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TOPIC 3
GEOMETRY AND TRIGONOMETRY

0 (https://intercom.help/kognity)

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The big picture

You know that the circumference of a circle is $2\pi r$ and its area is πr^2. But what is this mysterious irrational number π? Ancient people observed the stars, moon and sun as they moved across the sky, which they thought was a sphere. In the early years, astronomers were also mathematicians and believed that there was something divine in the shape of a circle. They divided circles into 360 regions, giving us the degree. This helped them to approximate the ratio between the circumference of a circle to its diameter, finding it to be the same for every circle, no matter how big.

This ratio is the well-known irrational number π.

In the 14th century, astronomers and mathematicians were already using the length of an arcof a circle to measure angles. Jamshīd al-Kāshī $(\text{c}.1400)$ calculated 2π correct to 17 decimal places using sexagesimaldigits. The term radianwas used first in examination questions set by James Thomson at Queen's College, Belfast, in 1873.

International Mindedness

Persian astronomer and mathematician Jamshīd al-Kāshī $(\text{c}.1400)$ set a goal for himself: to calculate the size of the universe with an error smaller than the width of a horse hair. To do so, he used a variety of instruments and he invented the azimuth -altitude instrument. After that, he computed sine and cosine tables for various angles.

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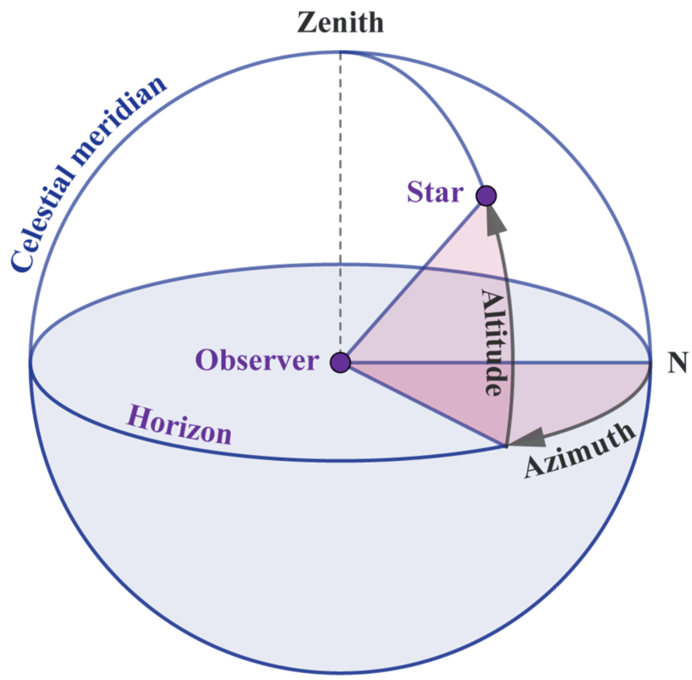
Why would small errors in the estimated value of π create large errors when calculating the path of a celestial object?



More information

The image is a spherical diagram representing the relationship between an observer, a star, and various celestial measurement terms. The sphere is sectioned into two halves, depicting a celestial observation setup. At the center of the base is the "Observer." From the observer, a line extends to the "Star," which is located at an angle inside the sphere, depicting the line of sight.

The "Celestial meridian" is a curved line that runs from the "Observer" upwards to the top marked as "Zenith," which is the point directly overhead.

The "Horizon" is marked at the base, separating the sky from the ground and aligning with the observer's line. The azimuth is represented by an angle at the base, showing the star's position horizontally relative to the observer.

The "Altitude" is depicted by a line that runs vertically from the "Horizon" to the "Star," representing the star's height above the horizon. Additional labels include "N" for north, positioned horizontally to the right of the observer, marking direction.

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Concept

In this subtopic, you will study the relationships between the parts of a circle and their corresponding angles, lengths and areas using radians. Why are all lengths

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and areas in circles approximate ? Why do we use the irrational number π ?

Radian measure of angles

Radius, arc length and angle

So far you have used degrees for measuring angles. A circle can be divided into 360 equal sectors so that the central angle of each sector is 1°. This unit of measure does not relate to the circle itself; it is used only to divide a circle into 360 equal parts. The degree measure does not appear in the formulae for area and circumference of the circle, instead you use π to find these.

Have you ever wondered why the circumference of a circle is $2\pi r$ or the area is πr^2? There is another unit of measure for angles. It is derived from the relation between the length of an arc and the radius of the circle. This unit of measure is called the radian and it is the SI unit for angles. It is the measure of a central angle subtended by a circular arc that has the same length as the radius.

This animation shows some angles measured in radians.

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Video 1. Visualizing Radian Measure.

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27628/print/) More information for video 1 Assign

This animation introduces radians as a unit for measuring angles by visually demonstrating the relationship between arc length and radius. It begins by drawing a radius from the center of a circle and constructing the full circle. The animation then sequentially highlights arc segments, showing how an angle of 1 radian corresponds to an arc length equal to the radius. It continues by marking 2 radians, 3 radians, and a smaller segment leading to π radians—completing half of the circle. The process then extends to the remaining half, culminating in a total angle of 2π radians.

Through this step-by-step visual representation, users gain an intuitive understanding of radian measure and how it naturally arises from the circle's geometry. This interactive helps learners connect radians to fundamental circle properties, reinforcing why a full revolution equals 2π radians and how this unit is integral to formulas for arc length, sector area, and trigonometric calculations.

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When the arc length is equal to half of the circumference, the corresponding angle is π radians. If it is a complete turn, then it is 2π radians.

So $2\pi\text{rad} \equiv 360°$ and $\pi\text{rad} \equiv 180°$.

Important

You can use the following ratio to convert between radians and degrees:

$$\frac{\text{radian measure}}{\pi} = \frac{\text{degree measure}}{180}$$

Be aware

It is important to understand the notation used in IB examinations so that you know if an angle is given in radians or degrees.

- Radians is often abbreviated to rad when used as a unit, e.g. 1.2 rad.
- However, the unit is often omitted, especially if the angle is given in terms of π, e.g. 1.2π .

Making connections

Why is π such a mysterious number?

To start with, it is an irrational number. This means you cannot write it as a fraction of two integers and its decimal places do not repeat or end. Computer scientists have calculated π to billions of digits. It starts $3.141592653\ldots$.

Apart from being irrational, there are many other reasons why people are fascinated by π . It appears wherever a curve and straight line are both involved, such as in the DNA double helix, waves, the meandering of rivers and many other places.

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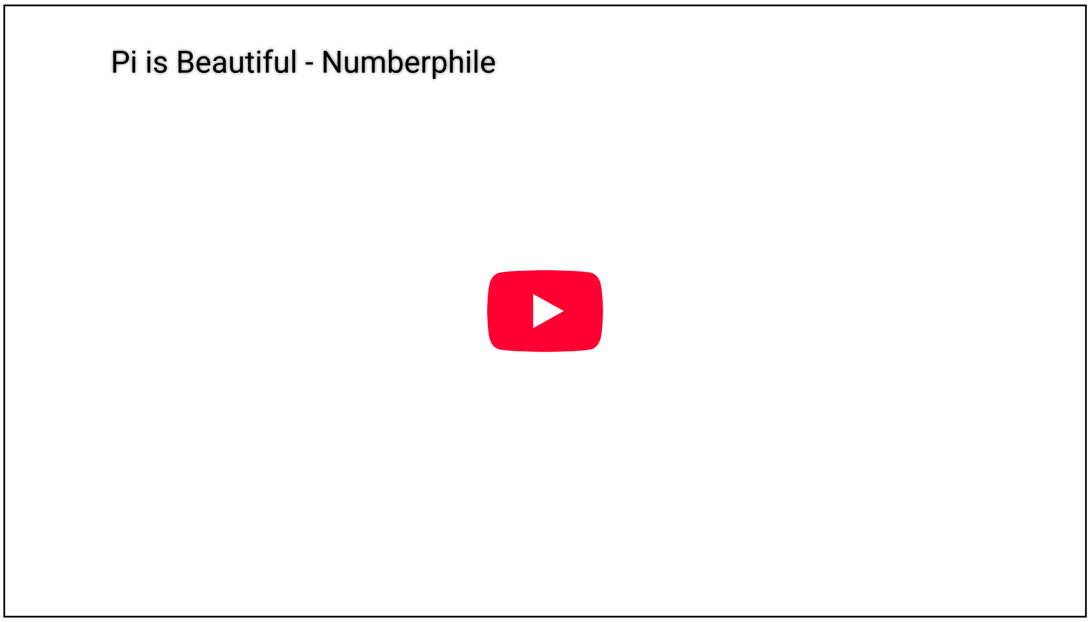
π in action
Credit: Stuart Sly EyeEm Getty Images

It also emerges in unlikely places. For example, in a group of random whole numbers, the probability of any two numbers being coprime (also known as relatively or mutually prime) is $\frac{6}{\pi^2}$.

You can watch a beautiful visualisation of the number π in the following video.



Exam tip

In examinations, if you are asked to find an exact value, always leave your answer as a multiple of π . When using your calculator, do not approximate π as 3.14 or $\frac{22}{7}$.

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Use the π button to minimise rounding errors.

Example 1

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An angle is given as 40°. Find its value in radians to 3 significant figures.

Steps	Explanation
$\frac{R}{40^{\circ}}=\frac{\pi}{180^{\circ}}$	As π radians is 180° , you can use this ratio to find the radian measure. Call the angle in radians R .
$R=\frac{40\pi}{180}=\frac{2}{9}\pi$	Rearranging the equation and simplifying.
$R=\frac{2}{9}\pi$	For an exact answer, leave as a multiple of π .
$R\approx 0.698 \text{ rad}$	Approximating to 3 significant figures ($\pi\approx 3.14159\ldots$)

Example 2

★☆☆

An angle is given as $\frac{3}{4}\pi$ radians. Find its value in degrees.

Steps	Explanation
$\frac{\frac{3}{4}\pi}{A}=\frac{\pi}{180^{\circ}}$	As π radians is 180° , you can use this ratio to find the value in degrees. Call the angle in degrees A .

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Steps	Explanation
$A = \dfrac{\frac{3}{4}\pi}{\frac{\pi}{180^\circ}} = 135^\circ$	Rearranging the equation and simplifying.
$A = 135^\circ$	

Exam tip

A description of the conversion process between degrees and radians is not included in the IB formula booklet.

Remember 180° is π radians. This means that $1 \text{ rad} = \dfrac{180}{\pi}$ degrees $\approx 57.3^\circ$.

It is useful to know how big 1 radian is in degrees when you are converting between the two forms.

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Length of an arc

Ratios of length

An arc is part of a circle's circumference. The diagram below shows a circle with centre O and radius r. The points A and B on the circumference form two arcs, both called AB. The minor arc AB is smaller than the half of the circumference and subtends the central angle θ. The other arc is called the major arc, as it is more than the half of the circumference of the circle.

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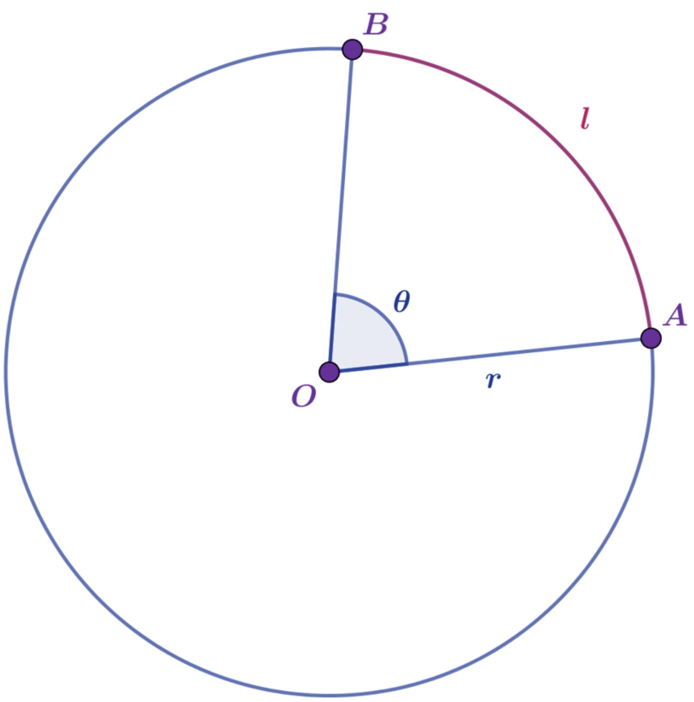

More information

The image is a diagram of a circle with center O. It illustrates a major and minor arc labeled AB along the circle's circumference. The radius from the center O to point A and point B is labeled r. The central angle subtended by the minor arc AB is labeled θ. The distance of the minor arc AB along the circumference is labeled l. Points A and B are marked where the arcs start and end on the circle's edge. The diagram visually demonstrates the relationship between the central angle, the radius, and the arc of a circle.

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Important

The ratio of θ to the full rotation, 2π , is the same as the ratio of the arc length l to the circumference, $2\pi r$.

Thus,

$$\frac{\theta}{2\pi} = \frac{l}{2\pi r}$$

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Rearranging and simplifying, you get the length of an arc in terms of the subtended angle and radius:

$$l = r\theta$$

Be aware

The formula $l = r\theta$ is true only when the angle θ is given in radians.

If the angle is given in degrees, use the formula $l = \frac{\theta}{360°} \times 2\pi r$.

Exam tip

In IB examinations, the formula booklet gives the formula for the length of an arc as $l = r\theta$ where r is the radius and θ is the angle measured in radians.

Example 1

★☆☆

A circle with radius 5 cm has an arc that subtends an angle of 1.2 radians.

Find the length of the arc.

$$l = 5 \times 1.2$$

Using the formula $l = r\theta$.

So, $l = 6 \text{ cm}$

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Example 2

★☆☆

A circle with a radius of 5 cm has an arc of length 8 cm which subtends an angle θ.

Find the size of θ in radians.

$$\theta = \frac{l}{r}$$

Rearrange the formula $l = r\theta$.

$$\theta = \frac{8}{5}$$

Substitute $l = 8$ and $r = 5$.

$$\theta = 1.6 \text{ rad}$$

3 section questions

3. Geometry and trigonometry / 3.7 The circle revisited

Area of a sector

Ratios of area

Just as the length of an arc is a fraction of the circle's circumference, so the area of a sector is a fraction of a circle's area.

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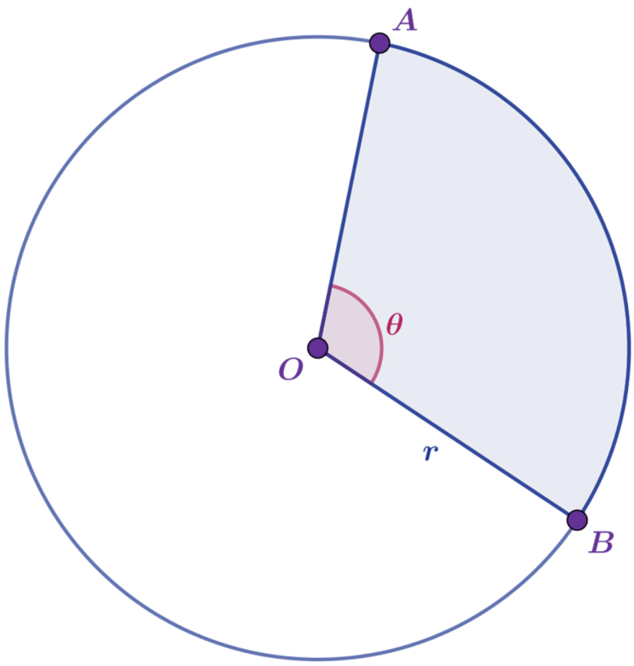

More information

The diagram illustrates a circle with a sector shaded in blue. The circle has a center labeled as 'O', and two radii labeled 'OA' and 'OB' extend to the edge of the circle, creating a sector with an angle (theta) at the center. Points on the circumference where the radii intersect the circle are labeled 'A' and 'B'. The angle between the two radii at the center 'O' is marked as theta (θ), and the radius of the circle is denoted by 'r'. The sector represents a fraction of the circle's total area, correlating with the given angular fraction of the total 360 degrees of the circle.

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Writing this as ratios gives:

$$\frac{\text{area of the sector}}{\pi r^2} = \frac{\theta}{2\pi}.$$

Rearrange to get

$$\text{area of the sector} = \frac{\pi r^2 \theta}{2\pi},$$

and simplify to get

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$$\text{area of the sector} = \frac{1}{2}r^2\theta.$$

Of course, the angle θ here is in radians.

Exam tip

In IB examinations, the formula booklet gives the formula for the area of a sector as

$$A = \frac{1}{2}r^2\theta$$

where r is the radius and θ is the angle measured in radians.

Be aware

If the angle is given in degrees, use the formula $A = \frac{\theta}{360} \times \pi r^2$.

Example 1

★☆☆

The radius of a circle is 5 cm.

Find the area of a sector of the circle with a central angle of 1.3 radians.

$$A = \frac{1}{2}\theta r^2$$

As the angle is given in radians you can use the formula A= $\frac{1}{2}\theta r^2$.

$$A = \frac{1}{2}1.3 \times 5^2$$

Substitute the values for the angle and the radius.

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$A = 16.25 \text{ cm}^2$

Example 2

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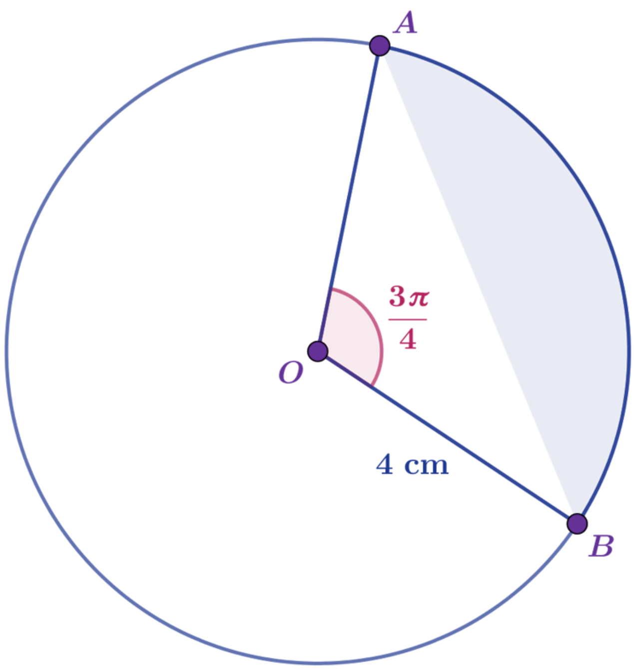
The diagram shows a circle with centre O and radius 4 cm.

If the measure of the central angle is $\frac{3\pi}{4}$, find exact area of the shaded segment.



More information

The image is a diagram of a circle with a central angle marked (\frac{3\pi}{4}) and a radius of 4 cm. The circle is centered at point O. There are two lines extending from the center which intersect the circumference at points A and B, forming a sector. The shaded segment is formed between the chord AB and the arc extending from A to B through the sector. The central angle of (\frac{3\pi}{4}) divides the circle into a sector marked by the arc connecting points A and B. The radius from O to B is labeled as 4 cm. The segment area bounded by arc AB and chord AB is shaded to indicate the area to be found.

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Steps	Explanation
$\text{area of segment} = \text{area of sector} - \text{area of triangle}$	To find the area of the shaded segment, find the area of the sector AOB and subtract the area of the triangle AOB .
$\text{area of sector} = \frac{1}{2} \times 4^2 \times \frac{3\pi}{4}$	As the angle is given in radians you can use the formula $$A = \frac{1}{2}r^2\theta$$ to find the sector area. Substitute $r = 4$ and $\theta = \frac{3}{4}\pi$.
$\text{area of sector} = 6\pi$	
$\text{area of triangle} = \frac{1}{2}4^2 \sin\left(\frac{3\pi}{4}\right) = 4\sqrt{2}$	Use the formula for the area of a triangle $$A = \frac{1}{2}ab\sin C.$$ In the triangle AOB, a and b are both $4\ \mathrm{cm}$ and C is $\frac{3\pi}{4}$.
$\text{area of segment} = \text{area of sector} - \text{area of triangle}$ $= 6\pi - 4\sqrt{2}$	Now subtract the area of the triangle from the area of the sector.
$\text{area of segment} = 6\pi - 4\sqrt{2}\text{cm}^2$	Exact value.

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Activity

Sectors to cones

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Using a sheet of paper, can you make a cone with a base of radius 9 cm and a perpendicular height of 12 cm?

- Draw a net of the cone.
- What is the shape of the curved surface?
- Calculate the dimensions of the net.
- Derive a general formula for the surface area of a cone.

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Checklist

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What you should know

By the end of this subtopic you should be able to

- convert degrees to radians
- convert radians to degrees
- find the length of an arc using the formula $l = r\theta$
- find the area of a sector using the formula $A = \frac{1}{2}r^2\theta$.

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Investigation

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Archimedes used the method of exhaustion to approximate π using the ratio of the diameter of the perimeter of inscribed polygonsto the diameter of the circle. Later, other mathematicians used a similar method with areas. The method approximates the area of a shape by inscribing it inside a sequence of polygons whose area converges to the area of the containing shape.

In this investigation you will follow the method used by these mathematicians. You will calculate the areas of inscribed and circumscribed polygonsto find upper and lower bounds of the area of a circle of radius 6 cm.

1. Start by using regular pentagons. Use the applet below to find:

- the area of the inscribed regular pentagon
- the area of the circumscribed regular pentagon
- the average of the two areas.

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Interactive 1. Calculate the Areas of Inscribed and Circumscribed Polygons.

More information for interactive 1

This interactive tool allows users to explore how to approximate the area of a circle using inscribed (interior) and circumscribed (exterior) regular polygons. A circle is displayed with both types of polygons, and users can adjust the radius from 1 to 8 and the number of polygon sides from 3 to 100 using sliders provided in the top right corner. The tool calculates the area of the interior and exterior polygons, as well as the average of these two areas.

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This average serves as an estimate of the circle's actual area, calculated using the formula πr^2. As users increase the number of sides, they observe how the average area converges toward the true area of the circle.

For example, when the radius is 5 and the polygons have 3 sides (triangles), the interior polygon area is approximately 32.476, the exterior is about 129.9038, and their average is 81.1899. The actual area of the circle is approximately 78.54, showing how close the estimate is.

By increasing the number of sides, users can visually and numerically see the improvement in accuracy and determine how many sides are needed for the average to match the circle's area to three significant figures. This hands-on approach demonstrates the principles of approximation and the method of exhaustion, offering an intuitive introduction to the foundational ideas of calculus.

2. Now, using regular hexagons, find:

- the area of the inscribed regular hexagon
- the area of the circumscribed regular hexagon
- the average of the two areas.

3. Create a table like the one below in a spreadsheet. Complete extra rows in the table for polygons with more sides.

Number of sides	Area of inscribed polygon	Area of circumscribed polygon	Average of the areas of polygons (AP)	Error = AP — area of circle, using πr^2
5				
6				

How many sides do you need so that the average of the areas of the inscribed and circumscribed polygons equals the area of the circle to 3 significant figures?

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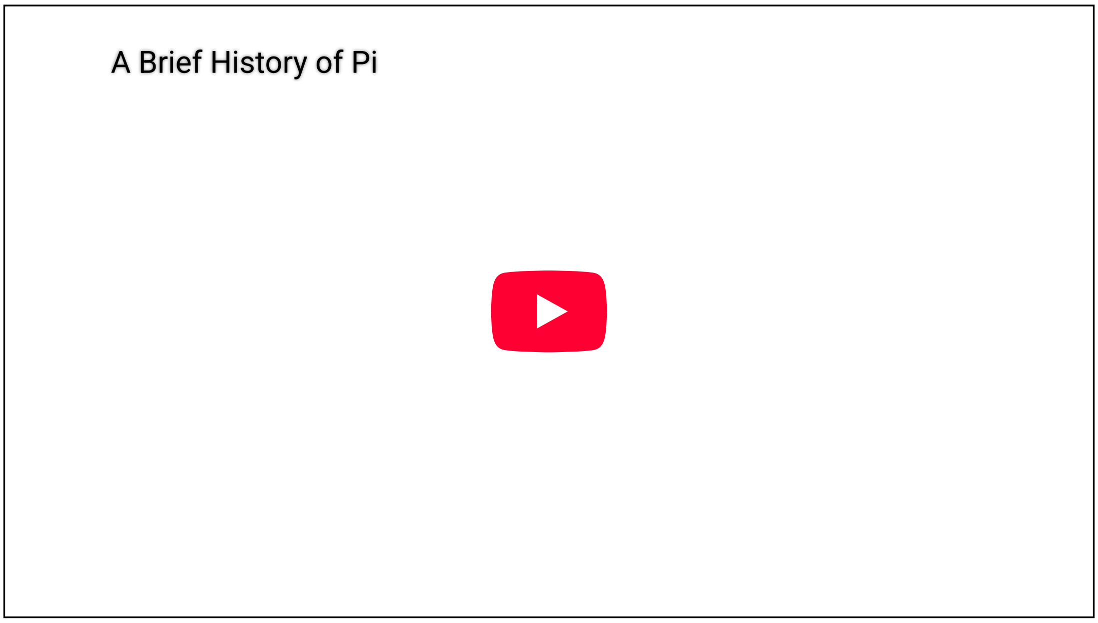
As this method is an approximation, why do you think the difference approaches zero?

In the video below, you will discover how ancient mathematicians used the method of exhaustion and limits to approximate π.



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