M647 Spring 2012 Assignment 7, due Friday March 30

1a. [5 pts] Suppose M grams of a certain heart medication are injected into a patient at time 0, and that whenever the drug is present in the body (excluding the heart) its rate of absorption out of the bloodstream is proportional to the concentration in the body (excluding the heart) with proportionality constant r_B L/s, while whenever the drug is in the heart its rate of absorption out of the bloodstream is proportional to the concentration in the heart with proportionality constant r_H L/s. If blood flows into the patient's heart with variable rate $r_I(t)$ L/s and out with variable rate $r_O(t)$ L/s, and if the initial volume of blood in the heart is V_H L while the initial volume of blood in the body (including the heart) is V_B L, develop a model for the amount of drug absorbed into heart tissue by t.

1b. [5 pts] Suppose M grams of a certain heart medication is injected into a patient at time 0, and that while the drug is beneficial for the heart it is detrimental to the liver. Assume the rates at which blood flows in and out of the heart are respectively $r_I(t)$ and $r_O(t)$, and that the rates at which blood flows in and out of the liver are respectively $R_I(t)$ and $R_O(t)$, and that the initial volume of blood in the heart is V_H while the initial volume of blood in the liver is V_L. Take the initial volume of blood in the entire body, including both the heart and the liver to be V_B. Finally, assume that whenever the drug is present in the heart it is absorbed out of the bloodstream at a rate proportional to the concentration of medication in the heart with proportionality constant r_H, while whenever the drug is present in the liver it is absorbed out of the bloodstream at a rate proportional to the concentration of medication in the liver with proportionality constant r_L. Assume that if the medication is not in either the heart or the liver then it is not absorbed out of the bloodstream. Write down a system of ODE that models the number of grams absorbed into the patient's heart at time t and also the number of grams absorbed into the patient's liver at time t.

2a. [5 pts] Airplane wings are typically designed so that the upper portion of the wing has a longer streamline than the bottom, meaning that the air flowing over the top has a longer path than does the air flowing over the bottom. This means that the velocity of air across the top of the wing v_+ is larger than the velocity of the air flowing over the bottom of the the wing v_-. Use Bernoulli's Principle to find the lift force l this effect has on an airplane wing with underside surface area A in the presence of air density ρ.

2b. [5 pts] Suppose a certain airplane has underside wing area 10 m^2 (both wings), and that while the plane is flying at a certain sustained height $v_+ = 48$ m/s, while $v_- = 40$ m/s. Assuming the density of air is $\rho = 1.25$ kg/m^3, estimate the mass of the airplane.

3a. [5 pts] Write down an ODE model for the chemical reaction described below.

$$\mathrm{2NO} \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{N_2O_2}$$

$$\mathrm{N_2O_2} + \mathrm{O_2} \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} \mathrm{2NO_2}.$$

3b. [5 pts] Write down an ODE model for the chemical reaction described below.

$$N_2O_5 \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} NO_2 + NO_3$$
$$NO + NO_3 \xrightarrow{k_2} 2NO_2.$$

4. [10 pts] A chemical reaction in which an enzyme E catalyzes the conversion of a substrate S to a product P (generally consisting of multiple chemicals) can be characterized as follows:

$$E + S \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{ES}$$
$$\mathrm{ES} \xrightarrow{k_2} \mathrm{EP}$$
$$\mathrm{EP} \xrightarrow{k_3} \mathrm{E} + \mathrm{P}.$$

(For example, the enzyme *sucrase* catalyzes the hydrolysis of the (substrate) *sucrose* (a.k.a., common table sugar) into the (products) *glucose* and *fructose*.) In many cases the rate of product release k_3 is much more rapid than the rate of chemical conversion k_2 (i.e., $k_3 \gg k_2$), and so the rate of the last two steps is approximately k_2. We'll assume that this is true for this problem.

4a. It's clear that the total amount of enzyme $[E_{tot}] = [E] + [ES]$ must be conserved. Explain where this appears in the mathematics for this system.

4b. Determine the rate at which this entire process proceeds in the case of substrate saturation (i.e., as $[S] \to \infty$). Write it in terms of the constant $[E_{tot}]$.

4c. Assuming the enzyme-substrate concentration is in equilibrium (i.e., [ES] = constant), show that the rate at which this entire process proceeds is

$$R = \frac{R_{max}[S]}{[S] + \frac{k_{-1}+k_2}{k_1}},$$

where R_{max} denotes the rate a substrate saturation from (b). (This last relation is known as the Michaelis-Menton equation.)

5a. [5 pts] When a raccoon is infected by the rabies virus, one of two (equally likely) things can happen:

1. The raccoon develops *furious rabies*, in which case he becomes hyperactive and is quick to attack other raccoons.

2. The raccoon develops *dumb rabies*, in which case he becomes paralyzed and does not spread the disease.

In either case, the raccoon will not recover, and will die within a week of becoming infected. Develop a model for the spread of the rabies virus through an isolated population of raccoons. Which parameter in your model is determined by the fact that infected raccoons will die within a week of becoming infected, and what is the value of this parameter?

5b. [5 pts] The West Nile Virus is carried predominately by birds and mosquitoes: When an infected mosquito bites a susceptible bird, the virus remains in the bird's blood for two or three days, long enough to infect a great number of mosquitoes. Assume that when a bird recovers it becomes susceptible again. That is, it neither dies nor becomes immune. Taking into account four variables—number of uninfected mosquitoes, $m(t)$, number of infected mosquitoes, $n(t)$, number of uninfected birds, $b(t)$, and number of infected birds, $c(t)$—develop a model for the spread of West Nile Virus through a population of birds and mosquitoes. While you should assume mosquitoes are born and die, since the time horizon researchers are primarily concerned with is, say, one summer, you should consider the total bird population $(b(t) + c(t))$ to be fixed.