Math 2700.009: Linear Algebra

Problem Set 14

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Question 1

Are the following matrices orthogonal?

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\vartheta) & -\sin(\vartheta) \\ 0 & 0 & \sin(\vartheta) & \cos(\vartheta) \end{bmatrix} \quad B = \begin{bmatrix} 1/\sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{bmatrix}$$

Where $\vartheta \in [0, 2\pi)$.

Note:-

Recall:
A matrix $A \in \mathbb{R}^{n\times n}$ is orthogonal if:

- The columns of A form an orthogonal basis for $\mathbb{R}^n$
- $A^\top A = I_n$
- $A^\top = A^{-1}$

Solution:
(a) To verify if A is orthogonal, we can check if $A^\top A = I_4$.

$$\begin{aligned}
A^\top A &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\vartheta) & \sin(\vartheta) \\ 0 & 0 & -\sin(\vartheta) & \cos(\vartheta) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\vartheta) & -\sin(\vartheta) \\ 0 & 0 & \sin(\vartheta) & \cos(\vartheta) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (\cos^2(\vartheta) + \sin^2(\vartheta)) & (\cos(\vartheta)\sin(\vartheta) - \cos(\vartheta)\sin(\vartheta)) \\ 0 & 0 & (\cos(\vartheta)\sin(\vartheta) - \cos(\vartheta)\sin(\vartheta)) & (\cos^2(\vartheta) + \sin^2(\vartheta)) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{By trigonometric identities and simplification} \\
&= I_4
\end{aligned}$$

Hence we can say A is orthogonal since $A^\top A = I_4$

(b) Similarly, we can check the orthagonality of B by finding $B^\top B$.

$$\begin{aligned}
B^\top B &= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
&\neq I_3
\end{aligned}$$

We can see that the product $B^\top B$ is not the identity matrix, so thus B is not orthogonal.

(c) Lastly, we can check the orthagonality of C by finding the product $C^\top C$ and checking if it is I_3.

$$C^\top C = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{bmatrix}$$

$$C^\top C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= I_3$$

Hence we can say C is orthogonal since $C^\top C = I_3$

Question 2

Verify that if $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then $Q^\top$ is an orthogonal matrix

Solution: Since Q is orthogonal, then we can conclude that $Q^\top Q = I$. This suggests that $Q^\top$ is a left inverse to Q. If we would like to show that $Q^\top$ is orthogonal when Q is orthogonal, then we must show that

$$(Q^\top)^\top Q^\top = QQ^\top = I$$

To do so, we must show that $Q^\top$ is also a right inverse to Q.

Proof. Suppose Q has some right inverse M. Let us prove that if $Q^\top$ is a left inverse to Q that it is also a right inverse.

$$Q^\top = Q^\top I = Q^\top(QM) = (Q^\top Q)M = IM = M$$
$$Q^\top = M$$

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The simple proof suggests that if Q has a left inverse $Q^\top$, then it is also a right inverse to Q. Hence we can say that $Q^\top$ is orthogonal since $QQ^\top = I$.

Question 3

Let $T : C([0,1]) \longrightarrow C([0,1])$ be defined by

$$T(f) = \sqrt{3}xf\left(x^3\right).$$

Verify that T is an orthogonal transformation where the inner-product on $C([0,1])$ is

$$\langle f \mid g\rangle = \int_0^1 f(x)g(x)\mathrm{d}x.$$

Note:-

Recall:
If V is an inner-product space and $T : V \longrightarrow V$ is a linear transformation, then T is called orthogonal if for all $\vec{v} \in V$, $||\vec{v}|| = ||T(\vec{v})||$, in other words, a linear transformation on an inner-product space is orthogonal if it is length preserving.

Solution: Let us check if $T(f)$ is length preserving on $C([0,1])$ with the given inner-product, first by checking the length of $f(x)$.

$$||f(x)|| = \sqrt{\langle f(x)|f(x)\rangle}$$
$$= \sqrt{\int_0^1 f(x)f(x)\mathrm{d}x}$$
$$= \sqrt{\int_0^1 f(x)^2\mathrm{d}x}$$

Now let us check the length of $T(f)$.

$$\begin{aligned}
||T(f)|| &= \sqrt{\langle T(f)|T(f)\rangle} \\
&= \sqrt{\langle \sqrt{3}xf(x^3)|\sqrt{3}xf(x^3)} \\
&= \sqrt{\int_0^1 (\sqrt{3}xf(x^3))^2 \mathrm{d}x} \\
&= \sqrt{\int_0^1 3x^2 f(x^3)^2 \mathrm{d}x} \qquad \begin{bmatrix} u = x^3 \\ \mathrm{d}u = 3x^2\mathrm{d}x \end{bmatrix} \\
&= \sqrt{\int_{0^3}^{1^3} f(u)^2 \mathrm{d}u} \\
&= \sqrt{\int_0^1 f(u)^2 \mathrm{d}u} \\
&= \sqrt{\int_0^1 f(x)^2 \mathrm{d}x} \\
&= ||f(x)||
\end{aligned}$$

Since we have demonstrated the equality of $||f(x)||$ and $||T(f)||$, then we have also demonstrated that T is length perserving on the inner-product space and hence T is orthogonal.

Question 4

Find the QR factorization of the following matrices.

(a)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

(c)

$$C = \begin{bmatrix} \cos(\vartheta) & -\sin(\vartheta) & 0 & 0 \\ \sin(\vartheta) & \cos(\vartheta) & 0 & 0 \\ 0 & 0 & \cos(\tau) & -\sin(\tau) \\ 0 & 0 & \sin(\tau) & \cos(\tau) \end{bmatrix}$$

where $\vartheta, \tau \in [0, 2\pi)$.

Solution:

(a) Let us call the columns of A as vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$. To find the QR factorization, we must first perform Gram-Schmidt, on the vectors.

$$\begin{aligned}\vec{y}_1 &= \frac{1}{||\vec{a}_1||}\vec{a}_1 \\ &= \frac{1}{\sqrt{1+4+9}}\vec{a}_1 \\ &= \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix} \\ &= \begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix}\end{aligned}$$

This also tells us that $\vec{a}_1 = (\sqrt{14})\vec{y}_1$. This will be important for when we construct R.

$$\begin{aligned}\vec{b}_2 &= \vec{a}_2 - (\vec{a}_2 \cdot \vec{y}_1)\vec{y}_1 \\ &= \vec{a}_2 - (1/\sqrt{14} + 3/\sqrt{14})\vec{y}_1 \\ &= \vec{a}_2 - (4/\sqrt{14})\vec{y}_1 \\ &= \begin{bmatrix}1\\0\\1\end{bmatrix} - 4/\sqrt{14}\begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix} \\ &= \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}4/14\\8/14\\12/14\end{bmatrix} \\ &= \begin{bmatrix}10/14\\-8/14\\2/14\end{bmatrix} = \begin{bmatrix}5/7\\-4/7\\1/7\end{bmatrix}\end{aligned}$$

$$\begin{aligned}||\vec{b}_2|| &= \sqrt{\frac{25}{49} + \frac{16}{49} + \frac{1}{49}} \\ &= \sqrt{\frac{42}{49}} \\ &= \frac{\sqrt{42}}{7}\end{aligned}$$

$$\begin{aligned}\vec{y}_2 &= \frac{1}{||\vec{b}_2||}\vec{b}_2 \\ &= 7/\sqrt{42}\begin{bmatrix}5/7\\-4/7\\1/7\end{bmatrix} \\ &= \begin{bmatrix}5/\sqrt{42}\\-4/\sqrt{42}\\1/\sqrt{42}\end{bmatrix}\end{aligned}$$

Similarly, we need to see the linear combination of $\vec{y}_i$'s that construct $\vec{a}_2$

$$\begin{aligned}\vec{b}_2 &= (\sqrt{42}/7)\vec{y}_2 \\ \vec{a}_2 - (4/\sqrt{14})\vec{y}_1 &= (\sqrt{42}/7)\vec{y}_2 \\ \vec{a}_2 &= (4/\sqrt{14})\vec{y}_1 + (\sqrt{42}/7)\vec{y}_2\end{aligned}$$

$$\begin{aligned}\vec{b}_3 &= \vec{a}_3 - (\vec{a}_3 \cdot \vec{y}_2)\vec{y}_2 - (\vec{a}_3 \cdot \vec{y}_1)\vec{y}_1 \\ &= \vec{a}_3 - (5/\sqrt{42} - 4/\sqrt{42} - 1/\sqrt{42})\vec{y}_2 \\ &\quad - (1/\sqrt{14} + 2/\sqrt{14} - 3/\sqrt{14})\vec{y}_1 \\ &= \vec{a}_3 - (0)\vec{y}_2 - (0)\vec{y}_1 \\ &= \vec{a}_3\end{aligned}$$

$$\begin{aligned}||\vec{b}_3|| &= \sqrt{1+1+1} \\ &= \sqrt{3}\end{aligned}$$

$$\begin{aligned}\vec{y}_3 &= \frac{1}{||\vec{b}_3||}\vec{b}_3 \\ &= 1/\sqrt{3}\begin{bmatrix}1\\1\\-1\end{bmatrix} \\ &= \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\-1/\sqrt{3}\end{bmatrix}\end{aligned}$$

Since $\vec{b}_3 = \vec{a}_3$, then we can quickly compute that $\vec{a}_3 = (\sqrt{3})\vec{y}_3$.

So now to construct the QR factorization of A, we will use the vectors that form the orthogonal basis as columns for Q, and the columns of R will be the linear combination of $\vec{y}_i$ vectors that construct our $\vec{a}_i$'s, so we see that

$$A = QR = \begin{bmatrix} 1/\sqrt{14} & 5/\sqrt{42} & 1/\sqrt{3} \\ 2/\sqrt{14} & -4/\sqrt{42} & 1/\sqrt{3} \\ 3/\sqrt{14} & 1/\sqrt{42} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{14} & 4/\sqrt{14} & 0 \\ 0 & \sqrt{42}/7 & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

(b) Like before, let us call the columns of B as vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ and then perform Gram-Schmidt on the vectors.

$$\begin{aligned} \vec{y}_1 &= \frac{1}{||\vec{a}_1||}\vec{a}_1 \\ &= \frac{1}{\sqrt{1+1+1}}\vec{a}_1 \\ &= 1/\sqrt{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \end{aligned}$$

We can see that $\vec{a}_1 = (\sqrt{3})\vec{y}_1$.

$$\begin{aligned} \vec{b}_2 &= \vec{a}_2 - (\vec{a}_2 \cdot \vec{y}_1)\vec{y}_1 \\ &= \vec{a}_2 - (2/\sqrt{3} - 1/\sqrt{3} - 1/\sqrt{3})\vec{y}_1 \\ &= \vec{a}_2 - (0)\vec{y}_1 \\ &= \vec{a}_2 \end{aligned}$$

$$\begin{aligned} ||\vec{b}_2|| &= \sqrt{4+1+1} \\ &= \sqrt{6} \end{aligned}$$

$$\begin{aligned} \vec{y}_2 &= \frac{1}{||\vec{b}_2||}\vec{b}_2 \\ &= 1/\sqrt{6}\begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \end{aligned}$$

So we can see that $\vec{a}_2 = (\sqrt{6})\vec{y}_2$

$$\begin{aligned} \vec{b}_3 &= \vec{a}_3 - (\vec{a}_3 \cdot \vec{y}_2)\vec{y}_2 - (\vec{a}_3 \cdot \vec{y}_1)\vec{y}_1 \\ &= \vec{a}_3 - (2/\sqrt{6} - 1/\sqrt{6})\vec{y}_2 - (1/\sqrt{3} + 1/\sqrt{3})\vec{y}_1 \\ &= \vec{a}_3 - (1/\sqrt{6})\vec{y}_2 - (2/\sqrt{3})\vec{y}_1 \\ &= \vec{a}_3 - (1/\sqrt{6})\begin{bmatrix} 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} - (2/\sqrt{3})\begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/6 \\ 0 \\ -1/6 \\ -1/6 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 0 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} ||\vec{b}_3|| &= \sqrt{1/4 + 1/4} \\ &= 1/2\sqrt{2} \end{aligned}$$

$$\begin{aligned} \vec{y}_3 &= \frac{1}{||\vec{b}_3||}\vec{b}_3 \\ &= 1/2\sqrt{2}\begin{bmatrix} 0 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix} \end{aligned}$$

We now need to construct $\vec{a}_3$ using a linear combination of $\vec{y}_i$'s

$$\begin{aligned}
\vec{b}_3 &= (1/2\sqrt{2})\vec{y}_3 \\
\vec{a}_3 - (1/\sqrt{6})\vec{y}_2 - (2/\sqrt{3})\vec{y}_1 &= (1/2\sqrt{2})\vec{y}_3 \\
\vec{a}_3 &= (2/\sqrt{3})\vec{y}_1 + (1/\sqrt{6})\vec{y}_2 + (1/2\sqrt{2})\vec{y}_3
\end{aligned}$$

$$\begin{aligned}
\vec{b}_4 &= \vec{a}_4 - (\vec{a}_4 \cdot \vec{y}_3)\vec{y} - (\vec{a}_4 \cdot \vec{y}_2)\vec{y}_2 - (\vec{a}_4 \cdot \vec{y}_1)\vec{y}_1 \\
&= \vec{a}_4 - (-\sqrt{2})\vec{y}_3 - (-1/\sqrt{6})\vec{y}_2 - (1/\sqrt{3})\vec{y}_1 \\
&= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + (\sqrt{2}) \begin{bmatrix} 0 \\ 0 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix} + (1/\sqrt{6}) \begin{bmatrix} 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \\
&\quad - (1/\sqrt{3}) \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2/6 \\ 0 \\ -1/6 \\ -1/6 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 0 \\ 1/3 \\ 1/3 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \\ -3/2 \\ 3/2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
||\vec{b}_4|| &= \sqrt{1 + 9/4 + 9/4} \\
&= \sqrt{22/4} \\
&= \sqrt{22}/2
\end{aligned}$$

$$\begin{aligned}
\vec{y}_4 &= \frac{1}{||\vec{b}_4||}\vec{b}_4 \\
&= 2/\sqrt{22} \begin{bmatrix} 0 \\ 1 \\ -3/2 \\ 3/2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 2/\sqrt{22} \\ -3/\sqrt{22} \\ 3/\sqrt{22} \end{bmatrix}
\end{aligned}$$

We now need to construct $\vec{a}_4$ using a linear combination of $\vec{y}_i$'s

$$\begin{aligned}
\vec{b}_4 &= (\sqrt{22}/2)\vec{y}_4 \\
\vec{a}_4 + (\sqrt{2})\vec{y}_3 + (1/\sqrt{6})\vec{y}_2 - (1/\sqrt{3})\vec{y}_1 &= (\sqrt{22}/2)\vec{y}_4 \\
\vec{a}_4 &= (1/\sqrt{3})\vec{y}_1 - (1/\sqrt{6})\vec{y}_2 - (\sqrt{2})\vec{y}_3 + (\sqrt{22}/2)\vec{y}_4
\end{aligned}$$

So now to construct the QR factorization of B, we will use the vectors that form the orthogonal basis as columns for Q, and the columns of R will be the linear combination of $\vec{y}_i$ vectors that construct our $\vec{a}_i$'s, so we see that

$$B = QR = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2/\sqrt{22} \\ 1/\sqrt{3} & -1/\sqrt{6} & -\sqrt{2} & -3/\sqrt{22} \\ 1/\sqrt{3} & -1/\sqrt{6} & \sqrt{2} & 3\sqrt{22} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & \sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \\ 0 & 0 & 1/2\sqrt{2} & -\sqrt{2} \\ 0 & 0 & 0 & \sqrt{22}/2 \end{bmatrix}$$