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Teacher view

0 (https://intercom.help/kognity)

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The big picture

Did you know that three-legged chairs never wobble while four-legged chairs might? Why is this? You would think that the more legs a chair has the more stable it would be. But even if all three legs are different lengths, a three-legged chair will not wobble, whereas if one leg of a four-legged chair is shorter than the others it will be wobbly. This is because three points determine a plane, so the points where the legs of a three-legged chair touch the ground are always in the same plane.

In this subtopic, you will define relative positions of points and vectors depending on the plane they are on.

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Three-legged stool

Credit: JanekWD

Concept

There are several ways in which you can represent a plane, none of which are unique. As you will discover in this subtopic, a plane can be represented using points, one point and a vector or by a Cartesian form equation. What is the relationship between these different forms?

Theory of Knowledge

It is important to consider the impact of culture on the individual knower. However, it seems at first glance that mathematics is immune from cultural bias. Is such a statement accurate? Is this immunity a key factor in creating mathematical authority in regard to knowledge validity? Or on the contrary, do you believe mathematics is also influenced by culture?

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view

Knowledge Question: Is it possible to abstract knowledge from the culture in which it was produced?

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3. Geometry and trigonometry / 3.17 Vector equations of a plane

Vector equation of a plane

In geometry, there is a unique line passing through any two given points. Using a similar approach to this, you can say that there is a unique plane passing through any three nonlinear points. Take a flat surface, for example a tray, and try to balance it using two fingers; you will find that this is not possible. However, if you try again using three fingers, it will be possible to carry the tray.

Activity

Use the following applet to explore a plane passing through three points.

Drag the points and the plane, observing the changes to the plane passing through the three points.

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Section

ssign

Interactive 1. Explore a Plane Passing Through Three Points.

More information for interactive 1

This interactive visualization allows users to explore how a plane is defined in three-dimensional space by selecting three points.

The display shows a 3D coordinate system with clearly labeled x-, y-, and z-axes and a light blue plane labeled "p" that passes through three movable red points labeled A, B, and C. Users can click and drag these points to new positions, and the plane updates in real-time to continue passing through the selected points. By rotating the entire view, users can examine how the orientation of the plane changes depending on the location of the points. The tool reinforces a key geometric principle: that three non-collinear points in 3D space determine exactly one unique plane. As users move the points, they can observe how the tilt and angle of the plane vary, while it still remains flat and infinite in extent.

This hands-on activity supports spatial reasoning and builds conceptual understanding of planes in vector geometry, offering a dynamic, visual, and tactile method to learn how planes behave in three-dimensional environments.

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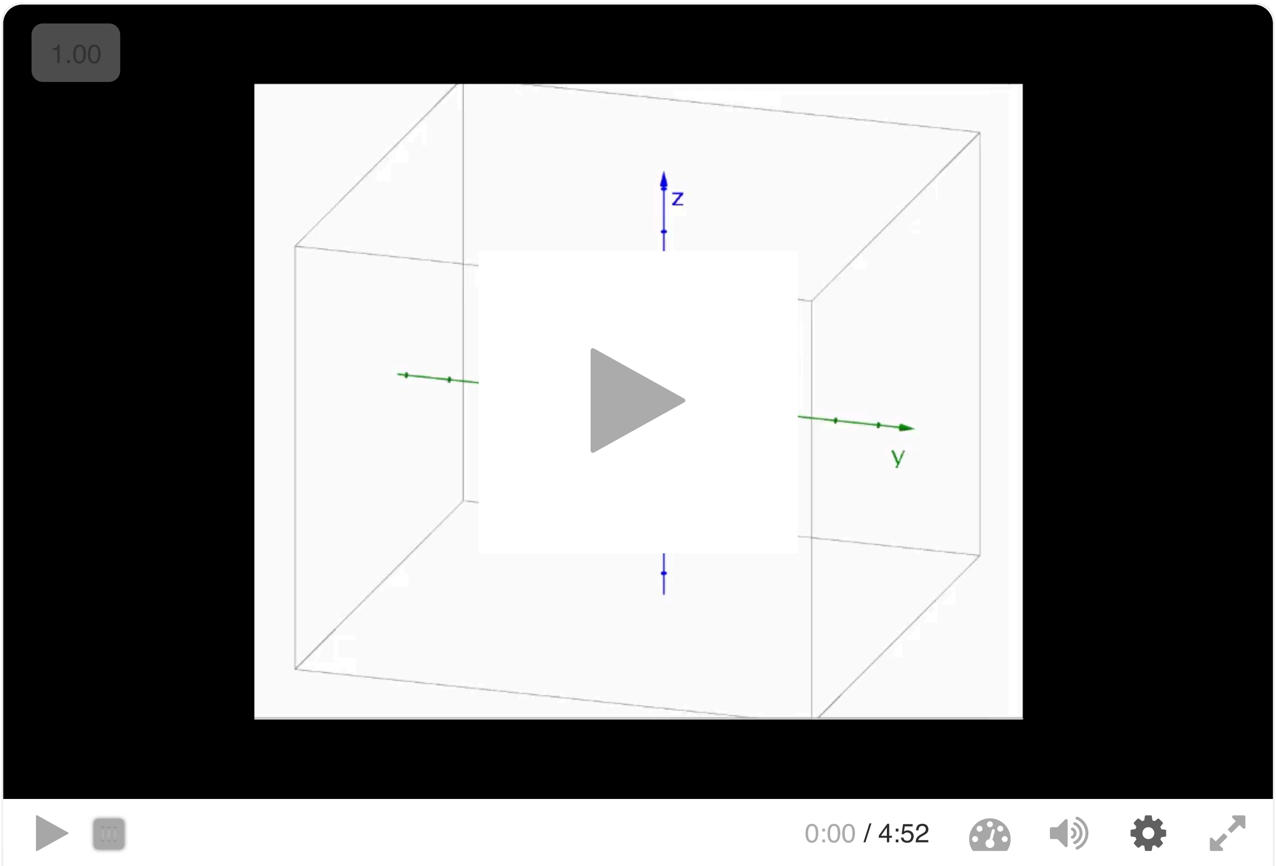
In subtopic 3.16 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-28028/)you saw that the vector product of two vectors results in a vector that is perpendicular to the plane containing the original two vectors. Hence, a plane is associated with two vectors. You can explore this in the following video.



Video 1. A Plane Associated with Two Vectors.

More information for video 1

1

00:00:00,367 --> 00:00:03,700

narrator: In this video, we're going

to use vectors to explore planes,

2

00:00:03,767 --> 00:00:06,967

after we've had quite

a bit of time looking at lines.

3

00:00:07,300 --> 00:00:12,233

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Now we've already encountered planes

and for that we had two vectors.

4

00:00:12,300 --> 00:00:16,700

So here we've called two vectors,

b and c in three dimensional space,

5

00:00:16,767 --> 00:00:19,733

and we've already seen

that the vector product

6

00:00:19,800 --> 00:00:23,767

between those is a vector

which is perpendicular to the plane

7

00:00:23,900 --> 00:00:25,933

that contains those two vectors,

8

00:00:26,000 --> 00:00:28,467

b and c, so we've already seen

somehow that two vectors

9

00:00:29,067 --> 00:00:32,500

can create

a plane in some form or another.

10

00:00:32,600 --> 00:00:34,233

And here we see that.

11

00:00:34,300 --> 00:00:38,267

However, this plane is of course

situated on the origin,

12

00:00:38,333 --> 00:00:39,733

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which is not always what we want.

13

00:00:39,800 --> 00:00:42,967

We would like to create

a plane anywhere in space,

14

00:00:43,033 --> 00:00:44,800

for example, at this point A.

15

00:00:45,400 --> 00:00:48,133

But that is very easy

because of course we now realize

16

00:00:48,200 --> 00:00:53,300

that we can take the vector b

and put it at A, it's parallel.

17

00:00:53,367 --> 00:00:54,933

So we can simply transport it there

18

00:00:55,000 --> 00:00:57,433

and we can do that

with the vector c of course.

19

00:00:57,500 --> 00:01:00,333

And now we've got the same

orientation of the earlier plane,

20

00:01:00,467 --> 00:01:02,833

but at an arbitrary point A.

21

00:01:02,933 --> 00:01:06,800

And here you can see that it is the same

orientation of the plane

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view

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22

00:01:06,867 --> 00:01:10,100

because it contains

the same vectors, b and c,

23

00:01:10,167 --> 00:01:11,967

but it is at the arbitrary point.

24

00:01:12,033 --> 00:01:14,133

Now all I need to do

is get to the point A,

25

00:01:14,200 --> 00:01:16,800

which of course I do

with a position vector, a little a,

26

00:01:17,067 --> 00:01:18,233

and now you've seen

27

00:01:18,500 --> 00:01:22,067

that I can create a plane

using those three vectors

28

00:01:22,133 --> 00:01:24,767

a, b, and c.

29

00:01:26,300 --> 00:01:30,400

Now quickly to remind ourselves

that planes have an orientation,

30

00:01:30,467 --> 00:01:34,300

if I recreate the earlier

two vectors b and c at the origin

31

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00:01:34,500 --> 00:01:39,100

and recreated plane, you can

see that those two blue planes,

32

00:01:39,167 --> 00:01:40,733

turquoise planes are parallel

33

00:01:40,800 --> 00:01:44,500

because they're created

by the same vectors b

34

00:01:45,100 --> 00:01:46,167

and c.

35

00:01:46,700 --> 00:01:51,033

So quickly then,

as a reminder, let's see how we get

36

00:01:51,100 --> 00:01:53,500

to create plane with those vectors.

37

00:01:53,567 --> 00:01:56,567

So we take a position vector

to get anywhere in space a,

38

00:01:56,800 --> 00:01:58,067

then we take two vectors,

39

00:01:58,133 --> 00:02:01,133

which are going

to create the planes orientation,

40

00:02:01,200 --> 00:02:02,667

let's call them band c.

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41

00:02:02,933 --> 00:02:05,333

And there I can create my plane,

42

00:02:05,400 --> 00:02:08,867

the turquoise plane,

which I've called d here.

43

00:02:09,833 --> 00:02:12,500

So that seems to be

a way to create planes.

44

00:02:12,667 --> 00:02:15,333

It uses three vectors

45

00:02:15,600 --> 00:02:17,100

and of course it is an object

46

00:02:17,167 --> 00:02:19,667

that lives in three dimensional

space necessarily.

47

00:02:20,633 --> 00:02:24,133

So planes, let's just simply

wave our hands a little bit

48

00:02:24,200 --> 00:02:29,000

and call them a collection

of points that lie on a sheet.

49

00:02:29,067 --> 00:02:31,133

But of course in some

sense that it's a circular

50

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00:02:31,733 --> 00:02:33,633

definition, but we'll leave it at that.

51

00:02:33,700 --> 00:02:35,833

Now, points of course have

a position vector

52

00:02:35,900 --> 00:02:37,633

and we've called it r before.

53

00:02:37,900 --> 00:02:40,300

And here we've seen,

we used three vectors,

54

00:02:40,367 --> 00:02:42,000

a, b, and c.

55

00:02:42,300 --> 00:02:44,533

So r is a + b+ c.

56

00:02:44,600 --> 00:02:47,300

However, since a, b and c

are unique vectors

57

00:02:47,367 --> 00:02:52,533

with this formulation, I only get to come

to one point, not a collection of point,

58

00:02:52,600 --> 00:02:54,933

but of course we already

seen a way around it in lines

59

00:02:55,000 --> 00:02:56,233

and we're gonna do the same here.

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view

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60

00:02:56,300 --> 00:02:57,833

But instead of having one

parameter lambda,

61

00:02:58,167 --> 00:02:59,600

we have two parameters,

62

00:02:59,667 --> 00:03:01,767

lambda and mu and lambda

and mu are just numbers.

63

00:03:01,900 --> 00:03:06,000

And they're the scale

of multiples of vectors b and c.

64

00:03:06,067 --> 00:03:07,833

So let's explore that again here and now,

65

00:03:07,900 --> 00:03:11,967

I've created two parameters, lambda and mu

66

00:03:12,400 --> 00:03:15,867

and multiplied b and c

by those parameters.

67

00:03:16,200 --> 00:03:18,833

Now I'm gonna take μc

68

00:03:19,033 --> 00:03:22,667

and put it at the head of b

so that I can create a point

69

00:03:22,867 --> 00:03:25,567

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that some position vector

r points to, of course,

70

00:03:25,633 --> 00:03:29,100

because it's a + λb+ μc.

71

00:03:29,233 --> 00:03:31,800

So I'm gonna leave those three vectors

72

00:03:31,867 --> 00:03:33,867

that you see right now,

and we're gonna see

73

00:03:33,933 --> 00:03:36,733

how indeed we can get

to any point on the plane

74

00:03:37,000 --> 00:03:42,300

by manipulating that is changing

the values of lambda and mu.

75

00:03:42,500 --> 00:03:44,100

Now here, change lambda,

76

00:03:44,167 --> 00:03:46,667

and you can always see,

of course I'm going in a straight line

77

00:03:46,733 --> 00:03:50,000

in the direction of b

and here I do it with mu.

78

00:03:50,233 --> 00:03:51,867

So let's trace our point

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79

00:03:51,933 --> 00:03:54,367

and then convince ourselves

that we can indeed get anywhere.

80

00:03:54,433 --> 00:03:57,533

So of course I can go up

and down and left and right,

81

00:03:58,033 --> 00:04:02,800

but since I can change

both lambda and mu arbitrarily,

82

00:04:03,067 --> 00:04:05,700

I can get anywhere along this plane

83

00:04:05,900 --> 00:04:10,800

with a grid system made out of $\mu \times c$

84

00:04:10,867 --> 00:04:12,700

plus $\lambda \times b$.

85

00:04:13,000 --> 00:04:15,367

Alright, so I hope

I've convinced you with that.

86

00:04:15,667 --> 00:04:20,033

So this $r = $ a vector

of $a + \lambda b + \mu c$,

87

00:04:20,367 --> 00:04:22,667

where a is the position vector,

88

00:04:23,100 --> 00:04:26,633

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which gets you onto a point in a plane.

89

00:04:26,700 --> 00:04:28,733

That is where you're

gonna create your plane.

90

00:04:29,433 --> 00:04:32,033

b and c are the vectors that

91

00:04:32,433 --> 00:04:35,333

determine the orientation of the plane.

92

00:04:35,600 --> 00:04:40,200

And then r is any point

on that actual plane,

93

00:04:40,700 --> 00:04:45,133

which you get to by choosing

particular values of lambda

94

00:04:45,500 --> 00:04:49,933

and mu to two parameters

in the vector equation of a plane.

In summary, the vector equation of a plane is given by

$$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors, and λ and μ are parameters, i.e. $\lambda, \mu \in \mathbb{R}$. Below is a 3D applet with similar functionality to that of the above video, i.e. by changing λ and μ arbitrarily, you can see that it is possible to get anywhere in the plane.

Student view

Interactive 2. Vector Equation of a Plane Visualization.

More information for interactive 2

This interactive visualization helps users explore the vector equation of a plane in three-dimensional space through hands-on manipulation. It demonstrates how a plane can be defined using a position vector and two non-parallel direction vectors, with adjustable parameters that dynamically affect the plane's orientation and extent.

The screen is divided into two sections. On the right, a 3D coordinate system is displayed with labeled x, y, and z axes. A blue plane is shown in space, defined by the vector equation $r = a + \lambda b + \mu c$, where a is the fixed position vector (marked in red), b and c are two non-parallel direction vectors (shown in green), λ and μ are scalar parameters and r (in dashed red) represents a vector that varies based on the chosen values of λ and μ, pointing to a specific point on the plane. On the left, there are two horizontal sliders that allow users to adjust the values of λ and μ (ranging from -5 to 5). Changing these values alters the linear combination of b and c, and therefore the position of r on the plane.

As users move the sliders:

- Increasing λ (with μ held constant) extends the plane further in the direction of vector b.
- Negative values of μ pull the plane in the opposite direction of vector c.
- The combined adjustment of both sliders allows exploration of the entire span of the plane.

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The visualization also supports rotation, allowing users to inspect the plane from multiple angles. This 3D interactivity highlights how the plane is situated in space and how the vectors interact to form its structure. This interactive experience gives users a visual and conceptual understanding of how planes behave in vector space, making it a powerful learning tool for vector geometry and linear algebra.

Exam tip

In the IB formula booklet, the vector equation of a plane will be given as

$$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$$

where $\boldsymbol{b}$ and $\boldsymbol{c}$ are two non-parallel vectors and $\boldsymbol{a}$ is a position vector of a point on the plane.

Important

Note that the equation of a plane is not unique because you could use any point or any two non-parallel vectors to write the equation of the plane. Remember that two non-parallel vectors define a plane.

Example 1

★☆☆

Consider the three points $\mathrm{A}(-1, 2, 0)$, $\mathrm{B}(3, 1, 1)$ and $\mathrm{C}(1, 0, 3)$ and find the vector equation of the plane through them.

Without any loss of generality, take $\boldsymbol{a}$ to be the position vector of point A, i.e. $\boldsymbol{a} = -\boldsymbol{i} + 2\boldsymbol{j}$. Then take $\boldsymbol{b} = \overrightarrow{\mathrm{AB}}$ and $\boldsymbol{c} = \overrightarrow{\mathrm{AC}}$, i.e.

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$$\boldsymbol{b} = \vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$$

and

$$\boldsymbol{c} = \vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

So the vector equation of this plane becomes

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

Example 2

★★☆

Find a vector equation of a plane containing the line $\boldsymbol{r} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and the point $\mathrm{P}(2, 2, 2)$.

Steps	Explanation
$\overrightarrow{\mathrm{AP}} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$	First, find a vector in the plane.The position point P is $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ and the point A on the line vector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

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Steps	Explanation
$\boldsymbol{r} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + k \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	The direction vector of the line, $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, is parallel to the plane.

3 section questions

Question 1

★☆☆

Select which of the following is a vector equation of a plane passing through the points $\mathrm{P}(-1, 1, 1)$, $\mathrm{Q}(-2, 2, 3)$ and $\mathrm{R}(1, -1, 0)$.

1 $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$ ✓

2 $\boldsymbol{r} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

3 $\boldsymbol{r} = \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$

4 $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

Explanation

You need position vectors of a point and two non-parallel vectors.

$$\boldsymbol{p} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

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Note that, as the equation is not unique, you could choose any combination of point and vectors using P, Q and R.

$$\overrightarrow{\mathrm{QR}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$$

$$\overrightarrow{\mathrm{PR}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

Using the vector equation of a plane

$$\boldsymbol{r} = \overrightarrow{\mathrm{OP}} + \lambda\overrightarrow{\mathrm{QR}} + \mu\overrightarrow{\mathrm{PR}}$$

Therefore the correct answer is

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

Question 2

★☆☆

Select which of the following is the vector equation of a plane through the point $(-1, 3, 2)$ that is parallel to the vectors $\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$.

1 $\boldsymbol{r} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$ ✓

2 $\boldsymbol{r} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$

3 $\boldsymbol{r} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$

4 $\boldsymbol{r} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

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Explanation

This follows from the definition of the vector equation of a plane as the sum of a position vector of a point on the plane and the scalar multiples of two vectors parallel to the plane.

Question 3

Select which of the following points lies on the plane with the vector equation

$$\boldsymbol{r} = \begin{pmatrix}0\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix} + \mu\begin{pmatrix}1\\0\\1\end{pmatrix}.$$

1 $(2, -1, 4)$ ✓

2 $(-1, -1, 3)$

3 $(0, 1, 3)$

4 $(2, 4, -1)$

Explanation

If the point lies on the plane it should satisfy the equation of the plane:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix} + \mu\begin{pmatrix}1\\0\\1\end{pmatrix}$$
$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}\lambda+\mu\\-\lambda\\2+\lambda+\mu\end{pmatrix}$$

So the coordinates of the point should satisfy

$$x = \lambda + \mu$$
$$y = -\lambda$$
$$z = 2 + \lambda + \mu$$

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Test the point $(2, -1, 4)$:

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$$2 = \lambda + \mu$$
$$-1 = -\lambda$$

Solving these equations simultaneously gives

$$\lambda = 1$$
$$\mu = 1$$

Check that these fit the third equation for z:

$$4 = 2 + \lambda + \mu$$
$$2 + \lambda + \mu = 2 + 1 + 1 = 4$$

Therefore $z = 4$.

So a point on the plane is $(2, -1, 4)$.

Normal vector and Cartesian equation of a plane

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/normal-vector-and-cartesian-equation-of-a-plane-id-28339/print/) Assign

So far, you have seen how to write the equation of a plane using points on the plane and two non-parallel vectors in the plane. Now, consider the diagram below, where $\mathrm{P}(x_0, y_0, z_0)$is a fixed point on the plane and $\boldsymbol{n}$ is a vector perpendicular to the plane. From now on, $\boldsymbol{n}$ will be referred to as a normal to the plane.

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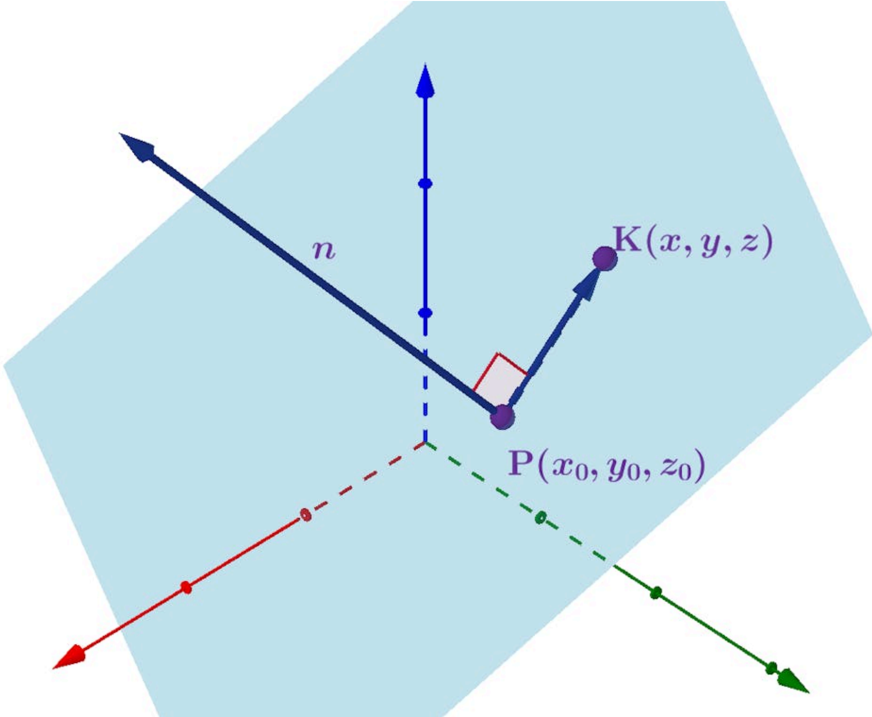

More information

The image is a 3D diagram depicting a plane in a blueish color. There is a fixed point labeled P(x0, y0, z0) on the plane and another point labeled K(x, y, z). A vector n, perpendicular to the plane, is shown extending outwards. Arrows are used to visualize the vectors. The vector from P to K is shown, and it forms a right angle with the normal vector n, indicated by a small red square. This represents the concept that the vector PK is parallel to the plane and perpendicular to the normal vector. The axes are delineated with dashed lines, where the x-axis is denoted in red, y-axis in green, and the z-axis in blue along with the perpendicular vector. The relationship between these points and vectors shows the normal vector concept in 3D geometry.

[Generated by AI]

If $\boldsymbol{p} = \overrightarrow{\mathrm{OP}}$ is a position vector to a fixed point on the plane and $\boldsymbol{r} = \overrightarrow{\mathrm{OK}}$ is a position vector to any other point, then $\boldsymbol{r} - \boldsymbol{p} = \overrightarrow{\mathrm{PK}}$ is parallel to the plane, so perpendicular to the normal vector. This means that the scalar product is 0.

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$$(\boldsymbol{r} - \boldsymbol{p}) \cdot \boldsymbol{n} = 0$$

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This equality can be rearranged.

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{p} \cdot \boldsymbol{n}$$

This form can be viewed as an equation of the plane.

Exam tip

In the IB formula booklet, the equation of a plane using the normal vector is given as

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n},$$

where $\boldsymbol{n}$ is normal to the plane and $\boldsymbol{a}$ is the position vector to a fixed point on the plane.

Let's rewrite this equation using coordinates. If $\mathrm{K}(x, y, z)$ is any arbitrary point on the plane and $\mathrm{P}(x_0, y_0, z_0)$ is a fixed point on the plane, then

$$\overrightarrow{\mathrm{PK}} = \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix}$$

Since $\boldsymbol{n}$ is a normal to the plane you can use the scalar product of $\boldsymbol{n}$ and $\overrightarrow{\mathrm{PK}}$ to write an equation for the plane.

Remember that if two vectors are perpendicular, then their scalar product is zero, so $\boldsymbol{n} \cdot \overrightarrow{\mathrm{PK}} = 0$.

If $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0$

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Multiplying the components of the two vectors and rearranging gives

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$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz = ax_0 + by_0 + cz_0$$

This is the **Cartesian** equation of the plane passing through $\mathrm{P}\left(x_0, y_0, z_0\right)$ that is perpendicular to the vector $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

If you let $d = ax_0 + by_0 + cz_0$, then the equation becomes

$$ax + by + cz = d$$

Exam tip

In the IB formula booklet, the Cartesian equation of a plane is given as

$$ax + by + cz = d$$

where vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is normal to the plane.

Example 1

★☆☆

Write a Cartesian equation of the plane that passes through $(3, -2, 1)$ and has normal vector $\boldsymbol{n} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

Student view

$\mathrm{K}\left(x, y, z\right)$ is any arbitrary point on the plane and $\mathrm{P} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

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$$\overrightarrow{\mathrm{PK}} = \begin{pmatrix} x-3 \\ y+2 \\ z-1 \end{pmatrix}$$

$\boldsymbol{n} \cdot \overrightarrow{\mathrm{PK}} = 0$ since the vector $\boldsymbol{n}$ is perpendicular to any vector in the plane.

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x-3 \\ y+2 \\ z-1 \end{pmatrix} = 0$$

Find the scalar product by multiplying the components and then rearrange.

Therefore, a Cartesian equation of the plane is $2x + y - z = 3$

Activity

How can the direction of a plane be defined? The answer is by a unique direction that describes the orientation of the plane. This is taken to be the direction perpendicular, i.e. normal, to its surface. So the direction of a plane is described by the normal vector $\boldsymbol{n}$ as shown in the applet below.

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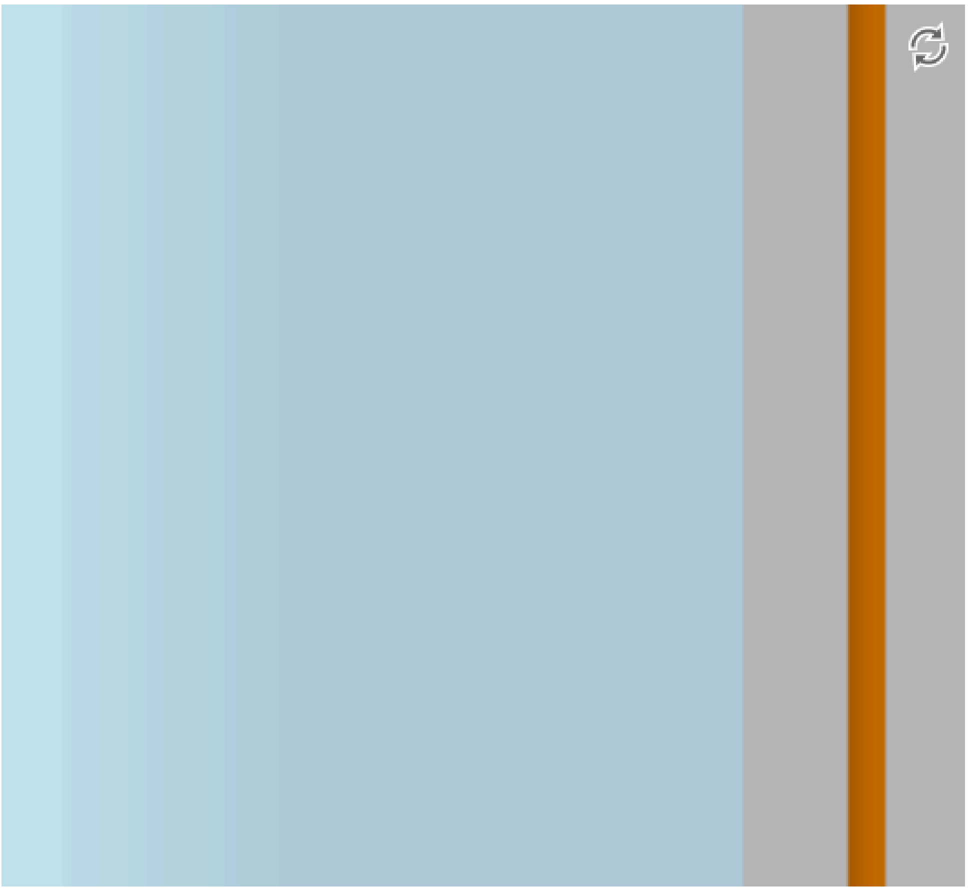

Interactive 1. Normal Vector and Plane Orientation.

More information for interactive 1

This interactive allows users to explore how a plane’s orientation in 3D space is determined by its normal vector.
The display shows a three-dimensional coordinate system with labeled x-, y-, and z-axes, a blue plane, and three movable points labeled A, B, and C. These points define the plane and can be dragged to change its shape and orientation. As the user moves these points, a purple arrow labeled "n" is dynamically updated to show the normal vector—this vector is always perpendicular to the plane and originates from point A. Vectors from A to B and A to C are also shown and labeled, along with their projections. As users rotate the 3D view, they can see how the direction of the normal vector reflects the plane’s tilt. This visual and interactive experience helps users understand that a normal vector uniquely describes a plane’s direction in space. Despite the plane being infinite, only one perpendicular vector is needed to define its orientation.
This hands-on tool enhances comprehension of the geometric concept that a plane in 3D space is determined by its normal vector and a point on the plane.

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⚠ Be aware

In the formula booklet three forms of the equation of a plane are given.

- Vector equation of a plane: $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
- Equation of a plane using normal vector: $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
- Cartesian equation of the plane: $ax + by + cz = d$

Example 2

★★☆

Write a Cartesian equation of the plane that passes through $\mathrm{A}(1, 1, 1)$, $\mathrm{B}(0, 2, 0)$ and $\mathrm{C}(-1, 2, 3)$.

Vectors AB and BC are parallel to the plane and their vector (cross) product will be perpendicular to the plane. This gives a normal vector $\boldsymbol{n}$.

$$\left.\begin{array}{l} \overrightarrow{\mathrm{AB}} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \\ \overrightarrow{\mathrm{BC}} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \end{array}\right\} \Rightarrow \overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{BC}} = \boldsymbol{n} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

Using the vector equation with normal vector $\boldsymbol{n}$.

$$\boldsymbol{r} \centerdot \boldsymbol{n} = \boldsymbol{a} \centerdot \boldsymbol{n}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \centerdot \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \centerdot \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

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Multiply the components and simplify.

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$$3x + 4y + z = 3 + 4 + 1$$
$$3x + 4y + z = 8$$

Therefore, a Cartesian equation is

$$3x + 4y + z = 8$$

Example 3

★★☆

Write a Cartesian equation of the plane with vector equation
$\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + k\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$.

Vectors parallel to the plane.

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \boldsymbol{v} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

The vector product of the two vectors will give a normal to the plane.

$$\boldsymbol{n} = \boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}$$

Using

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

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The position vector of a point on the plane is $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}$$

Multiply the components and simplify.

$$\begin{aligned} -2x - 2y + 4z &= -4 - 2 + 4 \\ -2x - 2y + 4z &= -2 \\ x + y - 2z &= 1 \end{aligned}$$

Therefore, a Cartesian equation is

$$x + y - 2z = 1$$

Shortest distance between a plane and the origin

According to the equation using the normal vector, if $\boldsymbol{a}$ is the position vector of a given point on a plane and $\boldsymbol{n}$ is a vector normal to that plane, then for any position vector $\boldsymbol{r}$ that points to a point on the plane the following is true.

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

Let's rewrite the right hand side using the definition of the scalar product.

$$\boldsymbol{r} \cdot \boldsymbol{n} = |\boldsymbol{a}||\boldsymbol{n}| \cos \theta,$$

where θ is the angle between the two vectors. Notice that in this expression $|\boldsymbol{a}| \cos \theta$ is the projection of $\boldsymbol{a}$ to $\boldsymbol{n}$. Since $\boldsymbol{n}$ is normal to the plane, this projection is the distance of the origin from the plane. If d denotes this distance, then the equation becomes

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$$\boldsymbol{r} \cdot \boldsymbol{n} = |\boldsymbol{n}|d, \text{ or } d = \boldsymbol{r} \cdot \boldsymbol{n} \times \frac{1}{|\boldsymbol{n}|}.$$

If you use the normal unit vector to the plane, $\hat{\boldsymbol{n}}$, then this equation simplifies to

$$\boldsymbol{r} \cdot \hat{\boldsymbol{n}} = d$$

This is illustrated in the diagram below .

Interactive 2. Normal Vector and Plane Equation Diagram.

More information for interactive 2

This interactive demonstrates how to calculate the shortest distance between a plane and the origin using vector mathematics. By visualizing the relationship between a plane's normal vector and position vectors, users can explore the geometric principles that determine this minimum distance. The tool brings to life the equation $r \cdot \hat{n} = d$, showing how the dot product connects algebraic formulas with spatial measurements in three-dimensional space.

The visualization displays a 3D coordinate system with a plane represented by a colored surface. Users can see several key elements: point A marking a position on the plane, the normal vector n perpendicular to the plane, the distance d shown as a perpendicular line to the origin, and a position vector r represented by a movable black line. By dragging the endpoint of vector r, users observe how its projection onto the normal vector always equals the constant distance d, regardless of where r

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view

touches the plane. The x, y, z axes provide spatial reference, and the display updates all components in real-time during interaction.

Through this interactive experience, users develop a deep understanding of several fundamental concepts in vector geometry. They learn how the dot product relates to perpendicular projections, why the normal vector determines the shortest distance to the origin, and how the unit normal vector simplifies distance calculations. The visualization makes clear why all points on the plane satisfy r · n̂ = d, reinforcing the connection between algebraic equations and geometric properties.

Example 4

★★☆

Given the Cartesian equation of a plane Π: $x - 2y + 3z = 2$, find the shortest distance between the plane and the origin.

Given the Cartesian equation, you have the normal vector to the plane $\boldsymbol{n} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$. Thus, the equation of the plane is

$$\boldsymbol{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 2$$

The length of the normal vector is $|\boldsymbol{n}| = \sqrt{14}$. Using that

$$\hat{\boldsymbol{n}} = \frac{1}{|\boldsymbol{n}|}\boldsymbol{n}$$

and

$$\boldsymbol{r} \cdot \hat{\boldsymbol{n}} = d,$$

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where d is the shortest distance between the plane and the origin, you get

$$\begin{aligned}\boldsymbol{r}\cdot\hat{\boldsymbol{n}}=\boldsymbol{r}\cdot\boldsymbol{n}\frac{1}{|\boldsymbol{n}|}&=\boldsymbol{r}\cdot\begin{pmatrix}1\\-2\\3\end{pmatrix}\times\frac{1}{\sqrt{14}}\\&=2\times\frac{1}{\sqrt{14}}\\&=\frac{2}{\sqrt{14}}\end{aligned}$$

Thus, the shortest distance between the plane and the origin is $\frac{2}{\sqrt{14}}$ units.

Example 5

★★☆

Consider the Cartesian equation of the plane Π: $x+6y+3z=14$, and the vector equation of the line L: $\boldsymbol{r}=\begin{pmatrix}2\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\-1\end{pmatrix}$. Show that the plane contains the line.

You can either show that

1. two points of the line also lie on the plane, or
2. the plane and the line have one point in common and the line is parallel to the plane.

For 1, set $\lambda=0$ and $\lambda=1$ to obtain the points of the line $(2,0,4)$ and $(2-3,0+1,4-1)=(-1,1,3)$, respectively. Then you can see that the coordinates of the two points satisfy the equation of the plane, i.e.

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$$2+6\times 0+3\times 4=14 \text{ and}$$
$$-1+6\times 1+3\times 3=14$$

Hence, the plane contains the line.

For 2, you can see that the point of the line L $(2, 0, 4)$, that you obtain for $\lambda = 0$ is also a point of the plane Π as it satisfies the equation since $2+6\times 0+3\times 4=14$.

Moreover, the direction vector $\begin{pmatrix}-3\\1\\-1\end{pmatrix}$ of the line L is perpendicular to the normal vector $\begin{pmatrix}1\\6\\3\end{pmatrix}$ of the plane Π as their scalar product is equal to zero since

$$\begin{pmatrix}1\\6\\3\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\-1\end{pmatrix}=1\times(-3)+6\times 1+3\times(-1)=0$$

Thus, the line is parallel to the plane.

Hence, the plane contains the line.

2 section questions

Question 1

★★☆

Select which of the following is the Cartesian equation of a plane containing the points $(0, 1, -2)$, $(3, -1, 2)$ and $(-1, -2, 3)$.

1 $2x - 19y - 11z = 3$ ✓

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2 $2x + 19y + 11z = 3$

3 $-2x + 19y - 11z = 3$

4 $-2x - 19y + 11z = 3$

Explanation

$$\left.\begin{aligned}\overrightarrow{AB} &= \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \\ \overrightarrow{BC} &= \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix}\end{aligned}\right\} \Rightarrow \overrightarrow{AB} \times \overrightarrow{BC} = \boldsymbol{n} = \begin{pmatrix} 2 \\ -19 \\ -11 \end{pmatrix}$$

Using the vector equation with normal vector

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -19 \\ -11 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -19 \\ -11 \end{pmatrix}$$

$$2x - 19y - 11z = -19 + 22$$
$$2x - 19y - 11z = 3$$

Therefore, a Cartesian equation is

$$2x - 19y - 11z = 3$$

Question 2

★☆☆

Select which of the following is a Cartesian equation of the plane given by $\boldsymbol{r} \cdot \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = 3.$

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1 $3x - 2y - z = -3$ ✓

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2 $3x - 2y - z = 3$

3 $x + y + z = -3$

4 $3x - 2y - z = 6$

Explanation

$$3 = \boldsymbol{r} \cdot \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = -3x + 2y + z \quad \Leftrightarrow \quad 3x - 2y - z = -3$$

3. Geometry and trigonometry / 3.17 Vector equations of a plane

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-28340/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- find a vector equation of a plane: $r = a + \lambda b + \mu c$
- find a Cartesian equation of a plane: $ax + by + cz = d$, where $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a normal vector of the plane, and $d = \boldsymbol{p} \cdot \boldsymbol{n}$, where $\boldsymbol{p}$ is a position vector to a point on the plane.
- find the shortest distance from the origin to a plane using $\boldsymbol{r} \cdot \widehat{n} = d$, where $\widehat{n}$ is the unit normal vector to the plane.

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3. Geometry and trigonometry / 3.17 Vector equations of a plane

Investigation

In this subtopic, you have used three forms of the equation of a plane: the vector equation, the Cartesian equation and the normal vector equation. What is the relationship between these different ways of representing a plane?

Can you show how each of these forms can be transformed into the other two?

Start with the general form of the vector equation $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$.

Show that this can be transformed into the normal vector form $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{r} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = d$, and then reverse the process and transform the normal vector form back to the Cartesian form.

Similarly, show that the vector equation $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$ can be transformed to the Cartesian form $ax + by + cz = d$ and that the Cartesian equation can be transformed to the vector equation.

Finally, show that the normal vector form $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{r} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = d$ can be transformed to the vector equation $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$ and that the vector equation can be transformed to the normal vector form.

There is another form of the equation of a plane that can be obtained from the Cartesian equation: this is the intercept form.

Let the coordinates of the points on the x-, y- and z-axes that are cut by the plane be $(A, 0, 0)$, $(0, B, 0)$ and $(0, 0, C)$, respectively.

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Show that the equation of the plane in intercept form can be written as $\frac{x}{A} + \frac{y}{B} + \frac{z}{C} = 1$.

Use this equation to find the axes intercepts of the plane with Cartesian equation $3x + 2y - z = 7$.

Rate subtopic 3.17 Vector equations of a plane

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