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5.16 Teacher view

Numerical solutions to differential equations

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5. Calculus / 5.16 Numerical solutions to differential equations

The big picture

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27933/print/) Assign

In subtopic 5.14 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27922/), you learned the basics about differential equations and some analytic techniques for finding solutions to them. However, there are times where finding an exact answer is either impossible or so time consuming that it is no longer relevant. In subtopic 5.15 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27928/), you studied some graphical techniques for estimating solution curves for differential equations. Although not particularly accurate, these techniques gave a rough idea of a solution.

Leonhard Euler made important contributions to many branches of mathematics. For example, in this course you willl learn about Euler forms in topic 1 (section 1.13.1 (/study/app/math-ai-hl/sid-132-cid-761618/book/polar-and-euler-forms-id-27569/)), Euler's number in topic 2 (section 2.5.7 (/study/app/math-ai-hl/sid-132-cid-761618/book/exponential-functions-id-27845/)) and Eulerian trails in topic 3 (subtopic 3.16 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-28238/)).

Source: "Leonhard Euler" by Emanuel Handmann is in public domain.With a hyperlink to https://commons.wikimedia.org/wiki/File:Leonhard_Euler.jpg (https://commons.wikimedia.org/wiki/File:Leonhard_Euler.jpg) at the "Leonhard Euler" part.

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Overview (/study/ap ai- hl/sid- 132- cid- 761618/ov

In this subtopic, you will learn a numerical technique, Euler's method, that will give approximate solutions with an improved degree of accuracy through an iterative process that can be carried out using technology. Leonhard Euler first described the method in his book *Institutionum calculi integralis* published in 1768. You can view the original text in Latin here (https://archive.org/details/institutionescal020326mbp/page/n1) or read an English translation by Ian Bruce here (http://www.17centurymaths.com/contents/integralcalculusvol1.htm).

Concept

A common question in the real world related to **approximation** is how close is close enough? More accuracy typically requires more time and money, both of which are limited resources. How do you balance the need for accuracy and the limitation of these resources?

5. Calculus / 5.16 Numerical solutions to differential equations

Numerical solution: Euler's method

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/numerical-solution-eulers-method-id-27934/print/) Assign

In subtopic 5.15 (/study/app/math-ai-hl/sid-132-cid-761618/book/the-big-picture-id-27928/), you investigated graphical methods of sketching solution curves to differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = F(x, y)$. While these methods nicely illustrate the behaviour of the function satisfying the equation, they do not really show how to find values to these solutions.

In the late 1700s, Leonhard Euler developed a numerical technique (Euler's method) to approximate the length travelled along the solution curve from a known starting point. The idea follows the concept of following the slope fields in the last section, but instead of drawing a curve by hand, you draw a curve based on the known point and the slope for a set distance, and then repeat the process.

Go back to the example of the leaf dropped into a stream.

Leaf in a stream

Credit: Eerik GettyImages

Student view

You know where the leaf starts, as well as the direction of the current at that point. Where is the leaf is going? Rather than plot a complex graph of a slope field or multiple isoclines, you can estimate the path algebraically. Given the direction and speed of the current at the initial point, you can predict where the leaf will be after 15 seconds. Using your equation and the new location, you can find the direction and speed of the leaf at that point and predict where it will be 15 seconds after that, at $t = 300$. As you continue the process, you can go along the curve to whatever point in time you are interested in, say $t = 60$.

Will your prediction be correct? Not exactly. As soon as you left your spot at $t = 0$, the slope field changed. Hopefully, it was not a dramatic change, but do expect some error. As you continue along the path, expect the error to compound, slowly getting worse. Your prediction for $t = 60$ should be considerably better than your prediction for $t = 300$.

So, how can you make your prediction better? The easiest way is to reduce the 'step size' from 15 seconds. Repeating the process every five seconds would be much closer to the actual curve, but would require three times as many iterations to get to your point of interest. Lowering it further would, of course, be both more accurate and more time consuming. At some point, you need to find a balance between the effort required and the accuracy you desire. With the advent of computer technology that was not available in the 18th century, you are able to use very small step sizes and let the computer do the calculation.

Now look at a numerical example.

Example 1

★☆☆

A solution of the differential equation $y' = x - y^2$ satisfies $y(1) = 1.5$. Use Euler's method with step size 1.0 to estimate $y(2)$.

With a step size of $\Delta x = 1$, you only need one step.

The diagram below shows the actual solution curve, and the points (x_k, y_k).

You have a point and a slope, so you should be able to move forward one step easily.

The new x-value can be found as $x_1 = x_0 + \Delta x = 1 + 1 = 2$.

To find the new y-value, you need the slope at that point: $\dfrac{\Delta y}{\Delta x} \approx \dfrac{dy}{dx} = x - y^2 = 1 - 1.5^2 = -1.25$.

Since $\dfrac{\Delta y}{\Delta x} = -1.25$, you can say that $\Delta y = -1.25\Delta x = -1.25(1) = -1.25$.

Therefore, $y_1 = y_0 + \Delta y = 1.5 - 1.25 = 0.25$.

The prediction for the solution using a step size of $\Delta x = 1$ is $(2, 0.25)$ as illustrated in the graph below .

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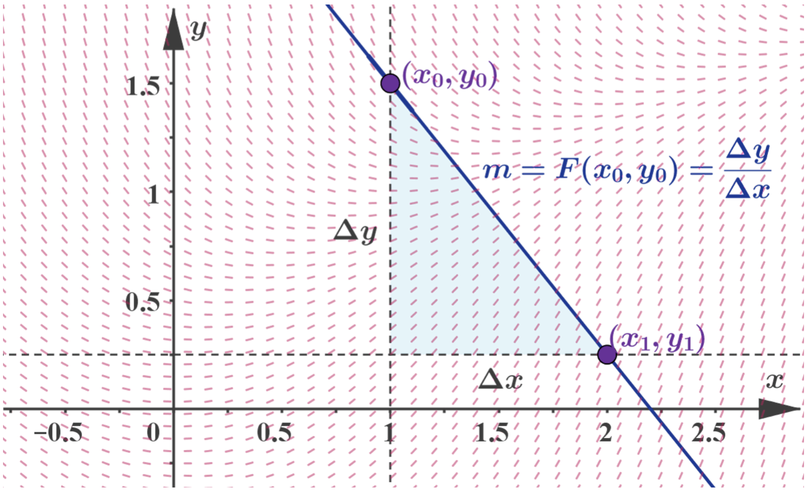
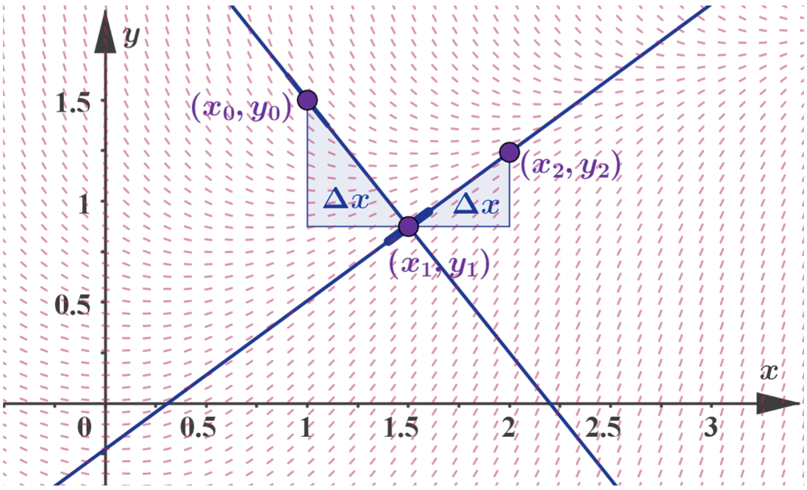


As you follow the straight-line approximation overlaid on the slope field, you can see that it starts off as a pretty good approximation, but as the solution continues, it should curve away from the line to the upper right of the diagram. The idea behind Euler's method is that you can use more steps to find a better approximation of $y(2)$. For example, if you set the step size to $\Delta x = 0.5$ and complete the process in two steps, you get:

$x_1 = x_0 + \Delta x = 1 + 0.5 = 1.5$
$\Delta y = (x - y^2)\Delta x = (1 - 1.5^2)(0.5) = -0.625$
$y_1 = y_0 + \Delta y = 1.5 - 0.625 = 0.875$

$x_2 = x_1 + \Delta x = 1.5 + 0.5 = 2.0$
$\Delta y = (x - y^2)\Delta x = (1.5 - 0.875^2)(0.5) = 0.367$
$y_2 = y_1 + \Delta y = 0.875 + 0.367 = 1.2421875 \approx 1.24$

The graph below shows a pproximating $y(2)$ in two steps at $\Delta x = 0.5$.



More information

The image is a graph that represents a slope field with several plotted points and annotations used to approximate a value of y(2) using two steps with a step size of Δx = 0.5. The X-axis is labeled with values from 0 to 3, while the Y-axis ranges from 0 to 1.5. The graph includes three significant points marked as (x0, y0), (x1, y1), and (x2, y2). These points are joined by straight lines forming two triangles between them. The change in x, denoted as Δx, is consistently labeled between the points. The vector field shows the approximate direction fields aiding in the visualization of the slope field. The caption notes how the new approximation of 1.24 is closer to the true value of y(2) than the previous 0.25, suggesting enhanced alignment with the slope field by following the paths marked on the diagram.

Student view

[Generated by AI]

If you look at the slope field, you can see that this new value of 1.24 is much closer to the actual (unknown) $y(2)$ than the previous 0.25. You see this because the blue 'path' is much more closely aligned with the slope field. If you were to continue to reduce the step size, what would you expect to happen to the time required to complete the process and the accuracy of the prediction?

Important

Euler's method is an iterative process for finding an approximate value for $y(x)$, where *y* is a solution of a differential equation of the form $y' = F(x, y)$ that also satisfies the initial condition $y(x_0) = y_0$.

During the process, points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are generated in *n* steps using the recursion:

- $x_{k+1} = x_k + \Delta x$
- $y_{k+1} = y_k + F(x_k, y_k) \times \Delta x,$

where $x_n = x$. Hence, $\Delta x = \dfrac{x - x_0}{n}$, and y_n is an approximation for $y(x)$.

Exam tip

The recursive formula for Euler's method is in section 5.16 of the formula booklet with

$y_{n+1} = y_n + h \times f(x_n, y_n); x_{n+1} = x_n + h$, where h is a constant (step length).

The applet below illustrates the iterative nature of Euler's method for the differential equation you have seen in this section. Move the point around to change the initial conditions. With the sliders, you can control the step size and the number of steps, n. You can also decide whether you want to see the actual solution curve or not.

Interactive 1. Euler's Method Visualization for Numerical Solutions to Differential Equations.

More information for interactive 1

This interactive tool demonstrates Euler's method for approximating solutions to differential equations through iterative steps.
The display shows an XY graph with an x-axis (-1 to 2.5) and y-axis (0 to 2), centered at $(0, 0)$. Two horizontal sliders on the top right control step size ranging from 0.1 to 0.5 and iteration count ranging from 0 to 39 (when step size sets to 0.1). A "Show guidelines" checkbox on the bottom right reveals the exact solution in red curve projecting on the graph, while the Euler approximation appears as connected blue segments.
Users can drag the initial point to set starting conditions. Users can adjust the parameters to understand how step size and iteration count affect the accuracy of the approximation, comparing the results against the actual solution curve when displayed.
For example, setting step size to 0.2 and $n = 10$ creates a piecewise linear approximation that diverges noticeably from the red curve. Reducing step size to 0.1 with $n = 20$ yields closer alignment. The tool visually demonstrates how smaller steps improve accuracy but require more computations, while larger steps sacrifice precision for simplicity.
Through this exploration, users grasp Euler's method as a fundamental numerical technique, understanding the trade-off between step size and accuracy. They learn to predict solution behavior from slope fields and recognize how iterative methods approximate differential equations in physics and engineering applications where exact solutions are unavailable. The comparison with actual solutions builds intuition for error analysis in numerical methods.

As you decrease the step size and increase the number of repetitions required, bookkeeping becomes an issue. Storing the values in a table can be a good way to keep up with these values.

Example 2

★★☆

A solution of the differential equation $y' = x - y^2$ satisfies $y(1) = 1.5$. Use Euler's method with step size 0.1 to estimate $y(2)$.

You can set up a table to store the repetition value (k), the independent and dependent variable values $(x^k$ and $y^k)$, and the slope (y'). Work your way across the table computing the new x, y, and (y') for each k value, then move on to the next row until you find the required value.

k	x_k	y_k	$y_k{}' = x_k - y_k{}^2$	
0	1.0	1.5	-1.25	$(x_0, y_0) = (1.0, 1.5)$ given
1	1.1	1.375	-0.790625	$y_k{}' = x_k - y_k{}^2$ given; $y_k{}'$ calculated
2	1.2	1.2959375	-0.479454004	$x_{k+1} = x_k + \Delta x$
3	1.3	1.2479921	-0.257484281	$y_{k+1} = y_k + y_k{}'\Delta x$
4	1.4	1.222243672	-0.093879593	
5	1.5	1.212855712	0.028981021	
6	1.6	1.215753814	0.121942663	

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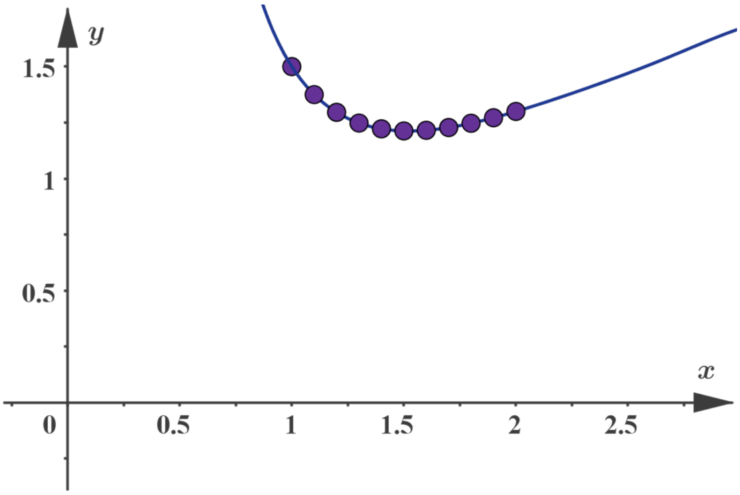
k	x_k	y_k	$y_k{}' = x_k - y_k{}^2$	
7	1.7	1.227948081	0.192143511	
8	1.8	1.247162432	0.244585869	
9	1.9	1.271621019	0.282979985	
10	2.0	1.299919017		

The diagram below shows the actual solution curve and the points (x_k, y_k).



Euler's method with a GDC

The following instructions for four GDCs show how to use Euler's method to solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y^2$, with initial condition $y(1) = 1.5$.

Student
view

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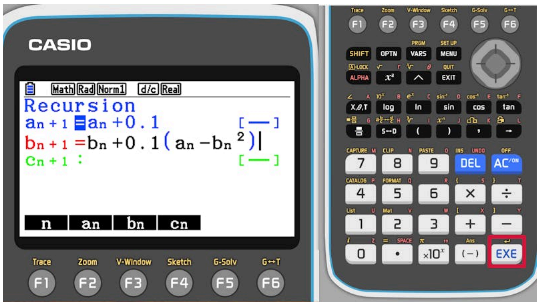
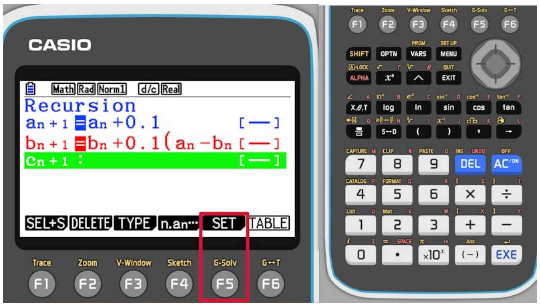
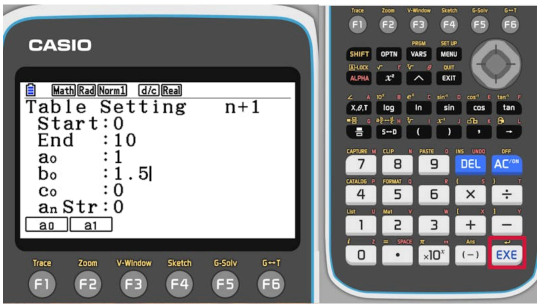
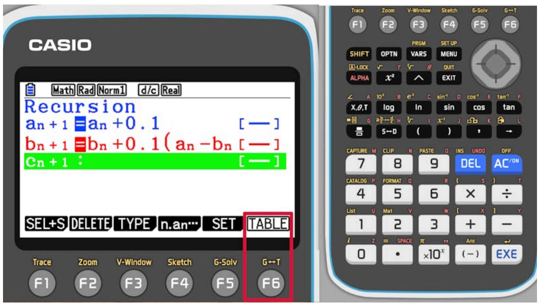
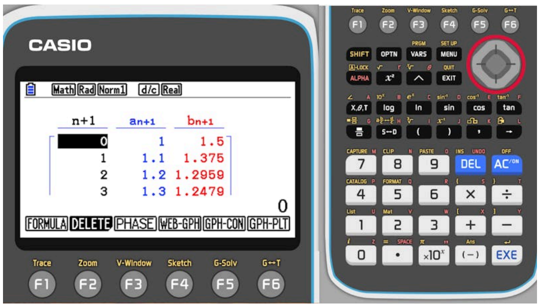
Steps	Explanation
The following instructions show you a way to use Euler's method with step size $h = 0.1$ to find an approximate value of $y(2)$, where $y(1) = 1.5$ and $y' = x - y^2$. You will also see a graphical illustration of the sequence of points leading to this approximation. Choose the option to work with recursive sequences.	
The recursive formula you need to use is $x_{n+1} = x_n + 0.1$ $y_{n+1} = y_n + 0.1(x_n - y_n^2)$ This calculator uses sequence names a and b instead of the names x and y in the formula. When you are done entering the recursion rules, press EXE.	
Press F5 to bring up the screen, where you can set the initial values for the recursion.	

Student view

Steps	Explanation
In this example the recursion starts with index 0. The condition $y(1) = 1.5$ means, that $x_0 = 1$ $y_0 = 1.5.$ Since with step size 0.1, the tenth term will be $x_{10} = 2$. Enter 10 as the last index. You can of course enter a larger number if you are interested in further values. Press EXE when all values are entered.	
The calculator now have all information needed, so press F6 to generate the table of values.	
You can see the starting x and y-values in the a and b sequences. Scroll down to see the rest of the sequence.	

Student view

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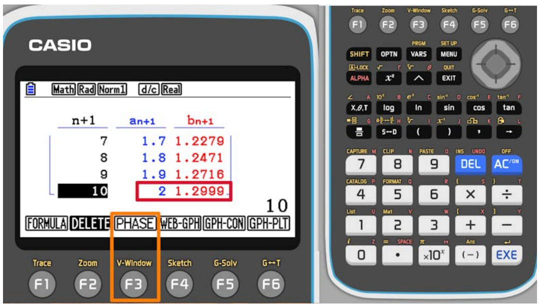
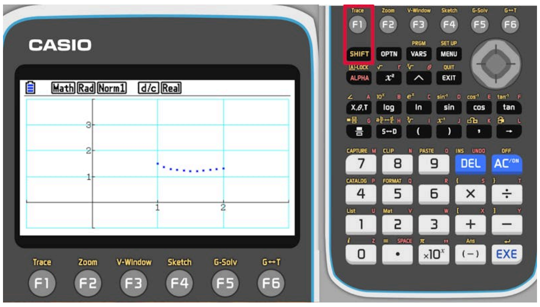
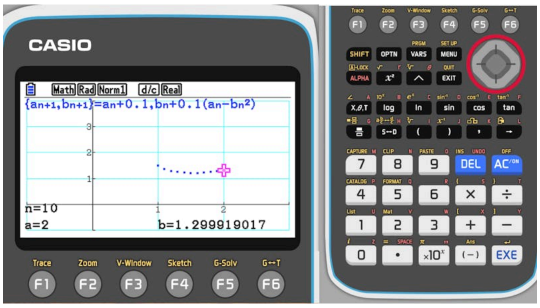
Steps	Explanation
The row corresponding to $x_{10} = 2$ gives y_{10}, the approximate value of $y(2)$. To see the graph of the sequence of the points leading to this approximation, press F3.	
You will probably need to adjust the viewing window to see the points. If you press shift/F1, you can also trace the points to see their coordinates.	
	

Student
view

Overview
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Steps	Explanation

The following instructions show you a way to use Euler's method with step size $h = 0.1$ to find an approximate value of $y(2)$, where $y(1) = 1.5$ and $y' = x - y^2$.

You will also see a graphical illustration of the sequence of points leading to this approximation.

Select the sequence application.

You can tell the calculator about the recursion in the symbolic view. The recursive formula you need to use is

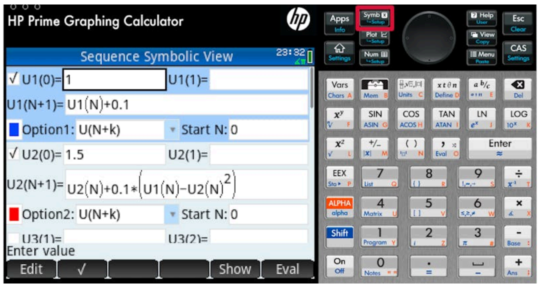
$$x_{n+1} = x_n + 0.1$$
$$y_{n+1} = y_n + 0.1(x_n - y_n^2)$$

This calculator uses sequence names U1 and U2 instead of the names x and y in the formula.

In this example the recursion starts with index 0. The condition $y(1) = 1.5$ means, that

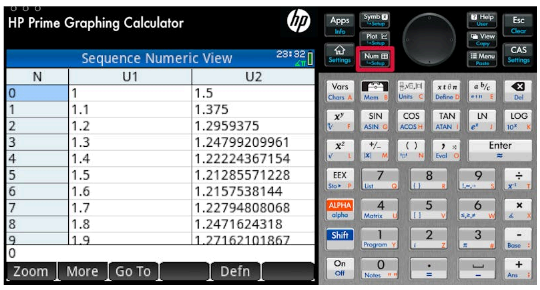
$$x_0 = 1$$
$$y_0 = 1.5.$$

Make sure you carefully enter all information in the appropriate places.



The calculator now have all information needed, so bring up the numeric view to generate the table of values.

You can see the starting x- and y-values in the U1 and U2 sequences. Scroll down to see the rest of the sequence.



Student view

Overview
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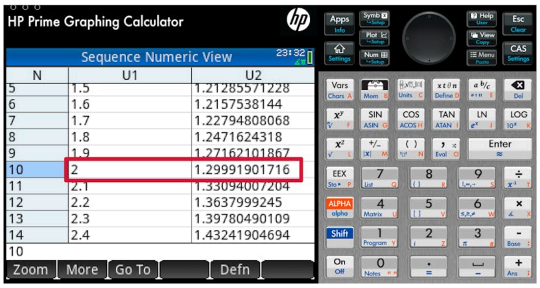
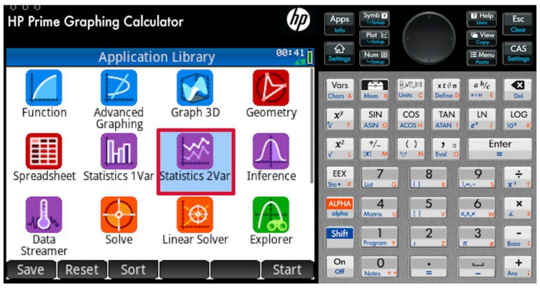
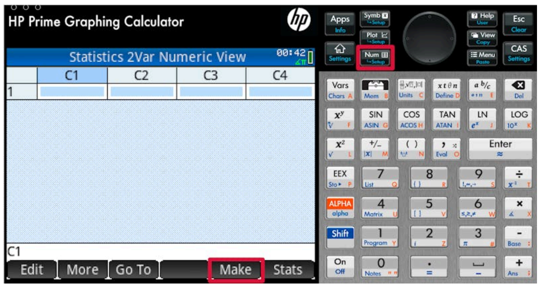
Steps	Explanation
The row corresponding to $x_{10} = 2$ gives y_{10}, the approximate value of $y(2)$.	
To generate a scatter plot of the sequence of the points leading to this approximation, go back to the application selector screen and choose the 2-variable statistics application.	
You need to copy the U sequences to this application, so in numeric view tap on Make.	

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view

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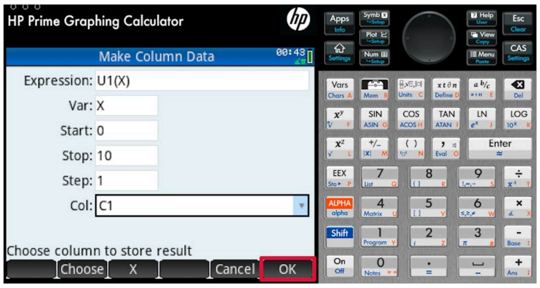
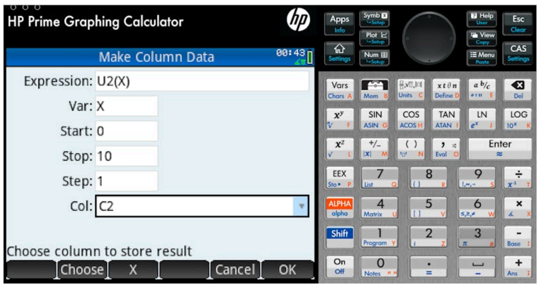
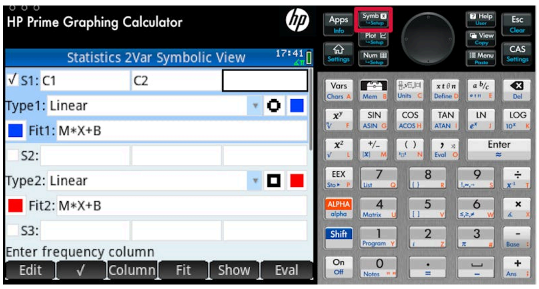
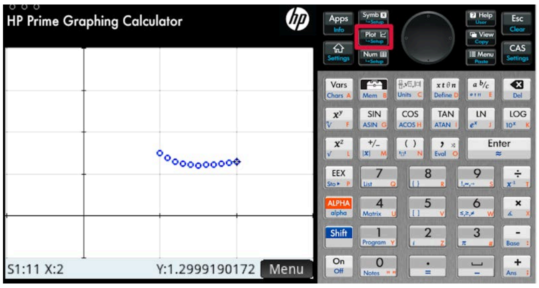
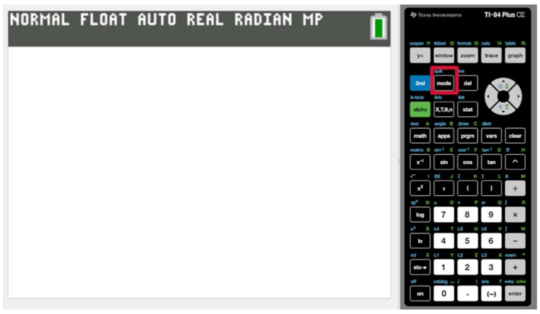
Steps	Explanation
Simply copy the first few terms of the U1 sequence to the C1 list. Since with step size 0.1, the tenth term is $x_{10} = 2$. Enter 10 as the last index. You can of course enter a larger number if you are interested in further values. Tap on OK, when you are done.	
Similarly, copy the U2 sequence to the C2 list.	
In symbolic view, make sure that the x and y lists are given as C1 and C2, and also make sure, that the frequency list is empty. The regression type does not matter now, all you want to see is the scatter plot. If you do not use the fit option, the calculator will not draw the best fit line, just the scatter plot.	

Student view

Overview (/study/ap ai-hl/sid-132-cid-761618/ov

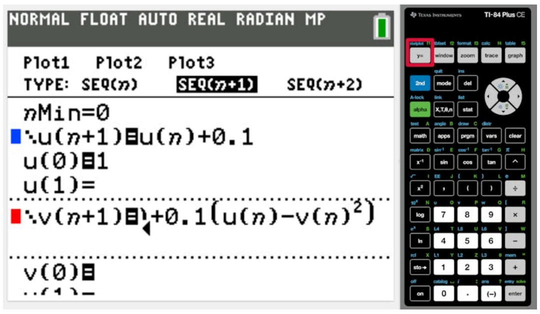
Steps	Explanation
In plot view you can see the scatter plot and you can trace the points to see the coordinates. You will probably need to adjust the viewing window to see the points.	

Steps	Explanation
The following instructions show you a way to use Euler's method with step size $h = 0.1$ to find an approximate value of $y(2)$, where $y(1) = 1.5$ and $y' = x - y^2$. You will also see a graphical illustration of the sequence of points leading to this approximation. First, you wil need to tell the calculator that you would like to work with sequences, so bring up the screen to change the settings.	
Change from function to sequence mode.	

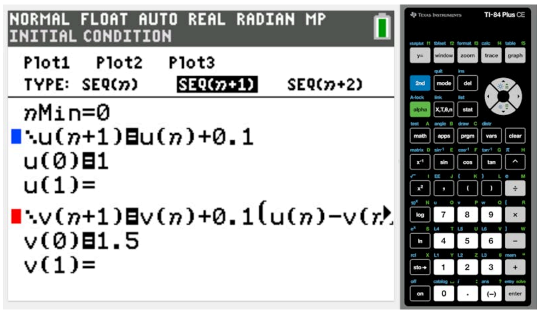
Student view

<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>
Press y= to bring up the screen to define the sequences.

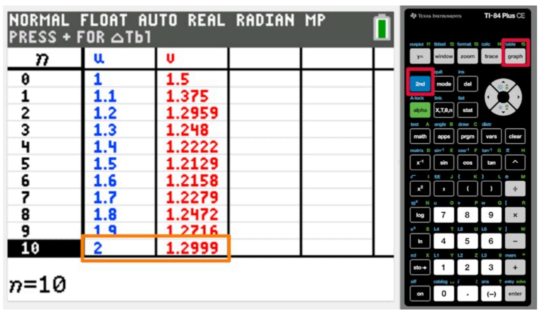
The recursive formula you need to use is

$$x_{n+1} = x_n + 0.1$$
$$y_{n+1} = y_n + 0.1(x_n - y_n^2)$$
This calculator uses sequence names u and v instead of the names x and y in the formula. In sequence mode, the variable button gives an n on the screen instead of x.
</td><td>


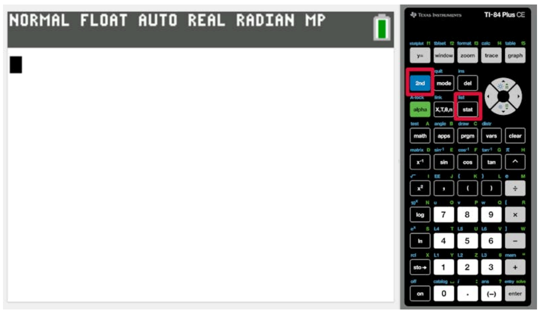
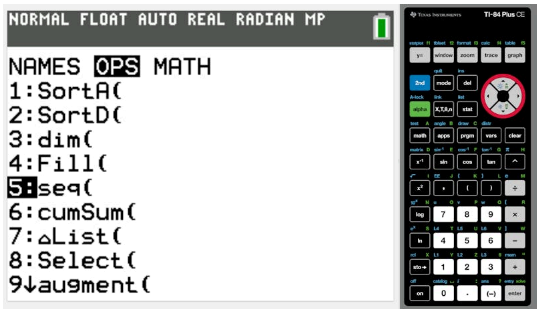
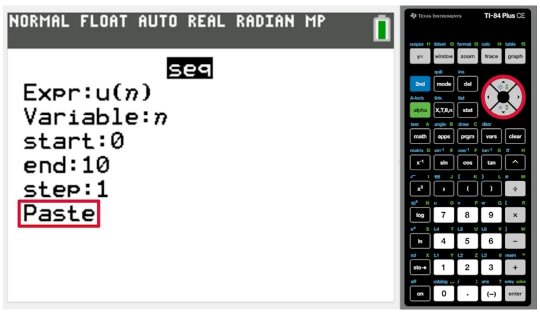
</td></tr>
<tr><td>
In this example the recursion starts with index 0. The condition $y(1) = 1.5$ means, that

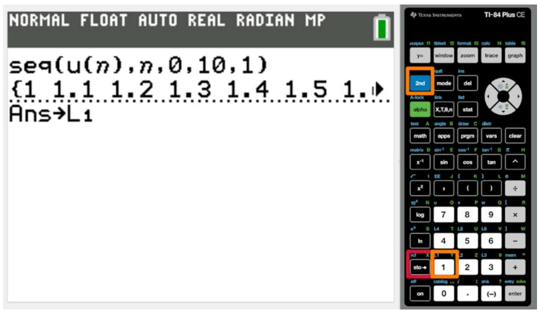
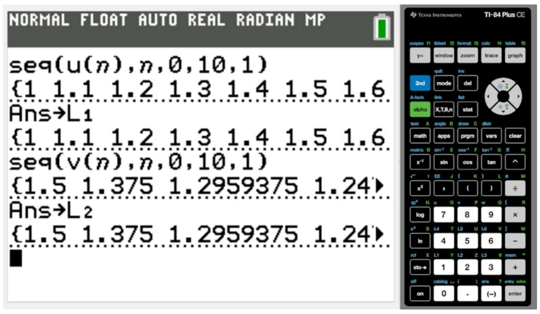
$$x_0 = 1$$
$$y_0 = 1.5.$$
</td><td>


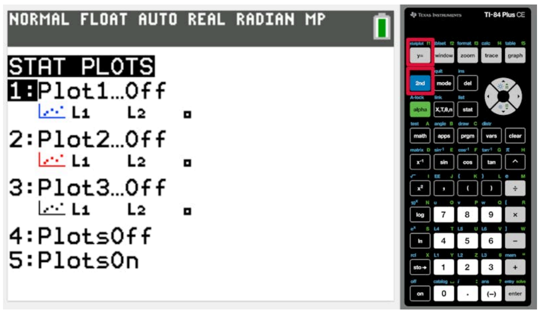
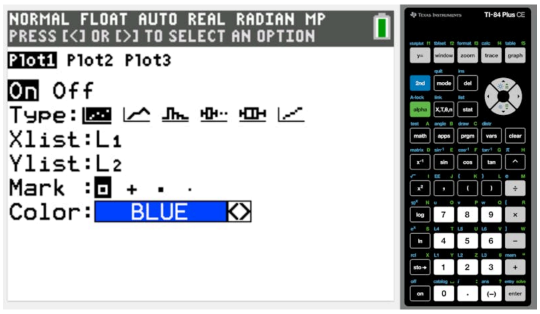
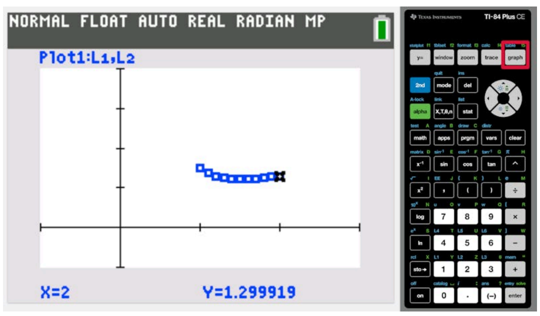
</td></tr>
<tr><td>
The calculator now have all information needed, so press 2nd/table to generate the table of values.

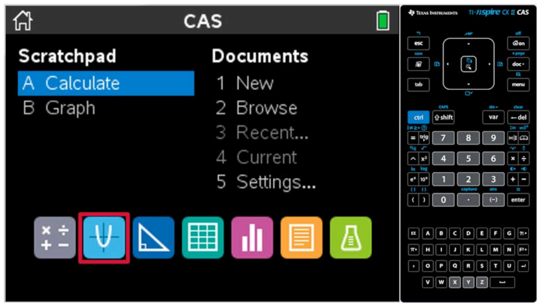
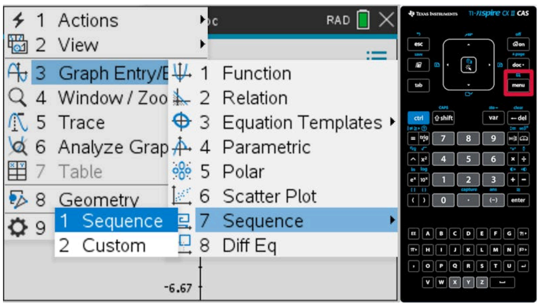
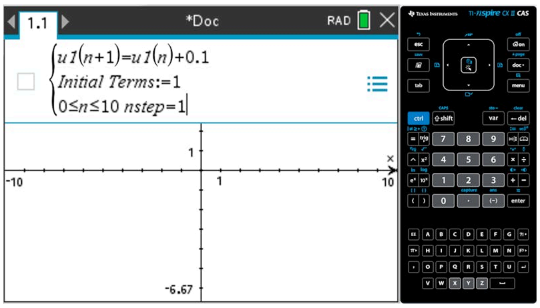
The row corresponding to $x_{10} = 2$ gives y_{10}, the approximate value of $y(2)$.
</td><td>


</td></tr>
</table>

Steps	Explanation
To generate a scatter plot of the sequence of the points leading to this approximation, go back to the calculator screen and choose the option to work with lists.	
Choose the option to generate a sequence.	
Copy the first few terms of the u sequence to the list you are generating now. Since with step size 0.1, the tenth term is $x_{10} = 2$. Enter 10 as the last index. You can of course enter a larger number if you are interested in further values. When you are done, move to Paste and press enter.	

Steps	Explanation
Store the resulting list in L1.	
Use similar steps to copy the v-list to L2. Now you have the x-coordinates of the approximating points in L1 and the y-coordinates in L2.	
Change from sequence mode back to function mode.	

Steps	Explanation
Bring up the screen to turn on a statistical plot, and choose any of the available plots.	
Turn the plot on, choose to see a scatter plot and make sure, that the x and y lists point to L1 and L2.	
Bring up the graph. You will probably need to adjust the viewing window to see the points. You can also use trace to move between the points and see the coordinates.	

Steps	Explanation
The following instructions show you a way to use Euler's method with step size $h = 0.1$ to find an approximate value of $y(2)$, where $y(1) = 1.5$ and $y' = x - y^2$. You will also see a graphical illustration of the sequence of points leading to this approximation. This can be done using the spreadsheet, but you will see a different approach here. Start with opening a graph page.	
Open the menu and choose to add a sequence.	
The recursive formula you need to use is $x_{n+1} = x_n + 0.1$ $y_{n+1} = y_n + 0.1(x_n - y_n^2)$ On this screen you enter the recursion for the x sequence. In these instructions the default sequence name, $u1$ is used. In this example the recursion starts with index 0. The condition $y(1) = 1.5$ means, that $x_0 = 1$ $y_0 = 1.5.$ Since with step size 0.1, the tenth term will be $x_{10} = 2$. Enter 10 as the last index. You can of course enter a larger number if you are interested in further values.	

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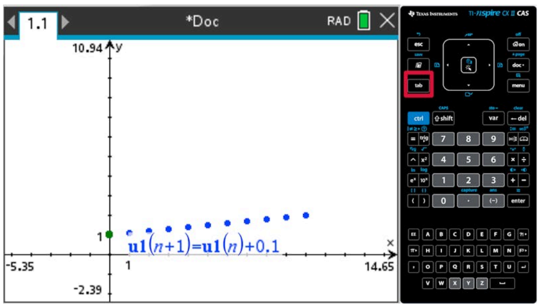
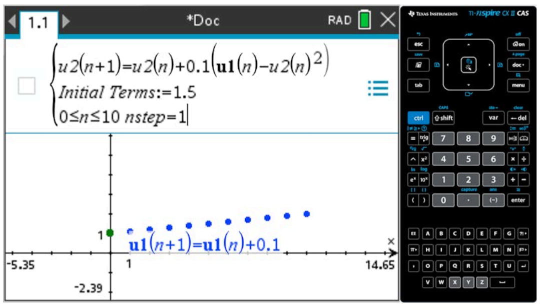
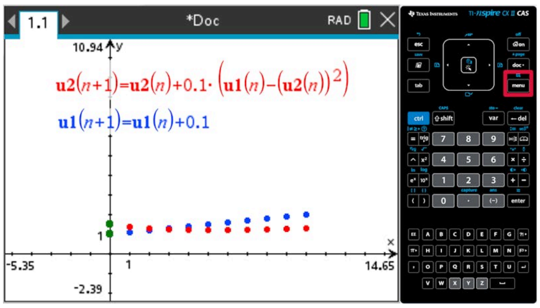
Steps	Explanation
To enter the other sequence, press tab.	
For the y-sequence the default name, $u2$ is used. Here are the formulas again for reference. $x_{n+1} = x_n + 0.1$ $y_{n+1} = y_n + 0.1(x_n - y_n^2)$ $x_0 = 1$ $y_0 = 1.5.$	
The two sequences are plotted. To see the value of the terms, you can trace the graph. Open the menu ...	

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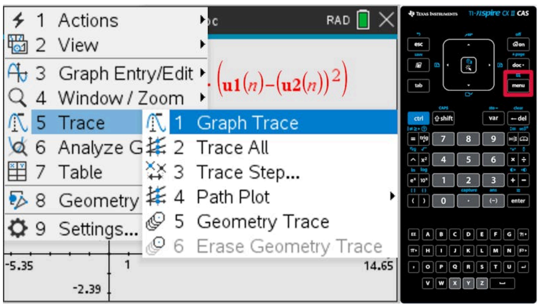
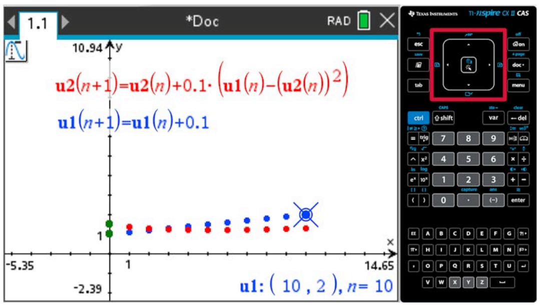
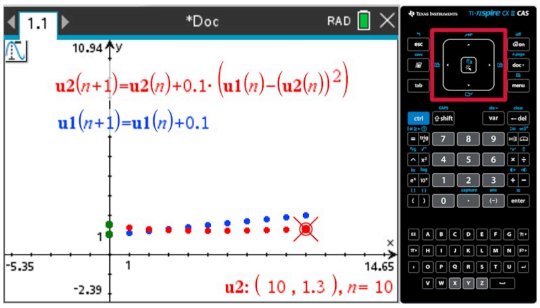
Steps	Explanation
... and choose to trace the graph.	
Confirm, that $x_{10} = 2$, ...	
... and check the tenth term of the $u2$ sequence, which is y_{10}, the approximation of $y(2)$.	

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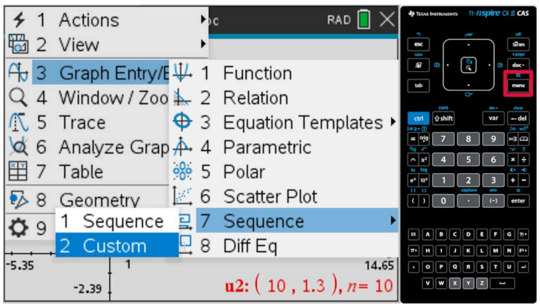
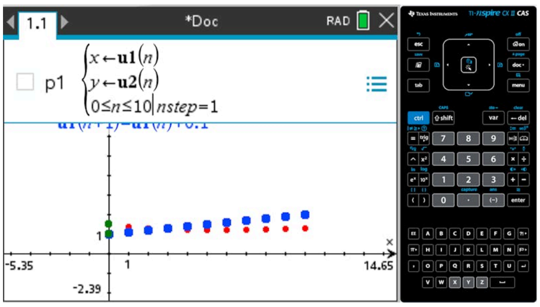
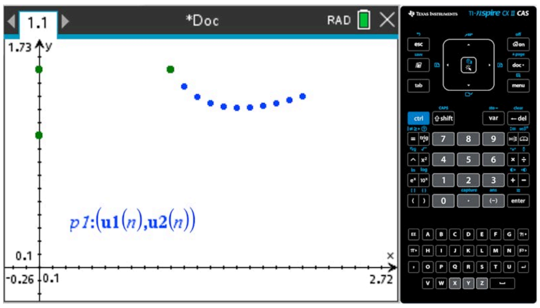
Steps	Explanation
To look at the plot of the (x_n, y_n) points, open the menu and select to add a custom sequence.	
Remember, the $u1$ list contains the x-values and the $u2$ list contains the y-values.	
After hiding the original two lists and adjusting the window, you can see the sequence of points leading to the approximation of $y(2)$.	

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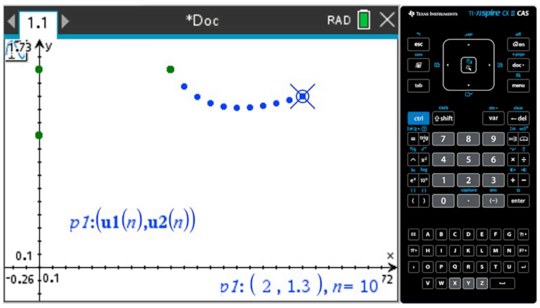
Steps	Explanation
Using the menu, you can turn on trace and investigate the coordinates of the points in the sequence.	

Activity

As you learned earlier in this section , Euler's method provides an approximation, not an exact solution. In order to improve the accuracy, decrease the step size and increase the number of computations.

In this investigation, you will investigate the impact of changing the step size.

Open up the applet at http://mathlets.org/mathlets/eulers-method/ (http://mathlets.org/mathlets/eulers-method/). There is a drop-down in the lower right-hand region to choose a function. The default is $F(x,y) = (0.5)y + 1$. The starting value is at the origin $(0,\ 0)$.

With the checkbox selected for Euler: 1.00, click the Start and Next Step buttons until the path goes off the screen. Without resetting, change the checkbox to Euler: 0.50 and repeat the process. Continue through all four step sizes. Then select the checkbox for Actual, and click the start button to see the actual curve. What do you notice about the accuracy and the number of iterations required to leave the screen?

Clear the screen. Using the drop-down menu, select the function $F\,(x,y) = y\sin(x)$. Select a starting value around $(-4,\ 1.5)$. Repeat the process above for all Euler step sizes and the actual solution. What do you notice about the accuracy and iterations required for finding the answer at $x = 4$?

Experiment with the other functions and other starting points on the screen. What kinds of functions are more susceptible to error and require smaller step sizes, and therefore more computations?

4 section questions

5. Calculus / 5.16 Numerical solutions to differential equations

Numerical solutions of coupled systems

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/numerical-solutions-of-coupled-systems-id-27935/print/) Assign

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Coupled systems

Not all events can be modelled by simple linear differential equations. There are scenarios that involve multiple variables that depend on each other. For example, the populations of wolves and deer are linked. As the population of deer increase, there is more food supply for the wolves, allowing the wolf population to increase. As the wolf population increases, the deer population decreases through hunting. This is the basis of the simple predator–prey model. As you can imagine, the differential equation describing the wolf population has the change in population being dependent on both the wolf and deer populations, just as the deer population is dependent on both the deer and wolf populations. This is an example of a coupled system.

✓ **Important**

A **coupled system** is a system formed by two differential equations with two variables and an independent variable:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(x, y, t)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f_2(x, y, t)$$

Usually, the independent variable is time, t . These are typically referred to as the Lotka—Volterra equations.

When analysing systems of differential equations, you can further classify these solutions based on how dependent the variables are on each other.

In a fully coupled system, the rates of change of both dependent variables are linked to both dependent variables. For example:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x - y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - 2y$$

The rate of change of x depends on y, and the rate of change of y depends on x.

In a partially decoupled system, the rate of change of one dependent variable is dependent on the other dependent variable, but the relationship does not go both ways. For example:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x - y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2y$$

The rate of change of x depends on y, but the rate of change of y is independent of x.

In a completely decoupled system, neither rate of change is linked to the other dependent variable. For example:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2y$$

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Furthermore, the systems covered in this course are **autonomous systems**. Autonomous systems are systems whose rates of change are not linked to the independent variable, typically time. The above examples are all autonomous. An example of a non-autonomous system would be:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = xt$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -yt^2$$

Exam tip

Only autonomous systems will be tested in this course.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(x, y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f_2(x, y)$$

Solutions of coupled systems

As with first-order linear differential equations already covered in this course, there are a variety of analytic, qualitative and numerical approaches available to solve these systems. Most of these techniques will be left for follow-on courses. Euler's method for finding a numerical solution is the first you will study.

Important

Euler's method for autonomous systems:

Given:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(x, y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f_2(x, y)$$

With initial condition (x_0, y_0) and step size Δt, Euler's approximation can be found through repeated calculations:

$$x_{k+1} = x_k + f_1(x_k, y_k)\Delta t$$
$$y_{k+1} = y_k + f_2(x_k, y_k)\Delta t$$

Example 1

★★☆

Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + (1 - x^2)y$$

with an initial condition of $x_0 = 1, y_0 = 1$ and a step size of $\Delta t = 0.25$. Find the approximate solution at $t = 1.0$.

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Using Euler's method, advance the approximation forward one step at a time, with $\Delta t = 0.25$, for four steps.

Steps	Explanation
$x_0 = 1$, Given	$y_0 = 1$, Given
$x_{k+1} = x_k + f_1(x_k, y_k)\Delta t$	$y_{k+1} = y_k + f_2(x_k, y_k)\Delta t$
$x_1 = x_0 + y_0\Delta t$ $= 1 + 1(0.25)$ $= 1.25$	$y_1 = y_0 + (-x_0 + (1 - x_0{}^2)y_0)\Delta t$ $= 1 + (-1 + (1 - 1^2)(1))(0.25)$ $= 0.75$
$x_2 = x_1 + y_1\Delta t$ $= 1.25 + 0.75(0.25)$ $= 1.4375$	$y_2 = y_1 + (-x_1 + (1 - x_1{}^2)y_1)\Delta t$ $= 0.75 + (-1.25 + (1 - 1.25^2)(0.75))(0.25)$ $= 0.332$
$x_3 = 1.4375 + 0.332(0.25)$ $= 1.5205$	$y_3 = 0.332 + (-1.4375 + (1 - 1.4375^2)(0.332))(0.25)$ $= -0.0115$
$x_4 = 1.492$	$y_4 = -0.458$
At $t = 1.0$, $(x_4, y_4) \approx (1.492, -0.458)$	

Two issues lead to efficiency problems. First, to increase accuracy, the easiest method is to decrease the step size, resulting in more computations. Second, to continue for a longer period of time, you also need more computations. Repeating these computations may not be mathematically challenging, but it is tedious and time-consuming. In the days of Leonhard Euler, hours of computations may have been acceptable, but today you can use computers.

Example 2

★★☆

Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + (1 - x^2)y$$

with an initial condition of $x_0 = 1, y_0 = 1$ and a step size of $\Delta t = 0.25$. Find the approximate solution at $t = 10.0$.

Using Euler's method, advance the approximation forward one step at a time, with $\Delta t = 0.25$, for 40 steps.

In using a spreadsheet, two things are important. First, the values must be organised. One method is to assign every variable to a column and every row to an iteration. The second challenge is to code the computations in a manner that allows the cells to be copied using relative referencing. By referring to cells rather than numbers, the reference cells of each computation will adjust automatically when copied. Consider the following cells:

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	A	B	C	D	E	F
1	k	t	x_k	y_k	x'_k	y'_k
2	0	0	1	1	1	-1
3	1	0.25	1.25	0.75	0.75	-1.67188

k is the iteration, from 0 (initial) to 40, t is time from 0 to 10.0 in increments of 0.25, x_k and y_k are the dependent variables and x'_k y'_k are the derivatives of the dependent variables. After the initial conditions are entered, formulae are built:

Cell E2: =D2	Compute x'
Cell F2: =-C2+(1-C2^2)*D2	Compute y'
Cell A3: =A2+1	Increment counter k
Cell B3: =B2+0.25	Increment time t
Cell C3: =C2+0.25*E2	Compute next x
Cell D3: =D2+0.25*F2	Compute next y

At this point, the cells can be copied down as far as needed:

k	t	x_k	y_k	x'_k	y'_k
0	0	1	1	1	-1
1	0.25	1.25	0.75	0.75	-1.67188
2	0.5	1.4375	0.332031	0.332031	-1.79158
3	0.75	1.520508	-0.11586	-0.11586	-1.3685
4	1	1.491542	-0.45799	-0.45799	-0.93064
5	1.25	1.377045	-0.69065	-0.69065	-0.75805
6	1.5	1.204382	-0.88016	-0.88016	-0.80784
7	1.75	0.984342	-1.08212	-1.08212	-1.01796
8	2	0.713811	-1.33661	-1.33661	-1.36938
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
38	9.5	1.859864	-0.64515	-0.64515	-0.27337

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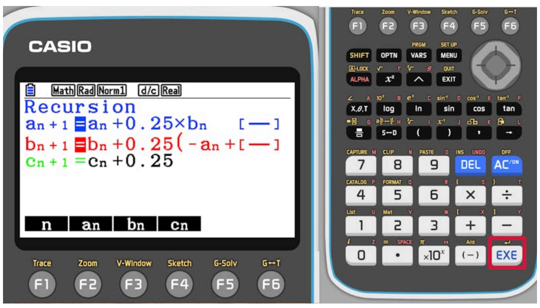
k	t	x_k	y_k	x'_k	y'_k
39	9.75	1.698576	-0.7135	-0.7135	-0.35352
40	10	1.520202	-0.80188	-0.80188	-0.46893

At $t = 10.0$, $(x_{40}, y_{40}) \approx (1.52, -0.802)$.

Learning to use some form of a spreadsheet program can be very valuable, not just in this course but throughout your future studies and in your profession. However, you are not allowed to use these on your exams. Some grahic display calculators can work with spreadsheets, some don't. However, all models that are allowed to use on IB exams can work with recursive sequences. In the instructions below you can see how to access these options to find the answer to the question in **Example 2**.

Steps	Explanation
The following instructions show you a way to use Euler's method with step size $h = 0.25$ to find an approximate solution of the coupled system $x' = y$ $y' = -x + (1 - x^2)y$ at $t = 10$, where $x(0) = 1$ and $y(0) = 1$. You will also see a graphical illustration of the sequence of points leading to this approximation. Choose the option to work with recursive sequences.	
The recursive formula you need to use is $x_{n+1} = x_n + 0.25y_n$ $y_{n+1} = y_n + 0.25(-x_n + (1 - x_n^2)y_n)$ $t_{n+1} = t_n + 0.25$ This calculator uses sequence names a, b and c instead of the names x, y and t in the formula. When you are done entering the recursion rules, press EXE.	

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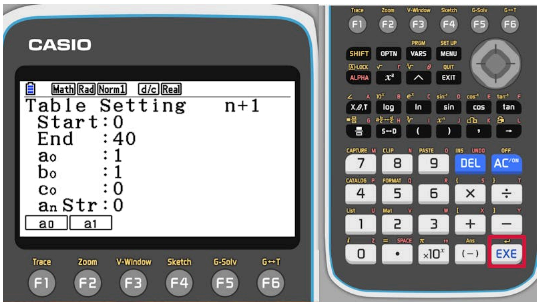
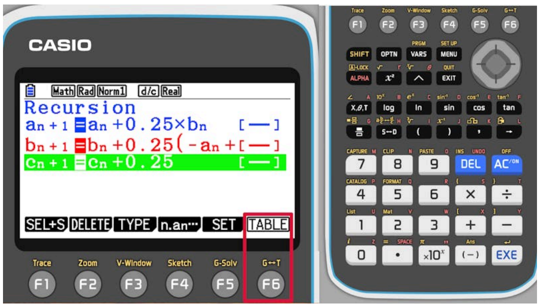
Steps	Explanation
Press F5 to bring up the screen, where you can set the initial values for the recursion.	
In this example the recursion starts with index 0. The initial conditions are: $x_0 = 1$ $y_0 = 1$ $t_0 = 0$ Since with step size 0.25, the 40th term will correspond to $t_{40} = 10$. Enter 40 as the last index. You can of course enter a larger number if you are interested in further values. Press EXE when all values are entered.	
The calculator now has all information needed, so press F6 to generate the table of values.	

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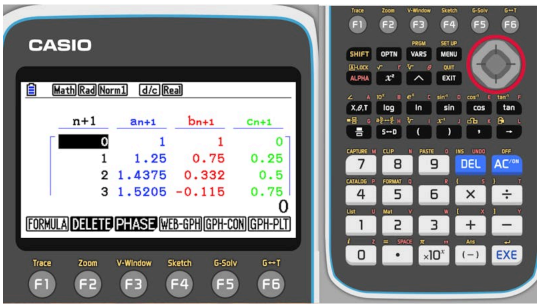
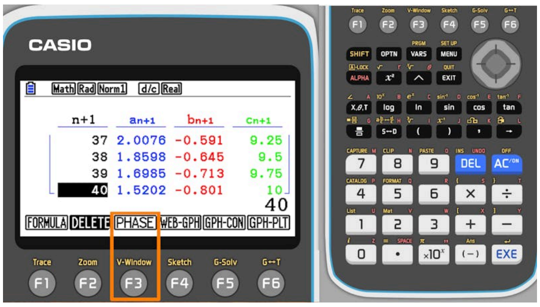
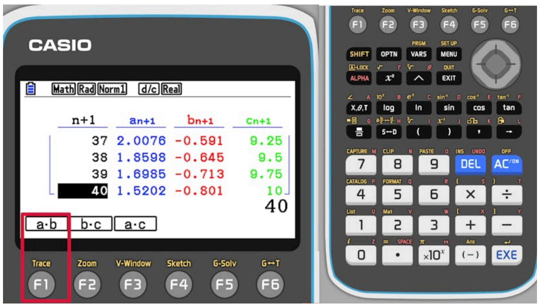
Steps	Explanation
You can see the starting x and y-values in the a and b sequences. Scroll down to see the rest of the sequence.	
The row corresponding to $t_{40} = 10$ gives the approximate x and y-values . To see the graph of the sequence of the points leading to this approximation, press F3.	
You need to choose the two variables you would like to use in the scatter plot. Press F1 to choose the x and y-values in the first two columns.	

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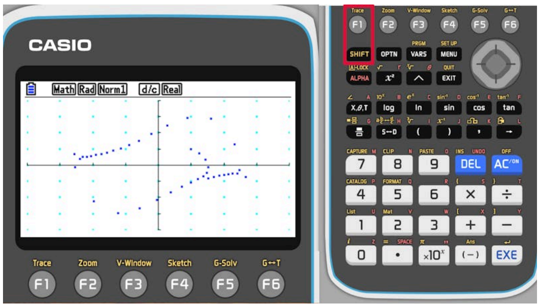
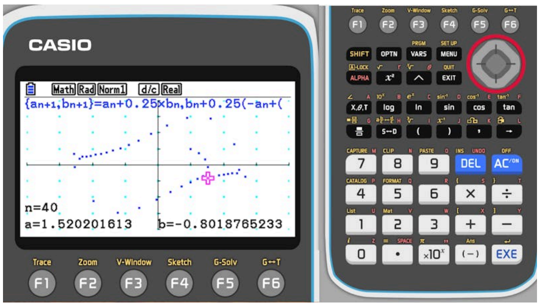
Steps	Explanation
You will probably need to adjust the viewing window to see the points. If you press shift/F1, you can also trace the points to see their coordinates.	
	

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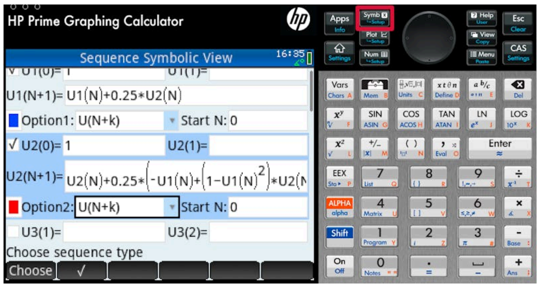
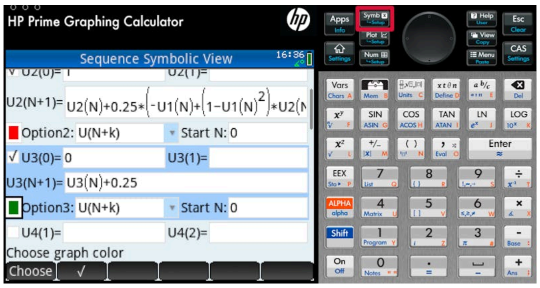
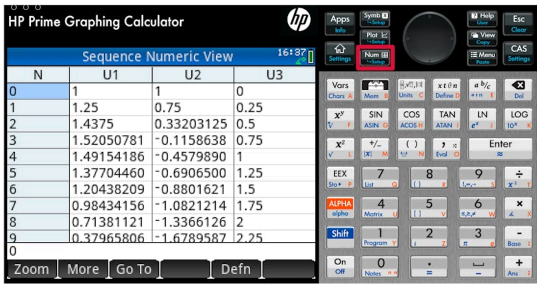
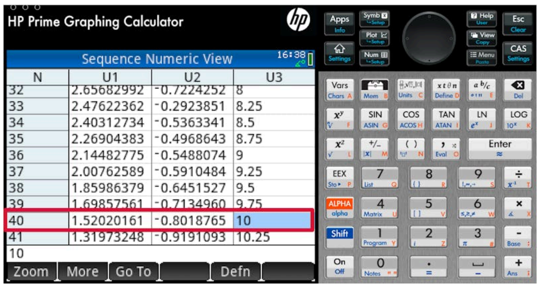
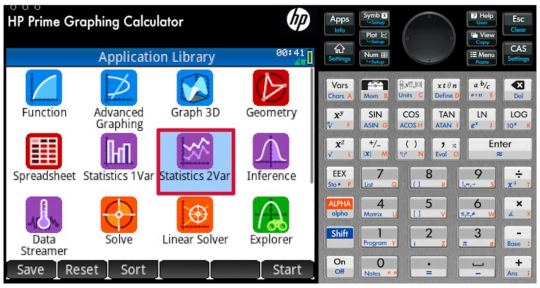
Steps	Explanation
The following instructions show you a way to use Euler's method with step size $h = 0.25$ to find an approximate solution of the coupled system $x' = y$ $y' = -x + (1 - x^2)y$ at $t = 10$, where $x(0) = 1$ and $y(0) = 1$. You will also see a graphical illustration of the sequence of points leading to this approximation. Select the sequence application.	
You can tell the calculator about the recursion in the symbolic view. The recursive formula you need to use is $x_{n+1} = x_n + 0.25y_n$ $y_{n+1} = y_n + 0.25(-x_n + (1 - x_n^2)y_n)$ $t_{n+1} = t_n + 0.25$ This calculator uses sequence names U1 and U2 instead of the names x and y in the formula. In this example the recursion starts with index 0. The initial condions are: $x_0 = 1$ $y_0 = 1.$ Make sure you carefully enter all information in the appropriate places.	
You can use the third sequence (U3) for the t -values, starting at $t_0 = 0$.	

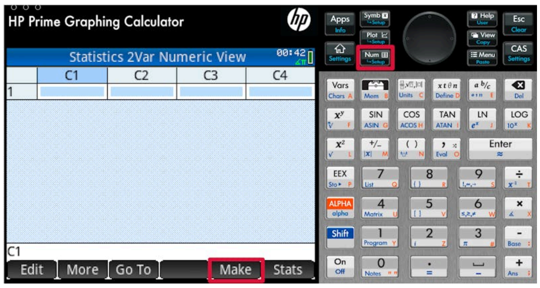
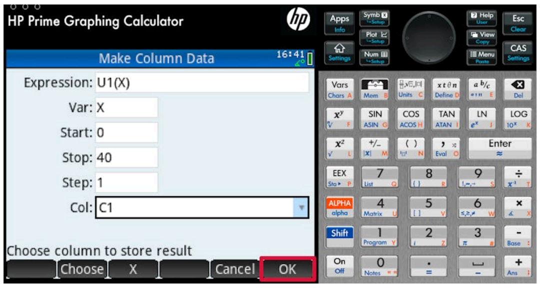
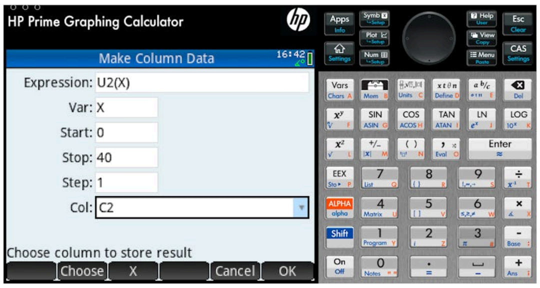
Student view

Overview
(/study/app/math-ai-hl/sid-132-cid-761618/ov

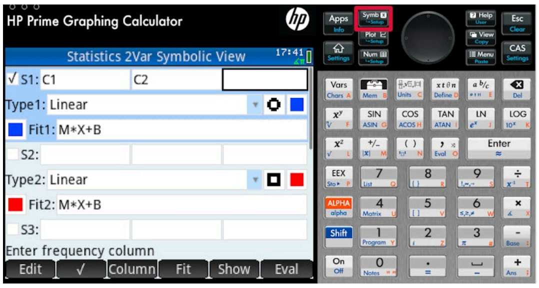
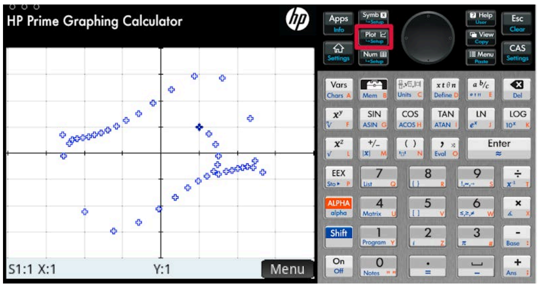
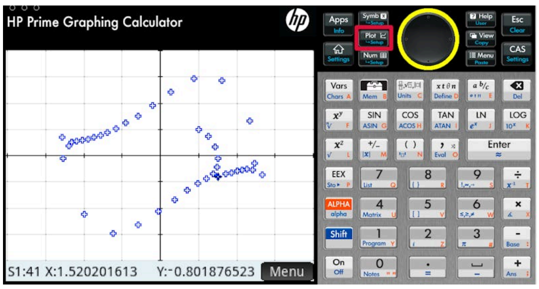
Steps	Explanation
The calculator now have all information needed, so bring up the numeric view to generate the table of values. You can see the starting x and y-values in the U1 and U2 sequences. Scroll down to see the rest of the sequence.	
The row corresponding to $t_{40} = 10$ gives the approximate x and y-values .	
To generate a scatter plot of the sequence of the points leading to this approximation, go back to the application selector screen and choose the 2-variable statistics application.	

Student view

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Steps	Explanation
You need to copy the U sequences to this application, so in numeric view tap on Make.	
Simply copy the first few terms of the U1 sequence to the C1 list. Since with step size 0.25, the 40th term will correspond to $t_{40} = 10$. Enter 40 as the last index. You can of course enter a larger number if you are interested in further values. Tap on OK, when you are done.	
Similarly, copy the U2 sequence to the C2 list.	

Student view

Steps	Explanation
In symbolic view, make sure that the x and y lists are given as C1 and C2, and also make sure, that the frequency list is empty. The regression type does not matter now, all you want to see is the scatter plot. If you do not use the fit option, the calculator will not draw the best fit line, just the scatter plot.	
In plot view you can see the scatter plot. You will probably need to adjust the viewing window to see the points.	
You can trace the points to see their coordinates.	

Steps	Explanation

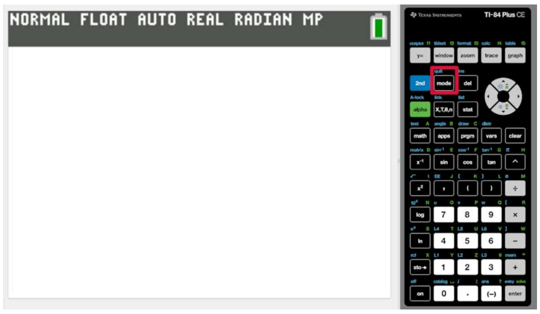
The following instructions show you a way to use Euler's method with step size $h = 0.25$ to find an approximate solution of the coupled system

$$x' = y$$
$$y' = -x + (1 - x^2)y$$

at $t = 10$, where $x(0) = 1$ and $y(0) = 1$.

You will also see a graphical illustration of the sequence of points leading to this approximation.

First, you wil need to tell the calculator that you would like to work with sequences, so bring up the screen to change the settings.



Change from function to sequence mode.

Press y= to bring up the screen to define the sequences.

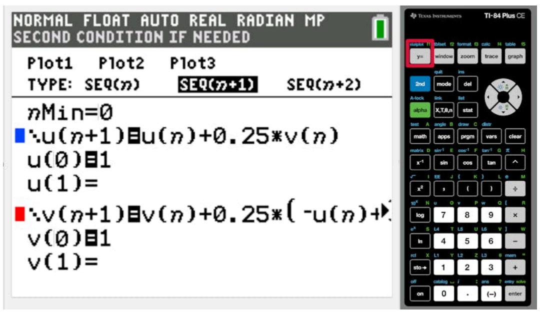
The recursive formula you need to use is

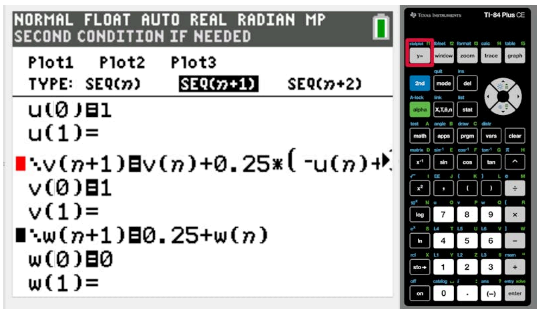
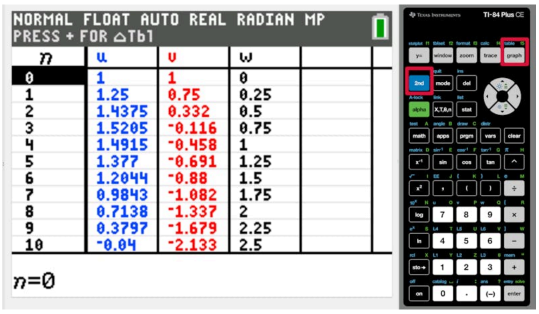
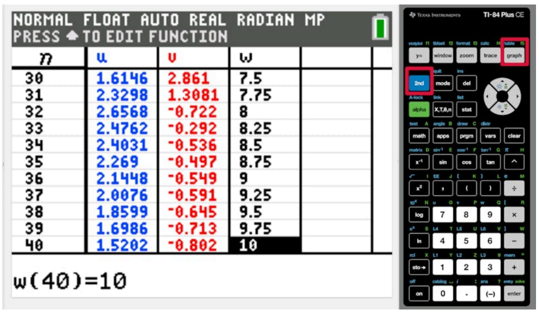
$$x_{n+1} = x_n + 0.25y_n$$
$$y_{n+1} = y_n + 0.25(-x_n + (1 - x_n^2)y_n)$$
$$t_{n+1} = t_n + 0.25$$

This calculator uses sequence names u and v instead of the names x and y in the formula. In sequence mode, the variable button gives an n on the screen instead of x.

In this example the recursion starts with index 0. The initial condions are:

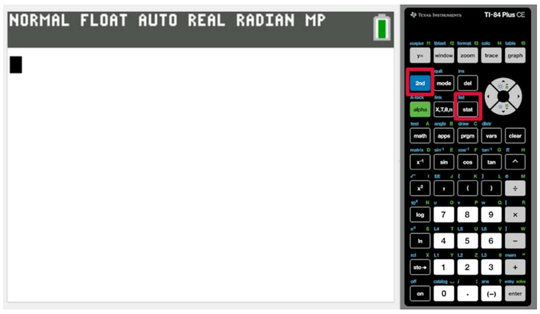
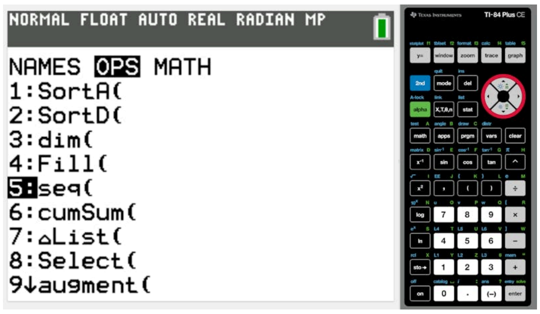
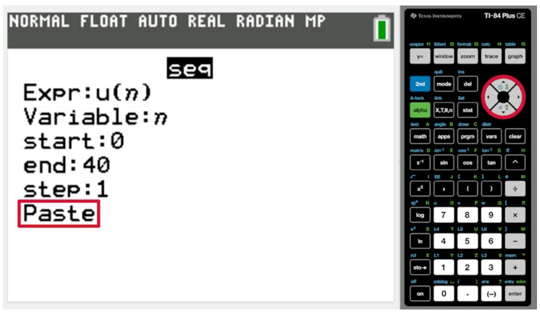
$$x_0 = 1$$
$$y_0 = 1.$$

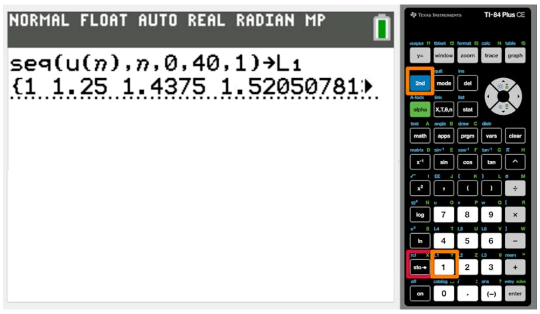
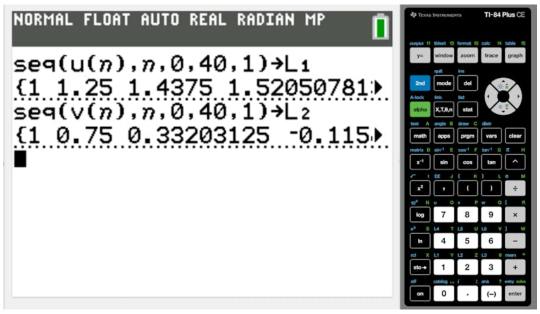


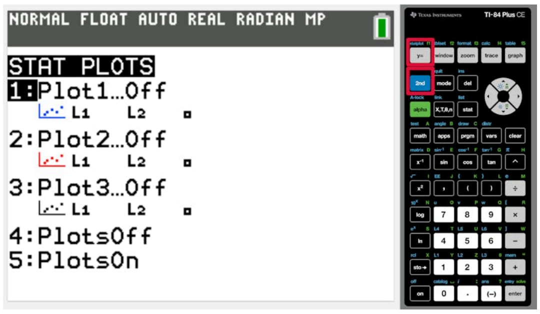
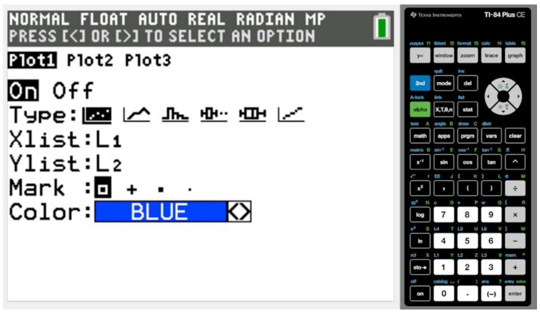
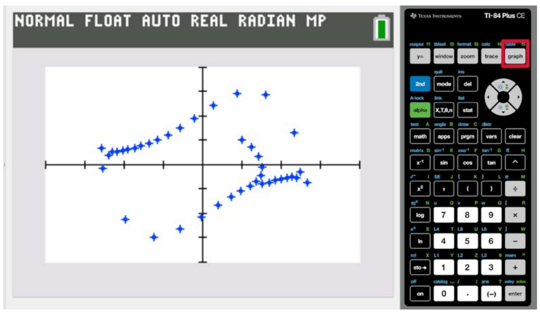
Steps	Explanation
You can use the third sequence (w) for the t-values, starting at $t_0 = 0$.	
The calculator now have all information needed, so press 2nd/table to generate the table of values. Scroll down to see more values.	
The row corresponding to $t_{40} = 10$ gives the approximate x and y-values .	

n	u	v	w
0	1	1	0
1	1.25	0.75	0.25
2	1.4375	0.332	0.5
3	1.5205	-0.116	0.75
4	1.4915	-0.458	1
5	1.377	-0.691	1.25
6	1.2044	-0.88	1.5
7	0.9843	-1.082	1.75
8	0.7138	-1.337	2
9	0.3797	-1.679	2.25
10	-0.04	-2.133	2.5

n	u	v	w
30	1.6146	2.861	7.5
31	2.3298	1.3081	7.75
32	2.6568	-0.722	8
33	2.4762	-0.292	8.25
34	2.4031	-0.536	8.5
35	2.269	-0.497	8.75
36	2.1448	-0.549	9
37	2.0076	-0.591	9.25
38	1.8599	-0.645	9.5
39	1.6986	-0.713	9.75
40	1.5202	-0.802	10

Steps	Explanation
To generate a scatter plot of the sequence of the points leading to this approximation, go back to the calculator screen and choose the option to work with lists.	
Choose the option to generate a sequence.	
Copy the first few terms of the u sequence to the list you are generating now. Since with step size 0.25, the 40th term will correspond to $t_{40} = 10$. Enter 40 as the last index. You can of course enter a larger number if you are interested in further values. When you are done, move to Paste and press enter.	

Steps	Explanation
Store the resulting list in L1.	
Use similar steps to copy the v-list to L2. Now you have the x-coordinates of the approximating points in L1 and the y-coordinates in L2.	
Change from sequence mode back to function mode.	

Steps	Explanation
Bring up the screen to turn on a statistical plot, and choose any of the available plots.	
Turn the plot on, choose to see a scatter plot and make sure, that the x and y lists point to L1 and L2.	
Bring up the graph. You will probably need to adjust the viewing window to see the points.	

Overview
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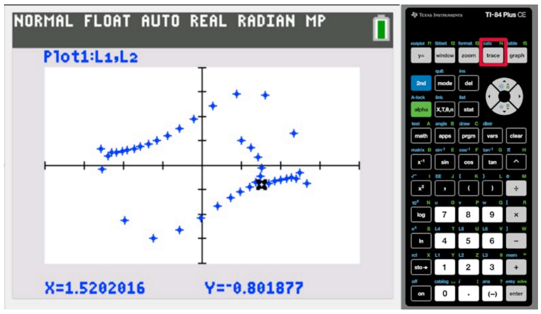
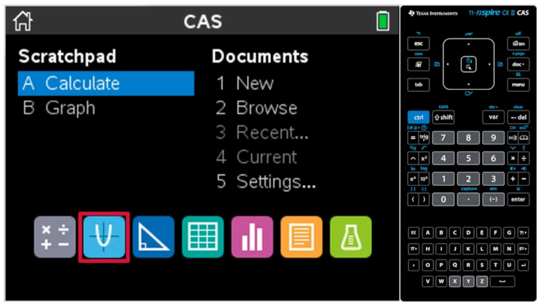
<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>You can also use trace to move between the points and see the coordinates.</td><td>
</td></tr>
</table>

<table>
<tr><th>Steps</th><th>Explanation</th></tr>
<tr><td>The following instructions show you a way to use Euler's method with step size $h = 0.25$ to find an approximate solution of the coupled system

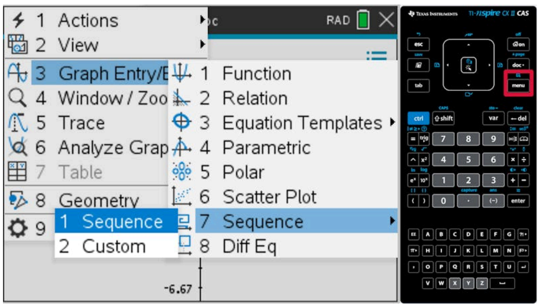
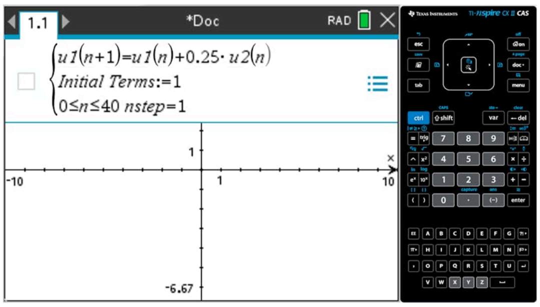
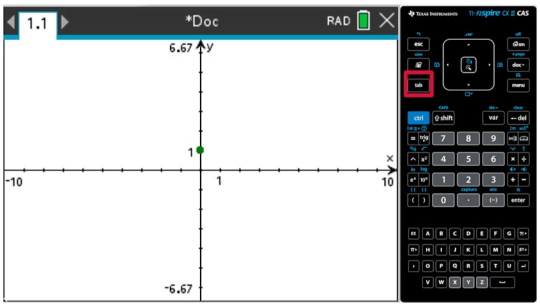
$$x' = y$$
$$y' = -x + (1 - x^2)y$$

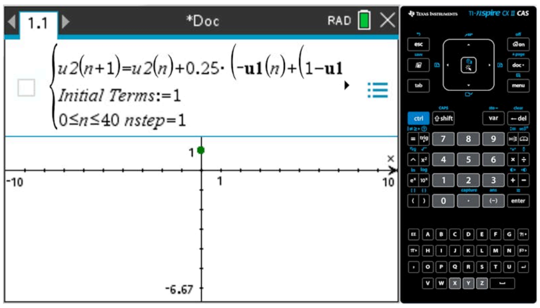
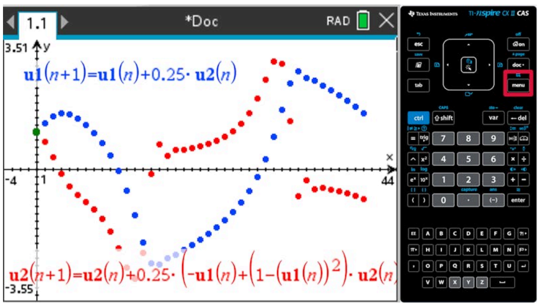
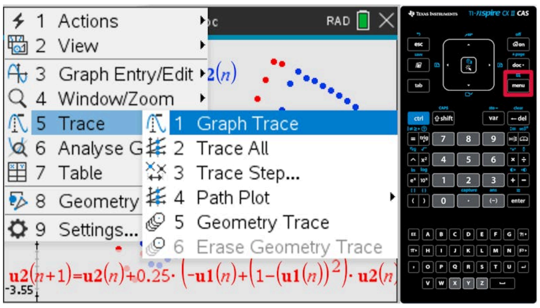
at $t = 10$, where $x(0) = 1$ and $y(0) = 1$.

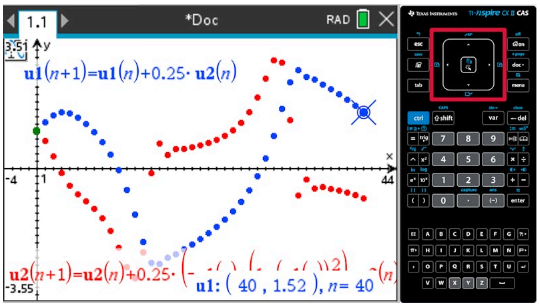
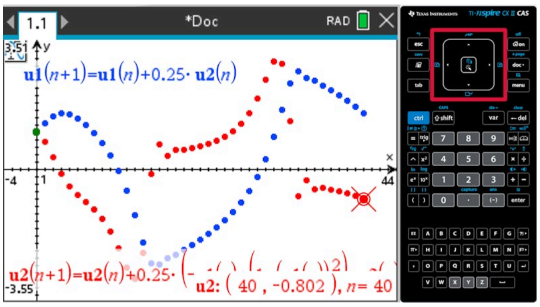
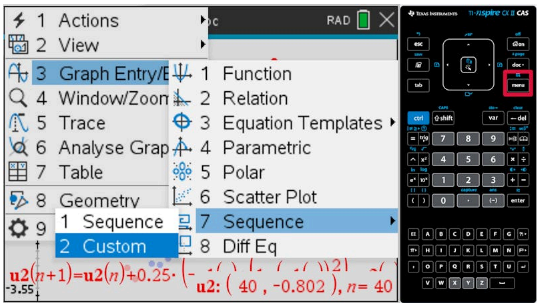
You will also see a graphical illustration of the sequence of points leading to this approximation.

This can be done using the spreadsheet, but you will see a different approach here. Start with opening a calculator page.</td><td>
</td></tr>
</table>

Student view

Steps	Explanation
Open the menu and choose to add a sequence.	
The recursive formula you need to use is $x_{n+1} = x_n + 0.25y_n$ $y_{n+1} = y_n + 0.25(-x_n + (1 - x_n^2)y_n)$ $t_{n+1} = t_n + 0.25$ On this screen you enter the recursion for the x sequence. In these instructions the default sequence name, $u1$ is used. In this example the recursion starts with index 0. The initial condions are: $x_0 = 1$ $y_0 = 1.$ Since with step size 0.25, the 40th term will correspond to $t_{40} = 10$. Enter 40 as the last index. You can of course enter a larger number if you are interested in further values.	
To enter the other sequence, press tab.	

Steps	Explanation
For the y-sequence the default name, $u2$ is used. Here are the formulas again for reference. $x_{n+1} = x_n + 0.25y_n$ $y_{n+1} = y_n + 0.25(-x_n + (1 - x_n^2)y_n)$ $x_0 = 1$ $y_0 = 1.$	
The two sequences are plotted. To see the value of the terms, you can trace the graph. Open the menu ...	
... and choose to trace the graph.	

Steps	Explanation
The second coordinate of the last point of the u1 sequence gives the x-value of the approximate solution ...	
... and the second coordinate of the last point of the u2 sequence gives the y-value of the approximate solution.	
To look at the plot of the (x_n, y_n) points, open the menu and select to add a custom sequence.	

Overview
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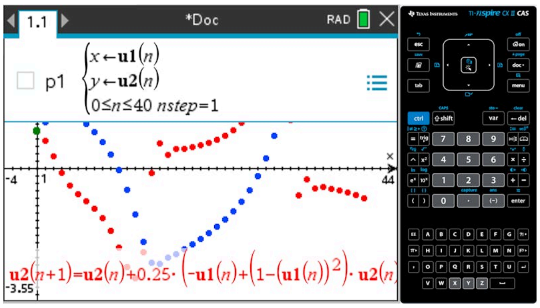
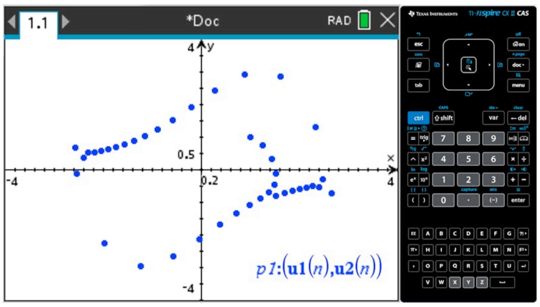
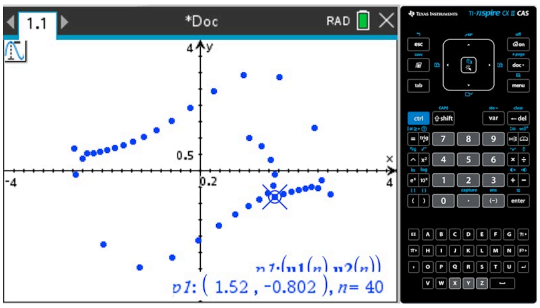
761618/ov

Steps	Explanation
Remember, the $u1$ list contains the x-values and the $u2$ list contains the y-values.	
After hiding the original two lists and adjusting the window, you can see the sequence of points leading to the approximation.	
Using the menu, you can turn on trace and investigate the coordinates of the points in the sequence.	

At the beginning of this section, the predator–prey model was mentioned. Consider the scenario with the following assumptions:

Student view

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- Wolves are predators; deer are prey.
- If no wolves are present, deer reproduce at a rate proportional to their population. All other resources are infinite; they are unaffected by overcrowding.
- Wolves eat deer, and the rate is proportional to the rate at which they interact.
- Without deer to eat, the wolf population decreases at a rate proportional to itself.
- The rate at which wolves are born is proportional to the number of deer eaten by wolves and the rate at which the two species interact.
- Variables and constants are as follows:
  - W – Wolf population
  - D – Deer population
  - a – Constant of proportionality for wolf–deer interaction affecting wolves
  - b – Death rate of wolves
  - c – Growth rate coefficient of deer
  - d – Constant of proportionality for wolf–deer interaction affecting deer.

This leads to a coupled system:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = aWD - bW$$
$$\frac{\mathrm{d}D}{\mathrm{d}t} = cD - dWD.$$

Example 3

★★★

Consider a typical predator–prey two-species system described above with:

- Initial wolf population of 200.
- Initial deer population of 120.
- Constant of proportionality for wolf–deer interaction affecting wolves of 0.0012.
- Death rate of wolves of 0.12.
- Growth rate coefficient of deer of 0.1.
- Constant of proportionality for wolf–deer interaction affecting deer of 0.0005.

Find the population range for both species over the next 150 units of time with a step size of 0.5.

$$\frac{\mathrm{d}W}{\mathrm{d}t} = 0.0012WD - 0.12W$$
$$\frac{\mathrm{d}D}{\mathrm{d}t} = 0.1D - 0.0005WD$$

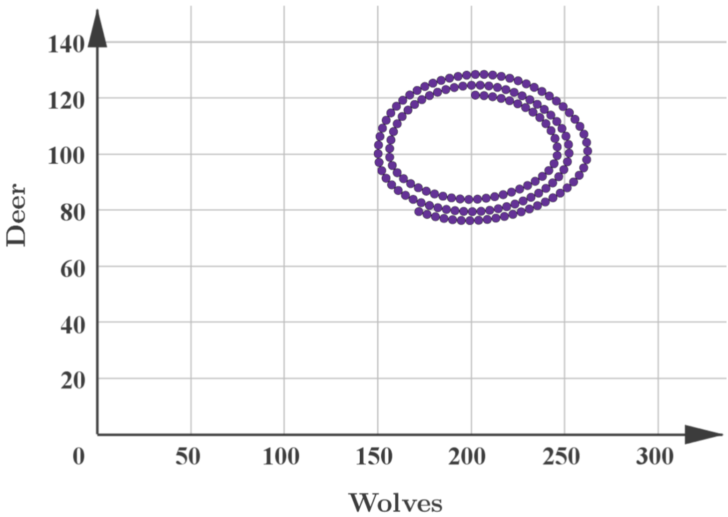
The table below shows the steps of Euler's method when a spreadsheet is used. You can get the same values for W_k and D_k when you use your calculator. Note though that on your calculator this will take a long time. On exams you will not get questions with 300 iterations, so the running time will be reasonable.

Student view

k	t	W_k	D_k	W'_k	D'_k
0	0	200	120	4.8	0
1	0.5	202.4	120	4.8576	−0.144
2	1	204.8288	119.928	4.898194	−0.28955
3	1.5	207.2779	119.7832	4.92075	−0.43588
4	2	209.7383	119.5653	4.924306	−0.58218
5	2.5	212.2004	119.2742	4.90799	−0.7276
⋮	⋮	⋮	⋮	⋮	⋮
299	149.5	162.6053	78.58785	−4.17807	1.469384
300	150	160.5163	79.32254	−3.98288	1.565975

Looking at the W_k and D_k columns, the wolf population varies cyclically with a range from 149–247 and the deer population varies cyclically from 74–129.

Below are two graphs depicting the wolf and deer populations over the course of the analysis, 300 units of time.



More information

The graph presents the relationship between deer and wolf populations over a period of time, represented as a polar plot. The X-axis represents the wolf population, ranging from 0 to 300. The Y-axis represents the deer population, ranging from 0 to 140. The data points form a tight concentric spiral, which suggests periodic cycles in the population dynamics between wolves and deer. As the step size decreases, the pattern converges towards an oval shape, indicating a closer, consistent oscillation of populations around certain values.

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Overview
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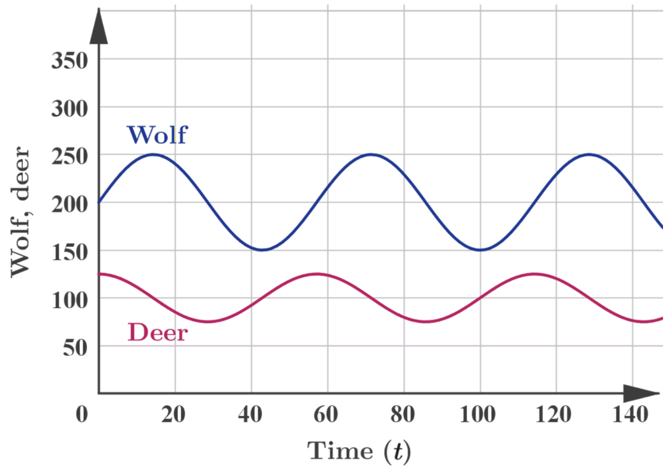
Notice the tight concentric spiral. As step size continues to decrease, the spiral would converge to an oval.



More information

The graph demonstrates the population trends of wolves and deer over time with two intersecting sinusoidal curves. The x-axis represents time, labeled 'Time (t)', ranging from 0 to 140. The y-axis indicates population sizes, labeled 'Wolf, deer', with a range from 0 to 350.

The blue curve represents the wolf population, beginning at approximately 250. It rises to about 300 at t=20, dips to around 200 at t=60, and then climbs back to 280 at t=100, following a sinusoidal pattern.

The red curve shows the deer population, starting near 100. It rises to roughly 150 at t=40, falls to around 80 at t=80, and then replicates the pattern to approximately 140 at t=120.

Both curves indicate cyclic patterns with increasing ranges over time. This pattern suggests that both populations are becoming periodic as the step size decreases, which aligns with the note suggesting the system will eventually converge to an oval due to tighter spirals and periodicity.

[Generated by AI]

Notice that the range of both populations is increasing slightly over time. As step size continues to decrease, the range would become constant and both would become periodic in nature.

This is a fairly significant range, particularly for the wolf population, which would indicate a feast-or-famine relationship. To reduce the suffering during famine caused by loss of food supply, game wardens might want to trap or allow hunting of wolves at specific times to approach an equilibrium relationship.

2 section questions

Student view

5. Calculus / 5.16 Numerical solutions to differential equations

Checklist

Overview (/study/app/math-ai-hl/sid-132-cid-761618/ov

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/checklist-id-27936/print/) Assign

What you should know

By the end of this subtopic you should be able to:

- approximate the solution to $y(x)$ if $y(x_0) = y_0$ is given and y is the solution of the first-order differential equation $y' = F(x, y)$ through the iterative process of Euler's method
- model and solve coupled systems of the form $\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(x, y, t)$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = f_2(x, y, t)$.

5. Calculus / 5.16 Numerical solutions to differential equations

Investigation

Section Student... (0/0) Feedback Print (/study/app/math-ai-hl/sid-132-cid-761618/book/investigation-id-27937/print/) Assign

Investigation 1

Can you imagine what it was like to use numerical techniques like Euler's method in the 1700s with nothing more than a slide rule and a pen (pencils were not developed until the turn of the 19th century)? There needs to be a balance between the effort involved in carrying out a computation and the accuracy. Today, with the development of calculators and computers, it is easy to carry out complex computations with a little bit of knowledge of programming.

For this investigation, open up a suitable software package and build a program that will approximate the solution of a differential equation that satisfies a given initial condition. Several spreadsheet programs are available – Microsoft Excel and Google Sheets are a couple of popular examples.

Look at the following problem.

A solution of the differential equation $y' = \frac{x}{y}$ satisfies $y(1) = -2$. Use Euler's method with varying step size to estimate $y(2)$.

Using the bookkeeping technique presented in the example and section questions from section 5.18.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/solving-secondorder-differential-equations-id-27944/) as a template, build a spreadsheet with relative referencing to complete the computations. If you are already comfortable with programming, go ahead. Otherwise, here are some sample instructions.

In cell A1, enter in the step size. You can start at 0.1.

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In cells A2, B2, C2, and D2, enter labels k, x_k, y_k and y' respectively. Subscripts are not important as long as you still know what they mean.

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In cell A3, enter 0 to indicate you are starting at $k = 0$.

In cell A4, you are going to automatically increase the counter with an equation. Enter =A3+1. Copy this cell down as far as you need. For a step size of 0.1, you will need it to go down at least as far as A13 where $k = 10$. Notice that as the cells go down, the formulae change where they point to. Although you had cell A4 point to A3, A5 will point to A4, A6 will point to A5, and so on. You call this relative referencing.

In cell B3, enter the initial x-value. For this problem, it is given as 1.

In cell C3, enter the initial y-value. For this problem, it is given as –2.

In cell D3, you are going to enter your first real equation. From the problem statement, you know that $y' = \frac{x}{y}$. To program this in the spreadsheet, enter =B3/C3. If you use this problem for another differential equation, you will need to change this equation.

In cell B4, you are going to increase your x-value by the step size. Enter =B3+$A$1. You could just enter in =B3+0.1, but then you would have to change the equation every time you changed the step size. By pointing to cell A1, it is much more obvious and helps you to remember what step size you are using. Also notice the $ signs. By placing the $ signs in front of the A and the 1, you can 'anchor' that value, preventing it from 'moving' as you copy the cell down later on. Go ahead and copy cell B4 down as far as you copied cell A4, at least to row 13.

In cell C4, you are going to increase your y-value. You know the slope, $\frac{dy}{dx}$, and you know the step size, Δx, so you just need to multiply those together and add them to the last value of y. Enter =C3+$A$1*D3. Notice that C3 (the old y) and D3 (the old slope) are not anchored as they will change as you move down, but cell A1 (the step size) is anchored as you do not want it to move. Copy cell C4 down to match columns A and B.

Copy cell D3 down to match the other columns.

If you followed these instructions, you should have something that looks like:

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	A	B	C	D
1	0.1	Step Size		
2	k	x_k	y_k	y'
3	0	1	-2	-0.5
4	1	1.1	-2.05	-0.54
5	2	1.2	-2.1	-0.57
6	3	1.3	-2.16	-0.6
7	4	1.4	-2.22	-0.63
8	5	1.5	-2.28	-0.66
9	6	1.6	-2.35	-0.68
10	7	1.7	-2.42	-0.7
11	8	1.8	-2.49	-0.72
12	9	1.9	-2.56	-0.74
13	10	2	-2.63	-0.76

More information

The image is a screenshot of a spreadsheet displaying four columns labeled A to D, each containing data relevant to solving a differential equation.

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Column A is titled "k" and contains integers from 0 to 10, representing discrete steps. Column B is labeled "x_k" with values starting from 1, increasing by 0.1 steps, indicating different x-values. Column C is labeled "y_k", showing initial and progressively decreasing values starting at -2 down to -2.63, representing y-values at each step. Column D titled "y'", portrays derivative values starting at -0.5, decreasing to -0.76.

The cells in the header row and the data columns A, B, and C include highlighted yellow fields, indicating editable or important values. Particularly, the header cells (B1, B2, C2, D2) indicate key points to alter step size, starting position, or differential equation parameters as described in the accompanying content.

[Generated by AI]

Cells A1, B3 and C3 are where you can alter the step size or the coordinates of the starting position. You can change the differential equation by altering the equation behind cell D3. The cells in yellow are those that will change value a s you vary the step size, starting position, or differential equation.

Now you are ready to study the effects of step size. By changing the step size and copying more rows as needed, determine approximations of $y(2)$ for the following step sizes:

Step size (Δx)	**Steps** (k)	$y(2)$
1		
0.5		
0.2		
0.1	10	-2.63455
0.05		
0.01		
Exact		$-\sqrt{7} \approx -2.645751311$

The exact answer was found using techniques found in section 5.14.2 (/study/app/math-ai-hl/sid-132-cid-761618/book/exact-solution-separable-equations-id-27924/) for separable equations. If you were not able to find an exact answer, how many steps would you be willing to do?

Investigation 2

Around 1900, German mathematicians Carl Runge and Wilhelm Kutta expanded on Euler's method by taking a weighted average of slopes over the step interval. These methods are referred to as Runge–Kutta methods. Technically, you have already learned one of the Runge–Kutta methods. The first-order Runge–Kutta method is nothing more than Euler's method with only one slope.

Although any number of averages can be used, the most common is the fourth-order, abbreviated as RK4. In this investigation, you will compare the efficiency of Euler's method with RK4.

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The formulae for the RK4 method are as follows:

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\begin{aligned}
lx_{n+1} &= x_n + h \\
y_{n+1} &= y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \\
k_1 &= f\left(x_k, y_k\right) \\
k_2 &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) \\
k_3 &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right) \\
k_4 &= f\left(x_n + h, y_n + h\,k_3\right)
\end{aligned}
$$

As a reminder, the formulae for Euler's method, or RK1, are:

$$
\begin{aligned}
x_{n+1} &= x_n + h \\
y_{n+1} &= y_n + h\,f\left(x_n, y_n\right)
\end{aligned}
$$

Consider finding an approximation to $y(1.4)$ for the solution of the differential equation $y' = 2xy, y(1) = 1$. The actual value is $y(1.4) = 2.6117$.

First, Euler's method as used previously with a step size of $h = 0.01$:

k	x_k	y_k	y'
0	1	1	2
1	1.01	1.02	2.0604
2	1.02	1.0406	2.1228
3	1.03	1.0618	2.1874
4	1.04	1.0837	2.2541
5	1.05	1.1062	2.3231
6	1.06	1.1295	2.3945
7	1.07	1.1534	2.4683
8	1.08	1.1781	2.5447
9	1.09	1.2036	2.6237
10	1.1	1.2298	2.7055
11	1.11	1.2568	2.7902
$\vdots$	$\vdots$	$\vdots$	$\vdots$
39	1.39	2.5024	6.9565
40	1.40	2.5719	7.2014

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Notice that, with a step size of 0.01, it took forty iterations, with three computations per iteration, for a total of 120 computations. The approximation is $y(1.4) \approx 2.5719$, resulting in an error of 1.5%.

In a spreadsheet program such as Microsoft Excel or Google Sheets, develop a spreadsheet that will compute approximations for $y' = 2xy, y(1) = 1$ with adjustable step sizes for Euler's method. What are the maximum step sizes that will find an approximation of $y(2.0)$ to within about 5% of the actual answer of $y(2.0) = 20.08554$?

Now find an approximation to $y(1.4)$ using RK4 with $h = 0.2$:

Iteration 1:

$$\begin{aligned}
x_{n+1} &= x_n + h = 1 + 0.2 = 1.2 \\
k_1 &= f(x_k, y_k) = 2x_1y_1 = 2(1)(1) = 2 \\
k_2 &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) = 2\left(1 + \frac{0.2}{2}\right)\left(1 + \frac{0.2}{2}(2)\right) = 2.64 \\
k_3 &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right) = 2\left(1 + \frac{0.2}{2}\right)\left(1 + \frac{0.2}{2}(2.64)\right) = 2.7808 \\
k_4 &= f(x_n + h, y_n + h\,k_3) = 2(1 + 0.2)(1 + 0.2(2.7808)) = 3.734784 \\
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{0.2}{6}(2 + 2(2.64) + 2(2.7808) + 3.734784) = 1.552546\ldots
\end{aligned}$$

Iteration 2:

$$\begin{aligned}
x_{n+1} &= 1.2 + 0.2 = 1.4 \\
k_1 &= 2(1.2)(1.552546) = 3.72611 \\
k_2 &= 2\left(1.2 + \frac{0.2}{2}\right)\left(1.552546 + \frac{0.2}{2}(3.72611)\right) = 5.00541 \\
k_3 &= 2\left(1.2 + \frac{0.2}{2}\right)\left(1.552546 + \frac{0.2}{2}(5.00541)\right) = 5.33803 \\
k_4 &= 2(1.2 + 0.2)(1.552546 + 0.2(5.33803)) = 7.33643 \\
y_{n+1} &= 1.55246 + \frac{0.2}{6}(3.72611 + 2(5.00541) + 2(5.33803) + 7.33643) \approx 2.6108
\end{aligned}$$

Notice that, with a step size of 0.2, it took only two iterations, with six computations per iteration, for a total of twelve computations. The approximation is $y(1.4) \approx 2.6108$, resulting in an error of 0.03%.

Now develop a spreadsheet that will compute approximations for $y' = 2xy, y(1) = 1$ with adjustable step sizes for RK4. What are the maximum step sizes that will find an approximation of $y(2.0)$ to within about 5% of the actual answer of $y(2.0) = 20.08554$?

Which method would you use? Even with a spreadsheet available, would you choose the more challenging method that can be summarised on one page of computations?

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