# Digital Signal Processing

## Module 8: Image Processing

# Module Overview:

- Module 8.1: Introduction to Images and Image Processing
- Module 8.2: Affine Transforms
- Module 8.3: 2D Fourier Analysis
- Module 8.4: Image Filters
- Module 8.5: Image Compression
- Module 8.6: The JPEG Compression Standard

# Digital Signal Processing

Module 8.1: Image Processing

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- Images as multidimensional digital signals
- 2D signal representations
- Basic signals and operators

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# Please meet ...

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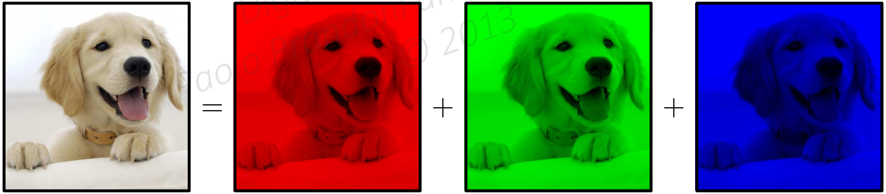

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- new manipulations: affine transforms
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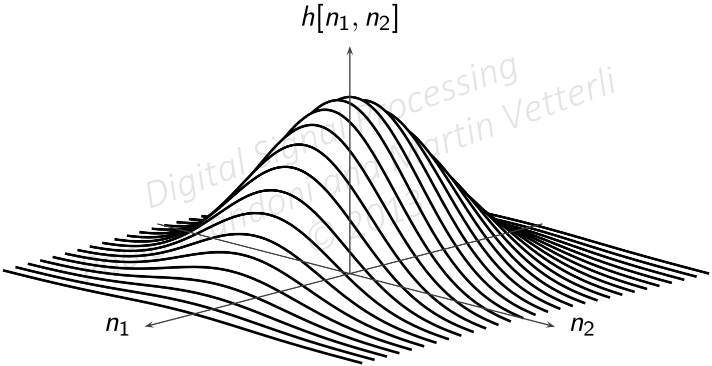
A two-dimensional discrete-space signal:

$$x[n_1, n_2], \qquad n_1, n_2 \in \mathbb{Z}$$

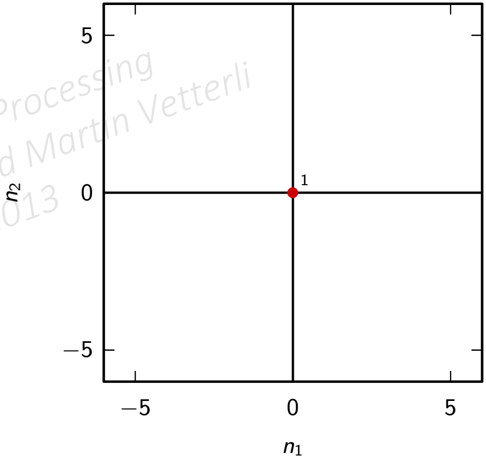
# 2D signals: Cartesian representation



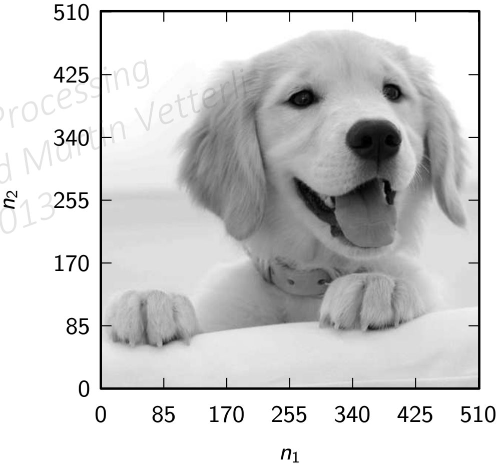
# 2D signals: support representation

- just show coordinates of nonzero samples
- amplitude may be written along explicitly
- example:

$$\delta[n_1, n_2] = \begin{cases} 1 & \text{if } n_1 = n_2 = 0 \\ 0 & \text{otherwise.} \end{cases}$$



- medium has a certain dynamic range (paper, screen)
- image values are quantized (usually to 8 bits, or 256 levels)
- the eye does the interpolation in space provided the pixel density is high enough



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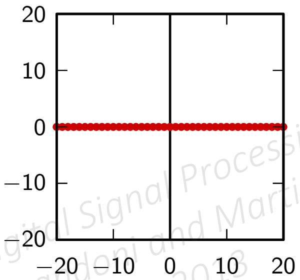
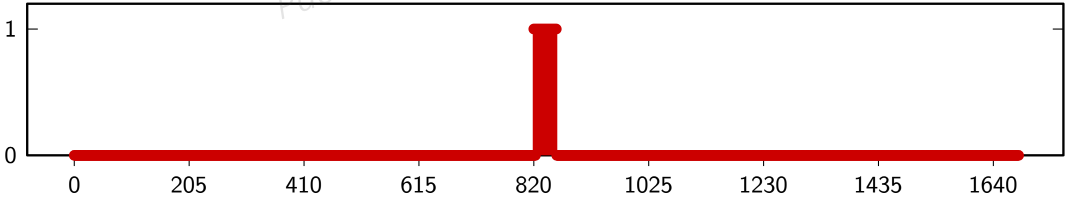
# 2D vs raster scan



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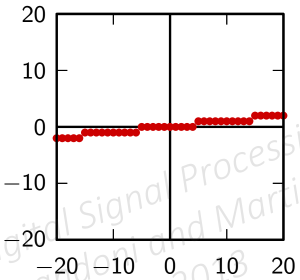


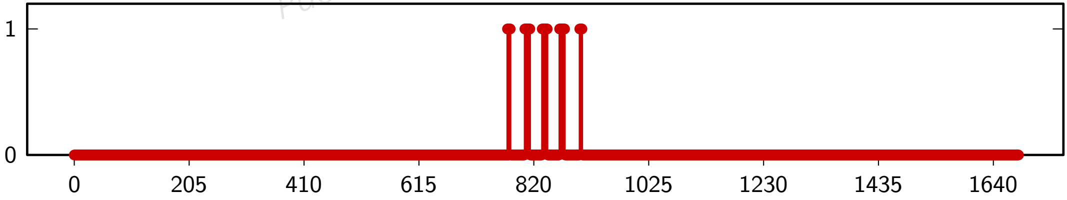

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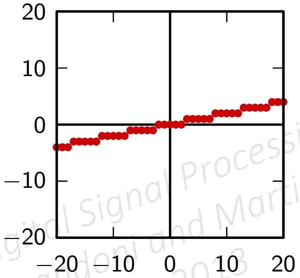


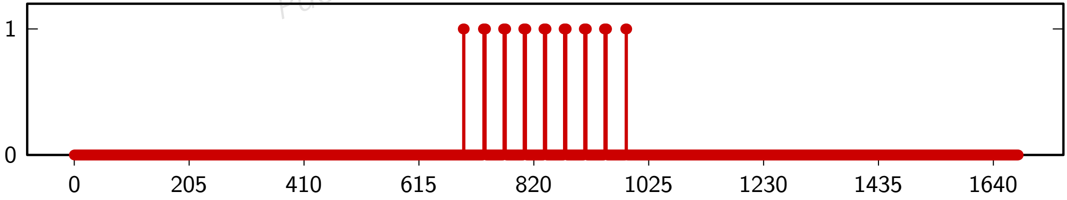

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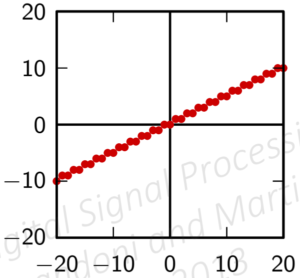




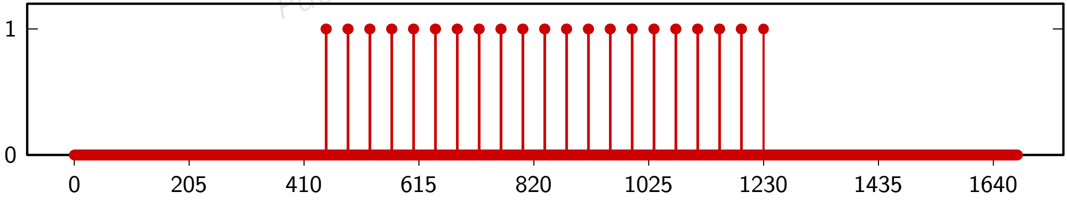

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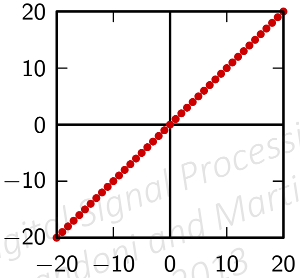




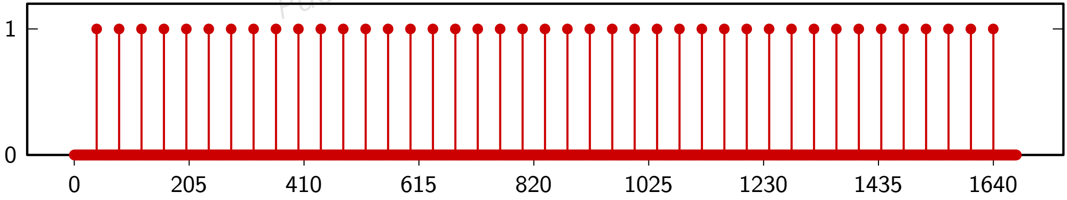

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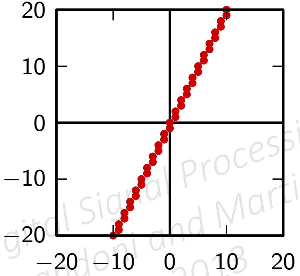


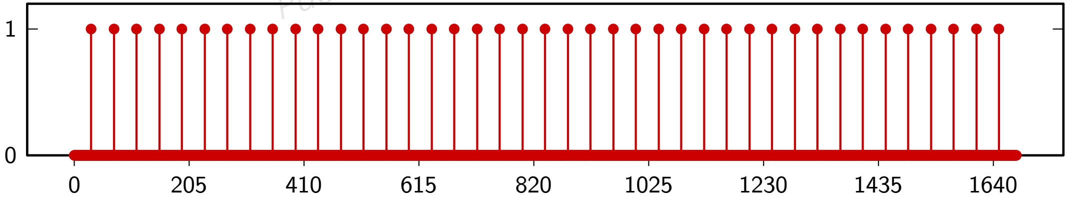

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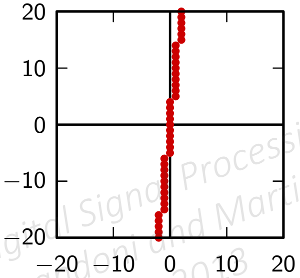


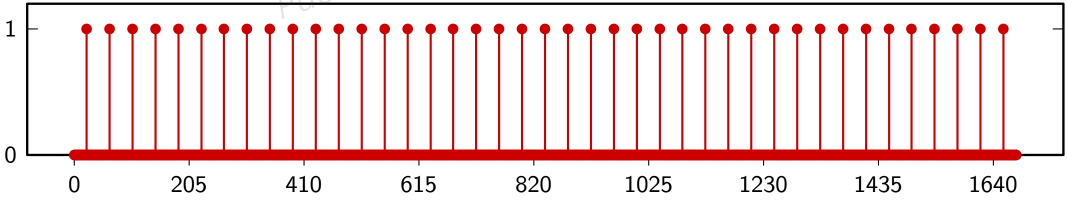

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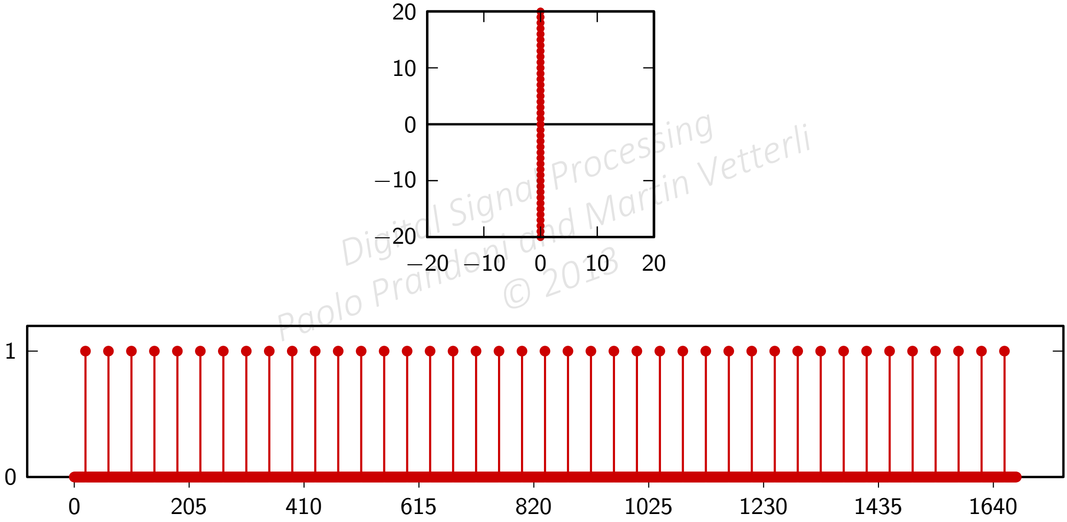
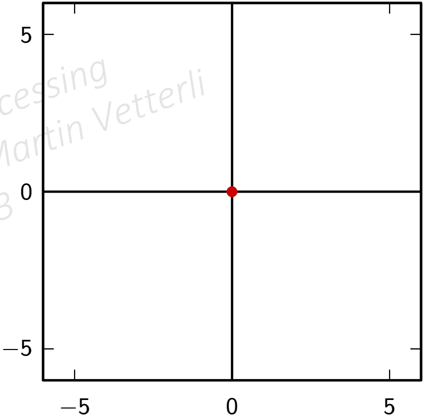


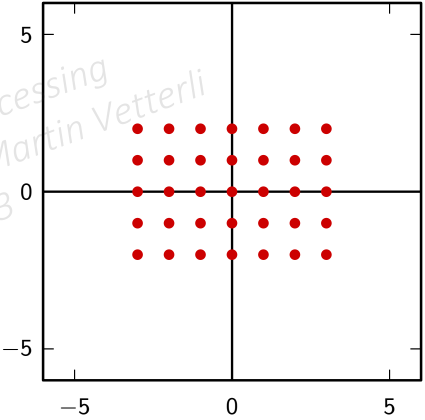
# Basic 2D signals: the impulse

$$\delta[n_1, n_2] = \begin{cases} 1 & \text{if } n_1 = n_2 = 0 \\ 0 & \text{otherwise.} \end{cases}$$



# Basic 2D signals: the rect

$$\text{rect}\left(\frac{n_1}{2N_1}, \frac{n_2}{2N_2}\right) = \begin{cases} 1 & \text{if } |n_1| < N_1 \\ & \text{and } |n_2| < N_2 \\ 0 & \text{otherwise;} \end{cases}$$



# Separability

$$x[n_1, n_2] = x_1[n_1]x_2[n_2]$$

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$$\delta[n_1, n_2] = \delta[n_1]\delta[n_1]$$

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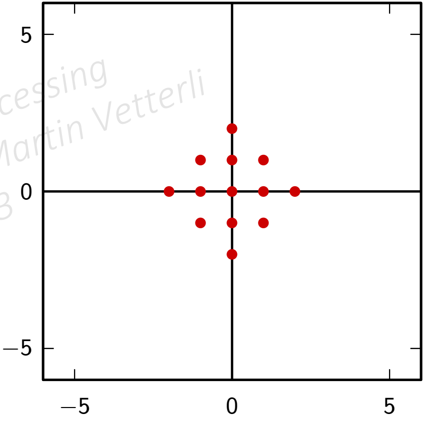
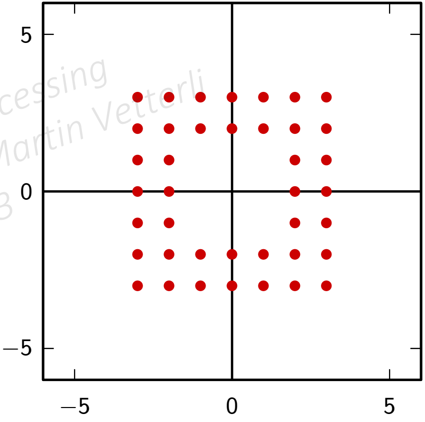
# Nonseparable signal

$$x[n_1, n_2] = \begin{cases} 1 & \text{if } |n_1| + |n_2| < N \\ 0 & \text{otherwise} \end{cases}$$



# Nonseparable signal

$$x[n_1, n_2] = \mathrm{rect}\left(\frac{n_1}{2N_1}, \frac{n_2}{2N_2}\right) - \mathrm{rect}\left(\frac{n_1}{2M_1}, \frac{n_2}{2M_2}\right)$$



# 2D convolution

$$x[n_1, n_2] * h[n_1, n_2] = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x[k_1, k_2] h[n_1 - k_1, n_2 - k_2]$$

# 2D convolution for separable signals

If $h[n_1, n_2] = h_1[n_1]h_2[n_2]$:

$$x[n_1, n_2] * h[n_1, n_2] = \sum_{k_1=-\infty}^{\infty} h_1[n_1 - k_1] \sum_{k_2=-\infty}^{\infty} x[k_1, k_2]h_2[n_2 - k_2]$$

$$= h_1[n_1] * (h_2[n_2] * x[n_1, n_2]).$$

# 2D convolution for separable signals

If $h[n_1, n_2]$ is an $M_1 \times M_2$ finite-support signal:

- non-separable convolution: $M_1 M_2$ operations per output sample
- separable convolution: $M_1 + M_2$ operations per output sample!

# END OF MODULE 8.1

# Digital Signal Processing

Module 8.2: Image Manipulations

- Affine transforms
- Bilinear interpolation

## Overview:

- Affine transforms
- Bilinear interpolation

# Affine transforms

mapping $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ that reshapes the coordinate system:

$$\begin{bmatrix} t_1' \\ t_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} - \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$



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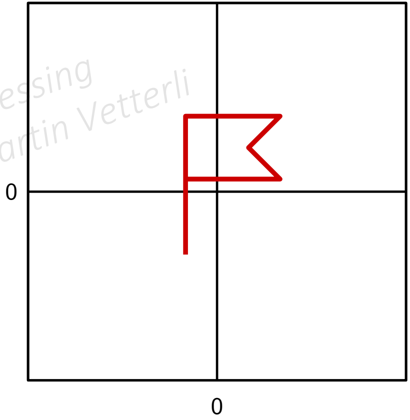
# Translation

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$





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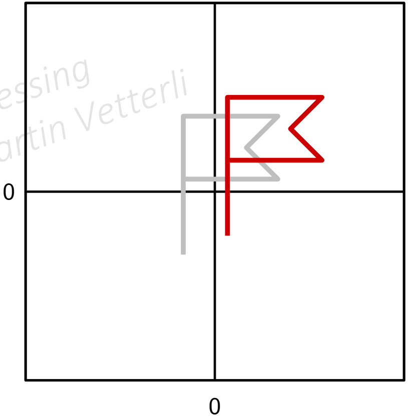


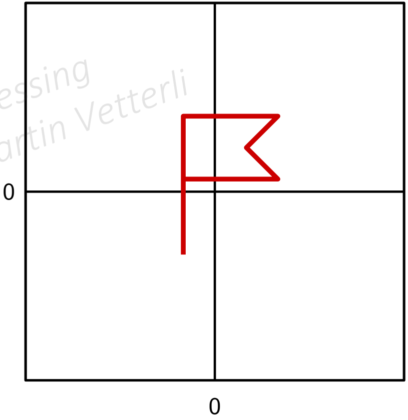
# Scaling

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

$$\mathbf{d} = 0$$

if $a_1 = a_2$ the *aspect ratio* is preserved





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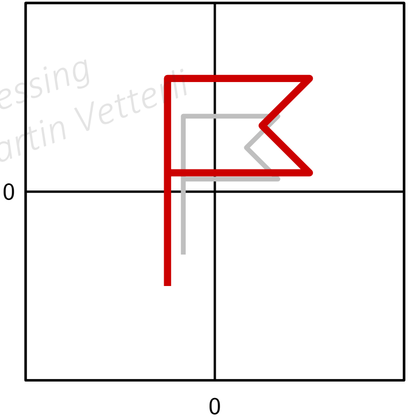
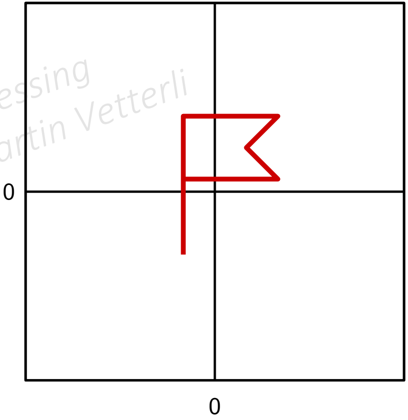


# Rotation

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{d} = 0$$





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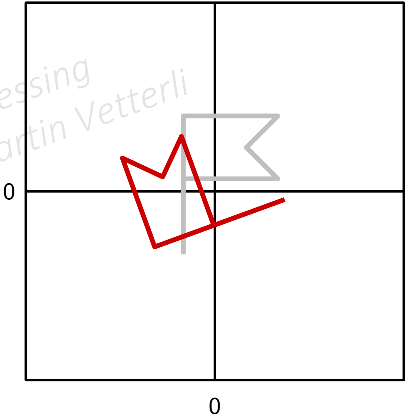

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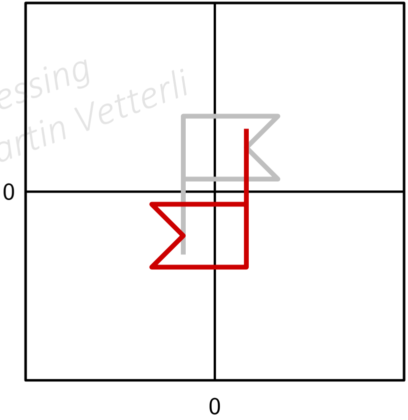


# Flips

Horizontal:

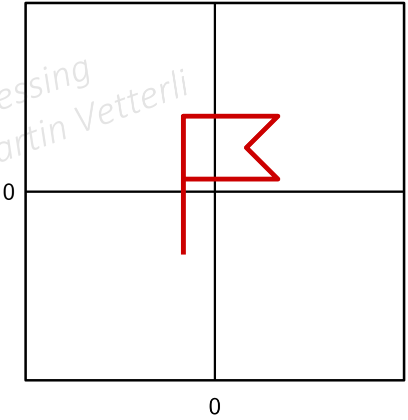
$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{d} = 0$$

Vertical:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{d} = 0$$



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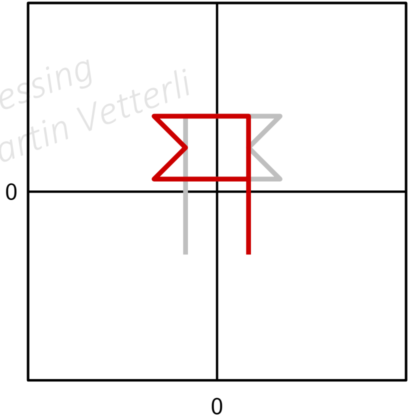

Horizontal:

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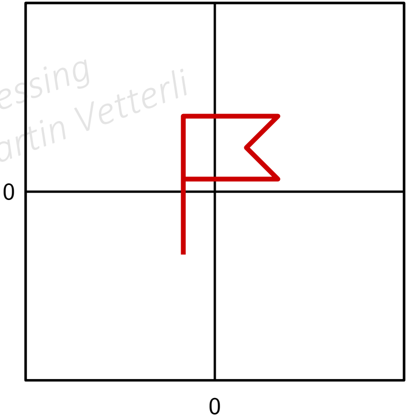

# Shear

Horizontal:

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$$\mathbf{d} = 0$$

Vertical:

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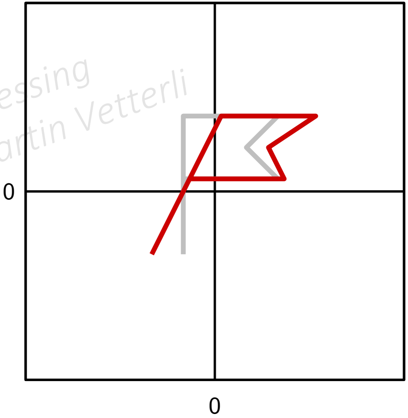

# Affine transforms in discrete-space

$$\begin{bmatrix} t_1' \\ t_2' \end{bmatrix} = \mathbf{A} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} - \mathbf{d} \in \mathbb{R}^2 \neq \mathbb{Z}^2$$

- apply the *inverse* transform:

$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} m_1 + d_1 \\ m_2 + d_2 \end{bmatrix};$$

- interpolate from the original grid point to the "mid-point"

$$(t_1, t_2) = (\eta_1 + \tau_1, \eta_2 + \tau_2), \qquad \eta_{1,2} \in \mathbb{Z}, \quad 0 \leq \tau_{1,2} < 1$$

# Solution for images

- apply the *inverse* transform:

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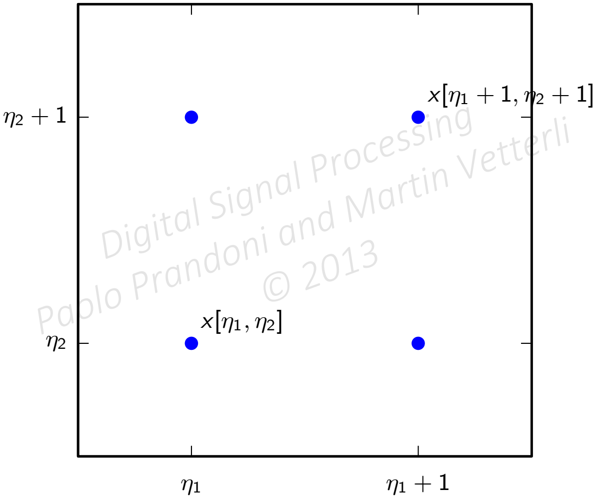
# Bilinear Interpolation



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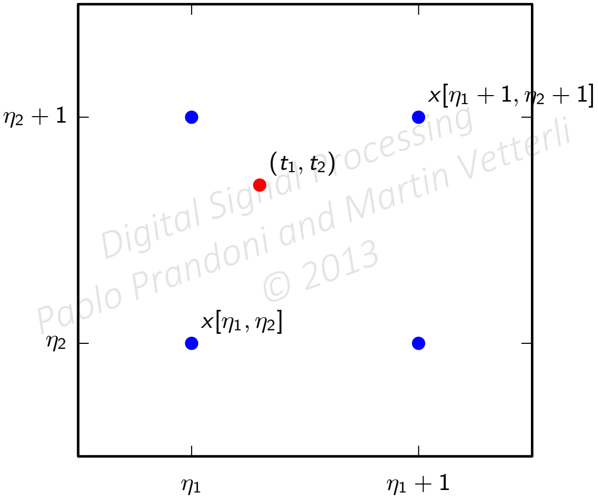

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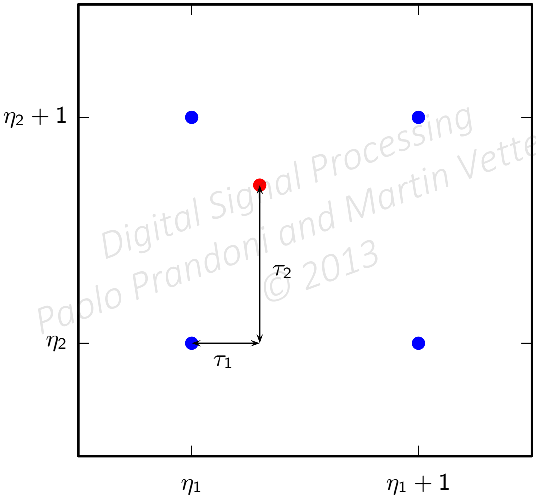

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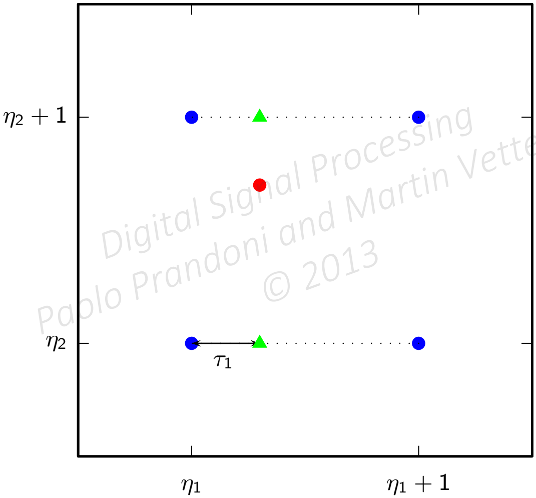

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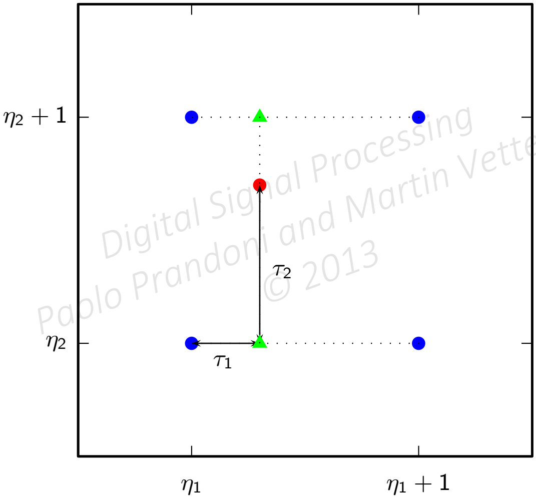

If we use a first-order interpolator:

$$y[m_1, m_2] = (1-\tau_1)(1-\tau_2)x[\eta_1, \eta_2] + \tau_1(1-\tau_2)x[\eta_1+1, \eta_2]$$
$$+ (1-\tau_1)\tau_2 x[\eta_1, \eta_2+1] + \tau_1\tau_2 x[\eta_1+1, \eta_2+1]$$

# Shearing

# END OF MODULE 8.2

# Digital Signal Processing

Module 8.3: Frequency Analysis

# Overview:

- DFT
- Magnitude and phase

# Overview:

- DFT
- Magnitude and phase

## 2D DFT

$$X[k_1, k_2] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[n_1, n_2] e^{-j\frac{2\pi}{N_1}n_1 k_1} e^{-j\frac{2\pi}{N_2}n_2 k_2}$$

$$x[n_1, n_2] = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X[k_1, k_2] e^{j\frac{2\pi}{N_1}n_1 k_1} e^{j\frac{2\pi}{N_2}n_2 k_2}$$

# 2D DFT

$$X[k_1, k_2] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[n_1, n_2] e^{-j\frac{2\pi}{N_1}n_1 k_1} e^{-j\frac{2\pi}{N_2}n_2 k_2}$$



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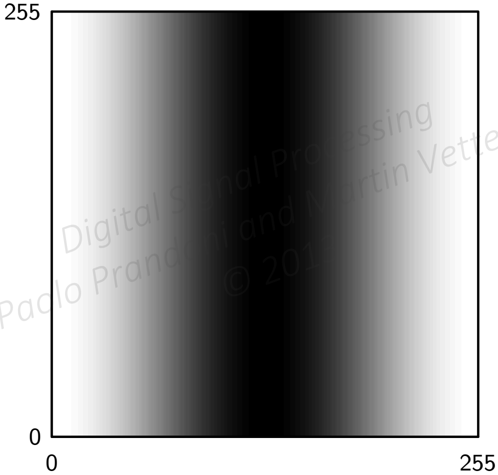
# 2D-DFT Basis Vectors

There are $N_1 N_2$ orthogonal basis vectors for an $N_1 \times N_2$ image:

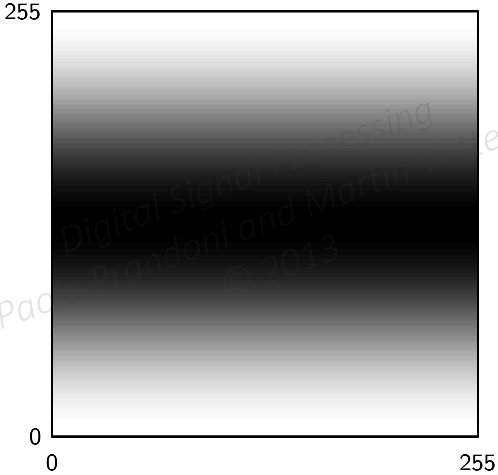
$$w_{k_1,k_2}[n_1, n_2] = e^{j\frac{2\pi}{N_1}n_1 k_1} e^{j\frac{2\pi}{N_2}n_2 k_2}$$

for $n_1, k_1 = 0, 1, \ldots, N_1 - 1$ and $n_2, k_2 = 0, 1, \ldots, N_2 - 1$

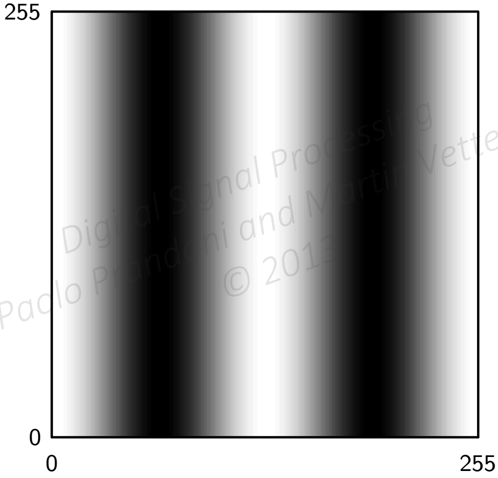
# 2D-DFT basis vectors for $k_1 = 1, k_2 = 0$ (real part)



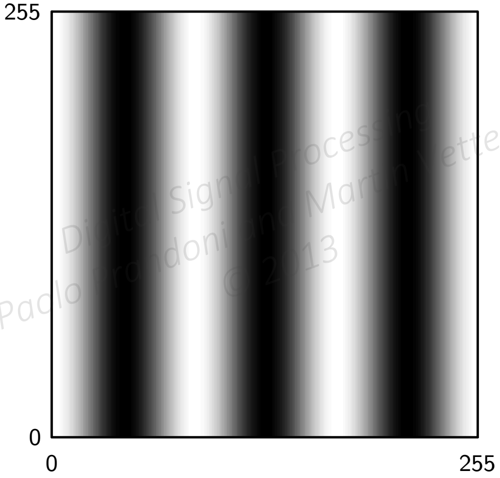
# 2D-DFT basis vectors for $k_1 = 0, k_2 = 1$ (real part)



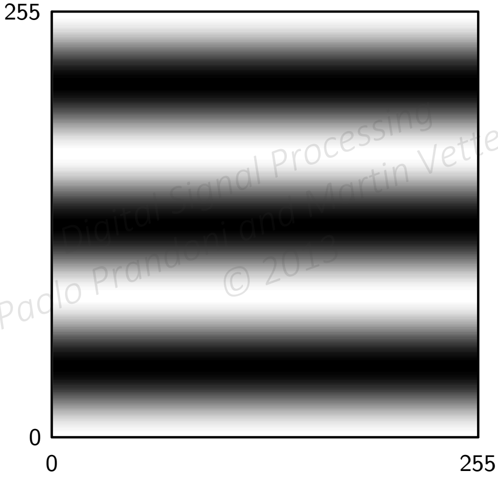
# 2D-DFT basis vectors for $k_1 = 2, k_2 = 0$ (real part)



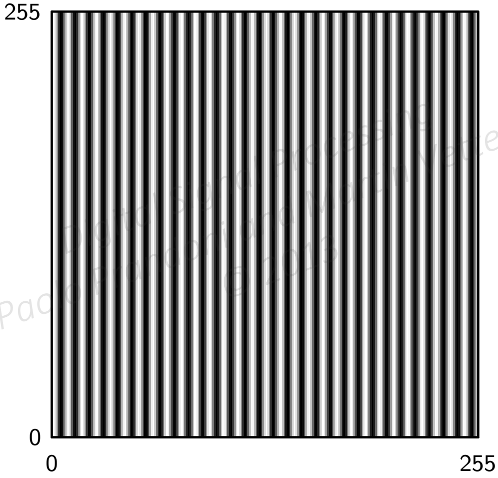
# 2D-DFT basis vectors for $k_1 = 3, k_2 = 0$ (real part)

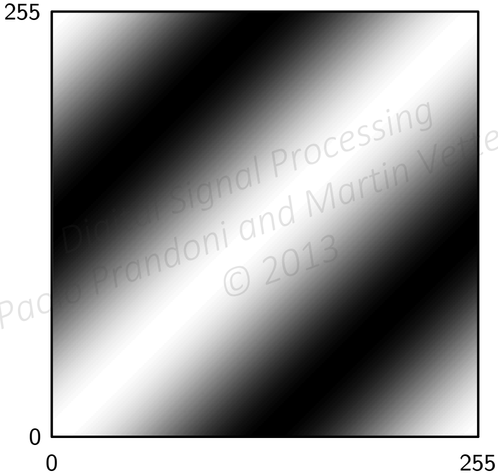


# 2D-DFT basis vectors for $k_1 = 0, k_2 = 3$ (real part)



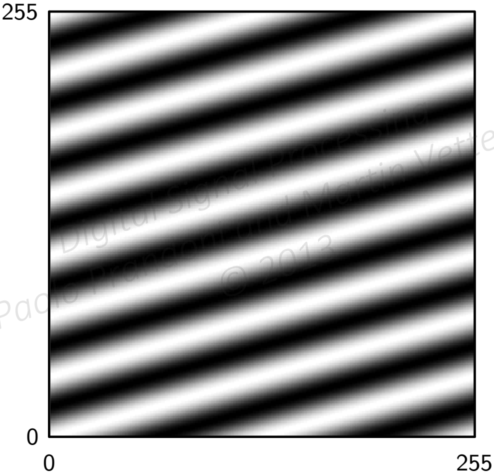
# 2D-DFT basis vectors for $k_1 = 30, k_2 = 0$ (real part)



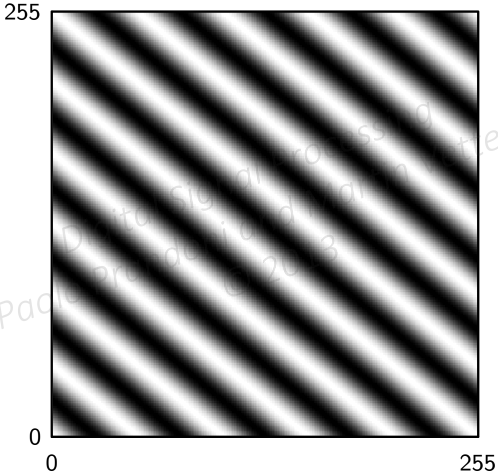


255
0
0
255

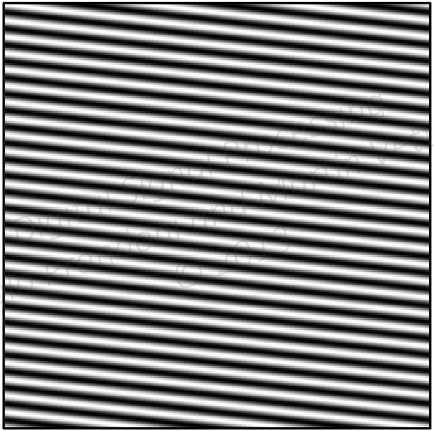
# 2D-DFT basis vectors for $k_1 = 2, k_2 = 7$ (real part)



# 2D-DFT basis vectors for $k_1 = 5, k_2 = 250$ (real part)



# 2D-DFT basis vectors for $k_1 = 3, k_2 = 230$ (real part)



# 2D DFT

2D-DFT basis functions are separable, and so is the 2D-DFT:

$$X[k_1, k_2] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[n_1, n_2] e^{-j\frac{2\pi}{N_1}n_1 k_1} e^{-j\frac{2\pi}{N_2}n_2 k_2}$$



- 1D-DFT along $n_2$ (the columns)
- 1D-DFT along $n_1$ (the rows)

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- 1D-DFT along $n_2$ (the columns)
- 1D-DFT along $n_1$ (the rows)

# 2D DFT in matrix form

- finite-support 2D signal can be written as a matrix $\mathbf{x}$
- $N_1 \times N_2$ image is an $N_2 \times N_1$ matrix ($n_1$ spans the columns, $n_2$ spans the rows)
- recall also the $N \times N$ DFT matrix (Module 4.2):

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W_N^1 & W_N^2 & W^3 & \dots & W_N^{N-1} \\ 1 & W_N^2 & W^4 & W_N^6 & \dots & W_N^{2(N-1)} \\ & & & \dots & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & W_N^{3(N-1)} & \dots & W_N^{(N-1)^2} \end{bmatrix}$$

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 & & & \dots & & \\
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$$\mathbf{V} = \mathbf{W}_{N_2}\,\mathbf{x}$$

$$\mathbf{V} \in \mathbb{R}^{N_2 \times N_1}$$

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$$\mathbf{X} = \mathbf{V}\, \mathbf{W}_{N_1}$$

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$$\mathbf{V} = \mathbf{W}_{N_2}\,\mathbf{x} \qquad \mathbf{X} = \mathbf{V}\,\mathbf{W}_{N_1}$$

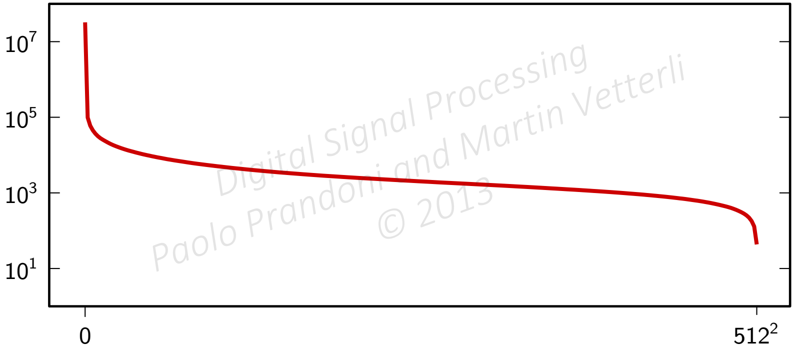
$$\mathbf{V} \in \mathbb{R}^{N_2\times N_1} \qquad \mathbf{X} \in \mathbb{R}^{N_2\times N_1}$$

$$\mathbf{X} = \mathbf{W}_{N_2}\,\mathbf{x}\,\mathbf{W}_{N_1}$$

# How does a 2D-DFT look like?

- try to show the magnitude as an image
- problem: the range is too big for the grayscale range of paper or screen
- try to normalize: $|X'[n_1, n_2]| = |X[n_1, n_2]| / \max |X[n_1, n_2]|$
- but it doesn't work...

# DFT coefficients sorted by magnitude



# Dealing with HDR images

if the image is high dynamic range we need to compress the levels

- remove flagrant outliers (e.g. $X[0,0] = \sum\sum x[n_1, n_2]$)
- use a nonlinear mapping: e.g. $y = x^{1/3}$ after normalization ($x \leq 1$)

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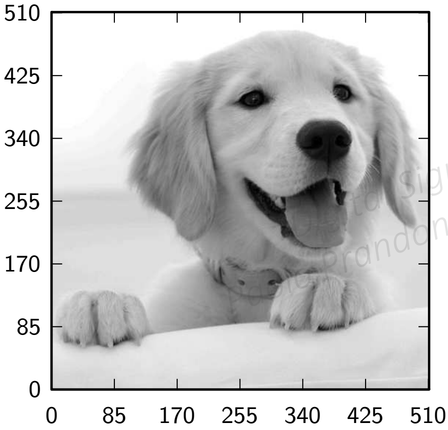
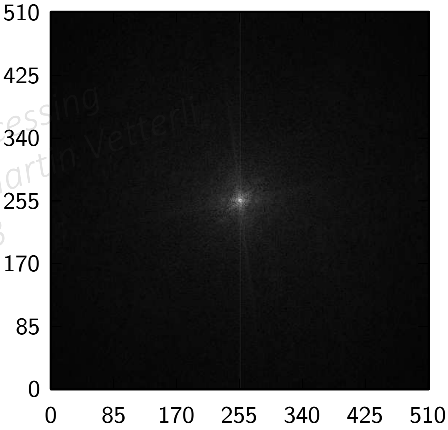
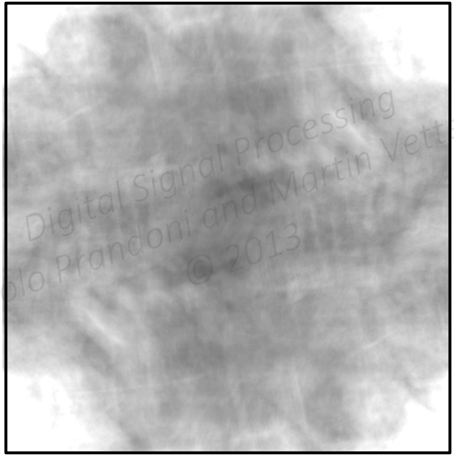


# DFT magnitude doesn't carry much information



# DFT phase, on the other hand...

# Image frequency analysis

- most of the information is contained in image's *edges*
- edges are points of abrupt change in signal's values
- edges are a "space-domain" feature $\rightarrow$ not captured by DFT's magnitude
- phase alignment is important for reproducing edges

# END OF MODULE 8.3

# Digital Signal Processing

Module 8.4: Filtering

# Overview:

- Filters for image processing
- Classification
- Examples

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# Analogies with 1D filters

- linearity
- *space* invariance
- impulse response
- frequency response
- stability
- 2D CCDE

# The problem with LSI operators

- interesting images contain lots of *semantics*: different information in different areas
- space-invariant filters process everything in the same way
- but we should process things differently
  - edges
  - gradients
  - textures
  - ...

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# Filter types

- IIR, FIR
- causal or noncausal
- highpass, lowpass, ...
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# The problems with 2D IIRs

- nonlinear phase (edges!)
- border effects
- stability: the fundamental theorem of algebra doesn't hold in multiple dimensions!
- computability

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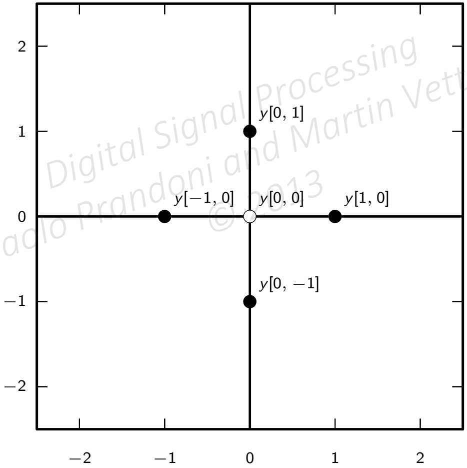
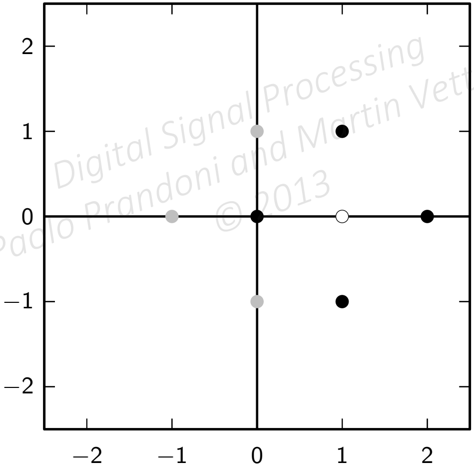
# A noncomputable CCDE

$$y[n_1, n_2] = a_0 y[n_1 + 1, n_2] + a_1 y[n_1, n_2 - 1] + a_2 y[n_1 - 1, n_2] + a_3 y[n_1, n_2 + 1] + x[n_1, n_2];$$



# A noncomputable CCDE

$$y[n_1, n_2] = a_0 y[n_1 + 1, n_2] + a_1 y[n_1, n_2 - 1] + a_2 y[n_1 - 1, n_2] + a_3 y[n_1, n_2 + 1] + x[n_1, n_2];$$



# Practical FIR filters

- generally zero centered (causality not an issue) $\Rightarrow$ odd number of taps in both directions
- per-sample complexity:
  - $M_1 M_2$ for nonseparable impulse responses
  - $M_1 + M_2$ for separable impulse responses
- obviously always stable

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$$y[n_1, n_2] = \frac{1}{(2N+1)^2} \sum_{k_1=-N}^{N} \sum_{k_2=-N}^{N} x[n_1 - k_1, n_2 - k_2]$$



$$h[n_1, n_2] = \frac{1}{(2N+1)^2} \mathrm{rect}\left(\frac{n_1}{2N}, \frac{n_2}{2N}\right)$$

# Moving Average

$$y[n_1, n_2] = \frac{1}{(2N+1)^2} \sum_{k_1=-N}^{N} \sum_{k_2=-N}^{N} x[n_1 - k_1, n_2 - k_2]$$



$$h[n_1, n_2] = \frac{1}{(2N+1)^2} \operatorname{rect}\left(\frac{n_1}{2N}, \frac{n_2}{2N}\right)$$

# Moving Average

$$h[n_1, n_2] = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

# Moving Average

11 × 11 MA

51 × 51 MA

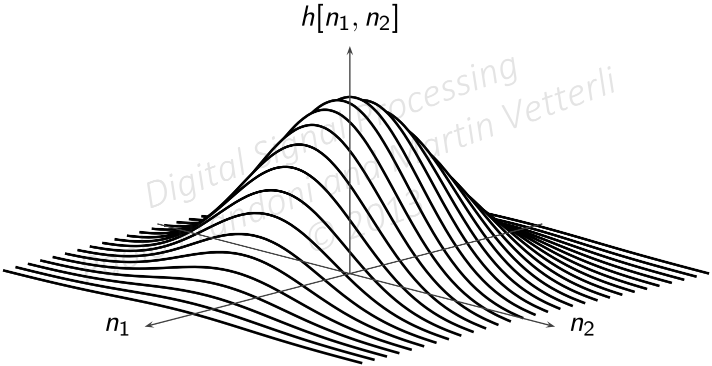
# Gaussian Blur

$$h[n_1, n_2] = \frac{1}{2\pi\sigma^2} e^{-\frac{n_1^2+n_2^2}{2\sigma^2}}, \qquad |n_1, n_2| < N$$

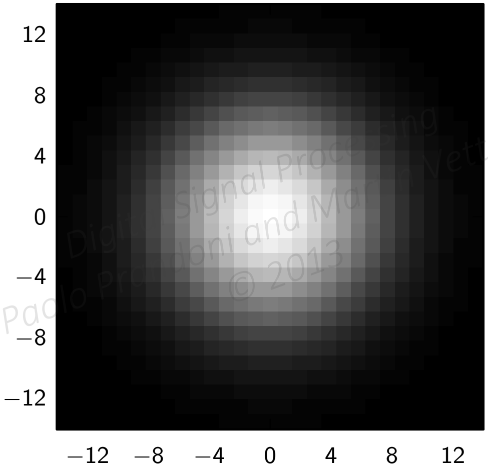


with $N \approx 3\sigma$

# Gaussian Blur



# Gaussian Blur




$\sigma = 5, N = 14$

$\sigma = 1.8, 11 \times 11$ blur

$\sigma = 8.7, 51 \times 51$ blur

# Sobel filter

approximation of the first derivative in the horizontal direction:

$$s_o[n_1, n_2] = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

separability and structure:

$$s_o[n_1, n_2] = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

# Sobel filter

approximation of the first derivative in the horizontal direction:

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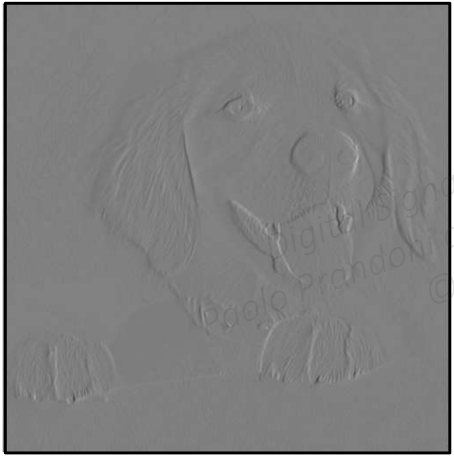
separability and structure:

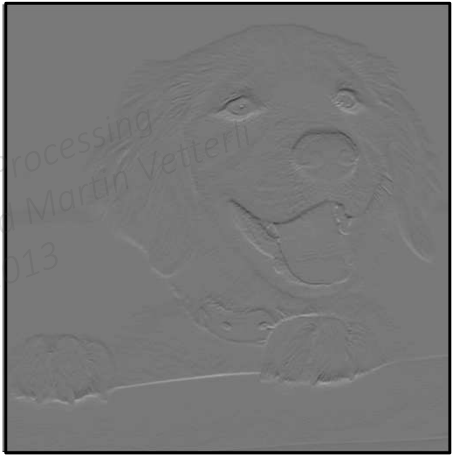
$$s_o[n_1, n_2] = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

approximation of the first derivative in the vertical direction:

$$s_v[n_1, n_2] = \begin{bmatrix} -1 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$



horizontal Sobel filter



vertical Sobel filter

# Sobel operator

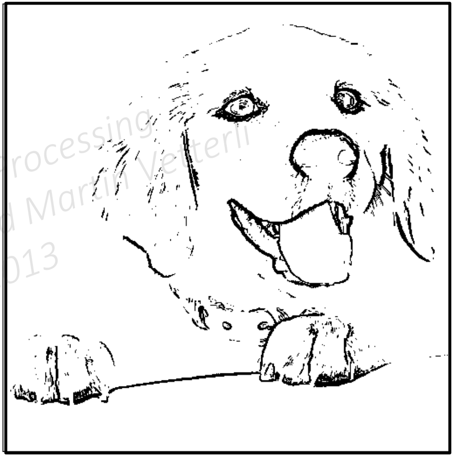
approximation for the square magnitude of the gradient:

$$|\nabla x[n_1, n_2]|^2 = |s_o[n_1, n_2] * x[n_1, n_2]|^2 + |s_v[n_1, n_2] * x[n_1, n_2]|^2$$

("operator" because it's nonlinear)

# Gradient approximation for edge detection

Sobel operator



thresholeded Sobel operator

# Laplacian operator

Laplacian of a function in continuous-space:

$$\Delta f(t_1, t_2) = \frac{\partial^2 f}{\partial t_1^2} + \frac{\partial^2 f}{\partial t_2^2}$$

approximating the Laplacian; start with a Taylor expansion

$$f(t+\tau) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t)}{n!}\tau^n$$

and compute the expansion in $(t+\tau)$ and $(t-\tau)$:

$$f(t+\tau) = f(t) + f'(t)\tau + \frac{1}{2}f''(t)\tau^2$$

$$f(t-\tau) = f(t) - f'(t)\tau + \frac{1}{2}f''(t)\tau^2$$

# Laplacian operator

by rearranging terms:

$$f''(t) = \frac{1}{\tau^2}\left(f(t-\tau) - 2f(t) + f(t+\tau)\right)$$

which, on the discrete grid, is the FIR $h[n] = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$

summing the horizontal and vertical components:

$$h[n_1, n_2] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

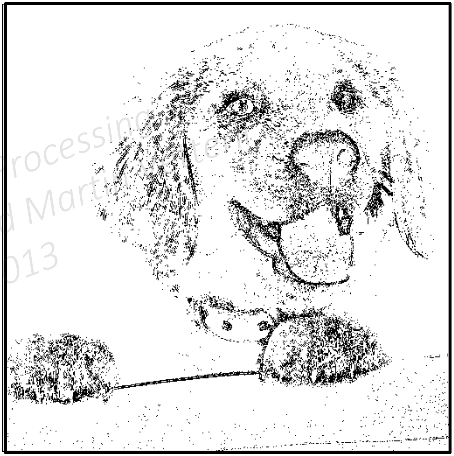
# Laplacian

If we use the diagonals too:

$$h[n_1, n_2] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

# Laplacian for Edge Detection

Laplacian operator



thresholeded Laplacian operator

# END OF MODULE 8.4

# Digital Signal Processing

Module 8.5: Image Compression

- Redundancy in natural images
- Image coding ingredients

# Overview:

- Redundancy in natural images
- Image coding ingredients

# A thought experiment

- consider all possible $256 \times 256$, 8bpp images
- each image is 524,288 bits
- total number of possible images: $2^{524,288} \approx 10^{157,826}$
- number of atoms in the universe: $10^{82}$

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# How many bits per image?

Another thought experiment

- take *all* images in the world and list them in an "encyclopedia of images"
- to indicate an image, simply give its number
- on the Internet: $M = 50$ billion
- raw encoding: 524,288 bits per image
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# Compression

Another approach:

- exploit "physical" redundancy
- allocate bits for things that matter (e.g. edges)
- use psychovisual experiments to find out what matters
- some information is discarded: *lossy* compression

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# Key ingredients

- compressing at block level
- using a suitable transform (i.e., a change of basis)
- smart quantization
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- reduce number bits per pixel
- equivalent to coarser quantization
- in the limit, 1bpp

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- divide the image in blocks
- code the average value with 8 bits
- $3 \times 3$ blocks at 8 bits per block gives less than 1bpp

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- compress remote regions independently

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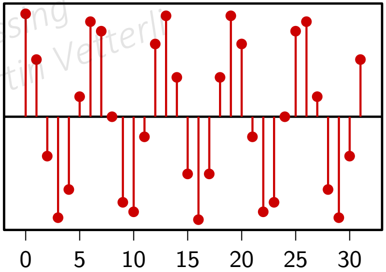
# Transform coding

A simple example:

- take a DT signal, assume $R$ bits per sample
- storing the signal requires $NR$ bits
- now you take the DFT and it looks like this
- in theory, we can just code the two nonzero DFT coefficients!



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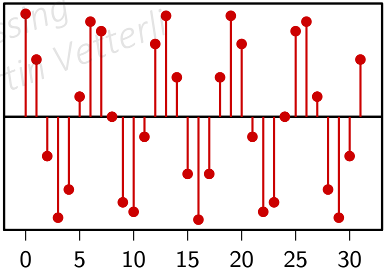

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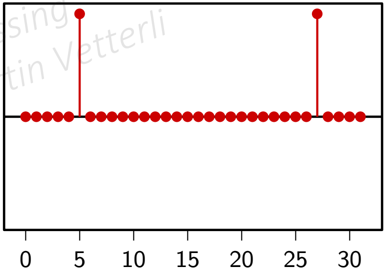

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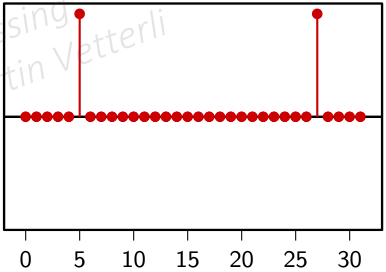

Ideally, we would like a transform that:

- captures the important features of an image block in a few coefficients
- is efficient to compute
- answer: the Discrete Cosine Transform

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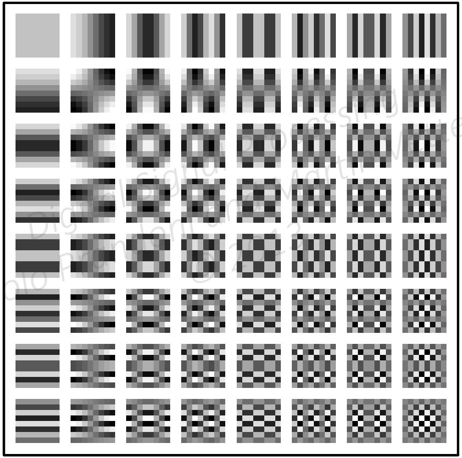
- is efficient to compute
- answer: the Discrete Cosine Transform

$$C[k_1, k_2] = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} x[n_1, n_2] \cos\left[\frac{\pi}{N}\left(n_1 + \frac{1}{2}\right) k_1\right] \cos\left[\frac{\pi}{N}\left(n_2 + \frac{1}{2}\right) k_2\right]$$

# DCT basis vectors for an 8 × 8 image



# Smart quantization

- deadzone
- variable step (fine to coarse)

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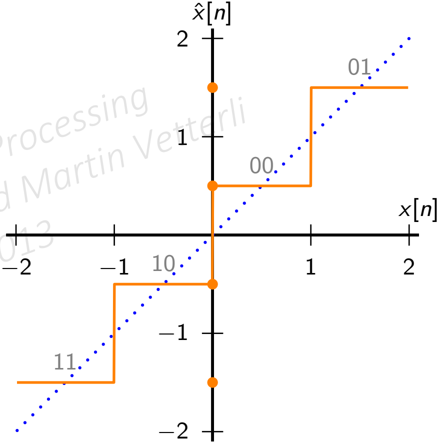
# Quantization

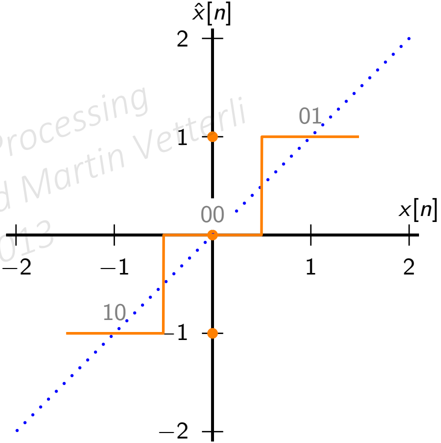
Standard quantization:

$$\hat{x} = \text{floor}(x) + 0.5$$



Deadzone quantization:

$$\hat{x} = \mathrm{round}(x)$$



# Entropy coding

- minimize the effort to encode a certain amount of information
- associate short symbols to frequent values and vice-versa
- if it sounds familiar it's because it is...

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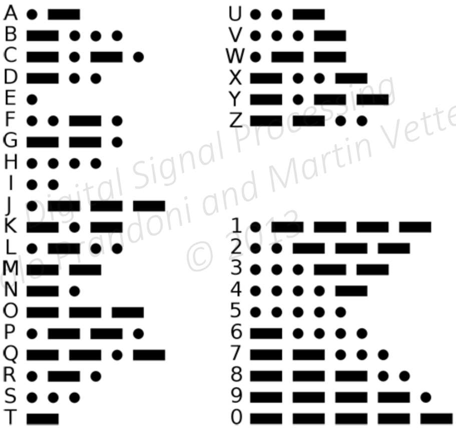
A
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# END OF MODULE 8.5

# Digital Signal Processing

## Module 8.6: The JPEG Compression Algorithm

# Key ingredients

- compressing at block level
- using a suitable transform (i.e., a change of basis)
- smart quantization
- entropy coding

# Key ingredients

- split image into $8 \times 8$ non-overlapping blocks
- using a suitable transform (i.e., a change of basis)
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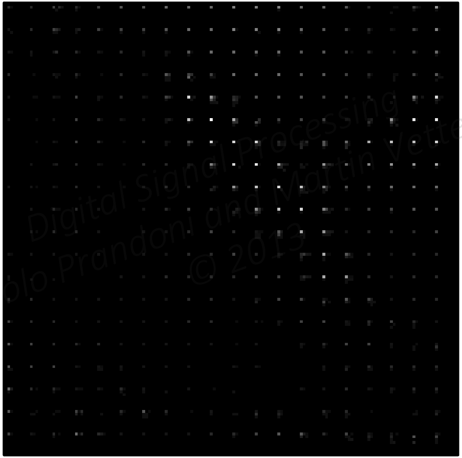
- split image into $8 \times 8$ non-overlapping blocks
- compute the DCT of each block
- quantize DCT coefficients according to psycovisually-tuned tables
- run-length encoding and Huffman coding

# DCT coefficients of image blocks (detail)



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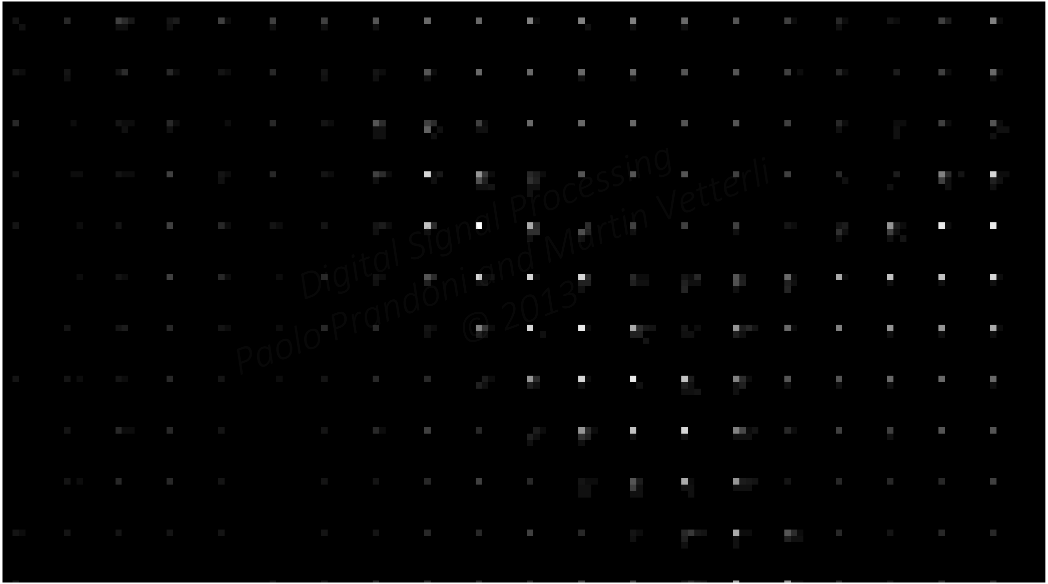

# Smart quantization

- most coefficients are negligible $\rightarrow$ captured by the deadzone
- some coefficients are more important than others
- find out the critical coefficients by experimentation
- allocate more bits (or, equivalently, finer quantization levels) to the most important coefficients

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# Psychovisually-tuned quantization table

$$\hat{c}[k_1, k_2] = \text{round}(c[k_1, k_2]/Q[k_1, k_2])$$

$$Q = \begin{bmatrix} 16 & 11 & 10 & 16 & 24 & 40 & 51 & 61 \\ 12 & 12 & 14 & 19 & 26 & 58 & 60 & 55 \\ 14 & 13 & 16 & 24 & 40 & 57 & 69 & 56 \\ 14 & 17 & 22 & 29 & 51 & 87 & 80 & 62 \\ 18 & 22 & 37 & 56 & 68 & 109 & 103 & 77 \\ 24 & 35 & 55 & 64 & 81 & 104 & 113 & 92 \\ 49 & 64 & 78 & 87 & 103 & 121 & 120 & 101 \\ 72 & 92 & 95 & 98 & 112 & 100 & 103 & 99 \end{bmatrix}$$

# Advantages of nonuniform bit allocation

uniform

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# Efficient coding

- most coefficients are small, decreasing with index
- use zigzag scan to maximize ordering
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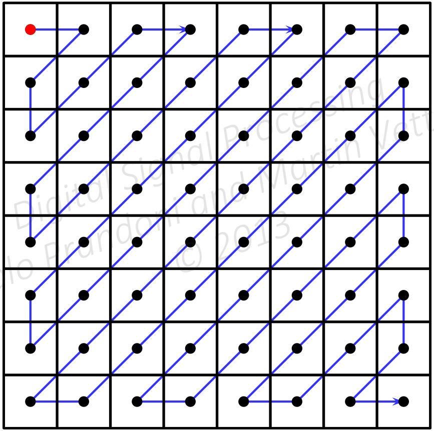
# Zigzag scan



# Example

$$
\begin{bmatrix}
100 & -60 & 0 & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
13 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$



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# Runlength encoding

Each nonzero value is encoded as the triple

$$[(r, s),\ c]$$

- $r$ is the *runlength* i.e. the number of zeros before the current value
- $s$ is the *size* i.e. the number of bits needed to encode the value
- $c$ is the actual value
- $(0, 0)$ indicates that from now on it's only zeros (end of block)

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$$[(0,7),\ 100],\ [(0,6),\ -60],\ [(4,3),\ 6],\ [(3,4),\ 13],\ [(8,1),\ -1], [(0,0)]$$

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# The runlength-size pairs

- by design, $(r, s) \in \mathcal{A}$ with $|\mathcal{A}| = 256$
- in theory, 8 bits per pair
- some pairs are much more common than others!
- a lot of space can be saved by being smart

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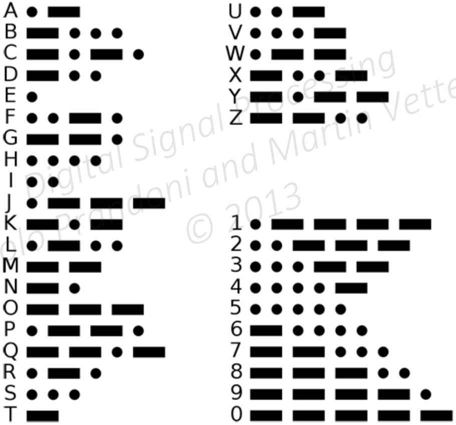
# Variable-length encoding

great idea: shorter binary sequences for common symbols



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however: if symbols have different lengths, we must know how to parse them!

- in English, spaces separate words $\rightarrow$ extra symbol (wasteful)
- in Morse code, pauses separate letters and words (wasteful)
- can we do away with separators?

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# Prefix-free codes

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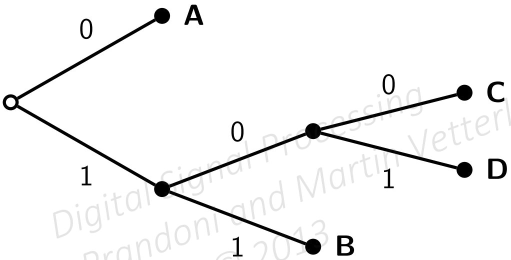
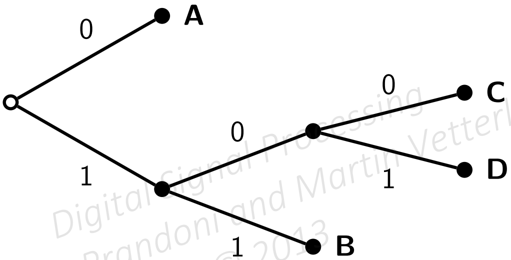





001100110101100

# Prefix-free code






001100110101100

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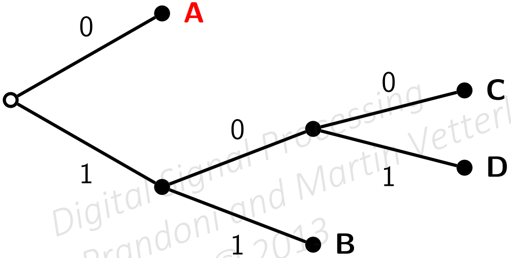
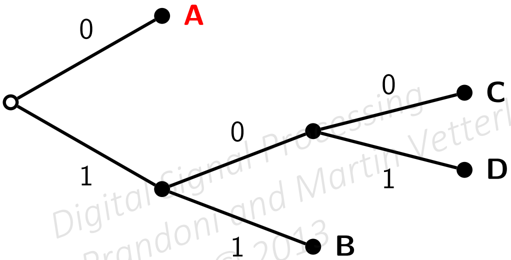





001100110101100

A

# Prefix-free code

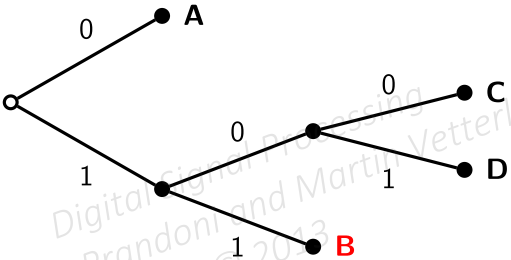





00110011010110 0

AA

# Prefix-free code

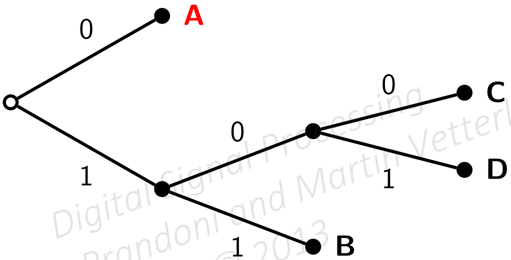




001100110101100

AAB

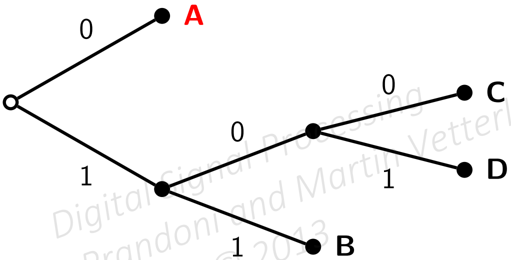
# Prefix-free code




0011001101011100

AABA

# Prefix-free code

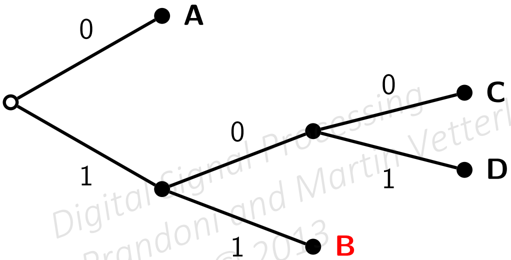




0011001101001100

AABAA

# Prefix-free code




001100110101100

AABAAB

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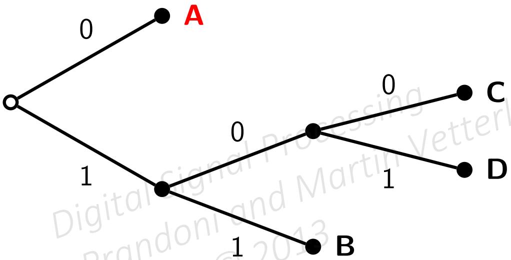
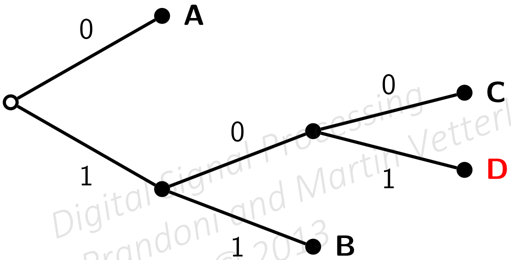




001100110101100

AABAABA

# Prefix-free code





00110011010**1**100

AABAABAD

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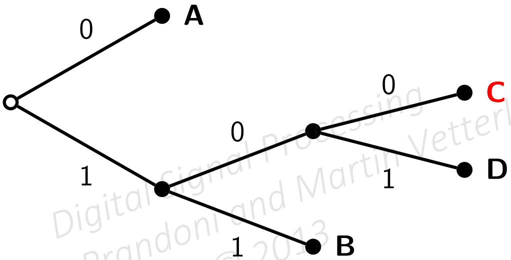




0011001101011100

AABAABADC

# Entropy coding

goal: minimize message length

- assign short sequences to more frequent symbols
- the Huffman algorithm builds the optimal code for a set of symbol probabilities
- in JPEG, you can use a "general-purpose" Huffman code or build your own (but then you pay a "side-information" price)

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# Example

- four symbols: A, B, C, D
- probability table:

$$p(A) = 0.38 \qquad p(B) = 0.32$$

$$p(C) = 0.1 \qquad p(D) = 0.2$$

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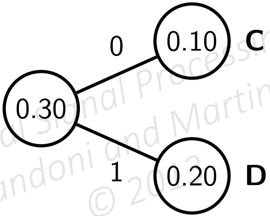
# Building the Huffman code

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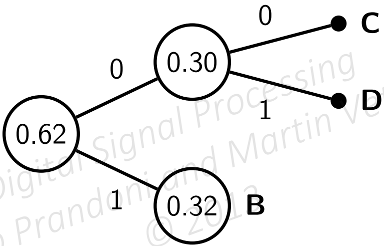
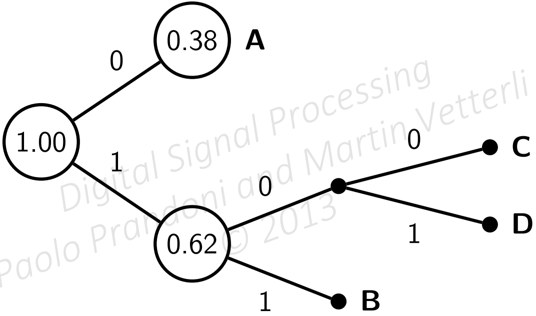



$p(A) = 0.38 \quad p(B) = 0.32 \quad p(C + D) = 0.3$

# Huffman Coding






$$p(A) = 0.38 \quad p(B + C + D) = 0.62$$

# END OF MODULE 8.6

# END OF MODULE 8