Index

The big picture

The second big concept in calculus is integration. There are two important aspects to our study of integration:

1. Integration as anti-differentiation
2. Integration to find the area under curves.

Although you have already covered rudimentary integration of polynomial functions in subtopic 5.5 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25533/), it is now time to more formally discuss integration and the various techniques to solve more complex problems.

Integration as differentiation

In earlier algebra topics, you have studied operations and their inverses, for example,

Function	Function
x^2	$\sqrt{x}$
$\sin x$	$\arcsin x$
e^x	$\ln x$

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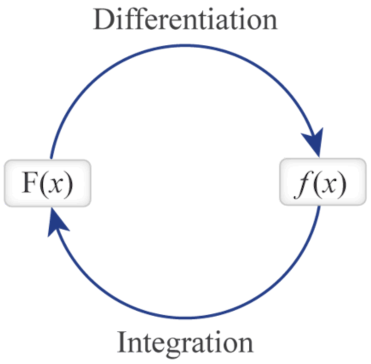
By using an operation and its inverse in series, you mathematically complete a circle. For example, $\ln(\mathrm{e}^x) = x$.

Integration and differentiation are inverse operations of each other. This means that if you start with a function, say $f(x)$, and if you integrate $f(x)$ and follow this by differentiating the result of the integration, you obtain again $f(x)$. Similarly, if you start with a function, say $g(x)$, and then you differentiate $g(x)$ and follow this by integrating the result of the differentiation, you obtain a function that belongs to the same family of functions as the function you started with, namely $g(x)$, possibly with a vertical translation. It is clear, therefore, that as differentiating a function gives another function (the derivative function), integrating a function also gives a function. You can show this inverse operation relationship as



More information

The diagram illustrates the inverse relationship between differentiation and integration using a circular flow chart. At the top of the circle, the word "Differentiation" is written, and at the bottom, "Integration" is written. There are two main components labeled "F(x)" and "f(x)" inside rectangular boxes on opposite sides of the circle. An arrow labeled "Differentiation" points clockwise from "F(x)" to "f(x)", indicating that differentiating F(x) results in f(x). Conversely, another arrow labeled "Integration" points counterclockwise from "f(x)" to "F(x)", indicating that integrating f(x) results in F(x). This visual representation clarifies the concept that integration and differentiation are inverse operations, resulting in related functions within the same family, possibly differing by a constant.

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This concept is important. But why? Integration is considerably more difficult than differentiation. Hence, after integrating a function, you can differentiate the result to see whether you get the original function back. If you do, then you probably got the right result.

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If not, you most likely made a mistake when integrating rather than in the process of checking your answer by differentiating. In fact, without seeing any of the rules for integrating functions, you can probably predict the result when integrating $f(x) = x^2$.

Concept

In the most basic sense, integration is nothing more than an antiderivative. Integration 'unwraps' the work of differentiation, just as differentiation 'unwraps' the work of integration. With that in mind, recall your knowledge of differentiation. How many functions do you think you can integrate already?

Theory of Knowledge

You have been studying various elements of calculus that are designed to provide you with knowledge of unknowns. In essence, you've been learning how to use mathematics to create knowledge. However, did you ever consider your work logically flawed because of circularity?

One of the most famous mathematicians who ever lived, Bertrand Russell, was very troubled by what he saw as a flaw in the knowledge production process of mathematics. He believed that if one could not prove that $2 + 2 = 4$, all the maths that flowed from arithmetic was flawed and invalidated. Russell and other colleges such as Alfred Whitehead spent their entire lives trying to work out a proof for basic arithmetic that avoided what he saw as the fatal flaw of circularity.

Knowledge Question: Can knowledge be valid without validation?

Integrals of power functions

Finding the integral, or antiderivative is like working backwards when compared with finding the derivative. As you found out in subtopic 5.5 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25533/), the derivatives of $f(x) = x^2$, $g(x) = x^2 + 1$, $h(x) = x^2 + 2$ and $j(x) = x^2 + \pi$ are the same, namely, $2x$. For that reason, indefinite

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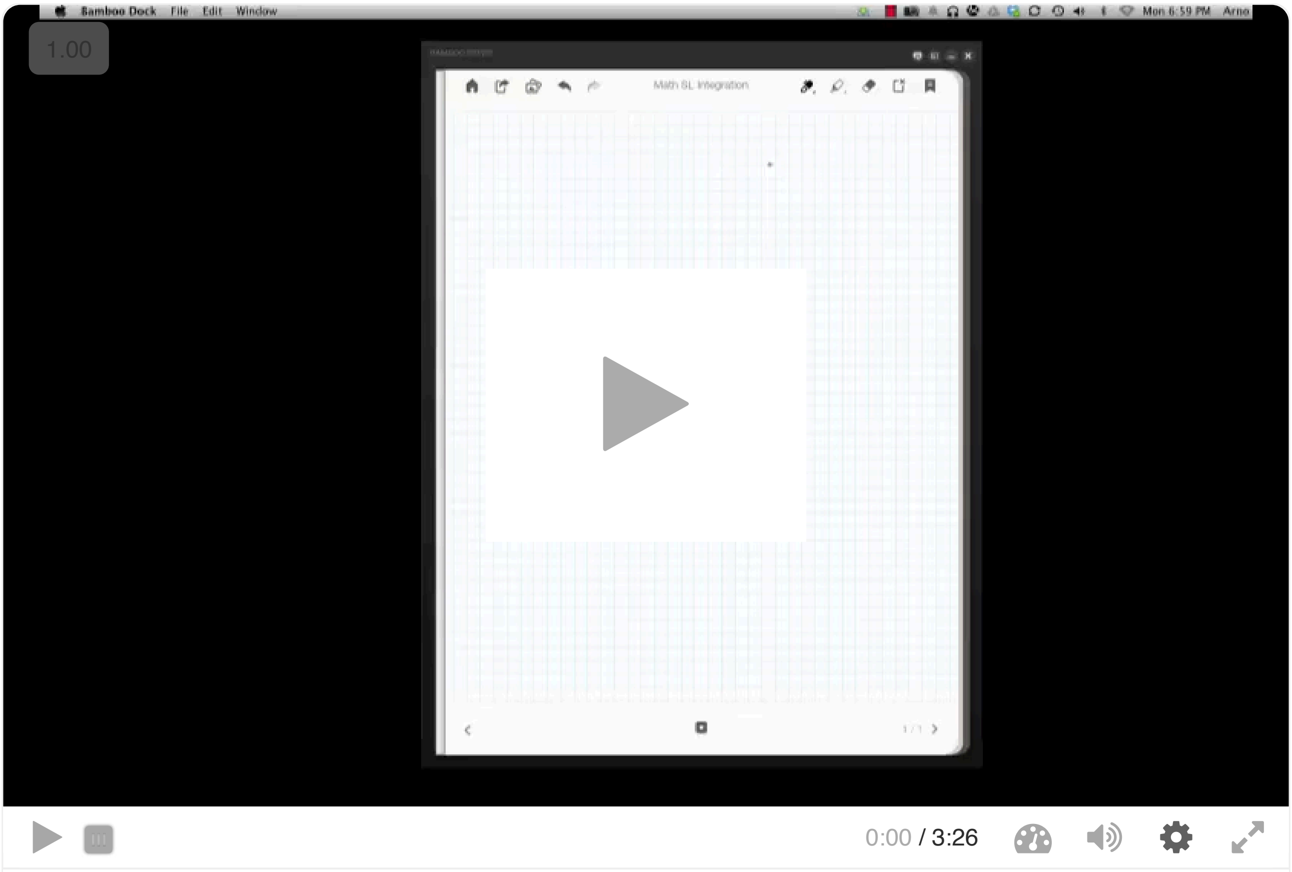
integrals include a ‘$+C$’ on the end. The C added on the end represents the possibility of any value being possible in the original function, such as the 1, 2, and π in the prior examples. The following video explores indefinite integration of power functions.



Video 1. Integrating Power Functions.

More information for video 1

1

00:00:00,567 --> 00:00:03,867

narrator: In this video

we're going to look at integration

2

00:00:04,100 --> 00:00:09,167

of power laws and in particularly

the indefinite integration of power laws.

3

00:00:09,800 --> 00:00:14,167

And the driving idea

is really the idea that integration

4

00:00:14,733 --> 00:00:18,933

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is anti differentiation, and we're gonna

use it over and over again to guide us

5

00:00:19,000 --> 00:00:20,433

and to check our answers.

6

00:00:20,867 --> 00:00:24,300

So let's start with a typical

power law x^2,

7

00:00:24,633 --> 00:00:27,100

and we already noted

if we differentiate x^2,

8

00:00:27,167 --> 00:00:30,900

then the outcome

will be $2 \times x^1$.

9

00:00:31,100 --> 00:00:32,600

So really this idea that means

10

00:00:32,667 --> 00:00:35,200

that if we integrate $2 \times x^1$,

11

00:00:35,267 --> 00:00:37,100

we ought to get x^2 back.

12

00:00:37,767 --> 00:00:42,333

Integration then should start

with $2x^1$

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27807/print/) Assign

13

00:00:42,467 --> 00:00:44,800

and take us to x^2.

14

00:00:44,867 --> 00:00:48,133

Alright, so we started

with $2x^1$,

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view

15

00:00:48,300 --> 00:00:49,600

if we balance the powers first,

16

00:00:49,667 --> 00:00:52,833

need to add 1 convert it

to multiplying 2 and divide by 2.

17

00:00:53,667 --> 00:00:56,033

Well, let's see,

again, let's take x^3.

18

00:00:56,100 --> 00:00:58,900

Differentiate $3x^2$

and the idea of anti differentiation,

19

00:00:59,000 --> 00:01:01,400

that should take us back

to x^3.

20

00:01:01,700 --> 00:01:04,400

So $3x^2$

should go to x^3

21-22

00:01:04,867 --> 00:01:11,400

$\frac{3x^{2+1}}{3}$

23

00:01:11,900 --> 00:01:13,967

Now let's check this for x^5.

24

00:01:14,033 --> 00:01:15,433

So if we integrate that,

25

00:01:15,700 --> 00:01:20,167

it goes to x^{5+1} divided by $5+1$.

26

00:01:20,933 --> 00:01:22,367

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Now let's check with differentiation.

27

00:01:22,500 --> 00:01:24,133

So differentiation of power law,

28

00:01:24,433 --> 00:01:28,233

and you subtract 1

from the power, all power comes down

29

00:01:28,333 --> 00:01:29,833

and divide by what ahead.

30

00:01:29,933 --> 00:01:32,400

So it seems indeed that this is the case.

31

00:01:32,500 --> 00:01:35,333

So we seem to have found

that if I integrate x^n,

32

00:01:35,433 --> 00:01:36,600

it goes to the power n,

33

00:01:36,867 --> 00:01:40,733

it goes to x^{n+1} times

$\frac{1}{n+1}$.

34

00:01:42,367 --> 00:01:44,533

But really, let's wait a minute here.

35

00:01:45,433 --> 00:01:48,500

If you let $y = x^2 +$ any number,

36

00:01:49,000 --> 00:01:53,800

then we also know that $\frac{dy}{dx} = 2x$ regardless of the number,

37

00:01:53,900 --> 00:01:55,100

so for all c.

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38

00:01:55,567 --> 00:01:59,367

And here we see that I've got an x^2

and a derivative is $2x$

39

00:01:59,500 --> 00:02:01,733

and I've created

a tangent line at a point.

40

00:02:02,067 --> 00:02:04,067

Now if I translate x^2

41

00:02:04,333 --> 00:02:06,633

up and down by arbitrary numbers,

42

00:02:06,700 --> 00:02:09,367

then you see that the steepness

of the tangent line doesn't change,

43

00:02:09,433 --> 00:02:11,467

the gradient function

doesn't change either.

44

00:02:12,267 --> 00:02:16,967

So we need to take this into consideration

when we do integration of power laws

45

00:02:17,033 --> 00:02:18,633

and indeed integration of anything.

46

00:02:20,100 --> 00:02:23,900

So really then x^n,

47

00:02:23,967 --> 00:02:28,033

if I integrate that, it seems to go to

x^{n+1}

48

00:02:28,100 --> 00:02:31,233

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times $\frac{1}{n+1}$ + c.

49

00:02:31,300 --> 00:02:33,900

And this constant is called

the constant of integration.

50

00:02:34,133 --> 00:02:37,800

It is, if you recall,

equal to a vertical translation

51

00:02:37,900 --> 00:02:39,233

of a function.

52

00:02:39,633 --> 00:02:42,000

Now this is a good time

to establish some notation.

53

00:02:42,233 --> 00:02:43,700

So if I have a function $f(x)$,

54

00:02:43,767 --> 00:02:46,933

and I integrate that respect to dx,

55

00:02:47,000 --> 00:02:49,367

again, another function plus a constant.

56

00:02:49,433 --> 00:02:52,833

So the signage, the $\int$ sign

57

00:02:52,967 --> 00:02:56,700

and dx is the way to write

the indefinite integral,

58

00:02:56,767 --> 00:02:58,633

you need to write it all the time,

59

00:03:00,267 --> 00:03:03,367

Nota bene, be very careful

Student view

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that the guiding idea

60

00:03:03,433 --> 00:03:05,833

has been that if you differentiate

what comes out of integration,

61

00:03:05,933 --> 00:03:08,600

you get back to function

that went into the integration.

62

00:03:08,667 --> 00:03:11,100

Little f (x) is sometimes

called the integrand.

63

00:03:11,667 --> 00:03:16,500

So we conclude that if I integrate the

power law x^n,

64

00:03:16,767 --> 00:03:22,500

then what I get $\frac{1}{n+1}x^{n+1} + c$.

65

00:03:22,933 --> 00:03:25,800

And that is our first standard integral.

Important

Here is the notation used for an indefinite integral.

The indefinite integral of $f(x)$ is given by $\int f(x)\,\mathrm{d}x = F(x) + C, C \in \mathbb{R}$, where $\frac{d(F(x))}{\mathrm{d}x} = f(x)$ and $F(x)$ is known as the antiderivative of $f(x)$.

As discussed in subtopic 5.5 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25533/), an indefinite integral is expressed without limits, while a definite integral is expressed with upper and lower limits of integration .

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In the video, you saw the following important standard integrals for power functions.

In subtopic 5.5 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25533/), you learned that $\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, n \neq -1$. In that section, this formula was applied almost exclusively to polynomials, but this is not a requirement. The process will work for any real powers.

✓ **Important**

$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, n \neq -1$

In addition to differentiation of basic polynomials in standard form, you will also study the following special case of the chain rule:

- If the derivative of $y = f(x)$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$, then if the function $f(x)$ is transformed by a linear transformation, i.e. $y = f(ax+b)$, you get $\frac{\mathrm{d}y}{\mathrm{d}x} = af'(ax+b)$.

For example, you can verify that $\frac{d}{\mathrm{d}x}(x^2) = 2x$.

Using either this special case or the chain rule, $\frac{d}{\mathrm{d}x}\left((5x+3)^2\right) = 10(5x+3)$, $\frac{d}{\mathrm{d}x}\left((-7x+3)^2\right) = -14(-7x+3)$, and $\frac{d}{\mathrm{d}x}\left((ax+b)^2\right) = 2a(ax+b)$.

Working backwards with the idea of finding the antiderivative, you can see that $\int 10(5x+3)\mathrm{d}x = (5x+3)^2 + C$, $\int -14(-7x+3)\mathrm{d}x = (-7x+3)^2 + C$, and $\int 2a(ax+b)\mathrm{d}x = (ax+b)^2 + C$.

More importantly, you can predict $\int (3x+4)\mathrm{d}x = \frac{1}{6}(3x+4)^2 + C$, $\int 7(5x+8)\mathrm{d}x = \frac{7}{10}(5x+8)^2 + C$, and $\int (ax+b)\mathrm{d}x = \frac{1}{2a}(ax+b)^2 + C$.

The same philosophy works for higher-order binomials as well as the sum rule for integration.

Student view

Be aware

The indefinite integral of x^n is given by $\frac{x^{n+1}}{n+1}+C, n\neq -1$. This equation is in the IB formula booklet.

Under a linear transformation, the indefinite integral of $(ax+b)^n$ is given by $\frac{(ax+b)^{n+1}}{a(n+1)}+C, n\neq -1$. This equation is **not** given in the IB formula booklet.

Example 1

★☆☆

Find the indefinite integral of $f(x)=5x^7$.

$$\int 5x^7\mathrm{d}x=\frac{5}{8}x^8+C$$

Example 2

★★☆

Find the indefinite integral of $g(x)=3(4x+9)^3$.

$$\int 3(4x+9)^3\mathrm{d}x=\frac{3}{4\times 4}(4x+9)^4+C=\frac{3}{16}(4x+9)^4+C$$

Example 3

★★☆

Find the indefinite integral of $h(x)=8x^{\frac{3}{2}}-2+\frac{1}{x^2}$.

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(/study/app/math-aa-hl/sid-134-cid-761926/o

$$\int\left(8x^{\frac{3}{2}}-2+\frac{1}{x^2}\right)\mathrm{d}x=\int\left(8x^{\frac{3}{2}}-2+x^{-2}\right)\mathrm{d}x=8\left(\frac{2}{5}\right)x^{\frac{5}{2}}-2x-x^{-1}=\frac{16}{5}$$

3 section questions

Question 1

Difficulty:

★☆☆

What is the indefinite integral of $f(x)=\frac{1}{3}x^2$?

1 $\frac{1}{9}x^3+C$ ✓

2 $\frac{1}{9}x^3$

3 x^3+C

4 $\frac{2}{3}x$

Explanation

$$\int\frac{1}{3}x^2\,\mathrm{d}x=\frac{1}{3}\times\frac{1}{3}x^{2+1}+C=\frac{1}{9}x^3+C$$

Question 2

Difficulty:

★★☆

Find the indefinite integral of $\frac{15}{2}x^{\frac{3}{2}}+\frac{4}{x^2}-8(5x-2)^3$.

1 $3x^{\frac{5}{2}}-\frac{4}{x}-\frac{2}{5}(5x-2)^4+C$ ✓

2 $3x^{\frac{5}{2}}-\frac{4}{3x^3}+\frac{2}{5}(5x-2)^4+C$

3 $3x^{\frac{5}{2}}+\frac{4}{x}-\frac{2}{5}(5x-2)^4+C$

4 $3x^{\frac{5}{2}}+\frac{4}{3x^3}+\frac{2}{5}(5x-2)^4+C$

Student view

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Explanation

$$\int \left(\frac{15}{2}x^{\frac{3}{2}} + \frac{4}{x^2} - 8(5x-2)^3\right) \mathrm{d}x$$

$$\int \frac{15}{2}x^{\frac{3}{2}}\mathrm{d}x + \int 4x^{-2}\mathrm{d}x + \int -8(5x-2)^3\mathrm{d}x$$

$$\frac{15}{2}\left(\frac{2}{5}\right)x^{\frac{5}{2}} + \frac{4}{-1}x^{-1} - \frac{8}{4\times 5}(5x-2)^4 + C$$

$$3x^{\frac{5}{2}} - \frac{4}{x} - \frac{2}{5}(5x-2)^4 + C$$

Question 3

Difficulty:

★★☆

Find the indefinite integral of $3(4x+5)^6$.

1. $\frac{3}{28}(4x+5)^7 + C$ ✓
2. $\frac{3}{4}(4x+5)^7 + C$
3. $\frac{3}{7}(4x+5)^7 + C$
4. $72(4x+5)^7 + C$

Explanation

Use the formula $\int (ax+b)^n \mathrm{d}x = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C$.

$$\int 3(4x+5)^6\mathrm{d}x$$

$$\frac{3}{4\times 7}(4x+5)^7 + C$$

$$\frac{3}{28}(4x+5)^7 + C$$

Student view

5. Calculus / 5.10 Indefinite integrals

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Integrals of reciprocal functions

In the last section (/study/app/math-aa-hl/sid-134-cid-761926/book/integrals-of-power-functions-id-27808/), you saw that $\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, n \neq -1$. Why is there a restriction that $n \neq -1$? Think through the power rule. Applying the power rule does not work well here as the denominator would become 0. Recall that, in section 5. 6 .3 (/study/app/math-aa-hl/sid-134-cid-761926/book/derivative-of-the-natural-logarithm-function-id-27781/), the derivative of the natural logarithm was defined as $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$. Therefore, it makes sense that the integral, or the antiderivative, is defined as $\int \frac{1}{x} \mathrm{d}x = \ln |x| + C$.

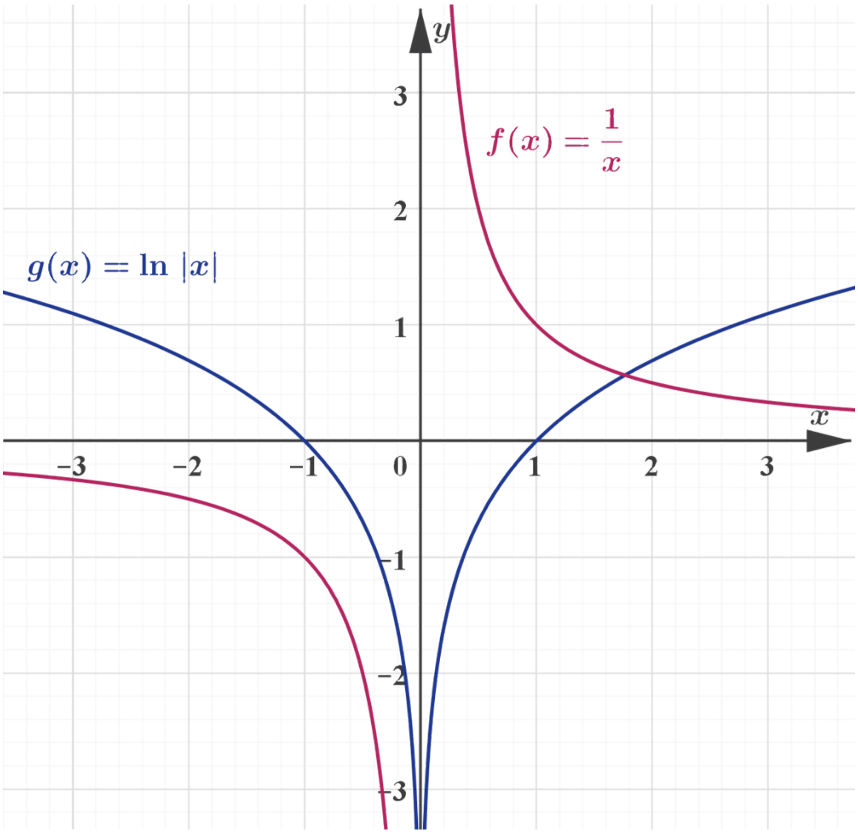
✓ **Important**

$$\int \frac{1}{x} \mathrm{d}x = \ln |x| + C$$

There is one small caveat to this definition. The natural logarithm, as with any logarithmic power function, has a limited domain of $x > 0$. This integral does not account for the presence of negative numbers. The easiest solution is to force them to be positive by using the absolute value, i.e., $\int \frac{1}{x} \, \mathrm{d}x = \ln |x| + C$. Since integration is merely anti-differentiation, comparing domains and ranges of the functions still works. Both functions are undefined at $x = 0$ and defined for $x > 0$. By using the absolute value, negative values that are defined in the reciprocal function will now be treated as positive values by the logarithmic function.

Does this make sense? Can you just put the absolute value sign in there and trust everything works out? The following graph shows why this works.

Student view



More information

The image is a graph showing the functions f(x) = -1/x and g(x) = ln|x| on a coordinate plane. The X-axis ranges from -4 to 4 and the Y-axis ranges from -4 to 4.

The red curve represents f(x) = -1/x, which approaches infinity as x approaches zero from the negative side and negative infinity from the positive side. The curve is asymptotic to both the X-axis and Y-axis, reflecting the fact that it never actually reaches these axes.

The blue curve represents g(x) = ln|x|, which is undefined at x = 0. For x > 0, the curve increases from negative infinity and levels off as it moves to the right. For x < 0, the curve reflects the positive side about the Y-axis, indicating symmetry.

Overall, the graph illustrates how g(x) changes sign based on the value of x and highlights the symmetry of the function. It shows that the slope of g(x) is negative when x < 0 and positive when x > 0, and how g(x) aligns with the anti-derivative of f(x) given the domain restrictions.

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Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

As you can see, the slope of $g(x)$ is negative when $x < 0$ and positive when $x > 0$. For $x > 0$, the relationship can be justified through the concept of anti-differentiation. For $x < 0$, the relationship can be justified based on the symmetry of the graph. Furthermore, the domain restriction of $f(x)$ lines up well with the domain restriction of $g(x)$, with the one-sided limits of $f'(x)$ at 0 approaching negative infinity from both sides.

Be aware

The indefinite integral of $\frac{1}{x}$ is given by $\ln|x| + C$. This equation is in the IB formula booklet.

Under a linear transformation, the indefinite integral of $\frac{1}{ax+b}$ is given by $\frac{1}{a}\ln|ax+b| + C$. This equation is **not** given in the IB formula booklet.

Example 1

★☆☆

Find the indefinite integral of $f(x) = \frac{1}{x^3} + \frac{1}{x}$.

$$\int \left(\frac{1}{x^3} + \frac{1}{x}\right) \mathrm{d}x = \int \left(x^{-3} + \frac{1}{x}\right) \mathrm{d}x = -\frac{1}{2}x^{-2} + \ln|x| + C = -\frac{1}{2x^2} + \ln|x| +$$

Example 2

★★☆

Find the indefinite integral of $g(x) = \frac{1}{3x+7}$.

$$\int \frac{1}{3x+7}\,\mathrm{d}x = \frac{1}{3}\ln|3x+7| + C$$

Student view

3 section questions

Integrals of exponential functions

As you learned in section 5. 6 .2 (/study/app/math-aa-hl/sid-134-cid-761926/book/derivative-of-the-exponential-function-id-27780/), finding the derivative of an exponential function with Euler's number, e, as a base is relatively simple, i.e. $\frac{d}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x$. Therefore, the antiderivative should be just as simple, $\int \mathrm{e}^x \,\mathrm{d}x = \mathrm{e}^x + C$. If you analyse the related special case from earlier sections, it should make sense that $\int \mathrm{e}^{ax+b} \,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax+b} + C$. Can you confirm this formula by using the chain rule to find the derivative?

Be aware

The indefinite integral of e^x is given by $\mathrm{e}^x + C$. This equation is in the IB formula booklet.

Under a linear transformation, the indefinite integral of e^{ax+b} is given by $\frac{1}{a}\mathrm{e}^{ax+b} + C$. This equation is **not** given in the IB formula booklet.

Example 1

★☆☆

Find the indefinite integral of $f(x) = \mathrm{e}^{5x}$.

$$\int \mathrm{e}^{5x} \,\mathrm{d}x = \frac{1}{5}\mathrm{e}^{5x} + C$$

Example 2

Student view

★★☆

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Find the indefinite integral of $f(x) = \mathrm{e}^{-\frac{x}{\pi}+\frac{1}{3}}$.

$$\int \mathrm{e}^{-\frac{x}{\pi}+\frac{1}{3}}\,\mathrm{d}x = -\pi\mathrm{e}^{-\frac{x}{\pi}+\frac{1}{3}} + C$$

2 section questions

Question 1

Difficulty:

★★☆

Evaluate the indefinite integral of $\mathrm{e}^{\frac{1}{3}x-2}$.

1. $3\,\mathrm{e}^{\frac{1}{3}x-2} + C$ ✓
2. $\frac{1}{3}\,\mathrm{e}^{\frac{1}{3}x-2} + C$
3. $\frac{1}{3}\,\mathrm{e}^{\frac{1}{3}x} + C$
4. $\mathrm{e}^{\frac{1}{3}x-2} + C$

Explanation

$$\int \mathrm{e}^{\frac{1}{3}x-2}\,\mathrm{d}x = \frac{1}{\frac{1}{3}}\,\mathrm{e}^{\frac{1}{3}x-2} + C = 3\,\mathrm{e}^{\frac{1}{3}x-2} + C$$

Question 2

Difficulty:

★★☆

Evaluate the indefinite integral of $\mathrm{e}^{-2x} - \mathrm{e}^{-\frac{3}{2}x}$.

1. $-\frac{1}{2}\mathrm{e}^{-2x} + \frac{2}{3}\,\mathrm{e}^{-\frac{3}{2}x} + C$ ✓
2. $2\,\mathrm{e}^{-2x} - \frac{3}{2}\,\mathrm{e}^{-\frac{3}{2}x} + C$
3. $-2\,\mathrm{e}^{-2x} + \frac{3}{2}\,\mathrm{e}^{-\frac{3}{2}x} + C$

Student view

4 $\frac{1}{2}e^{-2x} - \frac{2}{3}e^{-\frac{3}{2}x} + C$

Explanation

$$\int e^{-2x} - e^{-\frac{3}{2}x}\,dx = \frac{1}{-2}e^{-2x} - \frac{1}{-\frac{3}{2}}e^{-\frac{3}{2}x} + C$$
$$= -\frac{1}{2}e^{-2x} + \frac{2}{3}e^{-\frac{3}{2}x} + C$$

Integrals of trigonometric functions

Once again, you will see that for trigonometric functions integration is nothing more than anti-differentiation. In section 5.9.4 (/study/app/math-aa-hl/sid-134-cid-761926/book/velocity-and-acceleration-in-terms-of-position-id-27804/), you found that $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$. You can therefore conclude that $\int \sin x\,dx = -\cos x + C$ and $\int \cos x\,dx = \sin x + C$.

Be aware

The indefinite integrals $\int \sin x\,dx = -\cos x + C$ and $\int \cos x\,dx = \sin x + C$ are in the IB formula booklet.

Under a linear transformation, the indefinite integrals
$\int \sin(ax+b)\,dx = \frac{-\cos(ax+b)}{a} + C$ and
$\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + C$ are **not** given in the IB formula booklet.

Example 1

★★☆

Find the indefinite integral of $f(x) = 2\cos(3x+1)$.

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$$\int 2\cos(3x+1)\mathrm{d}x = \frac{1}{3} \times 2\sin(3x+1) + C = \frac{2}{3}\sin(3x+1) + C$$

Example 2

★★☆

Find the indefinite integral of $f(x) = \frac{1}{2}\sin\left(\frac{1}{3}x\right) - 3\cos\left(\frac{\pi}{2}x + 1.8\right)$.

$$\begin{aligned} &\int \left(\frac{1}{2}\sin\left(\frac{1}{3}x\right) - 3\cos\left(\frac{\pi}{2}x + 1.8\right)\right)\mathrm{d}x \\ &= -\frac{1}{1/3} \times \frac{1}{2}\cos\left(\frac{1}{3}x\right) - \frac{1}{\pi/2} \times 3\sin\left(\frac{\pi}{2}x + 1.8\right) + C \\ &= -\frac{3}{2}\cos\left(\frac{1}{3}x\right) - \frac{6}{\pi}\sin\left(\frac{\pi}{2}x + 1.8\right) + C \end{aligned}$$

Exam tip

- It is worth remembering the integrals of $\sin(ax+b)$ and $\cos(ax+b)$.
- Be careful where the minus sign goes: $\int \sin x \, \mathrm{d}x = -\cos x + C$.
- Do a quick check that your answer is correct by differentiating your integration result to see whether you get back to the function you integrated.

3 section questions

Question 1

Difficulty:

★★☆

What is the indefinite integral of $2\cos(2(x+1)) - \frac{1}{3}\sin\left(\frac{1}{3}x\right)$?

1 $\sin(2(x+1)) + \cos\left(\frac{1}{3}x\right) + C$ ✓

2 $4\sin(2(x+1)) + \frac{1}{9}\cos\left(\frac{1}{3}x\right) + C$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

3 $-\sin(2(x+1)) - \cos\left(\frac{1}{3}x\right) + C$

4 $-4\sin(2(x+1)) - \cos\left(\frac{1}{3}x\right) + C$

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/integrals-of-power-functions-id-27808/print/) Assign

Explanation

Using the standard integrals for the trigonometric functions, we obtain

$$\begin{aligned}\int\left(2\cos\left(2(x+1)\right) - \frac{1}{3}\sin\left(\frac{1}{3}x\right)\right)\,dx &= 2\int\cos\left(2(x+1)\right)\,dx - \frac{1}{3}\int\sin\left(\frac{1}{3}x\right)\,dx\\ &= 2\times\frac{1}{2}\sin\left(2(x+1)\right) - \frac{1}{3}\times -\frac{1}{\frac{1}{3}}\cos\left(\frac{1}{3}x\right) + C\\ &= \sin\left(2(x+1)\right) + \cos\left(\frac{1}{3}x\right) + C.\end{aligned}$$

Question 2

Difficulty: ★★☆

What is the indefinite integral for $\sin\left(\frac{x}{\pi}\right) - \cos\left(\frac{x}{2\pi}\right)$?

1 $-\pi\cos\left(\frac{x}{\pi}\right) - 2\pi\sin\left(\frac{x}{2\pi}\right) + C$ ✓

2 $+\frac{1}{\pi}\cos\left(\frac{x}{\pi}\right) + \frac{1}{2\pi}\sin\left(\frac{x}{2\pi}\right) + C$

3 $-\frac{1}{\pi}\cos\left(\frac{x}{\pi}\right) - \frac{1}{2\pi}\sin\left(\frac{x}{2\pi}\right) + C$

4 $+\pi\cos\left(\frac{x}{\pi}\right) + 2\pi\sin\left(\frac{x}{2\pi}\right) + C$

Explanation

$$\begin{aligned}\int\sin\left(\frac{x}{\pi}\right) - \cos\left(\frac{x}{2\pi}\right)\,dx &= -\frac{1}{\frac{1}{\pi}}\cos\left(\frac{x}{\pi}\right) - \frac{1}{\frac{1}{2\pi}}\sin\left(\frac{x}{2\pi}\right) + C\\ &= -\pi\cos\left(\frac{x}{\pi}\right) - 2\pi\sin\left(\frac{x}{2\pi}\right) + C.\end{aligned}$$

Question 3

Difficulty: ★★☆

Given that the indefinite integral $\int 12\sin\left(\frac{4}{3}x+7\right)\,dx = a\cos\left(\frac{4}{3}x+7\right) + C$, determine the value of a.

Student view

Accepted answers

-9, a=-9

Explanation

$$x = \int 12\sin\left(\frac{4}{3}x+7\right)\,\mathrm{d}x = -\frac{1}{\frac{4}{3}} \times 12\cos\left(\frac{4}{3}x+7\right) + C$$

$$= -9\cos\left(\frac{4}{3}x+7\right) + C$$

$$a = -9$$

Integration by substitution

Integration by inspection

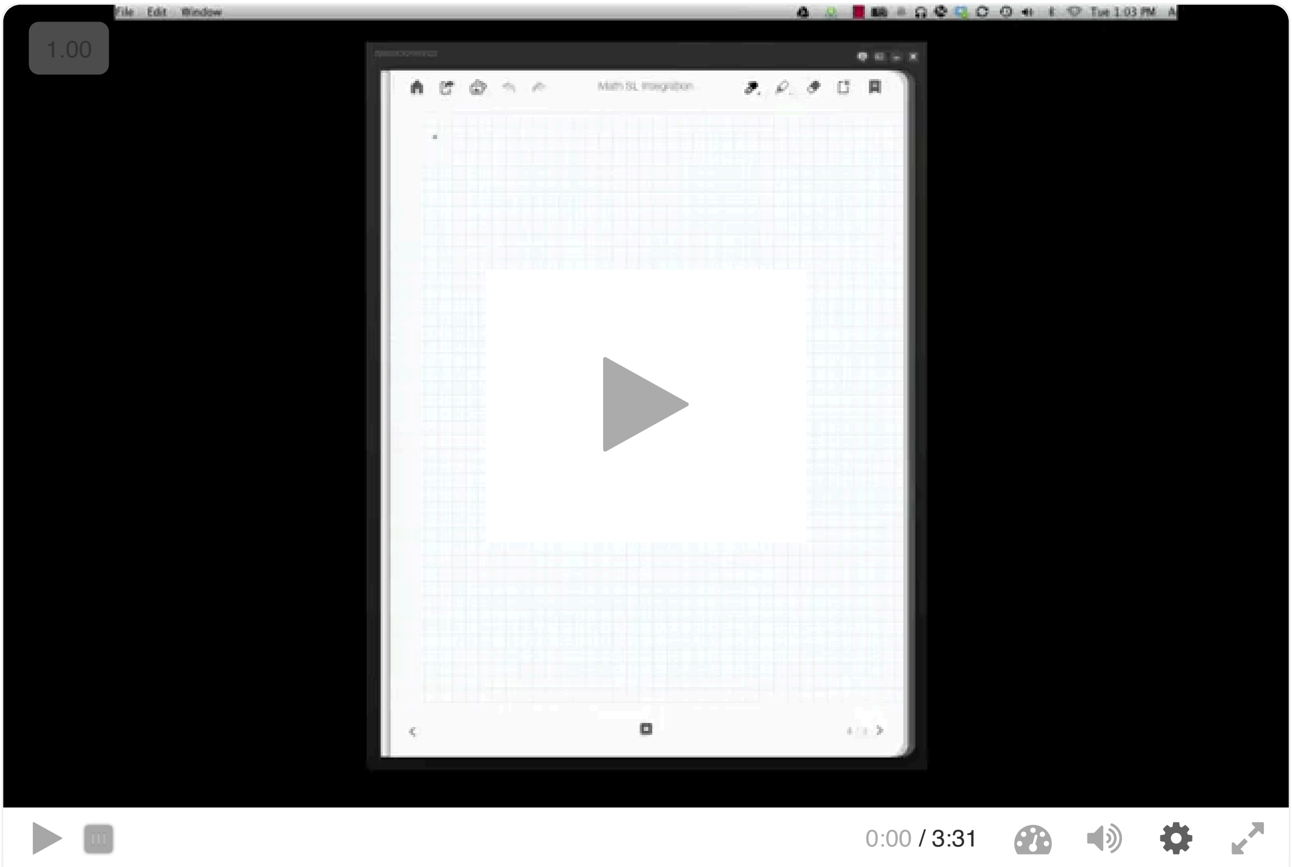
Subtopic 5. 6 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27778/) described some useful rules for differentiation:

- Chain rule If $y = g(u)$, where $u = f(x)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$
- Product rule If $y = uv$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$
- Quotient rule If $y = \frac{u}{v}$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Unfortunately, there are no similar rules for integration that allow you to integrate any combination of functions.

However, using the integration as anti-differentiation insight as well as the chain rule for differentiation, there are some integrands that involve products and quotients that can be integrated. The following video explores this and shows you a form of integrand involving a product that can be integrated.

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o



Video 1. Substitution Method in Integration.

More information for video 1

1

00:00:00,467 --> 00:00:01,401

narrator: In this short video,

2

00:00:01,468 --> 00:00:05,038

we're going to have a first look

at integration by substitution,

3

00:00:05,506 --> 00:00:07,741

which can be quite involving.

4

00:00:08,008 --> 00:00:10,377

The first part involves

doing it by inspection,

5

00:00:10,444 --> 00:00:14,147

which once again hinges

upon the idea that integration

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

6

00:00:14,448 --> 00:00:17,918

can be considered as anti differentiation.

7

00:00:18,418 --> 00:00:24,124

Let's consider function $\sin(5x)$

and let's differentiate that respect to x.

8

00:00:24,424 --> 00:00:27,895

Now, we already know what we get

is $5\cos(5x)$.

9

00:00:28,595 --> 00:00:31,031

Hence using the idea

of entity differentiation.

10

00:00:31,098 --> 00:00:34,902

We know that if we integrate

$5\cos(5x)$,

11

00:00:35,636 --> 00:00:39,239

then we should get $\sin(5x) + C$,

where C is the integration constant.

12

00:00:39,573 --> 00:00:43,443

Similarly, if I look at $(x^2 + 2)^6$,

13

00:00:43,510 --> 00:00:46,180

and if I differentiate that respect to x,

14

00:00:46,246 --> 00:00:50,884

then what I get is $6(x^2 + 2)^5$

15

00:00:50,951 --> 00:00:53,554

times $2x$ using the chain rule.

16

00:00:53,620 --> 00:00:57,191

Therefore, if I integrate $(x^2 + 2)^5$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

17

00:00:57,257 --> 00:00:59,359

times $2x$,

18

00:00:59,860 --> 00:01:02,329

I see that they're using portion

of the right hand side

19

00:01:02,396 --> 00:01:05,666

and therefore I took the left

hand side $(x^2 + 2)^6$

20

00:01:05,732 --> 00:01:08,035

to the power of 6 except

that I didn't have 6,

21

00:01:08,101 --> 00:01:10,871

so I need to divide by 6.

22

00:01:11,471 --> 00:01:13,407

So what we have, if the integrand,

23

00:01:13,473 --> 00:01:19,913

what I'm integrating is $f(u(x)) \times u'(x)\,dx$,

24

00:01:20,581 --> 00:01:23,317

then the result of that

will be $F(u(x)) + C$,

25

00:01:23,717 --> 00:01:25,018

if $F'(u) = f(u)$,

26

00:01:25,252 --> 00:01:30,591

if $F(u)$

is equal to the integration

27

00:01:31,491 --> 00:01:34,895

of $f(u)\,du$, which looks

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

perhaps a little bit strange

28

00:01:34,995 --> 00:01:36,964

but of course used just a variable.

29

00:01:37,331 --> 00:01:42,336

So I can say $F(x) = \int f(x)\,dx$.

30

00:01:42,669 --> 00:01:45,539

And this allows you

to use the standard integrals,

31

00:01:45,606 --> 00:01:50,444

which you find in the IB formula booklet.

32

00:01:51,712 --> 00:01:53,747

Let us return the original question.

33

00:01:54,014 --> 00:01:58,218

$5\cos(5x)$ can be written as

$\cos(5x) \times 5$.

34

00:01:58,352 --> 00:02:01,855

If I integrate that, then I can make the

35

00:02:02,523 --> 00:02:05,993

identification, $\cos(5x)$ is $f(u)$,

36

00:02:06,059 --> 00:02:09,763

which is $\cos(u)$,

but only if I let $u(x) = 5x$.

37

00:02:10,163 --> 00:02:12,766

Now if I differentiate $5x$ to get 5,

38

00:02:12,866 --> 00:02:15,936

and now I see that

the 5 appears over there.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

39

00:02:16,870 --> 00:02:20,240

Now since the integral of $\cos(u)$ du

40

00:02:20,307 --> 00:02:24,978

equals $\sin(u) + C$

, then I can conclude that

41

00:02:25,045 --> 00:02:31,118

$\int \cos(5x) \times 5 dx = \sin(5x) + C$.

42

00:02:33,654 --> 00:02:37,591

And of course I can check

this using differentiation

43

00:02:37,658 --> 00:02:40,427

to make sure that I get the integrand back

44

00:02:41,862 --> 00:02:43,163

always worth remembering.

45

00:02:43,230 --> 00:02:47,534

Similarly, here,

if I identify $(x^2 + 2)^5$

46

00:02:47,601 --> 00:02:49,770

to the power 5 as u^5,

47

00:02:49,837 --> 00:02:53,073

I can do that only if I let $u = x^2 + 2$,

48

00:02:53,140 --> 00:02:55,275

which of course differentiate becomes $2x$.

49

00:02:55,409 --> 00:02:58,879

And again, I noticed

that I have an $f(u)$ times

50

Student
view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

00:02:58,946 --> 00:03:00,581

u' times dx.

51

00:03:00,747 --> 00:03:06,019

Now if I remember that the integral

of u^5 is $\frac{1}{6}u^6$,

52

00:03:06,286 --> 00:03:10,591

then I know that $\int (x^2 + 2)^5 \times 2x \, dx$

53

00:03:11,091 --> 00:03:16,630

integrated is $\frac{1}{6}(x^2 + 2)^6 + C$.

54

00:03:17,231 --> 00:03:20,400

Now remember that this helps us very much

55

00:03:20,467 --> 00:03:23,904

by using an integrand

which has a product rule

56

00:03:23,971 --> 00:03:27,674

in that because unlike differentiation,

there is no such thing

57

00:03:27,741 --> 00:03:30,577

as a straight product rule in integration.

The video shows that:

$$\int f(u(x))u'(x)\,\mathrm{d}x = F(u(x)) + C\text{, where } F(u) = \int f(u)\,\mathrm{d}u$$

It investigates the following two integrals and evaluates them using this result:

$$\int 5\cos 5x\,\mathrm{d}x = \sin 5x + C\text{, and}$$

$$\int \left(x^2+2\right)^5 2x\,\mathrm{d}x = \frac{1}{6}\left(x^2+2\right)^6 + C\,.$$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

These integrals were found by inspection, using an understanding of anti-differentiation and working backwards from the derivative. They answered the question, “What function would allow me to take a derivative that results in this function?”

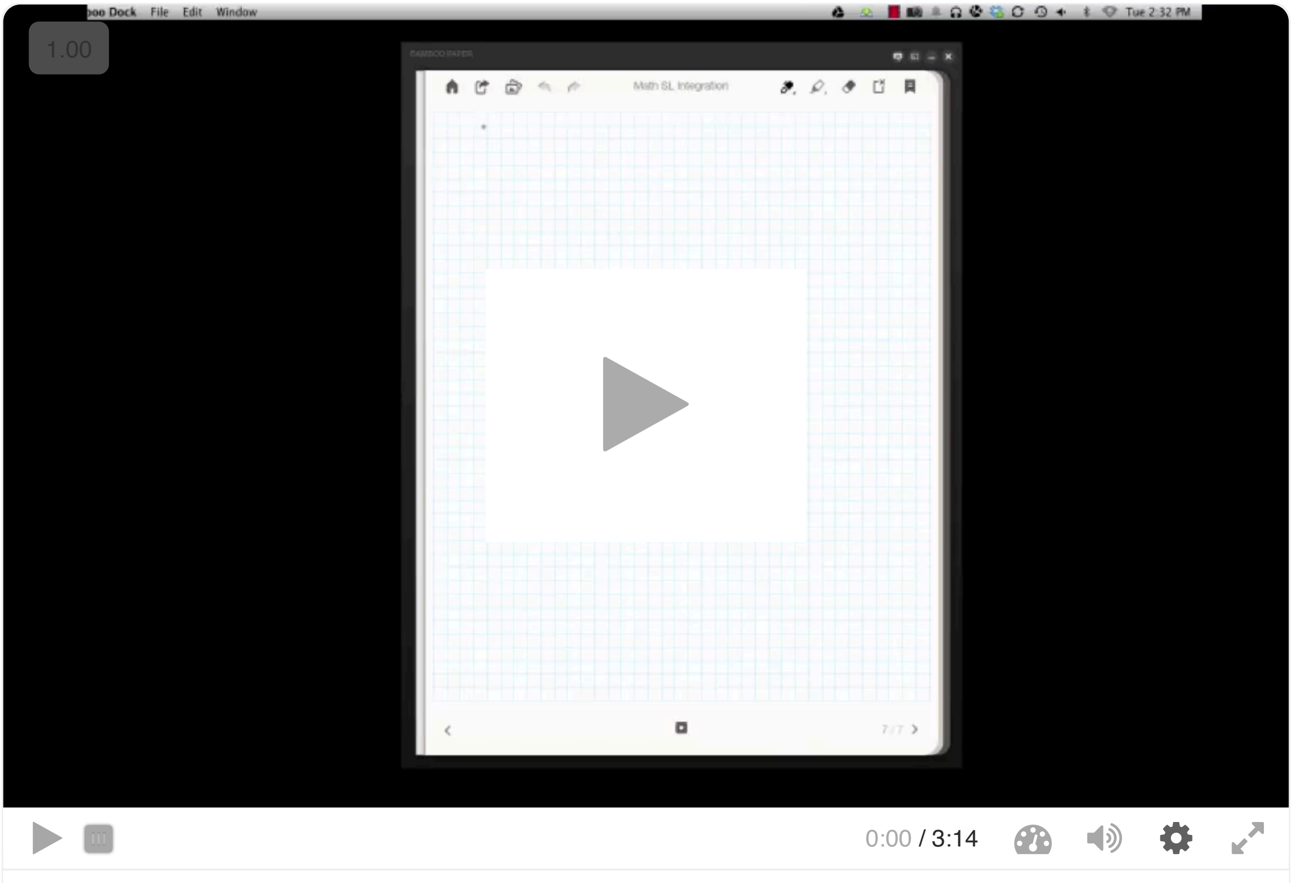
Notice that, as integration is working backwards from differentiation, this is really an example of anti-differentiation of the chain rule.

As always, upon differentiating the outcome of the integration, the result should be the original integrand. This gives a way of checking your answer.

This is commonly referred to as integration by inspection. As you gain more experience, you will get better at noticing patterns and be able to integrate a wider range of functions by inspection.

Integration by substitution

When integration by inspection does not seem to work, it is time to use a more formal approach, which is integration by substitution . This involves using a formal change of variable. The process is explored in the following video.



Video 2. Integration Using Substitution and the Chain Rule.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

More information for video 2

1

00:00:00,433 --> 00:00:03,900

narrator: Now integration by inspection

required you to see the relationship

2

00:00:03,967 --> 00:00:06,200

between the two components

in the integrand

3

00:00:06,600 --> 00:00:07,900

and that is not always very easy.

4

00:00:08,000 --> 00:00:10,933

So we're gonna do integration

by substitution proper

5

00:00:11,133 --> 00:00:13,167

by actually changing the variable.

6

00:00:13,633 --> 00:00:15,100

And let's look at an example first.

7

00:00:16,733 --> 00:00:20,633

So we're gonna look

at the integration of $x \sin(x^2)\, dx$,

8

00:00:20,833 --> 00:00:22,033

of $x^2\, dx$,

9

00:00:23,000 --> 00:00:24,967

and we're gonna make

a substitution formally,

10

00:00:25,133 --> 00:00:27,067

and I'm gonna say let u be the variable

11

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

00:00:27,133 --> 00:00:29,067

that makes this complicated,

which is x^2.

12

00:00:29,133 --> 00:00:31,467

And then we're gonna

differentiate u with respect to x,

13

00:00:31,533 --> 00:00:33,400

which is really $\frac{du}{dx}$,

14

00:00:33,600 --> 00:00:35,633

which is simple enough, it's $2x$.

15

00:00:35,733 --> 00:00:40,733

Now first of all, we are going to

rewrite $du = 2x\,dx$,

16

00:00:40,933 --> 00:00:45,767

so that actually we get

$\frac{1}{2x}\,du = dx$

17

00:00:45,833 --> 00:00:47,100

because what we need to do,

18

00:00:47,167 --> 00:00:49,733

we are actually going

to change all the x's into integration

19

00:00:49,800 --> 00:00:53,000

and that includes the measure dx.

20

00:00:53,467 --> 00:00:55,567

And then we're gonna make

all the substitutions.

21

00:00:55,633 --> 00:01:00,867

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

So we had $\int x \sin(x^2)\, dx$,

22

00:01:00,067 --> 00:01:04,167

which now becomes $\int x \sin(u)\, dx$

23

00:01:04,667 --> 00:01:10,167

of u times du by $2x$

and you see that the x's cancel out.

24

00:01:10,233 --> 00:01:14,600

So we left with a half times

$\int \sin(u)\, du$,

25

00:01:14,800 --> 00:01:16,500

which using our standard integrals

26

00:01:16,567 --> 00:01:20,233

becomes $-\frac{1}{2}\cos(u) + C$.

27

00:01:20,300 --> 00:01:23,667

Of course, now we need to go back to axis.

28

00:01:23,733 --> 00:01:27,100

So $-\frac{1}{2}\cos(x^2) + C$.

29

00:01:27,800 --> 00:01:29,200

Don't forget always to check.

30

00:01:29,667 --> 00:01:33,733

So if we differentiate minus half

cosine of x^2,

31

00:01:35,100 --> 00:01:38,400

if we differentiate that,

then you should get

32

00:01:38,700 --> 00:01:41,100

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

the integrand so minus a half times

33

00:01:41,167 --> 00:01:46,933

$-\sin(x^2) \times 2x$ using our chain rule.

34

00:01:47,000 --> 00:01:50,033

And you see that we left

with $\sin(x^2) \times x$,

35

00:01:50,100 --> 00:01:51,567

which was the integrand.

36

00:01:52,333 --> 00:01:55,633

The other example

is the integral of x divided

37

00:01:55,867 --> 00:01:59,800

by $\sqrt{x^2 - 4}$.

38

00:02:03,267 --> 00:02:07,400

So let's use as our substitution $u = x^2 - 4$ here,

39

00:02:07,467 --> 00:02:09,167

$u = x^2 - 4$ here,

40

00:02:09,233 --> 00:02:14,467

and then we're gonna differentiate

$\frac{du}{dx} = 2x$

41

00:02:14,833 --> 00:02:19,033

and that again we are going to

write dx as a function of du,

42

00:02:19,100 --> 00:02:20,367

which is the same as before.

43

00:02:20,433 --> 00:02:21,867

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Then we're gonna make a substitution.

44

00:02:21,933 --> 00:02:24,300

So we add the integral

of x divided a square root

45

00:02:24,367 --> 00:02:26,167

of $x^2 - 4dx$,

46

00:02:26,233 --> 00:02:30,000

which now becomes the integral

of $x \times u^{-1/2}$.

47

00:02:30,067 --> 00:02:32,767

We write explicitly as a power law times

48

00:02:33,000 --> 00:02:34,933

du divided by $2x$.

49

00:02:35,167 --> 00:02:36,300

So the x's cancel

50

00:02:36,400 --> 00:02:38,900

and we left with a half times the integral

51

00:02:38,967 --> 00:02:40,767

of $u^{-1/2} du$,

52

00:02:41,067 --> 00:02:45,933

which we can use as standard integrals

for becomes a half times 2

53

00:02:46,000 --> 00:02:48,600

times $u^{1/2} + C$.

54

00:02:48,667 --> 00:02:50,800

And now we substitute back in favor of x,

55

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

00:02:50,867 --> 00:02:52,833

so it becomes $(x^2 - 4)^{1/2} + C$.

56

00:02:52,900 --> 00:02:54,833

to the power of one half plus C.

57

00:02:55,067 --> 00:02:56,300

Now of course we are going to check

58

00:02:56,367 --> 00:02:59,200

after all this work,

so we are gonna differentiate

59

00:02:59,400 --> 00:03:01,533

$(x^2 - 4)^{1/2}$,

60

00:03:01,700 --> 00:03:04,567

and then that of course becomes one a half

61

00:03:04,900 --> 00:03:08,300

$(x^2 - 4)^{-1/2}$

62

00:03:08,367 --> 00:03:10,733

times $2x$ using the chain rule.

63

00:03:10,800 --> 00:03:13,433

And we find that we recover d integrant.

64

00:03:13,500 --> 00:03:14,667

And that was that.

The procedure laid out in the video to find integrals of the form $\int f(g(x))g'(x)\mathrm{d}x$can be summarized as follows:

Student view

1. Make the substitution for a part of the integrand, $g(x) = u$.

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

2. Differentiate: $\frac{\mathrm{d}u}{\mathrm{d}x} = g'(x)$.
3. Substitute u into the integrand, and also replace $g'(x)\mathrm{d}x$ with $\mathrm{d}u$.
4. The substitution is successful if no x is remaining in the expression and you can integrate with respect to the variable u.
5. In the final expression, change back to x by making use of $u = g(x)$.

Using this procedure, the following results are obtained in the video :

1. $\int x \sin x^2 \, \mathrm{d}x = -\frac{1}{2}\cos x^2 + C$, and
2. $\int \frac{x}{\sqrt{x^2-4}} \, \mathrm{d}x = \sqrt{x^2-4} + C$.

The most difficult part of the process is choosing u . This comes with experience, but in the most straightforward cases, u is the inside of the most complex part of the function. Often, u will be the argument of a trigonometric, logarithmic, exponential, rational or radical function. In more complex cases, you may have to do some algebra first to make it work.

Just because one mathematician integrates by inspection and another integrates by substitution does not make either of them wrong. Both methods, if done correctly, will yield the same result. It is quite normal for you to start out using substitution frequently and then to use inspection more as you gain experience.

⚠ **Be aware**

- You must remove of all the x after the substitution, and end up with an integral in one variable only, such as u . This includes replacing $g'(x)\mathrm{d}x$ with $\mathrm{d}u$.
- Do not forget to substitute back to obtain an expression in the original variable, usually x , after you have integrated using the substituted variable.

Example 1

★☆☆

Find the indefinite integral of $f(x) = 12x \cos 3x^2$.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Steps	Explanation
$\int 12x \cos 3x^2 \, \mathrm{d}x$	
$u = 3x^2$	Identify the substitution.
$\frac{\mathrm{d}u}{\mathrm{d}x} = 6x$ or $6x\mathrm{d}x = \mathrm{d}u$	Differentiate.
$\int 12x \cos 3x^2 \, \mathrm{d}x = \int 2 \cos 3x^2 \times 6x\mathrm{d}x$	Regroup the terms.
$= \int 2 \cos u \, \mathrm{d}u$	Make the substitution.
$= 2 \sin u + C$	Integrate
$= 2 \sin 3x^2 + C$	Substitute for u

Example 2

★★☆

Find the indefinite integral of $f(x) = \frac{x}{x^2+1}$.

Steps	Explanation
$\int \frac{x}{x^2+1} \, \mathrm{d}x$	
$u = x^2 + 1$	Identify the substitution.
$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ or $2x\mathrm{d}x = \mathrm{d}u$	Differentiate.
$\int \frac{x}{x^2+1} \, \mathrm{d}x = \int \frac{1}{2} \times \frac{1}{x^2+1} \times 2x\mathrm{d}x$	Regroup the terms.

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Steps	Explanation
$= \int \frac{1}{2u}\, \mathrm{d}u$	Make the substitution.
$= \frac{1}{2} \ln \lvert u \rvert + C$	I ntegrate
$= \frac{1}{2} \ln \lvert x^2 + 1 \rvert + C$	Substitute for u
$= \frac{1}{2} \ln(x^2 + 1) + C$	The absolute value is not required as $x^2 + 1$ is always positive.

This example leads to an interesting result:

✓ **Important**

- If $f(x) > 0$, then $\int \frac{f'(x)}{f(x)}\, \mathrm{d}x = \ln f(x) + C$.
- In general, $\int \frac{f'(x)}{f(x)}\, \mathrm{d}x = \ln \lvert f(x) \rvert + C$

Example 3

★★☆

Find the indefinite integral of $f(x) = \sin^3 x \cos x$.

Steps	Explanation
$\int \sin^3 x \cos x \mathrm{d}x$	
$u = \sin x$	Identify the substitution.
$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$, or $\cos x\, \mathrm{d}x = \mathrm{d}u$	Differentiate.

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Steps	Explanation
$\int u^3 \mathrm{d}u$	Make the substitution.
$= \frac{1}{4}u^4 + C$	Integrate
$= \frac{1}{4}\sin^4 x + C$	Substitute for u

There will be times when the substitution is too complex to recognise, especially when dealing with trigonometric functions. Often, a trigonometric identity is useful in identifying the substitution.

Example 4

★★★

Find the indefinite integral of $f(x) = \sin^3 x$.

Steps	Explanation
$\sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x)\sin x = \sin x - \cos^2 x \sin x$ $\begin{aligned}\int \sin^3 x \, \mathrm{d}x &= \int (\sin x - \cos^2 x \sin x) \, \mathrm{d}x \\ &= \int \sin x \, \mathrm{d}x - \int \cos^2 x \sin x \, \mathrm{d}x\end{aligned}$	Rewrite the integrand.
$\int \sin x \, \mathrm{d}x = -\cos x + C$	Find the first integral.
$u = \cos x$	Identify the substitition for the second integral.
$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$ or $-\sin x \, \mathrm{d}x = \mathrm{d}u$	Differentiate.

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Steps	Explanation
$\int \cos^2 x \sin x \, dx = -\int \cos^2 x(-\sin x) dx$	Rewrite the integral.
$-\int u^2 \, du$	Make the substitution.
$= -\frac{1}{3}u^3 + C$	Integrate
$= -\frac{1}{3}\cos^3 x + C$	Substitute for u
$\int \sin^3 x \, dx = -\cos x + \frac{1}{3}\cos^3 x + C$	Combine the two parts

4 section questions

Question 1

Difficulty:

★★☆

What is the indefinite integral $\int 10x\sqrt{5x^2+2}\,dx$?

1. $\frac{2}{3}\left(5x^2+2\right)^{\frac{3}{2}} + C$ ✓
2. $\frac{4}{3}\left(5x^2+2\right)^{\frac{3}{2}} + C$
3. $\frac{3}{2}\left(5x^2+2\right)^{\frac{3}{2}} + C$
4. $3\left(5x^2+2\right)^{\frac{3}{2}} + C$

Explanation

Use the substitution $u = 5x^2 + 2$, so

$$\frac{du}{dx} = 10x.$$

Thus,

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$$\begin{aligned}\int 10x\sqrt{u}\,\mathrm{d}u\,\frac{\mathrm{d}x}{\mathrm{d}u} &= \int 10x\sqrt{u}\,\mathrm{d}u\,\frac{1}{10x}\\ &= \int \sqrt{u}\,\mathrm{d}u\\ &= \frac{2}{3}u^{\frac{3}{2}} + C.\end{aligned}$$

Substitute back for u to give: $\frac{2}{3}\left(5x^2+2\right)^{\frac{3}{2}} + C.$

Question 2

Difficulty:

★★☆

What is the integral of $\frac{\sin x}{\cos x}$ $(= \tan x)$ for $0 < x < \frac{\pi}{2}$?

1 $-\ln\cos x + C$ ✓

2 $\sin^2 x + C$

3 $\cos^2 x + C$

4 $\ln\cos x + C$

Explanation

Use the substitution $u = \cos x$. As well as replacing $\cos x$ with u, we also need to replace $\mathrm{d}x$ with $\mathrm{d}u$ by differentiating our substitution, obtaining $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$. Substituting both of these expressions into the integral gives:

$$\begin{aligned}\int \frac{\sin x}{\cos x}\,\mathrm{d}x &= \int \frac{1}{u}\left(-\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x\\ &= -\int \frac{1}{u}\,\mathrm{d}u\\ &= -\ln|u| + c\\ &= -\ln|\cos x| + C.\end{aligned}$$

On the domain given in the question $\cos x$ is positive, so we do not need the absolute value.

$$\int \frac{\sin x}{\cos x}\,\mathrm{d}x = -\ln\cos x + C.$$

Question 3

Difficulty:

★☆☆

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Calculate the indefinite integral $\int \cos^3 x \sin x \, dx$.

1 $-\frac{\cos^4 x}{4} + C$ ✓

2 $-\frac{\cos^4 x}{4} \sin x + C$

3 $\frac{\cos^4 x}{4} + C$

4 $-\frac{\cos^3 x}{3} \sin x + C$

Explanation

Use the substitution $u = \cos x$.

Then, $\frac{du}{dx} = -\sin x \Rightarrow du = -\sin x \, dx$.

Hence,

$$\begin{aligned}\int \cos^3 x \sin x \, dx &= -\int \cos^3 x(-\sin x) \, dx \\ &= -\int u^3 \, du \\ &= -\frac{u^4}{4} + C,\end{aligned}$$

and substituting back for u we get

$$-\frac{\cos^4 x}{4} + C.$$

Question 4

Difficulty: ★★☆

What is $\int x^2 \left(x^3 + 2\right)^3 dx$?

1 $\frac{1}{12} \left(x^3 + 2\right)^4 + C$ ✓

2 $\frac{1}{4} \left(x^3 + 2\right)^4 + C$

3 $\frac{1}{12} \left(x^3\right)^4 + C$

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4 $\frac{3}{4}\left(x^3+2\right)^3+C$

Explanation

We let $u=x^3+2$, such that $\mathrm{d}x=\frac{1}{3x^2}\,\mathrm{d}u$.

Then we get

$$\begin{aligned}
&\int x^2\left(x^3+2\right)^3\mathrm{d}x = \int x^2(u)^3\frac{1}{3x^2}\,\mathrm{d}u\\
&= \frac{1}{3}\int u^3\,\mathrm{d}u\\
\Rightarrow\ &= \frac{1}{3}\frac{1}{4}u^4+C\\
\Rightarrow\ &= \frac{1}{12}\left(x^3+2\right)^4+C.
\end{aligned}$$

5. Calculus / 5.10 Indefinite integrals

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-27813/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- find the integral of a function using the following standard integrals:
  - $\int x^n\,\mathrm{d}x=\frac{x^{n+1}}{n+1}+C, n\neq -1$
  - $\int (ax+b)^n\,\mathrm{d}x=\frac{(ax+b)^{n+1}}{a(n+1)}+C, n\neq -1$
  - $\int \frac{1}{x}\,\mathrm{d}x=\ln|x|+C$
  - $\int \frac{1}{ax+b}\,\mathrm{d}x=\frac{1}{a}\ln|ax+b|+C$
  - $\int e^x\,\mathrm{d}x=e^x+C$
  - $\int e^{ax+b}\,\mathrm{d}x=\frac{1}{a}e^{ax+b}+C$
  - $\int \sin x\,\mathrm{d}x=-\cos x+C$
  - $\int \sin(ax+b)\,\mathrm{d}x=\frac{-\cos(ax+b)}{a}+C$
  - $\int \cos x\,\mathrm{d}x=\sin x+C$

Student view

- $\int \cos(ax+b)\,\mathrm{d}x = \frac{\sin(ax+b)}{a} + C$

- find the indefinite integral of functions by inspection and substitution.

Investigation

Throughout this subtopic, you studied the relationships between functions, their derivatives and their anti-derivatives (or integrals). Basically, a derivative of a function describes the slope of a tangent line to the curve at a specific x-value, and the integral describes the area between the curve and the x-axis along an interval. Explore the applet and consider these questions:

- Can you see the relationships between the slope of $f(x)$ and the value of $f'(x)$?
- How about the slope of $\int_0^x f(x')\,\mathrm{d}x'$ and the value of $f(x)$? What about the other direction?
- Does the area under $f(x)$ relate to the value of $\int_0^x f(x')\,\mathrm{d}x'$?
- Does the area under $f'(x)$ relate to the value of $f(x)$?

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Interactive 1. Relationships Between Functions, Their Derivatives and Their Anti-Derivatives.

More information for interactive 1

This interactive tool allows users to explore the relationships between a function, its derivative, and its antiderivative (integral) through dynamic visualizations. The graph of $y = f(x)$ represents the function, $y = f'(x)$ represents its derivative, and $\int_0^x f(x')\,dx'$ represents its antiderivative. By interacting with the graph, users can observe how changes in the function affect its derivative and integral, reinforcing key calculus concepts.

A derivative represents the **rate of change** of a function. As users move the red dot along the function $f(x)$, they will see how the slope at that point determines the corresponding value of $f'(x)$. When $f(x)$ is increasing, its derivative is positive, and when $f(x)$ is decreasing, its derivative is negative. At points where $f(x)$ has a local maximum or minimum, the slope is zero, meaning $f'(x)$ also equals zero at those points. This visually confirms that the derivative measures how steeply a function is changing at any given moment.

The integral, on the other hand, represents the **accumulated area** under the function's curve. As users move the red dot, they will notice that the integral increases when $f(x)$ is positive and decreases when $f(x)$ is negative. This means the integral continuously sums up the values of $f(x)$ over an interval, reflecting the net accumulation of the function's values over time. The integral graph also shows how an area under the x-axis contributes negatively to the accumulation, whereas an area above the x-axis contributes positively.

By observing these interactions, users will develop an intuitive understanding of how differentiation and integration are inverse operations. The derivative graph confirms how the slope of $f(x)$ translates into rate of change, while the integral graph reveals how accumulated areas relate to the original function. This interactive serves as a powerful visual aid, reinforcing the fundamental principles of calculus by linking function behavior, instantaneous rates of change, and accumulated quantities in a hands-on manner.

Student view

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