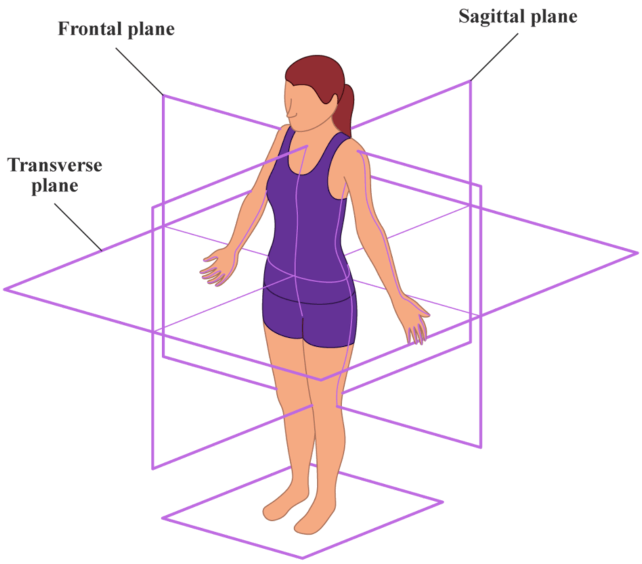
Index

The big picture

You have already explored the idea of relative positions of lines in 3D. Two lines can intersect at a point, they can be parallel, they can be skew (where they are neither parallel nor intersecting) or they can coincide and have infinitely many intersection points.

Similarly to lines in 3D, two planes can have intersection points or they can be parallel with no intersection points. In this section, you will be looking at the intersection of two or three planes and at the intersection of a plane and a line and calculating the angles between them.

Viewing a three-dimensional object in three orthogonal planes is useful in fields as diverse as medicine and engineering. The human body can be divided into three planes: the frontal plane, the transverse plane and the sagittal plane.



More information

The diagram shows a human figure annotated with three orthogonal planes, demonstrating how the body can be divided. The frontal plane, also known as the coronal plane, cuts vertically from side to side, dividing the body into anterior and posterior (front and back) sections. The transverse plane, or horizontal plane, divides the body into upper (superior) and lower (inferior) parts. The sagittal plane splits the body vertically down the middle into left and right portions. The planes intersect at the center of the body, forming a cross-section that highlights their spatial orientation and relationship to each other.

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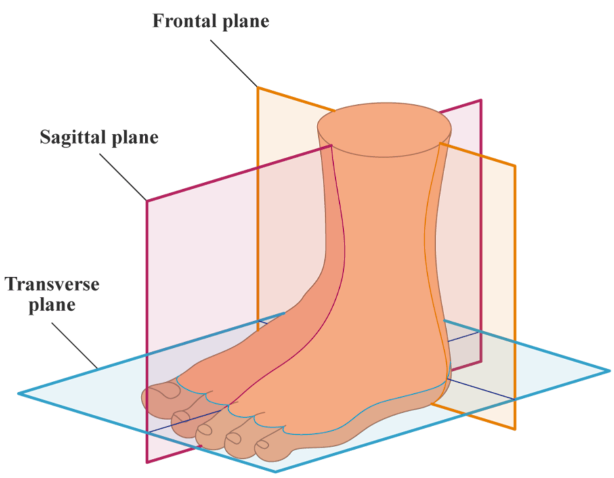

More information

The image is a diagram illustrating a human foot viewed from the side with three intersecting planes labeled as the Frontal plane, Sagittal plane, and Transverse plane. These planes are represented by colored rectangles intersecting through a model of the foot:

1. **Frontal plane**: An orange rectangle cutting through the foot from front to back, dividing the anatomical structure into front and back portions.
2. **Sagittal plane**: A pink rectangle passing vertically through the foot, separating it into left and right sections.
3. **Transverse plane**: A blue rectangle horizontally intersecting the foot, segmenting it into upper and lower parts.

These planes are positioned orthogonally to each other and are used to describe movements or actions in three-dimensional space, facilitating the analysis of interactions between these movements.

[Generated by AI]

This enables different movements or actions to be described in three dimensions and the interactions between these movements to be analysed.

By looking at the projections of a three-dimensional object in three orthogonal planes, an engineer or designer can describe and analyse its structure.

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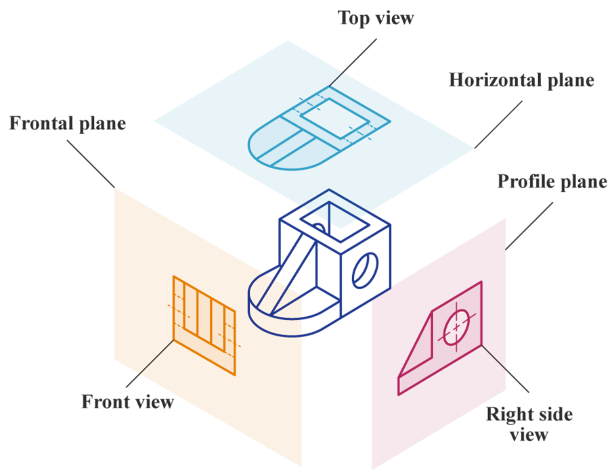

More information

The image is a diagram depicting a three-dimensional object shown in multiple orthogonal views. The object is positioned within a cube framework, with each face of the cube representing a different plane. The top view is on the horizontal plane, colored blue, and labeled 'Top view' and 'Horizontal plane.' The front view is on the frontal plane, colored orange, labeled 'Front view' and 'Frontal plane.' The right side view is exhibited on the profile plane, colored pink, labeled 'Right side view' and 'Profile plane.' The main structure of the object features simple geometric shapes, such as lines and circles, indicating possible cutouts or features on the object. The diagram is intended to help engineers or designers analyze the object's structure by visualizing it from different angles.

[Generated by AI]

Concept

There are different ways of representing lines and planes in 3D, for example, vector form, parametric form or with a Cartesian equation. How does each form enable you to find their relative positions and the ways in which they interact?

3. Geometry and trigonometry / 3.18 Intersections and angles between lines and planes

Relative positions of planes

Relative positions of two planes

Two planes in 3D can be parallel, intersecting or coincident as shown in below.

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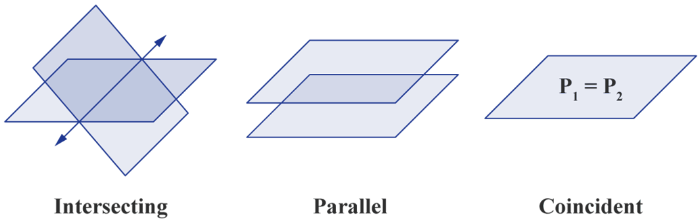


More information

The image shows three different configurations of 3D planes. On the left, two blue planes are intersecting, symbolized by arrows indicating the direction of overlap. The middle image displays two parallel planes that are aligned in the same direction but do not touch each other. On the right, the coincident planes overlap completely with the label P1 = P2, indicating they are the same plane. Labels underneath each plane configuration read "Intersecting," "Parallel," and "Coincident."

[Generated by AI]

Activity

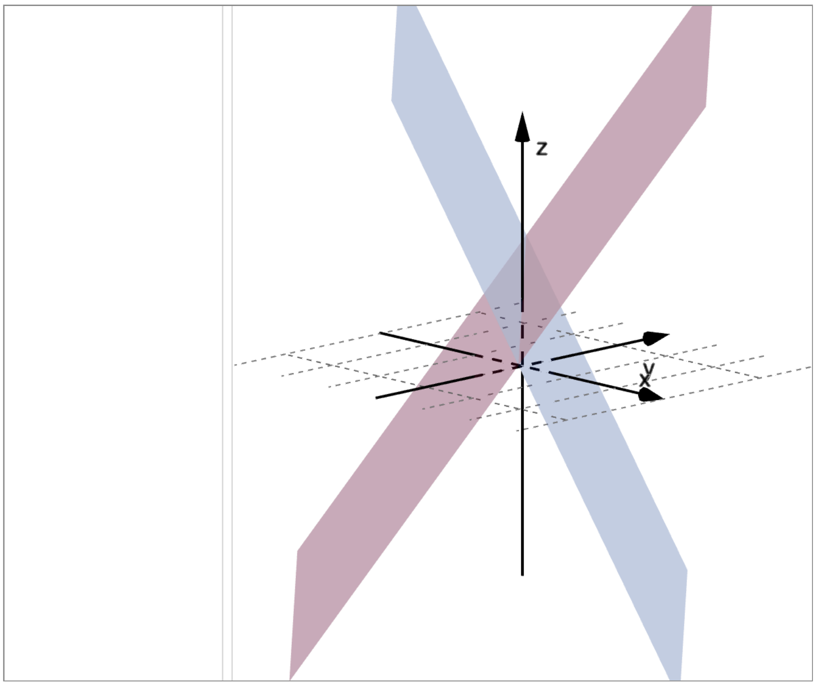
In the following applet, the Cartesian equations of two planes p and k are given in the form

$$ax + by + cz = d$$

How does their relative position change as you change the parameters?

For what values are the two planes are parallel?

When do they intersect?



Interactive 1. Relative Positions of Planes.

More information for interactive 1

Student
view

This interactive allows users to explore the relative positions of two planes in 3D space by manipulating their Cartesian equations. The visualization demonstrates how changing coefficients in the plane equations ax + by + cz = d affects whether planes become parallel, intersect, or coincide, providing an intuitive understanding of spatial relationships between planar surfaces through direct

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

parameter adjustment.

The display shows two distinct planes in a 3D coordinate system, each with their current equation visibly updating in real-time. Users can adjust four sliders for each plane (a, b, c, d) ranging from -5 to 5, immediately seeing how modifications affect the planes' orientations and positions. The planes are rendered in different colors for clear distinction, with their intersection lines (when present) highlighted. A draggable rotation control viewing the planes from all angles to better understand their three-dimensional relationship.

Users can systematically investigate three fundamental scenarios: First, by setting proportional coefficients (a,b,c) while keeping d different, they observe parallel planes that never intersect. Second, by creating non-proportional coefficients, they generate intersecting planes with clearly visible intersection lines. Third, by making all coefficients identical (including d), they see the planes coincide perfectly. For example, setting both planes to $2x - y + 3z = 4$ makes them identical, while changing one plane to $4x - 2y + 6z = 5$ creates parallel planes, and using $x + y + z = 1$ versus $x - y + z = 0$ produces intersecting planes.

Through this hands-on experimentation, users discover several key geometric principles: they learn to recognize when planes are parallel by comparing coefficient ratios ($a1 / a2 = b1 / b2 = c1 / c2 \neq d1 / d2$), understand how intersecting planes form linear solutions (when coefficients aren't proportional), and see how identical equations produce coincident planes.

Important

Recall from subtopic 3.17 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-28337/) that a plane can be defined by two non-parallel vectors in the plane and a vector that is normal to the plane.

When the equation of the plane is given in Cartesian form $ax + by + cz = d$, the coefficients a, b and c give the components of the normal vector $\boldsymbol{n}$, i.e. $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

As you saw in the activity, two planes are parallel if their normal vectors are parallel.

If their normal vectors are not parallel, then they intersect on a line.

How can you find the equation of this intersection line?

If the equation of plane is not given in Cartesian form, how can you decide whether or not two planes are coincident? You will need to be able to solve a system of equations with three unknowns, both with and without a calculator, to do this.

And if the planes are intersecting, how can you find the angle between them?

Making connections

You learned how to solve simultaneous equations using the method of Gaussian elimination in section 1.16.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/systems-with-three-equations-id-27379/) and solutions using a graphic display calculator were introduced in section 1.16.3 (/study/app/math-aa-hl/sid-134-cid-761926/book/using-technology-id-27380/).

Example 1

★★☆

Student view

Find

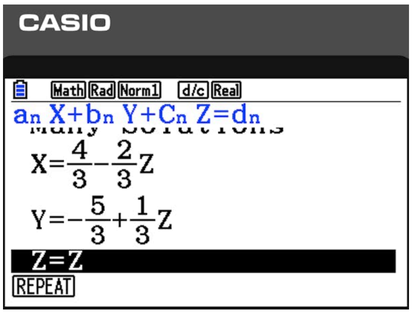
a) the equation of the line along which the two planes $2x + y + z = 1$ and $x - y + z = 3$ intersect

b) the angle between the planes. Give your answer in radians correct to 3 significant figures.

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Part	Explanation
a)	$x = \frac{4}{3} - \frac{2}{3}z$ $y = -\frac{5}{3} + \frac{1}{3}z$ $z = z$

The result screens from the calculators



Casio fx-CG50

The image shows the screen of a Casio calculator with a full view of the display area. The screen includes various mathematical equations and expression text. At the top, it has an example equation "an X+bn Y+Cn Z=dn" in blue. Below, it presents two expressions for X and Y in terms of Z: X = 4/3 - 2/3 Z and Y = -5/3 + 1/3 Z. The bottom part of the screen shows Z = Z and a button labeled "REPEAT." Above the main screen area are labels such as Math, Rad, Norm1, d/c, and Real, likely representing different modes or options related to mathematical operations.

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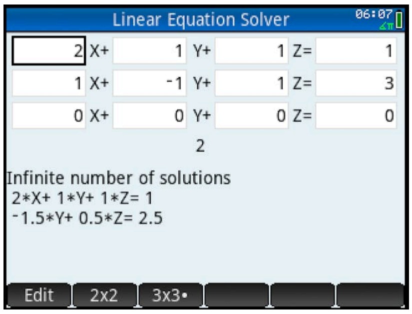


HP Prime

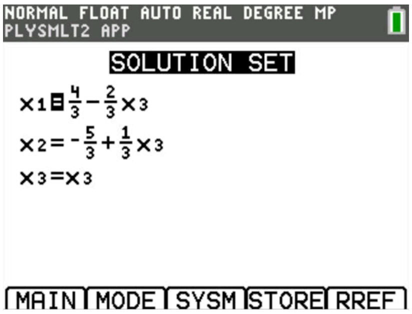
This is a screenshot of a linear equation solver interface. It contains a matrix representing a system of equations with coefficients and constants. The top section has labeled fields: "X+", "Y+", "Z=" followed by boxes with numerical input. The first row displays "2 X+ 1 Y+ 1 Z= 1", the second displays "1 X+ -1 Y+ 1 Z= 3", and the third displays "0 X+ 0 Y+ 0 Z= 0". Below this, there is a text indicating an "Infinite number of solutions," followed by two example equations: "2X+ *1Y+ 1Z= 1" and "-1.5Y+ 0.5Z= 2.5". There are three buttons at the bottom: "Edit", "2x2", and "3x3".* The interface has a blue header showing its title, "Linear Equation Solver."

[Generated by AI]

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

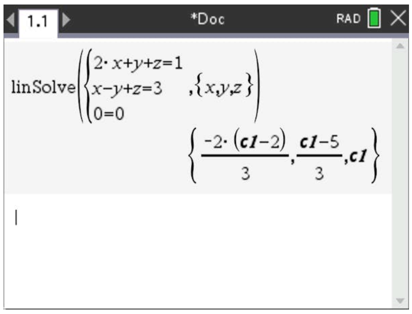
Part	Explanation



TI-84 plus CE

The image is a screenshot of a TI-84 calculator displaying a solution set screen. At the top, it shows "NORMAL FLOAT AUTO REAL DEGREE MP" followed by "PLYSMULT2 APP." Below that, there's a highlighted section labeled "SOLUTION SET." It lists three equations alternatively with fractions: "x1 = 4/3 - 2/3x3," "x2 = -5/3 + 1/3x3," and "x3 = x3." At the bottom, there are buttons labeled "MAIN," "MODE," "SYSM," "STORE," and "RREF."

[Generated by AI]



TI-nspire CX

The image is a screenshot of a TI-nspire calculator interface displaying a mathematical operation. The function 'linSolve' is used to solve a system of linear equations with three variables, x, y, and z. The equations shown are 2x + y + z = 1, x - y + z = 3, and an invalid equation 0 = 0. The solution is expressed in terms of another variable 'c1'. The solutions are given as {-2(c1 - 2)/3, (c1 - 5)/3, c1}. The interface includes typical features of a TI-nspire calculator like navigation indications and settings at the top.

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$z = \lambda$

$$x = \frac{4}{3} - \frac{2}{3}\lambda \Rightarrow \lambda = \frac{x - \frac{4}{3}}{-\frac{2}{3}}$$

$$y = -\frac{5}{3} + \frac{1}{3}\lambda \Rightarrow \lambda = \frac{y + \frac{5}{3}}{\frac{1}{3}}$$

Rearrange each equation to make λ the subject.

Student view

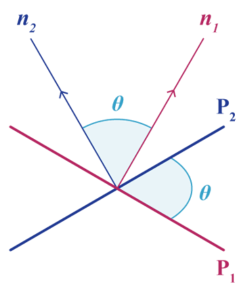
Part	Explanation		
	$\lambda = \dfrac{x - \frac{4}{3}}{-\frac{2}{3}} = \dfrac{y + \frac{5}{3}}{\frac{1}{3}} = z$	Equate all three expressions for λ to give the Cartesian equation of the line.	
	Therefore, the intersection of the two planes is line $\dfrac{x - \frac{4}{3}}{-\frac{2}{3}} = \dfrac{y + \frac{5}{3}}{\frac{1}{3}} = z$		
b)	$\boldsymbol{n}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is a vector normal to the plane $2x + y + z = 1$ and $\boldsymbol{n}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is a vector normal to the plane $x - y + z = 3$	If $ax + by + cz = d$, then a normal vector to the plane is $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$	
		The angle between two planes is found by calculating the angle between the two normal vectors. The formula for the angle between two vectors obtained from the scalar product is.	
	$\theta = 1.0799\ldots = 1.08 \text{ radians}$	Give the angle in radians to three significant figures.	

Important

If two planes are parallel, then their normal vectors will be parallel.

If two planes intersect, then the angle between the planes will be equal to the acute angle, θ, between their normal vectors, $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$.

Using the scalar product, $\cos\theta = \dfrac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1|\,|\boldsymbol{n}_2|}$.



More information

This diagram illustrates two intersecting planes, labeled as P1 and P2. Each plane has an associated normal vector labeled n1 for P1 and n2 for P2. The diagram shows the angle θ (theta) between the two normal vectors where the planes intersect. The normal vectors are depicted as arrows extending from the origin of the angle θ. The diagram effectively demonstrates how the planes' intersection creates the angle between their normals, highlighted in a shaded area, illustrating the spatial relationship between the planes and their normals.

[Generated by AI]

If two planes intersect, then the equation of the line along which they intersect can be found by solving the equations of the planes simultaneously.

Example 2

★★☆

The planes $x + y + z = 4$ and $x - y + 2z = k$ and the line $x - 1 = 2y + 4 = z$ have a common point.

Find the value of k.

The coordinates of a point on the line satisfy the parametric equation of this line.

$$\begin{aligned} &x - 1 = 2y + 4 = z = \lambda \\ &\Rightarrow x = \lambda + 1 \\ &\Rightarrow y = \frac{\lambda - 4}{2} \\ &\Rightarrow z = \lambda \end{aligned}$$

The point lies on the first plane $x + y + z = 4$, so substitute x, y and z into the equation.

$$(\lambda + 1) + \frac{\lambda - 4}{2} + \lambda = 4$$

Multiply this equation by 2 and solve for λ.

$$\begin{aligned} 2\lambda + 2 + \lambda - 4 + 2\lambda &= 8 \\ 5\lambda &= 10 \\ \lambda &= \frac{10}{5} = 2 \end{aligned}$$

Using this value you can get the coordinates of the intersection point.

$$\begin{aligned} &x = \lambda + 1 = 2 + 1 = 3 \\ &y = \frac{\lambda - 4}{2} = \frac{2 - 4}{2} = -1 \\ &z = \lambda = 2 \end{aligned}$$

This point also lies on the second plane $x - y + 2z = k$, so the coordinates satisfy this equation.

$$\begin{aligned} 3 - (-1) + 2(2) &= k \\ k &= 8 \end{aligned}$$

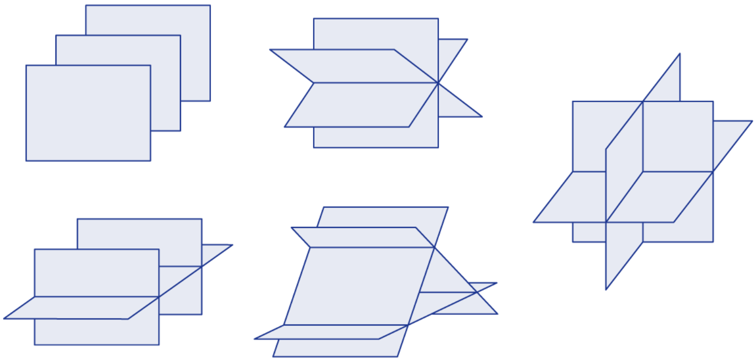
Student
view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Therefore, the value of k is 8.

Relative position of three planes

There are five possible relative positions for three planes, as shown below.



More information

The image displays five blue geometric shapes representing different relative positions of three planes in a 3D space. Each shape visually illustrates a unique possible configuration of how three planes can intersect or be oriented relative to each other. The individual shapes suggest scenarios such as parallel alignment, intersection at a point, or angular intersections, each showcasing possible spatial relationships. The configurations are intended to visually demonstrate the concept of plane positioning in geometric space.

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In this subtopic, you will be focussing on three of these:

1. The planes intersect at a point, so there is a single solution when the equations of the planes are solved simultaneously.
2. The planes are all parallel, so the simultaneous equations have no solutions.
3. The planes intersect along a line, so there are an infinite number of solutions.

Example 3

★★☆

Find the intersection, if any, of the following three planes:

$$\begin{aligned} \Pi_1 : \quad & 3x + y - z = 11 \\ \Pi_2 : \quad & x + y - 3z = 11 \\ \Pi_3 : \quad & x - y - z = 2 \end{aligned}$$

Student view

Use your graphic display calculator.

Overview
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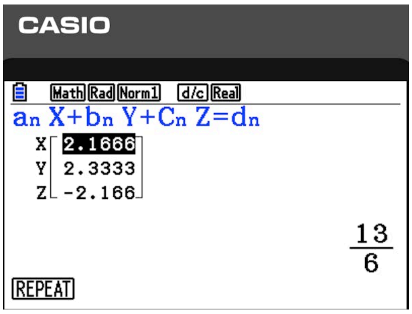
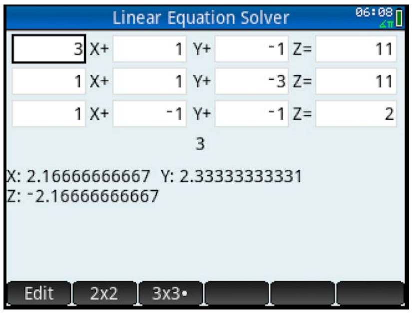
The result screens from the calculators



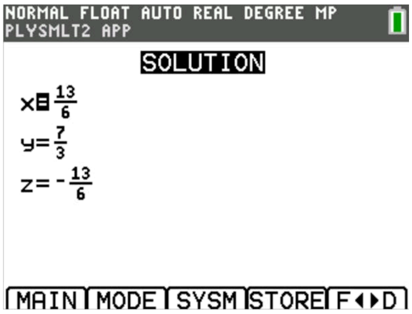
Casio fx-CG50



HP Prime

The image is a screenshot of a Linear Equation Solver application interface. At the top, it displays "Linear Equation Solver." Below this, there are input fields and labels for entering coefficients of variables in three linear equations. In the first row, the coefficients shown are: 3 for X, 1 for Y, and -1 for Z, equating to 11. The second row has coefficients 1 for X, 1 for Y, -3 for Z, equating to 11. The third row's coefficients are 1 for X, -1 for Y, -1 for Z, equating to 2. Below these inputs, the solution to the equations is displayed as X: 2.16666666667, Y: 2.33333333331, Z: -2.16666666667. At the bottom, there are buttons labeled "Edit", "2x2", and "3x3" with the "3x3" button selected, suggesting a 3x3 matrix is being solved. The interface also has a timestamp at the top right corner showing "06:08."

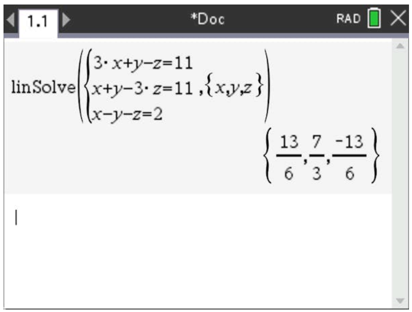
[Generated by AI]



TI-84 plus CE

The image is a screenshot of a calculator screen showing the word "SOLUTION" at the top. Below this, three equations are presented: x equals thirteen-sixths, y equals seven-thirds, and z equals negative thirteen-sixths. The top part of the screen lists modes like "NORMAL FLOAT AUTO REAL DEGREE MP," and a status indicator shows a battery icon at the top-right corner. The bottom of the screen displays buttons labeled "MAIN," "MODE," "SYSM," "STORE," with directional arrows and "F" written on a button, suggesting it's a function key.

[Generated by AI]



TI-nspire CX

The image is a screenshot of a TI-Nspire CX interface showing the solution to a set of linear equations. The equations listed are: 3x + y - z = 11, x + y - 3z = 11, and x - y - z = 2. Below the equations, the solution is presented in curly brackets, which shows the solved values for x, y, and z as fractions: x = 13/6, y = 7/3, and z = -13/6. The values are separated by commas and clearly indicated within the result braces.

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Student view

Overview
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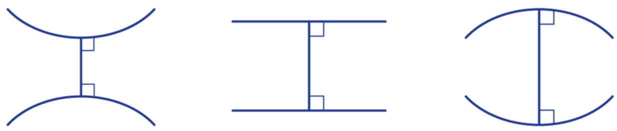
Hence, the three planes meet at the point with coordinates $\left(\frac{13}{6}, \frac{7}{3}, -\frac{13}{6}\right)$.

International Mindedness

All the formulae that you have defined in terms of vectors are based on Euclidian geometry.

However, there are two non-Euclidean 3D geometries: hyperbolic and elliptic.

Would it be possible to define vectors on non-Euclidian surfaces and in non-Euclidean space? If yes, how could you do so? What would be the purpose and applications?



More information

The image shows three geometric diagrams illustrating different types of spaces or surfaces, each labeled accordingly. The first diagram labeled "Hyperbolic" features two curved lines bowing outwards with perpendicular connecting lines forming right angles. The second diagram labeled "Euclidean" displays two straight parallel lines connected by perpendicular lines at right angles, illustrating a standard rectangular form. The third diagram labeled "Elliptic" shows two curved lines bowing inwards, also with perpendicular connecting lines at right angles. This set of diagrams visually represents the differences between the Hyperbolic, Euclidean, and Elliptic geometries.

[Generated by AI]

Example 4

★★☆

Find the intersection, if any, of the following three planes:

$$\begin{aligned} \Pi_1 :& \quad -2x + y + z = 3 \\ \Pi_2 :& \quad x - 2y + z = 2 \\ \Pi_3 :& \quad x + y - 2z = 2 \end{aligned}$$

Use your graphic display calculator.

The result screens from the calculators

Student view

Overview
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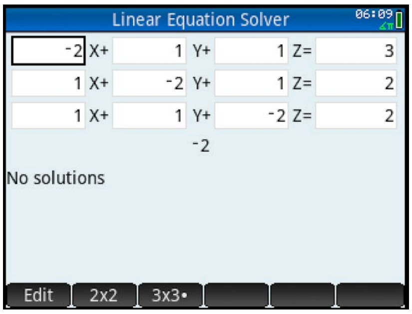

HP Prime

The image displays a screenshot of a Linear Equation Solver interface. At the top, there is a title bar labeled "Linear Equation Solver." Below it, there is a grid representing a matrix setup for solving three linear equations. The matrix format is structured with coefficients and constants in the following way: First row has coefficients -2, 1, 1 for variables X, Y, Z respectively, equated to 3. Second row has coefficients 1, -2, 1 for X, Y, Z, equated to 2. Third row has coefficients 1, 1, -2 for X, Y, Z, equated to 2. Below the matrix, the result "No solutions" is displayed, indicating that the set of equations has no solution. At the bottom, there are buttons labeled "Edit," "2x2," and "3x3" for selecting different matrix sizes or editing the current setup.

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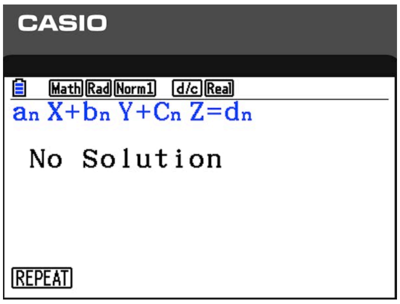

Casio fx-CG50

TI-84 plus CE

The image displays a screenshot of a calculator screen. At the top, there are several menu options: "NORMAL FLOAT AUTO REAL DEGREE MP PLYSMLT2 APP". Below, in the main section of the screen, a large box contains the message "NO SOLUTION FOUND". At the bottom, there are buttons labeled "MAIN", "MODE", "SYSM", and "RREF".

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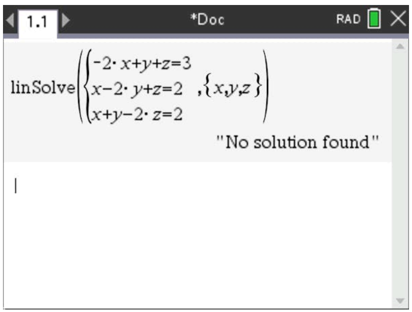

TI-nspire CX

The image depicts a screen, presumably from a calculator or software interface, that displays the result of solving a system of equations. The command used is 'linSolve' with the system of equations: -2x + y + z = 3, x - 2y + z = 2, and x + y - 2z = 2, with the variables {x, y, z}. The output states 'No solution found,' indicating that the system of equations does not have a solution.

[Generated by AI]

There is no solution. Therefore there are no common points between all three planes.

4 section questions

Student
view

Question 1

★☆☆

Find the common point(s) of the planes given by

$$\begin{aligned}\Pi_1: &\quad 3x+2y-4z=18\\ \Pi_2: &\quad y+z=3\\ \Pi_3: &\quad 3x-2y=-1\end{aligned}$$

1 $\left(\frac{9}{4}, \frac{31}{8}, -\frac{7}{8}\right)$ ✓

2 $\left(\frac{5}{2}, -\frac{7}{4}, \frac{1}{2}\right)$

3 $\boldsymbol{r} = \begin{pmatrix}-2\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\-1\end{pmatrix}$

4 No solution exists.

Explanation

There are several ways you can use your calculator to find the solution of this equation system. You can use the equation system solving application or you can ask the calculator to help you solving using row reduction.

The row reduced echelon form of the augmented matrix is

$$\begin{bmatrix}1 & 0 & 0 & \frac{9}{4}\\ 0 & 1 & 0 & \frac{31}{8}\\ 0 & 0 & 1 & -\frac{7}{8}\end{bmatrix}$$

More information

The image shows a 3x4 matrix in row reduced echelon form. The rows and columns are aligned as follows:

First row: The entries are 1, 0, 0, followed by the fraction 9/4. Second row: The entries are 0, 1, 0, followed by the fraction 31/8. Third row: The entries are 0, 0, 1, followed by the fraction -7/8.

This matrix indicates a unique solution for a system of linear equations aligning with the variables x, y, and z set to the corresponding constants 9/4, 31/8, and -7/8 respectively.

[Generated by AI]

Hence, the solution is unique at the point indicated.

Question 2

★☆☆

What is the intersection between the planes given by

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$$\begin{aligned} \Pi_1: &\quad x+y+z=-5 \\ \Pi_2: &\quad -2x-2y-2z=-9 \\ \Pi_3: &\quad 3x+4y+4z=-1 \end{aligned}$$

1 No solution exists. ✓

2 $(-1, 3, 4)$

3 $(0, 0, 0)$

4 Infinitely many solutions.

Explanation

The planes Π_1 and Π_2 are clearly parallel. No solution exists, as the row reduced echelon augmented matrix (by GDC) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

More information

Question 3

★☆☆

Consider the planes Π_1: $x-2y+3z=2$ and Π_2: $2x-y-3z=-1$.

Select the vector equation of the line of intersection.

1 $\boldsymbol{r} = \begin{pmatrix} 0 \\ -\frac{1}{3} \\ \frac{4}{9} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ \frac{1}{3} \end{pmatrix}$ ✓

2 $\boldsymbol{r} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$

3 $\boldsymbol{r} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$

4 $\boldsymbol{r} = \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{4}{9} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ \frac{1}{3} \end{pmatrix}$

Explanation

Method 1 (using a calculator)

Graphing calculators have applications to solve equation systems. Different calculators may give the answer in a different form. Here is one possibility:

$$x = -\frac{4}{3} + 3z$$
$$y = -\frac{5}{3} + 3z$$

Student view

Overview (/study/app/aa-hl/sid-134-cid-761926/o

In all answer options the x-coordinate of the point on the line is 0, so let's find the y and z-values our equation gives when $x = 0$.

The first equation gives $z = \frac{4}{9}$, and substituting this value in the second equation gives $y = -\frac{1}{3}$. The option that matches these values is

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -\frac{1}{3} \\ \frac{4}{9} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ \frac{1}{3} \end{pmatrix}$$

Method 2 (algebraic approach)

$$\begin{aligned} \Pi_1 + \Pi_2 : \quad & 3x - 3y = 1 \quad \Leftrightarrow \quad x = \frac{3y+1}{3} \\ \Pi_1 - 2\Pi_2 : \quad & -3x + 9z = 4 \quad \Leftrightarrow \quad x = \frac{9z-4}{3} \end{aligned}$$

These hold simultaneously, so you have the Cartesian equation of the line

$$x = \frac{3y+1}{3} = \frac{9z-4}{3}$$

The equations can be written in different form:

$$\begin{aligned} x = \frac{3y+1}{3} \quad & \Leftrightarrow \quad x = \frac{y - \left(-\frac{1}{3}\right)}{1} \\ x = \frac{9z-4}{3} \quad & \Leftrightarrow \quad x = \frac{z - \frac{4}{9}}{\frac{1}{3}} \end{aligned}$$

This gives the equation in vector form.

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -\frac{1}{3} \\ \frac{4}{9} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ \frac{1}{3} \end{pmatrix}$$

Question 4

★☆☆

Select which of the following is the angle (to one decimal place) between the planes $3x - 6y + 4z = -12$ and $x - 4y + 3z = 5$.

1 11.7° ✓

2 78.3°

3 168°

4 35.4°

Explanation

The angle is given by the angle between the normal vectors to these planes, i.e.

Student view

Overview
(/study/ap
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$$\cos\theta = \frac{\boldsymbol{n_1}\cdot\boldsymbol{n_2}}{|\boldsymbol{n_1}|\,|\boldsymbol{n_2}|}$$

$$\Rightarrow \quad = \frac{\begin{pmatrix}3\\-6\\4\end{pmatrix}\cdot\begin{pmatrix}1\\-4\\3\end{pmatrix}}{\left|\begin{pmatrix}3\\-6\\4\end{pmatrix}\right|\left|\begin{pmatrix}1\\-4\\3\end{pmatrix}\right|}$$

$$\Rightarrow \quad = \frac{3\times1+(-6)\times(-4)+4\times3}{\sqrt{3^2+(-6)^2+4^2}\sqrt{1^2+(-4)^2+3^2}}$$

$$\Leftrightarrow \quad \theta \approx 11.7^\circ$$

3. Geometry and trigonometry / 3.18 Intersections and angles between lines and planes

Intersection of a line and a plane

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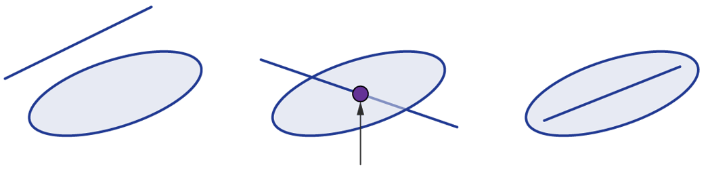
There are three possible relationships between a line and a plane, as shown below.



More information

The image is a diagram illustrating three possible relationships between a line and a plane. The first scenario shows a line above a plane with the label 'No intersection,' indicating that the line does not intersect with the plane. The second scenario depicts a line crossing through a plane at a single point, marked by a purple circle, with the label 'Point' and an arrow pointing to the intersection. This signifies that the line intersects the plane at precisely one point. The third scenario shows a line lying flat within the plane, labeled 'Line contained in plane,' meaning that the line is fully contained within the plane, with every point on the line also lying on the plane.

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Intersection of a line and a plane

Again, you can use a normal vector of the plane and the direction vector of the line to determine these relationships.

Example 1

★★☆

Student
view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Consider the Cartesian equation of the plane Π: $x + 6y + 3z = 14$, and the vector equation of the line L:

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} \frac{-9}{10} \\ \frac{3}{10} \\ \frac{-3}{10} \end{pmatrix}$$

Show that the line is parallel to the plane (so the line and the plane does not have a common point).

Given the Cartesian equation of the plane Π, you have the normal vector to the plane $\boldsymbol{n} = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$.

Thus, the vector equation of the plane is

$$\boldsymbol{r} \cdot \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} = 14$$

You need to show that the normal vector to the plane $\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$ and the direction vector of the line $\begin{pmatrix} \frac{-9}{10} \\ \frac{3}{10} \\ \frac{-3}{10} \end{pmatrix}$ are perpendicular and that the line L does not lie on the plane Π.

Firstly, $\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} \frac{-9}{10} \\ \frac{3}{10} \\ \frac{-3}{10} \end{pmatrix} = 1 \times \frac{-9}{10} + 6 \times \frac{3}{10} + 3 \times \frac{-3}{10} = 0.$

This means that the line is either parallel to the plane or in the plane.

Secondly, for $\lambda = 0$, you have the point $(3, 2, 2)$, which lies on the line. The line does not lie on the plane because this point does not satisfy the Cartesian equation of the plane as $3 + 6 \times 2 + 3 \times 2 = 21 \neq 14$.

Hence, the line L is parallel to the plane Π.

Example 2

★★☆

Consider the Cartesian equation of the plane Π: $x + 6y + 3z = 14$, and the point $\mathrm{A}(3, 4, 2)$ that is outside the plane. Find the vector equation of the line that passes through A and is perpendicular to the plane. Hence, find the point of intersection between the line and the plane.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

For a line to be perpendicular to the plane, its direction vector must be parallel to the normal vector $\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$ of the plane. Thus, the equation of a line that passes through A and is perpendicular to plane Π is

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$$

Let (x_0, y_0, z_0) be the point of intersection between the above line and the plane Π. Then,

$$x_0 + 6y_0 + 3z_0 = 14 \qquad (1)$$

and

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$$

which gives

$$\begin{cases} x_0 = 3 + \lambda \\ y_0 = 4 + 6\lambda \\ z_0 = 2 + 3\lambda \end{cases}$$

Substituting in (1)

$$\begin{aligned} x_0 + 6y_0 + 3z_0 &= 14 \\ 3 + \lambda + 6(4 + 6\lambda) + 3(2 + 3\lambda) &= 14 \\ 3 + \lambda + 24 + 36\lambda + 6 + 9\lambda &= 14 \\ \lambda &= -\frac{19}{46} \end{aligned}$$

Thus,

$$\begin{cases} x_0 = 3 + \lambda = 3 - \dfrac{19}{46} = \dfrac{138 - 19}{46} = \dfrac{119}{46} \\ y_0 = 4 + 6\lambda = 4 - 6 \times \dfrac{19}{46} = \dfrac{184 - 114}{46} = \dfrac{70}{46} = \dfrac{35}{23} \\ z_0 = 2 + 3\lambda = 2 - 3 \times \dfrac{19}{46} = \dfrac{92 - 57}{46} = \dfrac{35}{46} \end{cases}$$

Hence, the intersection point is

$$(x_0, y_0, z_0) = \left(\frac{119}{46}, \frac{35}{23}, \frac{35}{46}\right).$$

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Angle between a line and a plane

The orientation of a plane is given by the direction perpendicular to its surface, encoded in the normal vector to the plane, $\boldsymbol{n}$. The angle between a line and a plane is taken to be the angle that the surface makes with the line, say θ. The scalar product may be used to easily find the angle between a line, given by its direction vector in vector form, and the normal vector to the plane, say ϕ. However, generally $\theta \neq \phi$ (see below).

Consider the following.

Line L:

$$\boldsymbol{r_L} = \boldsymbol{a} + \lambda \boldsymbol{b}$$

and plane Π:

$$\boldsymbol{r_\Pi} \cdot \boldsymbol{n} = d$$

Then, using the scalar product:

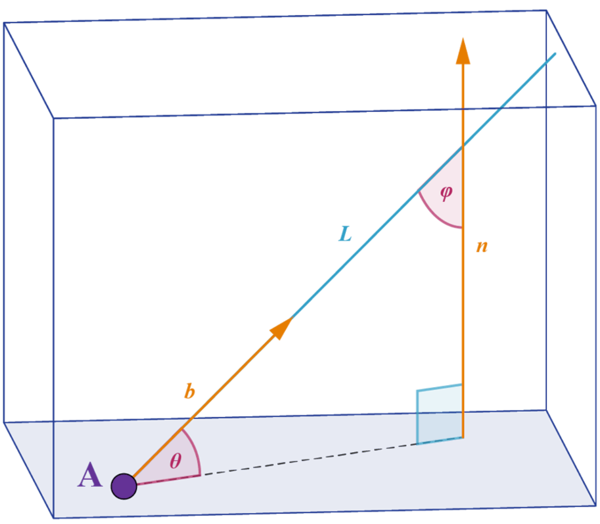
$$\cos \phi = \frac{\boldsymbol{b} \cdot \boldsymbol{n}}{|\boldsymbol{b}|\,|\boldsymbol{n}|}$$

such that the angle between the line and the plane, θ is

$$\theta = 90° - \phi$$

or, using the identity $\sin \alpha = \cos(90° - \alpha)$,

$$\sin \theta = \frac{\boldsymbol{b} \cdot \boldsymbol{n}}{|\boldsymbol{b}|\,|\boldsymbol{n}|}$$



More information

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

The image is a 3D diagram illustrating the angle between a line and a plane. The diagram shows a rectangular prism with a plane at its base. There is a line labeled 'L' that intersects the plane at point 'A'. This line 'L' is also represented by the direction vector 'b'. A normal vector 'n' to the plane is shown perpendicular to the base plane.

The angle θ is marked between the line 'b' and its projection on the plane. θ is the angle between the line and the plane. Another angle ϕ is implied between the line 'L' and the normal vector 'n', as noted by the accompanying text outside the image. The illustration helps visualize the angles in three-dimensional space, relevant to the equations and text provided.

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The angle between a line, L, and a plane, Π, is the angle θ. The angle ϕ is easily obtained by considering the scalar product between the vectors $\boldsymbol{b}$ (the direction vector of the line) and $\boldsymbol{n}$ (the normal to the plane).

Find the angle between the line $\frac{x-1}{3} = \frac{2-y}{2} = z+1$ and the plane $2x - 3y + 5z = 2$.

The direction vector of the line is given by $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ and the normal vector to the plane by $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$.

Then the angle between the line and the normal to the plane is

$$\begin{aligned} \cos\phi &= \frac{\boldsymbol{b}\cdot\boldsymbol{n}}{|\boldsymbol{b}|\,|\boldsymbol{n}|} \\ &= \frac{\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}}{\left|\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}\right|\left|\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}\right|} \\ &= \frac{3\times 2 + (-2)\times(-3) + 1\times 5}{\sqrt{3^2+(-2)^2+1^2}\sqrt{2^2+(-3)^2+5^2}} \\ &= \frac{17}{\sqrt{14}\sqrt{38}} \end{aligned}$$

Thus $\phi \approx 42.52^\circ$ and hence the angle between the line and the plane is

$$90^\circ - \phi \approx 47.5^\circ$$

✓ **Important**

The angle, θ, between a line and a plane is given by $90^\circ - \phi$, where ϕ is the angle between the direction vector of the line and the normal vector to the plane.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Exam tip

In IB examinations, remember to always give the acute angle when you are asked to find the angle between a line and a plane, or the angle between two planes.

4 section questions

Question 1

★★★

Consider the Cartesian equation of the plane Π: $2x - 3y + 5z = 8$, and the point $\mathrm{A}(2, -4, 2)$ that is outside the plane. Consider the line that passes through A and is perpendicular to the plane.

What is the point of intersection of this line and the plane?

1 $(1.05, -2.58, -0.37)$ ✓

2 $(0.11, -1.16, -2.74)$

3 $(1.23, -0.41, -2.39)$

4 $(2.90, -4.90, 3.50)$

Explanation

For a line to be perpendicular to the plane, its direction vector must be parallel to the normal vector $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ of the plane. Thus, the equation of a line that passes through A and is perpendicular to plane Π is

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

Let (x_0, y_0, z_0) be the point of intersection between the above line and the plane Π. Then,

$$2x_0 - 3y_0 + 5z_0 = 8 \qquad (1)$$

and

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

which gives

$$\begin{cases} x_0 = 2 + 2\lambda \\ y_0 = -4 - 3\lambda \\ z_0 = 2 + 5\lambda \end{cases}$$

Substituting in (1)

$$\begin{aligned} 2x_0 - 3y_0 + 5z_0 &= 8 \\ 4 + 4\lambda + 12 + 9\lambda + 10 + 25\lambda &= 8 \\ \lambda &= -\frac{9}{19} \end{aligned}$$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Thus,

$$\begin{cases} x_0 = 2 + 2\lambda & = 2 - 2 \times \frac{9}{19} & = 1.05 \\ y_0 = -4 - 3\lambda & = -4 + 3 \times \frac{9}{19} & = -2.58 \\ z_0 = 2 + 5\lambda & = 2 - 5 \times \frac{9}{19} & = -0.37 \end{cases}$$

Hence, the intersection point is

$$(x_0, y_0, z_0) = (1.05, -2.58, -0.37)$$

Question 2

★★☆

Consider the Cartesian equation of the plane Π: $2x - 3y + 5z = 8$, and the vector equation of the line L: $\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$. Select the true statement.

1 The line is parallel to the plane. ✓

2 The line intersects the plane.

3 The line lies on the plane.

4 We cannot determine from the given information.

Explanation

Given the Cartesian equation of the plane Π, you have the normal vector to the plane $\boldsymbol{n} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$. Thus, the vector equation of the plane is

$$\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = 8$$

Let's check the dot product of the direction vector of the line and the normal vector of the plane.

$$\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 2 \times 2 - 3 \times 3 + 5 \times 1 = 0$$

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This means that the line L is perpendicular to the normal vector of the plane, so the line is either in the plane or parallel to it.

Moreover, for $\lambda = 1$, you have the point $(2 + 2, -4 + 3, 2 + 1) = (4, -1, 3)$. It lies on the line and you can see that the line does not lie on the plane because the point does not satisfy the Cartesian equation of the plane as $2 \times 4 - 3 \times (-1) + 5 \times 3 = 26 \neq 8$.

Hence, the line L is parallel to the plane Π.

Student view

Question 3

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

★★★

Select the angle between the line $\frac{x}{3}=\frac{2-y}{4}=3z$ and the plane $-x+2y+3z=8$.

1 32.2° ✓

2 57.8°

3 148°

4 Does not exist

Explanation

The angle between the line and the normal to the plane is

$$\begin{aligned}
&& \cos\phi &= \frac{\boldsymbol{b}\cdot\boldsymbol{n}}{|\boldsymbol{b}|\,|\boldsymbol{n}|}\\
\Rightarrow && &= \frac{\begin{pmatrix}3\\-4\\\frac{1}{3}\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\3\end{pmatrix}}{\left|\begin{pmatrix}3\\-4\\\frac{1}{3}\end{pmatrix}\right|\left|\begin{pmatrix}-1\\2\\3\end{pmatrix}\right|}\\
\Rightarrow && &= \frac{3\times(-1)+(-4)\times 2+\dfrac{1}{3}\times 3}{\sqrt{3^2+(-4)^2+\left(\frac{1}{3}\right)^2}\sqrt{(-1)^2+2^2+3^2}}\\
\Rightarrow && &= \frac{-10}{\sqrt{\frac{226}{9}}\sqrt{14}}
\end{aligned}$$

Taking the acute angle:

$$\begin{aligned}
\Rightarrow && \cos\phi &= \left|\frac{-10}{\sqrt{\frac{226}{9}}\sqrt{14}}\right|\\
\Leftrightarrow && \phi &\approx 57.77^\circ
\end{aligned}$$

Thus, the angle between the line and plane is $90^\circ - 57.77^\circ \approx 32.2^\circ$

Question 4

★★☆

What is the angle between the line $\frac{x-1}{2}=\frac{2y-1}{3}=3-z$ and the plane $2x-3y+5z=1$?

Give your answer rounded to 2 decimal places without the degree symbol or the word 'degrees'.

19.35 ✓

Accepted answers

19.35, 19,35

Explanation

Rewrite the line as

$$\frac{x-1}{2}=\frac{y-\frac{1}{2}}{\frac{3}{2}}=\frac{z-3}{-1}$$

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

The direction vector of the line is given by $\begin{pmatrix} 2 \\ \frac{3}{2} \\ -1 \end{pmatrix}$ and the normal vector to the plane by $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$. Then the angle between the line and the normal to the plane is

$$\begin{aligned}\cos\phi &= \frac{\boldsymbol{b}\cdot\boldsymbol{n}}{|\boldsymbol{b}|\,|\boldsymbol{n}|}\\ &= \frac{\begin{pmatrix} 2 \\ \frac{3}{2} \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}}{\left|\begin{pmatrix} 2 \\ \frac{3}{2} \\ -1 \end{pmatrix}\right|\left|\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}\right|}\\ &= \frac{2\times 2+(-3)\times\frac{3}{2}+(-1)\times 5}{\sqrt{2^2+\left(\frac{3}{2}\right)^2+(-1)^2}\sqrt{2^2+(-3)^2+5^2}}\\ &= \frac{-\frac{11}{2}}{\frac{\sqrt{29}}{2}\sqrt{38}}\\ &= \frac{-11}{\sqrt{29}\sqrt{38}}\end{aligned}$$

Thus $\phi \approx 109.35^\circ$ and as we are interested in the acute angle that is formed by the normal vector and the line, this is $\approx 180^\circ - 109.35^\circ \approx 70.65^\circ$.

Hence, the angle between the line and the plane is

$$90^\circ - 70.65^\circ \approx 19.35^\circ.$$

3. Geometry and trigonometry / 3.18 Intersections and angles between lines and planes

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-28345/print/) Assign

What you should know

By the end of this subtopic you would be able to:

- understand that two planes in 3D can be parallel, intersecting or coincident
- recall that there are five possible relative positions for three planes in 3D
- recall that:
  - If three planes intersect at a point, then there is a single solution when the equations of the planes are solved simultaneously
  - If at least two of three planes are all parallel, then the simultaneous equations have no solutions
  - If three planes intersect along a line, then there are an infinite number of solutions when the equations of the planes are solved simultaneously
- recall that:
  - if two planes are parallel, then their normal vectors will be parallel
  - if two planes intersect, then the angle between two planes will be equal to the acute angle between the lines parallel to their normal vectors
  - if two planes intersect, then the equation of the line along which they intersect can be found by solving the equations of the planes simultaneously
- find θ, the angle between two planes using the scalar product of the normal vectors, $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, of the planes: $\cos\theta = \dfrac{|\boldsymbol{n}_1\cdot\boldsymbol{n}_2|}{|\boldsymbol{n}_1|\,|\boldsymbol{n}_2|}$
- recognise the relationship between a plane and a line:

Student view

Overview
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- if a line is parallel to the plane, then the normal vector of the plane will be perpendicular to the direction vector of the line
- if a line is contained in the plane, then the normal vector will be perpendicular to the direction vector of the line

- find θ, the angle between a line and a plane using $\sin\theta = \frac{|\boldsymbol{n}\cdot\boldsymbol{b}|}{|\boldsymbol{n}|\,|\boldsymbol{b}|}$, where $\boldsymbol{b}$ is the direction vector of the line and $\boldsymbol{n}$ is a normal vector to the plane.

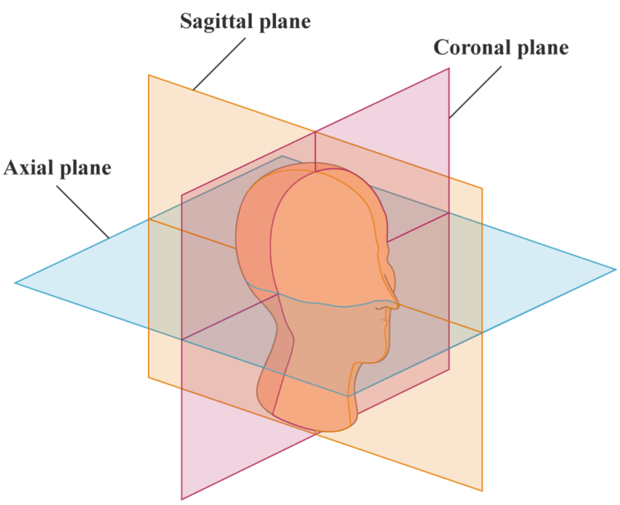
3. Geometry and trigonometry / 3.18 Intersections and angles between lines and planes

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-28346/print/) Assign

Computerised tomography (CT) scanning is used frequently in medicine. To obtain a CT scan, patients lie in a tunnel-shaped scanner and are exposed to X-rays.The X-ray tube and the detectors are on opposite sides of the scanner. Both of these rotate around the patient. The data are constructed by a computer and provide cross-sectional images in a single plane, which can be interpreted and analysed to make a diagnosis.

The figure directly below shows the three planes that are used to make images of the human head. Each plane will reveal different information about the patient depending on its orientation.



Imaging planes

More information

This diagram illustrates the three primary planes used in medical imaging of the human head: the sagittal plane, the coronal plane, and the axial plane. The sagittal plane divides the head into left and right sections and is represented as an orange vertical slice. It is positioned laterally, showing the side view of the head. The coronal plane, highlighted in pink, splits the head into front (anterior) and back (posterior) sections and is perpendicular to the sagittal plane. It shows the head as viewed from the front. Lastly, the axial plane, colored in blue, divides the head into upper and lower parts and is positioned horizontally, like the slices of a bread loaf viewed from above. The head is shown in profile, and each plane is labeled accordingly to indicate its orientation and the type of information it provides for medical imaging. This alignment of planes allows for comprehensive views of the brain and head structures for diagnostic purposes.

Student view

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More information

The image shows three MRI scan views of a human brain, each labeled beneath with its corresponding plane. The first view, labeled 'Axial plane,' displays a horizontal cross-section of the brain from above, showing the structure of the brain as cut horizontally. The second view, labeled 'Coronal plane,' presents a vertical cross-section from the front, illustrating the brain's interior structure from a frontal perspective. The third view, labeled 'Sagittal plane,' offers a side view of the brain, slicing it vertically from front to back to reveal the brain's side structure. These images highlight the anatomical differences and regions visible in each plane of the human brain.

[Generated by AI]

The illustration above shows images obtained from a CT scan of the human brain in the three orthogonal planes.

The pictures are obtained by measuring differences in X-ray absorption by different tissues, and the computer uses different shades of grey to distinguish between these.

Visit the two links below to find out more about the planes used in CT scanning and the images obtained.

- Link 1 (http://www.radtechonduty.com/2017/03/radiography-imaging-planes.html)
- Link 2 (https://www.ipfradiologyrounds.com/hrct-primer/image-reconstruction)

Write a report outlining the advantages of analysing an image in three different planes.

Rate subtopic 3.18 Intersections and angles between lines and planes

Help us improve the content and user experience.

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