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Teacher view

0 (https://intercom.help/kognity)

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The big picture

This family has four children; three girls and a boy. Does this affect the chance of having another girl if the parents have another child?

Credit: adamkaz GettyImages

In subtopic 4.6 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25650/) you learned how to find the probability that something would happen *given* that another event had already occurred, also known as conditional probability.

In this subtopic we will explore this concept further, including using it to determine whether two events are independent , meaning that the occurrence of one event has no impact on the probability that another event will occur. The binomial distribution, for example, involves repeated independent events.

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Think carefully about the concepts of independence and conditional probability. Can you think of what rule there might be to relate them?

Theory of Knowledge

The concept of independence can be related beyond mathematical relationships, and one area is in the realm of knowledge. In school, the curriculum is divided into studies of mathematics, natural science, social science, language and fine arts, among others. Do you think breaking knowledge into independent categories is appropriate? How does this relate to interdisciplinary education? Is it even possible to study one area of knowledge independent of others?

International Mindedness

Consider how mathematics is taught around the world. In some countries, like the United States, students often study algebra, geometry, calculus and trigonometry in separate years. In many other countries, however, mathematics is taught through an integrated approach that combines elements of various fields of mathematics. What impact do you think this has on how students in these countries view mathematics? Do you think it could affect how they solve problems?

Concept

Independence is just one of several relationships that can exist between events. Since some probability rules, including binomial probability, rely on independence, it is essential to be able to recognise when events are independent.

4. Probability and statistics / 4.11 Conditional probability and independence

Testing independence with conditional probability

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Reviewing conditional probability

Before we explore determining whether events are independent, we will review using the formula for conditional probability. The formula is $\mathrm{P}\left(A|B\right)=\dfrac{\mathrm{P}\left(A\cap B\right)}{\mathrm{P}\left(B\right)}$.

Example 1

★☆☆

If $\mathrm{P}\left(A\right)=0.7$ and $\mathrm{P}\left(A\cap B\right)=0.4$, then find $\mathrm{P}(B|A)$.

Use the formula $\mathrm{P}\left(B|A\right)=\dfrac{\mathrm{P}\left(A\cap B\right)}{\mathrm{P}\left(A\right)}$ to find the probability.

$$\begin{aligned}\mathrm{P}\left(B|A\right)&=\frac{\mathrm{P}\left(A\cap B\right)}{\mathrm{P}\left(A\right)}\\&=\frac{0.4}{0.7}\\&=\frac{4}{7}\ \approx 0.5714\end{aligned}$$

How reliable are medical tests?

Credit: Longhua Liao GettyImages

Student view

Example 2

★★☆

Suppose it is known that approximately 0.1% of a population has an infectious disease. A test is developed for the infection: if it detects the infection, it returns a positive result. Like all tests it is not infallible; out of all the people who are infected, 1% have a negative test (a false negative). The test returns a positive result for 1.098% of the people tested, some of which are false positives.

Find the probability that you are not infected given that you have a positive result.

It can be helpful to make a table to help organise the data you are given.

	Infected (%)	Not infected (%)	Total (%)
Positive test			1.098
Negative test	$0.01 \times 0.1 = 0.001$		
Total	0.1		100

Now you can complete the table by subtracting from the totals.

	Infected (%)	Not infected (%)	Total (%)
Positive test	0.099	0.999	1.098
Negative test	0.001	98.901	98.902
Total	0.1	99.9	100

Finally, use the formula to find the conditional probability.

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$$\begin{aligned}\mathrm{P}\,(\text{not infected}|\text{positive}) &= \frac{\mathrm{P}\,(\text{not infected} \cap \text{positive})}{\mathrm{P}\,(\text{positive})} \\ &= \frac{0.999}{1.098} \\ &= 0.909836 \\ &\approx 0.910\end{aligned}$$

Therefore, even if your test is positive, there is still approximately a 91% chance that you are not infected.

Making connections

The scenario in **Example 2** illustrates the probability of getting a *false positive* result in a test. You might wonder why doctors would use a test that was wrong so often.

The key is that it identifies virtually all of those who are infected. It may be that a more reliable test is very costly or invasive, while this test is inexpensive, harmless or easy to run.

This test might be used first so that only those who got a positive result would have to go through the more difficult testing.

Conditional probability of independent events

Earlier we learned that A and B are independent events if the occurrence of A has no effect on the probability of B occurring.

Mathematically, we can demonstrate this relationship with these equations:

$$\mathrm{P}\,(B|A) = \mathrm{P}\,(B) \text{ and } \mathrm{P}\,(A|B) = \mathrm{P}\,(A)\,.$$

Begin with the formula: $\mathrm{P}\,(A|B) = \dfrac{\mathrm{P}\,(A \cap B)}{\mathrm{P}\,(B)}$.

Multiply to get $\mathrm{P}\,(A \cap B) = \mathrm{P}\,(A)\,\mathrm{P}\,(B|A)\,.$

If A and B are independent, then you can substitute to get $\mathrm{P}\,(A \cap B) = \mathrm{P}\,(A)\,\mathrm{P}\,(B)\,.$

Therefore, A and B are independent if, and only if, $\mathrm{P}\,(A \cap B) = \mathrm{P}\,(A)\,\mathrm{P}\,(B)\,.$

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Important

Given that conditional probability can be used to express the relationship between independent variables multiple ways, there are different ways of expressing the test for independence:

- $\mathrm{P}(B|A) = \mathrm{P}(B)$
- $\mathrm{P}(A|B) = \mathrm{P}(A)$
- $\mathrm{P}(B|A) = \mathrm{P}(B|A')$
- $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$.

Activity

In subtopic 4.8 (/study/app/math-aa-hl/sid-134-cid-761926/book/the-big-picture-id-25661/) you explored several instances of independent events when working with the binomial theorem. Find the probabilities of two or three pairs of events and use the tests above to confirm they are independent. The first pair of events, for example, could be placing a number of different coloured items in a bag and selecting one item, then replacing it and choosing a second item. Is there any way you could modify the events to make them dependent?

Example 3

★★☆

If $\mathrm{P}(A) = 0.72$, $\mathrm{P}(B) = 0.5$ and $\mathrm{P}(A \cap B) = 0.36$, determine whether events A and B are independent.

Use the last of the tests listed above.

$$\mathrm{P}(A)\,\mathrm{P}(B) = 0.72 \times 0.5 = 0.36 = \mathrm{P}(A \cap B)$$

Therefore, A and B are independent events.

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Example 4

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The Ironman World Championship is an annual long-distance triathlon held in Hawaii that requires racers to swim 3.86 km, to ride a bike 180.25 km and to run a marathon without stopping in between.

The race is so demanding that just finishing it is a major accomplishment.

In 2018, the results were as follows:

	Finished	**Did not finish**	**Total**
Men	44	9	53
Women	32	7	39
Total	76	16	92

Determine whether the chance of finishing the race was independent of gender.

To determine whether finishing and gender are independent, find whether $\text{P}(\text{finishing} \cap \text{men}) = \text{P}(\text{finishing})\,\text{P}(\text{men})$.

$$\text{P}(\text{finishing} \cap \text{men}) = \frac{44}{92} \approx 0.478$$

$$\text{P}(\text{finishing}) = \frac{76}{92} \text{ and } \text{P}(\text{men}) = \frac{53}{92}\text{, so}$$

$$\text{P}(\text{finishing})\,\text{P}(\text{men}) = \frac{76}{92} \times \frac{53}{92} = \frac{1007}{2116} \approx 0.476$$

Since $\text{P}(\text{finishing} \cap \text{men}) > \text{P}(\text{finishing})\,\text{P}(\text{men})$, we cannot say finishing was independent of gender. However, since they are extremely close, we could say that gender and finishing are only slightly dependent on each other.

Student view

Exam tip

These formulae are given in the formula booklet.

Conditional probability: $\mathrm{P}\left(A|B\right) = \frac{\mathrm{P}\left(A \cap B\right)}{\mathrm{P}\left(B\right)}$

Independent events: $\mathrm{P}\left(A \cap B\right) = \mathrm{P}\left(A\right)\mathrm{P}\left(B\right)$

Be aware

Do not confuse the terms independent and mutually exclusive.

Mutually exclusive events have $\mathrm{P}\left(A \cap B\right) = 0$.

4 section questions

Question 1

Difficulty:

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Suppose you know that events A and B are independent and you have already calculated $\mathrm{P}\left(A\right) = 0.8$, $\mathrm{P}\left(A \cup B\right) = 0.825$, $\mathrm{P}\left(B\right) = 0.125$ and $\mathrm{P}\left(A|B\right) = 0.8$.

Find $\mathrm{P}\left(A \cap B\right)$.

Give your answer as a decimal.

0.1

Accepted answers

0.1, .1, 0,1, ,1

Explanation

Since A and B are independent events, you know that $\mathrm{P}\left(A \cap B\right) = \mathrm{P}\left(A\right)\mathrm{P}\left(B\right)$.

Using what you were given, $\mathrm{P}\left(A \cap B\right) = 0.8 \times 0.125 = 0.1$

Question 2

Difficulty:

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★★☆

Given two independent events A and B with $\mathrm{P}(A) = 0.8$ and $\mathrm{P}(B) = 0.6$, what is $\mathrm{P}(A'|B)$ and $\mathrm{P}(A \cap B')$?

1 $\mathrm{P}(A'|B) = 0.2, \mathrm{P}(A \cap B') = 0.32$ ✓

2 $\mathrm{P}(A'|B) = 0.2, \mathrm{P}(A \cap B') = 0.8$

3 $\mathrm{P}(A'|B) = 0.68, \mathrm{P}(A \cap B') = 0.32$

4 $\mathrm{P}(A'|B) = 0.8, \mathrm{P}(A \cap B') = 0.48$

Explanation

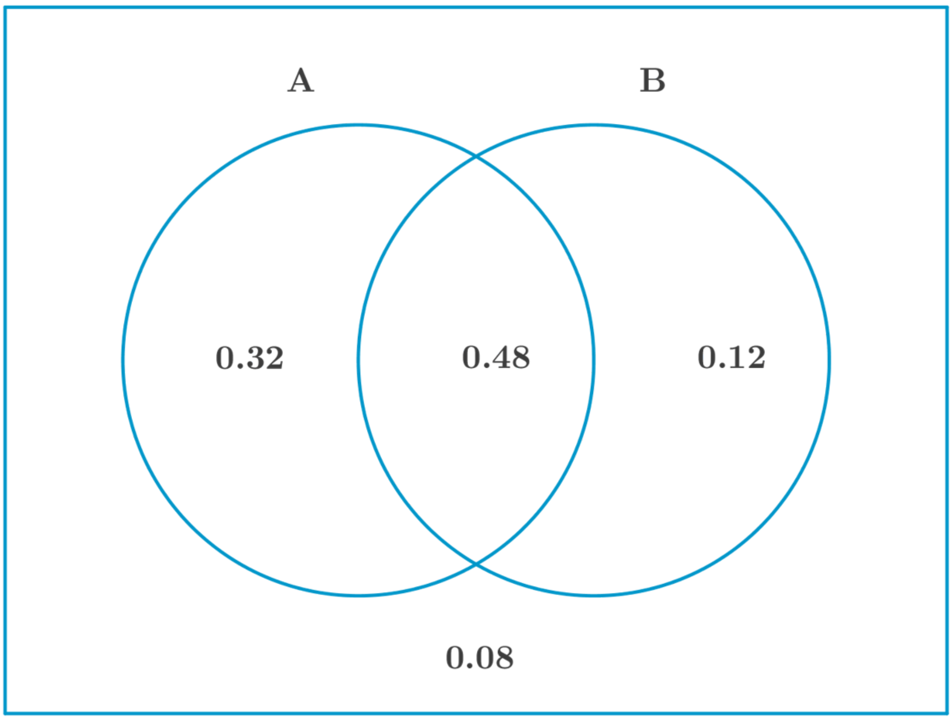
Since the two events A and B are independent, we have that

$$\mathrm{P}(A'|B) = \frac{\mathrm{P}(A' \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A') \times \cancel{\mathrm{P}(B)}}{\cancel{\mathrm{P}(B)}} = \mathrm{P}(A') = 1 - \mathrm{P}(A) = 0.2.$$

And so, also

$$\begin{aligned} \mathrm{P}(A \cap B') &= \mathrm{P}(A) \times \mathrm{P}(B') \\ &= \mathrm{P}(A) \times (1 - \mathrm{P}(B)) \\ &= 0.8 \times 0.4 \\ &= 0.32 \\ &= \mathrm{P}(A) - \mathrm{P}(A \cap B). \end{aligned}$$

This is shown in the Venn diagram below.



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More information

Question 3

Difficulty: ★★★

Two events A and B are such that $\mathrm{P}(A) = 0.3$, $\mathrm{P}(B) = 0.5$ and $\mathrm{P}(A \cup B) = 0.65$. What is $\mathrm{P}(A|B)$ and $\mathrm{P}(A'|B')$?

1 $\mathrm{P}(A|B) = 0.3, \mathrm{P}(A'|B') = 0.7$ ✓

2 $\mathrm{P}(A|B) = 0.5, \mathrm{P}(A'|B') = 0.7$

3 $\mathrm{P}(A|B) = 0.3, \mathrm{P}(A'|B') = 0.4$

4 $\mathrm{P}(A|B) = 0.5, \mathrm{P}(A'|B') = 0.4$

Explanation

First we find $\mathrm{P}(A \cap B)$:

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) = 0.3 + 0.5 - 0.65 = 0.15.$$

Using this value, we get

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{0.15}{0.5} = 0.3$$

and

$$\mathrm{P}(A'|B') = \frac{\mathrm{P}(A' \cap B')}{\mathrm{P}(B')} = \frac{1 - (\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B))}{\mathrm{P}(B')} = \frac{1 - (0.3 + 0.5 - 0.15)}{0.5} = \frac{0.35}{0.5} = 0.7.$$

This is represented in the Venn diagram below.

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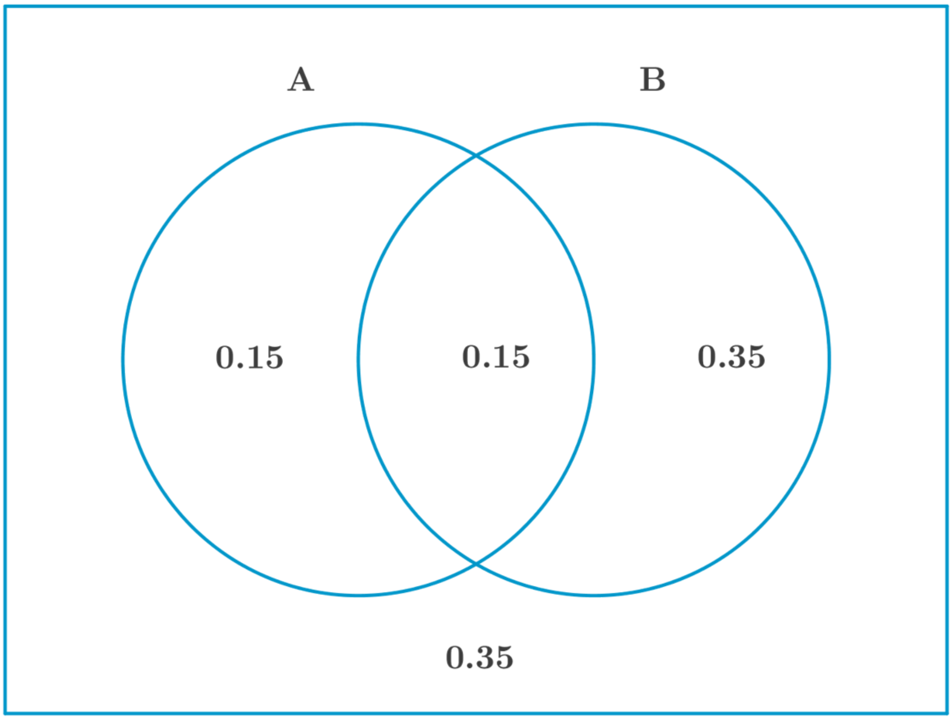

More information

Question 4

Difficulty:

★★★

Suppose $\mathrm{P}(A) = 0.75$, $\mathrm{P}(B) = \frac{2}{3}$ and $\mathrm{P}(A \cap B) = 0.5$.

Determine the value of $\mathrm{P}(B|A)$.

1 $\frac{2}{3}$ ✓

2 0.5

3 0.75

4 0.917

Explanation

You can see from the given information that $\mathrm{P}(A) \times \mathrm{P}(B) = 0.75 \times \frac{2}{3} = 0.5 = \mathrm{P}(A \cap B)$.

Since $\mathrm{P}(A)\,\mathrm{P}(B) = \mathrm{P}(A \cap B)$, events A and B must be independent.

If A and B are independent, then the occurrence of A has no impact on the probability of B, so $\mathrm{P}(B|A) = P(B) = \frac{2}{3}$.

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4. Probability and statistics / 4.11 Conditional probability and independence

Checklist

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/checklist-id-27771/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- find the probability of an event given that another event has occurred
- use conditional probability or related formulae to determine whether two events are independent.

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Investigation

Section Student... (0/0) Feedback Print (/study/app/math-aa-hl/sid-134-cid-761926/book/investigation-id-27772/print/) Assign

- In one of the practice questions for this subtopic, you were given the probabilities of two students passing their math tests, and the assumption was that the probability of one student passing is independent of the probability that another student passes. Do you think this assumption is valid? Why or why not?

- Test your hypothesis by pairing up with a classmate and compiling data from the tests you have both taken so you can test for independence.
  If you are not getting meaningful data because neither of you fails many tests (which is hopefully the case), you can use any grade boundary to categorise your performance as above or below that level.

Student view

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- Do the probabilities you find seem to support your original hypothesis? What factors might impact your results?

Share your findings with another pair of students and compare your results with theirs. Did you all come to the same conclusions?

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