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The big picture

Complex numbers can be written in a variety of forms. These include the Cartesian, $z = a + b\text{i}$, form that you studied in the previous subtopic, as well as polar and Euler forms that you will study now.

Defining complex numbers in Euler and polar forms leads to Euler's identity which is the famous equation $\text{e}^{\text{i}\pi} + 1 = 0$.

Many mathematicians call this equation the most beautiful in all of mathematics.

What do you think makes this equation beautiful? Do you find it beautiful?

International Mindedness

The idea of beauty and what is considered beautiful differs for different cultures. Think about what is considered beautiful in your culture and do some research to find out what is considered beautiful in one other culture. How well does the beauty found in Euler's identity correspond to the ideas of beauty in these two cultures?

Watch this video to see one mathematician explain his thinking about this equation.

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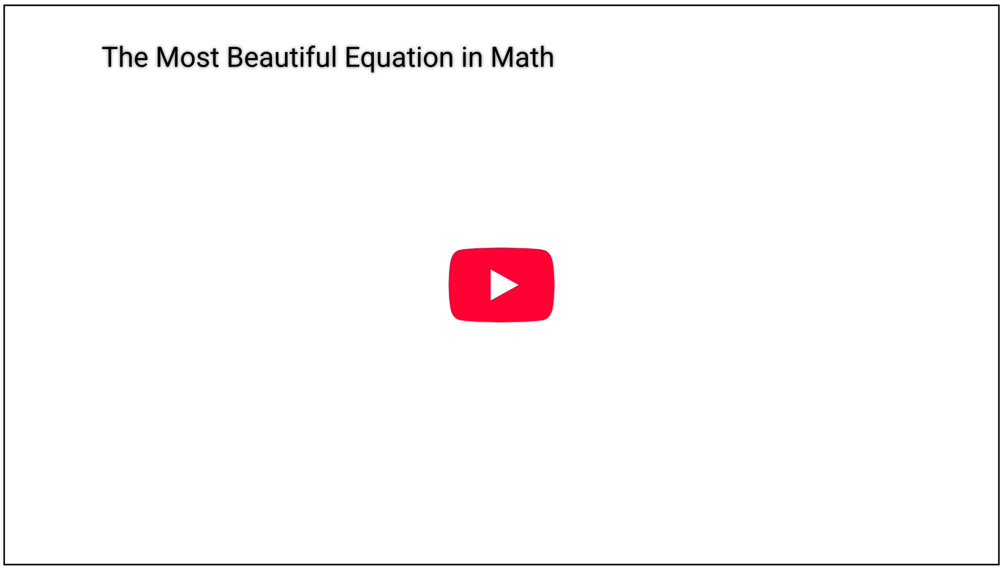

Concept

Complex numbers span a number of mathematical fields. The representationof complex numbers and understanding their properties involves the use of trigonometry, algebra, and geometry. How are connections between these fields made in the case of complex numbers? What do these connections tell you about the nature of complex numbers and the study of mathematics?

Theory of Knowledge

If imaginary numbers don't actually ‘exist’ in the real world, and complex numbers are formed using imaginary numbers, can complex numbers be said to exist?

On the flip side, if complex numbers do indeed exist in the real world (as demonstrated by their ability to be expressed as a Cartesian number), do they, by proxy, prove the existence of imaginary numbers?

This calls into question reason, the accuracy of deduction and role of perception in existence. A knowledge question that emerges is: ‘Is the application of deductive logic sufficient to establish truth?’

Theory of Knowledge

Euler's number relies on the concept of ‘imaginary’ numbers. As you know, these numbers are only ‘ real ’ in the sense that they work within a mathematical world.
Is that all it takes for something to be ‘ real ’ ? That it serves a function in some given context? Given this definition, your citadels in Fortnite are real; as is Thanos' Infinity Gauntlet for both serve a function in a context in order to make something ‘ fit ’.

Analyse the following knowledge question keeping in mind imaginary numbers: ‘ Must something exist in nature in order to be real? ’

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1. Number and algebra / 1.13 Complex numbers in polar and Euler forms

Polar and Euler forms of complex numbers

Polar form

In the previous subtopic, you learned how to find the modulus and argument of a complex number. Use what you know about the argument and modulus to complete the following activity.

Activity

You are given the modulus and argument for the following complex numbers. Write each number in Cartesian form.

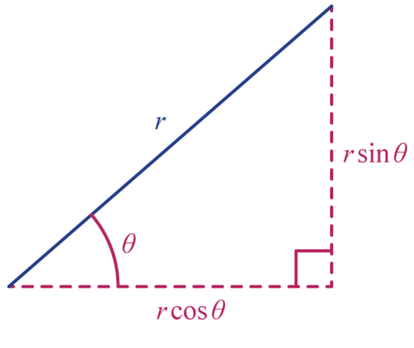
$$\arg(z) = 27^\circ,\ |z| = 3$$

$$\arg(w) = 65^\circ, |w| = 10$$

$$\arg(a) = 127^\circ, |a| = 5$$

Hence, write an expression for a and b in $z = a + b\mathrm{i}$ in terms of $\arg(z) = \theta$ and $|z| = r$.

You can use a right-angled triangle to express a and b in $z = a + b\mathrm{i}$ in terms of the argument (θ) and modulus (r) of the complex number. The diagram below shows this relationship.



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More information

The image is a diagram of a right-angled triangle used to represent a complex number in terms of its modulus and argument. The triangle has a horizontal side labeled 'r cos θ', a vertical side labeled 'r sin θ', and a hypotenuse labeled 'r'. The angle at the base of the triangle, adjacent to 'r cos θ', is marked as 'θ'. This diagram illustrates how the rectangular coordinates (r cos θ, r sin θ) relate to the polar coordinates in the complex plane.

[Generated by AI]

Since $a = r\cos\theta$ and $b = r\sin\theta$, a complex number $z = a + b\mathrm{i}$ can be written as $z = r\cos\theta + \mathrm{i}r\sin\theta$. This is called the polar form of a complex number and is also referred to as the modulus–argument form . It is called the polar form because the values of a and b give the polar coordinates of a point on a radial grid. However, you will not need to work with polar coordinate grids in this course.

Important

The polar (modulus—argument) form of a complex number is $z = r(\cos\theta + \mathrm{i}\sin\theta)$ where r is the modulus and θ is the argument of the complex number.

Be aware

You will often see $z = r\text{cis }\theta$ which is shorthand for $z = r(\cos\theta + i\sin\theta)$ Both of these forms of writing a complex numbers are given in the IB formula booklet.

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26989/print/) Assign

Why are the letters cis used in $z = r\text{cis }\theta$ notation?

Making connections

Generally, the argument of a complex number in polar form is given in radians rather than degrees unless specified by the question. If you are unfamiliar with radians you can learn about them by visiting subtopic 3.4 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27735/) .

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Example 1

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Write $z = 3 - 2\text{i}$ in polar form.

Steps	Explanation
$\|z\| = \sqrt{(3)^2 + (-2)^2} = \sqrt{13}$	Leave as an exact answer.
$\tan\theta = -\frac{2}{3} \quad \Leftrightarrow \quad \tan^{-1}\left(-\frac{2}{3}\right) = \theta$ $\Leftrightarrow \quad \theta = -0.588$	Use your calculator for the inverse of the tangent. Set the angle unit to radians. Round your answer to 3 significant figures. Angles measured in radians are usually not denoted with a unit symbol, but they can be denoted by rad or $^{\text{c}}$.
$z = \sqrt{13}(\cos(-0.588) + i\sin(-0.588))$ $= \sqrt{13}\,\text{cis}\ (-0.588)$	Either notation is acceptable for the answer.

Be aware

When you are using your calculator to evaluate trigonometric ratios and find the inverses make sure that the mode of the calculator (radians or degrees) matches the units in which the angle is measured in the question.

You can use the modulus and argument to write a complex number in Euler form which is also referred to as exponential form.

Important

If $\arg(z) = \theta$ and $|z| = r$, then the complex number z written in Euler (exponential) form is $z = r\text{e}^{\text{i}\theta}$.

There are a number of ways to derive Euler’s formula, $\text{e}^{\text{i}x} = \cos x + \text{i}\sin x$, θ and therefore show that a complex number can be written in Euler form but they require you to know Calculus, which you will study in topic 5 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25542/). One such

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derivation is given in the video in section 1.13.0 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26989/). You may want to come back to this section when you have finished the Calculus topic.

Activity

Consider these questions:

- How does Euler's formula show that $z = re^{i\theta}$ if $z = r(\cos\theta + i\sin\theta)$?
- How is Euler's formula related to Euler's identity which is $e^{i\pi} + 1 = 0$?

Example 2

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Write $w = -5 + i$ in Euler form.

Steps	Explanation
$\lvert w\rvert = \sqrt{(-5)^2 + (1)^2} = \sqrt{26}$	Leave as an exact answer.
$\tan\theta = -\frac{1}{5} \Leftrightarrow \tan^{-1}\left(-\frac{1}{5}\right) = \theta \Leftrightarrow \theta = -0.197$	Round to 3 significant figures.

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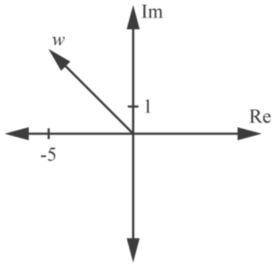
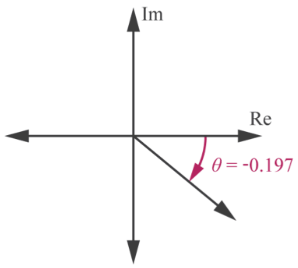
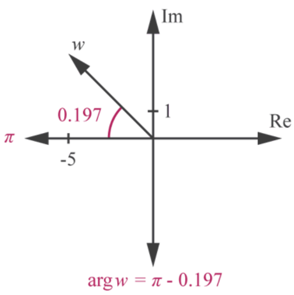
Steps	Explanation
Argand diagram for w: 	The calculator can only give you values for θ in the first and fourth quadrants. You should always make a sketch of a complex number on the Argand plane when finding the argument. You can see from the diagrams for w and $\theta=-0.197$ that arg $(w) \neq -0.197$. You should use the reference angle of 0.197 in the second quadrant to find $\arg(w)$.
Diagram for $\theta=-0.197$: 	
Finding $\arg(w)$: 	
$\arg(w)=\pi-0.197=2.94$	

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Steps	Explanation
$-5+\mathrm{i}=\sqrt{26}\mathrm{e}^{2.94\mathrm{i}}$	

⚠ **Be aware**

The calculator gives you values for tangent inverses in the first and fourth quadrants. You should always sketch an Argand diagram for your complex number when finding its argument. You can review this skill by visiting subtopic 1.12.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/graphing-in-the-complex-plane-id-26986/) .

Example 3

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Write $z=2\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}$ in Cartesian form.

Steps	Explanation
$z=2\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}$ $=2\left(\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}\right)$ $=2\left(0+\mathrm{i}\times(1)\right)=2\mathrm{i}$	Rewrite in polar form and evaluate sine and cosine.

The table below shows more examples of converting between Cartesian, polar and Euler forms.

Cartesian $z=x+iy$	$r=\|z\|$	$\theta=\operatorname{Arg}(z)$	Polar form $z=r\operatorname{cis}\theta$	Euler form $z=r\,\mathrm{e}^{\mathrm{i}\theta}$
$2+\mathrm{i}3$	$\sqrt{13}$	$0.983\,\mathrm{rad}$	$\sqrt{13}\operatorname{cis}(0.983)$	$\sqrt{13}\,\mathrm{e}^{\mathrm{i}0.983}$
$3+2\mathrm{i}$	$\sqrt{13}$	$0.588\,\mathrm{rad}$	$\sqrt{13}\operatorname{cis}(0.588)$	$\sqrt{13}\,\mathrm{e}^{\mathrm{i}0.588}$
$-2+\mathrm{i}$	$\sqrt{5}$	$2.68\,\mathrm{rad}$	$\sqrt{5}\operatorname{cis}(2.68)$	$\sqrt{5}\,\mathrm{e}^{\mathrm{i}2.68}$
$1-\mathrm{i}$	$\sqrt{2}$	$-\frac{\pi}{4}$	$\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$	$\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}$

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Cartesian $z = x + iy$	$r = \|z\|$	$\theta = \mathrm{Arg}(z)$	Polar form $z = r\,\mathrm{cis}\,\theta$	Euler form $z = r\,\mathrm{e}^{i\theta}$
$-\frac{3}{2} - \frac{3}{2}\mathrm{i}$	$\frac{3}{\sqrt{2}}$	$-\frac{3\pi}{4}$	$\frac{3}{\sqrt{2}}\mathrm{cis}\left(-\frac{3\pi}{4}\right)$	$\frac{3}{\sqrt{2}}\,\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}$

You can use your calculator to convert complex numbers from Euler to Cartesian form and from Cartesian to Euler form. If you need to practise more conversion questions, you can create your own examples and check your work using the calculator.

Steps	Explanation
To convert a complex number from Cartesian form to polar form, choose the options (OPTN) and press F3 to see the options related to complex numbers ...	
... press F6 to scroll to see more options ...	

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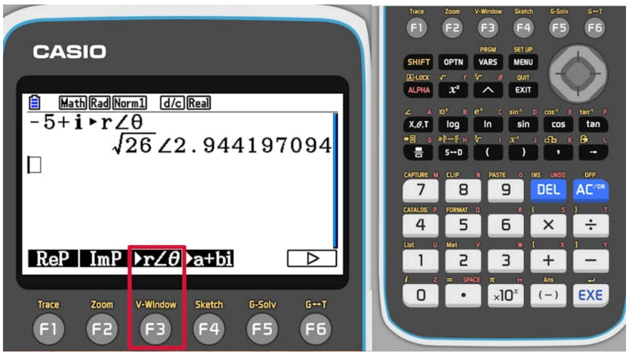
... and press F3 for the conversion.

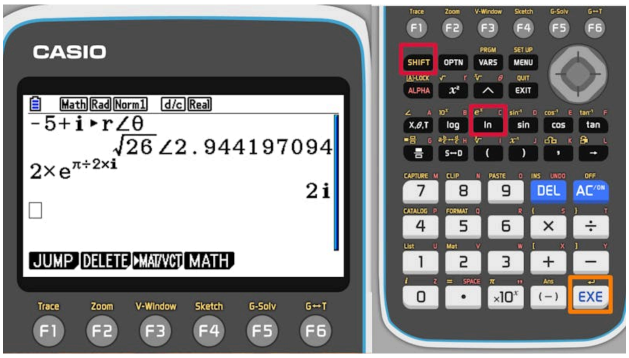
Note, that the calculator shows the modulus and the argument separated by an angle symbol ∠.

If you want to convert from polar form to Cartesian form, you need to press F4 at this point.



The calculator also understands the Euler form of a complex number.

Enter the expression and press EXE. The Cartesian form will be displayed.



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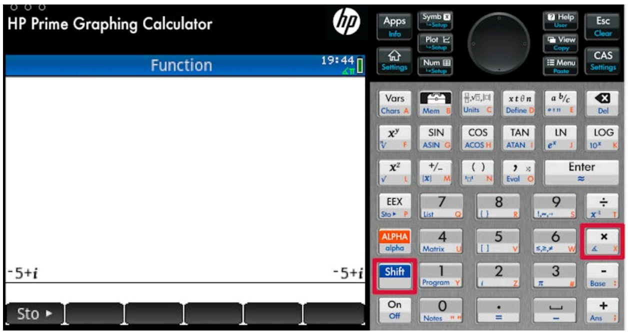
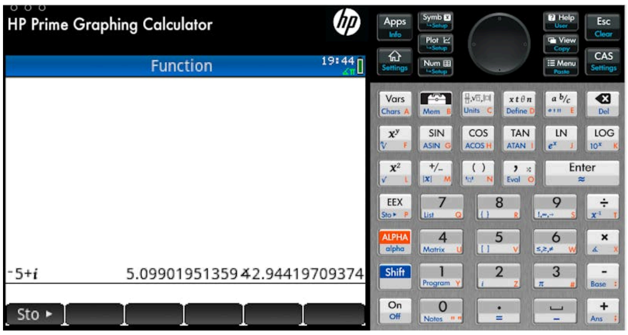
Steps	Explanation
To convert a complex number between Cartesian and polar form, press the angle symbol (Shift ∡). This will do the conversion both ways.	
Note, that the calculator shows the modulus and the argument separated by an angle symbol ∡.	

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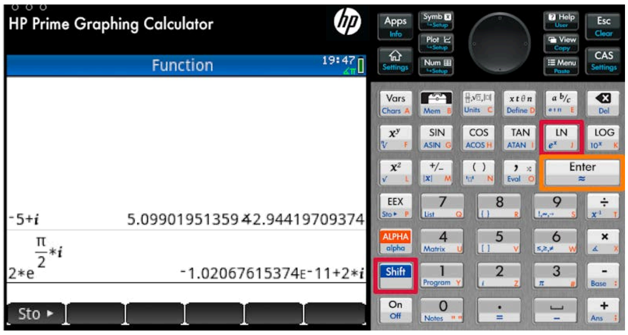
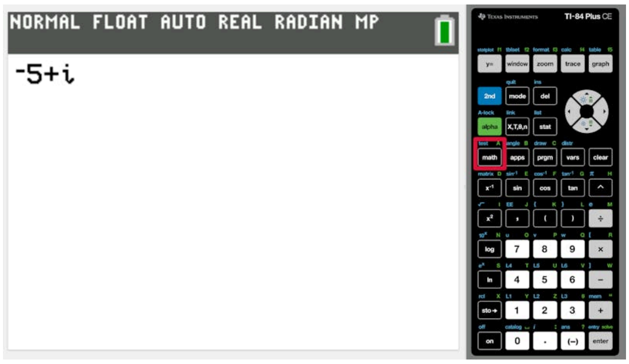
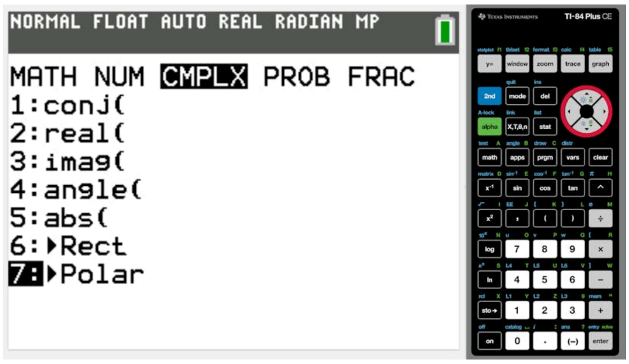
Steps	Explanation
The calculator also understands the Euler form of a complex number. Enter the expression and press enter. The Cartesian form will be displayed. Note, that this conversion is not exact. In this case the real part is exactly 0, but instead of the exact value the calculator gives a small number.	

Steps	Explanation
To convert a complex number from Cartesian form to polar form, choose the math options ...	

Steps	Explanation
... and find the option to convert to polar form. Note that if you want to convert from polar to Cartesian form, you need to choose option 6 here.	
You will get the modulus and the argument written using the Euler notation.	

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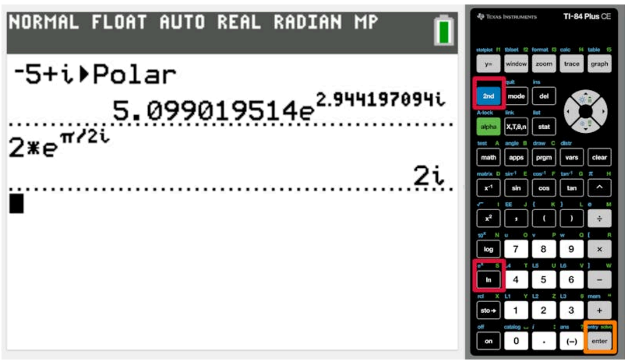
Steps	Explanation
If you enter the expression in Euler form and press enter, the Cartesian form will be displayed.	

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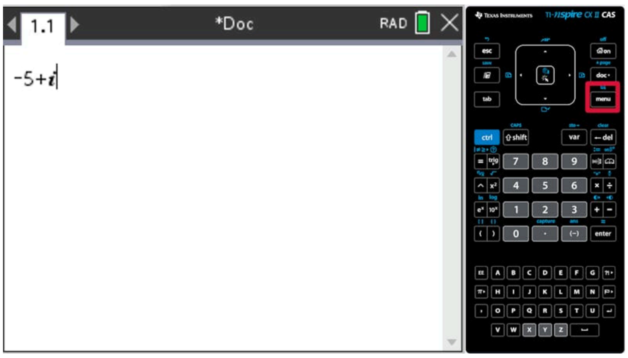
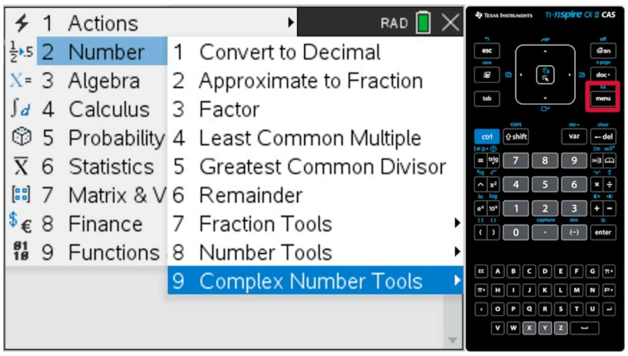
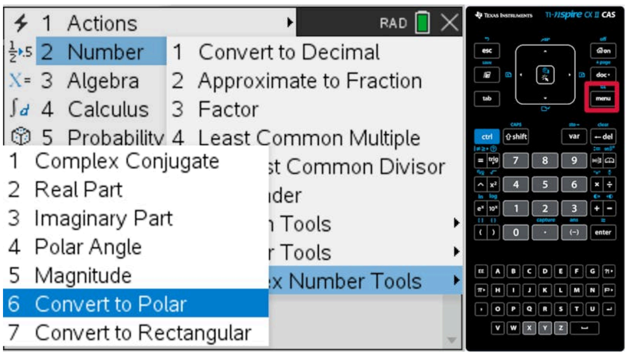
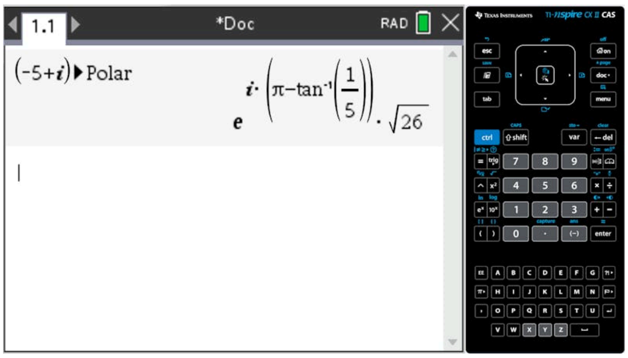
Steps	Explanation
To convert a complex number from Cartesian form to polar form, open the menu ...	
... navigate to the complex number tools, ...	

... and choose the option to convert to polar.

Note, that if you need to convert from polar form to Cartesian form, you need to choose option 7 here.

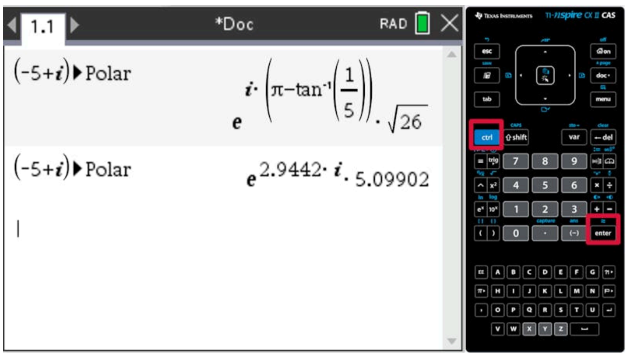


If you press enter, you get an exact expression. This may be useful in some ocasions, but more often it is not very helpful.



To get the approximate value, you need to press ctrl/ ≈.

You will get the modulus and the argument written using the Euler notation.



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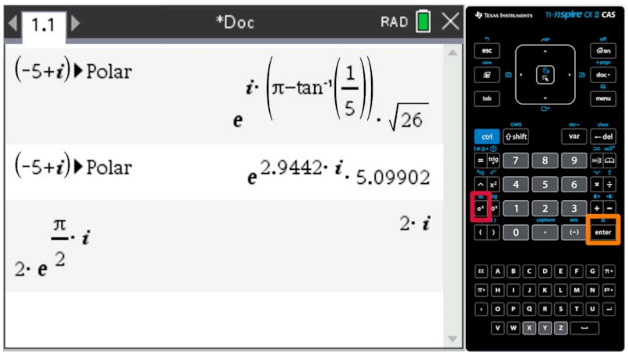
If you enter the expression in Euler form and press enter, the Cartesian form will be displayed.



Exam tip

In topic 3 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25407/) you will learn how to find exact ratios and inverses for a set of special angles using a table such as the one shown below.

On an exam you will be expected to apply this skill to your work with complex numbers in polar and Euler forms.

Most exam questions will require an exact value for the argument of the complex number.

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Degrees	0	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined

4 section questions

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Multiplication and division

Multiplication and division in Euler and polar forms

Working with complex numbers in Euler and polar forms simplifies calculations for multiplication and division.

Multiplication

Activity

Copy and complete the table.

z_1	z_2	$z_1 \times z_2$	Polar form of z_1	Polar form of z_2	Polar form of $z_1 \times z_2$
$2-\mathrm{i}$	$3+2\mathrm{i}$	$8+\mathrm{i}$			
$-3-4\mathrm{i}$	$7-\mathrm{i}$				
$1+\mathrm{i}$	$1-2\mathrm{i}$				

Comment on any patterns that you notice.

Important

Given that $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$, then

$$z_1 \times z_2 = r_1 \times r_2(\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2)).$$

And given $z_1 = r_1\mathrm{e}^{\mathrm{i}\theta_1}$ and $z_2 = r_2\mathrm{e}^{\mathrm{i}\theta_2}$, then

$$z_1 \times z_2 = r_1 \times r_2\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}.$$

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Activity

Show that $|z_1 \times z_2| = |z_1| \times |z_2|$ and that $\arg(z_1 \times z_2) = \arg z_1 + \arg z_2$.

Making connections

The result for multiplication of two complex numbers in polar form can be derived using the compound angle identities that you will study in Topic 3 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25407/). This is done as follows:

Compound angle identities are:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Multiplication in polar form:

Steps	Explanation
$z_1 \times z_2 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) \times r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$ $= r_1 \times r_2(\cos\theta_1\cos\theta_2 + \mathrm{i}\cos\theta_1\sin\theta_2 + \mathrm{i}\sin\theta_1\cos\theta_2 + \mathrm{i}^2\sin\theta_1\sin\theta_2)$	
$= r_1 \times r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + \mathrm{i}(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2))$	Group real terms together. Group imaginary terms together.
$\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \cos(\theta_1 + \theta_2)$ $\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 = \sin(\theta_1 + \theta_2)$	Using compound angle identities.
$z_1 \times z_2 = r_1 \times r_2(\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2))$	

Example 1

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Consider the complex numbers $z_1 = -1 + \mathrm{i}$ and $z_2 = 2\sqrt{3} - 2\mathrm{i}$. Find the product of z_1 and z_2 giving your answer in polar form.

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Steps	Explanation
$\|z_1\| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$ $\theta = \tan^{-1}\left(\frac{1}{-1}\right) = -\frac{\pi}{4} \approx -0.785$ Since z_1 is in the second quadrant, $\arg(z_1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \approx 2.36.$ $z_1 = \sqrt{2}\text{ cis }\left(\frac{3\pi}{4}\right)$	Converting z_1 to polar form.
$\|z_2\| = \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2} = 4$ $\theta = \tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right) = -\frac{\pi}{6} \approx -0.524$ Since z_2 is in the fourth quadrant, $\arg(z_2) = -\frac{\pi}{6} \approx -0.524.$ $z_2 = 4\text{ cis }\left(-\frac{\pi}{6}\right)$	Converting z_2 to polar form.
$z_1 \times z_2 = 4\sqrt{2}\text{cis }\left(\frac{3\pi}{4} + \left(-\frac{\pi}{6}\right)\right)$ $= 4\sqrt{2}\text{ cis }\left(\frac{7\pi}{12}\right)$	Multiplication. Remember to add the arguments.
$z_1 \times z_2 = 4\sqrt{2}\text{cis }(2.36 + (-0.524))$ $= 4\sqrt{2}\text{ cis }(1.84)$	The answer if you are doing this question without using the exact values.

Division

Rules for division of complex numbers in Euler and polar forms can be derived using the same methods as multiplication.

Important

Given $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)).$$

And given $z_1 = r_1\mathrm{e}^{\mathrm{i}\theta_1}$ and $z_2 = r_2\mathrm{e}^{\mathrm{i}\theta_2}$, then

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$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

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Activity

Show that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and that $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

Example 2

★☆☆

Given that $z = 3e^{2.17i}$ and $w = 4 \text{ cis } (-0.172)$ write $\frac{w}{z}$ in polar form.

Steps	Explanation
$\frac{w}{z} = \frac{4}{3} \text{ cis } (-0.172 - 2.17) = \frac{4}{3} \text{ cis } (-2.34)$	Divide the moduli, but subtract the arguments.

Example 3

★★★

Consider the complex numbers $z_1 = 1 - i$ and $z_2 = \sqrt{3} + i$.

a) Convert to polar form and find $\frac{z_1}{z_2}$.

b) Use your answer to part **a.** to find the *exact* values of $\cos\left(-\frac{5\pi}{12}\right)$ and $\sin\left(-\frac{5\pi}{12}\right)$.

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	Steps	Explanation
a)	$\|z_1\| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$ $\theta = \tan^{-1}\left(-\frac{1}{1}\right) = -\frac{\pi}{4}$ Since z_1 is in quadrant 4 $\arg(z_1) = -\frac{\pi}{4}$. $z_1 = \sqrt{2}\text{ cis }\left(-\frac{\pi}{4}\right)$	For this question you should use exact values of the inverse trig ratios because the second part of the question asks you for values of $-\frac{5\pi}{12}$. You may need to refer to the table of exact values given in **subtopic 1.13.1.**
	$\|z_2\| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2$ $\theta = \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$ Since z_2 is in the first quadrant $\arg(z_2) = \frac{\pi}{6}$. $z_2 = 2\text{ cis }\left(\frac{\pi}{6}\right)$	
	$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\text{ cis }\left(-\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}\text{ cis }\left(-\frac{5\pi}{12}\right)$	
b)	$\frac{z_1}{z_2} = \frac{1-\mathrm{i}}{\sqrt{3}+\mathrm{i}} = \left(\frac{1-\mathrm{i}}{\sqrt{3}+\mathrm{i}}\right) \times \left(\frac{\sqrt{3}-\mathrm{i}}{\sqrt{3}-\mathrm{i}}\right)$ $= \frac{\sqrt{3} - \mathrm{i} - \sqrt{3}\mathrm{i} + \mathrm{i}^2}{3+1} = \frac{\sqrt{3}-1}{4} + \frac{-\sqrt{3}-1}{4}\mathrm{i}$	Divide in Cartesian form.
	$\mathrm{Re}\left(\frac{z_1}{z_2}\right) = \frac{\sqrt{2}}{2}\cos\left(-\frac{5\pi}{12}\right) = \frac{\sqrt{3}-1}{4} \quad \Leftrightarrow$ $\cos\left(-\frac{5\pi}{12}\right) = \frac{\sqrt{3}-1}{4} \times \frac{2}{\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$ $\mathrm{Im}\left(\frac{z_1}{z_2}\right) = \frac{\sqrt{2}}{2}\sin\left(-\frac{5\pi}{12}\right) = \frac{-\sqrt{3}-1}{4} \quad \Leftrightarrow$ $\sin\left(-\frac{5\pi}{12}\right) = \frac{-\sqrt{3}-1}{4} \times \frac{2}{\sqrt{2}} = \frac{-\sqrt{6}-\sqrt{2}}{4}$	Equate real and imaginary parts in polar and Cartesian forms to find exact values.

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⚠ Be aware

The argument in your final answer should be given as the principle argument .

6 section questions

Geometric interpretations

Conjugates and negatives

Activity

Let $z = 2 + \mathrm{i}$. Sketch $z,\ -z,\ z^*,\ \text{and} - z^*$ on an Argand plane. Comment on any patterns that you notice.

Write $z,\ -z,\ z^*,\ \text{and} - z^*$ in polar form and closely examine the arguments of these numbers.

Given that $z = r\ \text{cis}\ \theta$, write an expression for $-z,\ z^*,\ \text{and} - z^*$ in terms of $r,\ \theta$ and π .

Important

If $z = r\ \text{cis}\ \theta$, then

- $-z = r\ \text{cis}\ (\theta + \pi)$
- $z^* = r\ \text{cis}\ (2\pi - \theta)$
- $-z^* = r\ \text{cis}\ (\pi - \theta)$

Why can't you write $-z = -r\ \text{cis}\ \theta$ instead of $-z = r\ \text{cis}\ (\theta + \pi)$?

Exam tip

The IB formula booklet does not give you any information about the geometric relationships for complex numbers. However, you do not need to memorise them so long as you can derive them using Argand diagrams.

Addition and subtraction

Activity

In the first applet below, add z_1 and z_2 algebraically and compare your result with the diagram shown. Move the purple points to change z_1 and z_2 and add them again.

Describe the geometrical properties of the sum of z_1 and z_2 .

In the second applet below, find $z_1 - z_2$ algebraically and compare your result with the diagram shown. Move the purple points to change z_1 and z_2 and repeat the calculations. Describe the geometrical properties of $-z_2$ and $z_1 - z_2$.

Interactive 1. Geometric Interpretation of Complex Number Addition.

Credit: GeoGebra (https://www.geogebra.org/m/FEqW6uZs) GeoGebra Materials Team

More information for interactive 1

The interactive applet allows users to explore the addition of complex numbers both visually and algebraically. It features an Argand plane with an x-axis ranging from -10 to 12 and a y-axis from -8 to 8. Two movable points, $Z1$ and $Z2$, are shown on the plane with vector arrows originating from the origin $(0, 0)$, representing the complex numbers. As users drag the purple points to change the values of $Z1$ and $Z2$, the applet dynamically updates to show the sum $Z1 + Z2$ as a new point on the plane. The numeric values of $Z1$, $Z2$, and their sum are also displayed, helping users connect algebraic addition to its geometric interpretation.

For example, $Z1$ is $(1.27, 4.15)$ and $Z2$ is $(4.86, -1.18)$ then $Z1 + Z2$ will be $(6.12, 2.96)$.

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Through this visual, learners can understand that adding complex numbers corresponds to vector addition — where placing the tail of Z_2 at the head of Z_1 (or vice versa) gives a diagonal representing the sum. This reinforces the concept that complex number addition can be viewed as the addition of directed line segments in the Argand diagram.

Interactive 2. Geometric Interpretation of Complex Numbers.

Credit: GeoGebra (https://www.geogebra.org/m/VZAzYFHh) GeoGebra Materials Team

More information for interactive 2

This interactive applet allows users to explore the subtraction of complex numbers both visually and algebraically. The display features an Argand plane with the x-axis ranging from -10 to 12 and the y-axis from -8 to 8. Two movable points, Z_1 and Z_2, are shown in blue on the plane with directional vector arrows starting from the origin $(0, 0)$, representing complex numbers.

As users drag the purple points to change the values of Z_1 and Z_2, the applet dynamically calculates and displays the difference $Z_2 - Z_1$ as a green vector. Additionally, the negative of Z_1, represented as $-Z_1$, is shown as a red vector pointing in the opposite direction of Z_1. These vectors provide a visual representation of how subtraction works geometrically in the complex plane. The diagram also uses dashed lines to connect the tip of Z_1 to Z_2, illustrating the resultant vector $Z_2 - Z_1$ as the diagonal of the triangle formed by the two original vectors.

For example, when $Z_1 = (1, 3)$ and $Z_2 = (4, 2)$, then $Z_2 - Z_1 = (3, -1)$, which is shown in green. The applet highlights how subtracting complex numbers is equivalent to vector subtraction: drawing a vector from the tip Z_1 of to the tip of Z_2 yields the result. This visualization reinforces the geometric interpretation of complex number subtraction as adding the inverse vector $-Z_1$ to Z_2.

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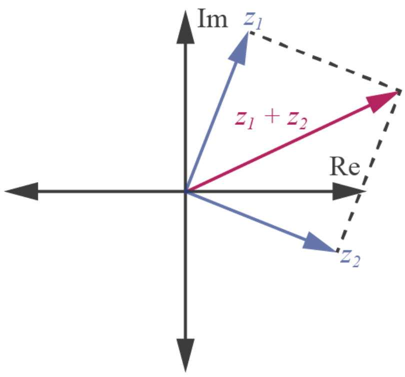
Making connections

Complex numbers can be represented as vectors on the Argand plane. The addition of complex numbers follows the parallelogram rule for addition of vectors. Subtraction can be done by adding a negative vector.

Important

The sum of two complex numbers can be represented in the Argand diagram as the diagonal of the parallelogram formed by the two complex numbers, as shown in the first graph, below.

The difference of two complex numbers can be represented in the Argand diagram as the sum of z_1 and $-z_2$, as seen in the second graph, below.



More information

This is a diagram illustrating the vector addition of two complex numbers on the complex plane. The horizontal axis is labeled 'Re' for the real part, and the vertical axis is labeled 'Im' for the imaginary part. There are three vectors shown:

1. A vector labeled 'z1', pointing upwards and right, indicating the position of the first complex number.
2. A vector labeled 'z2', pointing downwards and right, indicating the position of the second complex number.
3. A resultant vector labeled 'z1 + z2', shown as a dashed arrow originating from the start of 'z1' and pointing to the endpoint of 'z2', demonstrating the addition of the two vectors using the parallelogram rule.

The vectors 'z1' and 'z2' form a triangle with the resultant vector 'z1 + z2', illustrating a geometric interpretation of complex number addition.

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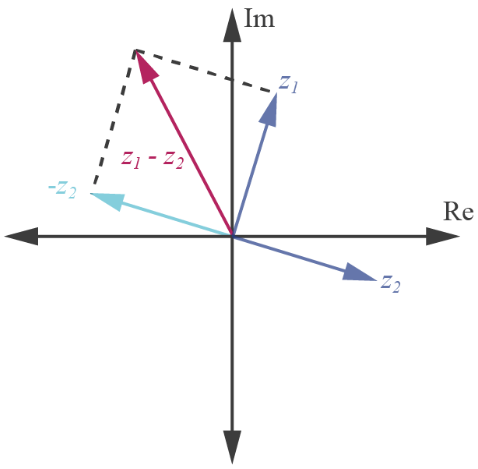

More information

The image is a diagram representing vectors on a complex plane with arrows indicating different directions.

- The horizontal axis is labeled 'Re' which represents the real part of complex numbers.
- The vertical axis is labeled 'Im' which represents the imaginary part.

There are three main vectors: 1. A vector labeled 'z_1' pointing upwards and slightly right. 2. A vector labeled 'z_2' pointing rightward. 3. Resultant vector 'z_1 - z_2', shown in color, pointing upwards and leftward from the head of vector 'z_2'.

Additionally, there is a vector '-z_2' pointing leftward, shown as a dashed line extending from the origin to the head of the 'z_2' vector in the opposite direction. The structure of the diagram highlights the subtraction of complex numbers by visually displaying the vectors and their resultant.

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Example 1

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Determine which vector, shown in the graph, represents

a) $z_1 + z_2$

b) $-z_1$

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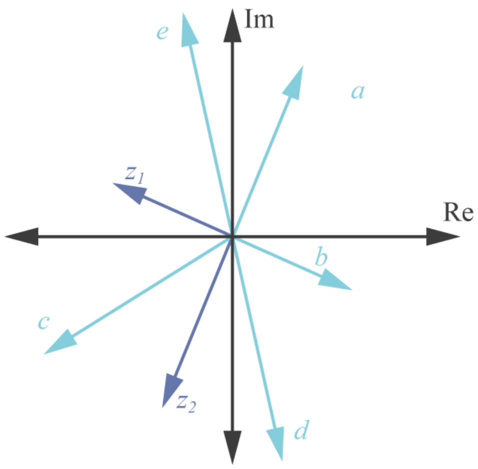
c) $z_2 - z_1$



More information

The image is a diagram of the complex plane with vectors originating from the origin. The horizontal axis is labeled 'Re' for the real part, and the vertical axis is labeled 'Im' for the imaginary part. There are several vectors: 'a', 'b', 'c', 'd', 'e', 'z1', and 'z2'. Each vector is labeled alongside its corresponding arrow. Vector 'z1' points in the direction of the negative imaginary axis, while vector 'z2' points in the direction of the positive imaginary axis. Vectors 'a' to 'e' are distributed around the plane, each pointing in different directions and showing different lengths.

[Generated by AI]

	Steps	Explanation
a)	$z_1 + z_2$ is represented by c	The sum should look like the diagonal of a parallelogram formed by z_1 and z_2 .
b)	$-z_1$ is represented by b	$-z_1$ should be the same length as z_1 and rotated $180°$ about the origin.
c)	$z_2 - z_1$ is represented by d	$z_2 - z_1$ is the sum of z_2 and $-z_1$ or z_2 and b .

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Multiplication by a real number

Activity

Let $z = 2\ \text{cis}\frac{\pi}{4}$. Sketch $2z,\ 3z,\ 5z,\ \frac{1}{2}z$ and $\frac{1}{4}z$ on an Argand diagram.

Describe the geometric significance of multiplying a complex number by a real number.

Important

If $z = r\ \text{cis}\ \theta$ and a is a positive real number, then $a \times z = a \times r\text{cis}\ \theta$.

It can be said that the modulus is stretched by a factor of a and the argument remains unchanged.

Example 2

★☆☆

Explain why it is unnecessary to describe the geometric significance of division by a real number separately from multiplication.

Division is the same as multiplication by a reciprocal which is already described under multiplication.

Multiplication and division between complex numbers

Example 3

★★☆

Given that $w = r\ \text{cis}\,\theta$ and $z = m\,\text{cis}\,\varphi$, find

a) $w \times z$

b) $\frac{w}{z}$

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Hence, describe geometrically how w is related to

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c) $w \times z$

d) $\frac{w}{z}$

a) $w \times z = (r \times m) \text{ cis } (\theta + \varphi)$

b) $\frac{w}{z} = \frac{r}{m} \text{ cis } (\theta - \varphi)$

c) When w was multiplied by z it was stretched by a factor of m and rotated anticlockwise through an angle φ.

d) When w was divided by z it was stretched by a factor of $\frac{1}{m}$ and rotated clockwise through an angle φ.

✓ **Important**

When a complex number is multiplied or divided by another complex number, the vector used to represent the complex number is stretched and rotated.

Example 4

★☆☆

Determine which of the vectors, in the graph shown, is the most appropriate representation of

a) $z \times \left(\frac{1}{2} \text{ cis} \frac{\pi}{4}\right)$

b) $\frac{z}{\frac{2}{3} \text{ cis } \frac{\pi}{2}}$

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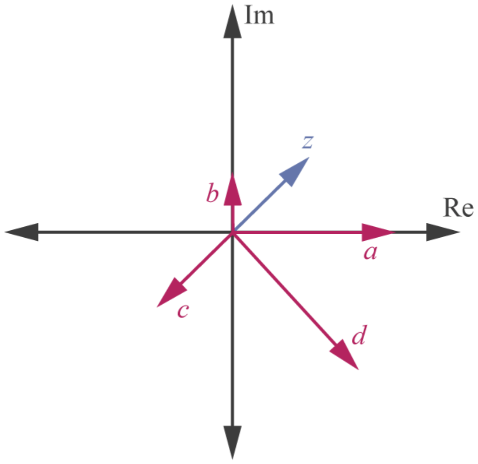

More information

This is a diagram of the complex plane. The horizontal axis is the real part (Re) and the vertical axis is the imaginary part (Im).

There are several vectors originating from the origin:

- Vector 'a' is pointing horizontally to the right along the real axis.
- Vector 'b' is pointing vertically upward along the imaginary axis.
- Vector 'c' has an angle downwards to the left, forming a negative slope.
- Vector 'd' angles down to the right, also forming a slope.
- Vector 'z' is pointing upwards and to the right, indicating a positive direction in both real and imaginary axes.

[Generated by AI]

a) Multiplying z by $\frac{1}{2}$ cis$\frac{\pi}{4}$ will stretch z by a factor of $\frac{1}{2}$ and rotate it anticlockwise by $\frac{\pi}{4}$.

This is most closely represented by vector b.

b) Dividing z by $\frac{2}{3}$ cis $\frac{\pi}{2}$ will stretch z by a factor of $\frac{3}{2}$ and rotate it clockwise by $\frac{\pi}{2}$.

This is most closely represented by vector d.

2 section questions

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Checklist

What you should know

By the end of this subtopic you should be able to:

- convert between Cartesian, modulus—argument (polar) and Euler forms of complex numbers
- multiply and divide complex numbers in modulus—argument (polar) and Euler forms
- understand and describe the geometric significance of operations with complex numbers as represented in the Argand diagram.

Investigation

Part 1

Create your own examples for values of a and b in $z = a + b\text{i}$ to investigate the relationships between the moduli and arguments of z and $\frac{1}{z}$.

Hence, write an expression for $\frac{1}{z}$ in terms of r and θ given that $z = r \text{ cis } \theta$.

Part 2

Let $z = 2 + 3\text{i}$ and $w = -4 + \text{i}$.

Show that the following properties are true for z and w :

$$|z \times w| = |z| \times |w|$$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

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Generalise to all complex numbers

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