Math 2700.009
Problem Set 6

Ezekiel Berumen

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Question 1

(a) Show that

$$W = \left\{ \begin{bmatrix} x+z \\ x-y \\ z+y \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^3$ by finding vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ so that $W = \operatorname{span}(v_1, v_2, v_3)$. Then (b) find a basis for W. (c) what is $\dim(W)$?

Solution: In order to find vectors $\vec{v_1}, \vec{v_2}, \vec{v_3} \in \mathbb{R}^3$, vectors from W can be be expressed as a linear combination of x, y, z.

$$\begin{bmatrix} x+z \\ x-y \\ z+y \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

In this sense, this also generates vectors that will span W, because any vector in W can be expressed as linear combination of the vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ that have been generated. Since the set of vectors spans W, then in order to find a basis for W, then it must be shown that these vectors are linearly independent. To find the set of vectors that forms a basis for W, linearly independence must be checked. With these vectors, it is immediately evident that they are not linearly independent. See:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

If we consider the set of vectors excluding this linearly dependent vector, it is clear that the two remaining vectors are linearly independent, and thus they serve to form a basis for W. Since the number of vectors in a basis for W is 2, then $\dim(W) = 2$.

Question 2

Let

$$A = \begin{bmatrix} 1 & 0 & 1 & -2 & 2 & 3 \\ 2 & 0 & -1 & 2 & 1 & 6 \\ -1 & 1 & 0 & 2 & 0 & -3 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \end{bmatrix}$$

and let $T : \mathbb{R}^6 \longrightarrow \mathbb{R}^5$ be defined by $T(\vec{x}) = A\vec{x}$.
(a) Find a basis for $\operatorname{ran}(T)$. What is $\dim(\operatorname{ran}(T))$?
(b) Use the Rank-Nullity theorem to find $\dim(\ker(T))$.
(c) Find a basis for $\ker(T)$, i.e. a linearly independent set of vectors from $\ker(T)$ of size $\dim(\ker(T))$.
(d) What are the solution sets to

$$A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} ?$$

Question 3

Let V_1, V_2, and V_3 be vector spaces and $T_1 : V_1 \longrightarrow V_2$ and $T_2 : V_2 \longrightarrow V_3$ be linear transformations. Show that $T_2 \circ T_1 : V_1 \longrightarrow V_3$ is a linear transformation by verifying:
(a) If $v, w \in V_1$ are vectors then $T_2 \circ T_1(v + w) = T_2 \circ T_1(v) + T_2 \circ T_1(w)$
(b) If $v \in V_1$ is a vector and $c \in \mathbb{R}$ is a scalar then $T_2 \circ T_1(cv) = c\,(T_2 \circ T_1(v))$.