M647, Spring 2012 Assignment 4, due Feb. 17

1. [10 pts] We said in class that if $\{X_k\}_{k=1}^N$ denote independent, identically distributed (iid) random variables with mean μ and variance σ^2, then the MLE estimator for variance

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{k=1}^N (X_k - \hat{\mu})^2,$$

satisfies

$$E[\hat{\sigma}^2] = \frac{N-1}{N}\sigma^2.$$

Show that this is true. (Recall $\hat{\mu} = \frac{1}{N}\sum_{k=1}^N X_k$.)

2. [10 pts] Show that if $X \sim N(0,1)$ then the PDF for X^2 is

$$f_{X^2}(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{\sqrt{2\pi x}} e^{-\frac{x}{2}} & x > 0. \end{cases}$$

3. [10 pts] Verify that the PDF for a chi-squared random variable with q degrees of freedom is

$$f(x;q) = \begin{cases} 0 & x \le 0 \\ \frac{1}{\Gamma(\frac{q}{2})}(\frac{1}{2})^{q/2} x^{q/2-1} e^{-\frac{x}{2}} & x > 0. \end{cases}$$

Notes. Recall that the chi-squared distribution is the distribution of the sum

$$Y = \sum_{k=1}^q X_k^2,$$

where for each k $X_k \sim N(0,1)$. In particular, Problem 2 gives the result for $q = 1$ (recall $\Gamma(1/2) = \sqrt{\pi}$). Proceed by induction, and the observation from Problem (3b) of Assignment 3 that the PDF of a sum of independent random variables $Z = X + Y$ can be expressed as

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-x) f_Y(x) dx.$$

4. [10 pts] In this problem we'll collect two useful observations about joint PDF's.

4a. Show that for any two random variables X and Y with individual PDF's $f_X(x)$ and $f_Y(y)$ and with joint PDF $f_{X,Y}(x,y)$ we have

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dy$$
$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dx.$$

4b. Suppose $f_{X,Y}(x,y)$ is the joint PDF for two random variables X and Y, and two new random variables are expressed in terms of X and Y

$$U = g(X,Y)$$
$$V = h(X,Y),$$

where the map is invertible so that there exist functions G and H with

$$X = G(U, V)$$
$$Y = H(U, V).$$

Show that the joint PDF for U and V will be

$$f_{U,V}(u, v) = f_{X,Y}(G(u, v), H(u, v))|J|,$$

where J denotes the Jacobian determinant

$$J = \det \begin{pmatrix} \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \\ \frac{\partial H}{\partial u} & \frac{\partial H}{\partial v} \end{pmatrix}.$$

Note. This problem does not ask that you re-prove the standard change of variables theorem from third-semester calculus; it only places that result in the context of PDF's.

5. [10 pts] If $X \sim N(0, 1)$ and $Y \sim \chi^2_q$ then the random variable

$$T := \frac{X}{\sqrt{Y/q}}$$

is said to be distributed according to a student's t distribution. Show that the PDF for T is

$$f_T(t) = \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q}{2})} \frac{1}{\sqrt{q\pi}} \frac{1}{(1 + t^2/q)^{\frac{q+1}{2}}}.$$