Math 2700.009
Problem Set 5

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Question 1

Show directly that the following vectors constitute a basis for $\mathbb{R}^4$, do not appeal to the dimension of $\mathbb{R}^4$ to do so, show that they are both linearly independent and span $\mathbb{R}^4$.

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\-2\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}3\\-1\\7\\0\end{bmatrix}$$

Solution: In order to show that the set of vectors is a basis of $\mathbb{R}^4$, I will show that the vectors span $\mathbb{R}^4$. In doing so it will also be proved that the vectors are linearly indepdent. To do so we must show that a given vector say

$$\vec{v} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \in \mathbb{R}^4$$

can be represented as some linear combination of the four vectors that we are given. That is to to say that there exists some scalars $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$c_1\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + c_2\begin{bmatrix}0\\-1\\0\\-2\end{bmatrix} + c_3\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + c_4\begin{bmatrix}3\\-1\\7\\0\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

This implies that it is possible to set up a system of equations, and further an augmented matrix to then perform row reduction operations to find whether a consistent solution exists for the system. In this case, a consistent solution will imply that there exists scalars such that any arbitrary vector from $\mathbb{R}^4$ can be found in the span of the vectors. See:

$$\begin{aligned} c_1 + 0c_2 + c_3 + 3c_4 &= x_1 \\ 2c_1 - c_2 + 0c_3 - c_4 &= x_2 \\ 3c_1 + 0c_2 + 1c_3 + 7c_4 &= x_3 \\ 4c_1 - 2c_2 + 0c_3 + 0c_4 &= x_4 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & x_1 \\ 2 & -1 & 0 & -1 & x_2 \\ 3 & 0 & 1 & 7 & x_3 \\ 4 & -2 & 0 & 0 & x_4 \end{array}\right]$$

The augmented matrix can now be row reduced to determine whether it is consistent. See:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & x_1 \\ 0 & -1 & -2 & -7 & -2x_1 + x_2 \\ 0 & 0 & -2 & -2 & -3x_1 + x_3 \\ 0 & -2 & -4 & -12 & -4x_1 + x_4 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \longrightarrow R_2 \\ R_3 - 3R_1 \longrightarrow R_3 \\ R_4 - 4R_1 \longrightarrow R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & x_1 \\ 0 & 1 & 2 & 7 & 2x_1 - x_2 \\ 0 & 0 & -2 & -2 & -3x_1 + x_3 \\ 0 & -2 & -4 & -12 & -4x_1 + x_4 \end{array}\right] \quad -R_2 \longrightarrow R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & x_1 \\ 0 & 1 & 2 & 7 & 2x_1 - x_2 \\ 0 & 0 & -2 & -2 & -3x_1 + x_3 \\ 0 & 0 & 0 & 2 & -2x_2 + x_4 \end{array}\right] \quad R_4 + 2R_2 \longrightarrow R_4$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & x_1 \\ 0 & 1 & 2 & 7 & 2x_1 - x_2 \\ 0 & 0 & 1 & 1 & \frac{3x_1 - x_3}{2} \\ 0 & 0 & 0 & 2 & -2x_2 + x_4 \end{array}\right] \quad -\frac{1}{2}R_3 \longrightarrow R_3$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -\frac{x_1 - x_3}{2} \\ 0 & 1 & 0 & 5 & -x_1 - x_2 + x_3 \\ 0 & 0 & 1 & 1 & \frac{3x_1 - x_3}{2} \\ 0 & 0 & 0 & 2 & -2x_2 + x_4 \end{array}\right] \quad \begin{array}{l} R_1 - R_3 \longrightarrow R_1 \\ R_2 - 2R_3 \longrightarrow R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{x_1 - x_3}{2} + 2x_2 - x_4 \\ 0 & 1 & 0 & 1 & -x_1 + x_2 + x_3 + x_4 \\ 0 & 0 & 1 & 0 & \frac{3x_1 + x_2 - x_3 - x_4}{2} \\ 0 & 0 & 0 & 2 & -2x_2 + x_4 \end{array}\right] \quad \begin{array}{l} R_1 - R_4 \longrightarrow R_1 \\ R2 - 2R_4 \longrightarrow R_2 \\ R_3 - \frac{1}{2}R_4 \longrightarrow R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{x_1 - x_3}{2} + 2x_2 - x_4 \\ 0 & 1 & 0 & 1 & -x_1 + x_2 + x_3 + x_4 \\ 0 & 0 & 1 & 0 & \frac{3x_1 + x_2 - x_3 - x_4}{2} \\ 0 & 0 & 0 & 1 & -x_2 + \frac{x_4}{2} \end{array}\right] \quad \frac{1}{2}R_4 \longrightarrow R_4$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{x_1 - x_3}{2} + 2x_2 - x_4 \\ 0 & 1 & 0 & 0 & -x_1 + 2x_2 + x_3 + \frac{x_4}{2} \\ 0 & 0 & 1 & 0 & \frac{3x_1 + x_2 - x_3 - x_4}{2} \\ 0 & 0 & 0 & 1 & -x_2 + \frac{x_4}{2} \end{array}\right] \quad R_2 - R_4 \longrightarrow R_4$$

Although the constants in the augmented matrix appear to be a bit complex, they are not necessarily important, since they represent values in $\mathbb{R}$. The main idea when analyzing this RREF augmented matrix is that it is max rank, that is to say that the system is consistent. In the context of this question, this implies that the vectors are linearly independent. Since the augmented matrix is consistent, it suffices to say that it is possible to generate any vector in the space $\mathbb{R}^4$ and thus the vectors span $\mathbb{R}^4$. Since the vectors are both linearly independent and span $\mathbb{R}^4$, it can be said that the vectors are a basis for $\mathbb{R}^4$.

Question 2

Are the following vectors linearly independent? Do they span $\mathbb{R}^3$? Justify your answers.

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$$

Solution: The vectors are not linearly independent. This is because there exists a vector in this set of vectors that can be expressed as a combination of other vectors in this set in particular we can see that:

$$\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

A new set of vectors can now be generated, denoted by S' by excluding the linearly dependent vector, defined by:

$$S' = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} \right\}$$

Assuming that this newly generated set of vectors is linearly dependent, it suffices to say then that the $span(S) = span(S')$ where S is the original set of vectors. This is particularly important because we can verify that S' is a spanning set of $\mathbb{R}^3$ and thus verify linear independence of the vectors in S'. Further if we can verify that S' is a spanning set of $\mathbb{R}^3$, then this must imply that S is also a spanning set of $\mathbb{R}^3$. In order to show that S' is a spanning set of $\mathbb{R}^3$, it must be shown that for some arbitrary vector $\vec{v} \in \mathbb{R}^3$, say:

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

can be represented as a linear combination of vectors from set S', that is to say that it must be shown that there exists scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This can be done by analyzing a system of equations generated by the scalar products of the vectors in set S' using an augmented matrix to show that there is a consistent solution for any $\vec{v} \in \mathbb{R}^3$.

$$\begin{array}{ccc|c} 1 & -2 & -1 & x \\ -1 & 1 & -1 & y \\ 2 & -1 & 4 & z \end{array}$$

To check whether the system is consistent for any $\vec{v} \in \mathbb{R}^3$, the augmented matrix must be row reduced, see:

$$\begin{array}{ccc|c} 1 & -2 & -1 & x \\ 0 & -1 & -2 & y+x \\ 0 & 3 & 6 & z-2x \end{array} \quad \begin{array}{c} R_2 + R_1 \longrightarrow R_2 \\ R_4 - 2R_1 \longrightarrow R_4 \end{array} \quad \begin{array}{ccc|c} 1 & -2 & -1 & x \\ 0 & 1 & 2 & -y-x \\ 0 & 3 & 6 & z-2x \end{array} \quad -R_2 \longrightarrow R_2$$

$$\begin{array}{ccc|c} 1 & -2 & -1 & x \\ 0 & 1 & 2 & -y-x \\ 0 & 0 & 0 & z+x+3y \end{array} \quad R_4 + 3R_2 \longrightarrow R_4$$

This would imply that there is not a consistent solution to the augmented matrix, because that would suggest that there is $0c_1 + 0c_2 + 0c_3$ could equal some non-zero real number. This further implies that there is another linearly dependent vector in the set S'

Question 3

What is the dimension of the following subspace of $\mathbb{R}^3$? Justify your answer.

$$\text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}\right)$$

Question 4

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ was a linear transformation so that

$$T\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

What is

$$T\left(\begin{bmatrix} 2 \\ 4 \\ 13 \end{bmatrix}\right)?$$

Question 5

Verify that the following is a linear transformation:

$$T : C([0,1]) \to C([0,1])$$
$$T(f) = \int x^2 f(x) \mathrm{d}x$$

where the constant of integration is always zero. Would this function be a linear transformation if the constant of integration was one instead?

Question 6