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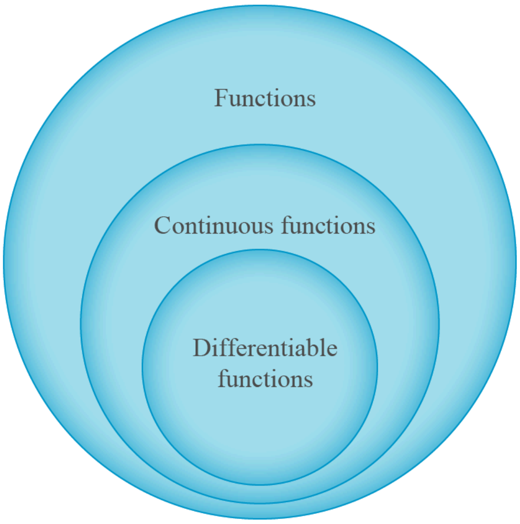
5. Calculus / 5.12 Limits and continuity

The big picture

In the earlier SL sections of calculus, you learned briefly about derivatives. At a very basic level, a derivative is the slope of the tangent at a given point. At the HL level, it is time to look more closely at the requirements for taking a derivative and how the derivative came to be understood.

As we move through this chapter, you will learn more formally how to classify functions based on continuity and differentiability. Both are required in order to find a derivative. The graph below is a handy way to visualise these classifications.



More information

The image is a Venn diagram consisting of three concentric circles. The largest outer circle is labeled "Functions." Inside this outer circle is a smaller circle labeled "Continuous functions," and within that is the smallest circle labeled "Differentiable functions." This illustrates the hierarchical relationship where differentiable functions are contained within continuous functions, which are in turn contained within the broader category of functions.

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Student
view

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Making connections

In subtopic 5.3 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25553/), you were introduced to the concept of a derivative as a function that depicts the instantaneous gradient of a function. In this section, you will look in depth at what a derivative really is and why you are able to compute it. You will also expand your understanding of higher derivatives, from the second derivative found in subtopic 5.7 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27788/) to higher-order derivatives.

Concept

Throughout this subtopic, think about how you the mathematics you are doing relates to relationships in the real world. You have already seen calculus applied to optimisation problems in subtopic 5.8 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27795/) and kinematic problems in subtopic 5.9 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27800/). Can you think of other areas where differentiation would be useful?

5. Calculus / 5.12 Limits and continuity

Limits and convergence

When investigating functions for the possible existence of horizontal asymptotes, you have already used statements such as ‘In the limit as x goes to infinity, the function $\frac{1}{x}$ goes to zero.’ Similarly, when studying infinite geometric series, you have said, ‘The infinite sum of $0.5 + 0.25 + 0.125 + \ldots$ converges to 1.’ These ideas of limits and convergence form the basis of a conceptual understanding of calculus and so you will touch briefly on them here.

Although you have met the notion of a limit already, you may not have seen the notation for it. Thus, the statement, ‘In the limit as x goes to infinity, the function $\frac{1}{x}$ goes zero’ can be written as

$$\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$$

The limit notation is not complicated. A few examples will highlight how it operates.

First, consider a limit where the measure, in this example n, goes to zero:

$$\lim_{n\to 0}\left(2n^2 - \frac{2}{3}n + 7\right) = \left(2(0)^2 - \frac{2}{3}(0) + 7\right) = 7$$

Student view

Note the following:

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- The limit $\lim\limits_{n \to 0}$ acts independently on all components that contain an n.
- If you think of this expression as a continuous function, i.e. $y = 2x^2 - \frac{2}{3}x + 7$, where n and x are variables, then this limit gives you the y-intercept.

This second example is a more interesting case that will enable you to learn a little more about the algebra involved with the limit notation.

Consider a rock dropped from rest falling freely to Earth. It will approximately follow the function $y = 5t^2$ metres in the first t seconds. Recall, from subtopic 2.1 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-24414/), that you can find the average velocity for the first 6 seconds using the gradient equation:

$$\frac{\Delta y}{\Delta t} = \frac{y_2 - y_1}{t_2 - t_1} = \frac{5(6)^2 - 5(0)^2}{6 - 0} = 30 \text{ ms}^{-1}$$

What if you want to know the instantaneous velocity at 6 seconds? Your equation would look something like:

$$\frac{\Delta y}{\Delta t} = \frac{y_2 - y_1}{t_2 - t_1} = \frac{5(6)^2 - 5(6)^2}{6 - 6} = \frac{0}{0}$$

So far, this is not something you have been allowed to do. You apply something called a difference quotient. The idea here is that you are going to look at the *average* gradient across a really small time span, and shrink that time span to 0. You can call the really small time span h. First, try this numerically:

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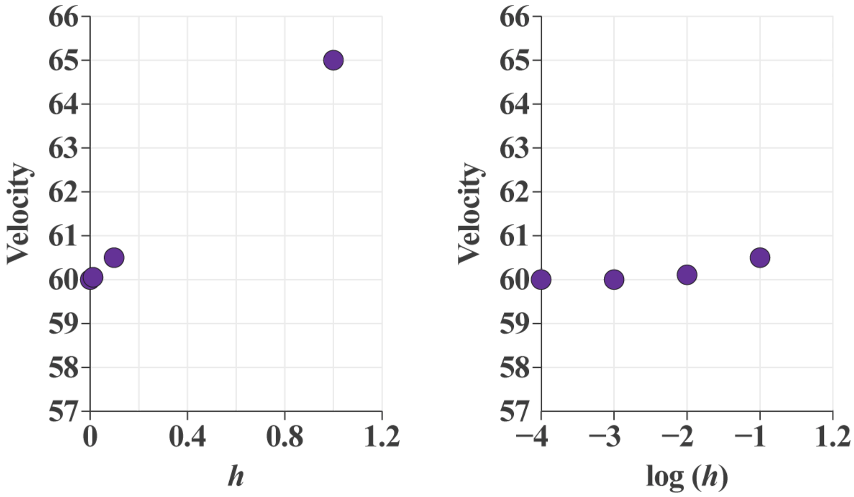
$h(\text{s})$	$t_1(\text{s})$	$t_2(\text{s})$	$y_1(\text{m})$	$y_2(\text{m})$	$v = \frac{\Delta y}{\Delta t}(\text{m s}^{-1})$
1	6	7	180	245	65
0.1	6	6.1	180	186.05	60.5
0.01	6	6.01	180	180.6005	60.05
0.001	6	6.001	180	180.060005	60.005
0.0001	6	6.0001	180	180.0060001	60.0005

To the observer, it looks like the velocity is fast approaching 60 m s^{-1} as h is approaching 0.

Student
view



More information

The image contains two graphs side by side.

1. The left graph plots "Velocity" on the Y-axis against "h" on the X-axis. The Y-axis ranges from 57 to 66, labelled at intervals (e.g., 57, 58, ... 66). The X-axis ranges from 0 to 1.2. There are three purple data points on the graph:
2. At h = 0, Velocity = 60
3. Near h = 0.2, Velocity = 60
4. At h = 1, Velocity = 65

5. The right graph plots "Velocity" on the Y-axis against "log(h)" on the X-axis. The Y-axis is the same as the left graph, ranging from 57 to 66. The X-axis ranges from -4 to 1.2, labelled at intervals (e.g., -4, -3, -2, -1). There are four purple data points on the graph:

6. At log(h) = -4, Velocity = 60
7. At log(h) = -3, Velocity = 60
8. At log(h) = -2, Velocity is still at 60
9. At log(h) = -1, Velocity stays at 60

In the graphs, the velocity appears to stabilize near 60 as h approaches 0.

[Generated by AI]

Now, to do this algebraically:

$$\frac{\Delta y}{\Delta t} = \frac{y_2 - y_1}{t_2 - t_1} = \frac{5(6+h)^2 - 5(6)^2}{(6+h) - 6} = \frac{180 + 60h + 5h^2 - 180}{h} = \frac{60h + 5h^2}{h} = 60 + 5h$$

So, now you have something you can work with. The average velocity from $t = 6$ to $t = 6 + h$ is $60 + 5h$. Therefore if you set $h = 0$, the instantaneous velocity at $t = 6$ is 60 m s^{-1}.

Student view

This confirms what you saw earlier.

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So, what exactly is a limit? The idea is that as you approach some value of x, whether $f(x)$ is defined at that point or not, you can get as close as you like to some limiting value.

The following definition is a little more formal.

✓ **Important**

Function $f(x)$ has a limit if it is defined in the near vicinity of $x = a$, and if, for any small number $\varepsilon > 0$, there is some $\delta > 0$ such that $|f(x) - L| < \varepsilon$ when $0 < |x - a| < \delta$, then $\lim_{x \to a} f(x) = L$.

Functions that have a limit are said to converge. Functions that do not have a limit are said to diverge.

Once you start working with limits, the following properties will be useful.

✓ **Important**

Properties of limits

- Sum rule: $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} (f(x)) + \lim_{x \to a} (g(x))$
- Difference rule: $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} (f(x)) - \lim_{x \to a} (g(x))$
- Product rule: $\lim_{x \to a} (f(x) \bullet g(x)) = \lim_{x \to a} (f(x)) \bullet \lim_{x \to a} (g(x))$
- Constant multiple rule: $\lim_{x \to a} (k \bullet g(x)) = k \bullet \lim_{x \to a} (g(x))$
- Quotient rule: $\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} (f(x))}{\lim_{x \to a} (g(x))}, \lim_{x \to a} (g(x)) \neq 0$

These properties are equally valid for $\lim_{x \to \pm\infty} (f(x))$.

In subtopic 2.8 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27718/), you learned that many rational functions have horizontal asymptotes which define their end behaviour. Using the definition and rules above, consider this example.

Example 1

★★☆

Evaluate $\lim_{x \to \infty} \left(\frac{3x - 4}{2x + 1} \right)$.

Student view

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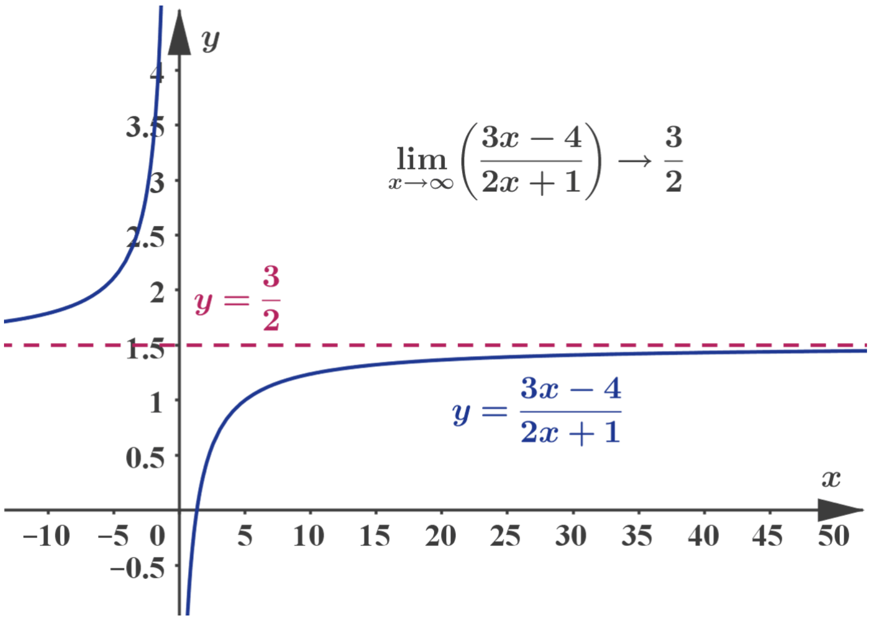
$$\lim_{x\to\infty}\left(\frac{3x-4}{2x+1}\right)=\lim_{x\to\infty}\left(\frac{x\left(3-\frac{4}{x}\right)}{x\left(2+\frac{1}{x}\right)}\right)=\lim_{x\to\infty}\left(\frac{3-\frac{4}{x}}{2+\frac{1}{x}}\right)=\frac{3-\lim_{x\to\infty}\left(\frac{4}{x}\right)}{2+\lim_{x\to\infty}\left(\frac{1}{x}\right)}=\frac{3-0}{2+0}=\frac{3}{2}$$

This limit behaviour is seen in the graph of the function $y=\frac{3x-4}{2x+1}$. A function converging on a limit as x approaches infinity corresponds to the horizontal asymptote of the graph of the function.



Since this function approaches a given value, it is said to converge.

Functions can also diverge. Consider this example.

Example 2

★★☆

Evaluate $\lim_{x\to\infty}\left(\frac{2x^2-5}{x}\right)$.

$$\lim_{x\to\infty}\left(\frac{2x^2-5}{x}\right)=\lim_{x\to\infty}\left(\frac{2x^2}{x}-\frac{5}{x}\right)=2\lim_{x\to\infty}(x)-\lim_{x\to\infty}\left(\frac{5}{x}\right)=2(\infty)-0=\infty$$

Student view

Since this function does not approach a given value, it is said to diverge.

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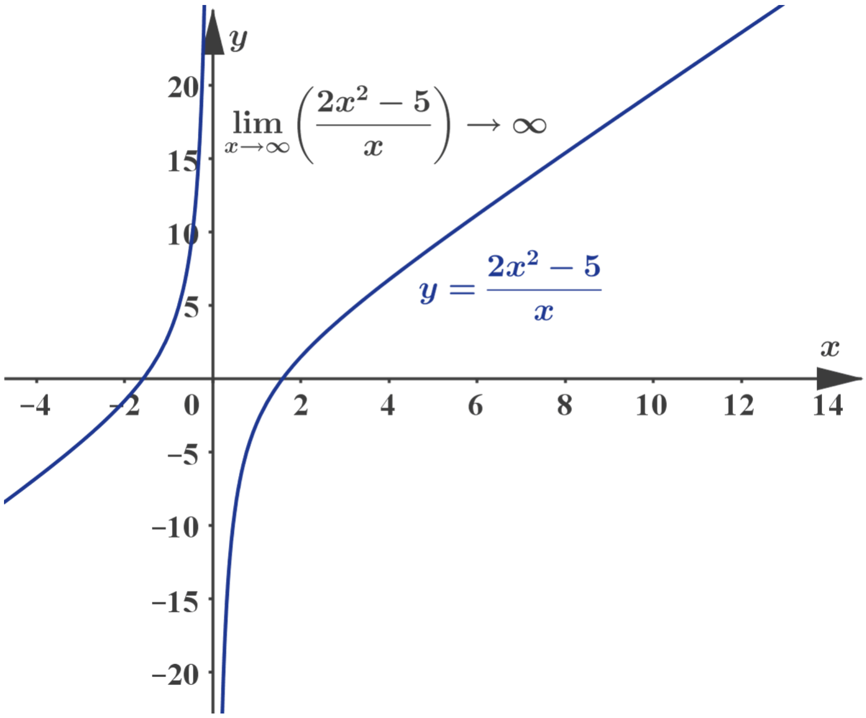


3 section questions

Question 1

Difficulty:

★★☆

Evaluate $\lim_{n\to 0}\left(\frac{3n^2+9n}{3n}\right)$

3

Accepted answers

3

Explanation

$$\lim_{n\to 0}\left(\frac{3n^2+9n}{3n}\right)=\lim_{n\to 0}\left(\frac{3n^2}{3n}+\frac{9n}{3n}\right)=\lim_{n\to 0}(n)+\lim_{n\to 0}(3)=0+3=3$$

Question 2

Difficulty:

★★☆

Student
view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Evaluate $\lim_{n\to 0}\left(\frac{(2+n)^2-4}{n}\right)$

4

Accepted answers

4

Explanation

$$\lim_{n\to 0}\left(\frac{(2+n)^2-4}{n}\right)=\lim_{n\to 0}\left(\frac{(4+4n+n^2)-4}{n}\right)=\lim_{n\to 0}\left(\frac{4n+n^2}{n}\right)$$

$$=\lim_{n\to 0}(4+n)=\lim_{n\to 0}4+\lim_{n\to 0}n=4+0=4$$

Question 3

Difficulty:

★★★

Evaluate $\lim_{x\to\infty}\left(\frac{5x^2+2x-3}{2x^2+8}\right)$

Give your answer as a decimal.

2.5

Accepted answers

2.5, 2,5, 2.50, 2,50

Explanation

$$\lim_{x\to\infty}\left(\frac{5x^2+2x-3}{2x^2+8}\right)=\lim_{x\to\infty}\left(\frac{x^2\left(5+\frac{2}{x}-\frac{3}{x^2}\right)}{x^2\left(2+\frac{8}{x^2}\right)}\right)=\lim_{x\to\infty}\left(\frac{5+\frac{2}{x}-\frac{3}{x^2}}{2+\frac{8}{x^2}}\right)$$

$$=\frac{\lim_{x\to\infty}\left(5+\frac{2}{x}-\frac{3}{x^2}\right)}{\lim_{x\to\infty}\left(2+\frac{8}{x^2}\right)}=\frac{\lim_{x\to\infty}(5)+\lim_{x\to\infty}\left(\frac{2}{x}\right)-\lim_{x\to\infty}\left(\frac{3}{x}\right)}{\lim_{x\to\infty}(2)+\lim_{x\to\infty}\left(\frac{8}{x}\right)}=\frac{5+0-0}{2+0}=\frac{5}{2}$$

5. Calculus / 5.12 Limits and continuity

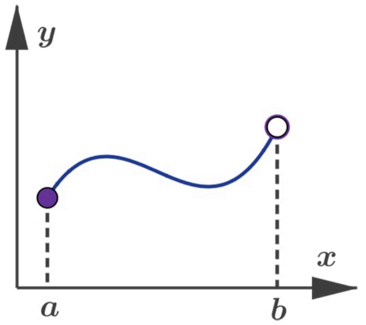
Continuity

What does it mean for a function, or even part of a function, to be a continuous function?

One of the most intuitive ways to find out whether a function f defined on an interval I is continuous is to see if you can draw its graph without taking your pencil off the paper.

Student view

For example, consider the two graphs shown below. The first is a function defined on the interval $[a, b)$ and is continuous. The second is a function defined on the interval $[a, b]$ and is also continuous.



More information

The image shows two graphs depicting mathematical functions on the x-y plane. \n\n**First Graph:**\n- X-axis represents values from 'a' to 'b'.\n- Y-axis is not labeled with specific units or scale but represents output values.\n- The curve starts at point 'a' with a solid dot, indicating the function includes the starting point at 'a'.\n- The curve is continuous and ends just before 'b', indicated by an open circle, showing the function does not include 'b'.\n\n**Second Graph:**\n- X-axis similarly spans from 'a' to 'b'.\n- Y-axis remains without specific units.\n- The graph begins at 'a' with a solid point and directly lines downwards and then upwards, forming a 'V' shape.\n- The function remains continuous throughout the interval and ends at 'b' with a solid dot, indicating inclusion of 'b'.

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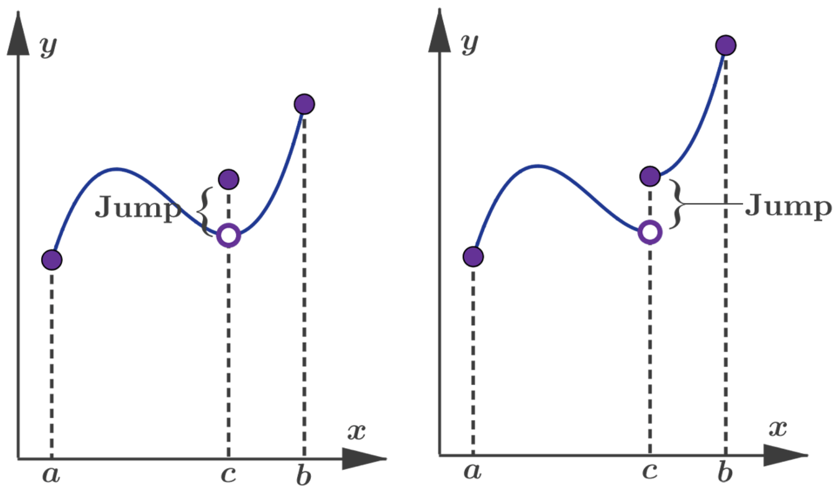
It is easy to identify that the graphs above have no gap over the interval, so the functions are continuous.

Now, look at the these next two graphs. The first is a function defined on the interval $[a, b]$ and is discontinuous at c. The second is a function defined on the interval $[a, b]$ and is also discontinuous at c.



More information

The image displays two separate graphs side by side. Each graph illustrates a function defined on an interval ([a,b]) and shows a discontinuity at the point (c).\n\nOn the left graph:\n- The X-axis represents the interval with labels from (a) to (b).\n- The Y-axis shows the range of the function.\n- The curve of the function starts at a low point near (a), rises sharply to a peak, and then descends gradually.\n- At point (c), there's a visible gap, indicating a discontinuity.

On the right graph:\n- Similarly, the X-axis is labeled from (a) to (b), and the Y-axis displays the function's range.\n- This function starts at a higher point closer to the Y-axis, with a steeper ascent, a peak, and then a sharp drop.\n- Like the left graph, there's a discontinuity at point (c), marked by a gap.

Both graphs visually demonstrate the concept of a function being discontinuous at a specific point within the interval ([a,b]).

[Generated by AI]

These graphs have gaps above point c . So these functions are discontinuous at c .

So, a basic understanding of continuity is that you can draw the graph of a continuous function without lifting your pencil off the paper, but if you need to lift your pencil to jump over some gap, the graph has a discontinuity.

However, you need to add more mathematical rigour to that definition and consider different types of discontinuity.

In the previous section (/study/app/math-aa-hl/sid-134-cid-761926/book/limits-and-convergence-id-26490/), you learned about limits. For a function to be continuous at a point, the point and limit both must exist, and they must be equal.

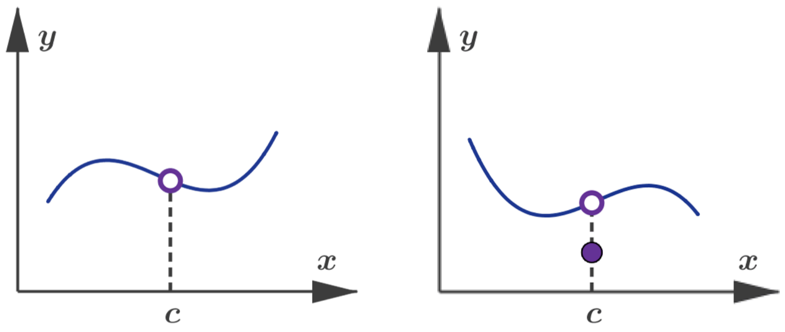
There are several basic types of discontinuity.

In some cases, a function looks very smooth, apart from at one point along the curve where it is not defined. When you were learning about rational functions, you called these holes. They are also called gap discontinuities as the function has an infinitely small gap in the curve. Other authors call these removable discontinuities as you could remove the discontinuity by simply plugging the hole.

Regardless of the words you use, the key point is that the limit exists at this point. The point of discontinuity can be either undefined or defined with a value not equal to the limit at that x-value.

The graphs below are examples of removable discontinuities. In both cases, the limit of $f(x)$ as $x \to c$ exists. In the first case, $f(c)$ does not exist. This often occurs with rational functions where a factor in the denominator cancels completely with a factor in the numerator. In the second case, $f(c) \neq \lim_{x \to c} f(x)$.

Student view



More information

The image features two graphs, each depicting a removable discontinuity in a function.

The first graph, on the left, shows a curve with an empty circle on the curve at a certain x-value, indicating a point where the function is not defined yet the limit exists. The y-axis and x-axis are not labeled, but the graph demonstrates how the curve approaches but does not reach the point.

The second graph, on the right, is more detailed and explicitly labeled with axes labeled 'x' and 'y'. The curve again approaches a point where an empty circle is placed, signifying the discontinuity. Below this point, along the vertical dashed line indicating the x-coordinate 'c' where the discontinuity occurs, there is a filled circle suggesting a defined value for the function at this x-value which is different from the limit the curve approaches.

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Another type of discontinuity is an asymptotic discontinuity, sometimes referred to as an infinite discontinuity. These were also covered in the subsections on rational functions. In this case, the limits from each side approach infinity and/or negative infinity asymptotically.

The graphs below are examples of asymptotic discontinuities. These often occur with rational functions where a factor in the denominator cannot be cancelled.

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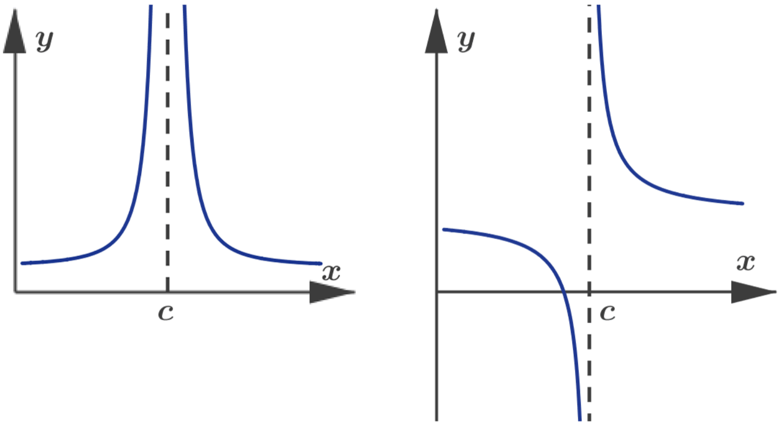

More information

The image contains two graphs demonstrating examples of asymptotic discontinuities. Both graphs deal with rational functions where a factor in the denominator cannot be cancelled.

1. The first graph on the left shows a typical discontinuity often associated with vertical asymptotes. It consists of an upward curve that approaches a vertical line labeled 'c' on the x-axis but never touches it. The x-axis is labeled as 'x' and the y-axis as 'y'. The curve rises sharply and becomes infinitely close to the vertical dashed line as it approaches 'c'.

2. The second graph on the right illustrates another scenario of an asymptotic discontinuity. This graph features a more complex shape, possibly involving more curves that approach vertical lines on both the positive and negative sides. The graph indicates that the values of the function increase or decrease without bound as they approach the line 'c'.

Both images demonstrate the key concept of an asymptote, where the graph heads towards a line but never actually intersects it, visually demonstrating the behavior of certain rational functions as they approach the excluded values.

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Making connections

Holes and asymptotes were introduced in subtopic 2.8 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27718/) and subtopic 2.13 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26748/) where you looked at graphs and the behaviour of rational functions.

As you learned earlier, the difference between holes and asymptotic discontinuities can be small. The applet below allows you to change one value in the function $f(x) = \frac{(x-2)(x-3)^2}{x-a}$.

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/limits-and-convergence-id-26490/print/) Assign

What happens as you change the value of a?

Student view

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Interactive 1. Exploring Asymptotic Discontinuities.

More information for interactive 1

This interactive program helps users understand the **continuity and vertical asymptotes** of a rational function. The graph displays a blue curve representing the function:

$$f(x) = \frac{\{(x-2)(x-3)^2\}}{x-a}$$

on a coordinate plane where the x-axis ranges from 0 to 5 and the y-axis from -12 to 16. A horizontal slider at the bottom allows users to adjust the value of the constant a from 0 to 5.

As users move the slider, the denominator of the function changes, which may introduce or **remove vertical asymptotes** depending on the value of a . A checkbox labeled "*Vertical asymptote*" can be toggled to display a dashed pink line at $x = a$ when the function becomes undefined there.

For $a = 1$: A vertical asymptote appears at $x = 1$, clearly visible on the graph. The function becomes undefined at this point due to division by zero. The graph splits into two branches on either side of the asymptote.

For $a = 2$: The factor $(x - 2)$ in the numerator and denominator cancels out, so the discontinuity at $x = 2$ becomes **removable**, and there is **no vertical asymptote.** The graph appears smooth and continuous, touching the x-axis at $x = 2$

For $a = 3$: Similarly, the factor $(x - 3)$ cancels out from both numerator and denominator (since it's squared in the numerator), resulting in **no vertical asymptote** at $x = 3$. The function is again continuous, touching the x-axis at both $x = 2$ and $x = 3$.

For $a = 4$ **and** $a = 5$: A vertical asymptote appears at $x = 4$ and $a = 5$ respectively, where the function is undefined. The graph splits into two separate curves, illustrating a discontinuity.

On either side of a vertical asymptote, the function shoots up to $+\infty$ or down to $-\infty$, creating a sharp break in the curve. This may make it appear as though there are two separate graphs, but it is actually **one function**, interrupted only where the function becomes undefined due to division by zero.

This applet provides a hands-on way to explore how **vertical asymptotes** and **removable discontinuities** arise in rational functions, helping users gain intuitive understanding of function continuity and asymptotic behavior based on algebraic structure.

Student view

A third type of discontinuity is a jump discontinuity, sometimes referred to as a finite discontinuity. With this type of discontinuity, there is a vertical gap in the function. Although the left-hand and right-hand limits must be defined and be finite, they are not equal, so the limit does not exist. These are common among piecewise

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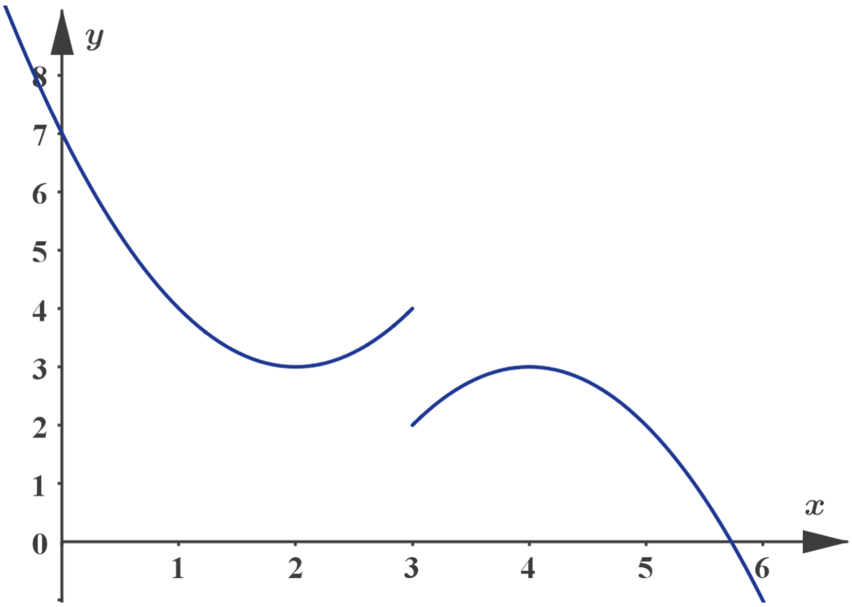
functions.

The graph below is an example of a jump discontinuity.

There are one-sided limits, $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$, but $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3c^+} f(x)$.

This particular function is defined by

$$f(x) = \{ \begin{matrix} x^2 - 4x + 7 & x < 3 \\ -x^2 + 8x - 13 & x \geqslant 3 \end{matrix}$$



More information

The image is a graph depicting a piecewise function. The graph consists of two distinct curves:

1. **First curve (for x < 3):** This segment shows a downward opening parabola that appears to follow the equation (y = x^2 - 4x + 7) as described in the text before the image. It starts above the x-axis, dipping downwards, and ends just before x = 3.
2. **Second curve (for x ≥ 3):** This segment shows an upward opening parabola that appears to follow the equation (y = -x^2 + 8x - 13). It starts immediately after x = 3, peaks, and gradually declines as x increases.

Axes Details: - The x-axis represents the independent variable (x), labeled from 0 to 6. - The y-axis represents the dependent variable (y), labeled from 0 to 8.

The curves illustrate transitions at x = 3, with the left curve ending abruptly and the right curve beginning from that point. This piecewise nature illustrates the concept discussed in the accompanying text, demonstrating different equations defined over separate domains for x.

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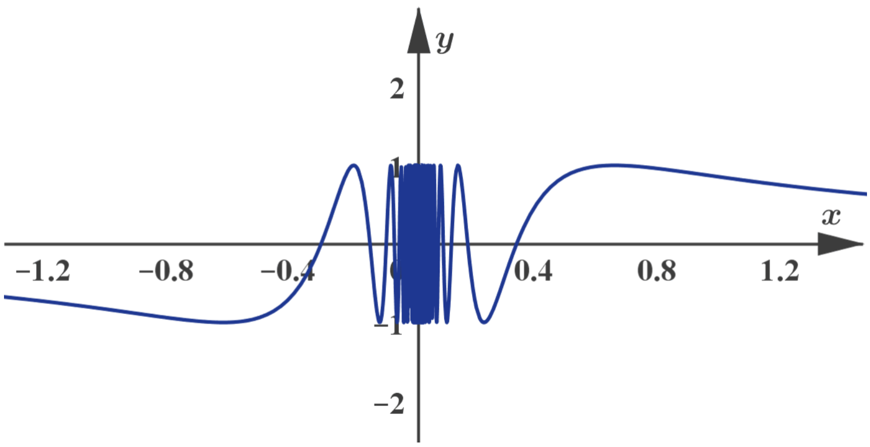
Note the use of the piecewise function above. This type of function is defined over specific restricted domains with different equations. For it to be a function, recall the restriction that no x-value (independent variable) can be mapped to more than one y-value (dependent variable).

Finally, an oscillating discontinuity occurs when a function appears to approach multiple values simultaneously.

The graph below is an example of an oscillating discontinuity. This particular function is $f(x) = \sin\left(\frac{1}{x}\right)$. As x approaches 0 from either side, the function continues to oscillate faster across the range $[-1, 1]$.



More information

The image is a graph demonstrating the function (f(x) = \sin\left(\frac{1}{x}\right)). The X-axis ranges approximately from -1.5 to 1.5, labeled at intervals of 0.4, while the Y-axis ranges from -2 to 2. The graph shows an oscillation that increases in frequency as it approaches (x = 0), thus illustrating an oscillating discontinuity. The oscillations occur within the range ([-1, 1]), becoming denser as (x) nears zero. Beyond this central oscillation, the graph stabilizes into smooth, less frequent waves.

[Generated by AI]

More complex functions can have multiple discontinuities of various types.

Exam tip

In examinations, you will not be asked to test for continuity.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Example 1

★☆☆

Determine whether the following function is continuous or discontinuous. If discontinuous, classify the type of discontinuity.

$$f(x) = \frac{(x-4)^2(x-2)}{(x-1)}$$

This function is not continuous. It has an asymptote at $x = 1$.

Important

- Factors in the denominator that do not completely cancel result in non-removable discontinuities, or vertical asymptotes.
- Factors in the denominator that completely cancel out result in removable discontinuities, or holes.

Example 2

★☆☆

Determine whether the following function is continuous or discontinuous. If discontinuous, classify the type of discontinuity.

$$f(x) = \frac{(x-4)^2(x-2)}{(x-2)}$$

This function is not continuous. It has hole at point $(2, 4)$.

Example 3

★★☆

Determine whether the following function is continuous or discontinuous. If discontinuous, classify the type of discontinuity.

$$f(x) = \frac{(x-4)^2(x-2)}{x^2+1}$$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

This function is continuous. The denominator has a range of $[1, \infty)$.

Example 4

★★☆

Determine whether the following function is continuous or discontinuous. If discontinuous, classify the type of discontinuity.

$$f(x) = \cos\left(\frac{5}{x}\right)$$

This function is not continuous. It has an oscillating discontinuity at $x = 0$. This can be confirmed by graphing.

Example 5

★★☆

Determine whether the following function is continuous or discontinuous. If discontinuous, classify the type of discontinuity.

$$f(x) = \begin{cases} x^2 - 2x + 3, & x \leqslant 1 \\ e^{x-1} + 1, & x > 1 \end{cases}$$

This function is continuous. Even though it is a piecewise function, the ends match up as $\lim_{x \to 1} f(x) = 2$. This can be confirmed by graphing.

Example 6

★★☆

Determine whether the following function is continuous or discontinuous. If discontinuous, classify the type of discontinuity.

$$f(x) = \begin{cases} \cos\left(\frac{\pi x}{2}\right), & x \leqslant 1 \\ x + 1, & x > 1 \end{cases}$$

Student view

This function is not continuous. $\lim_{x \to 1^-} f(x) = 0$ and $\lim_{x \to 1^+} f(x) = 2$. Since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$,

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

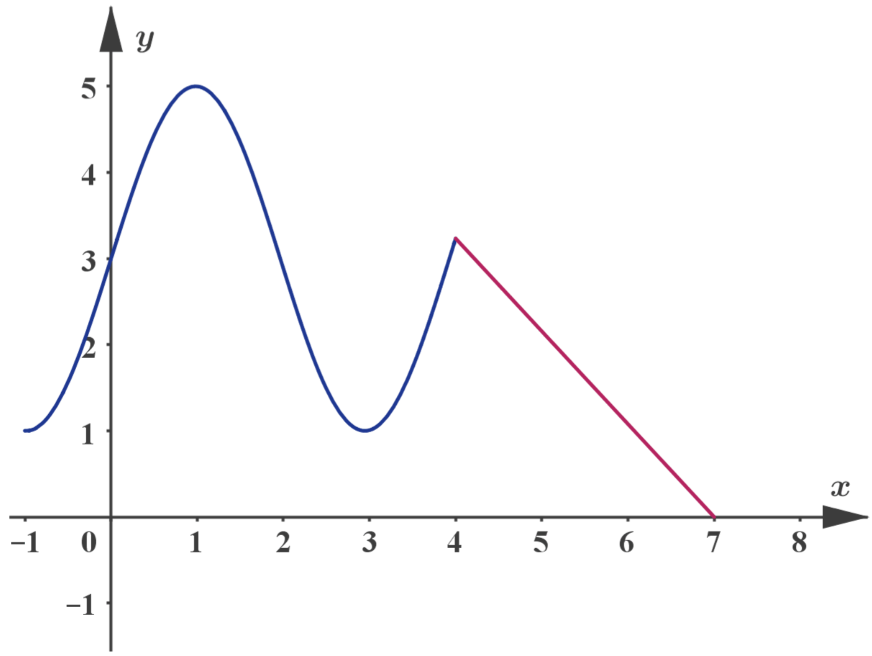
the limit does not exist. This is a jump discontinuity. This can be confirmed by graphing.

6 section questions

Question 1

Difficulty: ★☆☆

Determine whether the function graphed below is continuous or discontinuous.



More information

If discontinuous, classify the type of discontinuity.

Choose the correct answer from the given options.

1 Continuous. ✓

2 Discontinuous. Removable discontinuity (hole).

3 Discontinuous. Asymptotic discontinuity.

4 Discontinuous. Jump discontinuity.

Explanation

Although there is a sharp corner at $x = 4$, $\lim\limits_{x \to 4} f(x) = 3$.

Question 2

Difficulty: ★☆☆

Student view

Overview
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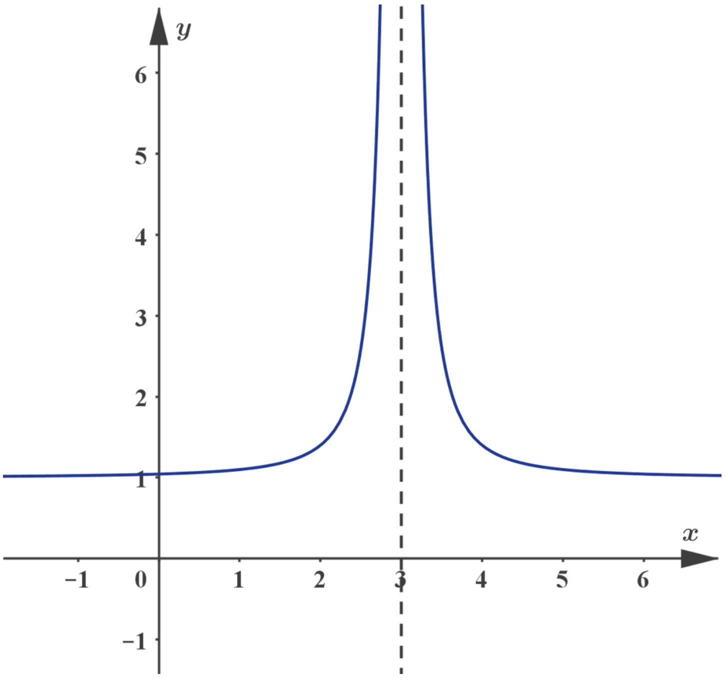
Determine whether the function graphed below is continuous or discontinuous.



More information

If discontinuous, classify the type of discontinuity.

Choose the correct answer from the given options.

1 Discontinuous. Asymptotic discontinuity. ✓

2 Discontinuous. Removable discontinuity (hole).

3 Continuous.

4 Discontinuous. Jump discontinuity.

Explanation

At $x = 3$, the limit approaches positive infinity.

Question 3

Difficulty:

★★☆

Determine whether this function is continuous or discontinuous.

$$f(x) = \begin{cases} (x-1)^3, & x \leqslant 2 \\ -(x-1)^2 + 2, & x > 2 \end{cases}$$

If discontinuous, classify the type of discontinuity.

Choose the correct answer from the given options.

Student
view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

1 Continuous. ✓

2 Discontinuous. Removable discontinuity (hole).

3 Discontinuous. Asymptotic discontinuity.

4 Discontinuous. Jump discontinuity.

Explanation

This function is continuous.

Even though it is a piecewise function, the ends match up as $\lim_{x \to 2} f(x) = 1$.

This can be confirmed by graphing.

Question 4

Difficulty: ★★☆

Determine whether this function is continuous or discontinuous.

$$f(x) = \begin{cases} \log x, & x \leqslant 10 \\ \dfrac{(x-1)^2}{40}, & x > 10 \end{cases}$$

If discontinuous, classify the type of discontinuity.

Choose the correct answer from the given options.

1 Discontinuous. Jump discontinuity ✓

2 Discontinuous. Removable discontinuity (hole)

3 Discontinuous. Asymptotic discontinuity

4 Discontinuous. Oscillating discontinuity

Explanation

$\lim_{x \to 10^-} f(x) = 1$ and $\lim_{x \to 10^+} f(x) = 2.025$.

Since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, the limit does not exist.

This is a jump discontinuity.

This can be confirmed by graphing.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Question 5

Difficulty:

★★☆

Determine whether this function is continuous or discontinuous.

$$f(x) = \frac{x^2 + 3x + 2}{x + 2}$$

If discontinuous, classify the type of discontinuity.

Choose the correct answer from the given options.

1. Discontinuous. Removable discontinuity (hole). ✓
2. Continuous.
3. Discontinuous. Asymptotic discontinuity.
4. Discontinuous. Jump discontinuity.

Explanation

Factorising:

$$\frac{x^2 + 3x + 2}{x + 2} = \frac{(x + 2)(x + 1)}{x + 2} = x + 1$$

There is a hole at point $(-2, -1)$.

Question 6

Difficulty:

★★☆

Determine whether this function is continuous or discontinuous.

$$f(x) = \frac{x^2 - 7x + 10}{x + 2}$$

If discontinuous, classify the type of discontinuity.

Choose the correct answer from the given options.

1. Discontinuous. Asymptotic discontinuity. ✓
2. Continuous.
3. Discontinuous. Jump discontinuity.
4. Discontinuous. Oscillating discontinuity.

Student view

Overview
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761926/o

Explanation

Factorising:

$$\frac{x^2 - 7x + 10}{x + 2} = \frac{(x-5)(x-2)}{x+2}$$

The denominator does not cancel, so there is an asymptote at $x = -2$.

Differentiability

Now that you have an understanding of continuity, it is time to look more formally at differentiability.

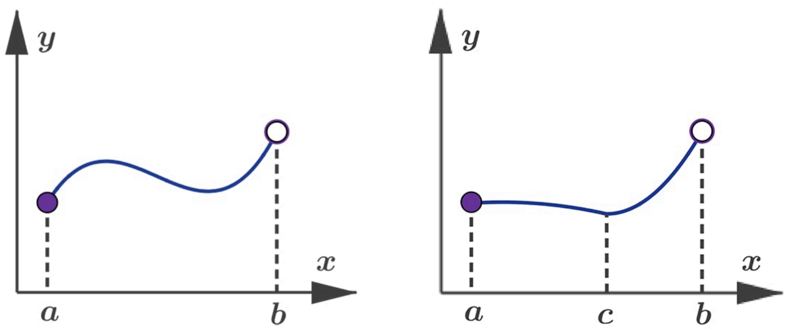
Making connections

You were introduced to an informal definition of a derivative as an instantaneous gradient in subtopic 5.1 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25542/).

Be aware

At this level of mathematics, the terms gradient and slope are often used interchangeably.

Consider the two functions graphed below.



More information

Student
view

The image contains two graphs comparing differentiable and non-differentiable functions, each plotted against x and y axes.\n\nOn the left, the graph shows a curve that begins at point 'a' with a filled circle, indicating the start of the function. It smoothly curves upwards and ends at point 'b' with an open circle, suggesting continuity but not differentiability at that point. This graph represents a differentiable function with smooth transitions between points.\n\nOn the right, the graph similarly starts at point 'a' with a filled circle, proceeds straight to point 'c', and then sharply moves upwards to end at point 'b' also with an open circle. The sharp transition at 'c' indicates a non-differentiable function due to the abrupt change in direction.

[Generated by AI]

You can immediately see that both functions are continuous on the intervals on which they are defined. However, looking closely, you may be able to recognise a difference: the graph of the function on the left seems to be smooth at each point, while the graph of the function on the right seems to be sharp just above c. This separates differentiable from non-differentiable functions.

A function is differentiable at a point c of its domain if, upon zooming in, its graph is sufficiently smooth at that point of the graph to approach linearity.

Finally, for a function to be differentiable, it must be continuous. Equivalently, every differentiable function is also continuous, although the opposite is not necessarily true, as shown in the second graph above.

Earlier in this subtopic, the derivative was introduced by studying secants of a curve. As the end points approach each other, the geometrical limit of the secants formed is the tangent. The gradient of the tangent at a point is the derivative.

The graphs below show the limit of the secants as point B approaches point A from the left and right. The black lines represent the final position of the secant and thus the tangent of f at A.

Interactive 1. The Limit of the Secants as Point B Approaches Point A from the Left.

More information for interactive 1

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

This interactive allows the user to understand the concept of the limit of the secant line approaching a tangent line for function f.

A graph is displayed with the XY-axis with a function $f(x)$ represented in blue. A pink line passes through the function with two points $B(b, f(b))$ and $A(a, f(a))$, representing the secant line. Here, '*a*' and '*b*' represent the x-coordinates of points *A* and *B* respectively, while the y-coordinate is the value of the function *f* at $x = a$ and $x = b$ respectively. Users can slide the value of point B from -0.4 to 1.2 using the horizontal slider located at the top middle of the graph.

As point *B* moves closer to point *A* along the function, *b* approaches *a*, and the secant line approaches the tangent line at point *A*. The graph helps the users to get a better view of how the secant line represented by the pink line pivots around point *A* as point *B* gets closer, eventually becoming the tangent line represented by a black line. This black line represents the final position of the secant and thus the tangent of function *f* at point *A*.

Interactive 2. The Limit of the Secants as Point B Approaches Point a from the Right.

More information for interactive 2

This interactive allows the user to understand the concept of the limit of secant and gradient of the tangent at the point of derivative. It explains how the secant line gets closer to the tangent line as the interval between point A and B reduces.
A graph is displayed with the XY-axis with a function *f* represented in blue. The function plots an upward parabola intersecting the y-axis and above the x-axis. An orange line intersects the parabola with two purple points, $A(a, f(a))$ and $B(b, f(b))$,, representing the secant line. Here, '*a*' and '*b*' represent the x-coordinates of points A and B, respectively, while the y-coordinate is the value of the function *f* at x=a and x=b, respectively. Users can slide the value of point *B* from 1.2 to 2.4 using the horizontal slider located at the top middle of the graph.
The users will notice that as point B moves closer to point A along the curve, the orange secant line approaches the tangent line and changes to black at point A. This interactive helps the user in understanding the fundamental idea behind limits and derivatives in calculus.

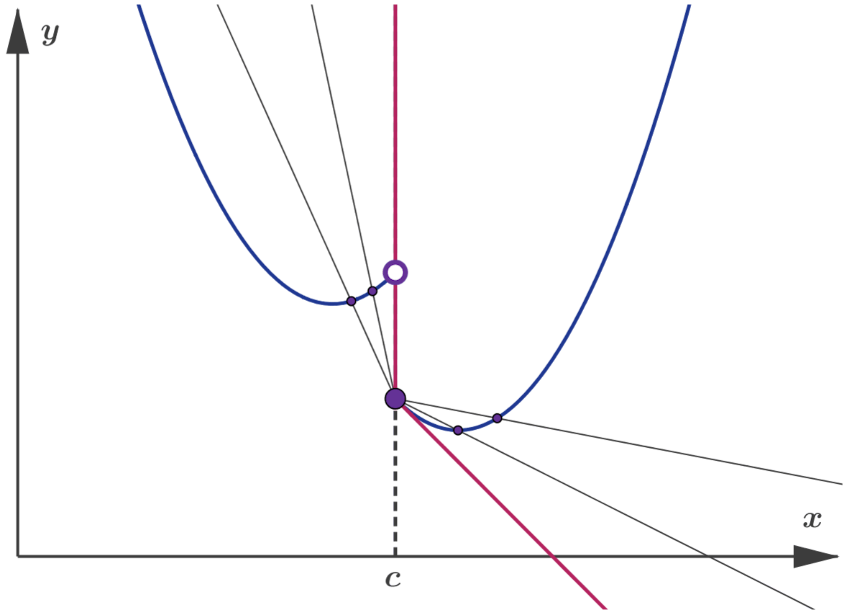
You are now going to look at differentiability.

Student view

First, consider the question: 'When can you find a derivative?'

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

There are two requirements for a function to be differentiable at a given point. First, the function must be continuous. If a function is not continuous at a given point, then you are not able to find the derivative. Although this argument will work for all discontinuities, the jump discontinuity is the easiest to see. Think about how the limit of the gradient of the secant changes depending on the direction from which you approach. There can be no definitive gradient and therefore no defined derivative.



More information

This image is a graph that illustrates a function represented by a blue curve. The x-axis is labeled 'x' and the y-axis is labeled 'y'. The point 'c' on the x-axis is marked, and a vertical dashed line extends through this point. There are several secant lines shown, intersecting the curve at multiple points. One tangent line, in purple, is shown as well, touching the curve precisely at point 'c'. The function appears to be continuous at 'c'. The shape of the curve indicates a downward-facing parabola, suggesting a point of inflection near 'c'.

[Generated by AI]

Example 1

★★☆

Show that the function is continuous but not differentiable at $x = 0$.

$$f(x) = \begin{cases} x^3, & x < 0 \\ 4x, & x \geqslant 0 \end{cases}$$

First, you can check for continuity.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

From the left side, $\lim_{x \to 0} x^3 = 0$.

From the right side, $\lim_{x \to 0} 4x = 0$. Since $\lim_{x \to 0} x^3 = \lim_{x \to 0} 4x$, the function is continuous.

Next, evaluate the derivatives (remember the power rule from subtopic 5.3). From the left side, $f'(x) = 3x^2$ and $f'(0) = 0$.

From the right side, $f'(x) = 4$ and $f'(0) = 4$.

Since the gradient coming from the left and the right are not equal, the function is not differentiable at $x = 0$.

The second requirement is that the function has local linearity. When you zoom in to look at a differentiable point of a function very closely, you will find that the function 'flattens out' and starts to look a lot like the tangent. This can be demonstrated on a graphing utility, such as a graphic display calculator or with an app such as Geogebra.

Important

For a function to be differentiable at a point, it must

- be continuous at the point
- have local linearity at the point.

For a function to be differentiable across a domain, it must be differentiable at every point within that domain.

Activity

1. Graph the functions $f(x) = |x|$ and $g(x) = \sqrt{x^2 + 0.0001} - 0.01$. What do these graphs look like?
2. Using the zoom function, zoom in at the point $(0, 0)$. What do you see?

Eventually, you should see $g(x)$ straightens out while $f(x)$ remains unchanged.

You have already seen some examples of functions that are not continuous at a specific point. In every one of these cases, this also means that they are not differentiable at that point. For continuous functions, there are several examples of non-differentiable points.

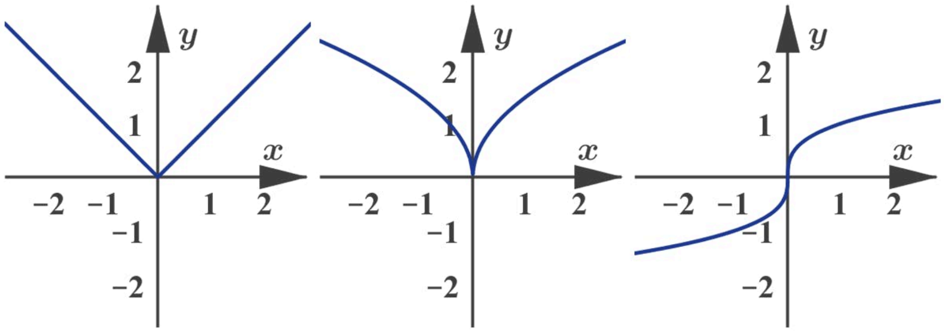
A corner is a point on a continuous function where the gradients coming from the left and the right are not equal. The simplest example is the absolute value function.

Student view

A cusp is a point on a continuous function where the gradients coming from the left and the right approach ∞ and $-\infty$.

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

A vertical tangent is a point on a continuous function where the gradient approaches ∞.



More information

The image features three distinct graphs side by side. Each graph consists of a set of axes: the x-axis (horizontal) and the y-axis (vertical), both labeled with values ranging from -2 to 2.

1. **First Graph (Left):**
2. The x-axis is labeled with values from -2 to 2, and the y-axis is similarly labeled.
3. A V-shaped curve passes through the origin (0,0) and extends towards (-2, 2) and (2, 2), representing the function (f(x) = |x|).
4. There is a visible corner at (x = 0).

5. **Second Graph (Middle):**

6. The same labeled axes as the first graph.
7. The graph shows a continuous curve starting from (-2, infinity) on the left, passing through an undefined point at the origin, and moving towards (-2, -infinity) at (x = 2).
8. This depicts the function with a vertical tangent at (x = 0).

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/continuity-id-26491/print/) Assign

9. **Third Graph (Right):**

10. Identical labeled axes as the previous graphs.
11. A smooth curve starts low on the left, crosses the origin, and ascends towards the top as (x) increases. It resembles the graph of a cubic function.

These graphs visually illustrate different behaviors of functions, such as corners and vertical tangents.

[Generated by AI]

In the first graph: A corner exists at $x = 0$ in the function $f(x) = |x|$.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

In the middle graph: A cusp exists at $x = 0$ in the function $f(x) = x^{\frac{2}{3}}$.

In the final graph: A vertical tangent exists at $x = 0$ in the function $f(x) = \sqrt[3]{x}$.

Exam tip

In the examination, you will not be asked to test for differentiability.

3 section questions

Question 1

Difficulty: ★☆☆

Which of the following is true about this function?

$$f(x) = \begin{cases} x^2, & x \leqslant 0 \\ 4x, & x > 0 \end{cases}$$

1 f is continuous but not differentiable at $x = 0$. ✓

2 f is both continuous and differentiable at $x = 0$.

3 f is neither continuous nor differentiable at $x = 0$.

4 f is differentiable but not continuous at $x = 0$.

Explanation

Continuity:

$$\lim_{x \to 0^-} f(0) = 0^2 = 0$$

$$\lim_{x \to 0^+} f(0) = 4(0) = 0$$

$$\lim_{x \to 0^-} f(0) = \lim_{x \to 0^+} f(0)$$

Therefore it is continuous.

Differentiability:

Left-hand limit

$$f'(x) = 2x$$

$$f'(0) = 2(0) = 0$$

Right-hand limit

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

$$f'(x) = 4$$

$$f'(0) = 4$$

$$0 \neq 4$$

Therefore it is not differentiable.

Question 2

Difficulty:

★★☆

Which of the following is true about this function?

$$f(x) = \begin{cases} -\dfrac{x^2 - 2x}{2}, & x \leqslant 0 \\ \sin x, & x > 0 \end{cases}$$

1 f is both continuous and differentiable at $x = 0$. ✓

2 f is continuous but not differentiable at $x = 0$.

3 f is neither continuous nor differentiable at $x = 0$.

4 f is differentiable but not continuous at $x = 0$.

Explanation

Continuity:

$$\lim_{x \to 0^-} f(x) = -\frac{(0)^2 - 2(0)}{2} = 0$$

$$\lim_{x \to 0^+} f(x) = \sin 0 = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

Therefore it is continuous.

Differentiability:

Left-hand limit

$$f'(x) = -x + 1$$

$$f'(0) = -(0) + 1 = 1$$

Right-hand limit

$$f'(x) = \cos x$$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$$f'(0) = \cos(0) = 1$$

$$1 = 1$$

Therefore it is differentiable.

Question 3

Difficulty:

★★☆

Which of the following is true about this function?

$$f(x) = \begin{cases} x^2 + 2x + 1, & x \leqslant 0 \\ 2x, & x > 0 \end{cases}$$

1 f is neither continuous nor differentiable at $x = 0$. ✓

2 f is both continuous and differentiable at $x = 0$.

3 f is continuous but not differentiable at $x = 0$.

4 f is differentiable but not continuous at $x = 0$.

Explanation

Continuity:

$$\lim_{x \to 0^-} f(0) = 0^2 + 2(0) + 1 = 1$$

$$\lim_{x \to 0^+} f(0) = 2(0) = 0$$

$$\lim_{x \to 0^-} f(0) \neq \lim_{x \to 0^+} f(0)$$

Therefore it is not continuous.

Since it is not continuous, it cannot be differentiable.

Note: The left-hand and right-hand gradients happen to both be equal to 2 in this case. The discontinuity is the driving factor for the non-differentiable point.

5. Calculus / 5.12 Limits and continuity

First principles

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Before you study the mathematics of differentiation, think about what you use differentiation for, namely, finding slopes. You differentiate a function to find an equation that describes the slope at any point along the curve.

Tangents

For a straight line $y = mx + b$, the gradient is very straightforward to calculate. It is just the value of m. This is because the gradient of a line is constant and you do not need any calculus to compute it.

However, most functions are curves. A curve does not have a constant gradient. However, graphically, you can estimate the gradient at a point on the curve with the help of the tangent to the curve at that point.

By drawing the tangent at that point of the curve, you can estimate the gradient at that point of the curve by finding the gradient of the tangent drawn. The diagram below shows the graph of $y = x^2$ and a tangent is drawn at the point $x = -2$.

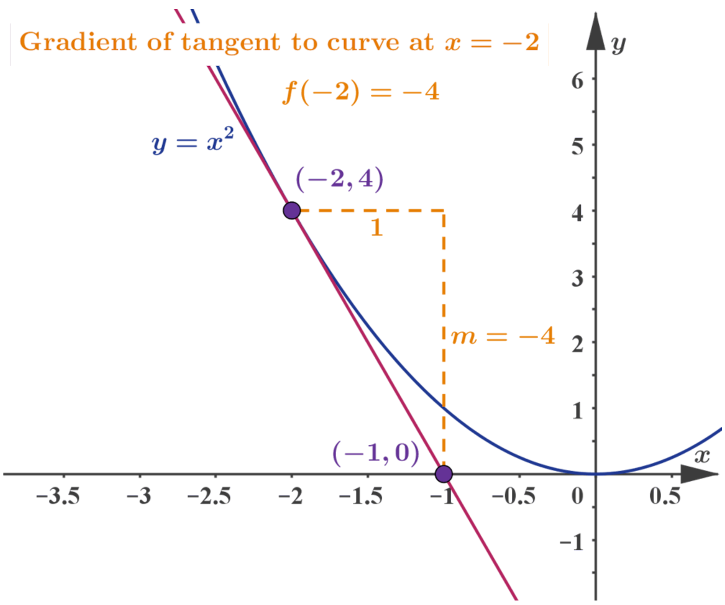
The gradient of the tangent can be found by using the equation:

$$\frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{0-4}{-1-(-2)} = \frac{-4}{1} = -4$$

or by using the difference quotient from section 5.12.1 (/study/app/math-aa-hl/sid-134-cid-761926/book/limits-and-convergence-id-26490/):

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-2+h)^2 - (-2)^2}{(-2+h) - (-2)} = \frac{4 - 4h + h^2 - 4}{h} = \frac{-4h + h^2}{h} = -4 + h = -4$$

The graph below shows the gradient of the curve $y = x^2$ using the gradient of the tangent (pink line) to the curve at $(-2, 4)$.



Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

More information

The image is a graph illustrating the curve y = x^2 and its tangent line at the point (-2,4). The X-axis ranges from -3.5 to 0.5 with markings at specific intervals, and the Y-axis ranges from -1 to 6 with numbered intervals. The curve of y = x^2 is represented in blue, showing a parabola that opens upwards. A pink tangent line is drawn touching the curve at the point (-2,4), marked as a purple dot. The tangent is identified with the label "Gradient of tangent to curve at x = -2." Additional annotations are included: f(-2) = -4 and m = -4, denoting the slope of the tangent. The graph shows a small dashed orange triangle illustrating the rise over run method, where the vertical leg is labeled "1" and the horizontal leg labeled from (-2,4) to (-1,0), emphasizing the concept of gradient measurement at the point of tangency.

[Generated by AI]

However, there are problems with using this method. You can calculate the gradient only at specific points and so need to repeat the calculation for each point. Also, if you have to draw the tangent by hand, it will probably not be perfect. Instead, you need an analytic method to calculate the gradient to a curve at any point along it.

Applying a limit to the difference quotient

You will now look more formally at the analytical method of the difference quotient. Start with two points on a curve, $f(x)$, and draw a line between them, which is called a secant. Using the difference quotient, you can take one point of interest to be at coordinates $(c, f(c))$ and the second point a little further along the curve, $(c+h, f(c+h))$.

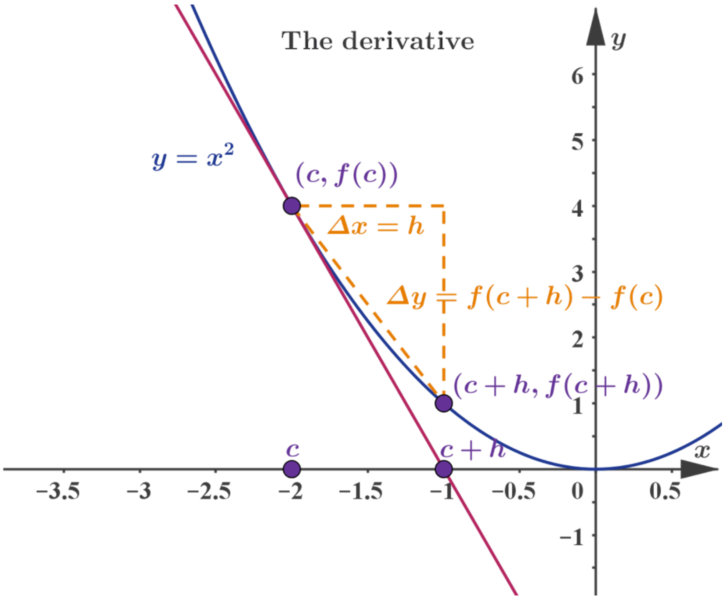
The gradient of the secant is

$$\frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{f(c+h)-f(c)}{(c+h)-c} = \frac{f(c+h)-f(c)}{h}$$

The graph below shown the gradient of the secant connecting the two points $(c, f(c))$ and $(c+h, f(c+h))$ on the curve.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o



More information

The graph illustrates a curve representing the function (y = x^2), with a focus on a specific segment where a secant line is connecting two points: ((c, f(c))) and ((c + h, f(c + h))). The X-axis denotes the variable (x) with a range from approximately -3.5 to 0.5, and the Y-axis represents (y) values ranging from 0 to 6. Two prominent points are labeled on the curve: ((c, f(c))) is at (x = -2) and ((c + h, f(c + h))) is at (x = -1). A pink tangent line touches the curve at ((c, f(c))) and a green secant line passes through both ((c, f(c))) and ((c + h, f(c + h))). Labeled on the graph are (\Delta x = h) and (\Delta y = f(c + h) - f(c)), representing the changes in x and y values between these two points. The graph visually demonstrates that as (h) decreases, ((c + h, f(c + h))) approaches ((c, f(c))), the secant gradually aligns with the tangent, illustrating the concept of derivative as a limit of secant slopes.

[Generated by AI]

It is clear by looking at the graph above that that the green secant and pink tangent do not have the same gradient; they are not parallel. However, can you imagine what happens when you move the point $(c+h, f(c+h))$ closer to the point $(c, f(c))$; in other words, when you decrease the value of h.

Activity

Consider the function $f(x) = x^2$ at point P, where $x = -2$. Choose a point fairly close to P, such as at $x = -1$. Calculate the slope of the secant between $(-2, f(-2))$ and $(-1, f(-1))$.

Now move the point close, say, $x = -1.9$. Calculate the slope of the secant between $(-2, f(-2))$ and $(-1, f(-1.9))$.

Now move the point close, say, $x = -1.99$. Calculate the slope of the secant between $(-2, f(-2))$ and $(-1, f(-1.99))$.

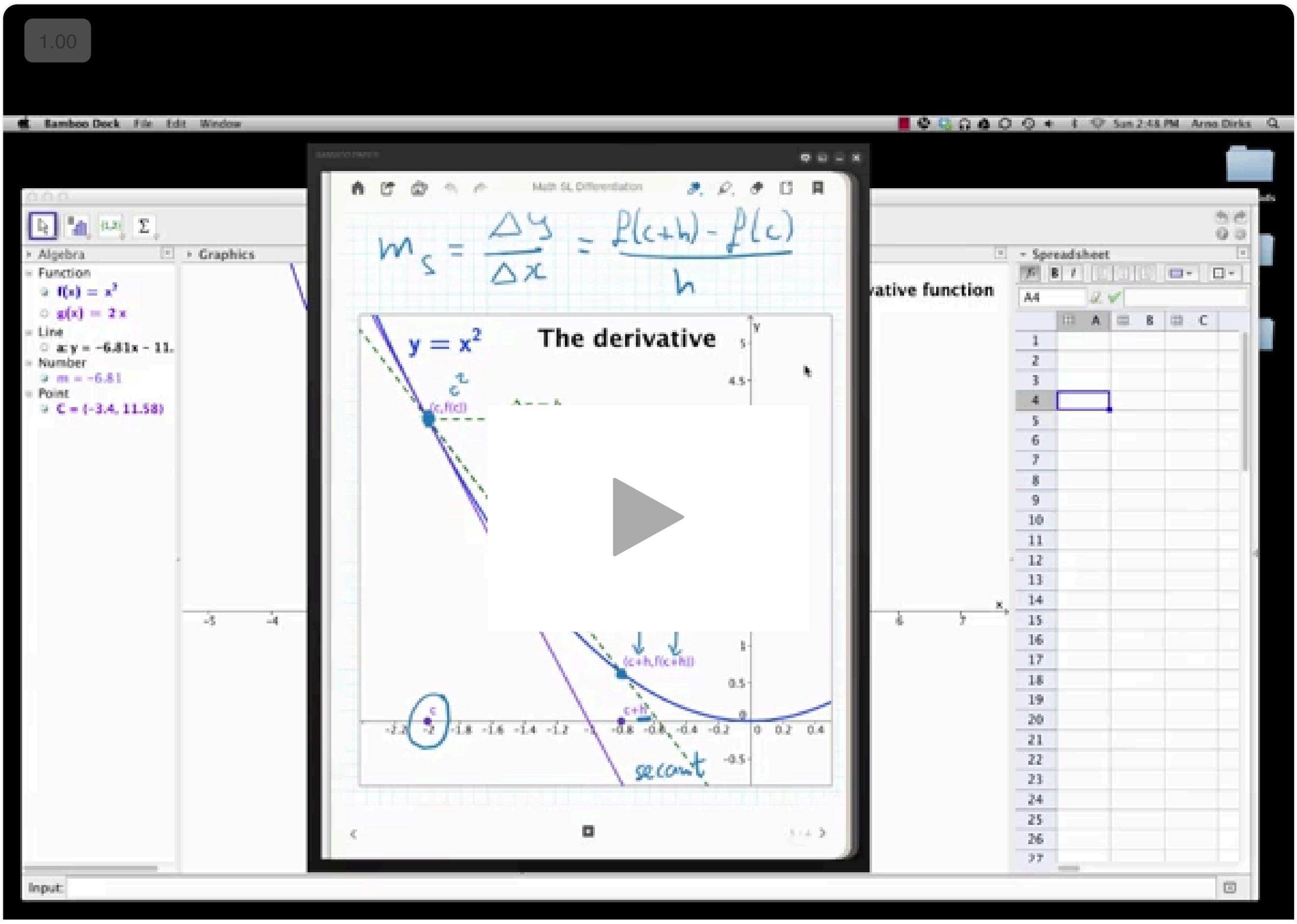
Keep repeating this until you see where the slope appears to be heading.

Does it match the difference quotient?

Student view

Overview (/study/ap aa-hl/sid-134-cid-761926/o

See the video below to see what happens if you repeat this for a general function.



Video 1. Differentiation from First Principles.

More information for video 1

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/differentiability-id-26492/print/) Assign

1

00:00:00,100 --> 00:00:02,503

[silence]

2

00:00:02,569 --> 00:00:04,872

narrator: Hello students!

And we're going to continue

3

00:00:04,938 --> 00:00:07,174

with our look at differential calculus.

4

00:00:07,508 --> 00:00:09,376

So the previous exercise

5

00:00:09,443 --> 00:00:13,814

we looked at how to find the gradient

to a curve at a particular point,

6

00:00:13,947 --> 00:00:15,749

using in fact the secant line,

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

7

00:00:15,816 --> 00:00:18,151

and then making

the step size indicated here

8

00:00:18,218 --> 00:00:19,853

by h, very small.

9

00:00:20,487 --> 00:00:25,259

Alright, so, and equation

that we explored was this:

10

00:00:25,392 --> 00:00:29,763

$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$.

11

00:00:30,564 --> 00:00:33,033

So it's the rise

of the run of the secant line

12

00:00:33,267 --> 00:00:34,735

when we let the step size go to zero.

13

00:00:34,801 --> 00:00:37,471

Now you realize that the outcome

of this will be the gradient,

14

00:00:37,671 --> 00:00:41,942

which is of course a value

because we are evaluating

15

00:00:42,943 --> 00:00:45,045

the function at two points.

16

00:00:45,345 --> 00:00:47,681

Hence, you're going to get a number.

17

00:00:47,748 --> 00:00:51,618

But now we're going to come

to a very important concept,

18

00:00:52,186 --> 00:00:54,755

which unfortunately is often overlooked.

19

00:00:54,922 --> 00:00:56,390

And that is that I can

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

use this also to find

20

00:00:57,891 --> 00:00:59,660

a derivative function.

21

00:01:00,527 --> 00:01:02,896

Now that is all very easy

because all we're going to do,

22

00:01:03,030 --> 00:01:05,766

instead of asking ourselves,

what is the gradient to a curve

23

00:01:05,832 --> 00:01:09,503

$f(x)$ at a particular point,

namely $x = c$,

24

00:01:09,703 --> 00:01:14,374

I'm going to get rid of the c

25

00:01:14,541 --> 00:01:15,909

and ask myself a question.

26

00:01:16,076 --> 00:01:17,845

What is the gradient function

27

00:01:18,011 --> 00:01:23,016

to this curve at any particular point x?

28

00:01:23,650 --> 00:01:27,554

And now you'll see that

with the same kind of tool,

29

00:01:27,654 --> 00:01:31,525

the same methodology

of how you found the gradient value,

30

00:01:31,625 --> 00:01:32,926

using first principles,

31

00:01:32,993 --> 00:01:34,995

you're actually going to

find a gradient function.

32

00:01:35,729 --> 00:01:37,631

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

And this is going to be very important.

33

00:01:38,699 --> 00:01:40,801

Now, when I say that

this is often overlooked,

34

00:01:40,868 --> 00:01:43,370

it's not that people don't find

derivative functions,

35

00:01:43,437 --> 00:01:47,074

they do it all the time,

but it is all too easy

36

00:01:47,140 --> 00:01:49,610

to forget that the derivative

function is related

37

00:01:49,676 --> 00:01:51,612

to the parent function $f(x)$.

38

00:01:51,678 --> 00:01:54,081

So what I've set up here is, in this case,

39

00:01:54,281 --> 00:01:55,616

a nice quadratic function.

40

00:01:55,716 --> 00:01:58,852

$f(x) = x^2$.

41

00:01:59,319 --> 00:02:03,056

And I'm going to march along this curve

42

00:02:03,590 --> 00:02:05,292

and find a gradient to the tension.

43

00:02:05,359 --> 00:02:07,794

And what I'm going to do,

I'm actually going to record

44

00:02:07,861 --> 00:02:08,962

that in my spreadsheet over here.

45

00:02:09,029 --> 00:02:12,533

So in the A column,

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

I'm going to record the x values

46

00:02:12,599 --> 00:02:15,836

of C in the B to y values,

not that I need them.

47

00:02:15,936 --> 00:02:19,406

And in the C column, I'm

going to cut to record D,

48

00:02:19,773 --> 00:02:21,942

actual gradient information.

49

00:02:22,109 --> 00:02:24,211

And what I'm going to do,

so I'm going to go down,

50

00:02:24,344 --> 00:02:27,314

I'm going to go down and take

little steps over here,

51

00:02:27,381 --> 00:02:30,851

and you can see definitely that

the gradient changes all the time, right?

52

00:02:30,951 --> 00:02:35,255

So here I've now created

a table with values.

53

00:02:35,422 --> 00:02:37,491

And you immediately realize

that of course,

54

00:02:37,558 --> 00:02:41,028

if I plot this table,

55

00:02:41,295 --> 00:02:43,697

the axis first device,

I get the quadratic function.

56

00:02:43,764 --> 00:02:44,598

But I'm not going to do that.

57

00:02:45,132 --> 00:02:50,103

Instead, what I'm going

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

to do is going to create a list of the x

58

00:02:50,304 --> 00:02:55,008

and the x values with associated

gradients at those points,

59

00:02:55,576 --> 00:02:58,612

create a list of points,

and here they are, right?

60

00:02:58,712 --> 00:03:02,049

So, and it turns out that if I

61

00:03:03,851 --> 00:03:07,354

plot a best fit line through those,

what I'm going to get,

62

00:03:07,421 --> 00:03:11,325

I'm going to get this function, which is $2x$,

63

00:03:11,391 --> 00:03:14,494

where $g(x)$ is then looks

like the behavior,

64

00:03:14,561 --> 00:03:17,764

the functional relationship

between the axis

65

00:03:17,831 --> 00:03:19,600

and the gradient of the parent function.

66

00:03:19,867 --> 00:03:22,169

Now let's first of all make sure

that this kind of makes sense.

67

00:03:22,236 --> 00:03:24,104

Well, if I'm going around here,

68

00:03:24,171 --> 00:03:27,441

then you definitely see

that I'm moving down the slope,

69

00:03:27,508 --> 00:03:30,110

and therefore my gradient

function should be negative.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

70

00:03:30,177 --> 00:03:31,612

In other words,

the gradients are negative.

71

00:03:31,678 --> 00:03:34,581

And here you see indeed,

that the gradient values,

72

00:03:34,715 --> 00:03:37,317

the values of the gradient

function are indeed negative.

73

00:03:37,784 --> 00:03:39,186

And here, once I've gone through zero,

74

00:03:39,253 --> 00:03:40,754

you can definitely see

that I'm stepping up

75

00:03:40,921 --> 00:03:45,259

and you can see that the associated

gradient function is positive indeed.

76

00:03:45,325 --> 00:03:49,463

The other thing you notice is that

over there I go exactly through zero.

77

00:03:49,596 --> 00:03:51,398

And you know that there's steepest here

78

00:03:51,798 --> 00:03:54,268

and shallow is there, but negative.

79

00:03:54,334 --> 00:03:57,704

So large negative values,

small negative values.

80

00:03:57,771 --> 00:04:01,441

And here I start very shallow,

small, positive number,

81

00:04:01,575 --> 00:04:04,545

and a very steep, large values.

82

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

00:04:04,611 --> 00:04:07,814

So here we can see

that this is indeed the case.

Formally, you say that, in the limit, the step size becomes zero, or infinitesimally small, and the gradient of the secant becomes equal to the gradient of the curve at $(c, f(c))$. This method is referred to as differentiation from first principles.

So, at point $x = c$ on a curve $y = f(x)$, the gradient at that point is defined as:

$$f'(c) = \lim_{h \to 0} \left(\frac{f(c+h) - f(c)}{h} \right)$$

Important

The derivative of the function f at $x = c$ is the limit $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$, where the limit exists.

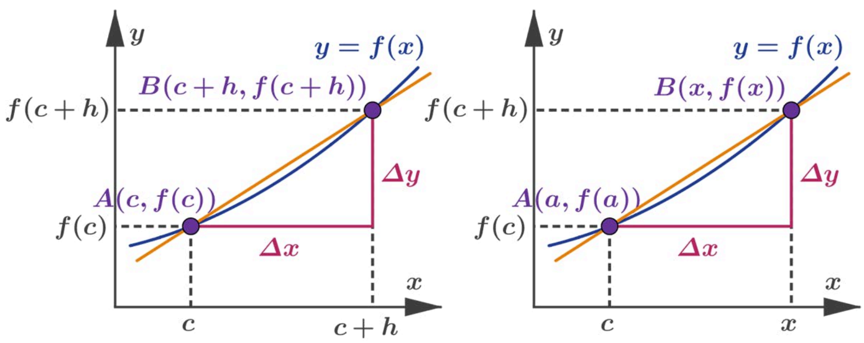
An alternative definition builds the secant with a set value c and variable x and reduces the secant to a tangent by taking x to c. So the following is also true.

Important

The derivative of the function f at $x = c$ is the limit $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$, where the limit exists.

These can be seen graphically below. The gradient of the secant connecting the two points AB is

$$\frac{\Delta y}{\Delta x} = \frac{f(c+h) - f(c)}{(c+h) - c} \text{ or } \frac{\Delta y}{\Delta x} = \frac{f(x) - f(c)}{x - c}$$



Student view

The image contains two similar graphs, each illustrating the concept of the difference quotient, which is a fundamental idea in calculus related to the derivative.

- **Graph on the Left:**
- The X-axis is labeled as 'x', with critical points marked at 'c' and 'c + h'.
- The Y-axis is labeled as 'y'.
- There is a curve labeled 'y = f(x)' and a tangent line touching the curve at point A.
- Point A is labeled as 'A(c, f(c))', and Point B is labeled as 'B(c + h, f(c + h))'.
- The change in x, denoted as 'Δx', is shown as the horizontal distance between 'c' and 'c + h'.
- The change in y, denoted as 'Δy', is the vertical distance between 'f(c)' and 'f(c + h)'.
- Horizontal and vertical dashed lines connect these coordinates, highlighting the changes in x and y.
- **Graph on the Right:**
- Similar structure to the left graph, with critical points marked at 'a' and 'x'.
- The tangent touches the curve at point A.
- Point A is labeled as 'A(a, f(a))', and Point B is labeled as 'B(x, f(x))'.
- 'Δx' and 'Δy' represent the changes along the x and y axes, respectively, with corresponding dashed lines indicating these changes.

Both graphs visually represent how the slope of the tangent line (the derivative) is calculated using limits of the difference quotient, visually represented as the ratio 'Δy/Δx'.

[Generated by AI]

Where $f'(c)$ is read as f-prime or f-dashed at $x = c$.

Below is an interactive applet with which you can visualise how the secants approach the tangent and how the gradient of the secants approach the gradient of the tangent. There are two sliders to show that the limiting process results in the same tangent from the left and right. Using the sliders, you can decrease the step size h to zero, and thus you force the respective point B to approach A , hence determining the gradient

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Interactive 1. Visualise How the Secants Approach the Tangent and How the Gradient of the Secants Approach the Gradient of the Tangent.

More information for interactive 1

This interactive allows the user to visualize how the secants approach the tangent and how the gradient (slope) of the secants converges toward the gradient of the tangent as one point approaches the other.

A graph is displayed with the XY-axis. The x-axis ranges from -1 to 3 and the y-axis from -0.5 to 3. The function is represented by a blue upward parabolic curve intersecting the y-axis at 1.5 and above the x-axis. Two lines pass through the function, one in pink-colored secant and the other in orange-colored secant, which intersect at $A(1.2, 0.5)$ on the parabola. Each secant has another point B on either side of B intersecting the parabola at $x = 0.4$ and $x = 2$. There is a slider on the x-axis positioned from 0.4 to 2. The slider of the pink secant ranges from 0.4 to 1.2, while the slider of the orange secant ranges from 1.2 to 2. Two perpendicular lines from points A and B join at a point, with a gradient value of negative 0.4 for the pink line and 1.22 for the orange line. A 2D image of the angle is shown above the value of gradients of the secants with angles $158.2°$ for pink and $50.63°$ for oranges. The angles change as the secants approach the tangent.

Using the sliders, users can decrease the step size h to zero, thus forcing the respective point *B* to approach point *A*, hence determining the gradient. The gradient of a line indicates its slopes. A positive gradient means the line slopes upwards from left to right as is the case with the orange line, while a negative gradient means it slopes downwards as is the case with the pink line. The users will understand how this type of graph could be used in finding tangents, optimizing functions or analyzing supply and demand curves.

Look back at the example in the first graph of this section. You can see how the limit converts the difference quotient to a derivative at a point.

$$f'(-2) = \lim_{h \to 0} \left(\frac{f(-2+h) - f(-2)}{h} \right) = \lim_{h \to 0} \frac{-4h + h^2}{h} = \lim_{h \to 0} (-4 + h) = -4$$

Exam tip

Differentiation from first principles may seem rather formal. Do not panic! This type of question does not come up frequently. You will learn many rules to make this faster.

Making connections

Some textbooks refer to differentiation from first principles as the limit definition of the derivative.

The gradient function

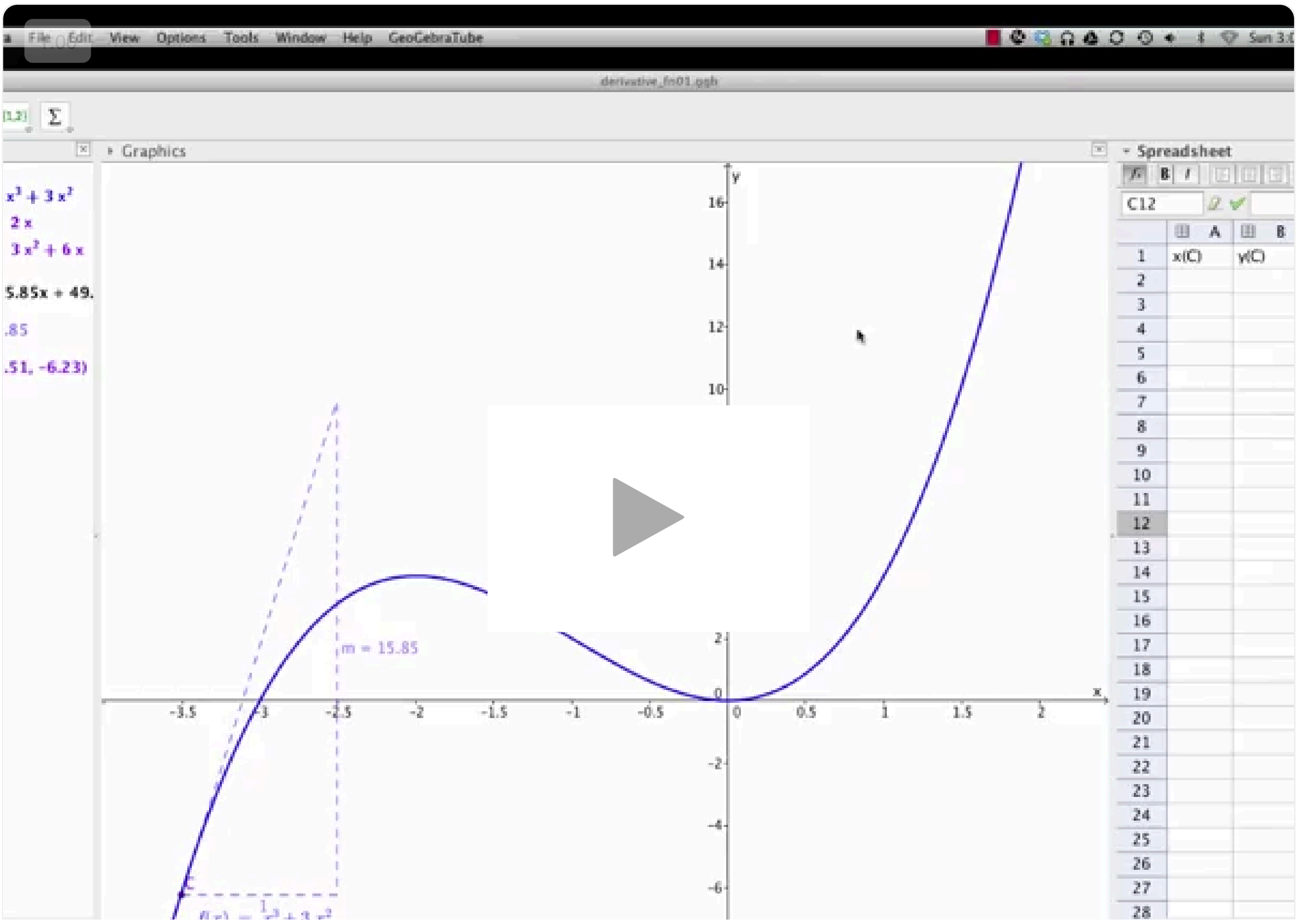
It will be much too cumbersome to find the gradient value each time you have a point along the curve of interest. Rather than starting again to find the value of the gradient each time, you can find a function that works across all values. This is the function that aligns the x values of an original function to the gradients of the function at the points of its graph having these x-coordinates.

Student view

Overview (/study/ap aa-hl/sid-134-cid-761926/o

Once you have found the gradient function, you have a description of the gradient, or the value of the steepness of the original function, anywhere, and this will assist you in understanding and analysing functional behaviour.

These two introductory videos look at the gradient function. In the first, the parent function is the quadratic $y = x^2$.



Video 2. Quadratic Gradient Function.

More information for video 2

1

00:00:00,267 --> 00:00:02,970

narrator: Let's do this once for more,

but for a different graph.

2

00:00:03,170 --> 00:00:05,272

And the blue graph now

is a cubic function,

3

00:00:05,439 --> 00:00:07,174

$x^3 + 3x^2$.

4

00:00:07,474 --> 00:00:11,011

But before I'm gonna trace the gradient

5

Student view

00:00:11,078 --> 00:00:14,848

through the curve

from here till about over there,

6

00:00:14,915 --> 00:00:18,952

let's ask ourselves further question,

what am I expecting my gradient to be?

7

00:00:19,019 --> 00:00:22,623

So I can certainly see

that in the beginning, my x, the y,

8

00:00:22,689 --> 00:00:24,925

step, the rise is positive.

9

00:00:24,992 --> 00:00:28,462

So I expect my gradient to

be positive until I peter out

10

00:00:28,529 --> 00:00:29,596

to about zero there.

11

00:00:29,796 --> 00:00:33,033

Then I'm stepping down until about here,

12

00:00:33,100 --> 00:00:34,434

and then I expect to step up against.

13

00:00:34,501 --> 00:00:41,241

So I expect my gradient to be positive

to the left of this negative,

14

00:00:41,308 --> 00:00:45,779

to the right of minus 2 till about here,

zero, and then again positive.

15

00:00:45,846 --> 00:00:47,915

So that's what my expectations are.

16

00:00:47,981 --> 00:00:50,918

So let's have a look what this

is actually going to be like.

17

00:00:51,018 --> 00:00:54,955

So let's add my x point step through this.

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

18

00:00:59,193 --> 00:01:01,495

And again, I'm going to create

19

00:01:06,500 --> 00:01:12,406

a list of points that tracing out a curve

that looks reasonably familiar over here.

20

00:01:12,573 --> 00:01:16,510

In fact, when I connect the points,

this is what I find, alright?

21

00:01:16,577 --> 00:01:19,346

And here is actually

the equation of the function

22

00:01:19,413 --> 00:01:23,317

that describes

the gradient of the blue curve.

23

00:01:23,650 --> 00:01:25,152

Okay? So what did we expect?

24

00:01:25,219 --> 00:01:26,520

We said, well, it should be positive

25

00:01:26,854 --> 00:01:27,888

to the left of minus 2.

26

00:01:27,955 --> 00:01:31,525

And it is, it's positive, it should be

negative between minus 2 and 0.

27

00:01:31,592 --> 00:01:34,361

And it is, it's the gradient function

in this case.

28

00:01:34,494 --> 00:01:37,898

$h(x)$ I've called it is negative

and it's possible there.

29

00:01:38,065 --> 00:01:40,334

So it goes through zero a couple times

30

00:01:40,801 --> 00:01:45,239

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

and then it is negative over here.

31

00:01:45,472 --> 00:01:47,040

Positive and positive.

32

00:01:47,107 --> 00:01:49,376

Okay, so notice the relationship

33

00:01:49,443 --> 00:01:52,579

we're gonna get to back this later

is that from a cubic function.

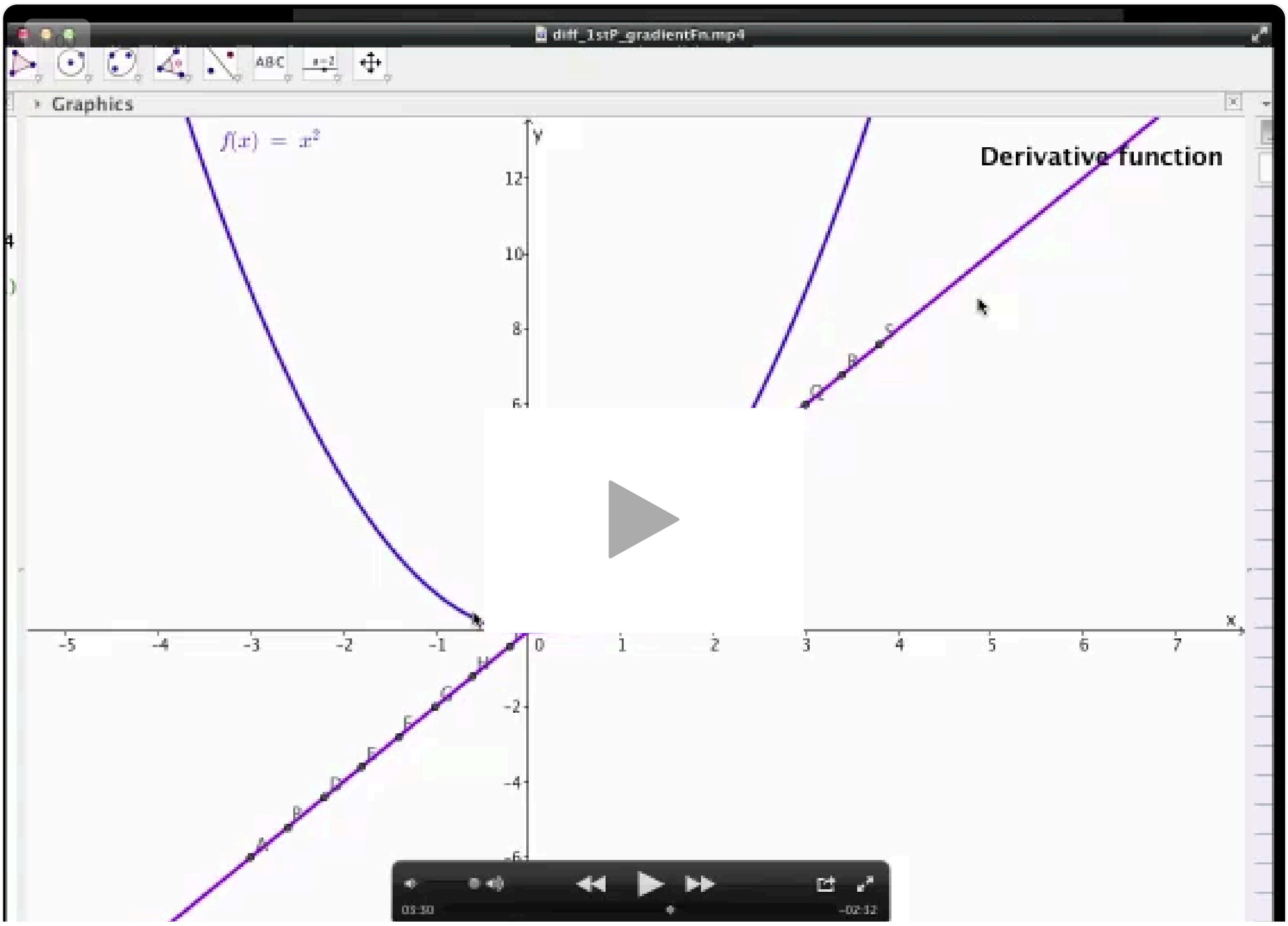
34

00:01:53,080 --> 00:01:56,817

Then when I plot the gradient function,

I'm gonna get a quadratic.

In the second, there is an investigation of the derivative function when the parent function is the cubic $y = x^3 + 3x^2$.



Video 3. Derivative Function with Parent Cubic Function.

More information for video 3

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

1

00:00:01,702 --> 00:00:05,405

narrator: So let's have a look

whether we can use the equation,

2

00:00:05,572 --> 00:00:08,475

the formula for differentiation

by first principle

3

00:00:08,909 --> 00:00:12,546

to find a gradient function just

like we've done with technology.

4

00:00:12,679 --> 00:00:15,282

So just a reminder,

the two functions we did it

5

00:00:15,349 --> 00:00:17,784

for was the x^2 function

6

00:00:18,318 --> 00:00:20,954

and we got the gradient function $2x$,

7

00:00:22,022 --> 00:00:26,827

and then we also looked

at a cubic function

8

00:00:26,894 --> 00:00:29,563

$x^3 + 3x^2$.

9

00:00:29,630 --> 00:00:32,199

And when technology differentiated that,

10

00:00:32,466 --> 00:00:37,704

it found a gradient function

that was equal to $3x^2 + 6x$.

11

00:00:37,771 --> 00:00:40,874

So let's see whether we can use analytical

methods to get the same.

12

00:00:42,809 --> 00:00:44,711

Now let's start with the function

13

00:00:44,778 --> 00:00:47,748

Student view

$f(x) = x^2$.

14

00:00:48,315 --> 00:00:50,250

So applying the formula that we have,

15

00:00:50,317 --> 00:00:53,720

I need to evaluate

that function, $f(x + h)$.

16

00:00:54,288 --> 00:00:55,923

So make sure you got the order right.

17

00:00:55,989 --> 00:00:57,257

You can use the binomial,

18

00:00:57,324 --> 00:01:00,794

but not necessary

for this simple quadratic function.

19

00:01:01,261 --> 00:01:02,496

Now, once I've done that,

20

00:01:02,563 --> 00:01:04,831

I'm gonna take a difference

of those two functions,

21

00:01:05,432 --> 00:01:09,169

$f(x + h) - f(x) = (x + h)^2 - x^2 = x^2 + 2xh + h^2 - x^2 = 2xh + h^2$,

22

00:01:09,269 --> 00:01:13,874

which I'm then gonna divide

by h, leaving $\frac{2xh+h^2}{h} = 2x + h$,

23

00:01:14,041 --> 00:01:17,544

and only now I'm gonna apply the limit

to find a derivative function.

24

00:01:17,611 --> 00:01:20,914

That's a limit as h goes

to 0 of the entire expression

25

00:01:20,981 --> 00:01:25,752

$\lim_{h\to 0}(2x + h)$, leaving only

$2x$ as we found with technology.

26

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

00:01:27,154 --> 00:01:29,389

And now let's apply to our cubic function,

27

00:01:29,456 --> 00:01:32,259

$g(x)$ I've called it,

which is x^3

28

00:01:32,326 --> 00:01:33,293

plus $3x^2$.

29

00:01:33,560 --> 00:01:37,164

So the first thing,

I'm gonna evaluate it as $g(x + h)$,

30

00:01:37,231 --> 00:01:40,267

which means

I've got two factors to expand.

31

00:01:41,134 --> 00:01:43,737

Now the first one,

I'm gonna use my binomial expression,

32

00:01:43,804 --> 00:01:48,675

$(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$,

gives me these four factors

33

00:01:49,476 --> 00:01:52,779

and then do the same thing

with a quadratic, keeping in mind

34

00:01:52,846 --> 00:01:54,348

to three out in front.

35

00:01:54,815 --> 00:01:56,850

Then I'm gonna subtract

the original function,

36

00:01:56,984 --> 00:01:58,285

$x^3 + 3x^2$.

37

00:01:58,352 --> 00:02:00,754

And you notice there are common factors

38

00:02:00,821 --> 00:02:05,092

that are going to disappear once

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

I take a difference,

39

00:02:05,158 --> 00:02:11,131

leaving just those factors

including x's and h's that you see here: $3x^2h + 3xh^2 + h^3 + 6xh + 3h^2$.

40

00:02:11,899 --> 00:02:13,567

Now I'm gonna divide by h,

41

00:02:13,634 --> 00:02:16,670

just like I've done before,

leaving the factors here: $3x^2 + 3xh + h^2 + 6x + 3h$.

42

00:02:16,737 --> 00:02:17,905

But you'll notice

43

00:02:17,971 --> 00:02:21,175

that among those factors there

are two without an h.

44

00:02:21,408 --> 00:02:24,645

And again, now I'm gonna apply

my limit as h goes to 0

45

00:02:24,711 --> 00:02:27,314

of entire expression for all the factors

46

00:02:28,515 --> 00:02:33,554

$\lim_{h\to 0}(3x^2 + 3xh + h^2 + 6x + 3h)$ so that those with h disappear

leaving just the ones without the h,

47

00:02:33,620 --> 00:02:35,355

$3x^2 + 6x$ as final.

As seen in the last section, finding a derivative at a point involves finding the gradient of a secant, and reducing the secant to a tangent by taking h to 0. To find the gradient function, you use x instead of a fixed value.

Important

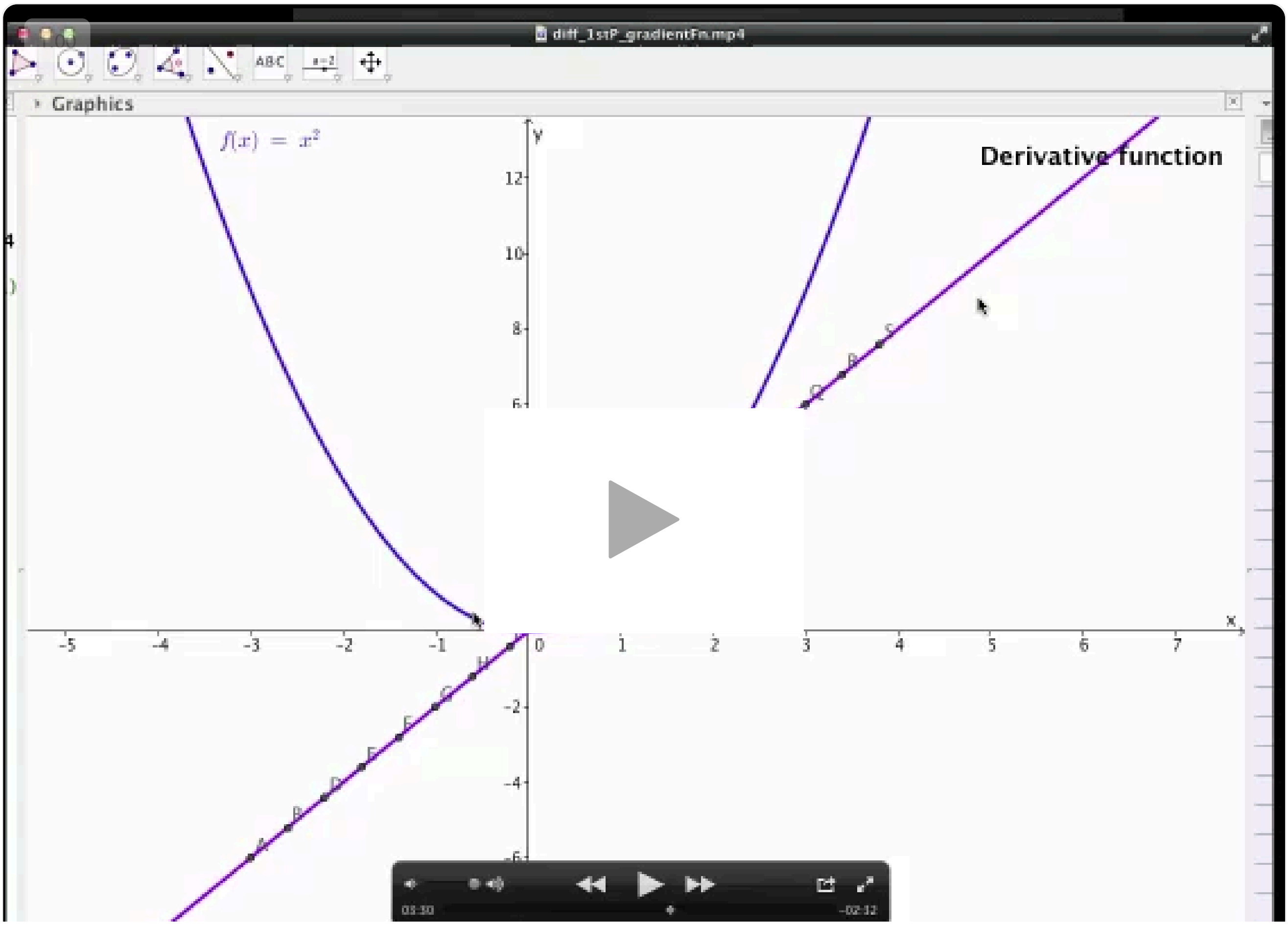
The derivative of the function f with respect to x is the function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$, where the limit exists.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

In the video below, the two gradient functions are found for the parent functions investigated earlier, namely, $y = x^2$ and $y = x^3 + 3x^2$.



Video 4. Derivative Function with Parent Cubic Function.

More information for video 4

1

00:00:01,702 --> 00:00:05,405

narrator: So let's have a look

whether we can use the equation,

2

00:00:05,572 --> 00:00:08,475

the formula for differentiation

by first principle

3

00:00:08,909 --> 00:00:12,546

to find a gradient function just

like we've done with technology.

4

00:00:12,679 --> 00:00:15,282

So just a reminder,

the two functions we did it

5

00:00:15,349 --> 00:00:17,784

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

for was the x^2 function

6

00:00:18,318 --> 00:00:20,954

and we got the gradient function $2x$,

7

00:00:22,022 --> 00:00:26,827

and then we also looked

at a cubic function

8

00:00:26,894 --> 00:00:29,563

$x^3 + 3x^2$.

9

00:00:29,630 --> 00:00:32,199

And when technology differentiated that,

10

00:00:32,466 --> 00:00:37,704

it found a gradient function

that was equal to $3x^2 + 6x$.

11

00:00:37,771 --> 00:00:40,874

So let's see whether we can use analytical

methods to get the same.

12

00:00:42,809 --> 00:00:44,711

Now let's start with the function

13

00:00:44,778 --> 00:00:47,748

$f(x) = x^2$.

14

00:00:48,315 --> 00:00:50,250

So applying the formula that we have,

15

00:00:50,317 --> 00:00:53,720

I need to evaluate

that function, $f(x + h)$.

16

00:00:54,288 --> 00:00:55,923

So make sure you got the order right.

17

00:00:55,989 --> 00:00:57,257

You can use the binomial,

18

00:00:57,324 --> 00:01:00,794

but not necessary

for this simple quadratic function.

Student view

19

00:01:01,261 --> 00:01:02,496

Now, once I've done that,

20

00:01:02,563 --> 00:01:04,831

I'm gonna take a difference

of those two functions,

21

00:01:05,432 --> 00:01:09,169

$f(x+h) - f(x) = (x+h)^2 - x^2 = x^2 + 2xh + h^2 - x^2 = 2xh + h^2$,

22

00:01:09,269 --> 00:01:13,874

which I'm then gonna divide

by h, leaving $\frac{2xh+h^2}{h} = 2x + h$,

23

00:01:14,041 --> 00:01:17,544

and only now I'm gonna apply the limit

to find a derivative function.

24

00:01:17,611 --> 00:01:20,914

That's a limit as h goes

to 0 of the entire expression

25

00:01:20,981 --> 00:01:25,752

$\lim_{h \to 0}(2x + h)$, leaving only

$2x$ as we found with technology.

26

00:01:27,154 --> 00:01:29,389

And now let's apply to our cubic function,

27

00:01:29,456 --> 00:01:32,259

$g(x)$ I've called it,

which is x^3

28

00:01:32,326 --> 00:01:33,293

plus $3x^2$.

29

00:01:33,560 --> 00:01:37,164

So the first thing,

I'm gonna evaluate it as $g(x+h)$,

30

00:01:37,231 --> 00:01:40,267

which means

I've got two factors to expand.

31

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

00:01:41,134 --> 00:01:43,737

Now the first one,

I'm gonna use my binomial expression,

32

00:01:43,804 --> 00:01:48,675

$(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$,

gives me these four factors

33

00:01:49,476 --> 00:01:52,779

and then do the same thing

with a quadratic, keeping in mind

34

00:01:52,846 --> 00:01:54,348

to three out in front.

35

00:01:54,815 --> 00:01:56,850

Then I'm gonna subtract

the original function,

36

00:01:56,984 --> 00:01:58,285

$x^3 + 3x^2$.

37

00:01:58,352 --> 00:02:00,754

And you notice there are common factors

38

00:02:00,821 --> 00:02:05,092

that are going to disappear once

I take a difference,

39

00:02:05,158 --> 00:02:11,131

leaving just those factors

including x's and h's that you see here: $3x^2h + 3xh^2 + h^3 + 6xh + 3h^2$.

40

00:02:11,899 --> 00:02:13,567

Now I'm gonna divide by h,

41

00:02:13,634 --> 00:02:16,670

just like I've done before,

leaving the factors here: $3x^2 + 3xh + h^2 + 6x + 3h$.

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00:02:16,737 --> 00:02:17,905

But you'll notice

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00:02:17,971 --> 00:02:21,175

that among those factors there

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

are two without an h.

44

00:02:21,408 --> 00:02:24,645

And again, now I'm gonna apply

my limit as h goes to 0

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00:02:24,711 --> 00:02:27,314

of entire expression for all the factors

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00:02:28,515 --> 00:02:33,554

$\lim_{h\to 0}(3x^2 + 3xh + h^2 + 6x + 3h)$ so that those with h disappear

leaving just the ones without the h,

47

00:02:33,620 --> 00:02:35,355

$3x^2 + 6x$ as final.

The results of differentiation from first principles for these functions are the following:

- When $f(x) = x^2$ then $f'(x) = 2x$.
- When $g(x) = x^3 + 3x^2$ then $g'(x) = 3x^2 + 6x$.

Notation

There are several accepted ways to write the derivative or gradient function. Given a function $y = f(x)$, derivatives can be depicted in multiple ways as shown in the table below.

Notation	Pronounced	Advantages / disadvantages	Developed by
y'	‘ *y* prime’	Efficient, but only identifies the dependent variable	Joseph Lagrange 1770
$f'(x)$	‘ *f* prime of *x* ’	Only identifies the independent variable	
$\frac{\mathrm{d}y}{\mathrm{d}x}$	‘d *y* d *x* ’ or ‘d *y* by d *x* ’	Identifies both variables and uses the ‘d’ for derivative	Gottfried Leibniz 1684
$\dot{y}$	‘ *y dot* ’	Typically used in physics, specifically with kinematic problems	Isaac Newton 1665

Example 1

Student view

★☆☆

Overview (/study/ap aa-hl/sid-134-cid-761926/o

Use differentiation from first principles to find the gradient function of $f(x) = x^3$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h\to 0}\frac{(x+h)^3-x^3}{(x+h)-x} = \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$
$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0}3x^2+3xh+h^2 = 3x^2$$

Example 2

★☆☆

Use differentiation from first principles to find the gradient function of $f(x) = x^3$.

Use the alternative definition.

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c}\frac{x^3-c^3}{x-c} = \lim_{x\to c}\frac{(x-c)\left(x^2+xc+c^2\right)}{x-c}$$
$$= \lim_{x\to c}x^2+xc+c^2 = c^2+c^2+c^2 = 3c^2 = 3x^2$$

This can easily be confirmed using the power rule from subtopic 5.3 .

Example 3

★★☆

Use differentiation from first principles to find the gradient function of $f(x) = \sqrt{x}$.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{(x+h)-x}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) = \lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$
$$= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

This can easily be confirmed using the power rule from subtopic 5.3 after rewriting the parent function $f(x) = \sqrt{x} = x^{\frac{1}{2}}$.

Example 4

★★☆

Student view

Use differentiation from first principles to find the gradient function of $f(x) = \sqrt{x}$.

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Use the alternative definition.

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c}\left(\frac{\sqrt{x}-\sqrt{c}}{x-c}\right)\left(\frac{\sqrt{x}+\sqrt{c}}{\sqrt{x}+\sqrt{c}}\right)=\lim_{x\to c}\frac{(x-c)}{(x-c)\left(\sqrt{x}+\sqrt{c}\right)}$$
$$=\lim_{x\to c}\frac{1}{\left(\sqrt{x}+\sqrt{c}\right)}=\frac{1}{\left(\sqrt{c}+\sqrt{c}\right)}=\frac{1}{\left(\sqrt{c}+\sqrt{c}\right)}=\frac{1}{2\sqrt{c}}=\frac{1}{2\sqrt{x}}$$

This can easily be confirmed using the power rule from subtopic 5.3 after rewriting the parent function $f(x)=\sqrt{x}=x^{\frac{1}{2}}$.

6 section questions

Question 1

Difficulty:

★★☆

Use differentiation from first principles to find the gradient of the function $f(x)=2x^2-3$ at $x=2$.

8

Accepted answers

8

Explanation

$$f'(x)=\lim_{h\to 0}\left(\frac{\left(2(x+h)^2-3\right)-\left(2x^2-3\right)}{h}\right)$$
$$f'(2)=\lim_{h\to 0}\left(\frac{\left(2(2+h)^2-3\right)-\left(2(2)^2-3\right)}{h}\right)$$
$$=\lim_{h\to 0}\left(\frac{\left(2(2)^2+4(2)h+2h^2-3\right)-\left(2(2)^2-3\right)}{h}\right)$$
$$=\lim_{h\to 0}\left(\frac{5+8h+2h^2-5}{h}\right)=\lim_{h\to 0}\left(\frac{8h+2h^2}{h}\right)=\lim_{h\to 0}(8+2h)=8$$

Question 2

Difficulty:

★★☆

Use differentiation from first principles to find the gradient of the function $f(x)=(x+3)^2$ at $x=-2$.

2

Accepted answers

2

Explanation

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

$$f'(x) = \lim_{h\to 0}\left(\frac{\left((x+3+h)^2\right) - \left((x+3)^2\right)}{h}\right)$$

$$f'(-2) = \lim_{h\to 0}\left(\frac{\left((1+h)^2\right) - \left((1)^2\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{(1+2h+h^2)-1}{h}\right) = \lim_{h\to 0}\left(\frac{2h+h^2}{h}\right) = \lim_{h\to 0}(2+h) = 2$$

Question 3

Difficulty:

★★★

Use differentiation from first principles to find the gradient of the function $f(x) = x^3 - 5x + 8$ at $x = 1$.

-2

Accepted answers

-2, −2

Explanation

$$f'(x) = \lim_{h\to 0}\left(\frac{\left((x+h)^3 - 5(x+h) + 8\right) - (x^3 - 5x + 8)}{h}\right)$$

$$f'(1) = \lim_{h\to 0}\left(\frac{\left((1+h)^3 - 5(1+h) + 8\right) - 4}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1+3h+3h^2+h^3-5-5h+8-4}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{-2h+3h^2+h^3}{h}\right) = \lim_{h\to 0}\left(-2+3h+h^2\right) = -2$$

Question 4

Difficulty:

★★☆

Use differentiation from first principles to find the gradient function of $f(x) = 2x^2 - 3$.

Give only the expression for the gradient function with no spaces.

4x

Accepted answers

4x, y=4x, y'=4x, f'(x)=4x

Explanation

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\left(2(x+h)^2 - 3\right) - (2x^2 - 3)}{h}$$

$$= \lim_{h\to 0}\frac{2x^2+4xh+2h^2-3-2x^2+3}{h}$$

$$= \lim_{h\to 0}\frac{4xh+2h^2}{h}$$

$$= \lim_{h\to 0} 4x + 2h = 4x$$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Question 5

Difficulty: ★★★

Use differentiation from first principles to find the gradient function of $f(x) = x^3 - 5x + 8$.

Choose the correct answer from the given options.

1 $3x^2 - 5$ ✓

2 $3x - 5$

3 $3x^2 - 10$

4 $x^3 - 5$

Explanation

$$
\begin{aligned}
f'(x) &= \lim_{x\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{x\to 0} \frac{\left((x+h)^3 - 5(x+h) + 8\right) - (x^3 - 5x + 8)}{h} \\
&= \lim_{x\to 0} \frac{\left(x^3 + 3x^2h + 3xh^2 + h^3 - 5x - 5h + 8\right) - \left(x^3 - 5x + 8\right)}{h} \\
&= \lim_{x\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 5x - 5h + 8 - x^3 + 5x - 8}{h} \\
&= \lim_{x\to 0} \frac{3x^2h + 3xh^2 + h^3 - 5h}{h} = \lim_{x\to 0} 3x^2 + 3xh + h^2 - 5 = 3x^2 - 5
\end{aligned}
$$

Question 6

Difficulty: ★★★

Use differentiation from first principles to find the gradient function of $f(x) = (x+3)^2$.

Give only the expression for the gradient function as a polynomial with no spaces, for example $5x - 7$.

1 $2x + 6$ ✓

2 $2x + 3$

3 $x + 3$

4 $4x + 12$

Explanation

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+3+h)^2 - (x+3)^2}{h} \\
&= \lim_{h\to 0} \frac{(x+3)^2 + 2(x+3)h + h^2 - (x+3)^2}{h} = \lim_{h\to 0} \frac{2(x+3)h + h^2}{h} \\
&= \lim_{h\to 0} 2(x+3) + h = 2(x+3)
\end{aligned}
$$

Student view

Higher derivatives

In subtopic 5.7 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27788/), you observed that the derivative of a function is itself a function and, as a function, it also has a gradient. The derivative of a derivative is called the second derivative.

In subtopic 5.8 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27795/), you used the second derivative to determine the concavity of a curve. When the second derivative is greater than zero, the function is concave up and opens up. When the second derivative is less than zero, the function is concave down and opens down. When the second derivative is equal to zero, then there is a possibility that the function is at an inflection point, but more research need to be done to confirm that the concavity changes.

In subtopic 5.9 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27800/), you applied the relationships of the function and the first two derivatives to solve kinematic problems, where $s(t)$ represents the displacement, $v(t) = \frac{\mathrm{d}s}{\mathrm{d}t}$ represents the velocity, and $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2s}{\mathrm{d}t^2}$ represents the acceleration.

Similarly, you can take any number of higher-order derivatives (providing the derivative function exists). You call this taking the nth derivative, and write it as follows:

Given $y = f(x)$, the nt h derivative is given by $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f^{(n)}(x)$.

✓ **Important**

You do not need to learn new rules for taking higher-order derivatives. All the methods that you use to obtain the first derivative can be used to find higher-order derivatives, since they all apply at any stage of differentiation.

Given a function $y = f(x)$, second and higher-order derivatives can be written in multiple ways as shown in the table below.

1st Derivative	2nd Derivative	3rd Derivative	4th Derivative	n th Derivative
y'	y''	y'''	$y^{(4)}$	$y^{(n)}$
$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$f^{(n)}(x)$

1st Derivative	2nd Derivative	3rd Derivative	4th Derivative	n th Derivative
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$	$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$	$\frac{\mathrm{d}^4y}{\mathrm{d}x^4}$	$\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$
$\dot{y}$	$\ddot{y}$	$\dddot{y}$	$\overset{4}{\dot{y}}$	$\overset{n}{\dot{y}}$

One example of a use of higher-order derivatives is in the study of kinematics. As mentioned earlier, if position is represented by $s(t)$, you can find the following functions:

$$\begin{array}{cccc} (t) & \text{displacement} & \mathrm{m} & \text{position} \\ v(t) = s'(t) & \text{velocity} & \mathrm{m\,s^{-1}} & \text{change in position over time} \\ a(t) = s''(t) & \text{acceleration} & \mathrm{m\,s^{-2}} & \text{change in velocity over time} \\ j(t) = s'''(t) & \text{jerk} & \mathrm{m\,s^{-3}} & \text{change in acceleration over time} \end{array}$$

Our inner-ears can detect jerk just as they can detect acceleration. When engineers design roller coasters, they study the jerk as it affects the feeling of motion sickness. In physics, there are higher-order derivatives of position, although they are rarely used.

Example 1

★☆☆

Consider the function $f\left(x\right) = 5x^4 + 3x^3 + 7x^2 + 4x + 12$.

Find multiple derivatives of the function. What pattern do you notice?

Using the power rule:

$$f'(x) = 20x^3 + 9x^2 + 14x + 4$$

$$f''(x) = 60x^2 + 18x + 14$$

$$f'''(x) = 120x + 18$$

$$f^{(4)}(x) = 120$$

$$f^{(5)}(x) = 0$$

Polynomials tend to become simpler with lower order in higher derivatives.

Example 2

★☆☆

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Consider the function $f\left(x\right)=\sin x$.

Find multiple derivatives of the function. What pattern do you notice?

Using the power rule:

$$f'\left(x\right)=\cos x$$

$$f''(x)=-\sin(x)$$

$$f'''(x)=-\cos x$$

$$f^{(4)}(x)=\sin x$$

$$f^{(5)}(x)=\cos x$$

Sinusoidal functions tend to cycle.

Example 3

★☆☆

Consider the function $f\left(x\right)=\mathrm{e}^{x}$.

Find multiple derivatives of the function. What pattern do you notice?

Using the power rule:

$$f'\left(x\right)=\mathrm{e}^{x}$$

$$f''(x)=\mathrm{e}^{x}$$

$$f'''(x)=\mathrm{e}^{x}$$

$$f^{(4)}(x)=\mathrm{e}^{x}$$

$$f^{(5)}(x)=\mathrm{e}^{x}$$

Exponential functions repeat.

3 section questions

Student view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

Question 1

Difficulty:

★★☆

Consider the function defined by $f(x) = 4x^5 + 4x^3 + 7x^2 + 5$.

Find $f^{(4)}(x)$, the fourth derivative of f.

480x ✓

Accepted answers

480x, 480*x

Explanation

$f'(x) = 20x^4 + 12x^2 + 14x$

$f''(x) = 80x^3 + 24x + 14$

$f'''(x) = 240x^2 + 24$

$f^{(4)}(x) = 480x$

Question 2

Difficulty:

★★★

Consider $f(x) = \sin(3x^2)$.

Find an expression for $f^{(4)}(x)$.

Choose the correct answer from the given options.

1 $-1296x^2 \cos(3x^2) + (1296x^4 - 108)\sin(3x^2)$ ✓

2 $1296x^2 \sin(3x^2)$

3 $1296x^2 \cos(3x^2)$

4 $-1296x^2 \sin(3x^2) + (1296x^4 - 108)\cos(3x^2)$

Explanation

You can use the chain rule to find the first derivative.

$f'(x) = 6x \cos(3x^2)$

For the higher order derivatives you will also need the product rule.

$f''(x) = 6\cos(3x^2) - 36x^2 \sin(3x^2)$

$f'''(x) = -216x^3 \cos(3x^2) - 108x \sin(3x^2)$

Student
view

$f^{(4)}(x) = -1296x^2 \cos(3x^2) + (1296x^4 - 108)\sin(3x^2)$

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Question 3

Difficulty:

★★☆

Consider $f(x) = \mathrm{e}^{cx}$ where c is a constant.

Find an expression for $f^{(n)}(x)$.

Choose the correct answer from the given options.

1 $\mathrm{c}^n\mathrm{e}^{cx}$ ✓

2 ce^{cx}

3 $cn\mathrm{e}^{cx}$

4 e^{cnx}

Explanation

$f'(x) = c\mathrm{e}^{cx}$

$f''(x) = c^2\mathrm{e}^{cx}$

$f'''(x) = c^3\mathrm{e}^{cx}$

$f^{(4)}(x) = c^4\mathrm{e}^{cx}$

As you continue, the power of the coefficient increases with every derivative.

5. Calculus / 5.12 Limits and continuity

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-26495/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- identify limits and convergence similar to horizontal asymptotes, as when analysing functions
- apply the properties of limits
- determine whether a function is continuous by visual inspection: if its graph can be drawn without taking the pencil off the paper it is continuous; otherwise it is discontinuous

Student view

- determine whether a function is differentiable by visual inspection; discontinuous points include discontinuities, corners, cusps and oscillating discontinuities
- understand that for a function to be differentiable, it must be continuous; but a continuous function is not necessarily differentiable
- understand that the tangent to a curve at a point is parallel to the curve at that point and thus it has the same gradient/slope
- differentiate from first principles polynomial functions, both at a point and for the entire function
- compute higher derivatives of a function beyond the second derivative.

Investigation

On a graphing tool such as your GDC, Geogebra, or Desmos, graph these functions:

- $f(x) = |x|$
- $g(x) = x^{2/3}$
- $h(x) = (4x)^2$

Using the zoom function, zoom in on the origin $(0, 0)$.

What do you see? Which functions approach linearity?

Rate subtopic 5.12 Limits and continuity

Help us improve the content and user experience.