Index

The big picture
Vector product
Algebraic properties
Geometric interpretations
Checklist
Investigation

The big picture

In subtopic 3.13 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-28034/), you studied the scalar product of two vectors and interpreted this as directional growth. In doing so, you considered the components of two vectors acting in the same dimension, parallel to the x-, y- or z-axes.

This subtopic considers the vector product (or cross product) of two vectors. Geometrically, the vector product provides a way of finding a vector perpendicular to a plane if you already know two vectors in the plane.

In physics, it can be used to calculate torque, angular momentum and the magnetic force on a moving charge.

You can think this as using a screwdriver to tighten a screw. You turn the screwdriver and the screw moves in a direction that is perpendicular to the force you apply. Watch the video to see an example. How could you model the forces applied to the screw and its motion?

Video 1. Vector Product in Action: The Screwdriver Example.

More information for video 1

This video illustrates the concept of the vector product through the action of manually tightening a screw using a screwdriver. Viewers can observe how the rotational motion of the hand occurs in one plane while the screw moves along both the horizontal and vertical planes. This demonstrates how vectors help describe combined movements and perpendicular forces. By analyzing this motion, users can explore the relationship between force and displacement, gaining insights into torque, angular momentum, and applications in physics and engineering, such as mechanical systems and electromagnetism.

Concept

The vector product can be used to find the area of a parallelogram whose sides are two vectors in a plane. In three dimensions, it can be used along with the scalar product to find the area of a parallelepiped, which is a six-sided solid whose faces are parallelograms. The vector product of two three-dimensional vectors represents a vector that is perpendicular to both of these vectors. Is it possible to generalise the vector product of n-dimensional vectors and, if so, what would this represent?

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

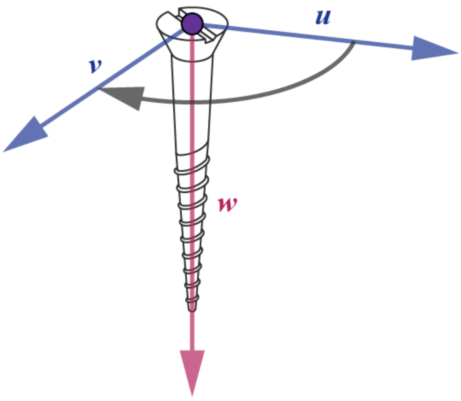
Vector product

Orthogonal vectors

The vector product, which is also known as the cross product, is a vector that represents the result of interactions between components of two vectors in different dimensions, x, y and z .

When would you need to find the result of interactions in different directions, or dimensions?

Consider the example of the screwdriver given in The big picture (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-28028/). If you turn the screwdriver in one direction it will tighten the screw, and if you turn it in the other direction it will loosen the screw, as shown in the diagram on the left below. You do not need to apply a force in the direction in which the screw moves as you must do when using a hammer to drive in a nail.



Turning in the direction
from $\boldsymbol{u}$ to $\boldsymbol{v}$ or $\boldsymbol{u} \times \boldsymbol{v}$

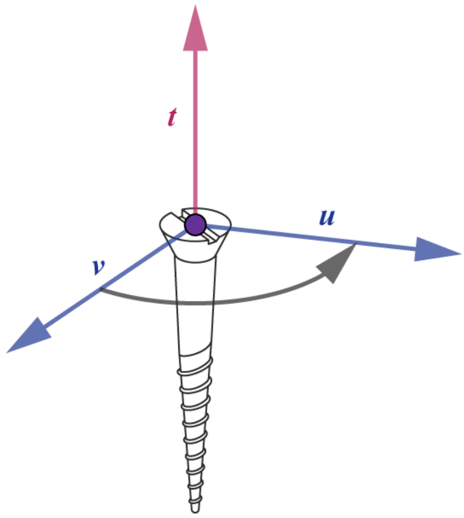
More information

The diagram illustrates a screw with arrows labeled u, v, and w. Arrow u points in one direction, indicating the initial motion. Arrow v points in a different direction, suggesting a rotation action or secondary motion. The arrow w points downward along the screw's axis, representing a downward force or resultant direction when rotating from u to v or u

Student view

cross v. The text "Turning in the direction from u to v or u × v" is shown at the bottom, explaining the rotational dynamics depicted in the diagram.

[Generated by AI]



Turning in the direction
from $\boldsymbol{v}$ to $\boldsymbol{u}$ or $\boldsymbol{v} \times \boldsymbol{u}$

More information

The image is a 3D illustration depicting three vectors: (\boldsymbol{u}), (\boldsymbol{v}), and (\boldsymbol{t}), set in a coordinate space. The vector (\boldsymbol{u}) is shown in blue, pointing to the right, while vector (\boldsymbol{v}) is also in blue, directed towards the viewer. These two vectors form a plane. The vector (\boldsymbol{t}) is drawn in red, extending upwards, perpendicular to the plane created by (\boldsymbol{u}) and (\boldsymbol{v}). At their intersection point, a circular arrow indicates the rotational direction from (\boldsymbol{v}) to (\boldsymbol{u}), supporting the concept of orthogonality and the cross product equation (\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{t}). This visual aids in understanding vector relationships and orthogonality through the screw analogy, where turning from (\boldsymbol{v}) to (\boldsymbol{u}) advances in the direction of (\boldsymbol{t}).

[Generated by AI]

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

The resulting vectors, $\boldsymbol{w}$ or $\boldsymbol{t}$, are perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$. They are also called orthogonal vectors. These relationships could be represented by the vector product as $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w}$ and $\boldsymbol{v} \times \boldsymbol{u} = \boldsymbol{t}$.

What is the connection between $\boldsymbol{w}$ and $\boldsymbol{t}$?

The direction in which you turn the screwdriver affects the direction of motion of the screw, so $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w}$ but $\boldsymbol{v} \times \boldsymbol{u} = \boldsymbol{t} = -\boldsymbol{w}$.

Therefore, the vector product is not commutative.

Be aware

As the vector product is also called the cross product, the notation used to denote it is

$$\boldsymbol{a} \times \boldsymbol{b}$$

Recall that the dot or scalar product is denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$.

The following video shows how the vector product is represented in 3D.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Video 1. Visualizing Vector Product.

More information for video 1

1

00:00:00,333 --> 00:00:02,067

narrator: So far we've taken two vectors

2

00:00:02,133 --> 00:00:05,400

and asked our ourself

what the scalar product was of those.

3

00:00:06,133 --> 00:00:09,300

And now we're gonna investigate

the other properties

4

00:00:09,367 --> 00:00:13,133

associated with vectors

and that is the vector product.

5

00:00:13,333 --> 00:00:15,600

Remember, the scalar product

is called the dot product

6

00:00:15,667 --> 00:00:18,767

and this one will be appropriately

called the cross product.

7

00:00:19,633 --> 00:00:22,467

Now here I've taken

the cross product of u and v,

8

00:00:22,533 --> 00:00:24,667

which are two vectors

in three dimensional space

9

00:00:24,867 --> 00:00:28,300

and there you can see the purple

vector is the cross product

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

10

00:00:28,500 --> 00:00:30,767

of u and v.

11

00:00:30,933 --> 00:00:32,733

Now you can see that it is another vector,

12

00:00:32,800 --> 00:00:35,267

unlike the scalar product,

which of course was a scale.

13

00:00:35,333 --> 00:00:37,333

So the vector product is a vector.

14

00:00:37,400 --> 00:00:41,600

Now if I plot a plane through

u and v, which I've called pi,

15

00:00:41,900 --> 00:00:43,933

then we're going to see

something interesting

16

00:00:44,000 --> 00:00:47,933

about the relative orientation

of u times v,

17

00:00:48,000 --> 00:00:52,667

the vector product with u and v relative

to that plane that goes through u and v.

18

00:00:52,733 --> 00:00:55,933

And you can see here that

it seems to be 90 degrees.

19

00:00:56,000 --> 00:00:59,367

And indeed if I do measure

that angle between the plane

20

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

00:00:59,733 --> 00:01:02,700

and the cross product of u and v,

21

00:01:02,767 --> 00:01:06,133

then it is indeed 90 degrees.

22

00:01:06,600 --> 00:01:09,267

So that is a property that

we need to keep in mind

23

00:01:09,333 --> 00:01:13,667

that the cross product of two vectors lies

perpendicular to the plane

24

00:01:13,733 --> 00:01:17,500

through those two vectors

or the plane containing those two vectors.

25

00:01:17,567 --> 00:01:19,567

Alright,

so that's something to keep in mind.

26

00:01:20,233 --> 00:01:22,633

Now here of course I've taken

the cross product

27

00:01:22,700 --> 00:01:25,933

between u and v, that is u times v,

28

00:01:26,533 --> 00:01:30,000

and now we're gonna take

the cross product of v and u

29

00:01:30,067 --> 00:01:33,333

and you can see that

those are not the same objects.

30

00:01:33,400 --> 00:01:36,767

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

So as 2 times 5 is the same as 5 times

2 with numbers,

31

00:01:36,833 --> 00:01:41,900

that does not appear to be the case

with the vector product.

32

00:01:41,967 --> 00:01:43,700

However you do see,

33

00:01:43,767 --> 00:01:46,833

I hope that there is a special

relationship between them.

34

00:01:46,900 --> 00:01:48,300

They lie in a line

35

00:01:48,367 --> 00:01:52,567

and indeed therefore v cross

with u is also perpendicular

36

00:01:52,633 --> 00:01:54,400

to the plane, including v and u.

37

00:01:55,033 --> 00:01:58,267

And v cross u is anti-parallel

38

00:01:58,333 --> 00:02:03,400

to u cross v as we can clearly

see in these cases.

39

00:02:03,500 --> 00:02:06,900

So in other words, the order in which you

40

00:02:07,300 --> 00:02:11,200

apply the cross product

in which you perform the cross product

41

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

00:02:11,267 --> 00:02:16,667

does make a difference in terms

of vector product multiplication.

42

00:02:16,733 --> 00:02:19,100

Alright, so this is the other

multiplication

43

00:02:19,167 --> 00:02:20,433

that we can do with two vectors.

44

00:02:20,500 --> 00:02:23,100

We have the cross product,

this one and a dot product.

45

00:02:23,667 --> 00:02:27,367

Now here is a two dimensional case

of a vector u and v,

46

00:02:27,867 --> 00:02:31,767

and you can now start

to wonder that where is u times v?

47

00:02:31,833 --> 00:02:33,533

I've just asked

this software to create it,

48

00:02:33,600 --> 00:02:35,000

but I cannot see it.

49

00:02:35,200 --> 00:02:38,267

But of course it's obvious that you cannot

see it in a two dimensional case

50

00:02:38,333 --> 00:02:41,600

because the cross product

lies perpendicular

51

Student
view

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o

00:02:41,733 --> 00:02:43,733

to the plane involving u and v.

52

00:02:43,800 --> 00:02:48,633

So if I project u and v in the xy plane,

53

00:02:48,700 --> 00:02:52,633

then the u cross v

is going to be perpendicular,

54

00:02:52,700 --> 00:02:53,800

which is in the z direction,

55

00:02:53,867 --> 00:02:57,000

which of course I need a three

dimensional plot as we can clearly

56

00:02:57,433 --> 00:02:58,967

see over here.

57

00:02:59,400 --> 00:03:01,733

So in other words, those factors, u and v

58

00:03:02,867 --> 00:03:05,633

do not have a component

in the z direction.

59

00:03:05,733 --> 00:03:07,200

I can still plot them of course,

60

00:03:07,300 --> 00:03:09,433

in three dimensional space,

kind of boring.

61

00:03:09,733 --> 00:03:11,900

so if I take the cross product,

62

00:03:11,967 --> 00:03:15,900

Student
view

I need to look at that geometrical object

63

00:03:15,967 --> 00:03:19,300

in three dimensional space

to see that product.

Finding the components of the vector product

Consider the two vectors $\boldsymbol{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$, and their vector product $w = u \times v = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$.

How would you find the components of the vector $\boldsymbol{w}$?

Write these vectors in terms of the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.

So $\boldsymbol{u} = u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}$ and $\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}$.

The vector product is

$$\begin{aligned}
\boldsymbol{u} \times \boldsymbol{v} =& (u_x\boldsymbol{i} + u_y\boldsymbol{j} + u_z\boldsymbol{k}) \times (v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}) \\
=& u_x\boldsymbol{i} \times (v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}) + u_y\boldsymbol{j} \times (v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}) + \\
& u_z\boldsymbol{k} \times (v_x\boldsymbol{i} + v_yj + v_z\boldsymbol{k}) \\
=& u_x\boldsymbol{i} \times v_x\boldsymbol{i} + u_x\boldsymbol{i} \times v_y\boldsymbol{j} + u_x\boldsymbol{i} \times v_z\boldsymbol{k} + u_y\boldsymbol{j} \times v_x\boldsymbol{i} + u_y\boldsymbol{j} \times v_y\boldsymbol{j} + \\
& u_y\boldsymbol{j} \times v_z\boldsymbol{k} + u_z\boldsymbol{k} \times v_x\boldsymbol{i} + u_z\boldsymbol{k} \times v_y\boldsymbol{j} + u_z\boldsymbol{k} \times v_z\boldsymbol{k} \\
=& u_xv_x\boldsymbol{i} \times \boldsymbol{i} + u_xv_y\boldsymbol{i} \times \boldsymbol{j} + u_xv_z\boldsymbol{i} \times \boldsymbol{k} + u_yv_x\boldsymbol{j} \times \boldsymbol{i} + u_yv_y\boldsymbol{j} \times \boldsymbol{j} + \\
& u_yv_z\boldsymbol{i} \times \boldsymbol{k} + u_zv_x\boldsymbol{k} \times \boldsymbol{i} + u_zv_y\boldsymbol{k} \times \boldsymbol{j} + u_zv_z\boldsymbol{k} \times \boldsymbol{k}
\end{aligned}$$

But this can be simplified.

When finding the components of the vector product you need to find the cumulative differences between components which are not in the same direction,

$$\boldsymbol{i} \times \boldsymbol{i} = 0, \boldsymbol{j} \times \boldsymbol{j} = 0 \text{ and } \boldsymbol{k} \times \boldsymbol{k} = 0$$

Overview
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761926/o

as each pair of vectors are parallel.

Because the vector product gives a vector that is perpendicular to the original vectors,

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}\,,\ \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i} \text{ and } \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$$

Because the vector product is not commutative,

$$\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}\,, \boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i} \text{ and } \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$$

So the expression above simplifies to

$$\begin{aligned}\boldsymbol{u} \times \boldsymbol{v} &= u_x v_y \boldsymbol{k} - u_x v_z \boldsymbol{j} - u_y v_x \boldsymbol{k} + u_y v_z \boldsymbol{i} + u_z v_x \boldsymbol{j} - u_z v_y \boldsymbol{i} \\ &= u_y v_z \boldsymbol{i} - u_z v_y \boldsymbol{i} + u_z v_x \boldsymbol{j} - u_x v_z \boldsymbol{j} + u_x v_y \boldsymbol{k} - u_y v_x \boldsymbol{k} \\ &= (u_y v_z - u_z v_y)\,\boldsymbol{i} + (u_z v_x - u_x v_z)\,\boldsymbol{j} + (u_x v_y - u_y v_x)\,\boldsymbol{k}\end{aligned}$$

This can also be shown in a multiplication table.

Multiplication	v_x	v_y	v_z
u_x	$u_x v_x$	$u_x v_y$	$u_x v_z$
u_y	$u_y v_x$	$u_y v_y$	$u_y v_z$
u_z	$u_z v_x$	$u_z v_y$	$u_z v_z$

The entries in the diagonal represent multiplication of the components in the same direction so $\boldsymbol{u} \cdot \boldsymbol{v} = u_x v_x + u_y v_y + u_z v_z$. The other entries represent multiplication in different directions.

Therefore, the components of the vector product $\boldsymbol{w}$ are:

$$w_x = u_y v_z - u_z v_y$$

$$w_y = u_z v_x - u_x v_z$$

$$w_z = u_x v_y - u_y v_x$$

Student
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Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Exam tip

The IB formula book gives the components of a vector product as

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}, \text{ where } \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ and } \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

How do you know the direction the cross product vector will act?

If the two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are in the plane of the page on the screen, the cross product will give a vector that is perpendicular to the plane. But how do you know whether the cross product vector will act into the screen or out of the screen?

You can use the right-hand rule to decide.

Right-hand rule

To find the direction of the vector product $\boldsymbol{u} \times \boldsymbol{v}$:

- hold your right hand so that your fingers point in the direction of vector $\boldsymbol{u}$
- then curl your fingers so that they point towards vector $\boldsymbol{v}$
- your thumb will now point in the direction of the vector product $\boldsymbol{u} \times \boldsymbol{v}$.

Student view

Overview
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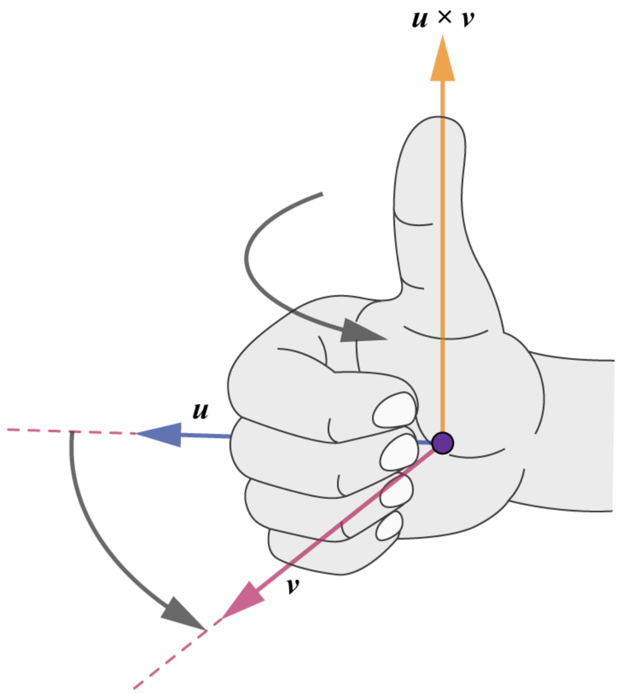
761926/o



More information

The image is a diagram of a right hand demonstrating the right-hand rule used in physics. The thumb is pointing upwards with an orange arrow, indicating the direction of the magnetic field. The fingers are curled in the direction of the current, depicted by a blue arrow on the left side of the hand. A red arrow at the bottom represents the force, and a purple dot indicates the axis of rotation. The arrows and labels show the relationship between magnetic field, current, and force according to the right-hand rule.

[Generated by AI]

Example 1

★☆☆

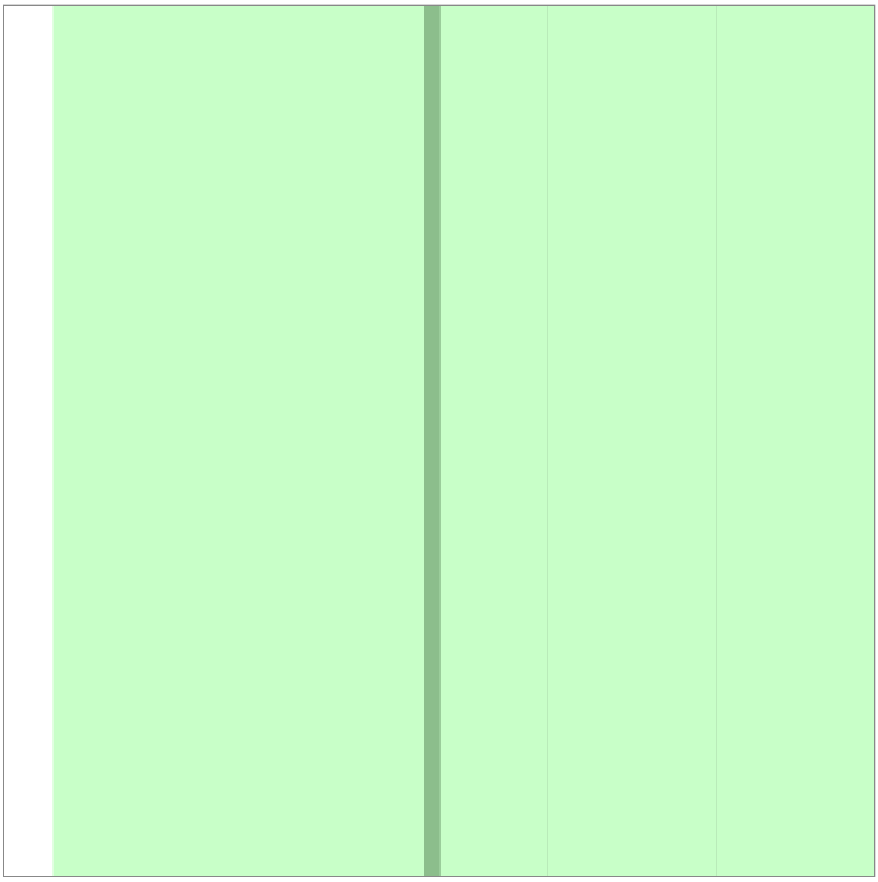
Let $\boldsymbol{v} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. Find the vector $\boldsymbol{v} \times \boldsymbol{w}$.

Student
view

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \times 1 - 3 \times (-2) \\ 3 \times 2 - (-2) \times 1 \\ (-2) \times (-2) - 1 \times 2 \end{pmatrix}$$
$$= \begin{pmatrix} 7 \\ 8 \\ 2 \end{pmatrix}$$

Activity

Use the applet below to explore how the cross product of two vectors changes as you change the vectors and the angle between them.You can drag the vectors, and also the plane they are on, to see the changes.



Credit: GeoGebra (https://www.geogebra.org/m/ckjq8dax) Nuriye Sirinoglu Singh

Overview
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761926/o

Important

The vector product of two vectors is itself a vector, and therefore it has a magnitude and a direction.

The vector product of two vectors is not commutative, i.e. $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{w} \times \boldsymbol{v}$.

In fact, $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$, i.e. they are parallel but point in opposite directions.

The vector product of two vectors $\boldsymbol{v} \times \boldsymbol{w}$ is oriented in a direction that is perpendicular to the plane containing $\boldsymbol{v}$ and $\boldsymbol{w}$.

Example 2

★★☆

Find a vector of length 3 that is perpendicular to both $-2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

You know that the vector product between these two vectors is perpendicular to the plane containing the vectors. Hence the vector product is perpendicular to each of these vectors in turn. Note that you found the vector product of these vectors in the previous example, so let $\boldsymbol{v} = -2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{w} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

Then,

$$\boldsymbol{v} \times \mathbf{w} = 7\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}$$

Clearly, this is not a vector of length 3 as $\sqrt{7^2 + 8^2 + 2^2} = \sqrt{117}$.

However, you can find a vector of length 1.

That is, a unit vector in the same direction:

$$\frac{1}{\sqrt{7^2 + 8^2 + 2^2}}\left(7\boldsymbol{i} + 8\boldsymbol{j} + 2\boldsymbol{k}\right) = \frac{1}{\sqrt{117}}\left(7\boldsymbol{i} + 8\boldsymbol{j} + 2\boldsymbol{k}\right).$$

Student
view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

This is a vector of length 1 that is parallel to $\boldsymbol{v} \times \boldsymbol{w}$ and therefore it is perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$. Now you can make it the required length of 3 by multiplying by 3.

So the vector $\frac{3}{\sqrt{117}}(7\boldsymbol{i} + 8\boldsymbol{j} + 2\boldsymbol{k})$ is of length 3 and is perpendicular to both $-2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.

Making connections

You found unit vectors in subtopic 3.12 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-28035/). Recall that a unit vector in the direction of vector $\boldsymbol{v}$ is given by $\widehat{\boldsymbol{v}} = \frac{1}{|\boldsymbol{v}|}\boldsymbol{v}$.

Example 3

★☆☆

Consider the three points $\mathrm{A}\,(1,1,1)$, $\mathrm{B}\,(-1,2,1)$ and $\mathrm{C}\,(-1,3,1)$. Find $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{CB}}$.

Steps	Explanation
$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} -1-1 \\ 2-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$	As $\overrightarrow{\mathrm{AB}} = \vec{\mathrm{B}} - \vec{\mathrm{A}} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\overrightarrow{\mathrm{CB}} = \vec{\mathrm{B}} - \vec{\mathrm{C}} = \begin{pmatrix} -1+1 \\ 2-3 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$	As $\overrightarrow{\mathrm{CB}} = \vec{\mathrm{B}} - \vec{\mathrm{C}} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - ($
$\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{CB}} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$	
$\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{CB}} = \begin{pmatrix} 1 \times 0 - 0 \times -1 \\ 0 \times 0 - (-2) \times 0 \\ (-2) \times (-1) - 1 \times 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$	Using $u \times w = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
Therefore, the answer is $$\overrightarrow{AB} \times \overrightarrow{CB} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$	

Be aware

In the IB examinations, examiner reports often identify that candidates make numerical mistakes when finding direction vectors or the cross product of vectors. It is always good practice to write each step of the calculation clearly to avoid such mistakes, even if you are using a calculator.

Making connections

You can use the determinant to find the components of the cross product of two vectors.

Let $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

Then,

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$
$$\boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{i}(v_2 w_3 - v_3 w_2) - \boldsymbol{j}(v_1 w_3 - v_3 w_1) + \boldsymbol{k}(v_1 w_2 - v_2 w_1)$$

Note the minus sign in front of the $\boldsymbol{j}$ component.

3 section questions

Question 1

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Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

If $\boldsymbol{a} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, what is $\boldsymbol{a} \times \boldsymbol{b}$?

1 $\begin{pmatrix} -5 \\ 3 \\ -7 \end{pmatrix}$ ✓

2 $\begin{pmatrix} -5 \\ -3 \\ -7 \end{pmatrix}$

3 $\begin{pmatrix} -5 \\ 5 \\ 0 \end{pmatrix}$

4 $\begin{pmatrix} -1 \\ 5 \\ 5 \end{pmatrix}$

Explanation

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \\ \Rightarrow \quad &= \begin{pmatrix} 3 \times (-1) - 2 \times 1 \\ 2 \times 2 - (-1) \times (-1) \\ (-1) \times 1 - 3 \times 2 \end{pmatrix} \\ \Rightarrow \quad &= \begin{pmatrix} -5 \\ 3 \\ -7 \end{pmatrix} \end{aligned}$$

Question 2

★★☆

What is a unit vector perpendicular to both $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$?

1 $\frac{1}{\sqrt{27}} \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$ ✓

2 $\frac{1}{\sqrt{42}} \begin{pmatrix} -4 \\ 5 \\ 1 \end{pmatrix}$

3 $\frac{1}{\sqrt{27}} \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

4 $\frac{1}{\sqrt{11}}\begin{pmatrix}1\\-3\\1\end{pmatrix}$

Explanation

The vector product of these two vector is

$$\begin{pmatrix}3\\-1\\2\end{pmatrix}\times\begin{pmatrix}1\\0\\-1\end{pmatrix}=\begin{pmatrix}1-0\\2+3\\0+1\end{pmatrix}=\begin{pmatrix}1\\5\\1\end{pmatrix}.$$

Thus, a unit vector in that direction is given by

$$\frac{1}{\sqrt{1^2+5^2+1^2}}\begin{pmatrix}1\\5\\1\end{pmatrix}=\frac{1}{\sqrt{27}}\begin{pmatrix}1\\5\\1\end{pmatrix}.$$

Question 3

★★☆

Find a vector which is perpendicular to both lines.

$$\mathrm{L}_1: \quad r_1=\begin{pmatrix}3\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\3\end{pmatrix}$$

$$\mathrm{L}_2: \quad r_2=\begin{pmatrix}-1\\2\\1\end{pmatrix}+t\begin{pmatrix}-1\\2\\2\end{pmatrix}$$

1 $\begin{pmatrix}-4\\-1\\-1\end{pmatrix}$ ✓

2 $\begin{pmatrix}-1\\-1\\-4\end{pmatrix}$

3 $\begin{pmatrix}1\\1\\4\end{pmatrix}$

4 $\begin{pmatrix}-1\\-4\\-1\end{pmatrix}$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Explanation

As the required vector is to be perpendicular to both lines, it will be perpendicular to the direction vectors of the lines.

$$\begin{pmatrix}-1\\1\\3\end{pmatrix}\times\begin{pmatrix}-1\\2\\2\end{pmatrix}=\begin{pmatrix}1\times2-3\times2\\3\times(-1)-(-1)\times2\\-1\times2-1\times(-1)\end{pmatrix}=\begin{pmatrix}-4\\-1\\-1\end{pmatrix}$$

3. Geometry and trigonometry / 3.16 Vector product

Algebraic properties

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/algebraic-properties-id-28030/print/) Assign

You need to be able to apply the following properties of the vector product of two 3D vectors.

✓ **Important**

For vectors $\boldsymbol{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$, $\boldsymbol{w}=\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}$ and $\boldsymbol{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$

1. $\boldsymbol{v}\times\boldsymbol{w}=-\boldsymbol{w}\times\boldsymbol{v}$ so the vector product is not commutative
2. $\boldsymbol{u}\times(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{u}\times\boldsymbol{v}+\boldsymbol{u}\times\boldsymbol{w}$ so the vector product is distributive
3. $(k\boldsymbol{v})\times\boldsymbol{w}=k(\boldsymbol{v}\times\boldsymbol{w}), k\in\mathbb{R}$
4. $\boldsymbol{v}\times\boldsymbol{v}=\mathbf{0}$

Can you prove these rules using the components of the vectors?

Example 1

★★☆

Prove that, for two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, $(\boldsymbol{u}-\boldsymbol{v})\times(\boldsymbol{u}+\boldsymbol{v})=2(\boldsymbol{u}\times\boldsymbol{v})$

Student view

Overview
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Steps	Explanation
$(\boldsymbol{u}-\boldsymbol{v}) \times (\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{u}\times\boldsymbol{u} + \boldsymbol{u}\times\boldsymbol{v} - \boldsymbol{v}\times\boldsymbol{u} - \boldsymbol{v}\times\boldsymbol{v}$	Using the distributive property of the cross product.
$\boldsymbol{u}\times\boldsymbol{u} + \boldsymbol{u}\times\boldsymbol{v} - \boldsymbol{v}\times\boldsymbol{u} - \boldsymbol{v}\times\boldsymbol{v} = \boldsymbol{u}\times\boldsymbol{v} - \boldsymbol{v}\times\boldsymbol{u}$	$\boldsymbol{u}\times\boldsymbol{u} = 0$ and $\boldsymbol{v}\times\boldsymbol{v} = 0$
$\boldsymbol{u}\times\boldsymbol{v} - \boldsymbol{v}\times\boldsymbol{u} = \boldsymbol{u}\times\boldsymbol{v} + \boldsymbol{u}\times\boldsymbol{v}$	$-\boldsymbol{v}\times\boldsymbol{u} = \boldsymbol{u}\times\boldsymbol{v}$
$\boldsymbol{u}\times\boldsymbol{v} + \boldsymbol{u}\times\boldsymbol{v} = 2(\boldsymbol{u}\times\boldsymbol{v})$	Simplify
Therefore, $(\boldsymbol{u}-\boldsymbol{v}) \times (\boldsymbol{u}+\boldsymbol{v}) = 2(\boldsymbol{u}\times\boldsymbol{v})$	

Example 2

★★☆

Prove that, for two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, $\boldsymbol{u}\cdot(\boldsymbol{u}\times\boldsymbol{v}) = 0$

$$\boldsymbol{u}\times\boldsymbol{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$

Let

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \text{ and } \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\boldsymbol{u}\times\boldsymbol{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$

$$\boldsymbol{u}\cdot(\boldsymbol{u}\times\boldsymbol{v}) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$

Student
view

Overview
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Find the scalar product.

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} = u_1\left(u_2v_3 - u_3v_2\right) + u_2\left(u_3v_1 - u_1v_3\right) + u_3\left(u_1v_2 - u_2v_1\right)$$

Expand the brackets and simplify.

$$\begin{aligned} & u_1\left(u_2v_3 - u_3v_2\right) + u_2\left(u_3v_1 - u_1v_3\right) + u_3\left(u_1v_2 - u_2v_1\right) \\ & = u_1u_2v_3 - u_1u_3v_2 + u_2u_3v_1 - u_2u_1v_3 + u_3u_1v_2 - u_3u_2v_1 \\ & = 0 \end{aligned}$$

Therefore, $\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0$

Example 3

★★☆

For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, simplify $(\boldsymbol{a} - 2\boldsymbol{b}) \times (\boldsymbol{a} + 2\boldsymbol{b})$.

Using the distributive property of the cross product and $(k\boldsymbol{v}) \times \boldsymbol{w} = \boldsymbol{k}\left(\boldsymbol{v} \times \boldsymbol{w}\right),\ k \in \mathbb{R}$.

$$\begin{aligned} (\boldsymbol{a} - 2\boldsymbol{b}) \times (\boldsymbol{a} + 2\boldsymbol{b}) & = \boldsymbol{a} \times \boldsymbol{a} + \boldsymbol{a} \times 2\boldsymbol{b} - 2\boldsymbol{b} \times \boldsymbol{a} - 2\boldsymbol{b} \times 2\boldsymbol{b} \\ & = 2(\boldsymbol{a} \times \boldsymbol{b}) - 2\left(\boldsymbol{b} \times \boldsymbol{a}\right) \\ & = 4(\boldsymbol{a} \times \boldsymbol{b}) \end{aligned}$$

As $-2\left(b \times a\right) = 2\left(a \times b\right)$, and $a \times a = 0$ and $b \times b = 0$. Note that 0 in both cases is the zero vector.

Therefore, the answer is $4\boldsymbol{a} \times \boldsymbol{b}$.

3 section questions ^

Student
view

Question 1

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

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For any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, what is $(\boldsymbol{v}+\boldsymbol{w}) \times (\boldsymbol{v}-\boldsymbol{w})$.

1. $2(\boldsymbol{w} \times \boldsymbol{v})$ ✓
2. $2\,\boldsymbol{v} \times \boldsymbol{w}$
3. $\mathbf{0}$
4. $2\,(\boldsymbol{v}+\boldsymbol{w})$

Explanation

$$
\begin{aligned}
& (\boldsymbol{v}+\boldsymbol{w}) \times (\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v} \times \boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{w} + \boldsymbol{w} \times \boldsymbol{v} - \boldsymbol{w} \times \boldsymbol{w} \\
\Rightarrow \quad & = 0 - \boldsymbol{v} \times \boldsymbol{w} + \boldsymbol{w} \times \boldsymbol{v} - 0 \\
\Rightarrow \quad & = -(-(\boldsymbol{w} \times \boldsymbol{v})) + \boldsymbol{w} \times \boldsymbol{v} \\
\Rightarrow \quad & = 2\,(\boldsymbol{w} \times \boldsymbol{v})
\end{aligned}
$$

Question 2

★★☆

For three vectors $\boldsymbol{u},\ \boldsymbol{v}$ and $\boldsymbol{w}$, you are given that $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}=0$.

By finding $\boldsymbol{u} \times (\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w})$ and $\boldsymbol{v} \times (\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w})$ determine which of the following statements is true.

1. $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{u} = \boldsymbol{v} \times \boldsymbol{w}$ ✓
2. $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{u} \times \boldsymbol{w} = \boldsymbol{w} \times \boldsymbol{v}$
3. $\boldsymbol{v} \times \boldsymbol{u} = \boldsymbol{w} \times \boldsymbol{u} = \boldsymbol{v} \times \boldsymbol{w}$
4. $\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{w} \times \boldsymbol{u} = -\boldsymbol{v} \times \boldsymbol{w}$

Explanation

Using the distributive property of the cross product

$$
\begin{aligned}
& \boldsymbol{u} \times (\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{u} + \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w} = \boldsymbol{u} \times 0 \\
& \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w} = 0 \\
& \boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{u} \times \boldsymbol{w} \\
& \boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{u} \qquad \mathbf{(1)}
\end{aligned}
\qquad \text{because } \begin{aligned} & \boldsymbol{u} \times \boldsymbol{u} = 0 \\ & -\boldsymbol{u} \times \boldsymbol{w} = \boldsymbol{w} \times \boldsymbol{u} \end{aligned}
$$

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Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$\boldsymbol{v} \times (\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{v} \times \boldsymbol{u} + \boldsymbol{v} \times \boldsymbol{v} + \boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{v} \times 0$

$\boldsymbol{v} \times \boldsymbol{u} + \boldsymbol{v} \times \boldsymbol{w} = 0$

$\boldsymbol{v} \times \boldsymbol{u} = -\boldsymbol{v} \times \boldsymbol{w}$ because $\boldsymbol{v} \times \boldsymbol{v} = 0$, $-\boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{w} \times \boldsymbol{v}$

$\boldsymbol{v} \times \boldsymbol{u} = \boldsymbol{w} \times \boldsymbol{v}$

$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{w} \times \boldsymbol{v}$

$\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{v} \times \boldsymbol{w}$ $\quad$ **(2)**

Combining **(1)** and **(2)** gives the answer is $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{u} = \boldsymbol{v} \times \boldsymbol{w}$

Question 3

★★☆

For any two vectors $\boldsymbol{u}$ and $\boldsymbol{w}$, select which of the following is equivalent to $(2\boldsymbol{u} + 3\boldsymbol{w}) \times (\boldsymbol{u} - 3\boldsymbol{w})$.

1 $9\boldsymbol{w} \times \boldsymbol{u}$ ✓

2 $9\boldsymbol{u} \times \boldsymbol{w}$

3 $3\boldsymbol{w} \times \boldsymbol{u}$

4 $-3\boldsymbol{w} \times \boldsymbol{u}$

Explanation

Using the distributive property of cross product

$$(2\boldsymbol{u} + 3\boldsymbol{w}) \times (\boldsymbol{u} - 3\boldsymbol{w}) = 2(\boldsymbol{u} \times \boldsymbol{u}) - 6(\boldsymbol{u} \times \boldsymbol{w}) + 3(\boldsymbol{w} \times \boldsymbol{u}) - 9(\boldsymbol{w} \times \boldsymbol{w})$$
$$(2\boldsymbol{u} + 3\boldsymbol{w}) \times (\boldsymbol{u} - 3\boldsymbol{w}) = -6(\boldsymbol{u} \times \boldsymbol{w}) + 3(\boldsymbol{w} \times \boldsymbol{u})$$
$$(2\boldsymbol{u} + 3\boldsymbol{w}) \times (\boldsymbol{u} - 3\boldsymbol{w}) = 6(\boldsymbol{w} \times \boldsymbol{u}) + 3(\boldsymbol{w} \times \boldsymbol{u}) = 9(\boldsymbol{w} \times \boldsymbol{u})$$

as

$$\boldsymbol{u} \times \boldsymbol{u} = 0$$
$$\boldsymbol{w} \times \boldsymbol{w} = 0$$
$$\boldsymbol{u} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{u}$$

Therefore, the correct answer is $9(\boldsymbol{w} \times \boldsymbol{u}) = 9\boldsymbol{w} \times \boldsymbol{u}$

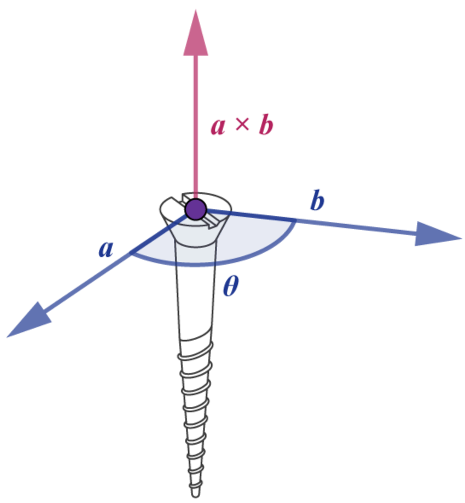
Student view

3. Geometry and trigonometry / 3.16 Vector product

Geometric interpretations

The cross product of two vectors, $\boldsymbol{a} \times \boldsymbol{b}$, gives a vector that is perpendicular to both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Vectors have both magnitude and direction, so what is the geometrical meaning of the magnitude of the cross product of two vectors?

Go back to the example of a screwdriver that you saw earlier. Consider the diagrams below. What could the magnitude of the cross product represent if you are loosening the screw?



More information

This diagram illustrates the concept of the cross product in relation to a screw. The screw is depicted vertically with two vectors labeled 'a' and 'b' in blue, extending outwards from the top. Vector 'a' points to the left, while vector 'b' points to the right, forming an angle, labeled as (\theta), between them. There is also a third vector, labeled (a \times b), in pink, perpendicular to the plane formed by 'a' and 'b', pointing upwards. The vectors emanate from a point at the top center of the screw representing the axis of rotation. The arrangement visually suggests how the magnitude and direction of the cross product relate to the axis along which a screw moves when turned.

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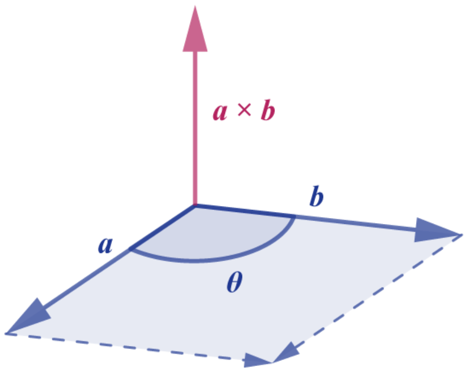

More information

The image is a diagram illustrating a 3D geometric interpretation of the cross product of two vectors (\boldsymbol{a}) and (\boldsymbol{b}). Vector (\boldsymbol{a}) is represented as an arrow lying along one axis in a 3D space, while vector (\boldsymbol{b}) lies along another axis, forming a plane. The cross product (\boldsymbol{a} \times \boldsymbol{b}) is shown as a third vector orthogonal to the plane formed by (\boldsymbol{a}) and (\boldsymbol{b}). The diagram uses arrows to represent the vectors and indicates the direction of each vector. The shaded area represents the plane in which vectors (\boldsymbol{a}) and (\boldsymbol{b}) lie, emphasizing that the cross product vector is perpendicular to this plane.

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In the first diagram above, $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$.

The second diagram above shows the area covered by both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

You can define the magnitude of the cross product as the area covered by both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. In this context, it will tell you how high the screw will move up.

The diagram below shows that the area covered by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a parallelogram.

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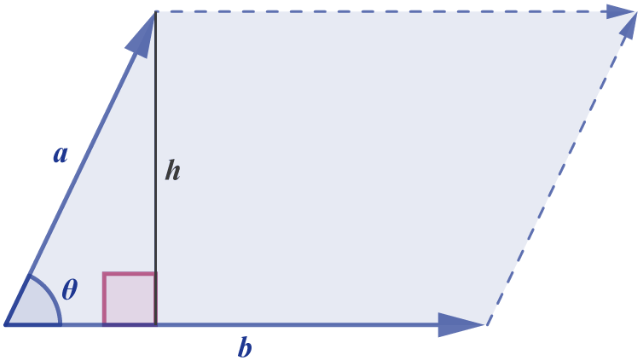

More information

The diagram illustrates a parallelogram with sides represented by vectors (\boldsymbol{a}) and (\boldsymbol{b}). The vector (\boldsymbol{a}) is on the left vertical side, and (\boldsymbol{b}) is along the bottom horizontal side. A vertical line extending from the top of (\boldsymbol{a}) to the base represents the height (h). The area of this parallelogram is calculated using the formula (|\boldsymbol{a} \times \boldsymbol{b}| = b \times h). The diagram visually shows these relationships with labeled components to indicate the height and base of the parallelogram, emphasizing the geometric view of area calculation.

[Generated by AI]

What is the formula for the area of a parallelogram?

In this case, $area = |\boldsymbol{b}|\ h$ and $\sin\theta = \dfrac{h}{|\boldsymbol{a}|} \Rightarrow h = |\boldsymbol{a}|\sin\theta$.

Combining both equations gives area $= |\boldsymbol{b}|\ |\boldsymbol{a}|\sin\theta$

Therefore the area of the parallelogram formed by the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\text{area} = |\boldsymbol{b}|\ |\boldsymbol{a}|\sin\theta$$

The magnitude of the cross product $\boldsymbol{a} \times \boldsymbol{b}$ is defined by the area covered by the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, so

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{b}|\ |\boldsymbol{a}|\sin\theta$$

where θ is the angle between the two vectors.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Note that the direction of the cross product does not matter here as you are finding the magnitude, i.e. the magnitude will be the same in both directions, i.e. $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{b} \times \boldsymbol{a}|$.

How could you adapt the formula for the area of a parallelogram to get a formula for the area of a triangle?

You are likely to have met this formula already in your IB course. How is it usually written?

Exam tip

In IB formula booklet, the components of a vector product and the area of a parallelogram are given as

Vector product

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}, \text{ where } \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ and } \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}|\,|\boldsymbol{w}| \sin\theta,$$

where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

Area of a parallelogram

$$A = |\boldsymbol{v} \times \boldsymbol{w}|,$$

where $\boldsymbol{v}$ and $\boldsymbol{w}$ form two adjacent sides of a parallelogram.

Exam tip

In IB examinations, although all the vectors will be written in bold in the formula booklet and in questions, you must use the hand written forms, $\underline{a}$ or $\vec{a}$, to identify vectors, otherwise you will be penalised.

Example 1

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Student view

Find the exact value of the sine of the angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ if:

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$|\boldsymbol{v}| = 2$, $|\boldsymbol{w}| = 3$ and $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Steps	Explanation
$\lvert\boldsymbol{v} \times \boldsymbol{w}\rvert = \sqrt{1^2 + 2^2 + 3^2}$	Using the formula for the magnitude of a vector.
$\lvert\boldsymbol{v} \times \boldsymbol{w}\rvert = \sqrt{14}$	
$\lvert\boldsymbol{v} \times \boldsymbol{w}\rvert = (2)(3)\sin\theta$	Use $\lvert\boldsymbol{v} \times \boldsymbol{w}\rvert = \lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\sin\theta$.
$\sqrt{14} = 6\sin\theta$	Simplify and solve for $\sin\theta$.
$\sin\theta = \frac{\sqrt{14}}{6}$	
Therefore, the answer is $\frac{\sqrt{14}}{6}$	

Important

Defining the cross product as $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}||\boldsymbol{w}|\sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$, gives some important results.

- If vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel, then $\boldsymbol{v} \times \boldsymbol{w} = 0$ because $\sin 0 = 0$.
- $\boldsymbol{v} \times \boldsymbol{v} = 0$
- If vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular, then $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}||\boldsymbol{w}|$

These results will not be given in the formula booklet but you can deduce them from the formulae that are given. Make sure you refer to the formula booklet so that you don’t mix up the formula for the vector product with the formula for the scalar product.

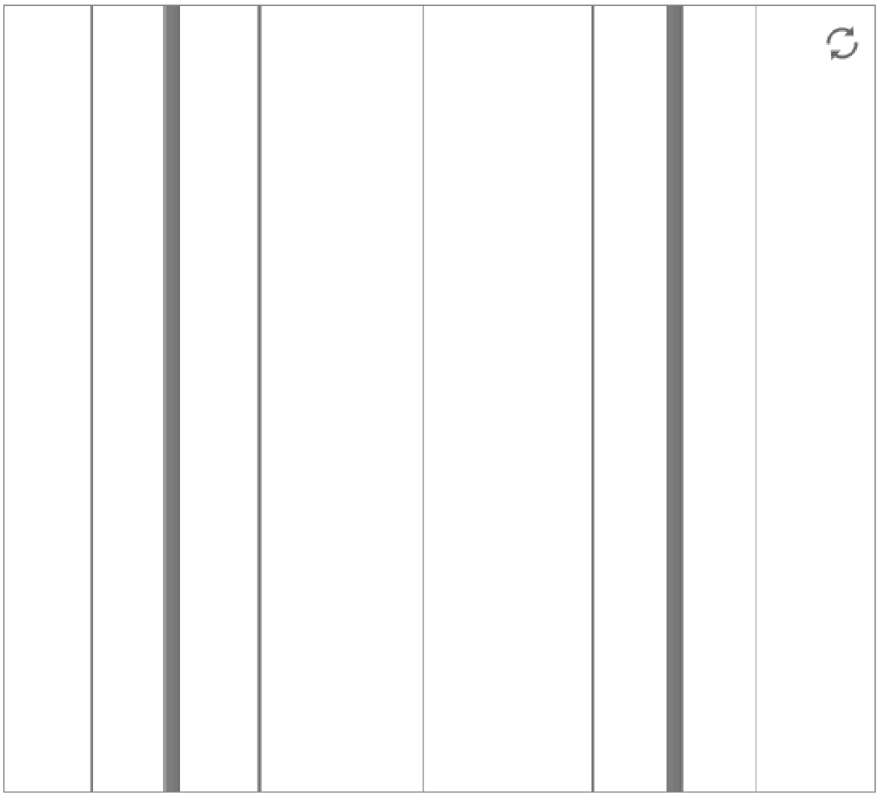
Activity

Student view

You can explore the relationship between two vectors and their cross product using the applet below.

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Drag the points and see how the cross product changes.What is the cross product when the angle between two vectors is 0 ° or 180 °?



Interactive 1. Visualizing the Cross Product.

Credit: GeoGebra (https://www.geogebra.org/m/RrDv9Wea) Tim Brzezinski

More information for interactive 1

This interactive allows user to explore the relationship between two vectors, u and v and their cross product.

The screen shows a grid with three red dots and three arrows pointing outwards with two vectors u and v pointing parallel to the base and another vector pointing upward.

As users drag the points to adjust the vectors, the cross product vector changes in real time. The length of the cross product vector dynamically adjusts as users move the vectors. The cross product vector is always perpendicular to the plane containing u and v, illustrating the fundamental property of the cross product in three-dimensional space.

When the angle between the two vectors is 0° or 180°, the vectors are either parallel or antiparallel. In these cases, the cross product is the zero. This means there is no unique perpendicular direction, and the cross product vector has no magnitude.

Student view

Example 2

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Overview
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Consider the quadrilateral ABCD with vertices $\mathrm{A}(2,0,4)$, $\mathrm{B}(5,1,1)$, $\mathrm{C}(-1,1,3)$ and $\mathrm{D}(-4,0,6)$. Show that the quadrilateral is a parallelogram and find its area.

$$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 5-2 \\ 1-0 \\ 1-4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$$

$$\overrightarrow{\mathrm{DC}} = \begin{pmatrix} -1-(-4) \\ 1-0 \\ 3-6 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$$

As $\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{DC}}$, it is implied that the quadrilateral ABCD is a parallelogram with AB and BC adjacent sides.

Find $\overrightarrow{\mathrm{BC}} = \begin{pmatrix} -1-5 \\ 1-1 \\ 3-1 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 2 \end{pmatrix}$

The vector product of $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{BC}}$ is

$$\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{BC}} = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} -6 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1\times 2-(-3)\times 0(-3)\times(-6)-3\times 2 \\ 3\times 0-1\times(-6) \end{pmatrix} = \begin{pmatrix} \\ 1 \\ \end{pmatrix}$$

Therefore the area is given by

$$\left|\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{BC}}\right| = \left|\begin{pmatrix} 2 \\ 12 \\ 6 \end{pmatrix}\right| = \sqrt{2^2+12^2+6^2} = \sqrt{184} = 2\sqrt{46}$$

Example 3

★★☆

The area of a parallelogram formed by the two adjacent vectors of $\boldsymbol{a} = x\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}, \boldsymbol{x}>0$ and $\boldsymbol{b} = \boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$ is $4\ \mathrm{unit}^2$. Find the value of x. Give your answer to three significant figures.

Student
view

Steps	Explanation
$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} x \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1-2x \\ x-1 \end{pmatrix}$	
$\lvert\boldsymbol{a} \times \boldsymbol{b}\rvert = \sqrt{9 + (-1-2x)^2 + (x-1)^2}$	Area of parallelogram = $\lvert\boldsymbol{a} \times \boldsymbol{b}$
$\sqrt{9 + (-1-2x)^2 + (x-1)^2} = 4$	
$9 + (-1-2x)^2 + (x-1)^2 = 16$	
$(-1-2x)^2 + (x-1)^2 = 7$	
$x = -1.220$ or $x = 0.8198$	Use your GDC correct to 4 significant figures
Therefore, $x = 0.820$ correct to 3 significant figures	Because $x > 0$

International Mindedness

The vector product has many applications in mathematics and physics, for example, when calculating torque and the current induced when a conductor attached to a circuit moves in a magnetic field.

Another application is the modelling of tornados. These are rotating winds that create funnels between the ground and clouds and cause devastation in the area in which they move. There are many destructive tornados reported each year around the world. Therefore, it is important to model them in order to understand their behaviour and be able to predict when they will occur. The vector product can be applied in more than three dimensions, so modelling can be carried out in higher dimensions.

Watch the video below and follow the link to learn more about the modelling of tornados.

Overview
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aa-
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134-
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761926/o

4 section questions

Question 1

★☆☆

Two adjacent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have $|\boldsymbol{a}| = 3$, $|\boldsymbol{b}| = 5$ and the angle between them is given by $\sin\theta = \dfrac{1}{7}$.

Find the area of the parallelogram formed by the two vectors.

1. $\dfrac{15}{7}$ ✓
2. $\dfrac{5}{21}$
3. $\dfrac{3}{35}$
4. $\dfrac{35}{3}$

Explanation

$\text{Area} = |\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta = 3 \times 5 \times \dfrac{1}{7} = \dfrac{15}{7}$

Therefore, the correct answer is $\dfrac{15}{7}$

Student
view

Question 2

★★☆

What is the area of the parallelogram formed by the vectors $\boldsymbol{i}+2\boldsymbol{k}$ and $3\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}$?

1 $\sqrt{45}\,\text{units}^2$ ✓

2 $\sqrt{69}\,\text{units}^2$

3 $\sqrt{23}\,\text{units}^2$

4 $6\,\text{units}^2$

Explanation

$$\begin{aligned} \left|\begin{pmatrix}1\\0\\2\end{pmatrix}\times\begin{pmatrix}3\\-2\\1\end{pmatrix}\right| &= \left|\begin{pmatrix}0+4\\6-1\\-2-0\end{pmatrix}\right| \\ \Rightarrow \quad &= \left|\begin{pmatrix}4\\5\\-2\end{pmatrix}\right| \\ \Rightarrow \quad &= \sqrt{4^2+5^2+(-2)^2} \\ \Rightarrow \quad &= \sqrt{45}=3\sqrt{5} \end{aligned}$$

Question 3

★★☆

What is the area of the triangle with vertices $(0,1,-2)$, $(3,-1,2)$ and $(-1,-2,3)$?

1 $9\sqrt{\frac{3}{2}}\,\text{units}^2$ ✓

2 $9\sqrt{6}\,\text{units}^2$

3 $\sqrt{\frac{483}{2}}\,\text{units}^2$

4 $2\sqrt{101}\,\text{units}^2$

Explanation

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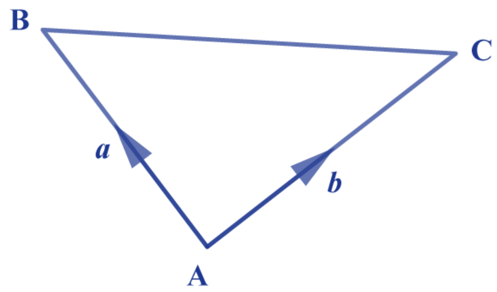

More information

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$$

$$\boldsymbol{b} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} (-2) \times 5 - 4 \times (-3) \\ 4 \times (-1) - 3 \times 5 \\ 3 \times (-3) - (-2) \times (-1) \end{pmatrix} = \begin{pmatrix} 2 \\ -19 \\ -11 \end{pmatrix}$$

$$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{2^2 + (-19)^2 + (-11)^2} = 9\sqrt{6}$$

Area = $\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}| = \frac{1}{2} \times 9\sqrt{6} = 9\sqrt{\frac{3}{2}}$

Question 4

★★☆

Consider two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ with $\boldsymbol{u} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $|\boldsymbol{v}| = \frac{5}{3}$ and $\boldsymbol{u} \cdot \boldsymbol{v} = -\frac{5}{2}$.

Find the area of the triangle formed by these two vectors. Give your answer to three significant figures.

2.17

Accepted answers

2.17, 2,17

Explanation

Find the magnitude of $\boldsymbol{u}$:

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aa-
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761926/o

$$|\boldsymbol{u}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$$

Now, we can find the angle θ formed by these two vectors:

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}\right) = \cos^{-1}\left(\frac{-\frac{5}{2}}{3 \times \frac{5}{3}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ$$

Therefore, the acute angle between the two vectors is $180 - 120 = 60$

Thus, the area of the triangle formed by the two vectors is

$$\frac{1}{2}|\boldsymbol{u} \times \boldsymbol{v}| = \frac{1}{2}|\boldsymbol{u}||\boldsymbol{v}| \sin 60 = \frac{1}{2} \times 3 \times \frac{5}{3} \times \frac{\sqrt{3}}{2} = 2.17$$

3. Geometry and trigonometry / 3.16 Vector product

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-28032/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- recall that the vector product of vectors $\boldsymbol{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ is denoted by $\boldsymbol{v} \times \boldsymbol{w}$
- calculate the vector product from the components of $\boldsymbol{u}$ and $\boldsymbol{v}$ using the formula

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

- recall the properties of the vector product:

  For vectors $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$:
  - $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$ so the vector product is not commutative
  - $\boldsymbol{u}\left(\boldsymbol{v} + \boldsymbol{w}\right) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$ so the vector product is distributive
  - $\left(k\boldsymbol{v}\right) \times \boldsymbol{w} = k\left(\boldsymbol{v} \times \boldsymbol{w}\right), k \in \mathbb{R}$

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Overview
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- recall that the area of a parallelogram can be calculated using $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}|\,|\boldsymbol{w}| \sin\theta$
- recall that the area of a triangle can be calculated using $\frac{1}{2}|\boldsymbol{v} \times \boldsymbol{w}| = \frac{1}{2}|\boldsymbol{v}|\,|\boldsymbol{w}| \sin\theta$.
- define the cross product as $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}|\,|\boldsymbol{w}| \sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
  - If vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel, then $\boldsymbol{v} \times \boldsymbol{w} = 0$
  - If vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular, then $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}|\,|\boldsymbol{w}|$.

3. Geometry and trigonometry / 3.16 Vector product

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-28033/print/) Assign

Vectors are widely used in computer animations to move objects on the screen and to draw the images. An interesting set of numbers called quaternions, which were invented by William Rowan Hamilton (1805–1865), are used in animations to move graphic objects and create motion.

Quaternions are numbers in the form $(a, x, y, z) = a + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ where a, x, y and z are real numbers, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are all different square roots of -1 and

$$\boldsymbol{ij} = \boldsymbol{k} = -\boldsymbol{ji},\ \boldsymbol{jk} = \boldsymbol{i} = -\boldsymbol{kj},\ \boldsymbol{ki} = \boldsymbol{j} = -\boldsymbol{ik}.$$

Similarly to complex numbers in 2D, you can describe quaternions geometrically as rotations and use them to represent rotations in three-dimensional space.

Read this article (https://plus.maths.org/content/os/issue42/features/lasenby/index) to learn more about how these numbers are used in animations.

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Draw a geometrical object such as a triangle in 3D space.

- Using the properties of quaternions, reflect the shape in the plane $x + y + z = 0$.

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- Then rotate the object in 3D.

Making connections

Vectors, graph theory and many more mathematical concepts are used in simulation games. You can read more here (https://nrich.maths.org/1374).

Rate subtopic 3.16 Vector product

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