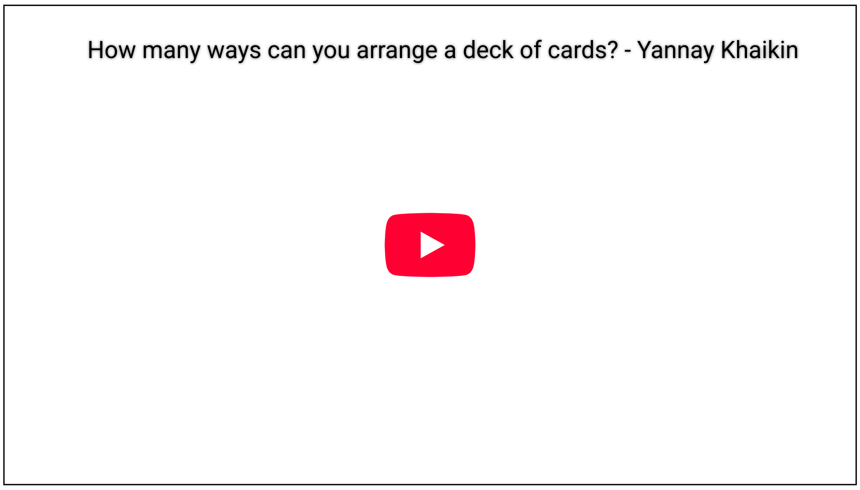
Index

The big picture

In this subtopic you will learn how to count the number of possible ways to arrange or select objects. At this point in your mathematical career you might be surprised that there is still more to learn about counting, but when it comes to selecting objects from a larger set or arranging objects in order, just listing all the possibilities one by one could be very time-consuming, therefore it is important to learn some different methods of counting.

Watch the video below to see how counting principles can be used to work out the number of ways in which the cards in a 52-card deck can be arranged.



According to the calculations in the above video, there are 8×10^{67} ways to arrange the 52 cards in a deck of cards. This number is so big that it would be impossible to check the count with an actual deck of cards, and even a computer-generated list of all the possible outcomes would take much longer than a human lifetime to simply read! Given the huge number of possibilities, when you shuffle a deck of cards, the particular arrangement you end up with is likely to never have appeared before and to never be repeated again.

Concept

As you work through the sections on permutations and combinations in this subtopic, reflect on how the **quantity** that you begin with (the size of a set of objects or the size of a subset that you have to select or arrange) gets magnified when you count the number of possible selections or arrangements. In this subtopic you will also study the extension of the binomial theorem to rational exponents. Think about how this might be helpful in calculating approximations to reciprocals and roots.

Counting principles: addition and multiplication

Addition and multiplication principles

Counting principles are used to calculate the number of ways in which objects can be arranged or selected. They form part of the area of mathematics known as combinatorics.

Example 1

★☆☆

An outfit must be selected that consists of a shirt and a pair of trousers. There are three choices for the shirt: a red, a blue and a green one. There are two choices for the trousers: jeans or corduroys.

a) List all the possible outfits.

b) Explain how the number of outfits can be found from the numbers of choices for the shirts and trousers.

c) Determine how the number of choices would change if you were told that an outfit consists of a shirt **or** a pair of trousers.

	Steps	Explanation
a)	List of outfits: $RJ \quad BJ \quad GJ$ $RC \quad BC \quad GC$	Let the shirt choices be denoted by letters $R,\ B$ and G, and the trouser choices by J and C.
b)	The total number of choices of outfit is found by multiplying the number of choices for the shirts by the number of choices for the trousers: $3 \times 2 = 6$	
c)	There are $3+2=5$ choices for an outfit that consists of a shirt or a pair of trousers.	In this case the numbers of choices are added instead of multiplied.

Section Student (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26973/print/) Assign

✓ **Important**

Addition principle: If one action can be done m ways and another action can be done n ways, then the number of ways that one **or** the other action is done is $m+n$.

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Multiplication principle: If one action can be done m ways and another action can be done n ways, then the number of ways that one action **and** the other are both done is $m \times n$.

Example 2

★☆☆

There are twelve flights that can be taken from Frankfurt to Amsterdam on a given day, and there are three buses that can be taken from the Amsterdam airport to a hotel.

Find the total number of ways in which the hotel in Amsterdam can be reached from Frankfurt using one of the flights and one of the buses.

Steps	Explanation
The number of ways is: $12 \times 3 = 36$	To get from Frankfurt to the hotel in Amsterdam you must fly **and** take the bus, so use the multiplication principle.

Example 3

★☆☆

There are x flights and $x + 7$ buses that can be taken from Mexico City to Guadalajara.

Find x given that there is a total of 19 possible ways to get to Guadalajara from Mexico City by plane or by bus.

Steps	Explanation
$x + x + 7 = 19 \quad \Leftrightarrow \quad 2x = 12$ $x = 6$	You can take a plane **or** a bus, so use the addition principle.

These multiplication and addition principles are illustrated using simple examples in this section, but you will see how useful they can be when applied to questions involving permutations and combinations in the next two sections.

Making connections

The addition and multiplication principles are also used in probability calculations.

Student view

3 section questions

Permutations

If you want to work out all the possible ways that you and two of your friends can sit in a row of aeroplane seats, you can write out all the possible arrangements.

These arrangements are called permutations .

Permutations are all the ways in which a certain number of objects can be arranged in order.

Be aware

Permutations are arrangements where the order matters. Be on the lookout for any information in a question that tells you that order is important. Examples might include arranging people in a line to pose for a photo, arranging pictures on a piece of paper, putting books in a certain order on a bookshelf, and organising letters and numbers to form a password.

Example 1

★☆☆

On a flight, Daisy, Hakim and Bo want to sit in a row with three seats.

List all the ways that they can arrange themselves.

Steps	Explanation
The possible arrangements are: $DHB \quad HDB \quad BHD$ $DBH \quad HBD \quad BDH$	Let D represent Daisy, H represent Hakim and B represent Bo.

If you want to know the total number of permutations, listing all the outcomes quickly becomes tedious and confusing as the number of objects that you want to arrange increases. You can see this for yourself by listing all the possible ways to arrange four people.

To find the total number of permutations, it is often easiest to consider the spots that you want to fill and the number of choices you have for each spot.

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Example 2

★☆☆

Find the number of different ways that four parked cars can be arranged in a straight line.

<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td><table><tr><td>4</td><td></td><td></td><td></td></tr></table></td><td>There are four spots to be filled. Represent each spot by a box.

The first spot can be filled by any of the four cars. You can write 4 in the first box to represent this.</td></tr>
<tr><td><table><tr><td>4</td><td>3</td><td>2</td><td>1</td></tr></table></td><td>Once you fill the first spot, there are 3 cars left to fill the second spot, 2 for the third spot, and 1 for the last spot.</td></tr>
<tr><td>Total number of permutations is:
$$4 \times 3 \times 2 \times 1 = 4! = 24$$</td><td>The numbers of possibilities in the boxes are multiplied together.</td></tr>
</table>

Why do you multiply together the number of choices for each spot that needs to be filled to find the total number of permutations?

Important

There are $n!$ ways to arrange n objects in a straight line when order is important.

Making connections

In subtopic 1.9 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27687/) you met $n!$, or 'n factorial', which is defined as

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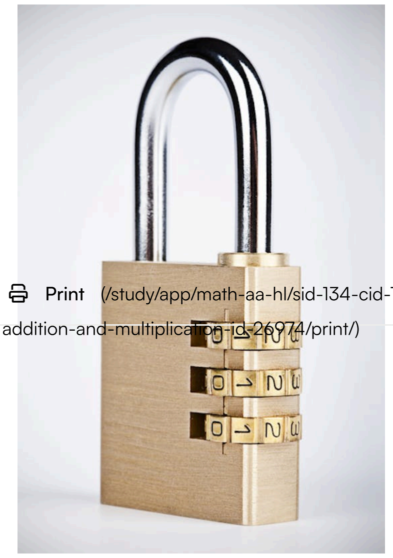
$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$$

The ‘boxes and choices’ method is also useful when arranging items that can be repeated, as in **Example 3**.



A three-digit combination lock

Credit: Getty Images mbbirdy

Example 3

★☆☆

The three-digit combination lock in the illustration above has three wheels, and on each wheel are all the integers from 0 to 9.

Find the total number of possible codes for the lock.

Steps	Explanation
10 | 10 | 10	Make three boxes for the three spots to be filled. There are 10 choices for the digit in each spot (digits can be repeated in a combination lock), so put 10 in each box.

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There are $10 \times 10 \times 10 = 1000$ possible codes.	Multiply together the number of possibilities in each box.

In many situations, the objects you want to arrange are taken from a larger set of items, so the number of choices for each spot may be greater than the number of spots in the arrangement.

Example 4

★★☆

a) Find the total number of different four-letter passwords that can be made from the 26 letters of the English alphabet if no letter can be used more than once.

b) Write your answer to part **a** in factorial notation.

	Steps	Explanati
a)	26 | 25 | 24 | 23	There are fou spots to be fi Letters cannc reused, so th number of ch decreases by from one spo the next.
	Total number of permutations is: $$26 \times 25 \times 24 \times 23 = 358\ 800$$	
b)	$26 \times 25 \times 24 \times 23 = \frac{26!}{22!}$	Note that $26 \times 25 \times 2$ looks like the beginning of The terms fro down to 1 are missing, whic be expressed dividing by 2

✓ **Important**

The number of ways to arrange r objects out of n distinct objects, where order is important, is:

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

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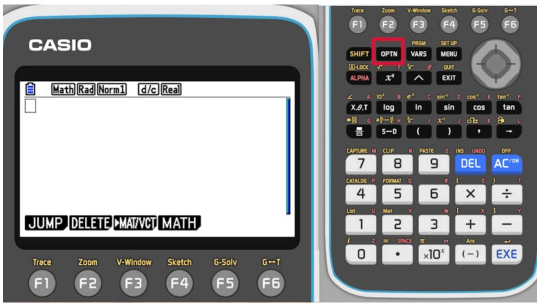
where the ‘P’ stands for ‘permutations’.

<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>To use the calculator to find the permutation number ${}^{26}P_4$, open the calculator mode, ...</td><td>
</td></tr>
<tr><td>... press OPTN to bring up some options, ...</td><td>
</td></tr>
<tr><td>... press F6 to scroll over to more options ...</td><td>
</td></tr>
</table>

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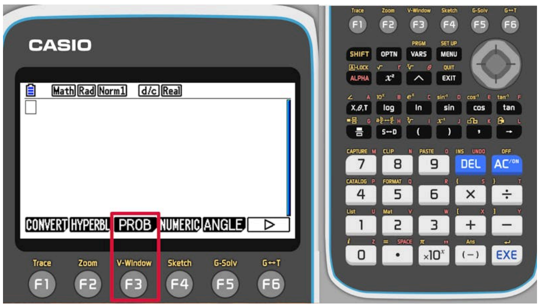
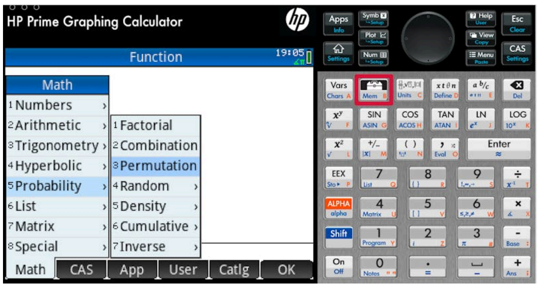
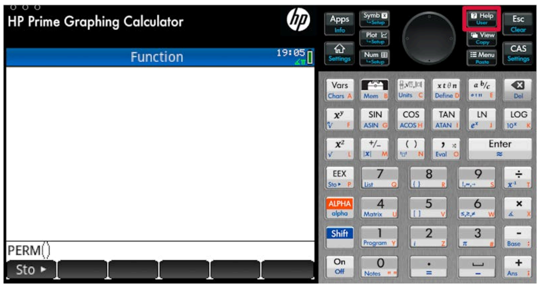
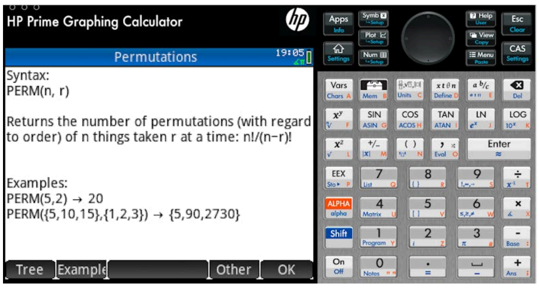
Steps	Explanation
... and press F3 for the probability related options.	
Type in the numbers and use F2 for the tool to find the permutation number. Press EXE to get the value of $^{26}P_4$.	

Steps	Explanation
To use the calculator to find the permutation number $^{26}P_4$, enter the home screen of any application.	

<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>Open the toolbox and find the option to find permutations.</td><td>
</td></tr>
<tr><td>You will need to tell the numbers 26 and 4 to the calculator. Remember, you can always bring up the help screen if you are not sure in which order the calculator expects these numbers.</td><td>
</td></tr>
<tr><td>This is the help screen explaining permutation numbers.</td><td>
</td></tr>
</table>

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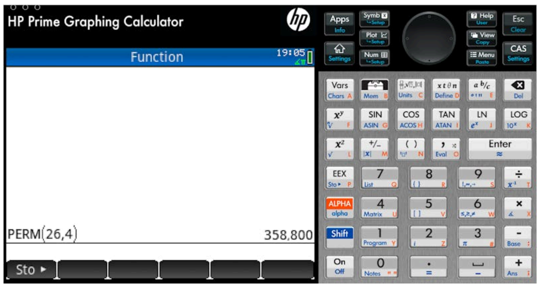
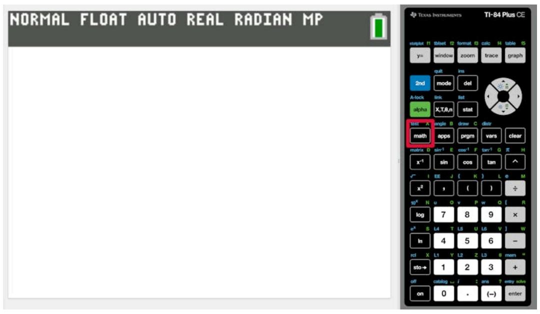
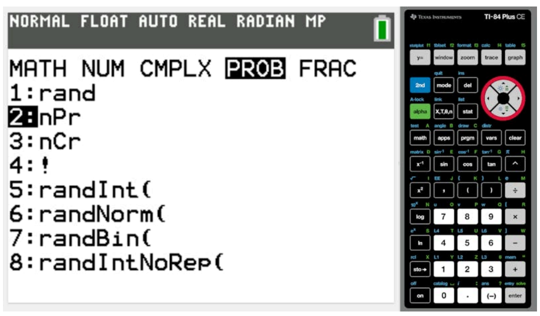
Steps	Explanation
Type in the numbers and press enter to get the value of ${}^{26}P_4$.	

Steps	Explanation
To use the calculator to find the permutation number ${}^{26}P_4$, open the math menu ...	
... and find the tool (nPr) to calculate permutation numbers.	

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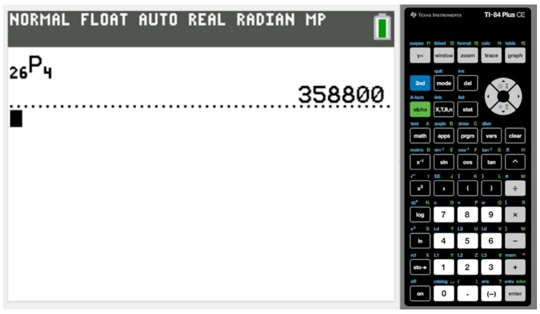
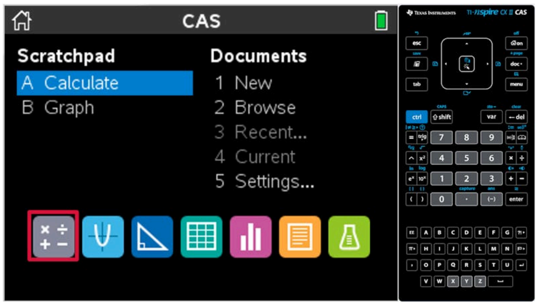
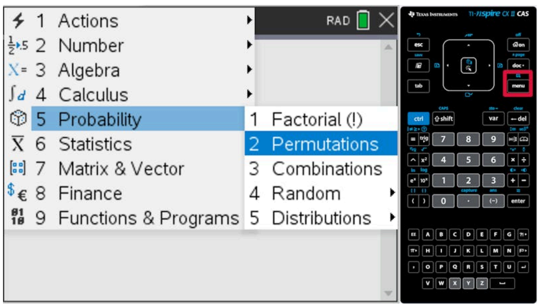
Steps	Explanation
Type in the numbers and press enter to get the value of ${}^{26}P_4$.	

Steps	Explanation
To use the calculator to find the permutation number ${}^{26}P_4$, open a calculator page.	
Open the menu and find the option to find permutations.	

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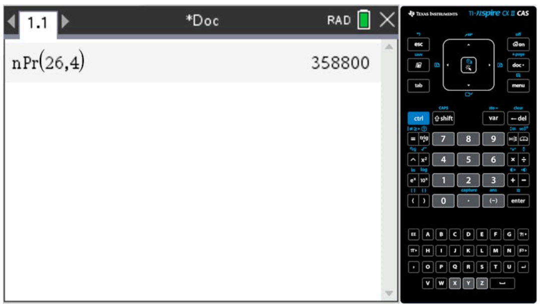
Steps	Explanation
You will need to remember the order in which you need to tell the two parameters to the calculator. Press enter to get the value of ${}^{26}P_4$.	

Exam tip

The permutations formula, ${}^{n}P_r = \dfrac{n!}{(n-r)!}$, is given in the IB formula booklet. This formula only applies to ordered arrangements of r objects chosen from n distinct (different) objects. Arrangements where some objects are identical, or circular arrangements such as people seated around a table, will not be tested on the IB exam.

Example 5

★★☆

An IB student studies six subjects and has six textbooks, one for each class.

a) Find how many ways there are to arrange three of the six textbooks on a shelf.

b) Find how many ways there are to arrange all six books on a shelf such that the book for maths and the book for science are next to each other.

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	Steps	
a)	The number of arrangements is: $${}^6P_3 = 120$$	Books are go You can also available for $6 \times 5 \times 4 =$
b)	${}^5P_5 = 120$ Or: 5 | 4 | 3 | 2 | 1 $5! = 120$	**The trick is t**
	Total number of permutations is: $$120 \times 2 = 240$$	You need to together.

Example 6

★★★

Find the number of four-digit numbers that can be made from the digits $0, 1, 3, 5, 8, 9$ if

a) each digit can only be used once

b) each digit is used once and the number must be bigger than 3000

c) each digit is used once and the number is divisible by 5.

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	Steps	Explanatic
a)	<table><tr><td>5</td><td>5</td><td>4</td><td>3</td></tr></table> The total number of permutations is: $5 \times 5 \times 4 \times 3 = 300$	A four-digit number canr start with 0. Each digit ca only be used once. 1st spot can u $1,\ 3,\ 5,\ 8$ o 2nd spot can 0 and the fou numbers remaining frc $1,\ 3,\ 5,\ 8$ a .
b)	Total number of permutations is: $4 \times 5 \times 4 \times 3 = 240$	To be bigger 3000 the first digit can be $3,\ 5,\ 8$ or 9. The next digi be any of the remaining fiv
c)	Number of ways to end in 5: <table><tr><td>4</td><td>4</td><td>3</td><td>1</td></tr></table> Number of permutations that end in 5 are: $4 \times 4 \times 3 \times 1 = 48$	A number divisible by 5 end in either 0. If the numbe ends in 5, the only one cho for the last di The choices remaining fo first digit are $1,\ 3,\ 8$ and For the secor digit the choi are 0 and the three digits th remain from $1,\ 3,\ 8$ and

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	Number of ways to end in 0: | 5 | 4 | 3 | 1 | Number of permutations that end in 0 are: $5 \times 4 \times 3 \times 1 = 60$	If the numbe ends in 0, the only one cho for the last di The choices remaining fo first digit are $1,\ 3,\ 5,\ 8$ a . For the secor digit the choi are any four digits that are from $1,\ 3,\ 5,\ 8$ a .
	Total number of permutations is: $48 + 60 = 108$	

4 section questions

1. Number and algebra / 1.10 Counting principles and the generalised binomial theorem

Combinations

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As opposed to asking in how many ways we can **arrange** 7 books on a shelf with only 4 spaces, we can ask the question: in how many ways can we **select** 4 books from 7 books that are on a shelf? We explore this and highlight the difference between combinations and permutations in the following video.

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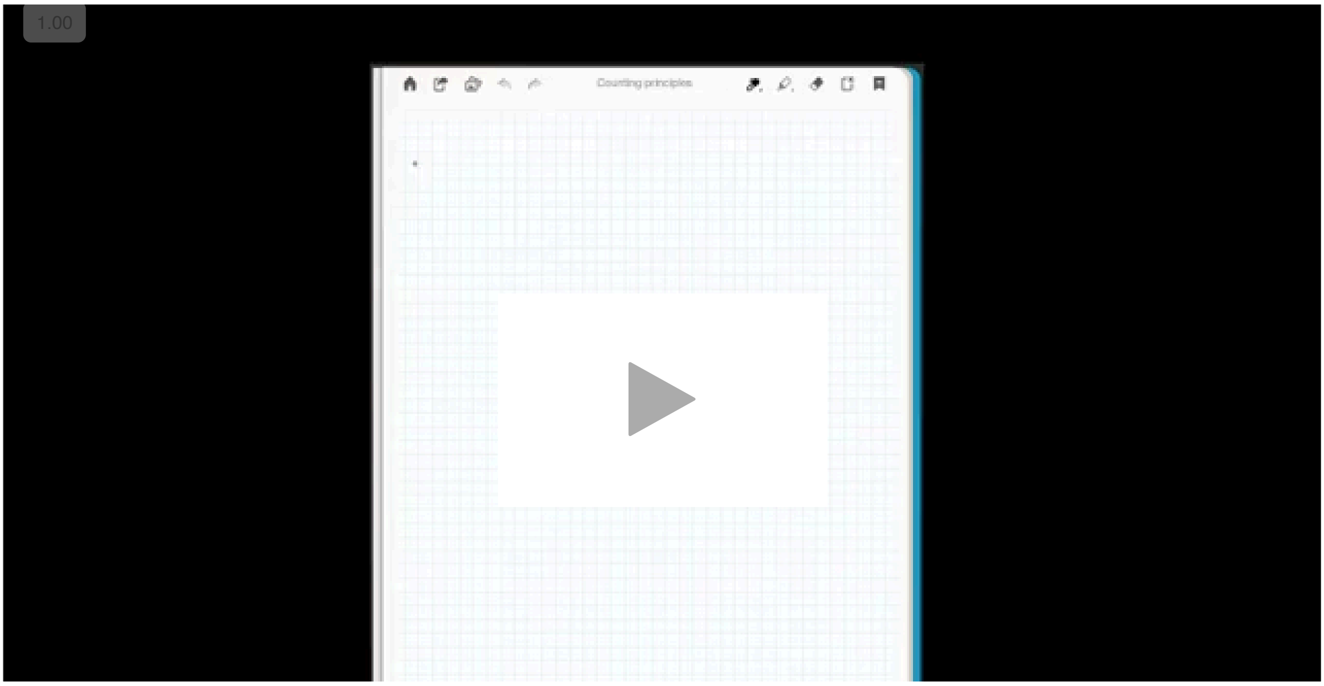

Video 1. Example for Combinations.

More information for video 1

1

00:00:02,239 --> 00:00:05,839

narrator: In this video, we're going

to look at counting principles,

2

00:00:06,373 --> 00:00:07,808

but a second version

of it, which is combinations.

3

00:00:07,875 --> 00:00:10,677

Now, combinations has all to do

4

00:00:10,677 --> 00:00:14,982

with selecting objects from,

let's say, a bag.

5

00:00:15,048 --> 00:00:18,719

So for example, the number of ways

to select four books

6

00:00:18,785 --> 00:00:19,720

from a set of seven

7

00:00:19,720 --> 00:00:21,855

would be a combination question.

8

00:00:21,922 --> 00:00:23,190

So let's have a look.

9

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00:00:23,323 --> 00:00:25,759

The first book that we're gonna choose,

well, there's seven to choose from.

10

00:00:25,826 --> 00:00:27,160

So I've got seven choices.

11

00:00:27,694 --> 00:00:30,264

Second book, of course, I've

got only six choices left

12

00:00:30,264 --> 00:00:35,702

because I'm not gonna return a book.

13

00:00:35,702 --> 00:00:37,905

Third book, I've got five choices left.

14

00:00:37,905 --> 00:00:43,443

And finally, the fourth book, I'm going

to have a choice out of four books.

15

00:00:43,443 --> 00:00:45,612

Now this we've already encountered.

16

00:00:45,612 --> 00:01:01,628

This was permutations,

which was $P(7, 4) = \frac{7!}{(7-4)!}$.

However, another selection

would be B, A, C, D,

and yet another one would be B, C, A, D,

and et cetera, et cetera.

17

00:01:01,895 --> 00:01:05,465

Now in combinations,

the order is not important

18

00:01:05,532 --> 00:01:12,406

and therefore all these ways

of ordering those four books,

which of course is $4!$,

19

00:01:12,472 --> 00:01:15,175

are identical and

therefore we need to take them out,

20

00:01:15,242 --> 00:01:16,944

if we looking at combinations.

21

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00:01:17,177 --> 00:01:21,515

And taking 'em out means we are going

to divide by the number.

22

00:01:21,615 --> 00:01:32,757

So the number of ways of selecting 4 books

out of 7 is $\frac{7!}{(7-4)!\cdot 4!}$.

23

00:01:30,924 --> 00:01:38,799

In other words, selecting n objects

r at a time is nCr

24

00:01:38,866 --> 00:01:45,739

or the binomial $\binom{n}{r}$ coefficient,

which is $\frac{n!}{r!\cdot(n-r)!}$.

25

00:01:45,806 --> 00:01:49,743

So let's have a look.

This $n = 7$ was n, $r = 4$ was r.

26

00:01:49,810 --> 00:01:54,748

And those you see appearing

in the equation of those calculations,

27

00:01:54,882 --> 00:01:58,452

and that is combinations

selecting objects.

Combinations are all the different selections that can be made of r objects out of n objects if order is not important.

Important

The number of ways to select r objects out of n distinct objects, where order is **not** important is:

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Exam tip

The combinations formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is given in the IB formula booklet along with the permutations formula.

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Making connections

In subtopic 1.9 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27687/) you used combination numbers, $\binom{n}{r}$, to find the binomial coefficients in the binomial expansion. You also learned how to evaluate $\binom{n}{r}$ by using Pascal's triangle, factorials and your GDC.

Example 1

★★☆

From a hockey team of 15 players, a committee of 5 players to represent the team is to be chosen. It must include the captain . Find the number of ways that the committee can be selected.

Steps	Explanation
Picking the captain: $$\binom{1}{1} = 1$$	There is one way to pick the captain.
Picking the remaining 4 players: $$\binom{14}{4} = 1001$$	There are 14 players left to fill 4 spots once the captain is chosen.
Total number of selections is: $$1 \times 1001 = 1001$$	

Example 2

★★☆

There are eight juniors and five seniors on a volleyball team.

Find the number of ways that the coach can chose a team of six players if it needs to have

a) Three players from each grade

b) At least two players from each grade.

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	Steps	Explanation
a)	Number of possible selections is: $$\binom{8}{3}\times\binom{5}{3}=560$$	You must pick 3 out of 8 juniors. $\binom{8}{3}$ And you need to pick 3 out of 5 seniors. $\binom{5}{3}$
b)	**Method 1**	
	$\binom{8}{2}\times\binom{5}{4}+\binom{8}{3}\times\binom{5}{3}+\binom{8}{4}\times\binom{5}{2}=1400$	The possible teams are: 2 juniors and 4 seniors, 3 juniors and 3 seniors, or 4 juniors and 2 seniors
	Method 2	
	$\binom{13}{6}-\binom{8}{1}\times\binom{5}{5}-\binom{8}{5}\times\binom{5}{1}-\binom{8}{6}\times\binom{5}{0}=1400$	You can find the total number of ways to make a team of 6 with no restrictions and then subtract the number of ways to make teams you do not want. The teams you don't want are: 1 junior and 5 seniors, 0 juniors and 6 seniors (but this is not possible because there only 5 seniors), 5 juniors and 1 senior, or 6 juniors and 0 seniors.

Example 3

★★☆

There are ten distinct labelled points on a circle.

a) Find the number of straight lines that can be drawn by connecting two points on the circle.

b) One of the points is labelled A. Find the number of straight lines that can be drawn which pass through point A and another point on the circle.

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	Steps	Explanation
a)	Total number of lines is: $\binom{10}{2} = 45$	Each line is formed by connecting two of the 10 points. The order in which the points are connected does not matter (for example, line AB is same as line BA). The question is really asking how many ways there are to pick two out of the 10 points.
b)	**Method 1**	
	Total number of lines is 9.	Point A has 9 other points that it can be connected to, so 9 lines can be drawn that pass through A.
	Method 2	
	Total number of lines is: $\binom{1}{1} \times \binom{9}{1} = 9$	Point A must be selected. There are $\binom{1}{1}$ ways to do that. One other point out of the remaining 9 must be selected. There are $\binom{9}{1}$ ways to do that.

International Mindedness

When people fill a lottery ticket, they need to choose certain numbers out of some given numbers. For example, they may be given 60 numbers and asked to pick six of these. Someone wins the grand prize if they picked the same numbers that appear on the draw. The order does not matter. They win if they pick the one winning combination out of the many possible. What does this mean for the chances of winning such a lottery?

There is a wide range of views on the moral and social implications of running a lottery and playing in one. Lotteries are banned in some countries, while others run national or state lotteries, with the proceeds being donated to charities. Bearing in mind the minute chances of winning a lottery, what do you think about the ethics of selling lottery tickets to people who may not understand the implications of the quantities and chances involved?

3 section questions

1. Number and algebra / 1.10 Counting principles and the generalised binomial theorem

Generalisation of the binomial theorem

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/generalisation-of-the-binomial-theorem-id-26977/print/) Assign

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Extension of the binomial theorem to negative and rational exponents

The binomial expansion that you learned about in subtopic 1.9 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27687/) was only for binomial expressions of the form $(a+b)^n$ with positive integer exponents.

Making connections

The binomial theorem says that the expansion of $(a+b)^n$ is given by:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b^1 + \cdots + \binom{n}{r} a^{n-r}b^r + \cdots + b^n$$

$$= \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r$$

Sir Isaac Newton, who developed the binomial theorem in the 17th century, also generalised it to exponents other than positive integer numbers.

The following activities and examples will show you how the binomial theorem can be extended to rational exponents, both positive and negative.

In the first activity you will need what you learned in subtopic 1.8 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27682/) about the sum of an infinite geometric sequence.

Making connections

An infinite geometric series is convergent (has a finite sum) if $|r| < 1$. The sum of a convergent infinite geometric sequence is:

$$S_\infty = \frac{u_1}{1-r}.$$

Activity

To understand some properties of the binomial expansion when $n = -1$, determine the range of values for which each of the following infinite sums is convergent.

Rewrite each infinite sum using the formula $S_\infty = \dfrac{u_1}{1-r}$.

$$1 + x + x^2 + x^3 + x^4 + \dots$$

$$1 + 2x + 4x^2 + 8x^3 + 16x^4 + \dots$$

$$1 - 3x + 9x^2 - 27x^3 + 81x^4 + \dots$$

$$1 + bx + b^2x^2 + b^3x^3 + b^4x^4 + \dots$$

Explain how your results can be used to write an expansion for:

$$(1-x)^{-1}$$

$$(1-2x)^{-1}$$

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$(1+3x)^{-1}$

$(1-bx)^{-1}$

✓ **Important**

The binomial expansion for $(1-bx)^{-1}$ is

$$(1-bx)^{-1} = 1 + bx + b^2\ x^2 + b^3x^3 + \cdots$$

where the interval of convergence is $-\frac{1}{|b|} < x < \frac{1}{|b|}$ or equivalently $|x| < \frac{1}{|b|}$.

The interval of convergence is the set of values of x for which the infinite sum converges. So the results you obtained from the activity are valid only for x values in the interval $-\frac{1}{|b|} < x < \frac{1}{|b|}$. Otherwise, the formula $S_\infty = \frac{u_1}{1-r}$ cannot be applied to the infinite sum.

Example 1

★☆☆

a) Write the first four terms in the expansion of $(1-4x)^{-1}$.

b) State the interval of convergence for the complete expansion.

	Steps	Explanation
a)	$(1-4x)^{-1} = 1 + 4x + 4^2x^2 + 4^3x^3 + \ldots$ $= 1 + 4x + 16x^2 + 64x^3 + \ldots$	Use $(1-bx)^{-1} = 1 + bx + b^2\ x^2 + b^3x^3 + \ldots$ with $b = 4$.
b)	The interval of convergence is $-\frac{1}{4} < x < \frac{1}{4}$	The interval of convergence is $-\frac{1}{b} < x < \frac{1}{b}$ where $b = 4$.

Having obtained a binomial expansion with negative exponent $n = -1$, the results can be extended to other values of n by rewriting the binomial coefficient in the binomial theorem. You can see how this is done in **Example 2**.

Example 2

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Show that $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r}\ a^{n-r}b^r$is equivalent to

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$$(a+b)^n = \sum_{r=0}^{n} \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} a^{n-r} b^r.$$

Steps	Explanation
$\binom{n}{r} = \frac{n!}{r!(n-r)!}$ $= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots(2)(1)}{r!(n-r)(n-r-1)\cdots(2)(1)}$ $= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$	Using the definition of the factorial, you can see that $(n-r)!$ cancels from the numerator and denominator.
So, $\sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r = \sum_{r=0}^{n} \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} a^{n-r} b^r$ and the two statements are equivalent.	

Using the expression $\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$ for the binomial coefficients, we have the following generalised binomial theorem.

✓ **Important**

The generalised binomial expansion formula for $n \in \mathbb{Q}$ is:

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \dots$$
$$+ \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}a^{n-r}b^r + \dots$$

You can check that this formula gives the previous result $(1-bx)^{-1} = 1 + bx + b^2 x^2 + b^3x^3 + \cdots$ when you take $a = 1$ and $n = -1$ and replace 'b' with '$(-bx)$'.

The generalised binomial expansion is more useful in the following form that contains the variable x (called a 'power series'):

$$(a+bx)^n = a^n + \frac{n}{1!}a^{n-1}(bx)^1 + \frac{n(n-1)}{2!}a^{n-2}(bx)^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}(bx)^3 + \dots$$
$$+ \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}a^{n-r}(bx)^r + \dots$$

Example 3

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Write out the first four terms in the expansion of $\left(1+\frac{x}{2}\right)^{-\frac{1}{2}}$.

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$$\left(1+\frac{x}{2}\right)^{-\frac{1}{2}} = 1^{-\frac{1}{2}} + \frac{-\frac{1}{2}}{1!}(1)^{-\frac{3}{2}}\left(\frac{x}{2}\right)^{1} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(1)^{-\frac{5}{2}}\left(\frac{x}{2}\right)^{2}$$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(1)^{-\frac{7}{2}}\left(\frac{x}{2}\right)^{3} + \ldots$$

$$= 1 - \frac{1}{4}x + \frac{3}{32}x^2 - \frac{5}{128}x^3 + \ldots$$

Use the power series form of the binomial expansion with

$$a = 1, b = \frac{1}{2}$$

and

$$n = -\frac{1}{2}.$$

What do you think is the benefit to rewriting the binomial coefficients without the factorials $n!$ and $(n-r)!$?

As demonstrated in **Example 3**, it is much easier to use the binomial expansion when $a = 1$, because all the powers of a will be 1.

For general binomial expansions, a simple algebraic manipulation

$$(a+bx)^n = \left(a\left(1+\frac{bx}{a}\right)\right)^n = a^n\left(1+\frac{bx}{a}\right)^n$$

leads to

$$(a+bx)^n = a^n\left(1+\frac{bx}{a}\right)^n$$

$$= a^n\left(1 + \frac{n}{1!}\left(\frac{bx}{a}\right)^1 + \frac{n(n-1)}{2!}\left(\frac{bx}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{bx}{a}\right)^3 + \ldots\right.$$

$$\left. + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\left(\frac{bx}{a}\right)^r + \ldots\right)$$

Activity

By using

$$(a+bx)^n = a^n\left(1+\frac{bx}{a}\right)^n$$

$$= a^n\left(1 + \frac{n}{1!}\left(\frac{bx}{a}\right)^1 + \frac{n(n-1)}{2!}\left(\frac{bx}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{bx}{a}\right)^3 + \ldots\right.$$

$$\left. + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\left(\frac{bx}{a}\right)^r + \ldots\right)$$

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write the first six terms in the expansion of:

$(3+2x)^4$

$(3+2x)^{-4}$

Describe what you observe about the last few terms in each expansion. Explain the significance of your findings to the expansion of $(a+bx)^n$ when n is not a positive integer

In the activity you observed that all binomial expansions form a geometric sum. When $n \in \mathbb{N}$, all terms after the $(n+1)$ th term are zero. When n is not a positive integer, the sum is infinite and may or may not converge. If the infinite sum involves powers of x, it may be convergent for only certain values of x, so you need to think about the interval of convergence.

✓ **Important**

The generalised binomial expansion can be written in the form

$$(a+bx)^n = a^n\left(1+\frac{bx}{a}\right)^n$$

$$= a^n\left(1+\frac{n}{1!}\left(\frac{bx}{a}\right)^1 + \frac{n(n-1)}{2!}\left(\frac{bx}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{bx}{a}\right)^3 + \ldots\right.$$

$$\left. + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\left(\frac{bx}{a}\right)^r + \ldots\right)$$

where $n \in \mathbb{Q}$.

The expansion converges for $-\left|\frac{a}{b}\right| < x < \left|\frac{a}{b}\right|$., which is called the interval of convergence.

The derivation of the interval of convergence for $(a+bx)^n$ is beyond the scope of this course, but it follows a similar logic to the derivation of the interval of convergence for $(1-bx)^{-1}$ which you found earlier.

Think about why it is necessary to include the absolute values for $\frac{a}{b}$ in the interval of convergence.

ⓘ **Exam tip**

The IB formula booklet gives the extension of the binomial theorem for $n \in \mathbb{Q}$ in the following form:

$$(a+b)^n = a^n\left(1+n\left(\frac{b}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+\ldots\right)$$

Example 4

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Expand $\frac{4}{1-3x}$ for $|x| < \frac{1}{3}$ up to and including the x^3 term.

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$$\frac{4}{1-3x} = 4(1-3x)^{-1}$$

Note that the interval of convergence is already given as $|x| < \frac{1}{3}$.

Method 1

$$(1-3x)^{-1} = 1 + (3x) + (3x)^2 + (3x)^3 + \ldots$$

$$= 1 + 3x + 9x^2 + 27x^3 + \ldots$$

Use

$$(1-bx)^{-1} = 1 + bx + b^2\, x^2 + b^3x^3 + \ldots$$

Method 2

$$(1-3x)^{-1} = 1 + \frac{-1}{1!}(-3x) + \frac{(-1)(-2)}{2!}(-3x)^2 + \frac{(-1)(-2)(-3)}{3!}(-3x)^3 + \ldots$$

$$= 1 + 3x + 9x^2 + 27x^3 + \ldots$$

Use

$$(a+bx)^n = a^n\left(1 + \frac{bx}{a}\right)^n$$

$$= a^n\left(1 + \frac{n}{1!}\left(\frac{bx}{a}\right)^1 + \frac{n(n-1)}{2!}\left(\frac{bx}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{bx}{a}\right)^3 + \ldots + \frac{n(n-1)(n-2)}{}\right.$$

$$\frac{4}{1-3x} = 4\left(1 + 3x + 9x^2 + 27x^3 + \ldots\right)$$

$$= 4 + 12x + 36x^2 + 108x^3 + \cdots$$

Example 5

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a) State the interval of convergence for the expansion of $\frac{1}{\sqrt{3+2x}}$.

b) Write the first four terms of the expansion.

a) The interval of convergence is:

$$-\frac{3}{2} < x < \frac{3}{2}$$

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$$\frac{1}{\sqrt{3+2x}} = (3+2x)^{-\frac{1}{2}}$$

$$= 3^{-\frac{1}{2}}\left(1+\frac{2}{3}x\right)^{-\frac{1}{2}}$$

$$= a^n\left(1+\frac{bx}{a}\right)^n$$

b)

$$\frac{1}{\sqrt{3+2x}} = 3^{-\frac{1}{2}}\left(1+\frac{2}{3}x\right)^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{3}}\left(1+\frac{\left(-\frac{1}{2}\right)}{1!}\left(\frac{2}{3}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{2}{3}x\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{2}{3}x\right)^3 -\right.$$

$$= \frac{1}{\sqrt{3}}\left(1-\frac{1}{3}x+\frac{1}{6}x^2-\frac{5}{54}x^3+\ldots\right) = \frac{1}{\sqrt{3}}-\frac{1}{3\sqrt{3}}x+\frac{1}{6\sqrt{3}}x^2-\frac{5}{54\sqrt{3}}x^3+\ldots$$

Use

$$(a+bx)^n = a^n\left(1+\frac{bx}{a}\right)^n$$

$$= a^n\left(1+\frac{n}{1!}\left(\frac{bx}{a}\right)^1+\frac{n(n-1)}{2!}\left(\frac{bx}{a}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{bx}{a}\right)^3+\ldots+\frac{n(n-}{}\right.$$

Example 6

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Given that $\frac{x^2}{(1+cx)^2} = x^2 - 2cx^3 + 6x^4 + \ldots$, find the value of c.

Steps	Explanation
$\frac{x^2}{(1+cx)^2} = x^2(1+cx)^{-2}$	The right-hand side of the given equation looks like the result of a binomial expansion.
$(1+cx)^{-2} = 1+\frac{(-2)}{1!}(cx)+\frac{(-2)(-3)}{2!}(cx)^2+\ldots$ $= 1-2cx+3c^2x^2+\ldots$	We can stop the expansion at the x^2 term because the given expansion only goes up to x^4 and $(1+cx)^{-2}$ is multiplied by x^2.
$x^2(1+cx)^{-2} = x^2\left(1-2cx+3c^2x^2+\ldots\right)$ $= x^2-2cx^3+3c^2x^4+\ldots$	

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Steps	Explanation
$x^2 - 2cx^3 + 3c^2x^4 + \ldots = x^2 - 2cx^3 + 6x^4 + \ldots$ $3c^2x^4 = 6x^4 \quad \Leftrightarrow \quad c^2 = 2$ So $c = \pm\sqrt{2}$.	

Making connections

The infinite sums generated in the binomial expansions for $n \in \mathbb{Q}$ are closely connected to the Maclaurin series that you will study in subtopic 5.19 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27009/).

4 section questions

1. Number and algebra / 1.10 Counting principles and the generalised binomial theorem

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-26978/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- apply the addition and multiplication principles to solve counting problems
- use the permutations formula to find the number of ways to arrange (where order is important) k distinct objects out of a set of n distinct objects, where $k \leq n$
- use the combinations formula to find the number of ways to select (where order is not important) k distinct objects out of a set of n distinct objects, where $k \leq n$
- write out a specified number of terms for the binomial expansion of $(a + bx)^n$ where $n \in \mathbb{Q}$
- determine the interval of convergence for the binomial expansion of $(a + bx)^n$ where $n \in \mathbb{Q}$.

1. Number and algebra / 1.10 Counting principles and the generalised binomial theorem

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-26979/print/) Assign

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Interactive 1. Exploring how the number of permutations and number of combinations change as the k value is increased from 0 to n.

Credit: GeoGebra (https://www.geogebra.org/m/y7hj9m8z) Heather Pierce

More information for interactive 1

An interactive visualizes how the number of permutations and combinations change based on different values of n and k. Users can manipulate the sliders to adjust the values of n and k and observe the mathematical relationships dynamically.

Permutations, $P_{n,\ k}$ in the left side of the interactive demonstrates how selecting k items from n matters when order is important. The formula of permutation is $P_{n,\ k} = \frac{n!}{(n-k)!}$

Combinations, $C_{n,\ k}$ in the right side shows the case where order does not matter, the formula of combination is $C_{n,\ k} = \frac{n!}{k!(n-k)!}$
The surrounding text in the interactive allows users to explore patterns, by increasing k and generalizing their observations across all integer values of n. The value of k can not exceed the value of n.

Read below for the solutions,

Permutations (Order Matters)

Let's calculate $P_{10,\ 2}$, which represents the number of ways to arrange 2 objects selected from a set of 10, where order matters.

Substituting values $n = 10$ and $k = 2$ in the formula of permutations $P_{n,\ k} = \frac{n!}{(n-k)!}$

$$P_{10,\ 2} = \frac{10!}{(10-2)!}$$
$$P_{10,\ 2} = \frac{10!}{8!}$$
$$P_{10,\ 2} = \frac{10\times 9\times 8!}{8!} = 10\times 9$$
$$P_{10,\ 2} = 90$$

Combinations (Order Doesn't Matter)

Let's calculate $C_{n,\ k}$ which represents the number of ways to choose 2 objects from a set of 10, where order does not matter.

Substituting values $n = 10$ and $k = 2$ in the formula of combination $C_{n,\ k} = \frac{n!}{(k!)(n-k)!}$

$$C_{10,\ 2} = \frac{10!}{2!(10-2)!}$$
$$C_{10,\ 2} = \frac{10\times 9\times 8!}{2!\times 8!}$$

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$C10, 2 = \frac{10\times 9}{2\times 1}$

$C10, 2 = 45.$

Part 1

Use the applet above to see how the number of permutations and number of combinations change as the k value is increased from 0 to n.

Determine if your observations hold true for all values of n in the applet. Do you think your observations extend to $n \in \mathbb{Z}$?

Describe how your findings could be relevant to real-life situations.

Part 2

In the Investigation of subtopic 1.9 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27687/) you explored how the binomial expansion of $(a+b)^n$ for $n \in \mathbb{Z}$ could be used to find the exact values of powers such as 3.02^4.

Now, use a binomial expansion for $n \in \mathbb{Q}$ to **approximate** the value of $\sqrt[3]{1.03}$. Compare your result to the value obtained from a calculator.

If you want to approximate $\sqrt[3]{1.03}$ correct to 4 decimal places, how many terms of the expansion are needed? How can you determine the necessary number of terms algebraically?

What happens when you try to use a binomial expansion to approximate $\sqrt{1+1}$? Why is this the case?

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