TOPIC 3
GEOMETRY AND TRIGONOMETRY

SUBTOPIC 3.9
PLANAR TRANSFORMATIONS

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Teacher view

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3. Geometry and trigonometry / 3.9 Planar transformations

The big picture

In section 1.14 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27428/), you studied matrices. What do matrices have to do with geometry? What happens to a set of points or a shape when you multiply each point by a matrix? What happens to a vector when you multiply it by a matrix? What happens when you add two vectors together?

In this section, you will be studying planar transformations. You can think of a matrix operation as a transformation of space, in this case the xy plane. These transformations can change either the shape or position of an object, or both.

Making connections

Before starting to study planar transformation, you might want to refresh your memory about transformations from prior knowledge topics.

We can group transformations under four headings:

1. Translations : Each point moves in the same direction by the same distance. The size of the object doesn't change.
2. Reflections : The object is reflected in a line to create a mirror image. The size of the object doesn't change.
3. Rotations : Each point on the object is rotated through an angle. The size of the object doesn't change.
4. Enlargements: The object is stretched or squeezed :
   - horizontal stretch : The y values are constant, but the x values are scaled by a factor.

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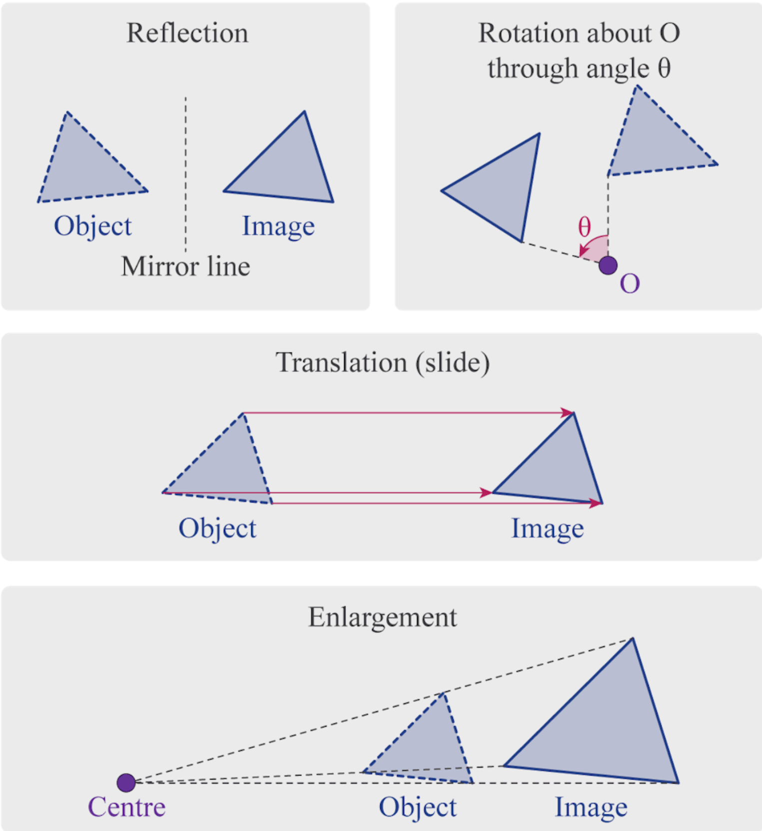
- vertical stretch : The x values are fixed but the y values are scaled by a factor
- enlargement: Both the x and y values are scaled by a factor.

Below are some examples of geometric transformation.



More information

The image contains four diagrams, each illustrating a different type of geometric transformation of a triangle.

1. **Reflection:** The top-left diagram shows a triangle labeled "Object" reflected across a vertical mirror line to form an "Image".
2. **Rotation:** The top-right diagram illustrates rotation. It shows a triangle rotating about a point labeled "O" through an angle θ, transforming the "Object" into an "Image".
3. **Translation:** The middle diagram demonstrates translation, where the "Object" triangle slides horizontally to become the "Image". Arrows indicate the direction and distance of movement.
4. **Enlargement:** The bottom diagram shows enlargement. The "Object" triangle is increased in size, centered on a point labeled "Centre", to create the larger "Image".

Each transformation is labeled with text indicating the type of transformation.

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You can use this link (https://www.mathsisfun.com/geometry/transformations.html) to revise transformations.

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Concept

Transformation matrices combined with translation vectors can help you to understand transformations of an object from one form to another in a plane. These transformations have many real-world applications, from modelling fluid flow and electromagnetism to computer graphics.

Theory of Knowledge

The key knowledge issue of formalism vs. Platonism in regard to mathematics is discussed in other Theory of Knowledge boxes throughout the course; however, it seems apropos to contemplate mathematics' rational origins in the context of rational functions.

Knower bias is a key factor in knowledge production and reception; however, at first glance, it seems that mathematics is immune to such biases because it is built on reason and has a very high level of real-world predictive validity.

Knowledge Question: To what extent can knowledge be free from bias?

Transformation matrices

Single transformations

When we multiply a 2×2 matrix and a 2×1 matrix (or column matrix or vector) together:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix},$$

what does this mean in terms of the cartesian plane?

Making connections

In section 1.14.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/matrix-multiplication-id-27431/) , you learned how to multiply matrices together. When you multiply a 2×2 matrix with a 2×1 matrix, what are the dimensions of the new matrix?

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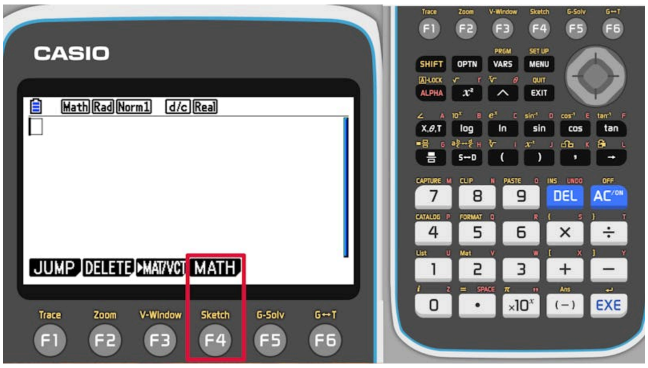
You might also wish to revisit how to use a graphic display calculator to multiply matrices.

Steps	Explanation
In section 1.14.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/matrix-multiplication-id-27431/) the calculator help showed you a way of storing matrices in the calculator and using their names in calculations. In case you do not need to use the matrix several times, you can work with them directly. This is the approach we use here to work out the product of the two matrices above. Open the calculator mode ...	
... press F4 to select the math options ...	

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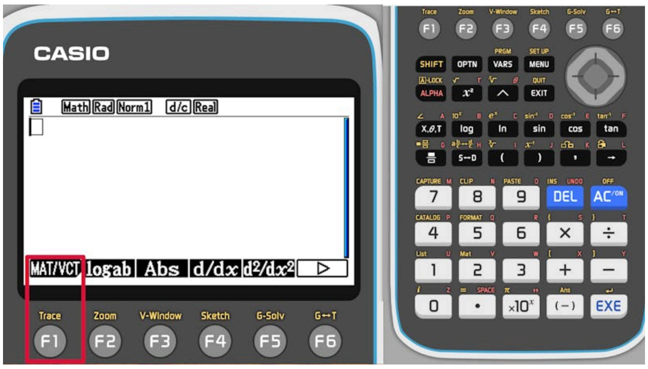
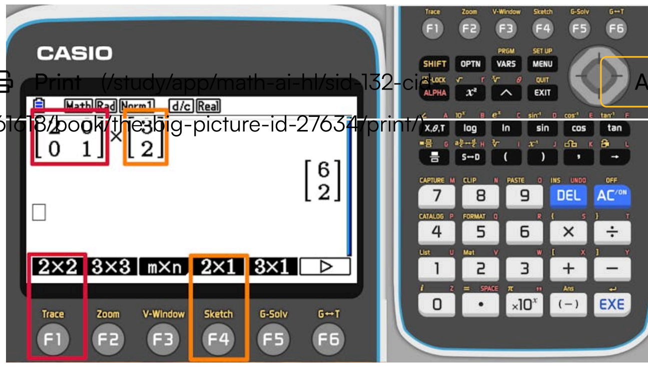
Steps	Explanation
... and press F1 to see the options related to matrices.	
You can choose between matrix templates of several dimensions. In this case you need F1 to choose the 2×2 template and F4 for the 2×1 matrix. Enter the numbers and press EXE to see the product.	

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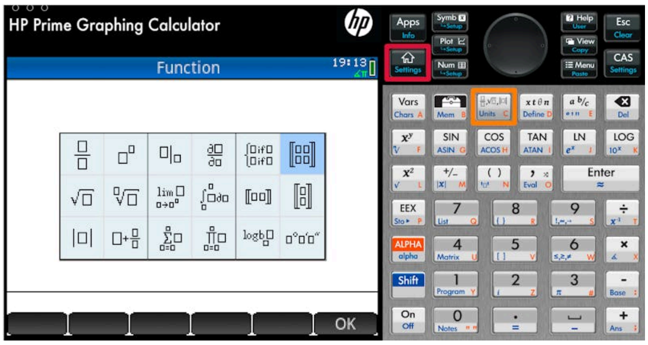
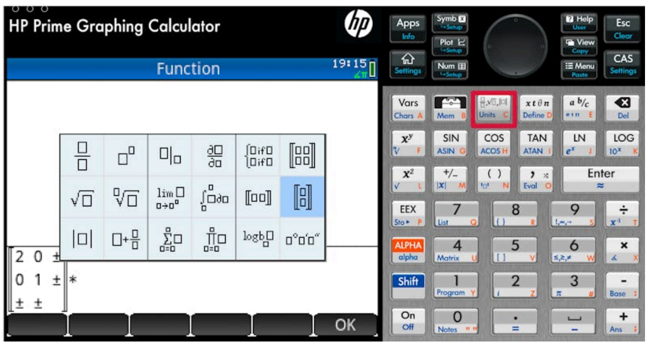
Steps	Explanation
In section 1.14.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/matrix-multiplication-id-27431/) the calculator help showed you a way of storing matrices in the calculator and using their names in calculations. In case you do not need to use the matrix several times, you can work with them directly. This is the approach we use here to work out the product of the two matrices above. On the home page of any application open the template menu and choose the matrix template.	
Enter the numbers in the first matrix, move out to the right and open the templates again for the second matrix. This time choose the column vector template.	

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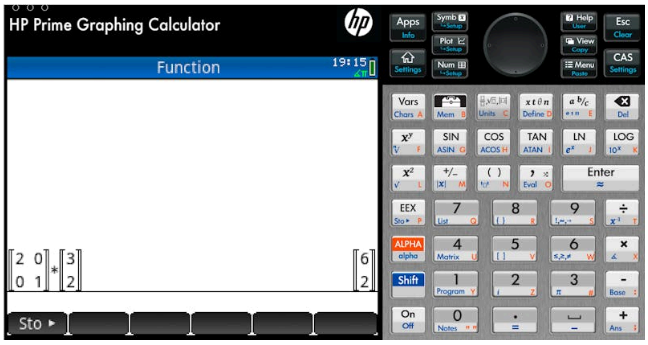
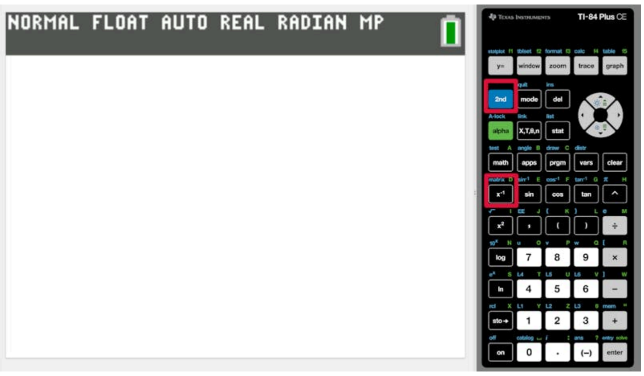
Steps	Explanation
Enter the numbers and press enter to see the product.	

Steps	Explanation
In [section 1.14.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/matrix-multiplication-id-27431/)] the calculator help showed you a way of storing matrices in the calculator and using their names in calculations. We do not repeat all the steps here. If you need help, refer to the other instruction set. To edit, and work with matrices, you need to open the matrix menu.	

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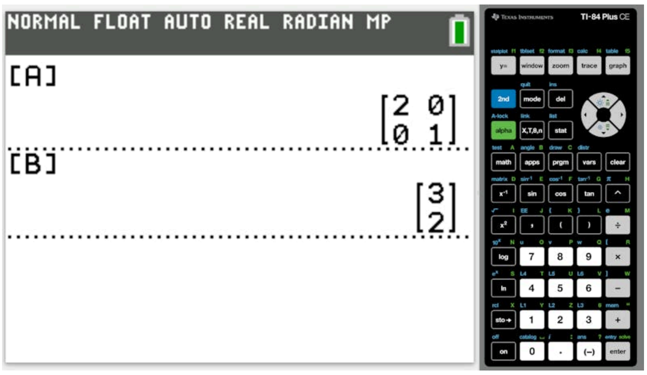
Steps	Explanation
You can edit matrices. In this case the matrices are already stored in the memory, so this step is skipped. You can also use their names.	
It is not needed for finding the product, but you can use the names and press enter to see the matrix stored.	

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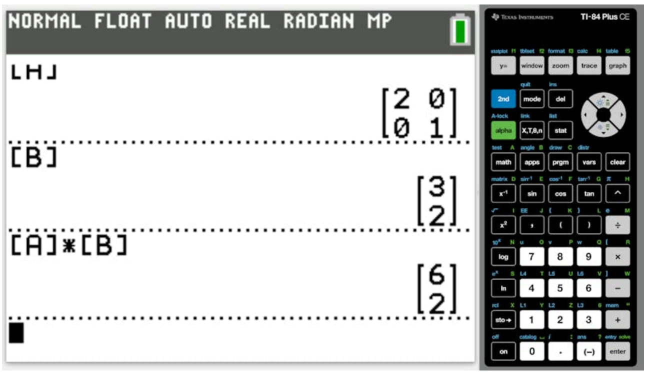
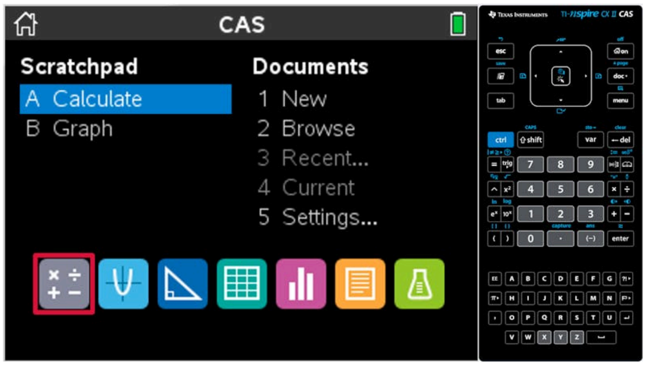
Steps	Explanation
You can also use the names to find the product.	

Steps	Explanation
In section 1.14.3 (/study/app/math-ai-hl/sid-132-cid-761618/book/matrix-multiplication-id-27431/) the calculator help showed you a way of storing matrices in the calculator and using their names in calculations. In case you do not need to use the matrix several times, you can work with them directly. This is the approach we use here to work out the product of the two matrices above. Open a calculator page.	

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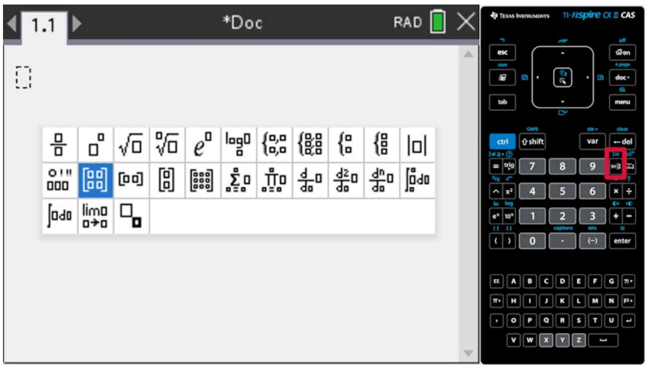
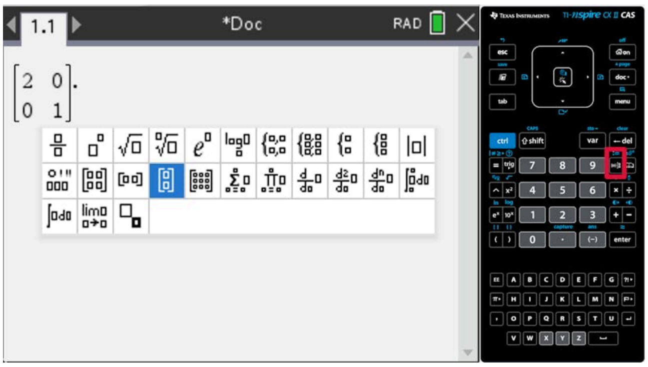
Steps	Explanation
Open the template menu and choose the matrix template.	
Enter the numbers in the first matrix, move out to the right and open the templates again for the second matrix. This time choose the column vector template.	

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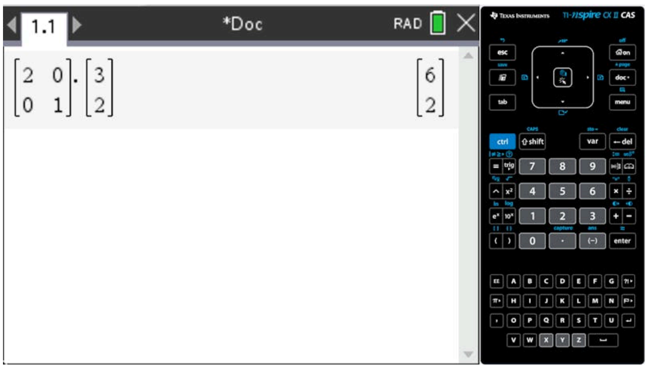
Steps	Explanation
Enter the numbers and press enter to see the product.	

Activity

You will be investigating the impact of multiplying a point with a 2×2 matrix.The applet below has a 2×2 matrix $\mathrm{m1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and sliders for the parameters a, b, c and d.

The matrix transforms the triangle BCD to triangle EFG.

See what happens when you move the sliders one by one.

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Interactive 1. Investigating the Impact of Multiplying a Point with a Matrix.

More information for interactive 1

The interactive applet allows users to explore the impact of multiplying a point by a 2×2 matrix.
A graph is displayed with the xy axes, where the x-axis ranges from 0 to 10 and y-axis ranges from -4 to 8. Two triangles, Triangle BCD in red points and Triangle EFG in purple points are projected on the graph.

The matrix is defined as $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and users can adjust the values of *a*, *b*, *c*, and *d* by dragging the horizontal sliders on the top left, with their values ranging from -5 to 5.

The applet demonstrates how the matrix transforms the triangle BCD into triangle EFG. Users can observe the effect of changing each parameter on the transformation of the triangle.

When a point (x, y) from Triangle BCD is multiplied by M, it moves to a new location (x', y') calculated as $x' = ax + by$ and $y' = cx + dy$. This linear transformation can produce various effects depending on the matrix entries. For instance, modifying a stretches or compresses the triangle horizontally, while changing *d* does the same vertically. Adjusting *b* introduces a horizontal shear, tilting the triangle diagonally, whereas altering *c* creates a vertical shear, lifting or lowering the shape. If *a* and *d* are set to zero while *b* and *c* are non-zero, the matrix can even rotate the triangle (left to right or up and down), demonstrating how different combinations of values lead to distinct geometric transformations.

The determinant of the matrix, ad - bc, plays a crucial role in determining whether the transformation preserves the triangle's area. A positive determinant scales the area proportionally, while a determinant of zero collapses the triangle into a line or a single point, indicating a loss of dimensionality. For example, setting $a = 2$ and $d = 1.5$ while keeping $b = c = 0$ will stretch the triangle horizontally without affecting

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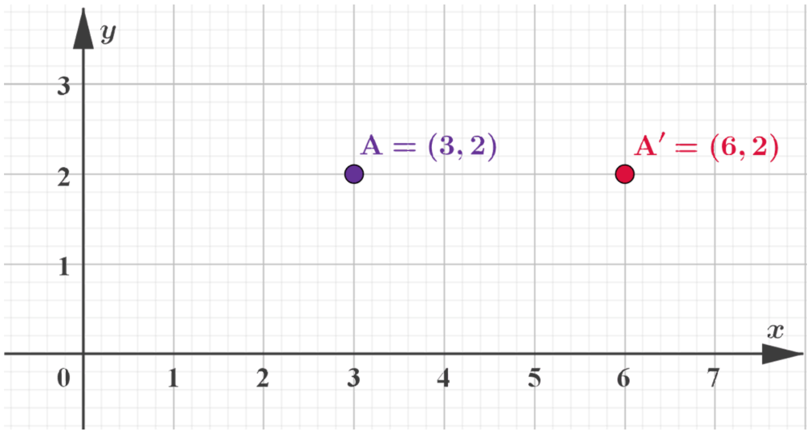
its height, whereas a = 1, b= 1, c = 0and d = 1 skews the triangle to the right without altering its area. By experimenting with different values, users can observe how matrices combine multiple effects, such as simultaneous scaling and shearing. This hands-on approach helps build an intuitive understanding of linear algebra concepts, bridging abstract mathematical operations with visual geometric changes. The applet serves as an engaging educational tool, making it easier to grasp how matrices function as powerful instruments for transforming shapes in two-dimensional space.

The position of a point $\mathrm{A}\ (x, y)$ can be represented by a position vector $\begin{pmatrix} x \\ y \end{pmatrix}$. When you multiply $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix}$ together as matrices, point $\mathrm{A}\ (\mathrm{x}, \mathrm{y})$ is transformed to a new location $\mathrm{A}'\ (x', y')$.

So, when you multiply $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, you are transforming $\mathrm{A}\ (3, 2)$ to $\mathrm{A}'\ (6, 2)$, as shown below.



More information

The image is a graph on a grid depicting a transformation in a 2D coordinate system. The X-axis is labeled as "x" and the Y-axis as "y," with both axes ranging from 0 to 8. There are two points marked: (A) at the coordinate ((3,2)) and (A') at the coordinate ((6,2)). The point (A) is represented with a purple dot and labeled "A = (3,2)," while the point (A') is shown with a red dot and labeled "A' = (6,2)." The grid helps illustrate the shift from point (A) to point (A'), showing the transformation caused by matrix multiplication from ((3,2)) to ((6,2)).

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Example 1

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Find the transformation of point $\text{A}\ (1, 1)$ by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Steps	Explanation
$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$	Position vector of the new point is $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
So the new point is $(2, 2)$.	The coordinates are $A'(x', y')$.

Important

When point $\text{A}\ (x, y)$ is transformed by the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the new position vector after the transformation is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

The new point after the transformation is $\text{A}'(x', y')$ where $x' = ax + by$ and $y' = cx + dy$.

Point A’ is often referred to as the **image** of point A under the transformation M.

Example 2

★★☆

The triangle in the diagram with the vertices $\text{B}\ (1, 1)$, $\text{C}\ (3, 1)$ and $\text{D}\ (2, 2)$ is transformed with the matrix $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$. Find the coordinates of the vertices of the image of the triangle and draw the transformed triangle on the same axes.

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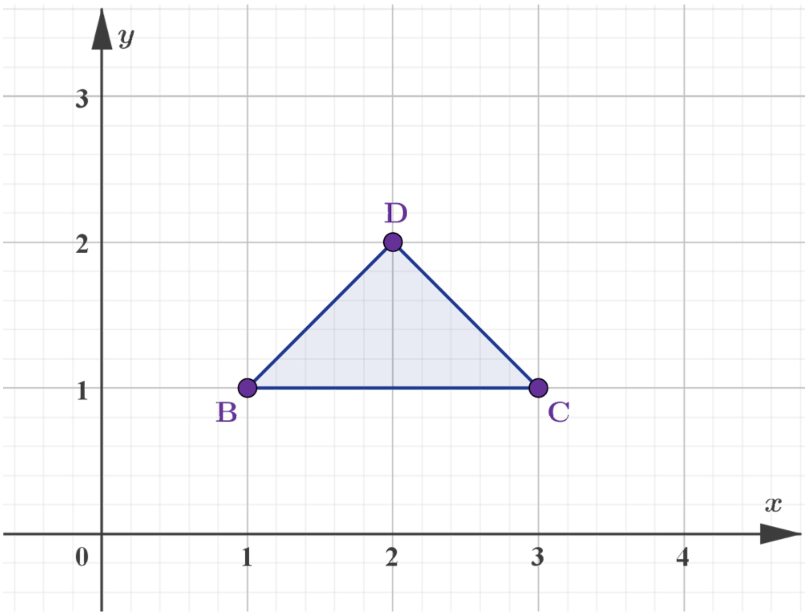

More information

The image is a diagram on a coordinate grid displaying a transformation of a triangle. The original triangle is defined by the vertices B(1,1), C(3,1), and D(2,2). The transformation applies a matrix (

$$(3 \quad 1\,0 \quad 2)$$

) to these coordinates. The new coordinates after transformation need calculation.

- The original triangle is located on a grid with a clear x-axis and y-axis.
- Point B is at (1,1), positioned near the bottom-left on the grid.
- Point C is at (3,1), directly to the right of B, forming the base of the triangle.
- Point D is at (2,2), situated above the midpoint of BC, forming the apex of the triangle.

Using the transformation matrix, the new coordinates are calculated as: - New B = ((3 \cdot 1 + 1 \cdot 1, 0 \cdot 1 + 2 \cdot 1) = (4, 2)) - New C = ((3 \cdot 3 + 1 \cdot 1, 0 \cdot 3 + 2 \cdot 1) = (10, 2)) - New D = ((3 \cdot 2 + 1 \cdot 2, 0 \cdot 2 + 2 \cdot 2) = (8, 4))

The transformed triangle is not drawn on the image, but these coordinates indicate its position relative to the original triangle, expanding and shifting its position as per the matrix applied.

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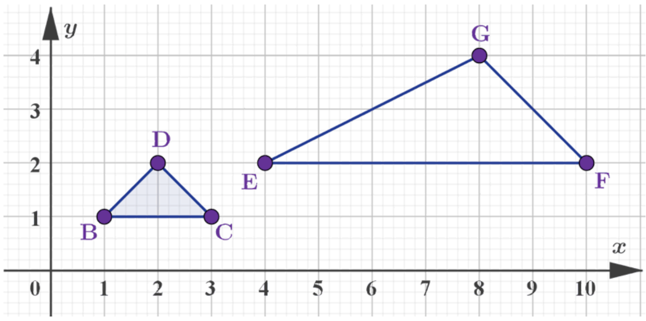
Steps	Explanation
$\begin{pmatrix}3 & 1\\0 & 2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}$ $\begin{pmatrix}3 & 1\\0 & 2\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}10\\2\end{pmatrix}$ $\begin{pmatrix}3 & 1\\0 & 2\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}=\begin{pmatrix}8\\4\end{pmatrix}$	Use the matrix multiplication for each vertex: $\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$
Therefore, the coordinates of the new triangle are $\mathrm{E}\,(4,2)$, $\mathrm{F}\,(10,2)$ and $\mathrm{G}\,(8,4)$.	Plotting the points 

In **Example 2**, instead of finding each coordinate of the transformation, you can create a new 2×3 matrix using the vertices of the triangle BCD:

$$M = \begin{pmatrix}1 & 3 & 2\\1 & 1 & 2\end{pmatrix}$$

where each column represents the x - and y - coordinates of one of the vertices of the triangle.Multiplying matrix $\boldsymbol{M}$ by the transformation matrix $\begin{pmatrix}3 & 1\\0 & 2\end{pmatrix}$ gives

$$\begin{pmatrix}3 & 1\\0 & 2\end{pmatrix} \times \begin{pmatrix}1 & 3 & 2\\1 & 1 & 2\end{pmatrix} = \begin{pmatrix}4 & 10 & 8\\2 & 2 & 4\end{pmatrix}.$$

You can see that the position vectors of the transformed vertices are the columns of the new matrix.

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✓ **Important**

When the vertices of an n -sided polygon (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , $\ldots$, (x_n, y_n) are transformed by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the transformed vectors are given by

$$\begin{pmatrix} x'_1 \; x'_2 \; x'_3 \; \ldots \; x'_n \\ y'_1 \; y'_2 \; y'_3 \; \ldots \; y'_n \end{pmatrix} = \begin{pmatrix} a \; b \\ c \; d \end{pmatrix} \begin{pmatrix} x_1 \; x_2 \; x_3 \; \ldots \; x_n \\ y_1 \; y_2 \; y_3 \; \ldots \; y_n \end{pmatrix}$$
$$= \begin{pmatrix} ax_1 + by_1 \; ax_2 + by_2 \; ax_3 + by_3 \; \ldots \; ax_n + by_n \\ cx_1 + dy_1 \; cx_2 + dy_2 \; cx_3 + dy_3 \; \ldots \; cx_n + dy_n \end{pmatrix}$$

where (x'_i , y'_i) are the coordinates of vertex *i* after the transformation.

Example 3

★★☆

The polygon with vertices $\mathrm{A}\,(-1, 1)$, $\mathrm{B}\,(-2, -1)$, $\mathrm{C}\,(2, 2)$ and $D\,(5, -5)$ is transformed with the matrix $\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$. Find the coordinates of the vertices of the image of ABCD.

Use the matrix multiplication for each vertex

$$\begin{pmatrix} x'_1 & x'_2 & x'_3 & \ldots & x'_n \\ y'_1 & y'_2 & y'_3 & \ldots & y'_n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 & \ldots & x_n \\ y_1 & y_2 & y_3 & \ldots & y_n \end{pmatrix}$$

$$\begin{pmatrix} x'_1 & x'_2 & x'_3 & x'_4 \\ y'_1 & y'_2 & y'_3 & y'_4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & -2 & 2 & 5 \\ 1 & -1 & 2 & -5 \end{pmatrix}$$

$$\begin{pmatrix} x'_1 & x'_2 & x'_3 & x'_4 \\ y'_1 & y'_2 & y'_3 & y'_4 \end{pmatrix} = \begin{pmatrix} -1 & -5 & 6 & 5 \\ 0 & -6 & 8 & 0 \end{pmatrix}$$

Each column represents a position vector of one of the new vertices.

Therefore, the resulting points are $\mathrm{A}'\,(-1, 0)$, $\mathrm{B}'\,(-5, -6)$, $\mathrm{C}'\,(6, 8)$ and $\mathrm{D}'\,(5, 0)$.

ⓘ **Exam tip**

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In IB examinations, you should write the matrices down before you use a graphic display calculator to avoid making typing mistakes. Getting the order of the numbers wrong can change the results.

✓ **Important**

A transformation of a point on the cartesian plane by a 2×2 matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is also called a linear transformation of the object, provided the transformation is about the origin (0,0).

Finding the transformation matrix

If the points A $(2, 3)$ and B $(3, -2)$ are transformed by a 2×2 matrix T and the resulting points are $(4, -5)$ and $(-7, 12)$ how can you find the transformation matrix T?

If you write this transformation using the matrix notation, $T \times \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 4 & -7 \\ -5 & 12 \end{pmatrix}$.

If we use the properties of matrices, $T = \begin{pmatrix} 4 & -7 \\ -5 & 12 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}^{-1}$ which, of course is possible as $\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}^{-1}$ is defined because its determinant is not 0.

Why does the determinant have to be non-zero?

Using a graphic display calculator, $T = \begin{pmatrix} 4 & -7 \\ -5 & 12 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 \\ -2 & -3 \end{pmatrix}$.

✓ **Important**

When the vertices of an n-sided polygon (x_1, y_1), (x_2, y_2), (x_3, y_3), $\ldots$, (x_n, y_n) are transformed by an unknown transformation matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then to find T, you first write the position vectors of the vertices as a matrix $A = \begin{pmatrix} x_1 & x_2 & x_3 & \ldots & x_n \\ y_1 & y_2 & y_3 & \ldots & y_n \end{pmatrix}$.

The matrix for the transformed positions is $A' = \begin{pmatrix} x'_1 & x'_2 & x'_3 & \ldots & x'_n \\ y'_1 & y'_2 & y'_3 & \ldots & y'_n \end{pmatrix}$.

Since

$$A' = T \times A$$

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then, you can find the transformation matrix from

$$T = A' \times A^{-1}$$

if A^{-1} exists.

Example 4

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Two points A $(-2, -2)$ and B $(1, -1)$ are transformed by a 2×2 matrix T. If the new points are D $(4, 4)$ and C $(2, 2)$, respectively, find the transformation matrix.

Steps	Explanation
$M = \begin{pmatrix} -2 & 1 \\ -2 & -1 \end{pmatrix}$ $M' = \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix}$	Writing the position vectors of each vertex
$T = \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}$	Use matrix operations and graphic display calculator $$M' = T \times M$$ $$T = M' \times M^{-1}$$
Therefore, the transformation matrix is $$\begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}$$	

5 section questions

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Reflections

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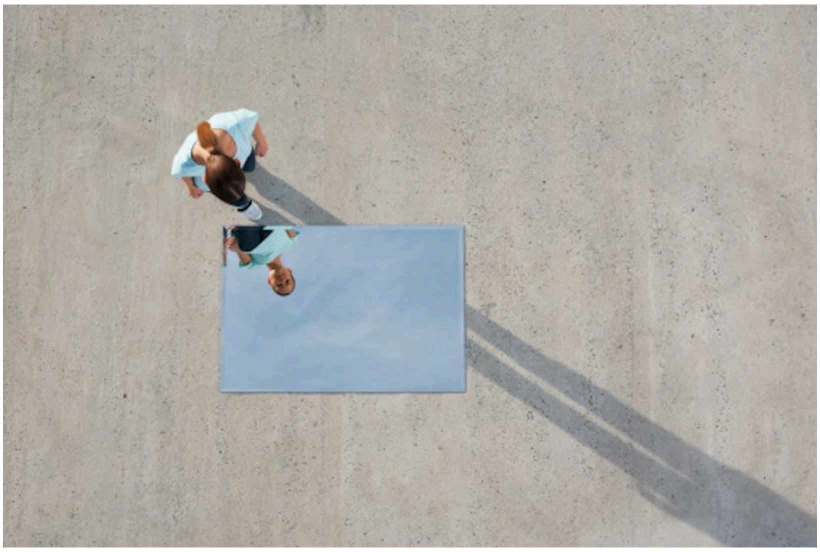
Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

When you look into a mirror you see your reflection.Your reflection will be the same size as you, of course, assuming it is not funfair mirror that distorts the image, but just a straighforward reflection on the other side of the mirror.



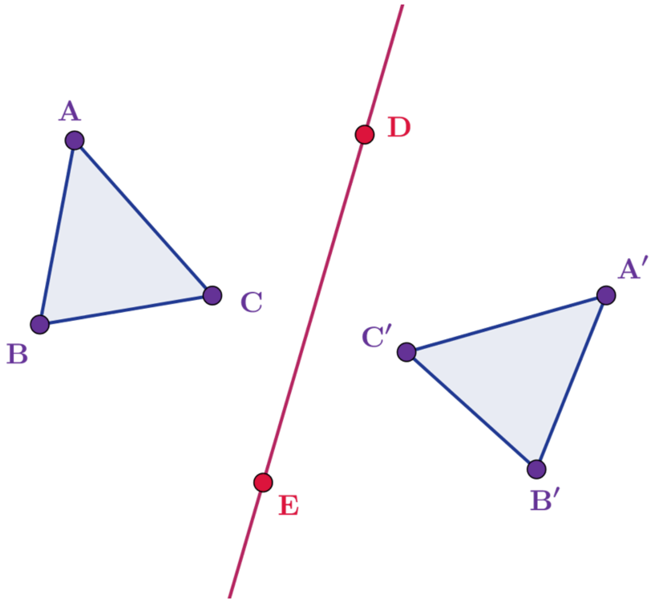
Examining your reflection

Credit: Martin Barraud Getty Images

In mathematics, reflections are like physical reflections in a mirror. Here we consider two dimensions, so that objects are reflected through a reflection line. How are objects reflected in three dimensions? Neither the shape nor the size of the object change and each point in the image is the same distance from the reflection line as its corresponding point on the object. In three dimensions, if the size and the shape are the same, is the image identical or congruent to the object? How does the image appear in two dimensions?

In the diagram shown below, ED is a reflection line and triangle ABC and its image are congruent. Corresponding points of the triangles are equidistant from the line ED.

Student view



More information

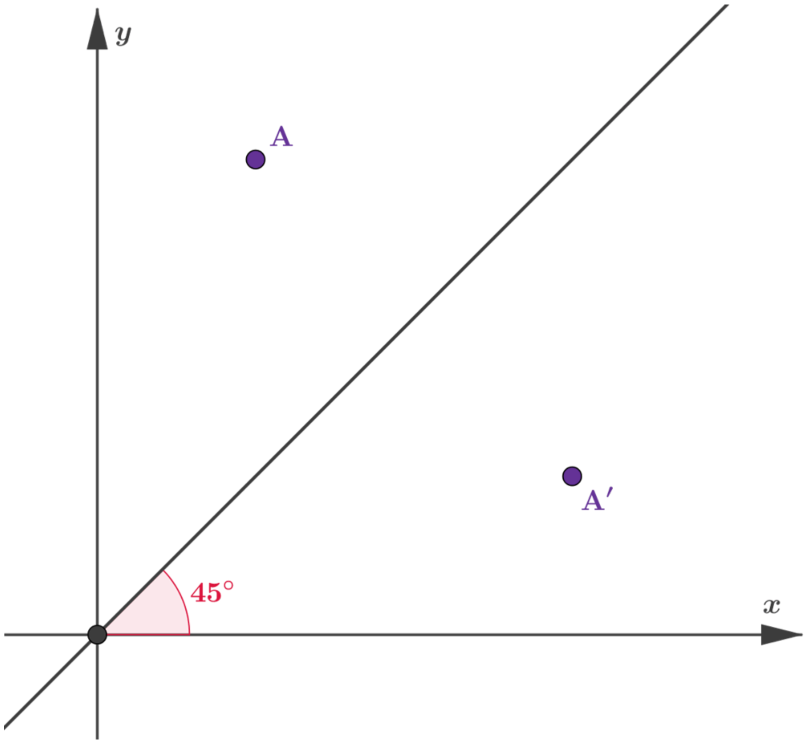
The diagram illustrates two congruent triangles, triangle ABC and its reflected image triangle A'B'C'. The reflection line ED lies vertically between the two triangles. Point D is situated on the upper part of the line ED, while E is below it. Triangle ABC is on the left side of line ED with A at the top-left, B at the bottom-left, and C at the bottom-right. Its reflected image A'B'C' is on the right side of the line reflecting each point from triangle ABC across line ED, making the two triangles congruent and equidistant from the line. The diagram indicates that corresponding points (such as A and A', B and B', C and C') are equidistant from the reflection line.

[Generated by AI]

Example 1

★★☆

Point $\mathrm{A}\ (1, 3)$ is reflected through the line $y = x$. Find the coordinates of its image, A'.

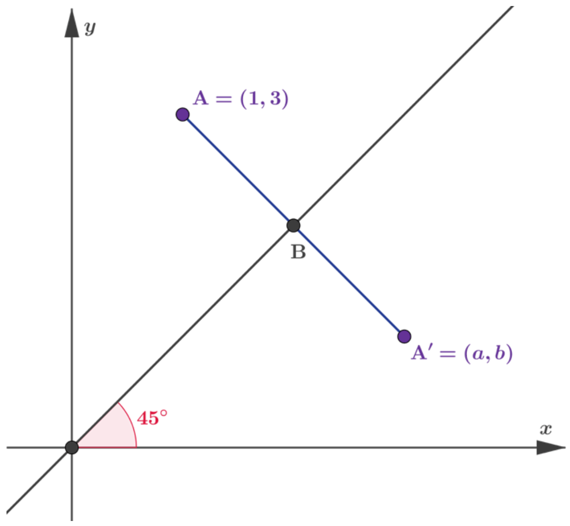


More information

The image is a graph showing the reflection of a point through the line y = x. The graph is on a Cartesian plane with x and y axes drawn, and the line y = x is depicted as a diagonal line intersecting the origin at a 45-degree angle. There are two points marked: Point A with coordinates (1, 3) and its image A' with coordinates (3, 1). Point A is above the line y = x, and its reflection, A', is below the line. The angle between the x-axis and the line y = x is marked as 45 degrees.

[Generated by AI]

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
 $\mathrm{B}\left(\frac{1+a}{2}, \frac{3+b}{2}\right)$	The reflection line, $y = x$, is the perpendicular bisector of AA'. Thus, the point B is the midpoint of AA'.
$\frac{1+a}{2} = \frac{3+b}{2}$	B lies on the line $y = x$.
$a = b + 2$ **(1)**	Rearrange.
$\frac{3-b}{1-a} = -1$	AA' is perpendicular to $y = x$, thus gradients are negative reciprocals.
$a = 4 - b$ **(2)**	Rearrange.
$a - b = 2$ $a + b = 4$ $a = 3$ and $b = 1$	Solve **(1)** and **(2)** using a graphic display calculator.
Thus, $\mathrm{A}'(3, 1)$	

Activity

In the following applet, the line passes through the point A and the origin $(0, 0)$.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

The image of the point C is transformed by matrix $\mathbf{M}$ to point C'.

θ is the anticlockwise angle between the x-axis and the line AO.

You can drag points A and C.

What is the equation of the line?

What do you notice when you drag the point A? Point C?

Interactive 1. Reflections in the Coordinate Plane.

More information for interactive 1

This interactive enables users to explore how matrix transformations affect points in a 2D coordinate system. The transformation matrix M is determined by the angle θ between this line and the positive x-axis.

A graph of XY axes is displayed, where x-axis ranges from -10 to 8 and y-axis ranges from -10 to 10. A blue line passes through point A and the origin $(0, 0)$, where users can drag point A to adjust the line OA and change angle θ. Point C and its transformed image C' are visible, with C' updating dynamically as users move point C. The transformation matrix M, defined as

$\mathrm{M} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$, adjusts automatically based on the current angle θ. All components - including the line, points, and matrix values - respond interactively to user inputs, providing immediate visual feedback.

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

For example, when users move point A to $(2, 2)$, creating $\theta = 45^\circ$, the matrix simplifies to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Dragging point C to $(3, 1)$ then shows C' transforming to $(1, 3)$, clearly demonstrating reflection across the line $y = x$. Adjusting point A to $(1, \sqrt{3})$ sets $\theta = 60^\circ$, making the matrix $\begin{bmatrix} 0.5 & \sqrt{\frac{3}{2}} \\ \sqrt{\frac{3}{2}} & 0.5 \end{bmatrix}$, and moving C to $(2, 0)$ transforms it to $(-1, \sqrt{3})$. These manipulations help users connect matrix operations with their geometric effects.

Users can manipulate points on the graph, gaining insight into the relationship between linear algebra and geometric transformations. Through this exploration, users develop an understanding of how reflection matrices are constructed from angles and how they transform points geometrically. They learn to predict transformation results based on angles and gain intuition about matrix-vector multiplication.

Exam tip

In IB examinations, the formula booklet gives the transformation matrix for a reflection in the line $y = (\tan\theta)x$ as

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}, \text{ reflection in the line } y = (\tan\theta)x$$

Example 2

★☆☆

If the transformation matrix for reflections in the line $y = ax$ is $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, find the value of a to 3 significant figures.

Steps	Explanation
$\cos 2\theta = -\frac{1}{2}$ and $sin 2\theta = \frac{\sqrt{3}}{2}$	Use the reflection matrix.
$2\theta = 120^\circ \Leftrightarrow \theta = 60^\circ$	Use a graphic display calculator or exact values.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$a = \tan 60° = \sqrt{3}$ or $a \approx 1.7320508$	Use a graphic display calculator or exact values.
Therefore, $a = 1.73$ correct to 3 significant figures.	

Example 3

★★☆

The vertices of triangle ABC are represented by the matrix $T = \begin{pmatrix} -2 & 1 & 3 \\ 1 & 1 & -2 \end{pmatrix}$. The triangle is reflected in the line $y = -x$. Find the vertices of the image.

Steps	Explanation
$\theta = 135° \left(or \frac{3}{4}\pi\right)$	As $y = (\tan\theta)x$ and $y = -x$. Thus $\tan\theta = -1$.
$\begin{pmatrix} \cos(270°) & \sin(270°) \\ \sin(270°) & -\cos(270°) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	Use the reflection matrix $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ and a graphic display calculator.
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} -2 & 1 & 3 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 2 \\ 2 & -1 & -3 \end{pmatrix}$	Multiplying transformation matrix with matrix T.
Therefore, the vertices of the image are $(-1, 2)$, $(-1, -1)$ and $(2, -3)$.	

3 section questions

Student view

3. Geometry and trigonometry / 3.9 Planar transformations

Stretches

Horizontal stretches

Think about taking a sheet of stretchy material, holding both ends and pulling them apart without changing the height. This would be a horizontal stretch. Like all the transformations, it can be represented by a transformation matrix.

Activity

You can use the applet below to investigate the impact of the transformation matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, on the image of triangle ABC, for different values of k .

Describe the similarities and differences between triangle ABC and its image $\mathrm{A'B'C'}$ for

$$k = 2, -2, 0.5, \text{ and } -0.5$$

Generalise your observations describing the relationship between ABC and its image $\mathrm{A'B'C'}$.

Interactive 1. Investigating the Impact of the Transformation Matrix on Triangle.

More information for interactive 1

This interactive tool enables users to explore how a transformation matrix affects the shape and position of a triangle. $M = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

A graph of XY axes is displayed, with the x-axis ranging from -5 to 5 and the y-axis ranging from -3 to 4. The parameter k can be adjusted by users using a horizontal slider in the top left corner, with a range from -5 to 5. This modifies the matrix $M = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$. After adjusting the value of k, the scaling of triangle *A'B'C''* and its transformed image,*A'B'C'*, stretches or compresses horizontally on the graph. The tool provides a dynamic way to visualize the effects of linear transformations, helping users connect algebraic concepts with geometric changes.

Point *A* and *B* can be moved in grid and Point *C* can be moved vertically on the y-axis, allowing users to test how the transformation behaves for different triangle configurations. The applet highlights the transformed triangle *A'B'C* in real-time, making it easy to compare similarities and differences with the original triangle *ABC*. This interactive enables user to understand how the matrix scales the horizontal dimension while preserving vertical positions.

Through experimentation, users discover how varying k alters the triangle's width without affecting its height, illustrating the concept of horizontal scaling. They learn to predict the effects of the transformation matrix and generalize the relationship between the original and transformed shapes.

What happens if $k = 0$? Or $k = 1$?

✓ **Important**

The transformation matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ stretches an object parallel to the x-axis.

This is often called a horizontal stretch. The scale factor is k. The width of the object is stretched by factor k in both horizontal directions from the vertical centre line.

Example 1

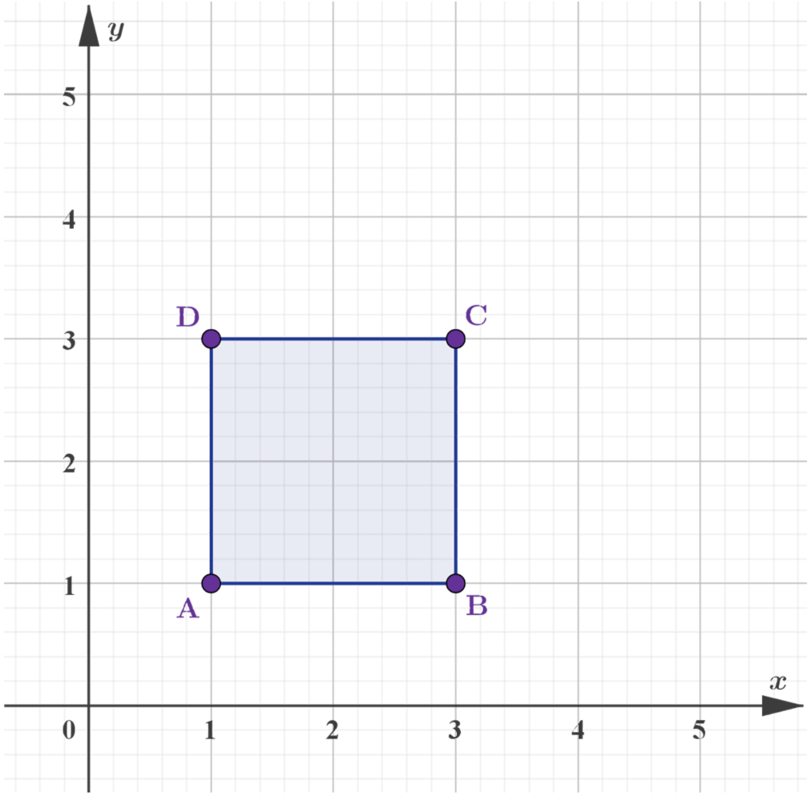
★☆☆

The square ABCD is transformed by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Copy the diagram and draw the image of the square after the transformation.

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov



More information

The image is a diagram of a square positioned on a coordinate grid. The square has its vertices labeled as A, B, C, and D.

- Point A is at (0, 0)
- Point B is at (3, 0)
- Point C is at (3, 3)
- Point D is at (0, 3)

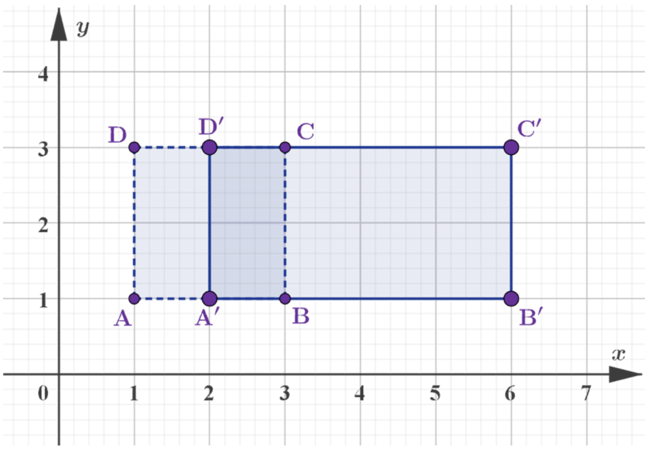
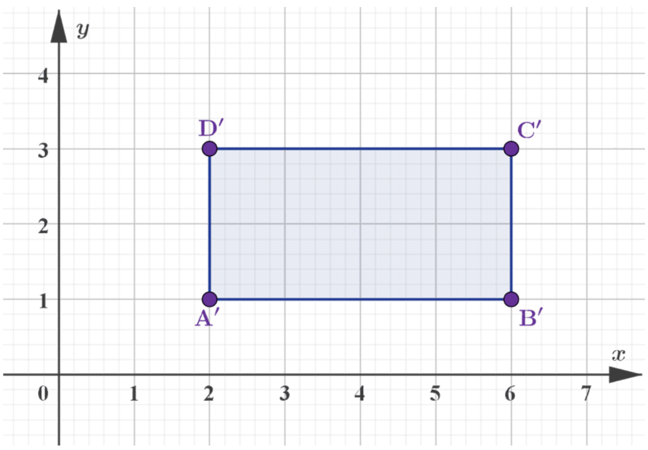
The square is centered on the grid, with each side measuring 3 units. The grid lines are visible, forming a series of squares, each representing one unit of measurement. The x-axis and y-axis are labeled, running horizontally and vertically, respectively.

The task appears to involve copying or transforming this square, as suggested by the text 'Copy the diagram and draw the image of the square after the transformation.'

[Generated by AI]

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$\text{A}\,(1,1) \rightarrow \text{A}'\,(2,1)$ $\text{B}\,(3,1) \rightarrow \text{B}'\,(6,1)$ $\text{C}\,(3,3) \rightarrow \text{C}'\,(6,3)$ $\text{D}\,(1,3) \rightarrow \text{D}'\,(2,3)$	As this is a horizontal stretch with a scale factor of 2, plot each image point twice as far from the y-axis as the original point.
	Connect the vertices in order.
	Image of ABCD under the transformation matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

Student view

Exam tip

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

In IB examinations, the transformation matrix for a horizontal stretch parallel to the x-axis is given in formula booklet:

$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, horizontal stretch/stretch parallel to x-axis with a scale factor of k.

Example 2

★★☆

Consider the line segment with end points $\text{A}(-1, 2)$ and $\text{B}(2, 4)$.

Find the coordinates of the end points after a horizontal stretch with the scale factor 3.

Hence, find the length of the image of AB after the transformation.

Steps	Explanation
$L = \begin{pmatrix} -1 & 2 \\ 2 & 4 \end{pmatrix}$	Write the matrix corresponding to the coordinates of the segment AB.
$T = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$	As the scale factor for the horizontal stretch is 3.
$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & 4 \end{pmatrix}$	Here, (x_1, y_1) and (x_2, y_2) are the end points of the image.
$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} -3 & 6 \\ 2 & 4 \end{pmatrix}$	Use a graphic display calculator.
Therefore, the corresponding end points of the image are $\text{A}'\ (-3, 2)$ and $\text{B}'\ (6, 4)$.	
$\lvert\text{A}'\text{B}'\rvert = \sqrt{(-3-6)^2 + (2-4)^2} = \sqrt{85}$	Use the formula for the distance between two points.

Vertical stretches

Student view

Like a horizontal stretch, a vertical stretch is like holding an object and pulling one end up and the other down.

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Activity

Use the following applet to investigate the impact of the transformation matrix $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ on the image of quadrilateral ABCD for different values of k .

Describe the similarities and differences between quadrilateral ABCD and its image $\text{A}'\text{B}'\text{C}'\text{D}'$ for

$k = 2, -2, 0.5$, and -0.5 .

Generalise your observations describing the relationship between quadrilateral ABCD and its image $\text{A}'\text{B}'\text{C}'\text{D}'$.

Interactive 2. Investigating the Impact of the Transformation Matrix on Quadrilateral.

More information for interactive 2

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

A coordinate grid with x-values ranging from -5 to 2 and y-values from -5 to 5 is displayed. A quadrilateral labeled *ABCD* is plotted on the grid. The transformation is defined by the matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, which users can manipulate using a horizontal slider in the top left corner. The slider allows the value of k to vary from -5 to 5. As k changes, the image of the quadrilateral, labeled *A'B'C'D'*, updates instantly on the graph to reflect the transformation.

This matrix transformation applies a **vertical scaling** to the quadrilateral: it multiplies the y-coordinates of all vertices by k, while the x-coordinates remain unchanged. Users can also drag the original vertices A, B, C, and D anywhere on the grid and observe how the transformation affects the resulting image.

The grid displays the original quadrilateral ABCD, with the following vertices:

- $A = (0, -1)$
- $B = (1, -1)$
- $C = (1, 1)$
- $D = (0, 1)$

k is a scaling factor that can be adjusted using a slider. This matrix affects only the y-coordinates of the shape, scaling them by a factor of k, while leaving the x-coordinates unchanged. The result is a vertical transformation of the quadrilateral into its image A'B'C'D', shown on the graph in red.

Example: When k=-2

- $A = (0, -1)$ transforms to $A' = (0,\ 2)$
- $B = (1, -1)$ transforms to $B' = (1,\ 2)$
- $C = (1, 1)$ transforms to $C' = (1,\ -2)$
- $D = (0, 1)$ transforms to $D' = (0,\ -2)$

This illustrates a **vertical reflection and stretch** — the quadrilateral is flipped across the x-axis and stretched because $|k| > 1$ and $k < 0$.

Each adjustment immediately updates the image A'B'C'D', allowing users to visualize:

- **Vertical stretch:** when $|k| > 1$
- **Vertical compression**: when $0 < |k| < 1$
- **Vertical reflection:** when $k < 0$
- **Collapse to a line**: when $k = 0$

This hands-on tool helps users understand how the transformation matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ alters only the vertical dimension of a shape. It reinforces key concepts in linear transformations and matrix operations, valuable in fields such as computer graphics, animation, and engineering simulations.

Student view

Overview
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761618/ov

Important

The transformation matrix $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ stretches an object parallel to the y-axis. This is often called a vertical stretch. Again, the scale factor is k. The height of the object is stretched by factor k in both vertical directions from the horizontal centre line.

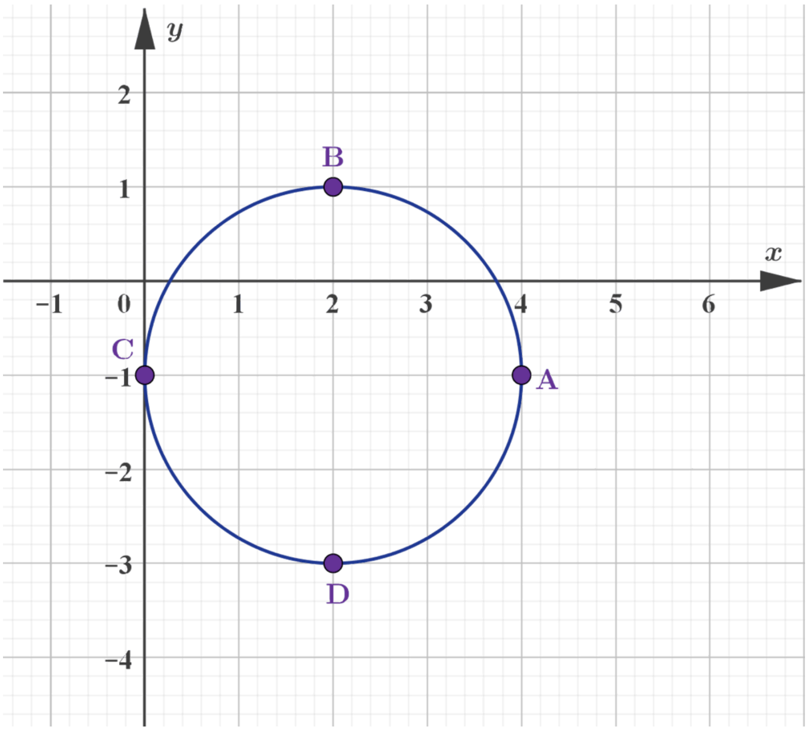
Example 3

★★☆

A circle with a radius of 2 units is transformed by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

If A, B, C and D points on the circumference, copy the diagram and draw the image of the circle after the transformation.

What is the image of the circle after the transformation?



More information

The image is a graph on a grid showing a circle with labeled points A, B, C, and D. The circle is centered at the origin (0,0) and intersects the axes at the points where these labels are located.

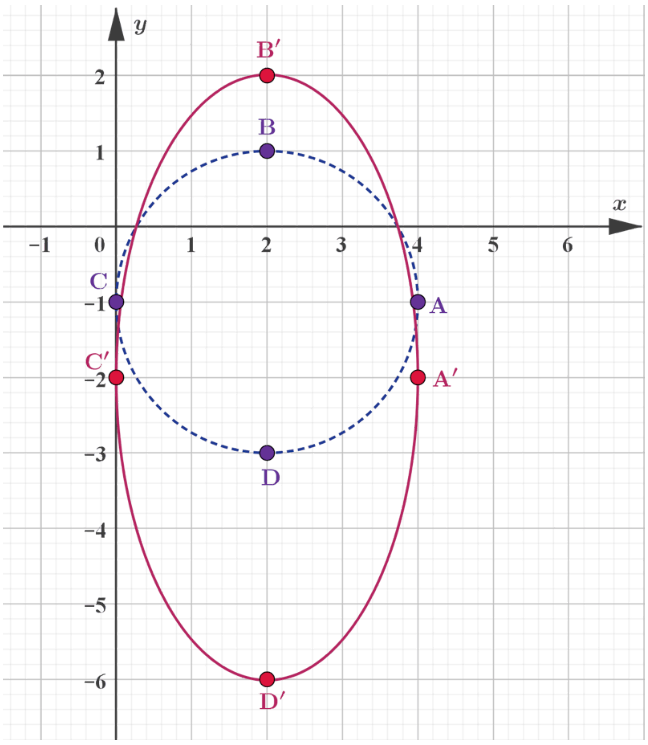
Student
view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

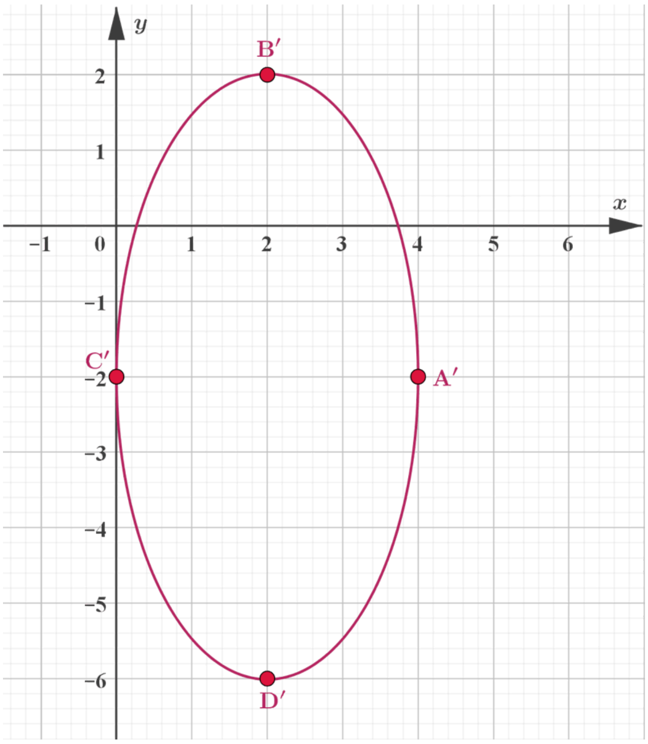
- Point A is on the positive x-axis at (3, 0).
- Point B is on the positive y-axis at (0, 3).
- Point C is on the negative x-axis at (-3, 0).
- Point D is on the negative y-axis at (0, -3).

The circle appears to be a standard circle with a radius of 3. The graph grid provides a clear set of coordinates and axes with numbered intervals for precise location data.

[Generated by AI]

Steps	Explanation
$\mathrm{A}\,(4,-1)\rightarrow\mathrm{A}'\,(4,-2)$ $\mathrm{B}\,(2,1)\rightarrow\mathrm{B}'\,(2,2)$ $\mathrm{C}\,(0,-1)\rightarrow\mathrm{C}'\,(0,-2)$ $\mathrm{D}\,(2,-3)\rightarrow\mathrm{D}'\,(2,-6)$	As this is a vertical stretch with a scale factor of 2, plot each image point twice as far from the x-axis as the original point.
	Plot the points and connect.

Student view

Steps	Explanation
 The image is an ellipse.	Draw the image of the circle after the transformation.

Example 4

★★☆

Consider the line segment with end points $\mathrm{A}(-1, 2)$ and $\mathrm{B}(2, 4)$.

Find the coordinates of the end points after a vertical stretch parallel to the y-axis with a scale factor of 3.

Hence, find the length of the image of AB after the transformation.

Steps	Explanation
$L = \begin{pmatrix} -1 & 2 \\ 2 & 4 \end{pmatrix}$	Write the matrix corresponding to the coordinates of the segment AB.

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
$T = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$	As the scale factor for the vertical stretch is 3.
$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & 4 \end{pmatrix}$	Here, (x_1, y_1) and (x_2, y_2)) are the end points of the image.
$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 6 & 12 \end{pmatrix}$	Use a graphic display calculator.
$\sqrt{(-1-2)^2 + (6-12)^2} = 3\sqrt{5}$	Use the distance formula.

3 section questions

3. Geometry and trigonometry / 3.9 Planar transformations

Enlargements

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/enlargements-id-27638/print/) Assign

Now that you have pulled objects either vertically or horizontally, what do you think would happen if you called a friend and both of you pulled an object in four directions: up and down and left and right? How would you represent the transformation matrix?

Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Interactive 1. Planar Transformations.

Credit: GeoGebra (https://www.geogebra.org/m/qhjuzuQz) GeoGebra Institute of Hong Kong

More information for interactive 1

This interactive GeoGebra tool visually demonstrates the concept of uniform scaling transformations through a dynamic display of concentric circles.

Set against a deep blue background, a series of orange circles radiate from a fixed point labeled “O”, marked with a cross. As users adjust the scaling factor kkk using the horizontal slider at the top left, ranging from 0 to 0.1, the circles undergo proportional enlargement, expanding symmetrically in all directions from the center.

The scaling transformation is centered at point O, and as the value of kkk increases, another cross-labeled point moves progressively farther from O, illustrating how each point on the shape scales outward uniformly.

A play button located at the bottom left initiates an animated loop, where the circles continuously grow and recede, simulating the effect of increasing and decreasing values of kkk. This motion offers a powerful visual cue of how distance from the origin scales linearly with kkk, reinforcing the concept of uniform enlargement.

Activity

Using the following applet, you will investigate the impact of the transformation matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ on the image of quadrilateral ABCD for different values of k.

What does the image of the quadrilateral ABCD look like?

Describe the similarities and differences between quadrilateral ABCD and its image after the transformation for

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

$k = 1, -2, 2, 0.5$ and -0.5

Generalise your observations describing the relationship between quadrilateral ABCD and its image after the transformation.

Interactive 2. Investigating the Impact of the Transformation Matrix on Quadrilateral.

More information for interactive 2

This interactive lets users explore how a uniform scaling transformation affects a quadrilateral.

A graph is displayed with xy axes centered at (0,0) where the original quadrilateral appears with purple vertex and blue side with its transformed version in red.

The transformation is controlled by matrix

M = [k 0]
[0 k]

with parameter k. This matrix scales the shape of the quadrilateral and its transformed version equally in both the x and y directions when the value of k is adjusted using a horizontal slider in the top left corner, with k ranging from -5 to 5. Users can observe how positive k values enlarge or shrink the quadrilateral while preserving its proportions, while negative k values additionally flip the shape across both axes.

As users adjust the k slider, the transformed version of the quadrilateral appears, clearly showing size changes and possible reflections. The current transformation matrix is ABCD displayed dynamically, updating instantly as k changes. Users can also drag points A, B, C, and D to reshape the original quadrilateral, with all transformations recalculating immediately based on the new shape and current k value. Black connecting lines between points help visualize the scaling effects more clearly.

For example, when users set k to 2.5, they see the quadrilateral grow to 2.5 times its original size in all directions while maintaining its shape and orientation. Setting k to −1 reflects the quadrilateral across both axes while keeping the same size, effectively rotating it 180 degrees about the origin. Trying k = 0.5

Student view

Overview
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shrinks the shape to half its dimensions, while $k = -3$ both triples the size and inverts the image. These manipulations allow users to experiment with different scaling factors and understand the corresponding matrix operations and geometric outcomes.

Through this exploration, users develop an intuitive understanding of uniform scaling transformations and how they're represented mathematically through diagonal matrices. They learn to predict how shapes will change under different scaling factors, including both positive and negative values, and see how matrix multiplication corresponds to geometric operations.

✓ **Important**

When an object is enlarged by the transformation matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, the distance of each point of the object from $(0, 0)$ changes by the scale factor k.

Example 1

★☆☆

A point $\mathrm{A}(1, 2)$ is transformed by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Find the image of A after the transformation. Hence, describe the transformation.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

The image is $(2, 4)$. Both the x- and y-coordinates are twice as big. Therefore, the image is twice as far from both axes.

✓ **Important**

When a line segment is enlarged by the transformation matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, the length of the image is increased by a factor of k.

Student view

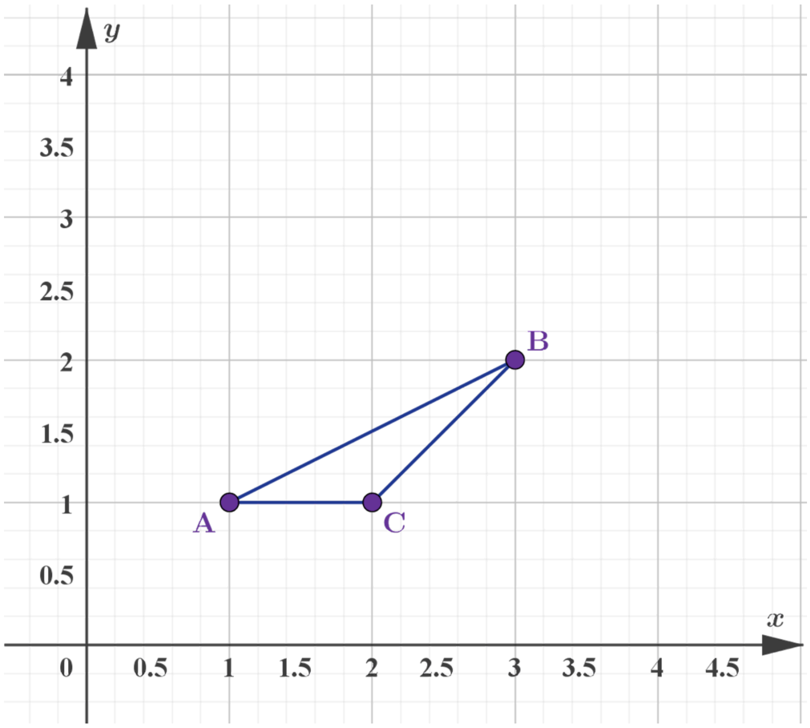
Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Example 2

★★☆

Draw the image of triangle ABC after an enlargement with $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Hence, compare the area of triangle ABC and with that of its image. What do you notice?

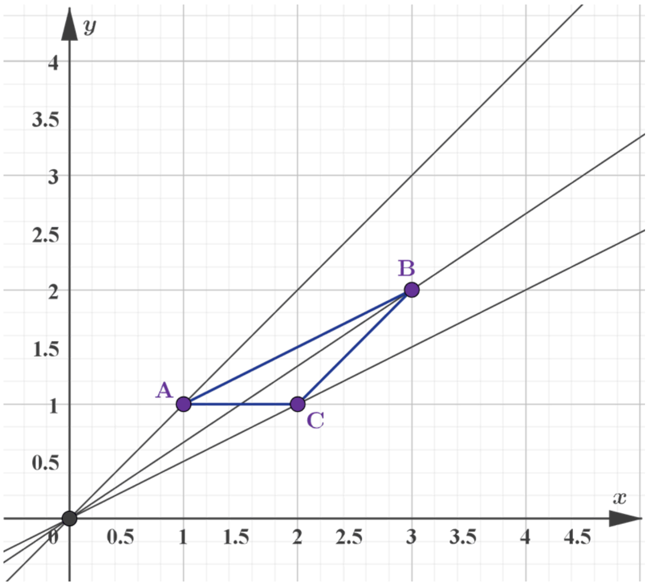
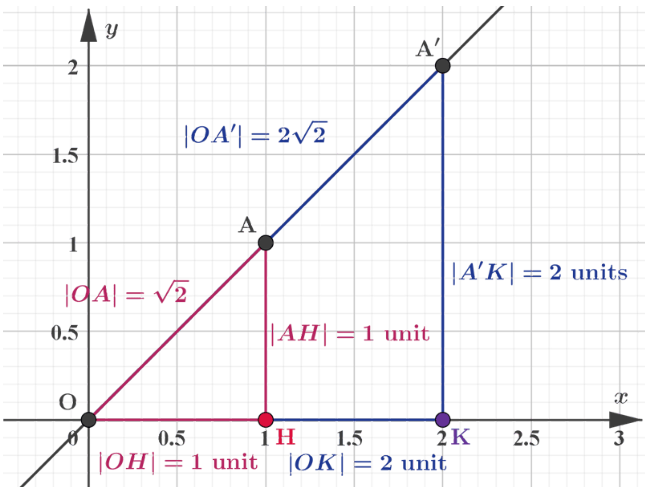


More information

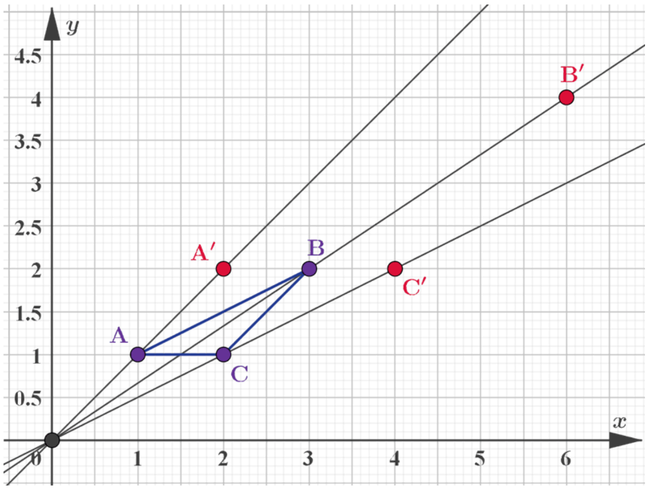
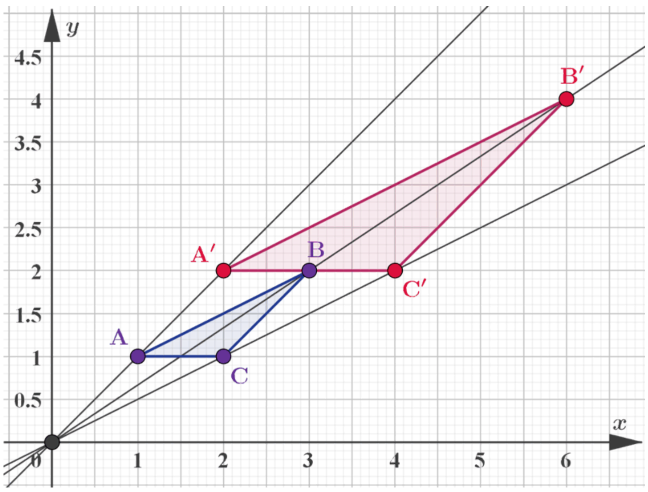
The image is a graph showing a triangle with vertices labeled A, B, and C. It's displayed on a coordinate grid with the x-axis labeled 't' ranging from 0 to 4.5 and the y-axis labeled 'y' ranging from 0 to 4.5. Point A is located at approximately (1,1), point B at (3,1.5), and point C at (2,1.2), forming a triangle. The grid lines are evenly spaced, and the intersection points create coordinates that define each vertex of the triangle. There are arrows indicating the positive direction on both axes. The triangle is outlined in blue lines connecting the three points.

[Generated by AI]

Student view

Steps	Explanation
	Draw lines passing through each vertex and the origin.
	As the scale factor is 2, the image of each vertex will be twice as far from the origin.

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
	Mark the image of each vertex.
Therefore, the image of the triangle ABC is 	
$\text{Area}\,(\mathrm{ABC}) = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}\ \text{unit}^2$ $\text{Area}\,(\mathrm{A'B'C'}) = \frac{1}{2} \times 2 \times 2 = 2\ \text{unit}^2$	Use area of a triangle formula.

Student view

Overview
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Steps	Explanation
Therefore, the area of the image is four times as big as the original triangle. $4 = 2^2 = k^2$. In general, the area of an image is k^2 that of the original object.	

Example 3

★★☆

The line segment with end points $(-1, 1)$ and $(3, -2)$ is enlarged with the scale factor 3.

Find the coordinates of the end points of the image.

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -3 & 9 \\ 3 & -6 \end{pmatrix}$$

Therefore, the coordinates of the end points of the image are

$$(-3, 3) \text{ and } (9, -6)$$

Exam tip

In IB examinations, the transformation matrix for enlargement with a scale factor of k and centre $(0, 0)$ will be given as $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, enlargement, with a scale factor of k , centre $(0, 0)$.

Important

When an object is enlarged by the transformation matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, the ratio of the area of the image to the area of the original object is k^2 .

3 section questions

Student view

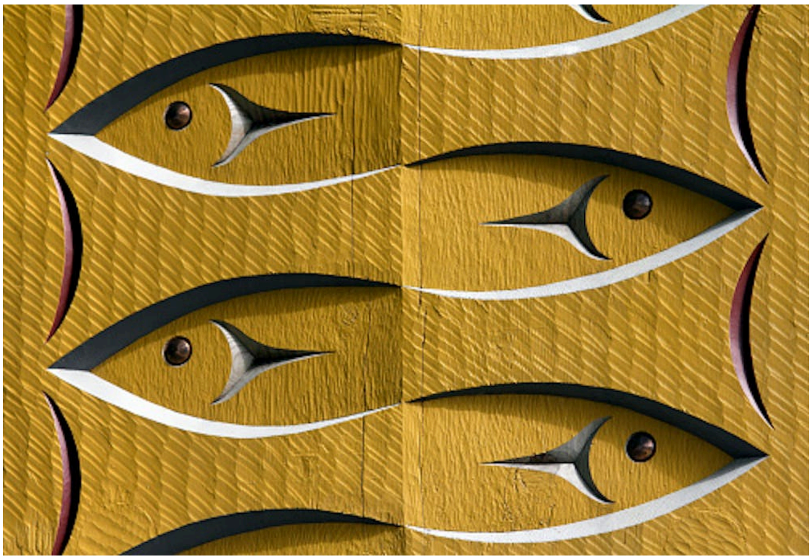
Rotations

Although there are many possible rotations with different centres other than the origin, you will be looking only at rotations anticlockwise through the origin.

International Mindedness

Maurits Cornelis Escher, more widely known as just Escher, was a Dutch graphics artist who used mathematical concepts in his art. His art included impossible objects, ideas of infinity, symmetry, transformations, tessellations and hyperbolic geometry. He interacted with famous mathematicians of his time, like Polya, Penrose and Coxeter. Although he believed that he did not have any mathematical talent, his artwork was based on mathematics and it often inspired mathematicians. Moreover, it was admired by the general public.

The diagram shows an image created using techniques like those used by Escher. It has rotational symmetry and uses various transformations. How would you create such an image using computer graphics and mathematics?



Fish in the style of Escher
Credit: Apexphotos Getty Images

More information

The image is an artwork depicting fish in the style of Escher. It features multiple stylized fish arranged in a pattern with rotational symmetry, creating a visually intriguing design. Each fish appears to be composed of geometric shapes and the entire pattern seems to flow in a continuous, interlocking loop. This creates a sense of movement and transformation throughout the artwork. This image is representative of the type of art that incorporates mathematical concepts like symmetry and transformation, similar to the works of M.C. Escher, known for his optical illusions and impossible constructions.

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

[Generated by AI]

Activity

Using the following applet, you will investigate the impact of the transformation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ θ on the image of a triangle for different values of θ

Describe the image of the triangle as you change the rotation angle.

Interactive 1. Investigating the Impact of the Transformation Matrix on Triangle.
Credit: GeoGebra (https://www.geogebra.org/m/hepjZFF7) GeoGebra Institute of MEI

More information for interactive 1

This interactive allows user to explore the effects of a rotation transformation on a triangle by adjusting the angle using a slider.

A graph is displayed with xy axes, x-axis ranging from -5 to 6 and y-axis ranging from -3 to 5.

On the graph there are two right triangles, one in red which is the base triangle, and one in blue which is a static. The base triangle in red, displayed on a 2D coordinate plane, undergoes a transformation defined by the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, which rotates the red triangle around the origin. As user change θ from 0° to 360°, the red triangle rotates counterclockwise, with its vertices tracing circular paths around the origin while preserving the shape and size of the triangle.

Student view

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For instance, at θ = 90°, the triangle is rotated a quarter turn counterclockwise, aligning its orientation perpendicular to its original position. The corresponding matrix updates in real-time i.e $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ displaying the values of cos and sin, to see how the matrix elements drive the rotation.

Exam tip

In IB examinations, the transformation matrix for the rotation by positive angle θ is in the formula booklet:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$, anticlockwise/counter-clockwise rotation of angle θ

about the origin $(\theta > 0)$.

Example 1

★☆☆

Write down the matrix that represents a rotation of $\frac{\pi}{3}$ anticlockwise about the origin, using exact values.

Use the exact values of $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

International Mindedness

In 2007, a group of US and European mathematicians and computer scientists managed to describe a well-known Lie group, E8. It took them about 4 years and they used a supercomputer to store 60 gigabytes of data.

If you would like to learn what a Lie group is, why this Lie group is important and how the collaboration helped to describe it, read the following article from PlusMath magazine.

Student view

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https://plus.maths.org/content/solving-symmetry (https://plus.maths.org/content/solving-symmetry)

Example 2

★★☆

Write down the matrix that represents a rotation of $\frac{\pi}{3}$ clockwise about the origin, using exact values.

The given rotation is clockwise. Use the unit circle, θ is in the fourth quadrant.

$$\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

Use the exact values of $\cos\frac{5\pi}{3} = \frac{1}{2}$ and $\sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$.

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

⚠ **Be aware**

The transformation matrix we are using is only for rotations anticlockwise with centre (0,0). If the rotation is defined clockwise or the angle is negative, you need to use the unit circle to find the positive angle θ for an anticlockwise rotation.

Example 3

★☆☆

Write down the coordinates of the image of $K\ (4, 3)$ after it has been rotated 150° anticlockwise about the origin.

Student view

Rotation matrix for 150° anticlockwise about the origin.

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$$\begin{pmatrix} \cos 150° & -\sin 150° \\ \sin 150° & \cos 150° \end{pmatrix} = \begin{pmatrix} -0.866 & -0.5 \\ 0.5 & -0.866 \end{pmatrix}$$

Use a graphic display calculator.

$$\begin{pmatrix} -0.866 & -0.5 \\ 0.5 & -0.866 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -4.964 \\ -0.598 \end{pmatrix}$$

Therefore, the coordinates of the image are

$$(-4.964, -0.598)$$

3 section questions

3. Geometry and trigonometry / 3.9 Planar transformations

Translations

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/translations-id-27640/print/) Assign

Along with reflections and rotations, translations are the third type of transformation where the shape of an object does not change. The original object and its image are congruent. Only its position changes, without flipping it over a line or a point. You simply move each point in the same direction, as shown below. A translation can be represented with a vector $\begin{pmatrix} a \\ b \end{pmatrix}$, where a is the translation parallel to the x-axis and b is the translation parallel to the y-axis.

Making connections

You will learn more about vectors in subtopic 3.10 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27907/).

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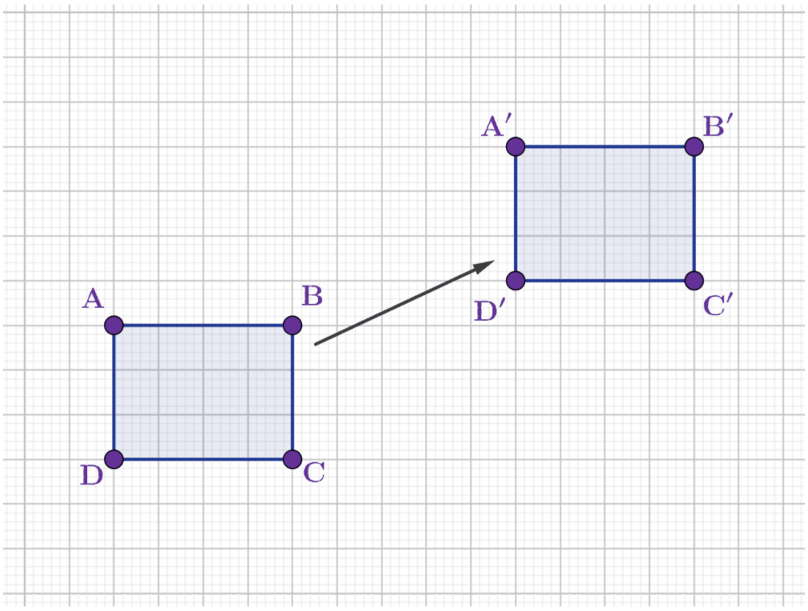
Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov



More information

The image features two blue squares overlaid on a grid background, each with four labeled corner points. The squares are connected by a dashed line, suggesting a relationship between them. The first square has points labeled A, B, C, and D, while the second square has points labeled A', B', C', and D'. This is a diagram illustrating a mapping or transformation of points from one geometric shape to another, possibly within a mathematical or graphical context.

[Generated by AI]

Exam tip

In IB examinations, a transformation matrix will not be given for translations. You need to remember that a translation a units parallel to the x -axis and b units parallel to the y-axis is given by the translation vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

Example 1

★☆☆

The point $\mathrm{A}(-2, 3)$ is translated by the vector $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$.

Write down the coordinates of the image under this translation.

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Adding the translation matrix to the matrix representing the point A:

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

Therefore, the coordinates of the image are:

$$(-4, 5)$$

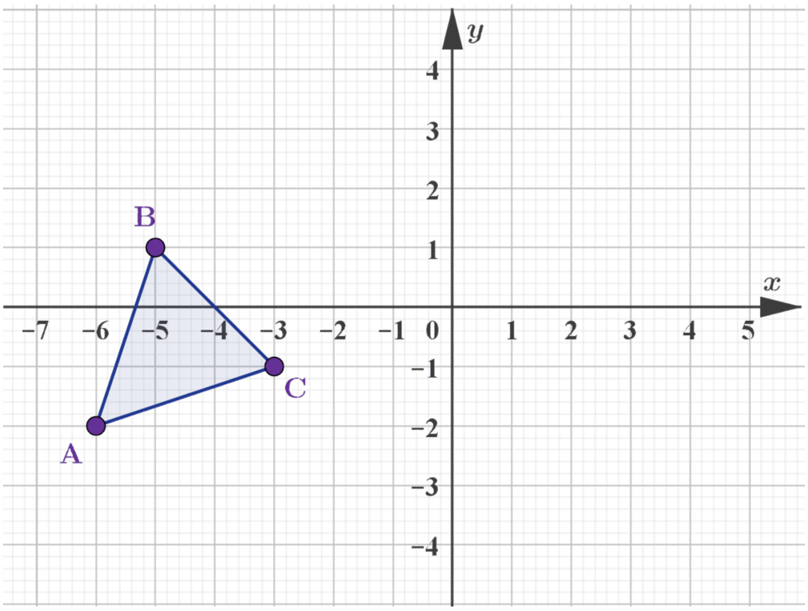
Example 2

★★☆

The triangle ABC is translated first by the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and then by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Write down the coordinates of the vertices of the final image under these translations.

Hence, find a single transformation from ABC to the final image.



More information

The image shows a geometric diagram of triangle ABC placed on a grid. The graph has labeled axes: the X-axis runs horizontally and ranges from -7 to 5, while the Y-axis runs vertically and ranges from -1 to 7. The triangle ABC is highlighted and appears to be translated or rotated on the grid. Point A is at the origin, point B is above point A on the left, and point C is

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to the right of points A and B. Each vertex is marked with a pink dot. The image visually depicts a potential transformation of triangle ABC within the coordinate plane.

[Generated by AI]

Adding the transformation matrix to the matrix representing the point A and then A'.

$$\mathrm{A} \to \mathrm{A}' : \begin{pmatrix} -6 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$
$$\mathrm{A}' \to \mathrm{A}'' : \begin{pmatrix} -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

Adding the transformation matrix to the matrix representing the point B and then B'.

$$\mathrm{B} \to \mathrm{B}' : \begin{pmatrix} -5 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$
$$\mathrm{B}' \to \mathrm{B}'' : \begin{pmatrix} -3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

Adding the transformation matrix to the matrix representing the point C and then C'.

$$\mathrm{C} \to \mathrm{C}' : \begin{pmatrix} -3 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$$\mathrm{C}' \to \mathrm{C}'' : \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

The coordinates of the vertices of the image are

$(-3, 2)$, $(-2, 5)$ and $(0, 3)$

As the addition of two vectors can be written as a single vector.

The single transformation from ABC to triangle $\mathrm{A}''\mathrm{B}''\mathrm{C}''$ is $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Student view

✓ **Important**

When a shape is translated by two or more translation vectors, the vectors can be summed together to give a single translation vector:

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} + \cdots = \begin{pmatrix} a+c+e+\cdots \\ b+d+f+\cdots \end{pmatrix}$$

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3 section questions

3. Geometry and trigonometry / 3.9 Planar transformations

Composite transformations

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/composite-transformations-id-27641/print/) Assign

Transformations can be combined and applied one after another to the same object. In the following applet, you will see how a sense of motion in still objects can be created using composite transformations.

Activity

In the applet below, different transformations are applied to the same object, triangle ABC.

Using the sliders, where k is the scale factor, θ is the angle for an anticlockwise rotation and n is the number of iterations, investigate how different images can be formed.

Student view

Interactive 1. Investigating Different Transformations Applied to Same Object.

More information for interactive 1

This interactive allows users to explore the effects of composite transformations on triangle ABC by adjusting the parameters k, θ, and n.

A graph is displayed with xy axes. A triangle ABC is projected on the graph. There are three horizontal sliders on the top left corner of the graph for the respective parameters k, θ, and n. The scale factor k can be varied from -1 to 2, enabling users to scale the triangle uniformly. Positive values of k enlarge or shrink the triangles, while negative values also reflect it across both axes. The rotation angle θ can be set from 0 to π radians, allowing you to rotate the triangle anticlockwise by the specified angle.

By fixing values for k and θ, and then increasing the number of iterations ‘n’ from 0 to 24, the triangle transforms step-by-step. Each iteration applies the combined scaling and rotation transformations to the triangle, creating a sequence of images that demonstrate the cumulative effect of these transformations. This process can generate intricate patterns and shapes, such as spirals or symmetrical designs, depending on the chosen values of k and θ.

Student view

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✓ **Important**

The order in which transformations are applied matters.

Example 1

★★☆

Point $\mathrm{A}\ (3, -2)$ is transformed to B by a transformation matrix that represents a horizontal stretch parallel to the x-axis with a scale factor of 2 . Point B is then transformed to C by a rotation matrix that represents a rotation of 60° anticlockwise about the origin.

Hence, write down a single transformation matrix representing the composite transformation that transforms A to C correct to 3 significant figures.

As the scale factor is 2

Horizontal stretch:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

$\mathrm{B}\ (6, -2)$

60° anticlockwise about the origin

Rotation:

$$\begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix}$$

$$\begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix} \begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 4.73 \\ 4.20 \end{pmatrix}$$

$\mathrm{C}\ (4.73, 4.20)$

Student view

Note the order of the transformations on $(3, -2)$

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$$\begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix} \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right)$$

Multiply the two transformation matrices in the order of the transformations

$$\begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -0.866 \\ 1.732 & 0.5 \end{pmatrix}$$

Correct to 4 significant figures.

The single matrix that represents the horizontal stretch followed by the rotation is

$$\begin{pmatrix} 1.000 & -0.866 \\ 1.732 & 0.500 \end{pmatrix}$$

Activity

You will investigate the impact of multiplying a point $\text{A}\ (2, 3)$ by two 2×2 matrices,

$$T \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \text{ and } M \begin{pmatrix} -2 & 0 \\ 2 & 1 \end{pmatrix}.$$

Find the image of A when it is

- transformed by T followed by M
- transformed by M followed by T .

Be aware

Matrix multiplication is not a commutative operation, which means that for two matrices A and B, $A \times B$ is not always equal to $B \times A$. Therefore, when you have composite transformations, you need to pay attention to the order.

Example 2

★★☆

A line segment with vertices $(2, 1)$ and $(1, 3)$ is reflected in the line $y = \sqrt{3}x$, and then rotated clockwise by $\frac{\pi}{3}$ about $(0, 0)$.

Student view

If F represents the reflection and T represents the rotation matrices:

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a) Write down the matrices F and T.

b) Write down a single matrix that represents F followed by T.

c) Hence, find the matrix representing the image of the segment under the two transformations.

	Steps	Explanation
a)	$\theta = 60^\circ$	$\tan\theta = \sqrt{3}$
	$F = \begin{pmatrix} -0.5 & 0.866 \\ 0.866 & 0.5 \end{pmatrix}$	Reflection matrix : $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
	$\alpha = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$	Rotation is given clockwise so convert it to an anticlockwise angle.
	$T = \begin{pmatrix} 0.5 & 0.866 \\ -0.866 & 0.5 \end{pmatrix}$	Use the rotation matrix formula $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$
b)	$\begin{pmatrix} 0.5 & 0.866 \\ -0.866 & 0.5 \end{pmatrix} \begin{pmatrix} -0.5 & 0.866 \\ 0.866 & 0.5 \end{pmatrix}$ $= \begin{pmatrix} 0.500 & 0.866 \\ 0.866 & -0.500 \end{pmatrix}$	Let S be the matrix representing the segment. Then the image is given by $T(\mathbf{F} \times \mathbf{S})$ So, here we have $T \times F$.
c)	$\begin{pmatrix} 0.500 & 0.866 \\ 0.866 & -0.500 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ $= \begin{pmatrix} 1.866 & 3.098 \\ 1.232 & -0.634 \end{pmatrix}$	$S = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ As the question uses 'hence' you must use the single transformation matrix you already found.

Affine transformations

Consider transformations applied in a sequence: first, multiply a point $\begin{pmatrix} x \\ y \end{pmatrix}$ by a transformation matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and then add a translation vector $\begin{pmatrix} e \\ f \end{pmatrix}$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$.

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In linear algebra, these transformations are called affine transformations. All the points, straight lines and planes still exist but the image may be enlarged and be in a different position, and any angles may have changed.

Amazing fractals of Romanesco broccoli. Each bud is similar to the whole head

Credit: BruceBlock Getty Images

By repeatedly applying particular affine transformations to an object and the resulting images, we can produce an overall object that is self-similar, such as a fractal.

That is, each part of the image resembles the whole image. And each part of each part resembles the part, all the way down to infinity. Have you ever seen Romanesco broccoli (shown above)?

These plants are a fine example of self-similarity. Each bud has been transformed and placed in such a way that the whole head of broccoli is like a bud.

The following video briefly explains why the Mandelbrot set is special and how it can be created by iterative transformations.

Student view

What's so special about the Mandelbrot Set? - Numberphile

Example 3

★★☆

An image is first transformed by an enlargement with a scale factor of 2 followed by an anticlockwise rotation of $180°$ about the origin. Finally, it is translated by the vector $\begin{pmatrix}1\\1\end{pmatrix}$.

a) If the transformations mapping (x, y) to (x', y') is written as

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}e\\f\end{pmatrix}$$

Find the 2×2 matrix and the 2×1 matrix representing the overall transformation.

b) Hence, find the image of $(3, -3)$ under these transformations.

	Steps	Explanation
a)	$\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$	Enlargement of scale 2.
	$\begin{pmatrix}\cos 180° & -\sin 180°\\\sin 180° & \cos 180°\end{pmatrix} = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}$
	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$	First enlargement then rotation.

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	Steps	Explanation
	$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$	Use a graphic display calculator.
	$\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	The translation vector.
	Therefore, $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$	
b)	$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$	
	So, the image is $$(-5, 7)$$	

3 section questions

3. Geometry and trigonometry / 3.9 Planar transformations

Area of an image produced by a transformation matrix

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/area-of-an-image-produced-by-a-transformation-id-27642/print/) Assign

So far, you have looked at what happens to points on an object when it undergoes various transformations. Now, you will study the relationship between the area of an object and the area of its image.

Example 1

★★☆

A triangle with vertices A $(1, 1)$, B $(1, 2)$ and C $(2, 3)$ is transformed with the matrix $M = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$.

Student view

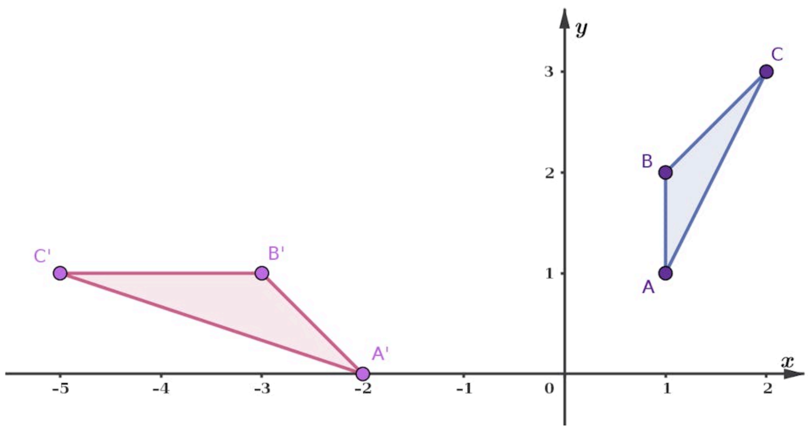
a) Put the triangle and the image on a coordinate plane.

b) Find the area of the triangle ABC.

c) Find the area of the image triangle.

d) Find the determinant of the transformation matrix.

e) Find a relationship between the results in parts (b), (c) and (d).

	Steps	Explanation
a)	$\begin{pmatrix}-1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 2\\1 & 2 & 3\end{pmatrix}=\begin{pmatrix}-2 & -3 & -5\\0 & 1 & 1\end{pmatrix}$	Multiply the transformation matrix with the matrix representing the vertices.
	Therefore, the transformed vertices are $\mathrm{A}'(-2,0)$, $\mathrm{B}'(-3,1)$ and $\mathrm{C}'(-5,1)$.	The product matrix contains the coordinates of the transformed vertices.
		
b)	$\text{area}\,(\mathrm{ABC})=\dfrac{1\times 1}{2}=0.5\ \text{unit}^2$	In triangle ABC the vertical side AB is 1 unit long and the horizontal distance of C from this line is also 1 unit.

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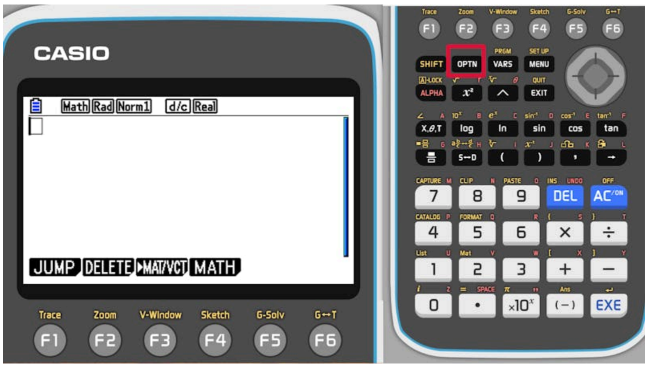
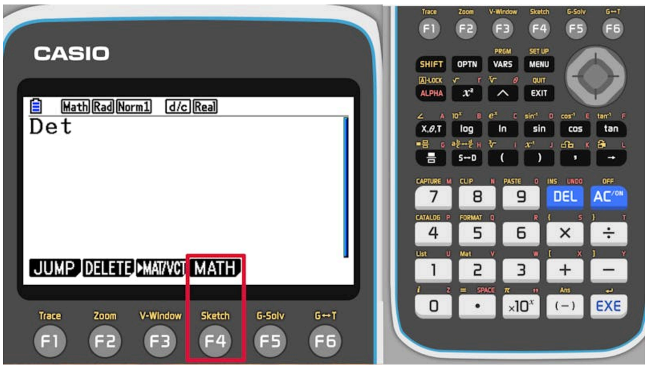
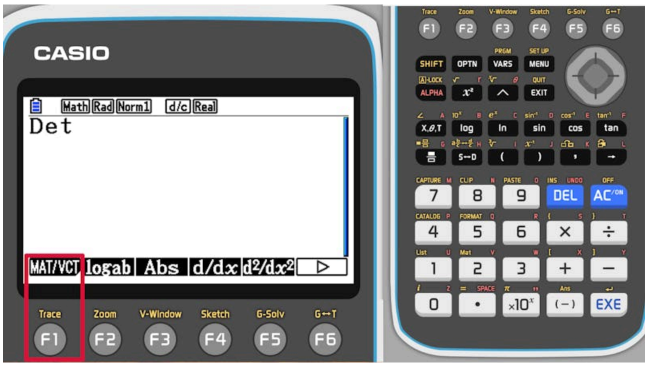
	Steps	Explanation
c)	$\text{area}\left(\mathrm{A'B'C'}\right)=\dfrac{2\times 1}{2}=1\text{ unit}^2$	In triangle $\mathrm{A'B'C'}$ the horizontal side $\mathrm{B'C'}$ is 2 units long and the vertical distance of $\mathrm{A'}$ from this line is 1 unit.
d)	$\det M=-2$	You can either find this determinant using the formula or using your calculator. For help on how to access this option on the calculator, see the instructions below this example.
e)	The area of the image triangle is the area of the original triangle multiplied by the absolute value of the determinant.	$0.5\times 2=1$

Steps	Explanation
In these instructions you will see how to find the determinant of a matrix not yet stored in the memory. If you need to store the matrix, see the instructions in <u>section 1.14.4 (/study/app/math-ai-hl/sid-132-cid-761618/book/matrix-algebra-id-27432/)</u>. In calculator mode, open the options (OPTN) ...	

Student
view

Steps	Explanation
... press F2 to access the matrix options ...	
... and F3 to tell the calculator that you are looking for a determinant. The next step is to enter the matrix, so exit the options menu.	

Steps	Explanation
Back in the main menu of the calculator mode press F4 for the math options ...	
... and F1 for the options related to matrices.	

Overview
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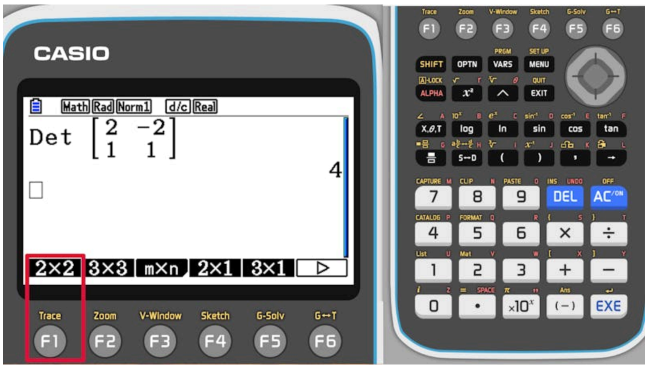
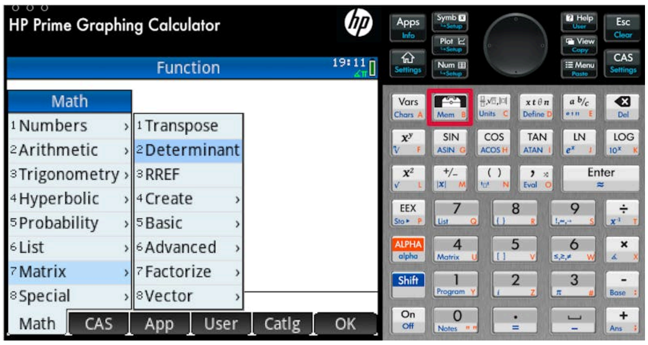
Steps	Explanation
Select the size of the matrix, enter the numbers and press EXE to see the determinant.	

Steps	Explanation
In these instructions you will see how to find the determinant of a matrix not yet stored in the memory. If you need to store the matrix, see the instructions in section 1.14.4 (/study/app/math-ai-hl/sid-132-cid-761618/book/matrix-algebra-id-27432/). On the home screen of any application open the toolbox and look for the tool to find the determinant of a matrix.	

Student view

Overview
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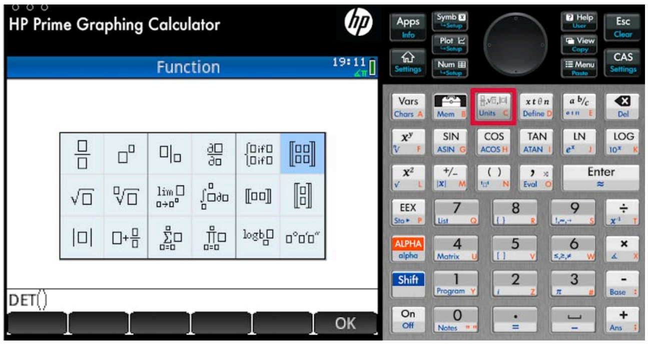
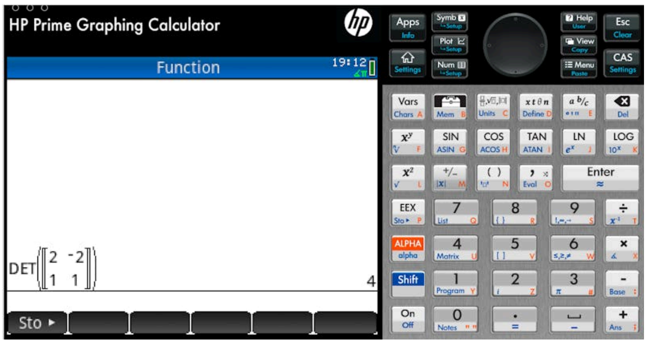
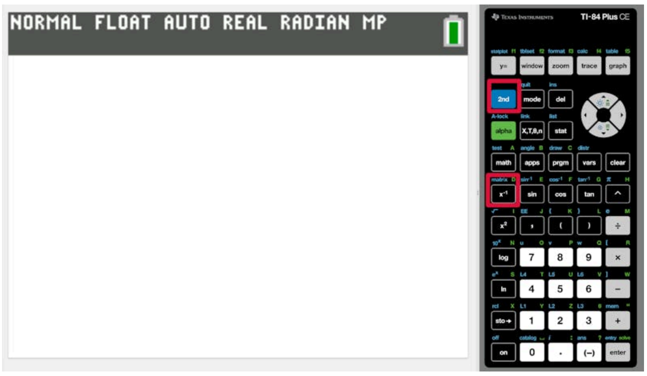
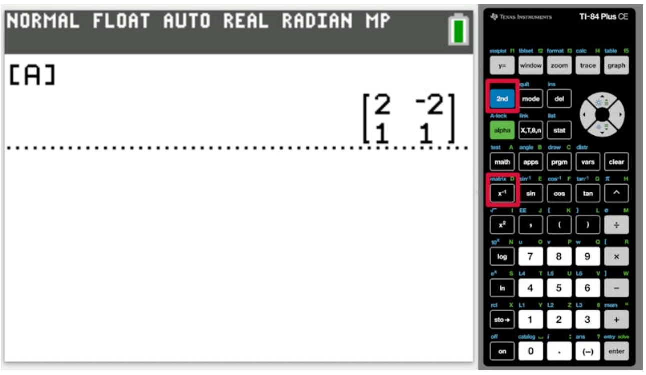
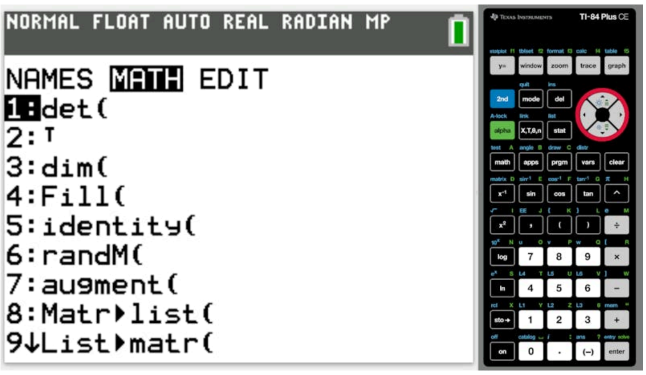
Steps	Explanation
Next, open the template menu and select the template to enter a matrix.	
Fill in the numbers and press enter to see the determinant.	

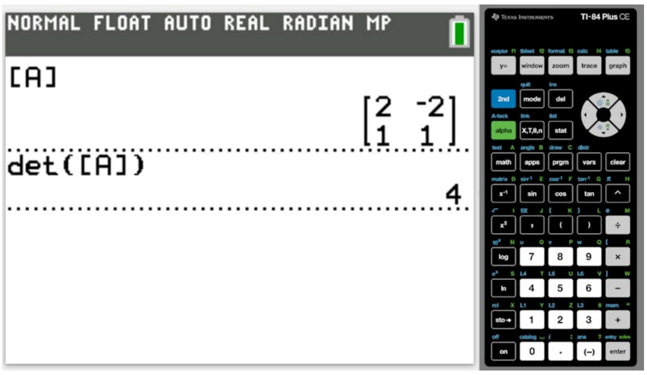
Student
view

Steps	Explanation
In these instructions you will see how to find the determinant of a matrix. There is a similar set of instructions in [section 1.14.4 (/study/app/math-ai-hl/sid-132-cid-761618/book/matrix-algebra-id-27432/)](/study/app/math-ai-hl/sid-132-cid-761618/book/matrix-algebra-id-27432/), where you can also see how to store the matrix in the memory. To work with matrices, you need to open the matrix menu.	
In the matrix menu you can choose to edit a matrix, use an already stored matrix or apply some operations. In this case the matrix is stored, To check the values, use the name ...	

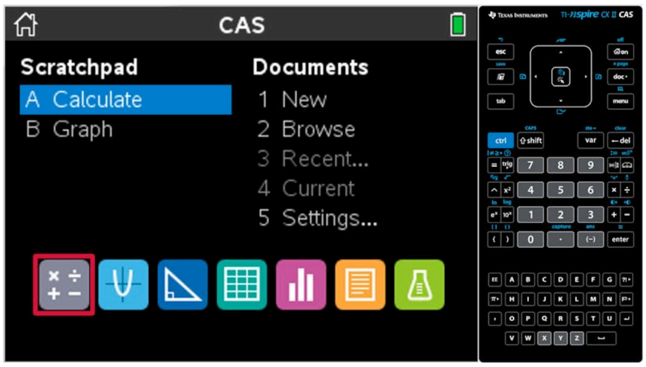
Student view

Steps	Explanation
... and press enter. You can see, that the matrix is the one you are interested in. To find the determinant, open the matrix menu again ...	
... navigate to the MATH options and choose the option to find the determinant.	

Overview (/study/ap ai- hl/sid- 132- cid- 761618/ov

Steps	Explanation
Copy the name and press enter to see the determinant.	

1200

Steps	Explanation
In these instructions you will see how to find the determinant of a matrix not yet stored in the memory. If you need to store the matrix, see the instructions in [section 1.14.4 (/study/app/math-ai-hl/sid-132-cid-761618/book/matrix-algebra-id-27432/)](/study/app/math-ai-hl/sid-132-cid-761618/book/matrix-algebra-id-27432/). Open a calculator page.	

Student view

Overview
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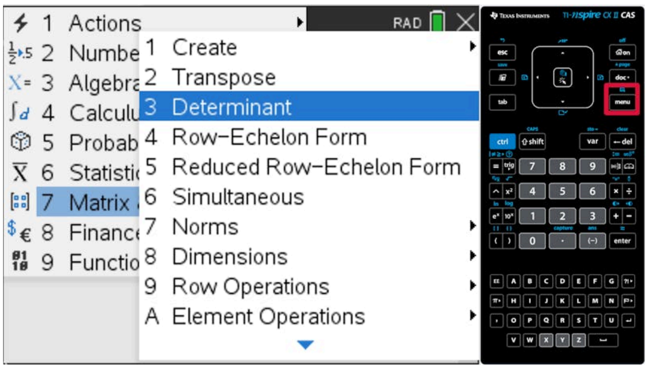
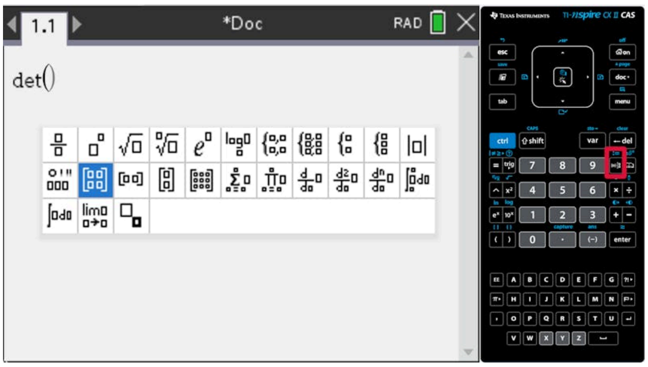
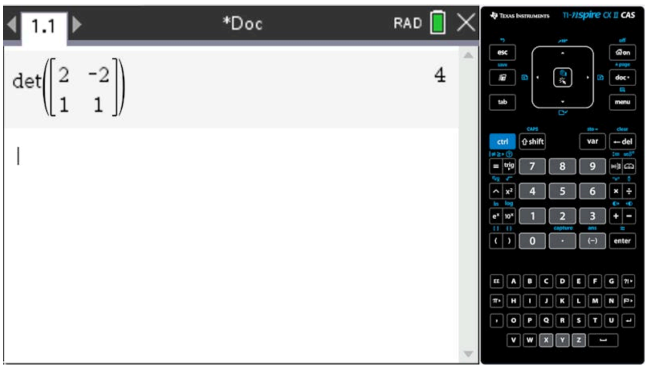
761618/ov

Steps	Explanation
Open the menu and look for the option to find the determinant of a matrix.	
Next, open the template menu and select the template to enter a matrix.	

Student
view

Steps	Explanation
Fill in the numbers and press enter to see the determinant.	

Important

The area of the image is proportional to the area of the object, and the ratio is equal to the absolute value of the determinant of the transformation matrix:

$$\text{area of image} = |\det A| \times \text{area of object}$$

Exam tip

In IB examinations, you need to remember the relationship between the area of the object and the area of the image because it will not be given.

Example 2

★☆☆

An object with area 3 unit^2 is transformed by the matrix $T = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$.

Find the area of the image.

Overview
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Using a graphic display calculator

$$\det T = 4$$

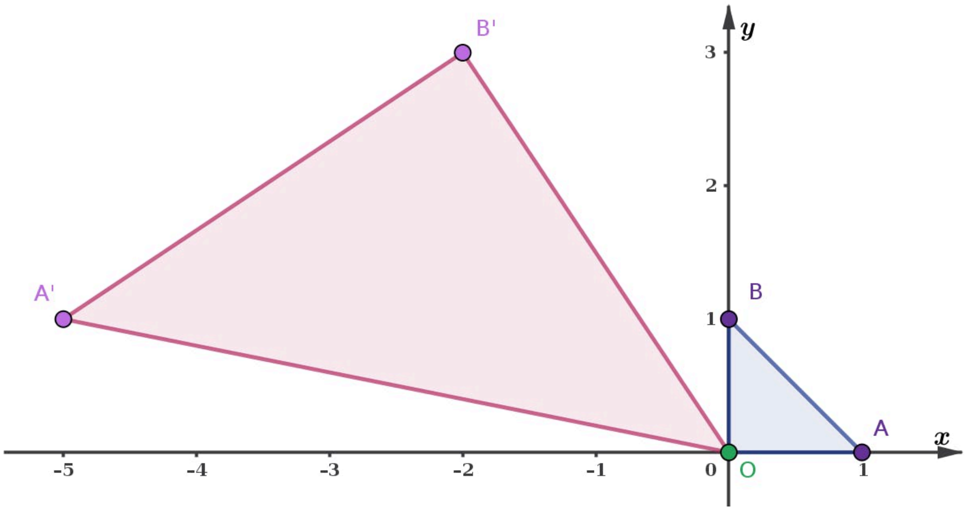
area of the image $= 4 \times 3 = 12$.

Example 3

★★☆

The diagram shows two triangles.

The vertices are $\mathrm{O}(0,0)$, $\mathrm{A}(1,0)$, $\mathrm{B}(0,1)$, $\mathrm{A}'(-5,1)$ and $\mathrm{B}'(-2,3)$.



More information

This image is a diagram showing two right triangles on a coordinate plane with x and y axes. The first triangle OAB is plotted with vertices at O(0,0), A(1,0), and B(0,1) forming a right triangle in the first quadrant. It is marked with a light blue shading and labeled respectively. The second triangle OA'B' is larger, covering more of the plane, with vertices at O(0,0), A'(-5,1), and B'(-2,3), filled with light pink shading. This larger triangle extends into the second quadrant. The x-axis ranges from -5 to 1, and the y-axis ranges from 0 to 3, representing the visual transformation of triangle OAB to OA'B'.

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Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

a) Find the transformation matrix that maps triangle OAB to triangle $\mathrm{OA'B'}$.

b) Find the area of triangle $\mathrm{OA'B'}$.

	Steps	Explanation
a)	The image of $(1,0)$ is $(-5,1)$. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$ Using the properties of matrix multiplication, this gives $a = -5$ and $c = 1$	Look for the transformation matrix in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
	The image of $(0,1)$ is $(-2,3)$. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ Using the properties of matrix multiplication, this gives $b = -2$ and $d = 3$	
	The transformation matrix is $T = \begin{pmatrix} -5 & -2 \\ 1 & 3 \end{pmatrix}$	Answer the question.
b)	Triangle OAB is an isosceles right triangle with legs 1 unit. The area is 0.5 units squared.	Find the area of triangle OAB.
	$\det T = -13$	Find the determinant of the transformation matrix.
	The area of triangle $\mathrm{OA'B'}$ is $13 \times 0.5 = 6.5$ units squared.	The area of the image triangle is the area of the original triangle multiplied by the abszolute value of the determinant.

Making connections

You can find the areas of simple polygons using the shoelace formula. This is a useful subject for further investigation, but is not part of the syllabus.

Here you will find the area of a triangle with vertices $(1,3)$, $(2,1)$ and $(-1,5)$.

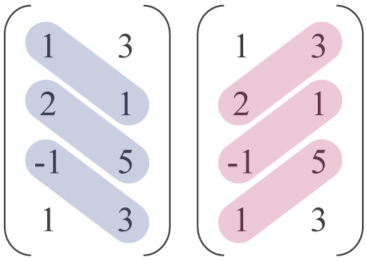
Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

First, write down the coordinates of the vertices as a matrix. The first column has the x values and the second column the y values. Add the first point once more as a final fourth row:

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \\ -1 & 5 \\ 1 & 3 \end{pmatrix}$$

Then multiply pairs of entries diagonally down and up:



More information

The image shows two vertical sets of numbers inside bracket-like shapes. The left set, in blue, contains the numbers: 1, 2, -1, and 1, arranged diagonally from top-left to bottom-right with the numbers 3 and 5 aligned vertically on the right of the diagonal arrangement. Similarly, the right set, in pink, mirrors this structure with the same numbers: 1, 2, -1, and 1 diagonally, and 3 and 5 vertically aligned. These pairs form a crisscross pattern surrounded by curves, suggesting a multiplication sequence or operation of diagonal entries.

[Generated by AI]

to give:

$$\begin{aligned} \text{area} &= \frac{1}{2}\left|((1\times 1)+(2\times 5)+(-1\times 3))-((1\times 5)+(-1\times 1)+(2\times 3))\right| \\ &= \frac{1}{2}\left|8-10\right| = 1 \end{aligned}$$

Therefore, the area of the triangle is 1 square unit.

In general, the area of a triangle with vertices (a,b), (c,d) and (e,f) is

$$\text{area} = \frac{1}{2}\left|(ad+cf+eb)-(af+ed+cb)\right|.$$

- Can you prove this formula?
- Think about how this is related to Example 3.

Student view

Checklist

What you should know

By the end of this subtopic you should be able to:

- represent the transformation from x, y to x', y' under transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$
- represent an anticlockwise (counter-clockwise) rotation of angle θ about the origin $(\theta > 0)$ using $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
- represent a horizontal stretch parallel to the x-axis with a scale factor of k as $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
- represent a vertical stretch parallel to the y-axis with a scale factor of k as $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
- represent an enlargement, with a scale factor k , centre $(0, 0)$, as $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
- represent a translation of a units parallel to the x-axis and b units parallel to the y-axis using vector $\begin{pmatrix} a \\ b \end{pmatrix}$
- represent affine transformations as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$
- find the area of a triangle with vertices ($a, b), (c, d)$, and (e, f) using

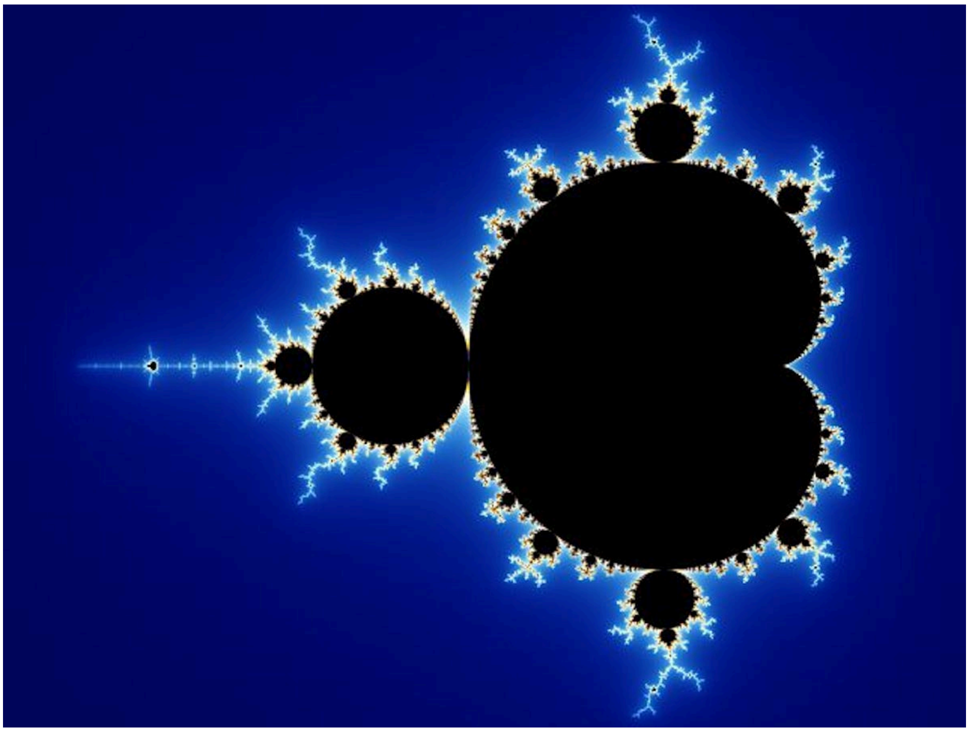
$$A = \frac{1}{2} \begin{vmatrix} a & b \\ c & d \\ e & f \\ a & b \end{vmatrix} = \frac{1}{2} \left| (ad + cf + eb) - (af + ed + cb) \right|$$

- find the area of an image using

$$\text{area of image} = |\text{det}A| \times \text{area of object} .$$

Investigation

Fractals are curves or geometrical figures where each part has the same characteristics as the whole. B. Mandelbrot (1924–2010) studied these complex geometrical shapes. In 1980, with the help of computer graphics, he was able to generate the image of a well-known fractal, the Mandelbrot set (shown below). He showed that with simple rules you can create such a visual complexity. Fractals can be used to model shorelines and clouds and are used for compressing data.



The Mandelbrot set within a continuously coloured environment

Source: "Mandel zoom 00 mandelbrot set" by Wolfgang Beyer is licensed under CC BY-SA 3.0

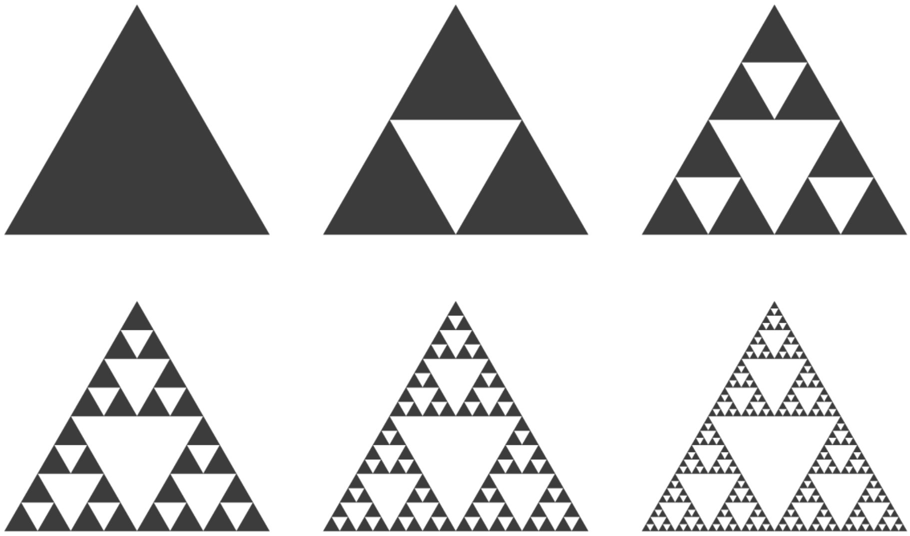
Activity

You can use the link below to investigate self-similarity using a well-known Mandelbrot set.

[http://math.hws.edu/eck/js/mandelbrot/MB.html (http://math.hws.edu/eck/js/mandelbrot/MB.html)](http://math.hws.edu/eck/js/mandelbrot/MB.html)

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

In this investigation you will be generating a well-known fractal, Sierpinski's triangle (shown below).



More information

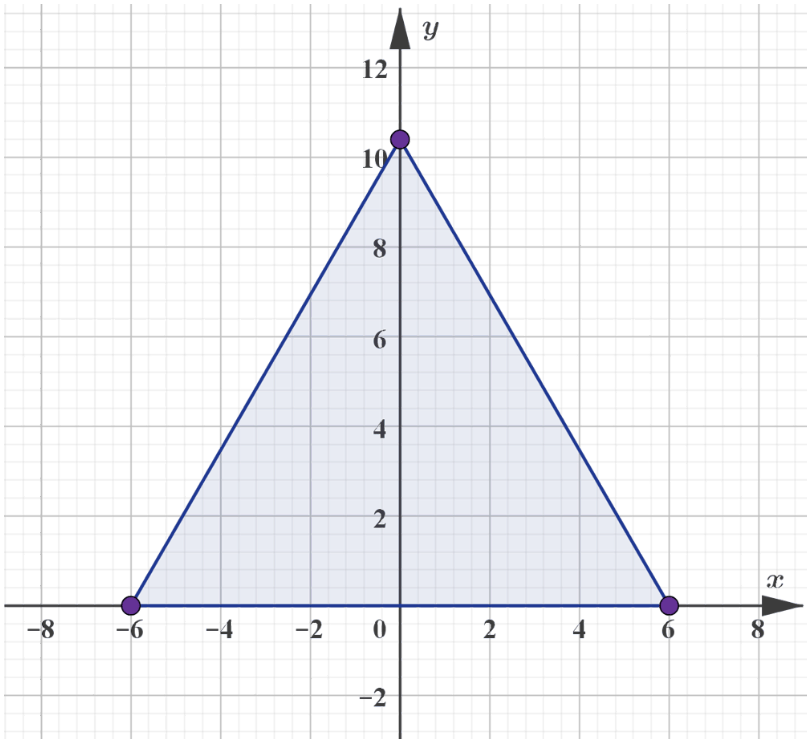
The image shows the step-by-step creation of the Sierpinski's triangle fractal. It consists of six panels arranged in two rows of three. Each panel depicts a stage of the fractal's development, starting with a large equilateral triangle. The process involves subdividing this triangle into smaller equilateral triangles and removing the central triangle to form a pattern. In each subsequent step, this process is repeated for each of the smaller triangles, generating more detailed fractal patterns. This iterative process produces a highly intricate design composed of numerous smaller triangles.

[Generated by AI]

1. Use graphing software, such as GeoGebra, to draw an equilateral triangle with a base having vertices at ((−6, 0)\) and $(6, 0)$.

Stage 1

Student view



More information

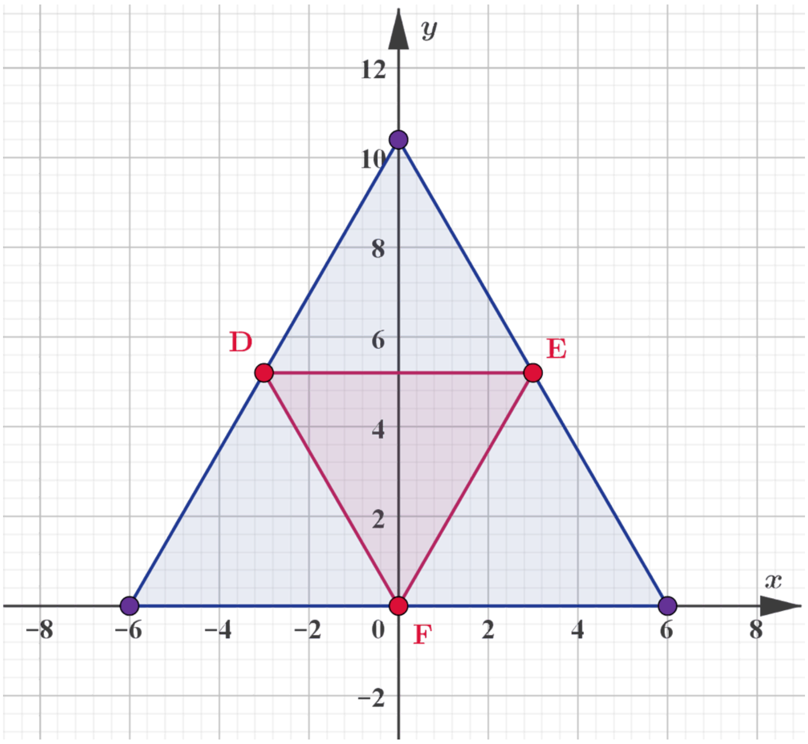
A diagram of a triangle plotted on a coordinate grid. The triangle's vertices are located at (-5,0), (6,0), and (0,7). The x-axis ranges from -7 to 7, and the y-axis ranges from -3 to 8. The grid lines intersect at integer values, and the triangle spans from left to right across the x-axis and upwards along the y-axis. The top vertex is centered on the y-axis, and the base of the triangle lies on the x-axis. Each vertex is marked by a purple dot.

[Generated by AI]

- D, E and F are the midpoints of the sides of the triangle.
- What series of transformations maps the original triangle into the triangle DEF?
- What is the relationship between the area of the triangle in Stage 1 and the area of the triangle DEF?

Stage 2

Student view



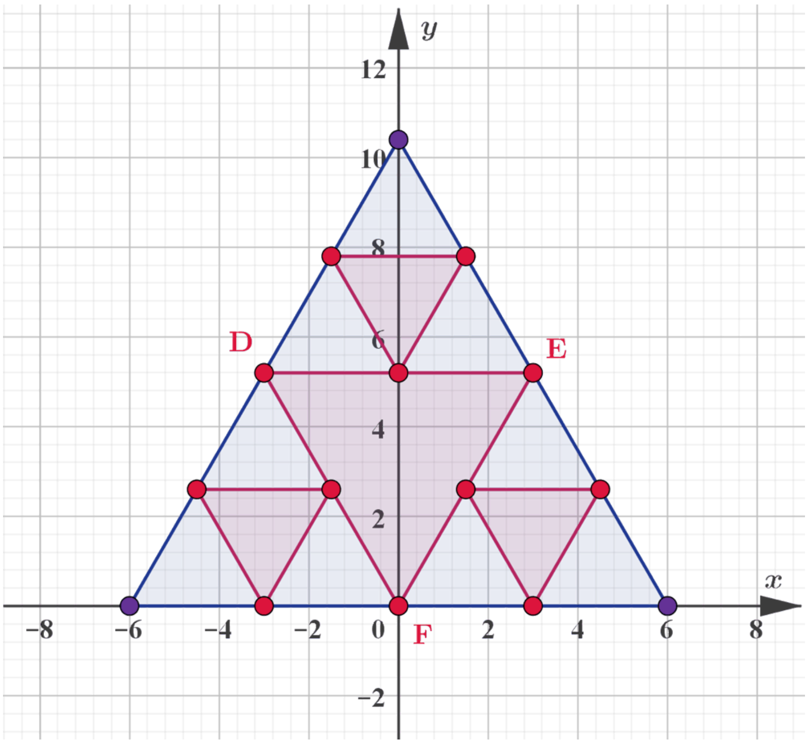
More information

This image features a geometric diagram presented on a grid. At the center of the diagram is an equilateral triangle, highlighted in blue, with its vertex at the origin of the graph. The grid has axes labeled with numbers ranging from -8 to 8 along both the X and Y axes, plotted in single-unit intervals. The main triangle is marked with letters A, D, E, and F, indicating specific points. A smaller, internal triangle is formed inside, highlighted in pink. This internal triangle also marks its vertices with points D and E, and the base pointing down at F. The points charted on the diagram are aligned symmetrically around the vertical axis.

[Generated by AI]

- What transformations are used to generate the triangles in Stage 3?

Stage 3



More information

The image is a triangular grid diagram placed on a square grid background. It appears as an equilateral triangle divided into smaller triangles and lines, with notable points labeled with letters such as D, E, and F. The triangle is positioned on a coordinate grid with visible axes labeled numerically. "Stage 3" is indicated at the top. Each vertex of the triangle and the intersections within the triangle are marked with red dots. The diagram possibly represents mathematical concepts related to geometry or lattice structures.

[Generated by AI]

- What transformation matrices are used to generate stage n?

2. Create a GeoGebra applet to generate Sierpinski's triangle for a given number of stages. What happens to the area coloured blue as the number of stages increases? What happens to the area coloured red?