Math 2700.009
Problem Set 13

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Question 1

Consider $C([-\pi,\pi])$ with the inner-product

$$\langle f \mid g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)\mathrm{d}x$$

and $S \subseteq C([-\pi,\pi])$ be

$$S = \{\sin(nx), \cos(mx) : m, n > 0\},$$

verify that S is an orthogonal set in $C([-\pi,\pi])$.

Note:-

Recall:

- To show a set S is an orthogonal set, we must show that for all $\vec{v},\vec{w} \in S$ the inner-product $\langle \vec{v} \mid \vec{w}\rangle = 0$, with $\vec{0} \notin S$

Solution: Let us begin by finding the inner product $\langle \sin(nx) \mid \cos(mx)\rangle$ when $m \neq n$:

$$\begin{aligned}
\langle \sin(nx) \mid \cos(mx)\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(nx+mx)+\sin(nx-mx)}{2}\mathrm{d}x \\
&= \frac{1}{4\pi}\left(\int_{-\pi}^{\pi}\sin(nx+mx)\mathrm{d}x + \int_{-\pi}^{\pi}\sin(nx-mx)\mathrm{d}x\right) \\
&= \frac{1}{4\pi}\left(-\frac{1}{n+m}\cos(nx+mx) - \frac{1}{n-m}\cos(nx-mx)\right)\Bigg|_{x=-\pi}^{\pi} \\
&= \begin{aligned}[t]\frac{1}{4\pi}\bigg(&-\frac{1}{n+m}\cos(n\pi+m\pi) - \frac{1}{n-m}\cos(n\pi-m\pi) \\
&+ \frac{1}{n+m}\cos(n\pi+m\pi) + \frac{1}{n-m}\cos(n\pi-m\pi)\bigg)\end{aligned} \\
&= \frac{1}{4\pi}(0) \\
&= 0
\end{aligned}$$

This "proof" only reveals if the vectors are orthogonal in instances where $m \neq n$, so let us review the case where $m = n$ by computing the inner-product $\langle \sin(nx) \mid \cos(nx)\rangle$

$$\begin{aligned}
\langle \sin(nx) \mid \cos(nx)\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(nx)\cos(nx)\mathrm{d}x \qquad \left[\begin{aligned} \text{Let } u &= \sin(nx) \\ \mathrm{d}u &= \frac{1}{n}\cos(nx)\mathrm{d}x \end{aligned}\right] \\
&= \frac{1}{2\pi}\int_{-n\pi}^{n\pi} u\frac{\mathrm{d}u}{n} \\
&= \frac{1}{2\pi n}\int_{-n\pi}^{n\pi} u\mathrm{d}u \\
&= \frac{u^2}{4\pi n}\bigg|_{u=-\pi n}^{\pi n} \\
&= \frac{\pi^2 n^2}{4\pi n} - \frac{\pi^2 n^2}{4\pi n} \\
&= 0
\end{aligned}$$

Since we have shown that the inner product $\langle \sin(nx) \mid \cos(mx)\rangle$ is zero regardless of the value of m and n, then we can say that S is an orthogonal set.

Question 2

Let V be an inner-product space with inner-product $\langle \cdot \mid \cdot \rangle$, and $\vec{u}_1, \ldots, \vec{u}_n \in V$. Define a function $T : V \longrightarrow V$ by

$$T(\vec{v}) = \sum_{k=1}^{n} \left\langle \vec{v} \mid \vec{u}_k \right\rangle \vec{u}_k$$

Verify that T is a linear transformation.

Note:-

Recall that if $T : V \longrightarrow V$ is a linear transformation, two properties must be verifiable:

- Additivity: A function $T : V \longrightarrow V$ is additive if for any $\vec{x}, \vec{y} \in V$, $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$
- Homogeneity: A function $T : V \longrightarrow V$ is homogeneous if for any $c \in \mathbb{R}$ and $\vec{x} \in V$, $T(c\vec{x}) = cT(\vec{x})$

Solution:

- Additivity: Let $\vec{w} \in V$

$$\begin{aligned} T(\vec{v} + \vec{w}) &= \sum_{k=1}^{n} \left\langle \vec{v} + \vec{w} \mid \vec{u}_k \right\rangle \vec{u}_k \\ &= \sum_{k=1}^{n} \left(\left\langle \vec{v} \mid \vec{u}_k \right\rangle + \left\langle \vec{w} \mid \vec{u}_k \right\rangle \right) \vec{u}_k \\ &= \sum_{k=1}^{n} \left\langle \vec{v} \mid \vec{u}_k \right\rangle \vec{u}_k + \left\langle \vec{w} \mid \vec{u}_k \right\rangle \vec{u}_k \\ &= \sum_{k=1}^{n} \left\langle \vec{v} \mid \vec{u}_k \right\rangle \vec{u}_k + \sum_{k=1}^{n} \left\langle \vec{w} \mid \vec{u}_k \right\rangle \vec{u}_k \\ &= T(\vec{v}) + T(\vec{w}) \end{aligned}$$

  Since T is additive for any $\vec{v}, \vec{w} \in V$, then it satisfies the first propery.

- Homogeneity: Let $c \in \mathbb{R}$

$$\begin{aligned} T(c\vec{v}) &= \sum_{k=1}^{n} \left\langle c\vec{v} \mid \vec{u}_k \right\rangle \vec{u}_k \\ &= \sum_{k=1}^{n} c \left\langle \vec{v} \mid \vec{u}_k \right\rangle \vec{u}_k \\ &= c \sum_{k=1}^{n} \left\langle \vec{v} \mid \vec{u}_k \right\rangle \vec{u}_k \\ &= cT(\vec{v}) \end{aligned}$$

Because T is both additive and homogeneous, then it is sufficient to say that it is a linear transformation.

Question 3

Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

(a) Find an ordered basis for ran(A)
(b) Perform the Gram-Schmidt algorithm on the basis you got in (a) to get an orthonormal basis for ran(A)
(c) Find a basis for ker(A)
(d) Extend the basis you found in (a) to a basis $\mathfrak{B}$ for $\mathbb{R}^3$
(e) Find $P_{\mathfrak{E}\to\mathfrak{B}}$.

Note:-

Recall:

Let $\langle \vec{x}_1, \ldots, \vec{x}_n \rangle$ be a basis for an n-dimensional inner-product space V. The Gram-Schmidt Algorithm:

$$\vec{v}_i = \vec{x}_i - \sum_{k=1}^{i-1} \langle \vec{x}_i \mid \vec{y}_k \rangle \vec{y}_k, \qquad \vec{y}_i = \vec{v}_i / ||\vec{v}_i||$$

Is used to find an orthogonal basis for V by subtracting all the pieces of $\vec{x}_i$ that are in the same direction as all of the $\vec{y}_k$, then renormalizing. This formula computes normalized vectors that are orthogonal to all other $\vec{y}_k$'s since the non-perpendicular pieces are removed.

Solution:

(a) We can see that $\vec{a}_1 = \vec{a}_3 - \vec{a}_2$ and thus $\vec{0} = -\vec{a}_1 - \vec{a}_2 + \vec{a}_3$. Since $\vec{a}_1$ is the only linearly dependent vector, then we can say that an ordered basis for the ran(A) is:

$$\left\langle \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\rangle$$

(b) We will apply the Gram-Schmidt formula to find the orthonormal basis vectors

$$\vec{b}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$||\vec{b}_1|| = \sqrt{\vec{b}_1 \cdot \vec{b}_1} = \sqrt{1 + 0 + 4} = \sqrt{5} \implies \frac{1}{||\vec{b}_1||} = \frac{1}{\sqrt{5}}$$

$$\vec{y}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}$$

$$\begin{aligned} \vec{b}_2 &= \vec{x}_2 - (\vec{y}_1 \cdot \vec{x}_2)\vec{y}_1 \\ &= \vec{x}_2 - (2/\sqrt{5} + 0 + 4/\sqrt{5})\vec{y}_1 \\ &= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - 6/\sqrt{5} \begin{bmatrix} 1\sqrt{5} \\ 0 \\ 2\sqrt{5} \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 6/5 \\ 0 \\ 12/5 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 1 \\ -2/5 \end{bmatrix} \end{aligned}$$

$$||\vec{b}_2|| = \sqrt{\vec{b}_2 \cdot \vec{b}_2} = \sqrt{16/25 + 1 + 4/25} = \sqrt{45/25} = \sqrt{9/5} = 3/\sqrt{5}$$

$$\implies \frac{1}{||\vec{b}_2||} = \frac{\sqrt{5}}{3}$$

$$\vec{y}_2 = \frac{\sqrt{5}}{3}\begin{bmatrix} 4/5 \\ 1 \\ -2/5 \end{bmatrix} = \begin{bmatrix} 4\sqrt{5}/15 \\ \sqrt{5}/3 \\ -2\sqrt{5}/15 \end{bmatrix}$$

The vectors we have found form an orthonormal basis for ran(A), which is

$$\left\langle \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 4\sqrt{5}/15 \\ \sqrt{5}/3 \\ -2\sqrt{5}/15 \end{bmatrix} \right\rangle$$

(c) To find a basis for the ker(A), we can use the linearly dependent vector we found in (a). In particular we see that $\vec{0} = -\vec{a}_1 - \vec{a}_2 + \vec{a}_3$. Rank-Nullity tells us that the dimension of the kernel will be one since the dimension of the range is 2. With this information we can say that a basis for ker(A) is

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(d) We can extend ordered basis we found in (a) to make a basis for $\mathbb{R}^3$ by using the standard basis vector $\vec{e}_3$. We find that

$$\mathfrak{B} = \left\langle \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

(e) In order to find $P_{\mathfrak{E}\to\mathfrak{B}}$ we need to find $P_{\mathfrak{B}\to\mathfrak{E}}$ and then find its inverse.

$$P_{\mathfrak{B}\to\mathfrak{E}} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

Now we can find $(P_{\mathfrak{B}\to\mathfrak{E}})^-1 = P_{\mathfrak{E}\to\mathfrak{B}}$ by using the row reduction method

$$\left.\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right. \quad R_1 - 2R_2 \to R_1 \quad \left.\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right. \quad R_3 - 2R_1 \to R_3$$

$$\left.\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & -2 & 4 & 1 \end{array}\right. \quad R_3 - 2R_2 \to R_3 \quad \left.\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right.$$

$$\implies P_{\mathfrak{E}\to\mathfrak{B}} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

Question 4

Perform the Gram-Schmidt method on $\left\langle \sin(x), \cos(x), x, x^2 \right\rangle$ in the inner product space $C([-\pi, \pi])$ with the inner-product from problem 1.

Solution:

$$\vec{b}_1 = \sin(x) \qquad ||\vec{b}_1|| = \sqrt{\langle \vec{b}_1 \mid \vec{b}_1 \rangle}$$

$$\begin{aligned}
\langle \sin(x) \mid \sin(x) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2(x) \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(2x)}{2} \mathrm{d}x \qquad \text{By half angle formula} \\
&= \frac{1}{2\pi} \left(\frac{1}{2} \int_{-\pi}^{\pi} 1 - \cos(2x) \mathrm{d}x \right) \\
&= \frac{1}{2\pi} \left[\frac{1}{2} \left(\int_{-\pi}^{\pi} 1 \mathrm{d}x - \int_{-\pi}^{\pi} \cos(2x) \mathrm{d}x \right) \right] \\
&= \frac{1}{2\pi} \left[\frac{1}{2} \left(x - \frac{1}{2} \sin(2x) \right) \right] \Bigg|_{x=-\pi}^{\pi} \\
&= \frac{1}{2\pi} \left[\frac{1}{2} \left(\pi - \frac{1}{2} \sin(2\pi) + \pi + \frac{1}{2} \sin(2\pi) \right) \right] \\
&= \frac{1}{2\pi} \left(\frac{1}{2} (2\pi) \right) \\
&= \frac{1}{2\pi} (\pi) \implies \int_{-\pi}^{\pi} \sin^2(x) \mathrm{d}x = \pi \qquad (1) \\
&= \frac{1}{2}
\end{aligned}$$

We can say that $||\vec{b}_1|| = 1/\sqrt{2}$, so if we normalize $\sin(x)$ in this inner-product space we get $\sqrt{2}\sin(x)$. This will serve as the first orthonormal basis vector.

$$\vec{b}_2 = \cos(x) - \langle \cos(x) \mid \sqrt{2}\sin(x) \rangle \sqrt{2}\sin(x)$$

$$\begin{aligned}
\langle \cos(x) \mid \sqrt{2}\sin(x) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(x)\sqrt{2}\sin(x) \mathrm{d}x \\
&= \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} \cos(x)\sin(x) \mathrm{d}x
\end{aligned}$$

Since $\cos(x)\sin(x)$ is odd, then the integral evaluates to zero. Thus $\vec{b}_2$ is just $\cos(x)$.

$$\begin{aligned}
\langle \cos(x) \mid \cos(x) \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2(x) \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + \cos(2x)}{2} \mathrm{d}x \qquad \text{By half angle formula} \\
&= \frac{1}{2\pi} \left(\frac{1}{2} \int_{-\pi}^{\pi} 1 + \cos(2x) \mathrm{d}x \right) \\
&= \frac{1}{2\pi} \left[\frac{1}{2} \left(\int_{-\pi}^{\pi} 1 \mathrm{d}x + \int_{-\pi}^{\pi} \cos(2x) \mathrm{d}x \right) \right] \\
&= \frac{1}{2\pi} \left[\frac{1}{2} \left(x + \frac{1}{2} \sin(2x) \right) \right] \Bigg|_{x=-\pi}^{\pi}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\left[\frac{1}{2}\left(\pi + \frac{1}{2}\sin(2\pi) + \pi + \frac{1}{2}\sin(2\pi)\right)\right] \\
&= \frac{1}{2\pi}\left(\frac{1}{2}(2\pi)\right) \\
&= \frac{1}{2\pi}(\pi) \implies \int_{-\pi}^{\pi} \cos^2(x)\mathrm{d}x = \pi \qquad (2) \\
&= \frac{1}{2}
\end{aligned}
$$

This means that the length of $\vec{b}_2$ is $1/\sqrt{2}$, and after normalizing $\cos(x)$, we get the second orthonormal basis vector $\sqrt{2}\cos(x)$.

$$\vec{b}_3 = x - \langle x \mid \sqrt{2}\sin(x)\rangle\sqrt{2}\sin(x) - \langle x \mid \sqrt{2}\cos(x)\rangle\sqrt{2}\cos(x)$$

$$
\begin{aligned}
\langle x \mid \sqrt{2}\sin(x)\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sqrt{2}\sin(x)\mathrm{d}x \\
&= \frac{\sqrt{2}}{2\pi}\int_{-\pi}^{\pi} x\sin(x)\mathrm{d}x
\end{aligned}
$$

We can evaluate this integral using tabular integration

$$
\begin{array}{ccc}
d & & I \\
x & & \sin x \\
 & \searrow^{+} & \\
1 & & -\cos x \\
 & \searrow^{-} & \\
0 & & -\sin x
\end{array}
$$

$$
\begin{aligned}
&= \frac{\sqrt{2}}{2\pi}(\sin(x) - x\cos(x))\Big|_{x=-\pi}^{\pi} \\
&= \frac{\sqrt{2}}{2\pi}(\sin(\pi) - \pi\cos(\pi) + \sin(\pi) - \pi\cos(-\pi)) \\
&= \frac{\sqrt{2}}{2\pi}(2\pi) \implies \int_{-\pi}^{\pi} x\sin(x)\mathrm{d}x = 2\pi \qquad (3) \\
&= \sqrt{2}
\end{aligned}
$$

$$
\begin{aligned}
\langle x \mid \sqrt{2}\cos(x)\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sqrt{2}\cos(x)\mathrm{d}x \\
&= \frac{\sqrt{2}}{2\pi}\int_{-\pi}^{\pi} x\cos(x)\mathrm{d}x
\end{aligned}
$$

Similarly, we can use tabular integration to compute this integral.

$$
\begin{array}{ccc}
d & & I \\
x & & \cos x \\
 & \searrow^{+} & \\
1 & & \sin x \\
 & \searrow^{-} & \\
0 & & -\cos x
\end{array}
$$

$$
\begin{aligned}
&= \frac{\sqrt{2}}{2\pi}(x\sin(x) + \cos(x))\Big|_{x=-\pi}^{\pi} \\
&= \frac{\sqrt{2}}{2\pi}(\pi\sin(\pi) + \cos(\pi) - \pi\sin(-\pi) - \cos(-\pi))
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sqrt{2}}{2\pi}(0) \\
&= 0
\end{aligned}
$$

Thus $\vec{b}_3 = x - \sqrt{2}\sqrt{2}\sin(x) = x - 2\sin(x)$. Now we must normalize it by finding its length and dividing by it.

$$
\begin{aligned}
\langle x - 2\sin(x) \mid x - 2\sin(x)\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(x - 2\sin(x))^2 \mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 - 4x\sin(x) + 4\sin^2(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} x^2\mathrm{d}x - 4\int_{-\pi}^{\pi} x\sin(x)\mathrm{d}x + 4\int_{-\pi}^{\pi}\sin^2(x)\mathrm{d}x\right) \\
&= \frac{1}{2\pi}\left(\frac{x^3}{3}\bigg|_{x=-\pi}^{\pi} - 4(2\pi) + 4(\pi)\right) \qquad \text{By (1) and (2)} \\
&= \frac{1}{2\pi}\left(\frac{2\pi^3}{3} - 4\pi\right) \\
&= \frac{\pi^2}{3} - 2 = \frac{\pi^2 - 6}{3}
\end{aligned}
$$

This means that $||\vec{b}_3|| = \sqrt{\frac{\pi^2-6}{3}}$, and normalizing our $\vec{b}_3$ we see that our third orthonormal basis vector is $\sqrt{\frac{3}{\pi^2-6}}(x - 2\sin(x))$.

$$
\vec{b}_4 = x^2 - \langle x^2 \mid \sqrt{2}\sin(x)\rangle\sqrt{2}\sin(x) - \langle x^2 \mid \sqrt{2}\cos(x)\rangle\sqrt{2}\cos(x) - \langle x^2 \mid \sqrt{\frac{3}{\pi^2-6}}(x - 2\sin(x))\rangle\sqrt{\frac{3}{\pi^2-6}}(x - 2\sin(x))
$$

Since $\sqrt{2}\sin(x)$ is odd and x^2 is even, their product will be odd. Thus the inner product will evaluate to zero.

$$
\begin{aligned}
\langle x^2 \mid \sqrt{2}\cos(x)\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\sqrt{2}\cos(x)\mathrm{d}x \\
&= \frac{\sqrt{2}}{2\pi}\int_{-\pi}^{\pi} x^2\cos(x)\mathrm{d}x
\end{aligned}
$$

We can evaluate the integral using tabular integration

$$
\begin{array}{ccc}
d & & I \\
x^2 & \xrightarrow{+} & \cos x \\
2x & \xrightarrow{-} & \sin x \\
2 & \xrightarrow{+} & -\cos x \\
0 & & -\sin x
\end{array}
$$

$$
\begin{aligned}
&= \frac{\sqrt{2}}{2\pi}(x^2\sin(x) + 2x\cos(x) - 2\sin(x))\bigg|_{x=-\pi}^{\pi} \\
&= \frac{\sqrt{2}}{2\pi}(2\pi\cos(\pi) + 2\pi\cos(-\pi)) \\
&= \frac{\sqrt{2}}{2\pi}(-4\pi) \implies \int_{-\pi}^{\pi} x^2\cos(x)\mathrm{d}x = -4\pi \qquad (4) \\
&= -2\sqrt{2}
\end{aligned}
$$

$$\langle x^2 \mid \sqrt{\frac{3}{\pi^2-6}}(x-2\sin(x))\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\sqrt{\frac{3}{\pi^2-6}}(x-2\sin(x))\mathrm{d}x$$
$$= \frac{1}{2\pi}\sqrt{\frac{3}{\pi^2-6}}\int_{-\pi}^{\pi} x^2(x-2\sin(x))\mathrm{d}x$$
$$= \frac{1}{2\pi}\sqrt{\frac{3}{\pi^2-6}}\int_{-\pi}^{\pi} x^3-2x^2\sin(x)\mathrm{d}x$$

We saw before that $x^2\sin(x)$ is odd, and x^3 is also odd, so the entire function is odd. Since the parity of the function is odd, that means that the integral will evaluate to 0. This means that $\vec{b}_4 = x^2+4\cos(x)$, now we must normalize it.

$$\langle x^2+4\cos(x) \mid x^2+4\cos(x)\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2+4cos(x))^2\mathrm{d}x$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4+8x^2\cos(x)+16\cos^2(x)\mathrm{d}x$$
$$= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} x^4\mathrm{d}x + 8\int_{-\pi}^{\pi} x^2\cos(x)\mathrm{d}x + 16\int_{-\pi}^{\pi}\cos^2(x)\mathrm{d}x\right)$$
$$= \frac{1}{2\pi}\left(\frac{x^5}{5}\Big|_{x=-\pi}^{\pi} + 8(-4\pi)+16(\pi)\right) \qquad \text{By (2) and (4)}$$
$$= \frac{1}{2\pi}\left(\frac{2\pi^5}{5}-16\pi\right)$$
$$= \frac{\pi^4}{5}-8 = \frac{\pi^4-40}{5}$$

Thus we can conclude that $||\vec{b}_4|| = \sqrt{\frac{\pi^4-40}{5}}$, and after normalizing $\vec{b}_4$ we find the the last orthonormal basis vector is $\sqrt{\frac{5}{\pi^4-40}}(x^3+4\cos(x))$.

Thus we can say that an orthonormal basis for the span$(\sin(x), \cos(x), x, x^2)$ is

$$\langle \sqrt{2}\sin(x), \sqrt{2}\cos(x), \sqrt{\frac{3}{\pi^2-6}}(x-2\sin(x)), \sqrt{\frac{5}{\pi^4-40}}(x^3+4\cos(x))\rangle$$

Question 5

Let $W \subseteq C([-\pi,\pi])$ be $W = \text{span}\left(\sin(x), \cos(x), x, x^2\right)$. Your answer from problem 4 should be an orthonormal basis for W. Compute:
(a) $\text{proj}_W(1)$
(b) $\text{proj}_W(x\sin(x))$

Note:-

Recall:
If $\langle \vec{u}_1, \ldots, \vec{u}_n\rangle$ is an orthonormal basis for W, then

$$\text{proj}_W(\vec{v}) = \sum_{k=1}^{n}\langle \vec{v} \mid \vec{u}_k\rangle\vec{u}_k$$

Solution:
(a) Using the formula from above, we can begin by computing the inner-products for each summand.

$$\langle 1 \mid \sqrt{2}\sin(x)\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{2}\sin(x)\mathrm{d}x = 0$$

This integral evaluates to zero since $\sin(x)$ is odd.

$$\begin{aligned}
\langle 1 \mid \sqrt{2}\cos(x)\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{2}\cos(x)\mathrm{d}x \\
&= \frac{\sqrt{2}}{2\pi}\int_{-\pi}^{\pi}\cos(x)\mathrm{d}x \\
&= \frac{\sqrt{2}}{2\pi}\sin(x)\Big|_{x=-\pi}^{\pi} \\
&= \frac{\sqrt{2}}{2\pi}(0) = 0
\end{aligned}$$

Since this integral evaluates to zero, then the summand is also nullified, so we will continue.

$$\langle 1 \mid \sqrt{\frac{3}{\pi^2-6}}(x-2\sin(x))\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{\frac{3}{\pi^2-6}}(x-2\sin(x))\mathrm{d}x$$

Like the first summand that we evaluated, the function is odd, so thus the integral will evaluate to zero.

$$\begin{aligned}
\langle 1 \mid \sqrt{\frac{5}{\pi^4-40}}(x^3+4\cos(x))\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{\frac{5}{\pi^4-40}}(x^3+4\cos(x))\mathrm{d}x \\
&= \frac{1}{2\pi}\sqrt{\frac{5}{\pi^4-40}}\int_{-\pi}^{\pi}x^3+4\cos(x)\mathrm{d}x \\
&= \frac{1}{2\pi}\sqrt{\frac{5}{\pi^4-40}}\left(\int_{-\pi}^{\pi}x^3\mathrm{d}x + 4\int_{-\pi}^{\pi}\cos(x)\mathrm{d}x\right)
\end{aligned}$$

Since x^3 is odd, that integral will evaluate to zero, and as we saw in previously the integral of $\cos(x)$ from $x=-\pi$ to π evaluates to zero as well. Since all the summands evaluate to zero, then $\mathrm{proj}_W(1) = 0$.

(b) Lets follow the same process as (a), by first evaluating the summands.

$$\langle x\sin(x) \mid \sqrt{2}\sin(x)\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}x\sin(x)\sqrt{2}\sin(x)\mathrm{d}x = 0$$

This evaluates to zero since we're multiplying an even function by an odd function. Thus the first summand will be nullified and is not necessary for the projection of $x\sin(x)$.

$$\begin{aligned}
\langle x\sin(x) \mid \sqrt{2}\cos(x)\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi}x\sin(x)\sqrt{2}\cos(x)\mathrm{d}x \\
&= \frac{\sqrt{2}}{2\pi}\int_{-\pi}^{\pi}x\sin(x)\cos(x)\mathrm{d}x \\
&= \frac{\sqrt{2}}{2\pi}\int_{-\pi}^{\pi}\frac{x\sin(2x)}{2}\mathrm{d}x \qquad \text{By double angle formula} \\
&= \frac{\sqrt{2}}{4\pi}\int_{-\pi}^{\pi}x\sin(2x)\mathrm{d}x
\end{aligned}$$

We can use tabular integration to complete this integral

$$\begin{array}{lcr}
d & & I \\
x & \searrow^{+} & \sin 2x \\
1 & \searrow^{-} & -\frac{1}{2}\cos 2x \\
0 & & -\frac{1}{4}\sin 2x
\end{array}$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{4\pi}\left(\frac{1}{4}\sin(2x) - \frac{1}{2}x\cos(2x)\right)\Big|_{x=-\pi}^{\pi} \\
&= \frac{\sqrt{2}}{8\pi}\left(-\pi\cos(2\pi) - \pi\cos(-2\pi)\right) = -\frac{\sqrt{2}}{4}
\end{aligned}$$

$$\langle x\sin(x) \mid \sqrt{\frac{3}{\pi^2-6}}(x-2\sin(x))\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin(x)\sqrt{\frac{3}{\pi^2-6}}(x-2\sin(x))\mathrm{d}x = 0$$

This inner-product evaluates to zero since we multiply an even function by an odd function. Thus this summand will be nullified and is not necessary for the projection of $x\sin(x)$.

$$\begin{aligned}
\langle x\sin(x) \mid \sqrt{\frac{5}{\pi^4-40}}(x^3+4\cos(x))\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin(x)\sqrt{\frac{5}{\pi^4-40}}(x^3+4\cos(x))\mathrm{d}x \\
&= \frac{1}{2\pi}\sqrt{\frac{5}{\pi^4-40}}\int_{-\pi}^{\pi} x\sin(x)(x^3+4\cos(x))\mathrm{d}x \\
&= \frac{1}{2\pi}\sqrt{\frac{5}{\pi^4-40}}\left(\int_{-\pi}^{\pi} x^4\sin(x)\mathrm{d}x + 4\int_{-\pi}^{\pi} x\cos(x)\sin(x)\mathrm{d}x\right) \\
&= \frac{1}{2\pi}\sqrt{\frac{5}{\pi^4-40}}(0+-2\pi) \\
&= -\sqrt{\frac{5}{\pi^4-40}}
\end{aligned}$$

Since we have computed all the inner-products, we can now find the projection of $x\sin(x)$.

$$\begin{aligned}
\operatorname{proj}_W(x\sin(x)) &= -\frac{\sqrt{2}}{4}\sqrt{2}\cos(x) - \sqrt{\frac{5}{\pi^4-40}}^{\,2}(x^3+4\cos(x)) \\
&= -\frac{1}{2}\cos(x) - \frac{5x^3+20\cos(x)}{\pi^4-40}
\end{aligned}$$

Question 6

Let A be as in problem 3.
(a) Compute the matrix of $\operatorname{proj}_{\operatorname{ran}(A)}$, that is, find the matrix M so that $M\vec{x} = \operatorname{proj}_{\operatorname{ran}(A)}(\vec{x})$ for all $\vec{x} \in \mathbb{R}^3$.
(b) For each of the following $\vec{b}_i$ compute $\operatorname{proj}_{\operatorname{ran}(A)}\left(\vec{b}_i\right)$:

$$\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{b}_4 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

(c) For each $\vec{b}_i$ from (b), let $\hat{b}_i = \operatorname{proj}_{\operatorname{ran}(A)}\left(\vec{b}_i\right)$, each $\hat{b}_i$ has $\hat{b}_i \in \operatorname{ran}(A)$ and so there is a solution to

$$A\hat{x} = \hat{b}_i.$$

Use $\ker(A)$ and the change of basis matrix you found in problem 3 to find full solution sets to

$$A\hat{x}_i = \hat{b}_i$$

for each $i = 1, 2, 3, 4$.

Note:-

Recall:
If $W \subseteq \mathbb{R}^n$, there exists a matrix for proj_W. If $\langle \vec{u}_1, \ldots, \vec{u}_n\rangle$ is an orthonormal basis for W then let

$$Q = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \ldots & \vec{u}_n \\ | & | & & | \end{bmatrix}$$

Then $\operatorname{proj}_W(\vec{x}) = QQ^\top\vec{x}$

Solution:
(a)

$$Q = \begin{bmatrix} 1/\sqrt{5} & 4\sqrt{5}/15 \\ 0 & \sqrt{5}/3 \\ 2/\sqrt{5} & -2\sqrt{5}/15 \end{bmatrix} \qquad Q^\top = \begin{bmatrix} 1/\sqrt{5} & 0 & 2\sqrt{5} \\ 4\sqrt{5}/15 & \sqrt{5}/3 & -2\sqrt{5}/15 \end{bmatrix}$$

$$QQ^\top = \begin{bmatrix} 5/9 & 4/9 & 2/9 \\ 4/9 & 5/9 & -2/9 \\ 2/9 & -2/9 & 8/9 \end{bmatrix}$$