

Math 2700.009  
Problem Set 13

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23 April 2024

### Question 1

Consider  $C([- \pi, \pi])$  with the inner-product

$$\langle f | g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$$

and  $S \subseteq C([- \pi, \pi])$  be

$$S = \{\sin(nx), \cos(mx) : m, n > 0\},$$

verify that  $S$  is an orthogonal set in  $C([- \pi, \pi])$ .

#### Note:-

Recall:

- To show a set  $S$  is an orthogonal set, we must show that for all  $\vec{v}, \vec{w} \in S$  the inner-product  $\langle \vec{v} | \vec{w} \rangle = 0$ , with  $\vec{0} \notin S$

**Solution:** Let us begin by finding the inner product  $\langle \sin(nx) | \cos(mx) \rangle$  when  $m \neq n$ :

$$\begin{aligned} \langle \sin(nx) | \cos(mx) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(nx) \cos(mx) dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(nx + mx) + \sin(nx - mx)}{2} dx \\ &= \frac{1}{4\pi} \left( \int_{-\pi}^{\pi} \sin(nx + mx) dx + \int_{-\pi}^{\pi} \sin(nx - mx) dx \right) \\ &= \frac{1}{4\pi} \left( -\frac{1}{n+m} \cos(nx + mx) - \frac{1}{n-m} \cos(nx - mx) \right) \Bigg|_{x=-\pi}^{\pi} \\ &= \frac{1}{4\pi} \left( -\frac{1}{n+m} \cos(n\pi + m\pi) - \frac{1}{n-m} \cos(n\pi - m\pi) \right. \\ &\quad \left. + \frac{1}{n+m} \cos(n\pi + m\pi) + \frac{1}{n-m} \cos(n\pi - m\pi) \right) \\ &= \frac{1}{4\pi} (0) \\ &= 0 \end{aligned}$$

This "proof" only reveals if the vectors are orthogonal in instances where  $m \neq n$ , so let us review the case where  $m = n$  by computing the inner-product  $\langle \sin(nx) | \cos(nx) \rangle$

$$\begin{aligned} \langle \sin(nx) | \cos(nx) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(nx) \cos(nx) dx \quad \left[ \begin{array}{l} \text{Let } u = \sin(nx) \\ du = \frac{1}{n} \cos(nx) dx \end{array} \right] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u \frac{du}{n} \\ &= \frac{1}{2\pi n} \int_{-\pi}^{\pi} u du \\ &= \frac{u^2}{4\pi n} \Bigg|_{u=-\pi}^{\pi} \\ &= \frac{\pi^2 n^2}{4\pi n} - \frac{\pi^2 n^2}{4\pi n} \\ &= 0 \end{aligned}$$

Since we have shown that the inner product  $\langle \sin(nx) | \cos(mx) \rangle$  is zero regardless of the value of  $m$  and  $n$ , then we can say that  $S$  is an orthogonal set.

## Question 2

Let  $V$  be an inner-product space with inner-product  $\langle \cdot | \cdot \rangle$ , and  $\vec{u}_1, \dots, \vec{u}_n \in V$ . Define a function  $T : V \rightarrow V$  by

$$T(\vec{v}) = \sum_{k=1}^n \langle \vec{v} | \vec{u}_k \rangle \vec{u}_k$$

Verify that  $T$  is a linear transformation.

### Note:-

Recall that if  $T : V \rightarrow V$  is a linear transformation, two properties must be verifiable:

- Additivity: A function  $T : V \rightarrow V$  is additive if for any  $\vec{x}, \vec{y} \in V$ ,  $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$
- Homogeneity: A function  $T : V \rightarrow V$  is homogeneous if for any  $c \in \mathbb{R}$  and  $\vec{x} \in V$ ,  $T(c\vec{x}) = cT(\vec{x})$

**Solution:**

- Additivity: Let  $\vec{w} \in V$

$$\begin{aligned} T(\vec{v} + \vec{w}) &= \sum_{k=1}^n \langle \vec{v} + \vec{w} | \vec{u}_k \rangle \vec{u}_k \\ &= \sum_{k=1}^n (\langle \vec{v} | \vec{u}_k \rangle + \langle \vec{w} | \vec{u}_k \rangle) \vec{u}_k \\ &= \sum_{k=1}^n \langle \vec{v} | \vec{u}_k \rangle \vec{u}_k + \sum_{k=1}^n \langle \vec{w} | \vec{u}_k \rangle \vec{u}_k \\ &= \sum_{k=1}^n \langle \vec{v} | \vec{u}_k \rangle \vec{u}_k + \sum_{k=1}^n \langle \vec{w} | \vec{u}_k \rangle \vec{u}_k \\ &= T(\vec{v}) + T(\vec{w}) \end{aligned}$$

Since  $T$  is additive for any  $\vec{v}, \vec{w} \in V$ , then it satisfies the first property.

- Homogeneity: Let  $c \in \mathbb{R}$

$$\begin{aligned} T(c\vec{v}) &= \sum_{k=1}^n \langle c\vec{v} | \vec{u}_k \rangle \vec{u}_k \\ &= \sum_{k=1}^n c \langle \vec{v} | \vec{u}_k \rangle \vec{u}_k \\ &= c \sum_{k=1}^n \langle \vec{v} | \vec{u}_k \rangle \vec{u}_k \\ &= cT(\vec{v}) \end{aligned}$$

Because  $T$  is both additive and homogeneous, then it is sufficient to say that it is a linear transformation.

### Question 3

Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

- Find an ordered basis for  $\text{ran}(A)$
- Perform the Gram-Schmidt algorithm on the basis you got in (a) to get an orthonormal basis for  $\text{ran}(A)$
- Find a basis for  $\ker(A)$
- Extend the basis you found in (a) to a basis  $\mathfrak{B}$  for  $\mathbb{R}^3$
- Find  $P_{\mathfrak{C} \rightarrow \mathfrak{B}}$ .

#### Note:-

Recall:

Let  $\langle \vec{x}_1, \dots, \vec{x}_n \rangle$  be a basis for an  $n$ -dimensional inner-product space  $V$ . The Gram-Schmidt Algorithm:

$$\vec{v}_i = \vec{x}_i - \sum_{k=1}^{i-1} \langle \vec{x}_i | \vec{y}_k \rangle \vec{y}_k, \quad \vec{y}_i = \vec{v}_i / \|\vec{v}_i\|$$

Is used to find an orthogonal basis for  $V$  by subtracting all the pieces of  $\vec{x}_i$  that are in the same direction as all of the  $\vec{y}_k$ , then renormalizing. This formula computes normalized vectors that are orthogonal to all other  $\vec{y}_k$ 's since the non-perpendicular pieces are removed.

#### Solution:-

(a) We can see that  $\vec{a}_1 = \vec{a}_3 - \vec{a}_2$  and thus  $\vec{0} = -\vec{a}_1 - \vec{a}_2 + \vec{a}_3$ . Since  $\vec{a}_1$  is the only linearly dependent vector, then we can say that an ordered basis for the  $\text{ran}(A)$  is:

$$\left\langle \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\rangle$$

(b) We will apply the Gram-Schmidt formula to find the orthonormal basis vectors

$$\vec{b}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\|\vec{b}_1\| = \sqrt{\vec{b}_1 \cdot \vec{b}_1} = \sqrt{1+0+4} = \sqrt{5} \implies \frac{1}{\|\vec{b}_1\|} = \frac{1}{\sqrt{5}}$$

$$\vec{y}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}$$

$$\begin{aligned} \vec{b}_2 &= \vec{x}_2 - (\vec{y}_1 \cdot \vec{x}_2) \vec{y}_1 \\ &= \vec{x}_2 - (2/\sqrt{5} + 0 + 4/\sqrt{5}) \vec{y}_1 \\ &= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - 6/\sqrt{5} \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 6/5 \\ 0 \\ 12/5 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 1 \\ -2/5 \end{bmatrix} \end{aligned}$$

$$\|\vec{b}_2\| = \sqrt{\vec{b}_2 \cdot \vec{b}_2} = \sqrt{16/25 + 1 + 4/25} = \sqrt{45/25} = \sqrt{9/5} = 3/\sqrt{5}$$

$$\implies \frac{1}{\|\vec{b}_2\|} = \frac{\sqrt{5}}{3}$$

$$\vec{y}_2 = \frac{\sqrt{5}}{3} \begin{bmatrix} 4/5 \\ 1 \\ -2/5 \end{bmatrix} = \begin{bmatrix} 4\sqrt{5}/15 \\ \sqrt{5}/3 \\ -2\sqrt{5}/15 \end{bmatrix}$$

The vectors we have found form an orthonormal basis for  $\text{ran}(A)$ , which is

$$\left\langle \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 4\sqrt{5}/15 \\ \sqrt{5}/3 \\ -2\sqrt{5}/15 \end{bmatrix} \right\rangle$$

(c) To find a basis for the  $\ker(A)$ , we can use the linearly dependent vector we found in (a). In particular we see that  $\vec{0} = -\vec{a}_1 - \vec{a}_2 + \vec{a}_3$ . Rank-Nullity tells us that the dimension of the kernel will be one since the dimension of the range is 2. With this information we can say that a basis for  $\ker(A)$  is

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(d) We can extend ordered basis we found in (a) to make a basis for  $\mathbb{R}^3$  by using the standard basis vector  $\vec{e}_3$ . We find that

$$\mathfrak{B} = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\rangle$$

(e) In order to find  $P_{\mathfrak{E} \rightarrow \mathfrak{B}}$  we need to find  $P_{\mathfrak{B} \rightarrow \mathfrak{E}}$  and then find its inverse.

$$P_{\mathfrak{B} \rightarrow \mathfrak{E}} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Now we can find  $(P_{\mathfrak{B} \rightarrow \mathfrak{E}})^{-1} = P_{\mathfrak{E} \rightarrow \mathfrak{B}}$  by using the row reduction method

$$\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 & 1 & 2 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 2 & 2 & 0 & 0 & 1 & 1 & 0 & -2 & -2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & -2 & 2 & 1 & 1 & 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline \end{array} \quad \begin{array}{l} R_1 \leftrightarrow R_3 \\ \\ R_1 - 2R_2 \rightarrow R_1 \\ \\ R_2 - 2R_3 \rightarrow R_2 \end{array} \quad \begin{array}{l} R_2 \leftrightarrow R_3 \\ \\ R_1 + 2R_3 \rightarrow R_1 \end{array}$$

$$\implies P_{\mathfrak{E} \rightarrow \mathfrak{B}} = \begin{bmatrix} -2 & 2 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

#### Question 4

Perform the Gram-Schmidt method on  $\langle \sin(x), \cos(x), x, x^2 \rangle$  in the inner product space  $C([- \pi, \pi])$  with the inner-product from problem 1.

**Solution:**

$$\vec{b}_1 = \sin(x) \quad ||\vec{b}_1|| = \sqrt{\langle \vec{b}_1 | \vec{b}_1 \rangle}$$

$$\begin{aligned} \langle \sin(x) | \sin(x) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2(x) dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(2x)}{2} dx \quad \text{By half angle formula} \\ &= \frac{1}{2\pi} \left( \frac{1}{2} \int_{-\pi}^{\pi} 1 - \cos(2x) dx \right) \\ &= \frac{1}{2\pi} \left[ \frac{1}{2} \left( \int_{-\pi}^{\pi} 1 dx - \int_{-\pi}^{\pi} \cos(2x) dx \right) \right] \\ &= \frac{1}{2\pi} \left[ \frac{1}{2} \left( x - \frac{1}{2} \sin(2x) \right) \right] \Big|_{x=-\pi}^{\pi} \\ &= \frac{1}{2\pi} \left[ \frac{1}{2} \left( \pi - \frac{1}{2} \sin(2\pi) + \pi + \frac{1}{2} \sin(2\pi) \right) \right] \\ &= \frac{1}{2\pi} \left( \frac{1}{2} (2\pi) \right) \\ &= \frac{1}{2\pi} (\pi) \implies \int_{-\pi}^{\pi} \sin^2(x) dx = \pi \quad (1) \\ &= \frac{1}{2} \end{aligned}$$

We can say that  $||\vec{b}_1|| = 1/\sqrt{2}$ , so if we normalize  $\sin(x)$  in this inner-product space we get  $\sqrt{2}\sin(x)$ . This will serve as the first orthonormal basis vector.

$$\vec{b}_2 = \cos(x) - \langle \cos(x) | \sqrt{2}\sin(x) \rangle \sqrt{2}\sin(x)$$

$$\begin{aligned} \langle \cos(x) | \sqrt{2}\sin(x) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(x) \sqrt{2}\sin(x) dx \\ &= \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} \cos(x) \sin(x) dx \end{aligned}$$

Since  $\cos(x)\sin(x)$  is odd, then the integral evaluates to zero. Thus  $\vec{b}_2$  is just  $\cos(x)$ .

$$\begin{aligned} \langle \cos(x) | \cos(x) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2(x) dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + \cos(2x)}{2} dx \quad \text{By half angle formula} \\ &= \frac{1}{2\pi} \left( \frac{1}{2} \int_{-\pi}^{\pi} 1 + \cos(2x) dx \right) \\ &= \frac{1}{2\pi} \left[ \frac{1}{2} \left( \int_{-\pi}^{\pi} 1 dx + \int_{-\pi}^{\pi} \cos(2x) dx \right) \right] \\ &= \frac{1}{2\pi} \left[ \frac{1}{2} \left( x + \frac{1}{2} \sin(2x) \right) \right] \Big|_{x=-\pi}^{\pi} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \left[ \frac{1}{2} \left( \pi + \frac{1}{2} \sin(2\pi) + \pi + \frac{1}{2} \sin(2\pi) \right) \right] \\
&= \frac{1}{2\pi} \left( \frac{1}{2} (2\pi) \right) \\
&= \frac{1}{2\pi} (\pi) \implies \int_{-\pi}^{\pi} \cos^2(x) dx = \pi \\
&= \frac{1}{2}
\end{aligned} \tag{2}$$

This means that the length of  $\vec{b}_2$  is  $1/\sqrt{2}$ , and after normalizing  $\cos(x)$ , we get the second orthonormal basis vector  $\sqrt{2} \cos(x)$ .

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$$\vec{b}_3 = x - \langle x | \sqrt{2} \sin(x) \rangle \sqrt{2} \sin(x) - \langle x | \sqrt{2} \cos(x) \rangle \sqrt{2} \cos(x)$$

$$\begin{aligned}
\langle x | \sqrt{2} \sin(x) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x \sqrt{2} \sin(x) dx \\
&= \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} x \sin(x) dx
\end{aligned}$$

We can evaluate this integral using tabular integration

$d$	$I$
$x$	$\sin x$
$1$	$-\cos x$
$0$	$-\sin x$

$$\begin{aligned}
&= \frac{\sqrt{2}}{2\pi} (\sin(x) - x \cos(x)) \Big|_{x=-\pi}^{\pi} \\
&= \frac{\sqrt{2}}{2\pi} (\sin(\pi) - \pi \cos(\pi) + \sin(\pi) - \pi \cos(-\pi)) \\
&= \frac{\sqrt{2}}{2\pi} (2\pi) \implies \int_{-\pi}^{\pi} x \sin(x) dx = 2\pi \\
&= \sqrt{2}
\end{aligned} \tag{3}$$

$$\begin{aligned}
\langle x | \sqrt{2} \cos(x) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x \sqrt{2} \cos(x) dx \\
&= \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} x \cos(x) dx
\end{aligned}$$

Similarly, we can use tabular integration to compute this integral.

$d$	$I$
$x$	$\cos x$
$1$	$\sin x$
$0$	$-\cos x$

$$\begin{aligned}
&= \frac{\sqrt{2}}{2\pi} (x \sin(x) + \cos(x)) \Big|_{x=-\pi}^{\pi} \\
&= \frac{\sqrt{2}}{2\pi} (\pi \sin(\pi) + \cos(\pi) - \pi \sin(-\pi) - \cos(-\pi))
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{2\pi}(0) \\
&= 0
\end{aligned}$$

Thus  $\vec{b}_3 = x - \sqrt{2}\sqrt{2}\sin(x) = x - 2\sin(x)$ . Now we must normalize it by finding its length and dividing by it.

$$\begin{aligned}
\langle x - 2\sin(x) | x - 2\sin(x) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (x - 2\sin(x))^2 dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 - 4x\sin(x) + 4\sin^2(x) dx \\
&= \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} x^2 dx - 4 \int_{-\pi}^{\pi} x\sin(x) dx + 4 \int_{-\pi}^{\pi} \sin^2(x) dx \right) \\
&= \frac{1}{2\pi} \left( \left. \frac{x^3}{3} \right|_{x=-\pi}^{\pi} - 4(2\pi) + 4(\pi) \right) \quad \text{By (1) and (2)} \\
&= \frac{1}{2\pi} \left( \frac{2\pi^3}{3} - 4\pi \right) \\
&= \frac{\pi^2}{3} - 2 = \frac{\pi^2 - 6}{3}
\end{aligned}$$

This means that  $||\vec{b}_3|| = \sqrt{\frac{\pi^2 - 6}{3}}$ , and normalizing our  $\vec{b}_3$  we see that our third orthonormal basis vector is  $\sqrt{\frac{3}{\pi^2 - 6}}(x - 2\sin(x))$ .

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$$\vec{b}_4 = x^2 - \langle x^2 | \sqrt{2}\sin(x) \rangle \sqrt{2}\sin(x) - \langle x^2 | \sqrt{2}\cos(x) \rangle \sqrt{2}\cos(x) - \langle x^2 | \sqrt{\frac{3}{\pi^2 - 6}}(x - 2\sin(x)) \rangle \sqrt{\frac{3}{\pi^2 - 6}}(x - 2\sin(x))$$

Since  $\sqrt{2}\sin(x)$  is odd and  $x^2$  is even, their product will be odd. Thus the inner product will evaluate to zero.

$$\begin{aligned}
\langle x^2 | \sqrt{2}\cos(x) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \sqrt{2}\cos(x) dx \\
&= \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} x^2 \cos(x) dx
\end{aligned}$$

We can evaluate the integral using tabular integration

$d$	$I$
$x^2$	$\cos x$
$\xrightarrow{+}$	$\sin x$
$2x$	$-\cos x$
$\xrightarrow{-}$	$-\sin x$
$2$	
$\xrightarrow{+}$	
$0$	

$$\begin{aligned}
&= \frac{\sqrt{2}}{2\pi} (x^2 \sin(x) + 2x \cos(x) - 2\sin(x)) \Big|_{x=-\pi}^{\pi} \\
&= \frac{\sqrt{2}}{2\pi} (2\pi \cos(\pi) + 2\pi \cos(-\pi)) \\
&= \frac{\sqrt{2}}{2\pi} (-4\pi) \implies \int_{-\pi}^{\pi} x^2 \cos(x) dx = -4\pi \\
&= -2\sqrt{2}
\end{aligned} \tag{4}$$



$$\begin{aligned}
\langle x^2 | \sqrt{\frac{3}{\pi^2 - 6}}(x - 2\sin(x)) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \sqrt{\frac{3}{\pi^2 - 6}}(x - 2\sin(x)) dx \\
&= \frac{1}{2\pi} \sqrt{\frac{3}{\pi^2 - 6}} \int_{-\pi}^{\pi} x^2(x - 2\sin(x)) dx \\
&= \frac{1}{2\pi} \sqrt{\frac{3}{\pi^2 - 6}} \int_{-\pi}^{\pi} (x^3 - 2x^2 \sin(x)) dx
\end{aligned}$$

We saw before that  $x^2 \sin(x)$  is odd, and  $x^3$  is also odd, so the entire function is odd. Since the parity of the function is odd, that means that the integral will evaluate to 0. This means that  $\vec{b}_4 = x^2 + 4\cos(x)$ , now we must normalize it.

$$\begin{aligned}
\langle x^2 + 4\cos(x) | x^2 + 4\cos(x) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^2 + 4\cos(x))^2 dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 + 8x^2 \cos(x) + 16\cos^2(x) dx \\
&= \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} x^4 dx + 8 \int_{-\pi}^{\pi} x^2 \cos(x) dx + 16 \int_{-\pi}^{\pi} \cos^2(x) dx \right) \\
&= \frac{1}{2\pi} \left( \frac{x^5}{5} \Big|_{x=-\pi}^{\pi} + 8(-4\pi) + 16(\pi) \right) \quad \text{By (2) and (4)} \\
&= \frac{1}{2\pi} \left( \frac{2\pi^5}{5} - 16\pi \right) \\
&= \frac{\pi^4}{5} - 8 = \frac{\pi^4 - 40}{5}
\end{aligned}$$

Thus we can conclude that  $\|\vec{b}_4\| = \sqrt{\frac{\pi^4 - 40}{5}}$ , and after normalizing  $\vec{b}_4$  we find the the last orthonormal basis vector is  $\sqrt{\frac{5}{\pi^4 - 40}}(x^3 + 4\cos(x))$ .

Thus we can say that an orthonormal basis for the  $\text{span}(\sin(x), \cos(x), x, x^2)$  is

$$\langle \sqrt{2}\sin(x), \sqrt{2}\cos(x), \sqrt{\frac{3}{\pi^2 - 6}}(x - 2\sin(x)), \sqrt{\frac{5}{\pi^4 - 40}}(x^3 + 4\cos(x)) \rangle$$

### Question 5

Let  $W \subseteq C([- \pi, \pi])$  be  $W = \text{span}(\sin(x), \cos(x), x, x^2)$ . Your answer from problem 4 should be an orthonormal basis for  $W$ . Compute:

- (a)  $\text{proj}_W(1)$
- (b)  $\text{proj}_W(x \sin(x))$

#### Note:-

Recall:

If  $\langle \vec{u}_1, \dots, \vec{u}_n \rangle$  is an orthonormal basis for  $W$ , then

$$\text{proj}_W(\vec{v}) = \sum_{k=1}^n \langle \vec{v} | \vec{u}_k \rangle \vec{u}_k$$

#### Solution:

(a) Using the formula from above, we can begin by computing the inner-products for each summand.

$$\langle 1 | \sqrt{2}\sin(x) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{2}\sin(x) dx = 0$$

This integral evaluates to zero since  $\sin(x)$  is odd.

$$\begin{aligned}\langle 1 | \sqrt{2} \cos(x) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{2} \cos(x) dx \\ &= \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} \cos(x) dx \\ &= \frac{\sqrt{2}}{2\pi} \sin(x) \Big|_{x=-\pi}^{\pi} \\ &= \frac{\sqrt{2}}{2\pi} (0) = 0\end{aligned}$$

Since this integral evaluates to zero, then the summand is also nullified, so we will continue.

$$\langle 1 | \sqrt{\frac{3}{\pi^2 - 6}} (x - 2 \sin(x)) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{\frac{3}{\pi^2 - 6}} (x - 2 \sin(x)) dx$$

Like the first summand that we evaluated, the function is odd, so thus the integral will evaluate to zero.

$$\begin{aligned}\langle 1 | \sqrt{\frac{5}{\pi^4 - 40}} (x^3 + 4 \cos(x)) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{\frac{5}{\pi^4 - 40}} (x^3 + 4 \cos(x)) dx \\ &= \frac{1}{2\pi} \sqrt{\frac{5}{\pi^4 - 40}} \int_{-\pi}^{\pi} x^3 + 4 \cos(x) dx \\ &= \frac{1}{2\pi} \sqrt{\frac{5}{\pi^4 - 40}} \left( \int_{-\pi}^{\pi} x^3 dx + 4 \int_{-\pi}^{\pi} \cos(x) dx \right)\end{aligned}$$

Since  $x^3$  is odd, that integral will evaluate to zero, and as we saw in previously the integral of  $\cos(x)$  from  $x = -\pi$  to  $\pi$  evaluates to zero as well. Since all the summands evaluate to zero, then  $\text{proj}_W(1) = 0$ .

(b) Lets follow the same process as (a), by first evaluating the summands.

$$\langle x \sin(x) | \sqrt{2} \sin(x) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \sin(x) \sqrt{2} \sin(x) dx = 0$$

This evaluates to zero since we're multiplying an even function by an odd function. Thus the first summand will be nullified and is not necessary for the projection of  $x \sin(x)$ .

$$\begin{aligned}\langle x \sin(x) | \sqrt{2} \cos(x) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x \sin(x) \sqrt{2} \cos(x) dx \\ &= \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} x \sin(x) \cos(x) dx \\ &= \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} \frac{x \sin(2x)}{2} dx \quad \text{By double angle formula} \\ &= \frac{\sqrt{2}}{4\pi} \int_{-\pi}^{\pi} x \sin(2x) dx\end{aligned}$$

We can use tabular integration to complete this integral

$$\begin{array}{rcl} & d & I \\ x & \xrightarrow{+} & \sin 2x \\ 1 & \xrightarrow{-} & -\frac{1}{2} \cos 2x \\ 0 & \xrightarrow{-} & -\frac{1}{4} \sin 2x \end{array}$$

$$\begin{aligned}&= \frac{\sqrt{2}}{4\pi} \left( \frac{1}{4} \sin(2x) - \frac{1}{2} x \cos(2x) \right) \Big|_{x=-\pi}^{\pi} \\ &= \frac{\sqrt{2}}{8\pi} (-\pi \cos(2\pi) - \pi \cos(-2\pi)) = -\frac{\sqrt{2}}{4}\end{aligned}$$

$$\langle x \sin(x) | \sqrt{\frac{3}{\pi^2 - 6}}(x - 2 \sin(x)) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \sin(x) \sqrt{\frac{3}{\pi^2 - 6}}(x - 2 \sin(x)) dx = 0$$

This inner-product evaluates to zero since we multiply an even function by an odd function. Thus this summand will be nullified and is not necessary for the projection of  $x \sin(x)$ .

$$\begin{aligned} \langle x \sin(x) | \sqrt{\frac{5}{\pi^4 - 40}}(x^3 + 4 \cos(x)) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x \sin(x) \sqrt{\frac{5}{\pi^4 - 40}}(x^3 + 4 \cos(x)) dx \\ &= \frac{1}{2\pi} \sqrt{\frac{5}{\pi^4 - 40}} \int_{-\pi}^{\pi} x \sin(x)(x^3 + 4 \cos(x)) dx \\ &= \frac{1}{2\pi} \sqrt{\frac{5}{\pi^4 - 40}} \left( \int_{-\pi}^{\pi} x^4 \sin(x) dx + 4 \int_{-\pi}^{\pi} x \cos(x) \sin(x) dx \right) \\ &= \frac{1}{2\pi} \sqrt{\frac{5}{\pi^4 - 40}} (0 + -2\pi) \\ &= -\sqrt{\frac{5}{\pi^4 - 40}} \end{aligned}$$

Since we have computed all the inner-products, we can now find the projection of  $x \sin(x)$ .

$$\begin{aligned} \text{proj}_W(x \sin(x)) &= -\frac{\sqrt{2}}{4} \sqrt{2} \cos(x) - \sqrt{\frac{5}{\pi^4 - 40}} (x^3 + 4 \cos(x)) \\ &= -\frac{1}{2} \cos(x) - \frac{5x^3 + 20 \cos(x)}{\pi^4 - 40} \end{aligned}$$

#### Question 6

Let  $A$  be as in problem 3.

- (a) Compute the matrix of  $\text{proj}_{\text{ran}(A)}$ , that is, find the matrix  $M$  so that  $M\vec{x} = \text{proj}_{\text{ran}(A)}(\vec{x})$  for all  $\vec{x} \in \mathbb{R}^3$ .  
 (b) For each of the following  $\vec{b}_i$  compute  $\text{proj}_{\text{ran}(A)}(\vec{b}_i)$ :

$$\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{b}_4 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

- (c) For each  $\vec{b}_i$  from (b), let  $\hat{b}_i = \text{proj}_{\text{ran}(A)}(\vec{b}_i)$ , each  $\hat{b}_i$  has  $\hat{b}_i \in \text{ran}(A)$  and so there is a solution to

$$A\hat{x} = \hat{b}_i.$$

Use  $\ker(A)$  and the change of basis matrix you found in problem 3 to find full solution sets to

$$A\hat{x}_i = \hat{b}_i$$

for each  $i = 1, 2, 3, 4$ .

#### Note:-

Recall:

If  $W \subseteq \mathbb{R}^n$ , there exists a matrix for  $\text{proj}_W$ . If  $\langle \vec{u}_1, \dots, \vec{u}_n \rangle$  is an orthonormal basis for  $W$  then let

$$Q = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \dots & \vec{u}_n \\ | & | & & | \end{bmatrix}$$

Then  $\text{proj}_W(\vec{x}) = QQ^T \vec{x}$

**Solution:**

(a)

$$Q = \begin{bmatrix} 1/\sqrt{5} & 4\sqrt{5}/15 \\ 0 & \sqrt{5}/3 \\ 2/\sqrt{5} & -2\sqrt{5}/15 \end{bmatrix} \quad Q^\top = \begin{bmatrix} 1/\sqrt{5} & 0 & 2\sqrt{5} \\ 4\sqrt{5}/15 & \sqrt{5}/3 & -2\sqrt{5}/15 \end{bmatrix}$$
$$QQ^\top = \begin{bmatrix} 5/9 & 4/9 & 2/9 \\ 4/9 & 5/9 & -2/9 \\ 2/9 & -2/9 & 8/9 \end{bmatrix}$$

(b) For each  $\vec{b}_i$ , we can apply the formula from my note.

$$\text{proj}_W(\vec{b}_1) = QQ^\top \vec{b}_1 = \begin{bmatrix} 5/9 & 4/9 & 2/9 \\ 4/9 & 5/9 & -2/9 \\ 2/9 & -2/9 & 8/9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

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$$\text{proj}_W(\vec{b}_2) = QQ^\top \vec{b}_2 = \begin{bmatrix} 5/9 & 4/9 & 2/9 \\ 4/9 & 5/9 & -2/9 \\ 2/9 & -2/9 & 8/9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 19/9 \\ 8/9 \\ 22/9 \end{bmatrix}$$

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$$\text{proj}_W(\vec{b}_3) = QQ^\top \vec{b}_3 = \begin{bmatrix} 5/9 & 4/9 & 2/9 \\ 4/9 & 5/9 & -2/9 \\ 2/9 & -2/9 & 8/9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11/9 \\ 7/9 \\ 8/9 \end{bmatrix}$$

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$$\text{proj}_W(\vec{b}_4) = QQ^\top \vec{b}_4 = \begin{bmatrix} 5/9 & 4/9 & 2/9 \\ 4/9 & 5/9 & -2/9 \\ 2/9 & -2/9 & 8/9 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

(c) We can use the change of basis matrix we found in problem 3 to find solutions to each  $\hat{b}_i$ .

$$\vec{x} = P_{\mathfrak{C} \rightarrow \mathfrak{B}} \hat{b}_i$$

And then use our basis for  $\ker(A)$  to construct full solution sets.

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$$\vec{x} = P_{\mathfrak{C} \rightarrow \mathfrak{B}} \hat{b}_1 = \begin{bmatrix} -2 & 2 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : c_1 \in \mathbb{R} \right\}$$

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$$\vec{x} = P_{\mathfrak{C} \rightarrow \mathfrak{B}} \hat{b}_2 = \begin{bmatrix} -2 & 2 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 19/9 \\ 8/9 \\ 22/9 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \\ 8/9 \end{bmatrix}$$
$$\left\{ \begin{bmatrix} 0 \\ 1/3 \\ 8/9 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : c_1 \in \mathbb{R} \right\}$$

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$$\vec{x} = P_{\mathfrak{C} \rightarrow \mathfrak{B}} \hat{b}_3 = \begin{bmatrix} -2 & 2 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 11/9 \\ 7/9 \\ 8/9 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/3 \\ 7/9 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 0 \\ -1/3 \\ 7/9 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : c_1 \in \mathbb{R} \right\}$$


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$$\vec{x} = P_{\mathfrak{C} \rightarrow \mathfrak{B}} \hat{b}_3 = \begin{bmatrix} -2 & 2 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : c_1 \in \mathbb{R} \right\}$$