Math 2700.009
Problem Set 5

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Question 1

Show directly that the following vectors constitute a basis for $\mathbb{R}^4$, do not appeal to the dimension of $\mathbb{R}^4$ to do so, show that they are both linearly independent and span $\mathbb{R}^4$.

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\-2\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}3\\-1\\7\\0\end{bmatrix}$$

Solution: In order to show that the set of vectors is a basis of $\mathbb{R}^4$, I will show that the vectors span $\mathbb{R}^4$. In doing so it will also be proved that the vectors are linearly indepdent. To do so we must show that a given vector say

$$\vec{v} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \in \mathbb{R}^4$$

can be represented as some linear combination of the four vectors that we are given. That is to to say that there exists some scalars $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$c_1\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + c_2\begin{bmatrix}0\\-1\\0\\-2\end{bmatrix} + c_3\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + c_4\begin{bmatrix}3\\-1\\7\\0\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

This implies that it is possible to set up a system of equations, and further an augmented matrix to then perform row reduction operations to find whether a consistent solution exists for the system. In this case, a consistent solution will imply that there exists scalars such that any arbitrary vector from $\mathbb{R}^4$ can be found in the span of the vectors. See:

$$\begin{aligned} c_1 + 0c_2 + c_3 + 3c_4 &= x_1 \\ 2c_1 - c_2 + 0c_3 - c_4 &= x_2 \\ 3c_1 + 0c_2 + 1c_3 + 7c_4 &= x_3 \\ 4c_1 - 2c_2 + 0c_3 + 0c_4 &= x_4 \end{aligned}$$

$$\left.\begin{array}{cccc|c} 1 & 0 & 1 & 3 & x_1 \\ 2 & -1 & 0 & -1 & x_2 \\ 3 & 0 & 1 & 7 & x_3 \\ 4 & -2 & 0 & 0 & x_4 \end{array}\right.$$

The augmented matrix can now be row reduced to determine whether it is consistent. See:

$$\left.\begin{array}{cccc|c} 1 & 0 & 1 & 3 & x_1 \\ 0 & -1 & -2 & -7 & -2x_1 + x_2 \\ 0 & 0 & -2 & -2 & -3x_1 + x_3 \\ 0 & -2 & -4 & -12 & -4x_1 + x_4 \end{array}\right. \quad \begin{array}{c} R_2 - 2R_1 \longrightarrow R_2 \\ R_3 - 3R_1 \longrightarrow R_3 \\ R_4 - 4R_1 \longrightarrow R_4 \end{array} \quad \left.\begin{array}{cccc|c} 1 & 0 & 1 & 3 & x_1 \\ 0 & 1 & 2 & 7 & 2x_1 - x_2 \\ 0 & 0 & -2 & -2 & -3x_1 + x_3 \\ 0 & -2 & -4 & -12 & -4x_1 + x_4 \end{array}\right. \quad -R_2 \longrightarrow R_2$$

$$\left.\begin{array}{cccc|c} 1 & 0 & 1 & 3 & x_1 \\ 0 & 1 & 2 & 7 & 2x_1 - x_2 \\ 0 & 0 & -2 & -2 & -3x_1 + x_3 \\ 0 & 0 & 0 & 2 & -2x_2 + x_4 \end{array}\right. \quad R_4 + 2R_2 \longrightarrow R_4 \quad \left.\begin{array}{cccc|c} 1 & 0 & 1 & 2 & x_1 \\ 0 & 1 & 2 & 7 & 2x_1 - x_2 \\ 0 & 0 & 1 & 1 & \frac{3x_1 - x_3}{2} \\ 0 & 0 & 0 & 2 & -2x_2 + x_4 \end{array}\right. \quad -\frac{1}{2}R_3 \longrightarrow R_3$$

$$\left.\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -\frac{x_1 - x_3}{2} \\ 0 & 1 & 0 & 5 & -x_1 - x_2 + x_3 \\ 0 & 0 & 1 & 1 & \frac{3x_1 - x_3}{2} \\ 0 & 0 & 0 & 2 & -2x_2 + x_4 \end{array}\right. \quad \begin{array}{c} R_1 - R_3 \longrightarrow R_1 \\ R_2 - 2R_3 \longrightarrow R_2 \end{array} \quad \left.\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{x_1 - x_3}{2} + 2x_2 - x_4 \\ 0 & 1 & 0 & 1 & -x_1 + x_2 + x_3 + x_4 \\ 0 & 0 & 1 & 0 & \frac{3x_1 + x_2 - x_3 - x_4}{2} \\ 0 & 0 & 0 & 2 & -2x_2 + x_4 \end{array}\right. \quad \begin{array}{c} R_1 - R_4 \longrightarrow R_1 \\ R2 - 2R_4 \longrightarrow R_2 \\ R_3 - \frac{1}{2}R_4 \longrightarrow R_3 \end{array}$$

Question 2

Are the following vectors linearly independent? Do they span $\mathbb{R}^3$? Justify your answers.

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$$

Question 3

What is the dimension of the following subspace of $\mathbb{R}^3$? Justify your answer.

$$\operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} \right)$$

Question 4

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ was a linear transformation so that

$$T\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

What is

$$T\left(\begin{bmatrix} 2 \\ 4 \\ 13 \end{bmatrix} \right)?$$

Question 5

Verify that the following is a linear transformation:

$$T : C([0,1]) \to C([0,1])$$
$$T(f) = \int x^2 f(x) \mathrm{d}x$$

where the constant of integration is always zero. Would this function be a linear transformation if the constant of integration was one instead?

Question 6