MATH 2700.009
PROBLEM SET 4

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Question 1

Verify that

$$W = \left\{ \begin{bmatrix} x \\ 0 \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^3$.

Solution:
(1) $\vec{0} \in W$, Take $x = y = 0$
(2) $a, b \in \mathbb{R}$, $\begin{pmatrix} x \\ 0 \\ y \end{pmatrix} + \begin{pmatrix} a \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} x + a \\ 0 \\ y + b \end{pmatrix} \in W$
(3) $r \in \mathbb{R}$, $r \begin{pmatrix} x \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} rx \\ 0 \\ ry \end{pmatrix} \in W$
(1), (2), (3) $\implies$ W is a subspace of $\mathbb{R}^3$

Question 2

Verify that $W = \left\{ tx^2 : t \in \mathbb{R} \right\}$ is a subspace of $\mathbb{P}_3$.

Solution:
(1) $\vec{0} \in W$, Take $t = 0 \implies \vec{0} = \deg(0)$ polynomial.
(2) $r \in \mathbb{R}$, $tx^2 + rx^2 = (t + r)x^2 \implies tx^2 + rx^2 \in W$ Since $t, r \in \mathbb{R}$
(3) $r \in \mathbb{R}$, $r(tx^2) = rtx^2 = (rt)x^2 \implies r(tx^2) \in W$ Since $t, r \in \mathbb{R}$
(1), (2), (3) $\implies$ W is a subspace of $\mathbb{P}_3$

Question 3

Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? Why or why not?

Solution: No, this is because $\mathbb{R}^2$ contains ordered pairs, whereas $\mathbb{R}^3$ contains ordered 3-tuples, i.e., the entry sizes do not match.

Question 4

Let

$$\mathbb{D}_3 = \left\{ \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} : d_1, d_2, d_3 \in \mathbb{R} \right\}.$$

Verify that $\mathbb{D}_3$ is a subspace of $\mathbb{R}^{3\times 3}$.

Solution:
(1) $\vec{0} \in \mathbb{D}_3$, take $d_1 = d_2 = d_3 = 0$
(2) $c_1, c_2, c_3 \in \mathbb{R}$, $\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} + \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} = \begin{bmatrix} d_1 + c_1 & 0 & 0 \\ 0 & d_2 + c_2 & 0 \\ 0 & 0 & d_3 + c_3 \end{bmatrix} \in \mathbb{D}_3$
(3) $r \in \mathbb{R}$, $r \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} rd_1 & 0 & 0 \\ 0 & rd_2 & 0 \\ 0 & 0 & rd_3 \end{bmatrix} \in \mathbb{D}_3$
(1), (2), (3) $\implies$ $\mathbb{D}_3$ is a subspace of $\mathbb{R}^{3\times 3}$

Question 5

Recall $\mathbb{E}$ from the last homework, let

$$W_1 = \left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} \quad \text{and} \quad W_2 = \left\{ \begin{bmatrix} 2t-1 \\ t-1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

(i) Is W_1 a subspace of $\mathbb{R}^2$? If it is, verify this, if not, explain why not.
(ii) Is W_2 a subspace of $\mathbb{R}^2$? If it is, verify this, if not, explain why not.
(iii) Is W_1 a subspace of $\mathbb{E}$? If it is, verify this, if not, explain why not.
(iv) Is W_2 a subspace of $\mathbb{E}$? If it is, verify this, if not, explain why not.

Note:-

Recall from Homework 03, Question 7:
$\mathbb{E}$ is a vector space that is generated when $\mathbb{R}^2$ is equiped with addition and a scalar multiplcation by

$$\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x+a+1 \\ y+b+1 \end{pmatrix} \quad \text{and} \quad c \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx+c-1 \\ cy+c-1 \end{pmatrix}$$

Solution:

(i) $W_1 \leq \mathbb{R}^2$

(1) $\vec{0} \in W_1$, take $t = 0$

(2) $r \in \mathbb{R}$, $\begin{bmatrix} t \\ 0 \end{bmatrix} + \begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} t+r \\ 0 \end{bmatrix} \in W_1$

(3) $r \in \mathbb{R}$, $r \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} rt \\ 0 \end{bmatrix} \in W_1$

(1), (2), (3) $\implies$ W_1 is a subspace of $\mathbb{R}^2$

(ii) $W_2 \leq \mathbb{R}^2$

W_2 is not a subspace of $\mathbb{R}^2$. This is because $\vec{0}$ of $\mathbb{R}^2$ is not an element of W_2

(iii) $W_1 \leq \mathbb{E}$

W_1 is not a subspace of $\mathbb{E}$. This is because the $\vec{0}^{\mathbb{E}}$ is $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$, which $\notin W_{\cdot}1$

(iv) $W_2 \leq \mathbb{E}$

(1) $\vec{0}^{\mathbb{E}} \in W_2$, take $t = 0$ because $\begin{bmatrix} 2(0)-1 \\ 0-1 \end{bmatrix} \in W_2$

(2) To show W_2 is closed under addition with regard to $\mathbb{E}$, take some $t, n, m \in \mathbb{R}$ such that $t = n + m$

$$\begin{aligned} \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix} \oplus \begin{bmatrix} 2m-1 \\ m-1 \end{bmatrix} &= \begin{bmatrix} 2n+2m-2+1 \\ n+m-2+1 \end{bmatrix} \\ &= \begin{bmatrix} 2(n+m)-1 \\ (n+m)-1 \end{bmatrix} \\ &= \begin{bmatrix} 2t-1 \\ t-1 \end{bmatrix} \end{aligned}$$

(3) To show W_2 is closed under multiplication with regard to $\mathbb{E}$, take some $t, r, m \in \mathbb{R}$ such that $t = r \times m$

$$\begin{aligned} r \odot \begin{bmatrix} 2m-1 \\ m-1 \end{bmatrix} &= \begin{bmatrix} r(2m-1)+r-1 \\ r(m-1)+r-1 \end{bmatrix} \\ &= \begin{bmatrix} 2rm-1 \\ rm-1 \end{bmatrix} \\ &= \begin{bmatrix} 2t-1 \\ t-1 \end{bmatrix} \end{aligned}$$

(1), (2), (3) $\implies$ W_2 is a subspace of $\mathbb{E}$

Question 6

Let

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

$S \subseteq \mathbb{R}^3$, is S a spanning set for $\mathbb{R}^3$? That is, does $\text{span}(S) = \mathbb{R}^3$?

Solution: Firstly, it can be shown that $\text{span}(S) = \text{span}(S')$ where we take

$$S' = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

This is because the vector that we have removed to generate S' can be generated using a combination of the remaining vectors, see:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

In order to show that S' is a spanning set for $\mathbb{R}^3$, it must be shown that an abitrary vector say

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

can be represented as a linear combination of vectors from set S', that is to say that it must be shown that there exists scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This can be done by analyzing a system of equations to find that $c_1 = c_2 = c_3 = 0$, which the system can be found by using the scalar products of the vectors in S'. My solution will use a more general solution by utilizing an augmented matrix to show that the vectors are linearly independent and span $\mathbb{R}^3$

$$\begin{aligned} c_1 + 0c_2 + c_3 &= 0 \\ -c_1 + 0c_2 + 0c_3 &= 0 \\ 0c_1 + 2c_2 + 1c_3 &= 0 \end{aligned}$$

$$\left.\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right.$$

Now, the augmented matrix can be row reduced to determine whether it is consistent, i.e., whether it has a solution, and further whether that solution implies that $c_1 = c_2 = c_3 = 0$

$$\left.\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right. \qquad R_2 + R_1 \longrightarrow R_2 \qquad \left.\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right. \qquad R_2 \leftrightarrow R_3$$

$$\left.\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right. \qquad \frac{1}{2}R_2 \longrightarrow R_2 \qquad \left.\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right. \qquad \begin{array}{l} R_1 - R_3 \longrightarrow R_1, \\ R_2 - \frac{1}{2}R_3 \longrightarrow R_2 \end{array}$$

Since the $\text{RREF}(S')$ is reduced to the identity matrix, this implies that S' contains only linearly independent vectors. This is to say that no vector in set S' is a combination of the others. As a result these vectors collectively cover all possible combinations in the space $\mathbb{R}^3$, i.e., $\text{span}(S') = \mathbb{R}^3$.

Question 7

Is S from the previous problem linearly independent? If it is not linearly independent, remove one vector from S to make it linearly independent. After you have removed one vector, verify the result is linearly independent.

Note:-

As discovered in Question 6, the set of vectors S is not linearly independent because it contains a vector which can be generated from a combination of the remaining vectors in the set. Further the result of the row reduced echelon form augmented matrix illustrates that the set S' which has the linearly dependent vector removed, is in fact linearly independent as evidenced by the identity matrix that is produced.

Solution: S is not linearly dependent, since it contains a vector that is a combination of other vectors in the set. More specifically set S contains the vector $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$. This vector can be removed from the set to be come linearly independent because it can be produced by adding the other vectors, see:

$$\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

This new resultant set of vectors, we will call it S' can now be tested for linear independece. As mentioned in the solution of Question 6, the system of linear equations found by multiplying the scalars c_1, c_2, c_3 by the vectors in set S' to find that $c_1 = c_2 = c_3 = 0$, however since I have already row reduced the augmented matrix generated by the same system of equations, it can be used to verify the linear independence of S', see

$$\left.\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right.$$

This indicates that the set of vectors with the linearly dependent vector removed is in fact linearly independent.

Question 8

Verify that

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

is linearly independent in $\mathbb{R}^3$.

Solution: Since I have already used a general approach to verifying linear independence and to verify a spanning set, I will use the more simple approach discussed in class. That is to generate a system of equations to equal the $\vec{0}$ of $\mathbb{R}$ for the vectors in the set, see:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -c_1 + 2c_2 - c_3 &= 0 \\ c_1 - c_2 &= 0 \end{aligned}$$

We can see from the third equation that:

$$c_1 = c_2$$

And further from the second equation:

$$\begin{aligned} -c_2 + 2c_2 - c_3 &= 0 \\ c_2 - c_3 &= 0 \\ c_2 &= c_3 \end{aligned}$$

and lastly from the first equation we see:

$$c_3 + c_3 + c_3 = 0$$
$$3c_3 = 0$$
$$c_3 = 0$$

Thus the solution from the system of equation suggests that $c_1 = c_2 = c_3 = 0$ and further the set of vectors are linearly independent in $\mathbb{R}^3$

Question 9

Let V be a vector space and W_1, W_2 be subspaces of V. Then verify that

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1 \text{ and } w_2 \in W_2\}$$

is a subspace of V by verifying the following:
(a) $\overrightarrow{0} \in W_1 + W_2$ (express the zero vector in the form $v + w$ where $v \in W_1$ and $u \in W_2$)
(b) Verify that if $v \in W_1 + W_2$ and $c \in \mathbb{R}$ is a scalar, then $cv \in W_1 + W_2$.
(c) Verify that if $v, u \in W_1 + W_2$ then $(v + w) \in W_1 + W_2$.

Solution: To verify that $W_1 + W_2$ is a subspace of V, three axioms must be verified:
(a)
Since $W_1 \leq V$ and $W_2 \leq V$, they both contain the zero vector, say $\vec{0}^{W_1} \in W_1$ and $\vec{0}^{W_2} \in W_2$, then $\vec{0} = \vec{0}^{W_1} + \vec{0}^{W_2}$, so thus $\vec{0} \in W_1 + W_2$
(b)
If there is a $v \in W_1 + W_2$, then this means that there exist $w_1 \in W_1$ and $w_2 \in W_2$ so that $v = w_1 + w_2$. This means that cv can be evaluated as

$$cv = c(w_1 + w_2) = cw_1 + cw_2$$

Because W_1 and W_2 are subspaces, they are closed under scalar multiplication which means $cw_1 \in W_1$ and $cw_2 \in W_2$, thus $cv \in W_1 + W_2$.
(c)
If there is a $v, u \in W_1 + W_2$, then this means that there exist some $w_{1v}, w_{1u} \in W_1$ and $w_{2v}, w_{2u} \in W_2$ so that $v = w_{1v} + w_{2v}$ and $u = w_{1u} + w_{2u}$. This means that $v + u$ can be evaluated as

$$v + u = (w_{1v} + w_{2v}) + (w_{1u} + w_{2u}) = (w_{1v} + w_{1u}) + (w_{2v} + w_{2u})$$

Because W_1 and W_2 are subspaces, they are closed under addition which means $(w_{1v} + w_{1u}) \in W_1$ and $(w_{2v} + w_{2u}) \in W_2$, thus $v + u \in W_1 + W_2$.
The results of (a), (b), (c) $\implies$ $W_1 + W_2$ is a subspace of V.