

Math 2700.009
Exam 2 Notes

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0.1 Vector Subspaces

Definition 0.1.1: Dimension

The dimension of V denoted $\dim(V)$ is the number of elements in a basis

By the definition of dimension, we know that it represents the number of linearly independent elements that also span the space. That is to say, that the size of basis for a space cannot exceed the dimension of that space. As a result, you cannot have more linearly independent elements in a space than that of the size of the basis. Spanning sets however differ, since the span of a set of vectors can potentially contain linearly dependent vectors. In particular we say that:

If $S \leq V$ and $\vec{v} \in \text{span}(S)$, $\vec{v} \in S$, then if S' is obtained by removing \vec{v} from S and $\vec{v} \in \text{span}(S')$, then $\text{span}(S) = \text{span}(S')$

In more simple terms, we can have two unique sets of vectors which can have equal spans. This is because the span of a linearly independent set of vectors is equal to that of a linearly dependent vectors which span the same space.

Theorem 0.1.1 Rank-Nullity

If $T : V \rightarrow W$ is a linear transformation:

$$\dim(V) = \dim(\ker(T)) + \dim(\text{ran}(T))$$

0.2 Linear Transformations

Recall that if V, W are vector spaces, to verify a linear transformation from V to W , two properties must be verified:

- Additivity: A function $T : V \rightarrow W$ is additive for any $\vec{x}, \vec{y} \in V$

$$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$$

- Homogeneity: A function $T : V \rightarrow W$ is homogenous if for any $c \in \mathbb{R}$ and $\vec{x} \in V$

$$T(c\vec{x}) = cT(\vec{x})$$

Definition 0.2.1: Kernel

The kernel (sometimes known as null-space) of a linear transformation is the subset of the domain that is transformed into the zero vector

0.3 Random Examples

Definition 0.3.1: Limit of Sequence in \mathbb{R}

Let $\{s_n\}$ be a sequence in \mathbb{R} . We say

$$\lim_{n \rightarrow \infty} s_n = s$$

where $s \in \mathbb{R}$ if \forall real numbers $\epsilon > 0 \exists$ natural number N such that for $n > N$

$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

Question 1

Is the set $x\text{-axis} \setminus \{\text{Origin}\}$ a closed set

Solution: We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

Note:-

We will do topology in Normed Linear Space (Mainly \mathbb{R}^n and occasionally \mathbb{C}^n) using the language of Metric Space

Claim 0.3.1 Topology

Topology is cool

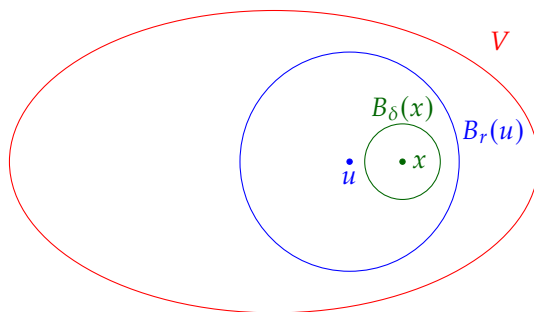
Example 0.3.1 (Open Set and Close Set)

- Open Set:
- ϕ
 - $\bigcup_{x \in X} B_r(x)$ (Any $r > 0$ will do)
 - $B_r(x)$ is open
- Closed Set:
- X, ϕ
 - $\overline{B_r(x)}$
- $x\text{-axis} \cup y\text{-axis}$

Theorem 0.3.1

If $x \in$ open set V then $\exists \delta > 0$ such that $B_\delta(x) \subset V$

Proof: By openness of V , $x \in B_r(u) \subset V$



Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u, x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u, y) < r$ but

$$d(u, y) \leq d(u, x) + d(x, y) < d + \delta < r$$

Corollary 0.3.1

By the result of the proof, we can then show...

Lemma 0.3.1

Suppose $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ is subspace of \mathbb{R}^n .

Proposition 0.3.1

$1 + 1 = 2$.

0.4 Random

Definition 0.4.1: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over \mathbb{R} (or \mathbb{C}). A norm on V is function $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$ satisfying

- ① $\|x\| = 0 \iff x = 0 \ \forall x \in V$
- ② $\|\lambda x\| = |\lambda| \|x\| \ \forall \lambda \in \mathbb{R}(\text{or } \mathbb{C}), x \in V$
- ③ $\|x + y\| \leq \|x\| + \|y\| \ \forall x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.

- Same definition works with V a vector space over \mathbb{C} (again $\|\cdot\| \rightarrow \mathbb{R}_{\geq 0}$) where ② becomes $\|\lambda x\| = |\lambda| \|x\|$ $\forall \lambda \in \mathbb{C}, x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

Example 0.4.1 (p -Norm)

$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geq 0}$. Define for $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \dots + |x_m|^p \right)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \dots + |x_m|$ is clearly a norm by usual triangle inequality.

Special Case $p \rightarrow \infty$ (\mathbb{R}^m with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_m|\}$

For $m = 1$ these p -norms are nothing but $|x|$. Now exercise

Question 2

Prove that triangle inequality is true if $p \geq 1$ for p -norms. (What goes wrong for $p < 1$?)

Solution: For Property ③ for norm-2

When field is \mathbb{R} :

We have to show

$$\begin{aligned}\sum_i (x_i + y_i)^2 &\leq \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2} \right)^2 \\ \Rightarrow \sum_i (x_i^2 + 2x_i y_i + y_i^2) &\leq \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2 \right] \left[\sum_i y_i^2 \right]} + \sum_i y_i^2 \\ \Rightarrow \left[\sum_i x_i y_i \right]^2 &\leq \left[\sum_i x_i^2 \right] \left[\sum_i y_i^2 \right]\end{aligned}$$

So in other words prove $\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$ where

$$\langle x, y \rangle = \sum_i x_i y_i$$

Note:-

- $\|x\|^2 = \langle x, x \rangle$
- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle \cdot, \cdot \rangle$ is \mathbb{R} -linear in each slot i.e.

$$\langle rx + x', y \rangle = r\langle x, y \rangle + \langle x', y \rangle \text{ and similarly for second slot}$$

Here in $\langle x, y \rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y \rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned}\langle x - \lambda y, x - \lambda y \rangle &= \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle \\ &= \langle x, x \rangle - \lambda \langle x, y \rangle - \lambda \langle y, x \rangle + \lambda^2 \langle y, y \rangle \\ &= \langle x, x \rangle - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle\end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y \rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is \mathbb{C} :

Modify the definition by

$$\langle x, y \rangle = \sum_i \bar{x}_i y_i$$

Then we still have $\langle x, x \rangle \geq 0$

0.5 Algorithms

Algorithm 1: what

Input: This is some input

Output: This is some output

/ This is a comment */*

```
1 some code here;
2  $x \leftarrow 0$ ;
3  $y \leftarrow 0$ ;
4 if  $x > 5$  then
5   |  $x$  is greater than 5 ;                                // This is also a comment
6 else
7   |  $x$  is less than or equal to 5;
8 end
9 foreach  $y$  in 0..5 do
10  |  $y \leftarrow y + 1$ ;
11 end
12 for  $y$  in 0..5 do
13  |  $y \leftarrow y - 1$ ;
14 end
15 while  $x > 5$  do
16  |  $x \leftarrow x - 1$ ;
17 end
18 return Return something here;
```
