Math 2700.009
Notes

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Note:-

- If V is a vector space, $\mathcal{B}$ V is a basis if it is both linearly independent and $span(\mathcal{B}) = V$.
- $dim(V)$ is the size of a basis for V.

Definition 0.0.1

If W is a subspace of V then $dim(W)$ is the size of a linearly independent set of vectors from which span W.

Definition 0.0.2

Vector space V is finite dimensional if it has a finite basis, and otherwise V is infinite dimensional

0.1 Rules of Basis'

- If $\mathcal{E} \subseteq V$ is a set of fewer than $dim(V)$ vectors, then $\mathcal{E}$ does not span V.
- If $\mathcal{E}$ has more vectors than $dim(V)$, it cannot be linearly independent. Otherwise V has a basis of size $\geq dim(V)$
- If $\mathcal{B}$ has $dim(V)$ vectors and is linearly independent, then it also spans.

Example 0.1.1

Show

$$\left\{\begin{pmatrix} x \\ x+y \\ y \end{pmatrix} : x, y \in \mathbb{R}\right\} \text{ is a subspace}$$

Solution:

$$= \left\{x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : x, y \in \mathbb{R}\right\}$$

$$= \text{span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$$

0.2 Linear Transformation

Definition 0.2.1

If V, W are vector spaces, a linear transformation from V to W is a function $T : V \longrightarrow W$ so that:
1) if $\vec{x}, \vec{y} \in V$ then $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$
2) if $\vec{x} \in V$ and $c \in \mathbb{R}$ then $T(c\vec{x}) = cT(\vec{x})$

Example 0.2.1

$$T : \mathbb{R} \longrightarrow \mathbb{R}$$

Example 0.2.2

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^k$$

$$T(\vec{x}) = A\vec{x} \text{ Where } A \in \mathbb{R}^{k\times n}$$

Example 0.2.3

$$T : C([0,1]) \to C([0,1])$$

Solution: $T(f) = \int f(x)dx$ where the constant of integration is zero

Note:-

The set of functions on [0, 1] which are differentiable on (0, 1) and continuous at the endpoints and whose derivative is continous is called $C^1([0,1])$

Proposition 0.2.1

If $T : V \to W$ is a linear transofrmation, then

$$T(\vec{0}^V) = \vec{0}^W$$

Proof: Prop 0.2.1
$T(\vec{0}^V) = T(\vec{0}^V + \vec{0}^V) = T(\vec{0}^V) + T(\vec{0}^V)$
So $T(\vec{0}^V) - T(\vec{0}^V) = T(\vec{0}^V) + T(\vec{0}^V) - T(\vec{0}^V)$
And thus $\vec{0}^W = \vec{0}^V$ ☺

Definition 0.2.2

If $T : V \to W$ is a linear transformation:
- $ker(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}$
- $ran(T) = \{\vec{w} \in W : \vec{w} = T(\vec{v})$ for some $\vec{v} \in V\}$

Note:-

$$T : \mathbb{R}^n \to \mathbb{R}^k$$

$$T(\vec{x}) = A\vec{x}$$

We sometimes write $ker(A)$ for $ker(T)$ and $ran(A)$ for $ran(T)$

Proposition 0.2.2

$T : V \to W$ is a linear transformation then $ker(T)$ is a subspace of V, and $ran(T)$ is a subspce of W

Proof: Prop 0.2.2
- $\vec{0}^V \in ker(T)$ because $T(\vec{0}^V) = \vec{0}^W$
- If $\vec{x}, \vec{y} \in ker(T)$, then $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}) = \vec{0}^W + \vec{0}^W = \vec{0}^W$ so $\vec{x} + \vec{y} \in ker(T)$
- If $\vec{x} \in ker(T)$ and $c \in \mathbb{R}$, $T(c\vec{x}) = cT(\vec{x}) = cT(\vec{x}) = c\vec{0}^W = \vec{0}^W$ so $\vec{x} \in ker(T)$ ☺

Definition 0.2.3: nullity

If $T : V \to W$ is a linear transformation, the nullity of T is $dim(ker(T))$ and the rank of T ($rank(T)$) is $dim(ran(T))$