

Math 2700.009  
Problem Set 10

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### Question 1

Compute the determinant of the following by doing a cofactor expansion about any row or column of your choosing.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 6 & -1 & 2 & 4 \\ 1 & -1 & 1 & 0 & 1 \end{bmatrix}$$

#### Note:-

Computing the determinant of a matrix by computing the cofactor expansion about a given row or column will always product the same result regardless of which column or row is picked. In this case it is a good idea to select the row or column with the most zeros, since it means a majority of the summands will be nullified.

**Solution:** For this cofactor expansion, we can iterate about  $i = 3$ .

$$\begin{aligned} & \det \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 6 & -1 & 2 & 4 \\ 1 & -1 & 1 & 0 & 1 \end{bmatrix} \\ &= 0 \det \begin{bmatrix} 2 & 3 & 4 & 5 \\ -1 & 1 & 3 & 2 \\ 6 & -1 & 2 & 4 \\ -1 & 1 & 0 & 1 \end{bmatrix} - 0 \det \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & 3 & 2 \\ 1 & -1 & 2 & 4 \\ 1 & 1 & 0 & 1 \end{bmatrix} + 0 \det \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & -1 & 3 & 2 \\ 1 & 6 & 2 & 4 \\ 1 & -1 & 0 & 1 \end{bmatrix} - 0 \det \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -1 & 1 & 2 \\ 1 & 6 & -1 & 4 \\ 1 & -1 & 1 & 1 \end{bmatrix} + 1 \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 1 & 3 \\ 1 & 6 & -1 & 2 \\ 1 & -1 & 1 & 0 \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 1 & 3 \\ 1 & 6 & -1 & 2 \\ 1 & -1 & 1 & 0 \end{bmatrix} \end{aligned}$$

We can see that performing a cofactor expansion about the 3rd row reduces the computation finding the determinant of just a single  $4 \times 4$  matrix. Now we can perform another cofactor expansion on this matrix.

$$\begin{aligned} & \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 1 & 3 \\ 1 & 6 & -1 & 2 \\ 1 & -1 & 1 & 0 \end{bmatrix} j = 1 \\ &= \det \begin{bmatrix} -1 & 1 & 3 \\ 6 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix} - 0 \det \begin{bmatrix} 2 & 3 & 4 \\ 6 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix} + \det \begin{bmatrix} 2 & 3 & 4 \\ -1 & 1 & 3 \\ -1 & 1 & 0 \end{bmatrix} - \det \begin{bmatrix} 2 & 3 & 4 \\ -1 & 1 & 3 \\ 6 & -1 & 2 \end{bmatrix} \\ &= \det \begin{bmatrix} -1 & 1 & 3 \\ 6 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix} + \det \begin{bmatrix} 2 & 3 & 4 \\ -1 & 1 & 3 \\ -1 & 1 & 0 \end{bmatrix} - \det \begin{bmatrix} 2 & 3 & 4 \\ -1 & 1 & 3 \\ 6 & -1 & 2 \end{bmatrix} \end{aligned}$$

Now we can find the determinant of these  $3 \times 3$  matrices.

$$\begin{aligned} & \det \begin{bmatrix} -1 & 1 & 3 \\ 6 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix} j = 3 \\ &= 3 \det \begin{bmatrix} 6 & -1 \\ -1 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + 0 \det \begin{bmatrix} -1 & 1 \\ 6 & -1 \end{bmatrix} \\ &= 3(5) - 2(0) \\ &= 15 \end{aligned}$$

$$\begin{aligned}
& \det \begin{bmatrix} 2 & 3 & 4 \\ -1 & 1 & 3 \\ -1 & 1 & 0 \end{bmatrix} j = 3 \\
&= 4 \det \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} - 3 \det \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} + 0 \det \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \\
&= 4(0) - 3(5) \\
&= -15
\end{aligned}$$

$$\begin{aligned}
& \det \begin{bmatrix} 2 & 3 & 4 \\ -1 & 1 & 3 \\ 6 & -1 & 2 \end{bmatrix} j = 2 \\
&= -3 \det \begin{bmatrix} -1 & 3 \\ 6 & 2 \end{bmatrix} + \det \begin{bmatrix} 2 & 4 \\ 6 & 2 \end{bmatrix} + \det \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} \\
&= -3(-20) + (-20) + (10) \\
&= 60 - 20 + 10 \\
&= 50
\end{aligned}$$

Since we have calculated the determinants of all the minor matrices, we can now find the sum to find the determinant of the original matrix, and we find that

$$\begin{aligned}
& \det \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 6 & -1 & 2 & 4 \\ 1 & -1 & 1 & 0 & 1 \end{bmatrix} = 15 - 15 - 50 \\
&= -50
\end{aligned}$$

## Question 2

Compute the determinant of each of the following

$$A = \begin{bmatrix} 1 & 47 & \pi & \ln(2) & \sqrt{2} \\ 0 & 2 & \sqrt{3} & 100 & -5 \\ 0 & 0 & 3 & -\pi & 2 \\ 0 & 0 & 0 & 4 & 47^2 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 11 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

### Note:-

Recall:

- If a matrix A is either triangular or diagonal, the determinant can be computed by taking the product of its diagonal elements

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}$$

$$\det(A) = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}$$

- Similarly, if a matrix A is either block triangular or block diagonal, the determinant can be computed

by taking the product of the determinants of its diagonal elements

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ & A_{22} & & A_{2n} \\ & & \ddots & \vdots \\ & & & A_{nn} \end{bmatrix}$$

$$\det(A) = \det(A_{11}) \cdot \det(A_{22}) \cdot \dots \cdot \det(A_{nn})$$

**Solution:** Since  $A$  is triangular, we can find its determinant by taking the product of its diagonal elements

$$\det \begin{bmatrix} 1 & 47 & \pi & \ln(2) & \sqrt{2} \\ 0 & 2 & \sqrt{3} & 100 & -5 \\ 0 & 0 & 3 & -\pi & 2 \\ 0 & 0 & 0 & 4 & 47^2 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

Since  $B$  is block triangular, we can find its determinant by taking the product of the determinants of its diagonal elements. Notice that I call this matrix block triangular instead of block diagonal. This is because the result will be the same however if we are to call it block diagonal then the largest matrix we need to compute the determinant of is  $3 \times 3$ , while if we call it block triangular then the largest matrix is  $2 \times 2$

$$\begin{aligned} \det \begin{bmatrix} 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 11 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} &= \det \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \cdot \det \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \det [1] \cdot [11] \cdot \det \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \\ &= (2 - 6) \cdot (1 - 1) \cdot 1 \cdot 11 \cdot (8 - 1) \\ &= -4 \cdot 0 \cdot 11 \cdot 7 \\ &= 0 \end{aligned}$$

### Question 3

(a) Compute the determinant of the following  $16 \times 16$  matrix  $E$ :

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 1 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & -1 & 2 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 2 & 1 & 0 & 2 & -3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 3 & 0 & 4 & 0 & 4 & 2 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 & 1 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -2 & -2 & 0 & -2 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -4 & 2 & -4 & -2 & 0 & -4 & 6 & 1 & 1 \end{bmatrix}$$

(b) Do the columns of the above matrix form a basis for  $\mathbb{R}^{16}$ ? What about the rows, do the rows of  $E$

form a basis for  $\mathbb{R}^{16}$ ?  
(c) What is  $\det(E^T)$ ?

**Note:-**

Recall:

- If  $E$  is a blocked off matrix

$$E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and  $A$  is an invertible matrix, then  $\det(E) = \det(A) \det(D - CA^{-1}B)$

- If a matrix  $A \in \mathbb{R}^{n \times n}$  is a diagonal matrix, then the inverse can be computed by taking the reciprocal of each diagonal element

$$A^{-1} = \begin{bmatrix} 1/a_{11} & & & \\ & 1/a_{22} & & \\ & & \ddots & \\ & & & 1/a_{nn} \end{bmatrix}$$

- $\det(A) = \det(A^T)$

**Solution:**

(a) We can divide this matrix into 4 unique  $8 \times 8$  matrices,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & -1 & 1 & 0 \\ 2 & -1 & 0 & 1 & -1 & 2 & -1 & 1 \\ 3 & 1 & 1 & 0 & 2 & 0 & 2 & 1 \\ 0 & -2 & 2 & 0 & 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & -1 & -1 \\ 2 & -1 & 2 & 1 & 0 & 2 & -3 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 & -1 & 1 & 0 \\ 2 & 0 & 0 & 1 & -1 & 2 & -1 & 1 \\ 6 & 2 & 3 & 0 & 4 & 0 & 4 & 2 \\ 0 & -3 & 3 & 1 & 0 & 3 & 0 & 3 \\ 1 & 1 & 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & -2 & -2 & 0 & -2 & 1 & -2 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & -2 & -1 \\ -4 & 2 & -4 & -2 & 0 & -4 & 6 & 1 \end{bmatrix}$$

As seen previously, we can find the determinant of a diagonal matrix by finding the product of the diagonal elements, so we can quickly find the  $\det(A)$

$$\begin{aligned} \det(A) &= 1 \cdot -1 \cdot 1 \cdot 2 \cdot 1 \cdot 1/2 \cdot 1 \cdot -1 \\ &= 1 \end{aligned}$$

Now we must find the inverse of  $A$  ( $A^{-1}$ ) This is easy since  $A$  is a diagonal matrix

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Now we can compute the operations to find the matrix that results from  $D - CA^{-1}B$ , Firstly lets find  $CA^{-1}B$ .

$$\begin{aligned}
CA^{-1} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}
\end{aligned}$$

With the matrix  $CA^{-1}$ , we can multiply by  $B$  to find  $CA^{-1}B$

$$\begin{aligned}
CA^{-1}B &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & -1 & 1 & 0 \\ 2 & -1 & 0 & 1 & -1 & 2 & -1 & 1 \\ 3 & 1 & 1 & 0 & 2 & 0 & 2 & 1 \\ 0 & -2 & 2 & 0 & 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & -1 & -1 \\ 2 & -1 & 2 & 1 & 0 & 2 & -3 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & -1 & 1 & 0 \\ 2 & -1 & 0 & 1 & -1 & 2 & -1 & 1 \\ 6 & 2 & 2 & 0 & 4 & 0 & 4 & 2 \\ 0 & -3 & 3 & 0 & 0 & 3 & 0 & 3 \\ 1 & 1 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & -2 & -2 & 0 & -2 & 0 & -2 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & -1 & -1 \\ -4 & 2 & -4 & -2 & 0 & -4 & 6 & 0 \end{bmatrix}
\end{aligned}$$

Now we can finally compute  $D - CA^{-1}B$ ,

$$D - CA^{-1}B = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 & -1 & 1 & 0 \\ 2 & 0 & 0 & 1 & -1 & 2 & -1 & 1 \\ 6 & 2 & 3 & 0 & 4 & 0 & 4 & 2 \\ 0 & -3 & 3 & 1 & 0 & 3 & 0 & 3 \\ 1 & 1 & 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & -2 & -2 & 0 & -2 & 1 & -2 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & -2 & -1 \\ -4 & 2 & -4 & -2 & 0 & -4 & 6 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & -1 & 1 & 0 \\ 2 & -1 & 0 & 1 & -1 & 2 & -1 & 1 \\ 6 & 2 & 2 & 0 & 4 & 0 & 4 & 2 \\ 0 & -3 & 3 & 0 & 0 & 3 & 0 & 3 \\ 1 & 1 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & -2 & -2 & 0 & -2 & 0 & -2 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & -1 & -1 \\ -4 & 2 & -4 & -2 & 0 & -4 & 6 & 0 \end{bmatrix}$$

Since we have found the matrix  $D - CA^{-1}B$ , and have also found it to be a diagonal matrix, the determinant will be the product of the diagonal elements

$$\begin{aligned}\det(D - CA^{-1}B) &= 1 \cdot 1 \cdot 1 \cdot 1 \cdot -1 \cdot 1 \cdot -1 \cdot 1 \\ &= 1\end{aligned}$$

So now we can compute the  $\det(E)$  by finding the product of the two determinants we have found, in particular

$$\begin{aligned}\det(E) &= \det(A) \det(D - CA^{-1}B) \\ &= (1)(1) \\ &= 1\end{aligned}$$

(b) The result of the determinant calculated in part (a) reveals all the information necessary to answer both of these questions. We know from the Invertible Matrix Theorem a few statements are equivalent for a given a matrix  $A \in \mathbb{R}^{n \times n}$ . In particular we see that

- $\det(A) \neq 0$
- The columns of  $A$  are linearly independent
- The columns of  $A$  span  $\mathbb{R}^n$
- The rows of  $A$  are linearly independent.
- The rows of  $A$  span  $\mathbb{R}^n$

We computed the  $\det(E)$  to be 1, meaning that the matrix is invertible though that is not important or information that is necessary to answer this question. By the Invertible Matrix Theorem, we know that the columns of  $E$  are linearly independent, as well as spanning of  $\mathbb{R}^{16}$ , so thus the columns form a basis for  $\mathbb{R}^{16}$ .

Further, the Invertible Matrix Theorem also tells us that since  $\det(E)$  is nonzero then the rows are linearly independent and also spanning of  $\mathbb{R}^{16}$ , and thus the rows also form a basis for  $\mathbb{R}^{16}$ .

(c)  $\det(E^T) = \det(E) = 1$

#### Question 4

- (a) Find the characteristic polynomial of  $B$  from problem 2.
- (b) Find all eigenvalues of  $B$
- (c) For each eigenvalue of  $B$ , find an eigenvector.

#### Note:-

Recall:

- The characteristic polynomial of  $A \in \mathbb{R}^{n \times n}$  is  $\det(\lambda I_n - A)$
- $\vec{0} = \lambda I_n \vec{x} - A \vec{x} = (\lambda I_n - A) \vec{x}$

#### Solution:

(a) Characteristic polynomial of  $B$

$$\det(\lambda I_8 - B) = \det \begin{bmatrix} \lambda - 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & \lambda - 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda - 1 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & \lambda - 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda - 11 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda - 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & \lambda - 2 \end{bmatrix}$$

We can use the same principle as before to find the determinant of a block triangular matrix, so that we can find the eigenvalues

$$\begin{aligned}
 &= ((\lambda - 2)(\lambda - 1) - 6)((\lambda - 1)(\lambda - 1) - 1)(\lambda - 1)(\lambda - 11)((\lambda - 4)(\lambda - 2) + 1) \\
 &= (\lambda^2 - 3\lambda - 4)(\lambda^2 - 2\lambda)(\lambda - 1)(\lambda - 11)(\lambda^2 - 6\lambda + 9) \\
 &= (\lambda - 4)(\lambda + 1)(\lambda)(\lambda - 2)(\lambda - 1)(\lambda - 11)(\lambda - 3)^2
 \end{aligned}$$

(b) With the characteristic polynomial of  $B$  found from part (a), We can find the zeros to get the eigenvalues of  $B$ . We find that

- $\lambda_1 = 4$
- $\lambda_2 = -1$
- $\lambda_3 = 0$
- $\lambda_4 = 2$
- $\lambda_5 = 1$
- $\lambda_6 = 11$
- $\lambda_7 = 3$