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TOPIC 3
GEOMETRY AND TRIGONOMETRY

SUBTOPIC 3.8
TRIGONOMETRIC RATIOS BEYOND ACUTE ANGLES

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3. Geometry and trigonometry / 3.8 Trigonometric ratios beyond acute angles

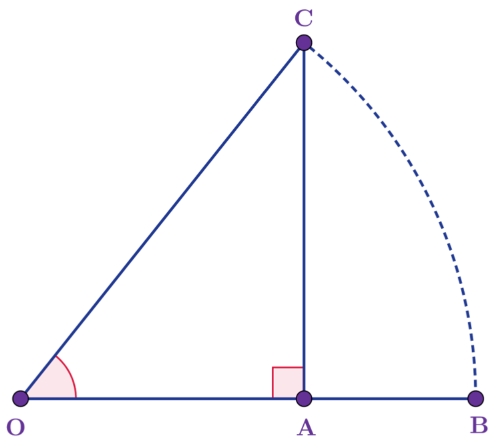
The big picture

So far, you have looked at trigonometric ratios only for right-angled triangles and angles of 90° or less. How can you extend the trigonometric ratios from right-angled triangles and to any angle?

Early mathematicians were also astronomers, which meant they dealt with spheres and circles to calculate the distances between earth and moon, earth and sun, and the circumference of the earth, as well as mapping the night sky or celestial sphere. The celestial sphere is an imaginary sphere with the earth at the centre. To do these calculations they used circles and chords of circles.

International Mindedness

The word ' sine ' comes from Greek sinus, which means *fold* . This word is derived from Indian-Persian and Arabic words for a half a chord.



More information

The image shows a geometric diagram featuring a right triangle OAC, which is inscribed in a circle. The triangle is formed by the segments OA, AC, and OC, with point O at the origin, point A on the horizontal axis, and point C on the vertical axis above point A. The hypotenuse OC is the radius of the circle. Point B lies on the circumference of the circle along the x-axis, extending from point O through A. The angle at O is marked as a right angle, and another angle is marked at point O, labeled for measurement. This diagram is used to illustrate the concept of the sine function in trigonometry, where angle O forms the basis for defining sine in the context of a circle.

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International Mindedness

During the 4th and 5th centuries in India, they used half chords, as shown above, and created the tables for the sine of any angle. The Jantar Mantar astronomical observatory in Jaipur, India has a structure which shows the sine of any angle.

The structure for sine ratios for any angle in Jantar Mantar astronomical observatory in Jaipur, India
Credit: TopPhotoImages Getty Images

More information

The image shows a historical astronomical structure from the Jantar Mantar observatory in Jaipur, India. It features a semicircular design with multiple evenly spaced radial beams converging towards a central pillar. These structures were used to study sine ratios and are part of ancient architectural instruments designed for astronomical observations dating back to the 4th and 5th centuries. The beams are arranged meticulously, allowing for precise angular calculations. The background includes lush greenery and parts of surrounding structures, indicating the observatory's location in an open and possibly verdant setting.

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Concept

In this subtopic, you will generalise the trigonometric ratios in right-angled triangles to trigonometric ratios for any angle. You will look at the relationships between the trigonometric ratios and identify equivalent expressions to form identities.

As you will see with the unit circle, trigonometric functions follow predictable recurring patterns.

How can you use these patterns to help you predict and approximate solutions to mathematical equations? How can circles help you to find the trigonometric ratios of any angle? How do identities help us prove further mathematical results?

Theory of Knowledge

The concepts in this section rely on what we could call 'a perfect circle'. Does a perfect circle exist in nature?

If you said yes, you may need to read this article from Carnegie Mellon University, *Do Perfect Circles Exist? Maybe* (https://www.cmu.edu/mcs/news-events/2019/0314_pi-day-perfect-circles.html) .

Student view

If perfect circles do not exist, then how could mathematics focused on geometry of circles be said to be valid?

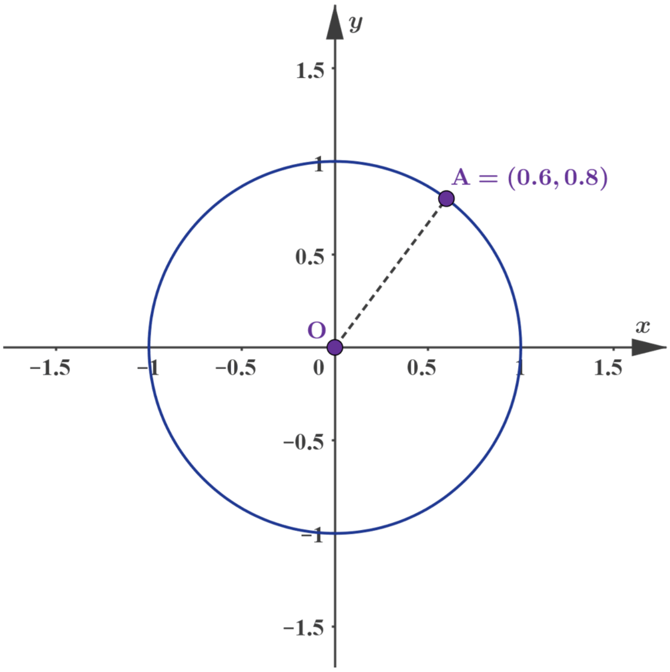
Knowledge Question: Must knowledge have a real-world corollary in order to be considered valid?

The unit circle

Symmetry of the unit circle

A circle centred at the origin $(0, 0)$ with radius 1 unit is called the unit circle.

The diagram below shows a unit circle.



More information

The image is a diagram of a unit circle centered at the origin O on a Cartesian coordinate plane. The circle is plotted with axes labeled x and y. The x-axis ranges from -1.5 to 1.5, while the y-axis ranges from -1.5 to 1.5, both in increments of 0.5. A point labeled A with coordinates (0.6, 0.8) is marked on the circumference of the circle, located in the first quadrant, where both x and y coordinates are positive. The line segment OA is drawn from the origin to the point A, indicating the radius of the circle. The diagram visually represents the concept of a unit circle, dividing the plane into four quadrants, each bounded by the x and y axes.

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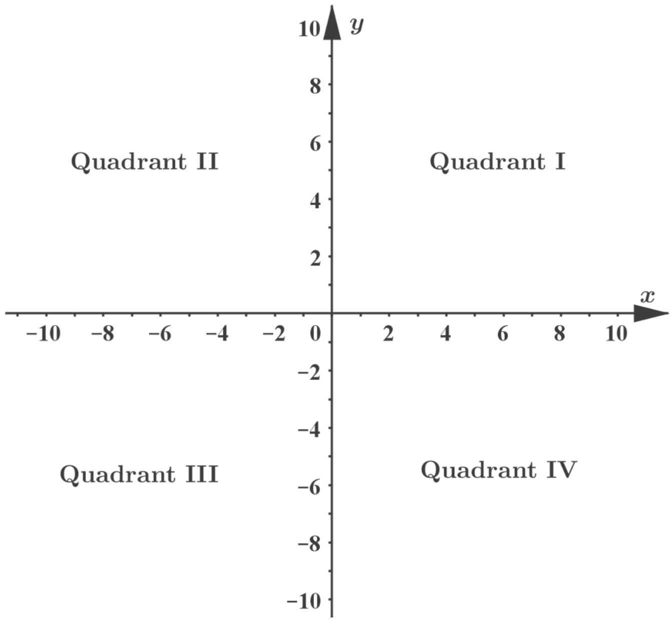
The perpendicular x -axis and y -axis divide the plane into four infinite areas, which are called quadrants. Each quadrant is bounded by two half axes and they are numbered anticlockwise using roman numerals I, II, III and IV, as illustrated in the diagram below. In quadrant I, both the x and y coordinates are positive. In quadrant II, the x coordinates are negative while the y coordinates are positive and so on. So, point A in the diagram above is in quadrant I and both its x and y values are positive.



More information

The image displays a Cartesian coordinate system with the x-axis and y-axis crossing at the origin (0,0). The plane is divided into four quadrants labeled in Roman numerals: Quadrant I is at the top right, Quadrant II at the top left, Quadrant III at the bottom left, and Quadrant IV at the bottom right. Each quadrant is bounded by two half-axes. The x-axis is labeled from -10 to 10, and the y-axis from -10 to 10. Both axes are marked at intervals of 2 units. Quadrant I has positive x and y values; Quadrant II has negative x and positive y values; Quadrant III has negative x and y values; Quadrant IV has positive x and negative y values. The whole diagram illustrates the division into and labeling of the quadrants.

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Example 1

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A rectangle with vertex A $(0.6, 0.8)$ is shown below.

Find the coordinates of the other three vertices.

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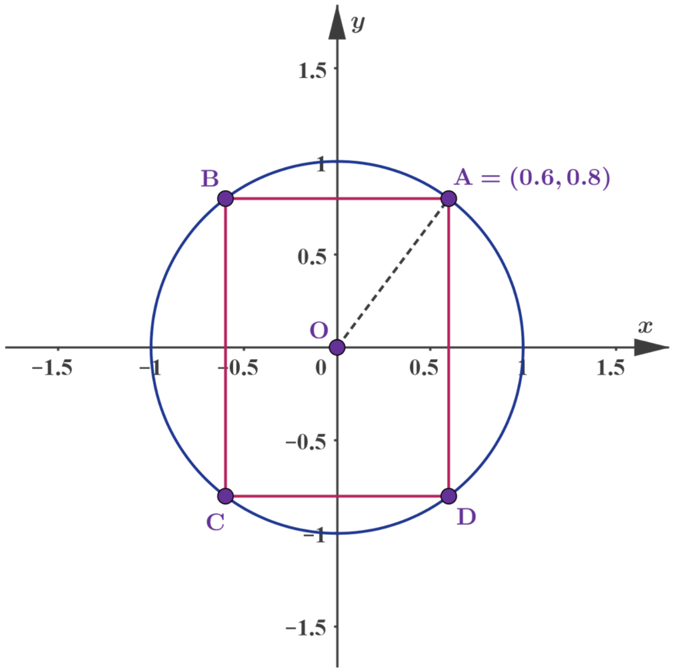

More information

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The image displays a coordinate plane with a circle centered at the origin O (0, 0) and the equation x^2 + y^2 = 1. Inside the circle, a square is inscribed with vertices labeled A, B, C, and D. The point A is labeled with coordinates (0.6, 0.8). The lines from A to B, B to C, C to D, and D to A form the perimeter of the square. There is also a dashed line from O to A, representing the radius from the origin to point A. The x-axis and y-axis both range from -1.5 to 1.5, with labeled intervals at every 0.5 mark. The diagram helps visualize the geometry needed to find the coordinates of the other three vertices: B, C, and D.

[Generated by AI]

Step	Explanation
	You can find the other points using symmetry.
$\mathrm{B}\,(-0.6, 0.8)$	As it is the reflection of A in the y -axis.
$\mathrm{C}\,(-0.6, -0.8)$	As it is the reflection of B in the x -axis.
$\mathrm{D}\,(0.6, -0.8)$	As it is the reflection of C in the y -axis or the reflection of A in the x -axis.

Important

If you know one of the vertices of an inscribed rectangle with vertical and horizontal sides parallel to the x - and y -axes, you can use symmetry to find the other three vertices of this rectangle.

You can often use symmetry arguments to derive facts.

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Exam tip

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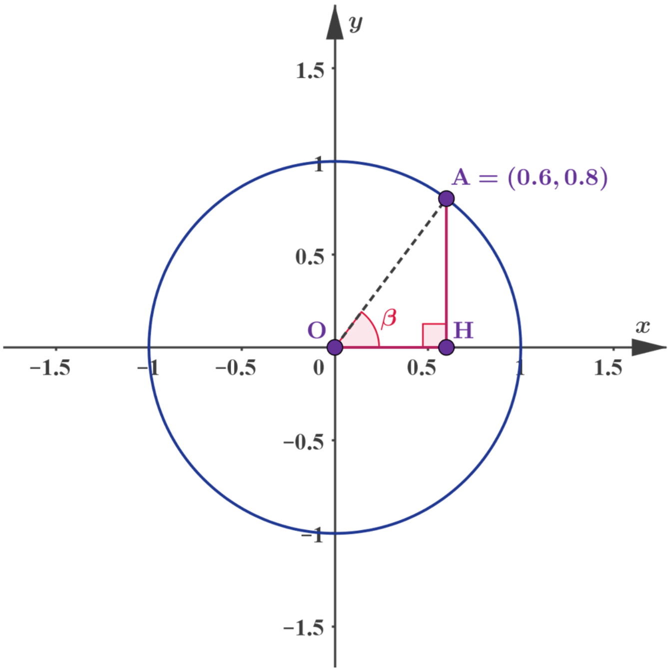
You can identify the sign of the x- and y-coordinates of a point by checking which quadrant the point is in.

You can also use the unit circle and right-angled triangles to write the coordinates of any point in terms of the angle formed between the x -axis and the segment OA, as in the diagram below.



More information

The image is a diagram showing a unit circle with a right triangle inside, placed on a set of axes. The circle is centered at the origin (0, 0) and has a radius of 1. The triangle is formed by the points O (0,0), A (0.6, 0.8), and H (0.6, 0). The segment OA represents the hypotenuse of the right triangle, and the angle (\beta) is formed between the x-axis and the segment (\text{OA}). The x-coordinate of point A is 0.6, and the y-coordinate is 0.8. The point H is directly below A on the x-axis. Angles, lines, and points are labeled in various colors for clarity.

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The trigonometric ratios in triangle OAH are

$$\sin\beta = \frac{AH}{OA} = \frac{0.8}{1} \Rightarrow \sin\beta = 0.8$$

$$\cos\beta = \frac{OH}{OA} = \frac{0.6}{1} \Rightarrow \cos\beta = 0.6$$

Therefore, for triangle OAH, you can say $\text{A}\,(0.6, 0.8) = \text{A}\,(\cos\beta, \sin\beta)$.

✓ **Important**

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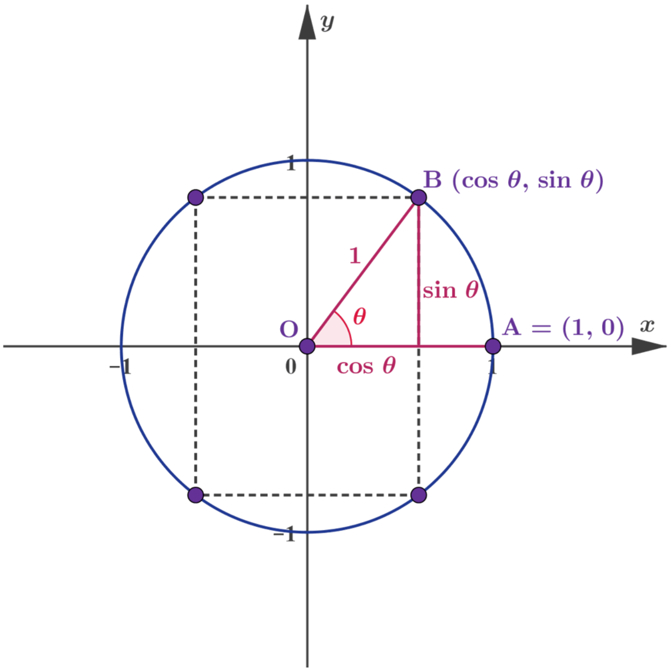
Angles on a unit circle are always measured from the point $(1, 0)$ on the x-axis in the anticlockwise direction. So, the anticlockwise direction is positive and the clockwise direction is negative.



More information

The image is a diagram of the unit circle illustrating the trigonometric functions sine and cosine. It includes a coordinate plane with labeled axes x and y. A circle with a radius of 1 is centered at the origin, O. A right triangle is formed inside the circle with the hypotenuse extending from O to a point B on the circle. The triangle has sides labeled as 1 (hypotenuse), cos θ (adjacent to angle θ on the x-axis), and sin θ (opposite to angle θ on the y-axis). The angle θ is marked in red within the triangle at the origin.

The coordinates of point A on the x-axis are labeled as (1, 0), and point B has coordinates marked as (cos θ, sin θ). There are dashed lines forming a square around the circle, with intersections at points where x and y equal 1 or -1.

[Generated by AI]

✓ **Important**

Any point $\mathrm{B}\,(x, y)$ on the circumference of the unit circle can be written with the trigonometric ratios of the angle AOB where O is the origin and A is the point $(1, 0)$.

As you can see in the diagram above, the coordinates of B are $(\cos\theta, \sin\theta)$.

Example 2

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Given that $\cos 57° \approx 0.545$ and $\sin 57° \approx 0.839$:

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a) Draw a unit circle and show the angle $57°$. Label the point on the circumference as A.

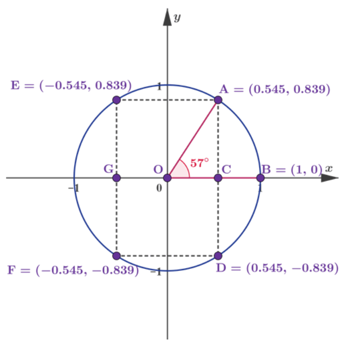
b) Draw an inscribed rectangle in the unit circle such that one of the vertices is A and with sides parallel to the x- and y-axes.

c) Use the symmetry of the rectangle to identify the coordinates of its other vertices.

d) Use the symmetry of the rectangle to determine the angles for its other vertices. Label the vertices with their coordinates using the cosine and the sine of the angles.

e) Complete the following table for the other three angles.

A ngle	Cosine	Sine
$57°$	0.545	0.839

	Steps	
a)		Mark poin to the x- a Mark the i and vertic and label Draw perp AD to fin F. Using sym each poin

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	Steps	
b)	$\mathrm{E}\left(\cos 123°, \sin 123°\right)$	OE is a re $\angle EOG =$ $\angle COE =$ 
c)	$\mathrm{F}\left(\cos 237°, \sin 237°\right)$	Similarly, ∠ $\angle GOF =$ always fro 
d)	$\mathrm{D}\left(\cos 303°, \sin 303°\right)$	Finally, ∠ $\angle BOD =$ 

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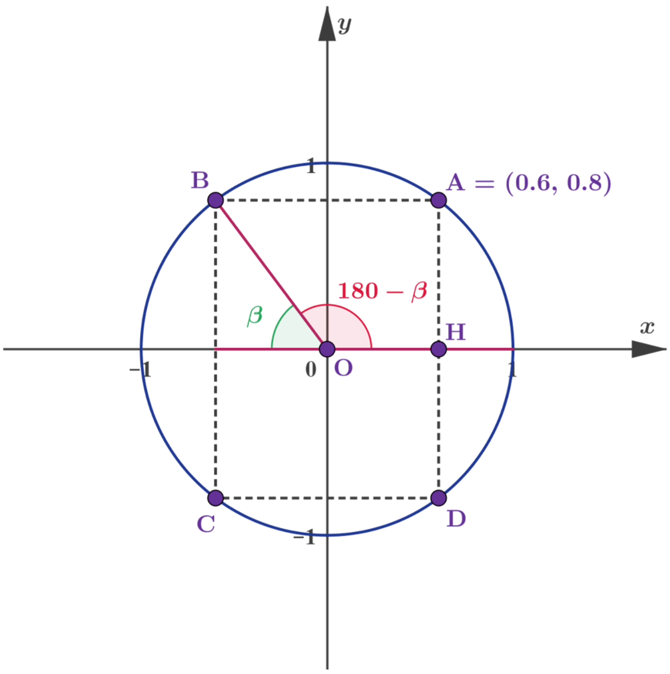
	Steps	
e)	(see table below)	Using the E $(-0.54$ Thus, $\cos 12$ and $\sin 12$ Then, follo other two Note the p

Angle	Cosine	Sine
$57°$	0.545	0.839
$123°$	-0.545	0.839
$237°$	-0.545	-0.839
$303°$	0.545	-0.839

Trigonometric ratios of any angle

As you saw in **Example 2**, you can find the sine and cosine of any angle by using the points and symmetry of the unit circle and writing down the coordinates of the relevant vertex.



More information

The image is a diagram of a unit circle centered at the origin (0, 0) of a Cartesian coordinate system. The circle is intersected by perpendicular lines, and key points are labeled, including points A (0.6, 0.8), B, C, D, H, and O at the center. The diagram shows an angle β (in green) and its supplementary angle 180-β (in red). The x-axis and y-axis are marked, with values from -1 to 1. The coordinates of A are specified, and the geometric symmetry helps demonstrate the trigonometric values for any angle on the unit circle.

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Using the symmetry and trigonometric ratios of the unit circle, you can see that the coordinates of point B are

$$(\cos(180° - \beta), \sin(180° - \beta)) = (-\cos\beta, \sin\beta)$$

which tells you that

$$\cos(180° - \beta) = -\cos\beta$$

and

$$\sin(180° - \beta) = \sin\beta$$

or, using radians,

$$\cos(\pi - \beta) = -\cos\beta$$

and

$$\sin(\pi - \beta) = \sin\beta$$

Example 3

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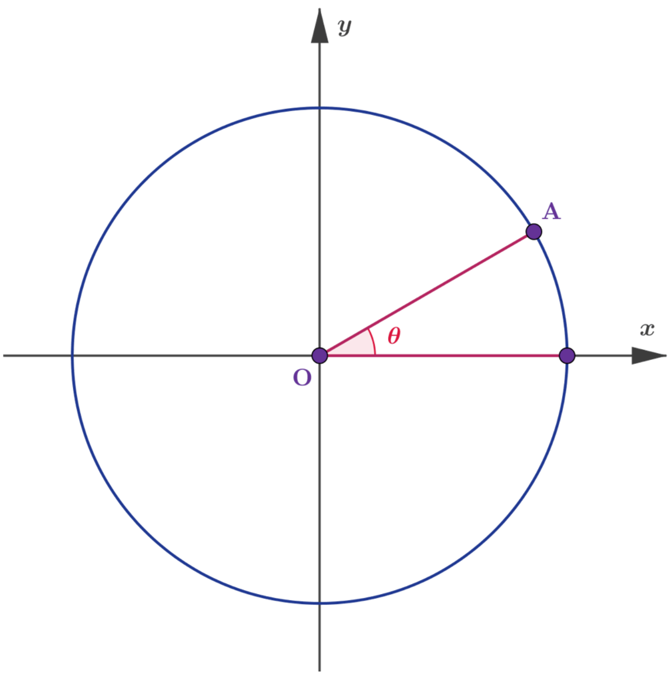

More information

Student view

The diagram depicts a unit circle with a center labeled O. Starting from the center O, a line extends to a point A on the circumference, creating an angle θ (theta) with the positive x-axis. This section of the circle is highlighted to indicate the angle. The circle is placed on a coordinate system with x and y axes labeled. The point A is marked on the circle, and an arc from O indicates the angle θ formed by the line segment OA with the x-axis. The diagram is

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intended to show the relationship between the angle and its corresponding point on the unit circle.

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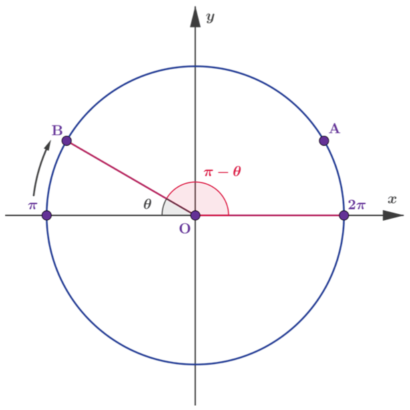
The diagram shows a point marked as A on a unit circle and the corresponding angle θ . Copy the diagram and mark the points corresponding to the following angles:

a) $\pi - \theta$

b) $2\pi - \theta$

c) $\pi + \theta$

d) $\theta + 2\pi$

	Steps	Explanation
a)		$\pi - \theta$ is in the second quadrant. Point B represents a turn of π minus the angle θ .

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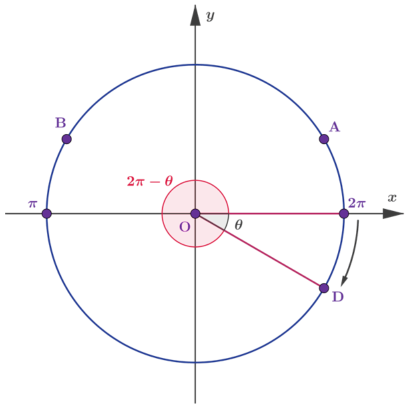
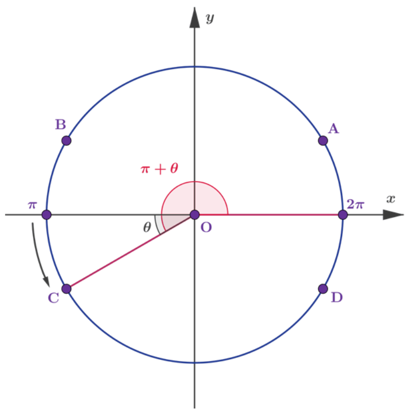
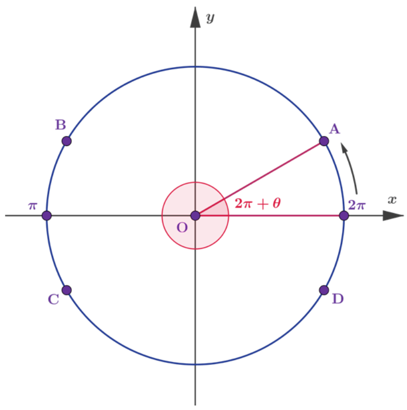
	Steps	Explanation
b)		$2\pi - \theta$ is in the fourth quadrant. Point D represents a turn of 2π minus the angle θ.
c)		$\pi + \theta$ is in the third quadrant. Point C represents a turn of π plus the angle θ .
d)		$\theta + 2\pi = 2\pi + \theta$ is in the first quadrant. Point E represents a full turn of 2π plus the angle θ . Note that θ and $2\pi + \theta$ can be represented by the same point.

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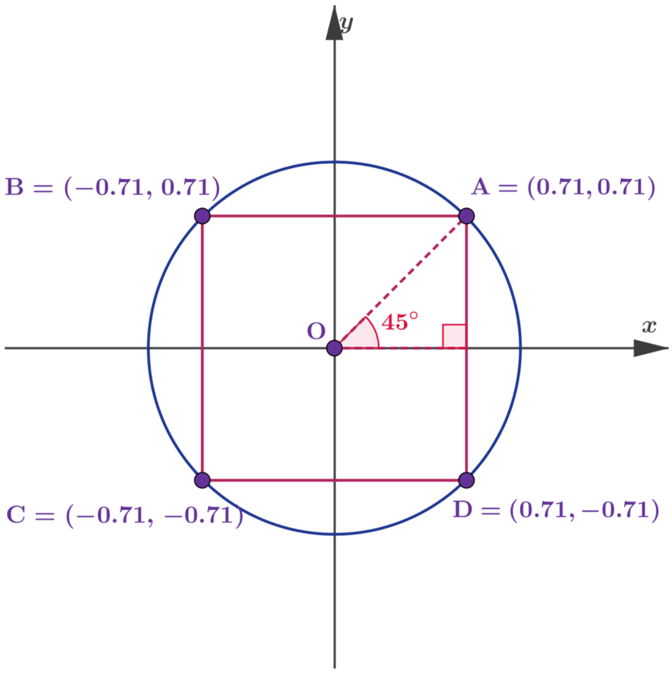
Activity

When determining the sine and cosine of any angle in the four quadrants, you can use the symmetry of the unit circle.

1. ABCD is a rectangle with sides parallel to the axes.
   Find $\sin 45°$, $\cos 45°$, $\sin 135°$ and $\cos 135°$.



More information

The image depicts a circle inscribed within a rectangle. The rectangle is denoted as ABCD, and its sides are parallel to the axes with specified points: A (0.71, 0.71), B (-0.71, 0.71), C (-0.71, -0.71), and D (0.71, -0.71). The center of the circle is marked as O, which is also the center of the rectangle. There is a dashed line forming a 45-degree angle from the center O to point A, indicating a right-angled triangle within the circle. This diagram serves to illustrate the cosine and sine values at 45 and 135 degrees based on the unit circle concept.

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2. Use the applet below to help you investigate further.
   - What do you notice about the coordinates of the rectangle?
   - How can you use the rectangle to find the sine and cosine of angles?

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Interactive 1. Using the Symmetry of Unit Circle to Find the Trignometric Ratios of Any Angle.

More information for interactive 1

This interactive helps users investigate the unit circle and how it connects to sine and cosine values. The red point can be dragged along the circle's circumference, and as it moves, the angle between the positive x-axis and the line segment *OA*, where O is the origin and A is the selected red point, is displayed.

A rectangle *ABCD*, with sides parallel to the coordinate axes, is dynamically drawn inside the unit circle, adjusting with the red point's position. The coordinates of all four vertices are shown, providing insight into how the sine and cosine values relate to the horizontal and vertical distances from the origin.

Since the coordinates of point A are $(\cos\theta, \sin\theta)$, where θ is the angle, students can observe how these values change with the angle and how symmetry in the unit circle affects the signs of sine and cosine in different quadrants. This helps reinforce the connection between geometric positioning and trigonometric values.

Use the interactive to explore how the rectangle's shape and coordinates reveal the sine and cosine of various angles and uncover patterns in the unit circle.

Important

Points that lie on the same horizontal line have angles with the same sine value.

Points that lie on the same vertical line have angles with the same cosine value.

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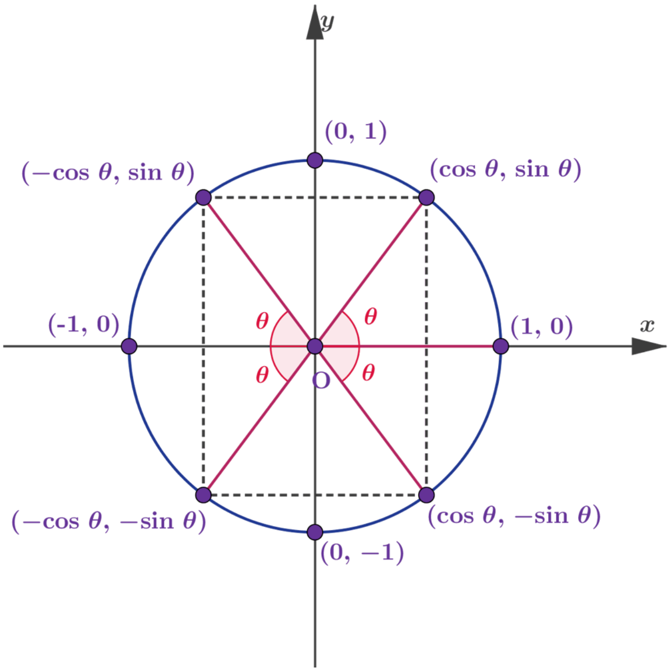

More information

The image is a diagram of the unit circle on a Cartesian coordinate system. It shows the X and Y axes meeting at the origin, labeled 'O'. The circle has a radius of 1, centered at the origin.

Key points on the circumference are marked with purple dots and labeled with their coordinates: - (1, 0) on the positive X-axis - (0, 1) on the positive Y-axis - (-1, 0) on the negative X-axis - (0, -1) on the negative Y-axis

Angles θ are represented in radians and are labeled at various symmetrical positions around the circle. The labels include trigonometric functions: - (cos θ, sin θ) on the circle's boundary in the first quadrant - (−cos θ, sin θ) in the second quadrant - (−cos θ, −sin θ) in the third quadrant - (cos θ, −sin θ) in the fourth quadrant

Dashed lines from the origin to points on the circle indicate the radius. The diagram illustrates key angles by showing sectors marked with the Greek letter theta (θ) to indicate the angle subtending from the origin to these points.

This unit circle diagram helps visualize trigonometric functions with respect to standard angle measurements.

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Example 4

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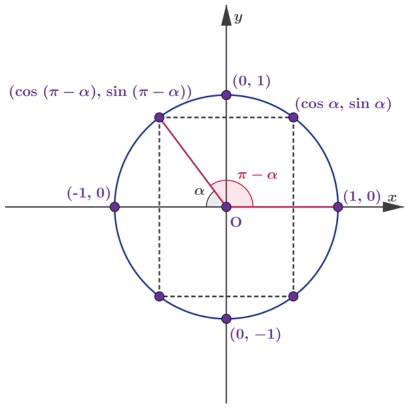
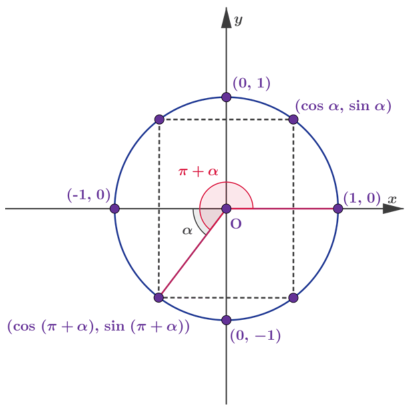
If $\sin \alpha = \frac{3}{5}$, find the exact values of

a) $\sin(\pi - \alpha)$

b) $\sin(\pi + \alpha)$

c) $\sin(2\pi - \alpha)$

Student view

	Steps	Explanation
a)		Mark angles α and $(\pi-\alpha)$ on the unit circle.
	$\sin(\pi-\alpha)=\sin\alpha$ Thus $\sin(\pi-\alpha)=\frac{3}{5}$	Both are on the same horizontal line. That means their sine values are equal. Alternatively, in the second quadrant, the y value is positive and this is the sine of the angle.
b)		Mark α and $\pi+\alpha$ on the unit circle.
	$\sin(\pi+\alpha)=-\sin\alpha$ Thus, $\sin(\pi+\alpha)=-\frac{3}{5}$	$\sin(\pi+\alpha)$ is in the third quadrant, so it will be negative.

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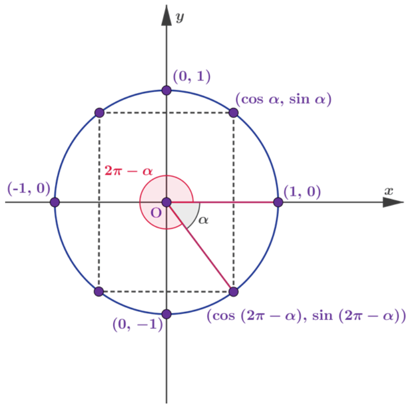
	Steps	Explanation
c)		Mark α and $2\pi - \alpha$ on the unit circle. Note that the positive direction is anticlockwise and the negative direction is clockwise.
	$\sin(2\pi - \alpha) = -\sin\alpha$ Thus, $\sin(2\pi - \alpha) = -\frac{3}{5}$	

Important

1. The coordinates of any point on the circle are given by $(\cos\theta, \sin\theta)$, i.e. $(x, y) \Longleftrightarrow (\cos\theta, \sin\theta)$, where θ is the angle measured from the x-axis in the anticlockwise direction. Hence, cos is the value of the x-coordinate of the corresponding point on the unit circle, and sin is the value of the y-coordinate of the corresponding point on the unit circle.
2. The unit circle is divided into four quadrants:
   a. **First quadrant** $0 < \theta < 90°$ or $0 < \theta < \frac{\pi}{2}$: Both $\sin\theta$ and $\cos\theta$ have positive values.
   b. **Second quadrant** $90° < \theta < 180°$ or $\frac{\pi}{2} < \theta < \pi$: $\sin\theta$ is positive but $\cos\theta$ is negative.
   c. **Third quadrant** $180° < \theta < 270°$ or $\pi < \theta < \frac{3\pi}{2}$: Both $\sin\theta$ and $\cos\theta$ have negative values.
   d. **Fourth quadrant** $270° < \theta < 360°$ or $\frac{3\pi}{2} < \theta < 2\pi$: $\sin\theta$ is negative and $\cos\theta$ is positive.

Exam tip

In the IB examination, if a question asks for the 'exact value', then keep any surds , fractions or multiples of π in your calculations, e.g. $\frac{1}{7}$, $\sqrt{3}$, 2π or $\frac{\pi}{3}$.

Decimal answers found with a calculator are not always exact values.

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Example 5

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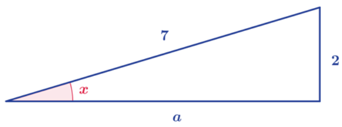
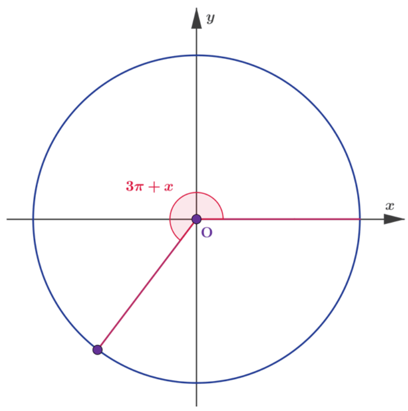
If $\sin x = \frac{2}{7}$ for an acute angle x , find the exact values of

a) $\cos x$

b) $\cos(3\pi + x)$

	Steps	Explanation
		Draw the right-angled triangle and label the sides and angle.
	$a^2 + 2^2 = 7^2$	Using Pythagoras' theorem.
	$a = \sqrt{45} = 3\sqrt{5}$	Solve for a . Leave in surd form as an exact answer is required.
a)	$\cos x = \frac{3\sqrt{5}}{7}$	
b)	$\cos(3\pi + x) = -\cos x$ so $\cos(3\pi + x) = -\frac{3\sqrt{5}}{7}$	Mark the angle $3\pi + x$ on the unit circle . This angle is bigger than 2π , so we've already completed one full cycle. 

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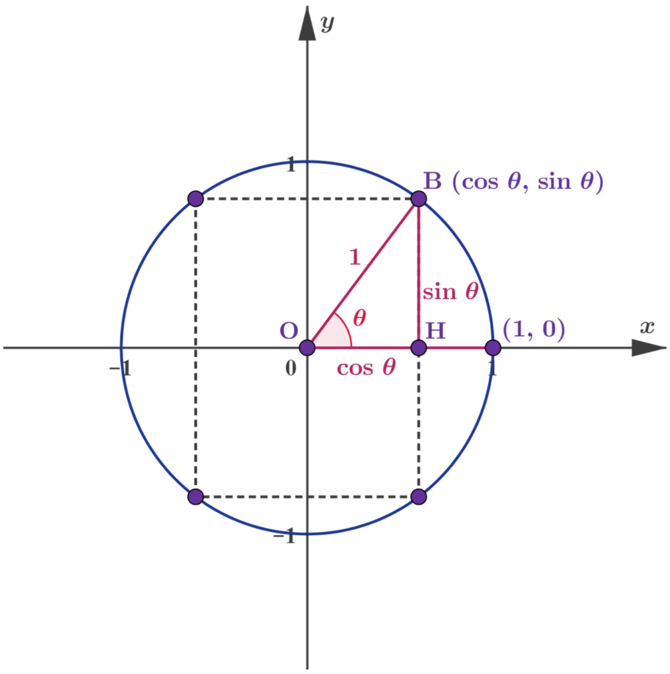
Tangents using the unit circle

In the diagram below,

$$\tan\theta = \frac{BH}{OH}$$

If you substitute the values of OH and BH in terms of the cosine and sine of the angle θ respectively, you would get the new identity:

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$



More information

The image shows a unit circle on a Cartesian coordinate system with x and y axes. The circle's radius is 1, and it is centered at the origin (O). A right triangle is inscribed in the circle, with the vertex O at the origin, vertex H at coordinates (1, 0), and vertex B at coordinates represented as (cos θ, sin θ).

The hypotenuse OB of the triangle has a length of 1 (the radius of the unit circle). The angle at the origin is labeled θ. Line OH is the adjacent side with a length of 'cos θ', and line HB is the opposite side with a length of 'sin θ'.

The angle θ is marked in red at the origin. The x-coordinate of point B is 'cos θ', and the y-coordinate is 'sin θ', meaning that B is the projection of the point (1, 0) when rotated by the angle θ around the origin. The diagram illustrates how sine and cosine represent projections of the unit circle onto the x and y axes respectively. Dotted lines are drawn parallel to the axes, forming the triangle supporting the angle's trigonometric representation.

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Exam tip

In the IB examination this identity for $\tan\theta$ will be in your formula booklet as

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Example 6

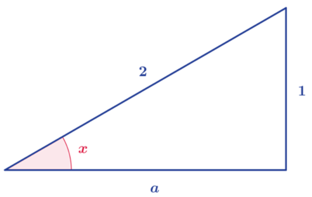
★★★

If $\sin x = \frac{1}{2}$, $0 < x < \frac{\pi}{2}$, find the exact value of

a) $\tan x$

b) $\tan(\pi - x)$

c) $\tan(\pi + x)$

	Steps	Explanation
		Draw and label the right-angled triangle.
	$a^2 + 1^2 = 2^2$	Using Pythagoras' theorem.
	$a = \sqrt{3}$	Solve for a. Leave it in surd form as an exact answer is required.
a)	$\tan x = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	Using $\tan x = \frac{\text{opp}}{\text{adj}}$

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	Steps	Explanation
b)	$\tan(\pi - x) = -\tan x$ Therefore, $$\tan(\pi - x) = -\frac{\sqrt{3}}{3}$$	$\tan(\pi - x) = \frac{\sin(\pi - x)}{\cos(\pi - x)}$ $(\pi - x)$ is in the second quadrant as $$0 < x < \frac{\pi}{2}$$ So, $$\tan(\pi - x) = \frac{\sin(x)}{-\cos(x)} = -\frac{\sin(x)}{\cos(x)}$$
c)	$\tan(\pi + x) = \tan x$ Therefore, $$\tan(\pi + x) = \frac{\sqrt{3}}{3}$$	$$\tan(\pi + x) = \frac{\sin(\pi + x)}{\cos(\pi + x)}$$ $(\pi + x)$ is in the third quadrant, so $$\tan(\pi + x) = \frac{-\sin(x) - \cos}{(x)} = \frac{\sin(x)}{\cos(x)}$$

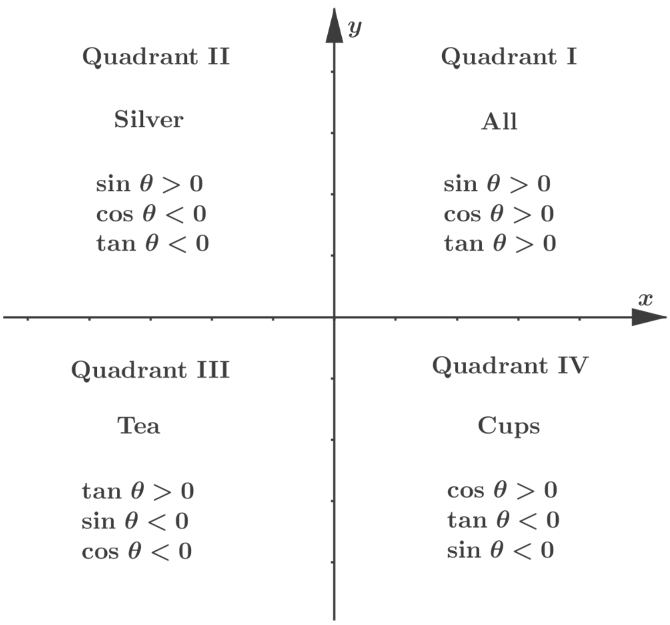
Exam tip

In IB examinations, these rules for the signs of trigonometric ratios for angles in the different quadrants are not given.

However, you can remember them by considering the signs of x and y in each quadrant.

Or you could use ‘ **All S** ilver **T** ea **C** ups’ to remind you:

- Quadrant I, **all** trigonometric values are positive
- Quadrant II, **sine** is positive
- Quadrant III, **tangent** is positive
- Quadrant IV, **cosine** is positive.



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More information

The image is a diagram showing a coordinate plane divided into four quadrants, labeled I through IV. Each quadrant has trigonometric function signs.

Quadrant I: - Label: "All" - Signs: sin θ > 0, cos θ > 0, tan θ > 0

Quadrant II: - Label: "Silver" - Signs: sin θ > 0, cos θ < 0, tan θ < 0

Quadrant III: - Label: "Tea" - Signs: tan θ > 0, sin θ < 0, cos θ < 0

Quadrant IV: - Label: "Cups" - Signs: cos θ > 0, tan θ < 0, sin θ < 0

The axes are labeled x and y, with arrows indicating positive directions.

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5 section questions

3. Geometry and trigonometry / 3.8 Trigonometric ratios beyond acute angles

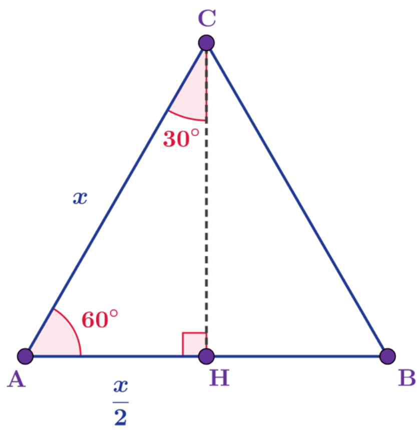
Exact values

Special triangles

In the diagram below. equilateral triangle ABC has side length x.

When you draw the height CH, you divide the triangle into two congruent right-angled triangles with interior angles $30^{\circ}, 60^{\circ}$ and 90°.

These are special anglesbecause we can calculate exact values for their trigonometric ratios.



More information

Student view

The image shows a triangle labeled as triangle AHC. Point C is at the top, point A is on the left bottom, and point B is on the right bottom. There's a perpendicular line from point C to point H, which is on line AB. Angle A is labeled 60 degrees, angle C is labeled 30 degrees. Line segment AH is labeled as x/2, and line segment AC is labeled as x. The hypotenuse is the line segment AB. The structure indicates special angles used for calculating trigonometric ratios, with the right angle at point H.

[Generated by AI]

Using Pythagoras' theorem for triangle AHC,

$$CH^2 + \left(\frac{x}{2}\right)^2 = x^2$$

$$CH^2 = x^2 - \left(\frac{x}{2}\right)^2$$

$$CH^2 = \frac{3x^2}{4}$$

$$CH = \sqrt{\frac{3x^2}{4}}$$

$$CH = \frac{x\sqrt{3}}{2}$$

Now that you have all the side lengths of triangle AHC, you can write the trigonometric ratios for $60°$:

$$\sin 60° = \frac{CH}{AC} = \frac{\frac{x\sqrt{3}}{2}}{x} = \frac{\sqrt{3}}{2} \qquad \Rightarrow \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{AH}{AC} = \frac{\frac{x}{2}}{x} = \frac{1}{2} \qquad \Rightarrow \cos 60° = \frac{1}{2}$$

$$\tan 60° = \frac{\mathrm{CH}}{\mathrm{AH}} = \frac{\frac{x\sqrt{3}}{2}}{\frac{x}{2}} = \frac{\sqrt{3}}{1} = \sqrt{3} \Rightarrow \tan 60° = \sqrt{3}$$

Following a similar approach, you can find the trigonometric ratios for $30°$:

$$\sin 30° = \frac{1}{2}$$

$$\cos 30° = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

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✓ **Important**

In IB examination, you are **not** required to know the exact values for the trigonometric ratios of these angles and you can use your calculator to find approximate values for them. However, special triangles are always helpful when you are solving geometry problems.

If you know these values, you can combine them to deduce other values.

Example 1

★★☆

Find the exact values of

a) $\sin\frac{7\pi}{6}$

b) $\tan\frac{7\pi}{6}$

c) $\cos\frac{5\pi}{3}$

	Steps	Explanation
a)	$\sin\frac{7\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}$ so $\sin\frac{7\pi}{6} = -\frac{1}{2}$	$\frac{7\pi}{6} = \pi + \frac{\pi}{6}$, which is in the third quadrant and so the sine of the angle is negative.
b)	$\tan\frac{7\pi}{6} = \tan\frac{\pi}{6}$ so $\tan\frac{7\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	$\frac{7\pi}{6} = \pi + \frac{\pi}{6}$, which is in the third quadrant and so the tangent of the angle is positive.
c)	$\cos\frac{5\pi}{3} = \cos\frac{\pi}{3}$ so $\cos\frac{5\pi}{3} = \frac{1}{2}$	$\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$, which is in the fourth quadrant and so the cosine of the angle is positive.

The second special triangle is the right-angled isosceles triangle.

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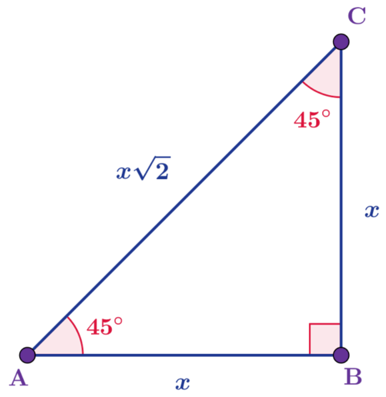

More information

The image shows a right-angled isosceles triangle labeled as triangle ABC. The right angle is at point B. Both angles at points A and C are 45 degrees each, making it an isosceles triangle. The legs of the triangle, AB and BC, are both labeled with a length of 'x'. The hypotenuse AC is labeled with the length 'x√2', which is consistent with the properties of a 45-45-90 triangle. The emphasis of the triangle is on understanding trigonometric ratios using this particular triangular configuration, where both legs are equal, and the hypotenuse is 'x√2'.

[Generated by AI]

Using the right-angled isosceles triangle ABC, you can find the trigonometric ratios of the angle 45° :

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = 1$$

Example 2

★★☆

What is $\cos \frac{5\pi}{4}$?

We note that $\frac{5\pi}{4} = \frac{\pi}{4} + \pi$ and so we are in the third quadrant where cos is negative. Using the unit circle, we go to $\frac{5\pi}{4}$ whose x value is the same as that of $\frac{\pi}{4}$ but with a negative sign, i.e. $-1 \times \frac{\sqrt{2}}{2}$. Thus, $\cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$.

Example 3

★★☆

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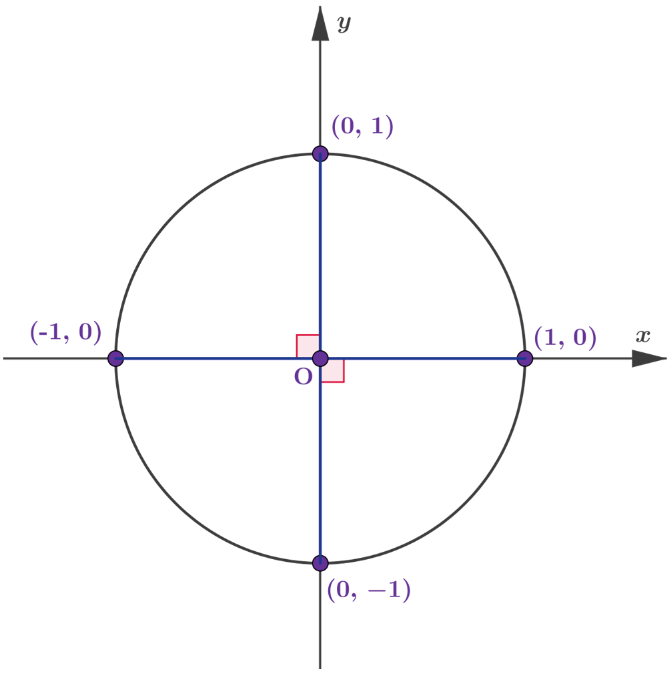
What is $\sin\dfrac{5\pi}{4}$?

Again we note that we are in the third quadrant. Using the unit circle, we go to $\dfrac{5\pi}{4}$ whose y value is the same as that of $\dfrac{\pi}{4}$ but negative, i.e. $-1\times\dfrac{\sqrt{2}}{2}$. Thus, $\sin\dfrac{5\pi}{4}=-\dfrac{\sqrt{2}}{2}$.

Other special angles

You have established that the coordinates of any point (x, y) on the unit circle can be written as $(\cos\theta, \sin\theta)$. You can use this to find the trigonometric ratios of the angles $0^\circ, 90^\circ, 180^\circ, 270^\circ$ and 360° .



More information

The image shows a unit circle on a Cartesian coordinate plane. The circle is centered at the origin (O), where the x-axis and y-axis intersect. Four points are marked and labeled at the intersections of the unit circle with the axes:\n\n1. On the positive x-axis, the point (1, 0) is labeled.\n2. On the positive y-axis, the point (0, 1) is labeled.\n3. On the negative x-axis, the point (-1, 0) is labeled.\n4. On the negative y-axis, the point (0, -1) is labeled.\n\nThe circle is used to represent the unit circle's property that any point on the circle can be written as (cos θ, sin θ). This diagram helps illustrate the trigonometric ratios at standard angles (0°, 90°, 180°, 270°, and 360°) and understand the coordinate positions on the unit circle.

[Generated by AI]

If you look at the coordinates of points where the unit circle intersects the x- or y-axis :

$$(1, 0):\ \text{angle } 0 \text{ radians} \Rightarrow \cos 0 = 1, \sin 0 = 0 \text{ and } \tan 0 = \frac{0}{1} = 0$$

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$(0, 1)$: angle $\frac{\pi}{2}$ radians $\Rightarrow \cos\frac{\pi}{2} = 0, \sin\frac{\pi}{2} = 1$ and $\tan\frac{\pi}{2} = \frac{1}{0}$ is undefined

$(-1, 0)$: angle πradians $\Rightarrow \cos\pi = -1, \sin\pi = 0$ and $\tan\pi = \frac{0}{1} = 0$

$(0, -1)$: angle $\frac{3\pi}{2}$ radians $\Rightarrow \cos\frac{3\pi}{2} = 0, \sin\frac{3\pi}{2} = -1$ and $\tan\frac{3\pi}{2} = \frac{1}{0}$ is undefined

$(1, 0)$: angle 2π radians $\Rightarrow \cos 2\pi = 1, \sin 2\pi = 0$ and $\tan 2\pi = \frac{0}{1} = 0$

Activity

- Draw a circle on graph paper with centre $(0, 0)$ and radius 5 cm.
- Using a protractor, mark on the circumference **(a)** $60°$ **(b)** $210°$.
- Draw lines from the points to the centre, and from the points to the x-axis to create right-angled triangles.
- Measure the sides.
- Calculate the three trigonometric ratios using the measured lengths.

Compare your calculated values to the exact values. Is there a difference? If yes, why?

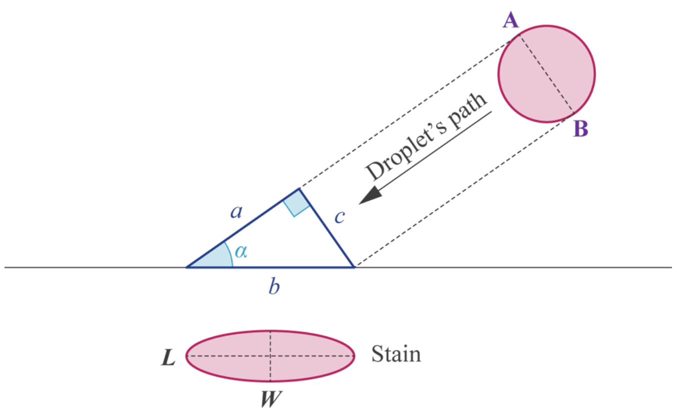
Making connections

Forensic analysis and trigonometry

Forensic scientist analyse the evidence left behind at crime scenes. Using biology, physics and mathematics, they can recreate the sequence of events from blood spatters.

The angle of impact α can be found using the sine ratio $\frac{c}{b}$ since $b = L$ and $c = W$ for a spherical droplet.

Moreover, experts can identify the height when the droplet started to fall, the angle of impact and also the path of the blood droplet as it fell using trigonometry, vectors and the equations of motion from physics.



More information

The diagram illustrates the path of a blood droplet as it falls and impacts a surface. It includes a triangle labeled with sides a, b, and c, and an angle α. The path of the droplet is shown with a dashed line and labeled "Droplet's path". The impact area is marked as a stain with dimensions L (length) and W (width). Points A and B are marked on the droplet demonstrating the direction and impact angle. This diagram helps demonstrate how trigonometry and the equations of motion can be used to analyze the droplet's path.

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If you would like to find out more about how mathematics is used in forensic science click here (https://science.howstuffworks.com/bloodstain-pattern-analysis3.htm)

3 section questions

3. Geometry and trigonometry / 3.8 Trigonometric ratios beyond acute angles

Extension of the sine rule

In section 3.2.4 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-sine-rule-id-26049/) you studied the sine rule. Consider triangle ABC with $AC = 12$, $CB = 7$ and $\hat{A} = 30°$.

As two sides and an angle are given, you could use the sine rule to solve this triangle.

But is there a unique triangle with these measurements? Is angle B acute or obtuse?

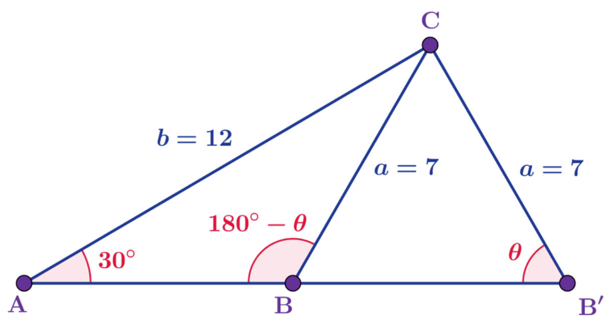
Both would give you the same sine value, as you learned in the previous section.

This is called the ambiguous case of the sine rule, as there are two triangles with these measurements.

Ambiguous case of the sine rule

Consider two given segments $[\mathrm{AC}]$, $[\mathrm{BC}]$ and a given angle $\angle\mathrm{BAC}$. As we can see in the diagram below, There are two different triangles $\triangle\mathrm{ABC}$ and $\triangle\mathrm{AB'C}$ that can be constructed with the above two segments and the angle.

The ambiguous part of the sine rule is illustrated by the diagram below . While the angles B and B' are not the same, given the sides a, b and angle A, two equally valid triangles can be drawn $\triangle\mathrm{ABC}$ and $\triangle\mathrm{AB'C}$. Note how the triangle $\triangle\mathrm{BCB'}$ is isosceles.



More information

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The diagram illustrates the ambiguous case of the sine rule using two triangles, labeled as (\triangle \text{ABC}) and (\triangle \text{AB'C}). The points (A), (B), and (C) define the first triangle, and (A), (B'), and (C) define the second triangle. The line segment (AB) measures 12 units, and both segments (AC) and (AB') are 7 units long. The angle at (A) is 30 degrees, while the angle at (B) is (180^\circ - \theta), and the angle at (B') is (\theta). The triangle (\triangle \text{BCB'}) is noted as isosceles, with the segments (BC) and (B'C) being equal.

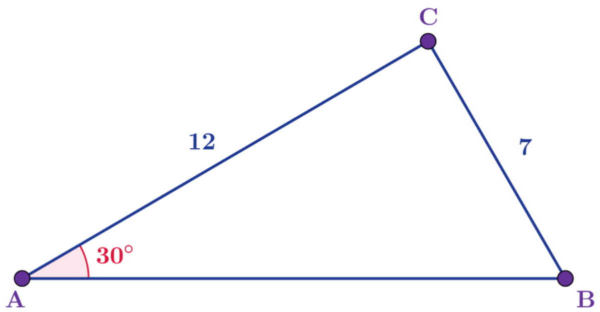
[Generated by AI]

So, given the sides a, b and the angle A, we may at times be able to build two triangles with different angles B and B'. We note that $\triangle \text{BCB}'$ is isosceles. Thus, regardless of whether we choose the side a from $\triangle \text{ABC}$ or $\triangle \text{AB}'\text{C}$, the value for $\frac{a}{\sin A}$ is the same. This implies that the following ratios are also the same

$$\frac{b}{\sin B} = \frac{b}{\sin B'}$$

However, be careful with generalising too quickly! It is not because $\frac{b}{\sin B} = \frac{b}{\sin B'}$ that $B = B'$. Instead, we have the relation $B = 180° - B'$; hence one angle is acute, the other is obtuse. So, given values for a, b and A, which angle is correct, B or B'? Both are correct, and so you must give both as the solution. Depending on the triangle $\triangle \text{ABC}$ or $\triangle \text{AB}'\text{C}$ the angle C is also different, as can clearly be seen in the diagram above. This is tricky and not obvious straightaway; however, a must be smaller than b for the ambiguous case. Hence, always ask yourself the question whether you have the ambiguous case.

Let us see an example based on the triangle shown in the diagram below.



More information

The diagram depicts a triangle labeled ABC. Vertex A is at the bottom left corner, vertex B is at the bottom right, and vertex C is at the top. The angle at vertex A is marked as 30 degrees. The side AC measures 12 units, and the side BC measures 7 units. The diagram is used as an example for an angle-based analysis.

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Activity

Use the following applet.

Drag the points to create new triangles and find the angle θ .

Interactive 1. Finding the Angle of the New Triangle.

Credit: GeoGebra (https://www.geogebra.org/m/MJYSH8q7) Mr Hardman

More information for interactive 1

This interactive applet helps users explore the **Sine Rule** and understand the **ambiguous case** in triangle construction. On the screen, triangle $\triangle$ABC is displayed with angle A, side a=BC, and side b=AC given. Using this information, the user is prompted to calculate angle B.

As users work through the problem, they can see the mathematical steps of applying the sine rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

The question in the applet is written as in the $\triangle$ABC, $A = 36.2^{\circ}$, a=7.4, and b=12.2, find angle B.

When we click on a show answer then this calculation comes up, $\frac{\sin \theta}{12.2} = \frac{\sin 36.2}{7.4}$

$\sin\theta = 12.2\frac{\sin 36.2}{7.4}$

$\theta = 77.6$

The interface is dynamic — users can interact with the triangle, adjust side lengths, and observe how different values can result in either **one or two possible triangles**, depending on the configuration. This provides an engaging way to visualize and understand the conditions that lead to the ambiguous case when using the sine rule.

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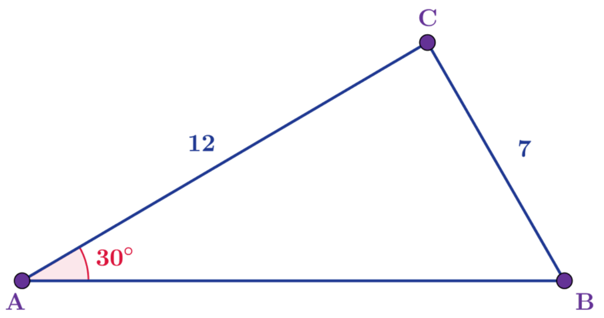
This tool supports both algebraic and visual understanding of triangle properties and trigonometric relationships, making it ideal for exploring real applications of the sine rule.

Example 1

★★★

In the triangle shown below, find all possible values for angles B and C and side AB.



More information

The image depicts a triangle labeled as triangle ABC. The vertex A is labeled with a red angle mark of 30 degrees. The side opposite this angle, AB, is the base of the triangle. The sides AC and BC are labeled with lengths of 12 and 7, respectively. Point C is at the apex of the triangle, opposite side AB. In this geometry problem, the angles at vertices B and C, as well as the potential values for side AB, need to be determined based on the given angle at A and the known side lengths of AC and BC.

[Generated by AI]

Steps	Explanation
$\frac{\sin B}{12} = \frac{\sin 30^\circ}{7}$ $\sin B = \frac{\frac{1}{2}}{7} \times 12$ $B = \sin^{-1}\left(\frac{6}{7}\right)$ $B = 59.0^\circ \quad$ [3 significant figures]	Solving to find angle B, we use the sine rule.

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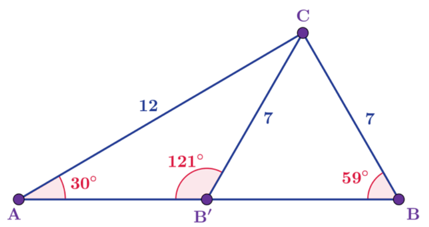
Steps	Explanation
For example, with the sine rule: $\frac{\sin B}{12} = \frac{\sin C}{c}$ $\frac{\sin 59^\circ}{12} = \frac{\sin 91^\circ}{c}$ $c = \frac{\sin 91^\circ}{\sin 59^\circ} \times 12$ $c = 14.0$	Now that we know angle $\angle \text{ABC}$, we also know angle $\angle \text{ACB}$, i.e. $C = 180^\circ - (30^\circ + 59.0^\circ) = 91.0^\circ$. And, thus, we can find the one remaining side (using the sine or cosine rule), which is given by $c = 14.0$.
	We, however, are in an ambiguous situation. Since $\sin 59.0^\circ = \sin(180^\circ - 59.0^\circ) = \sin 121^\circ$, we may build two triangles given the information in diagram in the question. The second triangle has angle $\angle \text{ABC} = 121^\circ$, which is obtuse as expected. This makes angle $\angle \text{ACB}$ also different, namely $C = 180^\circ - (30^\circ + 121^\circ) = 29^\circ$ .Consequently, side c would then be $c = 6.79$. These two ambiguous cases are shown schematically in the diagram opposite.

Be aware

When you are solving triangles using the sine rule, always check if it is an ambiguous case.

International Mindedness

The first mention of the sine function is found in the *Aryabhatiya* , which was written by Indian mathematician and astronomer Aryabhata (476—550 AD). He collected and expanded the works from Surya Siddhantas in which sine was defined using half an angle and half a chord. His work contains the earliest surviving tables of sine values and versine values $(1 - \text{cosine})$, in 3.75° intervals from 0° to 90°, to an accuracy of 4 decimal places.

Using trigonometric tables, Aryabhata was able to calculate the circumference of the Earth as $39\,968$ km. This is only 0.2 per cent smaller than the actual circumference. According to modern technology, the average circumference of the Earth is $40\,075$ km.

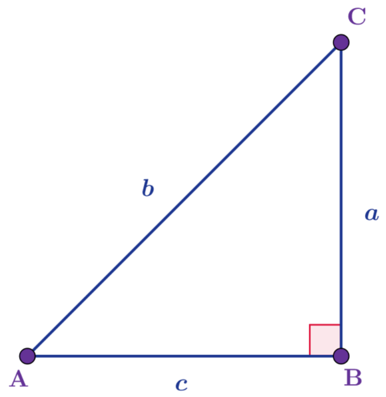
4 section questions

3. Geometry and trigonometry / 3.8 Trigonometric ratios beyond acute angles

Pythagorean identity

Student view

Revisiting right-angled triangles



More information

The diagram depicts a right triangle labeled (\text{ABC}). The letters (A), (B), and (C) are positioned at the triangle's vertices. The side opposite the right angle, (AB), is labeled (c). The side (BC) is labeled (a) and is perpendicular to (AB). The side (AC) is labeled (b), forming the hypotenuse of the triangle. There is a small red square at vertex (B) indicating it is a right angle. Each side is marked along its length with a lowercase letter: (a), (b), and (c), corresponding to the triangle's sides. The diagram illustrates the relationship and labeling of the sides in a right triangle configuration.

[Generated by AI]

In the diagram above, ABC is a right triangle.

For this triangle, you can write Pythagoras' theorem as

$$c^2 + a^2 = b^2$$

If you divide both sides of the identity by b^2 , you get

$$\frac{c^2}{b^2} + \frac{a^2}{b^2} = 1,$$

which is also

$$\left(\frac{c}{b}\right)^2 + \left(\frac{a}{b}\right)^2 = 1 \quad \textbf{(1)}$$

The trigonometric ratios are

$$\cos A = \frac{c}{b} \text{ and } \sin A = \frac{a}{b}$$

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so you can substitute them into identity (1) to get

$$(\cos A)^2 + (\sin A)^2 = 1$$

Therefore, for any right-angled triangle with an acute angle A ,

$$\cos^2 A + \sin^2 A = 1$$

Be aware

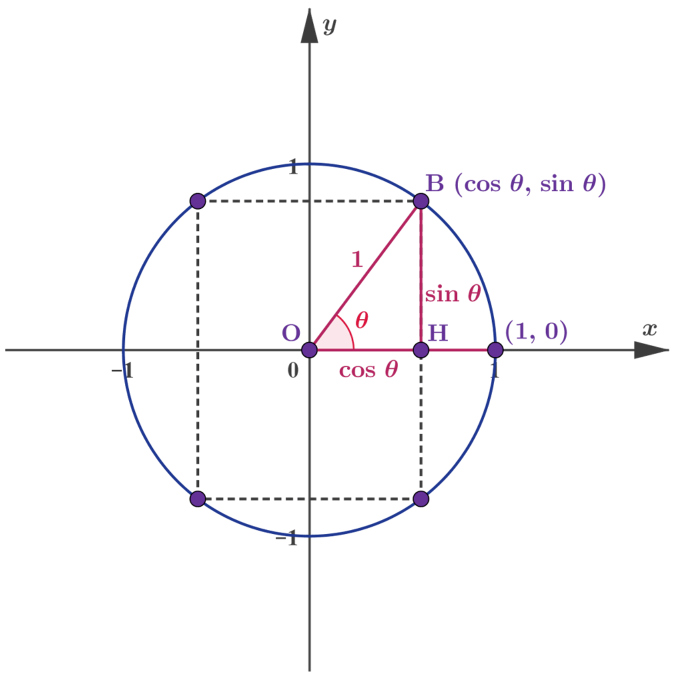
Squares of trigonometric ratios can be written as $(\cos A)^2$ or $\cos^2 A$.

The term $\cos A^2$ without the brackets is the cosine of the square of the angle A , i.e. $\cos(A^2)$.

Revisiting the unit circle

After defining the relation between the sine and cosine of an acute angle, do you think this identity holds true for any angle, meaning angles greater than 90° (or $\frac{\pi}{2}$)?

Solution Student (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/exact-values-id-28016/print/) Assign



More information

The image depicts a unit circle centered at the origin of a coordinate system, labeled with axes x and y. A right triangle, labeled O, B, H, is inscribed within the circle. Point O is at the origin, point H is on the x-axis at (cos θ, 0), and point B is on the circumference at (cos θ, sin θ).

The triangle O, B, H is right-angled at H. The angle at O is θ. Segment OH is labeled as 'cos θ', segment BH as 'sin θ', and the hypotenuse OB is 1 (the radius of the circle). The coordinates for point H are (1, 0) on the x-axis and (cos θ, sin θ) for point B on the circle.

The illustration is used to show the relation for sine and cosine in trigonometry, where sin θ and cos θ are the lengths of the sides of the right triangle, relating the angle θ to the unit circle.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

[Generated by AI]

As shown above, triangle OBH is a right-angled triangle so you can use Pythagoras' theorem:

$$OH^2 + BH^2 = OB^2$$

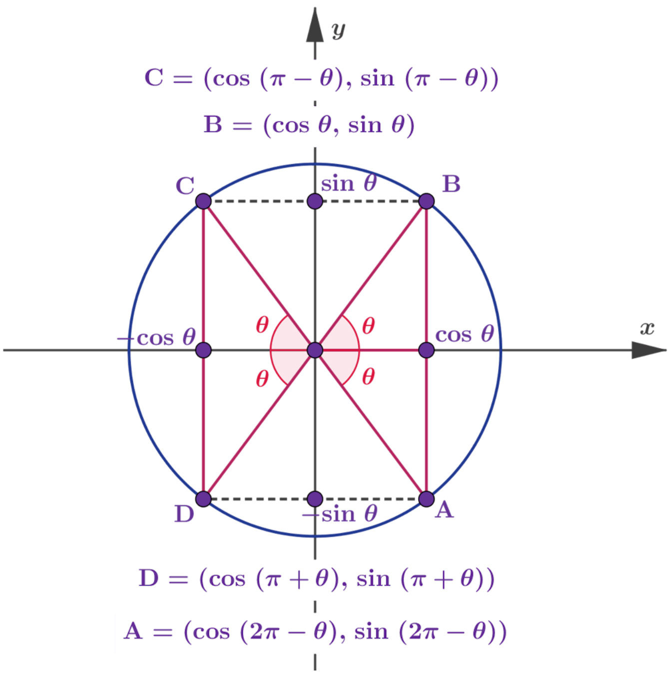
If you substitute the values of each length

$$\cos^2\theta + \sin^2\theta \equiv 1\,,$$

which is also known as the Pythagorean identity.

You can extend the idea to other quadrants using a unit circle and a rectangle, as shown below.

The points on the circumference of the unit circle are symmetrical. As the right triangles formed in each quadrant are congruent, each will have a base length of $\cos\theta$ and a height length of $\sin\theta$.



More information

This diagram illustrates a unit circle centered at the origin of an X and Y-axis with several points labeled as A, B, C, and D. The unit circle is divided into four quadrants, with lines connecting each labeled point to the center and to each other, creating four symmetrical right triangles within the circle. Each point on the circle is associated with specific cosine and sine values:

- Point A is at (cos(2π − θ), sin(2π − θ))
- Point B is at (cos(θ), sin(θ))
- Point C is at (cos(π − θ), sin(π − θ))
- Point D is at (cos(π + θ), sin(π + θ))

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

The diagram shows these coordinates next to their respective points and highlights the angle θ in each quadrant, demonstrating the symmetry and congruence of the right triangles formed by these positions.

[Generated by AI]

For example, point D has angle $\pi + \theta$. Here, $\cos(\pi + \theta) = -\cos\theta$ and $\sin(\pi + \theta) = -\sin\theta$

Therefore,

$$\cos^2(\pi + \theta) + \sin^2(\pi + \theta) = (-\cos\theta)^2 + (-\sin\theta)^2 = \cos^2\theta + \sin^2\theta$$

Finally, using the Pythagorean identity

$$\cos^2(\pi + \theta) + \sin^2(\pi + \theta) = 1$$

You can show that the identity holds for other quadrants as well following a similar approach.

Thus, the Pythagorean identity holds true for angles in quadrants I, II, III and VI.

✓ **Important**

An identity is always true.

Therefore, $\cos^2\theta + \sin^2\theta \equiv 1$ holds for all values of θ.

Exam tip

The Pythagorean identity can be found in the IB formula booklet as:

$$\cos^2\theta + \sin^2\theta = 1$$

Example 1

★★☆

Find the exact values of $\cos\alpha$ if $\sin\alpha = \frac{1}{3}$.

Steps	Explanation
$\cos^2\alpha + \left(\frac{1}{3}\right)^2 = 1$	Using the Pythagorean identity $\cos^2\alpha + \sin^2\alpha = 1$

Student view

Overview (/study/ap ai- hl/sid- 132- cid- 761618/ov

Steps	Explanation
$\cos^2\alpha = 1 - \frac{1}{9}$ $\cos\alpha = \sqrt{\frac{8}{9}}$ or $\cos\alpha = -\sqrt{\frac{8}{9}}$ Therefore, $\cos\alpha = \frac{2\sqrt{2}}{3}$ or $\cos\alpha = -\frac{2\sqrt{2}}{3}$	Solving for $\cos\alpha$. Use both positive and negative values of the square root as the range of the angle is not specified. You should leave your answer in surd form as the exact value is asked for.

Example 2

★★☆

Simplify the expression $\frac{1-\cos^2\theta 1-\mathrm{si}}{\mathrm{n}^2\theta}$

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/extension-of-the-sine-rule-id-28017/print/) Assign

Steps	Explanation
$\frac{1-\cos^2\theta 1-\mathrm{si}}{\mathrm{n}^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \left(\frac{\sin\theta}{\cos\theta}\right)^2$	Use the Pythagorean identity in the forms $\cos^2\theta = 1 - \sin^2\theta$ and $\sin^2\theta = 1 - \cos^2\theta$
Therefore $\frac{1-\cos^2\theta 1-\mathrm{si}=\mathrm{ta}}{\mathrm{n}^2\theta}\mathrm{n}^2\theta$	Using $\tan\theta = \frac{\sin\theta}{\cos\theta}$

International Mindedness

The Pythagorean identity is named after Pythagoras. Many other famous identities are named after famous mathematicians. For example, Euler's identity (which is also known as Euler's equation) is

$$\mathrm{e}^{\mathrm{i}\pi} + 1 = 0$$

where e is Euler's number, i is the imaginary number such that $\mathrm{i}^2 = -1$, and π is the ratio of the circumference of a circle to its diameter.

Although this identity is considered to be one of Euler's monumental results, some historians claim that he might have learned about it from another mathematician, Bernoulli. Like Pythagoras, he may have taken someone else's work and published it under his name. What do you think the consequences of such an action would be today?

Student view

2 section questions

Trigonometric equations

Finite interval equations: graphical method

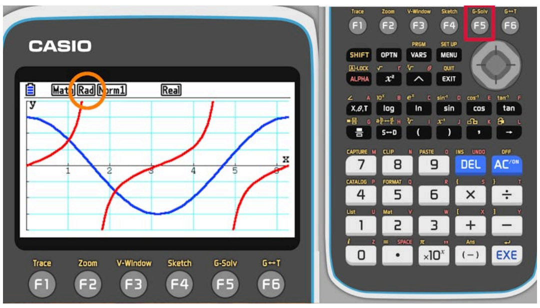
The graphical method for solving trigonometric equations often requires the use of your graphic display calculator.

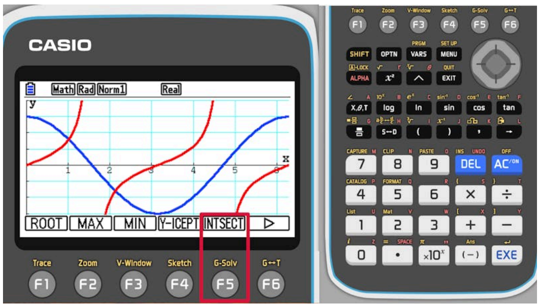
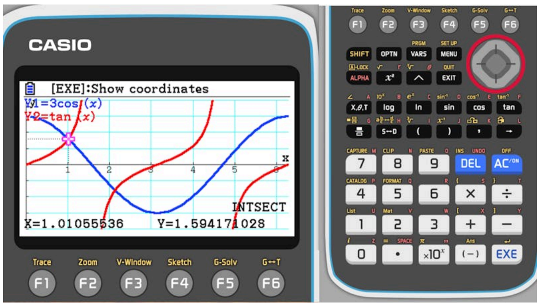
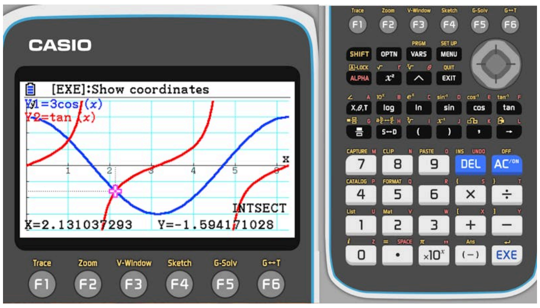
Let us consider the equation

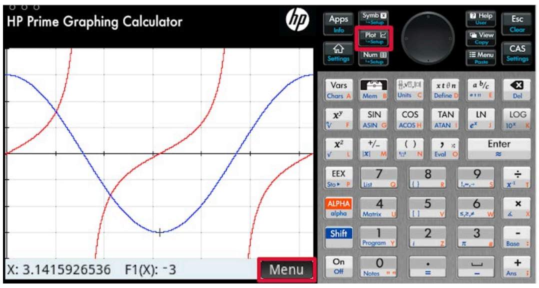
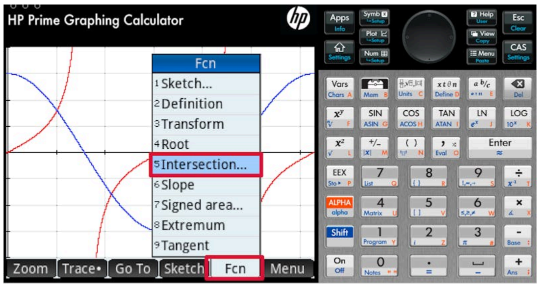
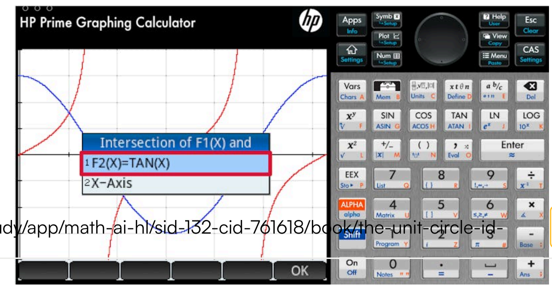
$$3\cos x = \tan x\text{ , } x \in [0, 2\pi]$$

If you define each side of the equation as a function, $y = 3\cos x$ and $y = \tan x$, the solution(s) of this equation will be the intersection point of two functions. That is, of course, if they intersect.

In the following, we show how to solve this equation with different graphic display calculator models.

Steps	Explanation
In these instructions you will see how to find the solutions of the equation $3\cos x = \tan x$ for $0 \le x \le 2\pi$ It is assumed, that you have the graph of $y = 3\cos x$ and $y = \tan x$ on the screen for $0 \le x \le 2\pi$. Make sure that the calculator is in radian mode. Press F5 (G_Solve) to bring up the options to analyse the graphs ...	

Steps	Explanation
... and F5 again to look for the intersection points of the graphs.	
The calculator moves the cursor to the first intersection point and displays its coordinates. The x-coordinate is the first solution of the equation. To find the other intersection point, move to the right.	
The cursor is moved to the second intersection point. The x-coordinate is the second solution of the equation.	

Steps	Explanation
In these instructions you will see how to find the solutions of the equation $$3\cos x = \tan x \text{ for } 0 \le x \le 2\pi$$ It is assumed, that you have the graph of $y = 3\cos x$ and $y = \tan x$ on the screen for $0 \le x \le 2\pi$. Make sure that the calculator is in radian mode. In the plot view of the function application tap on menu ...	
... and choose the option to find the intersection points of the two graphs.	
Choose the two graphs.	

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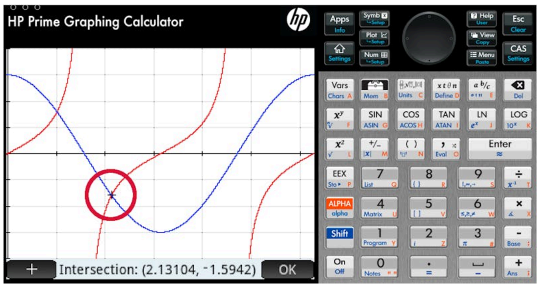
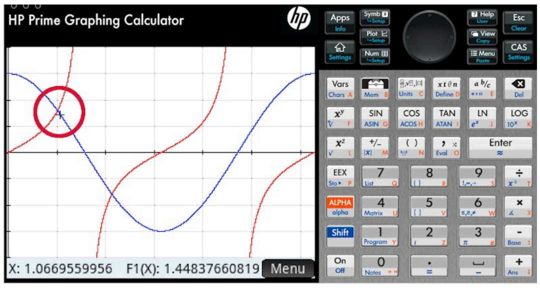
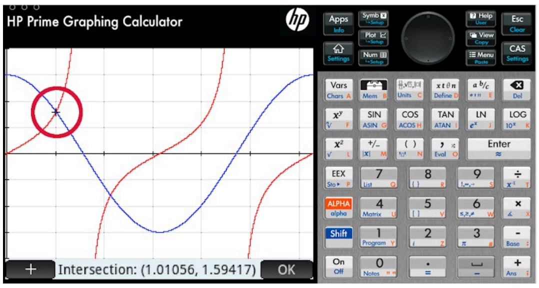
Steps	Explanation
The calculator moves the cursor to one of the intersection points and displays its coordinates. The x-coordinate is the first solution of the equation.	
To find the other intersection point, move the cursor close to it and repeat the process (menu ...).	
The cursor is moved to the second intersection point. The x-coordinate is the second solution of the equation.	

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<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>

In these instructions you will see how to find the solutions of the equation

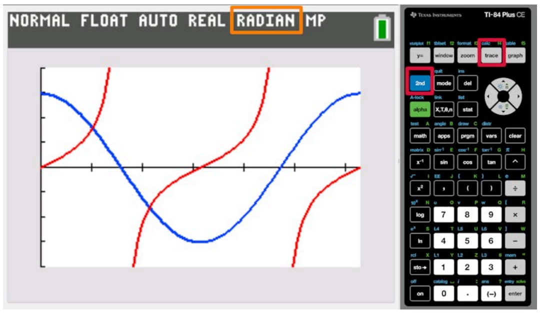
$$3\cos x = \tan x \text{ for } 0 \leq x \leq 2\pi$$

It is assumed, that you have the graph of $y = 3\cos x$ and $y = \tan x$ on the screen for $0 \leq x \leq 2\pi$.

Make sure that the calculator is in radian mode.

Press 2nd/calc to bring up the options to analyse the graphs ...

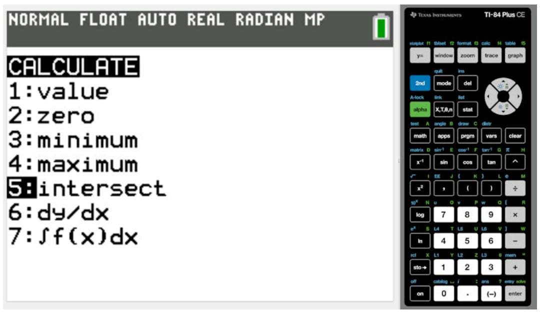
</td><td>



</td></tr>
<tr><td>

... and choose the option to find the intersection points of the two graphs.

</td><td>



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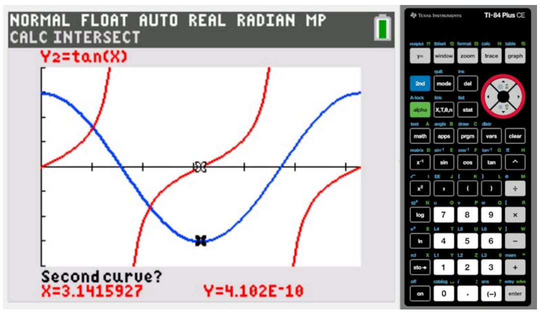
<tr><td>

The calculator will ask you to identify the first and the second curve (move the cursor to the curve and press enter to confirm) ...

</td><td>



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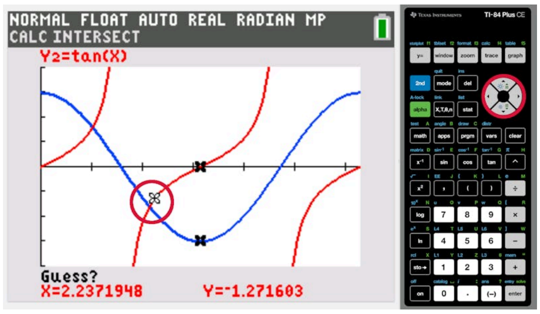
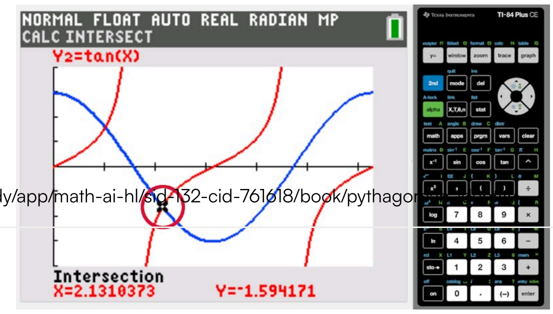
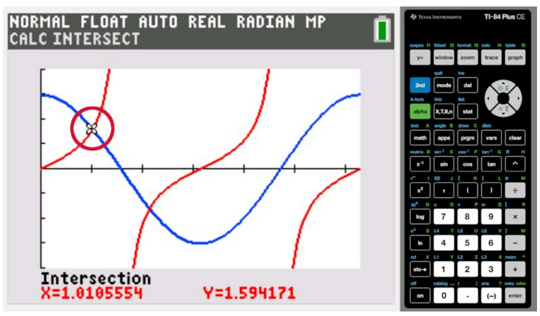
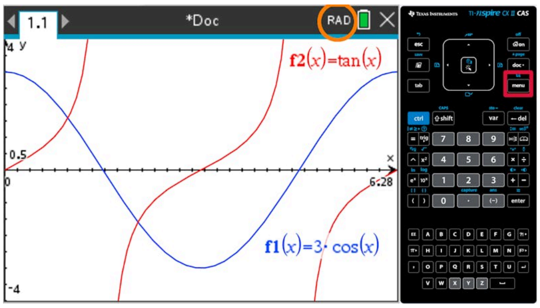
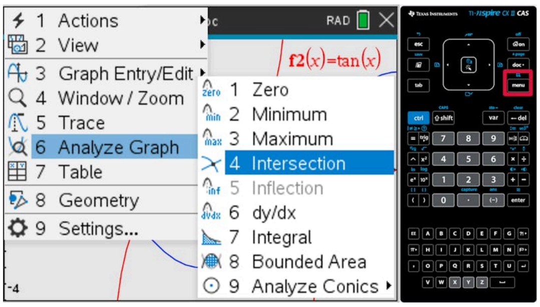
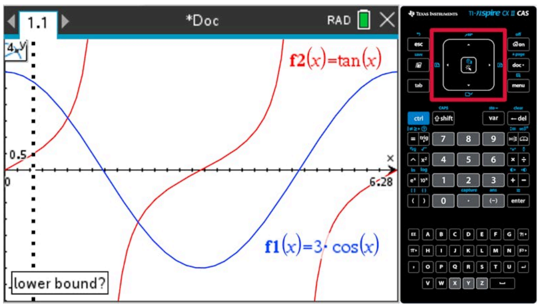
</table>

Student
view

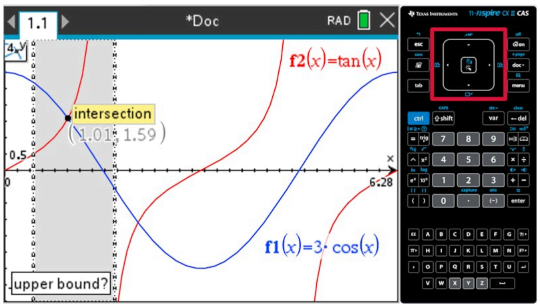
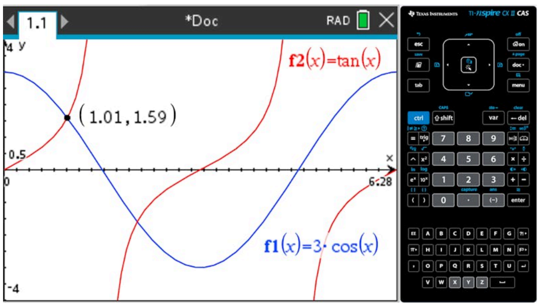
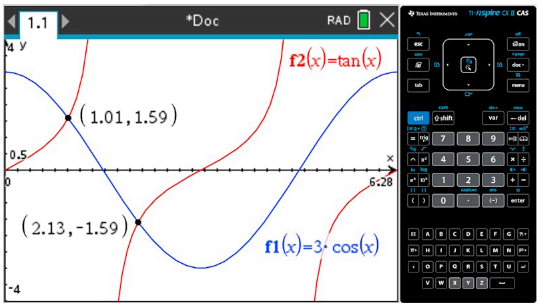
Steps	Explanation
... and the calculator asks you to make a guess. Move the cursor close to the intersection point.	
The calculator moves the cursor to the intersection point and displays its coordinates. The x-coordinate is the first solution of the equation.	
To find the other intersection point, repeat the process (2nd/calc, ...). This time type $x = 1$ when the calculator asks for a guess (since from the graphs you can see, that the intersection point is close to that value). The cursor is moved to the second intersection point. The x-coordinate is the second solution of the equation.	

Overview (/study/app/ai-hl/sid-132-cid-761618/ov

Steps	Explanation
In these instructions you will see how to find the solutions of the equation $$3\cos x = \tan x \text{ for } 0 \leq x \leq 2\pi$$ It is assumed, that you have the graph of $y = 3\cos x$ and $y = \tan x$ on the screen for $0 \leq x \leq 2\pi$. Make sure that the calculator is in radian mode. Press menu to bring up the options to analyse the graphs ...	
... and choose the option to find the intersection points of the two graphs.	
The calculator needs to know which intersection point you are looking for. Move to the left of the point and press enter to confirm the position ...	

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
... then move to the right of the intersection point and press enter.	
The calculator marks the intersection points and displays its coordinates. The x-coordinate is the first solution of the equation.	
Repeat the process (menu ...) to find the second intersection point. The x-coordinate is the second solution of the equation.	

In the above, we graphed both $3\cos x$ and $\tan x$ over an interval that included $[0, 2\pi]$, and found the points of intersection with the graphic display calculator. The graph is reproduced in the diagram below.

Student view

Overview
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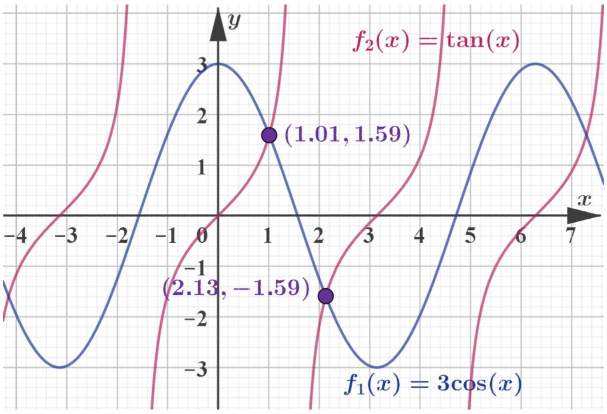

More information

The image features a graph depicting two trigonometric functions, 3cos(x) in blue and tan(x) in red, plotted over the interval [0, 2π]. The graph includes a grid and arrows indicating the axes. The x-axis is labeled with values from 0 to 7, while the y-axis ranges from -3 to 3. Two points of intersection are marked on the graph: one at approximately (1.01, 1.59) and another at around (2.13, -1.59). These points are highlighted to show where the curves meet. The function 3cos(x) oscillates between 3 and -3, completing several cycles, while tan(x) shows repeating asymptotes and crossing points due to its periodic nature. The intersections occur where the curves cross.

[Generated by AI]

As you will have experienced, finding intersection points with your GDC though easy, is time-consuming. Thus, if you are not required to find the values of the intersection points, but merely find the number of intersection points, all you need to do is plot the two functions and count the points of intersection.

Thus, if the question states: ‘Find the number of solutions to $\cos^2 x = x^5 + 4x^4 - x^3 - 6x^2 + x + 1.5$’, you need not find the values of the solutions. Hence, plotting these two functions and counting the points of intersection, we see that there are five solutions (see below).

Be aware

When solving trigonometric equations graphically with your GDC, make sure your GDC is set to the correct unit of angle, i.e. degrees or, usually, radians.

Student view

Overview
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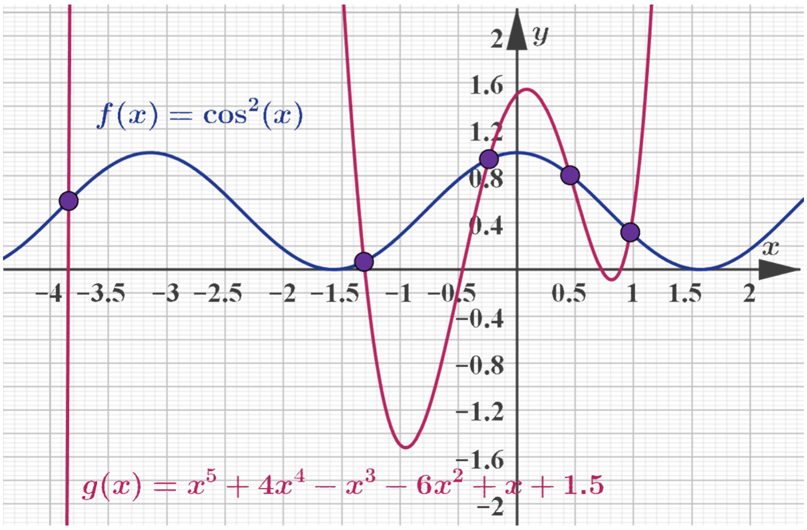


More information

This image is a graph showing two functions: f(x) = cos²(x) and g(x) = x³ + 4x² - x - 1.5. The x-axis ranges from -3 to 3 with increments marked every 0.5 units. The y-axis ranges from -2.5 to 1.6, also marked in increments of 0.5.

The blue curve represents the function f(x) = cos²(x), which oscillates between 0 and 1, showing periodic peaks and troughs at regular intervals.

The red curve represents g(x) and crosses the x-axis approximately at x = -3, x = -1.5, and x = 2.

Both functions intersect at several points, marked with purple dots, illustrating the intersections between f(x) and g(x). Overall, the graph displays a combination of trigonometric and polynomial function behavior, highlighting the periodic nature of f(x) compared to the algebraic expression of g(x).

[Generated by AI]

Important

When solving trigonometric equations, make sure you check your answers using graphical methods. Often, your calculator will give you only one of the possible answers.

Activity

Using a graphing app such as GeoGebra, graph the function $f(x) = 3 + 2\sin\left(\frac{\pi x}{6}\right)$, $0 \leqslant x \leqslant 3\pi$

Create a slider $0 \leqslant a \leqslant 6$.

Moving the slider, investigate when $f(x) = a$ has

- no solution
- one solution
- two or more solutions.

Explain algebraically when $f(x) = a$ has a solution and the nature of the roots.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Making connections

There are many real-life applications of trigonometry. It is used in heart rate monitors, in video games to produce smooth movements, in construction, in physics, in chemistry, etc.

Chemists use Bragg's law, which states that when an x-ray is incident onto a crystal surface with angle of incidence θ , the x-ray will be scattered at the same angle of scattering θ , which is given by

$$n\lambda = 2d\sin\theta$$

where λ is the wavelength of the x-ray, d is the spacing of the crystal layers (path difference), θ is the angle of incidence (the angle between the incident ray and the scattering plane), and n is an integer.

Based on this law, a Bragg spectrometer can be used to study the structure of crystals and molecules.

Ammonium dihydrogen phosphate crystals

Activity

Use the applet below to explore the number of solutions for some simple trigonometric equations.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Interactive 1. Exploring the Number of Solutions for Simple Trigonometric Equations.

Credit: GeoGebra (https://www.geogebra.org/m/NmnpVUmd) Paul Walter

More information for interactive 1

This interactive tool enables users to explore and solve trigonometric equations of the form $\sin(bx) = y$, $\cos(bx) = y$, or $\tan(bx) = y$ within a customizable domain.

The screen is divided into two halves. On the top side of the screen, a graph is displayed with an xy axis, the x-axis representing the angle (in degrees) ranging from -540 to 540, and the y-axis ranging from -2 to 2. A horizontal dashed line along the x-axis represents the constant value of y. On the bottom half of the screen, the “radians” checkbox allows users to toggle the x-axis units between degrees and radians. Below it, users can select a checkbox of one trigonometric function—sine, cosine, or tangent—and the tool will display its graph alongside a horizontal line representing a user-defined constant y-value.

The parameter b can be adjusted using a horizontal slider ranging from -3 to 3 to change the frequency of the trigonometric function, while the domain is set by dragging horizontal sliders to define the start and end points on the x-axis, which is labeled in terms of radians.

Intersection points between the trigonometric graph and the horizontal line, which represent the solutions to the equation, are marked with dots, and users can opt to display these solutions in degrees or radians by enabling a "Show solutions" feature. Additional options allow users to toggle the display of x-axis values in radians and switch between the trigonometric functions, with the graph dynamically updating as parameters like y, b, and the domain are modified, providing a hands-on way to visualize and understand trigonometric equations and their solutions.

Create some more complex trigonometrical equations and use your graphic display calculator to graph them.

Can you tell from the graph how many solutions they have? Can you find the values of these solutions from the graph?

2 section questions

3. Geometry and trigonometry / 3.8 Trigonometric ratios beyond acute angles

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/checklist-id-28020/print/) Assign

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

What you should know

By the end of this subtopic you should be able to:

- calculate trigonometric ratios in the unit circle
- recognise and use the exact values of trigonometric ratios for special angles
- recognise the ambiguous case of the sine rule
- use the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$
- use your graphic display calculator to identify the number of solutions of a trigonometric equation
- solve trigonometric equations using your graphic display calculator.

3. Geometry and trigonometry / 3.8 Trigonometric ratios beyond acute angles

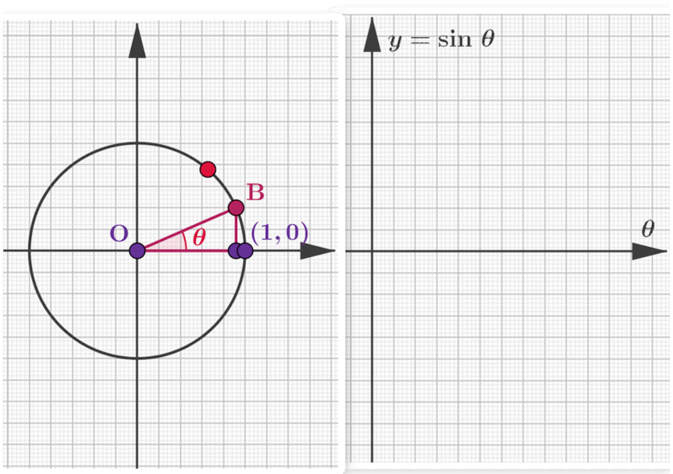
Investigation

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/investigation-id-28021/print/) Assign

Draw a unit circle on a sheet of graph paper using the scale 5 cm for 1 unit.

Draw two axes on a second sheet of graph paper, where the horizontal axis represents the angle θ and $y = \sin\theta$.

Place the two papers next to each, as shown below.



More information

The image consists of two parts side-by-side illustrating the relationship between a unit circle and the sine function.

The left part shows a unit circle on a set of X and Y axes. The center of the circle is labeled as point O. A line extends from the center, labeled as point O, to the circumference of the circle at point B. This line forms an angle (\theta) with the positive X-axis. The coordinates (1,0) indicate a point on the circle's edge on the horizontal axis. The length of the line OB is shown with a color gradient from purple to red, marking its terminals on the circle periphery.

The right part is a graph plotting (y = \sin \theta) against (\theta). The Y-axis is labeled as (y = \sin \theta) and the X-axis is labeled with (\theta). Each axis bears arrows indicating positive directions and grid lines facilitate perusal.

Student view

[Generated by AI]

1. Mark a point B on the unit circle. Then:
   - Connect the point B to the centre and draw the right-angled triangle.
   - Measure the angle θ using a protractor.
   - Measure the lengths of the right sides of the triangle.
   - Calculate the sine of the angle θ .
2. Plot the point $(\theta, \sin\theta)$ on the second graph.
3. Mark at least 18 more points spread around the circumference of the circle for $0° \leqslant \theta \leqslant 360°$ and repeat Step 1 so that you can complete the following table.

θ	$\sin\theta$	$\cos\theta$	Point $(\theta, \sin\theta)$
0	0	1	$(0, 0)$
$\ldots$	$\ldots$	$\ldots$	$\ldots$

4. Plot the points from the table onto the second graph. What do you notice?
5. For the same set of angles in the table, and using a different coloured pen, plot the points (θ , cos θ) on the second graph.

θ	$\cos\theta$	Point $(\theta, \cos\theta)$
0	0	$(0, 1)$
$\ldots$	$\ldots$	$\ldots$

6. What are the similarities and differences between the two curves?

Where do the two curves intersect?

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