M647 Spring 2012 Assignment 10, due Tuesday May 1

Note. Notice the unusual due date. This is the last day of classes, which is a re-defined Friday.

1a. Non-dimensionalize the system from Problem 1 of Assignment 8,

$$\begin{aligned}\frac{d^2x}{dt^2} &= -b\sqrt{x'^2+y'^2}x' \\ \frac{d^2y}{dt^2} &= -g-b\sqrt{x'^2+y'^2}y'.\end{aligned}$$

Write down initial conditions for your non-dimensional system, based on the original initial conditions,

$$\begin{aligned}x(0) &= 0 \\ y(0) &= h_0 \\ x'(0) &= |\vec{v}_0|\cos\theta \\ y'(0) &= |\vec{v}_0|\sin\theta.\end{aligned}$$

1b. [5 pts] An elastic beam of section modulus EI, resting on an elastic foundation of modulus k, is under a tension force F and a distributed downward force per length $p(s)$, where s denotes distance along the beam measured from some convenient point. The (small) vertical deflection w of the beam satisfies the ODE

$$EI\frac{d^4w}{ds^4}-F\frac{d^2w}{ds^2}+kw=p(s).$$

Assuming EI, T, and k are constant, show that this equation can be expressed in the non-dimensional form

$$\epsilon^2(\frac{d^4W}{d\tau^4}+W)-\frac{d^2W}{d\tau^2}=f(\tau),$$

for appropriate dimensionless choices of τ, W, ϵ, and f.

Notes. The constant E is Young's modulus, a measure of an object's tendency to deform under stress. It's a pressure, so $[E]=ML^{-1}T^{-2}$. The constant I is the area moment of inertia, or second moment of inertia of the beam. It's dimensions are $[I]=L^4$.

2. [10 pts] The coating of surfaces by thin fluid films is a critical industrial process that arises in applications such as the protection of microchips, de-icing of airplane wings, and the construction of photographic film. Consider, for example, painting a wall. What you would like to do is simply brush a single thick line of paint across the top of the wall and let the paint descend in a steady sheet to the floor. Unfortunately, in most circumstances the paint drips down in *fingers,* and the wall is not smoothly covered. Through the use of mathematical modeling, we can create laboratory situations in which the paint descends as a steady wall. In this problem, we will take the first step toward such a model by deriving a partial differential equation for the height $h(x,t)$ of a thin film moving along a surface (see Figure 1). Assuming the film has constant density ρ, that it is moving with velocity

$v(x,t)$ in the x-direction only, and that it is uniform in the y-direction (same height and velocity for all y over a steady width $y \in [0, L]$), derive a partial differential equation that takes a given $v(x,t)$ and describes the height $h(x,t)$. Discuss the types of initial values and boundary values your model would require, and what they mean physically.

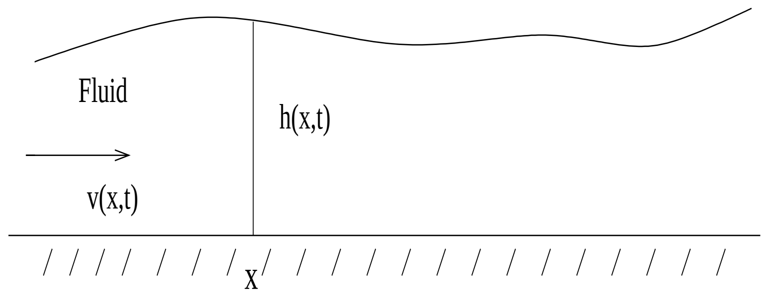

Figure 1: Flow of a thin film.

3a. [5 pts] Suppose $u(x,t)$ denotes traffic density (number of cars per unit length of road) along a certain stretch of road. In class, we discussed models in which the traffic flux depends only on traffic density u. One drawback of such models is that they do not capture a driver's reaction to what he sees ahead. For example, a driver who sees a higher density of traffic ahead will often slow down, while a driver who sees a lower density of traffic ahead will often speed up. Incorporate this idea to revise our model from class.

3b. [5 pts] Consider a mixture with two components contained in a long cylinder, and let $u(x,t)$ denote the volumetric concentration of one of the components. Assuming that mass is conserved, the concentration of the remaining component will be $1-u$. In 1958 John W. Cahn and John Hilliard suggested that under certain conditions the energy associated with such a mixture could be expressed as the functional

$$E[u] = \int_0^L F(u) + \frac{\kappa}{2} u_x^2 dx,$$

where F denotes the *bulk free energy density* of the mixture (the free energy density, assuming the entire mixture is homogeneously mixed with concentrations u and $1-u$), and $\frac{\kappa}{2}u_x^2$ is a measure of the energy associated with transitions from one concentration to another. The flux associated with u is

$$J = -M \frac{\partial}{\partial x} \frac{\delta E}{\delta u},$$

where $\frac{\delta E}{\delta u}$ is defined so that

$$E'[u][h] = \int_0^L \frac{\delta E}{\delta u} h(x) dx,$$

for

$$h \in \mathcal{V}_0 := \{h \in C^2([0,L]) : h(0) = 0, \quad h(L) = 0\}.$$

Here, M is *molecular mobility*, and plays a role in this context similar to the role thermal diffusivity k plays in heat transfer. Also, $E'[u][h]$ denotes variational derivative, as discussed in our section on Lagrangian mechanics. (We use J for the flux here both because it's the traditional letter in this context, and because I want to avoid confusion with the F (also traditional) we're using for bulk free energy density.) Compute $\frac{\delta E}{\delta u}$ and use it to write down a PDE for u. What is the order of your final equation?

4. [10 pts] Consider a fluid flowing through a cylindrical pipe with constant cross section A, velocity $v(x,t)$, density $\rho(x,t)$, specific internal energy $e(x,t)$, temperature $T(x,t)$, under pressure $p(x,t)$, and subject to viscous stress. (By *specific* internal energy, we mean internal energy per unit mass. Internal energy arises from intermolecular collisions in the fluid, and should be distinguished from kinetic energy, $\frac{1}{2}mv^2$. Potential energy will not play a role in this problem.) By conserving energy, show that

$$\left[\rho(\frac{v^2}{2}+e)\right]_t + \left[\rho ve + \frac{1}{2}\rho v^3 - \kappa(x)T_x + pv - \mu v v_x\right]_x = 0, \tag{1}$$

where κ is thermal diffusivity and μ is the viscosity coefficient discussed in class. This is called the Navier-Stokes energy equation.

Note. The energy density should be easy to identify. For the flux, consider each of the following, which correspond respectively with terms in (1): internal energy, kinetic energy, energy lost to heat, energy lost to work against pressure, energy lost to work against viscous stress.

5. Referring to the thin film described in problem 1, suppose the fluid is inviscid (i.e., $\mu = 0$).

5a. [5 pts] Show that v solves the PDE

$$v_t + vv_x = -gh_x.$$

5b. [5 pts] Suppose the film is moving along a surface that has been inclined from the horizontal with angle α (see figure). Show that in this case the momentum equation is

$$v_t + vv_x = -gh_x \cos\alpha + g\sin\alpha.$$

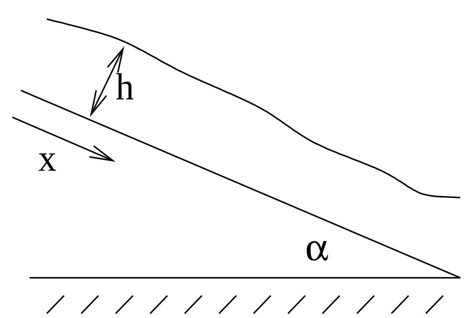

Figure 2: Thin film on an inclined plane.