

Math 2700.009: Linear Algebra

Problem Set 14

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Question 1

Are the following matrices orthogonal?

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\vartheta) & -\sin(\vartheta) \\ 0 & 0 & \sin(\vartheta) & \cos(\vartheta) \end{bmatrix} \quad B = \begin{bmatrix} 1/\sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{bmatrix}$$

Where $\vartheta \in [0, 2\pi)$.

Note:-

Recall:

A matrix $A \in \mathbb{R}^{n \times n}$ is orthogonal if:

- The columns of A form an orthogonal basis for \mathbb{R}^n
- $A^T A = I_n$
- $A^T = A^{-1}$

Solution:

(a) To verify if A is orthogonal, we can check if $A^T A = I_4$.

$$\begin{aligned} A^T A &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\vartheta) & \sin(\vartheta) \\ 0 & 0 & -\sin(\vartheta) & \cos(\vartheta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\vartheta) & -\sin(\vartheta) \\ 0 & 0 & \sin(\vartheta) & \cos(\vartheta) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (\cos^2(\vartheta) + \sin^2(\vartheta)) & (\cos(\vartheta)\sin(\vartheta) - \cos(\vartheta)\sin(\vartheta)) \\ 0 & 0 & (\cos(\vartheta)\sin(\vartheta) - \cos(\vartheta)\sin(\vartheta)) & (\cos^2(\vartheta) + \sin^2(\vartheta)) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{By trigonometric identities and simplification} \\ &= I_4 \end{aligned}$$

Hence we can say A is orthogonal since $A^T A = I_4$

(b) Similarly, we can check the orthogonality of B by finding $B^T B$.

$$\begin{aligned} B^T B &= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &\neq I_3 \end{aligned}$$

We can see that the product $B^T B$ is not the identity matrix, so thus B is not orthogonal.

(c) Lastly, we can check the orthogonality of C by finding the product $C^T C$ and checking if it is I_3 .

$$C^T C = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{bmatrix}$$

$$C^T C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ = I_3$$

Hence we can say C is orthogonal since $C^T C = I_3$

Question 2

Verify that if $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, then Q^T is an orthogonal matrix

Solution: Since Q is orthogonal, then we can conclude that $Q^T Q = I$. This suggests that Q^T is a left inverse to Q . If we would like to show that Q^T is orthogonal when Q is orthogonal, then we must show that

$$(Q^T)^T Q^T = Q Q^T = I$$

To do so, we must show that Q^T is also a right inverse to Q .

Proof. Suppose Q has some right inverse M . Let us prove that if Q^T is a left inverse to Q that it is also a right inverse.

$$Q^T = Q^T I = Q^T (QM) = (Q^T Q)M = IM = M \\ Q^T = M$$

⊙

The simple proof suggests that if Q has a left inverse Q^T , then it is also a right inverse to Q . Hence we can say that Q^T is orthogonal since $Q Q^T = I$.

Question 3

Let $T : C([0, 1]) \rightarrow C([0, 1])$ be defined by

$$T(f) = \sqrt{3}x f(x^3).$$

Verify that T is an orthogonal transformation where the inner-product on $C([0, 1])$ is

$$\langle f | g \rangle = \int_0^1 f(x)g(x)dx.$$

Note:-

Recall:

If V is an inner-product space and $T : V \rightarrow V$ is a linear transformation, then T is called orthogonal if for all $\vec{v} \in V$, $\|\vec{v}\| = \|T(\vec{v})\|$, in other words, a linear transformation on an inner-product space is orthogonal if it is length preserving.

Solution: Let us check if $T(f)$ is length preserving on $C([0, 1])$ with the given inner-product, first by checking the length of $f(x)$.

$$\|f(x)\| = \sqrt{\langle f(x) | f(x) \rangle} \\ = \sqrt{\int_0^1 f(x)f(x)dx} \\ = \sqrt{\int_0^1 f(x)^2 dx}$$

Now let us check the length of $T(f)$.

$$\begin{aligned}
||T(f)|| &= \sqrt{\langle T(f)|T(f) \rangle} \\
&= \sqrt{\langle \sqrt{3}xf(x^3)|\sqrt{3}xf(x^3) \rangle} \\
&= \sqrt{\int_0^1 (\sqrt{3}xf(x^3))^2 dx} \\
&= \sqrt{\int_0^1 3x^2 f(x^3)^2 dx} \quad \left[\begin{array}{l} u = x^3 \\ du = 3x^2 dx \end{array} \right] \\
&= \sqrt{\int_{0^3}^{1^3} f(u)^2 du} \\
&= \sqrt{\int_0^1 f(u)^2 du} \\
&= \sqrt{\int_0^1 f(x)^2 dx} \\
&= ||f(x)||
\end{aligned}$$

Since we have demonstrated the equality of $||f(x)||$ and $||T(f)||$, then we have also demonstrated that T is length perserving on the inner-product space and hence T is orthogonal.

Question 4

Find the QR factorization of the following matrices.

(a)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

(c)

$$C = \begin{bmatrix} \cos(\vartheta) & -\sin(\vartheta) & 0 & 0 \\ \sin(\vartheta) & \cos(\vartheta) & 0 & 0 \\ 0 & 0 & \cos(\tau) & -\sin(\tau) \\ 0 & 0 & \sin(\tau) & \cos(\tau) \end{bmatrix}$$

where $\vartheta, \tau \in [0, 2\pi)$.

Solution:

(a) Let us call the columns of A as vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$. To find the QR factorization, we must first perform Gram-Schmidt, on the vectors.

$$\begin{aligned}
\vec{y}_1 &= \frac{1}{\|\vec{a}_1\|} \vec{a}_1 \\
&= \frac{1}{\sqrt{1+4+9}} \vec{a}_1 \\
&= \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}
\end{aligned}$$

This also tells us that $\vec{a}_1 = (\sqrt{14})\vec{y}_1$. This will be important for when we construct R .

$$\begin{aligned}
\vec{b}_2 &= \vec{a}_2 - (\vec{a}_2 \cdot \vec{y}_1) \vec{y}_1 \\
&= \vec{a}_2 - (1/\sqrt{14} + 3/\sqrt{14}) \vec{y}_1 \\
&= \vec{a}_2 - (4/\sqrt{14}) \vec{y}_1 \\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 4/\sqrt{14} \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 4/14 \\ 8/14 \\ 12/14 \end{bmatrix} \\
&= \begin{bmatrix} 10/14 \\ -8/14 \\ 2/14 \end{bmatrix} = \begin{bmatrix} 5/7 \\ -4/7 \\ 1/7 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\|\vec{b}_2\| &= \sqrt{\frac{25}{49} + \frac{16}{49} + \frac{1}{49}} \\
&= \sqrt{\frac{42}{49}} \\
&= \frac{\sqrt{42}}{7}
\end{aligned}$$

$$\begin{aligned}
\vec{y}_2 &= \frac{1}{\|\vec{b}_2\|} \vec{b}_2 \\
&= 7/\sqrt{42} \begin{bmatrix} 5/7 \\ -4/7 \\ 1/7 \end{bmatrix} \\
&= \begin{bmatrix} 5/\sqrt{42} \\ -4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}
\end{aligned}$$

Similarly, we need to see the linear combination of \vec{y}_i 's that construct \vec{a}_2

$$\begin{aligned}
\vec{b}_2 &= (\sqrt{42}/7) \vec{y}_2 \\
\vec{a}_2 - (4/\sqrt{14}) \vec{y}_1 &= (\sqrt{42}/7) \vec{y}_2 \\
\vec{a}_2 &= (4/\sqrt{14}) \vec{y}_1 + (\sqrt{42}/7) \vec{y}_2
\end{aligned}$$

$$\begin{aligned}
\vec{b}_3 &= \vec{a}_3 - (\vec{a}_3 \cdot \vec{y}_2) \vec{y}_2 - (\vec{a}_3 \cdot \vec{y}_1) \vec{y}_1 \\
&= \vec{a}_3 - (5/\sqrt{42} - 4/\sqrt{42} - 1/\sqrt{42}) \vec{y}_2 \\
&\quad - (1/\sqrt{14} + 2/\sqrt{14} - 3/\sqrt{14}) \vec{y}_1 \\
&= \vec{a}_3 - (0) \vec{y}_2 - (0) \vec{y}_1 \\
&= \vec{a}_3
\end{aligned}$$

$$\begin{aligned}
\|\vec{b}_3\| &= \sqrt{1+1+1} \\
&= \sqrt{3}
\end{aligned}$$

$$\begin{aligned}
\vec{y}_3 &= \frac{1}{\|\vec{b}_3\|} \vec{b}_3 \\
&= 1/\sqrt{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}
\end{aligned}$$

Since $\vec{b}_3 = \vec{a}_3$, then we can quickly compute that $\vec{a}_3 = (\sqrt{3})\vec{y}_3$.

So now to construct the QR factorization of A , we will use the vectors that form the orthogonal basis as columns for Q , and the columns of R will be the linear combination of \vec{y}_i vectors that construct our \vec{a}_i 's, so we see that

$$A = QR = \begin{bmatrix} 1/\sqrt{14} & 5/\sqrt{42} & 1/\sqrt{3} \\ 2/\sqrt{14} & -4/\sqrt{42} & 1/\sqrt{3} \\ 3/\sqrt{14} & 1/\sqrt{42} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{14} & 4/\sqrt{14} & 0 \\ 0 & \sqrt{42}/7 & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

(b) Like before, let us call the columns of B as vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ and then perform Gram-Schmidt on the vectors.

$$\begin{aligned} \vec{y}_1 &= \frac{1}{\|\vec{a}_1\|} \vec{a}_1 \\ &= \frac{1}{\sqrt{1+1+1}} \vec{a}_1 \\ &= 1/\sqrt{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \end{aligned}$$

We can see that $\vec{a}_1 = (\sqrt{3})\vec{y}_1$.

$$\begin{aligned} \|\vec{b}_2\| &= \sqrt{4+1+1} \\ &= \sqrt{6} \end{aligned}$$

$$\begin{aligned} \vec{b}_2 &= \vec{a}_2 - (\vec{a}_2 \cdot \vec{y}_1) \vec{y}_1 \\ &= \vec{a}_2 - (2/\sqrt{3} - 1/\sqrt{3} - 1/\sqrt{3}) \vec{y}_1 \\ &= \vec{a}_2 - (0) \vec{y}_1 \\ &= \vec{a}_2 \end{aligned}$$

$$\begin{aligned} \vec{y}_2 &= \frac{1}{\|\vec{b}_2\|} \vec{b}_2 \\ &= 1/\sqrt{6} \begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \end{aligned}$$

So we can see that $\vec{a}_2 = (\sqrt{6})\vec{y}_2$

$$\begin{aligned} \vec{b}_3 &= \vec{a}_3 - (\vec{a}_3 \cdot \vec{y}_2) \vec{y}_2 - (\vec{a}_3 \cdot \vec{y}_1) \vec{y}_1 \\ &= \vec{a}_3 - (2/\sqrt{6} - 1/\sqrt{6}) \vec{y}_2 - (1/\sqrt{3} + 1/\sqrt{3}) \vec{y}_1 \\ &= \vec{a}_3 - (1/\sqrt{6}) \vec{y}_2 - (2/\sqrt{3}) \vec{y}_1 \\ &= \vec{a}_3 - (1/\sqrt{6}) \begin{bmatrix} 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} - (2/\sqrt{3}) \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/6 \\ 0 \\ -1/6 \\ -1/6 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 0 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \|\vec{b}_3\| &= \sqrt{1/4 + 1/4} \\ &= 1/2\sqrt{2} \end{aligned}$$

$$\begin{aligned} \vec{y}_3 &= \frac{1}{\|\vec{b}_3\|} \vec{b}_3 \\ &= 1/2\sqrt{2} \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix} \end{aligned}$$

We now need to construct \vec{a}_3 using a linear combination of \vec{y}_i 's

$$\begin{aligned}\vec{b}_3 &= (1/2\sqrt{2})\vec{y}_3 \\ \vec{a}_3 - (1/\sqrt{6})\vec{y}_2 - (2/\sqrt{3})\vec{y}_1 &= (1/2\sqrt{2})\vec{y}_3 \\ \vec{a}_3 &= (2/\sqrt{3})\vec{y}_1 + (1/\sqrt{6})\vec{y}_2 + (1/2\sqrt{2})\vec{y}_3\end{aligned}$$

$$\begin{aligned}\vec{b}_4 &= \vec{a}_4 - (\vec{a}_4 \cdot \vec{y}_3)\vec{y}_3 - (\vec{a}_4 \cdot \vec{y}_2)\vec{y}_2 - (\vec{a}_4 \cdot \vec{y}_1)\vec{y}_1 \\ &= \vec{a}_4 - (-\sqrt{2})\vec{y}_3 - (-1/\sqrt{6})\vec{y}_2 - (1/\sqrt{3})\vec{y}_1 \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + (\sqrt{2}) \begin{bmatrix} 0 \\ 0 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix} + (1/\sqrt{6}) \begin{bmatrix} 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \\ &\quad - (1/\sqrt{3}) \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2/6 \\ 0 \\ -1/6 \\ -1/6 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 0 \\ 1/3 \\ 1/3 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ -3/2 \\ 3/2 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\|\vec{b}_4\| &= \sqrt{1 + 9/4 + 9/4} \\ &= \sqrt{22/4} \\ &= \sqrt{22}/2\end{aligned}$$

$$\begin{aligned}\vec{y}_4 &= \frac{1}{\|\vec{b}_4\|} \vec{b}_4 \\ &= 2/\sqrt{22} \begin{bmatrix} 0 \\ 1 \\ -3/2 \\ 3/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 2/\sqrt{22} \\ -3/\sqrt{22} \\ 3/\sqrt{22} \end{bmatrix}\end{aligned}$$

We now need to construct \vec{a}_4 using a linear combination of \vec{y}_i 's

$$\begin{aligned}\vec{b}_4 &= (\sqrt{22}/2)\vec{y}_4 \\ \vec{a}_4 + (\sqrt{2})\vec{y}_3 + (1/\sqrt{6})\vec{y}_2 - (1/\sqrt{3})\vec{y}_1 &= (\sqrt{22}/2)\vec{y}_4 \\ \vec{a}_4 &= (1/\sqrt{3})\vec{y}_1 - (1/\sqrt{6})\vec{y}_2 - (\sqrt{2})\vec{y}_3 + (\sqrt{22}/2)\vec{y}_4\end{aligned}$$

So now to construct the QR factorization of B , we will use the vectors that form the orthogonal basis as columns for Q , and the columns of R will be the linear combination of \vec{y}_i vectors that construct our \vec{a}_i 's, so we see that

$$B = QR = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2/\sqrt{22} \\ 1/\sqrt{3} & -1/\sqrt{6} & -\sqrt{2} & -3/\sqrt{22} \\ 1/\sqrt{3} & -1/\sqrt{6} & \sqrt{2} & 3/\sqrt{22} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & \sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \\ 0 & 0 & 1/2\sqrt{2} & -\sqrt{2} \\ 0 & 0 & 0 & \sqrt{22}/2 \end{bmatrix}$$

(c) C is orthogonal, because it consists of only orthogonal block matrices along the diagonal, as we saw from problem 1. Since C is orthogonal, then the columns of C also form an orthogonal basis for \mathbb{R}^4 . If we consider $C = Q$, and $R = I$, then we have an orthogonal matrix Q and an upper triangular matrix R which when we find the product QR gives us C .

Question 5

Let $D \in \mathbb{R}^{n \times n}$ be the diagonal matrix:

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}.$$

For which values of the d_i is D an orthogonal matrix?

Solution: If D is orthogonal, then that means $D^T D = I$. Since D is diagonal, then it is also symmetric and thus $D^T = D$ so we have that $DD = I$ or $D^2 = I$. So then we can see that

$$D^T D = D^2 = \begin{bmatrix} (d_1)^2 & & & \\ & (d_2)^2 & & \\ & & \ddots & \\ & & & (d_n)^2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

So thus we can see that each $(d_i)^2 = 1$ and hence we can conclude that the value of the d_i 's must be -1 or 1 .

Question 6

For the following matrix A , find:

- $A^T A$
- All eigenvalues of $A^T A$
- An eigenbasis for $A^T A$
- All singular values of A
- An orthonormal eigenbasis for $A^T A$
- A singular value decomposition of A

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Solution:

(a)

$$A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

(b) $\det(\lambda I_3 - A^T A)$

$$\det \begin{bmatrix} \lambda - 5 & -2 & -4 \\ -2 & \lambda - 4 & 0 \\ -4 & 0 & \lambda - 4 \end{bmatrix}$$

If we perform a cofactor expansion about row 3

$$\begin{aligned} &= -4 \det \begin{bmatrix} -2 & -4 \\ \lambda - 4 & 0 \end{bmatrix} + (\lambda - 4) \det \begin{bmatrix} \lambda - 5 & -2 \\ -2 & \lambda - 4 \end{bmatrix} \\ &= -4(4(\lambda - 4)) + (\lambda - 4)((\lambda - 5)(\lambda - 4) - 4) \\ &= -16(\lambda - 4) + (\lambda - 4)(\lambda^2 - 9\lambda + 20 - 4) \\ &= (\lambda - 4)(\lambda^2 - 9\lambda) \\ &= (\lambda - 9)(\lambda - 4)(\lambda) \end{aligned}$$

We can see that $A^T A$ has eigenvalues $9, 4, 0$.

(c)

$$\begin{aligned} 9I - A^T A &= \begin{bmatrix} 4 & -2 & -4 \\ -2 & 5 & 0 \\ -4 & 0 & 5 \end{bmatrix} & \vec{v}_1 &= \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} \\ 4I - A^T A &= \begin{bmatrix} -1 & -2 & -4 \\ -2 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix} & \vec{v}_2 &= \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \\ 0I - A^T A &= \begin{bmatrix} -5 & -2 & -4 \\ -2 & -4 & 0 \\ -4 & 0 & -4 \end{bmatrix} & \vec{v}_3 &= \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \end{aligned}$$

From this then we can say that an eigenbases for $A^T A$

$$\left\{ \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \right\}$$

(d) We can find the singular values σ_i 's by taking the square root of each eigenvalue, we see that we have singular values 3, 2, 0

(e) Since an eigenbasis for a symmetric matrix is always orthogonal, then we need normalize each of the eigenbasis vectors to make it orthonormal. We wont need to perform Gram-Schmidt after normalizing since scalar multiples of eigenvectors are still eigenvectors, that is what the eigenspace is.

$$\begin{aligned} \|\vec{v}_1\| &= \sqrt{25 + 4 + 16} = \sqrt{45} = 3\sqrt{5} & \frac{1}{\|\vec{v}_1\|} \vec{v}_1 &= 1/3\sqrt{5} \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5/3\sqrt{5} \\ 2/3\sqrt{5} \\ 4/3\sqrt{5} \end{bmatrix} \\ \|\vec{v}_2\| &= \sqrt{4 + 1} = \sqrt{5} & \frac{1}{\|\vec{v}_2\|} \vec{v}_2 &= 1/\sqrt{5} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \\ \|\vec{v}_3\| &= \sqrt{4 + 1 + 4} = \sqrt{9} = 3 & \frac{1}{\|\vec{v}_3\|} \vec{v}_3 &= 1/3 \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} \end{aligned}$$

So we can say that an orthonormal eigenbasis for $A^T A$ is

$$\left\{ \begin{bmatrix} 5/3\sqrt{5} \\ 2/3\sqrt{5} \\ 4/3\sqrt{5} \end{bmatrix}, \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} \right\}$$

(f) $A = u \Sigma V^T$

$$V = \begin{bmatrix} 5/3\sqrt{5} & 0 & 2/3 \\ 2/3\sqrt{5} & -2/\sqrt{5} & -1/3 \\ 4/3\sqrt{5} & 1/\sqrt{5} & -2/3 \end{bmatrix}$$

We can construct Σ using the singular values 3 and 2

$$\Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

In order to find u , then we must calculate them using the non-zero singular values.

$$\vec{u}_1 = \frac{1}{\sigma_1} A \vec{v}_1 = 1/3 \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 5/3\sqrt{5} \\ 2/3\sqrt{5} \\ 4/3\sqrt{5} \end{bmatrix} = 1/3 \begin{bmatrix} 9/3\sqrt{5} \\ 18/3\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A \vec{v}_2 = 1/2 \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = 1/2 \begin{bmatrix} -4/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$u = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

So with these matrices that we have found, we can now construct A by taking the product

$$A = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 5/3\sqrt{5} & 2/3\sqrt{5} & 4/3\sqrt{5} \\ 0 & -2/\sqrt{5} & 1/\sqrt{5} \\ 2/3 & -1/3 & -2/3 \end{bmatrix}$$

Question 7

For the following matrix B , find:

- $B^T B$
- All eigenvalues of $B^T B$
- An eigenbasis for $B^T B$
- All singular values of B
- An orthonormal eigenbasis for $B^T B$
- A singular value decomposition of B .

$$B = \begin{bmatrix} 1 & 1 & 3\sqrt{7/15} \\ 1 & -1 & -\sqrt{7/15} \\ 0 & 0 & \sqrt{7/15} \\ 1 & 2 & -2\sqrt{7/15} \end{bmatrix}$$

Solution:

(a)

$$B^T B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 2 \\ 3\sqrt{7/15} & -\sqrt{7/15} & \sqrt{7/15} & -2\sqrt{7/15} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3\sqrt{7/15} \\ 1 & -1 & -\sqrt{7/15} \\ 0 & 0 & \sqrt{7/15} \\ 1 & 2 & -2\sqrt{7/15} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

(b) $\det(\lambda I - B^T B)$

$$\det \begin{bmatrix} \lambda - 3 & -2 & 0 \\ -2 & \lambda - 6 & 0 \\ 0 & 0 & \lambda - 7 \end{bmatrix}$$

We can perform cofactor expansion about the 3rd row

$$\begin{aligned} &= (\lambda - 7)((\lambda - 3)(\lambda - 6) - 4) \\ &= (\lambda - 7)(\lambda^2 - 9\lambda + 14) \\ &= (\lambda - 7)(\lambda - 7)(\lambda - 2) \end{aligned}$$

So then we can say $B^T B$ has eigenvalues 7, 2.

(c)

$$\begin{aligned} 7I - B^T B &= \begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \vec{v}_1 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \vec{v}_2 &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \\ 2I - B^T B &= \begin{bmatrix} -1 & -2 & 0 \\ -2 & -4 & 0 \\ 0 & 0 & -5 \end{bmatrix} & \vec{v}_3 &= \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \end{aligned}$$

From this then we can say that an eigenbases for $B^\top B$

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$$

(d) The singular values can be found by taking the square root of each eigenvalue, the singular values are $\sqrt{7}, \sqrt{7}, \sqrt{2}$

(e) The set of vectors is already orthogonal, so now we must normalize each vector.

$$\|\vec{v}_1\| = \sqrt{1} = 1 \quad \vec{v}_1 \text{ is already normalized}$$

$$\|\vec{v}_2\| = \sqrt{1+4} = \sqrt{5} \quad \frac{1}{\|\vec{v}_2\|} \vec{v}_2 = 1/\sqrt{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$$

$$\|\vec{v}_3\| = \sqrt{1+4} = \sqrt{5} \quad \frac{1}{\|\vec{v}_3\|} \vec{v}_3 = 1/\sqrt{5} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}$$

So then an orthonormal eigenbasis for $B^\top B$ is

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix} \right\}$$

(f) $B = u\Sigma V^\top$

$$V = \begin{bmatrix} 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 0 & 2/\sqrt{5} & -1/\sqrt{5} \\ 1 & 0 & 0 \end{bmatrix}$$

We can construct Σ using the singular values

$$\Sigma = \begin{bmatrix} \sqrt{7} & 0 & 0 \\ 0 & \sqrt{7} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

In order to find u , then we must calculate it using the non-zero singular values.

$$\vec{u}_1 = \frac{1}{\sigma_1} B \vec{v}_1 = 1/\sqrt{7} \begin{bmatrix} 1 & 1 & 3\sqrt{7/15} \\ 1 & -1 & -\sqrt{7/15} \\ 0 & 0 & \sqrt{7/15} \\ 1 & 2 & -2\sqrt{7/15} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1/\sqrt{7} \begin{bmatrix} 3\sqrt{7/15} \\ -\sqrt{7/15} \\ \sqrt{7/15} \\ -2\sqrt{7/15} \end{bmatrix} = \begin{bmatrix} 3/\sqrt{15} \\ -1/\sqrt{15} \\ 1/\sqrt{15} \\ -2/\sqrt{15} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} B \vec{v}_2 = 1/\sqrt{7} \begin{bmatrix} 1 & 1 & 3\sqrt{7/15} \\ 1 & -1 & -\sqrt{7/15} \\ 0 & 0 & \sqrt{7/15} \\ 1 & 2 & -2\sqrt{7/15} \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} = 1/\sqrt{7} \begin{bmatrix} 3/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \\ 5/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 3/\sqrt{35} \\ -1/\sqrt{35} \\ 0 \\ 5/\sqrt{35} \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\sigma_3} B \vec{v}_3 = 1/\sqrt{7} \begin{bmatrix} 1 & 1 & 3\sqrt{7/15} \\ 1 & -1 & -\sqrt{7/15} \\ 0 & 0 & \sqrt{7/15} \\ 1 & 2 & -2\sqrt{7/15} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix} = 1/\sqrt{2} \begin{bmatrix} 1/\sqrt{5} \\ 3/\sqrt{5} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \\ 0 \\ 0 \end{bmatrix}$$

But we need to extend our set of vectors since it needs to be a basis for \mathbb{R}^4 . An obvious candidate is the standard basis vector \vec{e}_3 , since it means we will have to do the least amount of work when we perform Gram-Schmidt

$$\begin{aligned}\vec{b}_4 &= \vec{a}_4 - (\vec{a}_4 \cdot \vec{u}_3)\vec{u}_3 - (\vec{a}_4 \cdot \vec{u}_2)\vec{u}_2 - (\vec{a}_4 \cdot \vec{u}_1)\vec{u}_1 \\ &= \vec{a}_4 - 0\vec{u}_3 - 0\vec{u}_2 - 1/\sqrt{5}\vec{u}_1 \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 3/15 \\ -1/15 \\ 1/15 \\ -2/15 \end{bmatrix} = \begin{bmatrix} -3/15 \\ 1/15 \\ 14/15 \\ 2/15 \end{bmatrix}\end{aligned}$$

Now to normalize \vec{b}_4 .

$$\|\vec{b}_4\| = \sqrt{\frac{9}{15} + \frac{196}{225} + \frac{1}{225} + \frac{4}{225}} = \sqrt{210/225} = \sqrt{14/15} \quad \vec{u}_4 = \frac{1}{\|\vec{b}_4\|}\vec{b}_4 = \sqrt{15/14} \begin{bmatrix} -3/15 \\ 1/15 \\ 14/15 \\ 2/15 \end{bmatrix} = \begin{bmatrix} -\sqrt{3/70} \\ 1/\sqrt{210} \\ \sqrt{14/15} \\ \sqrt{2/105} \end{bmatrix}$$

$$u = \begin{bmatrix} 3/\sqrt{15} & 3/\sqrt{35} & 1/\sqrt{10} & -\sqrt{3/70} \\ -1/\sqrt{15} & -1/\sqrt{35} & 3/\sqrt{10} & 1/\sqrt{210} \\ 1/\sqrt{15} & 0 & 0 & \sqrt{14/15} \\ -2/\sqrt{15} & 5/\sqrt{35} & 0 & \sqrt{2/105} \end{bmatrix}$$

Since we have found the matrices required for the singular value decomposition of B , then we can say

$$B = \begin{bmatrix} 3/\sqrt{15} & 3/\sqrt{35} & 1/\sqrt{10} & -\sqrt{3/70} \\ -1/\sqrt{15} & -1/\sqrt{35} & 3/\sqrt{10} & 1/\sqrt{210} \\ 1/\sqrt{15} & 0 & 0 & \sqrt{14/15} \\ -2/\sqrt{15} & 5/\sqrt{35} & 0 & \sqrt{2/105} \end{bmatrix} \begin{bmatrix} \sqrt{7} & 0 & 0 \\ 0 & \sqrt{7} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 2/\sqrt{5} & -1/\sqrt{5} & 0 \end{bmatrix}$$

Question 8

Consider $C([-\pi, \pi])$ with the inner-product

$$\langle f | g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx.$$

For each integer $n \geq 1$ let W_n be the subspace of $C([-\pi, \pi])$

$$W_n = \text{span}(1, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x), \dots, \sin(nx), \cos(nx)).$$

Take for granted (you pretty much verified this on the last problem set) that

$$\mathfrak{B}_n = \{1/2, \sin(x), \cos(x), \sin(2x), \cos(2x), \dots, \sin(nx), \cos(nx)\}$$

is an orthonormal basis for W_n . Then the n th Fourier approximation of $f \in C([-\pi, \pi])$ is:

$$\text{proj}_{W_n}(f) = \langle 1/2 | f \rangle + \sum_{k=1}^n \langle \sin(kx) | f \rangle \sin(kx) + \langle \cos(kx) | f \rangle \cos(kx)$$

and the Fourier series of f^1 is:

$$\lim_{n \rightarrow \infty} \text{proj}_{W_n}(f) = \langle 1/2 | f \rangle + \sum_{k=1}^{\infty} \langle \sin(kx) | f \rangle \sin(kx) + \langle \cos(kx) | f \rangle \cos(kx).$$

- (a) Compute the Fourier series of x^2 .
- (b) Compute the Fourier series of x^3 .

Solution:

(a)

$$\begin{aligned}
\text{proj}_{W_n}(x^2) &= \langle 1/2 | x^2 \rangle + \sum_{k=1}^n \langle \sin(kx) | x^2 \rangle \sin(kx) + \langle \cos(kx) | x^2 \rangle \cos(kx) \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 / 2 dx + \dots \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx + \dots \\
&= \frac{1}{2\pi} \left[\frac{x^3}{3} \right]_{x=-\pi}^{\pi} + \dots \\
&= \frac{1}{2\pi} \left[\frac{\pi^3}{3} + \frac{\pi^3}{3} \right] + \dots \\
&= \frac{\pi^2}{3} + \sum_{k=1}^n \langle \sin(kx) | x^2 \rangle \sin(kx) + \langle \cos(kx) | x^2 \rangle \cos(kx) \\
&= \frac{\pi^2}{3} + \sum_{k=1}^n \langle \cos(kx) | x^2 \rangle \cos(kx) \\
&= \frac{\pi^2}{3} + \sum_{k=1}^n \left(\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(kx) dx \right) \cos(kx) \\
&= \frac{\pi^2}{3} + \sum_{k=1}^n \frac{1}{\pi} \left(\left[\frac{x^2}{k} \sin(kx) + \frac{2x}{k^2} \cos(kx) - \frac{2}{k^3} \sin(kx) \right]_{x=-\pi}^{\pi} \right) \cos(kx) \\
&= \frac{\pi^2}{3} + \sum_{k=1}^n \frac{1}{\pi} \left(\frac{2\pi}{k^2} \cos(k\pi) - \frac{2(-\pi)}{k^2} \cos(k\pi) \right) \cos(kx) \\
&= \frac{\pi^2}{3} + \sum_{k=1}^n \frac{1}{\pi} \left(\frac{4\pi}{k^2} \cos(k\pi) \right) \cos(kx) \\
&= \frac{\pi^2}{3} + \sum_{k=1}^n \frac{1}{\pi} \left(\frac{4(-1)^k}{k^2} \right) \cos(kx)
\end{aligned}$$

(b)

$$\begin{aligned}
\text{proj}_{W_n} &= \langle 1/2 | x^3 \rangle + \sum_{k=1}^n \langle \sin(kx) | x^3 \rangle \sin(kx) + \langle \cos(kx) | x^3 \rangle \cos(kx) \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 / 2 dx + \dots \\
&= 0 + \sum_{k=1}^n \langle \sin(kx) | x^3 \rangle \sin(kx) \\
&= \sum_{k=1}^n \left(\frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \sin(kx) dx \right) \sin(kx) \\
&= \sum_{k=1}^n \frac{1}{\pi} \left[-\frac{x^3}{k} \cos(kx) - \frac{3x^2}{k^2} \sin(kx) + \frac{6x}{k^3} \cos(kx) + \frac{6}{k^4} \sin(kx) \right]_{x=-\pi}^{\pi} \sin(kx) \\
&= \sum_{k=1}^n \frac{1}{\pi} \left[-\frac{x^3}{k} \cos(kx) + \frac{6x}{k^3} \cos(kx) \right]_{x=-\pi}^{\pi} \sin(kx) \\
&= \sum_{k=1}^n \frac{1}{\pi} \left(-\frac{\pi^3}{k} \cos(k\pi) + \frac{6\pi}{k^3} \cos(k\pi) + \frac{(-\pi)^3}{k} \cos(kx) - \frac{6(-\pi)}{k^3} \cos(kx) \right) \sin(kx)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^n \frac{1}{\pi} \left(-\frac{2\pi^2}{k} \cos(k\pi) + \frac{12\pi}{k^3} \cos(k\pi) \right) \sin(kx) \\
&= \sum_{k=1}^n \left(-\frac{2\pi^2}{k} (-1)^k + \frac{12}{k^3} (-1)^k \right) \sin(kx)
\end{aligned}$$