Math 2700.009
Problem Set 08

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Question 1

Consider the following bases for $\mathbb{R}^4$

$$\mathfrak{B} = \left\langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \right\rangle \quad \text{and} \quad \mathfrak{D} = \left\langle \begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}3\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\3\\2\end{bmatrix}, \begin{bmatrix}0\\0\\4\\3\end{bmatrix} \right\rangle$$

and let $\mathfrak{E} = \left\langle \vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 \right\rangle$ be the standard basis for $\mathbb{R}^4$. Compute each of the following:

(a) $P_{\mathfrak{B}\to\mathfrak{E}}$
(b) $P_{\mathfrak{D}\to\mathfrak{E}}$
(c) $P_{\mathfrak{E}\to\mathfrak{B}}$
(d) $P_{\mathfrak{E}\to\mathfrak{D}}$
(e) $P_{\mathfrak{B}\to\mathfrak{D}}$
(f) $P_{\mathfrak{D}\to\mathfrak{B}}$

Note:-

- If $\begin{cases} \mathfrak{E} = \langle \vec{e}_1, \dots, \vec{e}_n \rangle \\ \mathfrak{B} = \langle \vec{b}_1, \dots, \vec{b}_n \rangle \end{cases}$, then $P_{\mathfrak{B}\to\mathfrak{E}}[\vec{x}]_{\mathfrak{B}} = [\vec{x}]_{\mathfrak{E}} = \vec{x}$
- $P_{\mathfrak{B}\to\mathfrak{E}} = \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_n \\ | & | & & | \end{bmatrix}$ Where $\mathfrak{E}$ is the standard basis.
- $(P_{\mathfrak{B}\to\mathfrak{E}})^{-1} = P_{\mathfrak{E}\to\mathfrak{B}}$
- If $\mathfrak{B}_1, \mathfrak{B}_2$, and $\mathfrak{B}_3$ are bases, then $P_{\mathfrak{B}_2\to\mathfrak{B}_3} P_{\mathfrak{B}_1\to\mathfrak{B}_2}[\vec{x}]_{\mathfrak{B}_1} = [\vec{x}]_{\mathfrak{B}_3}$

Solution:

(a) $P_{\mathfrak{B}\to\mathfrak{E}} = \begin{bmatrix}1&1&1&1\\1&1&1&0\\1&1&0&0\\1&0&0&0\end{bmatrix}$

(b) $P_{\mathfrak{D}\to\mathfrak{E}} = \begin{bmatrix}2&3&0&0\\1&2&0&0\\0&0&3&4\\0&0&2&3\end{bmatrix}$

(c) $P_{\mathfrak{E}\to\mathfrak{B}} = \begin{bmatrix}1&1&1&1\\1&1&1&0\\1&1&0&0\\1&0&0&0\end{bmatrix}^{-1}$

$$\left[\begin{array}{cccc|cccc} 1&1&1&1&1&0&0&0\\ 1&1&1&0&0&1&0&0\\ 1&1&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&1 \end{array}\right] \quad \begin{array}{c} R_1 \leftrightarrow R_4 \\ R_2 \leftrightarrow R_3 \end{array} \quad \left[\begin{array}{cccc|cccc} 1&0&0&0&0&0&0&1\\ 1&1&0&0&0&0&1&0\\ 1&1&1&0&0&1&0&0\\ 1&1&1&1&1&0&0&0 \end{array}\right] \quad R_4 - R_3 \to R_4$$

$$\left[\begin{array}{cccc|cccc} 1&0&0&0&0&0&0&1\\ 1&1&0&0&0&0&1&0\\ 1&1&1&0&0&1&0&0\\ 0&0&0&1&1&-1&0&0 \end{array}\right] \quad R_3 - R_2 \to R_3 \quad \left[\begin{array}{cccc|cccc} 1&0&0&0&0&0&0&1\\ 1&1&0&0&0&0&1&0\\ 0&0&1&0&0&1&-1&0\\ 0&0&0&1&1&-1&0&0 \end{array}\right] \quad R_2 - R_1 \to R_2$$

$$\left[\begin{array}{cccc|cccc} 1&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&1&-1\\ 0&0&1&0&0&1&-1&0\\ 0&0&0&1&1&-1&0&0 \end{array}\right] \implies P_{\mathfrak{E}\to\mathfrak{B}} = \begin{bmatrix}0&0&0&1\\0&0&1&-1\\0&1&-1&0\\1&-1&0&0\end{bmatrix}$$

(d) $P_{\mathfrak{E}\to\mathfrak{D}} = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 2 & 3 \end{bmatrix}^{-1}$

$$\left[\begin{array}{cccc|cccc} 2 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_2 - 2R_1 \to R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_1 + 2R_2 \to R_1 \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -3 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \quad -R_2 \to R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \quad \frac{1}{3}R_3 \to R_3 \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_4 - 2R_3 \to R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & -\frac{2}{3} & 1 \end{array}\right] \quad R_3 - 4R_4 \to R_3 \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & -\frac{2}{3} & 1 \end{array}\right] \quad 3R_4 \to R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 3 \end{array}\right] \implies P_{\mathfrak{E}\to\mathfrak{D}} = \begin{bmatrix} 2 & -3 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & -2 & 3 \end{bmatrix}$$

(e) $P_{\mathfrak{B}\to\mathfrak{D}} = P_{\mathfrak{E}\to\mathfrak{D}}P_{\mathfrak{B}\to\mathfrak{E}} = \begin{bmatrix} 2 & -3 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & 2 \\ 1 & 1 & 1 & -1 \\ -1 & 3 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix}$

(f) $P_{\mathfrak{D}\to\mathfrak{B}} = P_{\mathfrak{E}\to\mathfrak{B}}P_{\mathfrak{D}\to\mathfrak{E}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & -3 & -4 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

Question 2

The point of this problem is to see the effect of changing the order of the basis and the effect on the coordinate vectors. Let $\mathfrak{D}$ and $\mathfrak{E}$ be as in problem 1 and let

$$\mathfrak{F} = \left\langle \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle.$$

(a) What are $P_{\mathfrak{E}\to\mathfrak{F}}$ and $P_{\mathfrak{D}\to\mathfrak{F}}$? (note you have $P_{\mathfrak{D}\to\mathfrak{E}}$ from problem 1)
(b) Suppose $\vec{x} \in \mathbb{R}^4$ had

$$[\vec{x}]_{\mathfrak{D}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Use $P_{\mathfrak{D}\to\mathfrak{F}}$ to find $[\vec{x}]_{\mathfrak{F}}$.

Solution:

(a) $P_{\mathfrak{E}\to\mathfrak{F}} = \begin{bmatrix} 0 & 3 & 0 & 2 \\ 0 & 2 & 0 & 1 \\ 3 & 0 & 4 & 0 \\ 2 & 0 & 3 & 0 \end{bmatrix}^{-1}$

$$\left[\begin{array}{cccc|cccc} 0 & 3 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 4 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_4 \quad \left[\begin{array}{cccc|cccc} 2 & 0 & 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 2 & 1 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_3 - R_1 \to R_3 \\ R_4 - R_2 \to R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 2 & 0 & 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_1 - 2R_3 \to R_1 \\ R_2 - R_4 \to R_2 \end{array} \quad \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_4 - R_2 \to R_4 \\ R_3 - R_1 \to R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 1 & 2 & -3 & 0 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_3 \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 3 & -4 \\ 0 & 1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 1 & 2 & -3 & 0 & 0 \end{array}\right]$$

$$P_{\mathfrak{E}\to\mathfrak{F}} = \begin{bmatrix} 0 & 0 & 3 & -4 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & -2 & 3 \\ 2 & -3 & 0 & 0 \end{bmatrix} \text{ and } P_{\mathfrak{D}\to\mathfrak{F}} = P_{\mathfrak{E}\to\mathfrak{F}} P_{\mathfrak{D}\to\mathfrak{E}}$$

So then, $P_{\mathfrak{D}\to\mathfrak{F}} = \begin{bmatrix} 0 & 0 & 3 & -4 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & -2 & 3 \\ 2 & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(b) To find $[\vec{x}]_{\mathfrak{F}}$, then we we must find $P_{\mathfrak{E}\to\mathfrak{F}} P_{\mathfrak{D}\to\mathfrak{E}} [\vec{x}]_{\mathfrak{D}}$

$$\begin{aligned} [\vec{x}]_{\mathfrak{F}} &= P_{\mathfrak{E}\to\mathfrak{F}} P_{\mathfrak{D}\to\mathfrak{E}} [\vec{x}]_{\mathfrak{D}} \\ &= \begin{bmatrix} 0 & 0 & 3 & -4 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & -2 & 3 \\ 2 & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\ [\vec{x}]_{\mathfrak{F}} &= \begin{bmatrix} c \\ b \\ d \\ a \end{bmatrix} \end{aligned}$$

Question 3

In $\mathbb{R}^3$, let

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and let $W = \operatorname{span}\left(\vec{e}_1, \vec{e}_2\right)$. Let $T : \mathbb{R}^2 \to W$ be defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

(a) What is $\dim(W)$?
(b) What is $\ker(T)$? and what is $\text{ran}(T)$?
(c) Is T an isomorphism? (Recall that T is an isomorphism if and only if $\ker(T) = \{\overrightarrow{0}\}$ and $\text{ran}(T) = W$.
(d) Find a matrix $A \in \mathbb{R}^{2\times 2}$ so that

$$[T(\vec{x})]_{\mathfrak{B}} = A\vec{x}$$

where $\mathfrak{B} = \langle \vec{e}_1, \vec{e}_2 \rangle$ is the natural basis for W

Solution:
(a) The $\dim(W)$ is the number of linearly independent vectors that also span W. That is to say the dimension of W is the number of vectors in a basis for W. Since $\vec{e_1}$ and $\vec{e_2}$ are linearly independent, and they span W, then they also form a basis for W, so by definition of dimension, it must be 2.

(b) The $\ker(T)$ consists of the vectors from $\mathbb{R}^2$ that map to the zero vector of W which is the same as $\mathbb{R}^3$. If we set

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we can see that the only solution which results in the zero vector is when $x = y = 0$. This means that

$$\ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \{\vec{0}\}$$

The $\text{ran}(T)$ consists of all possible outputs of the linear transformation. The linear transformation T serves to map vectors from $\mathbb{R}^2$ to W. Since W is a subspace generated by taking the span of $\vec{e_1}$ and $\vec{e_2}$, then it is sufficient to say that the range of T covers all vectors from W. Thus we can say that $\text{ran}(T) = W$.

(c) As discovered in (b), We can say that T is an isomorphism, since it meets the biconditional statement.

(d) To find the matrix $A \in \mathbb{R}^{2\times 2}$, we can use some principles. Since the ordered basis we will be using in this example is standardized, then a useful property can be used, that is that $[\vec{x}]_{\mathfrak{E}} = \vec{x}$. Thus we can express our equation as $[T(\vec{x})]_{\mathfrak{B}} = A[\vec{x}]_{\mathfrak{B}}$ since $\mathfrak{B}$ is effectively the standard basis. We can find the matrix by taking the columns as the $\mathfrak{B}$-coordinate vectors of the calculated values from T so that we have

$$A = \begin{bmatrix} | & | \\ [T(e_1)]_{\mathfrak{B}} & [T(e_2)]_{\mathfrak{B}} \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Question 4

Let

$$\mathfrak{B} = \left\langle \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\rangle$$

and $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be given by

$$T(\vec{x}) = \begin{bmatrix} 5/2 & 3/2 & 0 & 0 \\ -1 & 1/2 & 0 & 0 \\ 0 & 0 & 5/4 & 1/4 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} \vec{x}.$$

Find the $\mathfrak{B}$-matrix of T, that is, find a matrix $A \in \mathbb{R}^{4\times 4}$ so that

$$[T(\vec{x})]_{\mathfrak{B}} = A[\vec{x}]_{\mathfrak{B}}$$

for all $\vec{x} \in \mathbb{R}^4$.

Solution: Since T is a linear transformation from $\mathbb{R}^4 \to \mathbb{R}^4$, then there exists some $A \in \mathbb{R}^{4\times 4}$ so that $[T(\vec{x})]_{\mathfrak{B}} = A[\vec{x}]_{\mathfrak{B}}$. This matrix A can be found by taking $P_{\mathfrak{C}\to\mathfrak{B}} M P_{\mathfrak{B}\to\mathfrak{C}}$, where M is the matrix where $T(\vec{x}) = M\vec{x}$. So $P_{\mathfrak{B}\to\mathfrak{C}}$ and $P_{\mathfrak{C}\to\mathfrak{B}}$ must be found.

$$P_{\mathfrak{B}\to\mathfrak{C}} = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \text{ and } P_{\mathfrak{B}\to\mathfrak{C}} = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}^{-1}$$

So then we must find $P_{\mathfrak{B}\to\mathfrak{C}}$ by inverting $P_{\mathfrak{B}\to\mathfrak{C}}$.

$$\left[\begin{array}{cccc|cccc} 2 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 - R_1 \to R_2 \\ R_3 - R_4 \to R_3 \end{array} \quad \left[\begin{array}{cccc|cccc} 2 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \frac{1}{2}R_1 \to R_1 \\ -\frac{1}{2}R_2 \to R_2 \\ R_4 + 2R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & -1 \end{array}\right] \quad \begin{array}{l} R_1 - \frac{3}{2}R_2 \to R_1 \\ -R_3 \to R_3 \end{array} \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & -1 \end{array}\right]$$

So now, we can find A by using the formula from before.

$$\begin{aligned} A = P_{\mathfrak{C}\to\mathfrak{B}} M P_{\mathfrak{B}\to\mathfrak{C}} &= \begin{bmatrix} -1/4 & 3/4 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 5/2 & 3/2 & 0 & 0 \\ -1 & 1/2 & 0 & 0 \\ 0 & 0 & 5/4 & 1/4 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -11/8 & 0 & 0 & 0 \\ 7/4 & 1/2 & 0 & 0 \\ 0 & 0 & -7/4 & 1/4 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \\ A &= \begin{bmatrix} -11/4 & -33/8 & 0 & 0 \\ 9/2 & 23/4 & 0 & 0 \\ 0 & 0 & -5/4 & -3/2 \\ 0 & 0 & 3 & 3 \end{bmatrix} \end{aligned}$$

Question 5

Let $T : \mathbb{P}_3 \to \mathbb{R}^{2\times 2}$ be defined by

$$T\left(a + bx + cx^2\right) = \begin{bmatrix} b & 2a \\ 2c & b \end{bmatrix}$$

and let $\mathfrak{B}$ be the basis for $\mathbb{P}_3$ and $\mathfrak{D}$ the basis for $\mathbb{R}^{2\times 2}$:

$$\mathfrak{B} = \left\langle 1, x, x^2 \right\rangle \quad \text{and} \quad \mathfrak{D} = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

Find a matrix A so that

$$[T(p)]_{\mathfrak{D}} = A[p]_{\mathfrak{B}}$$

for all $p \in \mathbb{P}_3$.

Solution: Since T is a linear transformation from $\mathbb{P}_3 \to \mathbb{R}^{2\times 2}$, and we are given a basis $\mathfrak{B}$ for $\mathbb{P}_3$ and $\mathfrak{D} for \mathbb{R}^{2\times 2}$, then there exists some matrix A such that $[T(p)]_{\mathfrak{D}} = A[p]_{\mathfrak{B}}$. The matrix can be constructed, first by computing the transformed elements in the ordered basis $\mathfrak{B}$, see:

$$T(1) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \implies \left[\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\right]_{\mathfrak{D}} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]_{\mathfrak{D}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(x^2) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \implies \left[\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right]_{\mathfrak{D}} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

The matrix can now be constructed taking the $\mathfrak{D}$-coordinate vectors that have been calculated as the columns to build a matrix so thus we have

$$A = \begin{bmatrix} | & | & & | \\ [T(b_1)]_{\mathfrak{D}} & [T(b_2)]_{\mathfrak{D}} & \dots & [T(b_n)]_{\mathfrak{D}} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Question 6

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \vec{x}$$

and $\mathfrak{E} = \left\langle \vec{e}_1, \vec{e}_2, \vec{e}_3 \right\rangle$ be the standard basis for $\mathbb{R}^3$. Let

$$\mathfrak{B} = \left\langle T\left(\vec{e}_1\right), T\left(\vec{e}_2\right), \vec{e}_2 \right\rangle$$

Find a matrix $A \in \mathbb{R}^{3\times 3}$ so that

$$[T(\vec{x})]_{\mathfrak{B}} = A[\vec{x}]_{\mathfrak{E}}.$$

Solution: To find the matrix A, firstly we can substitute some arbitrary vector into the fuction, take $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

Firstly we must find

$$T(\vec{v}) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + v_3 \\ v_1 + v_2 + 2v_3 \\ v_2 + v_3 \end{bmatrix}$$

Then when expressing this as the $\mathfrak{B}$-coordinate of the calculated vector, we find that

$$\left[\begin{bmatrix} v_1 + v_3 \\ v_1 + v_2 + 2v_3 \\ v_2 + v_3 \end{bmatrix}\right]_{\mathfrak{B}} = \begin{bmatrix} v_1 + v_3 \\ v_2 + v_3 \\ 0 \end{bmatrix}$$

We can conclude this because of the vectors contained in the orded basis. The result of the linear transformation suggests that we need at least $v_1 + v_3$ of the first element from the ordered basis. Further, we need $v_2 + v_3$ of the second element in the ordered basis because it is the only element which contains a value in the third component of the vector. Since we know what form the $\mathfrak{B}$-coordinates of the output of T will look like, we can generate a matrix.

$$A \begin{bmatrix} v_1 + v_2 \\ v_2 + v_3 \\ 0 \end{bmatrix} \implies A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$