the equation and see where the curves cross. The x-values of these intersection points are the solutions to the equation.

For example, consider the equation $x^3 - x^2 + 1 = 2x + 1$, where the left-hand side (LHS) is a function $f\left(x\right) = x^3 - x^2 + 1$ and the right-hand side (RHS) is a function $g\left(x\right) = 2x + 1$. Use your GDC to graph both functions and find the x-values of the points of intersection. The graphs of the functions are shown below.



More information

The image is a graph showing the functions y = x^3 - x^2 + 1 and y = 2x + 1. The x-axis ranges from approximately -3.5 to 4.5, while the y-axis ranges from approximately -4.5 to 6. The cubic function y = x^3 - x^2 + 1 is represented by a blue curve

while the linear function y = 2x + 1 is depicted as a magenta line. The graph highlights three points of intersection labeled as (-1, -1), (0, 1), and (2, 5), indicating where the two functions have the same x and y values. The blue curve passes through these points, showing a bend at (0, 1) before extending upward. The magenta line crosses the blue curve three times at these intersection points, demonstrating the solutions to the equation given in the text.

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Therefore, the solutions to the equation $x^3 - x^2 + 1 = 2x + 1$ are $x = -1, 0, 2$.

Alternatively, for the equation $x^3 - x^2 + 1 = 2x + 1$, you can make the RHS of the equation equal to zero and then consider the RHS of the equation as a function for which you can find the x-intercepts. That is:

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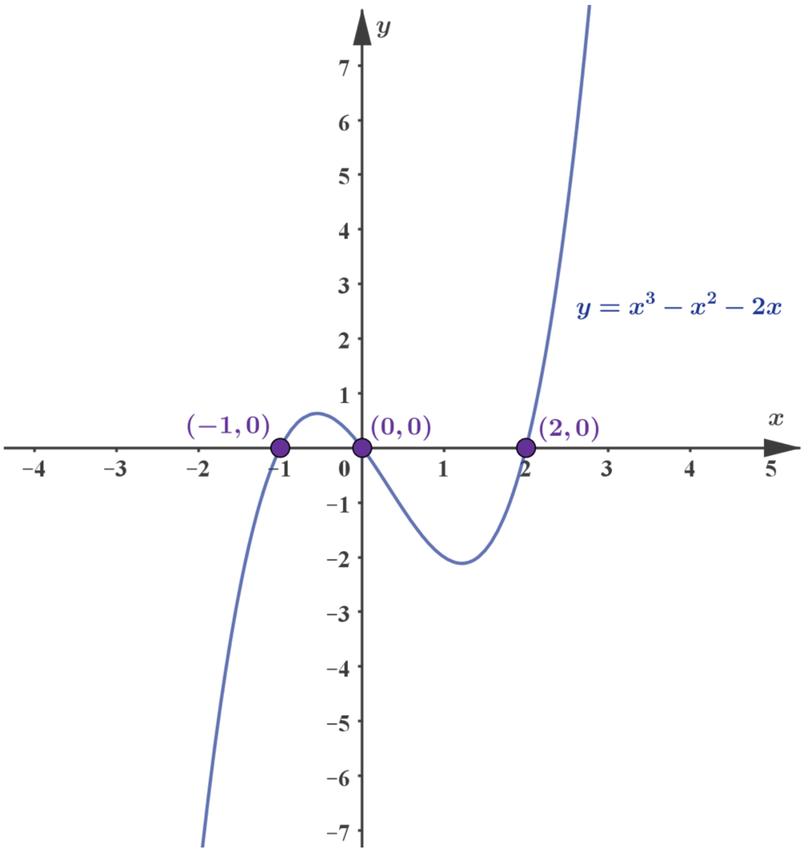
Steps	Explanation
$x^3 - x^2 + 1 = 2x + 1$	Make the RHS equal to zero.
$x^3 - x^2 - 2x + 1 - 1 = 0$	Simplify.
$x^3 - x^2 - 2x = 0$	Consider the LHS as a function and use your GDC to find the x -intercepts.

The graph of the function $y = x^3 - x^2 - 2x$ is shown below.



More information

The image shows the graph of the function (y = x^3 - x^2 - 2x). The X-axis is labeled from -4 to 5, with integer increments. The Y-axis ranges from -7 to 7, marked at integer increments. The graph passes through three notable intercepts, specifically at the points (-1, 0), (0, 0), and (2, 0), which are highlighted on the graph. These points indicate the solutions to the equation (x^3 - x^2 - 2x = 0), where the curve crosses the X-axis. The graph initially decreases and crosses the X-axis at (x = -1), then rises, crosses again at (x = 0), dips slightly, and finally rises sharply crossing at (x = 2).

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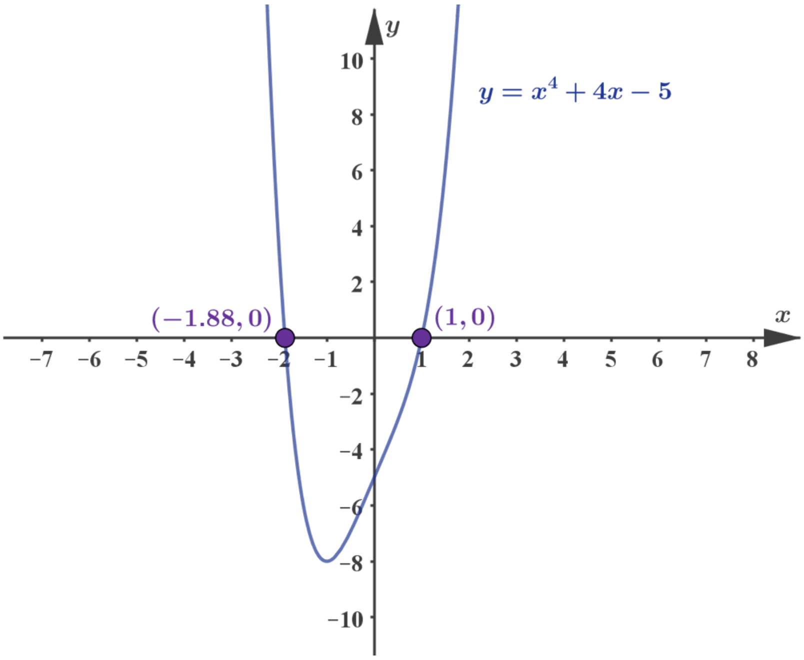
The solutions to the equation $x^3 - x^2 - 2x = 0$ are the x-values of the x-intercepts and thus the solutions are $x = -1, 0, 2$.

Example 1

★☆☆

Solve the equation $x^4 + 4x - 5 = 0$.

Use your GDC to obtain the graph of $y = x^4 + 4x - 5$ and find the x -values of the x -intercepts of the graph. The graph with its x -intercepts is shown below.



The solutions to the equation $x^4 + 4x - 5 = 0$ are $x = -1.88$ and $x = 1$.

Student view

Example 2

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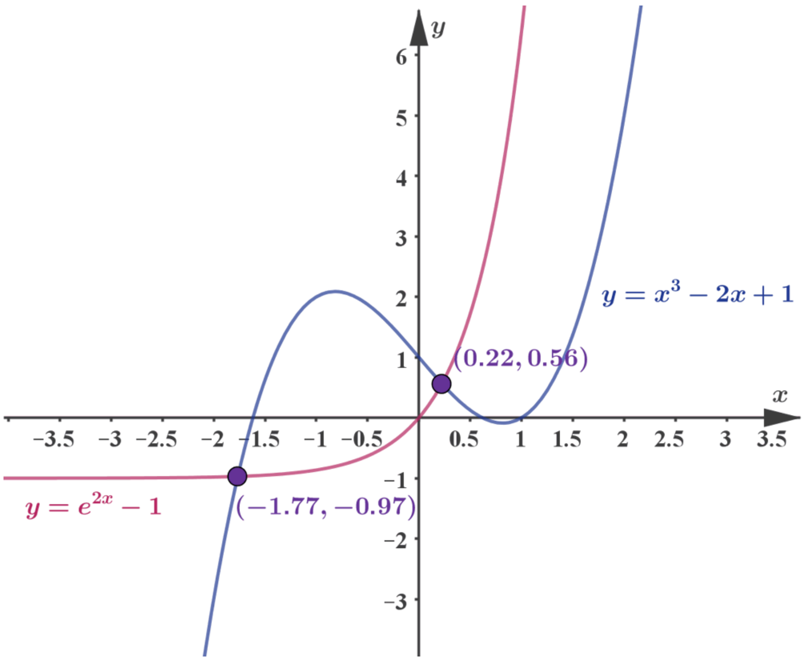
Solve the equation $x^3 - 2x + 1 = e^{2x} - 1$

Use your GDC to graph the functions $y = x^3 - 2x + 1$ and $y = e^{2x} - 1$ and find the x -values of the points of intersection of the graphs. The graphs of the functions are shown below.



The x -values of the points of intersection of the graphs are $x = -1.77$ and $x = 0.22$, which are the solutions to the equation $x^3 - 2x + 1 = e^{2x} - 1$.

Note that in this case the solutions are rounded to two decimal places. On your IB exam you will need to write the solution rounded to three significant figures.

4 section questions

2. Functions / 2.6 Modelling skills

Systems of equations

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Simultaneous equations

Sometimes you will come across two or more unknown quantities, and two or more equations relating them. These are called simultaneous equations.

In this section, you will investigate a real-life scenario of mountain hiking to form and solve simultaneous equations using a graphical method – use of a GDC is required.

International Mindedness

Professional mountain hikers and climbers need to carefully plan the routes they are going to take, especially when the **slope** of the mountain is challenging. Everest is one of the most challenging mountains to hike and variables such as slopes, hiking speeds at different slopes and altitudes, and the time required to cover each stretch of the trek are important factors that should be considered when planning a hike.

A topographic map of Mount Everest
Credit: MatthewBrosseau Getty Images

More information

This image is a topographic map of Mount Everest. It prominently displays contour lines that indicate the elevation and slope of different sections of the mountain. The label 'MOUNT EVEREST' is centrally placed on the map. The contour lines are dense in some areas, suggesting steep terrain, while other regions have more spaced-out lines, indicating gentler slopes. This type of map aids in visualizing the mountain's terrain and is crucial for planning hiking routes, as it shows the varying altitudes and gradients that hikers would encounter. No color-specific information is essential for understanding, but the layout and contour density provide key geographical details.

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Example 1

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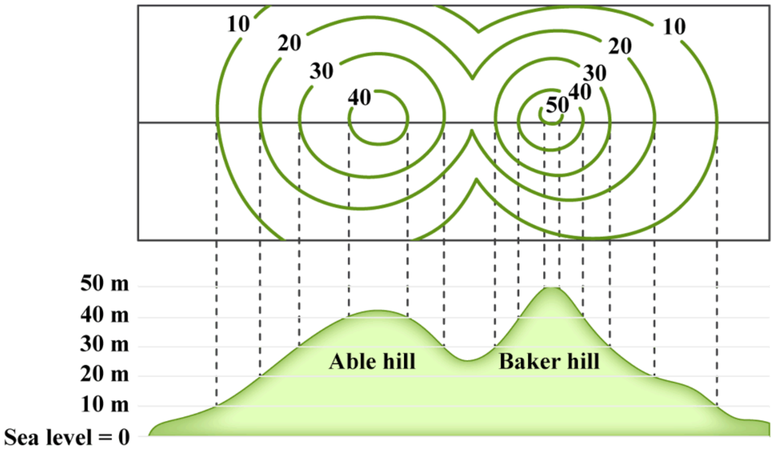
When navigating in wild terrain, the time taken to cover a kilometre can be modelled using a linear equation that depends on the number of metres of ascent.

Topographic map (with contour lines that show points that are on the same level)



Two hills seen from the side, with elevations marked and dotted lines pointing to the coresponding contour lines

More information

The image is a cross-section diagram depicting two hills: Able hill and Baker hill. The upper part of the image shows a top-down view of contour lines representing their arrangement. The lower part of the image provides a side view cross-section of the hills, with Able hill on the left and Baker hill on the right. The hills are labeled, and dotted lines link the corresponding sections of the contour map and the cross-section. The contour lines on the top part form concentric circles indicating elevation levels and terrain layout of the hills. This diagram illustrates the geographical features and elevation differences between the two hills by showing both the contour layout and the vertical profile of the terrain, enabling an understanding of the landscape structure.

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The linear model for the time taken to cover a kilometre as a function of the number of metres of ascent has equation:

$$t = ac + b$$

where

- t is the time to cover a kilometre over wild terrain in minutes
- c is the number of 10 m contours that must be crossed within that kilometre
- a is the time adjustment per contour line
- b is the time it takes someone to walk 1 km on flat terrain.

Hugo and Jay time themselves while climbing to the top of a hill on their walk to see the sea. Over 1 km, they climb eight contour lines (80 m of ascent). It took them 24 minutes. This can be written as $24 = 8a + b$. Then, they realise that there is another hill to walk up before they see the sea. To get to the top of it takes them 30 minutes and they cross 12 contour lines over one horizontal kilometre.

a) Write this information as an equation that relates a and b.

b) Solve these two equations simultaneously to find a, the time adjustment per contour line, and b, their time to walk 1 km on flat terrain.

Hugo and Jay plan another walk. This time they notice that over 1 km there are five contour lines.

c) Suggest a linear model that describes this situation and calculate how long will it take them to walk this kilometre, according to the model.

	Steps	Explanation
a)	$30 = 12a + b$	Substitute $t = 30$ and $c = 12$ into the formula $t = ac + b$.
b)	$24 = 8a + b$ $30 = 12a + b$ Refer to the calculator instructions as shown in section 1.8.2.	Use your GDC to solve the simultaneous equations.

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	Steps	Explanation
	$a = 1.5$ and $b = 12$.	So, it takes them 12 minutes to hike a kilometre on level ground, and they need to add 1.5 minutes for every 10 m of ascent.
c)	$t = 1.5c + 12$	Substitute $a = 1.5$ and $b = 12$ into the formula $t = ac + b$.
	$t = 19.5$ minutes	Substitute $c = 5$ into the formula.
	Therefore, to climb five contours, the time will be 19.5 minutes.	

Three equations in three unknowns

There are situations where you need to solve three equations in three unknowns. For example, consider a quadratic function of the form $f(x) = ax^2 + bx + c$, which passes through the points $(1, 0)$, $(2, 9)$ and $(-2, -3)$. To find the equation of the quadratic function, you can substitute the coordinates of the points into the general formula so as to obtain three equations in three unknowns, as follows.

$$f(1) = 0 \Leftrightarrow a(1)^2 + b(1) + c = 0 \Leftrightarrow a + b + c = 0$$

$$f(2) = 9 \Leftrightarrow a(2)^2 + b(2) + c = 9 \Leftrightarrow 4a + 2b + c = 9$$

$$f(-2) = -3 \Leftrightarrow a(-2)^2 + b(-2) + c = -3 \Leftrightarrow 4a - 2b + c = -3$$

This results in the following system of three equations in three unknowns:

$$a + b + c = 0$$

$$4a + 2b + c = 9$$

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$$4a - 2b + c = -3$$

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To find the values of a, b and c you can use your GDC and follow the calculator instructions as shown in section 1.8.2 (/study/app/m/sid-122-cid-754029/book/systems-of-linear-equations-id-27396/).

The solutions are $a = 2, b = 3$ and $c = -5$.

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2. Functions / 2.6 Modelling skills

Linear models

Linear functions

When the dependent variable y is a linear function of the independent variable x, the graph of the function is a straight line, which can be written in gradient–intercept form as

$$y = f(x) = mx + c$$

where m is the gradient of the line and c is the y-intercept.

Making connections

Recall from section 2.1 (/study/app/m/sid-122-cid-754029/book/the-big-picture-id-26160/) that a straight line can also be expressed in the forms:

- gradient—point form: $y - y_0 = m(x - x_0)$, where m is the slope and (x_0, y_0) is a point on the line
- standard form: $Ax + By + C = 0$, where $A,\ B$ and C are constants.

Important

A characteristic feature of **linear models** is that they grow or decrease at a **constant rate.**

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Example 1

★★☆

The table below shows the average carbon dioxide level in the atmosphere, measured in parts per million, from 1980 to 2016. Data obtained from climate.gov (https://www.climate.gov/news-features/understanding-climate/climate-change-atmospheric-carbon-dioxide).

Year	1980	1984	1988	1992	1996	2000	2004	2008	2012
CO_2 level (ppm)	337.5	343.7	349.9	355.6	361.4	370.7	376.5	384.5	392.3

In answering the following questions, give all values rounded to four significant figures.

a) Make a scatter plot of these data and decide whether a linear model is appropriate.

b) Find a linear model using the first and last data points.

c) Use your GDC to find a linear model that fits the data.

d) Using the linear model obtained in part c), estimate the average CO_2 level for 1998.

e) Predict the CO_2 level for the year 2030.

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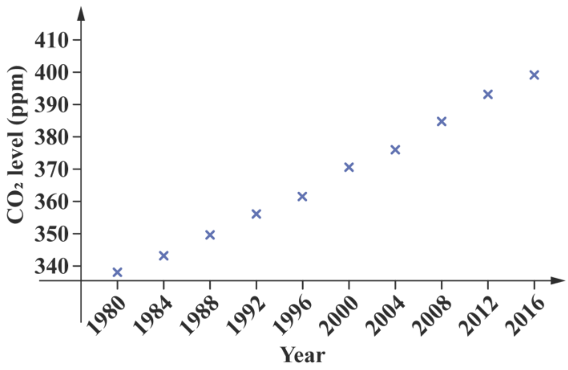
	Steps	Explanation
a)	Create an illustration, such as the one shown below. 	Use your GDC to plot the points in a scatterplot. Notice that the points appear to lie close to a straight line, so it is reasonable to choose a linear model in this case.
b)	Substituting point $(1980, 337.5)$ into the point—intercept form of a line gives $$y - 337.5 = 1.783\ldots(x - 1980)$$ which can be transformed into the gradient—intercept form as $$y = 1.783x - 3194$$	The slope of the line is $$\frac{401.7 - 337.5}{2016 - 1980} = 1.78333\ldots$$ Substitute either the first or last data point into the point—gradient form of a line $$y - y_0 = m(x - x_0)$$
c)	Your GDC gives the slope and y-intercept of the regression line as $$m = 1.762 \text{ and } c = -3153$$ So, the linear model is $y = 1.762x - 3153$	Use your GDC and the statistics utility — linear regression — to obtain the linear model.
d)	$y = 1.762(1998) - 3153 = 367.5$	Substitute $t = 1998$ in the linear equation of part c).
e)	$y = 1.762(2030) - 3153 = 423.9$	Substitute $t = 2030$ in the linear equation of part c).

Student view

In **Example 1**, parts d) and e) asked you to make predictions about the CO_2 levels in 1998 and 2030 respectively, using the linear model as obtained from your GDC.

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- Which prediction do you think is more reliable?
- Do you find it reasonable to use the linear model to predict the CO_2 levels in 2050?

Be aware

Interpolation is the process of estimating a value within the range of the observed values.

Extrapolation is the process of estimating a value outside the range of the observed values.

Suggest some reasons why interpolation is considered to be more reliable than extrapolation.

Making connections

Recall from section 4.4 (/study/app/m/sid-122-cid-754029/book/the-big-picture-id-26239/) that the line of best fit is also called least-squares regression line .

Example 2

★★☆

The American scientist Amos Dolbear observed a relationship between temperature and the rate at which the cricket (an insect related to the grasshopper) produces a chirping sound. The table shows the number of chirps per minute for a species of cricket at various temperatures.

Temperature ($^{\circ}F$)	Chirping rate (chirps/min)
55	45
60	80
65	92
70	114
75	141
80	174
85	202

Student view

Temperature ($^\circ F$)	Chirping rate (chirps/min)
90	226

a) Make a scatter plot of the data on your GDC and decide whether a linear model is appropriate.

b) Find a linear model using the first and last data points.

c) Use your GDC to find a linear model that fits the data.

d) Use the linear model of part c) to estimate the chirping rate at 82°F and 100°F and comment on your predictions.

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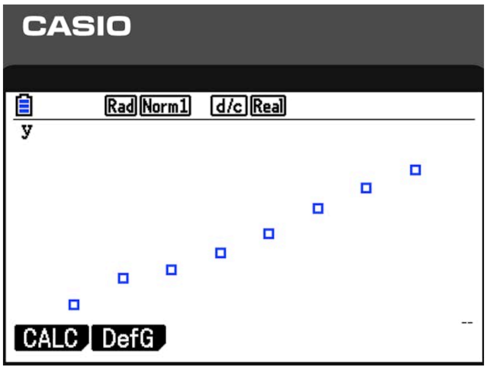
	Steps	Explanation
a)	The result screens from the calculators	Use your GDC to plot the points in a scatterplot. Notice that the points appear to lie close to a straight line, so it is appropriate to choose a linear model to fit the data.



Casio fx-CG50

The image shows a graph displayed on a Casio calculator screen. The graph consists of several blue square points arranged in an upward trend. At the top of the screen, there are calculator settings such as 'Rad', 'Norm1', 'd/c', and 'Real'. The letter 'y' is visible, indicating the y-axis label. Below the graph, there are buttons labeled 'CALC' and 'DefG'. The layout suggests a scatter plot or a simple linear trend being analyzed on the calculator.

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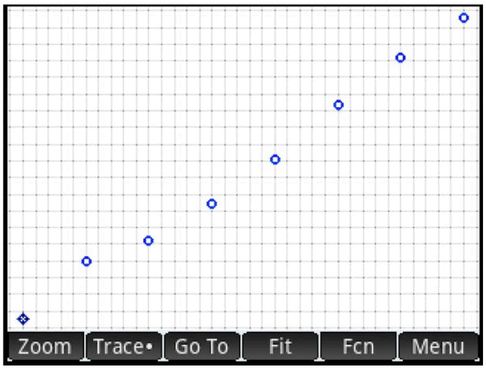

HP Prime

The image is a graph with a grid background, displaying a series of blue data points arranged in an upward trend from the bottom left to the top right. The graph has axes that represent a coordinate system, but no specific labels are present for the axes in this image. At the bottom of the image, there are menu options labeled "Zoom," "Trace," "Go To," "Fit," "Fcn," and "Menu." These buttons suggest interactions for adjusting the view or analysis of the graph data.

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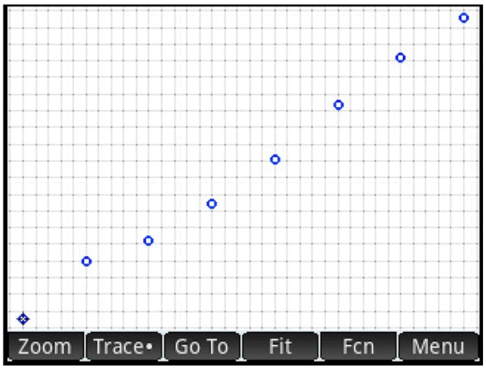
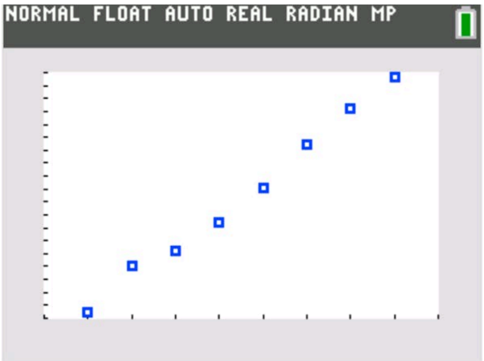
	Steps	Explanation



HP Prime



TI-84 plus CE

The image is a graph displayed on a screen, likely from a calculator. It shows a series of blue square data points forming an upward trend across a grid with dotted lines marking the axes. The graph area is bordered on the left and bottom by axes which lack labeling, making it unclear what units or values they represent. The screen above the graph reads "NORMAL FLOAT AUTO REAL RADIAN MP," likely indicating the mode or settings of the device. There is also a small battery icon on the right, illustrating remaining power. The graph suggests a linear or proportional relationship between two variables, but the absence of axis labels leaves the specific context undetermined.

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	Steps	Explanation
b)	Substituting point $(55, 45)$ into the point—intercept form of a line gives $$y - 45 = 5.17142857\ldots(x - 55)$$ which can be transformed into the gradient—intercept form as $$y = 5.17x - 239.$$	The slope of the line is $$m = \frac{226 - 45}{90 - 55} = 5.17142857\ldots.$$ Substitute either the first or last data point into the point—gradient form of a line $$y - y_0 = m\left(x - x_0\right).$$
c)	Your GDC gives the slope and y-intercept of the regression line as $m = 5.1190476\ldots$ and $c = -236.88095\ldots.$ So, the linear model is $$y = 5.12x - 237.$$	Use your GDC and the statistics utility — linear regression — to obtain the linear model.
d)	$y = 5.1190476\ldots(82) - 236.88095\cdots = 183$	Substitute $x = 82$ in the linear equation of part c).
	$y = 5.1190476\ldots(100) - 236.88095\cdots = 275$	Substitute $x = 100$ in the linear equation of part c).
	The estimate for the chirping rate at $82°\mathrm{F}$ is found by interpolation and thus it is expected to be more reliable than the estimate at $100°\mathrm{F}$, which is found by extrapolation.	

⚠ Be aware

For a linear model, there are no standardised letters to use for input and output. You can use letters that are associated with the context. For each linear relation, you may identify two variables (the input variable and the output variable) plus a coefficient of the input variable and a constant.

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Modelling with quadratic functions: optimisation

Maximising or minimising quadratic models

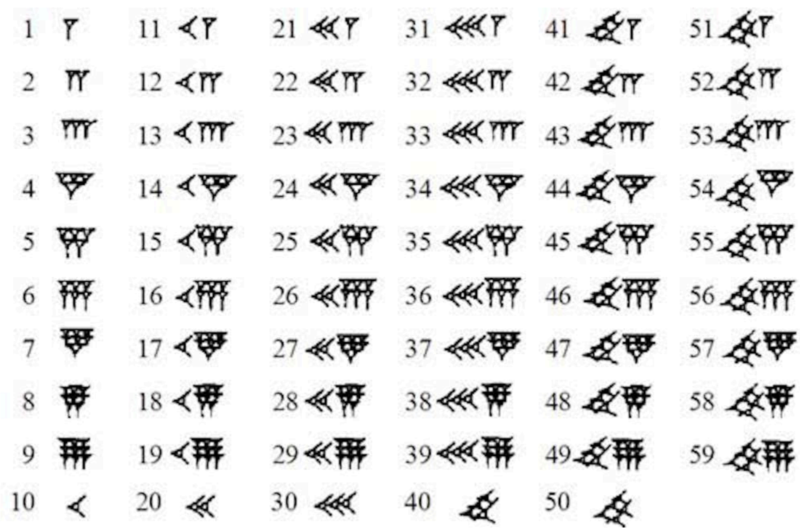
In modelling with quadratic functions, you are often interested in finding the 'optimal' solution (optimisation), which means finding the maximum or minimum point of a parabola.

Area division

Quadratic models have been extensively used in area division problems since ancient times.

International Mindedness

T he Sumerians and the Babylonians were early advanced civilisations based in Mesopotamia, a historical region of Western Asia. Sumerian and Babylonian mathematicians used a sexagesimal (base-60) counting system and inscribed notation on clay tablets that were baked in the sun.



Sexagesimal numbers in value notation
Source: (https://commons.wikimedia.org/wiki/File:Babylonian_numerals.jpg

More information

The image shows a table of numbers from 1 to 60 represented with Babylonian numerals. Each numeral is depicted in a series of grid lines, with unique symbols marking each number. The table is organized in six columns and ten rows, starting with the numeral 1 on the left and moving numerically to the right. Each Babylonian numeral is an arrangement of their traditional cuneiform symbols, distinct for each number.

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The image visually maps these cuneiform symbols to our understanding of the numbers 1 to 60 in a grid format, demonstrating the sexagesimal (base-60) counting system used by ancient Mesopotamian civilizations.

[Generated by AI]

The clay tablet shown in the photo below depicts a model for dividing the area of a 'Temen' (a large quadrangle that was staked out to approximate the area of a field). The mathematician Otto E. Neugebauer (1899—1990) discovered in the late 1920s that the notation on this clay tablet appears to give solutions to a series of quadratic equations.

Clay tablet showing a juridical field division of the area of a Temen
Credit: Atypeek Getty images

Example 1

★★★

Two rectangular fields are to be enclosed with 100 m of fence, as shown below.

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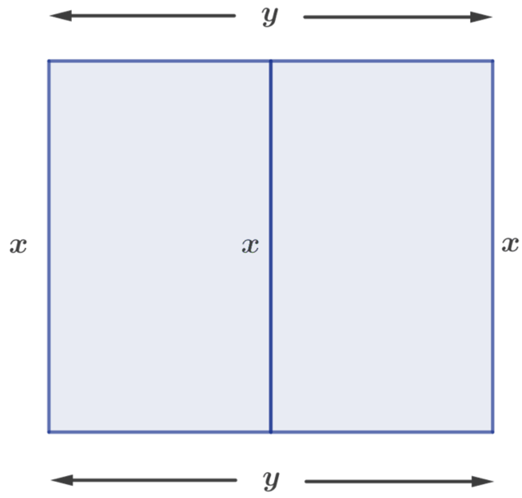
view



More information

The image depicts a diagram of two adjacent rectangular fields. Each field is a rectangle, and the diagram illustrates the arrangement of the fields and their dimensions. The total width of each field is labeled as 'x', and the total length of the fields is labeled as 'y'. The rectangles are placed side by side, so the combined width is 2x when considering both fields. The rectangle is surrounded by arrows indicating the dimensions: 'y' on the top and bottom sides, pointing horizontally, and 'x' on the left and right sides, pointing vertically.

[Generated by AI]

a) Express the total length, y, of the two rectangular fields in terms of the width, x.

b) Create a function that models the total area of the fields in terms of the width x.

c) Determine the value of the width x that results in the maximum area of the two fields.

d) Find the maximum area of the two fields.

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	Steps	Explanation
a)	$3x+2y=100$	The total length of fen equals 100 m , so you form an equation linki and y .
	$2y=100-3x$ $y=\dfrac{100-3x}{2}$	Solve the equation for
b)	$A=xy$	The total area of the tv fields is given by $A=\text{length}\times\text{width}$.
	$A\left(x\right)=x\left(\dfrac{100-3x}{2}\right)$	Express the area funct in terms of x only, by substituting the expres for y found in part a).
	$A\left(x\right)=x\left(50-\dfrac{3}{2}x\right)$ $A\left(x\right)=-\dfrac{3}{2}x^2+50x$	Simplify and expand th brackets.
c)	$A\left(x\right)=-\dfrac{3}{2}x^2+50x$	The area function is a quadratic function. Fir write it in standard for
	The x -coordinate of the vertex is $$-\frac{b}{2a}=-\frac{50}{2\left(-\frac{3}{2}\right)}=\frac{50}{3}$$ So $x=\dfrac{50}{3}$ will result in the maximum area.	Because $a=-\dfrac{3}{2}<($ the quadratic function concave down. Thus, t vertex will be the maxi point.

Student view

	Steps	Explanation
d)	The maximum area is $$A\left(\frac{50}{3}\right)=\frac{50}{3}\left(\frac{100-3\left(\frac{50}{3}\right)}{2}\right)$$ $$=\frac{50}{3}\left(\frac{100-50}{2}\right)=\frac{2500}{6}=\frac{1250}{3}\text{ m}^2.$$	The y -coordinate of t vertex is the maximum area. You may find it e to use the factorised f of $A(x)$ to evaluate th

Optimising product design

A team of designers has been asked to create a model that describes the area of a playground satisfying the following design requirements:

- The playground has the shape of a rectangle with a semicircle placed at one end of it.
- The playground should have a perimeter of 30 metres.

The model that describes the area of the playground is a quadratic function

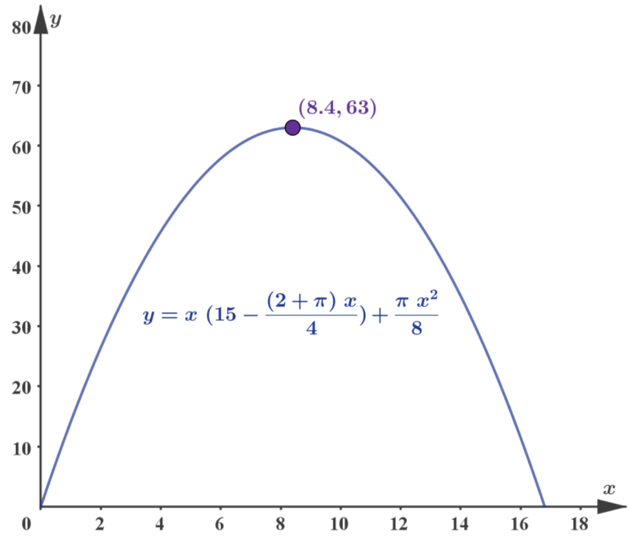
$$A=x\left[15-\frac{(2+\pi)x}{4}\right]+\frac{\pi x^2}{8},$$

where x is the width of the playground.

Example 2

★★☆

Use the quadratic function $A=x\left[15-\frac{(2+\pi)x}{4}\right]+\frac{\pi x^2}{8}$ to find the width x that gives the maximum area for the playground.



The vertex is $(8.40,\ 63.0)$ (correct to 3 significant figures), which means that when the width x is $8.40\ \text{m}$ the area of the playground takes a maximum value ($A = 63\ \text{m}^2$).

Use a GDC to obtain the graph of the function

$$y = x\left[15 - \frac{(2+\pi)\,x}{4}\right] + \frac{\pi x^2}{8}.$$

The parabola is concave down and so the vertex is a maximum point.

Fitting data with quadratic functions

In section 2.6.3 (/study/app/m/sid-122-cid-754029/book/linear-models-id-27471/), you used a GDC to fit data sets with linear models. In this section, you will use similar method to fit data that tends to show a parabolic behaviour.

Example 3

★★☆

A ball is dropped from a terrace, 450 metres above the ground, and its height h above the ground is recorded at 1-second intervals in the table below.

Overview
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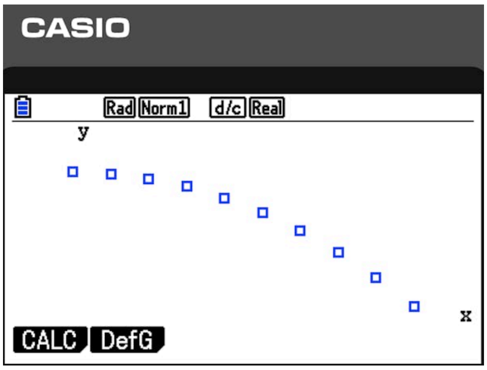
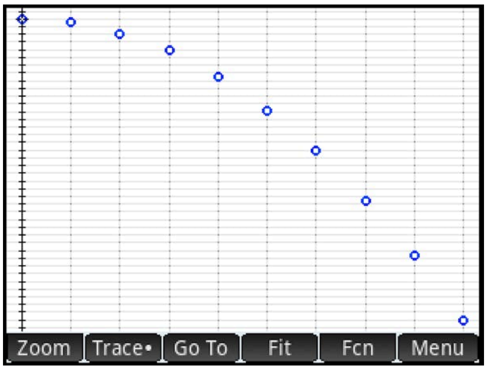
Time (seconds)	Height (metres)
0	450
1	445
2	430
3	409
4	375
5	331
6	280
7	215
8	144
9	59

a) Draw a scatter plot of the data and decide whether a quadratic model is appropriate.

b) Use your GDC to find a quadratic function that fits the data.

c) Use your model to estimate the time at which the ball hits the ground.

Student
view

	Steps	Explanation
a)	The result screens from the calculators  **Casio fx-CG50** The image shows a screenshot from a Casio calculator. The screen displays a graph with a series of blue square markers arranged in a descending pattern from left to right. The axes are labeled 'y' for the vertical axis and 'x' for the horizontal axis. The top of the screen has settings indicators like 'Rad', 'Norm1', 'd/c', and 'Real'. At the bottom, there are black buttons labeled 'CALC' and 'DefG'. [Generated by AI]  **HP Prime** The graph displays a series of data points plotted in a descending order. The X-axis is labeled with uniform intervals, representing the independent variable. The Y-axis represents the dependent variable, also labeled with consistent intervals. Data points are marked with blue circles across the graph. Starting from the top left, the trend shows a gradual decrease in the variable measured along the Y-axis as the X-axis values increase. This indicates a negative correlation between the X and Y values. [Generated by AI]	The points on the scatterplot resemble the shape of a parabola; therefore, it is appropriate to use a quadratic model to fit the data.

Overview
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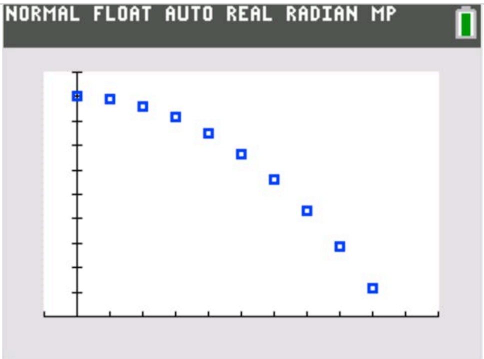
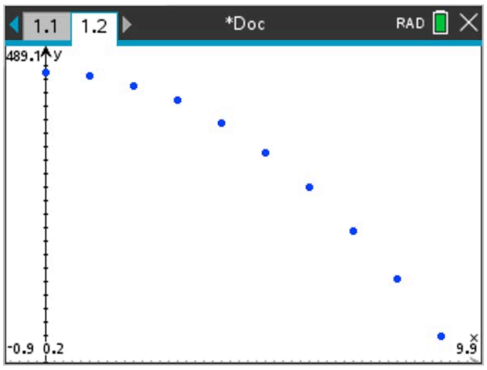
	Steps	Explanation
	 **TI-84 plus CE** The image shows a graph with a Y-axis featuring labeled intervals and an X-axis with smaller divisions, indicating a series of data points. The graph highlights a set of blue square data points decreasing linearly from left to right, suggesting a downward trend. Above the graph window, there are text indicators set to 'NORMAL FLOAT AUTO REAL RADIAN MP,' typical of graph calculator settings. [Generated by AI]  **TI-nspire CX** The image depicts a scatter plot on a TI-nspire graphing calculator screen. The X-axis is marked from -0.9 to 9.9, while the Y-axis ranges from approximately 0.2 to 489.1. Blue dots represent data points distributed diagonally across the graph, trending downward from the top left to the bottom right, indicating a negative correlation. [Generated by AI]	
b)	$y = -4.93x^2 + 1.23x + 449$	Use your GDC to obtain the quadratic model that fits the data.

Student view

Overview
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	Steps	Explanation
c)	The ball hits the ground after 9.66 seconds.	Use your GDC to find the positive x -intercept of the parabola.

Example 4

★★☆

In a rocket modelling competition, contestants launch their rockets and the rocket that flies the longest horizontal distance wins the first prize. You have been preparing for the competition by measuring the horizontal distance that your rocket can fly when it is launched at different angles. The table below shows some measurements of your experiment.

Angle (degrees)	20	30	40	50	60	70
Distance (feet)	371	465	511	498	439	325

a) Make a scatter plot of these data and decide whether a quadratic model is appropriate.

b) Use your GDC to find a model for the horizontal distance of the rocket.

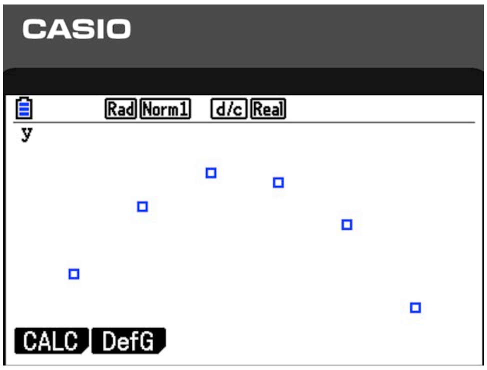
c) Using the quadratic model of part b), estimate the horizontal distance the rocket will travel when launched at $54°$.

d) Use your model to find at which launch angle the rocket reaches a maximum horizontal distance.

e) What is the maximum horizontal distance that the rocket can travel?

Student
view

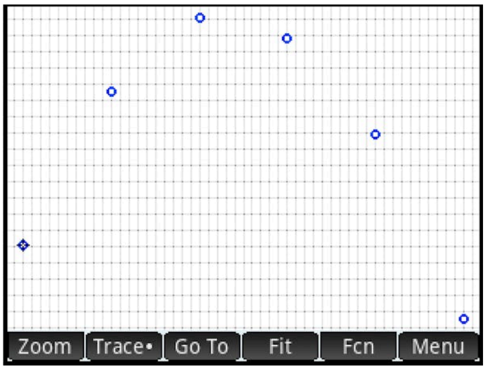
<table>
<tr><th></th><th>Steps</th><th>Explanation</th></tr>
<tr><td>a)</td><td>
The result screens from the calculators



Casio fx-CG50

The image displays a calculator screen from a Casio model. At the top, several options are visible: 'Rad,' 'Norm1,' 'd/c,' and 'Real.' Below these is a plot area with several small blue square symbols representing plotted data points scattered across the screen. On the left side, it shows the letter 'y,' presumably indicating the y-axis of the graph. At the bottom left, the words 'CALC' and 'DefG' are shown. This represents the result screen from the calculator showcasing a scattered data plot.

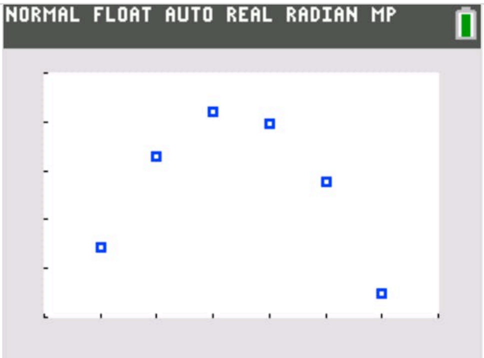
[Generated by AI]



HP Prime

The image depicts a graph on a grid with several points marked by blue circles. The X-axis represents the horizontal grid lines and the Y-axis represents the vertical grid lines. There are six data points placed randomly on the grid. At the bottom, there are six buttons labeled "Zoom," "Trace," "Go To," "Fit," "Fcn," and "Menu." The graph does not display specific numerical values on the axes or data points, but is interactive based on the button labels.

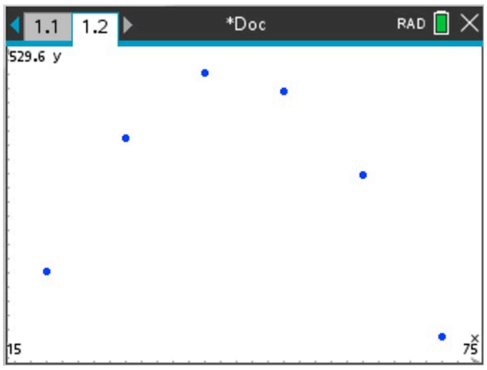
[Generated by AI]
</td><td>The data points on the scatterplot follow the shape of a parabola and thus a quadratic model would be appropriate.</td></tr>
</table>

	Steps	Explanation
		

TI-84 plus CE

The image is a screenshot of a graph displayed on a TI-84 calculator screen, featuring a scatter plot. The top of the screen shows modes such as "NORMAL FLOAT AUTO REAL RADIAN MP" and a battery indicator on the right. The graph itself is a scatter plot with five blue square markers arranged across the graph area. The X and Y axes have no visible numeric labels, but the ticks suggest several intervals marking them. The data points form a curve-like trend, starting from the lower left and progressing to the upper center before descending to the lower right.

[Generated by AI]



TI-nspire CX

The image is a scatter plot graph from a TI-nspire CX calculator display. The graph consists of blue data points plotted on a grid. The X-axis is labeled with numbers starting from 15 and the Y-axis is labeled starting from 529.6. Both axes have numerical markings. The data points are scattered across the plot area, suggesting a possible upward trend. The graph interface shows a tab labeled "*Doc" and an indication of the RAD (radian) mode.

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Overview
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	Steps	Explanation
b)	$y = -0.261x^2 + 22.5x + 24.2$	Use the quadratic regression utility in your GDC to obtain the quadratic model that fits the data.
c)	$y = 478$ feet. Note that if you use the more accurate values your calculator gives for the parameters, the answer is $y = 482$ feet.	Substitute $x = 54$ into the function of the question b).
d)	The rocket travels the maximum horizontal distance when it is launched at 43.1 degrees. Note that if you use the more accurate values your calculator gives for the parameters, the answer is 43.2 degrees.	Use your GDC to find the x -coordinate of the vertex of the quadratic function.
e)	The maximum horizontal distance the rocket can cover is 509 feet. Note that if you use the more accurate values your calculator gives for the parameters, the answer is 512 feet.	Use your GDC to find the y -coordinate of the vertex.

4 section questions

2. Functions / 2.6 Modelling skills

Direct and inverse variation

Direct variation

The simplest form of linear model is given by

$$y = mx$$

In this case, y is said to **vary directly** with x, or to be **directly proportional** to x.

Student
view

m is called the constant of variation or the **constant of proportionality**.

If the y-intercept is non-zero, the linear model is given by

$$y = mx + c, \quad c \neq 0.$$

Important

A linear model with formula $y = mx$ relates two variables, x and y, that are in direct variation (proportion).

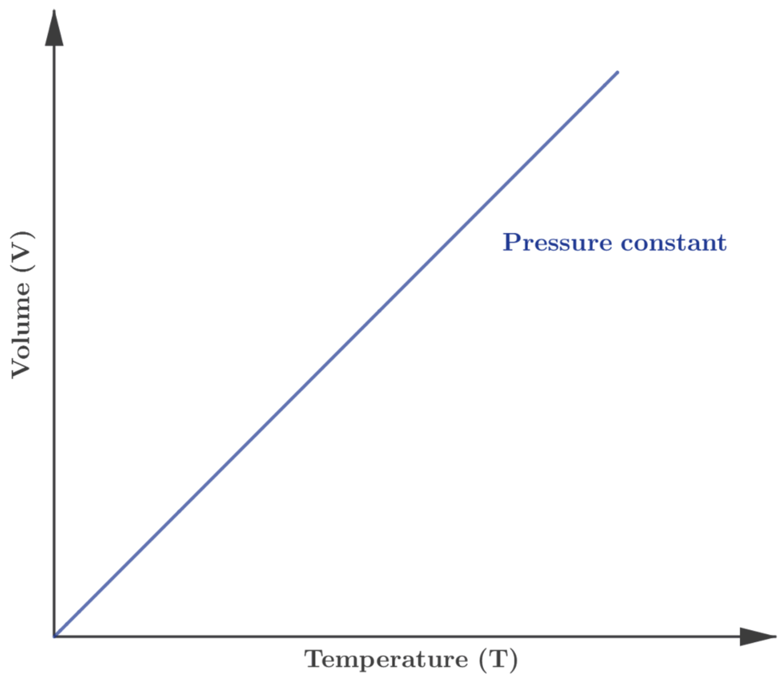
The gradient is the constant of proportionality; $m = \frac{y}{x}$ for any point (x, y) on the graph.

Making connections

Charles's law says that when temperature increases, the volume of a gas increases, and it increases at a constant rate. In other words, as a gas gets hotter, and its atoms move faster, the amount of space it takes up, or its volume, gets bigger too. Charles's law was named after Jacques Charles, who discovered the law around 1780.

Charles's law is an ideal gas law, which states that, at constant pressure, the volume V of an ideal gas is directly proportional to its absolute temperature T (in Kelvin). The simplest statement of the law is:

$$V = mT$$



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More information

The graph shows the relationship between Volume (V) on the Y-axis and Temperature (T) on the X-axis, with a straight diagonal line indicating the direct proportionality between V and T while pressure is constant. The X-axis is labeled 'Temperature (T)', and the Y-axis is labeled 'Volume (V)'. The line passing through the graph represents a constant pressure scenario and aligns with the equation (V = mT), where volume is directly proportional to temperature when pressure is constant.

[Generated by AI]

Example 1

★★☆

The volume of a sample of nitrogen is directly proportional to the temperature. The volume of the sample of nitrogen is 600 ml at a temperature of 300 Kelvin and it becomes 700 ml at 350 Kelvin.

a) Find the formula the describes the volume V as a function of the temperature T.

b) Use the formula to find the volume of the sample when the temperature is 400 Kelvin.

	Steps	Explanation
a)	$V = 2T$	The volume is directly proportional to the temperature so use the formula $V = mT$, where m represents the gradient (constant of proportionality) of the corresponding line. Use either point $(350, 700)$ or $(300, 600)$, to find the constant of proportionality as follows: $$\begin{aligned} 700 &= m\,(350) \\ m &= \frac{700}{350} = 2. \end{aligned}$$
b)	$V = 2\,(400) = 800 \text{ ml}$	Substitute $T = 400$ into the formula of part a).

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Overview (/study/ap 122- cid- 754029/

Direct variation as an $n\text{th}$ power

One variable can also vary directly to a **power** of another. For example, in the formula of the area of a circle

$$A = \pi r^2$$

the area A is directly proportional to the square of the radius r. In this case, π is the constant of proportionality.

✓ **Important**

The following statements are equivalent.

1. y varies directly with the $n\text{th}$ power of x.
2. y is directly proportional to the n th power of x .
3. $y = mx^n$ for some constant m .

Example 2

★★☆

The distance d (in metres) that a ball rolls down an inclined plane is directly proportional to the square of the time t (in seconds) it rolls. One second after release, the ball has rolled 0.9 metres.

a) Write an equation that relates the distance travelled to the time taken.

b) How far will the ball roll in the first 4 seconds?

	Steps	Explanation
a)	$d = kt^2$	The distance d (in metres) that a ball rolls is directly proportional to the square of the time t (in seconds) for which it rolls and thus you can use the formula $d = kt^2$
	$0.9 = k(1)^2$	Substitute $d = 0.9$ and $t = 1$ into the formula.
	$k = 0.9$	Solve for k .

Student view

	Steps	Explanation
	$d = 0.9t^2$	
b)	$d = 0.9(4)^2 = 0.9\,(16) = 14.4$ metres.	Substitute $t = 4$ into the formula.

Be aware

Do not assume that direct variation always implies that an increase in one variable results an increase in the other variable. For example, in the model $y = -4x$, an increase in x results a decrease in y , and yet y is said to vary directly with x .

Inverse variation

Direct variation describes a linear relationship between two variables where, as one quantity increases the other increases at a constant ratio; for example, when one quantity is doubled the other is doubled as well.

For two quantities with inverse variation, as one quantity increases, the other quantity decreases with a constant ratio. For example, when you travel to a particular location, as your speed increases, the time it takes to arrive at that location decreases. When you halve the speed, the time it takes to arrive at that location doubles. So, for a constant distance, speed and time taken are **inversely proportional** to each other **.**

Important

Inverse variation (or inverse proportionality) is represented by the equation $y = \frac{m}{x}$ or equivalently, $xy = m$.

That is, y varies inversely with x if there is some non-zero constant m such that $y = \frac{m}{x}$ or $xy = m$, where $x \neq 0$ and $y \neq 0$.

Be aware

The letter k is also used to represent the constant of variation (or constant of proportionality) between two quantities.

Overview
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Example 3

★★☆

On a balanced lever (seesaw), the weight varies inversely with the distance of the object from the fulcrum. If Mathieu, weighing 140 pounds, is sitting 7 feet from the fulcrum, where should his brother Harvey, who weighs 98 pounds, sit in order to balance the seesaw?

Steps	Explanation
$w = \frac{k}{d}$	The weight w varies inversely with the distance d , which means you can use the formula $w = \frac{k}{d}$, where k is the constant of variation.
$140 = \frac{k}{7}$	Find the value of k by substituting the given information $w = 140$ and $d = 7$ into the formula.
$k = 980$	Solve for k .
$w = \frac{980}{d}$	Substitute the value of k into the formula.
	Substitute $w = 98$ into the formula.
$98 = \frac{980}{d}$	
$d = 10$ feet	
Therefore, Harvey needs to sit 10 feet away from the fulcrum to balance the seesaw.	

Activity

Consider the following questions:

- How can you tell whether a situation or set of values represents an inverse variation?
- Give your own examples of direct and inverse variation and explain why your examples meet the criteria for direct and inverse variation, respectively.
- Explain how you can create an equation for inverse variation, given a context.

Student
view

- Compare and contrast direct and inverse variation.

2 section questions

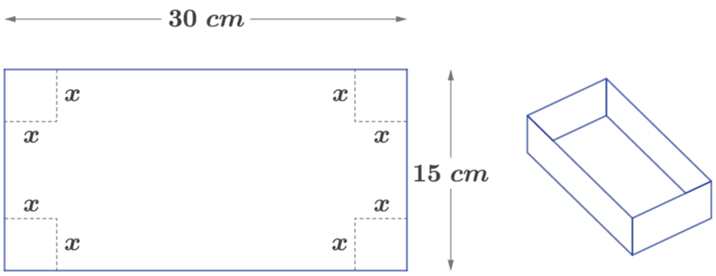
Modelling with cubic functions

Cubic functions are useful in modelling real-life phenomena that range from engineering to astronomy and product design. For example, engineers use cubic functions to find electrical resistance, or to describe how a device or material reduces the flow of electricity through it. Cubic functions are also commonly used to model three-dimensional objects, a process that allows the designer to identify a missing dimension or explore the result of changes to one or more dimensions.

Example 1

★★☆

A box with an open top can be made from a rectangular piece of cardboard with dimensions 15 cm by 30 cm by cutting out equal squares of side x cm at each corner and then folding up the sides, as shown below.



More information

Overview
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The image consists of two parts: a flat rectangular piece of cardboard with dimensions labeled as 15 cm in height and 30 cm in width, and an illustration of a 3D box made from this cardboard.

In the rectangular cardboard image, there are four squares labeled with 'x' at each corner, indicating the sections to be cut out. The remaining sections after the squares are removed show the areas that will be folded to form the sides of the box. The text 'x' is positioned at each cut out corner square, suggesting that these squares have equal sides of 'x' cm.

The transition from the flat cardboard to the 3D box is shown to the right of the image. This box has an open top and is structured by folding up the sides after cutting out the corner squares. The dimensions of the resulting box's length and width are depicted as '30 - 2x' and '15 - 2x' respectively, derived from the original piece by subtracting twice the 'x' dimension of the corner squares from the original dimensions.

[Generated by AI]

a) Explain why the length of the box is $30-2x$ and the width is $15-2x$.

b) Find a model function for the volume of the box in terms of its height.

c) Find the values of x for which your model is meaningful.

d) Plot the graph of the volume function.

e) Find the maximum point on the graph and interpret its coordinates.

	Steps	Explanation
a)	The length of the cardboard is 30 cm. When it is folded to form the box, $2x$ are cut out from the one side and thus the length of the box is $30-2x$. Similarly, the width of the box is $15-2x$.	

Student view

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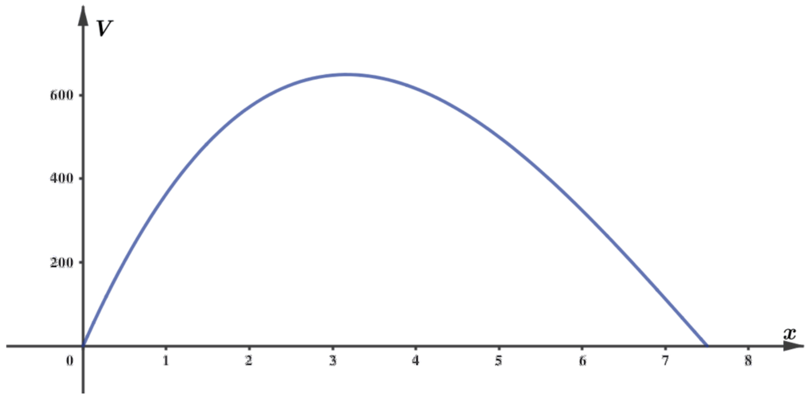
	Steps	Explanation
b)	$V = 4x^3 - 90x^2 + 450x$	$V = \text{length} \times \text{width} \times \text{height}$ $V = (30 - 2x)(15 - 2x)x$ $V = (30 - 2x)\left(15x - 2x^2\right)$ $V = 450x - 60x^2 - 30x^2 + 4x^3$ $V = 4x^3 - 90x^2 + 450x$
c)	• The height is positive, so $x > 0$. • The length is positive, so $30 - 2x > 0$. This is true for $x < 15$. • The width is positive, so $15 - 2x > 0$. This is true for $x < 7.5$. The meaningful values are $0 < x < 7.5$.	The height, the length and the width all need to be positive.
d)	 Use your GDC to graph the volume function.	

Student
view

Overview
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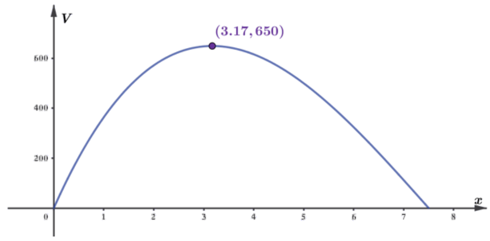
	Steps	Explanation
e)	 • The second coordinate of the maximum point tells that the maximum volume that can be achieved is approximately 650 cubic centimetres. • The first coordinate of the maximum point gives that the cut of 3.17 cm gives the box with maximum volume. • We can also tell the dimensions of this box of maximum volume: $\text{length: } 30 - 2 \times 3.17 = 23.66\text{ cm}$ $\text{width: } 15 - 2 \times 3.17 = 8.66\text{ cm}$ $\text{heigth: } 3.17\text{ cm}$	Use your GDC to find the maximum point.

Example 2

★★☆

You work in the design department of a soft-drink company and are experimenting with a new cylindrical container for beverages that is slightly narrower and taller than a standard container. For your experimental container, h, the height of the container measured in centimetres, is to be five times larger than the radius of the base, also measured in centimetres.

Student view

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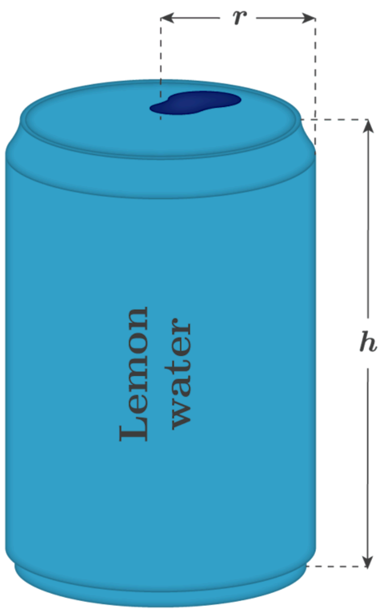

More information

The image is an illustration of a cylindrical beverage container. The container is slightly narrower and taller than a standard container. The height is labeled as "h" and the radius of the base as "r." The text on the container reads "Lemon water." The height is marked to be five times larger than the radius of the base, implementing the mathematical relationship between these dimensions. An arrow indicates the height measurement on the right side of the container while another arrow indicates the radius across the top of the can.

[Generated by AI]

Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/modelling-with-quadratic-functions-optimisation-id-27472/print/) Assign

a) Find the formula that describes the volume of the container as a function of the radius.

b) Find the volume of the container when the radius $r = 3$ cm.

c) If the volume of the container is to be 400 cm^3, explain why $80 = \pi r^3$.

Student
view

Overview
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	Steps	Explanation
a)	The variables in this problem are: V: volume of cylinder r: radius of cylinder h: height of cylinder	Identify the variables of the problem.
	$V = \pi r^2 h$	The formula for the volume of a cylinder.
	$V = \pi r^2 (5r)$	Substitute $h = 5r$ as the height is 5 times the radius.
	$V = 5\pi r^3$	Simplify the expression.
b)	$V = 424\ \text{cm}^3$	Substitute $r = 3\ \text{cm}$ into the formula $V = 5\pi r^3 = 5\pi(3)^3 = 424.115$ and give the answer correct to 3 significant figures.
c)	$5\pi r^3 = 400$	Set the expression for the volume equal to $400\ \text{cm}^3$.
	$\pi r^3 = 80$	Simplify the equation.

Fitting data with cubic models

Example 3

★★☆

The table gives the average speed y (in knots) of a boat for different engine speeds x (in hundreds of revolutions per minute, rpm).

Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/direct-and-inverse-variation-id-27473/print/) Assign

Student view

Overview (/study/ap 122- cid- 754029/

Engine speed (hundreds of rpm), x	9	11	13	15	17	19	21
Average boat speed (knots), y	6.45	7.44	8.88	9.66	10.98	12.56	15.44

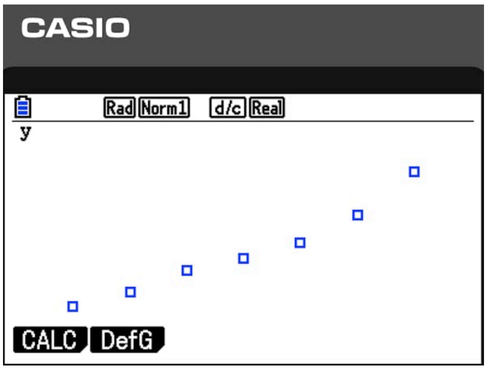
a) Draw a scatterplot of the data and decide whether a cubic model is appropriate.

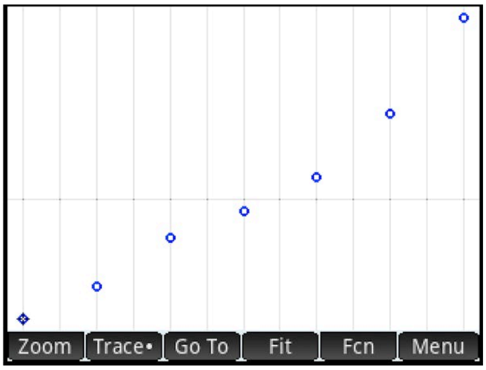
b) Use your GDC to find a cubic function that fits the data.

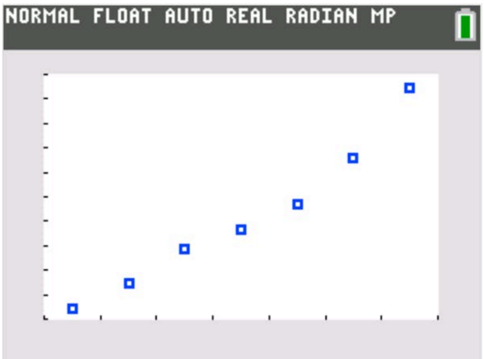
c) Use your model to estimate the average speed of the boat for an engine speed of 2500 rpm.

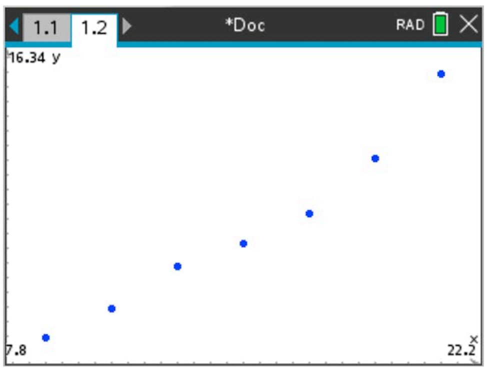
Student view

<table>
<tr><th></th><th>Steps</th><th>Explanation</th></tr>
<tr><td>a)</td><td>The result screens from the calculators









</td><td>The data points on the scatterplot follow the shape of a cubic curve and thus a cubic model would be appropriate.</td></tr>
</table>

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	Steps	Explanation
b)	$y = 0.00615x^3 - 0.243x^2 + 3.68x - 11.5$	Use the cubic regression utility to obtain the cubic model that fits the data.
c)	$y = 24.5$ knots	Substitute $x = 25$ into the cubic function of part b).

4 section questions

2. Functions / 2.6 Modelling skills

Modelling with exponential functions

Exponential growth and decay

Many real-life scenarios can be modelled by exponential functions. The growth of bacteria in a culture, the growth of your money in a bank account paying compound interest, or the growth of the world's population are all examples of exponential growth. Examples of exponential decay – when an amount decreases exponentially – include certain chemical reactions or the decay of radioactive substances.

Mathematical models that follow exponential growth describe situations where one quantity increases by the same factor over equal intervals of time. Exponential decay models describe situations where one quantity decreases by the same factor over equal intervals of time.

✓ Important

- After t years, the balance A in an account with principal P and annual interest r % is given by the following formulae $A = P\left(1 + \frac{r}{100n}\right)^{nt}$, where n is the number of times that the interest is compounded per year.

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- In other contexts, where it is not meaningful to talk about number of compounding periods, a simpler model of the form $A = P(1 \pm r)^x$, or $A = P \times a^x$ can be used.

Example 1

★★☆

Hugo deposits $\$900$ into an account that pays 4.55% interest per annum, compounded monthly.

a) Form an equation to model how much money Hugo has after x years.

b) Describe why the amount of money that Hugo will have in the bank after a period of time can be described as exponential growth.

c) Calculate how much money (to the nearest cent) Hugo has in the bank after 10 years.

d) How long will it be before Hugo has $\$3500$ in his account?

	Steps	Explanation
a)	Hugo's money in the bank after x years: $$y = 900\left(1 + \frac{4.55}{1200}\right)^{12x},$$	Use the formula $FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$ and substitute $r = 4.55$, $k = 12$ and $PV = 900$.
b)	The amount of money that Hugo will have in the bank after x years can be described by an exponential model because the independent variable x is the exponent.	
c)	$\$1417.34$	Substitute $x = 10$ into the formula or use your GDC.
d)	The x -coordinate of the point of intersection is 29.9 and thus it will take 30 years for Hugo's account to reach $\$3500$.	Use your GDC to plot the line $y = 3500$ and find the point of intersection between the graphs.

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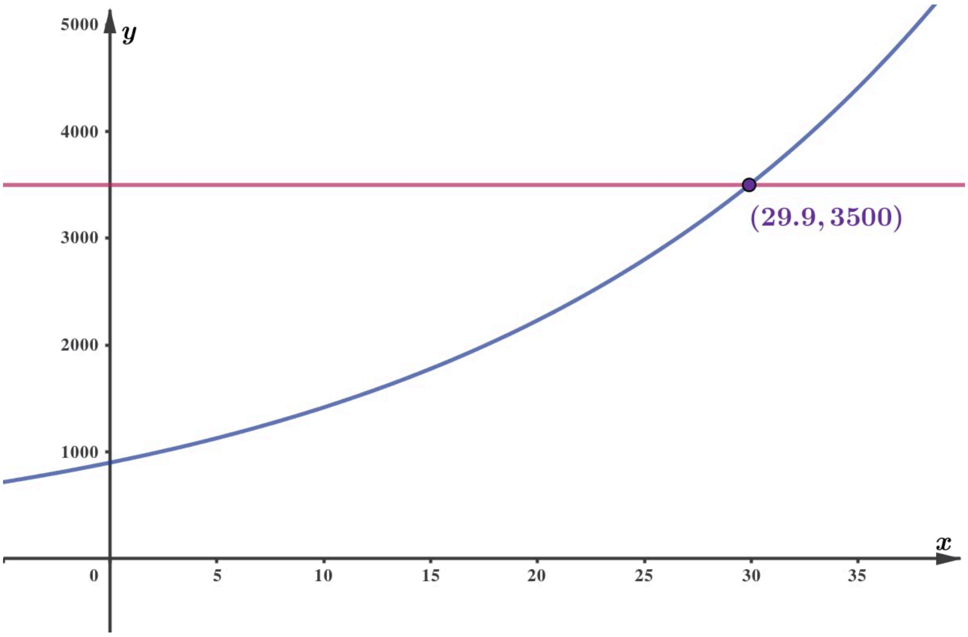


Example 2

★★★

The market value for a certain model of Volkswagen Golf is recorded for the first 5 years. The first 5 years can be modelled by the equation:

$$V\left(x\right)=20\,350(1.1409)^{-x},$$

where V is the value of the car in USD and x is the number of years after purchase.

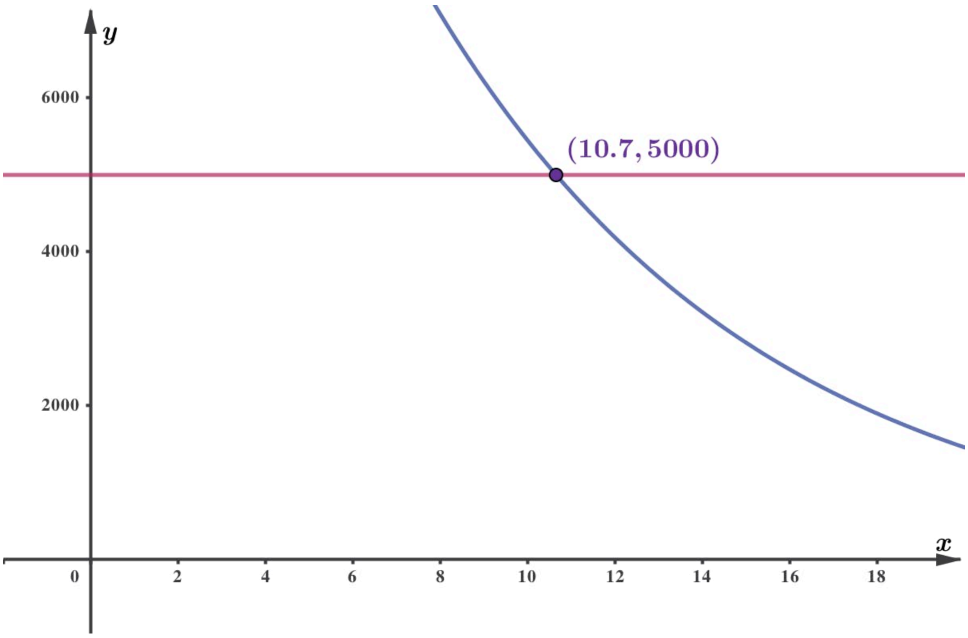
a) What was the value of the car when it was purchased for the first time?

b) According to the model, what is the expected value of a Volkswagen Golf that is 3.5 years old?

c) What is the minimum age (in complete years) of Volkswagen Golf that you can expect to be able to purchase for under $\$5000$?

Student
view

	Steps	Explanation
a)	$V(0) = 20\,350(1.1409)^{-0} = 20\,350\,(1) = 20350$ Therefore, the estimated value of the car when it is purchased for the first time is $\$20\,350$.	When the car is purchased for the first time, $x = 0$.
b)	$V(3.5) = 20\,350(1.1409)^{-3.5} = 12\,829$ Therefore, at 3.5 years of age, the estimated value of the car is $\$12\,829$.	When the car is 3.5 years old, $x = 3.5$.
c)	Therefore, an 11-year-old car can be purchased for less than $\$5000$.	Use your GDC to graph the functions $y = 20\,350(1.1409)^{-x}$ and $y = 5000$ and find the x -coordinate of the point of intersection.



Exam tip

In real life, a vast number of variables affect the price of a car after first purchase, such as distance driven, number of owners, care of engine and care of body to name a few.

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Every model is subject to some criticism. Remember that the creation of a model requires a balancing act between efficiency and accuracy. The purpose of the model will determine how accurate it needs to be.

One of the best-known applications of exponential models is in dating techniques applied by scientists in archaeology, palaeontology and geology to determine the age of an artefact.

Example 3

★★☆

The amount of radioactive material, M, in grams, is modelled according to the function

$M(t) = 250e^{-kt}$, where $k > 0$ and t is time measured in years. It is determined that after 20 years, the amount of radioactive material present is 50 grams.

a) Show that the amount of radioactive material, M, decays exponentially.

b) What is the initial amount of material present?

c) Find the value of k.

d) How much material is left after 80 years? Give your answer to 3 decimal places.

	Steps	Explanation
a)	$M(t) = 250e^{-kt} = 250\left(\frac{1}{e}\right)^{kt}$	Use exponent rules to express the model function without a negative exponent.
	$M(t)$ is a decreasing function and thus M models exponential decay.	As $\frac{1}{e} < 1$.
b)	$M(0) = 250\left(\frac{1}{e}\right)^{0} = 250$ and thus the initial amount, regardless the value of k, is 250 grams.	Substitute $t = 0$ into the function.
c)	$50 = 250e^{-20k}$	Substitute $t = 20$ into the function and use that it is given that $M(20) = 50$.

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	Steps	Explanation
	$k \approx 0.0805$	Use GDC to solve the equation.
d)	$M(t) = 250e^{-0.0805t}$	Substitute $k = 0.0805$ into the function.
	$M(80) = 250e^{-0.0805(80)} \approx 0.399$	Substitute $t = 80$ into the function.
	Thus, the amount of material remaining after 80 years is 0.399 grams.	

Be aware

All of the exponential models that you explored above have time as an independent variable. This is very often the case for exponential growth or decay.

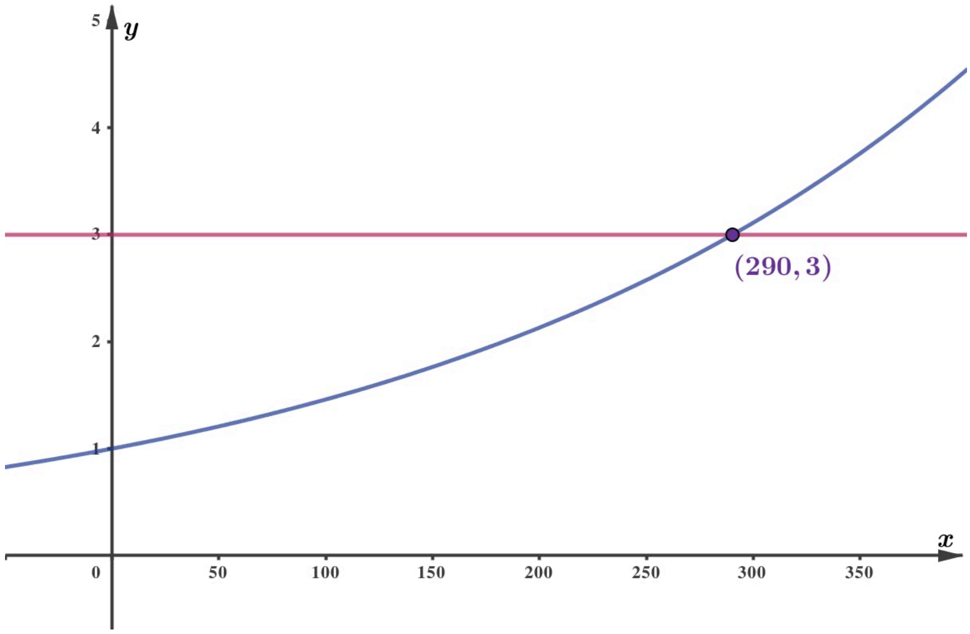
Example 4

★★☆

Hugo deposits $\$900$ into an account that pays 4.55% interest per annum compounded monthly. How long will it take Hugo to triple his money?

Steps	Explanation
$x = nt$	Let $x = nt$ be the number of monthly instalments until the initial capital is tripled.
$2700 = 900\left(1 + \frac{0.0455}{12}\right)^x$	Set up an equation with the balance of the account as 2700 (triple the initial amount) and simplify.
$3 = \left(1 + \frac{0.0455}{12}\right)^x$	Use a GDC to find the x -value of the point of intersection of the graphs $y = 3$ and $y = \left(1 + \frac{0.0455}{12}\right)^x$.
$x = 290.29\ldots$	

Steps	Explanation
Hence the solution is 291 monthly instalments, which is 24 years and 3 months.	



Example 5

★★★

A drug is developed to suppress the temperature of a patient during a fever. The model for the effect of the drug is given by the equation $T(t) = t \times 0.84^t, t \geq 0$, where T is the temperature of the patient above 37°C, and t is the time in hours after the drug has been given.

a) Determine the maximum temperature of the patient and the time at which it occurs.

b) How long does it take for a patient to return to within 0.5° of 37°C.

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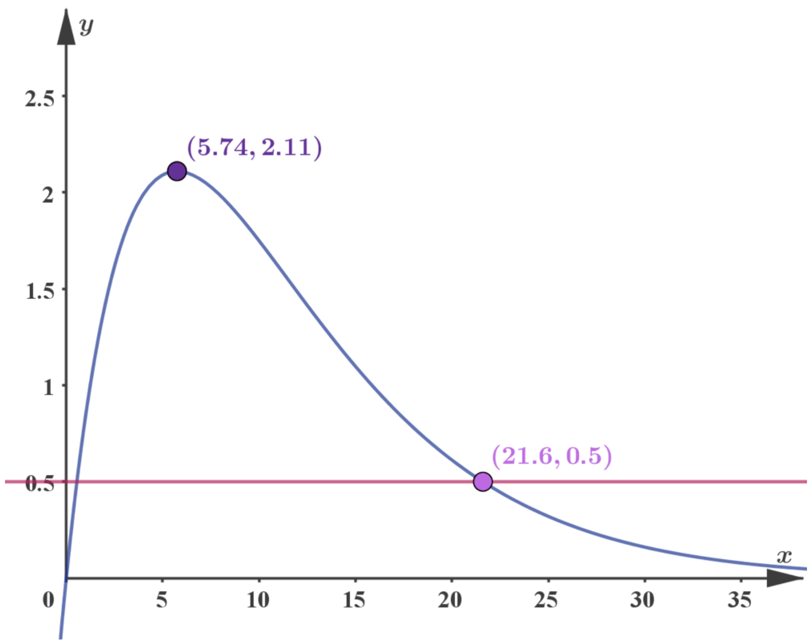
	Steps	Explanation
a)	The maximum point is $(5.74, 2.11)$ and thus the maximum temperature of the patient is $37 + 2.11 = 39.11°$ and occurs 5.74 hours after taking the drug.	Use your GDC to find the maximum point of the graph of the function $y = x \times 0.84^x$.
b)	The answer is the solution to the equation $0.5 = x \times 0.84^x$, where you accept the furthest point from $x = 0$.	Use a GDC to obtain the point of intersection for the graphs $y = 0.5$ and $y = x \times 0.84^x$.
	$x = 21.5985$	Give your answer correct to 3 significant figures.
	Thus, it takes 21.6 hours to return to within $0.5°$ of $37°\text{C}$.	

The solutions are shown as points in the figure below.



Student
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Fitting data with exponential functions

Exam tip

On standard level exams students will not be expected to use a calculator to fit a nonlinear model to a data set.

Example 6

★★☆

A chemist has a 250-gram sample of a radioactive material. She records the amount remaining in the sample every day for a week and obtains the following data.

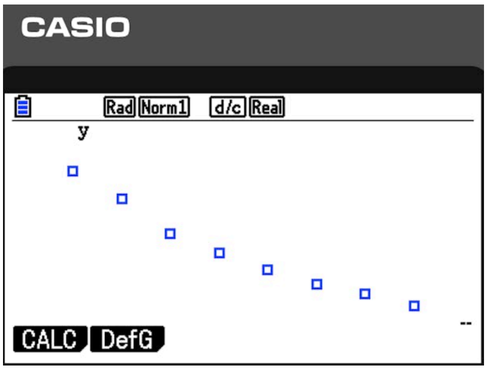
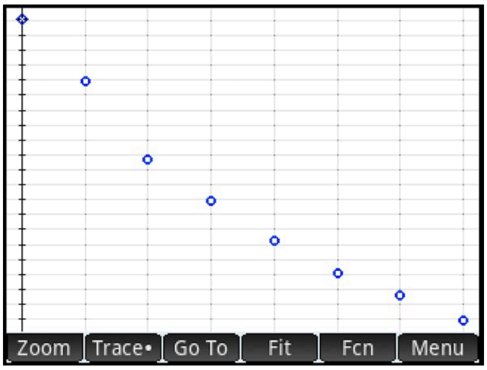
Day	0	1	2	3	4	5	6	7
Weight (g)	251	209	157	129	103	81	66	49

a) Enter the data in your GDC and draw a scatter plot.

b) Use your GDC to fit an exponential curve to the data.

c) Express the function in the form $A = A_0 e^{kt}$.

d) How much material will be left after 20 days?

	Steps	Explanation
a)	The result screens from the calculators  Casio fx-CG50 The image shows a graph screen from a Casio calculator. At the top, there are options labeled 'Rad', 'Norm1', 'd/c', and 'Real'. Below these options, a grid is displayed with marked data points forming a downward sloping line. On the left, there's a 'y' label indicating the vertical axis, though no other axes are labeled. At the bottom left, there are buttons labeled 'CALC' and 'DefG'. The graph consists of square data points plotted in blue, indicating specific values in a descending pattern from left to right across the screen, though exact values or units are not provided. [Generated by AI]  HP Prime The image is a graph that features scattered blue points on a grid, displayed on a screen with several navigation buttons at the bottom. The graph has a vertical Y-axis on the left side, which appears to have tick marks, and a horizontal X-axis, although specific labels and units are not visible. The data points form an overall descending trend from left to right, implying a negative correlation. Below the graph, there are	Enter the data in your GDC and use the graph utility to obtain a scatterplot.

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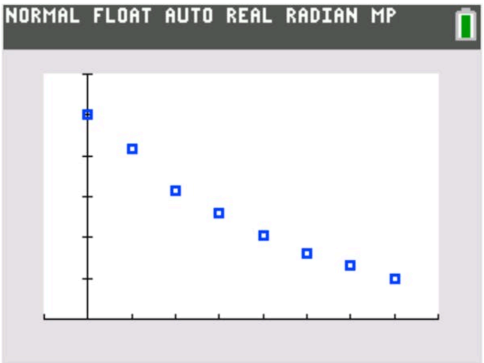
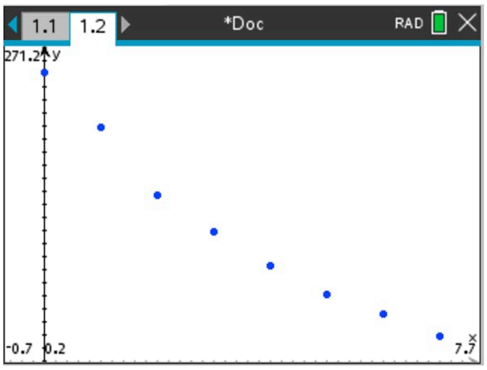
	Steps	Explanation
	buttons labeled 'Zoom', 'Trace', 'Go To', 'Fit', 'Fcn', and 'Menu', suggesting interactive functionality for adjusting or analyzing the graph. The graphical interface indicates this is likely from a graphing calculator or software tool. [Generated by AI]	



TI-84 plus CE

The image shows a graph on a calculator display with the labels "NORMAL FLOAT AUTO REAL RADIAN MP" at the top. The battery symbol is on the right side indicating battery status. The graph has a vertical axis with evenly spaced tick marks and a horizontal axis with similar tick marks. Data points are marked as blue squares decreasing in value from left to right, indicating a downward trend across the graph.

[Generated by AI]



TI-nspire CX

The image shows a scatter plot on the screen of a TI-nspire calculator. The X-axis is labeled with values ranging from 0.2 to 7.7, and the Y-axis ranges from -0.7 to 271.2. There are several blue data points plotted decreasingly from top left to bottom right, indicating a potential downward trend. The top of the

Student view

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	Steps	Explanation
	interface includes a tab labeled '1.1 1.2 *Doc', and a status bar shows the calculator in "RAD" mode, with a green battery indicator. [Generated by AI]	
b)	$y = 256.0910721(0.7936893921)^x$	Use your GDC to obtain the exponential function that fits the data.
c)	$k = -0.2310631$	Use your GDC to solve the equation $$0.7936893921 = \mathrm{e}^k.$$
	$A = 256.0910721\mathrm{e}^{-0.2310631t}$	Substitute the value of k into the equation $A = A_0\mathrm{e}^{kt}$, where $A_0 = 256.0910721$.
d)	$A = 256.0910721\mathrm{e}^{-0.2310631(20)} = 2.52003082$	Substitute $t = 20$ into the function of part c).

5 section questions

2. Functions / 2.6 Modelling skills

Modelling with sinusoidal functions

Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/modelling-with-sinusoidal-functions-id-27476/print/) Assign

General form of trigonometric functions

As you have seen, sine and cosine functions have a regular period and range. If you watch waves or ripples on a lake, you will observe that they resemble the sine and cosine functions. However, the basic graphs of $y = \sin x$ and $y = \cos x$ are rather limited in their use unless you consider transformations that allow you to change the period, amplitude and principal axis to any desired value.

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Important

A function that has the general shape of a sine or cosine function is known as **sinusoidal function.** The general forms of sinusoidal functions are:

$$y = A \sin (Bx) + D$$

and

$$y = A \cos (Bx) + D$$

In the activity below, you can explore the sinusoidal function of the form $y = A \sin (Bx) + D$.

Activity

Drag the three sliders to adjust the A, B and D parameters of the sinusoidal function $y = A \sin (Bx) + D$.

- What effect do the parameters A, B and D have on the graph of $y = \sin x$.
- Formulate a rule that predicts the amplitude, period and principal axis directly from the formula of the function.

Interactive 1. Sinusoidal Function of the Form $\mathrm{y = A\ \sin\ (Bx) + D}$.

More information for interactive 1

This interactive allows users to explore sinusoidal functions of the form y = A sin(Bx) + D, where A, B, and D are adjustable parameters ranging from -5 to 5. By dragging the sliders for A, B, and D, users can observe how these parameters transform the basic sine graph y = sin x. The amplitude |A| controls the

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height of the wave, determining how far the graph oscillates above and below the principal axis. The frequency B affects the period of the sine function, calculated as $T = \frac{360}{|B|}$, where a larger B results in a shorter period (more oscillations) and a smaller B results in a longer period (fewer oscillations). The vertical shift D moves the graph up or down along the y-axis, changing the principal axis (the midline of the wave) to $y = D$. Users can compare the transformed graph with the basic $y = \sin x$ graph to see how changes in A, B, and D affect the amplitude, period, and vertical shift.

For example, when you set the sliders to $A = 2.3$, $B = 2.5$, and $D = 3.2$, the equation $y = 2.3\sin(2.5x) + 3.2$ appears on the screen along with its graph. You'll notice a dotted line labeled 'Principal Axis' at $y = 3.2$, indicating the vertical shift. The graph also displays the amplitude of 2.3 and the period $T = 144$. By experimenting with different values, users can formulate rules to predict the amplitude, period, and principal axis directly from the equation, enhancing their understanding of sinusoidal transformations and their graphical representations.

✓ Important

Sinusoidal functions $y = A\sin(Bx) + D$, $A \neq 0$

- The amplitude of the graph is equal to $|A|$.
- The period of the graph is $T = \dfrac{360°}{B}$.
- The principal axis has equation $y = D$.

Example 1

★☆☆

For the sinusoidal function $f(x) = -3\sin(2x) - 1$, where x is in degrees, find the amplitude, period and principal axis.

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Steps	Explanation
The amplitude is 3 .	The amplitude is determined by the value of $\|A\| = \|-3\| = 3$.
The period is $T = 180°$.	The period is $T = \frac{360°}{B} = \frac{360°}{2} = 180°$.
The principal axis is $y = -1$.	The principal axis has equation $y = D = -1$.

In the activity below, you can explore the sinusoidal function of the form $y = A\cos(Bx) + D$.

Activity

Drag the three sliders to adjust the A, B and D parameters of the sinusoidal function $y = A\cos(Bx) + D$.

- What effect do the parameters A, B and D have on the graph of $y = A\cos(Bx) + D$.

Interactive 2. Sinusoidal Function of the Form $\mathrm{y} = \mathrm{A}\ \cos(\mathrm{Bx}) + \mathrm{D}$.

More information for interactive 2

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This interactive allows users to explore sinusoidal functions of the form $y = A\cos(Bx) + D$, where A, B, and D are adjustable parameters ranging from -5 to 5. By dragging the sliders for A, B, and D, users can observe how these parameters transform the basic cosine graph $y = \cos x$. The amplitude $|A|$ controls the height of the wave, determining how far the graph oscillates above and below the principal axis. The frequency B affects the period of the cosine function, calculated as $T = \frac{360}{|B|}$, where a larger B results in a shorter period (more oscillations) and a smaller B results in a longer period (fewer oscillations). The vertical shift D moves the graph up or down along the y-axis, changing the principal axis (the midline of the wave) to $y = D$. Users can compare the transformed graph with the basic $y = \cos x$ graph to see how changes in A, B, and D affect the amplitude, period, and vertical shift.

For example, when you set the sliders to $A = 2.3$, $B = 2.5$, and $D = 3.2$, the equation $y = 2.3\cos(2.5x) + 3.2$ appears on the screen along with its graph. You\textquotesingle ll notice a dotted line labeled \textquotesingle Principal Axis\textquotesingle at $y = 3.2$, indicating the vertical shift. The graph also displays the amplitude of 2.3 and the period $T = 144$. By experimenting with different values, users can formulate rules to predict the amplitude, period, and principal axis directly from the equation, enhancing their understanding of cosine transformations and their graphical representations.

Important

Sinusoidal functions $y = A\cos(Bx) + D$, $A \neq 0$.

- The amplitude of the graph is equal to $|A|$.
- The period of the graph is $T = \frac{360°}{B}$.
- The principal axis has equation $y = D$.

Be aware

In the case when parameter $A < 0$, the graph of both sinusoidal functions is reflected on the x -axis.

Example 2

★☆☆

Consider the sinusoidal function $f(x) = 4\cos\left(\frac{x}{5}\right) + \frac{1}{2}$. Find its amplitude, period and principal axis.

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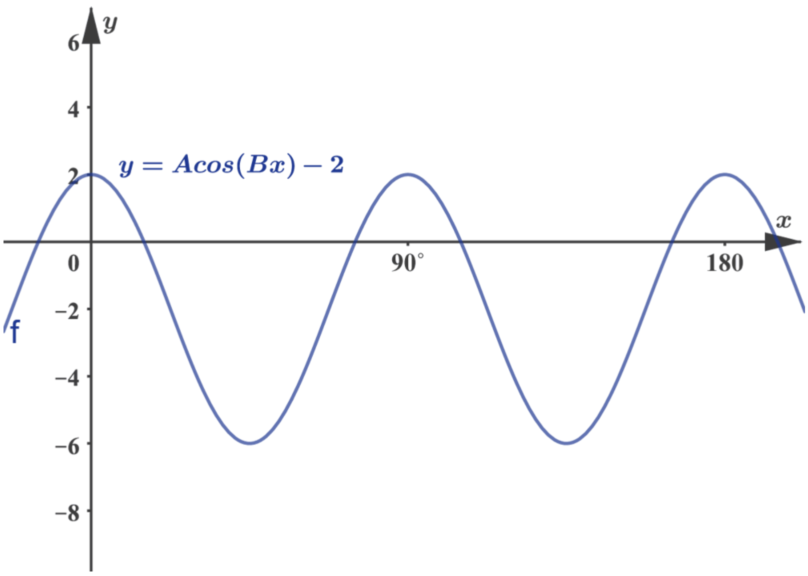
Steps	Explanation
The amplitude is 4.	The amplitude is determined by the value of $\|A\| = \|4\| = 4$.
The period is $T = 1800^\circ$.	The period is $T = \frac{360^\circ}{B} = \frac{360^\circ}{\frac{1}{5}} = 1800^\circ$.
Principal axis: $y = \frac{1}{2}$.	The principal axis has equation $y = D = \frac{1}{2}$.

Example 3

★★☆

The graph of the function $y = A\cos(Bx) - 2$ is shown below. Determine the numerical values of the unspecified parameters.



More information

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view

Steps	Explanation
$A = 4$	The amplitude is $A = \left\lvert \frac{\max - \min}{2} \right\rvert = \left\lvert \frac{2-(-6)}{2} \right\rvert = \left\lvert \frac{8}{2} \right\rvert = 4$.
$B = 4$	The period is $T = 90° = \frac{360°}{B} \Leftrightarrow B = 4$.

Example 4

★★☆

A Ferris wheel with a radius of 25 metres is rotating at a rate of three revolutions per minute. When $t = 0$, a cabin starts at the lowest point of the wheel, which is 5 metres above the ground. Write a model for the height h (in metres) of the cabin as a function of the time t (in seconds).

Steps	Explanation
Using the minimum and maximum value of the height, you can find the principal axis: $D = \frac{5+55}{2} = \frac{60}{2} = 30$	When the cabin is at the bottom of the Ferris wheel, it is 5 metres above the ground, so the minimum height is 5. When the wheel is at the top, the maximum height is $5 + 2\,(25) = 55$ metres above the ground.
$h = A\cos(Bx) + D$, $A < 0$	When $t = 0$, the height is at its minimum, so the model is a cosine function with $A < 0$.
$A = -25$	The amplitude is $\lvert A \rvert = \left\lvert \frac{\max - \min}{2} \right\rvert = \left\lvert \frac{55-5}{2} \right\rvert = \left\lvert \frac{50}{2} \right\rvert = 25$. As $A < 0$, the value of A in the equation must be -25.

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Steps	Explanation
$B = 18$	Since the Ferris wheel is rotating at three revolutions per minute, it completes one revolution in 20 seconds. The period is $\frac{360°}{B} = 20 \Leftrightarrow B = \frac{360}{20} = 18$.
A model for the height of the cabin as a function of time is: $h = -25\cos(18t) + 30$	

4 section questions

2. Functions / 2.6 Modelling skills

Checklist

Section Student... (0/0) Feedback Print (/study/app/m/sid-122-cid-754029/book/checklist-id-27477/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- solve various types of equations using a GDC
- solve systems of equations using a GDC
- find linear models that describe phenomena and use the models to make predictions
- find quadratic models that describe phenomena and use the models to make predictions
- find models that describe variables that are in direct or inverse proportion
- find cubic models that describe phenomena and use the models to make predictions
- find exponential models that describe phenomena and use the models to make predictions
- find sinusoidal models that describe phenomena and use the models to make predictions.

Student view

Investigation

Making connections

Prior to the Polish mathematician and astronomer Nicolaus Copernicus (1473-1543), the commonly held view was that the Sun and planets moved around the Earth, which was at the centre. Copernicus produced a model, heretical at the time, of the Sun at the centre of our solar system. Around a hundred years later, the German astronomer and mathematician Johannes Kepler made further advances to the theory behind planetary motion by providing a mathematical model that predicted the period of each planet's orbit depending on its mean distance from the Sun.

The planets of our solar system
Credit: alxpin Getty Images

The table shows the mean (average) distances d of the planets from the Sun (taking the unit of measurement to be the distance from the Earth to the Sun) and their periods, T (time of revolution in years).

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000

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Planet	d	T
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784

1. Investigate whether a power function is a good fit for the data.
2. Discover the power function that 'best' matches the data.

Kepler's third law of planetary motion states that:

'The square of the period of revolution of a planet is proportional to the cube of its mean distance from the Sun.'

3. Does your model verify Kepler's third law?

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