Index

The big picture

Poly means many and nomial means terms. So, in simple words, polynomial means an algebraic expression with 'many number of terms'. It could have multiple variables or a single variable.

Polynomials can also be considered as functions .

For example, $P(x, y, z) = 2xy - 3y^2 + z$ is a three variable trinomial function.

A roller coaster can be expressed as a polynomial function with a single variable. Understanding polynomials will help you to model the polynomial function that would fit a roller coaster picture given below:

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

This rollercoaster is shown in the graph below

Credit: Getty Images Ralf Geithe

In this subtopic, you will take a comprehensive look at polynomial expressions with a single variable. In particular, you will learn how to factorise them using various methods, including graphical, and will consider the theorems that assist in the process.

Concept

Modelling

Modelling using a polynomial function helps architects and engineers to build complex structures such as rollercoasters, bridges and curved terraces for buildings. Using a function to model the structure not only helps to define the shape of the structure but also to find out useful information such as the volume or surface area.

- How can you choose the degree of the polynomial function that will model the situation?
- Will using higher degree polynomials help to get the best fit model?

Theory of Knowledge

Look at the conditions given for division of polynomials. To what extent can you compare this with division of numbers. As numbers do not have 'degree', how can you interpret the above conditions with division of numbers? Are both the conditions stated for the division of polynomials always also necessary for the division of numbers? Or are there any exceptions?

Knowledge Question: Does the presence of rules strengthen or weaken mathematics as an area of knowledge (AOK)?

2. Functions / 2.12 Polynomial functions

Introduction to polynomials

Look at the following functions:

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$$f(x) = 2x + 3$$
$$g(x) = 3x^2 + 2x - 5$$
$$h(x) = x^3 - 2x^2 + 3x - 4$$

You are familiar with the above functions and you know that $f(x)$ is a linear function, $g(x)$ is a quadratic function and $h(x)$ is a cubic function.

What is the highest power of x in each of the above? What is the significance of this power of x ?

The highest power of x in $f(x)$ is 1, in $g(x)$ is 2 and in $h(x)$ is 3. These powers of x denote the **degree** of these polynomial functions.

Important

Definition

A **polynomial** of degree n can be written as:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots \ + \ a_1 x + a_0$$

where $a_n \neq 0,\ n \geq 0,\ n \in \mathbb{Z}$ and $a_0, a_1, a_2, \ldots a_n$ are real numbers.

A 4th degree polynomial, called a quartic function, would have the highest power of x as 4. For example, $3x^4 + 2x^3 - 5x^2 + x + 2$ is a 4th degree polynomial. Note that the functions below are all 4th degree polynomials:

$$f_1(x) = 3x^4$$
$$f_2(x) = 5x^4 - 3$$
$$f_3(x) = x^4 - x^2 + 7$$
$$f_4(x) = 3x - x^4 - 8$$

From the above, note that:

1. You can use any real number for coefficients. (Complex numbers are also allowed but it is beyond the scope of this section.)
2. If the degree of the polynomial is n , then the coefficient of $x^n \neq \ 0$. Any other coefficients could be 0. (In each of the above polynomials, find out which coefficients are 0.)

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

3. A polynomial is formed by adding different terms (a negative term would mean that you have added a term with a negative coefficient).

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26603/print/) Assign

Activity

What is the degree of the polynomial $P(x) = 4$ and $P(x) = 0$?

The coefficient of each term of a polynomial function forms the basis for the shape and position of its graph.

Below is an interactive activity from which you can find out how the coefficients of a **cubic function** affect the shape and position of its graph. You can also identify how some of these coefficients affect whether the graph does or does not cross the x -axis.

The polynomial function given is in the form $a_3x^3 + a_2x^2 + a_1x + a_0$ with sliders for a_3, a_2, a_1 and a_0. Using the sliders, you can change their values.

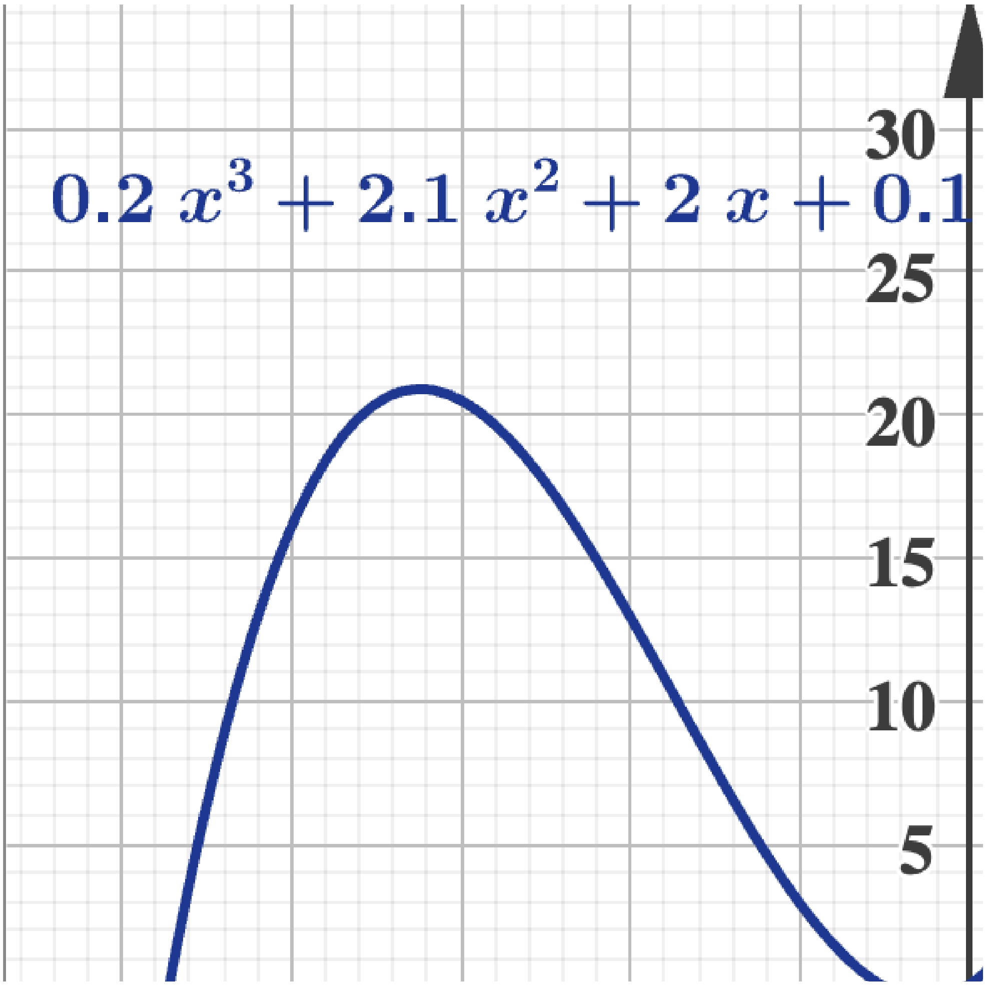
First, keep a_3, a_2, a_1 as 0 and change a_0 and see how the graph changes its shape and position.

Now keep all coefficients as 0 and change the values a_1 and observe.

Repeat the process with each of the coefficients and observe. Register your observation in your notebook and discuss within your class.

Student view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o



Interactive 1. How the Coefficients of a Cubic Function Affect the Shape and Position of Its Graph.

More information for interactive 1

This interactive allows the user to understand how the coefficient of each term of a polynomial function forms the basis for the shape and position of its graph.

A graph of XY axes is displayed, with the x-axis ranging from -10 to 10 and the y-axis ranging from -30 to 30. A curve is displayed on the graph, and the user is provided with parameters that can be adjusted using horizontal sliders for each given value:

a_3: Affects the overall steepness and direction of the curve at the extremes.

a_2: Affects the curvature and the location of the vertex or turning point.

a_1: Affects the slope of the curve, especially near the y-intercept.

a_0: Shifts the entire graph up or down along the y-axis (it's the y-intercept).

In this interactive, it's a cubic function of the form $f(x) = (a_3)x^3 + (a_2)x^2 + (a_1)x + (a_0)$.

Here, a_3, a_2, a_1, a_0 are the coefficients of x^3, x^2, x and constant terms respectively. The users can slide the value of a_3 from -2 to 2 and a_3, a_2, a_1 from -5 to 5.

Student
view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

Users will first keep the value of, $a3$, $a2$, $a1$ as 0 and on sliding the value of $a0$ from -5 to 5. At, $a0 = -5$ the graph is a straight line passing through $y = -5$ parallel to x axis and at $a0 = 5$ the graph is a straight line passing through $y = 5$ parallel to the x axis. This helps illustrate how the constant term affects the y-intercept without influencing the overall shape.

If all coefficients are set to 0 except for $a1$. As students increase or decrease $a1$, they witness the creation of a straight-line graph, showcasing how the linear term tilts the graph. If all coefficients are set to 0 except for $a2$. As students increase or decrease $a2$, they witnessed the creation of a parabolic graph. Since it is purely a quadratic function, the graph is symmetric about the y-axis (because there's no linear or cubic term). The vertex of the parabola is at the origin $(0, 0)$ because there's no constant or linear term to shift the graph. If all coefficients are set to 0 except for $a3$. As students increase or decrease $a3$, they witnessed the creation of an “S” shaped graph. The graph always passes through the origin because no constant or linear terms are present. The "S" curve gets sharper or flatter depending on the value of $a3$. The orientation of the "S" flips when the sign of $a3$ changes.
Through this interactive users can learn how the graph changes, giving them a better insight into how the coefficient of each term of a polynomial function forms the basis for the shape and position of its graph.

1 section question

2. Functions / 2.12 Polynomial functions

Algebra of polynomials

Any two polynomials can be added, subtracted, multiplied and divided as shown in this section.

Addition and subtraction of polynomial functions

This is done in the same way as you would add or subtract any two functions. In this process, you collect all like terms together and simplify to get a single polynomial.

If $P\left(x\right) = 2x + 3$ and $Q\left(x\right) = x^2 - 3x + 4$, then

Student
view

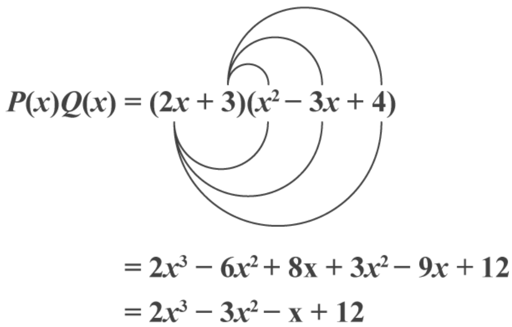
Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
$P(x)+Q(x)=2x+3+x^2-3x+4$ $=x^2-x+7$	[Collecting like terms]
$P(x)-3Q(x)=(2x+3)-3\left(x^2-3x+4\right)$ $=2x+3-3x^2+9x-12$ $=-3x^2+11x-9$	[Expanding $-3Q(x)$] [Collecting like terms]

Multiplication of polynomial functions

Two polynomial functions can be multiplied term by term, as you would do for any two algebraic expressions.

For example:



More information

The image is a diagram illustrating polynomial multiplication. At the top, it shows the expression P(x)Q(x) = (2x + 3)(x^2 - 3x + 4). Below this, there are two curved arrows pointing to successive expressions, breaking down the multiplication step-by-step. The first equation shown is 2x^3 - 6x^2 + 8x + 3x^2 - 9x + 12, indicating the result of distributing each term in the first polynomial across the second polynomial. The final expression simplifies to 2x^3 - 3x^2 - x + 12. These steps demonstrate the process of multiplying polynomials and combining like terms.

[Generated by AI]

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Example 1

★☆☆

Simplify $\left(x^2-2x+5\right)-2\left(x^2+5x-3\right)$

Steps	Explanation
$\left(x^2-2x+5\right)-2x^2-10x+6$	[Multiplying the second polynomial by -2]
$=-x^2-12x+11$	[Collecting like terms]

Example 2

★☆☆

Simplify $x\left(x^2-1\right)+3\left(x^2+3\right)$

Steps	Explanation
x^3-x+3x^2+9	[Multiplying the first polynomial by x and the second by 3]
$=x^3+3x^2-x+9$	[Arrange the terms in decreasing degree order]

There is no more simplification possible.

Example 3

★★☆

Simplify $\left(3x^2-2x+1\right)\left(x^2+3\right)$

Student view

Steps	Explanation
$3x^4 + 9x^2 - 2x^3 - 6x + x^2 + 3$	[Term-by-term multiplication]
$= 3x^4 - 2x^3 + 10x^2 - 6x + 3$	[Collecting like terms]

Division of polynomial functions

A polynomial function can be divided by another polynomial function in a similar way to division with natural numbers.

Work out $1794 \div 12$. Find out the quotient and remainder of this division.

Watch this video to understand the division algorithm with numbers.

The concept of quotient and remainder can also be introduced for polynomials.

Important

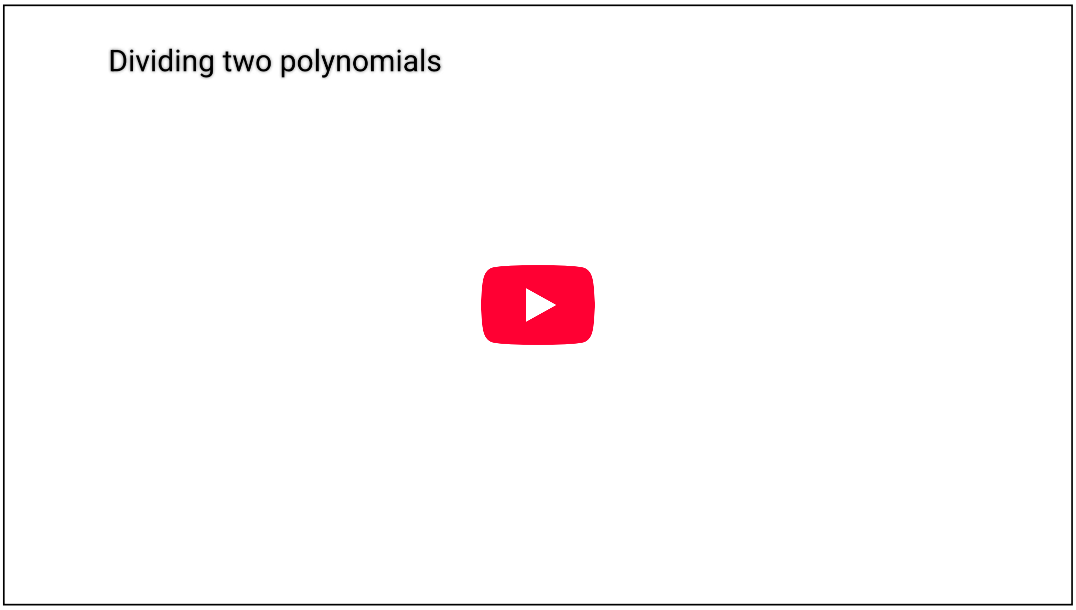
For two polynomials $P(x)$ and $D(x)$ there are two other polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = D(x)Q(x) + R(x).$$

Overview (/study/ap aa-hl/sid-134-cid-761926/o

The polynomials $Q(x)$ and $R(x)$ are unique if the degree of $R(x)$ is less than the degree of $D(x)$. In this case $P(x)$ is called the **dividend**, $D(x)$ is the **divisor**, $Q(x)$ is the **quotient** and $R(x)$ is the **remainder**.

You can use similar technique to the division of integer numbers to find the quotient and remainder when you divide polynomials. The video below shows you the process of dividing $x^2 - 3x + 1$ by $x + 2$.



In both the cases – dividing integer numbers and dividing polynomials – you can check the result by multiplication and addition.

$1794 = 12 \times 149 + 6$ [Simplify the right-hand side and verify]

In the same way,

$x^2 - 3x + 1 = (x + 2)(x - 5) + 11$ [Expand and simplify the right-hand side and verify]

Example 4

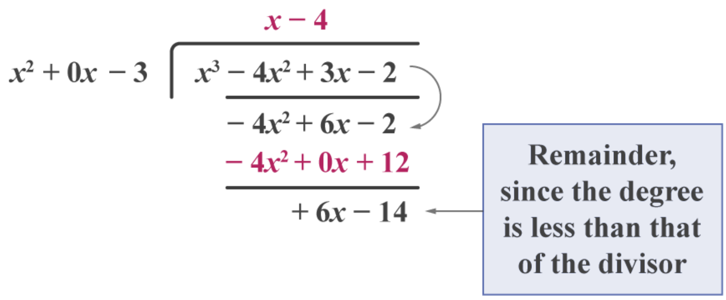
★★☆

Find the quotient and the remainder when you divide $x^3 - 4x^2 + 3x - 2$ by $x^2 - 3$.

Student view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

In this example, the divisor has the x term missing. Hence rewrite it as $x^2 + 0x - 3$, so that the terms line up correctly when doing the division. (This is similar to the use of a zero in numerical column arithmetic to preserve place value, for example when working out $7 - 2.456$ you can write $7.000 - 2.456$).



The quotient is $x - 4$ and the remainder is $6x - 14$.

$$x^3 - 4x^2 + 3x - 2 = (x^2 - 3)(x - 4) + (6x - 14)$$

Example 5

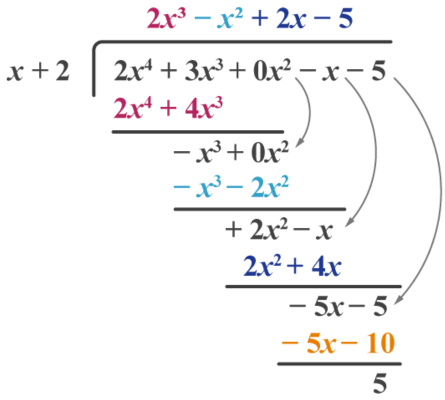
★★☆

Divide $2x^4 + 3x^3 - x - 5$ by $x + 2$.

Insert the missing term as $0x^2$ in the dividend

Student
view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o



so $\left(2x^4+3x^3-x-5\right)\div(x+2)=2x^3-x^2+2x-5+\dfrac{5}{x+2}$

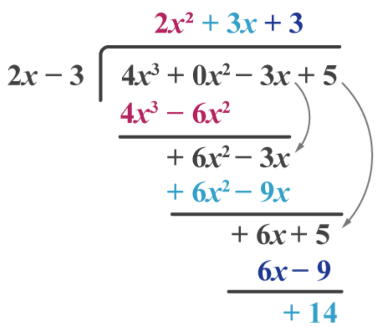
Example 6

★★☆

Find the quotient and the remainder when $5+4x^3-3x$ is divided by $2x-3$.

Rewrite the dividend in descending powers of x inserting the missing terms with coefficient 0.

Dividend: $4x^3+0x^2-3x+5$



Student
view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

Quotient: $2x^2 + 3x + 3$

Remainder: $+14$

Exam tip

In the next section you will see how to find the remainder without the long division process in case the divisor is a linear polynomial. When the quotient is not needed, the method you will see is more efficient than doing the long division.

3 section questions

2. Functions / 2.12 Polynomial functions

Theorems on polynomial functions

This is an activity-based learning section. Follow the instructions and answer the questions in the order given. By the end of this activity you will be able to state and apply the factor and remainder theorems which are used in factorising and solving polynomials with degrees higher than 2.

Activity

1. Find the quotient $(Q(x))$ and the remainder $(R(x))$ polynomials when each of the polynomials $(P(x))$ in **Table 1** below is divided by $(x - 1)$.

Table 1

	$P(x)$	$Q(x)$	I
a)	$x^2 + 5x + 3$		
b)	$x^3 + 2x^2 - x + 4$		

Student
view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

	$P(x)$	$Q(x)$	I
c)	$2x^4 + x^3 - 2x^2 - 3x + 1$		

2. Write $P(x)$ as $(x-1)Q(x) + R(x)$ for each of the above

   a) $x^2 + 5x + 3 =$
   b) $x^3 + 2x^2 - x + 4 =$
   c) $2x^4 + x^3 - 2x^2 - 3x + 1 =$

3. Find the value of $P(1)$ for each of the above

   a) $P(1) =$
   b) $P(1) =$
   c) $P(1) =$

   Which of the following will have the same values as $P(1)$ in all the cases?

   i. $Q(1)$ ii. $R(1)$ iii. $D(1)$

4. Divide all the three $P(x)$ by $(x-2)$ to find $Q(x)$ and $R(x)$ as before. Find $P(2)$ and $R(2)$ for each. What do you observe? Explain.

5. Divide each of the $P(x)$ by $2x - 3$ and express them in the form

$$P(x) = (2x-3)Q(x) + R(x)$$

6. What value of x will make $P(x) = R(x)$? Why?

Important

The remainder theorem

For any polynomial $P(x)$, the remainder when divided by $(x-a)$, $a \in \mathbb{R}$, is $P(a)$.

Student
view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

Activity

1. Divide each of the following by $(x-1)$ and fill in the remaining columns of **Table 2** below:

Table 2

	$P(x)$	$Q(x)$	$R(x)$	$(x-1$
a)	x^2-4x+3			
b)	x^3+2x^2-2x-1			
c)	$x^4-2x^3-7x^2+5x+3$			

2. What is common in all the above results? What does this indicate?
3. Find $P(1)$ as before, for each of the $P(x)$ above
4. If $(x-a)$ is a factor of $P(x)$ what will be the value of $P(a)$?
5. If $(ax+b)$ is a factor of $P(x)$ what will be the value of $P(-b/a)$?

Important

The factor theorem

For any polynomial $P(x)$, $(x-a)$ is a factor if and only if $P(a)=0$.

Example 1

★☆☆

Find the remainder when the cubic function

$$P(x)=2x^3+x^2-x+1 \text{ is divided by } x+1.$$

Student
view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$a = -1$ (make note of the minus sign) so the remainder is

$$P(-1) = 2 \times (-1)^3 + (-1)^2 - (-1) + 1 = -2 + 1 + 1 + 1 = 1$$

as can be readily shown through long division.

Example 2

★★☆

Given the polynomial function $P(x) = x^3 + x^2 - kx - k, k \in R$, find k so that $(x+2)$ is a factor of $P(x)$.

From the factor theorem, $(x+2) = (x-(-2))$ is a factor of $P(x)$ if and only if $P(-2) = 0$

so $P(-2) = (-2)^3 + (-2)\,2 - k(-2) - k = -8 + 4 + 2k - k \ = 0$

$k = 8 - 4 = 4$

Example 3

★★★

If the cubic expression $2x^3 + sx^2 - 5x + t$ is divisible by both $(x+1)$ and $(x-2)$, find s and t, and factorise the expression completely.

Let $P(x) = 2x^3 + sx^2 - 5x + t$

Since $(x+1)$ and $(x-2)$ are factors of $P(x)$ from the factor theorem,

$$\begin{aligned} P(-1) &= -2 + s + 5 + t \\ &= 3 + s + t = 0 \\ & \qquad s + t = -3 \qquad (1) \end{aligned}$$

Student view

and

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

$$\begin{aligned} P(2) &= 16 + 4s - 10 + t \\ &= 6 + 4s + t = 0 \\ &\qquad 4s + t = -6 \qquad (2) \end{aligned}$$

The solution to the simultaneous equations (1) and (2) is $s = -1$ and $t = -2$.

Hence, the cubic function is $2x^3 - x^2 - 5x - 2$

Two factors are the ones given, i.e. $(x + 1)$ and $(x - 2)$, leaving a linear factor as the third, which you can write as $(ax - b)$.

By comparing coefficients, you can find a and b,

i.e.

$$2x^3 - x^2 - 5x - 2 = (x + 1)(x - 2)(ax - b)$$

which gives for the coefficients of x^3,

$$a = 2$$

and for the constant terms,

$$2b = -2$$

$$b = -1$$

Thus the completely factorised expression is

$$(x + 1)(x - 2)(2x + 1).$$

3 section questions

2. Functions / 2.12 Polynomial functions

Roots, zeros and factors

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Zeros of a polynomial $P(x)$ are the values of x for which the graph crosses the x-axis.

Roots of a polynomial $P(x)$ are the solutions to $P(x) = 0$.

Activity

Investigate the difference between a root and a zero of a polynomial $P(x)$.

Making connections

Recall your knowledge on factorising and solving quadratic equations (subtopic 2.7 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27705/)) .

A polynomial of degree n can be written as a product of n factors

$$P(x) = (x - a_1)(x - a_2) \ldots . (x - a_n),$$

where $a_1, a_2, a_3 \ldots a_n$ are the zeros of the polynomial which could be real or complex .

The real zeros of the polynomial can be any one of the following:

1. all distinct
2. all equal
3. some equal and some distinct.

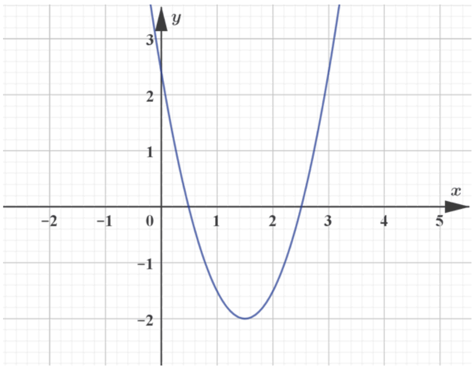
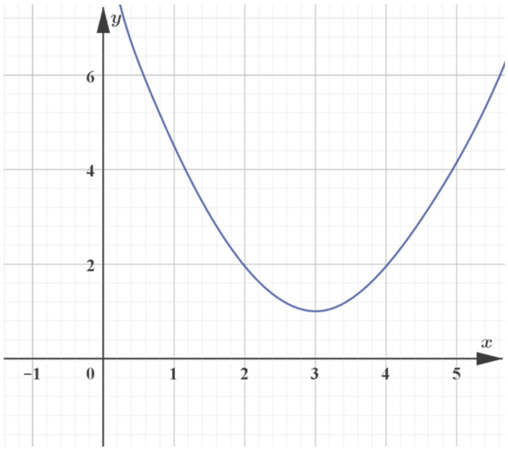
The complex roots of a polynomial occur in conjugate pairs. There can never be one complex root. (subtopic 1.14 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26837/)).

Hence if a polynomial has complex roots, they will follow a certain pattern. The graph crosses the x-axis at each of the real roots and stays above or below the x-axis if it has complex roots. Note the positions and shapes of each of the graphs given in the figures below.

Note the number of points at which the graphs cut the x-axis in each case.

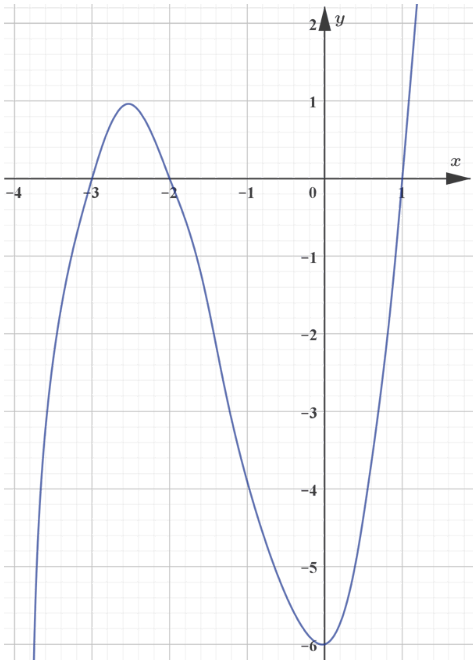
Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

	All real roots	Complex roots
Quadratic	 More information The image is a graph showing a parabolic curve plotted on a grid. The X-axis ranges from -3 to 5, and the Y-axis ranges from -3 to 3. The grid lines help indicate values along both axes. The curve crosses the X-axis around 2 and the Y-axis at 0, resembling a U-shaped parabola that opens upwards. The graph provides a visual representation of a mathematical function, likely a quadratic equation, showing how it behaves across the observed values. [Generated by AI]	Both are complex roots  More information The image shows a graph of a quadratic curve on a coordinate plane with a grid. The X-axis ranges from -1 to 5 with marks at each integer, while the Y-axis ranges from 0 to 5. A blue curve, representing a parabolic function, starts above the Y-axis at a positive value, curves down through the X-axis between x = 1 and x = 2, and continues upward. It reaches a minimum point at approximately x = 3, y = 2 and then rises again to intersect the X-axis near x = 4, continuing up past the top of the graph. This curve illustrates the concept of complex roots as part of the parabolic function. [Generated by AI]

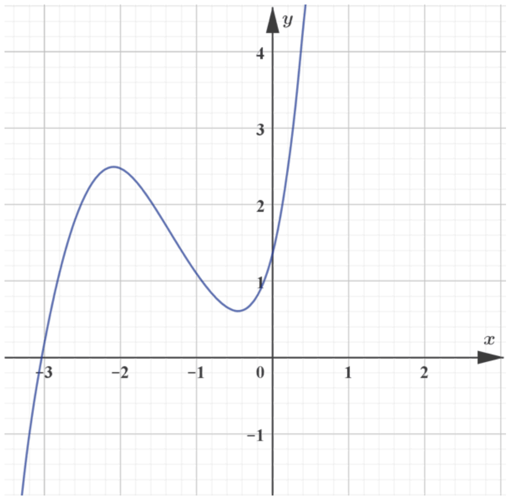
Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/algebra-of-polynomials-id-26605/print/) Assign

Student view

<table>
<tr><th></th><th>All real roots</th><th>Complex roots</th></tr>
<tr>
<td>Cubic</td>
<td>


More information

The image is a graph showing a mathematical function plotted on a grid. The X-axis ranges from -4 to 1 and represents numeric values. The Y-axis ranges from -6 to 3 and denotes another set of numeric values. The curved line on the graph starts from the left side, rising to a peak before descending and then rising steeply again towards the right. Key points on the line include a local maximum approximately at X=-2 and a local minimum near X=0. The function shows an upward then downward trend, followed by an upward incline towards the end.

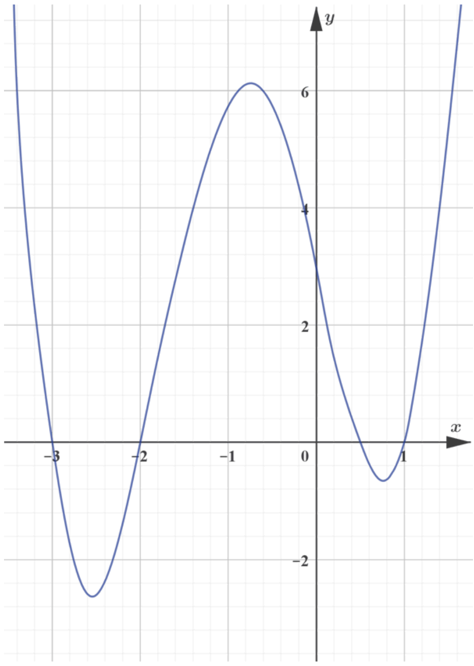
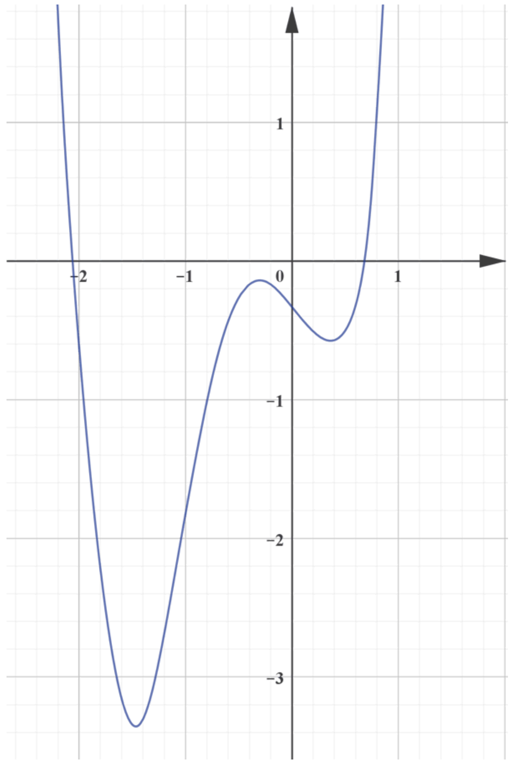
[Generated by AI]
</td>
<td>
One real and two complex roots


More information

This graph depicts a polynomial function with one real and two complex roots. The X-axis represents real numbers ranging from -3 to 3, while the Y-axis ranges from -1 to 4. The curve starts at the left end of the graph at a high point around Y=3, descends to a minimum near the origin, begins to rise again, and finally escalates sharply upwards as it moves to the right. This behavior indicates the presence of a real root near zero, with the curve not intersecting the X-axis elsewhere, suggesting the other roots are complex. The curve demonstrates one local maximum, one local minimum, and increases steeply on both sides of the graph representing the polynomial's behavior.

[Generated by AI]
</td>
</tr>
</table>

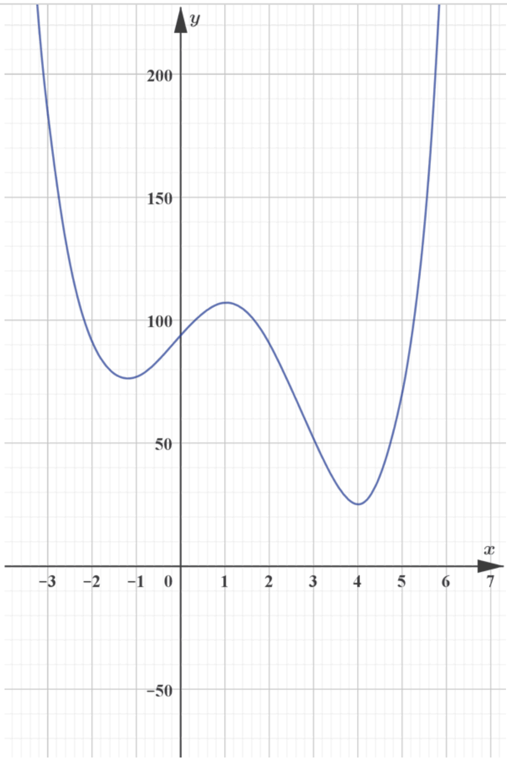
Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

	All real roots	Complex roots
Quartic	 More information The image is a graph displaying a sinusoidal curve. The X-axis ranges from -3 to 3, marked at single unit intervals. The Y-axis ranges from -3 to 3, also marked at single unit intervals. The curve crosses the X-axis several times: between -3 and -2, close to -1, and at 0 and 1. The peaks of the curve occur just under Y = 2 at around X = -2.2, and the troughs occur near Y = -2 at approximately X = 0.8. The graph indicates a wave-like pattern, typical of sine or cosine functions. [Generated by AI]	Two real and two complex roots  More information The image displays a graph of a polynomial function with the X-axis ranging from -3 to 3 and the Y-axis from -3 to 3. The graph shows a curve that crosses the X-axis at two points, indicating two real roots near -1 and 1. The curve begins from the top left and descends steeply across the X-axis, then dips to a local minimum before rising to a local maximum. It crosses again near 1, indicating the presence of complex roots where the curve does not cross the axis but changes direction. The function has two turning points above and below the X-axis, reflecting changes in the direction of the curve, suggesting two real roots and two complex roots based on curvature and inflection points. [Generated by AI] All four complex roots

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/theorems-on-polynomial-functions-id-26606/print/) Assign

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

	All real roots	Complex roots
		 More information The image displays a graph of a polynomial function overlaid on a grid. The X-axis ranges from -3 to 7, marked with integers. The Y-axis ranges from -200 to over 200, also marked with integers. A blue curve moves across these axes. It starts high, descends to a minimum point, rises to a local maximum, decreases slightly, and then rises sharply after crossing the X-axis. The graph shows changes in slope and crosses the X-axis multiple times, indicating the complex roots of the function as described in the accompanying text "All four complex roots." [Generated by AI]

In order to find the roots of higher-order polynomial equations, you can use the factor theorem (see section 2.12.3 (/study/app/math-aa-hl/sid-134-cid-761926/book/theorems-on-polynomial-functions-id-26606/)).

1. Use trial and error to find one root of the equation. (Usually this should be between -2 and $+2$ to be within the scope of this syllabus.)

   If $P\left(x\right)$ is the polynomial find a for which $P\left(a\right) = 0$.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

By the factor theorem, $(x-a)$ is a factor of $P(x)$ and hence $x=a$ is one of the roots of the equation.

2. Divide the given polynomial $P(x)$ by $(x-a)$ to find the other factor.

$$P(x)=(x-a)\,Q(x)$$

3. Factorise the quotient $Q(x)$, if possible.
4. Find all the factors of the polynomial and find the roots from each factor.

Important

If $Q(x)$ is a polynomial with degree higher than 2, the steps 1 to 3 should be repeated at least once more, in order to factorise it.

Example 1

★★★

Find the roots of the equation $x^3+2x^2-5x-6=0$.

This can be solved by letting $P(x)=x^3+2x^2-5x-6$

1. Use a trial and error method to find the first root: (find a for which $P(a)=0$)
   $$P(1)=1+2–5–6\neq 0$$
   $$P(-1)=-1+2+5-6=0$$
   Hence $x=-1$ is a root of the equation and so $(x+1)$ is a factor.

2. Divide x^3+2x^2-5x-6 by $x+1$ using long division:

Student view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

$$
\begin{array}{r|l}
 & x^2 + x - 6 \\
\hline
x+1 & x^3 + 2x^2 - 5x - 6 \\
 & \underline{x^3 + x^2} \\
 & \quad x^2 - 5x \\
 & \quad \underline{x^2 + x} \\
 & \qquad -6x - 6 \\
 & \qquad \underline{-6x - 6} \\
 & \qquad\qquad 0
\end{array}
$$

3. Factorise $x^2 + x - 6$

$$x^2 + x - 6 = (x - 2)(x + 3)$$

4. $P(x) = 0$

$$\Rightarrow (x + 1)(x - 2)(x + 3) = 0$$
$$\Rightarrow x + 1 = 0 \text{ or } x - 2 = 0 \text{ or } x + 3 = 0$$
$$\Rightarrow x = -1 \text{ or } x = 2 \text{ or } x = -3$$

Example 2

★★☆

Find the polynomial whose roots are:

$$1,\ 2,\ 2 \pm \mathrm{i}$$

Since there are four roots, the polynomial is quartic.

Any polynomial can be expressed as factors from its roots.

Hence, the polynomial for the above given roots would be:

$$P(x) = (x - 1)(x - 2)(x - (2 + \mathrm{i}))(x - (2 - \mathrm{i}))$$

Student view

Simplifying,

$$P(x) = x^4 - 7x^3 + 19x^2 - 23x + 10$$

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Example 3

★★★

Solve the equation: $x^3 - 2x^2 + 3x + 6 = 0$

By trial and error, $x = -1$ is a factor.

Dividing the polynomial by $(x+1)$ gives $x^2 - 3x + 6$

Solve $x^2 - 3x + 6 = 0$ using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{3 \pm \sqrt{3^2 - 4 \cdot (1) \cdot 6}}{2 \cdot 1}$$

$$x = \frac{3 \pm \sqrt{9 - 24}}{2}$$

$$\frac{1}{2}\left(3 \pm \mathrm{i}\sqrt{15}\right)$$

The roots are: $x = -1,\ x = \frac{1}{2}\left(3 \pm \mathrm{i}\sqrt{15}\right)$

Example 4

★☆☆

Solve the equation: $x^2 + 5x - 2 = 2x + \frac{3}{x}$

When the equation is simplified, it gives a cubic polynomial .

Student view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

Using trial and error in order to identify the first root shows that the roots are not integers. Hence, it is better to use graphing software to graph the equation and find the solutions.

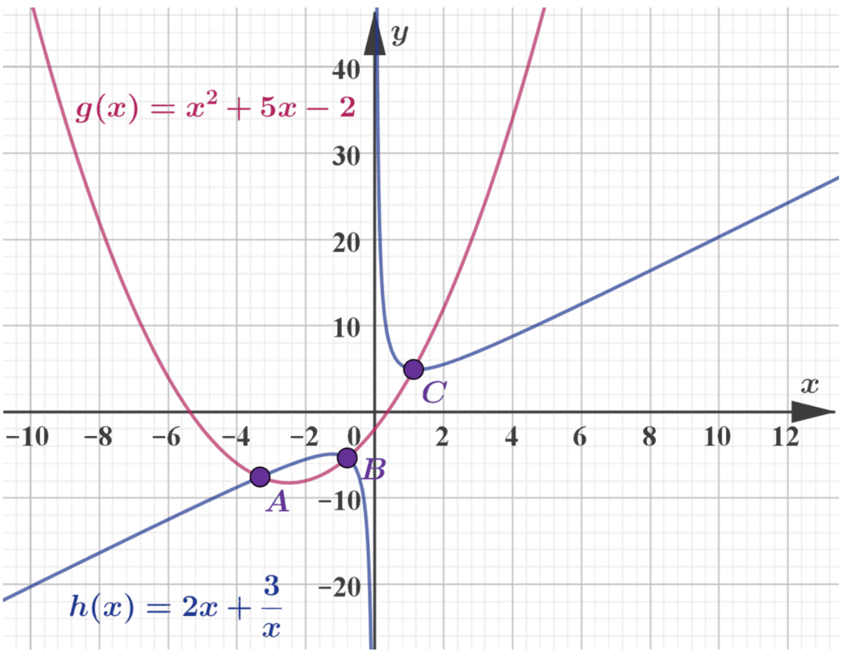
Graphing both the sides of the equation individually on the same set of axes, the first coordinates of the points of intersection (A, B, C) give the solutions.

$A\ (-3.33,\ -7.56)$

$B\ (-0.798,\ -5.35)$

$C\ (1.13,\ 4.92)$

Hence the solutions are $x \approx -3.33,\ x \approx -0.798$ and $x \approx 1.13$.



Refer to section 2.4.1 (/study/app/math-aa-hl/sid-134-cid-761926/book/turning-points-id-25402/) for appropriate calculator instructions.

Student view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

International Mindedness

There are other ways of dividing a polynomial by another polynomial. One such method is called synthetic division. Watch the following video to understand this method.

Which method came into use first and who invented it? How did the methods of division of polynomials improve over the years across the world?

3 section questions

2. Functions / 2.12 Polynomial functions

Sum and products of the roots of polynomial equations

Quadratic equations

An interesting feature of quadratic equations is that you can extract useful information about the roots without actually solving the equation to find numerical values for the roots.

Student
view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Activity

A general quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$, has two solutions x_1 and x_2.

Using the quadratic formula, find two expressions for the sum $x_1 + x_2$ and product x_1x_2 of the roots in terms of the coefficients $a,\ b$ and c.

International Mindedness

The expressions you derived in the activity for the sum and product of the roots of a quadratic equation are known as Viete's formulae. The French mathematician Francois Viete (1540—1603) invented a lot of the algebraic notation used up to the present day. He is credited with the original idea of using letters to represent unknown variables, which is fundamental to algebra.

In the following table, three quadratic equations are solved and their corresponding roots are found. Observe the last two columns.

Quadratic equations	Root 1 (x_1)	Root 2 (x_2)	Sum $(x_1 + x_2)$	Product (x_1x_2)
$2x^2 + 3x - 1 = 0$	$-3/4 + \sqrt{17}/4$	$-3/4 - \sqrt{17}/4$	$-3/2$	$-1/2$
$3x^2 + 5x - 2 = 0$	-2	$1/3$	$-5/3$	$-2/3$
$x^2 - x - 6 = 0$	3	-2	1	-6

The following claim is a generalisation of the examples above.

Important

Let x_1 and x_2 represent the roots of a quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Then

- $x_1 + x_2 = -\dfrac{b}{a}$ and
- $x_1x_2 = \dfrac{c}{a}$.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Example 1

★☆☆

For the quadratic equation $2x^2 - 3x + 1 = 0$, find the sum and the product of its roots.

Steps	Explanation
Let x_1 and x_2 be the roots of the equation. The equation has $$a = 2, b = -3, c = 1$$	Label the two roots. Identity the coefficients a, b and c.
$x_1 + x_2 = -\frac{-3}{2} = \frac{3}{2}$	Sum of roots $x_1 + x_2 = -\frac{b}{a}$
$x_1 x_2 = \frac{1}{2}$	Product of roots $x_1 x_2 = \frac{c}{a}$
Verification: $2x^2 - 3x + 1 = 0$ $(2x - 1)(x - 1) = 0$ $x_1 = \frac{1}{2}$ or $x_2 = 1$	To check that the formulae give the correct answers, you can solve the quadratic equation by factorisation.
$x_1 + x_2 = \frac{1}{2} + 1 = \frac{3}{2}$ and $x_1 x_2 = \frac{1}{2} \times 1 = \frac{1}{2}$	Add and multiply the roots.

Example 2

★★☆

The quadratic equation $3x^2 - 6x + 1 = 0$ has roots α and β. Without solving the equation, find the value of $\alpha^2 + \beta^2$.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
$(\alpha+\beta)^2=\alpha^2+2\alpha\beta+\beta^2$ $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$	Use the perfect square identity to express $\alpha^2+\beta^2$ in terms of the sum and product of α and β.
$3x^2-6x+1=0$ $a=3, b=-6, c=1$ $\alpha+\beta=-\frac{b}{a}-\frac{-6}{3}=2$ $a\beta=\frac{c}{a}=\frac{1}{3}$	Use the formulae to find the sum and product of the roots.
Hence, $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=2^2-2\cdot\frac{1}{3}=4-\frac{2}{3}=\frac{10}{3}$	Substitute the sum and product of roots into the expression derived in the first step.

Example 3

★★★

Let α and β be the roots of the quadratic equation $-2x^2+4x+\frac{5}{2}=0$. Find a quadratic equation whose roots are $\frac{2}{\alpha+1}$ and $\frac{2}{\beta+1}$.

Steps	Explanation
For $-2x^2+4x+\frac{5}{2}=0$: $a=-2,\ b=4,\ c=\frac{5}{2}$	

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
$\alpha+\beta=-\frac{b}{a}=-\frac{4}{-2}=2$ $\alpha\beta=\frac{c}{a}=\frac{\frac{5}{2}}{-2}=-\frac{5}{4}$	Use the formulae to find the sum and product of the roots.
For the new equation with roots $\frac{2}{\alpha+1}$ and $\frac{2}{\beta+1}$: $\frac{2}{\alpha+1}+\frac{2}{\beta+1}=\frac{2\beta+2+2\alpha+2}{(\alpha+1)(\beta+1)}=\frac{2(\alpha+\beta)+4}{\alpha\beta+(\alpha+\beta)+1}$	Express the sum of the roots of the new quadratic equation in terms of $\alpha+\beta$ and $\alpha\beta$.
$=\frac{2(2)+4}{-\frac{5}{4}+2+1}=\frac{8}{\frac{7}{4}}=\frac{32}{7}$	Substitute in the values of $\alpha+\beta$ and $\alpha\beta$.
Therefore, for the new quadratic equation we have $-\frac{b}{a}=\frac{32}{7}$	Now you know information about the coefficients 'a' and 'b' of the new quadratic equation.
$\frac{2}{\alpha+1}\cdot\frac{2}{\beta+1}=\frac{4}{(\alpha+1)(\beta+1)}=\frac{4}{\alpha\beta+(\alpha+\beta)+1}$	Express the product of the roots of the new quadratic equation in terms of $\alpha+\beta$ and $\alpha\beta$.
$=\frac{4}{-\frac{5}{4}+2+1}=\frac{4}{\frac{7}{4}}=\frac{16}{7}$	Substitute in the values of $\alpha+\beta$ and $\alpha\beta$.
Therefore, for the new equation we have $\frac{c}{a}=\frac{16}{7}$.	Now you know information about the coefficients 'a' and 'c' of the new quadratic equation.
So the new equation we are looking for is $x^2-\frac{32}{7}x+\frac{16}{7}=0$ or $7x^2-32x+16=0$	The new equation can be expressed as $x^2+\frac{b}{a}x+\frac{c}{a}=0$

Student view

Overview (/study/ap aa-hl/sid-134-cid-761926/o

Higher order equations

For a cubic equation $ax^3 + bx^2 + cx + d = 0$ and quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, use the table below to find the sum and product of the roots and compare them with the coefficients.

Cubic/ Quartic equations	Root 1 (x_1)	Root 2 (x_2)	Root 3 (x_3)	Root 4 (x_4)	Sum $(x_1 + x_2 + x_3 + x_4)$	P $(x_1$
$x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$	1	2	−1	3	5	
$2x^3 + x^2 - 5x + 2 = 0$	$\frac{1}{2}$	−2	1		$-\frac{1}{2}$	

For a cubic equation, $ax^3 + bx^2 + cx + d = 0$

- Sum $= -\frac{b}{a}$
- Product $= -\frac{d}{a}$

Be aware

Note the change in the sign for the product. The product is either positive or negative depending on the degree of the polynomial.

For a quartic equation, $ax^4 + bx^3 + cx^2 + dx + e = 0$

- Sum $= -\frac{b}{a}$
- Product $= \frac{e}{a}$

Extending this to polynomial of degree n.

Important

Consider the polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$.

- The sum of the roots is $-\frac{a_{(n-1)}}{a_n}$

Student view

- The product of the roots is $\frac{(-1)^n (a_0)}{a_n}$

Exam tip

Both of these formulae are in the formula booklet.

Example 4

★☆☆

Find the sum and product of the roots of the following equations:

1. $x^2 + 9x + 4 = 0$
2. $2x^3 - 3x^2 + x - 4 = 0$
3. $3x^5 - 2x^4 + 3x + 2 = 0$

Note: You need not solve the equations for roots. You just need to use the above concept and find the sum and product directly:

1. sum $= -9$; product $= 4$.
2. sum = $\frac{3}{2}$; product $= 2$.
3. sum = $\frac{2}{3}$; product = $-\frac{2}{3}$.

Example 5

★★☆

Find the roots of the quadratic equation $x^2 - 6x + k = 0$, given that one root is three times the other and that k is a constant. Also find the value of k.

In this case, if x_1 and x_2 are the roots of the equation, then $x_1 = 3x_2$

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$$x_1 + x_2 = 6$$
$$3x_2 + x_2 = 6$$
$$x_2 = \frac{6}{4} = \frac{3}{2} = 1.5$$

Hence $x_1 = 3\,(1.5) = 4.5$

To find the value of k,

$$x_1 \times x_2 = k$$
$$1.5 \times 4.5 = k$$
$$k = 6.75$$

3 section questions

2. Functions / 2.12 Polynomial functions

Sketching the graph of a polynomial function

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/sketching-the-graph-of-a-polynomial-function-id-26609/print/) Assign

Obtaining the equation from the graph

The rules explained within this subtopic connect the roots of the equation and its graph. Therefore, given a graph you can find its function.

1. Identify the number of different roots, n.
2. Identify whether there are any equal roots. If so, look at the shape at the equal root to identify the factor for this root.

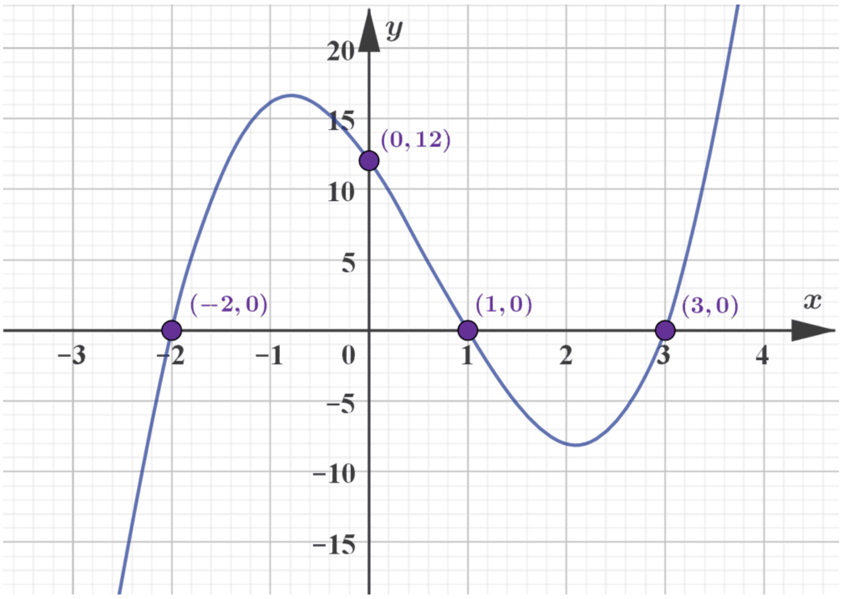
For example, it could be $(x-a)^2$or $(x-a)^3$ and the graph will have the corresponding shape depending upon the exponents of these factors.

3. Do not forget to find the leading coefficient using a given point on the graph.

For example, in this cubic curve, there are three different zeros, and hence the function will be of the form $f\,(x) = a\,(x-x_1)\,(x-x_2)\,(x-x_3)$.

Student view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o



More information

The image is a graph illustrating a cubic curve. The X-axis is labeled as 'x' and the Y-axis as 'y', both extending from -5 to 5 on the X-axis and -20 to 20 on the Y-axis. This scale is marked in intervals of 1 on the X-axis and 5 on the Y-axis.

The cubic curve passes through four key points marked with purple dots: (-2, 0), (0, 12), (1, 0), and (3, 0). These points represent zeros and a peak of the function.

Starting at the left, the curve rises steeply from x = -5 reaching its maximum at (0, 12), then it descends, crossing the X-axis at (1, 0), continues to a minimum point, and rises again, crossing the X-axis at (3, 0). The graph illustrates the typical "S" shape of a cubic curve, highlighting its turning points and zeros.

[Generated by AI]

From the graph you can see that, $x_1 = -2,\ x_2 = 1,\ x_3 = 3$.

Hence it is of the form: $y = \ a\left(x+2\right)\left(x-1\right)\left(x-3\right)$.

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/roots-zeros-and-factors-id-26607/print/) Assign

Substitute the y-intercept $(0, 12)$ to find the value of a in the equation:

$$a\left(2\right)\left(-1\right)\left(-3\right) = 12$$

$$\Rightarrow 6a = 12\ \Rightarrow a = 2$$

Student view

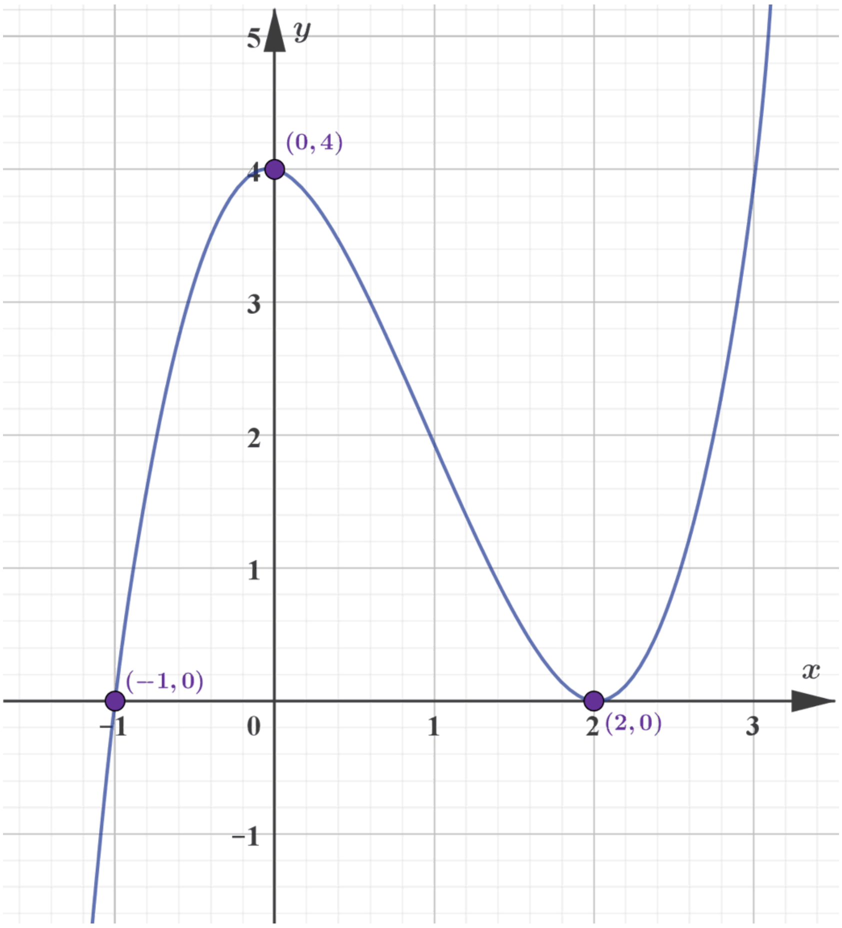
Hence the final equation will be: $y = 2\left(x+2\right)\left(x-1\right)\left(x-3\right)$.

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Example 1

★☆☆

Find the equation of the graph shown below:



More information

The image presents a graph of a polynomial function on a grid. The X-axis represents the horizontal axis, while the Y-axis is the vertical axis. The graph includes labeled points: (-1, 0), (2, 0), and (0, 4), indicating significant intersections and peaks on the curve. The curve begins from the bottom left, rising to a peak just above the Y-axis at point (0, 4). It then descends, crossing the X-axis at (-1, 0), dips down to a trough and ascends again, crossing the X-axis at (2, 0), and continuing upwards. The grid lines support measurement and indicate equal intervals for precise reading.

[Generated by AI]

This a cubic and you have one root touching the x-axis. At this root, the curve will have a quadratic expression $(x-2)^2$

Student view

Including the other root, (-1), the cubic equation should look like:

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

$$y = a(x-2)^2(x+1)$$

Now substitute $(0, 4)$ to find a:

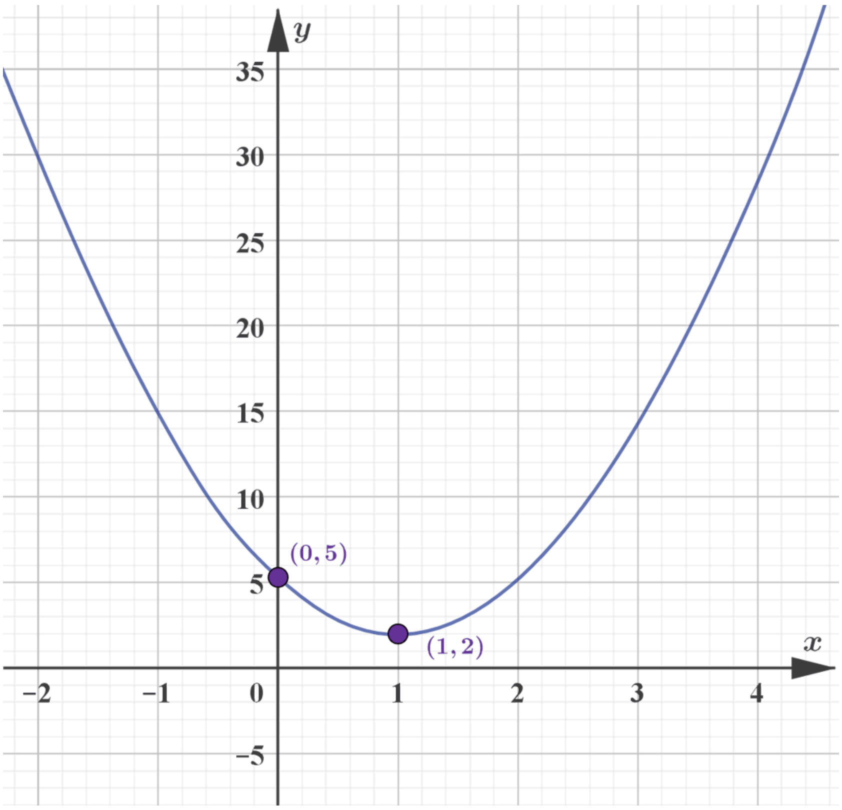
$$4 = a(-2)^2(1)$$
$$a = 1$$

Hence the equation is: $y = (x-2)^2(x+1)$.

Example 2

★★☆

Find the equation of the graph shown below:



More information

The image shows a quadratic graph plotted on a grid. The X-axis ranges from -4 to 4 and the Y-axis ranges from 0 to 35. There is a blue parabolic curve opening upwards. The graph passes through the points labeled (0, 5) and (-1, 2).

[Generated by AI]

Student
view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

This is a quadratic and the graph does not cross the x-axis. Only two points on the graph are given. (If three points are given, you can use simultaneous equations to find a, b and c in the general form $y = ax^2 + bx + c$.)

However, here one of the points is the vertex and hence you can use the vertex form of the quadratic (see subtopic 2.6):

$$y = a(x - h)^2 + k$$

In this case, it will be $y = a(x - 1)^2 + 2$

Now use the point $(0, 5)$ to find a:

$$\begin{aligned} a(-1)^2 + 2 &= 5 \\ a &= 3 \end{aligned}$$

Hence, the equation of the graph is $y = 3(x - 1)^2 + 2$.

3 section questions

2. Functions / 2.12 Polynomial functions

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-26610/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- know that a polynomial of degree n are real numbers and can be written as:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

- know how to add, subtract, multiply and divide polynomials
- apply the Factor and Remainder theorems:
  - when a polynomial $P(x)$ is divided by $(x - a)$
    - $P(a)$ is the remainder if $(x - a)$ is not a factor

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

    - $P(a) = 0$ if $(x - a)$ is a factor
  - when a polynomial $P(x)$ is divided by $(ax + b)$
    - $P(-\frac{b}{a})$ is the remainder if $(ax + b)$ is not a factor
    - $P(-\frac{b}{a}) = 0$ if $(ax + b)$ is a factor
- know that a polynomial of degree n can be written as a product of n factors, where $a_1,\ a_2,\ a_3 \ldots a_n$ are the zeros of the polynomial

$$\mathrm{P}(x) = (x - a_1)(x - a_2) \ldots . (x - a_n)$$

- use the sum and product of polynomial equations:

For $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

$$\text{Sum} = -\frac{a_{(n-1)}}{a_n}, \quad \text{Product} = \frac{(-1)^n (a_0)}{a_n}$$

- know that roots of a polynomial equation could be real or complex. If a polynomial has complex roots, they always occur in conjugate pairs $(a \pm ib)$.
- be able to sketch polynomial functions
- be able to find equations from graphs of polynomial functions.

2. Functions / 2.12 Polynomial functions

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-26611/print/) Assign

Investigate on various methods of using a polynomial to model a function.

Using a regression tool is one of the best methods to model a polynomial function.

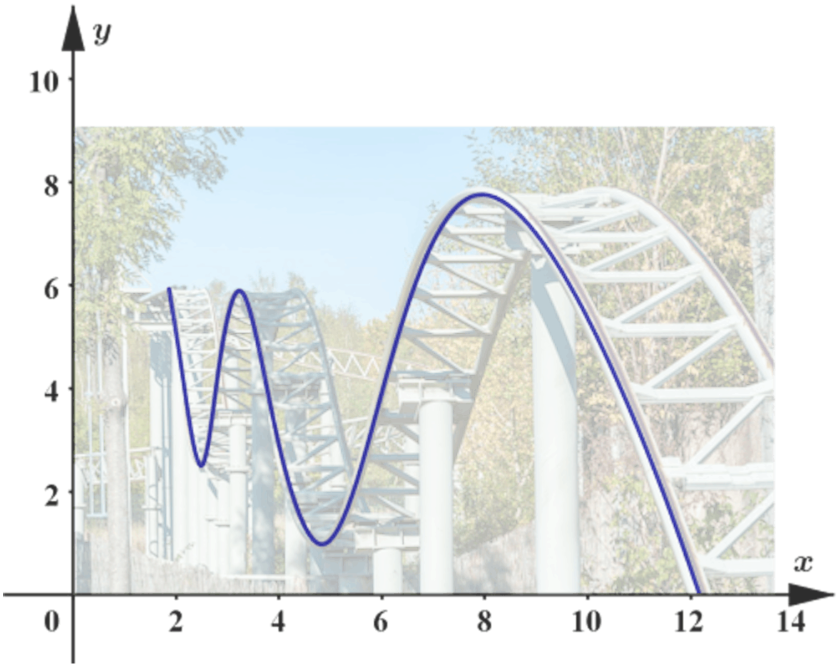
What software can help you with regression tools?

How can you check whether the model is a best fit or not?

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Do some research on these questions and come up with a method that would be appropriate to model the roller coaster given in The big picture (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26603/) section.



This rollercoaster is shown in the graph below

Credit: Ralf Geithe Getty Images

More information

The graph shows a rollercoaster structure with an x-axis labeled from 0 to 14 and a y-axis labeled from 0 to 10. The rollercoaster is a series of connected, wavy metal tracks on vertical supports, shown predominantly between the x-values of 0 and 14. The structure forms several peaks and troughs, simulating a rollercoaster ride's ups and downs. The image is set against a backdrop of trees and clear skies, but the focus is on the rollercoaster's geometric form, which is plotted similar to a graph curve. The entity suggests peaks at x-values around 3, 8, and 13, corresponding to the heights of the rollercoaster in the image.

[Generated by AI]

Consider the following while finding a polynomial function that best fits the picture:

1. Research how to create piecewise functions using different techniques.
2. Identify whether the curve fits into one polynomial function or whether you need piecewise functions.
3. If you are using a regression tool, make sure you choose an appropriate degree for the polynomial and give reasons for your choice.
4. Analyse the results and suggest limitations and scope of the model.

Student view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

5. Do not forget to cite all sources used.

Rate subtopic 2.12 Polynomial functions

Help us improve the content and user experience.

Student
view