

Quantum Decision, Quantum Logic, and Fuzzy sets

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Abstract

In the paper, we show that quantum logic of linear subspaces can be used for recognition of random signals by a Bayesian energy discriminant classifier. The energy distribution on linear subspaces is described by the correlation matrix of the probability distribution. We show that the correlation matrix corresponds to von Neumann density matrix in quantum theory. We suggest the interpretation of quantum logic as a fuzzy logic of fuzzy sets. The use of quantum logic for recognition is based on the fact that the probability distribution of each class lies approximately in a lower-dimensional subspace of feature space. We offer the interpretation of discriminant functions as membership functions of fuzzy sets. Also we offer the quality functional for optimal choice of discriminant functions for recognition from some class of discriminant functions.

Key words: recognition, quantum logic, discriminant function, fuzzy set, von Neumann density matrix, membership functions, subspace classifier, quality functional, quantum decision.

1 Introduction

A Bayesian probabilistic discriminant classifier is based on a classical probability theory using algebra of subsets. The decision rule of the probabilistic classifier maximizes the probability of “correct” recognition. A Bayesian energy discriminant classifier was briefly presented in [12]. The algebra of linear subspaces (quantum logic) is used instead of algebra of subsets. The decision rule of energy classifier maximizes the energy of “correct” recognition. The recognition of two classes is considered in detail. The use of quantum logic for recognition of signals is considered in [10].

The use of linear subspaces as class models is based on the assumption that the distribution of each class lies approximately in a lower-dimensional subspace of feature space. These spaces can be found by principal components analysis carried out individually on each class. An input vector from the unknown class is classified according to the greatest projection to the subspaces, each of which represents one class.

The subspace classifier was suggested by Watanabe (method CLAFIC [3], [4]). This method, however, has drawbacks: a priori probabilities of classes are not used; subspaces of classes can overlap. T. Kohonen has offered the Learning Subspace Method (LSM) [2], [3].

During the training LSM decreases the number of vectors that are included in subspaces of different classes. The recognition of handwritten signs by the subspace classifier is considered in [4]. The subspace classifier is applied to phonemes recognition in [5] and to speaker recognition in [6].

Y.C. Eldar and A.V. Oppenheim [7] draw a parallel between quantum measurements and algorithms in signal processing. They propose to exploit the rich mathematical structure of quantum theory in signal processing without realization of quantum processes. We suggest to consider energy processes instead of quantum processes because nature spends some energy to create any signal.

2 Quantum logic as an example of fuzzy logic

Let H be a Hilbert space. A fuzzy set A of H is a set of ordered pairs $A = \{x, \mu_A(x) : x \in H\}$ where $\mu_A(x) : H \rightarrow \{0, \infty\}$ is the membership function of the fuzzy set A . Suppose $\mu_A(x)$ be non necessarily normal: $\sup \mu_A(x) \neq 1, x \in H$. A set of membership functions is a partially ordered set equipped with a partial order relation: $\mu_A(x) \leq \mu_B(x)$ for all $x \in H$. The result of operations

$$\mu_A(x) \wedge \mu_B(x) = \inf(\mu_A(x), \mu_B(x)), \quad \mu_A(x) \vee \mu_B(x) = \sup(\mu_A(x), \mu_B(x))$$

is defined pointwise and the result is again a nonnegative function. Hence, the set of membership functions is a lattice.

Each closed linear subspace $M \subset H$ corresponds to an elementary logical proposition of quantum logic. Each linear subspace M has an orthogonal projection P_M onto M . So a proposition of quantum logic can be associated with the orthogonal projection. The set of all orthogonal projections is a lattice equipped with a partial order relation: $P \leq R$ if $\langle Px, x \rangle \leq \langle Rx, x \rangle$ for all $x \in H$. Hence every pair of projections P, R has a unique supremum (least upper bound) and a unique infimum (greatest lower bound):

$$P \wedge R = \inf(P, R), \quad P \vee R = \sup(P, R).$$

Operations $P \wedge R$, $P \vee R$, and $P^\perp = I - P$ are conjunction, disjunction, and negation of quantum logic, respectively.

Each projection P_M on the subspace M can be viewed as a filter [10] and it passes some energy $\mu_M(x) = \langle P_M x, x \rangle = \|P_M x\|^2$ of signal x (in quantum theory, a projection passes some quantum probability. This energy evaluates the value of membership of signal x to subspace M . So each linear subspace $M \subset H$ can be associated with the fuzzy set:

$$A_M = \{x, \mu_M(x) : x \in H, M \subset H\}, \quad \text{where } \mu_M(x) = \langle P_M x, x \rangle.$$

A set of all membership functions $\{\mu_M(x), M \in H\}$ is a lattice equipped with a partial order relation: $\langle P_M x, x \rangle \leq \langle P_N x, x \rangle$ for all $x \in H$. So operations supremum and infimum of that lattice can be used as a fuzzy logic conjunction and disjunction of fuzzy sets $\{A_M, M \in H\}$. A fuzzy logic negation of fuzzy set A_M with membership function $\mu_M(x)$ can be defined as a fuzzy set A_{M^\perp} using the following membership function: $\mu_{M^\perp}(x) = \langle P_{M^\perp} x, x \rangle = \langle P_M^\perp x, x \rangle = \langle (I - P_M)x, x \rangle$, where a subspace M^\perp is an orthogonal complement of subspace M . Thus fuzzy sets $\{A_M, M \in H\}$ form a fuzzy logic.

3 Discriminant functions as membership functions

If an object of recognition is described as a vector $x = (x_1, x_1, \dots, x_n)$, then the vector x is the pattern of the object in the feature space $H = R^n$. A membership of object to some class S_i , $i = 1 \dots l$, is an additional feature, which can be defined as the index i of the class, where $i \in I = \{1, 2, \dots, l\}$.

We use discriminant functions for the classifier of recognition. Discriminant functions are a set of functions $g_i(x)$, $i = 1 \dots l$, that determine the membership of the object with the pattern x to some class S_i according to the following decision rule: if the object with the pattern x satisfies $g_i(x) > g_j(x)$ for all $j \neq i$, then the object having the pattern x belongs to the class S_i .

Discriminant functions split the feature space H into disjoint sets:

$$A_i = \{x: g_i(x) > g_j(x), j = 1 \dots l, j \neq i\}.$$

Thus, if $x \in A_i$, then the object having the pattern x belongs to the class S_i . However, there are sets $\{x: g_i(x) = g_j(x), j \neq i\}$, $i = 1 \dots l$, whose elements it is impossible to include in some set A_i , $i = 1 \dots l$. Usually these sets are included in the sets A_i , $i = 1 \dots l$.

Using discriminant functions, the classifier determines only a “likelihood” value about the membership of the object with the pattern x to some class S_i . So discriminant functions $g_i(x)$, $i = 1 \dots l$, are membership functions. In the following, we assume that discriminant functions are negative and non-necessarily normal: $\sup g_i(x) \neq 1$, $x \in H$, $i = 1 \dots l$.

4 Quality functional for a choice of optimal decision rule

We shall use a probabilistic model for recognition. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space where a sample space Ω is a set of recognition objects. It is evident that the set of recognition classes S_1, S_2, \dots, S_l are a partition of Ω : $S_1 + S_2 + \dots + S_l = \Omega$ where $S_i \cap S_j = \emptyset$ for all $i \neq j$.

Following Zadeh [1], a fuzzy set A is called a fuzzy event if the corresponding membership function $\mu_A(\omega): \Omega \rightarrow \{0, \infty\}$ is \mathcal{A} -measurable. The probability of a fuzzy event is defined as

$$\mathbf{P}(A) = \mathbf{E}\mu_A = \int_{\Omega} \mu_A d\mathbf{P}. \quad (1)$$

Suppose that an object ω is described by the vector $\xi(\omega) = (\xi_1(\omega), \xi_2(\omega), \dots, \xi_n(\omega))$ of features where each $\xi_i(\omega): \Omega \rightarrow H$, $i = 1 \dots l$, is \mathcal{A} -measurable random variable. Since an object ω has the pattern x in the feature space H , there is a map $\xi(\omega): \Omega \rightarrow H$. If $\omega \in S_i$, then we can define an integer-valued random variable γ such that $\gamma(\omega) = i$ for all $\omega \in S_i$, where $i \in I = \{1, 2, \dots, l\}$. The sample space Ω of the objects usually not accessible to immediate observation, therefore it is necessary to deal with the feature space H . However, Ω can be identified with $I \times H$.

We use a Bayesian method which needs a priory probabilities $p_i = \mathbf{P}(S_i)$, $i = 1 \dots l$, and a conditional distributions $\mu_i(A) = \mathbf{P}(\xi \in A | S_i)$, $i = 1 \dots l$. Since $\mathbf{P}(S_i) = \mathbf{P}(\gamma = i)$, it follows that p_i , $i = 1 \dots l$, is the probability distribution of the random variable γ .

Let $\mu(B, A) = \mathbf{P}(\gamma \in B, \xi \in A)$ be a joint distribution of random variables γ, ξ , where $B = \{i_1, i_2, \dots, i_m\} \subset I$ and $A \subset H$. We have $\mu_i(A) = \mathbf{P}(\xi \in A | S_i)$, $i = 1 \dots l$. Since $S_i = (\gamma = i)$, we get

$$\mu(\{i\}, A) = \mathbf{P}(\gamma = i, \xi \in A) = \mathbf{P}(\xi \in A | \gamma = i) \mathbf{P}(\gamma = i) = \mathbf{P}(\xi \in A | S_i) \mathbf{P}(S_i) = p_i \mu_i(A).$$

Let us denote $\mu_1(\{i\}) = p_i$ and $\mu_2^1(i, A) = \mu_i(A)$. We have

$$\begin{aligned} \mu(B, A) &= \mathbf{P}\left(\sum_{k=1}^m (\gamma = i_k) \cap (\xi \in A)\right) = \sum_{k=1}^m \mathbf{P}(\xi \in A | \gamma = i_k) \mathbf{P}(\gamma = i_k) \\ &= \sum_{k=1}^m \frac{\mu(\{i_k\}, A)}{p_{i_k}} p_{i_k} = \sum_{k=1}^m \mu_2^1(i_k, A) \mu_1(\{i_k\}) = \int_B \mu_2^1(i, A) \mu_1(\mathrm{d}i). \end{aligned}$$

It follows that $\mu_2^1(i, A) = \mu_i(A)$ is the transition probability on $I \times \mathcal{B}$ [11], where \mathcal{B} is a σ -algebra of Borel subsets of feature space $H = R^n$.

Discriminant functions $g_i(x)$, $i = 1 \dots l$, define a random variable $g_\gamma(\xi) = g(\gamma, \xi)$. Since $\mu_2^1(i, A) = \mu_i(A)$ is the transition probability on $I \times \mathcal{B}$ [11], we have

$$\mathbf{E}g(\gamma, \xi) = \int_I \mu_1(\mathrm{d}i) \int_H g(i, x) \mu_2^1(i, \mathrm{d}x) = \sum_{i=1}^l p_i \int_H g_i(x) \mu_i(\mathrm{d}x). \quad (2)$$

Suppose $H = A_1 + A_2 + \dots + A_l$, where A_i , $i = 1 \dots l$, are disjoint sets. Let Φ be a class of discriminant functions which contain only indicator functions:

$$g_i(x) = 1_{A_i}(x) = \begin{cases} 1 & \text{if } x \in A_i, \\ 0 & \text{if } x \notin A_i. \end{cases}$$

It is evident that $g_{\gamma(\omega)}(\xi(\omega)) = g(\gamma(\omega), \xi(\omega))$ is the indicator function with a support:

$$G = \sum_{i=1}^l (\xi \in A_i) \cap (\gamma = i) = \sum_{i=1}^l (\xi \in A_i) \cap S_i.$$

We can say that the indicator function $1_G = g(\gamma, \xi)$ is the membership function of “correct” recognition, where G is a crisp event of “correct” recognition. By (2), we have

$$\mathbf{P}(G) = \mathbf{E}g(\gamma, \xi) = \sum_{i=1}^l p_i \int_H g(i, x) \mu_i(\mathrm{d}x) = \sum_{i=1}^l \mathbf{P}(\xi \in A_i | S_i) \mathbf{P}(S_i). \quad (3)$$

A Bayesian probabilistic discriminant classifier splits the feature space H on disjoint sets $H = A_1 + A_2 + \dots + A_l$ such that the probability (3) for the crisp event G of “correct” recognition would be maximal.

Let $g_i(x)$, $i = 1 \dots l$, be discriminant functions from some class Φ , where each function $g_i(x): H \rightarrow \{0, \infty\}$ is a Borel-measurable membership function of class S_i . Then the random variable $g_i(\xi(\omega))$, $i = 1 \dots l$ on Ω is a membership function such that the value $g_i(\xi(\omega))$ is a membership degree of object ω to a class S_i . We define a fuzzy event as follows: $G_i = \{\omega, g_i(\gamma(\omega), \xi(\omega)) : \omega \in \Omega\}$ for all $i = 1 \dots l$.

Let us define the membership function:

$$\mu_j(i, \omega) = 1_{S_j}(\omega)g_i(\xi(\omega)) = \begin{cases} g_i(\xi(\omega)) & \text{if } \omega \in S_j, \\ 0 & \text{if } \omega \notin S_j. \end{cases}$$

This membership function defines the fuzzy event $S_j G_i = \{\omega, \mu_j(i, \omega) : \omega \in \Omega\}$, which is an algebraic product [1] of events G_j and S_i . The value $\mu_j(i, \omega)$ is the membership degree of the object ω to the class S_i if the statement $\omega \in S_j$ is true. There can be two cases. First, if $j = i$, then $\mu_i(i, \omega)$ is the membership degree of the object ω to the class S_i when the object ω belongs to its own class S_i . We call the value $\mu_i(i, \omega)$ a “correct” degree of membership; we call the fuzzy event $S_i G_i$ a fuzzy event of “correct” recognition. Second, if $j \neq i$, then $\mu_j(i, \omega)$ is the membership degree of the object ω to the class S_i when the object ω belongs to other class S_j . We call the value $\mu_j(i, \omega)$, $j \neq i$, an “error” degree of membership; we call the fuzzy event $S_j G_i$, $j \neq i$, a fuzzy event of “error” recognition.

Since $1_{S_i} = 1_{(\gamma=i)}$ for all $i = 1 \dots l$, we can define a membership function:

$$g(\gamma, \xi) = \sum_{i=1}^l 1_{(\gamma=i)} g(\gamma, \xi) = \sum_{i=1}^l 1_{(\gamma=i)} g_i(\xi) = \sum_{i=1}^l 1_{S_i} g_i(\xi) = \sum_{i=1}^l \mu_j(i, \omega).$$

This membership function defines a degree of “correct” membership for all objects $\omega \in \Omega$. We call the random variable $g(\gamma, \xi)$ as a membership function of “correct” recognition and the fuzzy set $G = \{\omega, g(\gamma(\omega), \xi(\omega)) : \omega \in \Omega\}$ as a fuzzy event of “correct” recognition.

It is natural to choose discriminant functions $g_i(x)$, $i = 1 \dots l$ from the class Φ such that the probability of the fuzzy event G of “correct” recognition would be maximal. From (1) and (2), we have that the probability of the fuzzy event G is defined as

$$\mathbf{P}(G) = \mathbf{E}g(\gamma, \xi) = \sum_{i=1}^l p_i \int_H g(i, x) \mu_i(dx). \quad (4)$$

Also (4) defines a quality functional for choice of discriminant functions from the class Φ .

Let us show another interpretation of the quality functional (4). We define

$$1_{(k=j)}(k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Let us denote $\mu_1(\{i\}) = p_i$. Since $1_{S_i} = 1_{(\gamma=j)}$ and $\mu_2^1(i, A) = \mu_i(A)$ is a transition probability on $I \times \mathcal{B}$, it follows that [11]

$$\begin{aligned} \mathbf{E}(1_{S_j} g_i(\xi)) &= \mathbf{E}(1_{(\gamma=j)} g_i(\xi)) \\ &= \int_{I \times H} 1_{(k=j)} g_i(x) \mu(dk, dx) = \int_I \mu_1(dk) 1_{(k=j)} \int_H g_i(x) \mu_2^1(k, dx) \\ &= \sum_{k=1}^l \mu_1(\{k\}) 1_{(k=j)} \int_H g_i(x) \mu_2^1(k, dx) = p_j \int_H g_i(x) \mu_j(dx). \end{aligned}$$

Then the probability of the fuzzy event $S_j G_i = \{\omega, 1_{S_j}(\omega)g_i(\xi(\omega)) : \omega \in \Omega\}$ is defined as

$$r_j(i) = \mathbf{P}(S_j G_i) = \mathbf{E}(1_{S_j} g_i(\xi)) = p_j \int_H g_i(x) \mu_j(dx). \quad (5)$$

We call the value $r_j(i)$ a “correct” probability of recognition if $i = j$ and an “error” probability of recognition if $i \neq j$. The full sum of all the “correct” probability of recognition is defined as

$$\sum_{i=1}^l r_i(i) = \sum_{i=1}^l \mathbf{P}(S_i G_i) = \sum_{k=1}^l \mathbf{E}(1_{S_i} g_i(\xi)) = \sum_{i=1}^l p_i \int_H g_i(x) \mu_i(dx) = \mathbf{E}g(\gamma, \xi) = \mathbf{P}(G).$$

Let us define a conditional expectation of random variable relative to an event:

$$\mathbf{E}(g_i(\xi) 1_{S_i}) = \frac{\mathbf{E}(g_i(\xi) 1_{S_i})}{\mathbf{P}(S_i)} \mathbf{P}(S_i) = \mathbf{E}(g_i(\xi) | S_i) \mathbf{P}(S_i), \quad \text{where } i = 1 \dots l.$$

Then we get one more interpretation of the quality functional (4):

$$\mathbf{P}(G) = \mathbf{E}g(\gamma, \xi) = \mathbf{E}\left(\sum_{i=1}^l 1_{(\gamma=i)} g(\gamma, \xi)\right) = \sum_{i=1}^l \mathbf{E}(1_{S_i} g_i(\xi)) = \sum_{i=1}^l \mathbf{E}(g_i(\xi) | S_i) \mathbf{P}(S_i).$$

5 Basic formula

We consider the features vector $\xi(\omega): \Omega \rightarrow H$ as a random signal. Suppose μ is the probability distribution of the random signal ξ . Let us define one linear form and two bilinear forms for the random signal ξ

$$\langle m, y \rangle = \mathbf{E}\langle \xi, y \rangle = \int_H \langle x, y \rangle \mu(dx),$$

$$\langle Ky, z \rangle = \mathbf{E}(\langle \xi, y \rangle \langle \xi, z \rangle) = \int_H \langle x, y \rangle \langle x, z \rangle \mu(dx), \quad (6)$$

$$\langle Ry, z \rangle = \mathbf{E}(\langle \xi - m, y \rangle \langle \xi - m, z \rangle) = \int_H \langle x - m, y \rangle \langle x - m, z \rangle \mu(dx). \quad (7)$$

A non-random signal m , operator K , and operator R are called a mathematical expectation, correlation operator, and covariance operator, respectively.

From (6) and (7), we have $\langle Ky, z \rangle = \langle Ry, z \rangle + \langle m, y \rangle \langle m, z \rangle$. Then $\langle Ry, z \rangle + \langle m, y \rangle \langle m, z \rangle = \langle (R + p_m)y, z \rangle$, where $p_m y = \langle y, m \rangle m$ is a one-rank operator. It is evident that $p_m y = \|m\|^2 p_{\bar{m}} y$, where $\bar{m} = m/\|m\|$ and $p_{\bar{m}} y = \langle y, \bar{m} \rangle \bar{m}$ is a one-dimensional projection. Then

$$K = R + p_m = R + \|m\|^2 p_{\bar{m}}. \quad (8)$$

Let the signal $x = \xi(\omega)$ be the pattern of the object ω . An affine structure of Hilbert space H is used when realizations of random signal is considered as points. Using a vector structure H , it is possible to interpret a value $\|x\|^2$ as a physical value, for example, as energy, power, or intensity. The value $\|x\|^2$ is a measure of deviation of signal from the zero vector, and nature uses some energy for this deviation. In the following, let this value be energy.

Let $\langle A\xi, \xi \rangle$ be a bilinear form, where A is a linear operator. Then

$$\mathbf{E}\langle A\xi, \xi \rangle = \int_H \langle Ax, x \rangle \mu(dx) = \int_H \langle x, Ax \rangle \mu(dx) = \text{tr}KA = \text{tr}AK. \quad (9)$$

If P is an orthogonal projection, then $\langle P\xi, \xi \rangle$ is the membership function. We can define a fuzzy event $A_P = \{\omega, \langle P\xi(\omega), \xi(\omega) \rangle : \omega \in \Omega\}$. From (1) and (9), the probability of the fuzzy event A_P is defined as

$$\mathbf{P}(A_P) = \mathbf{E}\langle P\xi, \xi \rangle = \int_H \langle Px, x \rangle \mu(dx) = \text{tr}PK = \text{tr}KP.$$

We now prove formula (9). Let $\{e_i\}, i = 1 \dots n$, be an orthonormal basis in H . Using definitions of trace and correlation operator (6), we have

$$\begin{aligned} \text{tr}KA &= \sum_{i=1}^n \langle KAe_i, e_i \rangle = \sum_{i=1}^n \int_H \langle x, Ae_i \rangle \langle x, e_i \rangle \mu(dx) = \int_H \sum_{i=1}^n \langle A^*x, \langle x, e_i \rangle e_i \rangle \mu(dx) \\ &= \int_H \left\langle A^*x, \sum_{i=1}^n \langle x, e_i \rangle e_i \right\rangle \mu(dx) = \int_H \langle A^*x, x \rangle \mu(dx) = \int_H \langle x, Ax \rangle \mu(dx). \end{aligned}$$

Since the scalar product is symmetric in a real Hilbert space, $\langle x, y \rangle = \langle y, x \rangle$, we get $\langle Ax, x \rangle = \langle x, Ax \rangle$. Then

$$\begin{aligned} \text{tr}AK &= \sum_{i=1}^n \langle AKe_i, e_i \rangle = \sum_{i=1}^n \langle Ke_i, A^*e_i \rangle = \sum_{i=1}^n \int_H \langle x, e_i \rangle \langle x, A^*e_i \rangle \mu(dx) \\ &= \int_H \left\langle \sum_{i=1}^n \langle x, e_i \rangle x, A^*e_i \right\rangle \mu(dx) = \int_H \langle x, Ax \rangle \mu(dx) = \int_H \langle Ax, x \rangle \mu(dx) = \mathbf{E}\langle A\xi, \xi \rangle. \end{aligned}$$

Statistical states of quantum system are described by von Neumann density matrix [8]. In fact, von Neumann density matrix is the correlation matrix of the discrete probability distribution. The formula (9) enables to describe statistical states of quantum system with continuous probability distributions.

6 Recognition of two signal classes

K. Helstrom was first who considered recognition of two classes in the quantum theory [8]. We apply Helstrom's result for recognition of two classes of random signals; we only consider an energy distribution instead of quantum probability distribution on projections.

Assume that the object ω of recognition belongs to one of the classes S_i , $i = 1, 2$, and the pattern of object is the signal $x = \xi(\omega)$. Suppose that each class S_i , $i = 1, 2$, is matched with the orthogonal projection P_i , $i = 1, 2$, where $P_1 + P_2 = I$. Then the value $\langle P_i x, x \rangle = \langle P_i \xi(\omega), \xi(\omega) \rangle = g_i(\xi(\omega))$ is the membership of object ω to the class S_i , $i = 1, 2$. Therefore, the projections P_i , $i = 1, 2$, define a class Φ of discriminant functions $g_i(x) = \langle P_i x, x \rangle$, $i = 1, 2$.

Let $p_i = \mathbf{P}(S_i)$, $i = 1, 2$ be a priori probabilities of classes and the conditional distributions $\mu_i(A) = \mathbf{P}(\xi \in A | S_i)$, $i = 1, 2$, have the correlation operators K_i , $i = 1, 2$. We define a fuzzy event $G = \{\omega, g(\gamma(\omega), \xi(\omega)) : \omega \in \Omega\}$, where $g(\gamma, \xi) = \langle P_\gamma \xi, \xi \rangle$. By (4), we must maximize the probability of the fuzzy event G :

$$\mathbf{P}(G) = \mathbf{E}g(\gamma, \xi) = p_1 \int_H \langle P_1 x, x \rangle \mu_1(dx) + p_2 \int_H \langle P_2 x, x \rangle \mu_2(dx). \quad (10)$$

Let us suggest an energy interpretation of formula (10). Using (5) and (10), we have

$$r_j(i) = \mathbf{E}(1_{S_j} \langle P_i \xi, \xi \rangle) = p_j \int_H \langle P_i x, x \rangle \mu_j(dx) = p_j \text{tr} P_i K_j.$$

Each projection $P_i, i = 1, 2$, passes same energy of signals $x = \xi(\omega)$ from the own class $S_j, i = j$ and the other class $S_j, i \neq j$. We call energy $r_j(i)$ a “correct” energy if $i = j$ and an “error” energy if $i \neq j$. We also call a full “correct” energy, which passes projections of all classes, as an energy of “correct” recognition. This energy is defined as

$$\text{Enr}_C(P_1, P_2) = r_1(1) + r_2(2) = p_1 \text{tr} P_1 K_1 + p_2 \text{tr} P_2 K_2. \quad (11)$$

It is clear that we must find projections P_1, P_2 so that the value $\text{Enr}_C(P_1, P_2)$ would be the largest. In other words, projections P_1, P_2 together must pass the energy of signals from their own classes as much as possible.

Since $P_2 = I - P_1$, we have

$$\text{Enr}_C(P_1, P_2) = p_2 \text{tr} K_2 + \text{tr} P_1 (p_1 K_1 - p_2 K_2).$$

Here the first value is constant but the second value depends only on the projection P_1 . Hence we must find the projection P_1 such that the second value was the largest. Assume that $\lambda_i, i = 1 \dots n$, are eigenvalues and $y_i, i = 1 \dots n$, are the eigenvectors of the operator $p_1 K_1 - p_2 K_2$. Then

$$\begin{aligned} \text{tr} P_1 (p_1 K_1 - p_2 K_2) &= \sum_{i=1}^n \langle P_1 (p_1 K_1 - p_2 K_2) y_i, y_i \rangle = \sum_{i=1}^n \langle P_1 \lambda_i y_i, y_i \rangle \\ &= \sum_{i=1}^n \lambda_i \|P_1 y_i\|^2 = \sum_{\lambda_i > 0} \lambda_i \|P_1 y_i\|^2 + \sum_{\lambda_i \leq 0} \lambda_i \|P_1 y_i\|^2 = d_1 + d_2, \end{aligned}$$

where $\|P_1 y_i\|^2 \leq \|y_i\|^2$ for all $i = 1 \dots n$, $d_1 > 0$, $d_2 \leq 0$. Let P_1 be a projection onto a subspace spanned by the eigenvectors with positive eigenvalues. Then $\|P_1 y_i\|^2 = \|y_i\|^2$ if $\lambda_i > 0$ and $\|P_1 y_i\|^2 = 0$ if $\lambda_i \leq 0$. It follows that d_1 will be the largest and $d_2 = 0$. Hence the required projection P_1 is found and $P_2 = I - P_1$.

COMMENT 1. It is possible to minimize the energy of “error” recognition. The energy of “error” recognition is the following sum:

$$\text{Enr}_E(P_1, P_2) = p_1 r_1(2) + p_2 r_2(1) = p_1 \text{tr} P_2 K_1 + p_2 \text{tr} P_1 K_2.$$

If the projections P_1, P_2 maximize the energy of “correct” recognition, then they must minimize energy of “error” recognition. Indeed, we have

$$\begin{aligned} \text{Enr}_E(P_1, P_2) &= p_1 \text{tr}(P_2 K_1) + p_2 \text{tr}(P_1 K_2) = p_1 \text{tr}(I - P_1) K_1 + p_2 \text{tr}(I - P_2) K_2 \\ &= p_1 \text{tr} K_1 + p_2 \text{tr}(K_2) - p_1 \text{tr} P_1 K_1 - p_2 \text{tr}(P_2 K_2) \\ &= p_1 \text{tr} K_1 + p_2 \text{tr}(K_2) - \text{Enr}_C(P_1, P_2). \end{aligned} \quad (12)$$

There the values $p_1 \text{tr} K_1$ and $p_2 \text{tr} K_2$ are constant. Hence the value $\text{Enr}_E(P_1, P_2)$ will be the least if the value $\text{Enr}_C(P_1, P_2)$ is the greatest.

COMMENT 2. From (12) it follows that the sum energy of “correct” recognition and “error” recognition is a constant. Thus, increasing the energy of “correct” recognition, we decrease the energy of “error” recognition and vice versa.

7 Decision rule for recognition

Suppose there are two classes of objects $S_i, i = 1, 2$, and the signal $x = \xi(\omega)$ is the pattern of the object ω . If we use a probabilistic Bayesian classifier, then the feature space H is divided into the disjoint subsets: $L_1, L_2, L_1 \cup L_2 = H$, where the subset L_1 correspond to the class S_1 and the subset L_2 corresponds to the class S_2 . The decision rule that determines unambiguously to which class S_1 or S_2 belongs the object ω , is defined as follows: $\omega \in S_1$ if $x \in L_1$ and $\omega \in S_2$ if $x \in L_2$.

However, the situation is different when quantum logic is used. Suppose each class $S_i, i = 1, 2$, is matched with the orthogonal projection $P_i, i = 1, 2$, where $P_1 + P_2 = I$. Denote $L_1 = P_1 H, L_2 = P_2 H$, where $L_1 \oplus L_2 = H$. Then the pattern of the object $x = \xi(\omega)$ can be a sum of two signals: $x = P_1 x + P_2 x = x_1 + x_2$, where $x_1 \in L_1, x_2 \in L_2$. It is natural to accept that $\omega \in S_1$ if $P_1 x = x$ and $\omega \in S_2$ if $P_2 x = x$. If $x_1 \neq 0$ and $x_2 \neq 0$, then the pattern x belongs simultaneously to two subspaces: L_1 and L_2 . Hence we can not decide to which class belongs the object using subspaces of quantum logic. Therefore we must use discriminant functions $g_i(x) = \langle P_i x, x \rangle, i = 1, 2$, which unambiguously gives the decision about the membership of the object to one of the classes: S_1 or S_2 . By (11), we can find discriminant functions $g_1(x) = \langle P_1 x, x \rangle$ and $g_2(x) = \langle P_2 x, x \rangle$ such that they maximize the energy of “correct” recognition. Thus we have the following decision rule:

$$\omega \in S_1 \text{ if } \langle P_1 x, x \rangle > \langle P_2 x, x \rangle \text{ and } \omega \in S_2 \text{ otherwise.} \quad (13)$$

When the decision rule (13) is applied, the feature space H is divided into disjoint sets: $A_1 = \{x: \langle P_1 x, x \rangle > \langle P_2 x, x \rangle\}$ and $A_2 = \{x: \langle P_2 x, x \rangle \geq \langle P_1 x, x \rangle\}$. We put

$$\text{Enr}_C(A_1, A_2) = p_1 \int_{A_1} \langle P_1 x, x \rangle \mu_1(dx) + p_2 \int_{A_2} \langle P_2 x, x \rangle \mu_2(dx).$$

It is evident that

$$\text{Enr}_C(P_1, P_2) = \text{Enr}_C(A_1, A_2) + p_1 \int_{A_2} \langle P_1 x, x \rangle \mu_1(dx) + p_2 \int_{A_1} \langle P_2 x, x \rangle \mu_2(dx). \quad (14)$$

The object ω of recognition is chosen in a random way but we hope that the value of the discriminant function $g_i(x)$ of class S_i is maximal if statement $\omega \in S_i$ is true. Also it is natural to hope that $\text{Enr}_C(P_1, P_2)$ is approximately equal to $\text{Enr}_C(A_1, A_2)$. Using $\langle P_1 x, x \rangle \leq \langle P_2 x, x \rangle$ on G_2 and $\langle P_2 x, x \rangle < \langle P_1 x, x \rangle$ on G_1 , we get

$$\begin{aligned} p_1 \int_{A_2} \langle P_1 x, x \rangle \mu_1(dx) &\leq p_1 \int_{A_2} \langle P_2 x, x \rangle \mu_1(dx) \leq p_1 \int_H \langle P_2 x, x \rangle \mu_1(dx) = p_1 \text{tr} P_2 K_1, \\ p_2 \int_{A_1} \langle P_2 x, x \rangle \mu_2(dx) &\leq p_2 \int_{A_1} \langle P_1 x, x \rangle \mu_2(dx) \leq p_2 \int_H \langle P_1 x, x \rangle \mu_2(dx) = p_2 \text{tr} P_1 K_2. \end{aligned}$$

From (14) it follows that

$$0 \leq \text{Enr}_C(P_1, P_2) - \text{Enr}_C(A_1, A_2) \leq p_1 \text{tr} P_2 K_1 + p_2 \text{tr} P_1 K_2 = \text{Enr}_E(P_1, P_2). \quad (15)$$

If projections P_1, P_2 maximize the energy $\text{Enr}_C(P_1, P_2)$ of “correct” recognition, then from comment 1 it follows that projections P_1, P_2 minimize the energy $\text{Enr}_E(P_1, P_2)$ of

“error” recognition. If we have good recognition with projections P_1, P_2 , then the value $\text{Enr}_E(P_1, P_2)$ is small. Therefore from (15) it follows that $\text{Enr}_C(P_1, P_2)$ is approximately equal to $\text{Enr}_C(A_1, A_2)$.

EXAMPLE 1. Suppose the object of recognition ω belongs to one of the classes S_i , $i = 1, 2$. Assume that a priori probabilities of classes are equal $p_1 = p_2 = 1/2$; the conditional distributions $\mu_i(A) = \mathbf{P}(\xi \in A | S_i)$, $i = 1, 2$, have the identical covariance matrices equal to R and mathematical expectations m_1, m_2 are orthogonal as vectors.

We choose the orthonormal basis e_i , $i = 1 \dots n$, in H such that $e_1 = m_1/\|m_1\|$, $e_n = m_2/\|m_2\|$. We get from (8) that $K_1 = R + \|m_1\|^2 p_1$, $K_2 = R + \|m_2\|^2 p_2$, where $p_1 x = \langle x, e_1 \rangle e_1$, $p_2 x = \langle x, e_n \rangle e_n$. In the chosen basis, the matrix $p_1 K_1 - p_2 K_2 = 1/2(K_1 - K_2)$ is diagonal with eigenvalues $\|m_1\|^2/2, 0, \dots, 0, -\|m_2\|^2/2$. Then $P_1 x = \langle x, m_1 \rangle / \|m_1\|$, $P_2 x = \langle x, m_2 \rangle / \|m_2\|$. If $x = \xi(\omega)$ is the pattern of the object ω , then by (13) we have the following decision rule: $\omega \in S_1$ if $\langle m_1, x \rangle^2 / \|m_1\|^2 > \langle m_2, x \rangle^2 / \|m_2\|^2$ and $\omega \in S_2$ otherwise.

8 Normalization by trace

Suppose $x = \xi(\omega)$ is the pattern of the object ω and $\mathbf{E}(\langle P\xi, \xi \rangle | S_i) = \text{tr} P K_i$, $i = 1, 2$, are conditional energy distributions on projections. The conditional energy distributions on projections of different classes are not equivalent if the trace of the correlation operators K_i , $i = 1, 2$, are not equal. It is possible to normalize the conditional energy distribution on projections by normalizing the pattern of objects of each class as follows: $\eta_i = \xi / \sqrt{\text{tr} K_i}$, $i = 1, 2$. Then the correlation operators will be normalized as follows: $\bar{K}_1 = K_1 / \text{tr} K_1$, $\bar{K}_2 = K_2 / \text{tr} K_2$, where $\text{tr} \bar{K}_1 = \text{tr} \bar{K}_2 = 1$. Also it is necessary to normalize the object patterns $x = \xi(\omega)$ in the decision rule (13). So, we have the following decision rule: $\omega \in S_1$ if $\langle P_1 x, x \rangle / \text{tr} K_1 > \langle P_2 x, x \rangle / \text{tr} K_2$ and $\omega \in S_2$ otherwise.

EXAMPLE 2. We consider a classical recognition task of two classes: the class S_1 is a random signal $\xi = a + \eta$, where a is a non-random signal and η is a white noise; the class S_2 is a white noise η . Suppose $p_1 = p_2 = 1/2$.

The correlation matrix of white noise η is $\sigma^2 I$, where σ^2 is a constant and I is an identity matrix. The mathematical expectations of the random signals of classes S_i , $i = 1, 2$, are respectively $m_1 = a$, $m_2 = 0$. Applying the decision rule of example 1, the classifier always decide that all objects $\omega \in S_1$.

We normalize the correlation matrices of both classes by their trace. From (8), we have $K_1 = \sigma^2 I + \|a\|^2 p_{\bar{a}}$, where $p_{\bar{a}} x = \langle x, \bar{a} \rangle \bar{a}$, $\bar{a} = a/\|a\|$; we also have $K_2 = \sigma^2 I$. Then $\text{tr} K_1 = \sigma^2 \text{tr} I + \|a\|^2 \text{tr} p_{\bar{a}} = n\sigma^2 + \|a\|^2$ and $\text{tr} K_2 = n\sigma^2$. Since covariance matrices of both classes are $\sigma^2 I$, they are diagonal in any basis. We choose the basis in H such that $e_1 = \bar{a}$. Then the matrix $p_1 \bar{K}_1 - p_2 \bar{K}_2 = 1/2(K_1 / \text{tr} K_1 - K_2 / \text{tr} K_2)$ is diagonal in the chosen basis with following eigenvalues:

$$\frac{(n-1)\|a\|^2}{2n(n\sigma^2 + \|a\|^2)}, \dots, -\frac{\|a\|^2}{2n(n\sigma^2 + \|a\|^2)}, -\frac{\|a\|^2}{2n(n\sigma^2 + \|a\|^2)}.$$

Here the first eigenvalue is positive and the last $n-1$ eigenvalues are negative. So the projection P_1 is a one-dimensional projection: $P_1 x = \langle x, e_1 \rangle e_1$. Then $\langle P_1 x, x \rangle = \langle x, a \rangle^2 / \|a\|^2$ and $\langle P_2 x, x \rangle = \langle (I - P_1)x, x \rangle = \langle x, x \rangle - \langle x, a \rangle^2 / \|a\|^2$. By (9), the variance of the white

noise η is equal to $\mathbf{E}\|\eta\|^2 = \mathbf{E}\langle\eta, \eta\rangle = n\sigma^2$. So the signal-to-noise ratio is defined as $\text{SNR} = \|a\|^2/n\sigma^2$.

Normalizing the object pattern by the trace, we get from (13) the following decision rule: $\omega \in S_1$ if $\langle x, a \rangle^2/(1 + \text{SNR}) > \|x\|^2\|a\|^2 - \langle x, a \rangle^2$ and $\omega \in S_2$ otherwise.

We have $\text{tr}P_2K_1 = \sigma^2\text{tr}P_2 = (n-1)\sigma^2$ and $\text{tr}P_1K_2 = \sigma^2\text{tr}P_1 = \sigma^2$. Then

$$\text{Enr}_E(P_1, P_2) = \frac{p_1}{\text{tr}K_1}\text{tr}P_2K_1 + \frac{p_2}{\text{tr}K_2}\text{tr}P_1K_2 = \frac{(n-1)\sigma^2}{2(n\sigma^2 + \|a\|^2)} + \frac{\sigma^2}{2n\sigma^2} = \frac{1 - 1/n}{2(1 + \text{SNR})} + \frac{1}{2n}.$$

Thus the energy of “error” recognition is small if the SNR and the dimension n of the feature space H are large.

9 Normalization by signal norm

We can to normalize object pattern by normalizing each signal $x = \xi(\omega)$ as vector by its norm. In that case, ends of normalized random vectors are located on a unit sphere. Suppose $\mathbf{P}(\xi = 0) = 0$. Putting $\eta = \xi/\|\xi\|$, we have

$$\mathbf{E}\langle\eta, \eta\rangle = \mathbf{E}(\langle\xi, \xi\rangle/\|\xi\|^2) = \mathbf{E}(\|\xi\|^2/\|\xi\|^2) = 1. \quad (16)$$

Let \bar{K} be the correlation operator of the normalized random signal η . From (9) and (16), we have $\text{tr}\bar{K} = 1$. Hence, the energy distribution on projections is normalized.

If objects patterns of are normalized as $x = \xi(\omega)/\|\xi(\omega)\|$, then $g_i(x) = \langle P_i x, x \rangle \leq 1$, $i = 1, 2$. This yields that $\sup g_i(x) = 1$, where $i = 1, 2$. So the discriminant functions $g_i(x)$, $i = 1, 2$ are classical membership functions [1].

Vectors x and λx for any $\lambda > 0$ describe the same physical state in quantum mechanics. It means that states of quantum systems are rays, i.e. points of projective space. Due this fact, we can consider states with unit norm $\|x\| = 1$ only.

The same holds for sound signals and monochrome images. In fact, the sound signals x and λx for any $\lambda > 0$ differ in loudness only. The monochrome images can be described as a set of $l = nm$ real numbers corresponding to the intensity of the light in each pixel. Hence the space of the monochrome images can be described as a vector space of dimension $l = nm$. All the intensities of the monochrome image can be multiplied by a number $\lambda > 0$, but that does not change monochrome image.

10 Subtraction of mean

The following hypothesis is accepted in the recognition theory: the distribution of the patterns of a class is concentrated in a compact area of feature space. It is natural to assume that distribution of patterns is grouped around the mean (mathematical expectation) of this distribution. Then each object pattern $x = \xi(\omega)$ can be written as the sum $x = y + a$, where a is the mean and y is the random vector from the compact area such that its beginning is the end of the mean a .

On the other hand, linear subspaces that correspond to classes in feature space are intersect at the zero point of the space H (the origin of the coordinates). Therefore if quantum logic is used for recognition, then it is natural to combine compact areas with the origin of coordinates.

In this case, the energy distributions on projections are described by the covariance operators.

Suppose the conditional distributions $\mu_i(A) = \mathbf{P}(\xi \in A|S_i)$, $i = 1, 2$, have the covariance operators R_1, R_2 and means m_1, m_2 . Then it is necessary to find projections P_1, P_2 such that the value of energy $\text{Enr}_R(P_1, P_2) = p_1 \text{tr} P_1 R_1 + p_2 \text{tr} P_2 R_2$ would be the maximal. After subtracting from object patterns $x = \xi(\omega)$ their means, we get from (13) the following decision rule: $\omega \in S_1$ if $\langle P_1(x - m_1), x - m_1 \rangle > \langle P_2(x - m_2), x - m_2 \rangle$ and $\omega \in S_2$ otherwise.

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