

Digital Signal Processing

Digital Signal Processing

Module 3: from an and paolo Prandoni and Company 2013

Module Overview:



- ▶ Module 3.1: Signal processing as geometry or from Euclid to Hilbert spaces
- ▶ Module 3.2: Vectors, vector spaces, inner products, and Hilbert spaces
- ► Module 3.3: Bases for Hilbert spaces

3



Digital Signal Processing

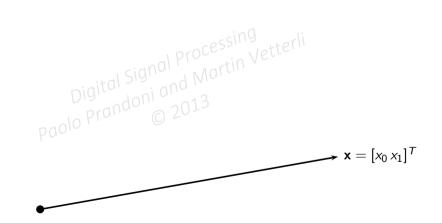
Digital Signal Processing

Module 3.1: a tale of two (and more) vectors

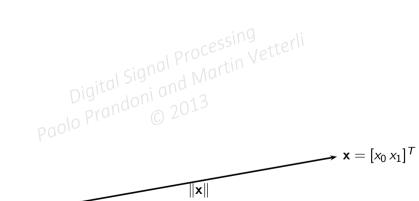


Digital Signal Processin Vetterni Paolo Prandoni and Martin Vetterni © 2013

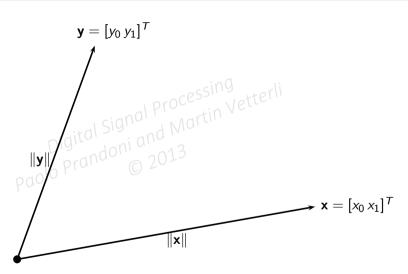




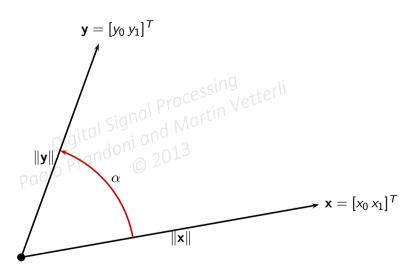




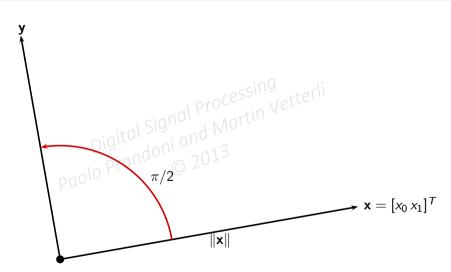








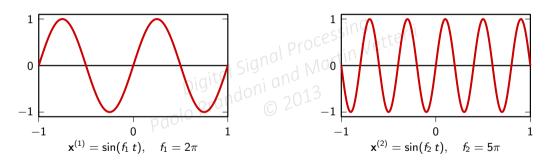




Vectors can be very general objects!



Example: space of square-integrable functions over [-1,1]: $L_2([-1,1])$

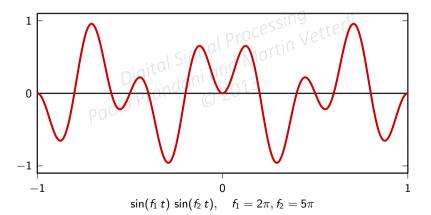


$$\langle \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \rangle = \int_{-1}^{1} \sin(f_1 t) \sin(f_2 t) dt$$

Orthogonality in a functional vector space.



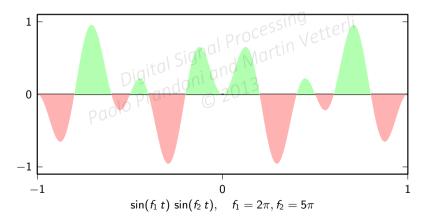
 $\mathbf{x}^{(1)} \perp \mathbf{x}^{(2)}$ if $f_1 \neq f_2$ and f_1, f_2 integer multiples of a fundamental (harmonically related)



Orthogonality in a functional vector space.



 $\mathbf{x}^{(1)} \perp \mathbf{x}^{(2)}$ if $f_1 \neq f_2$ and f_1, f_2 integer multiples of a fundamental (harmonically related)



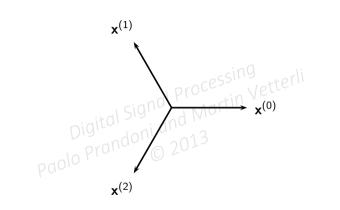
Vectors spanning a space





Too many vectors for the space



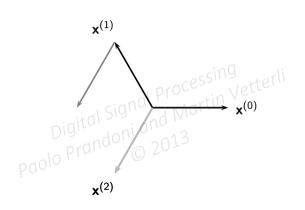


Linear dependence:

$$\exists \{a_0, a_1, a_2\} \text{ s.t. } a_0 \mathbf{x}^{(0)} + a_1 \mathbf{x}^{(1)} + a_2 \mathbf{x}^{(2)} = 0$$

Too many vectors for the space





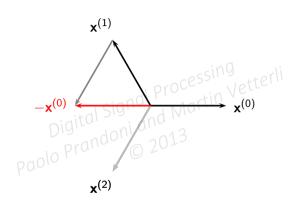
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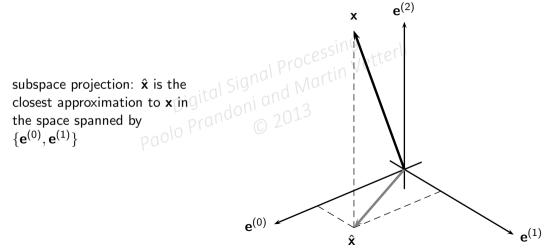


Linear dependence:

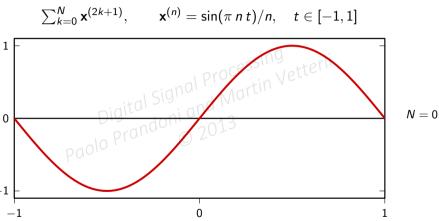
$$\exists \{a_0, a_1, a_2\} \text{ s.t. } a_0 \mathbf{x}^{(0)} + a_1 \mathbf{x}^{(1)} + a_2 \mathbf{x}^{(2)} = 0$$

Not enough vectors for the space

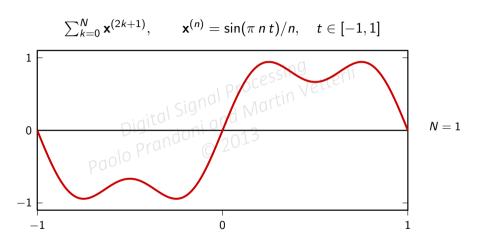




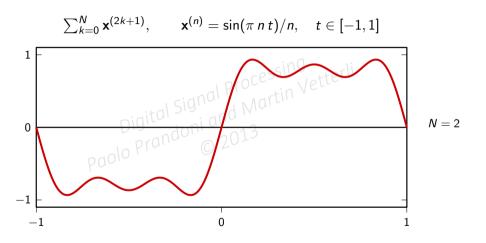




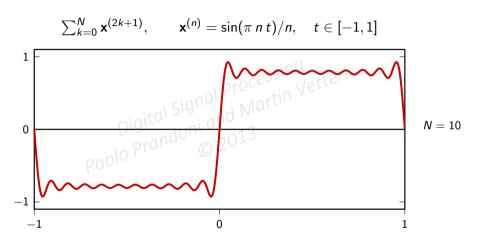




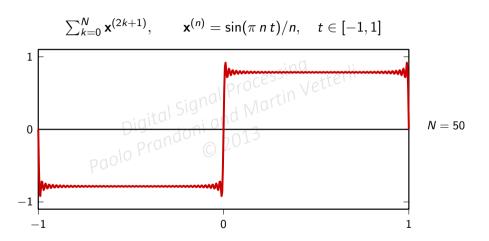




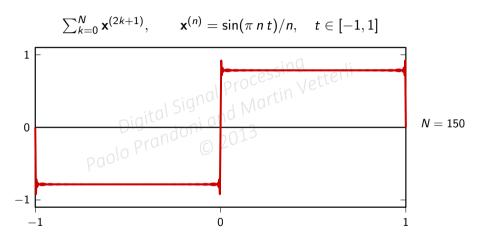












END OF MODULE 3.1

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Module 3.2: Hilbert Space, properties and bases

Overview:



- Definition of Hilbert space
- Examples

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Overview:



- ► Definition of Hilbert space
- Examples

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Hilbert Space – the ingredients:



2. an inner product: $\langle \cdot, \cdot \rangle$: V_{Aij} $V_{\text{alsignal Processing Digital Signal Processing Vetterli}$ 3. completeness

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Q 2013

Hilbert Space – the ingredients:



Hilbert Space – the ingredients:



2. an inner product: $\langle \cdot, \cdot \rangle$: $V \times V \to \mathbb{C}$ and Martin Vetterli

3. completeness

Paolo Prandoni and Martin \mathbb{C}

1) Vector space



- resize vectors: scalar multiplications ignal processing Vetterli

 combine vectors together: Pigital and Martin C 2013

 paolo Proposition C 2013

1) Vector space

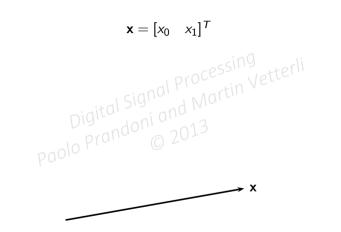


- resize vectors: scalar multiplications ignal processing

 combine vectors together: addition on and page 2013

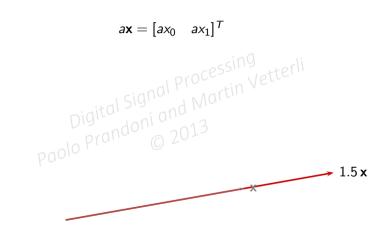
Scalar multiplication in \mathbb{R}^2





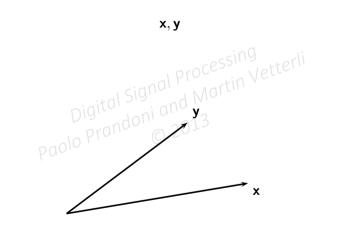
Scalar multiplication in \mathbb{R}^2





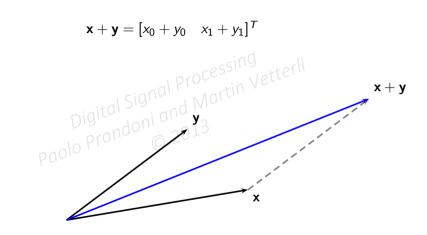
Addition in \mathbb{R}^2





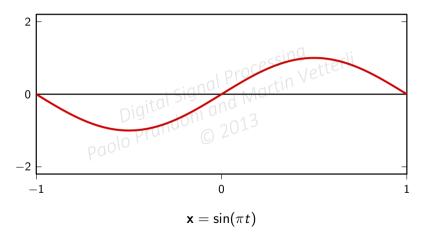
Addition in \mathbb{R}^2





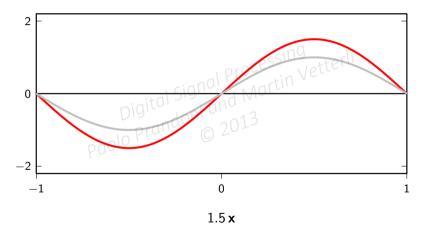
Scalar multiplication in $L_2[-1, 1]$





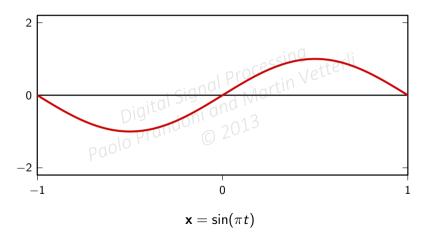
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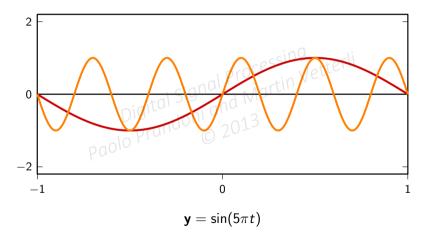
Addition in $L_2[-1,1]$





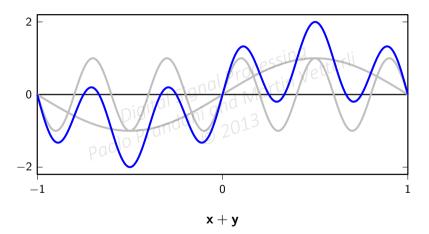
Addition in $L_2[-1,1]$





Addition in $L_2[-1,1]$







- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

- $\mathbf{y}_{0} = \alpha \mathbf{y} + \alpha \mathbf{x}$ $\mathbf{x}_{0} = \alpha \mathbf{y} + \alpha \mathbf{x}$ $\mathbf{x}_{0} = \alpha \mathbf{x} + \beta \mathbf{x}$

 - $\Rightarrow \exists 0 \in V \mid x + 0 = 0 + x = x$
 - $\forall x \in V \exists (-x) \mid x + (-x) = 0$



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$$\mathbf{r}$$
 $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

$$(x + y) + z = x + (y + z)$$

$$\alpha(x + y) = \alpha y + \alpha x$$

$$(\alpha + \beta)x = \alpha x + \beta x$$

$$\text{Digital Signal Processing Martin Ve}$$

$$\alpha(x + y) = \alpha x + \beta x$$

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$$(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$$

•
$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$$

• $\alpha(\mathbf{x} + \mathbf{y}) = \alpha \mathbf{y} + \alpha \mathbf{x}$
• $(\alpha + \beta)\mathbf{x} = \alpha \mathbf{x} + \beta \mathbf{x}$ Digital Signal Processing Vetterli
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• $\alpha(\beta \mathbf{x}) = (\alpha \beta)\mathbf{x}$ Paolo Prandoni and Martin Vetterli
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▶
$$\exists 0 \in V \mid \mathbf{x} + 0 = 0 + \mathbf{x} = \mathbf{x}$$

$$\qquad \forall \mathbf{x} \in V \ \exists (-\mathbf{x}) \quad | \quad \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$$



- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

- Digital Signal Processing $p_{\mathbf{j}}\mathbf{x} = \alpha \mathbf{x} + \beta \mathbf{x}$ Digital Signal Processiny Martin Vetterli $\alpha(\beta \mathbf{x}) = (\alpha \beta) \mathbf{x}$ Paolo Prandoni $0 \in V$

 - $\forall x \in V \exists (-x) \mid x + (-x) = 0$



$$\mathbf{r}$$
 $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

$$(x + y) + z = x + (y + z)$$

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• $\exists 0 \in V \mid \mathbf{x} + 0 = 0 + \mathbf{x} = \mathbf{x}$



- \triangleright x + y = y + x

- $\rho_{j} \mathbf{x} = \alpha \mathbf{x} + \beta \mathbf{x} \qquad \text{Digital Signal Processiny}$ $\mathbf{\alpha}(\beta \mathbf{x}) = (\alpha \beta) \mathbf{x} \qquad \text{Paolo Prandoni and } 0$ $\exists 0 \in V \quad | \quad \mathbf{x} \in V \quad | \quad \mathbf{x} \in V$

Vector subspace:



- addition and scaling in subspace remain and subspace Paolo Prand C 2013

Vector subspace:



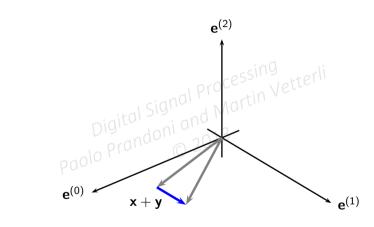
- → addition and scaling in subspace remain in subspace

 Paolo Prana

 2015

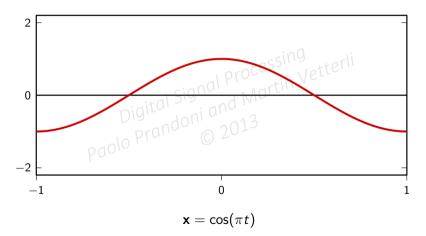
Addition in subspace:





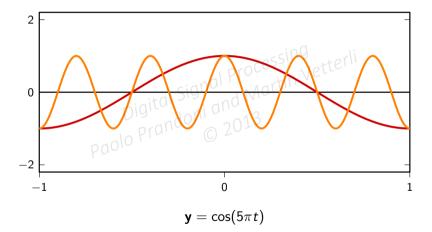
Subspace of symmetric functions over $L_2[-1,1]$





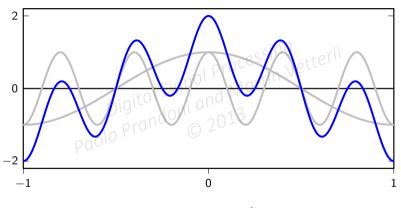
Subspace of symmetric functions over $L_2[-1,1]$





Subspace of symmetric functions over $L_2[-1,1]$





 $\mathbf{x}+\mathbf{y}$, symmetric

2) Inner product



- ► measure of similarity between vectors and Processing Vetterli

 ► when inner product is zero vectors are most different: orthogonal vectors

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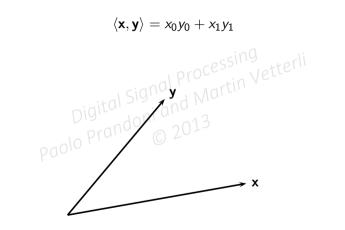
2) Inner product



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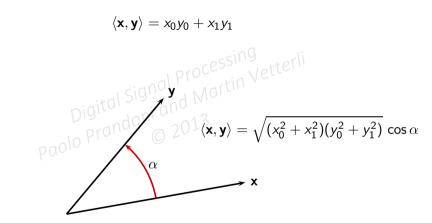
Inner product in \mathbb{R}^2





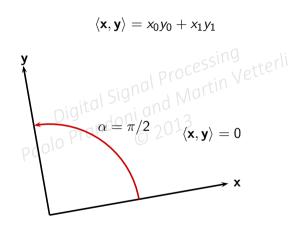
Inner product in \mathbb{R}^2





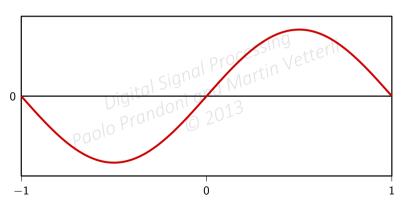
Inner product in \mathbb{R}^2







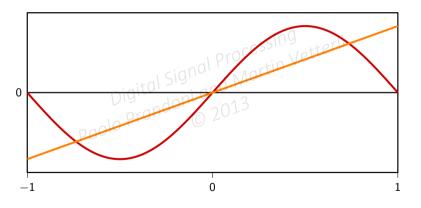
$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^{1} x(t) y(t) dt$$



$$\mathbf{x} = \sin(\pi t)$$



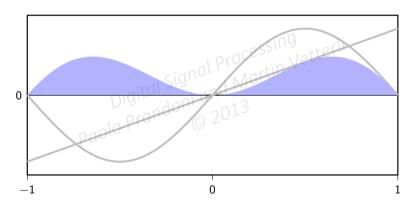
$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^{1} x(t) y(t) dt$$



$$\mathbf{y}=t$$

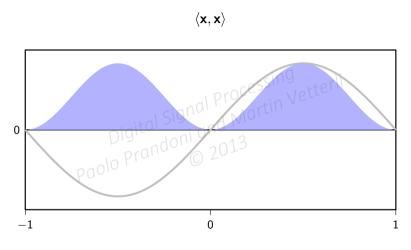


$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^{1} t \sin(\pi t) dt$$



$$\langle \mathbf{x}, \mathbf{y} \rangle = 2/\pi \approx 0.6367$$

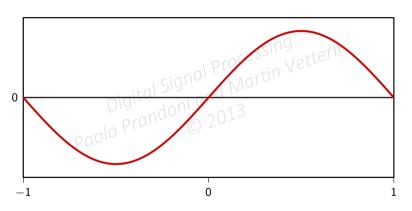




$$\mathbf{x} = \sin(\pi t), \ \langle \mathbf{x}, \mathbf{x} \rangle = 1$$



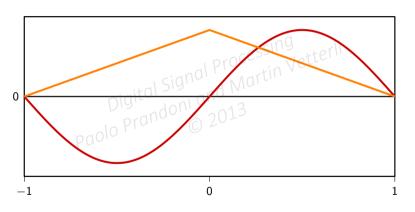
x, y from orthogonal subspaces



 $\mathbf{x} = \sin(\pi t)$, antisymmetric



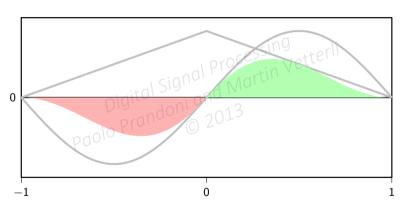
x, y from orthogonal subspaces



$$\mathbf{y} = 1 - |t|$$
, symmetric



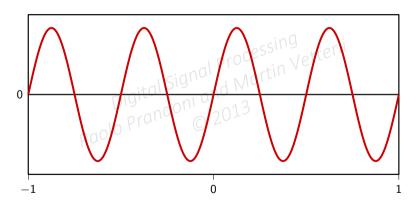
x, y from orthogonal subspaces



$$\langle \boldsymbol{x},\boldsymbol{y}\rangle=0$$



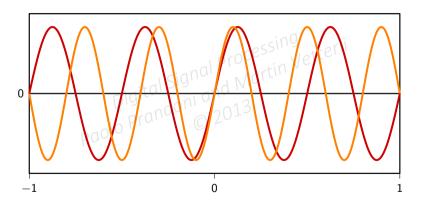
sinusoids with frequencies integer multiples of a fundamental



$$\mathbf{x} = \sin(4\pi t)$$



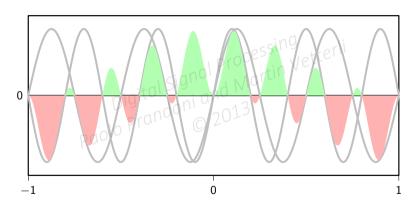
sinusoids with frequencies integer multiples of a fundamental



$$\mathbf{x} = \sin(4\pi t)$$
, $\mathbf{y} = \sin(5\pi t)$



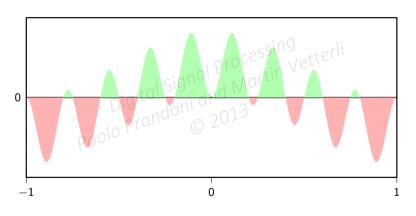
sinusoids with frequencies integer multiples of a fundamental



$$\mathbf{x} = \sin(4\pi t)$$
, $\mathbf{y} = \sin(5\pi t)$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$



sinusoids with frequencies integer multiples of a fundamental



$$\langle \boldsymbol{x},\boldsymbol{y}\rangle=0$$

Formal properties of the inner product



- \land $\langle x, y \rangle = \langle y, x \rangle^*$
- Digital Signal Processing

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 Paolo Prandoni and Martin Vetterli $\langle \mathbf{x}, \alpha \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
- $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$
- ightharpoonup if $\langle \mathbf{x}, \mathbf{v} \rangle = 0$ and $\mathbf{x}, \mathbf{v} \neq \mathbf{0}$ then \mathbf{x} and \mathbf{v} are called orthogonal

Formal properties of the inner product



- Digital Signal Processing

 Digital Signal Processing

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For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha \in \mathbb{C}$:

- ., Digital Signal Processing

 Digital Signal Processing

 Martin Vetterli

 Paolo Prandoni and Martin
- $ightharpoonup \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle^*$
- $ightharpoonup \langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha^* \langle \mathbf{x}, \mathbf{y} \rangle$ $\langle \mathbf{x}, \alpha \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
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 Digital Signal Processing

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- ., Digital Signal Processing

 Digital Signal Processing

 Martin Vetterli

 Paolo Prandoni and Martin
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- Digital Signal Processing

 Digital Signal Martin Vetterli

 Paolo Prandoni and
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Inner product for signals



$$Digit\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=0}^{N-1} x^*[n]y[n]$$

well defined for all finite-length vectors (i.e. vectors in \mathbb{C}^N)

Inner product for signals



$$\begin{array}{c} \text{Digi}\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=-\infty}^{\infty} dx^*[n]y[n] \\ \text{Prodo} \\ \text{careful: sum may explode!} \end{array}$$

Inner product for signals



$$\langle \mathbf{x},\mathbf{y}\rangle = \sum_{n=+\infty}^{\infty} x^*[n]y[n]$$
 We require sequences to be square-summable: $\sum |x[n]|^2 < \infty$

Space of square-summable sequences: $\ell_2(\mathbb{Z})$

Norm



- ► inner product defines a norm: $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ occessing vetterli

 ► norm defines a distance: $d(\mathbf{x}, \mathbf{x}) + \sin \alpha$ which was a distance of α and α and α and α and α and α and α are α and α and α are α and α and α are α are α and α are α are α are α and α are α and α are α and α are α and α are α are α are α and α are α are α are α and α are α and α are α and α are α

Norm



- ► inner product defines a norm: $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ occessing vetterli

 ► norm defines a distance: $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} \mathbf{y}\| d$ Martin

Norm and distance in \mathbb{R}^2



$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_0^2 + x_1^2}$$

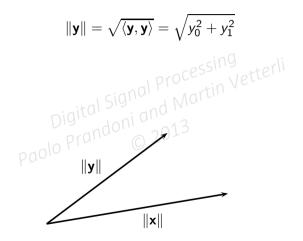
$$\text{Digital Signal Processing Wattin Vetterli}$$

$$\text{Paolo Prandoni and Martin}$$

$$\text{Digital Signal Processing Wattin}$$

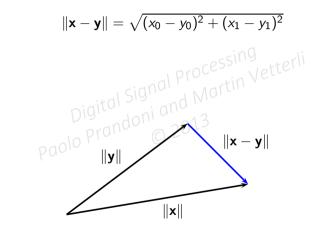
Norm and distance in \mathbb{R}^2





Norm and distance in \mathbb{R}^2

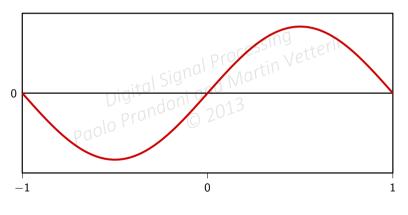




Norm and distance in $L_2[-1,1]$



$$\|\mathbf{x} - \mathbf{y}\|^2 = \int_{-1}^{1} |x(t) - y(t)|^2 dt$$
 (MSE)

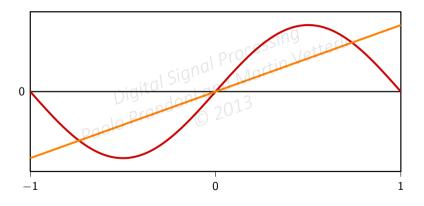


$$\mathbf{x} = \sin(\pi t)$$

Norm and distance in $L_2[-1, 1]$



$$\|\mathbf{x} - \mathbf{y}\|^2 = \int_{-1}^{1} |x(t) - y(t)|^2 dt$$
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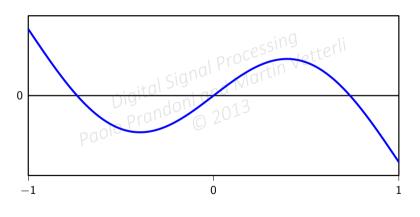


$$\mathbf{y} = t$$

Norm and distance in $L_2[-1,1]$



$$\|\mathbf{x} - \mathbf{y}\|^2 = \int_{-1}^{1} |x(t) - y(t)|^2 dt$$
 (MSE)

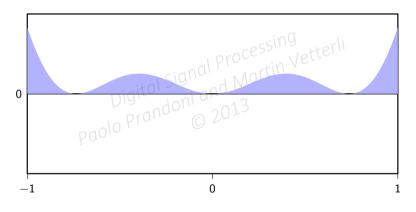


$$x - y$$

Norm and distance in $L_2[-1,1]$



$$\|\mathbf{x} - \mathbf{y}\|^2 = \int_{-1}^{1} |x(t) - y(t)|^2 dt$$
 (MSE)



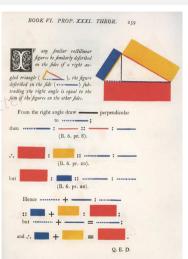
$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{5/3 - 4/\pi} \approx 0.6272$$

A familiar result



Pythagorean theorem:

ean theorem:
$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \text{ for } \mathbf{x} \perp \mathbf{y}$$



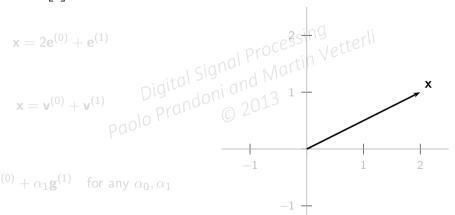
From Euclid's elements by Oliver Byrne (1810 - 1880)



$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$

$$\mathbf{x} = 2\mathbf{e}^{(0)} + \mathbf{e}^{(1)}$$

 $\mathbf{x} \neq \alpha_0 \mathbf{g}^{(0)} + \alpha_1 \mathbf{g}^{(1)}$ for any α_0, α_1





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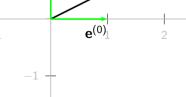
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$$\mathbf{paolo Prandoni} \text{ and Martin Vetterli}$$

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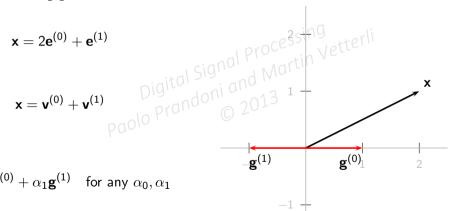


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- vector space H

▶ set of K vectors from H: $W = \{\mathbf{w}^{(k)}\}_{k=0,1,...,K-1}$ sing V is a basis for V if:

▶ we can write for all $\mathbf{x} \in H$. Pigital Signal Propagation Vetterlians V is a basis for V if:

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 \triangleright the coefficients α_k are unique



- vector space H

W is a basis for H if:

set of
$$K$$
 vectors from H : $W = \{\mathbf{w}^{(k)}\}_{k=0,1,\dots,K-1}$ sing \mathbb{V} is a basis for H if:

• we can write for all $\mathbf{x} \in H$:

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}, \quad \alpha_k \in \mathbb{C}$$

• the coefficients α_k are unique



Unique representation implies linear independence:

Unique representation implies linear independence:
$$\sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} = 0 \quad \Leftrightarrow \quad \alpha_k = 0, \ k = 0, 1, \dots, K-1$$

Special bases



Orthogonal basis:

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = 0$$
 for $k \neq n$

Digital Signal Process

and Martin Vetterli

paolo Pranction or pathonomy basis:

 $\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = \delta[n-k]$

We can always orthonormalize a basis via the Gram-Schmidt algorithm

Special bases



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We can always orthonormalize a basis via the Gram-Schmidt algorithm.

Basis expansion



$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} \sin \theta$$

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Basis expansion



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Basis expansion



$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}$$
 recessing

how do we find the lpha's ?

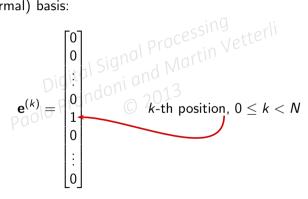
Orthonormal bases are the best:

$$\alpha_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle$$

Example: bases for \mathbb{C}^N



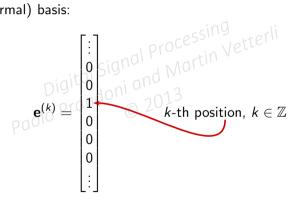
- ► a basis will contain *N* vectors
- canonical (orthonormal) basis:



Example: bases for sequences in $\ell_2(\mathbb{Z})$



- ▶ a basis will contain infinite vectors
- canonical (orthonormal) basis:



Completeness



limiting operations must yield vector space elements

Example of an *incomplete* space: the set of rational numbers
$$x_n = \sum_{k=0}^n \frac{1}{k!} \in \text{ gign But} \text{ Matthin} x_n = e \not\in \mathbb{Q}$$

$$\text{Paolo Prandoni and } 0$$

Completeness



limiting operations must yield vector space elements

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Completeness



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$$x_n = \sum_{k=0}^n \frac{1}{k!} \in \mathbb{Q} \text{ gn but } \min_{n \to \infty} x_n = e \not\in \mathbb{Q}$$



END OF MODULE 3.2

Digital Sign and Martin Et 3.2

Paolo Prandoni and Martin Et 3.2



Digital Signal Processing

Digital Signal Processing

Module 3.3: Hilbert Space and approximation

Overview:



- Approximation by projection gital Signal Martin Vetterli

 Examples

 Paolo Prandoni and Martin Vetterli

 O 2013

Overview:



- Approximation by projection and Martin Vetterli

 Examples

 Parseval

 Processing
 Vetterli

 One of the property of the processing Vetterli

 Parseval

 Processing
 Vetterli

 One of the processing

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Overview:



- Norm conservation, Parseval
- Jection Digital Signal Processing Vetterli Digital Signal Martin Vetterli Paolo Prandoni and Martin Paolo Prandoni © 2013 Approximation by projection
- Examples

Parseval's Theorem



$$\mathbf{x} = \sum_{k=0}^{K-1} |\alpha_k \mathbf{w}(k) \leq \sin \theta$$

$$\text{Digital Signal Martin Vetterli}$$

$$\text{Digital Martin Vette$$

Parseval's Theorem

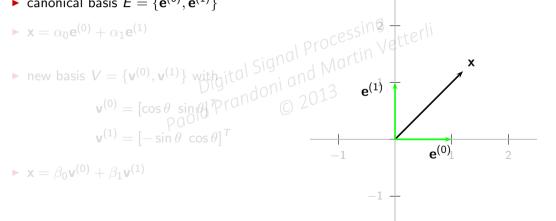


$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k) \text{SSING}} \text{Vettern}$$
 For an orthonormal basis:
$$\|\mathbf{x}\|^2 = \sum_{k=0}^{K-1} |\alpha_k|^2$$



- ► canonical basis $E = \{\mathbf{e}^{(0)}, \mathbf{e}^{(1)}\}$

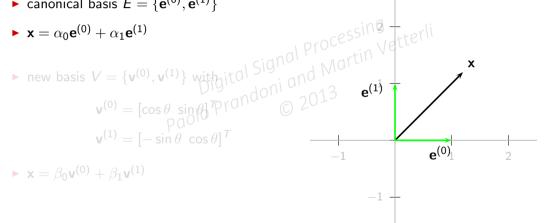
- $\mathbf{x} = \beta_0 \mathbf{v}^{(0)} + \beta_1 \mathbf{v}^{(1)}$





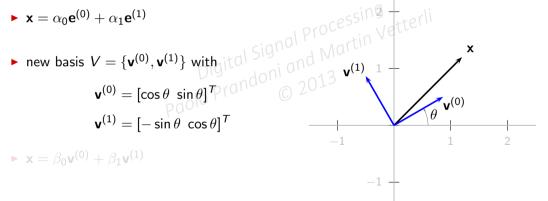
- ightharpoonup canonical basis $E = \{ \mathbf{e}^{(0)}, \mathbf{e}^{(1)} \}$

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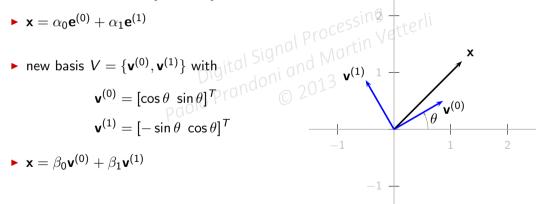


- ► canonical basis $E = {\mathbf{e}^{(0)}, \mathbf{e}^{(1)}}$
- $\mathbf{x} = \alpha_0 \mathbf{e}^{(0)} + \alpha_1 \mathbf{e}^{(1)}$

$$\mathbf{v}^{(0)} = [\cos\theta \ \sin\theta]^T$$

$$\mathbf{v}^{(1)} = [-\sin\theta \cos\theta]^T$$

 $\mathbf{x} = \beta_0 \mathbf{v}^{(0)} + \beta_1 \mathbf{v}^{(1)}$





new basis is orthonormal:

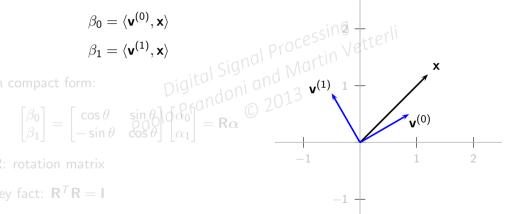
$$\beta_0 = \langle \mathbf{v}^{(0)}, \mathbf{x} \rangle$$

$$\beta_1 = \langle \mathbf{v}^{(1)}, \mathbf{x} \rangle$$
I Signal Processing - Vetterli

▶ in compact form:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \mathbf{R} \alpha$$

- ▶ R: rotation matrix
- \triangleright key fact: $\mathbf{R}^T \mathbf{R} = \mathbf{I}$





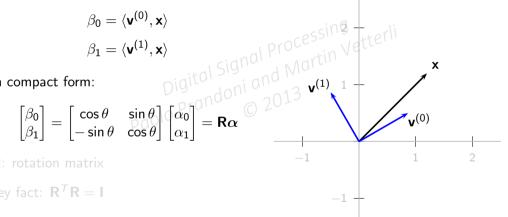
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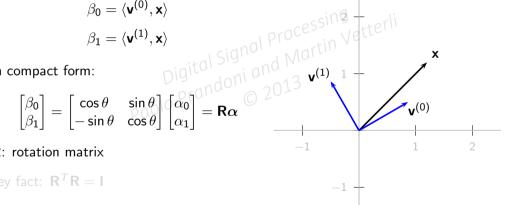
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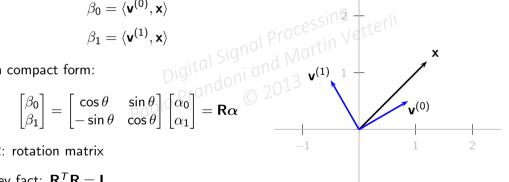
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- square norm in canonical basis: $\|\mathbf{x}\|^2 = \alpha_0^2 + \alpha_1^2$
- square norm in rotated basis: $\|\mathbf{x}\|^2 = \beta_0^2 + \beta_1^2$
- ▶ let's verify Parseval:



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Digital Processing
$$\begin{array}{c} \text{Digital Processing} \\ \text{Digital Processing} \\ \text{Paolo Prandoni} & \text{and Martin Vetterli} \\ \text{Paolo Prandoni} & \text{and Prandoni} \\ \text{Paolo Prandoni} \\ \text{Paolo Prandoni} & \text{Paolo Prandoni} \\ \text{Pao$$



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$$\beta_0^2 + \beta_1^2 = \beta^T \beta r \sin \theta$$

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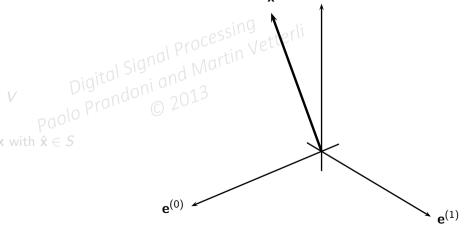
Approximation



Problem:

- ightharpoonup vector $\mathbf{x} \in V$
- ▶ subspace $S \subseteq V$





 $e^{(2)}$

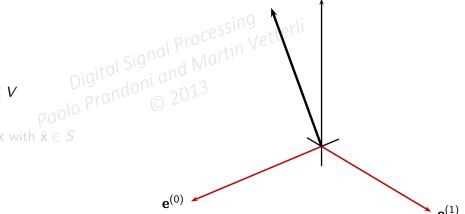
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- ightharpoonup vector $\mathbf{x} \in V$
- ▶ subspace $S \subseteq V$

ightharpoonup approximate $m {f x}$ with $m {f \hat x} \in S$



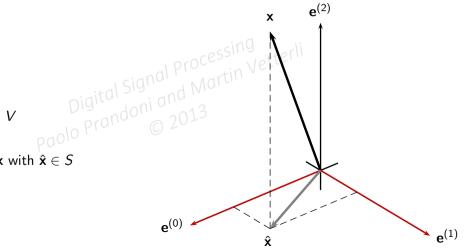
 $e^{(2)}$

Approximation



Problem:

- ightharpoonup vector $\mathbf{x} \in V$
- ▶ subspace $S \subseteq V$
- lacktriangle approximate f x with $\hat{f x}\in S$





- $\{\mathbf{s}^{(k)}\}_{k=0,1,\dots,K-1}$ orthonormal basis for S



- $\{\mathbf{s}^{(k)}\}_{k=0,1,\dots,K-1}$ orthonormal basis for S
- orthogonal projection:

ojection:

Digital
$$\hat{\mathbf{x}} = \sum_{k=0}^{K-1} \langle \mathbf{s}^{(k)}, \mathbf{x} \rangle \mathbf{s}^{(k)}$$

Page 1



- $\{\mathbf{s}^{(k)}\}_{k=0,1,\dots,K-1}$ orthonormal basis for S
- orthogonal projection:

ojection:

Digital
$$\hat{\mathbf{x}} = \sum_{k=0}^{K-1} \langle \mathbf{s}^{(k)}, \mathbf{x} \rangle \mathbf{s}^{(k)}$$

Paolo Prandon $\hat{\mathbf{x}} = \sum_{k=0}^{K-1} \langle \mathbf{s}^{(k)}, \mathbf{x} \rangle \mathbf{s}^{(k)}$

orthogonal projection is the "best" approximation over S



▶ orthogonal projection has minimum-norm error:
$$\arg\min_{\mathbf{y} \in \mathbf{S}} \|\mathbf{x} - \mathbf{y}\|_{\mathbf{S}} = \hat{\mathbf{x}}^{g}$$

$$\gcd_{\mathbf{y} \in \mathbf{S}} \text{ Provention}$$
 error is orthogonal to approximation: © 2013
$$\langle \mathbf{x} - \hat{\mathbf{x}}, \hat{\mathbf{x}} \rangle = 0$$

$$\langle \mathbf{x} - \hat{\mathbf{x}}, \, \hat{\mathbf{x}} \rangle = 0$$



orthogonal projection has minimum-norm error:

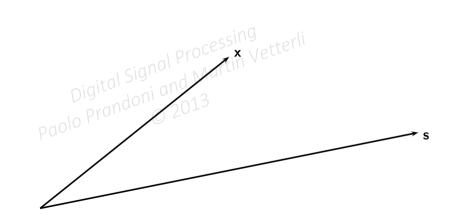
$$\underset{\mathbf{y} \in S}{\arg\min} \|\mathbf{x} - \mathbf{y}\|_{S} = \hat{\mathbf{x}}^{Q}$$

$$\underset{\mathbf{y} \in S}{\operatorname{production}} \text{ Nartin Vetterli}$$

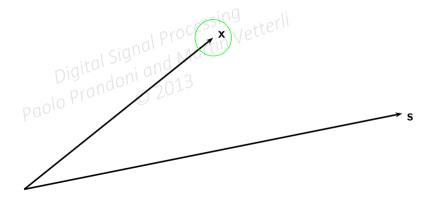
$$\underset{\mathbf{y} \in S}{\operatorname{production}} \text{ on and Martin } \text{ on a pproximation: } \mathbb{C}^{2013}$$

$$\langle \boldsymbol{x} - \hat{\boldsymbol{x}},\, \hat{\boldsymbol{x}} \rangle = 0$$

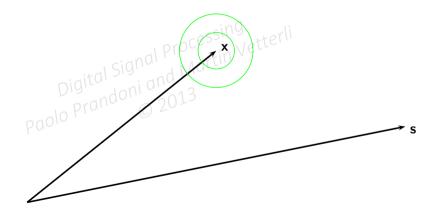




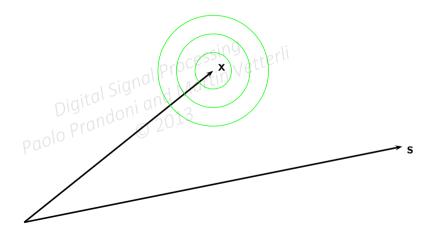




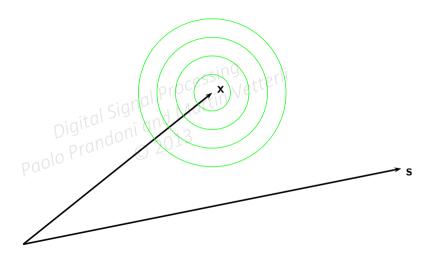




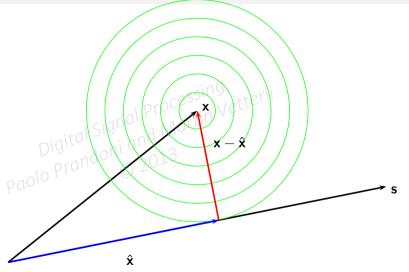
















- $\begin{array}{c} \dots & \text{$_{N[-1,1]} \subset L_2[-1,1]$} \\ \text{$^{\text{basis:}} \ \textbf{s}^{(k)} = t^k, \quad k=0,1,\dots,N-1$} \\ \text{$^{\text{naive basis is not orthonormal randon}} \\ & \text{$^{\text{colo}}$} \end{array}$



- a self-evident, naive basis: $\mathbf{s}^{(k)} = t^k$, $k = 0, 1, \dots, N-1$ ▶ naive basis is not orthonormal and $\mathbf{s}^{(k)} = \mathbf{s}^{(k)} =$



goal: approximate $\mathbf{x} = \sin t$ over $P_3[-1, 1]$

- build orthonormal basis from naive basisal Processing
 project x over the orthonomial basis on and Martin Vetterli
 compute approximation charge C 2013



goal: approximate $\mathbf{x} = \sin t$ over $P_3[-1, 1]$

- build orthonormal basis from naive basisal Processing

 project x over the orthonormal basis on and Martin

 compute approximation cropped a 2013

Example: polynomial approximation



goal: approximate $\mathbf{x} = \sin t$ over $P_3[-1, 1]$

- build orthonormal basis from naive basis of Processing Vetterli
 project x over the orthonormal basis of and Martin compute approximation from C 2013

Example: polynomial approximation



goal: approximate $\mathbf{x} = \sin t$ over $P_3[-1, 1]$

- ▶ build orthonormal basis from naive basis | Processing Vetter
- ▶ project **x** over the orthonormal basis
- ► compute approximation error
- compare error to Taylor approximation (well known but not optimal over the interval)

Example: polynomial approximation



goal: approximate $\mathbf{x} = \sin t$ over $P_3[-1,1]$

- build orthonormal basis from naive basis
- project x over the orthonormal basis
- ► compute approximation error
- compare error to Taylor approximation (well known but not optimal over the interval)



The Gram-Schmidt algorithm leads to an orthonormal basis for $P_N([-1,1])$

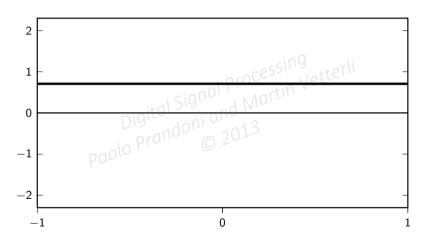
schmidt algorithm leads to an orthonormal basis for (see appendix if interested in details)
$$\mathbf{u}^{(0)} \equiv \sqrt{1/2}$$

$$\mathbf{u}^{(1)} = \sqrt{3/2}\,t$$

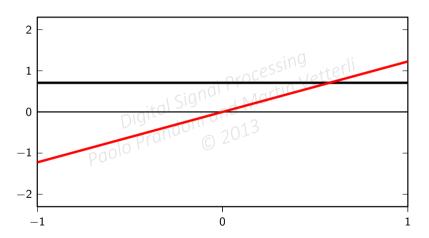
$$\mathbf{u}^{(2)} = \sqrt{5/8}(3t^2-1)$$

$$\mathbf{u}^{(3)} = \dots$$

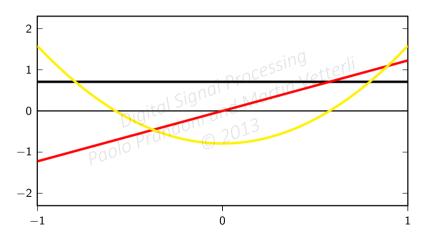




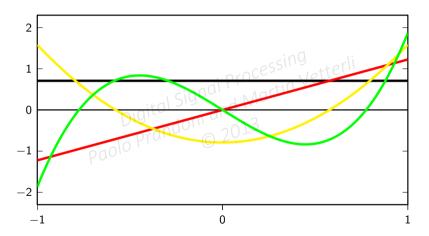




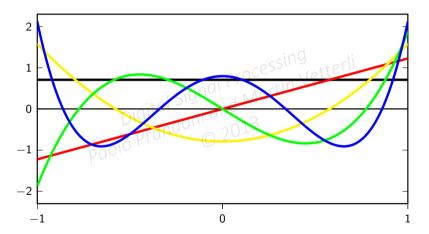




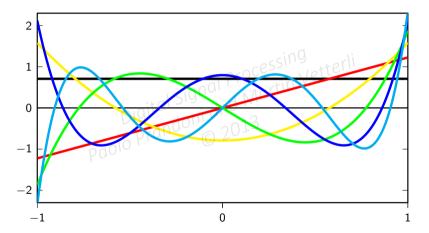












Orthogonal projection over $P_3[-1, 1]$



$$\alpha_{k} = \langle \mathbf{u}^{(k)}, \mathbf{x} \rangle = \int_{-1}^{1} u_{k}(t) \sin t \, dt \text{ terli}$$

$$\boldsymbol{\alpha}_{0} = \langle \sqrt{1/2}, \sin t \rangle = 0 \quad \text{ pigital Signal Martin}$$

$$\boldsymbol{\alpha}_{1} = \langle \sqrt{3/2} t, \sin t \rangle \approx 0.737774000 \text{ (a)}$$

$$\boldsymbol{\alpha}_{2} = \langle \sqrt{5/8}(3t^{2} - 1), \sin t \rangle = 0$$

Orthogonal projection over $P_3[-1, 1]$



$$\alpha_k = \langle \mathbf{u}^{(k)}, \mathbf{x} \rangle = \int_{-1}^1 u_k(t) \sin t \, dt \text{ and } \mathbf{v}$$

$$\alpha_0 = \langle \sqrt{1/2}, \sin t \rangle = 0 \qquad \text{ and } \mathbf{v}$$

$$\alpha_1 = \langle \sqrt{3/2} \, t, \sin t \rangle \approx 0.7377 \quad \text{ and } \mathbf{v}$$

$$\alpha_2 = \langle \sqrt{5/8}(3t^2 - 1), \sin t \rangle = 0$$

Orthogonal projection over $P_3[-1, 1]$



Approximation



Using the orthogonal projection over $P_3[-1,1]$:

$$\sin t \rightarrow \alpha_1 \mathbf{u}^{(1)} \approx 0.9035 t^9$$

$$\text{Digital Signal Properties}$$

$$\text{Digital Signal Martin Vetterlie}$$

$$\text{Digital Signal Properties}$$

$$\text{Digital Properties$$

Approximation



Using the orthogonal projection over $P_3[-1,1]$:

$$\sin t \rightarrow \alpha_1 \mathbf{u}^{(1)} \approx 0.9035 \, t^9$$

$$\text{Digital Signal Properties}$$

$$\text{Digital Signal Martin Vetterlies}$$

$$\text{Properties}$$

$$\text{Properties}$$

$$\text{Properties}$$

$$\text{Page 1}$$

$$\text{Properties}$$

$$\text{Page 2}$$

$$\text{Properties}$$

$$\text{Page 3}$$

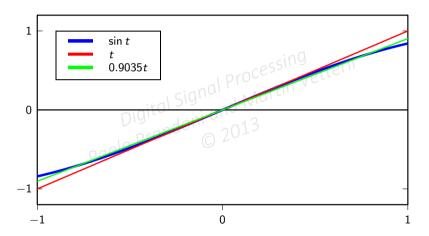
$$\text{Properties}$$

3.3

 $\sin t \approx t$

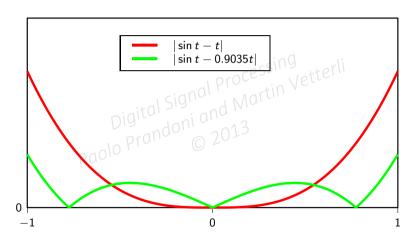
Sine approximation





Approximation error





Error norm



Orthogonal projection over $P_3[-1,1]$:

$$\|\sin t - \alpha_1 \mathbf{u}^{(1)}\| \approx 0.0337$$

Digital Signal Properties Digital Signal Martin Vetterli

Paolo Prandoni and Martin Vetterli

Error norm



Orthogonal projection over $P_3[-1,1]$:

$$\|\sin t - \alpha_1 \mathbf{u}^{(1)}\| \approx 0.0337$$
Digital Signal Properties Digital Signal Martin Vetterli
Paolo Prandoni and 13
Taylor series:

3.3

 $\|\sin t - t\| \approx 0.0857$



Why do we do all this?

- infinite-length signals live in $\ell_2(\mathbb{Z})$ Signal Processing Vetterli Digital and Martin Digital and Martin Digital and Servation (Locals for signals subspace projections are useful: "



Why do we do all this?

- rinite-length and periodic signals live in C^N

 infinite-length signals live in ℓ₂(ℤ) signal Processing Vetterli
 Digliand Martin

 different bases are different diservation codes for signals



Why do we do all this?

- infinite-length signals live in \mathbb{C}^N infinite-length signals live in $\ell_2(\mathbb{Z})$ Signal Processing Vetterli Digital and Martin Vetterli Digital and Digit



Why do we do all this?

- infinite-length signals live in \mathbb{C}^N infinite-length signals live in $\ell_2(\mathbb{Z})$ Signal Processing Vetterli Digital and Martin Vetterli Digital and Digit
- subspace projections are useful in filtering and compression

END OF MODULE 3.3

Digital Signa and Martin Paolo Prandoni and Paolo Prandoni and

Appendix: orthonormalization of the naive polynomial basis



Gram-Schmidt orthonormalization procedure:

$$\{\mathbf{s}^{(k)}\} \xrightarrow{} \{\mathbf{u}^{(k)}\}^{g}$$
 original set orthonormal set
$$\text{Algorithmic procedure: at each step } k$$

1.
$$\mathbf{p}^{(k)} = \mathbf{s}^{(k)} - \sum_{n=0}^{k-1} \langle \mathbf{u}^{(n)}, \mathbf{s}^{(k)} \rangle \mathbf{u}^{(n)}$$

2.
$$\mathbf{u}^{(k)} = \mathbf{p}^{(k)} / \|\mathbf{p}^{(k)}\|$$



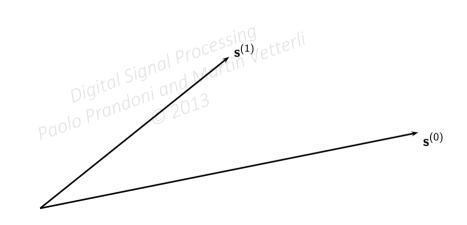
Gram-Schmidt orthonormalization procedure:

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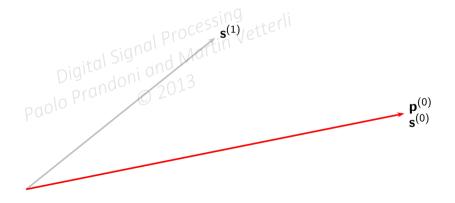
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$$\mathbf{u}^{(k)} = \mathbf{p}^{(k)} / \|\mathbf{p}^{(k)}\|$$

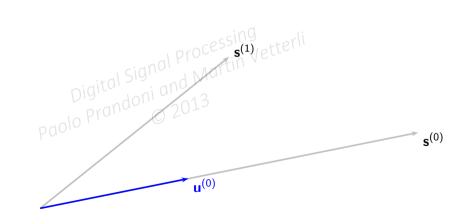




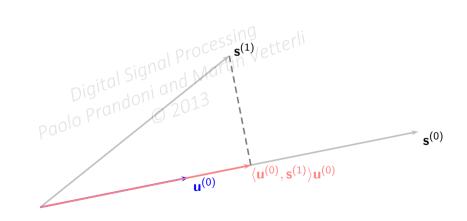




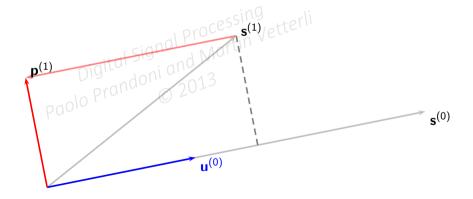




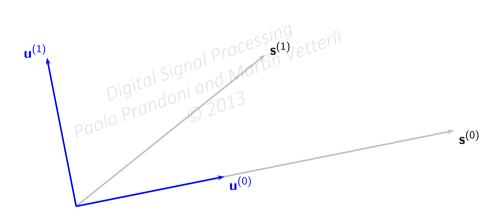














Gram-Schmidt orthonormalization of the naive basis: $\{\mathbf{s}^{(k)}\} \rightarrow \{\mathbf{u}^{(k)}\}$

$$\mathbf{s}^{(0)} = 1$$

•
$$\mathbf{p}^{(0)} = \mathbf{s}^{(0)} = 1$$

•
$$\|\mathbf{p}^{(0)}\|^2 = 2$$

$$\begin{aligned} \mathbf{s^{(0)}} &= \mathbf{1} \\ &\bullet \mathbf{p^{(0)}} = \mathbf{s^{(0)}} = \mathbf{1} \\ &\bullet \|\mathbf{p^{(0)}}\|^2 = 2 \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{p^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{u^{(0)}}/\|\mathbf{p^{(0)}}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u^{(0)}} = \mathbf{u^{(0)}}/\|\mathbf{u^{(0)}}\| = \mathbf{u^{(0)}}/\|\mathbf{u^{(0)}}$$

$$s^{(1)} = t$$

•
$$\langle \mathbf{u}^{(0)}, \mathbf{s}^{(1)} \rangle = \int_{-1}^{1} t / \sqrt{2} = 0$$

•
$$p^{(1)} = s^{(1)} = t$$

•
$$\|\mathbf{p}^{(1)}\|^2 = 2/3$$

•
$$\mathbf{u}^{(1)} = \sqrt{3/2} t$$

•
$$\mathbf{p}^{(2)} = \mathbf{s}^{(2)} - (2/3\sqrt{2})\mathbf{u}^{(0)} = t^2 - 1/3$$

$$\|\mathbf{p}^{(2)}\|^2 = 8/45$$

•
$$\mathbf{u}^{(2)} = \sqrt{5/8(3t^2 - 1)}$$



Gram-Schmidt orthonormalization of the naive basis: $\{\mathbf{s}^{(k)}\} \rightarrow \{\mathbf{u}^{(k)}\}$

- $\begin{aligned} \mathbf{u}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Processing} \\ \mathbf{u}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{v}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{v}^{(0)}\| = \sqrt{1/2} & \text{igital Signal Martint} \\ \mathbf{v}^{(0)} &= \mathbf{p}^{(0)}/\|\mathbf{v}^{(0)}\| = \sqrt{1/2} & \text{igital Signa$



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$$\begin{aligned} \mathbf{s}^{(0)} &= 1 \\ &\bullet \mathbf{p}^{(0)} = \mathbf{s}^{(0)} = 1 \\ &\bullet \|\mathbf{p}^{(0)}\|^2 = 2 \\ &\bullet \mathbf{u}^{(0)} = \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u}^{(0)} = \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u}^{(0)} = \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} \text{igital Signal Processing} \\ &\bullet \mathbf{u}^{(0)}, \mathbf{s}^{(2)} = \mathbf{p}^{(1)} + \mathbf{p}^{(2)} = \mathbf{p}^{(2)} + \mathbf{p}^{(2)} = \mathbf{p}^{(2)}$$

•
$$\langle \mathbf{u}^{(0)}, \mathbf{s}^{(1)} \rangle = \int_{-1}^{1} t / \sqrt[4]{2} = 0$$

•
$$\mathbf{p}^{(1)} = \mathbf{s}^{(1)} = t$$

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$$\mathbf{s}^{(0)} = 1$$

•
$$\mathbf{p}^{(0)} = \mathbf{s}^{(0)} = 1$$

•
$$\|\mathbf{p}^{(0)}\|^2 = 2$$

▶
$$\mathbf{s}^{(0)} = 1$$
• $\mathbf{p}^{(0)} = \mathbf{s}^{(0)} = 1$
• $\|\mathbf{p}^{(0)}\|^2 = 2$
• $\mathbf{u}^{(0)} = \mathbf{p}^{(0)} / \|\mathbf{p}^{(0)}\| = \sqrt{1/2} \text{igital signal Processing}$
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• $\langle \mathbf{u}^{(0)}, \mathbf{s}^{(1)} \rangle = \int_{-1}^{1} t^2 / \sqrt{2} = 0$
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• $\|\mathbf{p}^{(0)}\|^2 = 2$
• $\mathbf{u}^{(0)} = \mathbf{p}^{(0)} / \|\mathbf{p}^{(0)}\| = \sqrt{1/2} \text{ igital Signal Programmes}$
• $\mathbf{s}^{(1)} = t$
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$$s^{(1)} = t$$

•
$$\langle \mathbf{u}^{(0)}, \mathbf{s}^{(1)} \rangle = \int_{-1}^{1} t / \sqrt{2} = 0$$

•
$$\mathbf{p}^{(1)} = \mathbf{s}^{(1)} = t$$

•
$$\|\mathbf{p}^{(1)}\|^2 = 2/3$$

•
$$\mathbf{u}^{(1)} = \sqrt{3/2} t$$



$$\mathbf{s}^{(0)} = 1$$

•
$$\mathbf{p}^{(0)} = \mathbf{s}^{(0)} = 1$$

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$$\|\mathbf{p}^{(0)}\|^2 = 2$$

$$\mathbf{s}^{(0)} = 1$$
• $\mathbf{p}^{(0)} = \mathbf{s}^{(0)} = 1$
• $\|\mathbf{p}^{(0)}\|^2 = 2$
• $\mathbf{u}^{(0)} = \mathbf{p}^{(0)} / \|\mathbf{p}^{(0)}\| = \sqrt{1/2} \text{ igital Signal Prosessing}$
• $\mathbf{s}^{(1)} = t$
• $\langle \mathbf{u}^{(0)}, \mathbf{s}^{(1)} \rangle = \int_{-1}^{1} t / \sqrt{2} = 0$

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• $\|\mathbf{p}^{(0)}\|^2 = 2$
• $\mathbf{u}^{(0)} = \mathbf{p}^{(0)} / \|\mathbf{p}^{(0)}\| = \sqrt{1/2} \text{ igital Signal Proof.}$

$$\mathbf{s}^{(1)} = t$$
• $\langle \mathbf{u}^{(0)}, \mathbf{s}^{(1)} \rangle = \int_{-1}^{1} t / \sqrt{2} = 0$

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$$s^{(1)} = t$$

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- $\mathbf{s}^{(1)} = t$
- ... || = 2• $\mathbf{u}^{(0)} = \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2} \text{igital Signal Processing}$ = t• $\mathbf{u}^{(0)}, \mathbf{s}^{(1)} = \int_{-\infty}^{1} t^{1/2} \mathbf{p}^{(0)} \mathbf{p}^{$
 - $\mathbf{p}^{(1)} = \mathbf{s}^{(1)} = t$
 - $\|\mathbf{p}^{(1)}\|^2 = 2/3$
 - $\mathbf{u}^{(1)} = \sqrt{3/2} t$



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$$\mathbf{u}^{(0)} = \mathbf{p}^{(0)} / \|\mathbf{p}^{(0)}\| = \sqrt{1/2} \text{ igital signal and } \mathbf{M} \cdot \langle \mathbf{u}^{(0)}, \mathbf{s}^{(2)} \rangle = \int_{-1}^{1} t^{2} / \sqrt{2} = 2/3\sqrt{2}$$

$$\mathbf{s}^{(1)} = t$$

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$$ightharpoonup {\bf s}^{(1)} = t$$

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$$O_{\mathbf{S}^{(2)} = t^2}^{(2)}$$

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Legendre polynomials



$$\mathbf{u}^{(0)} = \sqrt{1/2} \operatorname{processing}$$

$$\mathbf{u}^{(1)} = \sqrt{3/2} t \operatorname{Martin}$$

$$\mathbf{v}^{(2)} = \sqrt{5/8} (3t^2 - 1)$$

$$\mathbf{u}^{(3)} = \dots$$

END OF MODULE'3 Digital Sign and Martin E'3 Paolo Prandoni and 2013