M647 Spring 2012 Assignment 8, due Friday April 13

1. [10 pts] Suppose an object with mass m is launched at an angle θ from the ground, with initial height h_0 and initial speed $|\vec{v}_0|$ (see Figure 1). Use Newton's second law of motion to write down a system of ODE describing the flight of this object, and also write down appropriate initial conditions. Do not ignore air resistance.

Note. Recall that we obtained an expression for force due to air resistance earlier in the semester, using dimensional analysis.

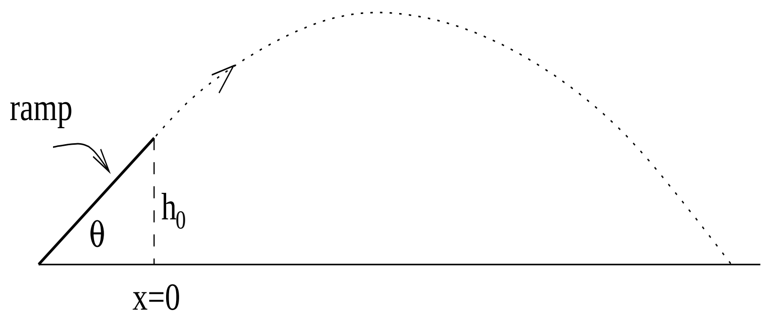

Figure 1: Object in Problem 3.

2a. [5 pts] We know from first-semester calculus that the length of the graph of a C^1 function $y(t)$ connecting the point $(0, \alpha)$ to the point (T, β) is given by the arclength formula

$$L[y(t)] = \int_0^T \sqrt{1 + y'(t)^2} dt.$$

Compute $L'[y(t)]$, and use your result to show that the shortest such curve must be a straight line. Work in the function space

$$y \in \mathcal{S} := \{y \in C^2[0, T] : y(0) = \alpha, y(T) = \beta\}.$$

2b. [5 pts] As observed in class, the derivative $L'[y(t)]$ is a linear mapping on the space

$$\mathcal{S}_0 := \{h \in C^2[0, T] : h(0) = h(T) = 0\}.$$

We can compute the derivative of $L'[y(t)]$ (and so the second derivative of $L[y(t)]$) as the *bilinear form*

$$L''[y(t)](k(t), h(t)) := \lim_{\tau \to 0} \frac{L'[y(t) + \tau k(t)] - L'[y(t)]}{\tau}(h(t)).$$

For the arclength functional from Part (a), compute $L''[y(t)]$, where $y(t)$ is taken as the minimizer found in Part (a), and show that $L''[y(t)]$ is a positive definite bilinear form: that is, show that for any $h \in \mathcal{S}_0$, h not identically 0,

$$L''[y(t)](h(t), h(t)) > 0.$$

This is the variational second-order derivative condition for a minimizer, and ensures that $y(t)$ is in fact a local minimizer.

Note. This isn't necessary for the current problem, but FYI another way to look at this is as follows: we say $L[y(t)]$ is twice differentiable if there exist a linear functional $L'[y(t)](h)$, a bilinear functional $L''[y(t)](h,h)$, and a functional $\epsilon[y(t)](h)$ so that

$$L[y+h] = L[y] + L'[y](h) + \frac{1}{2}L''[y](h,h) + \epsilon[y](h),$$

with

$$\lim_{\|h\|\to 0} \frac{\epsilon[y](h)}{\|h\|^2} = 0,$$

where $\|\cdot\|$ denotes Holder norm (as discussed in class) on $\mathcal{S}_0$. I.e., functionals have Taylor expansions.

3. [10 pts] Consider the system depicted in Figure 2 in which a pendulum with mass m_2 is attached to a block with mass m_1 that is attached to a wall by a spring with spring constant k. Take $x = 0$ to be the position of the block's center when the spring is neither stretched nor compressed, and denote the length of the pendulum by l. Write down the Lagrangian for this system and also the Euler-Lagrange equations. Ignore friction, which for practical purposes would make the system non-conservative. (The system is still conservative, but energy is being transferred into molecular motion.)

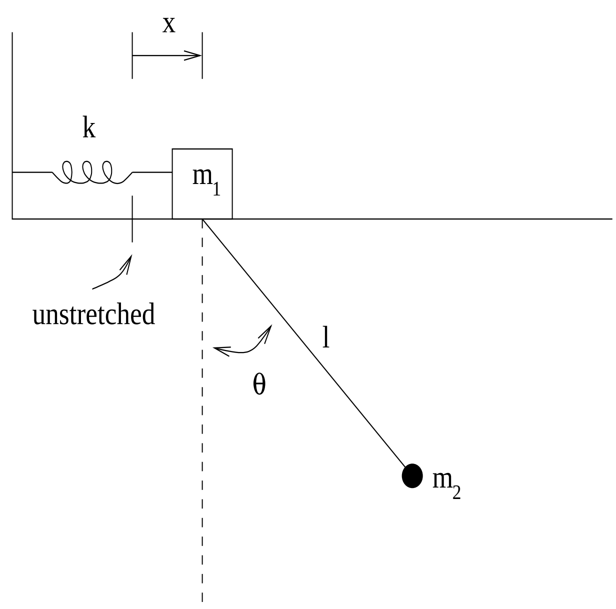

Figure 2: Spring-pendulum system for Problem 4.

4. [10 pts] Consider the mechanism depicted in Figure 3 in which a mass m is attached by a spring with spring constant k to a frictionless pivot. Let $r(t)$ denote the length of the pendulum at time t, and let r_0 denote its length when the spring is neither stretched nor compressed. Write down the Hamilton ODE system for this arrangement.

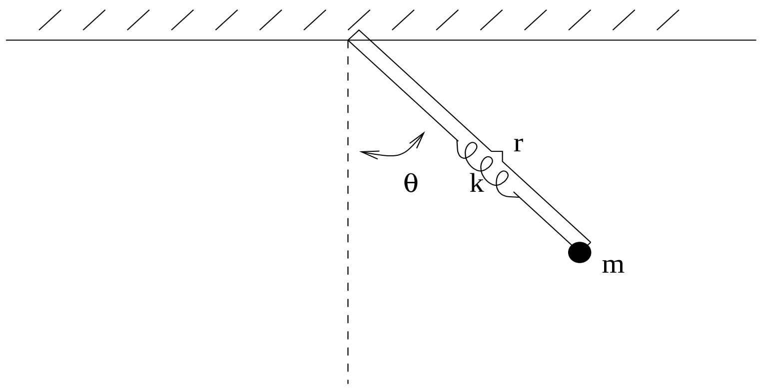

Figure 3: Spring-Pendulum for Problem 5.

5. [10 pts] Show that if potential energy P depends only on the generalized coordinates $\vec{q}$ (i.e., $P = P(\vec{q})$), and kinetic energy is quadratic in $\vec{q}\,'$,

$$K = K(\vec{q}\,') = \sum_{i,j=1}^{n} a_{ij} q_i' q_j',$$

then the Hamiltonian

$$H(\vec{p}, \vec{q}, t) = \vec{p} \cdot \vec{q}\,' - L(\vec{q}, \vec{q}\,', t)$$

is total system energy.

Note. Assume here, as in class, that

$$\vec{p} = D_{q'} L(\vec{q}, \vec{q}\,', t).$$