Math 2700.009
Problem Set 11

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Question 1

Let

$$A = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

(a) Find the characteristic polynomial of A
(b) Find all eigenvalues of A
(c) For each eigenvalue λ of A, what is the algebraic and geometric multiplicities of λ ?
(d) For each eigenvalue λ of A, find a basis for the eigenspace $E_\lambda(A)$.
(e) Is A diagonalizable? If so, give an eigenbasis for A.

Note:-

Recall:

- If $T : V \longrightarrow V$ is a linear transformation with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then set $d_i = \dim(E_{\lambda_i}(T))$. T is diagonalizable exactly when $d_1 + d_2 + \ldots + d_n = \dim(V)$
- An eigenvalues algebraic multiplicity is always $\geq$ geometric multiplicity

Solution:
(a) To find the characteristic polynimal we must solve $\det(\lambda I_4 - A)$

$$\det\left(\begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda - 1 & -1 & -3 & 0 \\ 0 & \lambda + 2 & -2 & -4 \\ 0 & 0 & \lambda + 1 & 2 \\ 0 & 0 & 0 & \lambda + 2 \end{bmatrix}\right)$$

In the previous homework, we found a unique property of the determinant of triangular matricies. In particular we found that the determinant of a triangular matrix is the product of the diagonal elements. We can quickly solve for the characteristic polynomial using this method.

$$\det\left(\begin{bmatrix} \lambda - 1 & -1 & -3 & 0 \\ 0 & \lambda + 2 & -2 & -4 \\ 0 & 0 & \lambda + 1 & 2 \\ 0 & 0 & 0 & \lambda + 2 \end{bmatrix}\right) = (\lambda - 1)(\lambda + 1)(\lambda + 2)^2$$

Solving for the roots, we quickly see that the eigenvalues for this matrix A are: $\lambda = 1, \lambda = -1, \lambda = -2$

Question 2

Let

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 7 & -3 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

(a) Find the characteristic polynomial of A and all eigenvalues of A.
(b) What are the algebraic and geometric multiplicites of each eigenvalue. (Keep in mind the geometric multiplicity is always at least one and at most the algebraic multiplicity)
(c) Is A diagonalizable?

Question 3

Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(a) Find the characteristic polynomial of A and all eigenvalues of A.
(b) What are the algebraic and geometric multiplicites of each eigenvalue. (Keep in mind the geometric multiplicity is always at least one and at most the algebraic multiplicity)
(c) Is A diagonalizable?