Math 2700.009
Problem Set 11

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Question 1

Let

$$A = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

(a) Find the characteristic polynomial of A
(b) Find all eigenvalues of A
(c) For each eigenvalue λ of A, what is the algebraic and geometric multiplicities of λ ?
(d) For each eigenvalue λ of A, find a basis for the eigenspace $E_\lambda(A)$.
(e) Is A diagonalizable? If so, give an eigenbasis for A.

Note:-

Recall:

- If $T : V \longrightarrow V$ is a linear transformation with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then set $d_i = \dim(E_{\lambda_i}(T))$. T is diagonalizable exactly when $d_1 + d_2 + \ldots + d_n = \dim(V)$
- An eigenvalue's algebraic multiplicity is always $\geq$ geometric multiplicity
- $\dim(E_\lambda(A)) = \dim(\ker(\lambda I - A))$
- If we have $A \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and $\{v_1, \ldots, v_n\}$ as a basis for $E_{\lambda_1}(A)$, $\{u_1, \ldots, u_k\}$ as a basis for $E_{\lambda_2}(A)$ and $\{w_1, \ldots, w_j\}$ as a basis for $E_{\lambda_3}(A)$, then we find that $\{v_1, \ldots, v_n, u_1, \ldots, u_k, w_1, \ldots, w_j\}$ are linearly independent vectors. In other words, vectors from a basis of a particular eigenspace are linearly independent from the vectors that form a basis for another eigenspace. This is particularly important because we can use this concept when determining whether a matrix or transformation is diagonalizable.

Solution:
(a) To find the characteristic polynimal we must solve $\det(\lambda I_4 - A)$

$$\det\left(\begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda-1 & -1 & -3 & 0 \\ 0 & \lambda+2 & -2 & -4 \\ 0 & 0 & \lambda+1 & 2 \\ 0 & 0 & 0 & \lambda+2 \end{bmatrix}\right)$$

In the previous homework, we found a unique property of the determinant of triangular matricies. In particular we found that the determinant of a triangular matrix is the product of the diagonal elements. We can quickly solve for the characteristic polynomial using this method.

$$\det\left(\begin{bmatrix} \lambda-1 & -1 & -3 & 0 \\ 0 & \lambda+2 & -2 & -4 \\ 0 & 0 & \lambda+1 & 2 \\ 0 & 0 & 0 & \lambda+2 \end{bmatrix}\right) = (\lambda-1)(\lambda+1)(\lambda+2)^2$$

(b) Solving for the roots, we quickly see that the eigenvalues for this matrix A are: $\lambda = 1, \lambda = -1, \lambda = -2$.

(c) The algebraic multiplicities of $\lambda = -1$ and $\lambda = 1$ are both 1. This implies that geometric multiplicites for these eigenvalues is also 1. The alebraic multiciplity when $\lambda = -2$ doesn't immediately tell us the value, but we do know that it will be either 1 or 2. To find its geometric multiplicity we need to find a basis for the eigenspace $E_{-2}(A)$

$$-2I - A = \begin{bmatrix} -3 & -1 & -3 & 0 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v}_1 = 3\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4 \implies \vec{0} = -\vec{v}_1 + 3\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4$$
$$\vec{v}_3 = 0\vec{v}_1 + 3\vec{v}_2 + \frac{1}{2}\vec{v}_4 \implies \vec{0} = 0\vec{v}_1 + 3\vec{v}_2 - \vec{v}_3 + \frac{1}{2}\vec{v}_4$$

We can say that a basis for the kernel of $-2I - A$ is

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ \frac{1}{2} \end{bmatrix} \right\}$$

Since our basis contains two vectors, so that means $\dim(E_{-2}(A)) = 2$. Thus since we know the dimension of the eigenspace, we know also that the geometric mutiplicity of -2 is 2.

(d) In part (c) we already found an eigenbasis for $E_{-2}(A)$ to be

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ \frac{1}{2} \end{bmatrix} \right\}$$

So now we must find bases for eigenspaces $E_{-1}(A)$ and $E_1(A)$. We can do so by finding the null space of $\lambda I - A$

$$I - A = \begin{bmatrix} 0 & 1 & -3 & 0 \\ 0 & 3 & -2 & -4 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\vec{v}_1 = 0\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4$$
$$\vec{0} = -\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4$$

So then an basis for $E_1(A)$ is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Lastly we must find a basis for $E_{-1}(A)$

$$-I - A = \begin{bmatrix} -2 & -1 & -3 & 0 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\vec{v}_3 = \frac{5}{2}\vec{v}_1 - 2\vec{v}_2 + 0\vec{v}_4$$
$$\vec{0} = \frac{5}{2}\vec{v}_1 - 2\vec{v}_2 - \vec{v}_3 + 0\vec{v}_4$$

So we can see that a basis for the kernel of $E_{-1}(A)$ is

$$\left\{ \begin{bmatrix} \frac{5}{2} \\ -2 \\ -1 \\ 0 \end{bmatrix} \right\}$$

(e) Since we know that the vectors that form a basis for the different eigenspaces of A are all linearly independent from one another, we can appeal to the definition of dimension.

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{5}{2} \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ \frac{1}{2} \end{bmatrix} \right\}$$

We can see that we have 4 eigenvectors, which we found by taking a basis of each eigenspace. For A to be diagonalizable, then we must be able to find a basis for $\mathbb{R}^4$ using only eigenvectors. Hence we can conclude that A is diagonalizable because the size of this set of vectors matches that of the dimension of $\mathbb{R}^4$, so we call A diagonalizable. Further, since this set of vectors forms a basis for $\mathbb{R}^4$, and since it consisting of only eigenvectors, then we can call it an eigenbasis.

Question 2

Let

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 7 & -3 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

(a) Find the characteristic polynomial of A and all eigenvalues of A.
(b) What are the algebraic and geometric multiplicites of each eigenvalue. (Keep in mind the geometric multiplicity is always at least one and at most the algebraic multiplicity)
(c) Is A diagonalizable?

Solution:
(a)

$$\begin{aligned} \det(\lambda I_4 - A) &= \det \begin{bmatrix} \lambda - 1 & 2 & 0 & 0 \\ -1 & \lambda - 4 & 0 & 0 \\ 0 & 0 & \lambda - 7 & 3 \\ 0 & 0 & -2 & \lambda - 2 \end{bmatrix} \\ &= ((\lambda - 4)(\lambda - 1) + 2)((\lambda - 7)(\lambda - 2) + 6) \\ &= (\lambda^2 - 5\lambda + 6)(\lambda^2 - 9\lambda + 20) \\ &= (\lambda - 2)(\lambda - 3)(\lambda - 4)(\lambda - 5) \end{aligned}$$

A has eigenvalues $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 4, \lambda_4 = 5$.

(b) Each eigenvalue has an algebraic multiplicity of one. Because of this we can also conclude that each eigenvalue has a geometric multiplicity of one also. We can conclude this because the geometric multiplicity $\leq$ algebraic multiplicity and always $\geq$ one.

(c) Since the sum of the dimension of each eigenspace equals the sum of $\mathbb{R}^4$ we can say that A is diagonalizable. We know that the sum of the dimension of A's eigenspaces is equal to the dimension of $\mathbb{R}^4$ because each of the four eigenvalues has a geometric multiplicity of one.

Question 3

Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(a) Find the characteristic polynomial of A and all eigenvalues of A.
(b) What are the algebraic and geometric multiplicites of each eigenvalue. (Keep in mind the geometric multiplicity is always at least one and at most the algebraic multiplicity)
(c) Is A diagonalizable?

Solution:
(a)

$$\begin{aligned} \det(\lambda I_6 - A) &= \det \begin{bmatrix} \lambda - 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda - 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda - 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & -1 & \lambda & 0 \\ 0 & 0 & 0 & 0 & -1 & \lambda \end{bmatrix} \\ &= (\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda)^3 \end{aligned}$$

A has eigenvalues $\lambda = 1, \lambda = 2, \lambda = 3, \lambda = 0$

(b) $\lambda = 1, \lambda = 2, \lambda = 3$ have algebraic and geometric multiplicity of 1. $\lambda = 0$ has an algebraic multiplicty of 3 but to find its geometric multiplicity we must find a basis for the eigenspace $E_0(A)$.

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{aligned} \vec{v}_6 &= 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4 + 0\vec{v}_5 \\ \vec{0} &= 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4 + 0\vec{v}_5 - \vec{v}_6 \end{aligned}$$

Thus a basis for the eigenspace $E_0(A)$ is

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

So this makes the dimension of the eigenspace 1, and further the geometric multiplicity is also 1.

(c) We know now the dimension of each eigenspace. This tells us whether or not A is diagonalizable, in particular we see that the sum of the dimensions of each of the eigenspaces is equal to 4. This means that A is not diagonalizable since the sum would need to equal the dimension of $\mathbb{R}^6$ which it does not.