TOPIC 1
NUMBER AND ALGEBRA

SUBTOPIC 1.15
EIGENVALUES AND EIGENVECTORS

Index

The big picture

Eigenvaluesand eigenvectorsare usually introduced to students in the context of linear algebra and matrices. However, their applications originated in work with differential equations and analysis of rotational motion of rigid bodies.

International Mindedness

Eigenvalues and eigenvectors were named by a German mathematician, David Hilbert. The prefix *eigen* can be translated as *own* or *characteristic* and is used to signify the special nature of these values and vectors in relation to a matrix. Sometimes eigenvalues and eigenvectors are referred to as characteristic values and may be called that in other languages such as in Spanish where they are called *autovalores* and *autovectores* .

Currently, eigenvalues and eigenvectors have a wide range of applications, from analysis of stock market trends, to reducing car braking noise, to bridge design.

One of the most profitable uses of eigenvalues and eigenvectors is in Google's PageRank algorithm which allows the search engine to list the most relevant results first.In this subtopic you will learn how eigenvalues and eigenvectors are used to find powers of matrices. This application greatly reduces the amount of work that you must do in an example such as $\begin{pmatrix} a & 3b \\ 7 & 2 \end{pmatrix}^{10}$.

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Stock market trends

Credit: Tetra Images Getty Images

Concept

In this section you will learn how to diagonalise a matrix by using its eigenvalues and eigenvectors. Rewriting a matrix in an equivalent diagonalised form allows you to easily find powers of this matrix. How do patterns in matrix multiplication make it is easier to find powers of diagonalised matrices? Why can eigenvalues and eigenvectors be used to write a matrix in diagonalised form? How are the eigenvalues and eigenvectors of a matrix related to the eigenvalues and vectors of that matrix raised to a power?

1. Number and algebra / 1.15 Eigenvalues and eigenvectors

Eigenvalues and eigenvectors

Eigenvalues of a matrix are scalar quantities and are labelled with the letter λ. A 2×2 matrix will have two eigenvalues labelled λ_1 and λ_2. Eigenvectors are written in column matrix form. For a 2×2 matrix they are shown as $x_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ and $x_2 = \begin{pmatrix} c \\ d \end{pmatrix}$.

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Exam tip

You will only be asked for eigenvalues and eigenvectors of 2×2 matrices in the exam.

Eigenvectors of matrix A have a special relationship with that matrix. You can explore this relationship in the activity below.

Activity

Let $A = \begin{pmatrix} 5 & 1 \\ 4 & 5 \end{pmatrix}$. The eigenvalues of A are $\lambda_1 = 7$ and $\lambda_2 = 3$. The eigenvectors of A are $x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $x_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Find Ax_1 and Ax_2 and comment on your results.

Consider the following column matrices:

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -3 \\ -6 \end{pmatrix}, \begin{pmatrix} -10 \\ 20 \end{pmatrix}, \begin{pmatrix} -0.5 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}.$$

How are these matrices related to x_1 and x_2?

What patterns do you observe when you multiply A by these new column matrices?

Important

If $Ax = \lambda x$, then x is an eigenvector of A and λ is an eigenvalue. Any scalar multiple of an eigenvector of A is also an eigenvector of A.

To find the eigenvalues and their eigenvectors you can rearrange $Ax = \lambda x$ as follows:

$$\begin{aligned} Ax - \lambda x &= \lambda x - \lambda x \\ Ax - \lambda x &= O \\ (A - \lambda I)x &= O. \end{aligned}$$

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Solutions to this equation can be found if:

- $\det(A-\lambda I) \neq 0$; then the solution is $(A-\lambda I)(A-\lambda I)^{-1}x = O(A-\lambda I)^{-1} \Leftrightarrow x = O$. This is a solution, but not an interesting one.
- $\det(A-\lambda I) = 0$; this allows for other solutions. This is the case that will yield the eigenvalues and eigenvectors that you are looking for.

Example 1

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Find the eigenvalues of $\begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix}$.

Steps	Explanation
$A-\lambda I = \begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-\lambda & 5 \\ 3 & 1-\lambda \end{pmatrix}$ $\det(A-\lambda I) = (3-\lambda)(1-\lambda) - 15 = 0$	You want to consider case where $\det(A-\lambda I) = 0$.
$(3-\lambda)(1-\lambda) - 15 = \lambda^2 - 4\lambda - 12 = 0$ $\lambda_1 = -2,\ \lambda_2 = 6$	Solve the quadratic equation for λ.

✓ **Important**

The characteristic polynomial of matrix A is $P(\lambda) = \det(A-\lambda I)$.

The eigenvalues are the solutions to $P(\lambda) = \det(A-\lambda I) = 0$.

Example 2

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Find the eigenvalues of $\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$.

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Steps	Explanatic
$A - \lambda I = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix}$ $\det(A - \lambda I) = (3-\lambda)(-1-\lambda) + 8 = 0$	You want to cons case where $\det(A - \lambda I) =$
$(3-\lambda)(-1-\lambda) + 8 = \lambda^2 - 2\lambda + 5 = 0$ $\lambda_1 = 1 - 2i,\ \lambda_2 = 1 + 2i$	Solve the quadra equation for λ.

✓ **Important**

For a 2×2 matrix there may be:

- two distinct real eigenvalues
- one repeated real eigenvalue
- two complex eigenvalues.

Once you know the eigenvalues you can use $(A - \lambda I)x = O$ to find the eigenvectors.

Example 3

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a) Find one eigenvector for each eigenvalue of $A = \begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix}$.

b) Show that $Ax = \lambda x$ is true for the eigenvalues and eigenvectors that you found in part **a**.

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	Steps	Explanation
a)	$\lambda_1 = -2,\ \lambda_2 = 6$	You already found the eigenvalues in **Exampl** for this matrix.
	$A - \lambda I = \begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-\lambda & 5 \\ 3 & 1-\lambda \end{pmatrix}$ Let $x = \begin{pmatrix} a \\ b \end{pmatrix}$: $$(A - \lambda I)\, x = O$$ $$\begin{pmatrix} 3-\lambda & 5 \\ 3 & 1-\lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$	You can let $x = \begin{pmatrix} a \\ b \end{pmatrix}$ c $\begin{pmatrix} x \\ y \end{pmatrix}$ or you can use ar other variables that ha not already been used the question.
	For $\lambda_1 = -2$: $$\begin{pmatrix} 3-(-2) & 5 \\ 3 & 1-(-2) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow$$ $$5a + 5b = 0$$ $$3a + 3b = 0$$ $$a = -b$$ Let $b = t$ where $t \neq 0$. $$a = -t$$ $$x_1 = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$ For $t = 1$: $$x_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$	Find x_1 for $\lambda_1 = -2$. Note that both equatio should give you the sar relationship between a b. Remember that any sc multiple of x_1 is also ar eigenvector of matrix A When you write $x_1 = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, you are giving the eigenvector general form, where t i scalar.

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	Steps	Explanation
	For $\lambda_2 = 6$: $$\begin{pmatrix} 3-(6) & 5 \\ 3 & 1-(6) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$$ $$-3a+5b=0$$ $$3a-5b=0$$ $$a=\frac{5}{3}b$$ Let $b = 3t \text{ where } t \neq 0.$ $$a = 5t$$ $$x_2 = t\begin{pmatrix} 5 \\ 3 \end{pmatrix}$$ For $t = 1$: $$x_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$	Note that both equatio should give you the sar relationship between a b.
b)	For $\lambda_1 = -2$ and $x_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$: $$Ax = \begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = -2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \lambda x$$	
	For $\lambda_2 = 6$ and $x_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$: $$Ax = \begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 30 \\ 18 \end{pmatrix} = 6\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \lambda x$$	

Example 4

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Find two eigenvectors for each value of the eigenvalues of $\begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix}$.

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Steps	Explanation
$\det(A-\lambda I)=(4-\lambda)(6-\lambda)-3=\lambda^2-10\lambda+21=0$ $\lambda_1=3,\ \lambda_2=7$	Find the eigenvalues.
For $\lambda_1=3$: Let $x=\begin{pmatrix}x\\y\end{pmatrix}$: $$(A-\lambda I)\,x=O$$ $$\begin{pmatrix}4-(3) & 1\\3 & 6-(3)\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$ $$x+y=0$$ $$3x+3y=0$$ $$x=-y$$ Let $y=t$, $t\neq 0$: $x=-t$ $x_1=t\begin{pmatrix}-1\\1\end{pmatrix}$ Let $t=1$: $x_1=\begin{pmatrix}-1\\1\end{pmatrix}$ Let $t=2$: $x_1=\begin{pmatrix}-2\\2\end{pmatrix}$ Two eigenvectors for $\lambda_1=3$ are $\begin{pmatrix}-1\\1\end{pmatrix}$ and $\begin{pmatrix}-2\\2\end{pmatrix}$.	Note that both equatioı should give you the sar relationship between x y. You can chose other va of t and get other acceptable answers for eigenvectors for λ_1.

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Steps	Explanation
For $\lambda_2 = 7$: Let $x = \begin{pmatrix} x \\ y \end{pmatrix}$ $$(A - \lambda I)\, x = O$$ $$\begin{pmatrix} 4-(7) & 1 \\ 3 & 6-(7) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ $$-3x + y = 0$$ $$3x - y = 0$$ $$x = \frac{1}{3}y$$ Let $y = 3t$, $t \neq 0$: $$x = t$$ $$x_2 = t\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$ Let $t = 1$: $$x_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$ Let $t = 2$: $$x_2 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$ Two eigenvectors for $\lambda_2 = 7$ are $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$.	Note that both equatio should give you the sar relationship between x y.

4 section questions

1. Number and algebra / 1.15 Eigenvalues and eigenvectors

Powers of matrices

Student view

Activity

Let $A = \begin{pmatrix} 2 & 7 \\ -3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 & -1 \\ 0 & 9 & 7 \end{pmatrix}$, $C = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and $E = \begin{pmatrix} 2 & 0 & 7 \\ -3 & 0 & 1 \\ 2 & -4 & 5 \end{pmatrix}$.

Whenever possible, find the result of each matrix raised to the second power, third power, and fourth power. If this is not possible, explain why.

Describe the characteristics of matrix F such that F^n is defined.

Activity

Let $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Find $A^2,\ A^3,\ A^4,$ and A^n for $n \in \mathbb{Z}^+$.

Important

For a diagonal matrix $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$.

Due to the nature of matrix multiplication, only a square matrix can be raised to a power. Calculating powers of square matrices by hand is a time consuming and tedious process unless you are working with a diagonal matrix in the form $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

In this section you will learn how the properties of eigenvalues and eigenvectors allow you to diagonalise a 2×2 matrix and simplify the process of raising the matrix to a power.

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Example 1

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Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ and let $x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ be two eigenvectors of A.

a) Find the corresponding eigenvalue for each eigenvector.

b) Let $P = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Show that $AP = PD$.

c) Explain why $AP = PD$ for any 2×2 matrix A when the columns of matrix P are x_1 and x_2 and matrix $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

	Steps	Explanation
a)	For x_1: $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda_1 = 3$ For x_2: $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} = -2\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ $\lambda_2 = -2$	Use $Ax = \lambda x$. You can also use det $(A -$ to find the eigenvalues.
b)	$AP = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 3 & 6 \end{pmatrix}$ $PD = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 3 & 6 \end{pmatrix}$ $AP = PD$	

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	Steps	Explanation
c)	Since $$P = (x_1 \quad x_2)$$ $$AP = (Ax_1 \quad Ax_2) = (\lambda_1 x_1 \quad \lambda_2 x_2).$$ Then, $$PD = (x_1 \quad x_2)\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} = (\lambda_1 x_1 \quad \lambda_2 x_2).$$ So $$AP = PD \text{ for any } 2\times 2 \text{ matrix } A.$$	In this case, $P = (x_1 \quad x_2$ 2×2 matrix where x_1 and eigenvectors in the form x $Ax_1 = \lambda_1 x_1$ and $Ax_2 =$ $\Big($ $Using\ the\ defintio$ $eigenvalues\ and\ eigen$

You can use the fact that $AP = PD$ for any 2×2 matrix A when the columns of matrix P are x_1 andx_2 and matrix $D = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ to find powers of A.

Example 2

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Let $A = \begin{pmatrix}2 & -12\\ -3 & 2\end{pmatrix}$.

a) Show that $A = PDP^{-1}$ for matrix P where the columns are x_1 and x_2 and matrix $D = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$.

b) Hence, write an equation for A^5 in terms of P, P^{-1} and D and use it to find A^5.

Student
view

	Steps	Explanatio
a)	$AP = PD \quad \Leftrightarrow \quad APP^{-1} = PDP^{-1} \quad \Leftrightarrow$ $AI = PDP^{-1} \quad \Leftrightarrow \quad A = PDP^{-1}$	If the columns of P are x_1 and x_2 a matrix $D = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}$ then $AP = PD$.
b)	$\begin{aligned} A^5 &= \left(PDP^{-1}\right)^5 \\ &= PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1}PDP^{-1} \\ &= PDIDIDIDIDP^{-1} \\ &= PD^5P^{-1} \end{aligned}$	
	$\det(A - \lambda I) = 0$ $(2-\lambda)(2-\lambda) - (-12)(-3) = \lambda^2 - 4\lambda - 32 = 0$ $\lambda_1 = -4,\ \lambda_2 = 8$ For $\lambda_1 = -4$: $(A - \lambda I)x = 0$ $\begin{pmatrix} 2-(-4) & -12 \\ -3 & 2-(-4) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $6x - 12y = 0$ $-3x + 6y = 0$ $x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ For $\lambda_2 = 8$: $\begin{pmatrix} 2-(8) & -12 \\ -3 & 2-(8) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $-6x - 12y = 0$ $-3x - 6y = 0$ $x_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $P = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix},\ D = \begin{pmatrix} -4 & 0 \\ 0 & 8 \end{pmatrix},\ P^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix}$	Find eigenvectors eigenvalues to de and D. You can also use y calculator to find eigenvalues and eigenvectors.

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	Steps	Explanatio
	$A^5 = PD^5P^{-1}$ $= \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} (-4)^5 & 0 \\ 0 & (8)^5 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix}$ $= \begin{pmatrix} 15\,872 & -33\,792 \\ -8448 & 15\,872 \end{pmatrix}$	Using the fact tha $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$,the $A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ You can check yo by finding A^5 on calculator.

✓ **Important**

If $M = PDP^{-1}$, where P is a matrix of eigenvectors and D a diagonal matrix of eigenvalues, then PDP^{-1} is the diagonalisation of matrix M .

$M^n = PD^nP^{-1}$, where P is a matrix of eigenvectors and D a diagonal matrix of eigenvalues.

Exam tip

The power formula for a matrix, $M^n = PD^nP^{-1}$, is given in the IB formula booklet. You should be able to find powers of matrices both using the calculator and the power formula.

Example 3

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Let $A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$. Write A^4 as a multiple of three matrices with no exponents.

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Steps	Explanation
$\lambda_1 = 3,\ x_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\lambda_2 = -7,\ x_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ $P = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix},\ P^{-1} = \begin{pmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{1}{10} & \frac{3}{10} \end{pmatrix},$ $D = \begin{pmatrix} 3 & 0 \\ 0 & -7 \end{pmatrix}$	Find eigenvalues and eigenvectors usir the calculator. Be sure to pair λ_1 with x and λ_2 with x_2.
$A^4 = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 81 & 0 \\ 0 & 2401 \end{pmatrix} \begin{pmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{1}{10} & \frac{3}{10} \end{pmatrix}$	

Writing matrix powers in PDP^{-1} form lends itself to application problems involving population movement.

Example 4

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A town has two schools, School A and School B. At the start of each school year, 10% of students move from School A to School B and 5% move from B to A. At the start of this school year, School A has 280students and School B has 320 students.

a) Let $S_1 = \begin{pmatrix} a \\ b \end{pmatrix}$represent the number of students in each school at the start of the next school year, where a is the number of students in School A and b the number in School B. Show that $S_1 = \begin{pmatrix} 268 \\ 332 \end{pmatrix}$.

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b) Let $T = \begin{pmatrix} 0.90 & 0.05 \\ 0.10 & 0.95 \end{pmatrix}$. Show that S_2, which represents the number of students in each school at the start of the school year two years from now, is $S_2 = T^2 \begin{pmatrix} 280 \\ 320 \end{pmatrix}$.

c) Write an equation for S_n in terms of matrix multiplication of the diagonalised matrix for T.

d) Hence, find the number of students in each school as n approaches infinity. Explain the significance of this result.

	Steps	Expla
a)	The number of students in School A at the start of next year: $280 \times 0.90 + 320 \times 0.05 = 268$ The number of students in School B at the start of next year: $320 \times 0.95 + 280 \times 0.10 = 332$ $S_1 = \begin{pmatrix} 0.90 & 0.05 \\ 0.10 & 0.95 \end{pmatrix}\begin{pmatrix} 280 \\ 320 \end{pmatrix} = \begin{pmatrix} 280 \times 0.90 + 320 \times 0.05 \\ 280 \times 0.10 + 320 \times 0.95 \end{pmatrix} = \begin{pmatrix} 268 \\ 332 \end{pmatrix}$	
b)	$S_2 = TS_1 = TT\begin{pmatrix} 280 \\ 320 \end{pmatrix} = T^2\begin{pmatrix} 280 \\ 320 \end{pmatrix}$.	$S_1 = T$ accord part a.

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	Steps	Expla
c)	$S_n = T^n \begin{pmatrix} 280 \\ 320 \end{pmatrix}$. $T^n = PD^nP^{-1}$ where P is a matrix of eigenvectors and D a diagonal matrix of eigenvalues for matrix T. $D = \begin{pmatrix} 0.85 & 0 \\ 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} -1 & -1 \\ 1 & -2 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$ $S_n = \begin{pmatrix} -1 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0.85^n & 0 \\ 0 & 1^n \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 280 \\ 320 \end{pmatrix}$.	
d)	As $n \to \infty$, $D^n \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. So $S_n \to \begin{pmatrix} -1 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 280 \\ 320 \end{pmatrix} = \begin{pmatrix} 200 \\ 400 \end{pmatrix}$ In the long run the populations of the two schools tend towards 200 and 400 students.	

Important

A matrix that can be used to find the new state of a system is called a transition matrix. In a problem dealing with movement between two populations, A and B, the general format of a transition matrix is

$$T = \begin{pmatrix} \text{\% staying in A} & \text{\% moving to A} \\ \text{\% moving to B} & \text{\% staying in B} \end{pmatrix}.$$

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Exam tip

You do not need to memorise the general form of a transition matrix for movement between two populations. You should be able to derive it using a similar process to part a of **Example 4**.

Making connections

Application question involving transition matrices are closely related to Markov chains which you will study in subtopic 4.19 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27443/).

Example 5

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Two companies provide internet service to a particular area. The Zipternet company gains 12% of the customers of its rival, Fast URL, every year and loses 4% of its customers to Fast URL.

a) Write down a transition matrix, T, that represents the movement of customers between the two companies.

b) Write down matrices P and D such that $T = PDP^{-1}$.

c) Initially the Zipternet company has $45\ 000$ customer and Fast URL has $52\ 000$. Write an expression for the number of customers the Zipternet company will have after n years.

d) Find the number of customers Zipternet can expect to have in the long run.

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	Steps	Explanation
a)	$T = \begin{pmatrix} 0.96 & 0.12 \\ 0.04 & 0.88 \end{pmatrix}$	The general pattern for a transition mat $\begin{pmatrix} \text{\% staying in A} & \text{\% moving t} \\ \text{\% moving to B} & \text{\% staying i} \end{pmatrix}$
b)	$\lambda_1 = 1,\ \lambda_2 = 0.84$ $x_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix},\ x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$	Find the eigenvalues and eigenvectors you need a reminder on how to do this look at the examples in the previous se (/study/app/math-ai-hl/sid-132-cid-761618/book/eigenvalues-and-eigenvec 27437/).
	$P = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix},\ D = \begin{pmatrix} 1 & 0 \\ 0 & 0.84 \end{pmatrix}$	

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	Steps	Explanation
c)	Let C_n be the matrix that represents the number of customers in each company a years. $$C_n = T^n \begin{pmatrix} 45\,000 \\ 52\,000 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & 0.84^n \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 45\,00 \\ 52\,00 \end{pmatrix}$$ $$= \begin{pmatrix} 3 \times 1^n & -0.84^n \\ 1 \times 1^n & 0.84^n \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 45\,000 \\ 52\,000 \end{pmatrix}$$ $$= \begin{pmatrix} 3 \times 1^n & -0.84^n \\ 1 \times 1^n & 0.84^n \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 45\,000 \\ 52\,000 \end{pmatrix}$$ $$= \begin{pmatrix} \frac{3}{4} + \frac{1}{4} \times 0.84^n & \frac{3}{4} - \frac{3}{4} \times 0.84^n \\ \frac{1}{4} - \frac{1}{4} \times 0.84^n & \frac{1}{4} + \frac{3}{4} \times 0.84^n \end{pmatrix} \begin{pmatrix} 45\,000 \\ 52\,000 \end{pmatrix}$$ $$= \begin{pmatrix} \left(\frac{3}{4} + \frac{1}{4} \times 0.84^n\right) \times 45\,000 + \left(\frac{3}{4} - \frac{3}{4} \times 0.84^n\right) \times 52\,00 \\ \left(\frac{1}{4} - \frac{1}{4} \times 0.84^n\right) \times 45\,000 + \left(\frac{1}{4} + \frac{3}{4} \times 0.84^n\right) \times 52\,00 \end{pmatrix}$$ After n years the number of customers at Zipternet is given by: $$\left(\frac{3}{4} + \frac{1}{4} \times 0.84^n\right) \times 45\,000 + \left(\frac{3}{4} - \frac{3}{4} \times 0.84^n\right) \times 52\,000$$	
d)	In the long run $n \to \infty$ and $0.84^n \to 0$ So in the long run Zipternet can expect to have: $$\frac{3}{4} \times 45\,000 + \frac{3}{4} \times 52\,000 = 72\,750 \text{ customers}$$	

International Mindedness

Application questions involving movement between two populations can be used to model population movements between different countries. What assumptions are made in the models used in **Examples 4** and **5** ? How would

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these assumptions affect the validity of the results generated by the transition matrix model when applied to population movement between two countries?

4 section questions

Checklist

What you should know

By the end of this subtopic you should be able to:

- find eigenvalues and eigenvectors for a 2×2 matrix by hand and using a calculator
- determine whether a given vector is the eigenvector of a 2×2 matrix
- write the characteristic polynomial of a 2×2 matrix
- use eigenvalues and eigenvectors to diagonalise a matrix
- find powers of 2×2 matrices using a diagonalisation
- solve population movement questions using matrix powers and diagonalisation.

Investigation

Part 1

Overview
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Let A be a 2×2 matrix with eigenvalues λ_1 and λ_2 and eigenvectors x_1 and x_2 . Investigate the relationship between the eigenvalues and vectors of A and powers of A . Generalise your findings for A^n .

Show that your result for A^n holds true for any square matrix A .

Part 2

Eigenvalues and eigenvectors are used in bridge design to prevent collapse due to oscillation at the natural frequency of the bridge structure. The collapse of the Tacoma Narrows Bridge is a famous example of such collapse. Do some research to find more information about the Tacoma Narrows Bridge and to learn more about the use of eigenvectors and eigenvalues in bridge design.

Part 3

Matrix A is a 2×2 matrix such that $A^n = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1}$ for positive integer values of n. Investigate whether it is it possible to define A^n for rational values of n.

What meaning can $\sqrt{A}$ have? Can rational exponents be used to solve $B^3 = A$?

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