Index

The big picture

Differential equations are very important in science and mathematics, both theoretically and because they have many applications. For example:

- Can you predict the temperature of a hot liquid as it cools over time to room temperature (Newton's law of cooling)?
- Can you predict the age of an artefact by analysing the amount of carbon-14 remaining in the material?
- Can you predict the future population of bacteria (exponential growth)?
- Can you predict the future radioactivity of uranium (exponential decay)?
- Can you predict the future reaction rate of a first-order chemical reaction?

Scientists can predict the exponential decay of uranium using differential equations

Credit: abadonian Getty Images

The answer to all of these questions is yes, by using differential equations.

In general, the solution to a differential equation is not unique, as there may be several functions satisfying the same equation. For example, $y = \sin x + c$ is a solution of $y' = \cos x$, for any real constant c. Sometimes, however, you are interested in finding a specific solution satisfying some additional conditions.

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

You could study an entire course on ordinary differential equations. In college, many of you will do just that. But, for this course you are going to learn about the following topics only.

1. Numerical approximations of first-order differential equations using Euler's method.
2. Exact solutions of separable differential equations using separation of variables.
3. Exact solutions of homogeneous differential equations using substitution.
4. Exact solutions of first-order linear differential equations using an integrating factor.

Concept

Throughout this subtopic, you will be expanding on your understanding of differential and integral calculus to solve problems from science and mathematics questions involving rates of **change**. Just as you solved related rate problems in section 5.14.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/related-rates-of-change-id-26503/), you will now solve more complex problems with more advanced techniques. As you go through this subtopic, think about how you can apply these techniques in science.

Theory of Knowledge

Your goal throughout this course has been to get the 'right' answer. Did it ever occur to you that mathematics may be unique to all the other areas of knowledge in this regard? Is there a right answer in history? Art? Human sciences? Consider the methodology by which areas of knowledge establish validity of knowledge.

Knowledge Question: What is unique about the methodology of mathematics that allows for such certainty?

First-order differential equations

In this section, you will formally define differential equations, learn to classify them by order and linearity, and sketch solutions through the use of slope fields and isoclines.

First, consider this example. The function $y = \mathrm{e}^{x^2}$ is differentiable on the interval $(-\infty, \infty)$. Using the chain rule, you can find $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\mathrm{e}^{x^2}$. If you replace e^{x^2} with y, the derivative becomes $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy$.

But what would you do if you only had the derivative function $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy$ and were asked to solve for the unknown function $y = f(x)$?

Classifying differential equations

As you learned in subtopic 5.3 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25553/), the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ of a function $y = f(x)$ is itself a function $f'(x)$. The equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy$ is an example of a differential equation.

Student view

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Important

A differential equation is an equation containing the derivatives of one or more dependent variable with respect to one or more independent variables.

Recall that there are several ways of writing an ordinary derivative, for example, $\frac{\mathrm{d}y}{\mathrm{d}x},\ f'(x),\ y',\ \dot{y}$.

Also, there are many ways to classify, or describe, differential equations. The most common classifications are by type, order and linearity. You will only be working with a limited range of differential equations in this course.

There are two main types of differential equations: ordinary differential equations (ODEs) and partial differential equations (PDEs).

Ordinary differential equations deal with functions of a single variable and ordinary derivatives. You have already met these in this course.

Partial differential equations deal with multivariable equations and their partial derivatives. For example, if you have a multivariate function

$$f(x,y) = x^2y + 2y$$

you can find partial derivatives by treating other variables as constant. If you treat y as a constant, you can find the partial derivative with respect to x,

$$\frac{\partial f}{\partial x} = 2xy$$

and the partial derivative with respect to y,

$$\frac{\partial f}{\partial y} = x^2 + 2$$

You touched on this topic in subtopic 5.14 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26501/) when dealing with implicit differentiation. However, you will only deal with ordinary differential equations in this course. Some examples of ordinary differential equations include:

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 8y = \sin x$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = 5y + x$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t} = x + y$$

The order of a differential equation is the highest order **derivative** in the equation. Recall that in subtopic 5.7 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-27788/) and subtopic 5.12 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-26489/) you used higher derivatives. Some examples include:

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Student view

Overview (/study/app/aa-hl/sid-134-cid-761926/o

Steps	Explanation
$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \mathrm{e}^x$	First-order ODE
$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 8y = 3x$	First-order ODE (the first derivative is squared)
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = 5y + x$	Second-order ODE

This course will deal exclusively with first-order ordinary differential equations.

Finally, you can classify differential equations by their **linearity**. A differential equation is said to be linear if all of the terms with dependent variables are first-order. Some examples of linear and nonlinear differential equations include:

Steps	Explanation
$y + 4x\frac{\mathrm{d}y}{\mathrm{d}x} = x$	First-order linear ODE
$(1-y)\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \sin x$	First-order nonlinear ODE (the coefficient of $\frac{dy}{dx}$ contains y)
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0$	Second-order linear ODE
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \cos y = 0$	Second-order nonlinear ODE ($\cos y$ is trigonometric, not a first-order polynomial)
$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + x\frac{\mathrm{d}y}{\mathrm{d}x} + 7y = \ln x$	Third-order linear ODE

You will learn how to solve first-order linear ODEs as well as separable and homogeneous ODEs later in this subtopic.

✓ **Important**

An **ordinary differential equation** is an equation containing only ordinary derivatives of one or more dependent variables with respect to a single independent variable.

The **order of a differential equation** is the highest order derivative in the equation.

An ordinary differential equation is said to be **linear** if:

- the dependent variable *y* and all of its derivatives are of first degree
- the coefficients of all terms in the dependent variable and its derivatives depend only on the independent variable x .

So the initial example, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy$, is a first-order linear ordinary differential equation.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Slope fields

Exam tip

Slope fields and isoclines will not be included in Analysis and Approaches examinations. They have been included here as additional illustrations of differential equations.

You will now investigate graphical methods that help you to sketch solution curves even if you do not know how to solve the differential equation. The idea behind slope fields and isoclines is that for a differential equation of the form $y' = F(x, y)$, if a solution curve goes through a point (x_0, y_0) , then the equation tells you the slope of the tangent line to the solution curve at this point, $m = y'(x_0) = F(x_0, y_0)$.

A slope field , sometimes called a **direction field**, is a graphical representation of the rate-of-change across all values of x and y over a given domain and range. To build a slope field, you will draw little line segments at points on a rectangular grid to illustrate the tangent line to a solution curve passing through the points of the grid. If you choose a dense enough grid, these line segments will guide you to sketch solution curves of the differential equation.

Important

The slope field corresponding to the differential equation $y' = F(x, y)$ is the collection of line segments with slope $F(x, y)$ drawn at all points (x, y) of the plane. If $F(x, y)$ is not defined at a particular point, you draw a vertical line segment.

Of course, you cannot draw the full slope field because it has infinitely many line segments, so you choose a grid of points and draw only the corresponding segments. Note that, in some resources, slope fields are vector fields (they contain vectors instead of line segments).

You will now see how to draw and use slope fields to find approximate solutions to differential equations using the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -y\frac{1+xy}{1-xy}$.

But you will not learn how to find exact solutions for this equation. You will attempt only to sketch solution curves.

Example 1

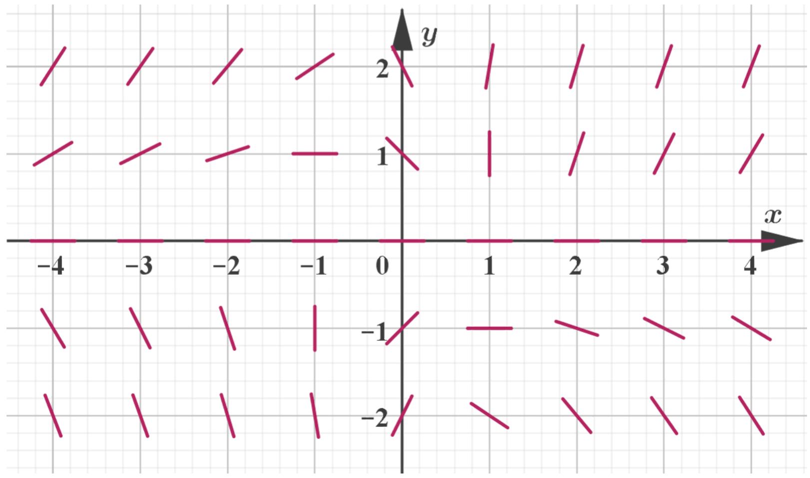
★★☆

Draw the slope field for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -y\frac{1+xy}{1-xy}$ using the grid of points with integer coefficients such that $-4 \leqslant x \leqslant 4$ and $-2 \leqslant y \leqslant 2$.

In an exam, it is unlikely that you will get a question like this. The method is not difficult but it is very repetitive, because it involves doing the same process over and over again. You need to calculate the value of the slope for any possible (x, y) pair.

Student view

For example, for $(x, y) = (-4, -2)$, $m = -(-2)\dfrac{1+(-4)(-2)}{1-(-4)(-2)} = -\dfrac{18}{7} = -2.57$. Hence, you need to draw a line segment through the point $(-4, -2)$ with approximate slope -2.57. This segment, and others, are illustrated in the diagram below.



N ote, that at $(-1, -1)$ and at $(1, 1)$, the lines are vertical. At these points the denominator of the quotient is 0, so the gradient is not defined.

The applet below shows the slope field for the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -y\dfrac{1+xy}{1-xy}$. Y ou can move the point around and draw the line segment at any point.

Interactive 1. Slope Field for the Differential Equation.

More information for interactive 1

This interactive tool enables users to explore the behavior of solutions to the differential equation

$$\frac{dy}{dx} = -y\frac{1+xy}{1-xy}$$

by visualizing its slope field. The graph displays short directional segments across the plane, representing the slope of the solution curve at each point (x, y), based on the equation. These segments form a slope field, helping users understand the local direction of any solution passing through a particular point.

Student
view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

A key feature of the interactive is a draggable purple point that allows users to select different initial conditions. As this point is moved, the corresponding solution curve (in red) is dynamically generated to follow the direction field, offering immediate visual feedback on how different starting values affect the trajectory of the solution.

The x-axis ranges from approximately -4 to 4, and the y-axis from -2 to 2. Points where the denominator becomes zero, such as $(x, y) = (-1, -1)$ and $(1, 1)$, lead to an undefined slope (division by zero), which is visually represented as abrupt changes or vertical slopes in the field.

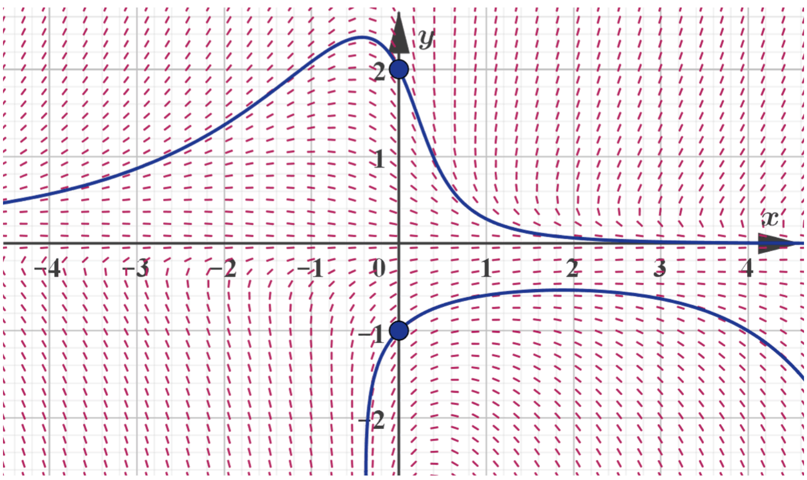
Although interesting, the power of the slope field is not to see how the function tends to act at every point. You can think of this like a river with currents. If you were to drop a leaf in at a specific starting point, where would it go? By mapping the currents, you could get a pretty good idea where they would lead. The same is true with slope fields. If you start at a given point with predicted slopes or directions, where would you expect the solution to take you? When given a starting point, follow the direction of the slope in both directions, adjusting the solution curve along with the change in the slope as you do so.

Example 2

★★☆

Using the slope field above, sketch a solution curve to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -y\frac{1+xy}{1-xy}$ with initial condition $y(0) = 2$. Sketch another curve with initial condition $y(0) = -1$.

The first curve should go through the point $(0, 2)$ and the second curve should go through the point $(0, -1)$. Both curves should follow the slope field. The line segments of the slope field should be close to the tangents of the solution curves at any point on the curve. The diagram below shows the curves. Did you get a curve close to these?



Example 3

★★☆

Draw some more solution curves for the differential equation above.

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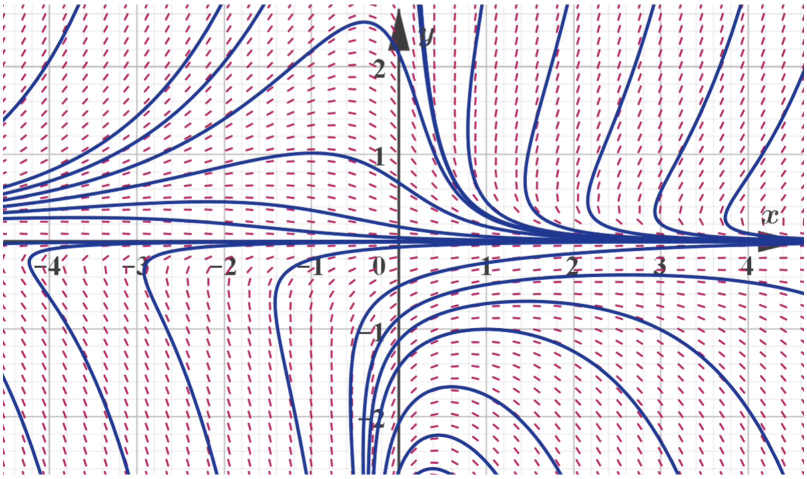
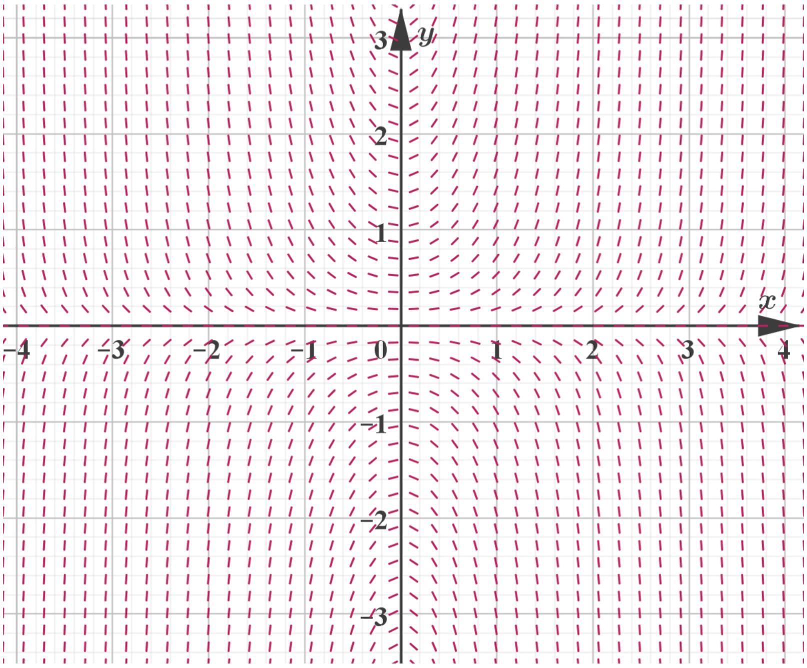


Look again at the initial example. The equation $\frac{dy}{dx} = 2xy$ has a slope field that looks something like this:



If you were to give an initial condition of $f(x, y) = (1, 1)$, the solution curve would look something like:

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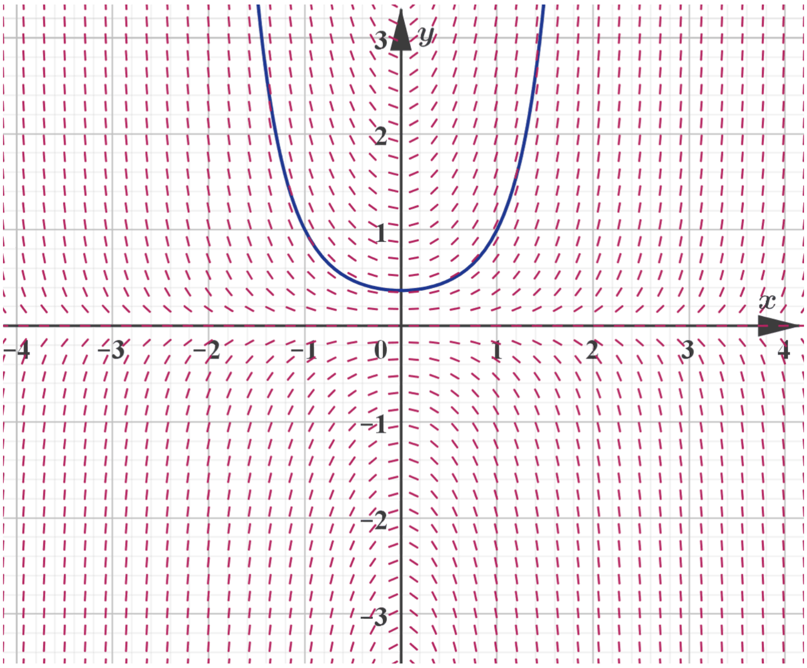

7 section questions

Question 1

Difficulty:

★☆☆

Classify the ordinary differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y^2x = 2x$

1 First-order nonlinear ✓

2 First-order linear

3 Second-order linear

4 Second-order nonlinear

Explanation

The highest (and only) derivative is a first derivative. Therefore it is first-order.

The y^2 term makes it nonlinear.

Question 2

Difficulty:

★☆☆

Classify the ordinary differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$

1 Second-order linear ✓

Student
view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

2 First-order nonlinear

3 First-order linear

4 Second-order nonlinear

Explanation

The highest derivative is a second derivative. Therefore it is second-order.

There is no term with a nonlinear y or a coefficient depending on y. It is linear.

Question 3

Difficulty:

★☆☆

Classify the ordinary differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - \sin y = -x$

1 First-order nonlinear ✓

2 First-order linear

3 Second-order linear

4 Second-order nonlinear

Explanation

The highest (and only) derivative is a first derivative. Therefore it is first-order.

The $\sin y$ term makes it nonlinear.

Question 4

Difficulty:

★☆☆

Classify the ordinary differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

1 Second-order nonlinear ✓

2 First-order linear

3 First-order nonlinear

4 Second-order linear

Explanation

The highest derivative is a second derivative. Therefore it is second-order.

Student view

The $y\frac{dy}{dx}$ term makes it nonlinear.

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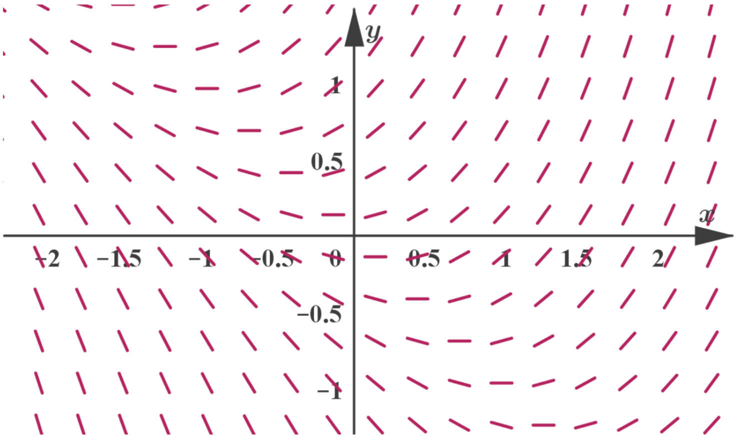
Question 5

Difficulty: ★☆☆

The diagram below illustrates the slope field of one of the differential equations below. Which one?



More information

1 $y' = x + y$ ✓

2 $y' = x - y$

3 $y' = y - x$

4 $y' = xy$

Explanation

The slope at the point $(x, y) = (0, 1)$ is positive, which rules out $y' = xy$ and $y' = x - y$.

The slope at the point $(x, y) = (1, 0)$ is also positive, which rules out $y' = y - x$.

Hence, we conclude that the slope field is for the differential equation $y' = x + y$.

Question 6

Difficulty: ★☆☆

Which one of the diagrams below illustrates the slope field of the differential equation $y' = y - x$?

1 ✓

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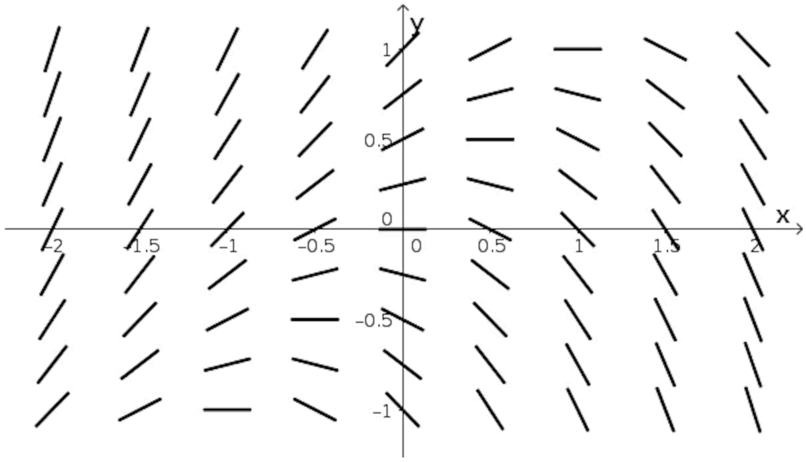
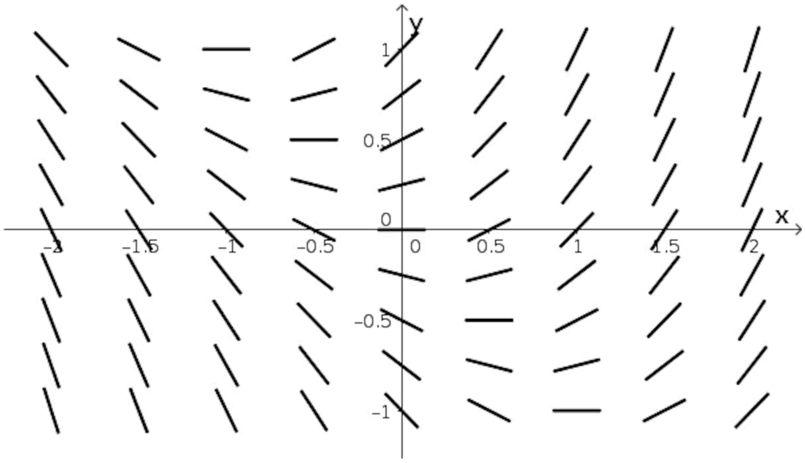


More information

2



More information

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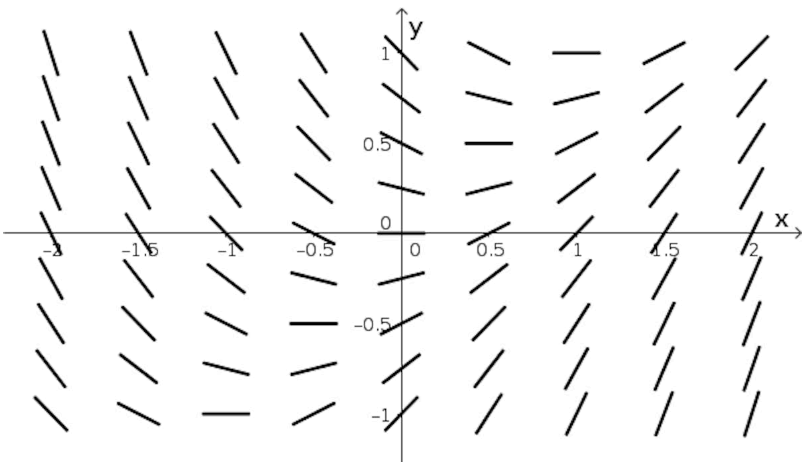

More information

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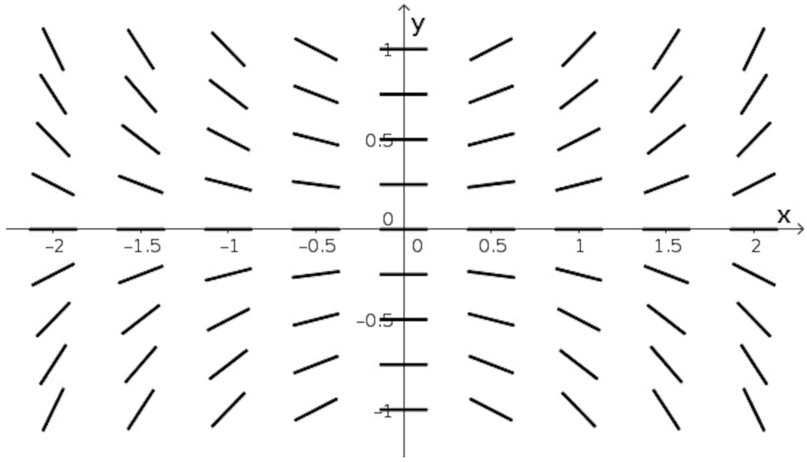


More information

Explanation

The slopes at $(0, -1)$ and $(1, 0)$ must be negative. This rules out the other possibilities.

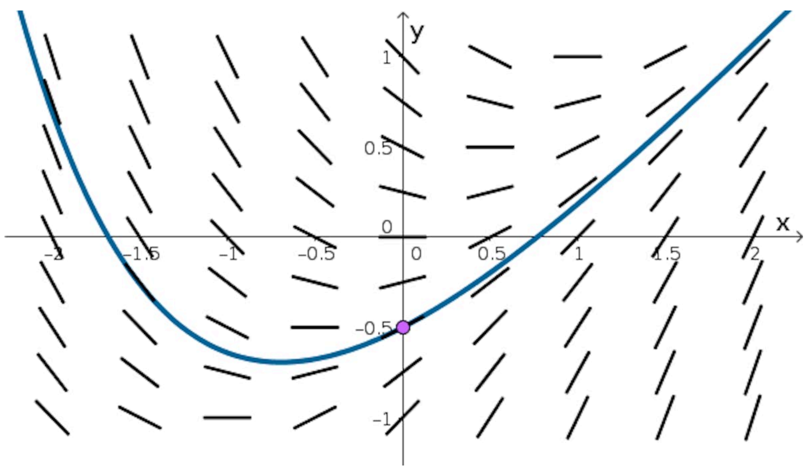
Question 7

Difficulty: ★☆☆

The diagrams below illustrate the slope field of a first-order differential equation of the form $y' = F(x, y)$. A curve is also drawn on each diagram, passing through the point $(0, -0.5)$.

On only one of the diagrams is this curve a solution curve to the differential equation represented by the slope field. Which one is this diagram?

1



✓

More information

2

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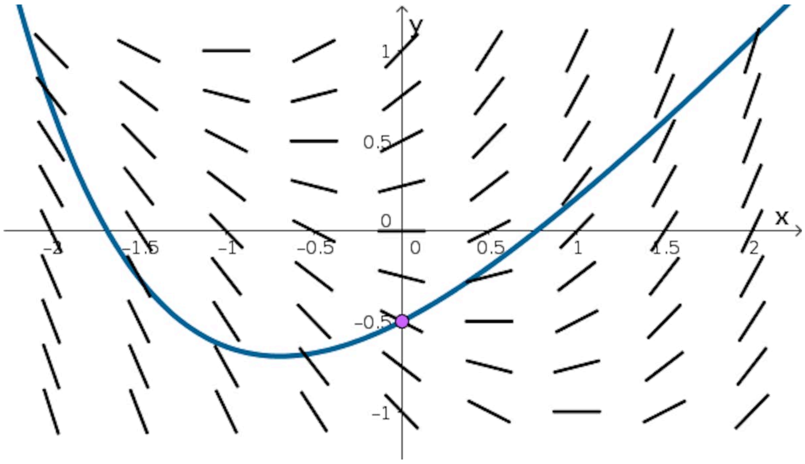

More information

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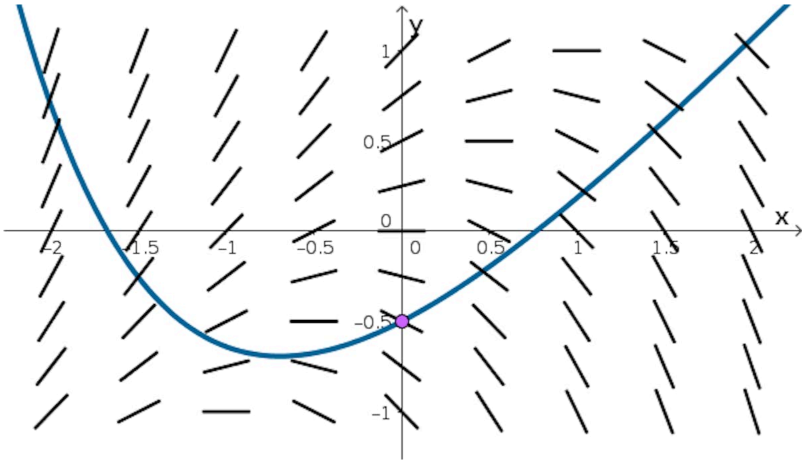

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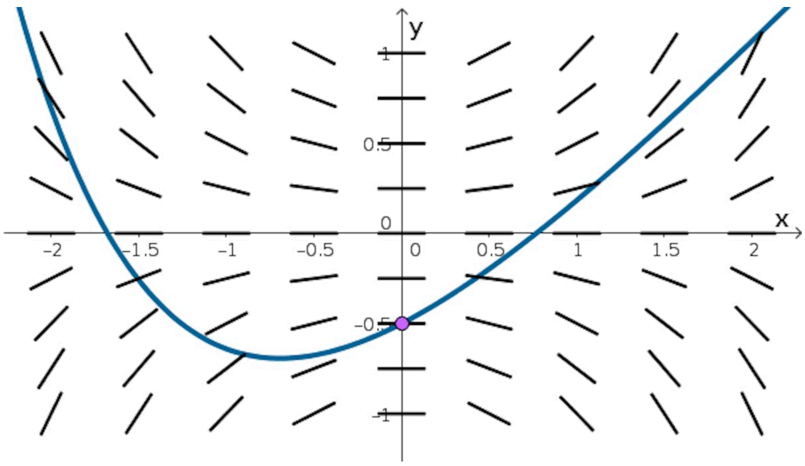


More information

Explanation

The curves on the other diagrams do not go in the direction of the slope field at the point $(0, -0.5)$.

5. Calculus / 5.18 Differential equations

Isoclines

Student
view

Exam tip

Slope fields and isoclines will not be included in Analysis and Approaches examinations. They have been included here as additional illustrations of differential equations.

Building entire slope fields by hand, especially with any meaningful density, is extremely time consuming. Although there is technology that can help you, it is useful to explore other graphical techniques that are more efficient to help you study the slopes. Another such technique is the development of isoclines.

Important

An **isocline**, corresponding to the differential equation $y' = F(x, y)$, is a curve with the (implicit) equation $F(x, y) = c$, for some real number c.

Alternatively, an **isocline** is a curve on a graph connecting points of equal slope c.

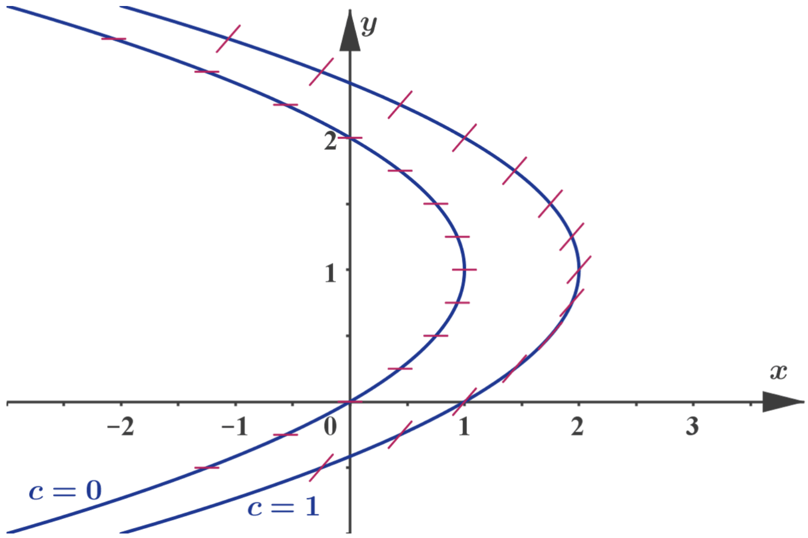
The name isocline comes from the Greek words 'isos' and 'klinein' and roughly means 'same slope'. Can you see how the definition reflects this meaning? If a solution curve of the differential equation crosses the isocline corresponding to c, then the tangent line to the solution curve at the intersection point (x_0, y_0) has slope $y'(x_0) = F(x_0, y_0) = c$.

To see how to use isoclines, work out the equation of a few isoclines for the differential equation $y' = x - 2y + y^2$.

For $c = 0$, the equation is $x - 2y + y^2 = 0$ or, after rearrangement, $x = -y^2 + 2y$. Notice that, since the slope $c = 0$, this equation is related to the horizontal line segments from the slope field. The graph of this equation is a parabola with a horizontal axis of symmetry and a vertex at $(1, 1)$.

For $c = 1$, the equation is $x - 2y + y^2 = 1$ or, after rearrangement, $x = -y^2 + 2y + 1$. The graph of this equation is a parabola with a horizontal axis of symmetry and a vertex at $(2, 1)$.

The following diagram shows the two isoclines corresponding to $c = 0$ and $c = 1$ for the differential equation $y' = x - 2y + y^2$. It also indicates the directions in which a solution curve crosses these isoclines.



More information

The diagram presents two isoclines for the differential equation (y' = x - 2y + y^2). It features a standard Cartesian coordinate system with labeled x and y axes. The graph has two curved blue lines representing isoclines. One isoclinal curve is labeled (c = 0), which runs from the bottom left to the top right, passing through points approximately at ((-2, 2)) and ((3, 0.5)). The other curve is labeled (c = 1) and appears above the first one, starting from the bottom left and heading towards the upper right, passing through approximately ((-2, 1)) to ((2, 2.5)). Pink dashed lines are perpendicular to the curves, illustrating the direction in which the solution curve intersects the isoclines. The x-axis ranges roughly from -3 to 3, and the y-axis ranges roughly from -1 to 3. This diagram is used for estimating solution curves by analyzing how they cross the isoclines.

[Generated by AI]

Once you have the isoclines and the starting point, you can estimate the solution curve by sketching a curve that crosses the isoclines at a slope equal to the constant of each isocline.

Example 1

★★☆

Sketch a solution curve for the differential equation $y' = x - 2y + y^2$ that starts at the point $(0, 1)$, crosses the isocline corresponding to $c = 0$ and reaches the isocline corresponding to $c = 1$.

At $(0, 1)$, $y'(0) = F(0, 1) = 0 - 2 \times 1 + 1^2 = -1$. Hence, the tangent line to the solution curve at $(0, 1)$ has gradient -1. This shows that the solution curve is decreasing at this point. When it crosses the isocline corresponding to $c = 0$, it has a horizontal tangent line, and when it reaches the isocline corresponding to $c = 1$, it is increasing with gradient 1. The diagram below illustrates this part of the solution curve.

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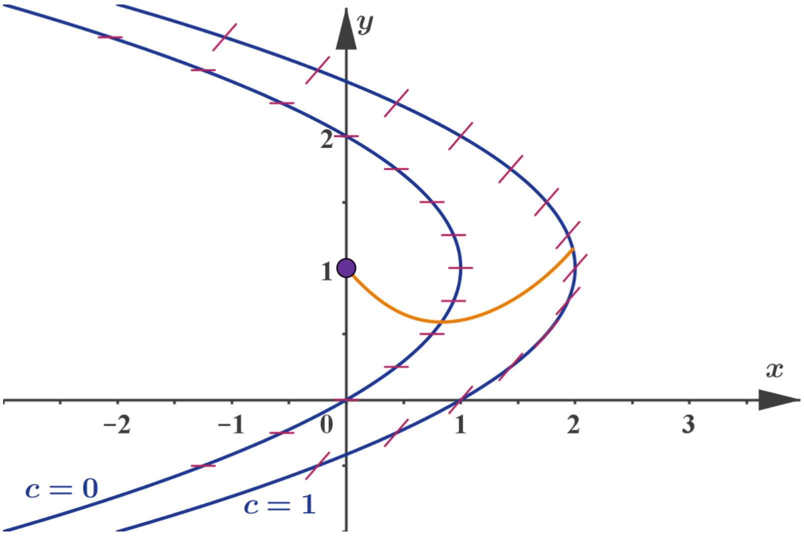


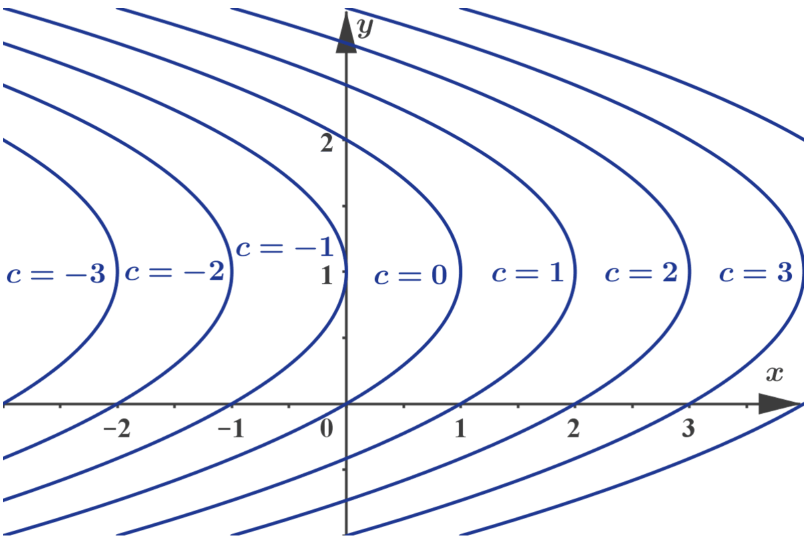
To extend the solution curve from the previous example, you can draw more isoclines.

Example 2

★★☆

Use the isoclines from the previous example to sketch a solution curve defined on $[-3, 3]$ for the differential equation $y' = x - 2y + y^2$ that passes through the point $(0, 1)$.

First, you need more isoclines. As in the calculations above, the isoclines are parabolas with equations $x = -y^2 + 2y + c$. The labelled isoclines without the slope field look something like this:



Student view

You already have part of this sketch from **Example 1** above.

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

To extend it to the right, notice that the gradient is increasing (since the isoclines to the right correspond to gradients greater than 1).

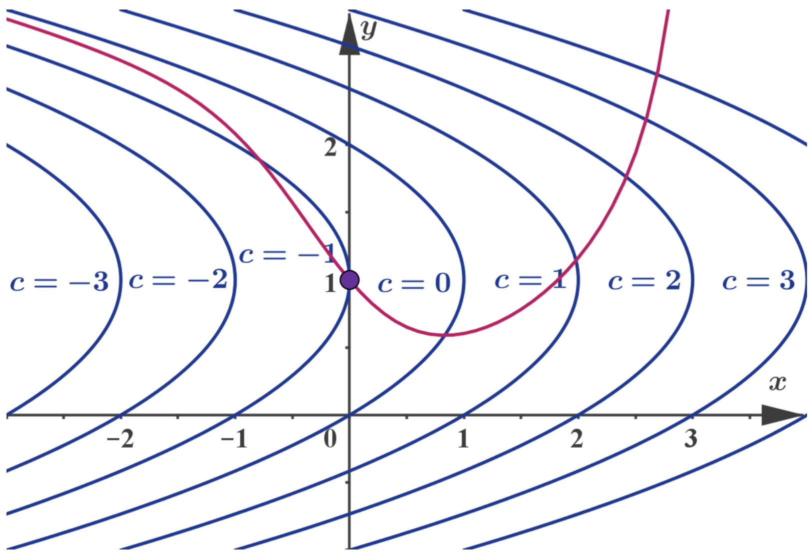
Extending the curve to the left is more difficult. The explanation below gives you the reasons in an informal way. In an exam, if a question like this is asked, then a correct sketch of the curve is expected, along with a few words of explanation.

First, notice that the curve will not reach the isocline corresponding to $c = -2$. This is not a precise observation but, informally, the isocline for $c = -2$ is far from the point $(0, 1)$ on the isocline corresponding to $c = -1$. A solution curve needs to be quite flat to reach it. On the other hand, the gradient of the solution curve is less than -1 to the left of the isocline $c = -1$, so it cannot be too flat. As mentioned before, this is not a precise argument, but for an approximate sketch, you will not need any more details.

In addition, similar arguments (considering slopes) will also show that the curve does not stay below the isocline $c = -1$ forever, so it crosses it again somewhere to the left of the y-axis.

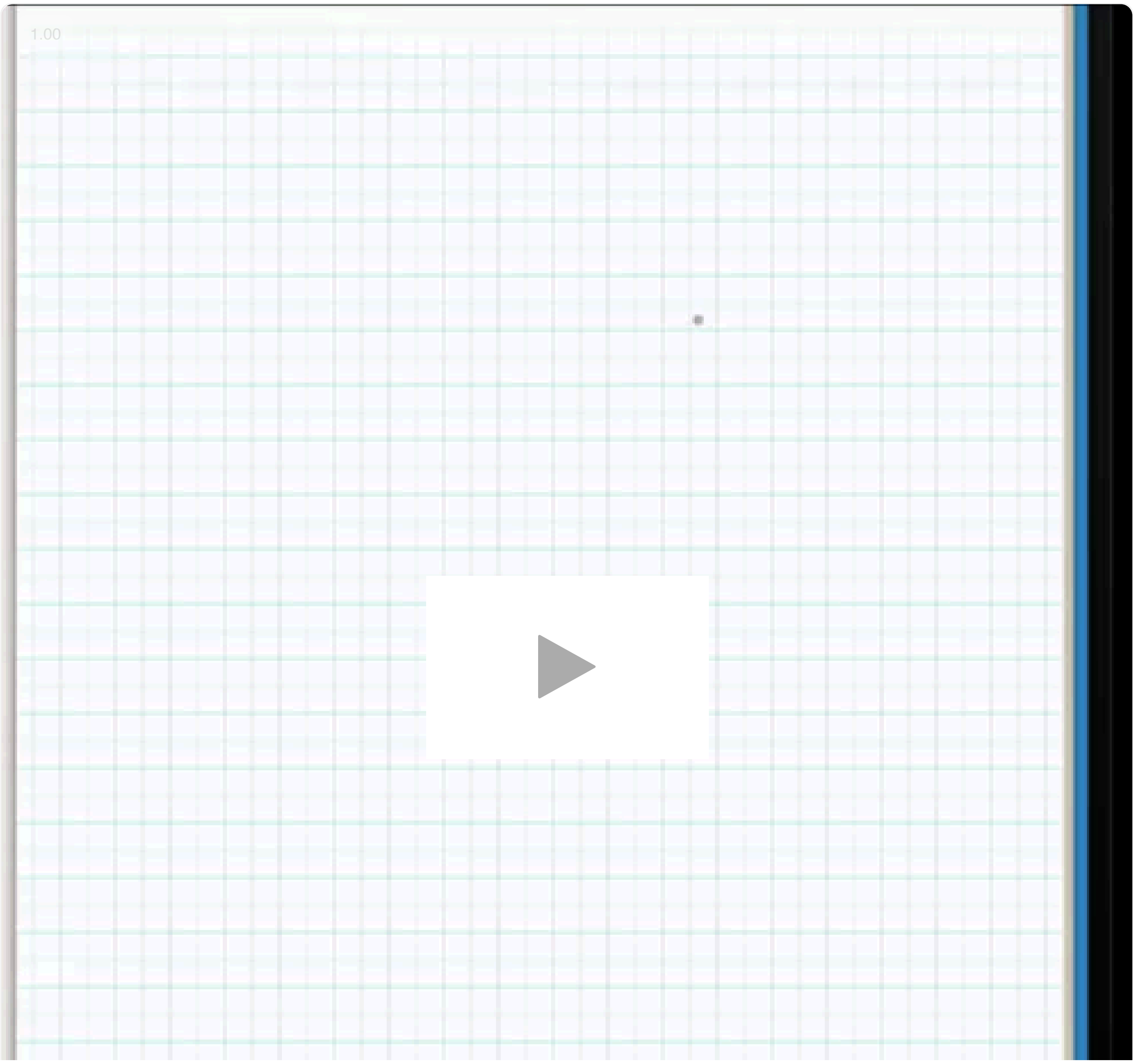
To the left of this second crossing point, it will now always stay between the isoclines $c = -1$ and $c = 0$. Again, you will not need to prove this precisely, but informal arguments using slopes can be used to justify this claim. For example, it does not cross the isocline corresponding to $c = 0$ because at the crossing point the slope should be 0, but the shape of the isocline contradicts this.

The diagram below shows the actual solution curve.



In the following video, you can explore the differential equation $y' = x - 2y + y^2$ and the application of isoclines as laid out in **Example 1** and **Example 2** above.

Student view



Video 1. Isoclines.

More information for video 1

1

00:00:00,033 --> 00:00:01,933

narrator: In this video,

we're going to study isoclines

2

00:00:02,000 --> 00:00:04,533

and how they can help

us giving the solution

3

00:00:04,633 --> 00:00:07,733

to an equation y prime equals

4

00:00:08,000 --> 00:00:11,400

F of x and y.

That is a differential equation.

5

00:00:12,267 --> 00:00:14,900

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

Now let us choose

the differential equation y prime

6

00:00:14,967 --> 00:00:18,000

is x minus 2y plus y squared.

7

00:00:18,233 --> 00:00:22,533

Then the isoclines

are solutions to the y prime equals

8

00:00:22,600 --> 00:00:25,000

C, where C is a constant.

9

00:00:25,933 --> 00:00:28,933

For our case,

that then means of course, that

10

00:00:29,433 --> 00:00:33,567

x minus 2y plus y square

is C, where C is a constant,

11

00:00:33,633 --> 00:00:36,467

which in fact is the gradient

12

00:00:36,533 --> 00:00:40,667

at certain points on the xy plane,

which of course gives you slope field.

13

00:00:42,467 --> 00:00:47,700

Now this means that x is minus y square

plus 2y plus a constant C.

14

00:00:48,333 --> 00:00:51,633

Now we need to analyze

these to give us the isocline curves.

15

00:00:51,700 --> 00:00:53,500

But let's first consider a simpler one,

16

00:00:53,567 --> 00:00:57,100

and that is y is minus

x square plus 2x plus C,

17

00:00:57,167 --> 00:00:59,933

which of course we can complete

a square quite easily.

18

00:01:00,133 --> 00:01:03,100

Make sure that you be careful

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

with the minus signs,

19

00:01:03,167 --> 00:01:07,333

which gives you y equals minus x minus 1

squared plus 1 plus C.

20

00:01:07,433 --> 00:01:09,800

So it is a parabola upside down,

21

00:01:09,900 --> 00:01:13,933

and the vertex is given

by 1 comma 1 plus C,

22

00:01:14,233 --> 00:01:17,167

and you have drawn

this quadratic curve y equal minus x

23

00:01:17,233 --> 00:01:18,867

minus 1 square plus 1 plus C.

24

00:01:19,167 --> 00:01:20,633

Now I'm going to manipulate C,

25

00:01:20,700 --> 00:01:23,800

which of course amounts to nothing

more than a vertical translation.

26

00:01:23,967 --> 00:01:28,700

So you can see here that the green point

of vertex remains at the same x position,

27

00:01:28,767 --> 00:01:30,467

but simply change this y.

28

00:01:30,533 --> 00:01:32,933

as I change the C.

29

00:01:33,067 --> 00:01:36,600

Of course,

I can move the vertex up by increasing C,

30

00:01:36,700 --> 00:01:39,467

but of course I can also

move the vertex down

31

00:01:39,533 --> 00:01:40,800

by decreasing C.

32

00:01:41,167 --> 00:01:43,200

Student view

Now, let's create a family of curves

33

00:01:43,267 --> 00:01:46,733

by tracing all those

quadratic functions as I step

34

00:01:46,800 --> 00:01:49,167

through C in steps of one,

35

00:01:49,233 --> 00:01:51,800

and this creates

this family of quadratic curves.

36

00:01:52,200 --> 00:01:53,233

Now here I've shown it again.

37

00:01:53,333 --> 00:01:57,133

Now of course, this was not

the original function we had there.

38

00:01:57,200 --> 00:01:58,867

The y's and the x's were interchanged,

39

00:01:58,933 --> 00:02:02,667

but that amounts nothing more

than rotating the axis system

40

00:02:02,933 --> 00:02:05,833

y to x, which I'm simply gonna do by

41

00:02:05,900 --> 00:02:07,700

rotating physically this picture.

42

00:02:07,800 --> 00:02:11,067

So here I've done

that just rotated clockwise,

43

00:02:11,233 --> 00:02:15,400

and all I need to do now

is rename the axis, y vertical,

44

00:02:15,467 --> 00:02:17,800

x horizontal, and of course equation.

45

00:02:18,100 --> 00:02:20,833

Then I need to also

interchange the x and y,

46

00:02:20,900 --> 00:02:24,900

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

and there is our quadratic

with the isoclines.

47

00:02:24,967 --> 00:02:28,567

So here I've redrawn

the isoclines x equals minus y,

48

00:02:28,667 --> 00:02:34,567

square plus 2y plus C for various values

of C ranging from minus 3 to 3.

49

00:02:34,667 --> 00:02:37,267

Now, let's make our solution

go through point 0 comma 1.

50

00:02:37,333 --> 00:02:39,933

So I'm gonna create

that point right there.

51

00:02:40,000 --> 00:02:42,167

And now let's remember

what those values of C means.

52

00:02:42,233 --> 00:02:44,700

That means that the gradient,

53

00:02:44,767 --> 00:02:47,300

anywhere along those

isocline is that value.

54

00:02:47,400 --> 00:02:51,167

So minus 1 at a point 0 comma 1,

55

00:02:51,233 --> 00:02:53,833

then 0 for C0, 1 at C1,

56

00:02:53,900 --> 00:02:56,433

greater 2 at C2 and 3 at C3.

57

00:02:56,700 --> 00:02:58,633

Now I can approximate

58

00:02:58,733 --> 00:03:01,100

the solution for this

curve going through the point

59

00:03:01,533 --> 00:03:05,367

by connecting the gradient slopes,

60

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

00:03:05,567 --> 00:03:08,667

the gradient segments at those isoclines.

61

00:03:08,867 --> 00:03:10,033

So you have shown a few of them

62

00:03:10,100 --> 00:03:13,133

and then a smooth curve

should connect them,

63

00:03:13,667 --> 00:03:16,733

which looks roughly like so.

64

00:03:17,400 --> 00:03:19,733

Now the one to the right

of the point is rather easy.

65

00:03:19,800 --> 00:03:21,633

The one to the left is a

little bit more tricky,

66

00:03:21,967 --> 00:03:25,967

so it has a gradient of minus

1 at the point 0 comma 1,

67

00:03:26,233 --> 00:03:27,567

and then a little bit to the left of that.

68

00:03:27,633 --> 00:03:30,500

It's a little bit steeper,

but negative of course.

69

00:03:30,733 --> 00:03:35,000

So you can see that it reaches

the C minus line again,

70

00:03:35,100 --> 00:03:36,567

roughly where I've drawn it.

71

00:03:36,833 --> 00:03:39,800

And of course, the next

isocline up is the C equal 0.

72

00:03:39,867 --> 00:03:40,767

So it's horizontal.

73

00:03:40,833 --> 00:03:44,933

So you see that the red line never

becomes horizontal

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

74

00:03:45,000 --> 00:03:47,033

before it can hit the blue line,

75

00:03:47,100 --> 00:03:49,400

and so therefore it never quite

crosses it.

76

00:03:49,467 --> 00:03:52,833

Now let's trace this with a bigger

font size, and there you've got it.

77

00:03:53,267 --> 00:03:56,367

We can, of course,

verify this using technology.

78

00:03:56,500 --> 00:03:58,567

So we have drawn the isocurves,

79

00:03:58,700 --> 00:04:01,133

and now I'm going to draw the slope field

80

00:04:01,200 --> 00:04:03,033

using technology over here.

81

00:04:03,367 --> 00:04:06,233

Of course, remember

that the little segments

82

00:04:06,333 --> 00:04:09,367

that make up a slope

field take on the values

83

00:04:09,467 --> 00:04:11,867

of C when crosses those isoclines.

84

00:04:12,033 --> 00:04:13,600

Now the point was 0 comma 1.

85

00:04:13,667 --> 00:04:16,367

So the orange point is

what the solution needs to go through.

86

00:04:16,500 --> 00:04:18,933

And let's draw

the right hand branch first.

87

00:04:19,033 --> 00:04:22,000

And there you can see

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

this is the right hand branch across

88

00:04:22,067 --> 00:04:24,800

the isoclines at the

appropriate gradients.

89

00:04:25,100 --> 00:04:28,100

And on the left hand side,

the solution is here roughly

90

00:04:28,167 --> 00:04:29,667

what we saw.

91

00:04:30,233 --> 00:04:32,067

So this is using technology

92

00:04:32,133 --> 00:04:34,867

and let's just compare

to what we had by hand.

93

00:04:34,933 --> 00:04:39,433

And this is our solution by hand,

which agrees quite well,

94

00:04:40,200 --> 00:04:43,333

and that is how to use isoclines

to find approximate solutions,

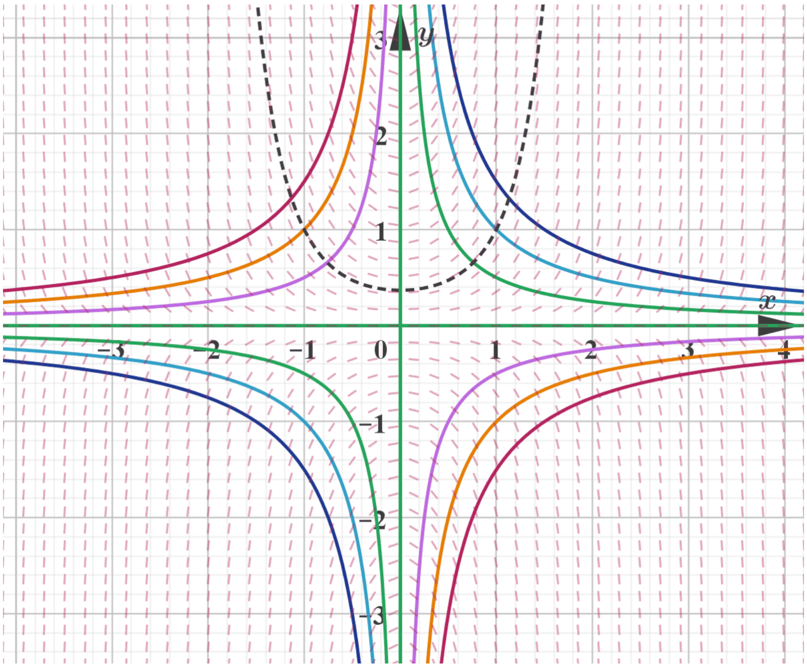
95

00:04:43,467 --> 00:04:46,333

curves to a differential equation.

Look again at the initial example, $\frac{dy}{dx} = 2xy$. A handful of isoclines with the solution curve would look something like this:

Student view



More information

The image features a graphical representation of isoclines and a solution curve on a grid. The background consists of a grid with intersecting vertical and horizontal lines. Overlaid on this grid are multiple curved lines, representing the solution curves to a differential equation displayed as (\frac{dy}{dx} = 2xy). These curves exhibit a symmetrical pattern, intersecting mostly around the origin, and span outwards with varied curvatures, indicating different solutions or isoclines. Arrows, possibly representing direction fields or slopes, are densely packed throughout the grid, indicating the slope at various points on these solution paths. The color-coded lines and varying curvatures visually demonstrate the relationship between the slope field, the isoclines, and the solution curves, offering insight into the dynamics of the equation in question.

[Generated by AI]

As mentioned earlier, you typically would not take the time to develop both the slope field and the isoclines, but this does help to show the relationship between the slope field, the isoclines and the solution curve. They are all related.

5. Calculus / 5.18 Differential equations

Numerical solution: Euler's method

In the previous section (/study/app/math-aa-hl/sid-134-cid-761926/book/firstorder-differential-equations-id-27274/), you investigated graphical methods of sketching solution curves to differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = F(x, y)$. While these methods nicely illustrate the behaviour of the function satisfying the equation, they do not really show how to find values to these solutions. Up to now, you have been content with being able to approximate a solution with a hand-drawn curve, but you had credible information on the accuracy of the process or a method to improve that accuracy. Basically, you have been estimating an answer to a complex mathematical problem without using any maths.

In the late 1700s, Leonhard Euler developed a numerical technique (Euler's method) to approximate the length travelled along the solution curve from a known starting point. The idea follows the concept of following the slope fields in the last section, but instead of drawing a curve by hand, you draw a curve based on the known point and the slope for a set

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

distance, and then repeat the process.

Go back to the example of the leaf dropped into a stream.

Leaf in a stream

Credit: Eerik GettyImages

You know where the leaf starts, as well as the direction of the current at that point. Where is the leaf going? Rather than plot a complex graph of a slope field or multiple isoclines, you can estimate the path algebraically. Given the direction and speed of the current at the initial point, you can predict where the leaf will be after 15 seconds. Using your equation and the new location, you can find the direction and speed of the leaf at that point and predict where it will be 15 seconds after that, at $t = 30$. As you continue the process, you can go along the curve to whatever point in time you are interested in, say $t = 60$.

Will your prediction be correct? Not exactly. As soon as you left your spot at $t = 0$, the slope field changed. Hopefully, it was not a dramatic change, but do expect some error.

Now look at a numerical example.

Example 1

★☆☆

A solution of the differential equation $y' = x - y^2$ satisfies $y(1) = 1.5$. Use Euler's method with step size 1.0 to estimate $y(2)$.

With a step size of $\Delta x = 1$, you only need one step.

The diagram below shows the actual solution curve, and the points (x_k, y_k).

You have a point and a slope, so you should be able to move forward one step easily.

The new x-value can be found as $x_1 = x_0 + \Delta x = 1 + 1 = 2$.

Student view

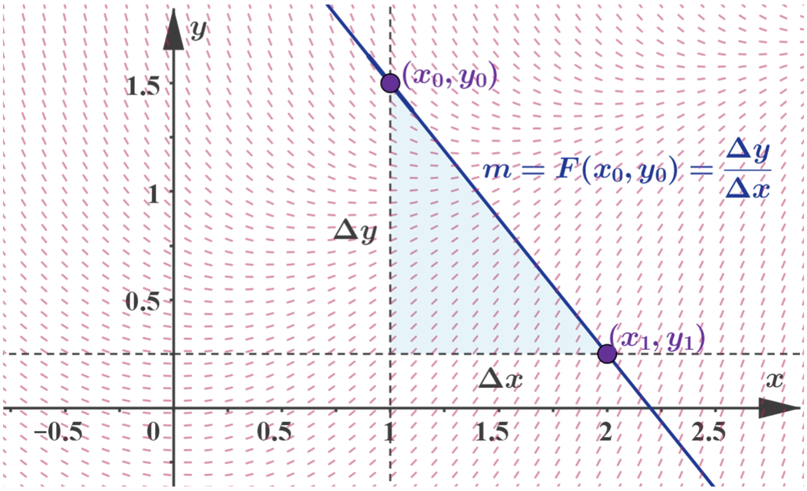
To find the new y-value, you need the slope at that point: $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} = x - y^2 = 1 - 1.5^2 = -1.25$.

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Since $\frac{\Delta y}{\Delta x} = -1.25$, you can say that $\Delta y = -1.25\Delta x = -1.25(1) = -1.25$.

Therefore, $y_1 = y_0 + \Delta y = 1.5 - 1.25 = 0.25$.

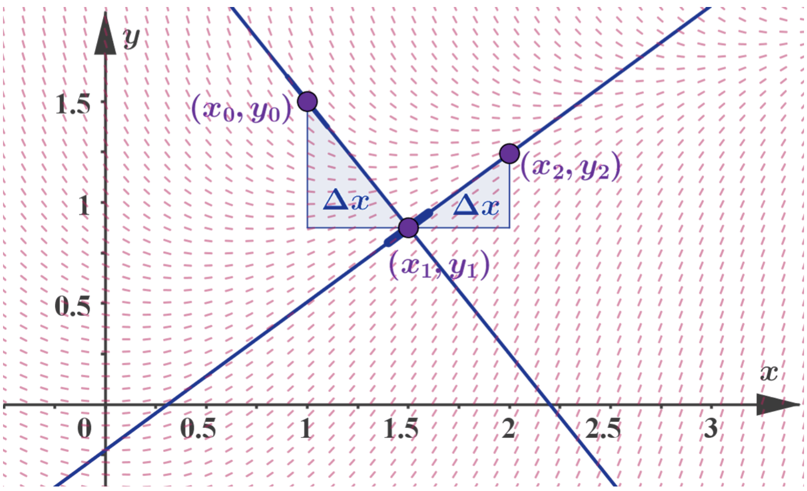
The prediction for the solution using a step size of $\Delta x = 1$ is $(2, 0.25)$ as illustrated in the graph below .



As you follow the straight-line approximation overlaid on the slope field, you can see that it starts off as a pretty good approximation, but as the solution continues, it should curve away from the line to the upper right of the diagram. The idea behind Euler's method is that you can use more steps to find a better approximation of $y(2)$. For example, if you set the step size to $\Delta x = 0.5$ and complete the process in two steps, you get:

$x_1 = x_0 + \Delta x = 1 + 0.5 = 1.5$	$x_2 = x_1 + \Delta x = 1.5 + 0.5 = 2.0$
$\Delta y = (x - y^2)\Delta x = (1 - 1.5^2)(0.5) = -0.625$	$\Delta y = (x - y^2)\Delta x = (1.5 - 0.875^2)(0.5) = 0.367$
$y_1 = y_0 + \Delta y = 1.5 - 0.625 = 0.875$	$y_2 = y_1 + \Delta y = 0.875 + 0.367 = 1.2421875 \approx 1.24$

The graph below shows approximating $y(2)$ in two steps at $\Delta x = 0.5$.



Student view

More information

The graph illustrates a slope field with two approximation steps to estimate (y(2)) using a step size of (\Delta x = 0.5). The X-axis is labeled from 0 to 3.5, and the Y-axis is labeled from 0 to 2. The grid is covered with red sloped lines indicating the direction of the slope field. A blue path begins at point ((x_0, y_0)), moves to ((x_1, y_1)), and finally points to ((x_2, y_2)). Triangular shapes represent the change in x between points, labeled as (\Delta x). This visual representation suggests that the blue path increasingly aligns with the slope field, giving a closer approximation of the actual value of (y(2)).

[Generated by AI]

If you look at the slope field, you can see that this new value of 1.24 is much closer to the actual (unknown) $y(2)$ than the previous 0.25. You see this because the blue “path” is much more closely aligned with the slope field. Usually, further reducing the step size Δx (and at the same time increasing the number of steps and therefore the amount of work) will result in a better approximation.

✓ **Important**

Euler's method is an iterative process for finding an approximate value for $y(x)$, where y is a solution of a differential equation of the form $y' = F(x, y)$ that also satisfies the initial condition $y(x_0) = y_0$.

During the process, points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are generated in n steps using the recursion:

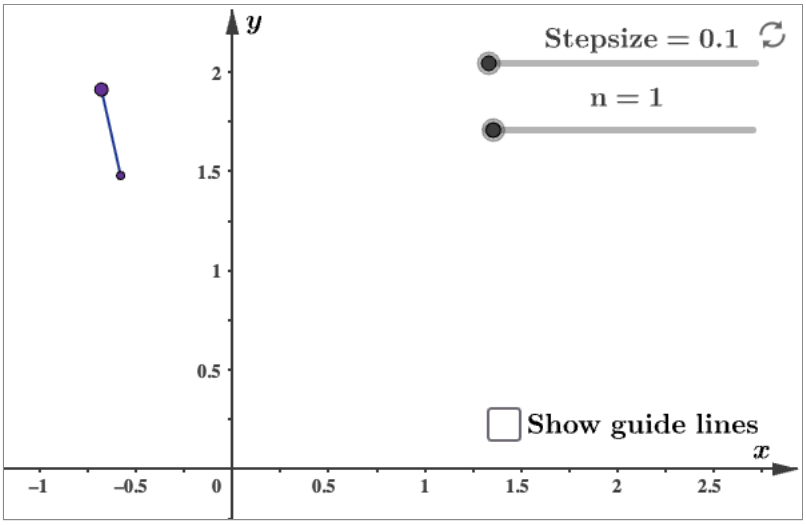
- $x_{k+1} = x_k + \Delta x$
- $y_{k+1} = y_k + F(x_k, y_k) \times \Delta x$,

where $x_n = x$. Hence, $\Delta x = \frac{x - x_0}{n}$, and y_n is an approximation for $y(x)$.

✓ **Important**

The recursive formula for Euler's method is in section 5.18 of the formula booklet with $y_{n+1} = y_n + h \times f(x_n, y_n); x_{n+1} = x_n + h$, where h is a constant (step length).

The applet below illustrates the iterative nature of Euler's method for the differential equation you looked at in this section. Move the point around to change the initial conditions. With the sliders, you can control the step size and the number of steps, n. You can also decide whether or not you want to see the actual solution curve.



Interactive 1. The Iterative Nature of Euler's Method for a Differential Equation.

More information for interactive 1

This interactive graph allows users to explore Euler's method for approximating solutions to first-order differential equations in a visual and intuitive way. The coordinate plane ranges from -1 to 2.5 on the x-axis and from 0 to 2 on the y-axis. Users can begin by dragging a purple point to set the initial condition, which serves as the starting point for the approximation. A step size slider lets users control the increment for each iteration (e.g., 0.1 or 0.14), while another slider allows them to adjust the number of steps, from 1 to 35 or more.

The applet dynamically constructs the Euler approximation using straight line segments connecting successive points, forming a piecewise-linear path (in blue) that approximates the actual solution curve (in red). A "Show guide lines" checkbox toggles the visibility of vertical guide markers, which help visualize the slope-based updates at each step. As users decrease the step size and increase the number of steps, they observe the blue approximation curve closely converging to the red solution curve, reinforcing the core concept of Euler's method—using tangent lines at each point to estimate the next point on the curve.

This hands-on tool highlights the balance between computational effort and accuracy and effectively illustrates how small changes in parameters influence the overall solution. It's particularly useful for simulating real-world phenomena like motion, growth, or decay when exact solutions are difficult to obtain.

As you decrease the step size and increase the number of repetitions required, bookkeeping becomes an issue. Storing the values in a table or spreadsheet can be a good way to keep up with these values.

Example 2

★★☆

A solution of the differential equation $y' = x - y^2$ satisfies $y(1) = 1.5$. Use Euler's method with step size 0.1 to estimate $y(2)$.

You can set up a table to store the repetition value (k), the independent and dependent variable values $(x_k andy\_k)$ and the slope (y'). Work your way across the table computing the new x, y, and y' for each k value, then move on to the next row until you find the required value.

k	x_k	y_k	${y_k}' = x_k - {y_k}^2$	
0	1.0	1.5	-1.25	$(x_0, y_0) = (1.0, 1.5)$ given

Overview
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aa-
hl/sid-
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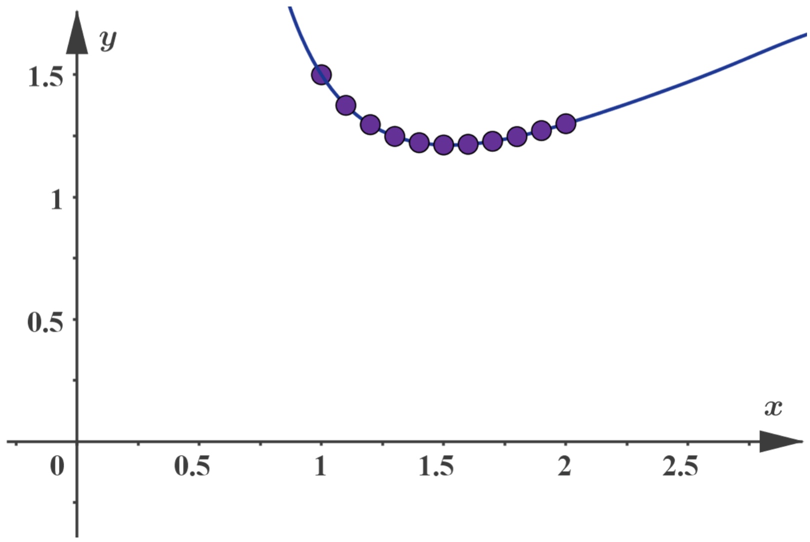
cid-
761926/o

k	x_k	y_k	${y_k}' = x_k - {y_k}^2$	
1	1.1	1.375	-0.790625	${y_k}' = x_k - {y_k}^2$ given; ${y_k}'$ calculated
2	1.2	1.2959375	-0.479454004	$x_{k+1} = x_k + \Delta x$
3	1.3	1.2479921	-0.257484281	$y_{k+1} = y_k + {y_k}'\Delta x$
4	1.4	1.222243672	-0.093879593	
5	1.5	1.212855712	0.028981021	
6	1.6	1.215753814	0.121942663	
7	1.7	1.227948081	0.192143511	
8	1.8	1.247162432	0.244585869	
9	1.9	1.271621019	0.282979985	
10	2.0	1.299919017		

The diagram below shows the actual solution curve, and the points (x_k, y_k) .



The diagram depicts a coordinate plane graph with an x-axis and a y-axis. The y-axis is labeled 'y' and ranges from 0 to 1.5 in increments of 0.5, while the x-axis is labeled 'x' with a similar range, going from 0 to 2.5.

On this graph, a blue curve is plotted which represents a solution curve. It starts from above the y-axis and arcs downward before rising again. Along this curve, there are a series of purple points plotted, labeled as (({x_k},{y_k})), indicating specific points of interest along the solution curve. The points are clustered in the middle of the curve where it dips, suggesting an analysis of this region. The curve and points demonstrate the behavior of the solution and the critical points under study.

Student
view

Overview
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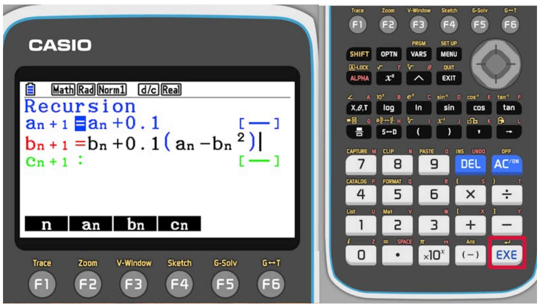
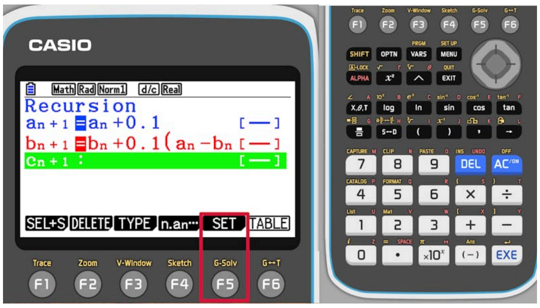
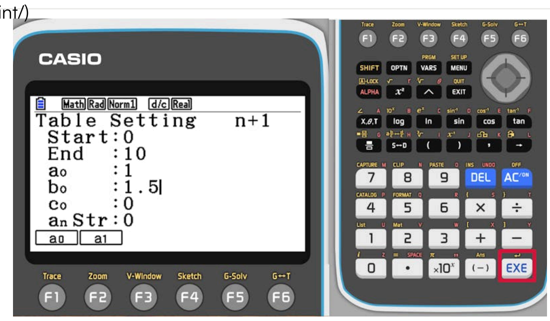
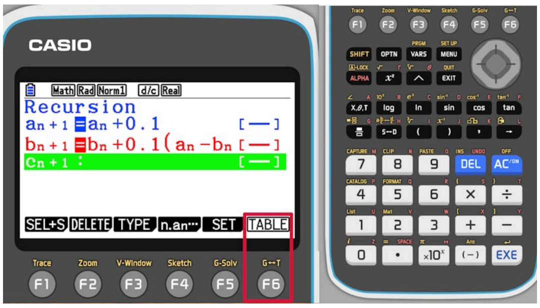
[Generated by AI]

Euler's method with a GDC

You will see how to use Euler's method to solve the differential equation $\frac{dy}{dx} = x - y^2$, with initial condition $y(1) = 1.5$ for three different GDCs.

Steps	Explanation
The following instructions show you a way to use Euler's method with step size $h = 0.1$ to find an approximate value of $y(2)$, where $y(1) = 1.5$ and $y' = x - y^2$. You will also see a graphical illustration of the sequence of points leading to this approximation. Choose the option to work with recursive sequences.	
The recursive formula you need to use is $x_{n+1} = x_n + 0.1$ $y_{n+1} = y_n + 0.1(x_n - y_n^2)$ This calculator uses sequence names a and b instead of the names x and y in the formula. When you are done entering the recursion rules, press EXE.	

Student
view

Steps	Explanation
Press F5 to bring up the screen, where you can set the initial values for the recursion.	
In this example the recursion starts with index 0. The condition $y(1) = 1.5$ means, that $$x_0 = 1$$ $$y_0 = 1.5$$ Since with step size 0.1, the tenth term will be $x_{10} = 2$. Enter 10 as the last index. You can of course enter a larger number if you are interested in further values. Press EXE when all values are entered.	
The calculator now have all information needed, so press F6 to generate the table of values.	

Overview
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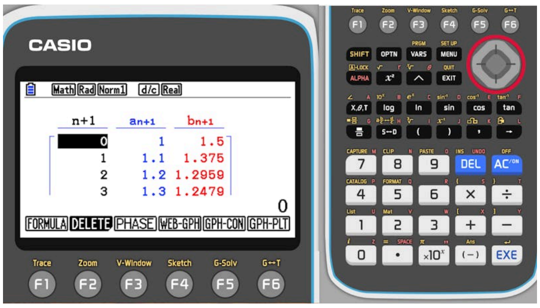
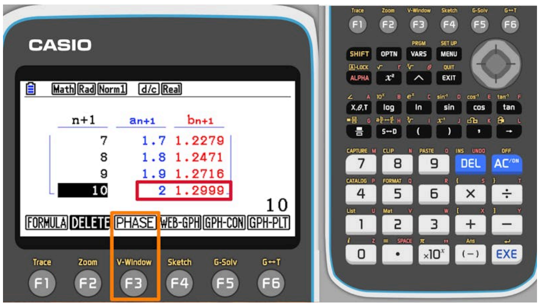
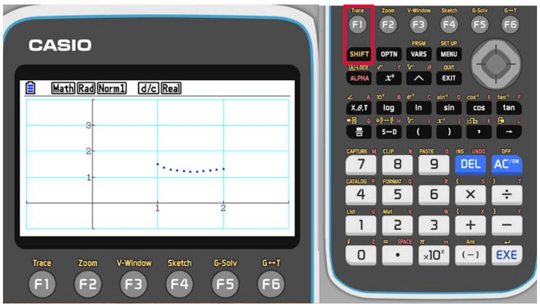
<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>You can see the starting x and y-values in the a and b sequences. Scroll down to see the rest of the sequence.</td><td>


</td></tr>
<tr><td>The row corresponding to $x_{10} = 2$ gives y_{10}, the approximate value of $y(2)$.

To see the graph of the sequence of the points leading to this approximation, press F3.</td><td>


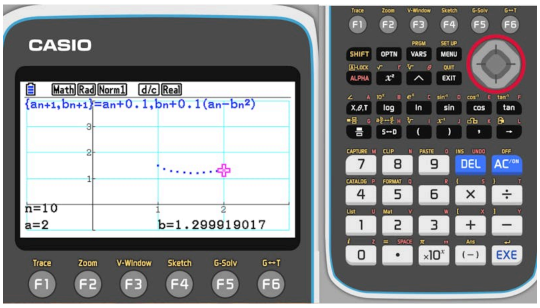
</td></tr>
<tr><td>You will probably need to adjust the viewing window to see the points.

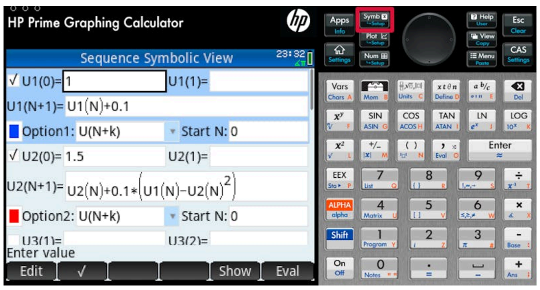
If you press shift/F1, you can also trace the points to see their coordinates.</td><td>


</td></tr>
</table>

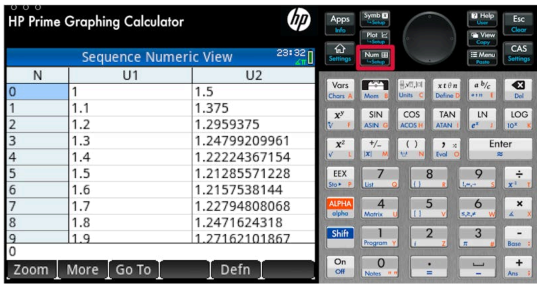
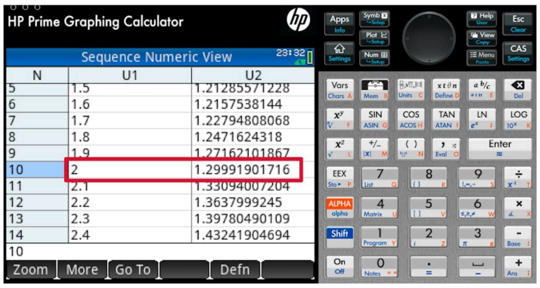
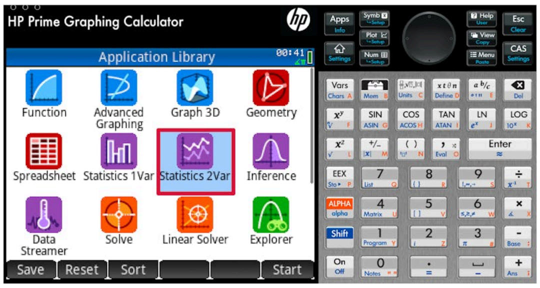
Student view

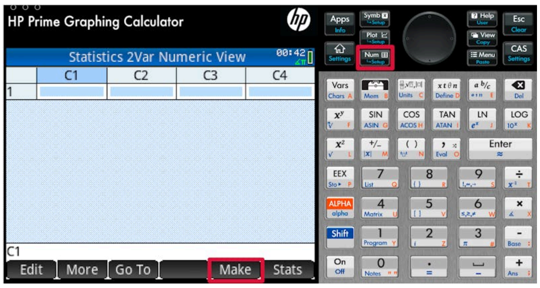
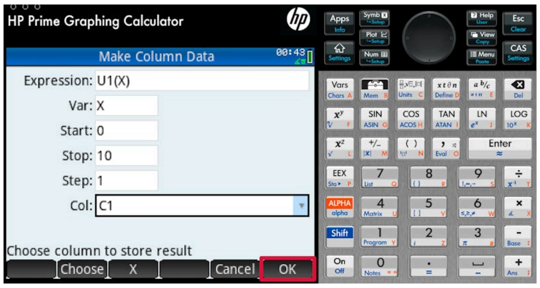
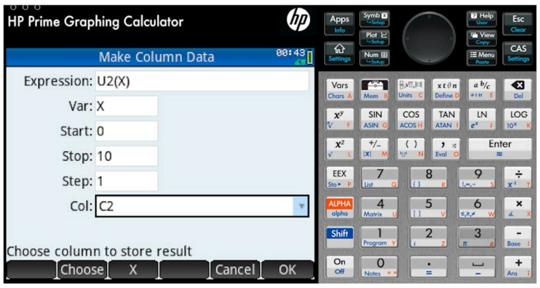
Overview
(/study/app/aa-hl/sid-134-cid-761926/o

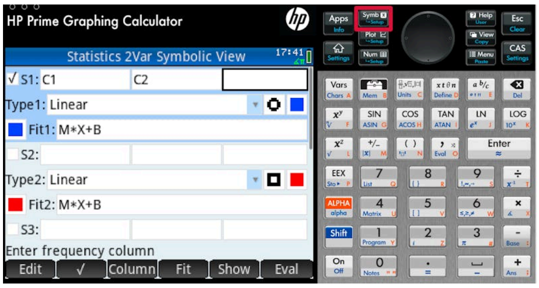
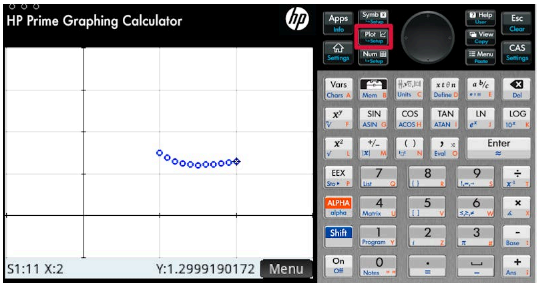
Steps	Explanation
	

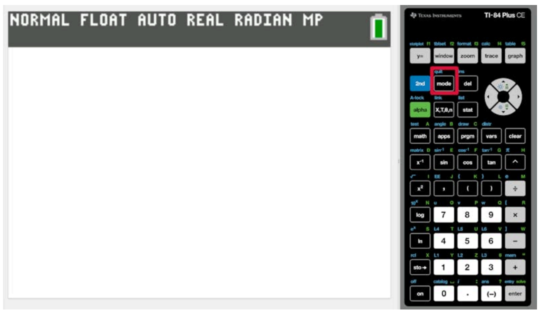
Steps	Explanation
The following instructions show you a way to use Euler's method with step size $h = 0.1$ to find an approximate value of $y(2)$, where $y(1) = 1.5$ and $y' = x - y^2$. You will also see a graphical illustration of the sequence of points leading to this approximation. Select the sequence application.	
You can tell the calculator about the recursion in the symbolic view. The recursive formula you need to use is $$x_{n+1} = x_n + 0.1$$ $$y_{n+1} = y_n + 0.1(x_n - y_n^2)$$ This calculator uses sequence names U1 and U2 instead of the names x and y in the formula. In this example the recursion starts with index 0. The condition $y(1) = 1.5$ means, that $$x_0 = 1$$ $$y_0 = 1.5$$ Make sure you carefully enter all information in the appropriate places.	

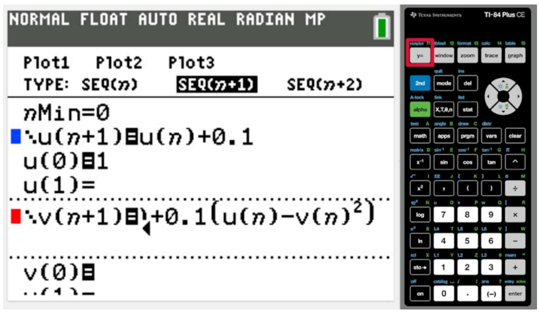
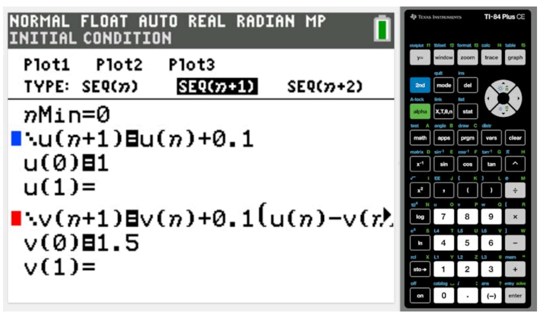
Student view

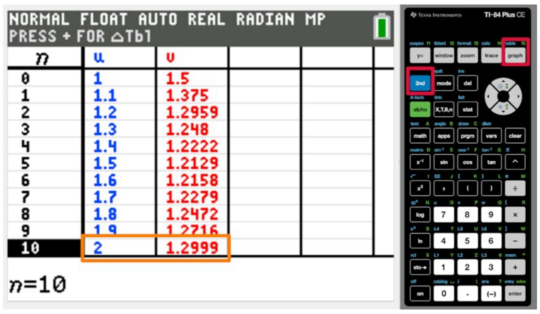
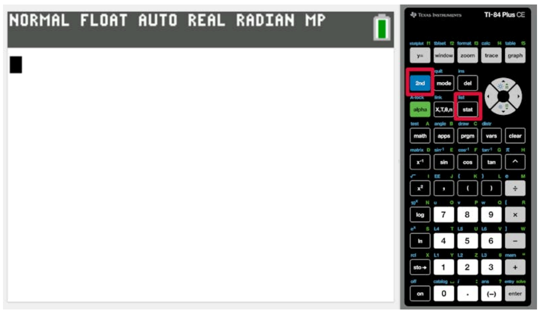
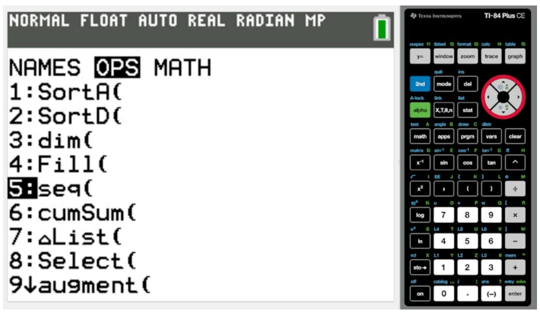
Steps	Explanation
The calculator now have all information needed, so bring up the numeric view to generate the table of values. You can see the starting x and y-values in the U1 and U2 sequences. Scroll down to see the rest of the sequence.	
The row corresponding to $x_{10} = 2$ gives y_{10}, the approximate value of $y(2)$.	
To generate a scatter plot of the sequence of the points leading to this approximation, go back to the application selector screen and choose the 2-variable statistics application.	

Steps	Explanation
You need to copy the U sequences to this application, so in numeric view tap on Make.	
Simply copy the first few terms of the U1 sequence to the C1 list. Since with step size 0.1, the tenth term is $x_{10} = 2$. Enter 10 as the last index. You can of course enter a larger number if you are interested in further values. Tap on OK, when you are done.	
Similarly, copy the U2 sequence to the C2 list.	

Steps	Explanation
In symbolic view, make sure that the x and y lists are given as C1 and C2, and also make sure, that the frequency list is empty. The regression type does not matter now, all you want to see is the scatter plot. If you do not use the fit option, the calculator will not draw the best fit line, just the scatter plot.	
In plot view you can see the scatter plot and you can trace the points to see the coordinates. You will probably need to adjust the viewing window to see the points.	

Steps	Explanation
The following instructions show you a way to use Euler's method with step size $h = 0.1$ to find an approximate value of $y(2)$, where $y(1) = 1.5$ and $y' = x - y^2$. You will also see a graphical illustration of the sequence of points leading to this approximation. First, you wil need to tell the calculator that you would like to work with sequences, so bring up the screen to change the settings.	

Steps	Explanation
Change from function to sequence mode.	
Press y= to bring up the screen to define the sequences. The recursive formula you need to use is $$x_{n+1} = x_n + 0.1$$ $$y_{n+1} = y_n + 0.1(x_n - y_n^2)$$ This calculator uses sequence names u and v instead of the names x and y in the formula. In sequence mode, the variable button gives an n on the screen instead of x.	
In this example the recursion starts with index 0. The condition $y(1) = 1.5$ means, that $$x_0 = 1$$ $$y_0 = 1.5$$	

Steps	Explanation
The calculator now have all information needed, so press 2nd/table to generate the table of values. The row corresponding to $x_{10} = 2$ gives y_{10}, the approximate value of $y(2)$.	
To generate a scatter plot of the sequence of the points leading to this approximation, go back to the calculator screen and choose the option to work with lists.	
Choose the option to generate a sequence.	

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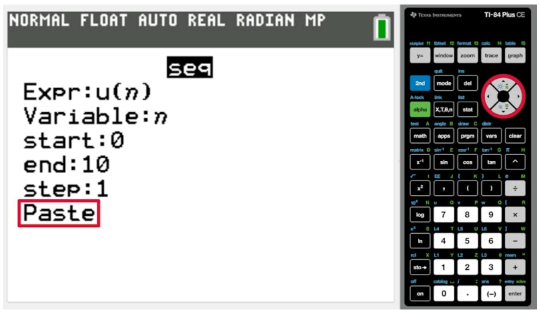
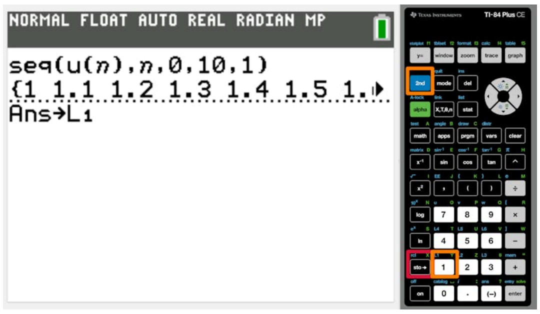
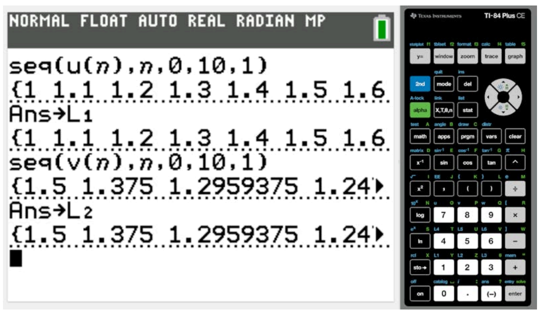
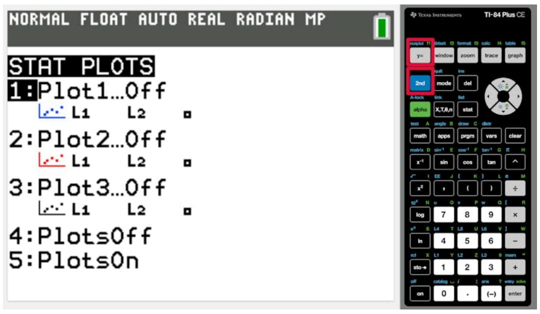
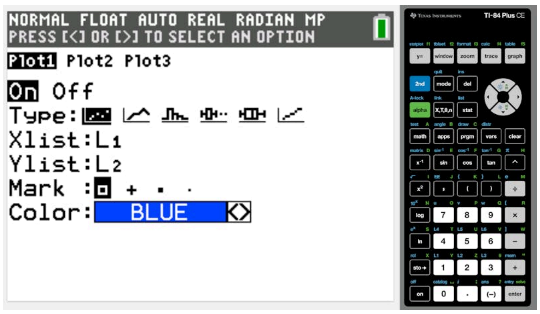
Steps	Explanation
Copy the first few terms of the u sequence to the list you are generating now. Since with step size 0.1, the tenth term is $x_{10} = 2$. Enter 10 as the last index. You can of course enter a larger number if you are interested in further values. When you are done, move to Paste and press enter.	
Store the resulting list in L1.	
Use similar steps to copy the v-list to L2. Now you have the x-coordinates of the approximating points in L1 and the y-coordinates in L2.	

Student view

Steps	Explanation
Change from sequence mode back to function mode.	
Bring up the screen to turn on a statistical plot, and choose any of the available plots.	
Turn the plot on, choose to see a scatter plot and make sure, that the x and y lists point to L1 and L2.	

Overview
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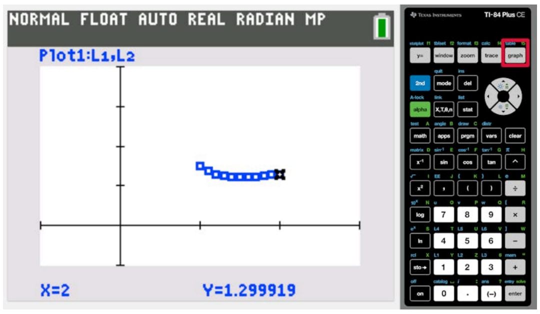
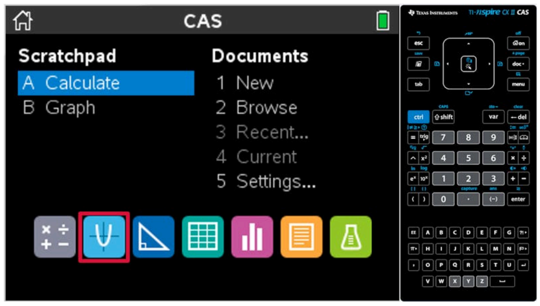
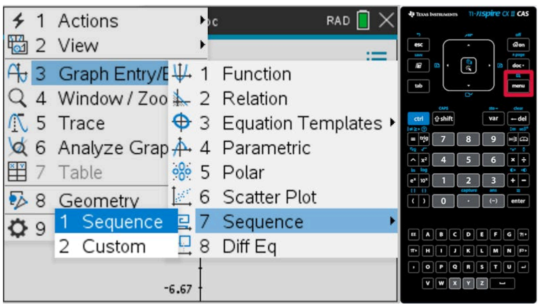
Steps	Explanation
Bring up the graph. You will probably need to adjust the viewing window to see the points. You can also use trace to move between the points and see the coordinates.	

Steps	Explanation
The following instructions show you a way to use Euler's method with step size $h = 0.1$ to find an approximate value of $y(2)$, where $y(1) = 1.5$ and $y' = x - y^2$. You will also see a graphical illustration of the sequence of points leading to this approximation. This can be done using the spreadsheet, but you will see a different approach here. Start with opening a graph page.	
Open the menu and choose to add a sequence.	

Student view

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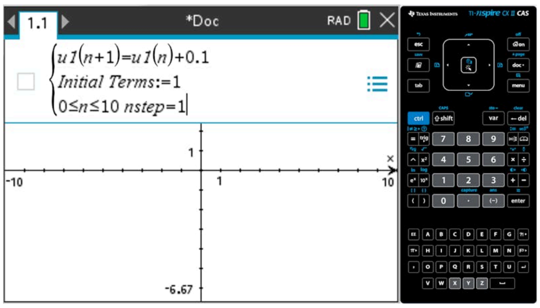
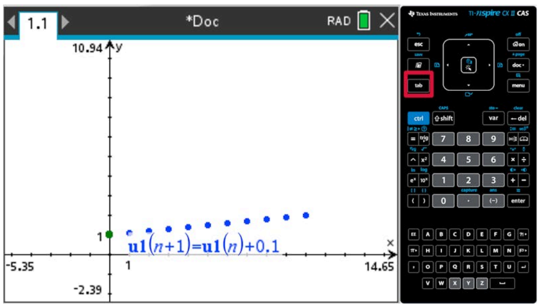
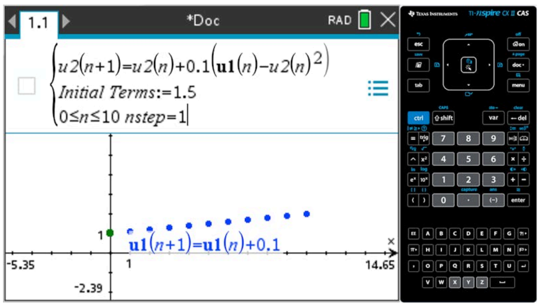
Steps	Explanation
The recursive formula you need to use is $x_{n+1} = x_n + 0.1$ $y_{n+1} = y_n + 0.1(x_n - y_n^2)$ On this screen you enter the recursion for the x sequence. In these instructions the default sequence name, $u1$ is used. In this example the recursion starts with index 0. The condition $y(1) = 1.5$ means, that $x_0 = 1$ $y_0 = 1.5$ Since with step size 0.1, the tenth term will be $x_{10} = 2$. Enter 10 as the last index. You can of course enter a larger number if you are interested in further values.	
To enter the other sequence, press tab.	
For the y-sequence the default name, $u2$ is used. Here are the formulas again for reference. $x_{n+1} = x_n + 0.1$ $y_{n+1} = y_n + 0.1(x_n - y_n^2)$ $x_0 = 1$ $y_0 = 1.5$	

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view

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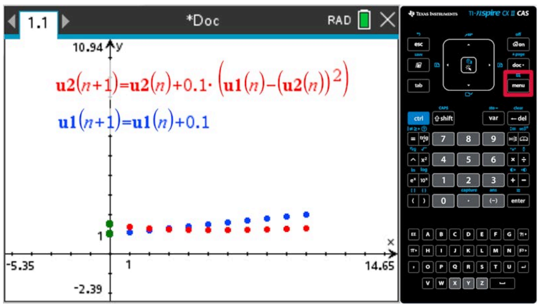
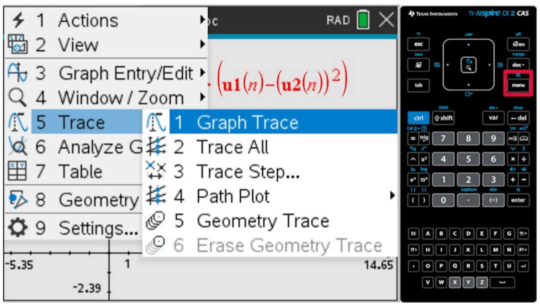
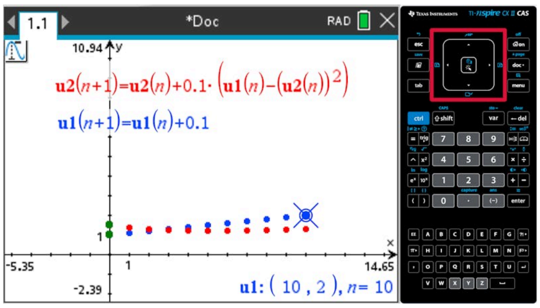
Steps	Explanation
The two sequences are plotted. To see the value of the terms, you can trace the graph. Open the menu ...	
... and choose to trace the graph.	
Confirm, that $x_{10} = 2$ ...	

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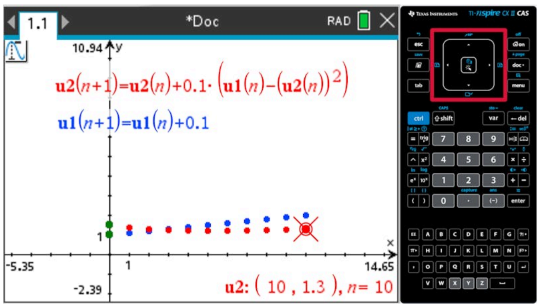
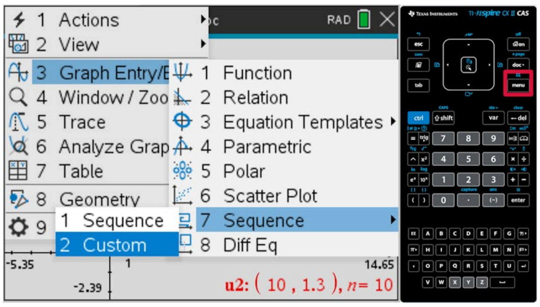
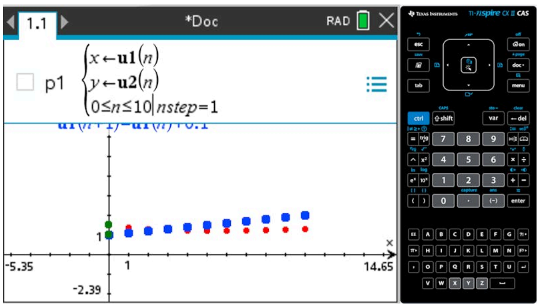
Steps	Explanation
... and check the tenth term of the $u2$ sequence, which is y_{10}, the approximation of $y(2)$.	
To look at the plot of the (x_n, y_n) points, open the menu and select to add a custom sequence.	
Remember, the $u1$ list contains the x-values and the $u2$ list contains the y-values.	

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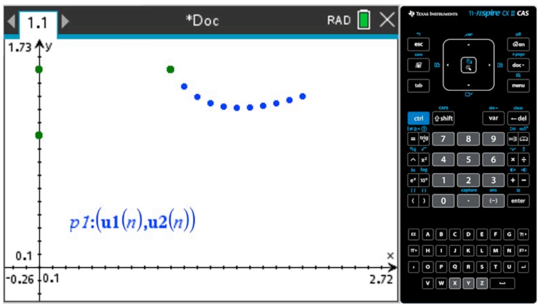
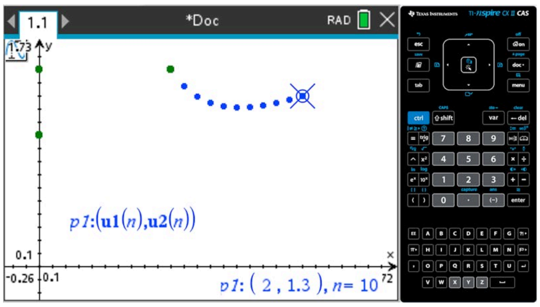
Steps	Explanation
After hiding the original two lists and adjusting the window, you can see the sequence of points leading to the approximation of $y(2)$.	
Using the menu, you can turn on trace and investigate the coordinates of the points in the sequence.	

4 section questions

Question 1

Difficulty:

★★☆

A solution of the differential equation $y' = xy$ satisfies $y(1) = 0.5$.

Use Euler's method with step size 0.1 to estimate $y(1.5)$.

Give your answer rounded to three significant figures. Give the numerical value only.

0.881

Accepted answers

0.881, 0.881, .881

Explanation

The following table shows the calculation.

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k	x_k	y_k	${y_k}' = x_k y_k$
0	1	0.5	0.5
1	1.1	0.55	0.605
2	1.2	0.6105	0.7326
3	1.3	0.68376	0.888888
4	1.4	0.7726488	1.08170832
5	1.5	0.880819632	

$0.880819632 \approx 0.881$

Question 2

Difficulty: ★★☆

A solution of the differential equation $y' = x + y$ satisfies $y(-1) = 1$.

Use Euler's method with step size 0.2 to estimate $y(1)$.

Give your answer rounded to three significant figures. Give the numerical value only.

4.19

Accepted answers

4.19, 4,19

Explanation

The following table shows the calculation.

k	x_k	y_k	${y_k}' = x_k + y_k$
0	-1	1	0
1	-0.8	1	0.2
2	-0.6	1.04	0.44
3	-0.4	1.128	0.728
4	-0.2	1.2736	1.0736
5	0	1.48832	1.48832
6	0.2	1.785984	1.985984
7	0.4	2.1831808	2.5831808
8	0.6	2.69981696	3.29981696

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k	x_k	y_k	${y_k}' = x_k + y_k$
9	0.8	3.359780352	4.159780352
10	1	4.191736422	

$4.191736422 \approx 4.19$

Question 3

Difficulty:

★★☆

A solution of the differential equation $y' = y^2 - x^2$ satisfies $y(2) = 1$.

Use Euler's method with step size -0.2 to estimate $y(0)$.

Give your answer rounded to three significant figures. Give the numerical value only.

0.655

Accepted answers
0.655, 0,655, .655

Explanation
The following table shows the calculation.

k	x_k	y_k	${y_k}' = {y_k}^2 - {x_k}^2$
0	2	1	-3
1	1.8	1.6	-0.68
2	1.6	1.736	0.453696
3	1.4	1.6452608	0.7468831
4	1.2	1.49588418	0.79766948
5	1	1.336350284	0.785832082
6	0.8	1.179183868	0.750474594
7	0.6	1.029088949	0.699024065
8	0.4	0.889284136	0.630826274
9	0.2	0.763118881	0.542350427
10	0.0	0.654648796	

$0.654648796 \approx 0.655$

Student
view

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Question 4

Difficulty:

★★☆

A solution of the differential equation $y' = y\cos x$ satisfies $y(0) = 1$.

Use Euler's method with step size 0.1 to estimate $y(1)$.

Give your answer rounded to three significant figures. Give the numerical value only.

2.29

Accepted answers

2.29, 2,29

Explanation

The following table shows the calculation.

k	x_k	y_k	${y_k}' = y_k \cos x_k$
0	0	1	1
1	0.1	1.1	1.0945045818
2	0.2	1.209450458	1.185341972
3	0.3	1.3279846551	1.268672198
4	0.4	1.454851875	1.340007314
5	0.5	1.588852607	1.394349341
6	0.6	1.728287541	1.42641726
7	0.7	1.870929267	1.430965633
8	0.8	2.01402583	1.403185309
9	0.9	2.154344361	1.33916193
10	1	2.288260554	

$2.288260554 \approx 2.29$

5. Calculus / 5.18 Differential equations

Exact solution: separable equations

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/exact-solution-separable-equations-id-27276/print/) Assign

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Solving through separation of variables

In the last section (/study/app/math-aa-hl/sid-134-cid-761926/book/numerical-solution-eulers-method-id-27275/), you used Euler's method to find a numerical approximation to a first-order ordinary differential equation. With the advent of computers, you may find that numerical methods are close enough as long as you continue to shrink the step size down, but there may come a time when you really want the exact answer without error. As you move away from your starting point, error becomes more significant. A closed-form exact answer may be much more valuable than a numerical solution that slowly diverges from the correct solution.

First, you will learn how to find exact solutions of separable equations ; specifically, first-order separable equations.

✓ **Important**

A first-order differential equation is **separable** if it can be written in the form $y' = f(x)g(y)$.

For separable equations, using the $\frac{\mathrm{d}y}{\mathrm{d}x}$ notation for the derivative is more helpful, so it is a good idea to write the equations as $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$. That allows you to separate the equation completely with all of the x-variables on one side and the y-variables on the other.

More formally, the steps leading to exact solutions of a separable equation are:

1. **Separate the variables** to write the equation in the form $\frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$.
2. **Integrate** both the left-hand side and right-hand side (with respect to x) to get $\int \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int f(x)\mathrm{d}x$ or, after substitution, on the left-hand side $\int \frac{1}{g(y)}\mathrm{d}y = \int f(x)\mathrm{d}x$.
3. **Find the integrals** of both sides and do not forget the constant of integration. This will lead to an implicit equation for y.
4. Most of the time, you will be asked to express y in terms of x, but sometimes this is not necessary (or possible), and an implicit equation is an acceptable answer.

You will use this process without going into detail about the conditions on f and g under which this method gives all the solutions of the differential equation. This would be beyond the scope of the syllabus. For example, if $g(y)$ is 0, then it is not valid to divide by it, but these subtleties are left for a future course.

When you first studied integration, you learned about indefinite and definite integrals. Indefinite integrals result in a family of functions, all looking very similar apart from a constant added at the end. To account for all of these functions, you added a "$+c$". In contrast, for a definite integral, you have some initial condition that allows you to find the value of the constant and produce one correct result.

Differential equations are similar. A general solution is one that still has a constant resulting from the integration in step 3. A particular solution is obtained from the general solution by specifying some set of initial conditions.

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Important

A **general solution** to the differential equation $y' = F(x, y)$ is a solution that involves an essential arbitrary constant.

A **particular solution** to the differential equation $y' = F(x, y)$ is the specific solution that also satisfies some initial condition $y(x_0) = y_0$.

The method you are using here to solve separable first-order differential equations requires you to find two integrals. Sometimes, these integrals are not too difficult to find, but in exams you may get questions in which a particular method is required to carry out the integration.

Exam tip

Make sure you review the techniques of integration. You will need to be efficient in finding anti-derivatives when you are asked to solve differential equations.

Example 1

★★☆

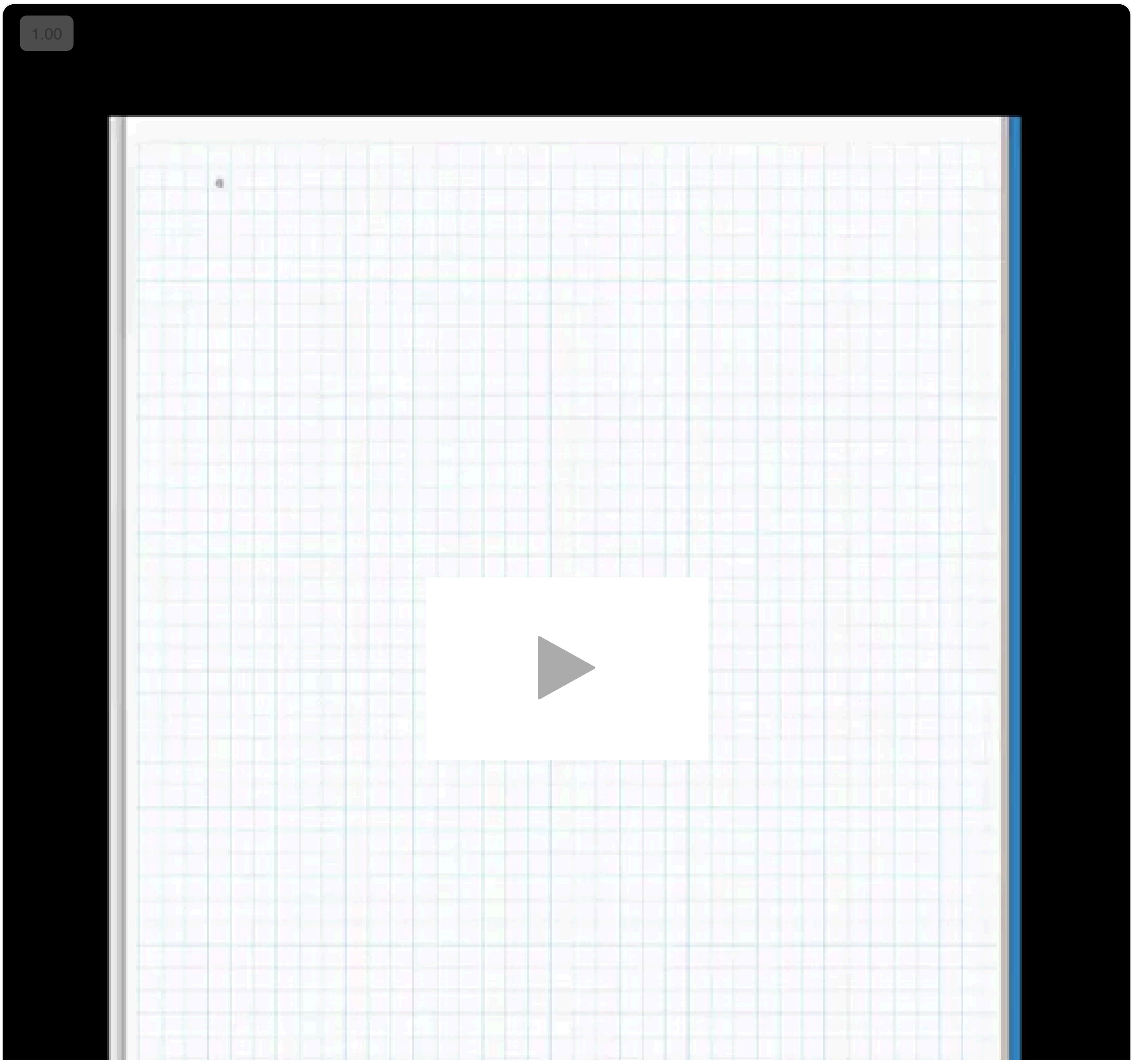
Write down the integrals you need to find to solve the following separable differential equations:

1. $\dfrac{y'}{x} = \dfrac{e^x}{y}$
2. $y' = \cos^2 x \cos^2 y$
3. $xy' \ln y = \ln x$
4. $\sqrt{1 - x^2} y' = 1 + y^2$

1. $yy' = xe^x$
   $\int y \, dy = \int xe^x dx$
2. $\dfrac{1}{\cos^2 y} y' = \cos^2 x$
   $\int \sec^2 y \, dy = \int \cos^2 x \, dx$
3. $y' \ln y = \dfrac{\ln x}{x}$
   $\int \ln y \, dy = \int \dfrac{\ln x}{x} dx$
4. $\int \dfrac{1}{1 + y^2} dy = \int \dfrac{1}{\sqrt{1 - x^2}} dx$

It is not the goal of this section to revise integration. However, it is a good exercise to find these integrals. You may want to revisit the sections on integration, in particular, indefinite integrals of exponential functions, integration by substitution and integration by parts. You can always check your answers using WolframAlpha (http://www.wolframalpha.com) or Symbolab (http://www.symbolab.com). Some hints on how to perform these integrals as well as the solutions are shown in the following video.

Student view



Video 1. Solutions to Differential Separable Equations.

More information for video 1

1

00:00:00,033 --> 00:00:02,233

narrator: In this video we're going to

take a look at exact solutions

2

00:00:02,300 --> 00:00:05,200

to differential equations

of separable form,

3

00:00:05,267 --> 00:00:09,933

which means that they, of the form

$y' = f(x) \cdot g(y)$,

4

00:00:10,000 --> 00:00:12,933

so that the x and y parts

are completely separated.

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5

00:00:13,200 --> 00:00:15,333

Now y' is of course $\frac{dy}{dx}$.

6

00:00:15,400 --> 00:00:18,400

Now this becomes two

integrals, $\int \frac{1}{g(y)} dy$,

7

00:00:18,467 --> 00:00:20,133

and $\int f(x) dx$.

8

00:00:20,200 --> 00:00:22,367

Now we're going to take a look

at the four examples

9

00:00:22,433 --> 00:00:28,000

in the text, which are $\int y dy = \int x e^x dx$,

$\int \sec^2 y dy =$

10

00:00:28,100 --> 00:00:32,000

$\int \cos^2 x dx$,

$\int \ln y dy = \int \frac{\ln x}{x} dx$

11

00:00:32,067 --> 00:00:34,367

and $\int \frac{1}{1+y^2} dy$,

12

00:00:34,433 --> 00:00:37,367

and $\int \frac{1}{\sqrt{1-x^2}} dx$.

13

00:00:37,433 --> 00:00:42,300

So the first one is the integral

of $\int y dy =$ and $\int x e^x dx$.

14

00:00:42,367 --> 00:00:44,667

Now the righthand one

is an integration by part,

15

00:00:44,733 --> 00:00:48,167

so we identify x as u

and then $v' = e^x$,

16

00:00:48,233 --> 00:00:51,967

$uv - u'v$

so that if we take v from v'

$xe^x - \int 1 e^x dx$

becomes e^x of course,

17

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00:00:52,033 --> 00:00:55,833

$xe^x - e^x + c$

and $u' = 1$ so that has become

an integral very simple

18

00:00:55,900 --> 00:00:58,800

$e^x(x - 1) + C$.

19

00:00:58,900 --> 00:01:02,500

Of course y is easily integrated

to $\frac{1}{2}y^2$.

20

00:01:02,900 --> 00:01:05,267

And this then gives a solution

$\frac{1}{2}y^2$

21

00:01:05,333 --> 00:01:08,200

is $e^x(x - 1) + C$.

22

00:01:08,333 --> 00:01:09,700

The second difference equations,

23

00:01:09,767 --> 00:01:13,300

$\frac{1}{\cos^2 y}y^1 = \cos^2 x$

24

00:01:13,533 --> 00:01:17,967

So this takes on the integral

of $\int \sec^2 y dy = \int \cos^2 x dx$.

25

00:01:18,133 --> 00:01:20,167

Let's start with $\int \cos^2 x dx$.

26

00:01:20,233 --> 00:01:23,233

We know that $\cos 2x = \cos^2 x - \sin^2 x$,

27

00:01:23,300 --> 00:01:25,267

which is also $2\cos^2 x - 1$.

28

00:01:25,333 --> 00:01:30,167

So that $\cos^2 x$

is $\frac{1}{2}(\cos 2x + 1)$.

29

00:01:30,667 --> 00:01:33,667

So the right hand side integral

becomes $\int \frac{1}{2}(\cos 2x + 1) dx$,

30

00:01:33,733 --> 00:01:34,833

Student
view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

plus 1 dx,

31

00:01:34,900 --> 00:01:39,400

which is equal to $\frac{1}{2}(\frac{1}{2}\sin 2x + x + C)$.

32

00:01:39,667 --> 00:01:44,133

And of course, we can take away the $2\sin x.\cos x$

$2x$ in favor of $\sin x$ and $\cos x$,

33

00:01:44,200 --> 00:01:45,333

so that's the right hand side.

34

00:01:45,800 --> 00:01:47,067

Now you see in your formula book,

35

00:01:47,133 --> 00:01:50,033

there is no standard

integral for $\sec^2 x$,

36

00:01:50,100 --> 00:01:54,600

however, the derivative

of $\tan x$ is $\sec^2 x$.

37

00:01:54,867 --> 00:01:56,900

So derivative of $\tan$

of x is $\sec^2 x$,

38

00:01:56,967 --> 00:01:59,100

and therefore the integral

$\tan x + c = \int \sec^2 x dx$

of $\sec^2 x dx$

39

00:01:59,167 --> 00:02:00,933

is of course $\tan x + C$.

40

00:02:01,333 --> 00:02:05,433

So the left hand side

of the integral becomes $\tan y$,

41

00:02:05,533 --> 00:02:07,333

and so the solution is $\tan y$

42

00:02:07,400 --> 00:02:10,933

is $\frac{1}{2}(\sin x \cos x + x + C)$.

43

00:02:11,633 --> 00:02:14,000

Now the third difference

in the equation was $\ln y$

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44

00:02:14,067 --> 00:02:16,467

$y' = \frac{\ln x}{x}$,

45

00:02:16,533 --> 00:02:21,033

so that becomes $\int \ln y dy$ and $\int \frac{\ln x}{x} dx$ integrated.

46

00:02:21,100 --> 00:02:22,433

Let's start with the right hand side.

47

00:02:22,533 --> 00:02:24,633

We make a substitution $x = e^z$,

48

00:02:24,700 --> 00:02:29,300

so that $dx = e^z dz$

or in other words, $\frac{dx}{x} = dz$,

49

00:02:29,667 --> 00:02:32,067

and of course $\ln x$ becomes z itself.

50

00:02:32,133 --> 00:02:35,867

So in other words,

$\int \frac{\ln x}{x} dx$ becomes $\int z dz$,

51

00:02:35,933 --> 00:02:38,500

which is easy enough

as $\frac{1}{2} z^2 + C$.

52

00:02:38,567 --> 00:02:42,167

Substitute back is $\frac{1}{2} (\ln x)^2 + C$.

53

00:02:42,500 --> 00:02:46,133

Now, for the left hand side, we'd make

a substitution again, $y = e^v$.

54

00:02:46,200 --> 00:02:50,633

So the $dy = e^v dv$,

and of course $\ln y = v$.

55

00:02:51,067 --> 00:02:54,833

Therefore the integral

becomes $\int ve^v dv$.

56

00:02:54,900 --> 00:02:57,333

But of course, we've already

seen what that integral is.

57

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00:02:57,400 --> 00:03:00,700

We already integrated $\int xe^x dx$

58

00:03:00,767 --> 00:03:03,600

and therefore this becomes

$e^v(v-1)$.

59

00:03:03,833 --> 00:03:08,033

And when you substitute back in favor y,

it becomes $y(\ln y - 1)$,

60

00:03:08,100 --> 00:03:11,167

so that the solution is $y(\ln y - 1)$

61

00:03:11,233 --> 00:03:13,933

equals $\frac{1}{2}(\ln x)^2 + C$.

62

00:03:14,700 --> 00:03:18,567

The fourth one was $\frac{1}{1+y^2}y'$

63

00:03:18,700 --> 00:03:20,867

equals $\frac{1}{\sqrt{1-x^2}}$.

64

00:03:20,933 --> 00:03:22,533

And when you separate 'em out,

$\arctan y = \arcsin x + c, |x| < 1$

65

00:03:22,600 --> 00:03:26,700

you immediately see

that you have a $\arctan$ and an $\arcsin$.

66

00:03:27,133 --> 00:03:29,533

So the left hand side becomes $\arctan y$,

67

00:03:29,633 --> 00:03:33,367

and the right hand side

becomes $\arcsin x + C$,

68

00:03:33,533 --> 00:03:36,833

where the absolute value

of x must be less than 1

69

00:03:37,500 --> 00:03:40,767

or $y = \tan(\arcsin x + C)$.

70

00:03:41,200 --> 00:03:43,500

And those are the four examples

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

71

00:03:43,800 --> 00:03:46,800

for separable differential equations.

Example 2

★☆☆

Determine whether $y' = x + y$ is separable. If so, find the general solution to the differential equation.

$y' = x + y$ is not separable.

Example 3

★★☆

Determine whether $y' = xy^2$ is separable. If so, find the general solution to the differential equation.

Steps	Explanation
$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2$	
$\frac{\mathrm{d}y}{y^2} = x\,\mathrm{d}x$	Separate
$\int \frac{1}{y^2}\mathrm{d}y = \int x\,\mathrm{d}x$	Integrate
$-\frac{1}{y} = \frac{x^2}{2} + c_1 = \frac{x^2 + 2c_1}{2}$	Find integrals
$y = -\frac{2}{x^2 + 2c_1}$	Solve for y
$y = -\frac{2}{x^2 + c}$	Substitute for a simpler constant

Notice the last step. Although not required, it is a good idea to simplify constants when possible. Any value you have for c_1 maps directly to a specific value for c.

Example 4

★★☆

Determine if $xy' = 1 + y^2$ is separable. If so, find the general solution to the differential equation.

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Steps	Explanation
$x\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$	
$\frac{\mathrm{d}y}{1+y^2} = \frac{\mathrm{d}x}{x}$	Separate
$\int \frac{1}{1+y^2}\mathrm{d}y = \int \frac{1}{x}\mathrm{d}x$	Integrate
$\arctan y = \ln x + c$	Find the integrals
$y = \tan(\ln x + c)$	Solve for y

Example 5

★★☆

Determine whether $xy' - y = 1 - y'$ is separable. If so, find the general solution to the differential equation.

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Steps	Explanation
$xy' + y' = 1 + y$	
$(x+1)y' = 1 + y$	Factorise
$\frac{\mathrm{d}y}{1+y} = \frac{\mathrm{d}x}{1+x}$	Separate
$\int \frac{1}{1+y}\mathrm{d}y = \int \frac{1}{1+x}\mathrm{d}x$	Integrate
$\ln\|1+y\| = \ln\|x+1\| + c$	Find the integrals
$\|1+y\| = \mathrm{e}^{\ln\|x+1\|+c}$	Solve for y
$\|y+1\| = \mathrm{e}^{c}\mathrm{e}^{\ln\|x+1\|}$	
$\|y+1\| = A\|x+1\|$	Here, $A = \mathrm{e}^{\mathrm{c}}$, and since c represents any constant, A can be any positive constant.
$y + 1 = \pm A(x+1)$	Two expressions in absolute value can be equal only if they are equal or if they are opposite of each other.
$y + 1 = A(x+1)$	You can ignore the plus/minus if A is now allowed to be any nonzero constant.
$y = A(x+1) - 1$, where A is any constant.	We can also include the solution for $A = 0$, which is the constant -1 function. This is lost when we divided by $y+1$, but it is the solution of the original equation.

Example 6

★★☆

Find the particular solution of the differential equation $xy' - y = 1 - y'$ that satisfies $y(1) = -2$.

From **Example 5**, you have already found the general solution to be $y = A(x+1) - 1$. By applying the initial conditions, you get

$$\begin{aligned} -2 &= A(1+1) - 1 \\ -1 &= 2A \\ A &= -\frac{1}{2} \\ y &= -\frac{1}{2}(x+1) - 1 \\ y &= -\frac{1}{2}x - \frac{3}{2} \end{aligned}$$

The following diagram illustrates the slope field and some solution curves of this differential equation, with the initial condition and the particular solution you just found highlighted.

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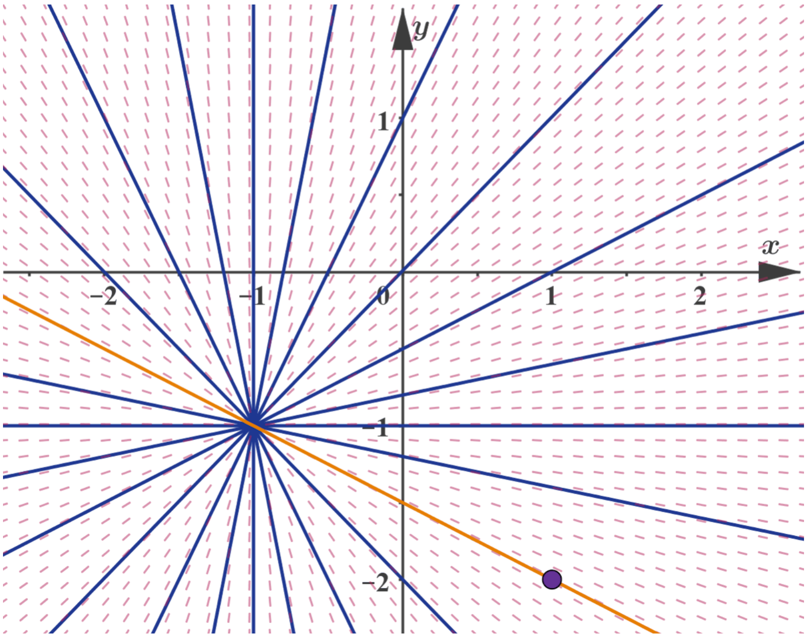


3 section questions

Question 1

Difficulty:

★☆☆

Three of the following differential equations are separable, and one is not. Find the equation that is not separable.

1 $y' = x + y$ ✓

2 $y' = xy$

3 $xy' = y$

4 $xy' = 1 + y$

Explanation

We separate the variables for the other three:

$y' = xy$ becomes $\frac{1}{y}y' = x$

$xy' = y$ becomes $\frac{1}{y}y' = \frac{1}{x}$

$xy' = 1 + y$ becomes $\frac{1}{1+y}y' = \frac{1}{x}$

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Question 2

Difficulty:

★★☆

The solution of the differential equation $xy' = y$ satisfies the initial condition $y(1) = 2$.

Find the **exact** value of $y(3)$. Give the numerical value only.

6 ✓

Accepted answers

6

Explanation

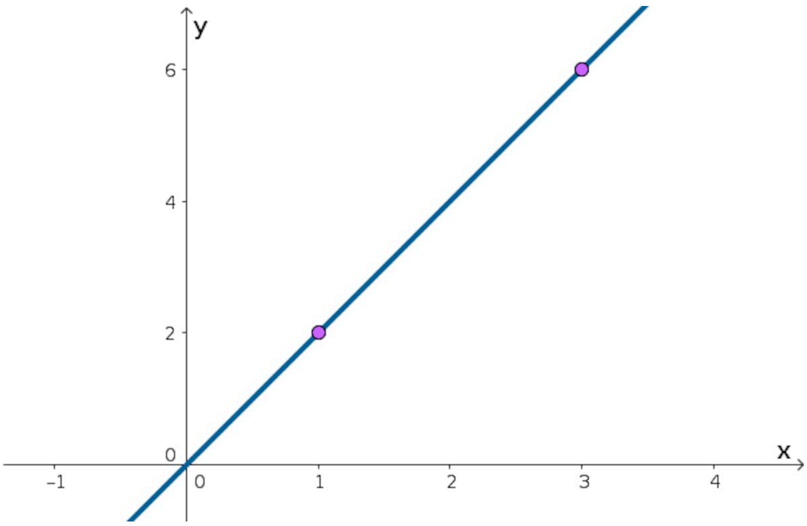
We separate and integrate:

$$\begin{aligned} xy' &= y \\ \frac{1}{y}y' &= \frac{1}{x} \\ \int \frac{1}{y}\,\mathrm{d}y &= \int \frac{1}{x}\,\mathrm{d}x \\ \ln|y| &= \ln|x| + c \\ |y| &= \mathrm{e}^c|x| \\ y &= Cx \end{aligned}$$

Since $y(1) = 2$, so $C = 2$.

Hence, $y(3) = 2 \times 3 = 6$.

The diagram below shows the solution curve with the two points mentioned in the question.



More information

Question 3

Difficulty:

★★★

A solution curve of the differential equation $y' = \dfrac{x}{y}$ goes through the point $(1, -2)$. It also goes through the point $(\sqrt{6}, k)$.

Find the **exact** value of k. Give the numerical value only.

Student view

-3 ✓

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Accepted answers

-3

Explanation

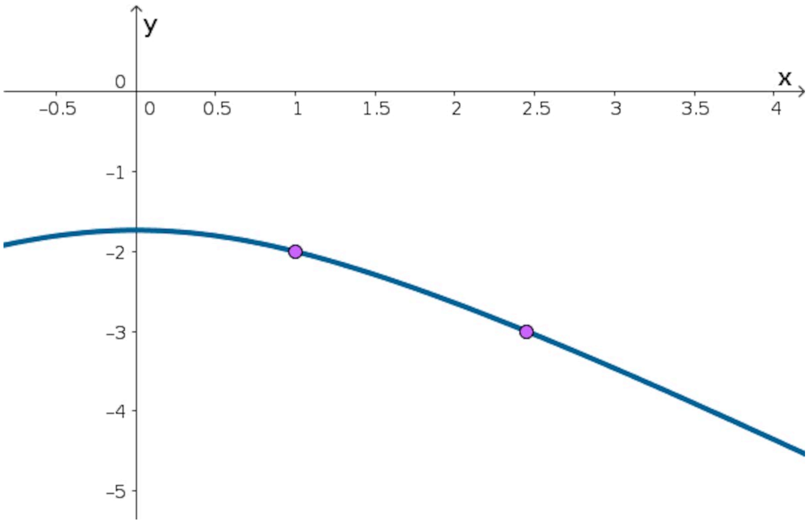
We separate and integrate:

$$\begin{aligned} y' &= \frac{x}{y} \\ yy' &= x \\ \int y\,\mathrm{d}y &= \int x\,\mathrm{d}x \\ \frac{y^2}{2} &= \frac{x^2}{2} + c \\ y^2 &= x^2 + 2c \\ y &= \pm\sqrt{x^2 + 2c} \end{aligned}$$

Since $y(1) = -2$, we choose the negative square root, and $-2 = -\sqrt{1^2 + 2c}$ implies that $2c = 3$, so $y = -\sqrt{x^2 + 3}$.

Hence, $y(\sqrt{6}) = -\sqrt{6 + 3} = -3$.

The diagram below shows the solution curve with the two points mentioned in the question.



More information

5. Calculus / 5.18 Differential equations

Exact solution: homogeneous equations

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/exact-solution-homogeneous-equations-id-27277/print/) Assign

In this section, you will learn how to solve homogeneous equations. First, you will need a formal definition of homogeneous differential equations.

Important

A first-order differential equation is **homogeneous** if it can be written in the form $y' = F\left(\frac{y}{x}\right)$.

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Note that equations of the form $p_n(x)y^{(n)} + \cdots + p_1(x)y' + p_0(x)y = 0$ are also called homogeneous, but we will not deal with these higher-order equations.

Solving homogeneous equations through substitution

The trick of solving a homogeneous differential equation is to use the substitution $y = vx$. Using the product rule, you can find the derivative of this substitution $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x}x + \frac{\mathrm{d}x}{\mathrm{d}x}v$, which can be simplified and rewritten as $y' = v'x + v$. With these substitutions, you can rewrite the homogeneous equation:

$$y' = F\left(\frac{y}{x}\right) \Leftrightarrow v'x + v = F(v)$$

Notice that this new equation is separable, which means that once you have applied the substitution, you can use the techniques learned in the previous section.

Is there any requirement to substitute $v = \frac{y}{x}$? Technically, no. Substituting $u = \frac{x}{y}$ works equally well, but, for simplicity, you will just use one option for substitution in this course.

The following steps summarise this process:

1. **Rearrange** the equation into the form $y' = F\left(\frac{y}{x}\right)$.
2. Use the **substitution** $y = vx$ and $y' = v'x + v$.
3. Solve the resulting **separable** equation $xv' + v = F(v)$.
4. **Replace** v with y/x to get an implicit equation for y, which can sometimes be rearranged to make for y the subject.

The first step in solving a homogeneous equation is recognising that it is indeed homogeneous. After doing several examples, you will be able to tell this quite easily by inspection. You need to rearrange the equation into the form $y' = f(x, y)$ and try to rearrange $f(x, y)$ as a function of $\frac{y}{x}$. If you succeed, it is a homogeneous equation, but if not, then it is probably not homogeneous.

Example 1

★☆☆

The following are homogeneous equations. Rearrange them into the form $y' = F\left(\frac{y}{x}\right)$.

1. $x^2y - y^3y' = x^3 + xy^2y'$
2. $y' + \ln y = \ln x$
3. $xy' = \sqrt{x^2 + y^2}$

1. $x^2y - y^3y' = x^3 + xy^2y'$

$$y^3y' + xy^2y' = x^2y - x^3$$

$$y'\left(y^3 + xy^2\right) = x^2y - x^3 = \frac{x^2y - x^3}{y^3 + xy^2}\left(\frac{1/x^3}{1/x^3}\right) = \frac{\left(\frac{y}{x}\right) - 1}{\left(\frac{y}{x}\right)^3 + \left(\frac{y}{x}\right)^2}$$

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2. $y' + \ln y = \ln x$

$$y' = \ln x - \ln y = \ln\left(\frac{x}{y}\right) = \ln\left(\frac{y}{x}\right)^{-1} = -\ln\left(\frac{y}{x}\right)$$

3. $xy' = \sqrt{x^2 + y^2}$

$$y' = \frac{\sqrt{x^2 + y^2}}{x}$$

If $x > 0$:

$$y' = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2}} = \sqrt{\frac{x^2 + y^2}{x^2}} = \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

If $x < 0$:

$$y' = \frac{\sqrt{x^2 + y^2}}{-\sqrt{x^2}} = -\sqrt{\frac{x^2 + y^2}{x^2}} = -\sqrt{1 + \left(\frac{y}{x}\right)^2}$$

Example 2

★★★

Find all solutions of the differential equation $xy' = x + 2y$.

1. Rearrange

$$\begin{aligned} xy' &= x + 2y \\ y' &= \frac{x + 2y}{x} \\ y' &= 1 + 2\left(\frac{y}{x}\right) \end{aligned}$$

2. Substitute $y = vx$ and $y' = xv' + v$

$$\begin{aligned} y' &= 1 + 2\left(\frac{y}{x}\right) \\ xv' + v &= 1 + 2v \end{aligned}$$

3. Solve the separable equation

Student view

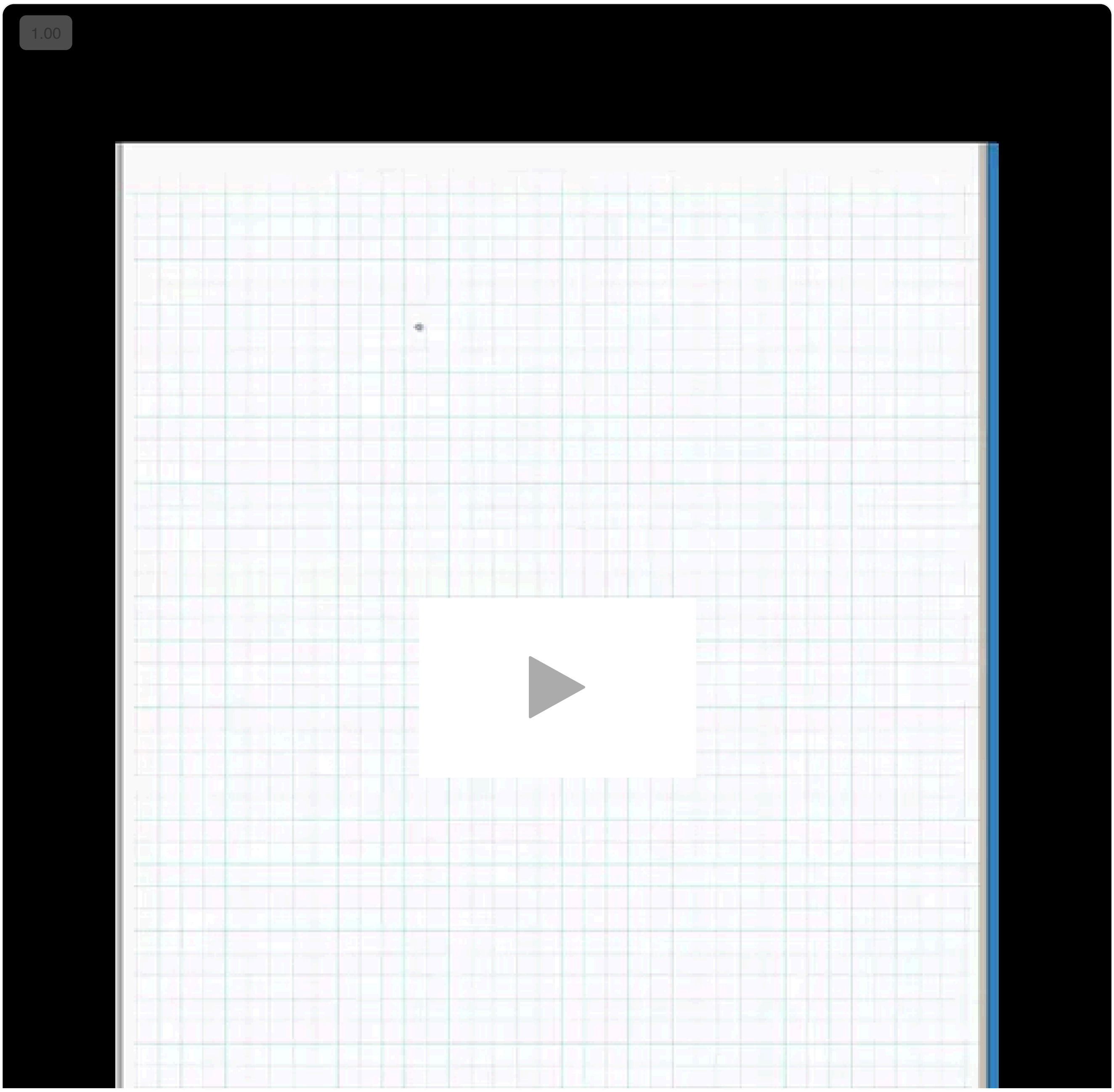
$$\begin{aligned} xv' + v &= 1 + 2v \\ xv' &= 1 + v \\ \frac{1}{1+v} v' &= \frac{1}{x} \\ \int \frac{1}{1+v} dv &= \int \frac{1}{x} dx \\ \ln|1+v| &= \ln|x| + c \\ |1+v| = e^{\ln|x|+c} &= e^{\ln|x|} e^c = e^c|x| \\ 1 + v &= \pm e^c x \\ v &= \pm e^c x - 1 \\ v &= Ax – 1 \end{aligned}$$

4. Replace $v = \frac{y}{x}$

$$\begin{aligned} v &= Ax - 1 \\ \frac{y}{x} &= Ax - 1 \\ y &= Ax^2 – x \end{aligned}$$

A worked solution to **Example 2** is shown in the video below.

Overview
(/study/ap
aa-
hl/sid-
134-
cid-
761926/o



Video 1. Solution for a Differential Equation.

More information for video 1

1

00:00:00,067 --> 00:00:02,400

narrator: In this video,

we're going to find exact solutions

2

00:00:02,467 --> 00:00:05,533

to differential equations

in the homogeneous form.

3

00:00:06,033 --> 00:00:08,033

So the homogeneous form is a form

4

00:00:08,167 --> 00:00:11,033

where we have $y' = F\left(\frac{y}{x}\right)$.

5

Student
view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

00:00:11,133 --> 00:00:13,000

of y divided by x.

6

00:00:13,100 --> 00:00:15,200

The way to deal with them

is to make a substitution

7

00:00:15,267 --> 00:00:19,167

that $\frac{y}{x} = v$,

or other words, $y = vx$.

8

00:00:19,233 --> 00:00:22,933

Then you immediately see that y'

is $v'x + v$

9

00:00:23,000 --> 00:00:25,200

using the product rule of differentiation.

10

00:00:25,600 --> 00:00:30,767

Hence, we get that $v'x + v = F(v)$.

11

00:00:31,133 --> 00:00:34,567

Now we can rearrange

that $v'x = F(v) - v$.

12

00:00:34,633 --> 00:00:38,867

In other words, $v' = (F(v) - v) \cdot \frac{1}{x}$.

13

00:00:38,933 --> 00:00:42,267

And now you see

that we have a product where

14

00:00:42,333 --> 00:00:44,367

one part is function of v

15

00:00:44,433 --> 00:00:47,600

and the other part

is a function of x equal to v'.

16

00:00:47,867 --> 00:00:50,400

Hence we have the separable form

which we can solve.

17

00:00:50,733 --> 00:00:54,533

So $v' = (F(v) - v) \cdot \frac{1}{x}$,

18

00:00:54,600 --> 00:00:58,367

such that $\frac{1}{F(v) - v} v' = \frac{1}{x}$.

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

19

00:00:58,433 --> 00:00:59,800

v' is $\frac{1}{x}$.

20

00:00:59,867 --> 00:01:02,233

But of course, $v' = \frac{dv}{dx}$.

21

00:01:02,300 --> 00:01:07,433

So we can say that $\frac{1}{F(v) - v}dv = \frac{1}{x}dx$,

22

00:01:07,500 --> 00:01:09,600

which we can then integrate on both sides,

23

00:01:09,900 --> 00:01:11,133

hence the separable form.

24

00:01:11,200 --> 00:01:14,667

Now let's apply this

to the example two in the text.

25

00:01:14,933 --> 00:01:17,000

So let's have a look at example

26

00:01:17,267 --> 00:01:18,633

two in a little bit of detail.

27

00:01:18,833 --> 00:01:22,633

So $xy' = x + 2y$.

28

00:01:22,933 --> 00:01:27,567

If we divide through by x,

we got $y' = 1 + \frac{2y}{x}$,

29

00:01:27,733 --> 00:01:29,167

which is in homogeneous form.

30

00:01:29,233 --> 00:01:33,567

So let $y = v(x)x$ such

at $y' = v'x + v$.

31

00:01:34,267 --> 00:01:38,033

Now the equation becomes $v'x + v = 1 + 2v$

32

00:01:38,100 --> 00:01:42,600

such that $v'x = 1 + v$.

33

00:01:42,667 --> 00:01:46,000

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

And in other words,

$v' = \frac{1}{1+v} \cdot \frac{1}{x}$.

34

00:01:46,133 --> 00:01:49,267

And of course, $\frac{1}{1+v}v' = \frac{1}{x}$,

35

00:01:49,633 --> 00:01:53,433

such that $\frac{1}{1+v}dv = \frac{1}{x}dx$,

36

00:01:53,500 --> 00:01:55,700

which we then integrate on both sides.

37

00:01:56,700 --> 00:01:59,633

So we had the integral

of $\int \frac{1}{1+v}dv$

38

00:01:59,700 --> 00:02:01,933

equals the integral of $\int \frac{1}{x}dx$,

39

00:02:02,000 --> 00:02:05,167

we can see that we have natural

logarithm as a result.

40

00:02:05,233 --> 00:02:09,200

So $\ln|1 + v| = \ln|x| + C$.

41

00:02:09,267 --> 00:02:11,133

Taking the natural exponent on both sides

42

00:02:11,200 --> 00:02:15,967

gives such the absolute value of $1 + v$

is $e^{\ln|x|+C}$,

43

00:02:16,033 --> 00:02:19,633

which of course use exponent losses

$e^{\ln|x|} \cdot e^{C}$,

44

00:02:19,700 --> 00:02:22,400

which is $e^{C} \cdot |x|$,

45

00:02:22,467 --> 00:02:26,333

such as $1 + v = \pm e^{C}x$,

46

00:02:26,400 --> 00:02:27,733

where C is are constant.

47

00:02:27,800 --> 00:02:28,933

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

In other words, if C are constant,

48

00:02:29,000 --> 00:02:31,900

then e^C is a constant,

which you might as well call A,

49

00:02:31,967 --> 00:02:36,933

such as $1 + v = Ax$,

and $v = Ax - 1$.

50

00:02:37,267 --> 00:02:39,533

So $v = Ax - 1$,

51

00:02:39,600 --> 00:02:42,033

but of course v was $\frac{y}{x}$

52

00:02:42,100 --> 00:02:44,867

and therefore we get $\frac{y}{x} = Ax - 1$.

53

00:02:44,933 --> 00:02:50,567

In other words, $y = Ax^2 - x$,

where A is some real number.

54

00:02:50,733 --> 00:02:52,767

So let's check this, it might be worth it.

55

00:02:52,833 --> 00:02:55,300

So $y' = 2Ax - 1$,

56

00:02:55,933 --> 00:02:58,933

such that $xy' = 2Ax^2 - x$,

57

00:02:59,000 --> 00:03:02,700

but $y = Ax^2 - x$.

58

00:03:02,800 --> 00:03:05,967

Therefore $Ax^2 = y + x$

59

00:03:06,100 --> 00:03:08,667

and $2Ax^2 = 2y + 2x$

60

00:03:08,733 --> 00:03:12,800

such that we get $xy' = x + 2y$,

61

00:03:12,867 --> 00:03:14,667

which if we look at the original question,

62

00:03:14,733 --> 00:03:18,133

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

was indeed the differential equation

that we had to start off with.

63

00:03:18,200 --> 00:03:19,933

So that works quite nicely

64

00:03:20,100 --> 00:03:23,800

and that was the homogeneous form

with this exact solution.

Example 3

★★★

Find the particular solution of the differential equation $x^2y' = x^2 + y^2 + xy$ that satisfies $y(1) = 0$.

1. Rearrange

$$\begin{aligned} x^2y' &= x^2 + y^2 + xy \\ y' &= \frac{x^2 + y^2 + xy}{x^2} \\ &= 1 + \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) \end{aligned}$$

2. Substitute $y = vx$ and $y' = xv' + v$

$$\begin{aligned} y' &= 1 + \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) \\ xv' + v &= 1 + v^2 + v \end{aligned}$$

3. Solve the separable equation

$$\begin{aligned} xv' + v &= 1 + v^2 + v \\ xv' &= 1 + v^2 \\ \frac{1}{1+v^2}v' &= \frac{1}{x} \\ \int \frac{1}{1+v^2}\mathrm{d}v &= \int \frac{1}{x}\mathrm{d}x \\ \arctan v &= \ln|x| + c \end{aligned}$$

Recall from subtopic 5.15:

$$\int \frac{1}{a^2 + x^2}\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

In this case, $a = 1$.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o

4. Replace $v = \dfrac{y}{x}$

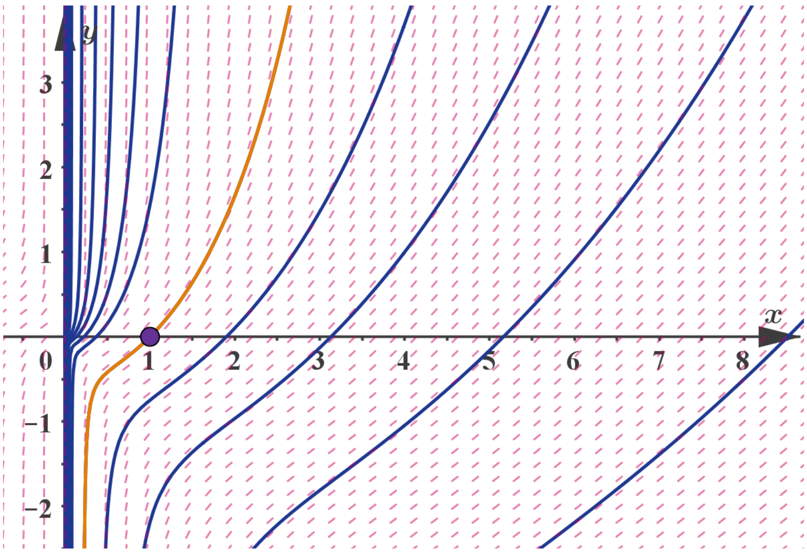
$$\begin{aligned}\arctan v &= \ln|x| + c\\ \arctan\left(\frac{y}{x}\right) &= \ln|x| + c\\ \frac{y}{x} &= \tan(\ln|x| + c)\\ y &= x\tan(\ln|x| + c)\end{aligned}$$

Apply $y(1) = 0$

$$\begin{aligned}&0 = 1\tan(\ln|1| + c) = \tan(0 + c) = \tan c\\ &c = \tan^{-1}(0) = 0\\ &y = x\tan(\ln|x|)\end{aligned}$$

As with most problems in maths, there are alternative paths to the correct solution. Although this example waited until step 4 to evaluate $y(1) = 0$, this could have been done during step 3 instead.

The following diagram illustrates the slope field and some solution curves of this differential equation, with the initial condition and the particular solution you just found highlighted.



More information

This diagram is a slope field for a differential equation with additional solution curves. The X-axis is labeled with values from 0 to 8, and the Y-axis runs vertically, increasing upwards. The diagram features a series of short, dashed pink lines indicating the slope direction at various points in the field. Overlaid are several curved lines, some in blue and others in orange, representing particular solution curves of the differential equation. One particular line and an initial condition point is highlighted. The pattern shows how the slopes guide the curve behavior in the two-dimensional space.

[Generated by AI]

4 section questions

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

Question 1

Difficulty: ★☆☆

Only one of the following differential equations is homogeneous. Which one?

1 $y^2 + x^2y' = xy - xyy'$ ✓

2 $xy + xy' = x^2 - yy'$

3 $xy + x^2y' = x - y^2y'$

4 $y + x^2y' = xy - y^2y'$

Explanation

Rewriting:

$$\begin{aligned} y^2 + x^2y' &= xy - xyy' \\ x^2y' + xyy' &= xy - y^2 \\ y'\left(x^2 + xy\right) &= xy - y^2 \\ y' &= \frac{xy - y^2}{x^2 + xy} \end{aligned}$$

An equation is **homogeneous** if $f(tx, ty) = t^n f(x, y)$.

$$f'(tx, ty) = \frac{(tx)(ty) - (ty)^2}{(tx)^2 + (tx)(ty)} = \frac{t^2xy - t^2y^2}{t^2x^2 + t^2xy} = \frac{t^2\left(xy - y^2\right)}{t^2\left(x^2 + xy\right)} = t^0\left(\frac{xy - y^2}{x^2 + xy}\right)$$

This means the differential equation is homogeneous of degree 0, while both the numerator and denominator are homogeneous of degree 2.

Alternatively, you could continue solving:

$$y' = \left(\frac{xy - y^2}{x^2 + xy}\right)\left(\frac{1/x^2}{1/x^2}\right) = \frac{\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2}{1 + \left(\frac{y}{x}\right)}$$

Question 2

Difficulty: ★★☆

The solution of the differential equation $xy' = 2x + 3y$ satisfies the initial condition $y(1) = 0$.

Find the **exact** value of $y(2)$. Give the numerical value only.

6 ✓

Accepted answers

6

Explanation

1. Dividing the equation by x gives $y' = 2 + 3\dfrac{y}{x}$.
2. Then we make the substitutions $y = vx$ and $y' = v'x + v$ to give the equation $v'x + v = 2 + 3v$.
3. Now, solving this equation:

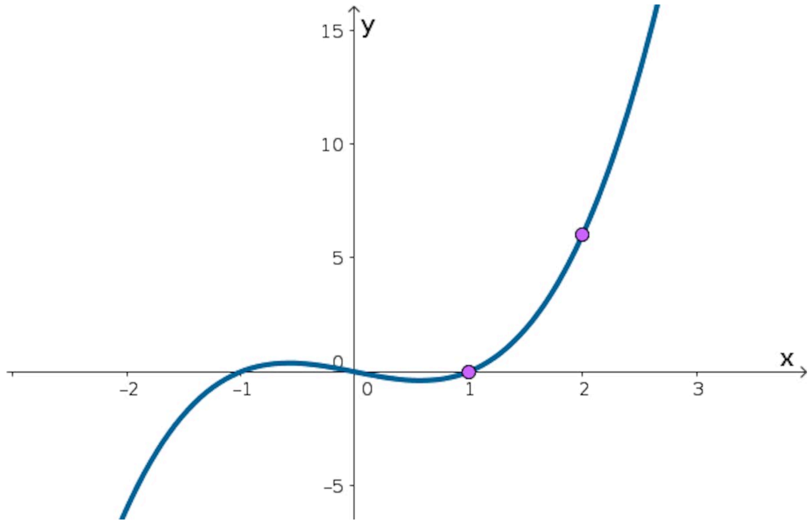
Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

$$\begin{aligned} v'x + v &= 2 + 3v \\ v'x &= 2 + 2v \\ &= 2(1+v) \\ \frac{1}{1+v}v' &= \frac{2}{x} \\ \int \frac{1}{1+v}\,\mathrm{d}v &= \int \frac{2}{x}\,\mathrm{d}x \\ \ln|1+v| &= 2\ln|x| + c \\ |1+v| &= x^2\mathrm{e}^c \\ 1+v &= Cx^2 \\ v &= Cx^2 - 1 \end{aligned}$$

4. Since $y = vx$, this means that $y = Cx^3 - x$.
   Using the initial condition, $y(1) = 0$, we get $C = 1$, so $y = x^3 - x$.
5. Hence, $y(2) = 2^3 - 2 = 6$.

The diagram below shows the solution curve with the two points mentioned in the question.



More information

Question 3

Difficulty:

★★★

The solution of the differential equation $xy^2y' = y^3 - x^3$ satisfies $y(1) = 2$. Evaluate $y(2)$ giving the answer to three significant figures. Give the numerical answer only.

3.62

Accepted answers

3.62, 3,62

Explanation

1. Rearrange

$$\begin{aligned} xy^2y' &= y^3 - x^3 \\ y' &= \frac{y^3 - x^3}{xy^2} = \left(\frac{y}{x}\right) - \left(\frac{x^2}{y^2}\right) = \left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^{-2} \end{aligned}$$

2. Substitute

$$xv' + v = v - \frac{1}{v^2}$$

3. Solve

Student view

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761926/o

$$\begin{aligned} xv' + v &= v - \frac{1}{v^2} \\ xv' &= -\frac{1}{v^2} \\ v^2 v' &= -\frac{1}{x} \\ \int v^2 \mathrm{d}v &= \int -\frac{1}{x} dx \\ \frac{v^3}{3} &= -\ln|x| + c \\ v^3 &= -3\ln|x| + c \end{aligned}$$

4. Replace

$$\begin{aligned} \left(\frac{y}{x}\right)^3 &= -3\ln|x| + c \\ y^3 &= -3x^3\ln|x| + cx^3 \end{aligned}$$

Apply $y(1) = 2$

$$\begin{aligned} (2)^3 &= -3(1)^3\ln|1| + c(1)^3 \\ 8 &= c \\ y^3 &= -3x^3\ln|x| + 8x^3 \end{aligned}$$

Evaluate $y(2)$

$$\begin{aligned} y^3 &= -3(2)^3\ln|2| + 8(2)^3 \\ y^3 &= 47.3644676\ldots \\ y &= 3.61813044\ldots \approx 3.62 \end{aligned}$$

Question 4

Difficulty:

★★★

A solution curve of the differential equation $xyy' = x^2 + y^2$ is defined for $x > 1$ and goes through the point $(2, \sqrt{4\ln 4})$.

Find $\lim\limits_{x\to 1^+} y(x)$.

0

Accepted answers

0

Explanation

1. Dividing the equation by xy gives $y' = \frac{x}{y} + \frac{y}{x}$.
2. The substitution $y = vx$ and $y' = v'x + v$ gives the equation $v'x + v = \frac{1}{v} + v$.
3. Solving this equation,

$$\begin{aligned} v'x + v &= \frac{1}{v} + v \\ v'x &= \frac{1}{v} \\ vv' &= \frac{1}{x} \\ \int v\,\mathrm{d}v &= \int \frac{1}{x}\,\mathrm{d}x \\ \frac{v^2}{2} &= \ln|x| + c \\ v^2 &= 2\ln|x| + 2c \\ &= \ln x^2 + 2c \end{aligned}$$

4. Since $y = vx$, this means that $y^2 = x^2(\ln x^2 + 2c)$.
5. Using the initial condition, $y(2) = \sqrt{4\ln 4}$, we get $4\ln 4 = 2^2\ln 2^2 + 2^2 2c$ and, hence, $c = 0$.
6. Hence, $y^2 = 2x^2\ln x$,
   so $\lim\limits_{x\to 1^+} y^2(x) = \lim\limits_{x\to 1^+} 2x^2\ln x = 2\times 1^2\times\ln 1 = 0$ and hence $\lim\limits_{x\to 1^+} y(x) = 0$.

Student
view

The diagram below shows the graph of this solution curve.

Overview
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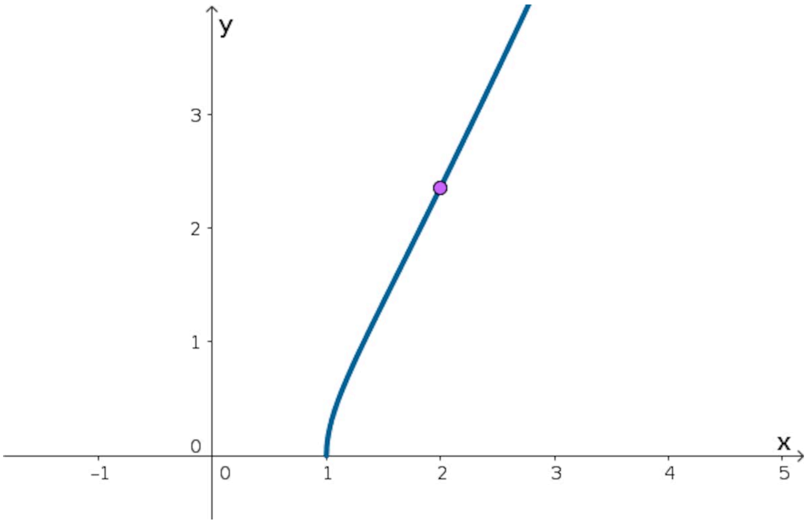

More information

5. Calculus / 5.18 Differential equations

Exact solution: linear equations

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/exact-solution-linear-equations-id-27278/print/) Assign

Identifying first-order linear equations

Now, turn your attention to first-order linear differential equations.

In section 5.18.1 (/study/app/math-aa-hl/sid-134-cid-761926/book/firstorder-differential-equations-id-27274/), you learned how to classify differential equations. You learned that an ordinary differential equation is linear if:

1. The dependent variable y and all of its derivatives are of first degree.
2. The coefficients of all terms in the dependent variable and its derivatives depend only on the independent variable x.

In a very general sense, that means an nth order ordinary differential equation is of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0y - g(x) = 0$$

Or, alternatively,

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0y = g(x)$$

Although exploration of higher-order differential equations is interesting, and you may study them if you pursue maths to a higher level, in this course you will only solve first-order linear equations of the form

Student view

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0y = g(x)$$

If you were to divide all terms by the coefficient associated with the derivative, you would get

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{a_0}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

This is the more useful **standard form** of a linear differential equation.

✓ **Important**

A first-order differential equation is **linear** if it can be written in the form $y' + P(x)y = Q(x)$.

Solving through the use of an integrating factor

Of the differential equations you will solve, this is probably the easiest type to recognise, but this involves more steps than the other types.

Consider the differential equation $y' = y + \mathrm{e}^x$. As this is not separable, you need another technique.

✓ **Important**

In general, differential equations of the form $y' + P(x)y = Q(x)$ can be solved through the use of an integrating factor.

Here are the recommended steps to use:

1. Write the equation in the form $y' + P(x)y = Q(x)$, and identify $P(x)$ and $Q(x)$.
2. Find $\int P(x)\mathrm{d}x$, i.e. find any anti-derivative of P.
3. Find (and simplify) $I(x) = \mathrm{e}^{\int P(x)\mathrm{d}x}$ (this is called the **integrating factor**).
4. Multiply the equation by the integrating factor to get $I(x)y'(x) + I(x)P(x)y(x) = I(x)Q(x)$.
5. Integrate both sides of the equation to get $I(x)y(x) = \int I(x)Q(x)\mathrm{d}x$.
6. Hence, the solution of the equation is $y = \dfrac{\int I(x)Q(x)\mathrm{d}x}{I(x)}$.

To justify the transition from step 5 to step 6, you can check the integral relationship by going backwards. If you were to differentiate $I(x)y(x)$ using the product rule, you would get

$$\frac{d}{dx}\left(I(x)y(x)\right) = I(x)y'(x) + I'(x)y(x).$$

Studying the integrating factor and the chain rule for differentiating, you further find

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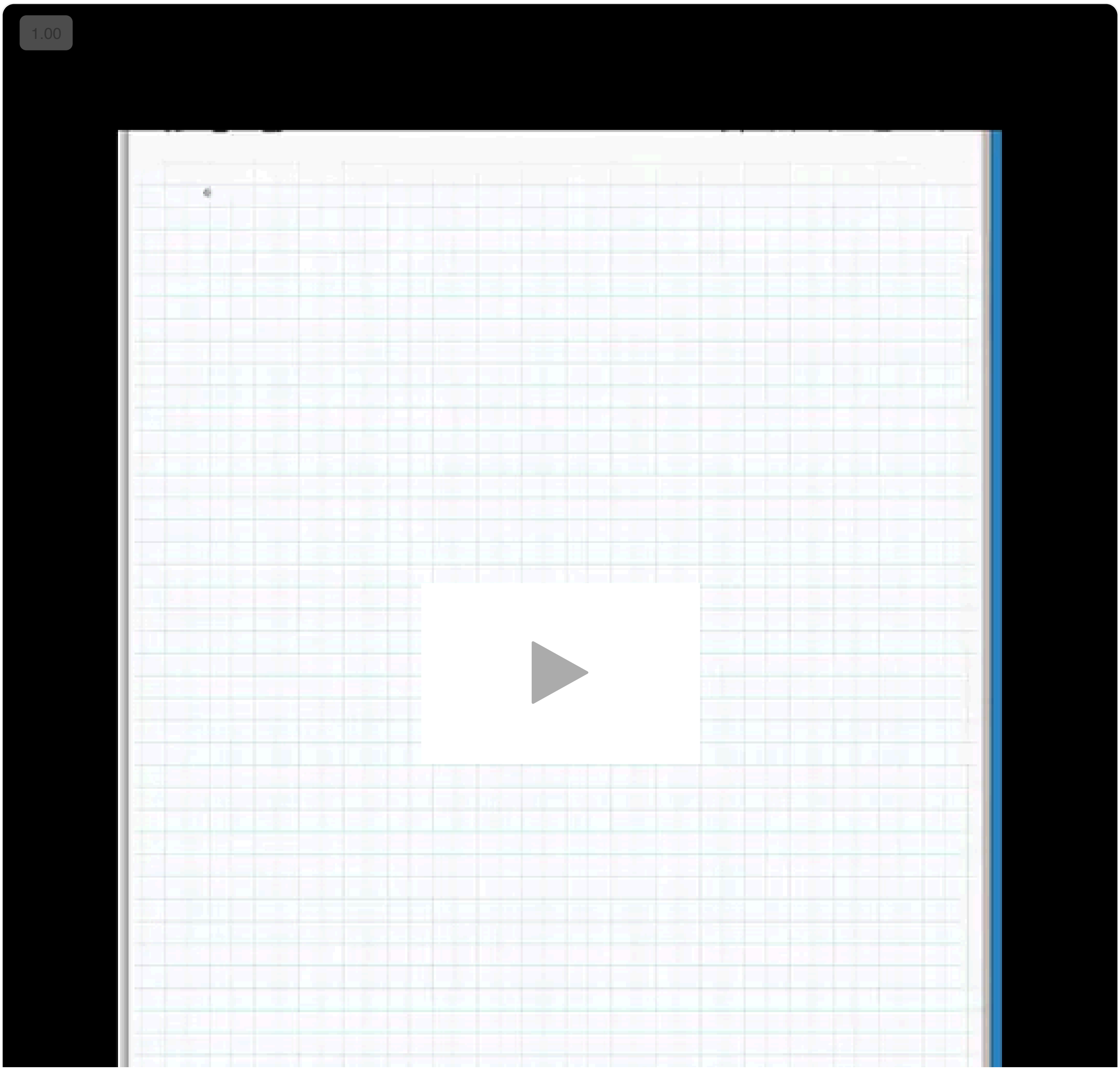
$$\begin{aligned} I(x) &= \mathrm{e}^{\int P(x)\mathrm{d}x} \\ I'(x) &= \mathrm{e}^{\int P(x)\mathrm{d}x} P(x) \\ &= I(x)P(x) \end{aligned}$$

Therefore,

$$(Iy)'(x) = I(x)y'(x) + I'(x)y(x) = I(x)y'(x) + I(x)P(x)y(x)$$

The video below demonstrates this concept.



Video 1. Solving a Differential Equation with an Integrating Factor.

More information for video 1

1

00:00:00,067 --> 00:00:01,667

narrator: In this video we're going to

take a brief look

2

00:00:01,733 --> 00:00:05,333

Student
view

at exact solutions to differential

equations in linear form,

3

00:00:05,400 --> 00:00:08,933

which are $y' + P(x)y = Q(x)$.

4

00:00:09,000 --> 00:00:10,800

Now, why is this called the linear form?

5

00:00:11,267 --> 00:00:14,467

$y' + P(x)y = Q(x)$ can be rearranged

6

00:00:14,533 --> 00:00:18,433

to give us $y' = -P(x)y + Q(x)y^0$.

7

00:00:18,500 --> 00:00:21,533

So the right hand side is linear in y,

8

00:00:21,600 --> 00:00:22,733

hence the linear form.

9

00:00:23,200 --> 00:00:26,933

So, $y' + P(x)y = Q(x)$.

10

00:00:27,000 --> 00:00:30,233

Now we're going to introduce

new function $I(x)$,

11

00:00:30,300 --> 00:00:32,333

which is $e^{\int P(x)dx}$

12

00:00:32,400 --> 00:00:35,167

and this is called the integrating factor.

13

00:00:35,667 --> 00:00:37,400

Then we're gonna take

the integrating factor

14

00:00:37,467 --> 00:00:40,033

and multiplies

through the entire differential equation

15

00:00:40,100 --> 00:00:44,933

$I(x)y' + I(x)P(x)y = I(x)Q(x)$.

16

00:00:45,500 --> 00:00:48,200

Now let's consider the left hand side

and we're gonna investigate

Student view

Overview (/study/app/math-aa-hl/sid-134-cid-761926/o

17

00:00:48,267 --> 00:00:50,833

a total derivative of $I(x)y$,

18

00:00:50,900 --> 00:00:52,767

which using the product rule

of differentiation

19

00:00:52,833 --> 00:00:58,733

becomes $\frac{d}{dx}(I(x))y + I(x)\frac{d}{dx}(y)$,

20

00:00:58,800 --> 00:01:00,100

which of course is y'.

21

00:01:00,400 --> 00:01:02,933

Now the total derivative of $I(x)$ then

22

00:01:03,000 --> 00:01:08,200

becomes the derivative

of $e^{\int P(x)dx}$,

23

00:01:08,267 --> 00:01:11,433

which applying what we know about

differentiation of exponential functions

24

00:01:11,500 --> 00:01:13,833

becomes $e^{\int P(x)dx}$

25

00:01:14,200 --> 00:01:16,867

times the derivative

of the integral $\int P(x)dx$.

26

00:01:16,933 --> 00:01:19,333

This then becomes $I(x)P(x)$

27

00:01:19,400 --> 00:01:22,333

using again the idea

that integrations anti differentiation.

28

00:01:22,400 --> 00:01:25,800

Therefore the left hand side

$I(x)P(x)y$

29

00:01:25,867 --> 00:01:31,033

plus $I(x)y'$ equals

the derivative of $I(x)y$

Student view

30

00:01:31,100 --> 00:01:32,933

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and that's how we use

the integrating factor.

31

00:01:33,000 --> 00:01:37,567

So $I(x)y' + I(x)P(x)y$

32

00:01:37,633 --> 00:01:41,967

equals $I(x)Q(x)$

33

00:01:42,033 --> 00:01:47,367

then becomes

$\frac{d}{dx}(I(x)y)$

34

00:01:48,333 --> 00:01:51,000

equals $I(x)Q(x)$.

35

00:01:51,167 --> 00:01:58,133

Then the total derivative

of $I(x)y$ equals $I(x)Q(x)$.

36

00:01:58,333 --> 00:02:01,333

Now let's integrate

both sides with respect to dx.

37

00:02:01,400 --> 00:02:05,100

The left hand side of course

becomes just a product $I(x)y$

38

00:02:05,167 --> 00:02:08,700

and the right hand side

is the integral of $\int I(x)Q(x)dx$,

39

00:02:08,867 --> 00:02:10,667

such that the solution y

40

00:02:10,733 --> 00:02:14,400

is equal to $\frac{\int I(x)Q(x)dx}{I(x)}$.

41

00:02:14,467 --> 00:02:17,500

divided by the integrating vector $I(x)$.

42

00:02:17,567 --> 00:02:20,533

Hence, if you have a differential

equation in the form

43

00:02:20,600 --> 00:02:23,700

$y' + P(x)y = Q(x)$,

44

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00:02:24,067 --> 00:02:26,667

then introduce

the integrating factor $I(x)$

45

00:02:26,733 --> 00:02:29,133

is $e^{\int P(x)dx}$,

46

00:02:29,433 --> 00:02:32,433

and the solution $y = \frac{\int I(x)Q(x)dx}{I(x)}$

and that is how we deal with this.

48

00:02:38,033 --> 00:02:41,867

Let's apply this to the example $y' = y + e^x$.

49

00:02:41,933 --> 00:02:43,733

So $y' = y + e^x$,

50

00:02:43,800 --> 00:02:47,400

first put it in linear form,

so $y' - y = e^x$,

51

00:02:47,467 --> 00:02:51,167

and we can immediately see that

$P(x) = -1$,

52

00:02:51,233 --> 00:02:55,800

such that if we integrate $P(x)$,

it becomes $\int -1dx$,

53

00:02:55,867 --> 00:02:57,667

which of course is $-x + C$.

54

00:02:58,133 --> 00:02:59,100

We can choose any C.

55

00:02:59,167 --> 00:03:01,967

Let's make a life easy,

so choose $C = 0$,

56

00:03:02,033 --> 00:03:05,367

therefore the integrating factors

is $e^{\int P(x)dx}$,

57

00:03:05,467 --> 00:03:09,067

which then becomes e^{-x}.

58

00:03:09,633 --> 00:03:11,067

And then we've seen

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in the previous section

59

00:03:11,133 --> 00:03:16,533

that solution $y = \frac{\int I(x)Q(x)dx}{I(x)}$,

60

00:03:16,600 --> 00:03:18,867

where $I(x) = e^{-x}$

61

00:03:18,933 --> 00:03:22,400

and $Q(x)$ seen from the original equation

is e^{x}.

62

00:03:22,467 --> 00:03:25,667

So $y = \frac{\int e^{-x}e^{x}dx}{e^{-x}}$,

63

00:03:25,733 --> 00:03:27,833

which of course

becomes $\frac{\int 1dx}{e^{-x}}$,

64

00:03:27,900 --> 00:03:30,400

which is $\frac{x+C}{e^{-x}}$,

65

00:03:30,467 --> 00:03:33,433

divided by e^{-x},

which is $e^{x}(x + C)$.

66

00:03:33,500 --> 00:03:36,000

which is $e^{x}(x + C)$.

67

00:03:36,067 --> 00:03:40,100

Therefore, $y = e^{x}(x + C)$

where C is a constant.

68

00:03:40,333 --> 00:03:41,500

Now let's check,

69

00:03:41,567 --> 00:03:45,933

so y' then becomes the derivative

of $e^{x}(x + C)$,

70

00:03:46,000 --> 00:03:50,433

using a product rule becomes derivative

of e^{x} times $x + C$

71

00:03:50,500 --> 00:03:53,067

plus e^{x} times

the derivative of $x + C$,

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72

00:03:53,133 --> 00:03:57,133

which of course then becomes

$e^x(x + C)$

73

00:03:57,200 --> 00:04:00,533

plus $e^x \cdot 1$,

which is $y + e^x$.

74

00:04:00,667 --> 00:04:01,733

And you can see that

75

00:04:01,800 --> 00:04:04,233

was indeed the differential

equation that we started off with.

76

00:04:04,300 --> 00:04:07,433

And so this solution satisfies

the differential equation.

Important

To solve a first-order linear differential equation using an integrating factor, you need to know:

- **How to start** (find the integrating factor).
  - The formula is in section 5.18 of the Formula booklet.
- **What to do** (multiply the equation by the integrating factor).
- **Why this is helpful** (because the left-hand side becomes the derivative of $(Ix)y(x)$).

Example 1

★★★

Find all solutions of the differential equation $y' = y + e^x$.

Steps	Explanation
Step 1: $y' - y = e^x$, $P(x) = -1$, $Q(x) = e^x$	
Step 2: $\int P(x)\,dx = \int -1\,dx = -x + c = -x$	Choose $c = 0$
Step 3: $I(x) = e^{\int P(x)dx} = e^{-x}$	Substitution from Step 2
Step 4: $y' - y = e^x$	

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Steps	Explanation
$\mathrm{e}^{-x}y' - \mathrm{e}^{-x}y = \mathrm{e}^{-x}\mathrm{e}^{x}$	Multiply the equation by the integrating factor
$\mathrm{e}^{-x}y' - \mathrm{e}^{-x}y = 1$	
Step 5: $\int (\mathrm{e}^{-x}y' - \mathrm{e}^{-x}y)\,\mathrm{d}x = \int \mathrm{d}x$	
Step 6: $\mathrm{e}^{-x}y = x + c$ since $(\mathrm{e}^{-x}y' - \mathrm{e}^{-x}y) = \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x}y)$	

The video above shows the solution of this problem in detail.

Example 2

★★★

Find the particular solution of $y' + y\tan x = \sec x$ that satisfies the initial condition $y(0) = 1$.

Steps	Explanation
Step 1: $y' + y\tan x = \sec x$, $P(x) = \tan x$, $Q(x) = \sec x$.	
Step 2: $\int P(x)dx = \int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{-\sin x}{\cos x}\, dx = -\ln(\cos x)$	
After replacing $\tan x$ with $\frac{\sin x}{\cos x}$, the integration was completed through substitution. $u = \cos x$, $du = -\sin x\, dx$. For further review, see sections 5.10.6 and 5.16.1. Furthermore, absolute values are not needed, since the initial condition is given for $x = 0$ and around this value the cosine function is positive.	
Step 3: $I(x) = \mathrm{e}^{-\ln(\cos x)} = \frac{1}{\mathrm{e}^{\ln(\cos x)}} = \frac{1}{\cos x} = \sec x$	Substitution from Step 2, then simplification
Step 4: $y' + y\tan x = \sec x$ $y'\sec x + y\tan x\sec x = \sec^2 x$	
Step 5: $\int (y'\sec x + y\tan x\sec x)\,\mathrm{d}x = \int \sec^2 x\,\mathrm{d}x$ $y\sec x = \tan x + c$ since $y'\sec x + y\tan x\sec x = \frac{\mathrm{d}}{\mathrm{d}x}(y\sec x)$	
Apply $y(0) = 1$: $1\sec 0 = \tan 0 + c$ $1 = 0 + c$ $c = 1$ $y\sec x = \tan x + 1$	

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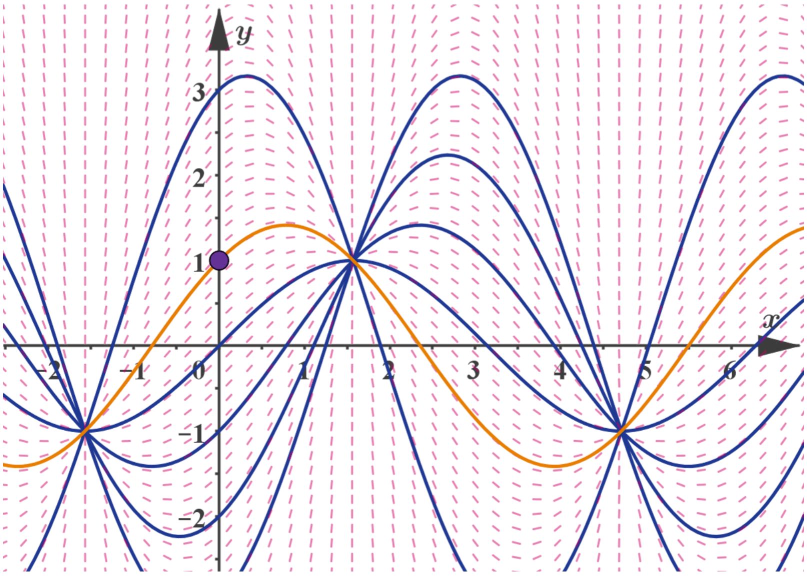
Steps	Explanation
Step 6: $y = \frac{\tan x + 1}{\sec x} = \left(\frac{\sin x}{\cos x} + 1\right)\cos x = \sin x + \cos x$	

The following diagram illustrates the slope field and some solution curves of this differential equation, with the initial condition and the particular solution you just found highlighted.



Example 3

★★★

Consider a hot metal object at 200°C placed in a large tank of water at 20°C to cool. After 1 minute, the temperature of the object is 150°C. What will the temperature be after 3 minutes?

Note: Newton's law of cooling states that $T'(t) = k\left(T(t) - T_0\right)$, for some constant k, where t is the time in minutes and T_0 is the temperature of the surroundings, in this case the tank of water.

Steps	Explanation
Step 1: $T'(t) - kT(t) = -kT_0$ or $T' - kT = -kT_0$, $P(t) = -k$, $Q(t) = -kT_0$	Note: $-kT_0$ is a constant made up of two constants.

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Steps	Explanation
Step 2: $\int P(t)\mathrm{d}t = \int -k\mathrm{d}t = -kt$	Notice the different notation. t is for time while T is for the function and T_0 is the room temperature.
Step 3: $I(x) = \mathrm{e}^{-kt}$	
Step 4: $T' - kT = -kT_0$ $\mathrm{e}^{-kt}T' - k\mathrm{e}^{-kt}T = -kT_0\mathrm{e}^{-kt}$	
Step 5: $\int \left(\mathrm{e}^{-kt}T' - k\mathrm{e}^{-kt}T\right) dt = \int -kT_0\mathrm{e}^{-kt}\mathrm{d}t$ $\mathrm{e}^{-kt}T = T_0\mathrm{e}^{-kt} + c$ since $\mathrm{e}^{-kt}T' - k\mathrm{e}^{-kt}T = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-kt}T\right)$	
Apply $T_0 = 20$ and $T(0) = 200$: $\mathrm{e}^{-k(0)}200 = 20\mathrm{e}^{-k(0)} + c$ $200 = 20 + c$ $c = 180$	
Step 6: $\mathrm{e}^{-kt}T = 20\mathrm{e}^{-kt} + 180$ $T = 20 + 180\mathrm{e}^{kt}$ Apply $T(1) = 150$ $150 = 20 + 180\mathrm{e}^{k(1)}$ $\frac{130}{180} = \mathrm{e}^{k}$ $\ln(0.7222) = k$ $k = -0.32542\ldots$ $T = 20 + 180\mathrm{e}^{-0.32542t}$ Therefore $t = 3$ $T(20) = 20 + 180\mathrm{e}^{-0.32542(3)} \approx 87.8^\circ\ \mathrm{C}$.	

3 section questions

Question 1

Difficulty:

★☆☆

An integrating factor for the first-order linear differential equation $xy' + x^2 = 3xy$ is e^{kx} for some real number k.

Find the **exact** value of k. Give the numerical value only.

-3

Student view

Accepted answers

-3, −3

Overview
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Explanation

Dividing by x and rearranging the equation gives $y' - 3y = -x$.

The integrating factor is $e^{\int P(x)\,dx}$, where $P(x) = -3$.

$\int -3\,dx = -3x + c$, so choosing $c = 0$, the integrating factor becomes e^{-3x}

Hence, $k = -3$.

Question 2

Difficulty:

★★☆

A solution of the differential equation $x^2y' + 2xy = 1$ satisfies the initial condition $y(1) = 1$.

Find the **exact** value of $y(2)$. Give the numerical value only.

0.5

Accepted answers

0.5, 1/2, 0,5, 1 /2, 1 /2, 1 / 2

Explanation

Dividing the equation by x^2 gives $y' + \frac{2}{x}y = \frac{1}{x^2}$

This is a linear equation, so we look for the integrating factor by finding $\int \frac{2}{x}\,dx = 2\ln|x| + c$ first.

Choosing $c = 0$, an integrating factor is $e^{2\ln|x|} = x^2$.

Multiplying the equation by the integrating factor, noticing the derivative on the left-hand side and then integrating both sides,

$$\begin{aligned} x^2y' + 2xy &= 1 \\ (x^2y)' &= 1 \\ x^2y &= \int 1\,dx \\ x^2y &= x + c \end{aligned}$$

Since $y(1) = 1$, this implies that $1 = 1 + c$ so $c = 0$.

Hence, $y = \frac{1}{x}$, so $y(2) = 0.5$.

The diagram below shows the solution curve with the two points mentioned in the question.

Student
view

Overview
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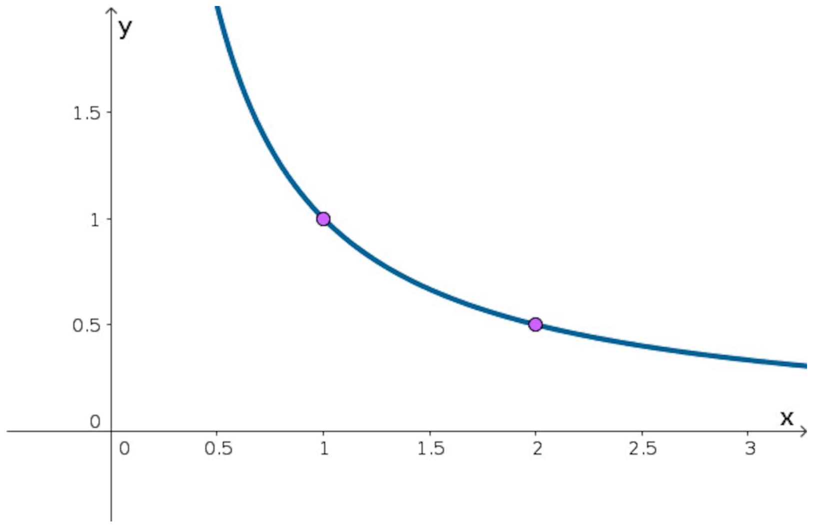


More information

Question 3

Difficulty: ★★★

A solution curve of the differential equation $y' \cos x - y \sin x = 1$ has x-intercept $(1, 0)$.

Find the **exact** value of the y-intercept of this solution curve. Give the numerical value only.

-1 ✓

Accepted answers

-1, −1

Explanation

First, we divide the equation by $\cos x$ to get $y' - y \tan x = \sec x$.

This is a linear equation, so we look for the integrating factor by finding $\int -\tan x \, dx = \ln|\cos x| + c$ first

Choosing $c = 0$, an integrating factor is $e^{\ln|\cos x|} = \cos x$ (we ignore the absolute value, because close to the initial condition, $\cos x$ is positive).

Multiplying the equation by the integrating factor, noticing the derivative on the left-hand side and then integrating both sides:

$$\begin{aligned} y' \cos x - y \sin x &= \sec x \cos x = 1 \\ (y \cos x)' &= 1 \\ y \cos x &= \int 1 \, dx \\ y \cos x &= x + c \end{aligned}$$

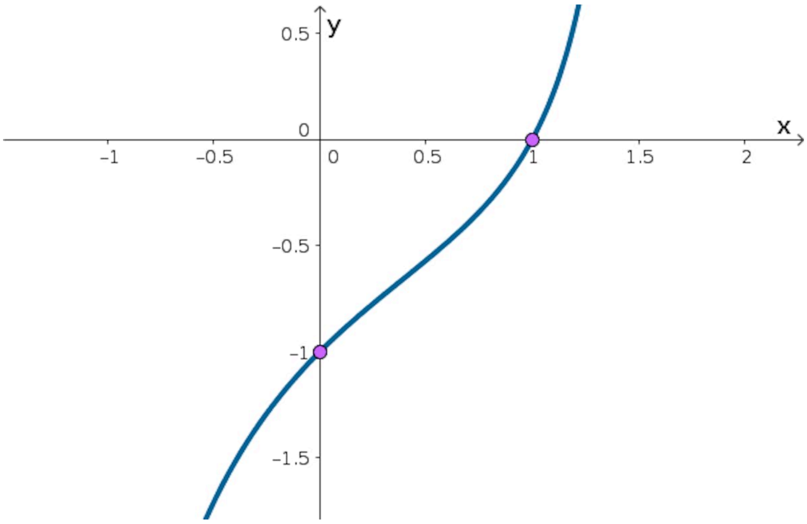
Since $y(1) = 0$, this implies that $0 = 1 + c$ so $c = -1$.

Hence, $y = \dfrac{x-1}{\cos x}$, so $y(0) = \dfrac{0-1}{\cos 0} = -1$.

The diagram below shows the solution curve with the two points mentioned in the question.

Student view

Overview
(/study/app/math-aa-hl/sid-134-cid-761926/o



More information

We could have noticed the derivative on the left-hand side of the original differential equation in the first place, so finding the integrating factor was not really necessary.

5. Calculus / 5.18 Differential equations

Applications

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/applications-id-27279/print/) Assign

At the beginning of this subtopic, several real-world scenarios were mentioned. The following examples will address a few of these topics.

Newton’s law of cooling

Newton’s law of cooling states that a body approaches the temperature of the surroundings asymptotically, with the rate of change of the temperature directly proportional to the difference in temperature between the body and the surroundings. The name implies that the body is warmer and its temperature is dropping to match the surroundings, but the same principle applies to warming. The temperature difference is an example of exponential decay.

Example 1

★★★

A hard-boiled egg at 100°C is cooled in running water at 20°C. After 5 minutes, the temperature of the egg is 43°C. When will the temperature reach 30°C?

Steps	Explanation
Variable definition:	
T	Temperature
$T_S = 20$	Temperature of surroundings

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Steps	Explanation
$T_0 = 100$	Starting temperature
r	Time constant
$\frac{dT}{dt} = -r(T - T_s)$	Change in temperature over time
$\frac{dT}{T - 20} = -rdt$	Separate the variables and substitute known value
$\ln(T - 20) = -rt + C^*$	Integrate, C^* is a constant that will be replaced later
$T = \mathrm{e}^{-rt+C^*} + 20$	Solve for T
$T = C\mathrm{e}^{-rt} + 20$	Let $C = \mathrm{e}^{C^*}$
$100 = Ce^{-r(0)} + 20$	Apply initial conditions $T_0 = 100$
$C = 80$	Solve for C
$T = 80\mathrm{e}^{-rt} + 20$	
$43 = 80\mathrm{e}^{-r(5)} + 20$	Apply initial conditions $T(5) = 43$
$r = 0.2493$	Solve for r
$T = 80\mathrm{e}^{-0.2493t} + 20$	
$30 = 80\mathrm{e}^{-0.2493t} + 20$	Apply values for question
$t = 8.34$ min	Solve for t

This is known as Newton's law of cooling. It is typically written as $T - T_S = (T_0 - T_S)\mathrm{e}^{-rt}$ where T_0 is the initial temperature, T_S is the surrounding temperature, and t is time.

Half-life

Defined by Ernest Rutherford when he was studying radioactive material, a half-life, $t_{\frac{1}{2}}$, is defined as the time required for the activity of a radioactive substance to reduce to half its original value. This also represents the time required for half of the radioactive material to decay. For example, carbon-14 naturally decays to nitrogen- 14 in a predictable manner. The concept can be applied in other areas, such as the time required for half of a medication to be processed by a person's metabolism. As in Newton's law of cooling, the rate of change of the substance is directly proportional to the amount present at any given time. This is an example of exponential decay.

Example 2

★★★

Carbon-14 has a half-life of about 5700 years. Find the age of a fossil that has 80% of the carbon- remaining.

Student view

Steps	Explanation
Variable definition:	
A	Amount
A_0	Initial amount
k	Time constant
$\frac{\mathrm{d}A}{\mathrm{d}t} = -kA$	Change in amount over time
$\frac{\mathrm{d}A}{A} = -k\mathrm{d}t$	Separate the variables and substitute known value
$\ln(A) = -kt + C^*$	Integrate
$A = \mathrm{e}^{-kt+C^*}$	Solve for A
$A = C\mathrm{e}^{-kt}$ Let $C = \mathrm{e}^{C^*}$	
$A_0 = C\mathrm{e}^{-k(0)}$	Apply initial conditions $A(0) = A_0$
$C = A_0$	Solve for C
$A = A_0\mathrm{e}^{-kt}$	
$\frac{1}{2}A_0 = A_0\mathrm{e}^{-k(5700)}$	Apply half-life conditions $A(5700) = \frac{1}{2}A_0$
$k = \frac{\ln 2}{5700}$	Solve for k
$A = A_0\mathrm{e}^{-\frac{t\ln 2}{5700}}$	Apply values for question
$0.8A_0 = A_0\mathrm{e}^{-\frac{t\ln 2}{5700}}$ $0.8 = \mathrm{e}^{-\frac{t\ln 2}{5700}}$ $\ln 0.8 = -\frac{t\ln 2}{5700}$ $t = -\frac{5700\ln 0.8}{\ln 2} \approx 1835$	Substitute $0.8A_0$ for A and solve for the corresponding time, t

This specific example is known as the radioactive half-life formula. More generally, it is referred to as the exponential decay model. It can be used for carbon dating or prediction of the decay of other radioactive elements. It is commonly written as $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$ where A is the amount remaining, A_0 is the initial amount, t is the time and h is the half-life. Can you follow the algebraic conversion from the equation in the example to this one using the laws of exponents?

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Exponential growth

As with exponential decay, there are examples of exponential growth, where the rate of change of the substance is directly proportional to the amount present at any given time, but the amount of the substance is increasing. Population growth (including bacteria) can be modelled in this manner when resource limitations are negligible. Compound interest on a financial account is another common example, as the interest earned increases over time as the total amount of money in the account grows.

Example 3

★★★

A colony of bacteria grows exponentially over time. At the end of 3 hours, there are 5000 bacteria. At the end of 5 hours, there are 20 000 bacteria. How many were there initially?

Steps	Explanation
Variable definition:	
P	Amount
P_0	Initial amount
r	Growth rate
$\frac{dP}{dt} = rP$	Change in amount over time
$\frac{dP}{P} = rdt$	Separate the variables and substitute known value
$\ln(P) = rt + C^*$	Integrate
$P = \mathrm{e}^{rt+C^*}$	Solve for P
$P = C\mathrm{e}^{rt}$ Let $C = \mathrm{e}^{C^*}$	
$P_0 = C\mathrm{e}^{r(0)}$	Apply initial conditions $P(0) = P_0$
$C = P_0$	Solve for C
$P = P_0\mathrm{e}^{rt}$	
$5000 = P_0\mathrm{e}^{r(3)}$	Apply first condition $P(3) = 5000$
$20\,000 = P_0\mathrm{e}^{r(5)}$	Apply second condition $P(5) = 20000$
$\frac{20\,000}{5000} = \frac{\mathrm{e}^{5r}}{\mathrm{e}^{3r}}$	Divide second equation by first
$4 = \mathrm{e}^{5r}\mathrm{e}^{-3r} = \mathrm{e}^{2r}$	Simplify

Student view

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Steps	Explanation
$r = 0.6931$	Solve for r
$5000 = P_0\mathrm{e}^{0.6931(3)}$	Substitute into first equation
$P_0 = 625$	Solve for P_0

This is generally referred to as the exponential growth model. It is not only used in modelling population growth, but is also used in economics when modelling continuous compounding. In this field of study, it is colloquially referred to as the PERT model, $A = P\mathrm{e}^{rt}$. If the number of compoundings per year increases, there is more of an advantage in the compounding. For example,

- 1000 at 12% simple interest for one year results in $1000(1.12) = 1120$
- 1000 at 12% interest compounded quarterly for one year results in $1000(1.03)^4 = 1125.51$
- 1000 at 12% interest compounded monthly for one year results in $1000(1.01)^{12} = 1126.83$
- 1000 at 12% interest compounded daily for one year results in $1000(1.00033)^{12} = 1127.47$
- 1000 at 12% interest compounded continuously for one year results in $1000e^{0.12(1)} = 1127.50$.

Logistic model

An exponential function can model the growth of a population for a limited time. However, it certainly is not correct for very long time intervals, since a population cannot grow without bound. A more realistic model for population growth is the logistic model.

✓ **Important**

The logistic differential equation that models a population growth has the following form.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(a - P)$$

Notice, that the constant functions, $P(t) = 0$ and $P(t) = a$ are both solutions of this differential equation. In general, with an initial condition between 0 and a the solution curve will increase from the horizontal asymptote $P = 0$ to the other horizontal asymptote $P = a$. According to this model the population cannot increase above a, the number a is called the carrying capacity of the model.

Example 4

★★★

Rangers would like to reintroduce beavers in a certain national park, so they take 20 beavers there. The differential equation that models the growth of the population is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.0005P(240 - P),$$

where P is the size of the population t years after the introduction of the first 20 beavers.

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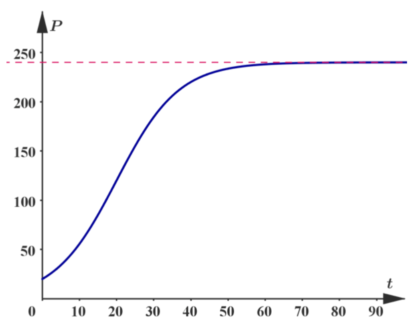
Find $P(t)$ and find an upper limit on the number of beavers in the national park that the model predicts.

Steps	Explanation
$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.0005P(240 - P)$	Use the given equation
$\int \frac{1}{P(240-P)}\mathrm{d}P = \int 0.0005\mathrm{d}t$	Separate the variables
$\int \frac{1}{240P}\mathrm{d}P + \int \frac{1}{240(240-P)}\mathrm{d}P = \int 0.0005\mathrm{d}t$	Partial fraction decomposition. To review how to do this, see subtopic 1.11.
$\frac{1}{240}\ln\|P\| - \frac{1}{240}\ln\|240 - P\| = 0.0005t + C$	Integrate
$\ln\left\|\frac{P}{240-P}\right\| = 0.12t + 240C$	Multiply by 240 and use the laws of logarithm.
$\ln\frac{20}{220} = 240C$ $C = \frac{1}{240}\ln\frac{1}{11} = -\frac{1}{240}\ln 11$	Apply initial conditions $P(0) = 20$
$\ln\left\|\frac{P}{240-P}\right\| = 0.12t - \ln 11$	Substitute this value of C in the solution.
$\frac{P}{240-P} = \mathrm{e}^{0.12t - \ln 11} = \frac{\mathrm{e}^{0.12t}}{11}$ $11P = 240\mathrm{e}^{0.12t} - P\mathrm{e}^{0.12t}$ $P(\mathrm{e}^{0.12t} + 11) = 240\mathrm{e}^{0.12t}$ $P = \frac{240\mathrm{e}^{0.12t}}{\mathrm{e}^{0.12t} + 11}$	Solve for P
According to this model, an upper limit of the beaver population in the national park is 240.	$\frac{240\mathrm{e}^{0.12t}}{\mathrm{e}^{0.12t} + 11} < \frac{240\mathrm{e}^{0.12t}}{\mathrm{e}^{0.12t}} = 240$
	The graph has a horizontal asymptote at $P = 240$.

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3 section questions

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Overview
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Question 1

Difficulty: ★★☆

A hamburger is placed on a grill. The hamburger starts at $40°\mathrm{F}$ and the grill is maintained at $400°\mathrm{F}$. After 10 minutes, the hamburger is at $107°\mathrm{F}$. When will the hamburger have a temperature of $160°\mathrm{F}$? Give your answer to the nearest tenth of a minute without units, for example 12.7.

19.7 ✓

Accepted answers

19.7, 19,7

Explanation

Variable definition:	
T	Temperature
$T_s = 400$	Surrounding temperature
$T_0 = 40$	Starting temperature
r	Time constant
$\frac{\mathrm{d}T}{\mathrm{d}t} = -r(T - T_s)$	Change in temperature over time
$\frac{\mathrm{d}T}{T-400} = -r\mathrm{d}t$	Separate the variables and substitute known value
$\ln(T - 400) = -rt + C^*$	Integrate
$T = \mathrm{e}^{-rt+C^*} + 400$	Solve for T
$T = C\mathrm{e}^{-rt} + 400$	Let $C = e^{C^*}$
$40 = Ce^{-r(0)} + 400$	Apply initial conditions $T(0) = 40$
$C = -360$	Solve for C
$T = -360\mathrm{e}^{-rt} + 400$	
$107 = -360\mathrm{e}^{-r(10)} + 400$	Apply initial conditions $T(10) = 107$
$r = 0.02059$	Solve for r
$T = -360\mathrm{e}^{-0.02059t} + 400$	
$160 = -360\mathrm{e}^{-0.02059t} + 400$	Apply values for question
$t = 19.7$ min	Solve for t

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Question 2

Difficulty:

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★★☆

Uranium-235 has a half-life of 700 million years. To the nearest thousand years, find the age of a sample in which 1% has decayed.

10150000 ✓

Accepted answers

10150000

Explanation

Variable definition:	
A	Amount
*T*O	Initial amount
k	Time constant
$\frac{\mathrm{d}A}{\mathrm{d}t} = -kA$	Change in amount over time
$\frac{\mathrm{d}A}{A} = -k\mathrm{d}t$	Separate the variables and substitute known value
$\ln(A) = -kt + C^*$	Integrate
$A = \mathrm{e}^{-kt+C^*}$	Solve for A
$A = C\mathrm{e}^{-kt}$ Let$C = \mathrm{e}^{C^*}$	
$A_0 = C\mathrm{e}^{-k(0)}$	Apply initial conditions $A(0) = A_0$
$C = A_0$	Solve for C
$A = A_0\mathrm{e}^{-kt}$	
$\frac{1}{2}A_0 = A_0\mathrm{e}^{-k(700)}$	Apply half-life conditions $A(700) = \frac{1}{2}A_0$
$k = \frac{\ln 2}{700}$	Solve for k
$A = A_0\mathrm{e}^{-\frac{t\ln 2}{700}}$	
$0.99A_0 = A_0\mathrm{e}^{-\frac{t\ln 2}{700}}$	Apply values for question
$t = 10.150$ million years	Solve for t
$t = 10\ 150\ 000$	

Question 3

Difficulty:

★★☆

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The world population can be modelled as growing exponentially over time. In 1804, the population was one billion. In 1922, the population was two billion. Following this pattern, when would the world population reach six billion? Enter your answer to the nearest year.

2109

Accepted answers

2109

Explanation

Variable definition:	
P	Amount (in billions)
P_0	Initial amount
r	Growth rate
$\frac{dP}{dt} = Pr$	Change in amount over time
$\int \frac{1}{P} dP = \int r dt$	Separate the variables
$\ln(P) = rt + C^*$	Integrate
$P = \mathrm{e}^{rt+C^*}$	Solve for P
$P = C\mathrm{e}^{rt}$	Let $C = \mathrm{e}^{C^*}$
$P_0 = C\mathrm{e}^{r(0)}$	Apply initial conditions $P(0) = P_0$
$C = P_0$	Solve for C
$P = P_0\mathrm{e}^{rt}$	
$1 = P_0\mathrm{e}^{r(1804)}$	Apply first condition $P(1804) = 1$
$2 = P_0\mathrm{e}^{r(1922)}$	Apply second condition $P(1922) = 2$
$\frac{2}{1} = \frac{\mathrm{e}^{1922r}}{\mathrm{e}^{1804r}}$	Divide second equation by first
$2 = \mathrm{e}^{1922r}\mathrm{e}^{-1804r} = \mathrm{e}^{118r}$	Simplify
$r = 0.005874$	Solve for r
$1 = P_0\mathrm{e}^{0.005874(1804)}$	Substitute into first equation
$P_0 = 2.4993 \times 10^{-5}$	Solve for P_0
$6 = (2.4993 \times 10^{-5})\mathrm{e}^{0.005874t}$	Substitute required value
$t = 2109$	Solve for t (nearest year)

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Checklist

What you should know

By the end of this subtopic you should be able to:

- identify and classify a differential equation by type, order and linearity
- set up a differential equation to model situations where information about rate of change is given
- find an approximate solution of the first-order differential equation $y' = F(x, y)$, if $y(x_0) = y_0$ is given, using the iterative process of Euler's method
- find an exact solution of a separable differentiable equation by rearranging the equation into the form $y' = f(x)g(y)$ and separating the variables
- find an exact solution of a homogeneous differentiable equation by rearranging the equation into the form $y' = F\left(\frac{y}{x}\right)$ and using substitutions
- find an exact solution of a first-order linear differentiable equation by rearranging the equation into the form $y' + P(x)y = Q(x)$ and finding the integrating factor.

Investigation

Can you imagine what it was like to use numerical techniques like Euler's method in the 1700s with nothing more than a slide rule and a pen (pencils were not developed until the turn of the 19th century)? There needs to be a balance between the effort involved in carrying out a computation and the accuracy. Today, with the development of calculators and computers, it is easy to carry out complex computations with a little bit of knowledge of programming.

For this investigation, open up a suitable software package and build a program that will approximate the solution of a differential equation that satisfies a given initial condition. Several spreadsheet programs are available – Microsoft Excel and Google Sheets are a couple of popular examples.

Look at the following problem.

A solution of the differential equation $y' = \frac{x}{y}$ satisfies $y(1) = -2$. Use Euler's method with varying step size to estimate $y(2)$.

Using the bookkeeping technique presented in the example and section questions from section 5.18.2 (/study/app/math-aa-hl/sid-134-cid-761926/book/numerical-solution-eulers-method-id-27275/) as a template, build a spreadsheet with relative referencing to complete the computations. If you are already comfortable with programming, go ahead. Otherwise, here are some sample instructions.

In cell A1, enter in the step size. You can start at 0.1.

In cells A2, B2, C2, and D2, enter labels k, x_k, y_k and y' respectively. Subscripts are not important as long as you still know what they mean.

In cell A3, enter 0 to indicate you are starting at $k = 0$.

In cell A4, you are going to automatically increase the counter with an equation. Enter =A3+1. Copy this cell down as far as you need. For a step size of 0.1, you will need it to go down at least as far as A13 where $k = 10$. Notice that as the cells go down, the formulae change where they point to. Although you had cell A4 point to A3, A5 will point to A4, A6 will point to A5, and so on. You call this relative referencing.

In cell B3, enter the initial x-value. For this problem, it is given as 1.

In cell C3, enter the initial y-value. For this problem, it is given as –2.

In cell D3, you are going to enter your first real equation. From the problem statement, you know that $y' = \frac{x}{y}$. To program this in the spreadsheet, enter =B3/C3. If you use this problem for another differential equation, you will need to change this equation.

In cell B4, you are going to increase your x-value by the step size. Enter =B3+$A$1. You could just enter in =B3+0.1, but then you would have to change the equation every time you changed the step size. By pointing to cell A1, it is much more obvious and helps you to remember what step size you are using. Also notice the $ signs. By placing the $ signs in front of the A and the 1, you can 'anchor' that value, preventing it from 'moving' as you copy the cell down later on. Go ahead and copy cell B4 down as far as you copied cell A4, at least to row 13.

In cell C4, you are going to increase your y-value. You know the slope, $\frac{dy}{dx}$, and you know the step size, Δx, so you just need to multiply those together and add them to the last value of y. Enter =C3+$A$1*D3. Notice that C3 (the old y) and D3 (the old slope) are not anchored as they will change as you move down, but cell A1 (the step size) is anchored as you do not want it to move. Copy cell C4 down to match columns A and B.

Copy cell D3 down to match the other columns.

If you followed these instructions, you should have something that looks like:

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	A	B	C	D
1	0.1	Step Size		
2	k	x_k	y_k	y'
3	0	1	-2	-0.5
4	1	1.1	-2.05	-0.54
5	2	1.2	-2.1	-0.57
6	3	1.3	-2.16	-0.6
7	4	1.4	-2.22	-0.63
8	5	1.5	-2.28	-0.66
9	6	1.6	-2.35	-0.68
10	7	1.7	-2.42	-0.7
11	8	1.8	-2.49	-0.72
12	9	1.9	-2.56	-0.74
13	10	2	-2.63	-0.76

More information

The image is a screenshot of a spreadsheet displaying a table. The table has four columns labeled: 'k', 'x_k', 'y_k', and 'y''. The first row contains the header with a step size of 0.1 highlighted in yellow in cell B1. The second row displays initial values: 'k' as 0, 'x_k' as 1, 'y_k' as -2, and 'y'' as -0.5. Subsequent rows (3 to 13) show the progression of values: column A for 'k' increments from 1 to 10, column B 'x_k' ranges from 1.1 to 2, column C 'y_k' has values from -2.05 to -2.63, and column D 'y'' records values starting at -0.54 down to -0.76. Columns C and D are highlighted in yellow, indicating that they change as the step size, starting position, or differential equation is altered. Cell D3 contains an equation that can be adjusted.

[Generated by AI]

The cells in green are where you can alter the step size or the coordinates of the starting position. You can change the differential equation by altering the equation behind cell D3. The cells in yellow are those that will change value a s you vary the step size, starting position, or differential equation.

Now you are ready to study the effects of step size. By changing the step size and copying more rows as needed, determine approximations of $y(2)$ for the following step sizes:

Step size (Δx)	**Steps** (k)	$y(2)$
1		
0.5		
0.2		
0.1	10	-2.63455
0.05		
0.01		
Exact		$-\sqrt{7} \approx -2.645751311$

The exact answer was found using techniques found in [section 5.18.3 (/study/app/math-aa-hl/sid-134-cid-761926/book/exact-solution-separable-equations-id-27276/)] for separable equations. If you were not able to find an exact answer, how many steps would you be willing to do?

Student view

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