M647 Spring 2011 Assignment 3, due Friday Feb. 11

Note. This assignment was revised due to the cancellation of classes, Fri. Feb. 4. The last three problems will be moved to Assignment 4.

1. [10 pts] A random variable X is said to be distributed according to a *Rayleigh* distribution provided its PDF has the form

$$f(x;\theta) \quad = \quad \begin{cases} \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} & x > 0 \\ 0 & x \leq 0 \end{cases},$$

where we generally take $\theta > 0$. (Since it only appears squared, its sign is irrelevant.)

1a. Compute $E[X]$ and $\mathrm{Var}[X]$. In these calculations, you should feel free to use the standard integral

$$\int_0^{+\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \quad a > 0$$

without justification. (Though it's well worth looking up how to compute this.)

1b. Given data $\{x_k\}_{k=1}^N$, find a maximum likelihood estimate for θ^2.

1c. Write down the MLE estimator for θ^2, and compute its expected value.

2. [10 pts] In this problem we'll collect a few straightforward observations about PDFs that we've found useful in class.

2a. Show that if X is a random variable with PDF $f_X(x)$, and we set $Y = cX$ for any constant $c > 0$, then Y has PDF $f_Y(y) = \frac{1}{c} f_X(\frac{y}{c})$.

2b. Show that if X and Y are any two independent continuous random variables with respective PDFs f_X and f_Y, then the PDF for $Z = X + Y$ is

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-x) f_Y(x) dx.$$

Note. This integral is a *convolution*, and we often write

$$f_X * f_Y(z) = \int_{-\infty}^{+\infty} f_X(z-x) f_Y(x) dx.$$

You can take as your starting point the observation

$$\begin{aligned} \int_a^b f_Z(z) dz = P(a \leq Z \leq b) &= P(a \leq X + Y \leq b) \\ &= \iint\limits_{a \leq x+y \leq b} f_X(x) f_Y(y) dx dy. \end{aligned}$$

2c. Show that if X and Y are independent Gaussian $N(\mu, \sigma^2)$ random variables, then $Z = X + Y$ is $N(2\mu, 2\sigma^2)$.