

Digital Signal Processing

Digital Signal Processing

Module 6: Interpolation and Sampling

Module Overview:



- ▶ Module 6.1: Continuous-time signals
- ► Module 6.2: Interpolation
- ► Module 6.3: Sampling
- ► Module 6.4: Aliasing
- ► Module 6.5: Interpolation and sampling in practice
- ▶ Module 6.6: Discrete-time processing of continuous-time signals



Digital Signal Processing

Digital Signal Processing

Module 6.1: The Continuous-Time Paradigm

Overview:



- Continuous-time signals Digital Signal Martin Vetterli

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Overview:



- Continuous-time signals Digital Signal Processing Vetterli

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Two views of the world



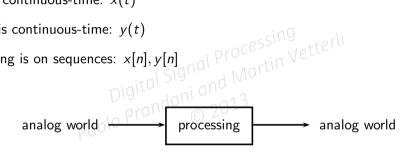


Analog/continuous versus discrete/digital

Digital processing of signals from/to the analog world



- ightharpoonup input is continuous-time: x(t)
- \triangleright output is continuous-time: y(t)
- processing is on sequences: x[n], y[n]

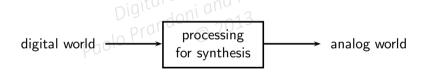


examples: MP3, digital photography

Digital processing of signals to the analog world



- ▶ input is discrete-time: x[n]
- ightharpoonup output is continuous-time: y(t)
- ▶ processing is on sequences: x[n], y[n]

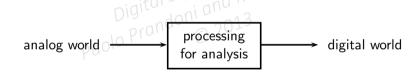


examples: computer graphics, video games

Digital processing of signals from the analog world



- ▶ input is continuous-time: x(t)
- ightharpoonup output is discrete-time: y[n]
- ▶ processing is on sequences: x[n], y[n]



examples: control systems, monitoring

Two views of the world



digital worldview:

- combinatorics Digital Signal Processing
 computer science Prandoni and Martin calculus
 DSP

analog worldview:

- distributions
- system theory

Two views of the world



digital worldview:

- sequences $x[n] \in \ell_2(\mathbb{Z})$ gital Signal Problem Functions $x(t) \in L_2(\mathbb{R})$ Frequency $\omega \in [-\pi, \pi]$ Problem Functions $x(t) \in L_2(\mathbb{R})$

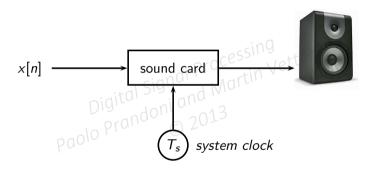
 - ▶ DTFT: $\ell_2(\mathbb{Z}) \mapsto L_2([-\pi, \pi])$

analog worldview:

- frequency $\Omega \in \mathbb{R}$ (rad/sec)
- ightharpoonup FT: $L_2(\mathbb{R}) \mapsto L_2(\mathbb{R})$

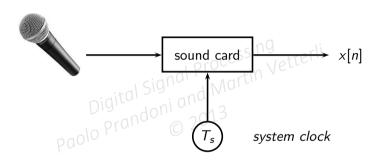
Bridging the gap





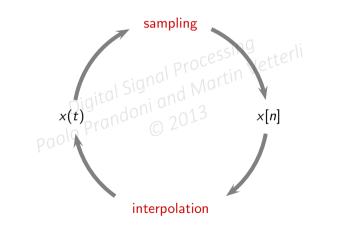
Bridging the gap





Bridging the gap







- time: real variable t

- ⇒ signal x(t): complex functions of a real variable essing

 Finite energy: $x(t) \in L_2(\mathbb{R})$ Finite energy: $x(t) \in$
- energy: $||x(t)||^2 = \langle x(t), x(t) \rangle$



- time: real variable t

- ▶ signal x(t): complex functions of a real variable essing Vetterli Finite energy: $x(t) \in L_2(\mathbb{R})$ | Signal Properties Vetterli Finite energy: $x(t) \in L_2(\mathbb{R})$ | Digital Signal Martin Vetterli Vetterli
- energy: $||x(t)||^2 = \langle x(t), x(t) \rangle$



- time: real variable t

- ▶ signal x(t): complex functions of a real variable essing

 ▶ finite energy: $x(t) \in L_2(\mathbb{R})$ ▶ inner product in $L_2(\mathbb{R})$ Prandoni and Martin

 Prandoni and $L_2(\mathbb{R})$ Prandoni $L_2(\mathbb{R})$
- energy: $||x(t)||^2 = \langle x(t), x(t) \rangle$



- time: real variable t

▶ signal
$$x(t)$$
: complex functions of a real variable essing vetter y .

▶ finite energy: $x(t) \in L_2(\mathbb{R})$

▶ inner product in $L_2(\mathbb{R})$
 $\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x^*(t)y(t)dt$

• energy: $||x(t)||^2 = \langle x(t), x(t) \rangle$



- time: real variable t

▶ signal
$$x(t)$$
: complex functions of a real variable essing

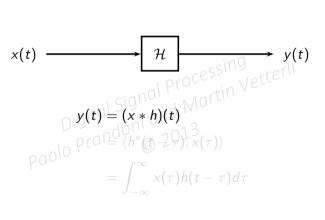
▶ finite energy: $x(t) \in L_2(\mathbb{R})$

▶ inner product in $L_2(\mathbb{R})$
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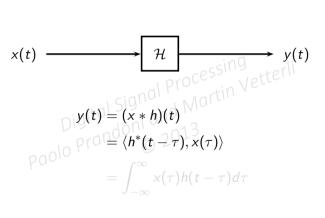
Analog LTI filters





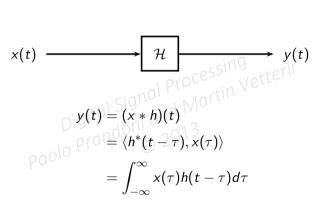
Analog LTI filters





Analog LTI filters





Fourier analysis



- lacktriangle in discrete time max angular frequency is $\pm\pi$
- \blacktriangleright in continuous time no max frequency: $\Omega \in \mathbb{R}$
- concept is the same:

$$X(j\Omega)$$
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Fourier analysis



- \blacktriangleright in discrete time max angular frequency is $\pm\pi$
- \blacktriangleright in continuous time no max frequency: $\Omega \in \mathbb{R}$

same:
$$X(j\Omega) = \frac{1}{2} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

$$\times (t) = \frac{1}{2} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

Fourier analysis



- \blacktriangleright in discrete time max angular frequency is $\pm\pi$
- \blacktriangleright in continuous time no max frequency: $\Omega \in \mathbb{R}$
- concept is the same:

ime no max frequency:
$$\Omega \in \mathbb{R}$$
 same:
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \qquad \leftarrow \textit{not periodic!}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

Real-world frequency

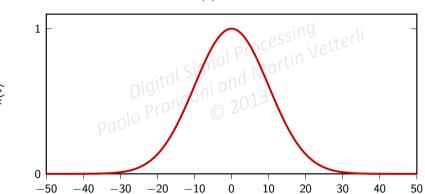


- ► $F = \frac{\Omega}{2\pi}$, expressed in Hertz (1/s) Signal Processing Vetterli ► period $T = \frac{1}{F} = \frac{2\pi}{\Omega}$ paolo Prandoni and (2013)

Example

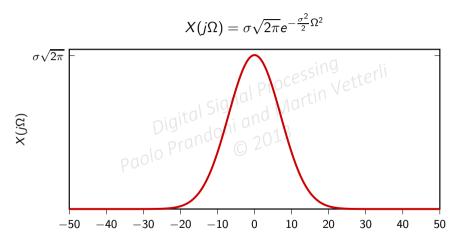






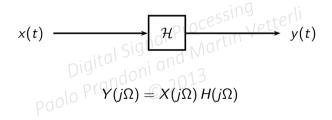
Example





Convolution theorem





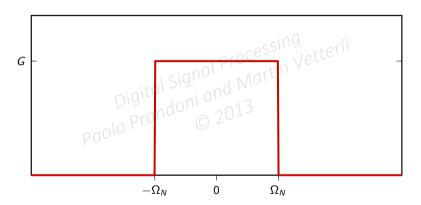
A new concept: bandlimited functions



$$\Omega_N$$
-bandlimitedness:

 $Vetterli$
 $Vett$







$$\Phi(j\Omega) = G \operatorname{rect}\left(\frac{\Omega}{2\Omega_N}\right)$$

$$\operatorname{Digital Signal Processing}$$

$$\operatorname{Digital Signal Processing}$$

$$\operatorname{Partin Vetterli}$$

$$\operatorname{See Module 5.5}$$

$$\operatorname{G}\frac{\Omega_N}{\pi}\operatorname{sinc}\left(\frac{\Omega_N}{\pi}t\right)$$



$$\Phi(j\Omega) = G \operatorname{rect}\left(\frac{\Omega}{2\Omega_N}\right)$$

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(j\Omega) e^{j\Omega t} d\Omega$$

$$= \dots \quad \text{see Module 5.5}$$

$$= G \frac{\Omega_N}{\pi} \operatorname{sinc}\left(\frac{\Omega_N}{\pi}t\right)$$



- Ω_{N} total bandwidth: $\Omega_{B} = 2\Omega_{N}$ define $T_{s} = \frac{2\pi}{\Omega_{B}} = \frac{\pi}{\Omega_{N}}$ | Operation of the processing vertex is processing to the processing vertex and processing vertex is processing vertex.

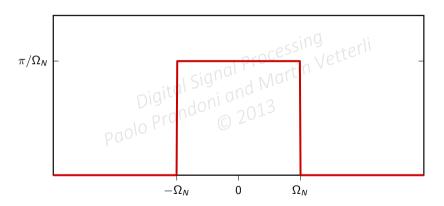


$$\Phi(j\Omega) = \frac{\pi}{\Omega_N} \operatorname{rect}\left(\frac{e\Omega}{2\Omega_N}\right)^{\gamma} \operatorname{etterli}$$

$$\operatorname{pigital Sign}(\alpha) = \frac{\pi}{\Omega_N} \operatorname{rect}\left(\frac{e\Omega}{2\Omega_N}\right)^{\gamma} \operatorname{etterli}$$

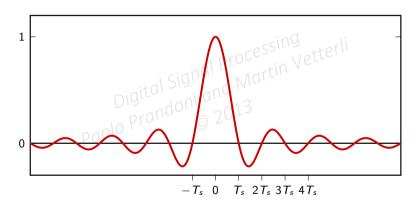
$$\varphi(t) = \operatorname{sinc}\left(\frac{t}{T_s}\right)$$





The prototypical bandlimited function





END OF MODULE 6.1

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Module 7

Paolo Prandoni and Module 7

Overview:



- Polynomial interpolation
- Local interpolation
- Sinc interpolation

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Overview:



- ▶ Polynomial interpolation
- Local interpolation
- Sinc interpolation

on Digital Signal Processing

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Digital Signal Processing

Overview:



- ► Polynomial interpolation
- Local interpolation
- Sinc interpolation

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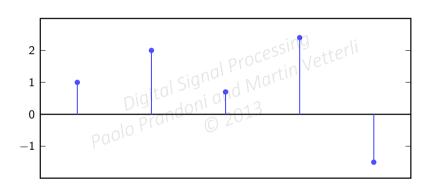
Interpolation



 $x[n] \xrightarrow{\times} x(t)^{\text{essing}}$ Pfill the gaps" between samples

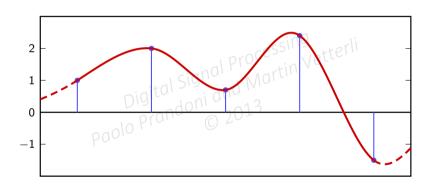
Example





Example





Interpolation requirements



- ► make sure $x(nT_s) = x[n]$ Digital Signal Processing

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 Martin Vetterli

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 Paolo Prandoni and Martin Vetterli

 Paolo Prandoni and Martin Vetterli

Interpolation requirements



- ► make sure $x(nT_s) = x[n]$ Digital Signal Processing

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 Martin Vetterli

 Paolo Prandoni and Martin

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Interpolation requirements



- ► make sure $x(nT_s) = x[n]$ Digital Signal Processing Vetterli

 The make sure x(t) is smooth Prandoni and Martin Paolo Prandoni and Prandoni



- ▶ jumps (1st order discontinuities) would require the signal to move "faster than light"...
- ➤ 2nd order discontinuities would require infinite acceleration terli

 ...

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 the interpolation should be infinitely differentiable

 - ► "natural" solution: polynomial interpolation



- ▶ jumps (1st order discontinuities) would require the signal to move "faster than light"...
- ▶ 2nd order discontinuities would require infinite acceleration
- ► ...

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 The interpolation should be infinitely differentiable partial partia
 - "natural" solution: polynomial interpolation



- ▶ jumps (1st order discontinuities) would require the signal to move "faster than light"...
- ▶ 2nd order discontinuities would require infinite acceleration
- Digital Signal Production Nartin Digital Signal Martin V

 the interpolation should heinfinitely differentiable page 13 page 13 page 14 page 1
 - "natural" solution: polynomial interpolation



- ▶ jumps (1st order discontinuities) would require the signal to move "faster than light"...
- ▶ 2nd order discontinuities would require infinite acceleration Digital Signal Produceleral
- ▶ the interpolation should be infinitely differentiable
- "natural" solution: polynomial interpolation



- ▶ jumps (1st order discontinuities) would require the signal to move "faster than light"...
- ▶ 2nd order discontinuities would require infinite acceleration terms Digital Signal Proaccelera Digital Signal Martin
- ▶ the interpolation should be infinitely differentiable
- "natural" solution: polynomial interpolation



- lacktriangleright N points o polynomial of degree (N-1)
- $p(t) = a_0 + a_1t + a_2t^2 + \ldots + a_{N-1}t^{(N-1)}$
- ► "naive" approach:

```
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- ▶ N points \rightarrow polynomial of degree (N-1)
- $p(t) = a_0 + a_1t + a_2t^2 + \ldots + a_{N-1}t^{(N-1)}$

"naive" approach:

```
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X = \frac{1}{2}

Digital Signal Processing

X = \frac{1}{2}

X = \frac{1}{2}

X = \frac{1}{2}

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```



- ightharpoonup N points ightharpoonup polynomial of degree (N-1)
- \triangleright $p(t) = a_0 + a_1 t + a_2 t^2 + ... + a_{N-1} t^{(N-1)}$
- "naive" approach:

$$+ a_2t^2 + \dots + a_{N-1}t^{(N-1)}$$
n:
$$pado = \begin{cases} Signal \ Processing \\ P(0) = x[0] \end{cases}$$

$$p(T_s) = x[1]$$

$$p(2T_s) = x[2]$$

$$\dots$$

$$p((N-1)T_s) = x[N-1]$$



Without loss of generality:

- ightharpoonup consider a symmetric interval $I_N = [-N, \dots, N]$
- ightharpoonup set $T_s=1$

symmetric interval
$$I_N = [-N, ..., N]$$

$$\begin{cases} p(-N) = x[-N] \\ p(-N+1) = x[-N+1] \\ ... \\ p(0) = x[0] \\ ... \\ p(N) = x[N] \end{cases}$$



Without loss of generality:

- ightharpoonup consider a symmetric interval $I_N = [-N, \dots, N]$
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symmetric interval
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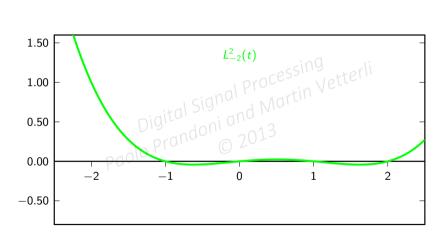
$$\begin{cases} p(-N) = x[-N] \\ p(-N+1) = x[-N+1] \\ ... \\ p(0) = x[0] \\ ... \\ p(N) = x[N] \end{cases}$$



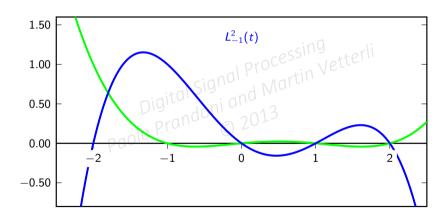
- ▶ P_N : space of degree-2N polynomials over I_N
- ▶ a basis for P_N is the family of 2N + 1 Lagrange polynomials

$$L_n^{(N)}(t) = \prod_{\substack{k=-N\\k\neq n}}^{N} \frac{t-k}{n-k} \qquad n = -N, \dots, N$$

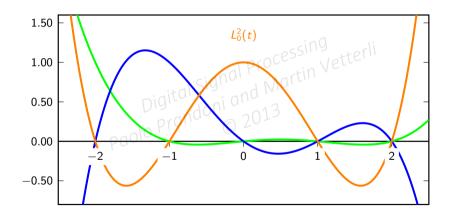




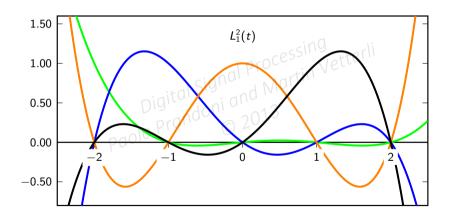




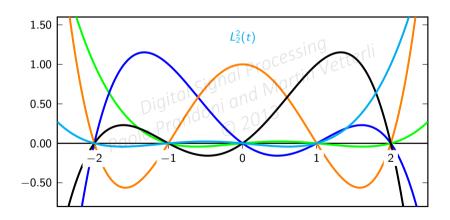














$$\begin{array}{c} \text{Dig}_{R}(t) \stackrel{\text{dig}}{=} \sum_{n=1}^{M} \sum_{n=1}^{N} L_{n}^{(n)}(t) \\ \text{Paolo Prando} \stackrel{\text{dig}}{=} 2013 \end{array}$$



The Lagrange interpolation is the sought-after polynomial interpolation:

- \triangleright polynomial of degree 2N through 2N+1 points is unique
- ▶ the Lagrangian interpolator satisfies

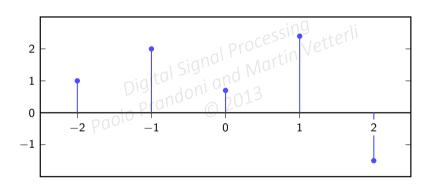
$$p(n) = x[n]$$
 for $-N \le n \le N$

since

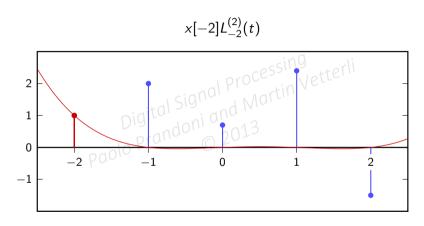
interpolation is the sought-after polynomial interpolation: all of degree
$$2N$$
 through $2N+1$ points is unique ngian interpolator satisfies
$$p(n)=x[n] \qquad \text{for } -N \leq n \leq N$$

$$L_n^{(N)}(m)=\left\{\begin{array}{ll} 1 & \text{if } n=m \\ 0 & \text{if } n\neq m \end{array}\right. \qquad N\leq n, m\leq N$$

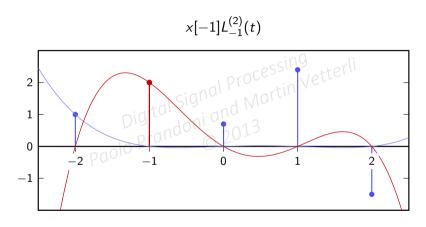




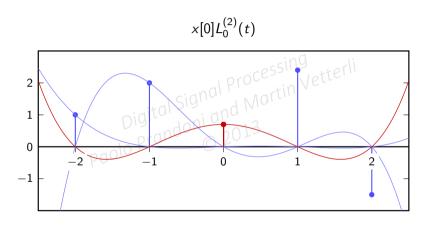




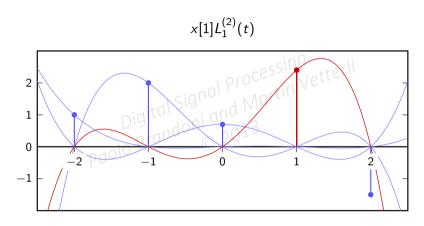




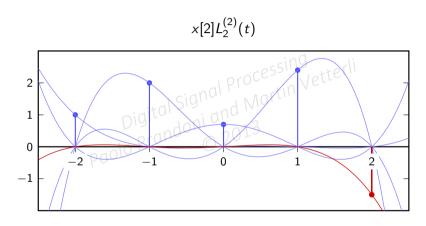






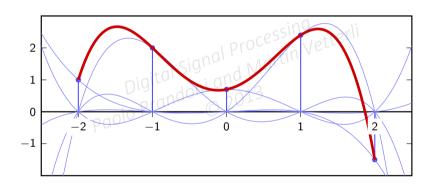






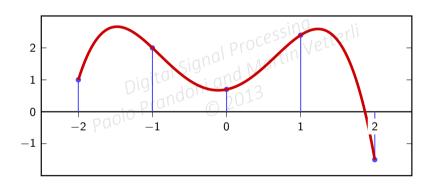
Lagrange interpolation





Lagrange interpolation





Polynomial interpolation



key property:

▶ maximally smooth (infinitely many continuous derivatives)
 Irawback:
 ▶ interpolation "bricks" depend on N

Relaxing the interpolation requirements



- ► make sure $x(nT_s) = x[n]$ The make sure x(t) is smooth page 2013

Relaxing the interpolation requirements



- ► make sure $x(nT_s) = x[n]$ Digital Signal Processing

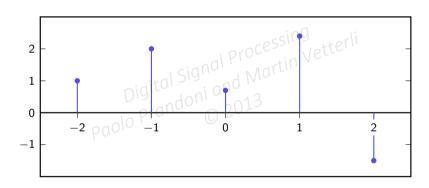
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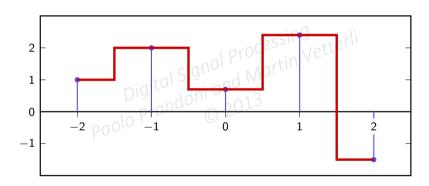
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 Paolo Prandoni and Pra











$$ightharpoonup x(t) = x[\lfloor t + 0.5 \rfloor], \qquad -N \le t \le N$$

$$x(t) = \sum_{n=-N}^{N} x[n] \operatorname{rect}(t-n)$$

$$= \inf_{n=-N} x[n] \operatorname{rect}(t-n) = \inf$$

- ▶ interpolator's support is 1



►
$$x(t) = \sum_{n=-N}^{N} x[n] \operatorname{rect}(t-n)$$

► interpolation kernel: $i_0(t)$ Digital Signal Processing

► $i_0(t)$: "zero-order hope of the processing of the processing vetterling in the processing vetterling in the processing vetterling is $i_0(t)$: "zero-order hope of the processing vetterling is $i_0(t)$: "zero-order hope of the processing vetterling is $i_0(t)$: "zero-order hope of the processing vetterling is $i_0(t)$."

- ▶ interpolator's support is 1
- ▶ interpolation is not even continuous



►
$$x(t) = \sum_{n=-N}^{N} x[n] \operatorname{rect}(t-n)$$

► interpolation kernel: $i_0(t) = \operatorname{rect}(t) \operatorname{oni}$ and Martin Vetterli
► $i_0(t)$: "zero-order hoped $i_0(t)$ or $i_0(t)$ "zero-order hoped $i_0(t)$ " "zero-order hoped $i_0(t)$

- ▶ interpolator's support is 1
- ▶ interpolation is not even continuous



►
$$x(t) = \sum_{n=-N}^{N} x[n] \operatorname{rect}(t-n)$$

► interpolation kernel: $i_0(t) = \operatorname{rect}(t) \operatorname{oni}$ and Martin Vetterli

- $ightharpoonup i_0(t)$: "zero-order hold"
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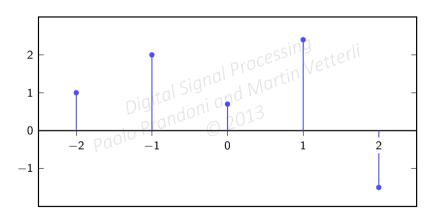


►
$$x(t) = \sum_{n=-N}^{N} x[n] \operatorname{rect}(t-n)$$

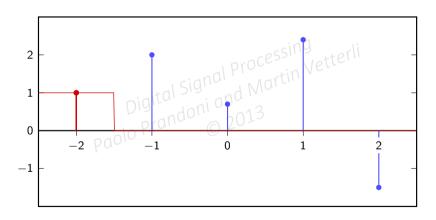
► interpolation kernel: $i_0(t) = \operatorname{rect}(t)$ on and Martin Vetterli

- interpolator's support is 1
- interpolation is not even continuous

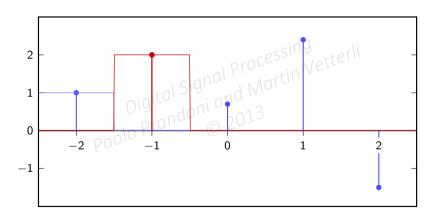




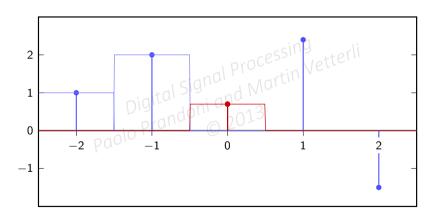




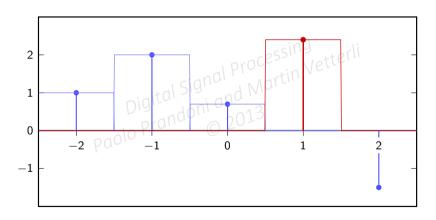




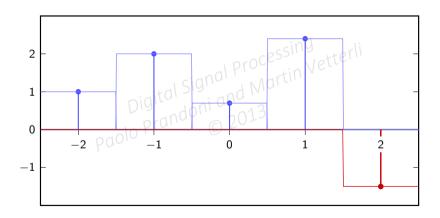




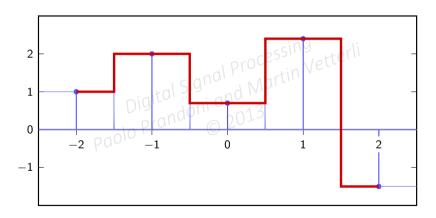




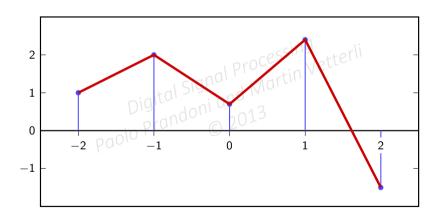














"connect the dots" strategy

interpolation kernel: Digital and a prandoni and t paolo $\underset{i_1(t)}{\text{Prandoni}} \underset{i_2(t)}{\text{and}} = \begin{cases} 29t \\ 0 \end{cases} |t| \leq 1$ otherwise

- ▶ interpolator's support is 2
- ▶ interpolation is continuous but derivative is not



"connect the dots" strategy

(t-n)

Example 1. Digital Signal Processing Vetterli Processing V ▶ interpolation kernel:



"connect the dots" strategy

▶ interpolation kernel:

$$i_1(t) = egin{cases} 1 - |t| & |t| \leq 1 \ 0 & ext{otherwise} \end{cases}$$



"connect the dots" strategy

► interpolation kernel:

$$i_1(t) = egin{cases} 1 & 2 & |t| & |t| \leq 1 \ 0 & ext{otherwise} \end{cases}$$

- interpolator's support is 2



"connect the dots" strategy

►
$$x(t) = \sum_{n=-N}^{N} x[n] i_1(t-n)$$

► interpolation kernel: Digital Signal Processing Vetterli

$$paolo Prandoni and Martin Vetterli$$

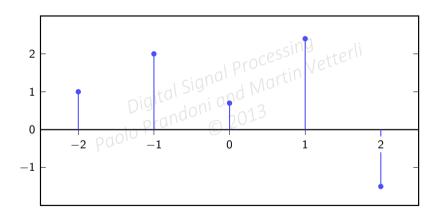
$$i_1(t) = \begin{cases} 1 & \text{otherwise} \end{cases}$$

► interpolator's support is 2

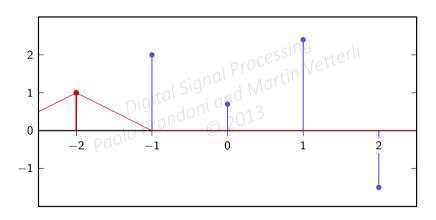
$$i_1(t) = egin{cases} 1 - |t| & |t| \leq 1 \ 0 & ext{otherwise} \end{cases}$$

- interpolation is continuous but derivative is not

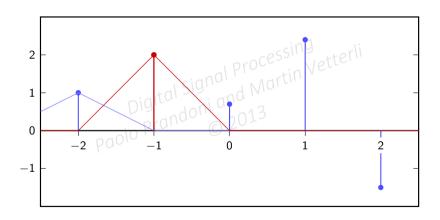




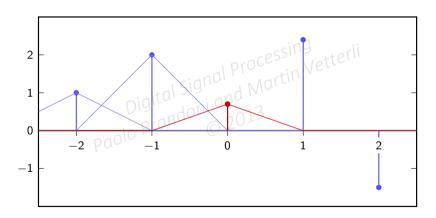




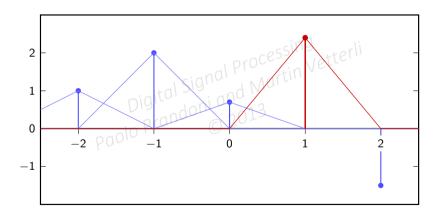




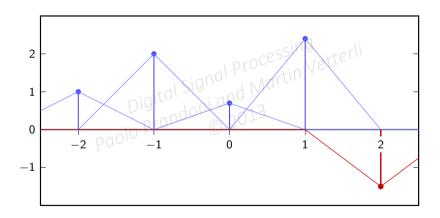




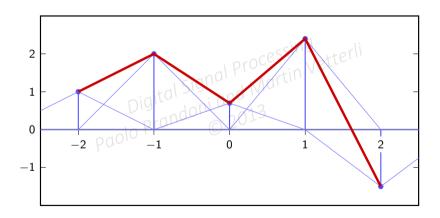












Third-order interpolation



►
$$x(t) = \sum_{n=-N}^{N} x[n] i_3(t-n)$$

• interpolation kernel obtained by splicing two delibric polynomials

• interpolator's support is 4 prandon © 2013

• interpolation is continuous up to second derivative

Third-order interpolation



►
$$x(t) = \sum_{n=-N}^{N} x[n] i_3(t-n)$$

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► interpolator's support is 4 promotion is continuous up to second derivative

Third-order interpolation



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► interpolation kernel obtained by splicing two cubic polynomials

- ► interpolator's support is 4 prondon © 2013

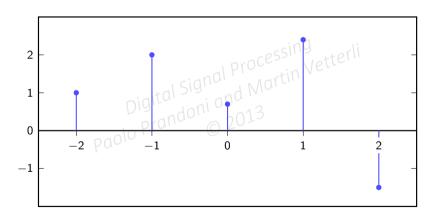


►
$$x(t) = \sum_{n=-N}^{N} x[n] i_3(t-n)$$

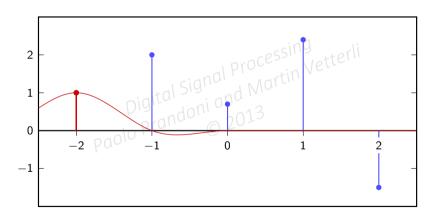
• interpolation kernel obtained by splicing two cubic polynomials

- ► interpolator's support is 4 prondon © 2013
- ▶ interpolation is continuous up to second derivative

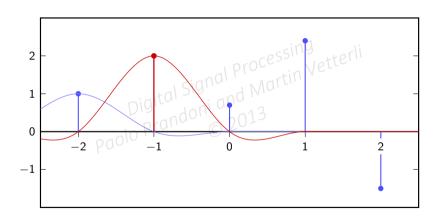




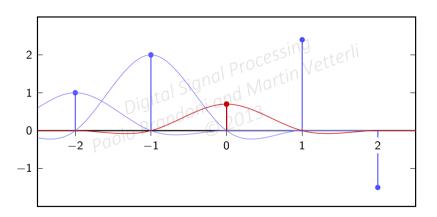




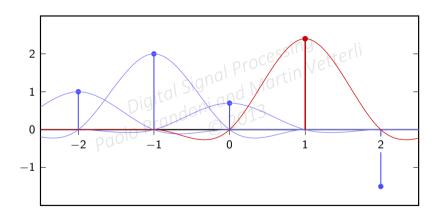




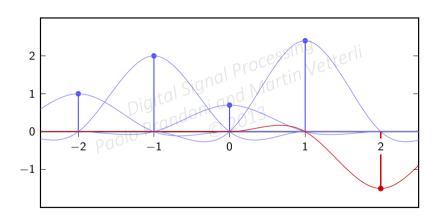




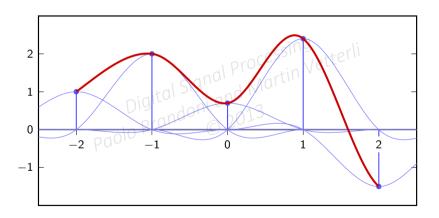












Local interpolation schemes



$$x(t) = \sum_{\substack{n = -N \\ \text{Proposition}}}^{N} x[n] i_{c}(t \le n)$$
ents:

Paolo Prandoni and Martin

2013

 $i_c(0) = 1$

 $i_c(t) = 0$ for t a nonzero integer.

Local interpolation schemes



$$x(t) = \sum_{\substack{n = -N \\ \text{proof on and Martin}}}^{N} x[n] i_c(t \le n)$$
etterli
nts:

paolo Prandoni and Martin

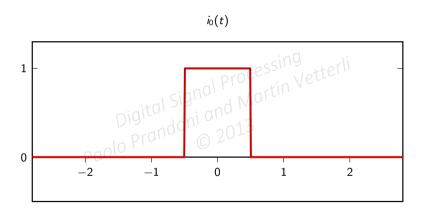
© 2013

Interpolator's requirements:

- $i_c(0) = 1$
- $ightharpoonup i_c(t) = 0$ for t a nonzero integer.

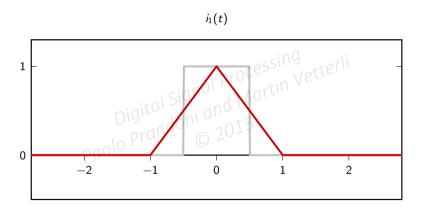
Local interpolators





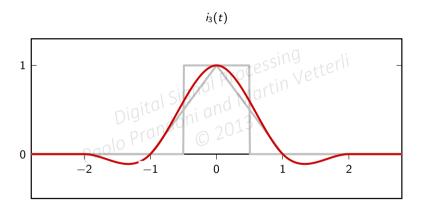
Local interpolators





Local interpolators





Local interpolation



A remarkable result:



$$\lim_{N\to\infty} L_n^{(N)}(t) = \operatorname{sinc}(t-n)^{\operatorname{rerli}}$$

$$\operatorname{Digital Signal Marian}(t-n)^{\operatorname{rerli}}$$

$$\operatorname{Digital Signal Marian}(t-n)^{\operatorname{rerli}}$$

$$\operatorname{Digital Signal Marian}(t-n)^{\operatorname{rerli}}$$
in the pional, Prandoni and global interpolation are the same!

A remarkable result:



$$\lim_{N\to\infty} L_n^{(N)}(t) = \operatorname{sinc}(t-n)^{\text{terli}}$$

$$\operatorname{Digital Sign and Martin}(t-n)^{\text{terli}}$$

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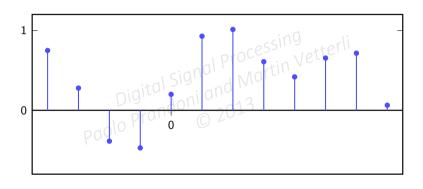
in the limit, local and global interpolation are the same!

Sinc interpolation formula

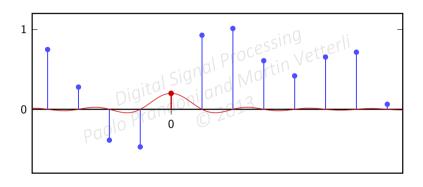


$$x(t) = \sum_{n=1}^{\infty} \frac{\sum_{s=1}^{\infty} x[n] \operatorname{sinc}\left(\frac{t-nT_s}{T_s}\right)}{\sum_{s=1}^{\infty} x[n] \operatorname{sinc}\left(\frac{t-nT_s}{T_s}\right)}$$

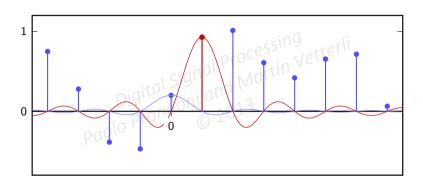




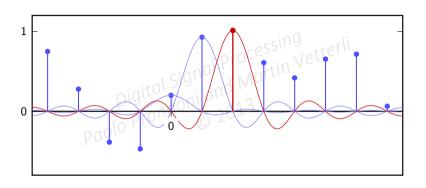




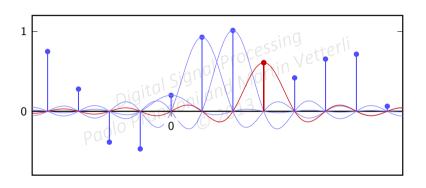




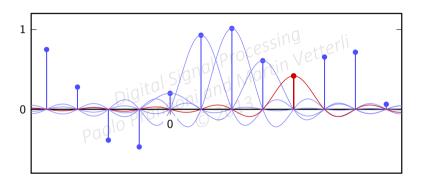




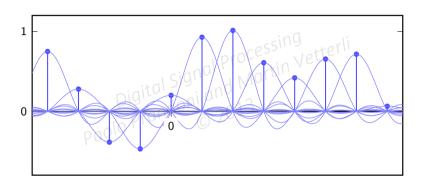






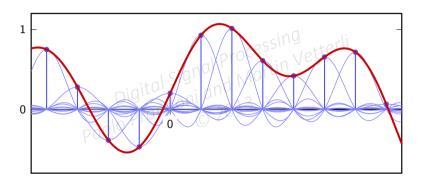




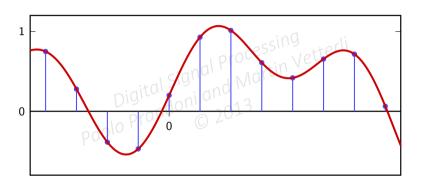


6.2 53









"Proof" that $L_n^{(N)}(t) \rightarrow \operatorname{sinc}(t-n)$



- real proof is rather technical (see the book)

intuition:
$$\operatorname{sinc}(t-n)$$
 and $L_n^{(\infty)}(t)$ share an infinite number of zeros:
$$\operatorname{sinc}(m-n) = \delta[m-n] = \delta[m-n] = m, n \in \mathbb{Z}, \quad -N \leq n, m \leq N$$

"Proof" that $L_n^{(N)}(t) \rightarrow \operatorname{sinc}(t-n)$



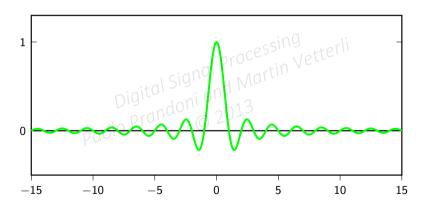
- real proof is rather technical (see the book)
- intuition: $\operatorname{sinc}(t-n)$ and $L_n^{(\infty)}(t)$ share an infinite number of zeros:

$$\operatorname{sinc}(m-n) = \delta[m-n] \quad m, n \in \mathbb{Z}$$

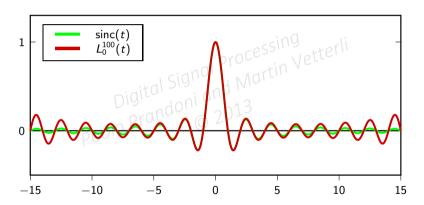
$$L_n^{(N)}(m) = \delta[m-n] \quad m, n \in \mathbb{Z}, \quad -N \leq n, m \leq N$$

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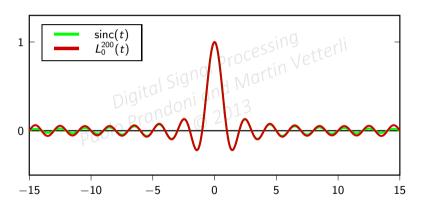




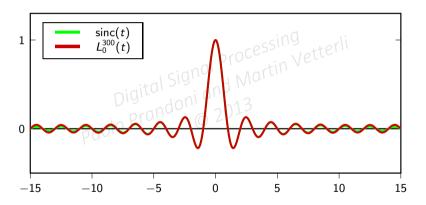












END OF MODULE 6.2 Digital Signa and Martin Et 6.2 Paolo Prandoni and Martin Et 6.2



Digital Signal Processing

Digital Signal Processing

Module 6.3: The space of bandlimited signals

Overview:



- Inctions Signal Processing

 Signal Processing

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 The sampling Digital © 2013

 The sampling theoremaolo Prandoni © 2013

Overview:



- ... sampling Digital Signal Processing

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Overview:



- ... sampling Digital Signal Processing

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 Digital Signal Processing

 One of the sampling theorem of the sampling the sampl

Overview:



- ... sampling Digital Signal Processing

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 Digital Signal Processing

 → Sampling Digital Signal Processing

 → Sampling Digital Signal Processing

 → Signal Processing

 → Comparison Of Prandoni and Martin Vetterli

 → Sampling Digital Signal Processing

 → Signal Processing

 → Office Signal Processing

 → Office



the ingredients:

- Digital Signal Processing

 Digital Signal Martin Vetterli

 Paolo Prandoni and Martin Vetterli

 Paolo Prandoni and Martin Vetterli

 Paolo Prandoni and Martin Vetterli ▶ discrete-time signal x[n], $n \in \mathbb{Z}$ (with DTFT $X(e^{j\omega})$)
- \triangleright interpolation interval T_s
- the sinc function

ightharpoonup a smooth, continuous-time signal $x(t), t \in \mathbb{R}$



the ingredients:

- Digital Signal Processing

 Digital Signal Processing

 Martin Vetterli

 Paolo Prandoni and Martin Vetterli

 2013

 Paolo Prandoni C 2013 ▶ discrete-time signal x[n], $n \in \mathbb{Z}$ (with DTFT $X(e^{j\omega})$)
- \triangleright interpolation interval T_s
- the sinc function

the result:

a smooth, continuous-time signal $x(t), t \in \mathbb{R}$



the ingredients:

- Digital Signal Processing

 Digital Signal Processing

 Martin Vetterli

 Paolo Prandoni and Martin

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 Paolo Prandoni

 Time signal • discrete-time signal $x[n], n \in \mathbb{Z}$ (with DTFT $X(e^{j\omega})$)
- \triangleright interpolation interval T_s
- the sinc function

the result:

a smooth, continuous-time signal $x(t), t \in \mathbb{R}$

what does the spectrum of x(t) look like?

Key facts about the sinc



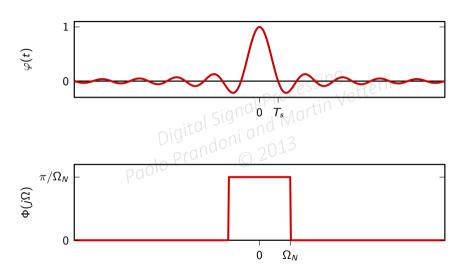
$$\varphi(t) = \operatorname{sinc}\left(\frac{t}{T_s}\right) \longleftrightarrow \Phi(j\Omega) = \frac{\pi}{\Omega_N} \operatorname{rect}\left(\frac{\Omega}{2\Omega_N}\right)$$

$$T_s = \frac{\pi}{\Omega_N} \operatorname{rect}\left(\frac{1}{2\Omega_N}\right)$$

$$\Omega_N = \frac{\pi}{T_s}$$

Key facts about the sinc







$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t + nT_s}{T_s}\right)$$
paolo Pranco (2013)



$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}}^{\infty} \sum_{\substack{i,j \in \Omega_{n} = -\infty \\ n = -\infty}$$



$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t + nT_s}{T_s}\right) e^{-j\Omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{\pi}{\Omega_N}\right) \operatorname{rect}\left(\frac{\Omega}{2\Omega_N}\right) e^{-jnT_s\Omega}$$



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$$= \left(\frac{\pi}{2\Omega_N}\right) \operatorname{rect}\left(\frac{\pi}{2\Omega_N}\right) \operatorname{rect}\left(\frac{\pi}{2\Omega_N}\right) e^{-jnT_s\Omega}$$

$$= \left(\frac{\pi}{2\Omega_N}\right) \times \left(e^{j\pi(\Omega/\Omega_N)}\right) \quad \text{for } |\Omega| \le \Omega_N$$
otherwise



$$X(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{\pi}{\Omega_N}\right) \operatorname{rect}\left(\frac{\Omega}{2\Omega_N}\right) e^{-jnT_s\Omega}$$

$$= \left(\frac{\pi}{\Omega_N}\right) \operatorname{rect}\left(\frac{\Omega}{2\Omega_N}\right) \sum_{n=-\infty}^{\infty} x[n] e^{-j(\pi/\Omega_N)\Omega_n}$$

$$= \begin{cases} (\pi/\Omega_N) \times (e^{j\pi(\Omega/\Omega_N)}) & \text{for } |\Omega| \leq \Omega_N \\ 0 & \text{otherwise} \end{cases}$$

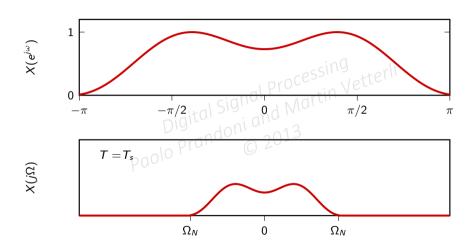


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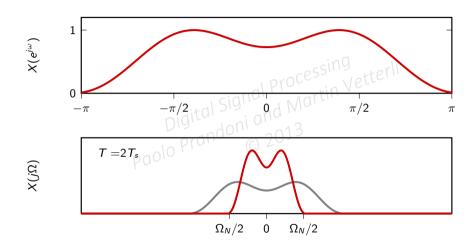
$$= \left(\frac{\pi}{\Omega_N}\right) \operatorname{rect}\left(\frac{\Omega}{2\Omega_N}\right) \sum_{n=-\infty}^{\infty} x[n] e^{-j(\pi/\Omega_N)\Omega n}$$

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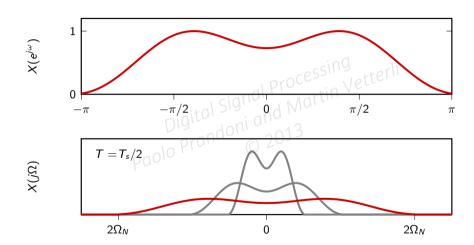














- ► $X(j\Omega)$ is Ω_N -bandlimited, with $\Omega_N = \pi/T_s$ processing

 fast interpolation (T_s small) This is performed and property of the slow interpolation (T_s large) promotion of the spectrum.

 - (for those who remember...) it's like changing the speed of a record player



- $X(j\Omega)$ is Ω_N -bandlimited, with $\Omega_N = \pi/T_{sprocessing}$ vertering fast interpolation $(T_s \text{ small}) \rightarrow \text{wider spectrum}$
- ▶ slow interpolation $(T_s | \text{large})_{promarrower}$ spectrum
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- ightharpoonup slow interpolation (T_s large) ightharpoonup narrower spectrum
- ▶ (for those who remember...) it's like changing the speed of a record player

Space of bandlimited functions



Space of bandlimited functions



$$x[n] \in \ell_2(\mathbb{Z})$$

Digital Signal And Markin $x(t) \in L_2(\mathbb{R})$
 Ω_N -BL

Let's lighten the notation



Digital Signal Processing Vetterli And Martin Vetterli and Martin Vetterli and Martin Vetterli and Prandoni and Martin 2013 (derivations in the general case are in the book)

The road to the sampling theorem



claims:

- the space of π -bandlimited functions is a Hilbert space Vetterli
- the functions $\varphi^{(n)}(t) = \text{sinc}(t)$, with $\eta \in \mathbb{Z}$, form a basis for the space if x(t) is π -BL, the sequence $\varphi(n) = x(n)$, with $n \in \mathbb{Z}$, is a sufficient representation (i.e. we can reconstruct Q(t) from x[n])

The road to the sampling theorem



claims:

- the space of π -bandlimited functions is a Hilbert space $\sqrt{e^{tterli}}$
- ► the functions $\varphi^{(n)}(t) = \operatorname{sinc}(t+n)$, with $n \in \mathbb{Z}$, form a basis for the space ► if x(t) is π -BL, the sequence x(n) = x(n), with $n \in \mathbb{Z}$, is a sufficient representation (i.e. we can reconstruct Q(t) from x[n])

The road to the sampling theorem



claims:

- the space of π -bandlimited functions is a Hilbert space $\sqrt{e^{tterli}}$
- ▶ the functions $\varphi^{(n)}(t) = \operatorname{sinc}(t-n)$, with $n \in \mathbb{Z}$, form a basis for the space
- if x(t) is π -BL, the sequence x[n] = x(n), with $n \in \mathbb{Z}$, is a sufficient representation (i.e. we can reconstruct x(t) from x[n])

The space π -BL



- lacktriangle clearly a vector space because $\pi ext{-BL}\subset L_2(\mathbb{R})$ (and linear combinations of $\pi ext{-BL}$ functions ► inner product is standard in ingital Signal Production

 Completeness... that's more delicate © 2013

The space π -BL



- ▶ clearly a vector space because π -BL $\subset L_2(\mathbb{R})$ (and linear combinations of π -BL functions are π -BL functions)
- lacktriangle inner product is standard inner product in $L_2(\mathbb{R})$
- completeness... that's more delicate

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- ▶ completeness... that's more delicate

The space of π -BL functions



recap:

inner product:

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x^*(t)y(t)dt$$

$$(x*y)(t) = \langle x^*(\tau), y(t-\tau) \rangle$$

convolution:

$$(x*y)(t) = \langle x^*(\tau), y(t-\tau) \rangle$$



$$\varphi^{(n)}(t) = \operatorname{sinc}(t-n), \qquad n \in \mathbb{Z}$$

$$\langle \varphi^{(n)}(t), \varphi^{(m)}(t) \rangle = \langle \varphi^{(0)}(t, m), \varphi^{(0)}(t, m) \rangle^{(n)}$$

$$\text{Digital Signor } (h, h), \varphi^{(0)}(m-t) \rangle$$

$$\text{Paralogorizant} (m-t) dt$$

$$= \int_{-\infty}^{\infty} \operatorname{sinc}(\tau) \operatorname{sinc}((m-n) - \tau) d\tau$$

$$= (\operatorname{sinc} * \operatorname{sinc})(m-n)$$



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now use the convolution theorem knowing that:

$$\begin{aligned} \mathsf{FT} \left\{ \mathsf{sinc}(t) \right\} &= \mathsf{rect} \left(\frac{\Omega}{2\pi} \right) \\ (\mathsf{sinc} * \mathsf{sinc}) (m + n) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\mathsf{rect} \left(\frac{\Omega}{2\pi} \right) \right]^2 e^{j\Omega(m-n)} d\Omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega(m-n)} d\Omega \\ &= \begin{cases} 1 & \mathsf{for} \ m = n \\ 0 & \mathsf{otherwise} \end{cases} \end{aligned}$$



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for any $x(t) \in \pi$ -BL:

$$\langle \varphi^{(n)}(t), x(t) \rangle = \langle \operatorname{sinc}(t-n), x(t) \rangle = \langle \operatorname{sinc}(n-t), x(t) \rangle$$

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$$= (\operatorname{sinc} * x)(n)$$

$$= (\operatorname{sinc} *$$



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$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{rect}\left(\frac{\Omega}{2\pi}\right) X(j\Omega) e^{j\Omega n} d\Omega$$

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Sampling as a basis expansion, π -BL



Analysis formula:

Analysis formula:
$$x[n] = \langle \operatorname{sinc}(t-n), x(t) \rangle$$

Sampling as a basis expansion, Ω_N -BL



Analysis formula:

$$x[n] = \langle \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right), x(t) \rangle = T_s x(nT_s)$$

$$x(t) = \frac{1}{T_s} \sum_{n = -\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

$$x(t) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

The sampling theorem



- \triangleright the space of Ω_N -bandlimited functions is a Hilbert space
- set $T_s=\pi/\Omega_N$ the functions $\varphi^{(n)}(t)=\mathrm{sinc}((t-nT_s)/T_s)$ form a basis for the space
- for any $x(t) \in \Omega_N$ -BL the coefficients in the sinc basis are the (scaled) samples $T_s \times (nT_s)$

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- for any $x(t) \in \Omega_N$ -BL the coefficients in the sinc basis are the (scaled) samples $T_s \times (nT_s)$

for any $x(t) \in \Omega_N$ -BL, a sufficient representation is the sequence $x[n] = x(nT_s)$

The sampling theorem, corollary



for any $x(t) \in \Omega_N$ -BL redshifticienty essentation is the sequence $Pao\{\Phi_n\} = x(nT_s) \text{ for any } T_s \leq \pi/\Omega_N$ $ightharpoonup \Omega_N$ -BL $\subset \Omega$ -BL for any $\Omega > \Omega_N$

The sampling theorem, corollary



 $ightharpoonup \Omega_N$ -BL $\subset \Omega$ -BL for any $\Omega > \Omega_N$

 $x[n] = x(nT_s)$ for any $T_s \le \pi/\Omega_N$

The sampling theorem, in hertz



any signal x(t) bandlimited to F_N Hz can be sampled with no loss of information using a sampling frequency $F_s \geq 2F_N$ (i.e. a sampling period $T_s \leq 1/2F_N$)

END OF MODULE 6.3

Digital Signa and Martin Paolo Prandoni and 2013



Digital Signal Processing

Digital Signal Processing

Module 6.4: Sampling and Aliasing - Introduction

Overview:



- "Raw" sampling
- Sinusoidal aliasing

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Digital Signal Processing
Vetterli

Paolo Prandoni and Martin Vetterli

Paolo Prandoni and Martin

Overview:



- "Raw" sampling
- Sinusoidal aliasing

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Digital Signal Processing
Vetterli

Paolo Prandoni and Martin Vetterli

O 2013

Sinc Sampling



$$x[n] = \langle \operatorname{sinc} \left(\frac{t - nT_s}{T_s} \right) s x(t) \rangle$$

$$pigital \ signal \ Processor \ Vetter II$$

$$paolo \ Prandoni \ and \ Martin \ Vetter II$$

$$paolo \ Prandoni \ 2013$$

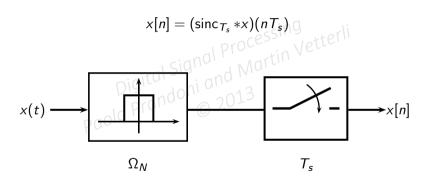
Sinc Sampling



 $x[n] = (\operatorname{sinc}_{T_s} * x)(nT_s)_{i,j}$ $\operatorname{Digital Signal Processin Vetterli}_{i,j}$ $\operatorname{Digital Signal Martin Vetterli}_{i,j}$ $\operatorname{Digital Signal Processin Vetterli}_{i,j}$

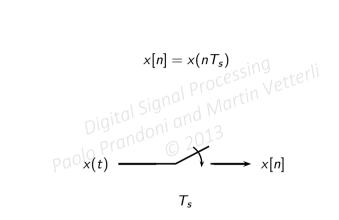
Sinc Sampling





"Raw" Sampling





Remember the wagonwheel effect?



Digital Signal Processin Vetterli Paolo Prandoni and Martin Vetterli © 2013



$$x(t) = e^{j\Omega_0 t}$$

- $x(t)=e^{j\Omega_0t}$ $* always periodic, period $T=2\pi/\Omega_0$ signal Processing Vetterli <math display="block"> * all angular speeds are allowed random = 0.013$ $* FT \{e^{j\Omega_0t}\} = 2\pi\delta(\Omega-\Omega_0)$ $* bandlimited to $\Omega_0$$



$$x(t)=e^{j\Omega_0t}$$

- $x(t)=e^{j\Omega_0t}$ $\Rightarrow \text{ always periodic, period } T=2\pi/\Omega_0 \text{ and Martin Vetterli}$ $\Rightarrow \text{ all angular speeds are allowed random}$ $\Rightarrow \text{ all } T=2\pi\delta(\Omega) \text{ and } T=2\pi\delta(\Omega)$



$$x(t) = e^{j\Omega_0 t}$$

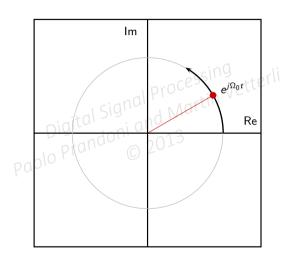
- $x(t)=e^{j\Omega_0t}$ always periodic, period $T=2\pi/\Omega_0$ and Martin Vetterli all angular speeds are allowed random (2013) $\mathrm{FT}\left\{e^{j\Omega_0t}\right\}=2\pi\delta(\Omega-\Omega_0)$ addimited to Ω



$$x(t) = e^{j\Omega_0 t}$$

- $x(t)=e^{j\Omega_0t}$ always periodic, period $T=2\pi/\Omega_0$ and Martin Vetterli all angular speeds are allowed random =20.3 FT $\left\{e^{j\Omega_0t}\right\}=2\pi\delta(\Omega-\Omega_0)$ and with vetterli always periodic, period =20.3 relative to =20.3





Raw samples of the continuous-time complex exponential



$$x[n] = e^{i\Omega_0 T_s n} ssing$$

$$pigital Signal Processing Vetter II$$

$$pigital Signal Martin Vetter II$$

$$pigital Signal Martin Vetter II$$

- raw samples are snapshots at regular intervals of the rotating point
- resulting digital frequency is $\omega_0 = \Omega_0 T_s$

Raw samples of the continuous-time complex exponential



$$x[n] = e^{i\Omega_0 T_s n} ssing$$

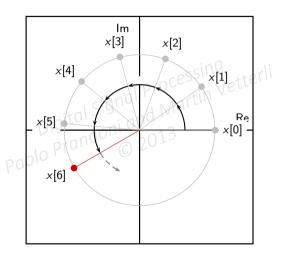
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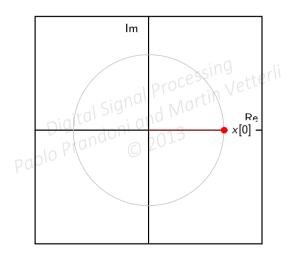
When $T_s < \pi/\Omega_0$, $\omega_0 < \pi$...





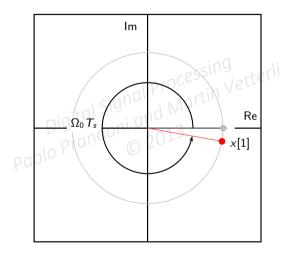
When $\pi/\Omega_0 < T_s < 2\pi/\Omega_0$, $\pi < \omega_0 < 2\pi$...





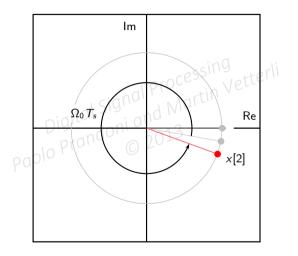
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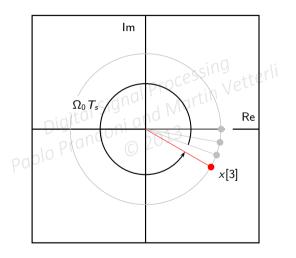
When $\pi/\Omega_0 < T_s < 2\pi/\Omega_0, \ \pi < \omega_0 < 2\pi...$





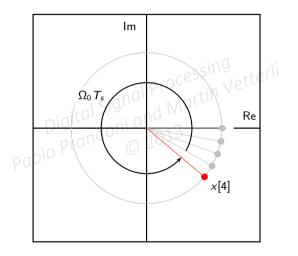
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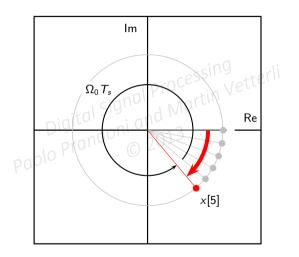
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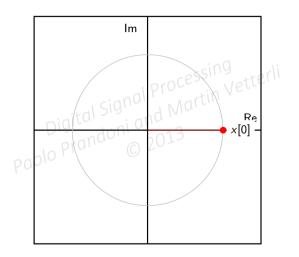


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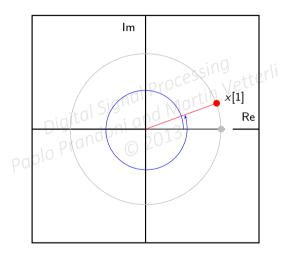




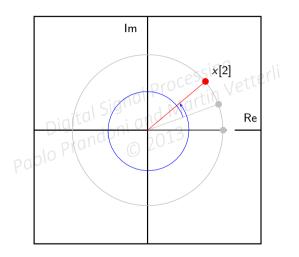




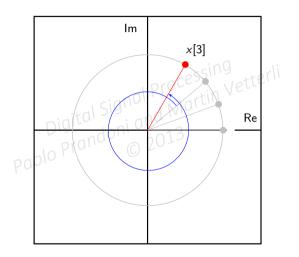






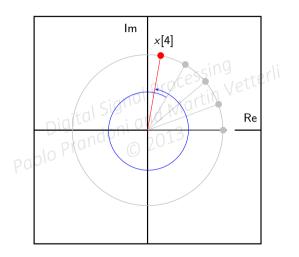






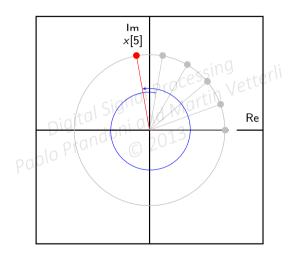
When $T_s > 2\pi/\Omega_0$, $\omega_0 > 2\pi$...





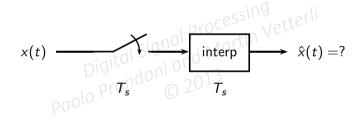
When $T_s > 2\pi/\Omega_0$, $\omega_0 > 2\pi$...





Aliasing





Aliasing



$$x(t) = e^{j\Omega_0 t}$$

sampling period

 $x(t)=\mathrm{e}^{j\Omega_0t}$ digital frequency $\hat{x}(t)$

$$\begin{array}{lll} T_s < \pi/\Omega_0 & 0 < \omega_0 < \pi & e^{j\Omega_0 t} \\ \pi/\Omega_0 < T_s < 2\pi/\Omega_0 & \pi < \omega_0 < 2\pi & e^{j\Omega_1 t}, & \Omega_1 = \Omega_0 - 2\pi/T_s \\ T_s > 2\pi/\Omega_0 & \omega_0 > 2\pi & e^{j\Omega_2 t}, & \Omega_2 = \Omega_0 \mod(2\pi/T_s) \end{array}$$

Again, with a simple sinusoid and using hertz



$$x(t) = \cos(2\pi F_0 t)$$
 $x[n] = x(nT_s) = \cos(\omega_0 n)$
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Sampling a Sinusoid

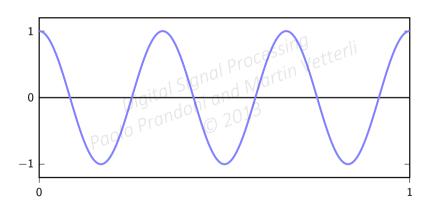


sampling frequency	digital frequency result
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$F_s > 2F_0$	$0<\omega_0<\pi$
$F_s = 2F_0$	$\omega_0 = \pi$ max digital frequency: $x[n] = (-1)^n$
$F_0 < F_s < 2F_0$	$\pi < \omega_0 < 2\pi$ negative frequency $\omega_0 - 2\pi$
$F_s < F_0$	$\omega_0 > 2\pi$ full aliasing: $\omega_0 \mod 2\pi$

4

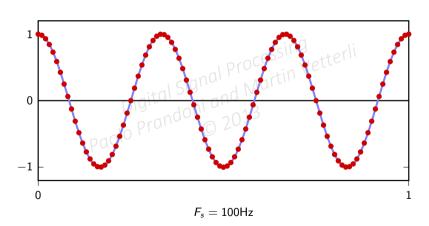


$$x(t) = \cos(6\pi t) \qquad (F_0 = 3Hz)$$



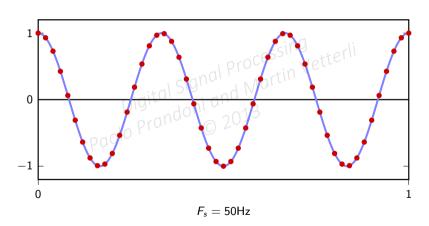


$$x(t) = \cos(6\pi t) \qquad (F_0 = 3Hz)$$



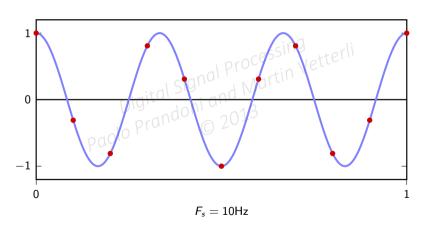


$$x(t) = \cos(6\pi t) \qquad (F_0 = 3Hz)$$



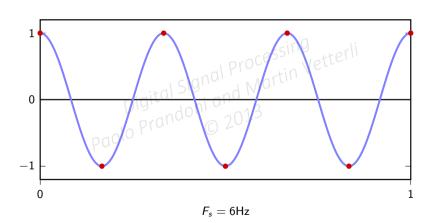


$$x(t) = \cos(6\pi t) \qquad (F_0 = 3Hz)$$



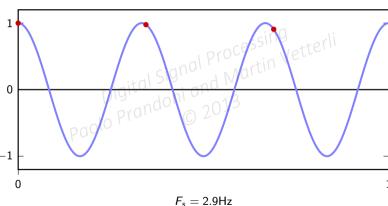


$$x(t) = \cos(6\pi t) \qquad (F_0 = 3Hz)$$





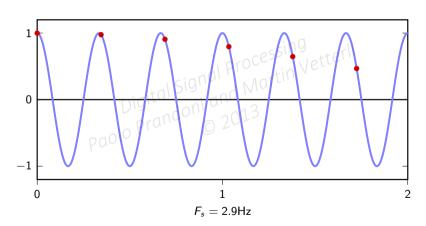
$$x(t) = \cos(6\pi t) \qquad (F_0 = 3Hz)$$



 $\Gamma_s = 2.9 \Pi Z$

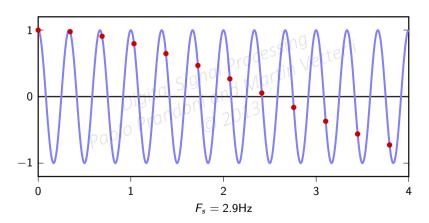


$$x(t) = \cos(6\pi t) \qquad (F_0 = 3Hz)$$



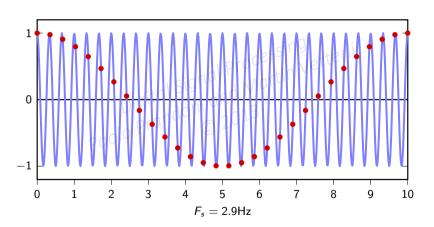


$$x(t) = \cos(6\pi t) \qquad (F_0 = 3Hz)$$





$$x(t) = \cos(6\pi t) \qquad (F_0 = 3Hz)$$



END OF MODULE 6.4

Digital Signard Martine 6.4

Paolo Prandoni and Martine 2013



Digital Signal Processing

Digital Signal Processing

Module 6.5: Same

Overview:



- Aliasing for arbitrary spectra
- Examples

Digital Signal Processing

Digital Signal Martin Vetterli

and Martin Vetterli

Paolo Prandoni and Martin

© 2013

Overview:



- ► Aliasing for arbitrary spectra
- Examples

Digital Signal Processing

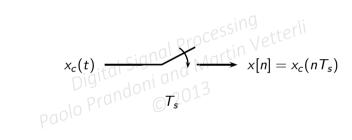
Digital Signal Martin Vetterli

Digital Signal Martin Vetterli

O 2013

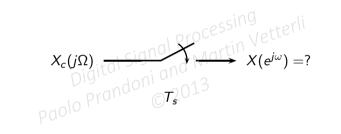
Raw-sampling an arbitrary signal





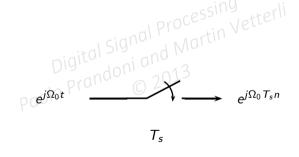
Raw-sampling an arbitrary signal





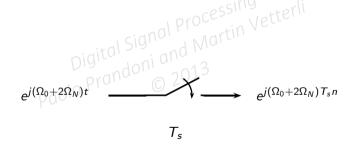


- lacksquare pick T_s (and set $\Omega_N=\pi/T_s$)
- ▶ pick $\Omega_0 < \Omega_N$



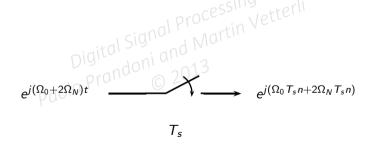


- lacksquare pick T_s (and set $\Omega_N=\pi/T_s$)
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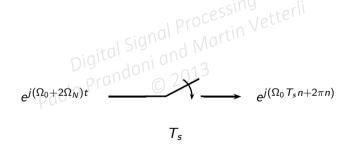


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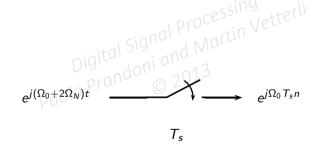


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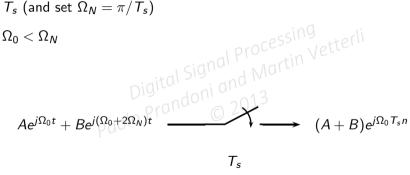


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- pick T_s (and set $\Omega_N = \pi/T_s$)
- ▶ pick $\Omega_0 < \Omega_N$



Spectrum of raw-sampled signals



start with the inverse Fourier Transform

Fourier Transform
$$x[n] = x_c(nT_s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(j\Omega) e^{j\Omega nT_s} d\Omega$$

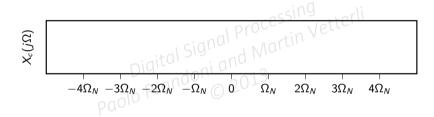
Spectrum of raw-sampled signals



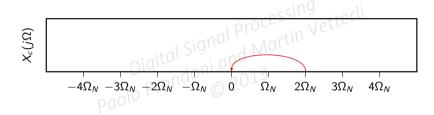
frequencies $2\Omega_{N}$ apart will be aliased, so split the integration interval

$$x[n] = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{(2k+1)\Omega_N}^{(2k+1)\Omega_N} X_c(j\Omega) e^{j\Omega n T_s} d\Omega$$

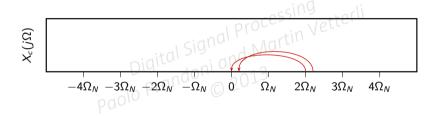




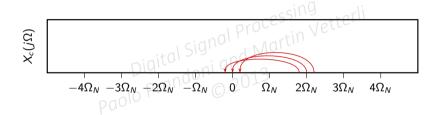




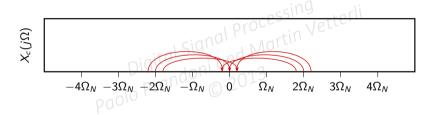




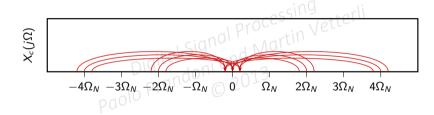














with a change of variable and using $e^{j(\Omega+2k\Omega_N)T_sn}=e^{j\Omega T_sn}$:

$$x[n] = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\Omega_N}^{\Omega_N} X_c(j(\Omega - 2k\Omega_N)) e^{j\Omega nT_s} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_N}^{\Omega_N} \left[\sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2k\Omega_N)) \right] e^{j\Omega nT_s} d\Omega$$



periodization of the spectrum; define:

$$ilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega + 2k\Omega_N))$$
 $ilde{X}_c(j\Omega + 2k\Omega_N)$
 $ilde{X}_c(j\Omega + 2k\Omega_N)$
 $ilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j\Omega) e^{j\Omega n T_s} d\Omega$

so that:

$$x[n] = rac{1}{2\pi} \int_{-\Omega_N}^{\Omega_N} \tilde{X}_c(j\Omega) e^{j\Omega \, n T_s} d\Omega$$



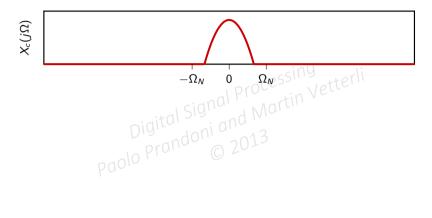
set $\omega = \Omega T_s$:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{T_s} \tilde{X}_c \left(j \frac{\omega}{T_s} \right) e^{j\omega n} d\omega$$

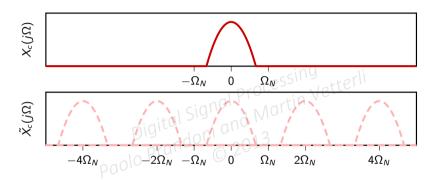
$$= IDTFT \left\{ \frac{1}{T_s} \tilde{X}_c \left(j \frac{\omega}{T_s} \right) \right\}$$

$$X(e^{j\omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c \left(j \frac{\omega}{T_s} - j \frac{2\pi k}{T_s} \right)$$

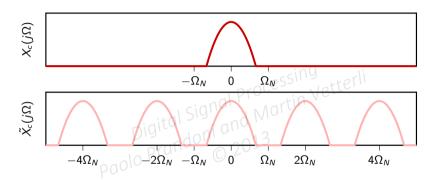




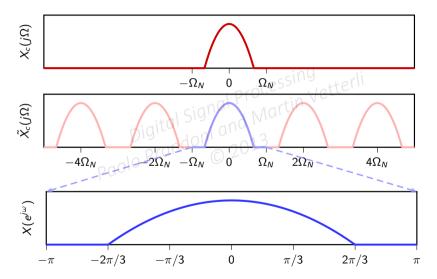




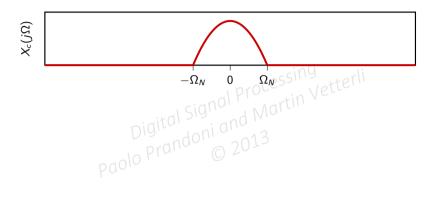




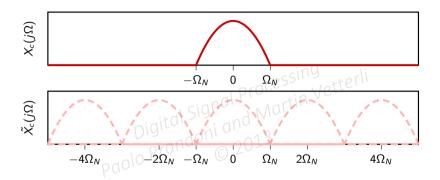




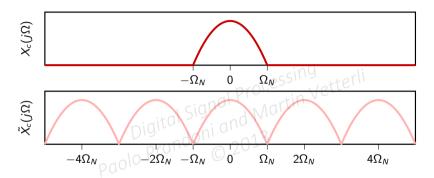




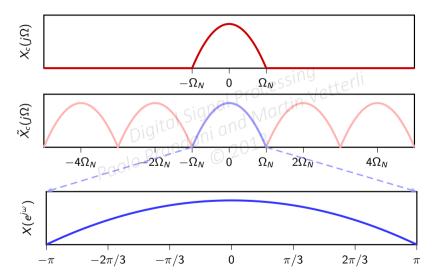




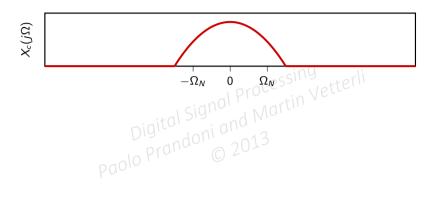




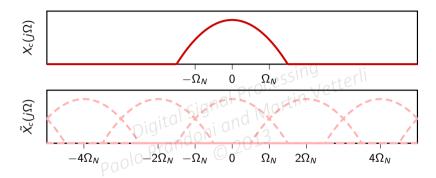




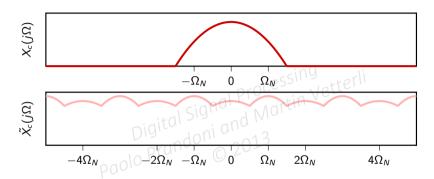




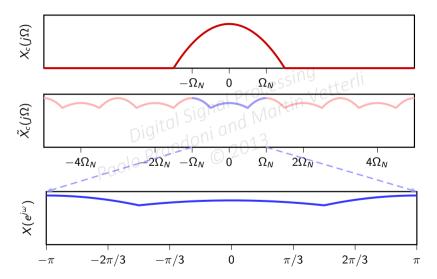




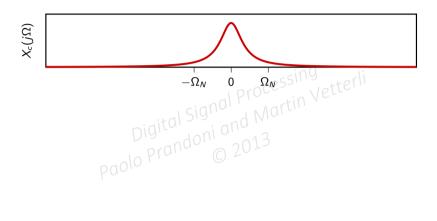




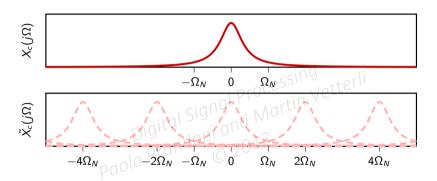




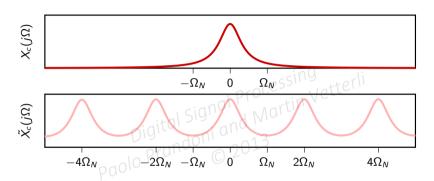




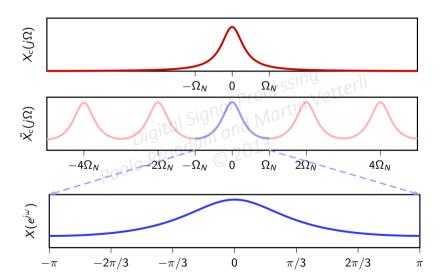














given a sampling period T_s

- \triangleright if the signal is bandlimited to π/T_s or less, raw sampling is fine (i.e. equivalent to sinc if the signal is not bandlimited, two isolices: Martin Vettern

 bandlimit via a lowpass Dight in the production of the sampling)

 or, raw sample the production of the sampling of the samplin



given a sampling period T_s

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 • bandlimit via a lownassiality of the signal is not bandlimit via a lownassiality of the signal is not bandlimit via a lownassiality of the signal is not bandlimit via a lownassiality of the signal is not bandlimit via a lownassiality of the signal is not bandlimit via a lownassiality of the signal is not bandlimited.
- - bandlimit via a lowpass Biller in the point involves time domain before sampling (i.e. sinc sampling)
 - or, raw sample the signal and incur aliasing



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 ▶ bandlimit via a lowness fill
- - bandlimit via a lowpass filter in the continuous-time domain before sampling (i.e. sinc sampling)
 - or, raw sample the signal and incur aliasing
- aliasing sounds horrible, so usually we choose to bandlimit in continuous time

Sinc Sampling and Interpolation

$$\hat{x}[n] = \langle \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right), x(t) \rangle = (\operatorname{sinc}_{T_s} * x)(nT_s)$$

$$\begin{array}{c} \operatorname{pigital Signal Processing} \\ \operatorname{pigital Signal Martin} \\ \operatorname{paolo Prandoni} \\ \operatorname{c} 2013 \end{array}$$

Sinc Sampling and Interpolation



$$\hat{x}[n] = \langle \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right), x(t) \rangle = (\operatorname{sinc}_{T_s} * x)(nT_s)$$

$$\hat{x}(t) = \sum_{n} x[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right) \text{ detain}$$

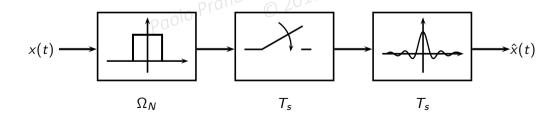
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Sinc Sampling and Interpolation



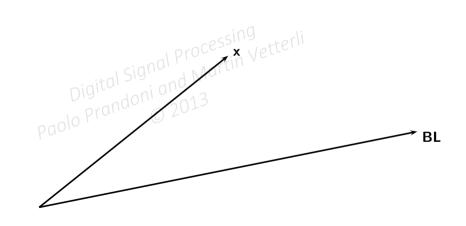
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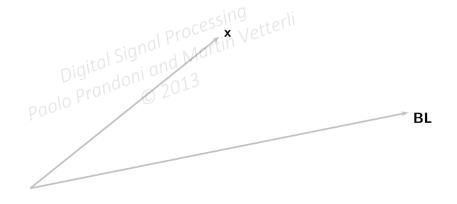
Least squares approximation with sinc sampling and interpolation





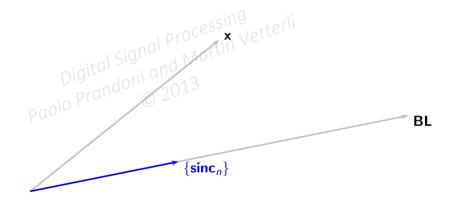
Least squares approximation with sinc sampling and interpolation



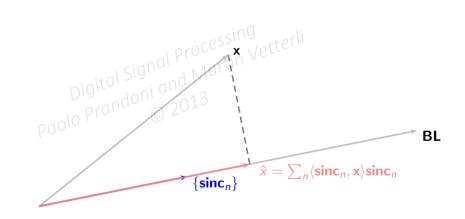


Least squares approximation with sinc sampling and interpolation

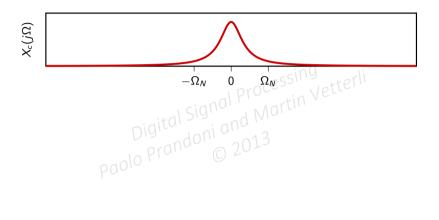




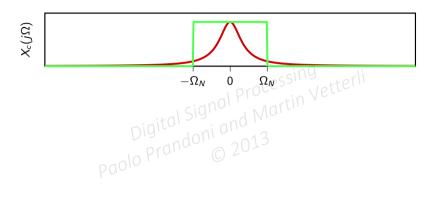




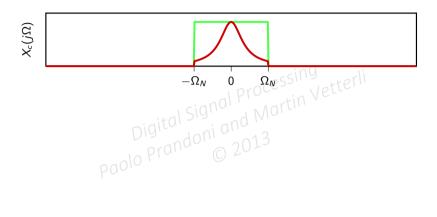




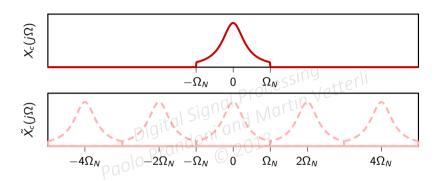




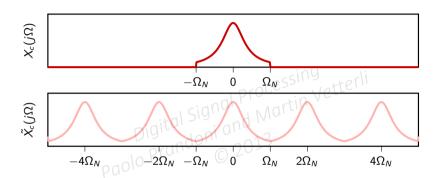




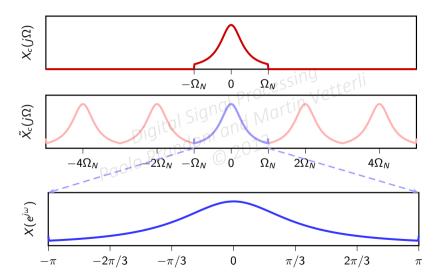




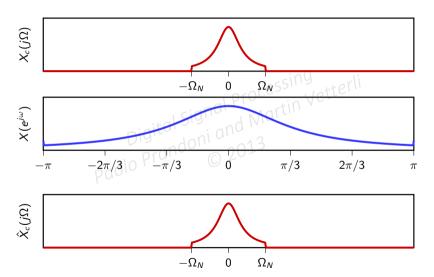












END OF MODULE 6.5

Digital Signa and Martine Prandoni and Martine Prando



Digital Signal Processing

Module 6.6: Discrete-time Processing and Continuous-time Signals

Overview:



- ► Impulse invariance
- Duality
- Examples

Digital Signal Processing

Digital Signal Processing

Nartin Vetterli

and Martin Vetterli

Paolo Prandoni and Martin

2013

Overview:



- ► Impulse invariance
- Duality
- Examples

nce

Digital Signal Processing

Digital Signal Martin Vetterli

and Martin Vetterli

Paolo Prandoni and Martin

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Overview:



- ► Impulse invariance
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- Examples

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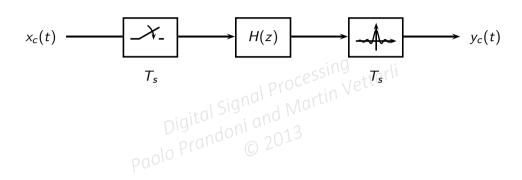
Digital Signal Processing

Digital Signal Martin Vetterli

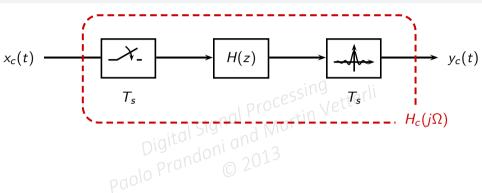
and Martin

Paolo Prandoni and 2013

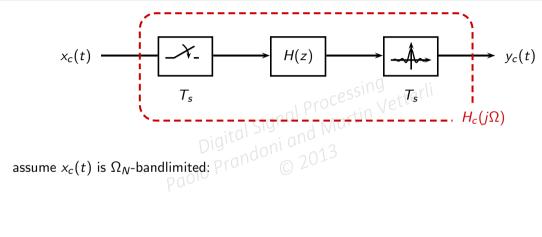




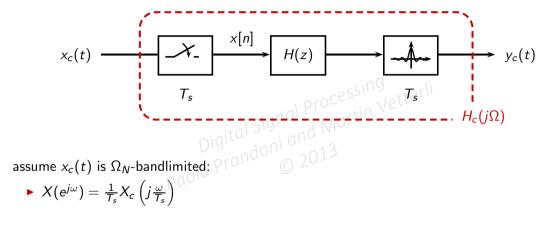






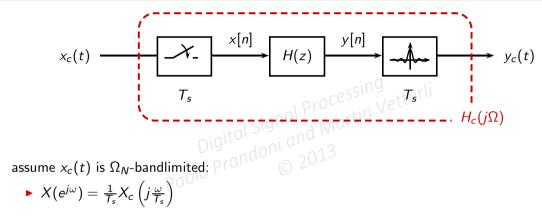






$$X(e^{j\omega}) = \frac{1}{T_s} X_c \left(j \frac{\omega}{T_s} \right)$$

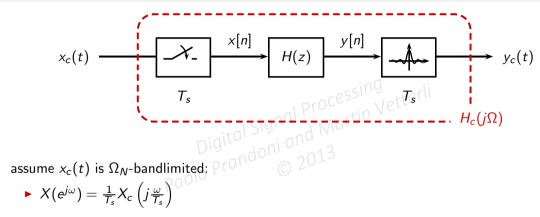




$$X(e^{j\omega}) = \frac{1}{T_s} X_c \left(j \frac{\omega}{T_s} \right)$$

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$





$$X(e^{j\omega}) = \frac{1}{T_s} X_c \left(j \frac{\omega}{T_s} \right)$$

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

$$Y_c(j\Omega) = T_s Y(e^{j\Omega T_s})$$



 $Y_c(j\Omega) = X_c(j\Omega) H(e^{j\pi\Omega/\Omega_N})$ Paolo Prandoni

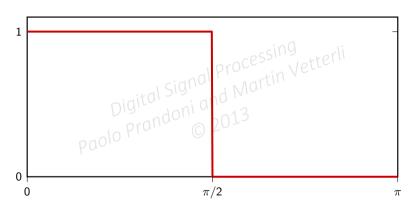
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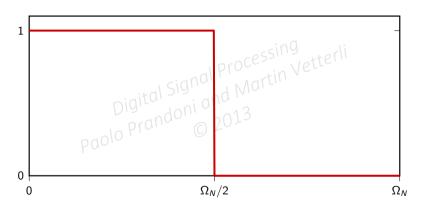
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Digit H_c(j\Omega) = H(e^{j\pi\Omega/\Omega_N})

Paolo Prandoni al (2013
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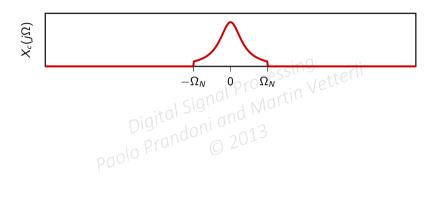




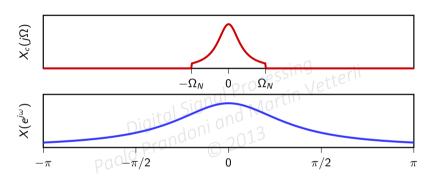




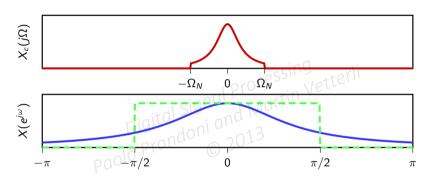




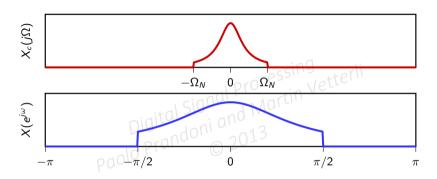




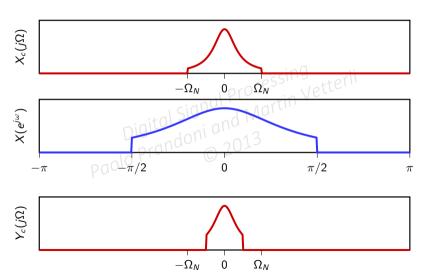














design a discrete-time filter to isolate a band of frequencies between 4000 and 5000Hz; input signals are bandlimited to 7KHz.



- ightharpoonup 7KHz band limit \Rightarrow we can use any sampling frequency above 14KHz



- ightharpoonup 7KHz band limit \Rightarrow we can use any sampling frequency above 14KHz

- we need a bandpass with a lowpass with a lowpass with property bandwidth start with a lowpass with property bandwidth property bandwidth and bandwidth property bandwidth between the start with a lowpass with curoff 500Hz © 2013



- ▶ 7KHz band limit ⇒ we can use any sampling frequency above 14KHz
- ullet pick $F_s=16 extit{KHz}$ so that $\Omega_N=2\pi\cdot 8000$ rad/s
- ▶ we need a bandpass with a 1000Hz bandwidth
- start with a lowpass with cutoff 500Hz © 2013
- ▶ modulate it to center it around 4500Hz

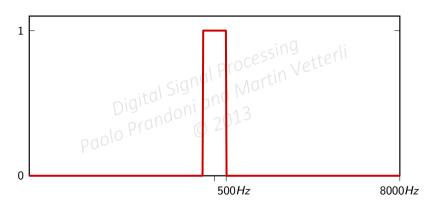


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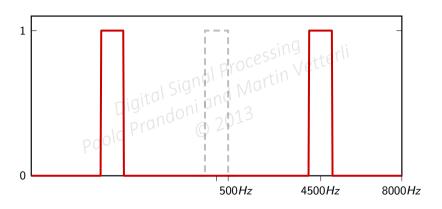


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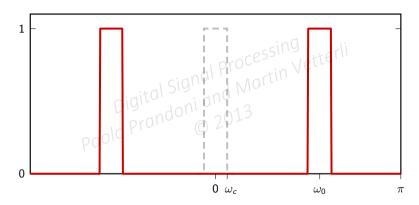






Impulse invariance







$$\omega_c = \pi \frac{\Omega_c}{\Omega_N} = \pi \frac{500}{8000} = 0.0625\pi$$

- $\omega_0 = \pi \frac{4500}{8000} = 0.5625\pi$ $\Rightarrow \text{ design an FIR lowpass with cutoff and Martin Practice of the lasing your favorite method multiply the impulse perponse by <math>2\cos\omega_0 n$



$$\omega_c = \pi \frac{\Omega_c}{\Omega_N} = \pi \frac{500}{8000} = 0.0625\pi$$

$$\omega_0 = \pi \frac{4500}{8000} = 0.5625\pi$$



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$$\omega_c = \pi \frac{\Omega_c}{\Omega_N} = \pi \frac{500}{8000} = 0.0625\pi$$

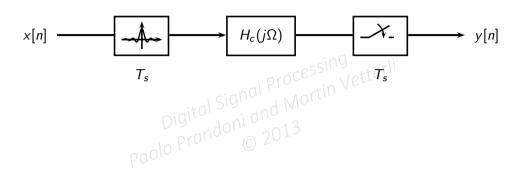
- $\omega_c = \pi \frac{\Omega_c}{\Omega_N} = \pi \frac{500}{8000} = 0.0625\pi$ $\omega_0 = \pi \frac{4500}{8000} = 0.5625\pi$ design an FIR lowpass with cutoff ω_c using your favorite method



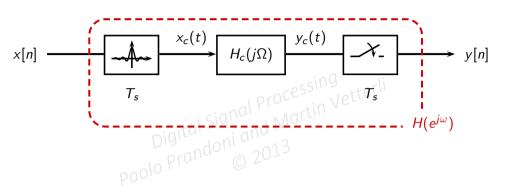
$$\omega_c = \pi \frac{\Omega_c}{\Omega_N} = \pi \frac{500}{8000} = 0.0625\pi$$

- $\omega_c = \pi \frac{\Omega_c}{\Omega_N} = \pi \frac{500}{8000} = 0.0625\pi$ $\omega_0 = \pi \frac{4500}{8000} = 0.5625\pi$ design an FIR lowpass with cutoff ω_c using your favorite method
- multiply the impulse response by $2\cos\omega_0 n$

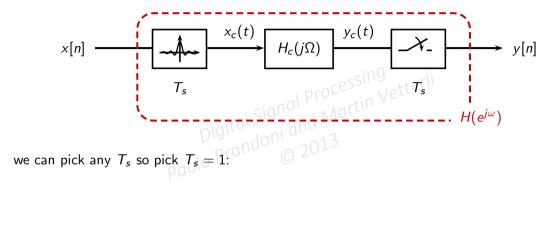




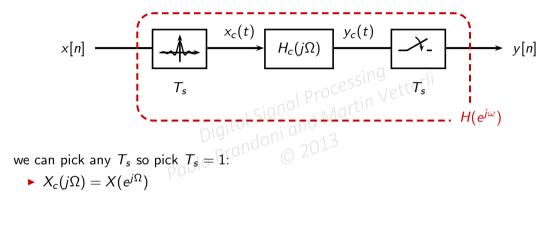






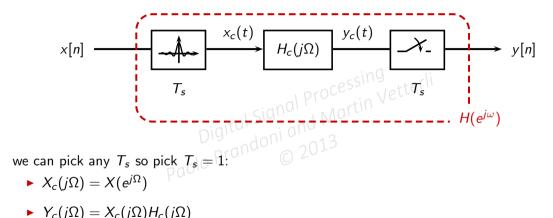






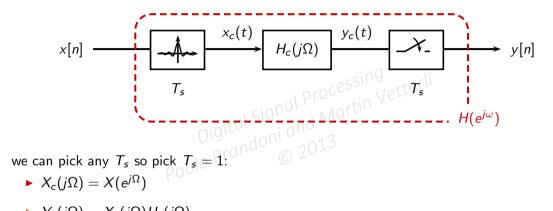
$$X_c(j\Omega) = X(e^{j\Omega})$$





- $Y_c(j\Omega) = X_c(j\Omega)H_c(j\Omega)$





- $Y_c(j\Omega) = X_c(j\Omega)H_c(j\Omega)$
- ▶ LTI systems cannot change the bandwidth $\Rightarrow Y(e^{j\omega}) = Y_c(j\omega)$

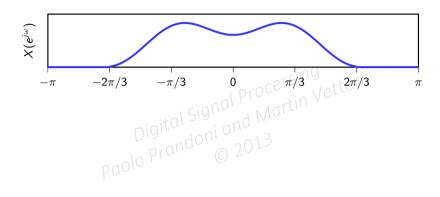


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\begin{array}{c} \text{Digi} Y(e^{j\omega}) = X(e^{j\omega}) H_c(j\omega) \\ \text{Paolo Prandoni} & 2013 \end{array}
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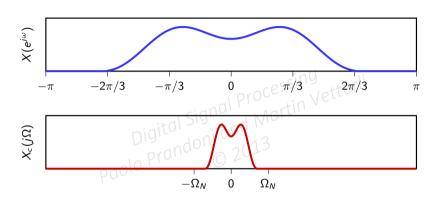


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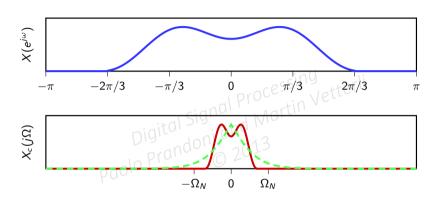




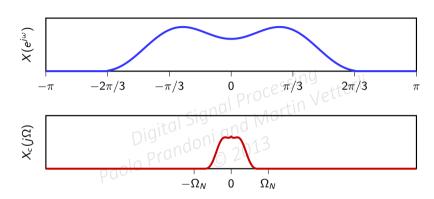




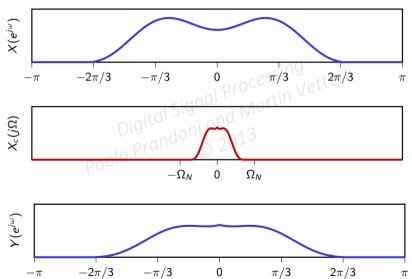












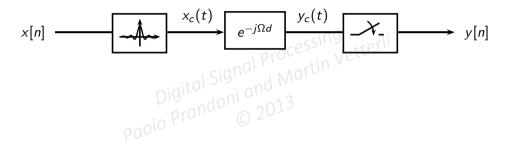


$$H(e^{j\omega}) = e^{-j\omega d}$$
 vetter

- $H(e^{j\omega}) = e^{-j\omega d} \sin \theta$ $\downarrow \text{ if } d \in \mathbb{Z}, \text{ simple delay } \text{ Digital Signal Martin}$ $\downarrow \text{ if } d \notin \mathbb{Z}, h[n] = \text{sinc}(n-d).\text{Prandoni}$

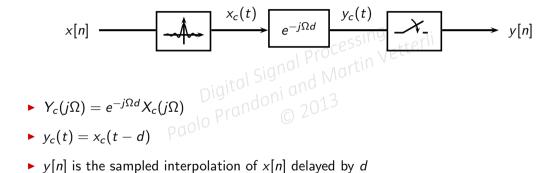
By duality





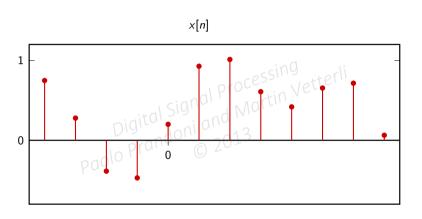
By duality



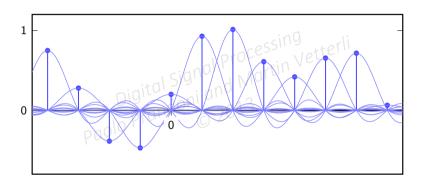


- \triangleright y[n] is the sampled interpolation of x[n] delayed by d

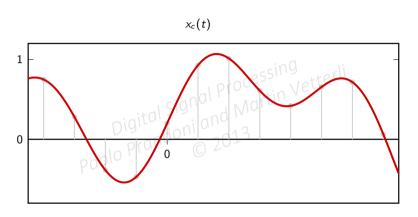




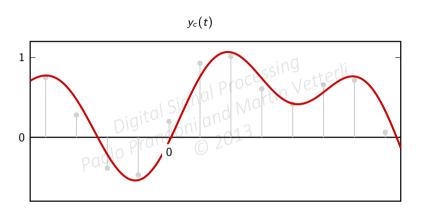




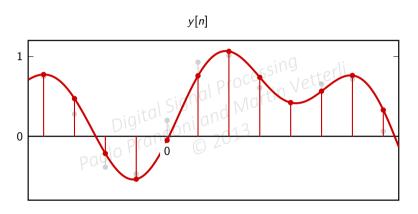




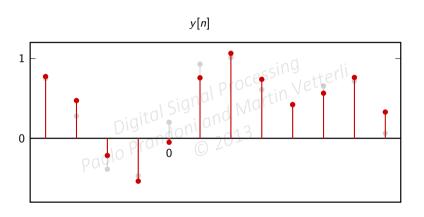














- ▶ to delay a discrete-time signal by a fraction of a sample we need an ideal filter!
- ▶ efficient time-variant approximations exist (see Module 11)

Example: differentiator



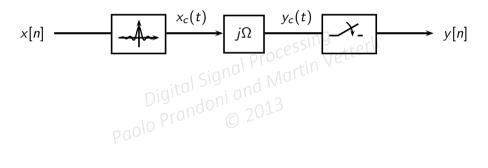
$$H(e^{j\omega})=j\omega_{\rm essing}$$

$$Vetter is signal Processing Vetter in Vetter is signal Processing Vetter is signal Processing Vetter in Vetter in Vetter is signal Processing Vetter in Vetter in Vetter is signal Processing Vetter in Ve$$

- in discrete time...

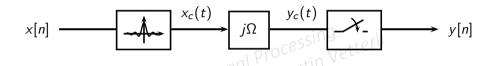
By duality





By duality



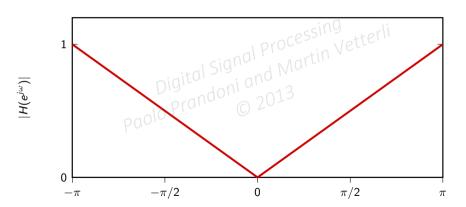


- $Y_c(j\Omega) = j\Omega X_c(j\Omega)$
- $y_c(t) = x'_c(t)$
- y[n] is the sampled interpolation of x[n], differentiated

Digital differentiator, magnitude response



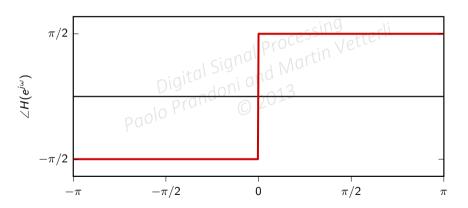




Digital differentiator, phase response



$$H(e^{j\omega})=j\omega$$



Digital differentiator, impulse response

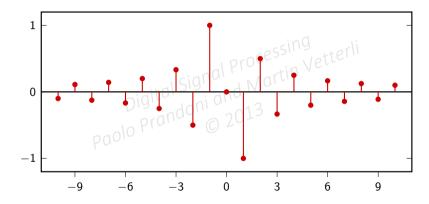


$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} j\omega e^{j\omega n} d\omega$$

$$= \begin{cases} 0 & n = 0 \\ \frac{(-1)^n}{n} & n \neq 0 \end{cases}$$

Digital differentiator, impulse response





Digital differentiator



- the digital differentiator is again an ideal filter!
 many approximations exist with a second Martin Paolo Prando © 2013

Wrap up



- ► Continuous-time processing of discrete-time sequences
- ▶ Discrete-time processing of continuous-time signals
- Jumping back and forth using sampling and interpolation
- ▶ In practice: Many applications of processing continuous-time signals in discrete time!

END OF MODULE 6.6

Digital Signa Martine 6.6

Paolo Prandoni and Martine 2013

END OF MODULE 6 Digital Sign and Martin People Prandoni and Martin People Prandoni and Martin People Peopl