Math 2700.009: Linear Algebra

Problem Set 14

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Question 1

Are the following matrices orthogonal?

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\vartheta) & -\sin(\vartheta) \\ 0 & 0 & \sin(\vartheta) & \cos(\vartheta) \end{bmatrix} \quad B = \begin{bmatrix} 1/\sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{bmatrix}$$

Where $\vartheta \in [0, 2\pi)$.

Note:-

Recall:
A matrix $A \in \mathbb{R}^{n\times n}$ is orthogonal if:

- The columns of A form an orthogonal basis for $\mathbb{R}^n$
- $A^\top A = I_n$
- $A^\top = A^{-1}$

Solution:
(a) To verify if A is orthogonal, we can check if $A^\top A = I_4$.

$$\begin{aligned} A^\top A &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\vartheta) & \sin(\vartheta) \\ 0 & 0 & -\sin(\vartheta) & \cos(\vartheta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\vartheta) & -\sin(\vartheta) \\ 0 & 0 & \sin(\vartheta) & \cos(\vartheta) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (\cos^2(\vartheta) + \sin^2(\vartheta)) & (\cos(\vartheta)\sin(\vartheta) - \cos(\vartheta)\sin(\vartheta)) \\ 0 & 0 & (\cos(\vartheta)\sin(\vartheta) - \cos(\vartheta)\sin(\vartheta)) & (\cos^2(\vartheta) + \sin^2(\vartheta)) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{By trigonometric identities and simplification} \\ &= I_4 \end{aligned}$$

Hence we can say A is orthogonal since $A^\top A = I_4$

(b) Similarly, we can check the orthagonality of B by finding $B^\top B$.

$$\begin{aligned} B^\top B &= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &\neq I_3 \end{aligned}$$

We can see that the product $B^\top B$ is not the identity matrix, so thus B is not orthogonal.

(c) Lastly, we can check the orthagonality of C by finding the product $C^\top C$ and checking if it is I_3.

$$C^\top C = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{bmatrix}$$

$$
\begin{aligned}
C^\top C &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= I_3
\end{aligned}
$$

Hence we can say C is orthogonal since $C^\top C = I_3$

Question 2

Verify that if $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then $Q^\top$ is an orthogonal matrix

Solution: Since Q is orthogonal, then we can conclude that $Q^\top Q = I$. This suggests that $Q^\top$ is a left inverse to Q. If we would like to show that $Q^\top$ is orthogonal when Q is orthogonal, then we must show that

$$(Q^\top)^\top Q^\top = QQ^\top = I$$

To do so, we must show that $Q^\top$ is also a right inverse to Q.

Proof. Suppose Q has some right inverse M. Let us prove that if $Q^\top$ is a left inverse to Q that it is also a right inverse.

$$
\begin{aligned}
Q^\top = Q^\top I = Q^\top(QM) = (Q^\top Q)M = IM = M \\
Q^\top = M
\end{aligned}
$$

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The simple proof suggests that if Q has a left inverse $Q^\top$, then it is also a right inverse to Q. Hence we can say that $Q^\top$ is orthogonal since $QQ^\top = I$.

Question 3

Let $T : C([0, 1]) \longrightarrow C([0, 1])$ be defined by

$$T(f) = \sqrt{3}xf\left(x^3\right).$$

Verify that T is an orthogonal transformation where the inner-product on $C([0, 1])$ is

$$\langle f \mid g\rangle = \int_0^1 f(x)g(x)\mathrm{d}x.$$

Note:-

Recall:
If V is an inner-product space and $T : V \longrightarrow V$ is a linear transformation, then T is called orthogonal if for all $\vec{v} \in V$, $||\vec{v}|| = ||T(\vec{v})||$, in other words, a linear transformation on an inner-product space is orthogonal if it is length preserving.

Solution: Let us check if $T(f)$ is length preserving on $C([0, 1])$ with the given inner-product, first by checking the length of $f(x)$.

$$
\begin{aligned}
||f(x)|| &= \sqrt{\langle f(x)|f(x)\rangle} \\
&= \sqrt{\int_0^1 f(x)f(x)\mathrm{d}x} \\
&= \sqrt{\int_0^1 f(x)^2\mathrm{d}x}
\end{aligned}
$$

Now let us check the length of $T(f)$.

$$\begin{aligned}
||T(f)|| &= \sqrt{\langle T(f)|T(f)\rangle} \\
&= \sqrt{\langle \sqrt{3}xf(x^3)|\sqrt{3}xf(x^3)} \\
&= \sqrt{\int_0^1 (\sqrt{3}xf(x^3))^2 \mathrm{d}x} \\
&= \sqrt{\int_0^1 3x^2 f(x^3)^2 \mathrm{d}x} \qquad \begin{bmatrix} u = x^3 \\ \mathrm{d}u = 3x^2\mathrm{d}x \end{bmatrix} \\
&= \sqrt{\int_{0^3}^{1^3} f(u)^2 \mathrm{d}u} \\
&= \sqrt{\int_0^1 f(u)^2 \mathrm{d}u} \\
&= \sqrt{\int_0^1 f(x)^2 \mathrm{d}x} \\
&= ||f(x)||
\end{aligned}$$

Since we have demonstrated the equality of $||f(x)||$ and $||T(f)||$, then we have also demonstrated that T is length perserving on the inner-product space and hence T is orthogonal.

Question 4

Find the QR factorization of the following matrices.

(a)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

(c)

$$C = \begin{bmatrix} \cos(\vartheta) & -\sin(\vartheta) & 0 & 0 \\ \sin(\vartheta) & \cos(\vartheta) & 0 & 0 \\ 0 & 0 & \cos(\tau) & -\sin(\tau) \\ 0 & 0 & \sin(\tau) & \cos(\tau) \end{bmatrix}$$

where $\vartheta, \tau \in [0, 2\pi)$.

Solution:

(a) Let us call the columns of A as vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$. To find the QR factorization, we must first perform Gram-Schmidt, on the vectors.

$$
\begin{aligned}
\vec{y}_1 &= \frac{1}{||\vec{a}_1||}\vec{a}_1 \\
&= \frac{1}{\sqrt{1+4+9}}\vec{a}_1 \\
&= \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix} \\
&= \begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix}
\end{aligned}
$$

This also tells us that $\vec{a}_1 = (\sqrt{14})\vec{y}_1$. This will be important for when we construct R.

$$
\begin{aligned}
\vec{b}_2 &= \vec{a}_2 - (\vec{a}_2 \cdot \vec{y}_1)\vec{y}_1 \\
&= \vec{a}_2 - (1/\sqrt{14} + 3/\sqrt{14})\vec{y}_1 \\
&= \vec{a}_2 - (4/\sqrt{14})\vec{y}_1 \\
&= \begin{bmatrix}1\\0\\1\end{bmatrix} - 4/\sqrt{14}\begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix} \\
&= \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}4/14\\8/14\\12/14\end{bmatrix} \\
&= \begin{bmatrix}10/14\\-8/14\\2/14\end{bmatrix} = \begin{bmatrix}5/7\\-4/7\\1/7\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
||\vec{b}_2|| &= \sqrt{\frac{25}{49} + \frac{16}{49} + \frac{1}{49}} \\
&= \sqrt{\frac{42}{49}} \\
&= \frac{\sqrt{42}}{7}
\end{aligned}
$$

$$
\begin{aligned}
\vec{y}_2 &= \frac{1}{||\vec{b}_2||}\vec{b}_2 \\
&= 7/\sqrt{42}\begin{bmatrix}5/7\\-4/7\\1/7\end{bmatrix} \\
&= \begin{bmatrix}5/\sqrt{42}\\-4/\sqrt{42}\\1/\sqrt{42}\end{bmatrix}
\end{aligned}
$$

Similarly, we need to see the linear combination of $\vec{y}_i$'s that construct $\vec{a}_2$

$$
\begin{aligned}
\vec{b}_2 &= (\sqrt{42}/7)\vec{y}_2 \\
\vec{a}_2 - (4/\sqrt{14})\vec{y}_1 &= (\sqrt{42}/7)\vec{y}_2 \\
\vec{a}_2 &= (4/\sqrt{14})\vec{y}_1 + (\sqrt{42}/7)\vec{y}_2
\end{aligned}
$$

$$
\begin{aligned}
\vec{b}_3 &= \vec{a}_3 - (\vec{a}_3 \cdot \vec{y}_2)\vec{y}_2 - (\vec{a}_3 \cdot \vec{y}_1)\vec{y}_1 \\
&= \vec{a}_3 - (5/\sqrt{42} - 4/\sqrt{42} - 1/\sqrt{42})\vec{y}_2 \\
&\quad - (1/\sqrt{14} + 2/\sqrt{14} - 3/\sqrt{14})\vec{y}_1 \\
&= \vec{a}_3 - (0)\vec{y}_2 - (0)\vec{y}_1 \\
&= \vec{a}_3
\end{aligned}
$$

$$
\begin{aligned}
||\vec{b}_3|| &= \sqrt{1+1+1} \\
&= \sqrt{3}
\end{aligned}
$$

$$
\begin{aligned}
\vec{y}_3 &= \frac{1}{||\vec{b}_3||}\vec{b}_3 \\
&= 1/\sqrt{3}\begin{bmatrix}1\\1\\-1\end{bmatrix} \\
&= \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\-1/\sqrt{3}\end{bmatrix}
\end{aligned}
$$

Since $\vec{b}_3 = \vec{a}_3$, then we can quickly compute that $\vec{a}_3 = (\sqrt{3})\vec{y}_3$.

So now to construct the QR factorization of A, we will use the vectors that form the orthogonal basis as columns for Q, and the columns of R will be the linear combination of $\vec{y}_i$ vectors that construct our $\vec{a}_i$'s, so we see that

$$A = QR = \begin{bmatrix} 1/\sqrt{14} & 5/\sqrt{42} & 1/\sqrt{3} \\ 2/\sqrt{14} & -4/\sqrt{42} & 1/\sqrt{3} \\ 3/\sqrt{14} & 1/\sqrt{42} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{14} & 4/\sqrt{14} & 0 \\ 0 & \sqrt{42}/7 & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

(b) Like before, let us call the columns of B as vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ and then perform Gram-Schmidt on the vectors.

$$\begin{aligned} \vec{y}_1 &= \frac{1}{||\vec{a}_1||}\vec{a}_1 \\ &= \frac{1}{\sqrt{1+1+1}}\vec{a}_1 \\ &= 1/\sqrt{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \end{aligned}$$

We can see that $\vec{a}_1 = (\sqrt{3})\vec{y}_1$.

$$\begin{aligned} \vec{b}_2 &= \vec{a}_2 - (\vec{a}_2 \cdot \vec{y}_1)\vec{y}_1 \\ &= \vec{a}_2 - (2/\sqrt{3} - 1/\sqrt{3} - 1/\sqrt{3})\vec{y}_1 \\ &= \vec{a}_2 - (0)\vec{y}_1 \\ &= \vec{a}_2 \end{aligned}$$

$$\begin{aligned} ||\vec{b}_2|| &= \sqrt{4+1+1} \\ &= \sqrt{6} \end{aligned}$$

$$\begin{aligned} \vec{y}_2 &= \frac{1}{||\vec{b}_2||}\vec{b}_2 \\ &= 1/\sqrt{6}\begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \end{aligned}$$

So we can see that $\vec{a}_2 = (\sqrt{6})\vec{y}_2$

$$\begin{aligned} \vec{b}_3 &= \vec{a}_3 - (\vec{a}_3 \cdot \vec{y}_2)\vec{y}_2 - (\vec{a}_3 \cdot \vec{y}_1)\vec{y}_1 \\ &= \vec{a}_3 - (2/\sqrt{6} - 1/\sqrt{6})\vec{y}_2 - (1/\sqrt{3} + 1/\sqrt{3})\vec{y}_1 \\ &= \vec{a}_3 - (1/\sqrt{6})\vec{y}_2 - (2/\sqrt{3})\vec{y}_1 \\ &= \vec{a}_3 - (1/\sqrt{6})\begin{bmatrix} 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} - (2/\sqrt{3})\begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/6 \\ 0 \\ -1/6 \\ -1/6 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 0 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} ||\vec{b}_3|| &= \sqrt{1/4 + 1/4} \\ &= 1/2\sqrt{2} \end{aligned}$$

$$\begin{aligned} \vec{y}_3 &= \frac{1}{||\vec{b}_3||}\vec{b}_3 \\ &= 1/2\sqrt{2}\begin{bmatrix} 0 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix} \end{aligned}$$

We now need to construct $\vec{a}_3$ using a linear combination of $\vec{y}_i$'s

$$\begin{aligned} \vec{b}_3 &= (1/2\sqrt{2})\vec{y}_3 \\ \vec{a}_3 - (1/\sqrt{6})\vec{y}_2 - (2/\sqrt{3})\vec{y}_1 &= (1/2\sqrt{2})\vec{y}_3 \\ \vec{a}_3 &= (2/\sqrt{3})\vec{y}_1 + (1/\sqrt{6})\vec{y}_2 + (1/2\sqrt{2})\vec{y}_3 \end{aligned}$$

$$\begin{aligned} \vec{b}_4 &= \vec{a}_4 - (\vec{a}_4 \cdot \vec{y}_3)\vec{y} - (\vec{a}_4 \cdot \vec{y}_2)\vec{y}_2 - (\vec{a}_4 \cdot \vec{y}_1)\vec{y}_1 \\ &= \vec{a}_4 - (-\sqrt{2})\vec{y}_3 - (-1/\sqrt{6})\vec{y}_2 - (1/\sqrt{3})\vec{y}_1 \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + (\sqrt{2}) \begin{bmatrix} 0 \\ 0 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix} + (1/\sqrt{6}) \begin{bmatrix} 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \\ &\quad - (1/\sqrt{3}) \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2/6 \\ 0 \\ -1/6 \\ -1/6 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 0 \\ 1/3 \\ 1/3 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ -3/2 \\ 3/2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} ||\vec{b}_4|| &= \sqrt{1 + 9/4 + 9/4} \\ &= \sqrt{22/4} \\ &= \sqrt{22}/2 \end{aligned}$$

$$\begin{aligned} \vec{y}_4 &= \frac{1}{||\vec{b}_4||}\vec{b}_4 \\ &= 2/\sqrt{22} \begin{bmatrix} 0 \\ 1 \\ -3/2 \\ 3/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 2/\sqrt{22} \\ -3/\sqrt{22} \\ 3/\sqrt{22} \end{bmatrix} \end{aligned}$$

We now need to construct $\vec{a}_4$ using a linear combination of $\vec{y}_i$'s

$$\begin{aligned} \vec{b}_4 &= (\sqrt{22}/2)\vec{y}_4 \\ \vec{a}_4 + (\sqrt{2})\vec{y}_3 + (1/\sqrt{6})\vec{y}_2 - (1/\sqrt{3})\vec{y}_1 &= (\sqrt{22}/2)\vec{y}_4 \\ \vec{a}_4 &= (1/\sqrt{3})\vec{y}_1 - (1/\sqrt{6})\vec{y}_2 - (\sqrt{2})\vec{y}_3 + (\sqrt{22}/2)\vec{y}_4 \end{aligned}$$

So now to construct the QR factorization of B, we will use the vectors that form the orthogonal basis as columns for Q, and the columns of R will be the linear combination of $\vec{y}_i$ vectors that construct our $\vec{a}_i$'s, so we see that

$$B = QR = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2/\sqrt{22} \\ 1/\sqrt{3} & -1/\sqrt{6} & -\sqrt{2} & -3/\sqrt{22} \\ 1/\sqrt{3} & -1/\sqrt{6} & \sqrt{2} & 3\sqrt{22} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & \sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \\ 0 & 0 & 1/2\sqrt{2} & -\sqrt{2} \\ 0 & 0 & 0 & \sqrt{22}/2 \end{bmatrix}$$

(c) C is orthogonal, because it consists of only orthogonal matrices, as we saw from problem 1. Since C is orthogonal, then the columns of C also from an orthogonal basis for $\mathbb{R}^4$. If we consider $C = Q$, and $R = I$, then we have an orthogonal matrix Q and a triangular matrix R which when we find the product QR gives us C.

Question 5

Let $D \in \mathbb{R}^{n\times n}$ be the diagonal matrix:

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}.$$

For which values of the d_i is D an orthogonal matrix?

Solution: If D is orthogonal, then that means $D^\top D = I$. Since D is diagonal, then it is also symmetric and thus $D^\top = D$ so we have that $DD = I$ or $D^2 = I$. So then we can see that

$$D^\top D = D^2 = \begin{bmatrix} (d_1)^2 & & & \\ & (d_2)^2 & & \\ & & \ddots & \\ & & & (d_n)^2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

So thus we can see that each $(d_i)^2 = 1$ and hence we can conclude that the value of the d_i's must be -1 or 1.

Question 6

For the following matrix A, find:
(a) $A^\top A$
(b) All eigenvalues of $A^\top A$
(c) An eigenbasis for $A^\top A$
(d) All singular values of A
(e) An orthonormal eigenbasis for $A^\top A$
(f) A singular value decomposition of A

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Solution:
(a)

$$A^\top A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

(b) $\det(\lambda I_3 - A^\top A)$

$$\det \begin{bmatrix} \lambda - 5 & -2 & -4 \\ -2 & \lambda - 4 & 0 \\ -4 & 0 & \lambda - 4 \end{bmatrix}$$

If we perform a cofactor expansion about row 3

$$\begin{aligned} &= -4 \det \begin{bmatrix} -2 & -4 \\ \lambda - 4 & 0 \end{bmatrix} + (\lambda - 4) \det \begin{bmatrix} \lambda - 5 & -2 \\ -2 & \lambda - 4 \end{bmatrix} \\ &= -4(4(\lambda - 4)) + (\lambda - 4)((\lambda - 5)(\lambda - 4) - 4) \\ &= -16(\lambda - 4) + (\lambda - 4)(\lambda^2 - 9\lambda + 20 - 4) \\ &= (\lambda - 4)(\lambda^2 - 9\lambda) \\ &= (\lambda - 9)(\lambda - 4)(\lambda) \end{aligned}$$

We can see that $A^\top A$ has eigenvalues $9, 4, 0$.

(c)

$$9I - A^\top A = \begin{bmatrix} 4 & -2 & -4 \\ -2 & 5 & 0 \\ -4 & 0 & 5 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$$

$$4I - A^\top A = \begin{bmatrix} -1 & -2 & -4 \\ -2 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$0I - A^\top A = \begin{bmatrix} -5 & -2 & -4 \\ -2 & -4 & 0 \\ -4 & 0 & -4 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

From this then we can say that an eigenbases for $A^\top A$

$$\left\{ \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \right\}$$

(d) We can find the singular values σ_i's by taking the square root of each eigenvalue, we see that we have singular values 3, 2, 0

(e) Since an eigenbasis for a symmetric matrix is always orthogonal, then we need normalize each of the eigenbasis vectors to make it orthonormal.

$$||\vec{v}_1|| = \sqrt{25+4+16} = \sqrt{45} = 3\sqrt{5} \qquad \frac{1}{||\vec{v}_1||}\vec{v}_1 = 1/3\sqrt{5}\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5/3\sqrt{5} \\ 2/3\sqrt{5} \\ 4/3\sqrt{5} \end{bmatrix}$$

$$||\vec{v}_2|| = \sqrt{4+1} = \sqrt{5} \qquad \frac{1}{||\vec{v}_2||}\vec{v}_2 = 1/\sqrt{5}\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$||\vec{v}_3|| = \sqrt{4+1+4} = \sqrt{9} = 3 \qquad \frac{1}{||\vec{v}_3||}\vec{v}_3 = 1/3\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}$$

So we can say that an orthonormal eigenbasis for $A^\top A$ is

$$\left\{ \begin{bmatrix} 5/3\sqrt{5} \\ 2/3\sqrt{5} \\ 4/3\sqrt{5} \end{bmatrix}, \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} \right\}$$

(f) $A = u \sum V^\top$

$$V = \begin{bmatrix} 5/3\sqrt{5} & 0 & 2/3 \\ 2/3\sqrt{5} & -2/\sqrt{5} & -1/3 \\ 4/3\sqrt{5} & 1/\sqrt{5} & -2/3 \end{bmatrix}$$

We can construct $\sum$ using the singular values 3 and 2

$$\sum = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

In order to find u, then we must calculate them using the non-zero singular values.

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = 1/3\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 5/3\sqrt{5} \\ 2/3\sqrt{5} \\ 4/3\sqrt{5} \end{bmatrix} = 1/3\begin{bmatrix} 9/3\sqrt{5} \\ 18/3\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = 1/2 \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = 1/2 \begin{bmatrix} -4/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$u = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

So with these matrices that we have found, we can now construct A by taking the product

$$A = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 5/3\sqrt{5} & 2/3\sqrt{5} & 4/3\sqrt{5} \\ 0 & -2/\sqrt{5} & 1/\sqrt{5} \\ 2/3 & -1/3 & -2/3 \end{bmatrix}$$

Question 7

For the following matrix B, find:
(a) $B^\top B$
(b) All eigenvalues of $B^\top B$
(c) An eigenbasis for $B^\top B$
(d) All singular values of B
(e) An orthonormal eigenbasis for $B^\top B$
(f) A singular value decomposition of B.

$$B = \begin{bmatrix} 1 & 1 & 3\sqrt{7/15} \\ 1 & -1 & -\sqrt{7/15} \\ 0 & 0 & \sqrt{7/15} \\ 1 & 2 & -2\sqrt{7/15} \end{bmatrix}$$

Solution:
(a)

$$B^\top B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 2 \\ 3\sqrt{7/15} & -\sqrt{7/15} & \sqrt{7/15} & -2\sqrt{7/15} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3\sqrt{7/15} \\ 1 & -1 & -\sqrt{7/15} \\ 0 & 0 & \sqrt{7/15} \\ 1 & 2 & -2\sqrt{7/15} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

(b) $\det(\lambda I - B^\top B)$

$$\det \begin{bmatrix} \lambda - 3 & -2 & 0 \\ -2 & \lambda - 6 & 0 \\ 0 & 0 & \lambda - 7 \end{bmatrix}$$

We can perform cofactor expansion about the 3rd row

$$\begin{aligned} &= (\lambda - 7)((\lambda - 3)(\lambda - 6) - 4) \\ &= (\lambda - 7)(\lambda^2 - 9\lambda + 14) \\ &= (\lambda - 7)(\lambda - 7)(\lambda - 2) \end{aligned}$$

So then we can say $B^\top B$ has eigenvalues 7, 2.

(c)

$$7I - B^\top B = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$2I - B^\top B = \begin{bmatrix} -1 & -2 & 0 \\ -2 & -4 & 0 \\ 0 & 0 & -5 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

From this then we can say that an eigenbases for $B^\top B$

$$\left\{\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}2\\-1\\0\end{bmatrix}\right\}$$

(d) The singular values can are found by taking the square root of each eigenvalue, the singular values are $\sqrt{7}, \sqrt{7}, \sqrt{2}$

(e) The set of vectors is already orthogonal, so now we must normalize each vector.

$$||\vec{v}_1|| = \sqrt{1} = 1 \qquad \vec{v}_1 \text{ is already normalized}$$

$$||\vec{v}_2|| = \sqrt{1+4} = \sqrt{5} \qquad \frac{1}{||\vec{v}_2||}\vec{v}_2 = 1/\sqrt{5}\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix}$$

$$||\vec{v}_3|| = \sqrt{1+4} = \sqrt{5} \qquad \frac{1}{||\vec{v}_2||}\vec{v}_3 = 1/\sqrt{5}\begin{bmatrix}2\\-1\\0\end{bmatrix} = \begin{bmatrix}2/\sqrt{5}\\-1/\sqrt{5}\\0\end{bmatrix}$$

So then an orthonormal eigenbasis for $B^\top B$ is

$$\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix}\begin{bmatrix}2/\sqrt{5}\\-1/\sqrt{5}\\0\end{bmatrix}\right\}$$

(f) $B = u\Sigma V^\top$

$$V = \begin{bmatrix}0 & 1/\sqrt{5} & 2/\sqrt{5}\\0 & 2/\sqrt{5} & -1/\sqrt{5}\\1 & 0 & 0\end{bmatrix}$$

We can construct Σ using the singular values

$$\Sigma = \begin{bmatrix}\sqrt{7} & 0 & 0\\0 & \sqrt{7} & 0\\0 & 0 & \sqrt{2}\\0 & 0 & 0\end{bmatrix}$$

In order to find u, then we must calculate it using the non-zero singular values.

$$\vec{u}_1 = \frac{1}{\sigma_1}B\vec{v}_1 = 1/\sqrt{7}\begin{bmatrix}1 & 1 & 3\sqrt{7/15}\\1 & -1 & -\sqrt{7/15}\\0 & 0 & \sqrt{7/15}\\1 & 2 & -2\sqrt{7/15}\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = 1/\sqrt{7}\begin{bmatrix}3\sqrt{7/15}\\-\sqrt{7/15}\\\sqrt{7/15}\\-2\sqrt{7/15}\end{bmatrix} = \begin{bmatrix}3/\sqrt{15}\\-1/\sqrt{15}\\1/\sqrt{15}\\-2/\sqrt{15}\end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2}B\vec{v}_2 = 1/\sqrt{7}\begin{bmatrix}1 & 1 & 3\sqrt{7/15}\\1 & -1 & -\sqrt{7/15}\\0 & 0 & \sqrt{7/15}\\1 & 2 & -2\sqrt{7/15}\end{bmatrix}\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix} = 1/\sqrt{7}\begin{bmatrix}3/\sqrt{5}\\-1/\sqrt{5}\\0\\5/\sqrt{5}\end{bmatrix} = \begin{bmatrix}3/\sqrt{35}\\-1/\sqrt{35}\\0\\5/\sqrt{35}\end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\sigma_3}B\vec{v}_2 = 1/\sqrt{7}\begin{bmatrix}1 & 1 & 3\sqrt{7/15}\\1 & -1 & -\sqrt{7/15}\\0 & 0 & \sqrt{7/15}\\1 & 2 & -2\sqrt{7/15}\end{bmatrix}\begin{bmatrix}2/\sqrt{5}\\-1/\sqrt{5}\\0\end{bmatrix} = 1/\sqrt{2}\begin{bmatrix}1/\sqrt{5}\\3/\sqrt{5}\\0\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{10}\\3/\sqrt{10}\\0\\0\end{bmatrix}$$

But we need to extend our set of vectors since it needs to be a basis for $\mathbb{R}^4$. An obvious candidate is the standard basis vector $\vec{e}_3$, since it means we will have to do the least amount of work when we perform Gram-Schmidt

$$\begin{aligned}\vec{b}_4 &= \vec{a}_4 - (\vec{a}_4 \cdot \vec{u}_3)\vec{u}_3 - (\vec{a}_4 \cdot \vec{u}_2)\vec{u}_2 - (\vec{a}_4 \cdot \vec{u}_1)\vec{u}_1 \\ &= \vec{a}_4 - 0\vec{u}_3 - 0\vec{u}_2 - 1/\sqrt{5}\vec{u}_1 \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 3/15 \\ -1/15 \\ 1/15 \\ -2/15 \end{bmatrix} = \begin{bmatrix} -3/15 \\ 1/15 \\ 14/15 \\ 2/15 \end{bmatrix}\end{aligned}$$

Now to normalize $\vec{b}_4$.

$$||\vec{b}_4|| = \sqrt{\frac{9}{15} + \frac{196}{225} + \frac{1}{225} + \frac{4}{225}} = \sqrt{210/225} = \sqrt{14/15} \qquad \vec{u}_4 = \frac{1}{||\vec{b}_4||}\vec{b}_4 = \sqrt{15/14}\begin{bmatrix} -3/15 \\ 1/15 \\ 14/15 \\ 2/15 \end{bmatrix} = \begin{bmatrix} -\sqrt{3/70} \\ 1/\sqrt{210} \\ \sqrt{14/15} \\ \sqrt{2/105} \end{bmatrix}$$

$$u = \begin{bmatrix} 3/\sqrt{15} & 3/\sqrt{35} & 1/\sqrt{10} & -\sqrt{3/70} \\ -1/\sqrt{15} & -1/\sqrt{35} & 3/\sqrt{10} & 1/\sqrt{210} \\ 1/\sqrt{15} & 0 & 0 & \sqrt{14/15} \\ -2/\sqrt{15} & 5/\sqrt{35} & 0 & \sqrt{2/105} \end{bmatrix}$$

Since we have found the matrices required for the singular value decomposition of B, then we can say

$$B = \begin{bmatrix} 3/\sqrt{15} & 3/\sqrt{35} & 1/\sqrt{10} & -\sqrt{3/70} \\ -1/\sqrt{15} & -1/\sqrt{35} & 3/\sqrt{10} & 1/\sqrt{210} \\ 1/\sqrt{15} & 0 & 0 & \sqrt{14/15} \\ -2/\sqrt{15} & 5/\sqrt{35} & 0 & \sqrt{2/105} \end{bmatrix}\begin{bmatrix} \sqrt{7} & 0 & 0 \\ 0 & \sqrt{7} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 2/\sqrt{5} & -1/\sqrt{5} & 0 \end{bmatrix}$$

Question 8

Consider $C([-\pi, \pi])$ with the inner-product

$$\langle f \mid g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\mathrm{d}x.$$

For each integer $n \geqslant 1$ let W_n be the subspace of $C([-\pi, \pi])$

$$W_n = \operatorname{span}(1, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x), \dots, \sin(nx), \cos(nx)).$$

Take for granted (you pretty much verified this on the last problem set) that

$$\mathfrak{B}_n = \{1/2, \sin(x), \cos(x), \sin(2x), \cos(2x), \dots, \sin(nx), \cos(nx)\}$$

is an orthonormal basis for W_n. Then the nth Fourier approximation of $f \in C([-\pi, \pi])$ is:

$$\operatorname{proj}_{W_n}(f) = \langle 1/2 \mid f \rangle + \sum_{k=1}^{n} \langle \sin(kx) \mid f \rangle \sin(kx) + \langle \cos(kx) \mid f \rangle \cos(kx)$$

and the Fourier series of f[1] is:

$$\lim_{n\to\infty} \operatorname{proj}_{W_n}(f) = \langle 1/2 \mid f \rangle + \sum_{k=1}^{\infty} \langle \sin(kx) \mid f \rangle \sin(kx) + \langle \cos(kx) \mid f \rangle \cos(kx).$$

(a) Compute the Fourier series of x^2.
(b) Compute the Fourier series of x^3.

Solution:

(a)

$$
\begin{aligned}
\mathrm{proj}_{W_n}(x^2) &= \langle 1/2|x^2\rangle + \sum_{k=1}^{n} \langle \sin(kx)|x^2\rangle \sin(kx) + \langle \cos(kx)|x^2\rangle \cos(kx) \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2/2\mathrm{d}x + \dots \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\mathrm{d}x + \dots \\
&= \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{x=-\pi}^{\pi} + \dots \\
&= \frac{1}{2\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right] \\
&= \frac{\pi^2}{3} + \sum_{k=1}^{n} \langle \sin(kx)|x^2\rangle \sin(kx) + \langle \cos(kx)|x^2\rangle \cos(kx) \\
&= \frac{\pi^2}{3} + \sum_{k=1}^{n} \langle \cos(kx)|x^2\rangle \cos(kx) \\
&= \frac{\pi^2}{3} + \sum_{k=1}^{n} \left(\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(kx)\right)\cos(kx) \\
&= \frac{\pi^2}{3} + \sum_{k=1}^{n} \frac{1}{\pi}\left(\left[\frac{x^2}{k}\sin(kx) + \frac{2x}{k^2}\cos(kx) - \frac{2}{k^3}\sin(kx)\right]_{x=-\pi}^{\pi}\right)\cos(kx) \\
&= \frac{\pi^2}{3} + \sum_{k=1}^{n} \frac{1}{\pi}\left(\frac{2\pi}{k^2}\cos(k\pi) - \frac{2(-\pi)}{k^2}\cos(k\pi)\right)\cos(kx) \\
&= \frac{\pi^2}{3} + \sum_{k=1}^{n} \frac{1}{\pi}\left(\frac{4\pi}{k^2}\cos(k\pi)\right)\cos(kx) \\
&= \frac{\pi^2}{3} + \sum_{k=1}^{n} \frac{1}{\pi}\left(\frac{4(-1)^k}{k^2}\right)\cos(kx)
\end{aligned}
$$

(b)