M647 Spring 2011 Assignment 11, due Fri. Apr. 29

1. [10 pts] Consider a fluid flowing through a cylindrical pipe with constant cross section A, velocity $v(x,t)$, density $\rho(x,t)$, specific internal energy $e(x,t)$, temperature $T(x,t)$, under pressure $p(x,t)$, and subject to viscous stress. (By *specific* internal energy, we mean internal energy per unit mass. Internal energy arises from intermolecular collisions in the fluid, and should be distinguished from kinetic energy, $\frac{1}{2}mv^2$. Potential energy will not play a role in this problem.) By conserving energy, show that

$$\Big[\rho(\frac{v^2}{2}+e)\Big]_t + \Big[\rho ve + \frac{1}{2}\rho v^3 - \kappa(x)T_x + pv - \mu v v_x\Big]_x = 0, \qquad (1)$$

where κ is thermal diffusivity and μ is the viscosity coefficient discussed in class. This is called the Navier-Stokes energy equation.

Note. The energy density should be easy to identify. For the flux, consider each of the following, which correspond respectively with terms in (1): internal energy, kinetic energy, energy lost to heat, energy lost to work against pressure, energy lost to work against viscous stress.

2a. [3 pts] The Navier-Stokes momentum equation is often written in the form

$$(\rho v)_t + (\rho v^2 + p)_x = \mu v_{xx} + f.$$

Show that this is equivalent to our form from class.

2b. [3 pts] Recall from our derivation in class of the Navier-Stokes momentum equation that if the fluid is incompressible then $\Delta x(t)$ is constant in t. Show that in this case we must have

$$v_x = 0.$$

2c. [4 pts] Euler's equations of gas dynamics comprise a system of three equations that are immediate from the continuity equation, the Navier-Stokes momentum equation, and the Navier-Stokes energy equation. In particular, these are the equations you obtain if you ignore effects due to viscosity, body forces, and temperature. Write down Euler's equations.

3. Recall from Problem 1 of Assignment 10 that if a thin film is moving over a flat surface its height h can be modeled by the equation

$$h_t + (vh)_x = 0,$$

where v denotes fluid velocity.

3a. [5 pts] Show that in the invscid case (no viscosity) v solves the PDE

$$h(v_t + vv_x) = -ghh_x.$$

3b. [5 pts] Suppose the film is moving along a surface that has been inclined from the horizontal with angle α (See figure 1). Show that in this case (and still assuming no viscosity) the momentum equation is

$$h(v_t + vv_x) = -ghh_x \cos\alpha + gh\sin\alpha.$$

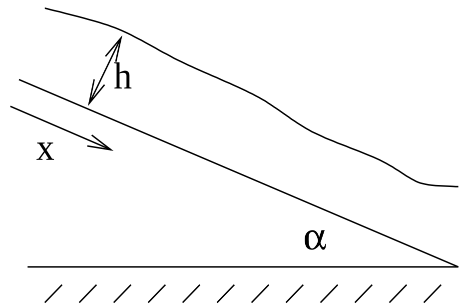

Figure 1: Thin film on an inclined plane.

4a. [5 pts] According to Newton's law of cooling, the flux of temperature across a surface is proportional to the temperature difference across the surface. For example, if the temperature at the left endpoint of a heat-conducting cylinder is $u(0,t)$ and the temperature outside the cylinder is T_0, then we expect to have the relation

$$f = -h(u(0,t) - T_0),$$

where the constant h is referred to as the heat transfer coefficient. (Notice that if the temperature inside is greater than the temperature outside, the flux will be to the left.) Since the flux can be expressed as $f = -k(x)u_x$, this gives a boundary condition

$$-k(0)u_x(0,t) = -h(u(0,t) - T_0).$$

Assume that for a particular heat-conducting cylinder we have cylinder length $L = 1$ m, thermal diffusivity $k(x) \equiv .1\ m^2 s^{-1}$, ambient temperature $T_0 = 25^o$ C, and heat transfer coefficient $h = .03\ m^{-2}s^{-1}$. Suppose the bar is initially cooled uniformly to 0^o C, and that the sides are insulated so that heat only enters or escapes at the ends. Plot a temperature profile after 10 seconds, and also create a mesh plot of your solution for all $t \in [0, 10]$.

4b. [5 pts] One general three-species competition model can be written as

$$\begin{aligned}
u_t &= r_1 u\Big(1 - \frac{u + s_1 v + s_2 w}{K_1}\Big) + (b_{11}(x)u_x)_x + (b_{12}(x)v_x)_x + (b_{13}(x)w_x)\\
v_t &= r_2 v\Big(1 - \frac{s_3 u + v + s_4 w}{K_2}\Big) + (b_{21}(x)u_x)_x + (b_{22}(x)v_x)_x + (b_{23}(x)w_x)\\
w_t &= r_3 u\Big(1 - \frac{s_5 u + s_6 v + w}{K_3}\Big) + (b_{31}(x)u_x)_x + (b_{32}(x)v_x)_x + (b_{33}(x)w_x).
\end{aligned}$$

Solve this system in MATLAB with parameter values $r_1 = .02$, $r_2 = .03$, $r_3 = .05$, $s_1 = 2$, $s_2 = 2$, $s_3 = \frac{1}{2}$, $s_4 = 1$, $s_5 = \frac{1}{2}$, $s_6 = 1$, $K_1 = 20$, $K_2 = 10$, $K_3 = 10$, and

$$B = \begin{pmatrix} 1 & 2 & 1 \\ 5 & 3 & 2 \\ 10 & .8 & 30 \end{pmatrix} \times 10^{-5}.$$

Use no-flux boundary conditions and the initial conditions

$$
\begin{aligned}
u(x,0) &= 20(1+\cos(\pi x)) \\
v(x,0) &= 10(1+\sin(2\pi x - \frac{\pi}{2})) \\
w(x,0) &= 10(1+\sin(\pi x - \frac{\pi}{2})).
\end{aligned}
$$

Plot a stacked plot of solution profiles for $t = 25$.

5. [10 pts] Consider a drum head stretched with tension $T(x,y,t)$, in units force per unit length, and with mass density $\rho(x,y)$ with units mass per unit area. Assuming small displacement, derive the two-dimensional wave equation,

$$\rho(x,y)u_{tt} = \Big((T(x,y,t)u_x)_x + (T(x,y,t)u_y)_y\Big).$$

Note. Consider a square strip of membrane with sidelengths (approximately) Δx and Δy.