Math 2700.009: Linear Algebra

Problem Set 14

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Question 1

Are the following matrices orthogonal?

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\vartheta) & -\sin(\vartheta) \\ 0 & 0 & \sin(\vartheta) & \cos(\vartheta) \end{bmatrix} \quad B = \begin{bmatrix} 1/\sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{bmatrix}$$

Where $\vartheta \in [0, 2\pi)$.

Note:-

Recall:
A matrix $A \in \mathbb{R}^{n\times n}$ is orthogonal if:

- The columns of A form an orthogonal basis for $\mathbb{R}^n$
- $A^\top A = I_n$
- $A^\top = A^{-1}$

Solution:
(a) To verify if A is orthogonal, we can check if $A^\top A = I_4$.

$$\begin{aligned} A^\top A &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\vartheta) & \sin(\vartheta) \\ 0 & 0 & -\sin(\vartheta) & \cos(\vartheta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\vartheta) & -\sin(\vartheta) \\ 0 & 0 & \sin(\vartheta) & \cos(\vartheta) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (\cos^2(\vartheta) + \sin^2(\vartheta)) & (\cos(\vartheta)\sin(\vartheta) - \cos(\vartheta)\sin(\vartheta)) \\ 0 & 0 & (\cos(\vartheta)\sin(\vartheta) - \cos(\vartheta)\sin(\vartheta)) & (\cos^2(\vartheta) + \sin^2(\vartheta)) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{By trigonometric identities and simplification} \\ &= I_4 \end{aligned}$$

Hence we can say A is orthogonal since $A^\top A = I_4$

(b) Similarly, we can check the orthagonality of B by finding $B^\top B$.

$$\begin{aligned} B^\top B &= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &\neq I_3 \end{aligned}$$

We can see that the product $B^\top B$ is not the identity matrix, so thus B is not orthogonal.

(c) Lastly, we can check the orthagonality of C by finding the product $C^\top C$ and checking if it is I_3.

$$C^\top C = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{bmatrix}$$

$$
\begin{aligned}
C^\top C &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= I_3
\end{aligned}
$$

Hence we can say C is orthogonal since $C^\top C = I_3$

Question 2

Verify that if $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then $Q^\top$ is an orthogonal matrix

Question 3

Let $T : C([0,1]) \to C([0,1])$ be defined by

$$T(f) = \sqrt{3}xf\left(x^3\right).$$

Verify that T is an orthogonal transformation where the inner-product on $C([0,1])$ is

$$\langle f \mid g \rangle = \int_0^1 f(x)g(x)\mathrm{d}x.$$

Question 4

Find the QR factorization of the following matrices.

(a)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

(c)

$$C = \begin{bmatrix} \cos(\vartheta) & -\sin(\vartheta) & 0 & 0 \\ \sin(\vartheta) & \cos(\vartheta) & 0 & 0 \\ 0 & 0 & \cos(\tau) & -\sin(\tau) \\ 0 & 0 & \sin(\tau) & \cos(\tau) \end{bmatrix}$$

where $\vartheta, \tau \in [0, 2\pi)$.