Digital Signal Processing

Module 6: Interpolation and Sampling

Module Overview:

- Module 6.1: Continuous-time signals
- Module 6.2: Interpolation
- Module 6.3: Sampling
- Module 6.4: Aliasing
- Module 6.5: Interpolation and sampling in practice
- Module 6.6: Discrete-time processing of continuous-time signals

Digital Signal Processing

Module 6.1: The Continuous-Time Paradigm

Overview:

- Models of the world
- Continuous-time signals

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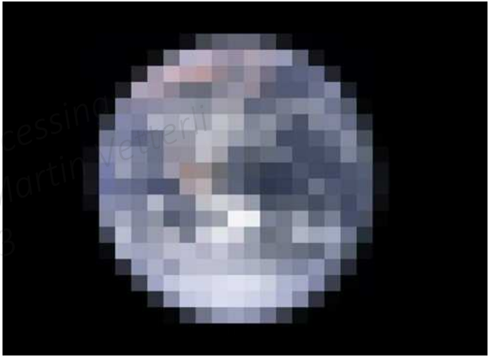
- Continuous-time signals



Analog/continuous versus discrete/digital

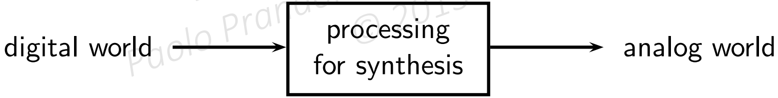
Digital processing of signals from/to the analog world

- input is continuous-time: $x(t)$
- output is continuous-time: $y(t)$
- processing is on sequences: $x[n], y[n]$

examples: MP3, digital photography

Digital processing of signals to the analog world

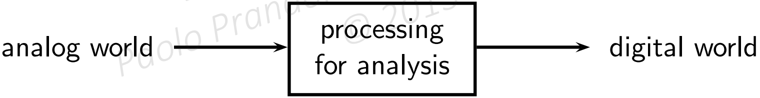
- input is discrete-time: $x[n]$
- output is continuous-time: $y(t)$
- processing is on sequences: $x[n], y[n]$



examples: computer graphics, video games

Digital processing of signals from the analog world

- input is continuous-time: $x(t)$
- output is discrete-time: $y[n]$
- processing is on sequences: $x[n], y[n]$



examples: control systems, monitoring

Two views of the world

digital worldview:

- arithmetic
- combinatorics
- computer science
- DSP

analog worldview:

- calculus
- distributions
- system theory
- electronics

Two views of the world

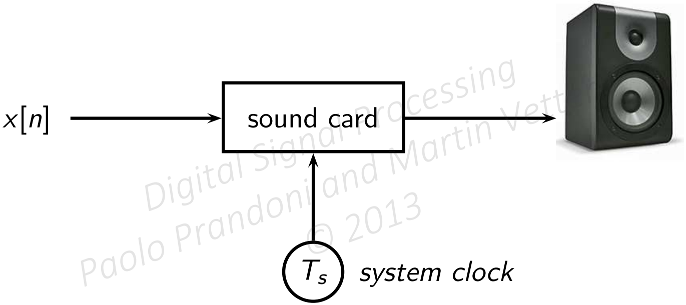
digital worldview:

- countable integer index n
- sequences $x[n] \in \ell_2(\mathbb{Z})$
- frequency $\omega \in [-\pi, \pi]$
- DTFT: $\ell_2(\mathbb{Z}) \mapsto L_2([-\pi, \pi])$

analog worldview:

- real-valued time t (sec)
- functions $x(t) \in L_2(\mathbb{R})$
- frequency $\Omega \in \mathbb{R}$ (rad/sec)
- FT: $L_2(\mathbb{R}) \mapsto L_2(\mathbb{R})$

Bridging the gap



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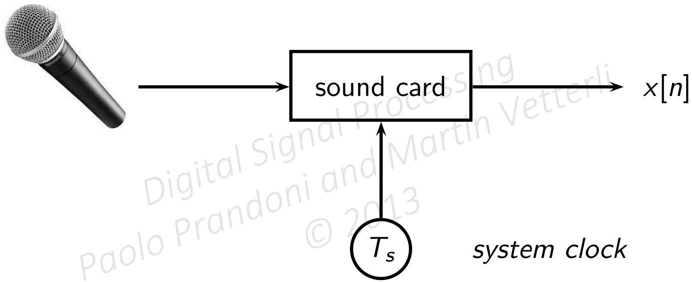

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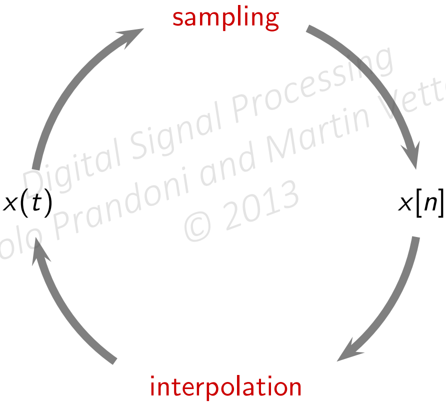

About continuous time

- time: real variable t
- signal $x(t)$: complex functions of a real variable
- finite energy: $x(t) \in L_2(\mathbb{R})$
- inner product in $L_2(\mathbb{R})$

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x^*(t) y(t) dt$$

- energy: $||x(t)||^2 = \langle x(t), x(t) \rangle$

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Analog LTI filters

$$x(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(t)$$

$$
\begin{aligned}
y(t) &= (x * h)(t) \\
&= \langle h^*(t-\tau), x(\tau) \rangle \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau
\end{aligned}
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Fourier analysis

- in discrete time max angular frequency is $\pm\pi$
- in continuous time no max frequency: $\Omega \in \mathbb{R}$
- concept is the same:

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt \qquad \leftarrow \textit{not periodic!}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}\,d\Omega$$

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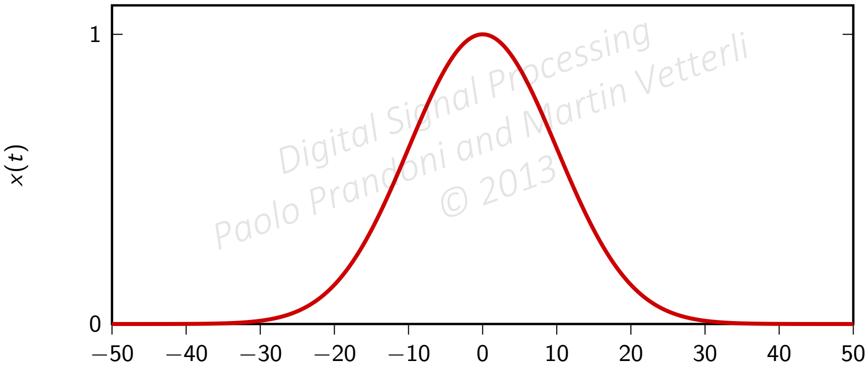
Real-world frequency

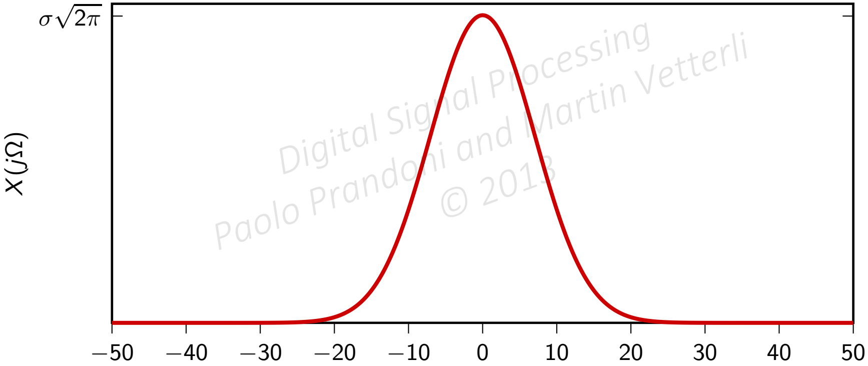
- Ω expressed in rad/s
- $F = \dfrac{\Omega}{2\pi}$, expressed in Hertz ($1/s$)
- period $T = \dfrac{1}{F} = \dfrac{2\pi}{\Omega}$

$$x(t) = e^{-\frac{t^2}{2\sigma^2}}$$



Example

$$X(j\Omega) = \sigma\sqrt{2\pi}e^{-\frac{\sigma^2}{2}\Omega^2}$$



Convolution theorem

$$x(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(t)$$

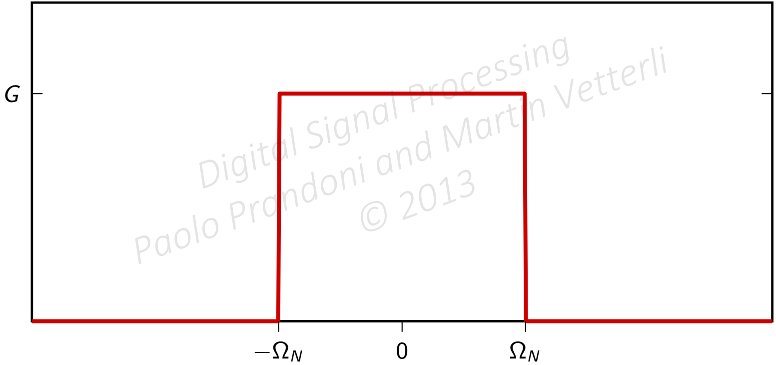
$$Y(j\Omega) = X(j\Omega)\, H(j\Omega)$$

A new concept: bandlimited functions

Ω_N-bandlimitedness:

$$X(j\Omega) = 0 \quad \text{for } |\Omega| > \Omega_N$$

Prototypical bandlimited function



The prototypical bandlimited function

$$\Phi(j\Omega) = G \text{ rect}\left(\frac{\Omega}{2\Omega_N}\right)$$

$$\varphi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(j\Omega)e^{j\Omega t}\,d\Omega$$

$$= \ldots \quad \text{see Module 5.5}$$

$$= G\,\frac{\Omega_N}{\pi}\,\text{sinc}\left(\frac{\Omega_N}{\pi}t\right)$$

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The prototypical bandlimited function

- normalization: $G = \dfrac{\pi}{\Omega_N}$
- total bandwidth: $\Omega_B = 2\Omega_N$
- define $T_s = \dfrac{2\pi}{\Omega_B} = \dfrac{\pi}{\Omega_N}$

The prototypical bandlimited function

$$\Phi(j\Omega) = \frac{\pi}{\Omega_N}\,\mathrm{rect}\left(\frac{\Omega}{2\Omega_N}\right)$$

$$\varphi(t) = \mathrm{sinc}\left(\frac{t}{T_s}\right)$$

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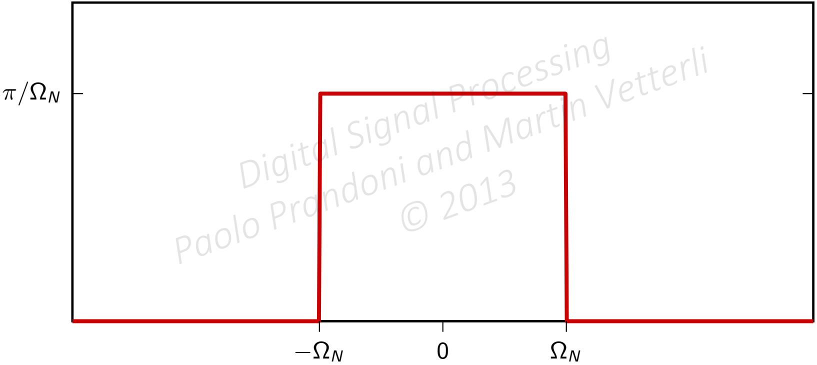

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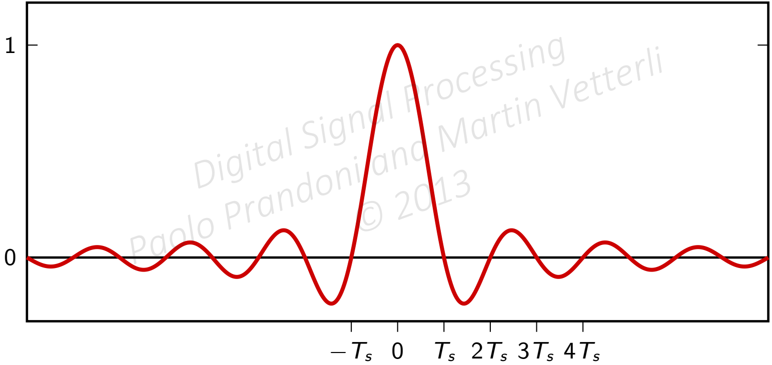

END OF MODULE 6.1

Digital Signal Processing

Module 6.2: Interpolation

Overview:

- Polynomial interpolation
- Local interpolation
- Sinc interpolation

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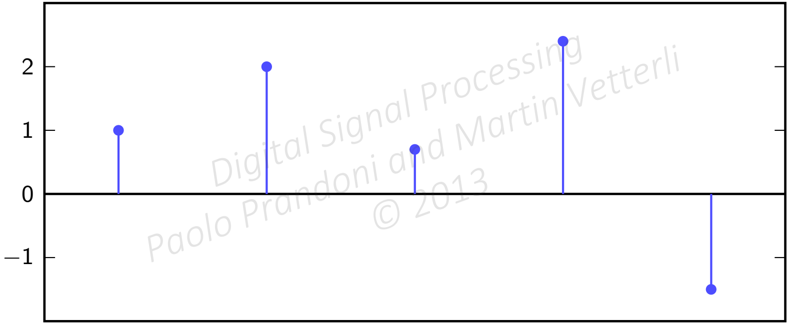
Interpolation

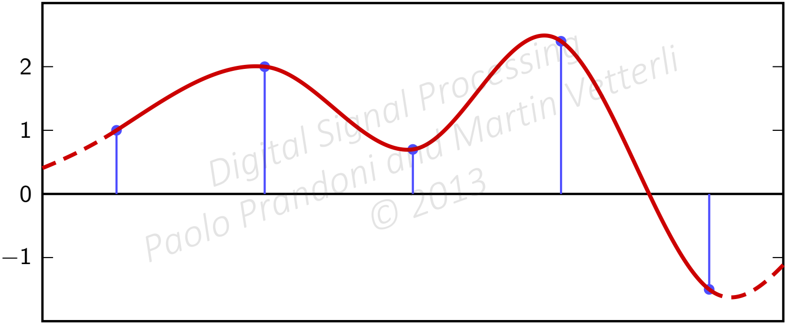
$$x[n] \longrightarrow x(t)$$

"fill the gaps" between samples

Example



Example



Interpolation requirements

- decide on T_s
- make sure $x(nT_s) = x[n]$
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Why smoothness?

- jumps (1st order discontinuities) would require the signal to move "faster than light"...
- 2nd order discontinuities would require infinite acceleration
- ...
- the interpolation should be infinitely differentiable
- "natural" solution: polynomial interpolation

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Polynomial interpolation

- N points $\rightarrow$ polynomial of degree $(N-1)$
- $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_{N-1} t^{(N-1)}$
- "naive" approach:

$$\begin{cases} p(0) = x[0] \\ p(T_s) = x[1] \\ p(2T_s) = x[2] \\ \ldots \\ p((N-1)T_s) = x[N-1] \end{cases}$$

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Polynomial interpolation

Without loss of generality:

- consider a symmetric interval $I_N = [-N, \ldots, N]$
- set $T_s = 1$

$$\begin{cases} p(-N) = x[-N] \\ p(-N+1) = x[-N+1] \\ \ldots \\ p(0) = x[0] \\ \ldots \\ p(N) = x[N] \end{cases}$$

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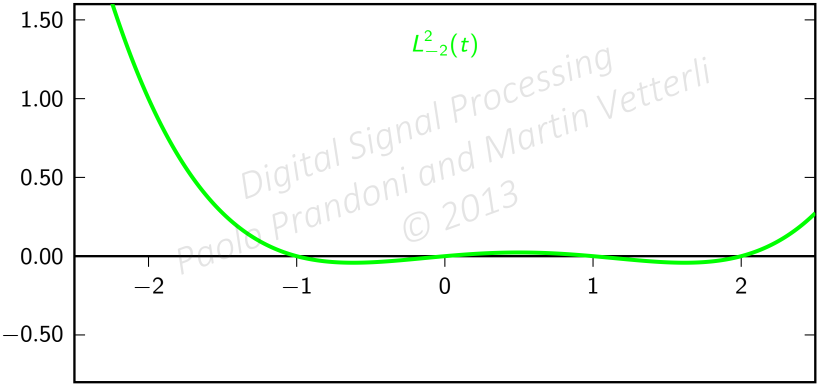
Lagrange interpolation

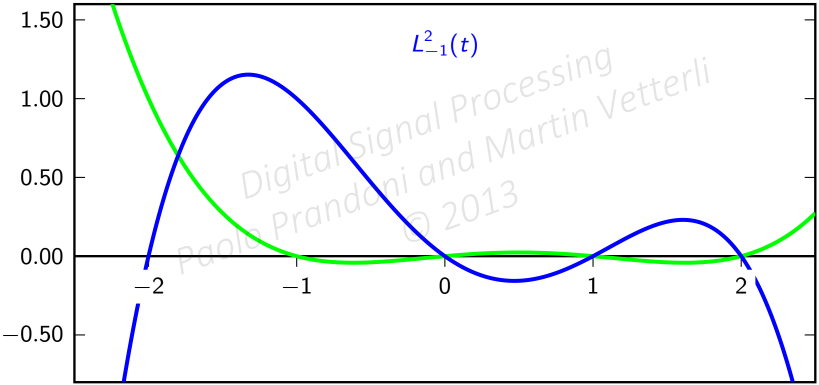
- P_N: space of degree-$2N$ polynomials over I_N
- a basis for P_N is the family of $2N+1$ Lagrange polynomials

$$L_n^{(N)}(t) = \prod_{\substack{k=-N \\ k \neq n}}^{N} \frac{t-k}{n-k} \qquad n = -N, \ldots, N$$

Lagrange interpolation polynomials



Lagrange interpolation polynomials



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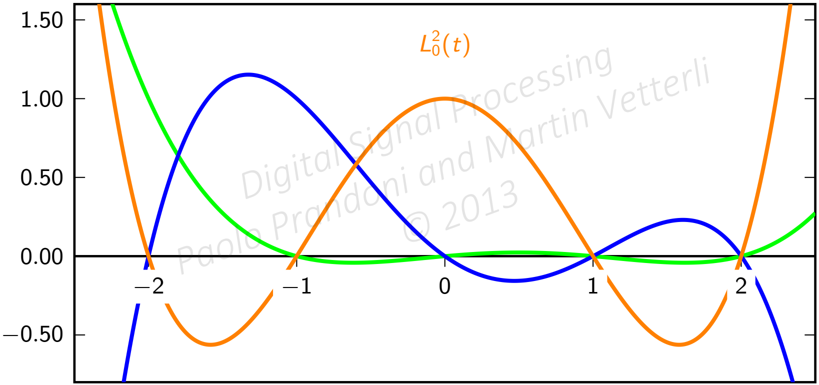

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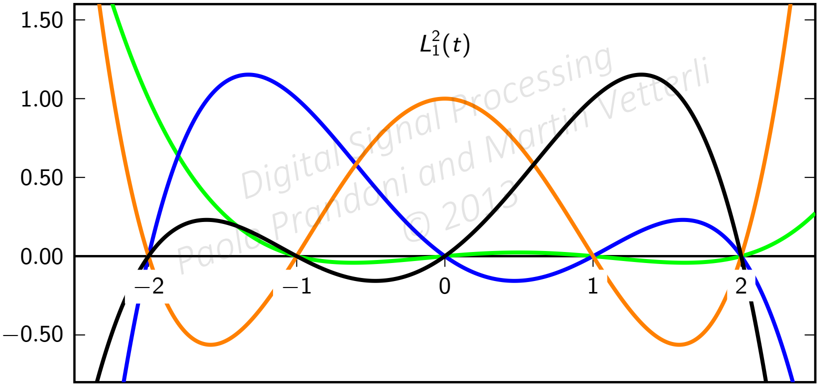

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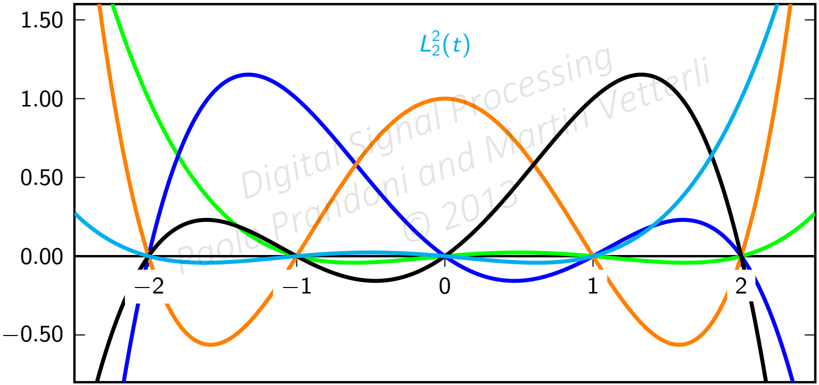


Lagrange interpolation

$$p(t) = \sum_{n=-N}^{N} x[n] L_n^{(N)}(t)$$

Lagrange interpolation

The Lagrange interpolation *is* the sought-after polynomial interpolation:

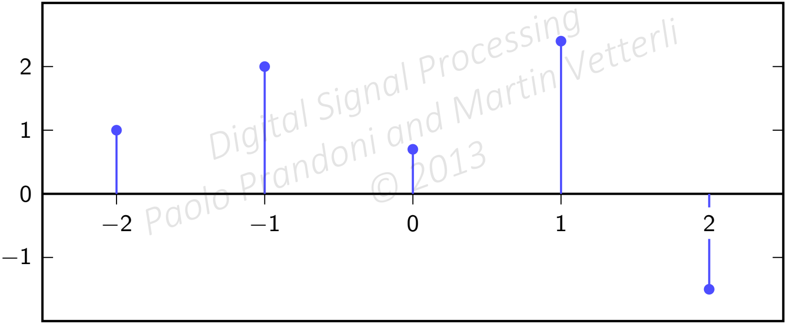
- polynomial of degree $2N$ through $2N+1$ points is unique
- the Lagrangian interpolator satisfies

$$p(n) = x[n] \qquad \text{for } -N \leq n \leq N$$

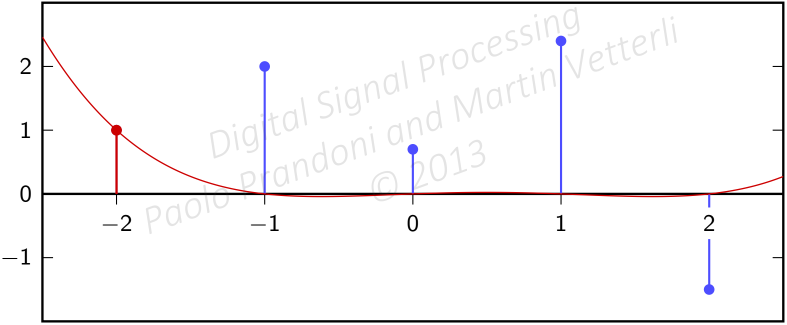
since

$$L_n^{(N)}(m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \qquad -N \leq n, m \leq N$$

Lagrange interpolation



$$x[-2]L_{-2}^{(2)}(t)$$



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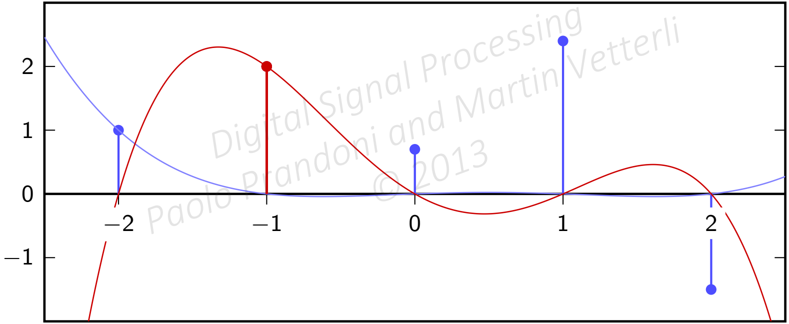

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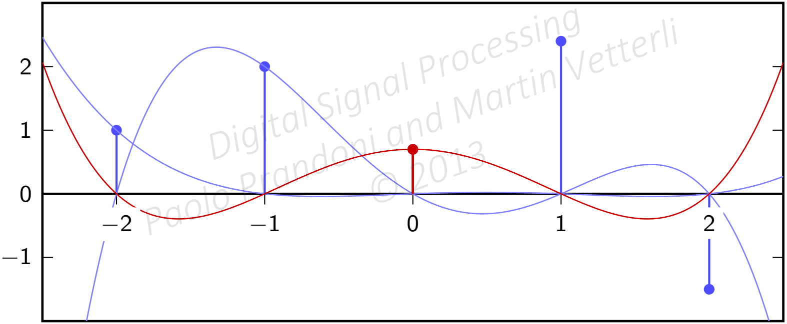

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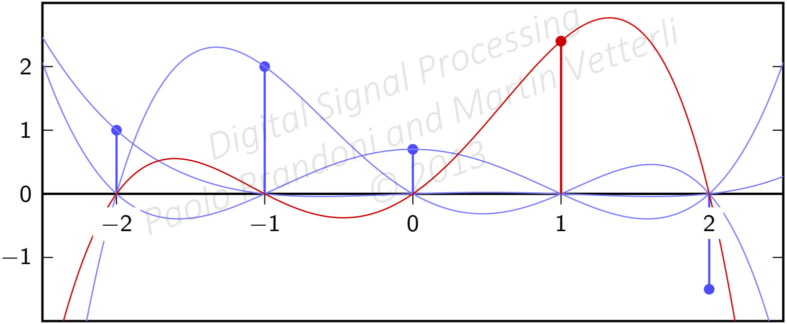

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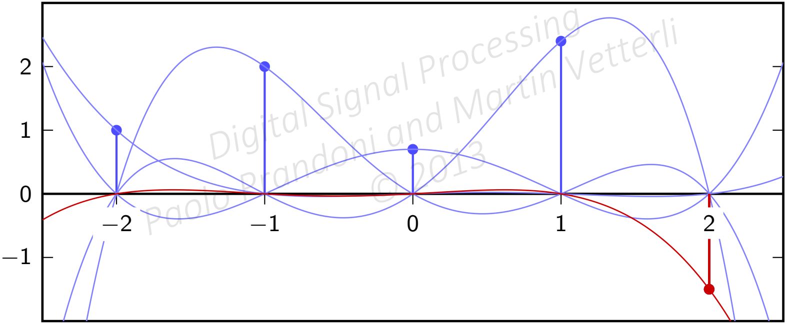

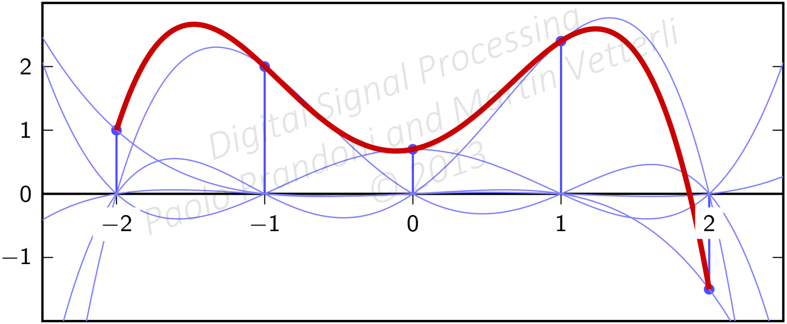
2
1
0
−1
−2
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2

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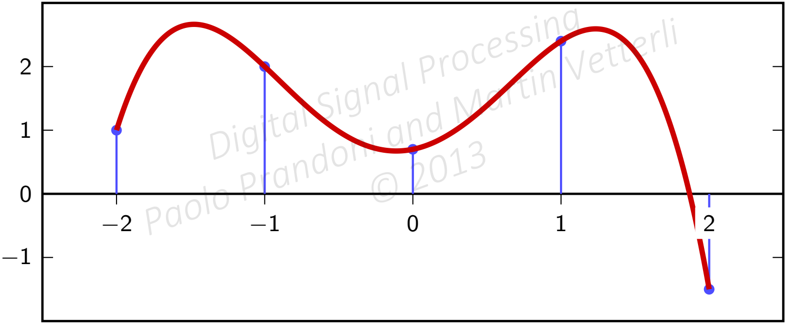

Polynomial interpolation

key property:

- maximally smooth (infinitely many continuous derivatives)

drawback:

- interpolation "bricks" depend on N

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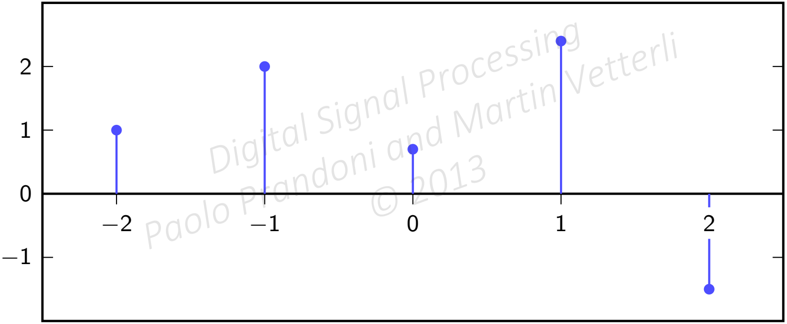
Zero-order interpolation



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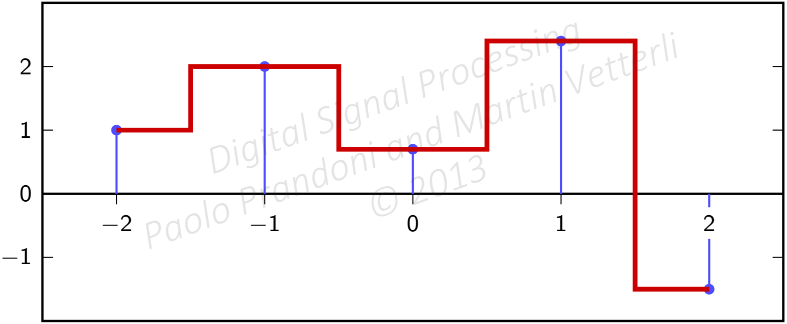

Zero-order interpolation

- $x(t) = x[\lfloor t + 0.5 \rfloor], \qquad -N \leq t \leq N$
- $x(t) = \sum_{n=-N}^{N} x[n] \operatorname{rect}(t - n)$
- interpolation kernel: $i_0(t) = \operatorname{rect}(t)$
- $i_0(t)$: “zero-order hold”
- interpolator’s support is 1
- interpolation is not even continuous

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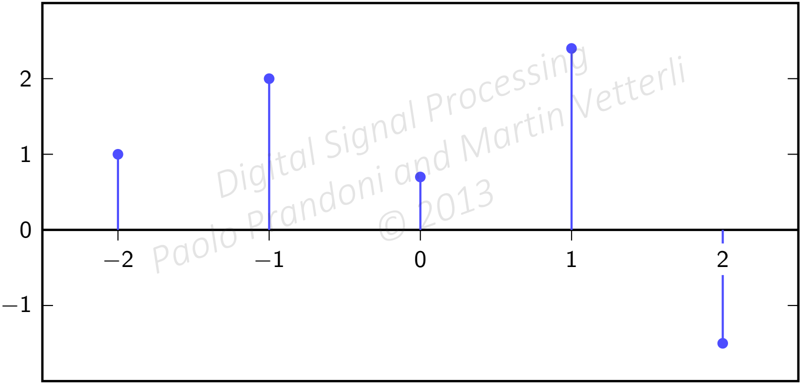

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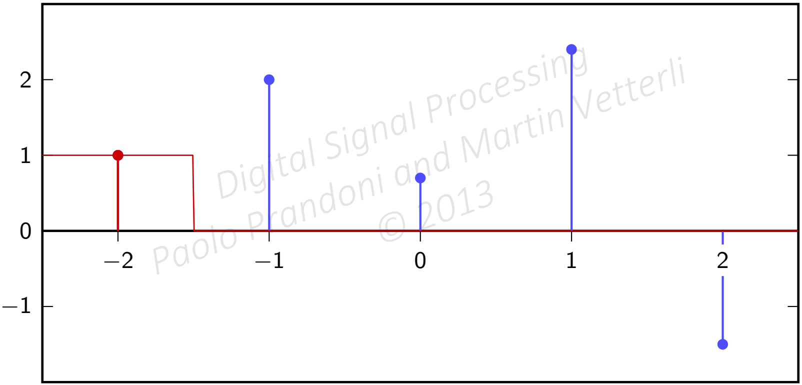
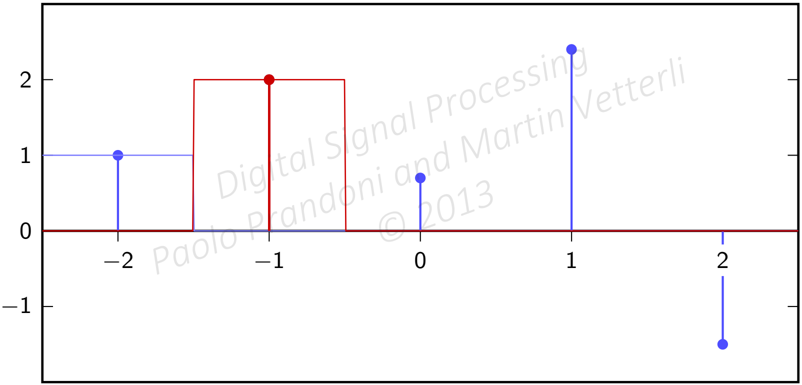


Zero-order interpolation



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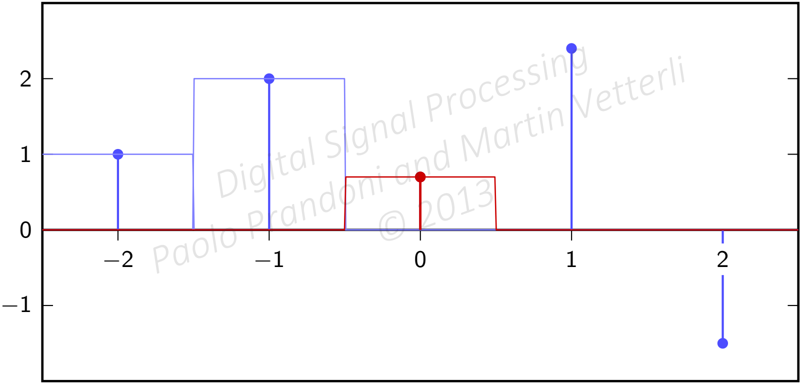

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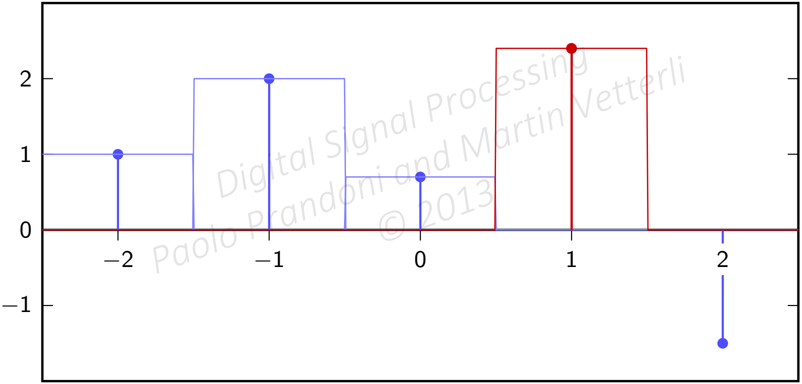

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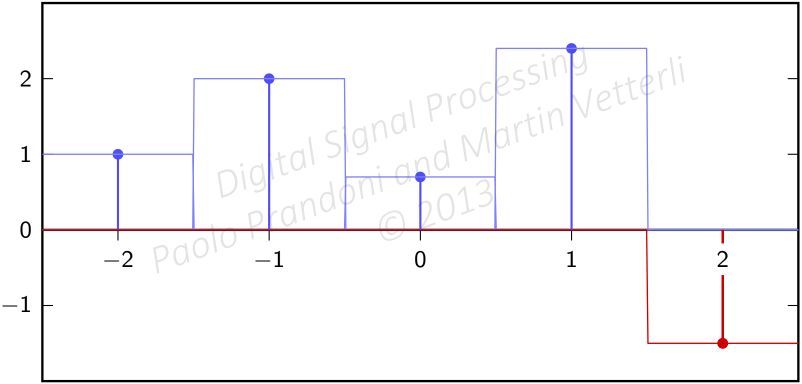

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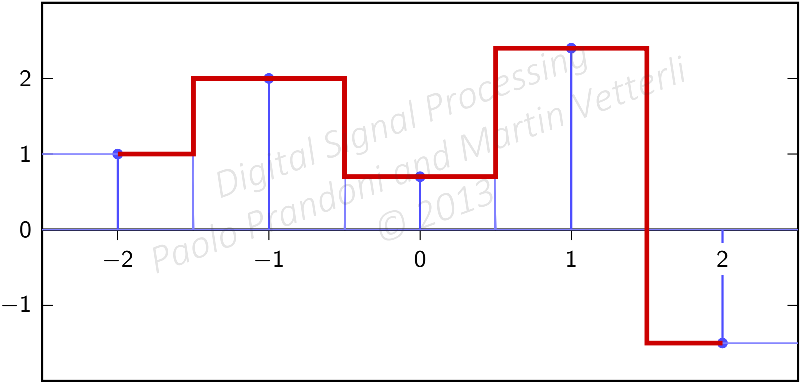

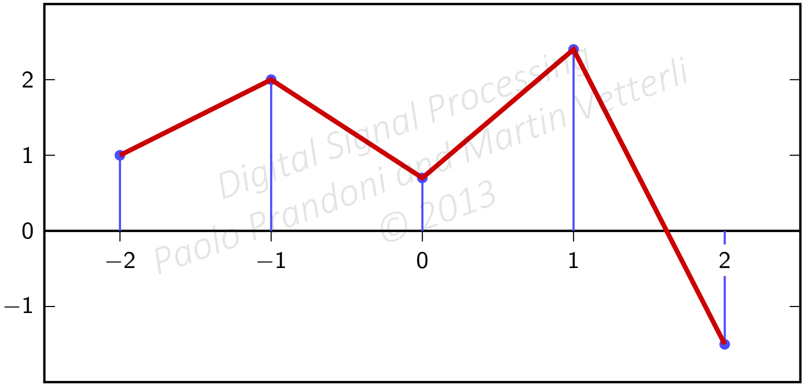
2
1
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−1
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2
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First-order interpolation

- "connect the dots" strategy
- $x(t) = \sum_{n=-N}^{N} x[n]\, i_1(t-n)$
- interpolation kernel:

$$i_1(t) = \begin{cases} 1 - |t| & |t| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

- interpolator's support is 2
- interpolation is continuous but derivative is not

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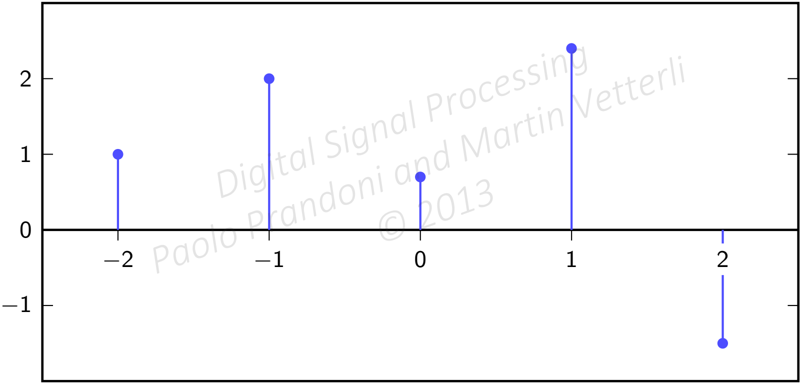

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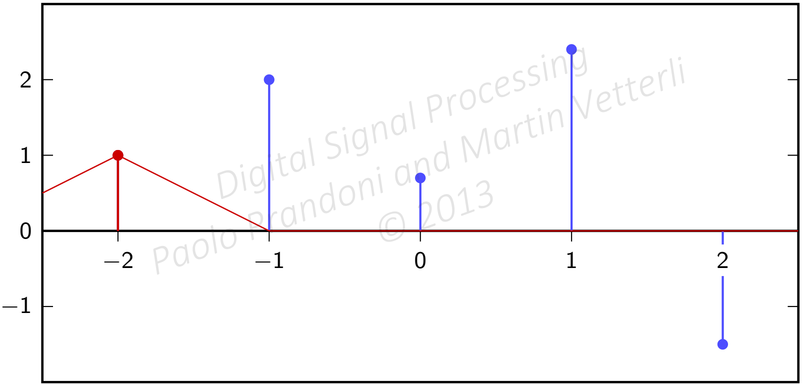

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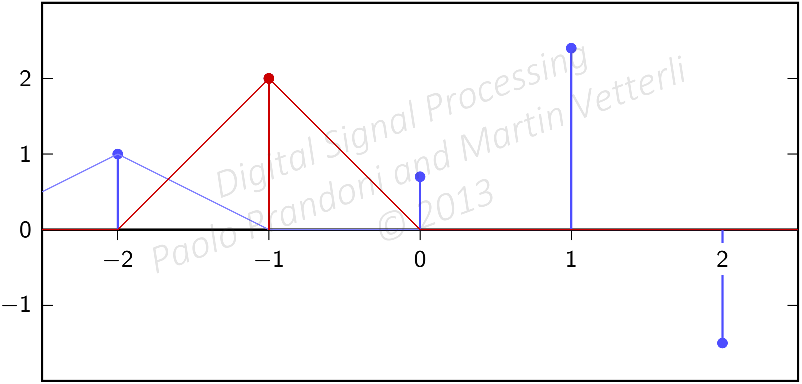

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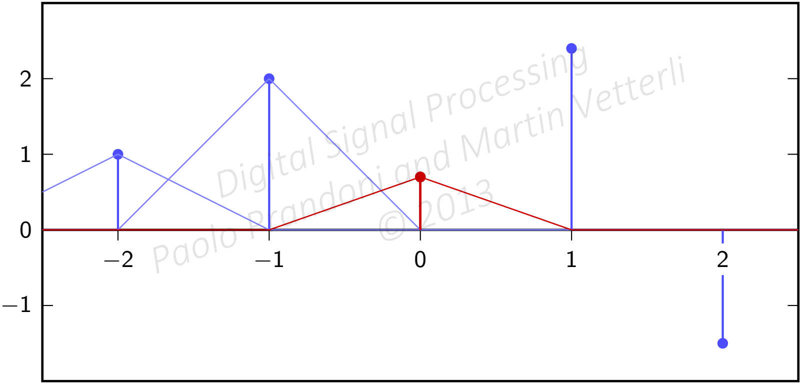

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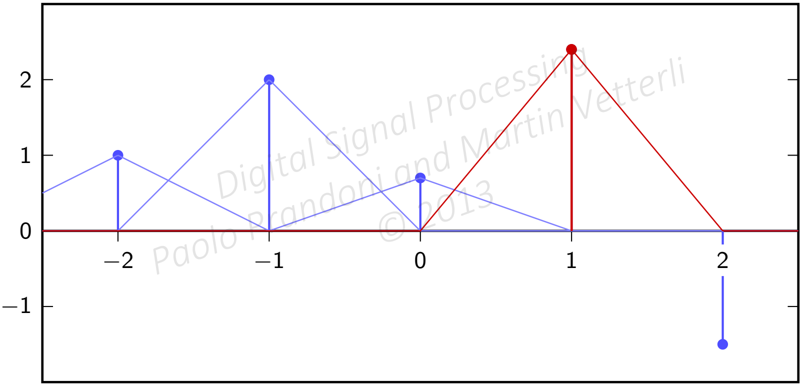

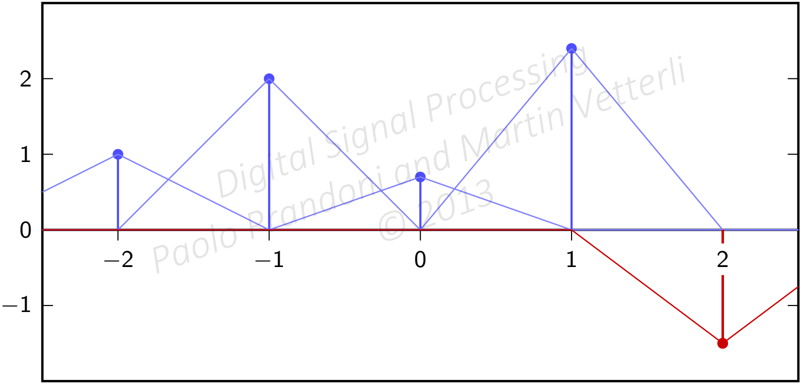
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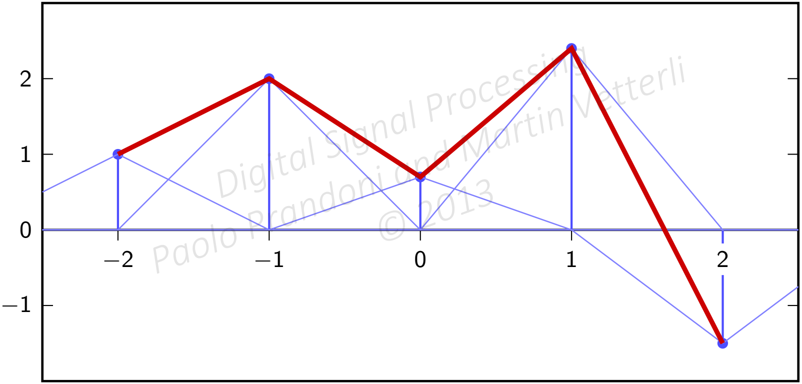

Third-order interpolation

- $x(t) = \sum_{n=-N}^{N} x[n]\, i_3(t-n)$
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- interpolator's support is 4
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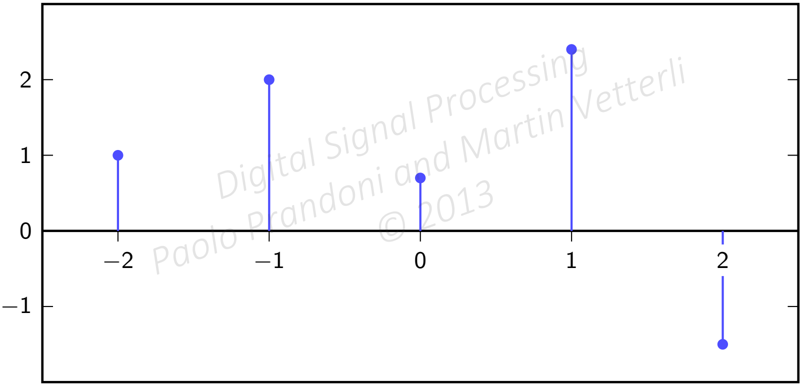

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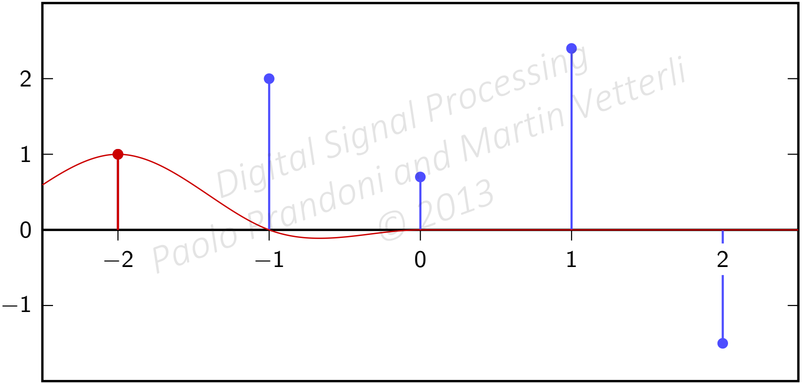

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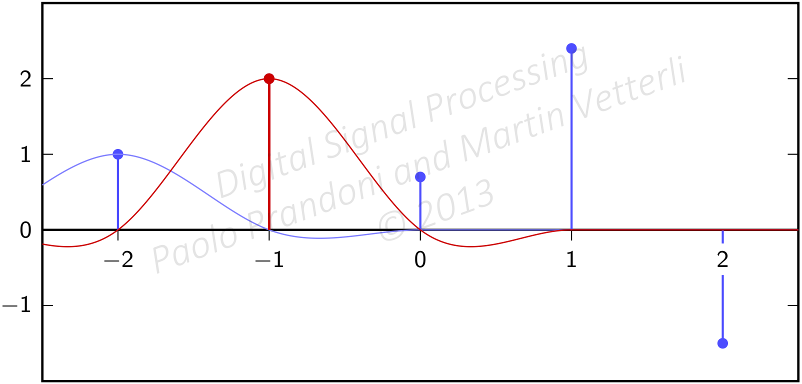

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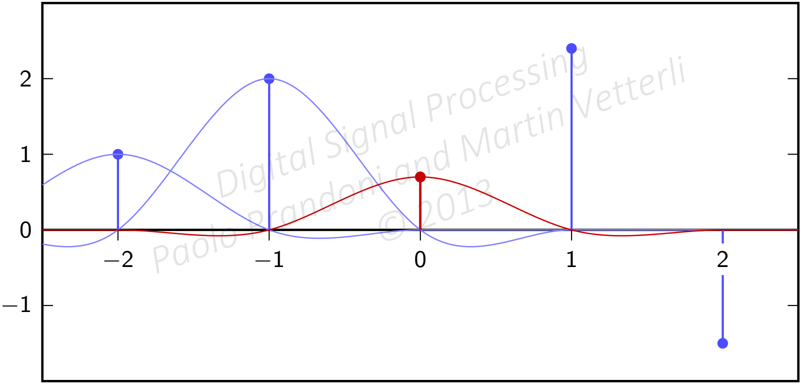

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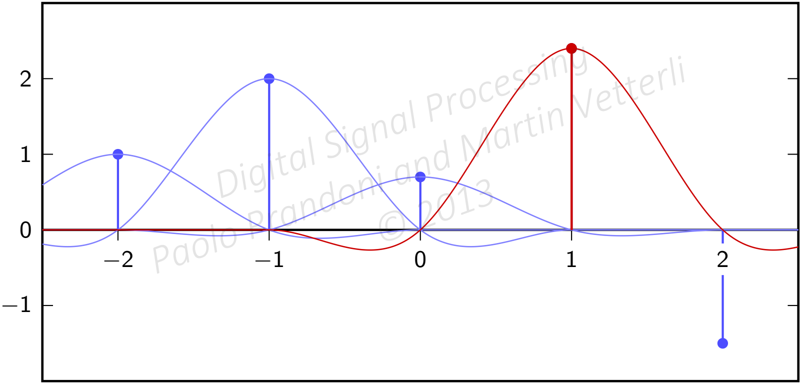

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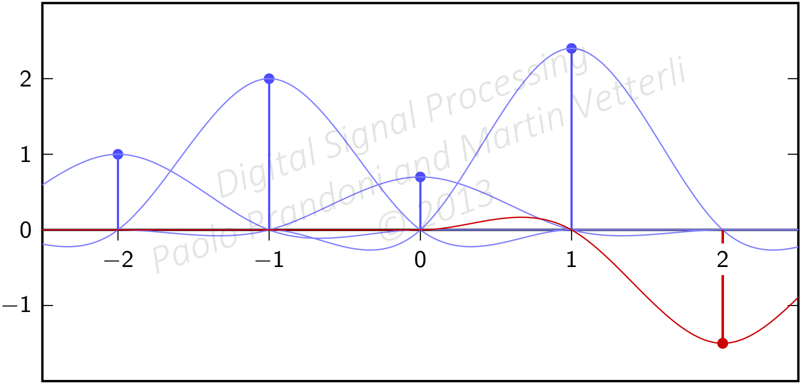

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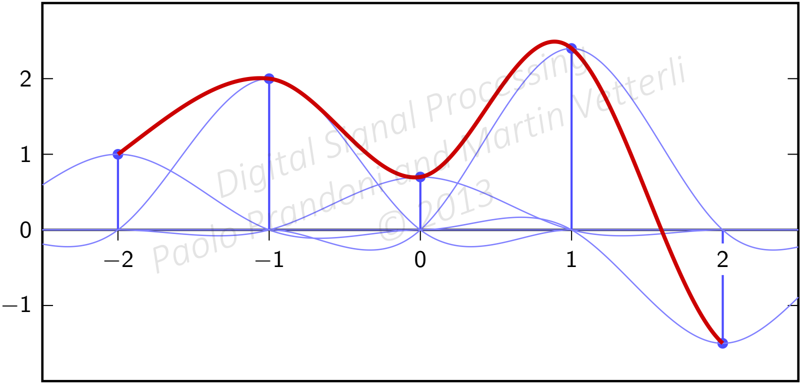

Local interpolation schemes

$$x(t) = \sum_{n=-N}^{N} x[n]\, i_c(t-n)$$

Interpolator's requirements:

- $i_c(0) = 1$
- $i_c(t) = 0$ for t a nonzero integer.

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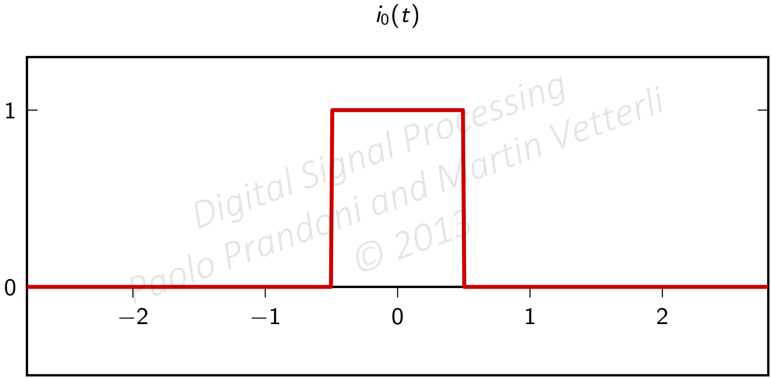

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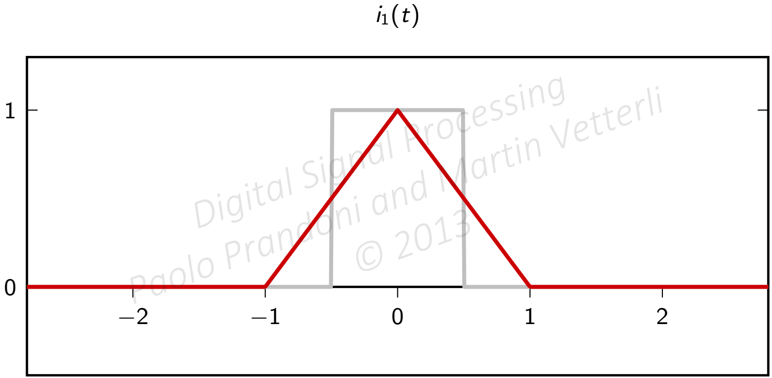

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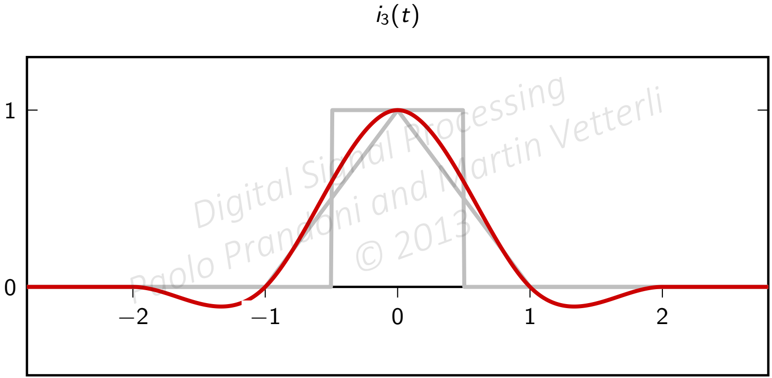

Local interpolation

key property:

- same interpolating function independently of N

drawback:

- lack of smoothness

A remarkable result:

$$\lim_{N\to\infty} L_n^{(N)}(t) = \mathrm{sinc}\,(t-n)$$

in the limit, local and global interpolation are the same!

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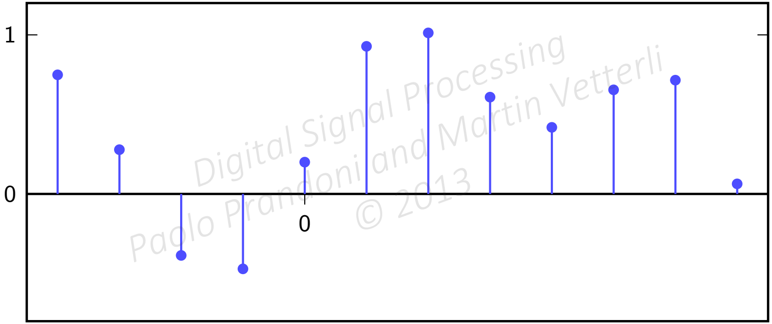
Sinc interpolation formula

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \, \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

Sinc interpolation



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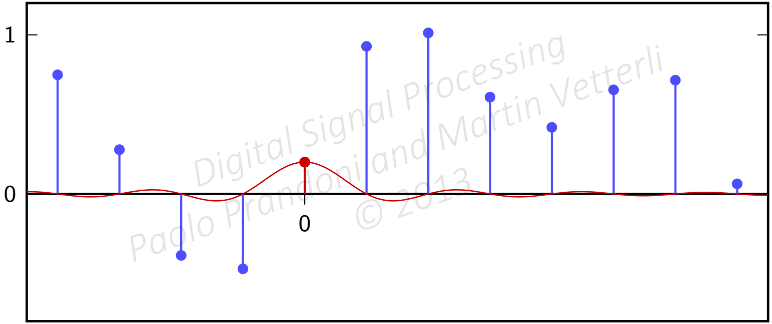

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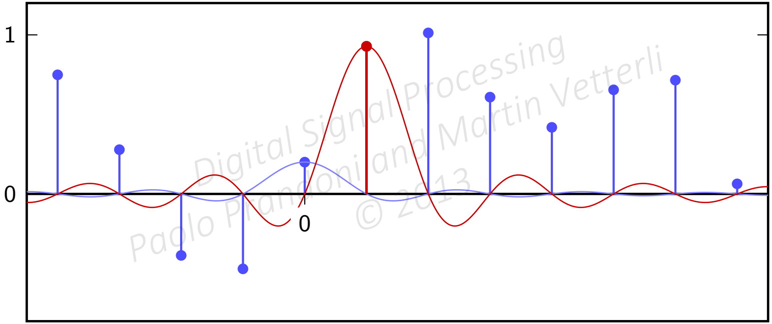

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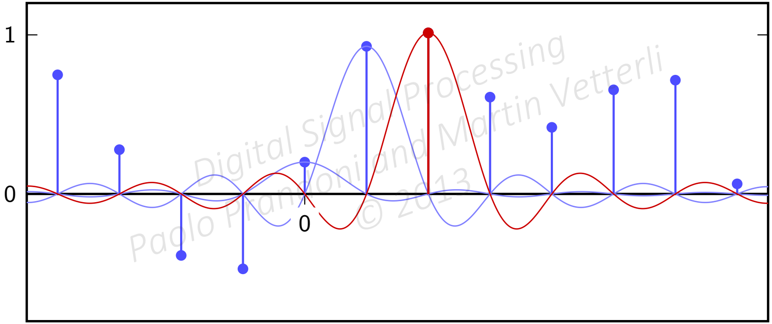

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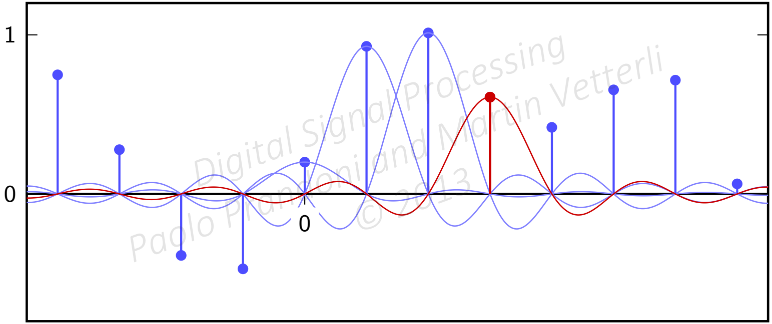

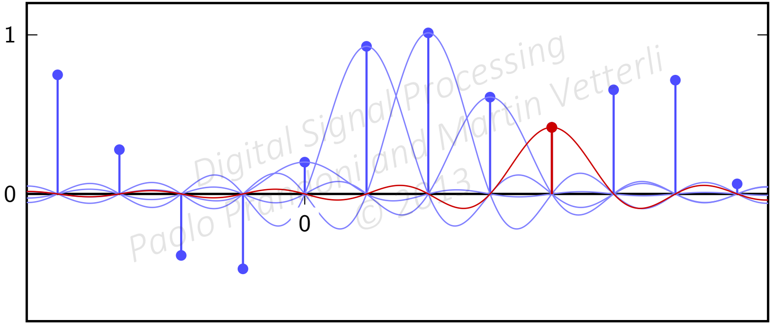
1
0
0
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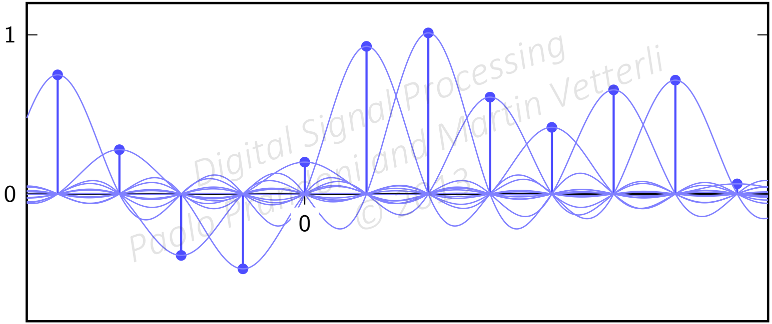
Sinc interpolation



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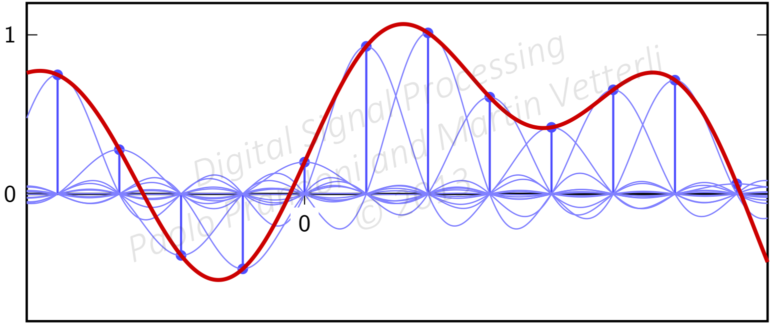

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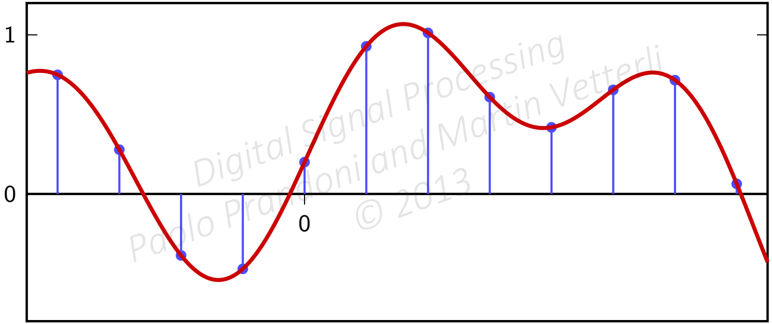

"Proof" that $L_n^{(N)}(t) \to \mathrm{sinc}\,(t - n)$

- real proof is rather technical (see the book)
- intuition: $\mathrm{sinc}(t-n)$ and $L_n^{(\infty)}(t)$ share an infinite number of zeros:

$$\mathrm{sinc}(m-n) = \delta[m-n] \qquad m, n \in \mathbb{Z}$$

$$L_n^{(N)}(m) = \delta[m-n] \qquad m, n \in \mathbb{Z}, \quad -N \leq n, m \leq N$$

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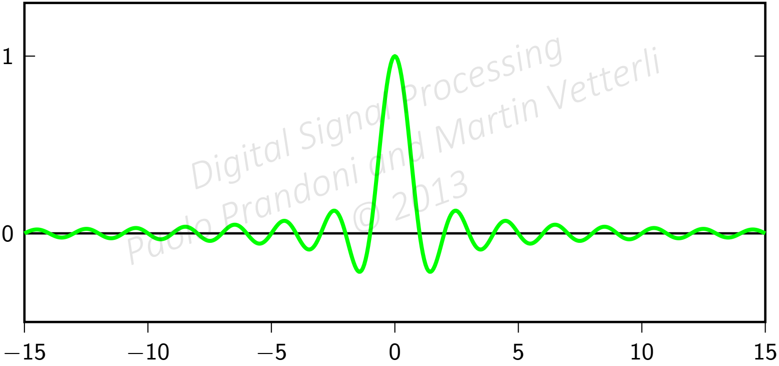
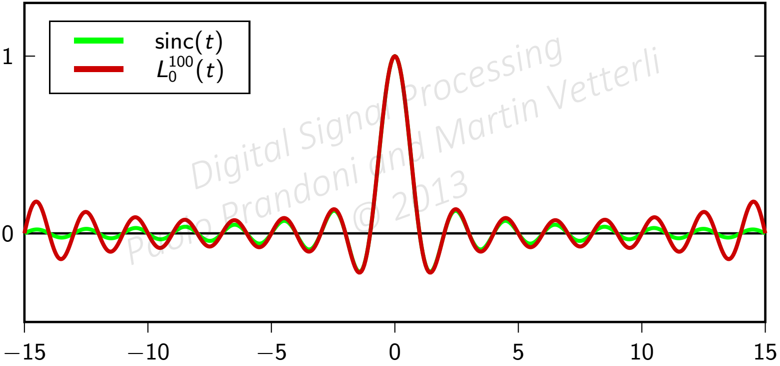
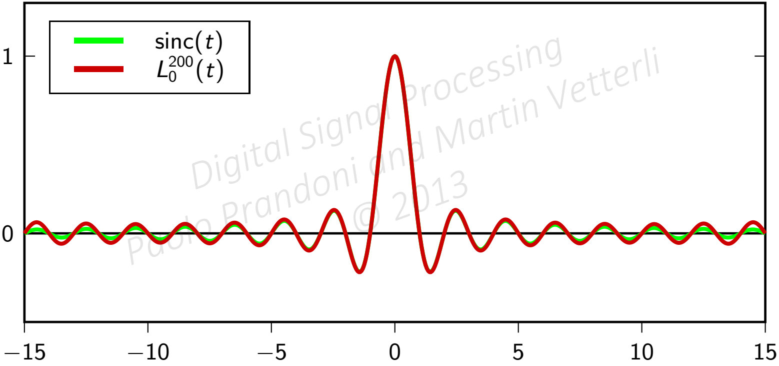
Graphically

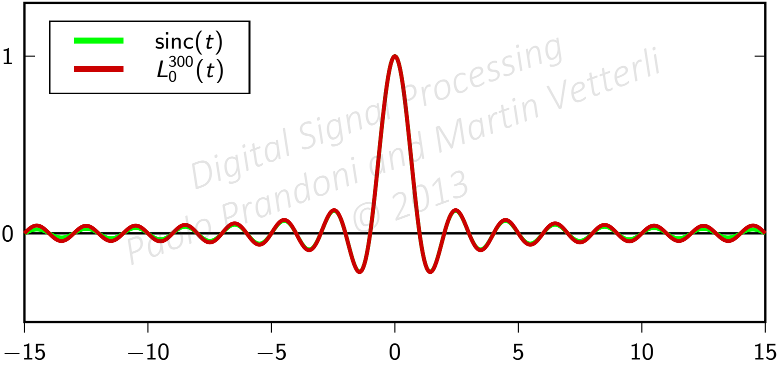


Graphically



Graphically




sinc(t)
$L_0^{300}(t)$
1
0
−15
−10
−5
0
5
10
15

END OF MODULE 6.2

Digital Signal Processing

Module 6.3: The space of bandlimited signals

Overview:

- Spectrum of interpolated signals
- Space of bandlimited functions
- Sinc sampling
- The sampling theorem

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the ingredients:

- discrete-time signal $x[n]$, $n \in \mathbb{Z}$ (with DTFT $X(e^{j\omega})$)
- interpolation interval T_s
- the sinc function

the result:

- a smooth, continuous-time signal $x(t)$, $t \in \mathbb{R}$

what does the spectrum of $x(t)$ look like?

Sinc interpolation

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Key facts about the sinc

$$\varphi(t) = \mathrm{sinc}\left(\frac{t}{T_s}\right) \quad \longleftrightarrow \quad \Phi(j\Omega) = \frac{\pi}{\Omega_N}\,\mathrm{rect}\left(\frac{\Omega}{2\Omega_N}\right)$$

$$T_s = \frac{\pi}{\Omega_N} \qquad\qquad \Omega_N = \frac{\pi}{T_s}$$

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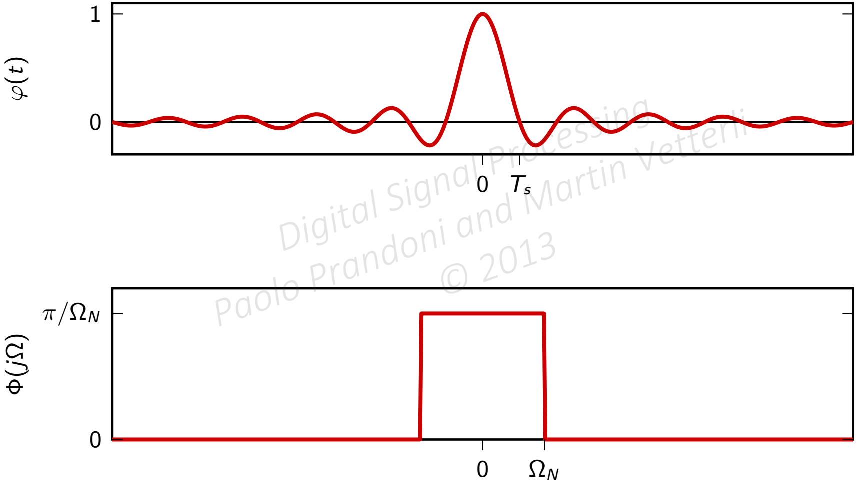

Sinc interpolation

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

Spectral representation (I)

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right) e^{-j\Omega t} dt$$

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&= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \operatorname{sinc}\left(\frac{t-nT_s}{T_s}\right) e^{-j\Omega t} dt \\
&= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{\pi}{\Omega_N}\right) \operatorname{rect}\left(\frac{\Omega}{2\Omega_N}\right) e^{-jnT_s\Omega}
\end{aligned}$$

Spectral representation (II)

$$X(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{\pi}{\Omega_N}\right) \mathrm{rect}\left(\frac{\Omega}{2\Omega_N}\right) e^{-jnT_s\Omega}$$

$$= \left(\frac{\pi}{\Omega_N}\right) \mathrm{rect}\left(\frac{\Omega}{2\Omega_N}\right) \sum_{n=-\infty}^{\infty} x[n] e^{-j(\pi/\Omega_N)\Omega n}$$

$$= \begin{cases} (\pi/\Omega_N) X(e^{j\pi(\Omega/\Omega_N)}) & \text{for } |\Omega| \leq \Omega_N \\ 0 & \text{otherwise} \end{cases}$$

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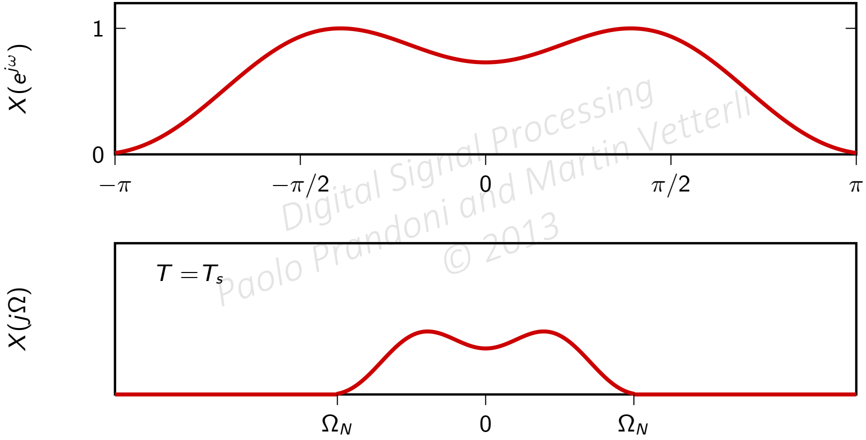
Spectrum of interpolated signals



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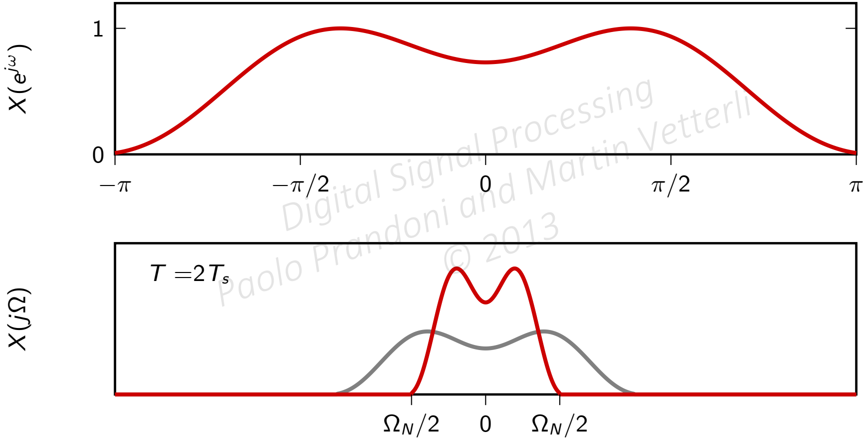

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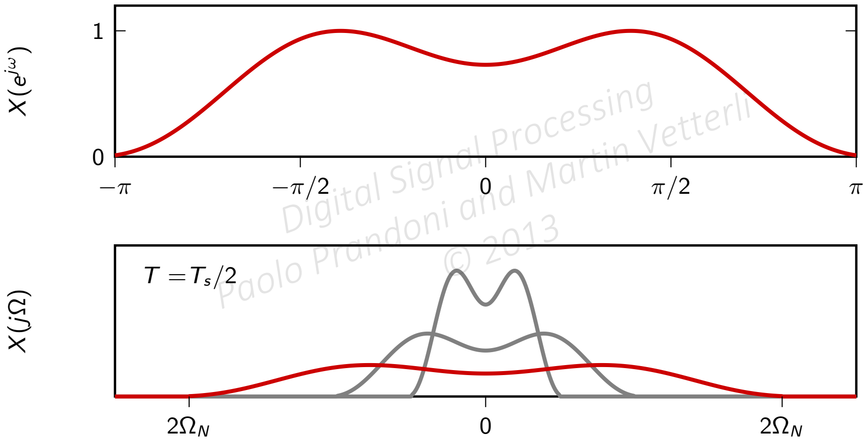

Spectrum of interpolated signals

pick interpolation period T_s:

- $X(j\Omega)$ is Ω_N-bandlimited, with $\Omega_N = \pi / T_s$
- fast interpolation (T_s small) $\rightarrow$ wider spectrum
- slow interpolation (T_s large) $\rightarrow$ narrower spectrum
- (for those who remember...) it's like changing the speed of a record player

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$$x[n] \in \ell_2(\mathbb{Z}) \xrightarrow{\quad T_s \quad} \begin{array}{c} x(t) \in L_2(\mathbb{R}) \\ \Omega_N\text{-BL} \end{array}$$

Space of bandlimited functions

$$x[n] \in \ell_2(\mathbb{Z}) \quad \overset{T_s}{\underset{?}{\longleftrightarrow}} \quad x(t) \in L_2(\mathbb{R})$$

$$\Omega_N\text{-BL}$$

Let's lighten the notation

for a while we will proceed with

- $T_s = 1$
- $\Omega_N = \pi$

(derivations in the general case are in the book)

claims:

- the space of π-bandlimited functions is a Hilbert space
- the functions $\varphi^{(n)}(t) = \mathrm{sinc}(t - n)$, with $n \in \mathbb{Z}$, form a basis for the space
- if $x(t)$ is π-BL, the sequence $x[n] = x(n)$, with $n \in \mathbb{Z}$, is a sufficient representation (i.e. we can reconstruct $x(t)$ from $x[n]$)

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- inner product is standard inner product in $L_2(\mathbb{R})$
- completeness... that's more delicate

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- completeness... that's more delicate

The space of π-BL functions

recap:

- inner product:

$$\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x^*(t)y(t)dt$$

- convolution:

$$(x * y)(t) = \langle x^*(\tau), y(t-\tau)\rangle$$

A basis for the π-BL space

$$\varphi^{(n)}(t) = \mathrm{sinc}(t-n), \qquad n \in \mathbb{Z}$$

$$\begin{aligned}
\langle \varphi^{(n)}(t), \varphi^{(m)}(t) \rangle &= \langle \varphi^{(0)}(t-n), \varphi^{(0)}(t-m) \rangle \\
&= \langle \varphi^{(0)}(t-n), \varphi^{(0)}(m-t) \rangle \\
&= \int_{-\infty}^{\infty} \mathrm{sinc}(t-n)\,\mathrm{sinc}(m-t)\,dt \\
&= \int_{-\infty}^{\infty} \mathrm{sinc}(\tau)\,\mathrm{sinc}((m-n)-\tau)\,d\tau \\
&= (\mathrm{sinc} * \mathrm{sinc})(m-n)
\end{aligned}$$

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A basis for the π-BL space

now use the convolution theorem knowing that:

$$\text{FT}\left\{\text{sinc}(t)\right\} = \text{rect}\left(\frac{\Omega}{2\pi}\right)$$

$$\begin{aligned}(\text{sinc} * \text{sinc})(m-n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\text{rect}\left(\frac{\Omega}{2\pi}\right)\right]^2 e^{j\Omega(m-n)}d\Omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(m-n)}d\Omega \\ &= \begin{cases} 1 & \text{for } m = n \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

A basis for the π-BL space

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\end{aligned}$$

Sampling as a basis expansion

for any $x(t) \in \pi$-BL:

$$\langle \varphi^{(n)}(t), x(t) \rangle = \langle \mathrm{sinc}(t-n), x(t) \rangle = \langle \mathrm{sinc}(n-t), x(t) \rangle$$

$$= (\mathrm{sinc} * x)(n)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{\Omega}{2\pi}\right) X(j\Omega) e^{j\Omega n} d\Omega$$

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Sampling as a basis expansion, π-BL

Analysis formula:

$$x[n] = \langle \mathrm{sinc}(t-n), x(t) \rangle$$

Synthesis formula:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{sinc}(t-n)$$

Sampling as a basis expansion, Ω_N-BL

Analysis formula:

$$x[n] = \left\langle \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right), x(t) \right\rangle = T_s\, x(nT_s)$$

Synthesis formula:

$$x(t) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} x[n]\, \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

The sampling theorem

- the space of Ω_N-bandlimited functions is a Hilbert space
- set $T_s = \pi/\Omega_N$
- the functions $\varphi^{(n)}(t) = \text{sinc}((t - nT_s)/T_s)$ form a basis for the space
- for any $x(t) \in \Omega_N$-BL the coefficients in the sinc basis are the (scaled) samples $T_s\, x(nT_s)$

for any $x(t) \in \Omega_N$-BL, a sufficient representation is the sequence $x[n] = x(nT_s)$

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The sampling theorem, corollary

- Ω_N-BL $\subseteq$ Ω-BL for any $\Omega \geq \Omega_N$

for any $x(t) \in \Omega_N$-BL, a sufficient representation is the sequence

$$x[n] = x(nT_s) \text{ for any } T_s \leq \pi/\Omega_N$$

The sampling theorem, corollary

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for any $x(t) \in \Omega_N$-BL, a sufficient representation is the sequence
$x[n] = x(nT_s)$ for any $T_s \leq \pi/\Omega_N$

any signal $x(t)$ bandlimited to F_N Hz can be sampled with no loss of information using a sampling frequency $F_s \geq 2F_N$ (i.e. a sampling period $T_s \leq 1/2F_N$)

END OF MODULE 6.3

Digital Signal Processing

Module 6.4: Sampling and Aliasing - Introduction

Overview:

- “Raw” sampling
- Sinusoidal aliasing

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- "Raw" sampling
- Sinusoidal aliasing

Sinc Sampling

$$x[n] = \left\langle \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right), x(t) \right\rangle$$

Sinc Sampling

$$x[n] = (\mathrm{sinc}_{T_s} * x)(nT_s)$$

Sinc Sampling

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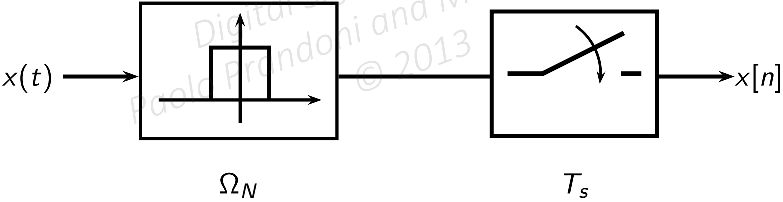
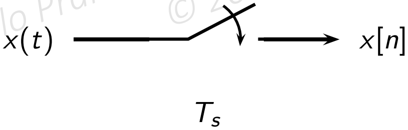


"Raw" Sampling

$$x[n] = x(nT_s)$$



Remember the wagonwheel effect?

The continuous-time complex exponential

$$x(t) = e^{j\Omega_0 t}$$

- always periodic, period $T = 2\pi/\Omega_0$
- all angular speeds are allowed
- $\mathrm{FT}\left\{e^{j\Omega_0 t}\right\} = 2\pi\delta(\Omega - \Omega_0)$
- bandlimited to Ω_0

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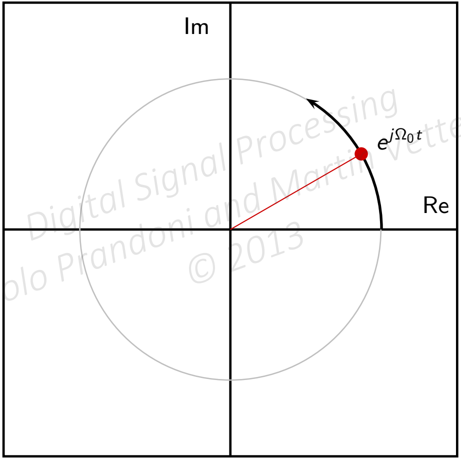

Raw samples of the continuous-time complex exponential

$$x[n] = e^{j\Omega_0 T_s n}$$

- raw samples are snapshots at regular intervals of the rotating point
- resulting digital frequency is $\omega_0 = \Omega_0 T_s$

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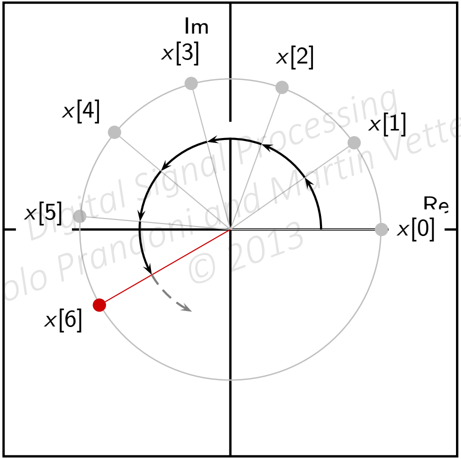
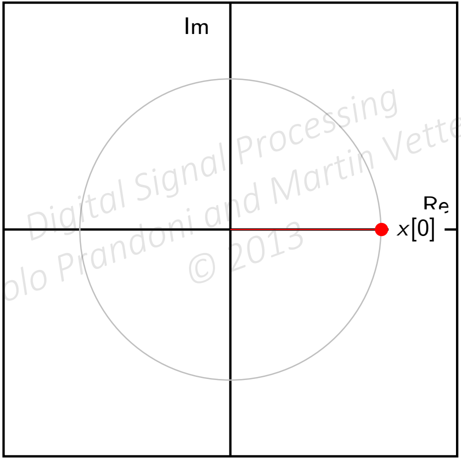
When $T_s < \pi/\Omega_0$, $\omega_0 < \pi \ldots$



When $\pi/\Omega_0 < T_s < 2\pi/\Omega_0$, $\pi < \omega_0 < 2\pi \ldots$



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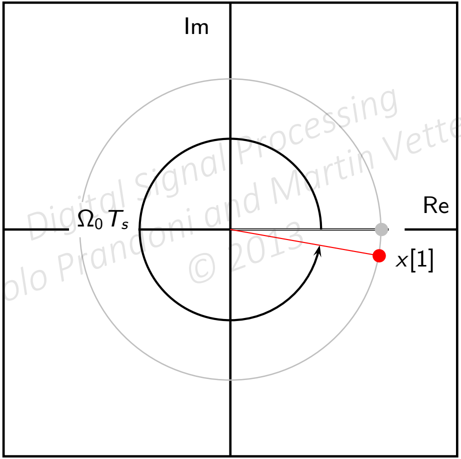

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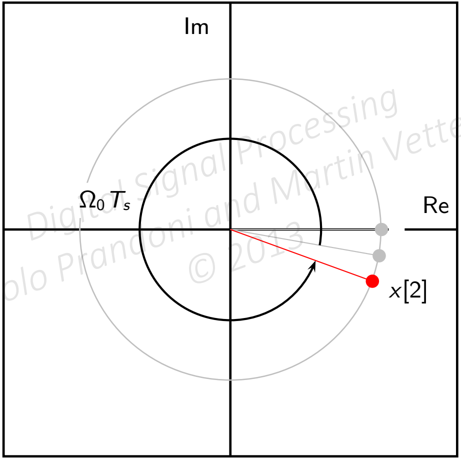

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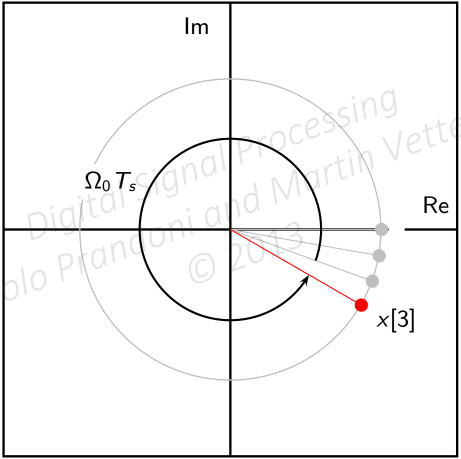

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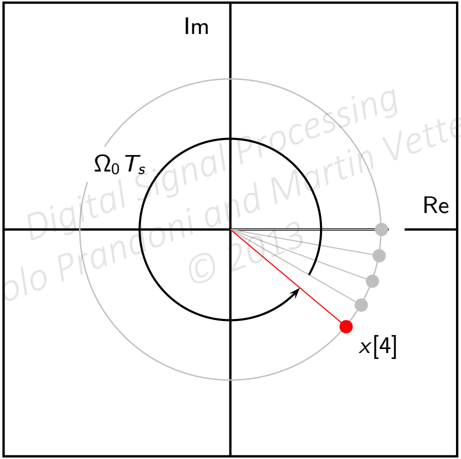

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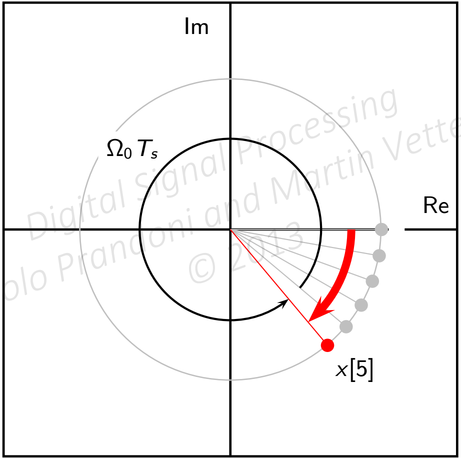
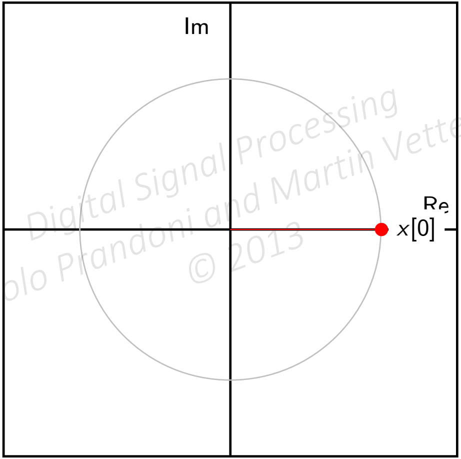


When $T_s > 2\pi/\Omega_0$, $\omega_0 > 2\pi \ldots$



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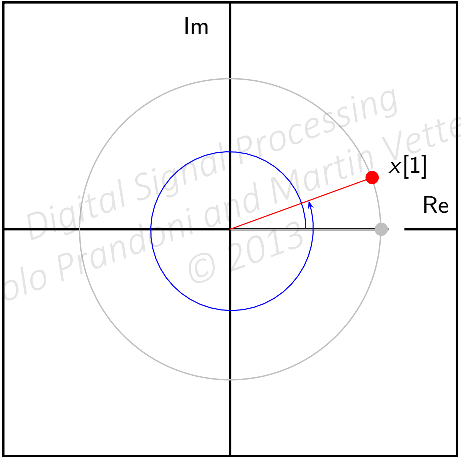

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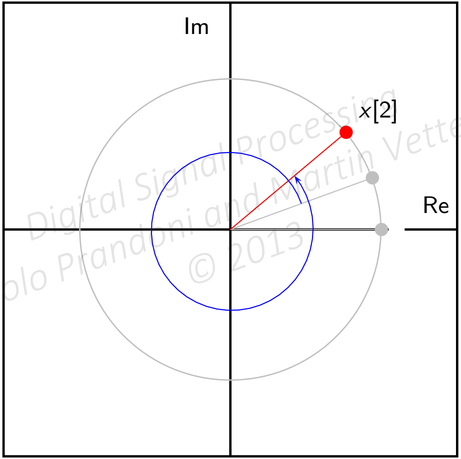

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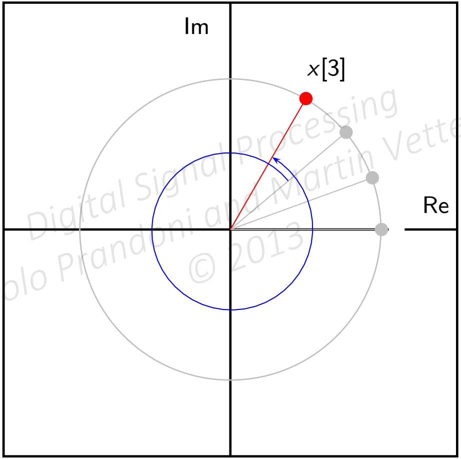

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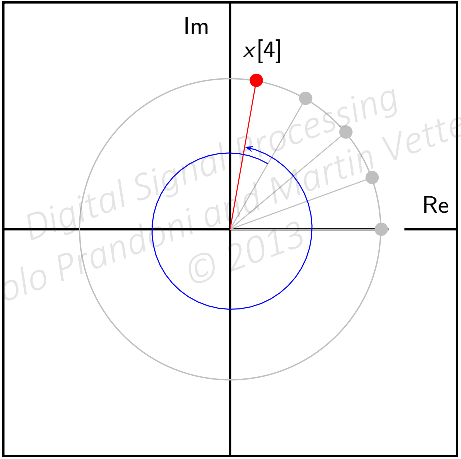
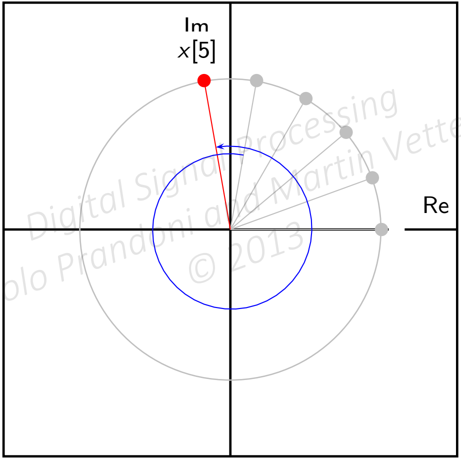
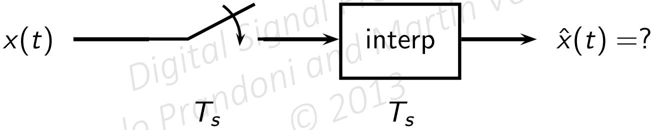




Im
$x[5]$
Re


$x(t)$
T_s
interp
T_s
$\hat{x}(t) = ?$

$$x(t) = e^{j\Omega_0 t}$$

sampling period	digital frequency	$\hat{x}(t)$
$T_s < \pi/\Omega_0$	$0 < \omega_0 < \pi$	$e^{j\Omega_0 t}$
$\pi/\Omega_0 < T_s < 2\pi/\Omega_0$	$\pi < \omega_0 < 2\pi$	$e^{j\Omega_1 t}, \quad \Omega_1 = \Omega_0 - 2\pi/T_s$
$T_s > 2\pi/\Omega_0$	$\omega_0 > 2\pi$	$e^{j\Omega_2 t}, \quad \Omega_2 = \Omega_0 \mod (2\pi/T_s)$

Again, with a simple sinusoid and using hertz

$$x(t) = \cos(2\pi F_0 t)$$

$$x[n] = x(nT_s) = \cos(\omega_0 n)$$

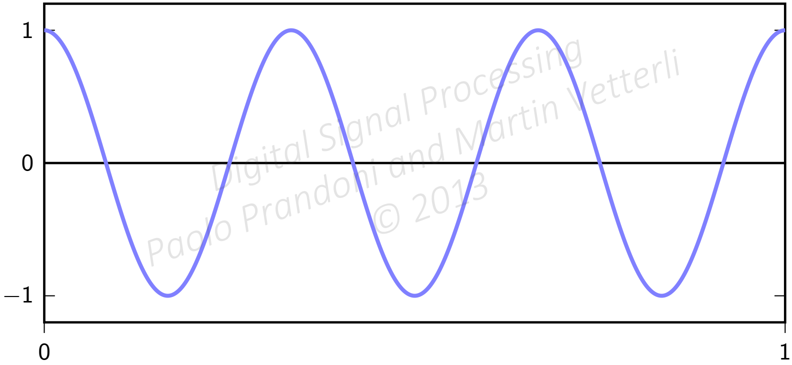
$$F_s = 1/T_s$$

$$\omega_0 = 2\pi(F_0/F_s)$$

sampling frequency	digital frequency	result
$F_s > 2F_0$	$0 < \omega_0 < \pi$	OK
$F_s = 2F_0$	$\omega_0 = \pi$	max digital frequency: $x[n] = (-1)^n$
$F_0 < F_s < 2F_0$	$\pi < \omega_0 < 2\pi$	negative frequency $\omega_0 - 2\pi$
$F_s < F_0$	$\omega_0 > 2\pi$	full aliasing: $\omega_0 \mod 2\pi$

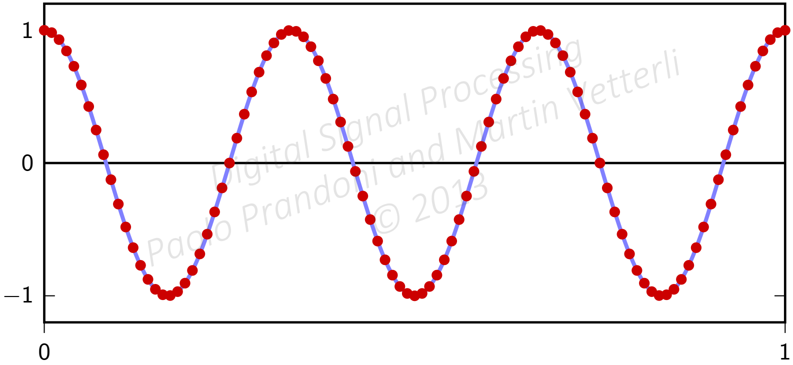
Aliasing: Sampling a Sinusoid

$$x(t) = \cos(6\pi t) \qquad (F_0 = 3\text{Hz})$$



Aliasing: Sampling a Sinusoid

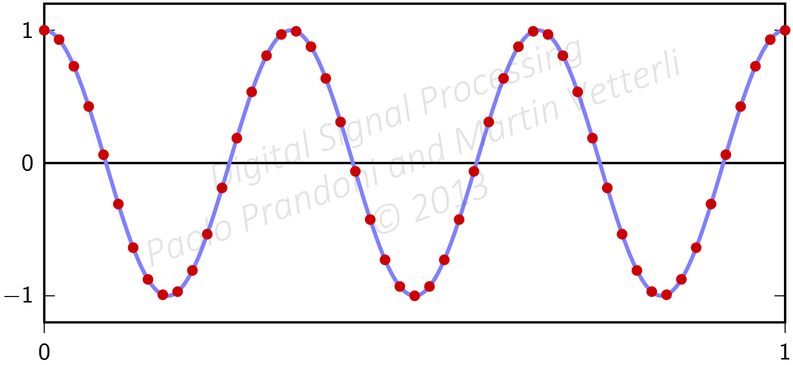
$$x(t) = \cos(6\pi t) \qquad (F_0 = 3\text{Hz})$$



$F_s = 100$Hz

Aliasing: Sampling a Sinusoid

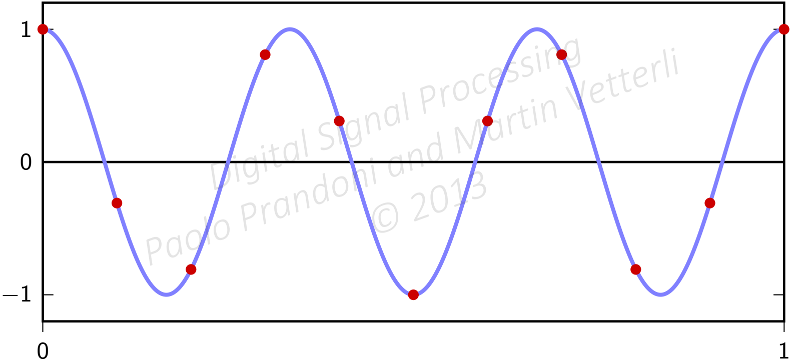
$$x(t) = \cos(6\pi t) \qquad (F_0 = 3\text{Hz})$$



$F_s = 50\text{Hz}$

Aliasing: Sampling a Sinusoid

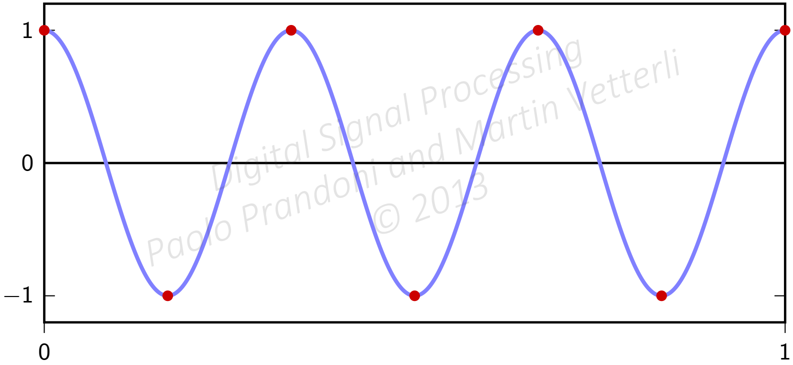
$$x(t) = \cos(6\pi t) \qquad (F_0 = 3\text{Hz})$$



$F_s = 10\text{Hz}$

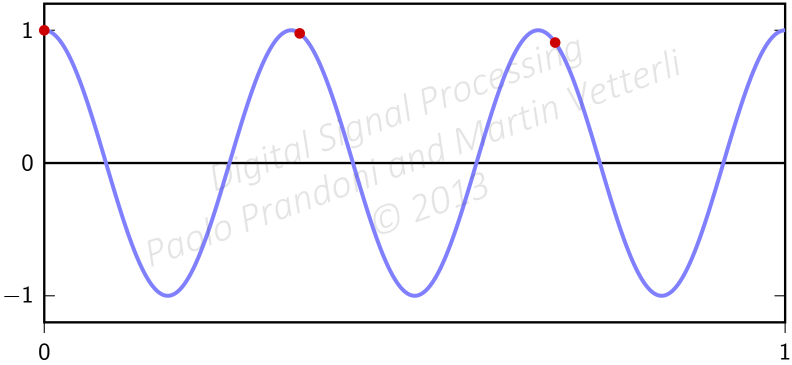
Aliasing: Sampling a Sinusoid

$$x(t) = \cos(6\pi t) \qquad (F_0 = 3\text{Hz})$$



$F_s = 6\text{Hz}$

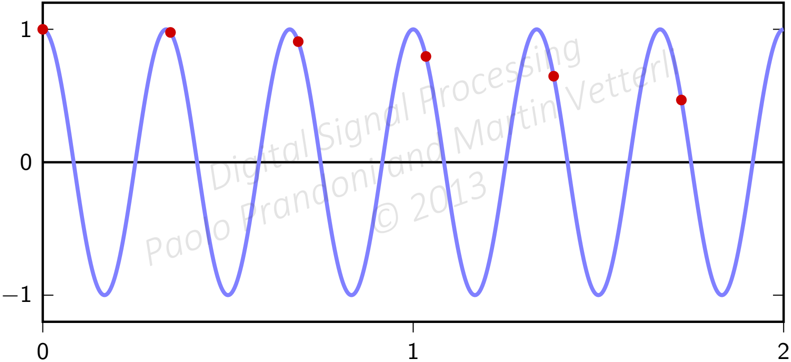
$$x(t) = \cos(6\pi t) \qquad (F_0 = 3\text{Hz})$$



$F_s = 2.9\text{Hz}$

Aliasing: Sampling a Sinusoid

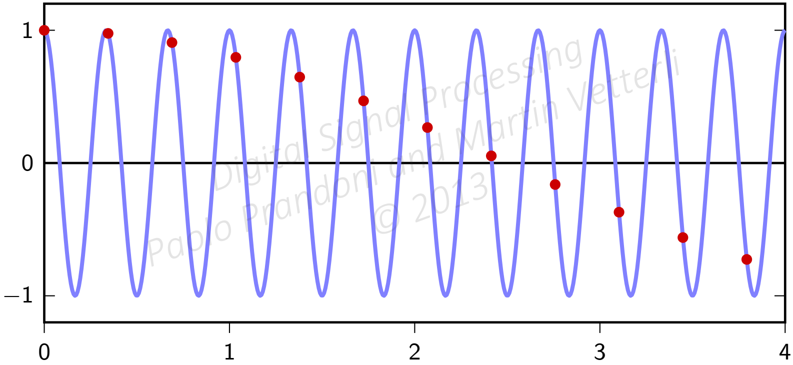
$$x(t) = \cos(6\pi t) \qquad (F_0 = 3\text{Hz})$$



$F_s = 2.9\text{Hz}$

Aliasing: Sampling a Sinusoid

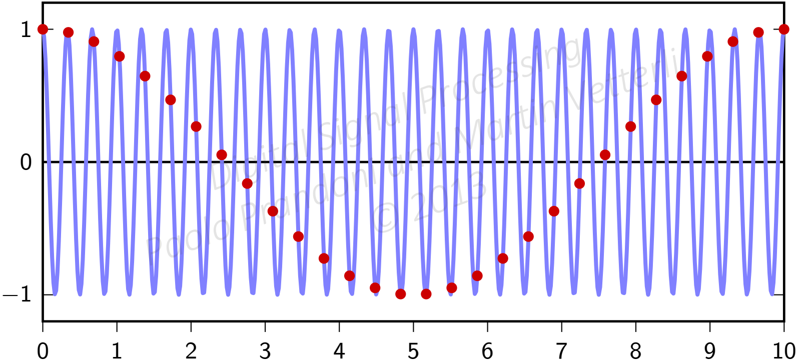
$$x(t) = \cos(6\pi t) \qquad (F_0 = 3\text{Hz})$$



$F_s = 2.9\text{Hz}$

Aliasing: Sampling a Sinusoid

$$x(t) = \cos(6\pi t) \qquad (F_0 = 3\text{Hz})$$



$F_s = 2.9\text{Hz}$

END OF MODULE 6.4

Digital Signal Processing

Module 6.5: Sampling and Aliasing

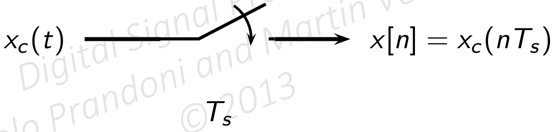
Overview:

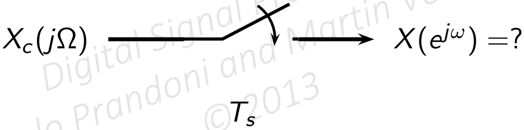
- Aliasing for arbitrary spectra
- Examples

Overview:

- Aliasing for arbitrary spectra
- Examples

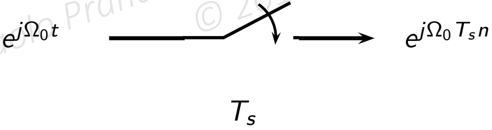
Raw-sampling an arbitrary signal




$X_c(j\Omega)$
T_s
$X(e^{j\omega}) = ?$

Key idea

- pick T_s (and set $\Omega_N = \pi / T_s$)
- pick $\Omega_0 < \Omega_N$



Key idea

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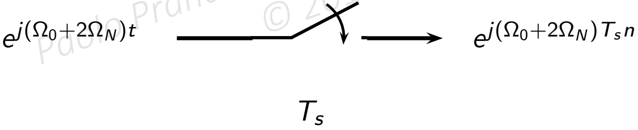


Key idea

- pick T_s (and set $\Omega_N = \pi / T_s$)
- pick $\Omega_0 < \Omega_N$

$e^{j(\Omega_0 + 2\Omega_N)t}$

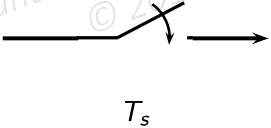
$e^{j(\Omega_0 T_s n + 2\Omega_N T_s n)}$

T_s

Key idea

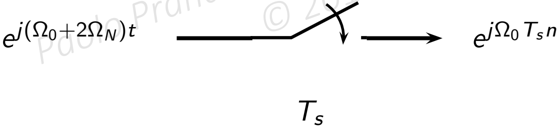
- pick T_s (and set $\Omega_N = \pi/T_s$)
- pick $\Omega_0 < \Omega_N$

$e^{j(\Omega_0 + 2\Omega_N)t}$



$e^{j(\Omega_0 T_s n + 2\pi n)}$

- pick T_s (and set $\Omega_N = \pi / T_s$)
- pick $\Omega_0 < \Omega_N$



Key idea

- pick T_s (and set $\Omega_N = \pi / T_s$)
- pick $\Omega_0 < \Omega_N$

$$A e^{j\Omega_0 t} + B e^{j(\Omega_0 + 2\Omega_N)t}$$

$$(A + B) e^{j\Omega_0 T_s n}$$

T_s

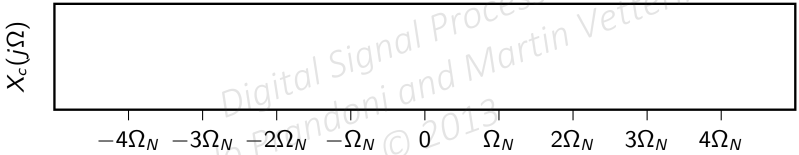
start with the inverse Fourier Transform

$$x[n] = x_c(nT_s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\Omega)e^{j\Omega\, nT_s}\, d\Omega$$

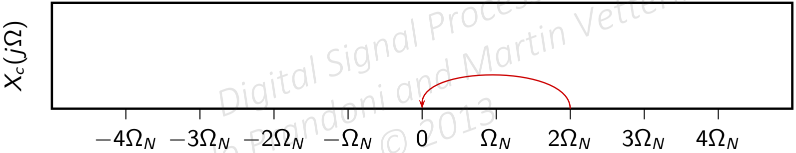
frequencies $2\Omega_N$ apart will be aliased, so split the integration interval

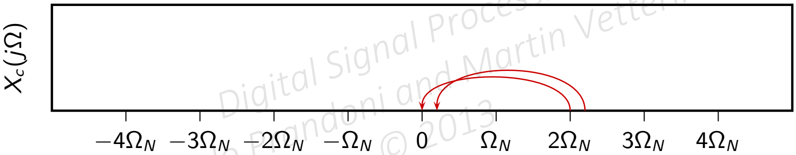
$$x[n] = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{(2k-1)\Omega_N}^{(2k+1)\Omega_N} X_c(j\Omega) e^{j\Omega\, nT_s} d\Omega$$

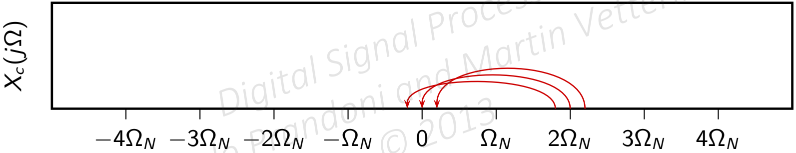
Spectrum of raw-sampled signals

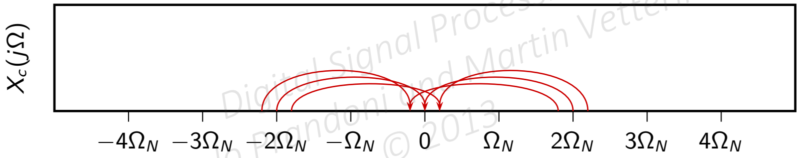


Spectrum of raw-sampled signals

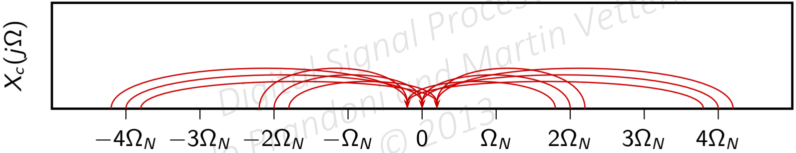



$X_c(j\Omega)$
$-4\Omega_N$ $-3\Omega_N$ $-2\Omega_N$ $-\Omega_N$ 0 Ω_N $2\Omega_N$ $3\Omega_N$ $4\Omega_N$


$X_c(j\Omega)$
$-4\Omega_N$ $-3\Omega_N$ $-2\Omega_N$ $-\Omega_N$ 0 Ω_N $2\Omega_N$ $3\Omega_N$ $4\Omega_N$


$X_c(j\Omega)$
$-4\Omega_N$
$-3\Omega_N$
$-2\Omega_N$
$-\Omega_N$
0
Ω_N
$2\Omega_N$
$3\Omega_N$
$4\Omega_N$

Spectrum of raw-sampled signals



Spectrum of raw-sampled signals

with a change of variable and using $e^{j(\Omega+2k\Omega_N)T_s n} = e^{j\Omega T_s n}$:

$$x[n] = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\Omega_N}^{\Omega_N} X_c(j(\Omega - 2k\Omega_N))e^{j\Omega\, nT_s}\,d\Omega$$

$$= \frac{1}{2\pi}\int_{-\Omega_N}^{\Omega_N}\left[\sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2k\Omega_N))\right] e^{j\Omega\, nT_s}\,d\Omega$$

Spectrum of raw-sampled signals

periodization of the spectrum; define:

$$\tilde{X}_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2k\Omega_N))$$

so that:

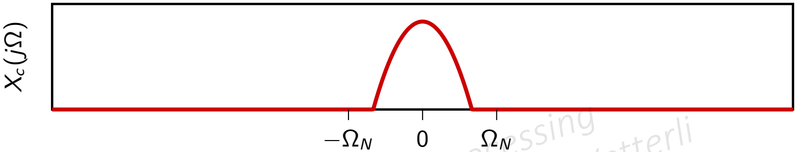
$$x[n] = \frac{1}{2\pi} \int_{-\Omega_N}^{\Omega_N} \tilde{X}_c(j\Omega) e^{j\Omega\, nT_s} d\Omega$$

set $\omega = \Omega T_s$:

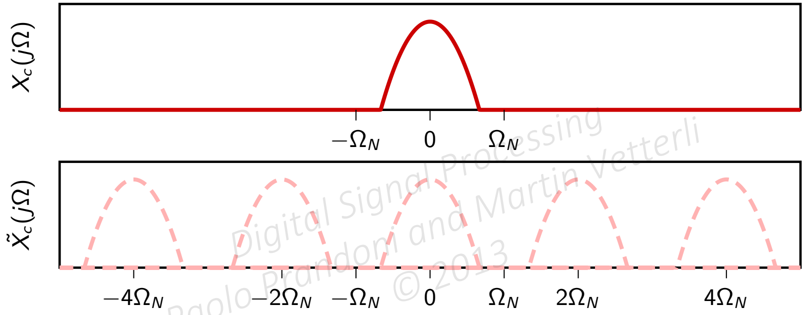
$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{T_s}\tilde{X}_c\left(j\frac{\omega}{T_s}\right)e^{j\omega n}d\omega \\
&= \mathrm{IDTFT}\left\{\frac{1}{T_s}\tilde{X}_c\left(j\frac{\omega}{T_s}\right)\right\}
\end{aligned}
$$

$$
X(e^{j\omega}) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s}\right)
$$

Example: signal bandlimited to Ω_0 and $\Omega_N > \Omega_0$



Example: signal bandlimited to Ω_0 and $\Omega_N > \Omega_0$



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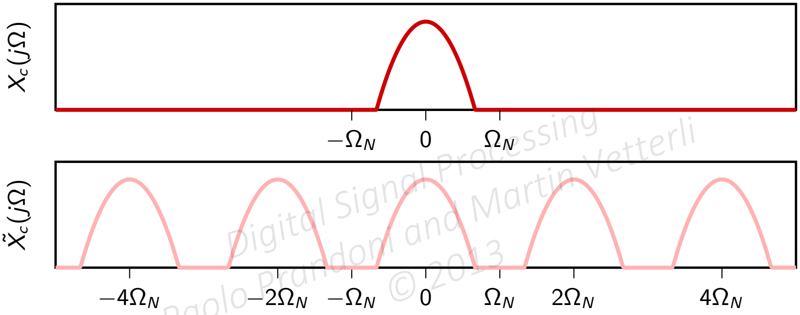
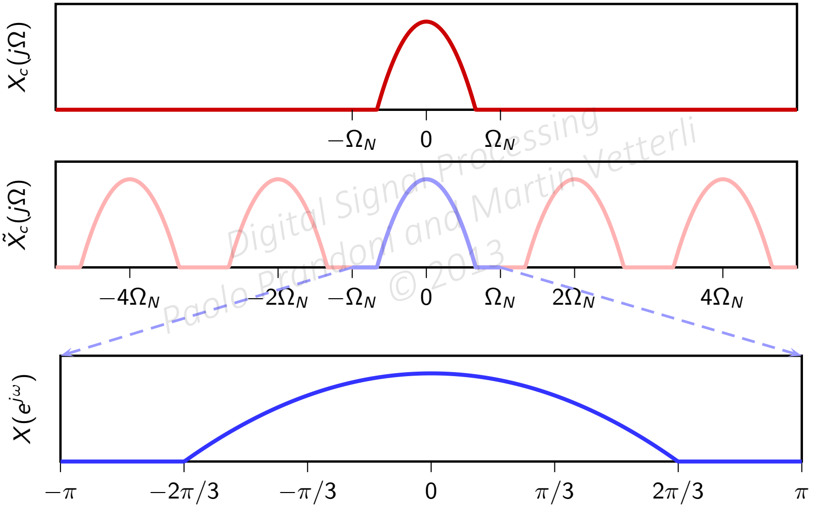
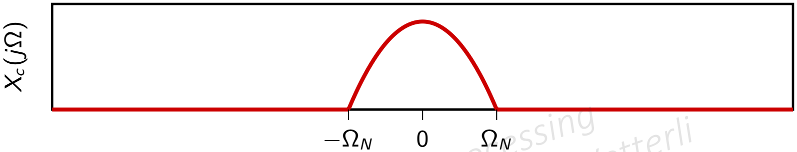


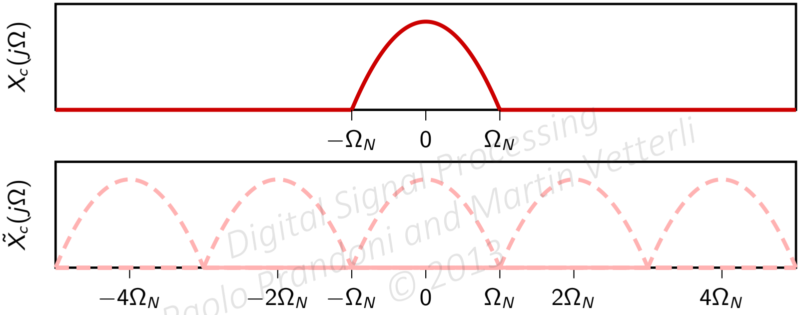
Example: signal bandlimited to Ω_0 and $\Omega_N > \Omega_0$



Example: signal bandlimited to Ω_0 and $\Omega_N = \Omega_0$



Example: signal bandlimited to Ω_0 and $\Omega_N = \Omega_0$



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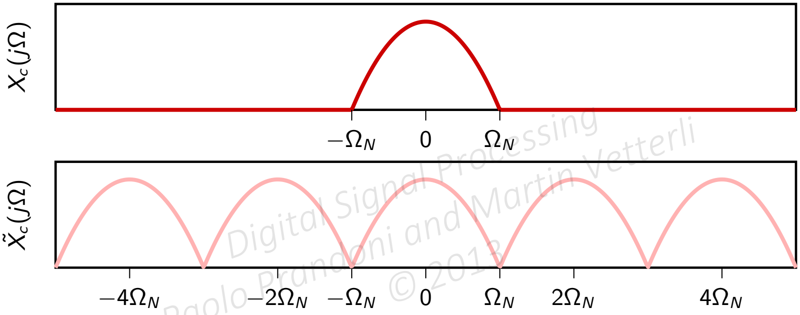

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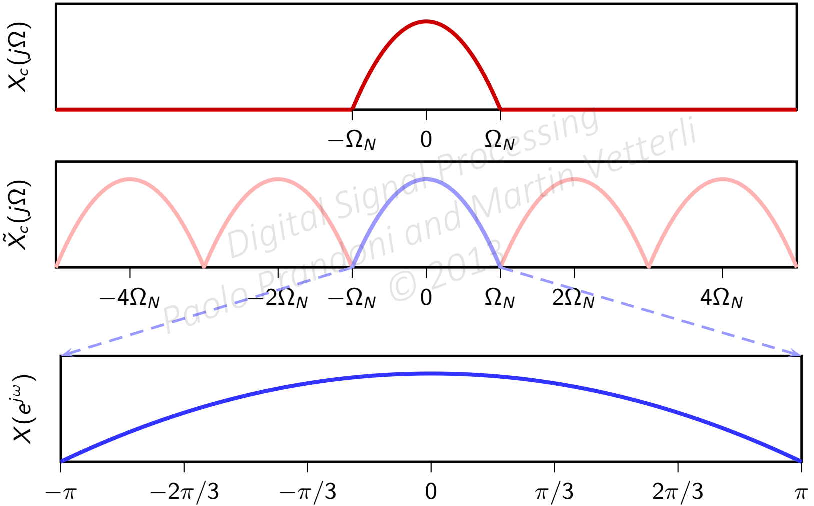
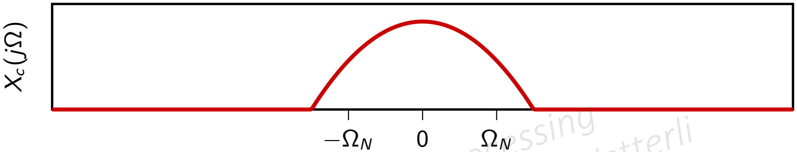
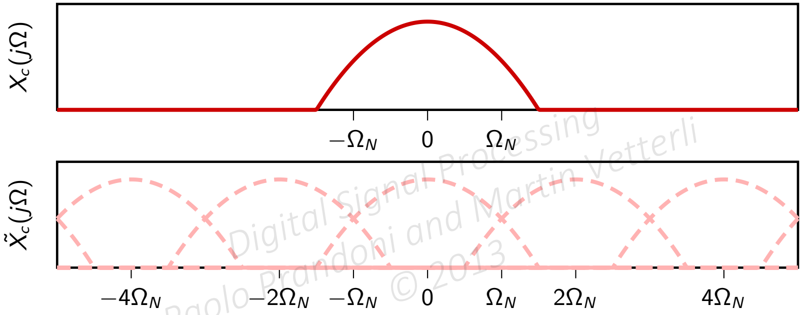


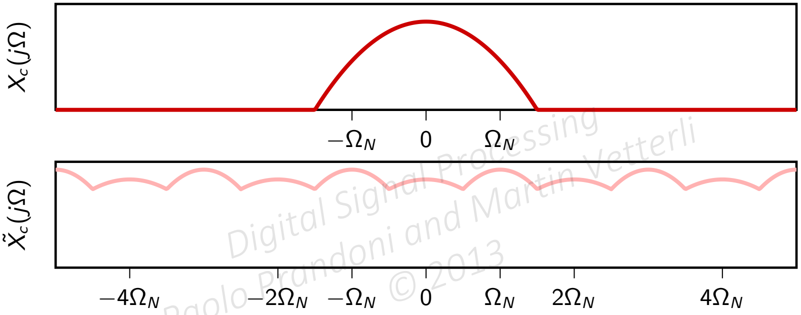
Example: signal bandlimited to Ω_0 and $\Omega_N < \Omega_0$



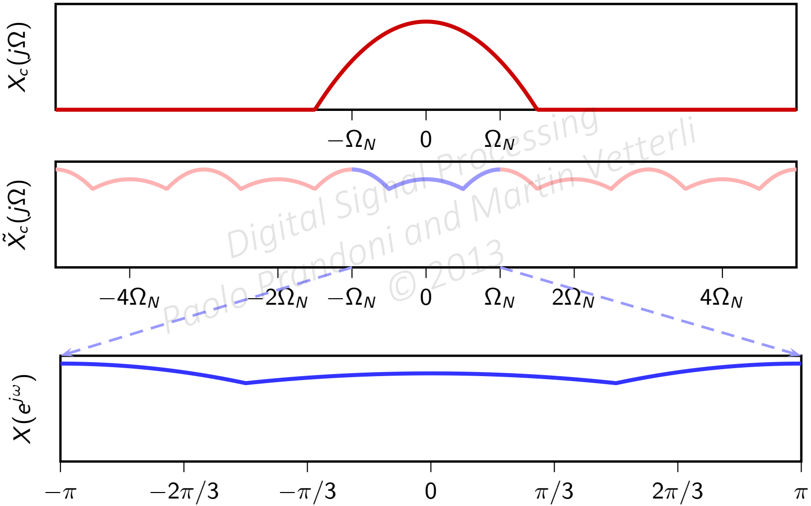
Example: signal bandlimited to Ω_0 and $\Omega_N < \Omega_0$



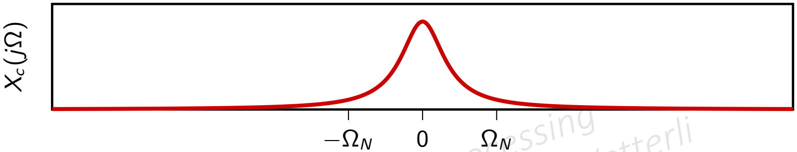
Example: signal bandlimited to Ω_0 and $\Omega_N < \Omega_0$



Example: signal bandlimited to Ω_0 and $\Omega_N < \Omega_0$



Example: non-bandlimited signal



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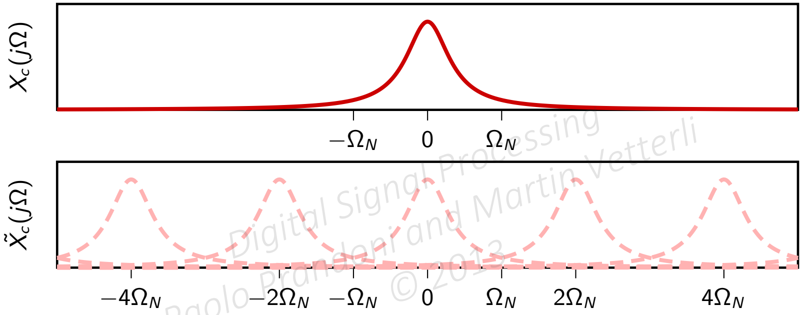

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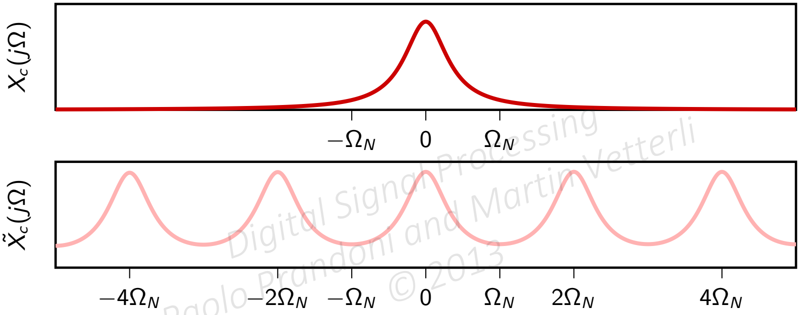

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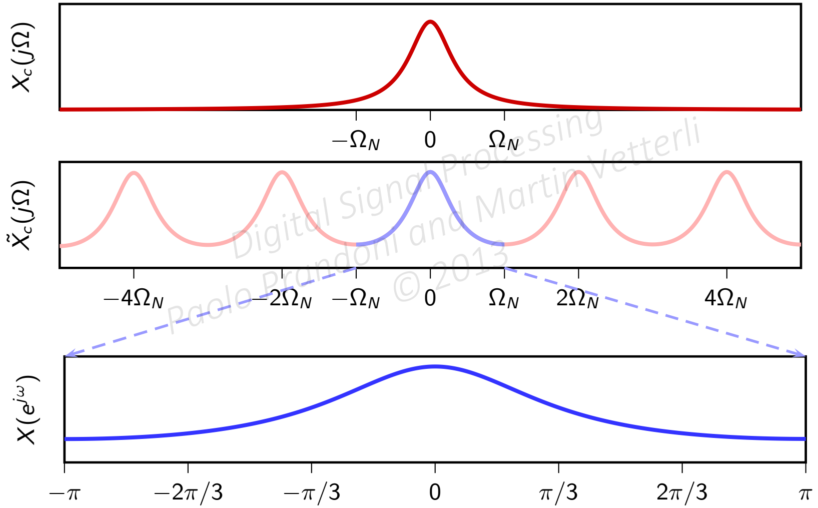

Sampling strategies

given a sampling period T_s

- if the signal is bandlimited to π/T_s or less, raw sampling is fine (i.e. equivalent to sinc sampling up to a scaling factor T_s)
- if the signal is not bandlimited, two choices:
  - bandlimit via a lowpass filter *in the continuous-time domain* before sampling (i.e. sinc sampling)
  - or, raw sample the signal and incur aliasing
- aliasing sounds horrible, so usually we choose to bandlimit in continuous time

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Sinc Sampling and Interpolation

$$\hat{x}[n] = \left\langle \text{sinc}\left(\frac{t-nT_s}{T_s}\right), x(t) \right\rangle \;=\; (\text{sinc}_{T_s} * x)(nT_s)$$

$$\hat{x}[n] = \left\langle \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right), x(t) \right\rangle = (\operatorname{sinc}_{T_s} * x)(nT_s)$$

$$\hat{x}(t) = \sum_n x[n] \operatorname{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

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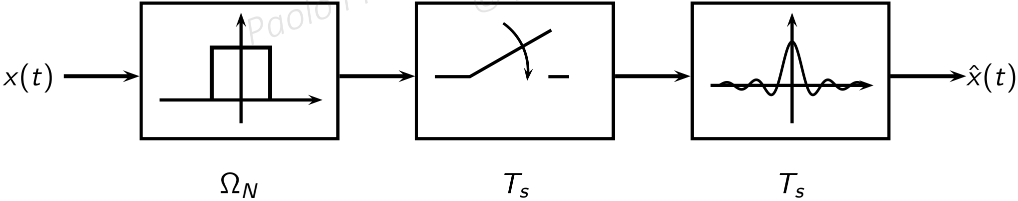
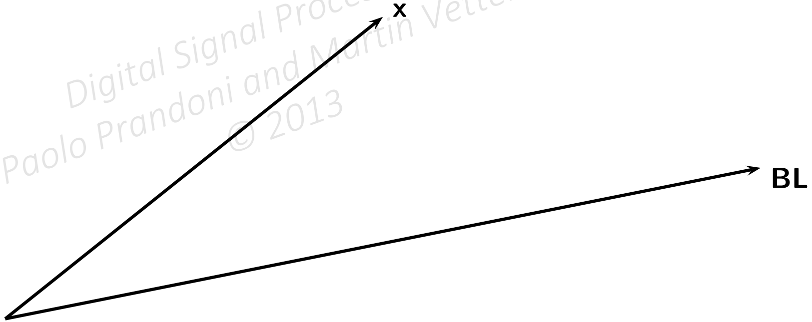


Least squares approximation with sinc sampling and interpolation



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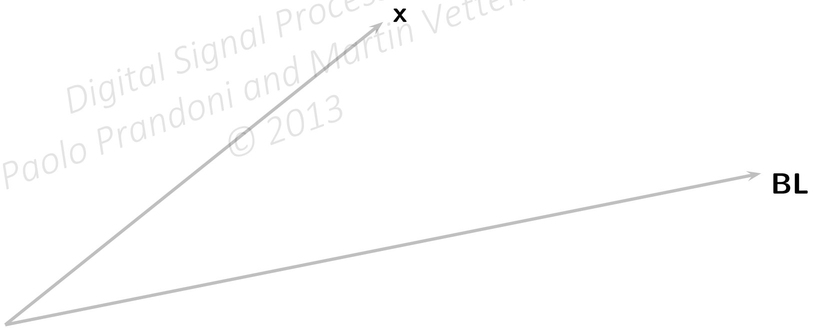
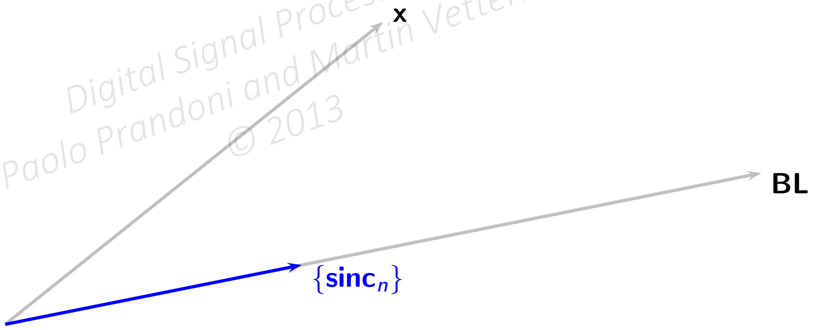
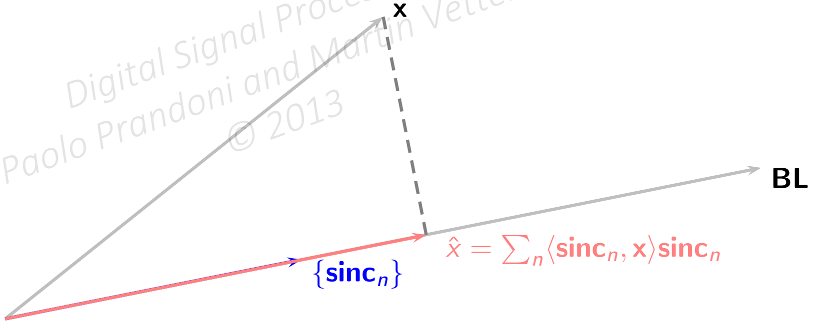
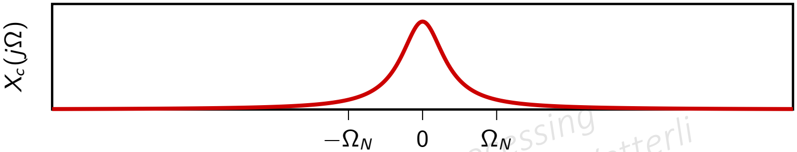


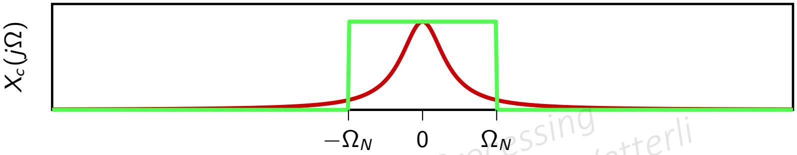

x
BL
$\{\mathbf{sinc}_n\}$

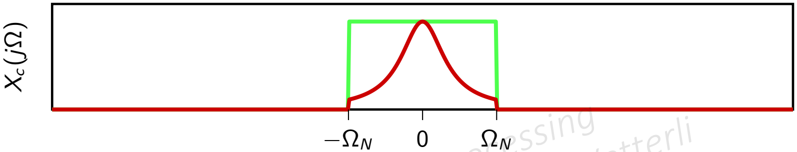
Least squares approximation with sinc sampling and interpolation

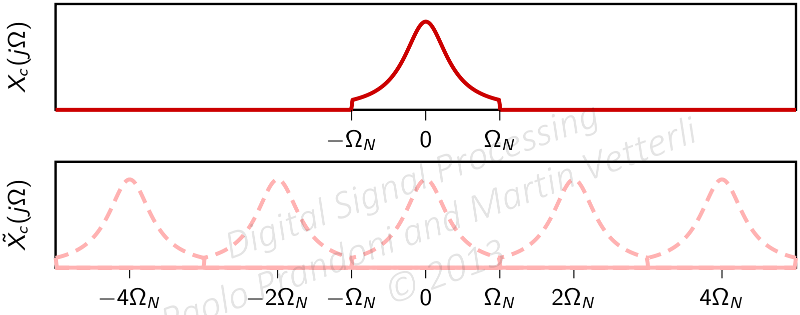



$X_c(j\Omega)$
$-\Omega_N$
0
Ω_N

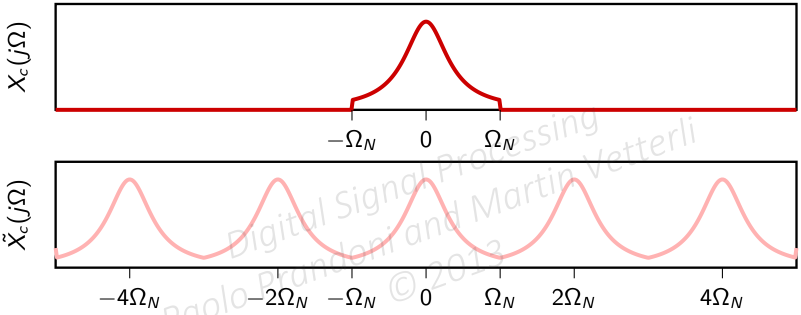
Least squares approximation with sinc sampling and interpolation

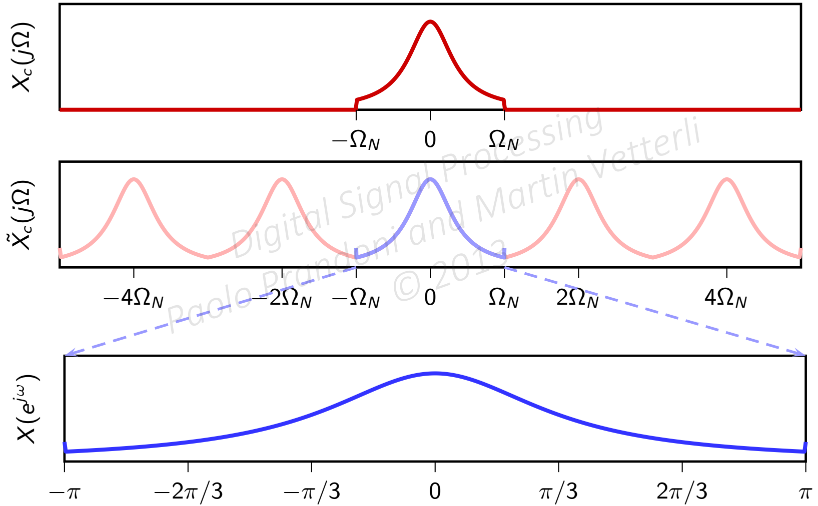



$X_c(j\Omega)$
$-\Omega_N$
0
Ω_N

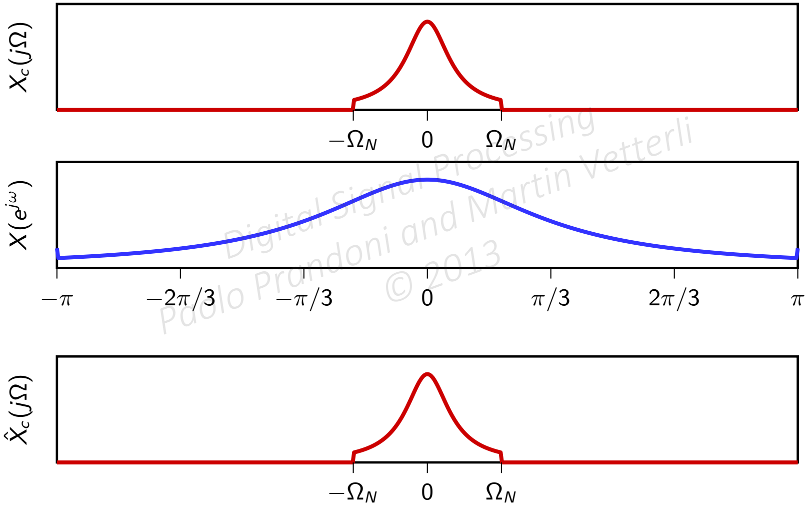


$X_c(j\Omega)$
$-\Omega_N$
0
Ω_N
$\tilde{X}_c(j\Omega)$
$-4\Omega_N$
$-2\Omega_N$
$-\Omega_N$
0
Ω_N
$2\Omega_N$
$4\Omega_N$


$X_c(j\Omega)$
$-\Omega_N$
0
Ω_N
$\tilde{X}_c(j\Omega)$
$-4\Omega_N$
$-2\Omega_N$
$-\Omega_N$
0
Ω_N
$2\Omega_N$
$4\Omega_N$


$X_c(j\Omega)$
$-\Omega_N$
0
Ω_N
$\tilde{X}_c(j\Omega)$
$-4\Omega_N$
$-2\Omega_N$
$-\Omega_N$
0
Ω_N
$2\Omega_N$
$4\Omega_N$
$X(e^{j\omega})$
$-\pi$
$-2\pi/3$
$-\pi/3$
0
$\pi/3$
$2\pi/3$
π

Least squares approximation with sinc sampling and interpolation



END OF MODULE 6.5

Digital Signal Processing

Module 6.6: Discrete-time Processing and Continuous-time Signals

Overview:

- Impulse invariance
- Duality
- Examples

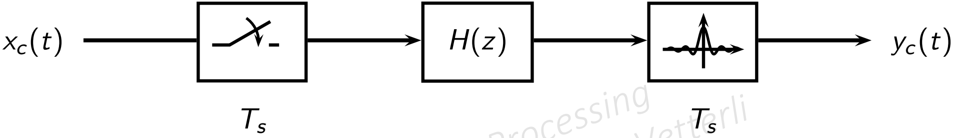
Overview:

- Impulse invariance
- Duality
- Examples

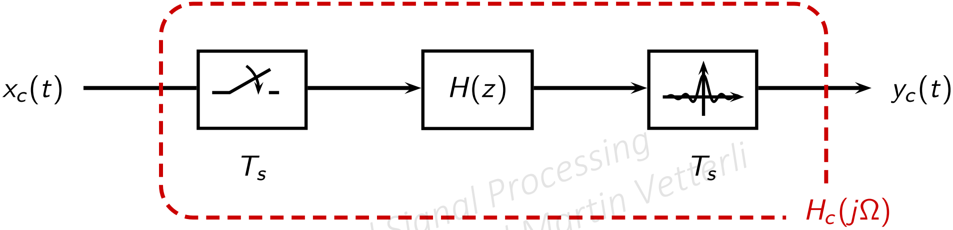
Overview:

- Impulse invariance
- Duality
- Examples

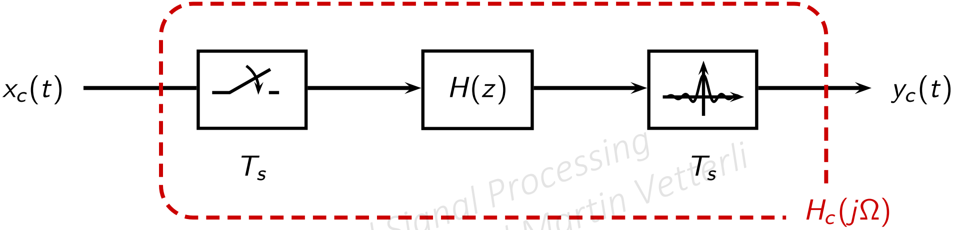
Basic setup



Basic setup

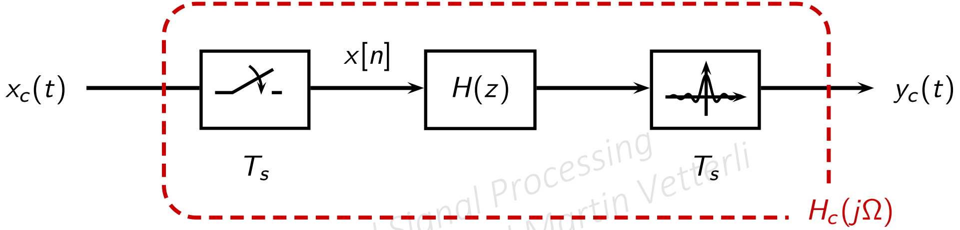


Basic setup



assume $x_c(t)$ is Ω_N-bandlimited:

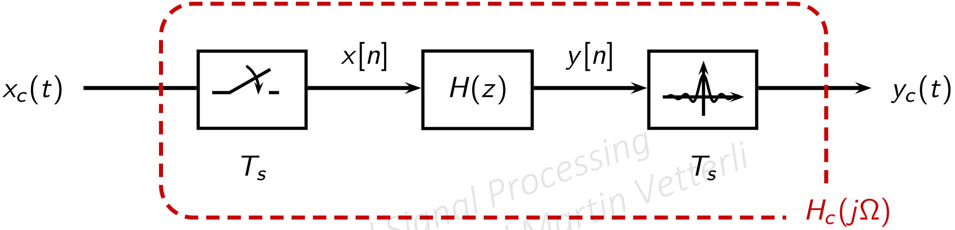
Basic setup



assume $x_c(t)$ is Ω_N-bandlimited:

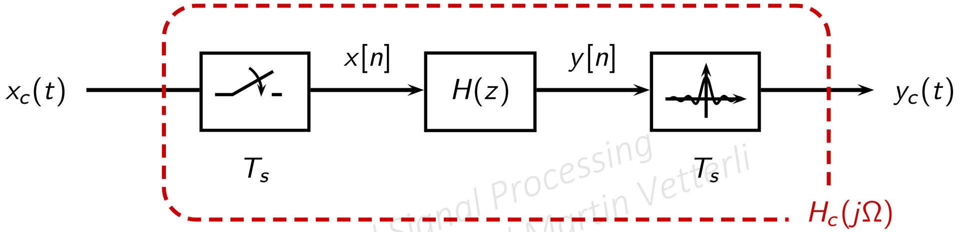
- $X(e^{j\omega}) = \frac{1}{T_s} X_c\left(j\frac{\omega}{T_s}\right)$

Basic setup



assume $x_c(t)$ is Ω_N-bandlimited:

- $X(e^{j\omega}) = \frac{1}{T_s} X_c\left(j\frac{\omega}{T_s}\right)$
- $Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$



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- $Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$
- $Y_c(j\Omega) = T_s\, Y(e^{j\Omega T_s})$

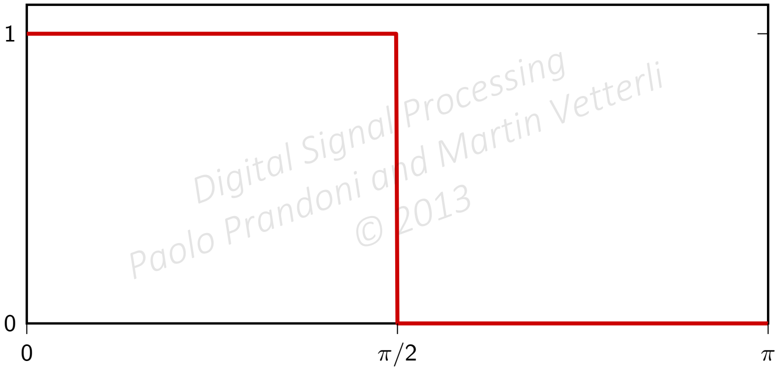
Impulse invariance

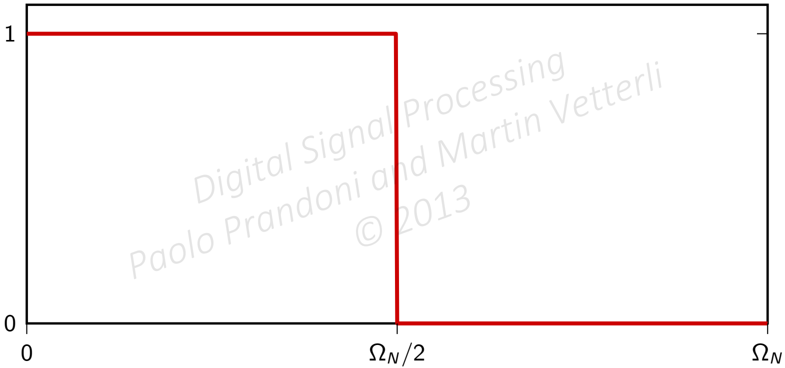
$$Y_c(j\Omega) = X_c(j\Omega)\, H(e^{j\pi\Omega/\Omega_N})$$

Impulse invariance

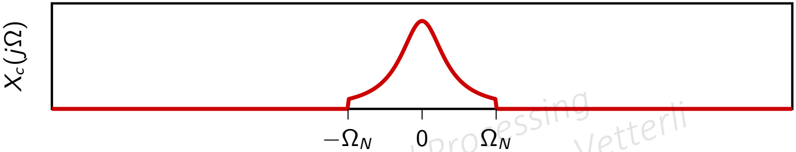
$$H_c(j\Omega) = H(e^{j\pi\Omega/\Omega_N})$$

Impulse invariance

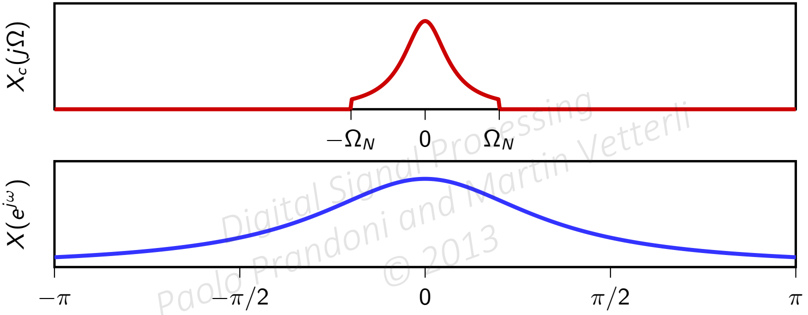



1
0
0
$\Omega_N/2$
Ω_N

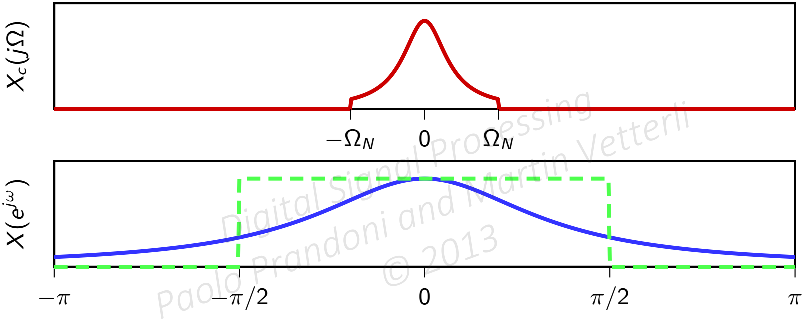
DT processing of CT signals

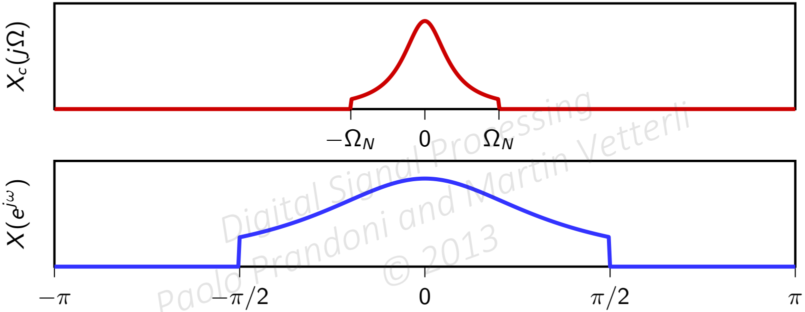


DT processing of CT signals

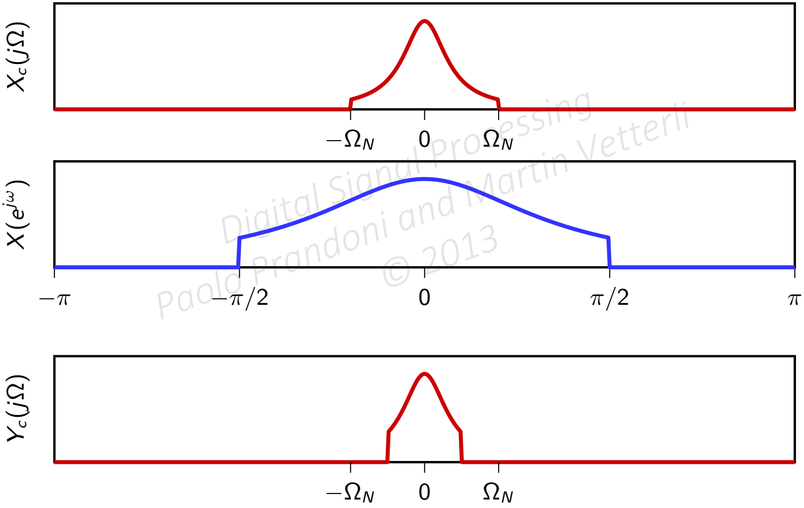


DT processing of CT signals




$X_c(j\Omega)$
$-\Omega_N$
0
Ω_N
$X(e^{j\omega})$
$-\pi$
$-\pi/2$
0
$\pi/2$
π

DT processing of CT signals



Example

design a discrete-time filter to isolate a band of frequencies between 4000 and 5000Hz; input signals are bandlimited to 7KHz.

Example

- 7KHz band limit $\Rightarrow$ we can use any sampling frequency above 14KHz
- pick $F_s = 16KHz$ so that $\Omega_N = 2\pi \cdot 8000$ rad/s
- we need a bandpass with a 1000Hz bandwidth
- start with a lowpass with cutoff 500Hz
- modulate it to center it around 4500Hz

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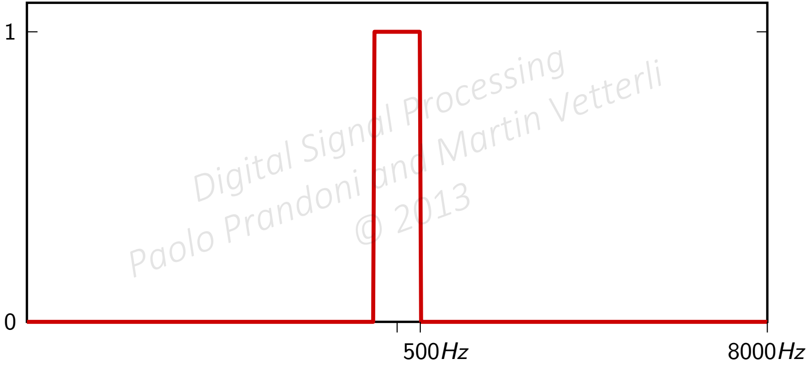
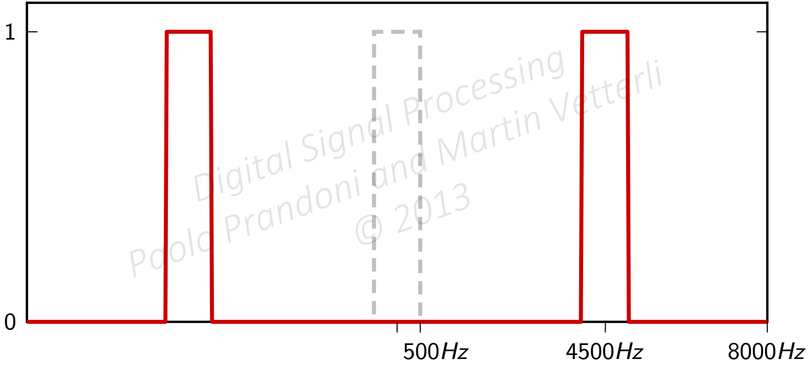
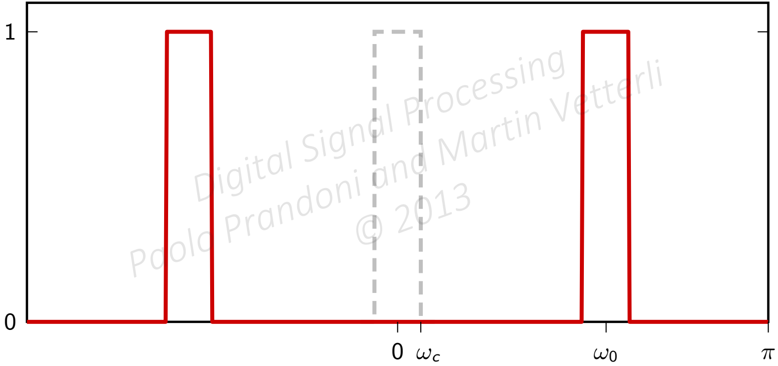

1
0
500 Hz
8000 Hz
Digital Signal Processing
Paolo Prandoni and Martin Vetterli
© 2013

Impulse invariance




1
0
0
ω_c
ω_0
π

Example

- $\omega_c = \pi \frac{\Omega_c}{\Omega_N} = \pi \frac{500}{8000} = 0.0625\pi$
- $\omega_0 = \pi \frac{4500}{8000} = 0.5625\pi$
- design an FIR lowpass with cutoff ω_c using your favorite method
- multiply the impulse response by $2 \cos \omega_0 n$

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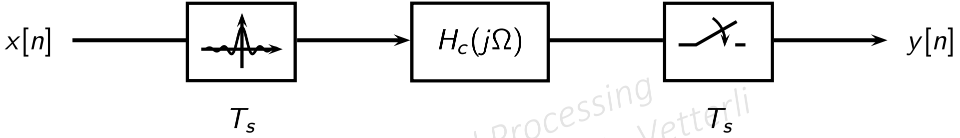
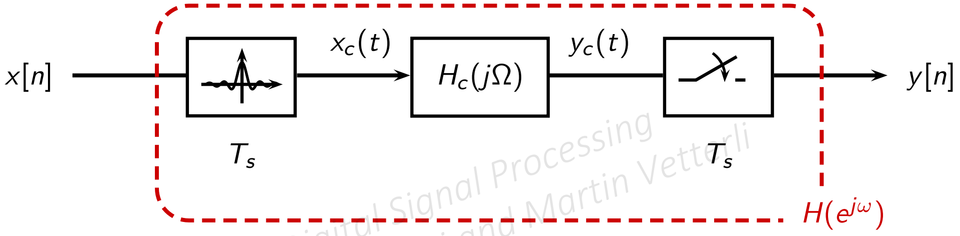
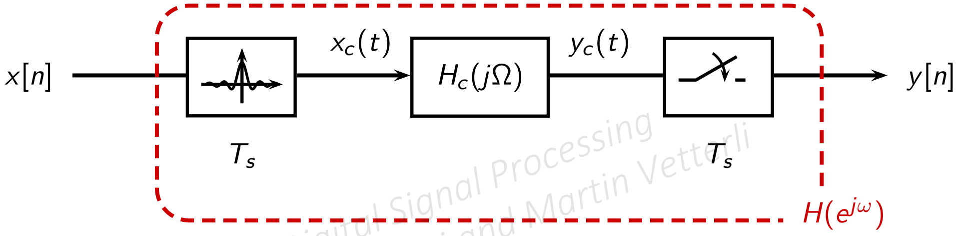
Duality

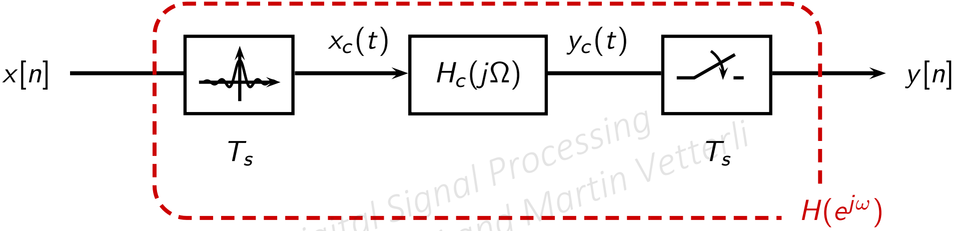




$x[n]$
$x_c(t)$
$H_c(j\Omega)$
$y_c(t)$
$y[n]$
T_s
T_s
$H(e^{j\omega})$

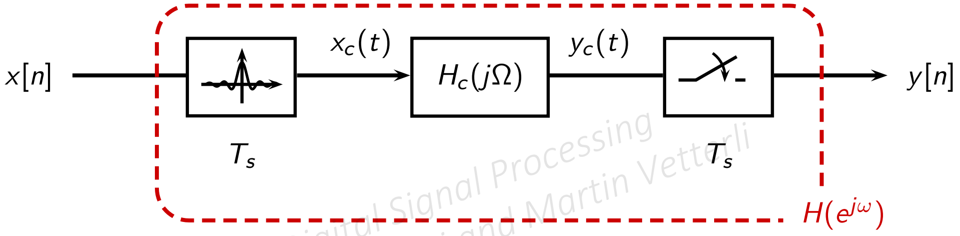


we can pick any T_s so pick $T_s = 1$:



we can pick any T_s so pick $T_s = 1$:

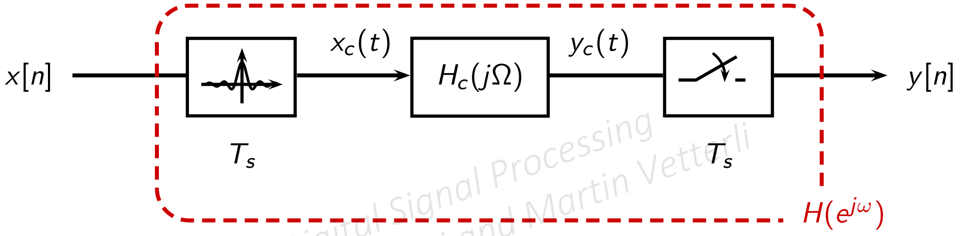
- $X_c(j\Omega) = X(e^{j\Omega})$



we can pick any T_s so pick $T_s = 1$:

- $X_c(j\Omega) = X(e^{j\Omega})$
- $Y_c(j\Omega) = X_c(j\Omega)H_c(j\Omega)$

Duality



we can pick any T_s so pick $T_s = 1$:

- $X_c(j\Omega) = X(e^{j\Omega})$
- $Y_c(j\Omega) = X_c(j\Omega)H_c(j\Omega)$
- LTI systems cannot change the bandwidth $\Rightarrow Y(e^{j\omega}) = Y_c(j\omega)$

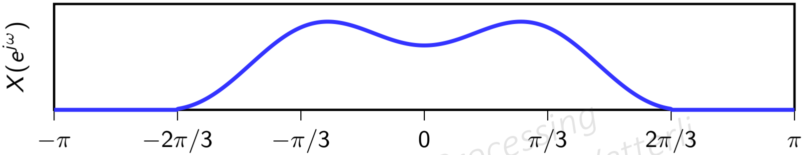
Duality

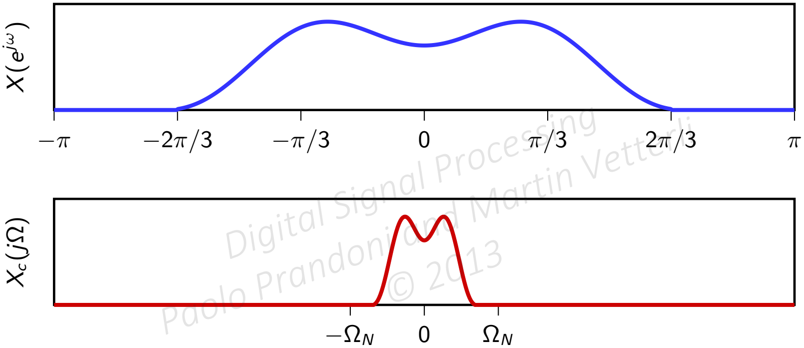
$$Y(e^{j\omega}) = X(e^{j\omega})\, H_c(j\omega)$$

Duality

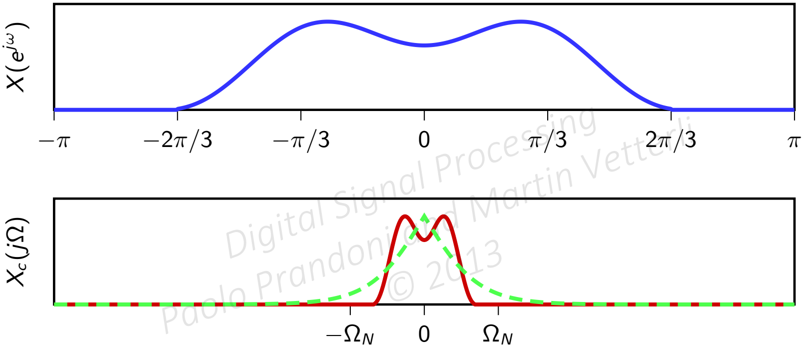
$$H(e^{j\omega}) = H_c(j\omega)$$

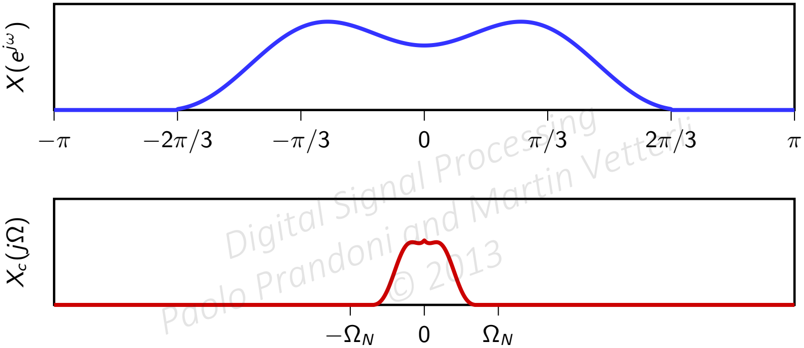
CT processing of DT signals



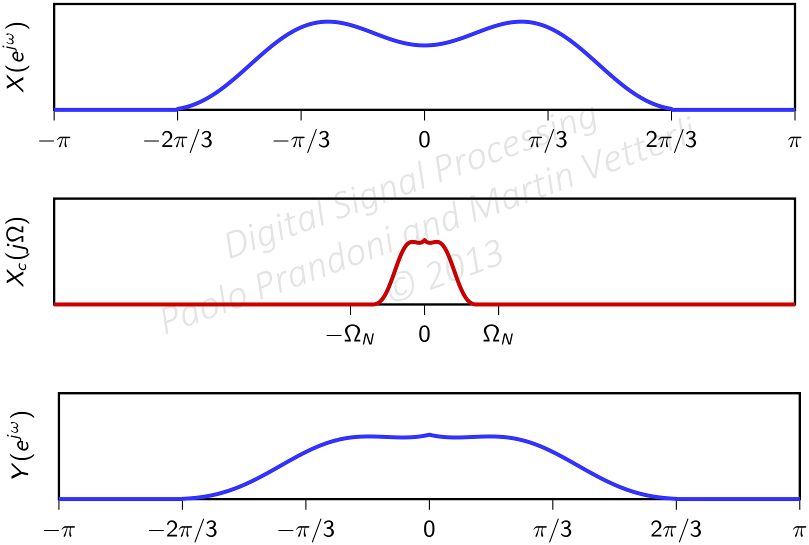

$X(e^{j\omega})$
$-\pi$
$-2\pi/3$
$-\pi/3$
0
$\pi/3$
$2\pi/3$
π
$X_c(j\Omega)$
$-\Omega_N$
0
Ω_N

CT processing of DT signals




$X(e^{j\omega})$
$-\pi$
$-2\pi/3$
$-\pi/3$
0
$\pi/3$
$2\pi/3$
π
$X_c(j\Omega)$
$-\Omega_N$
0
Ω_N

CT processing of DT signals

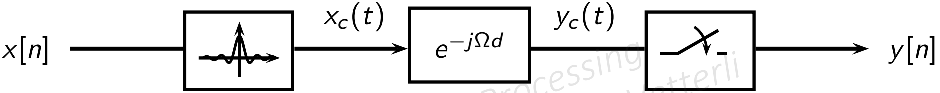


Example: fractional delay

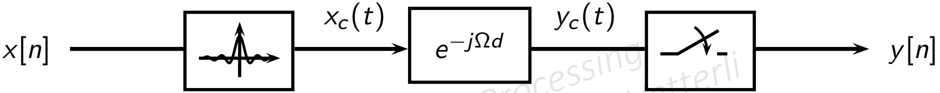
$$H(e^{j\omega}) = e^{-j\omega d}$$

- if $d \in \mathbb{Z}$, simple delay
- if $d \notin \mathbb{Z}$, $h[n] = \text{sinc}(n - d)$...

By duality

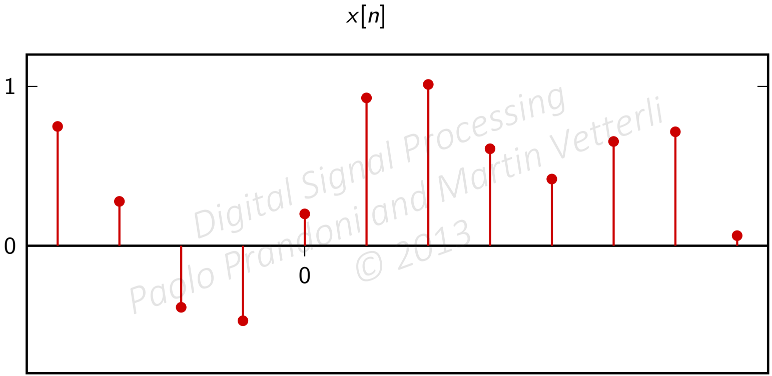


By duality

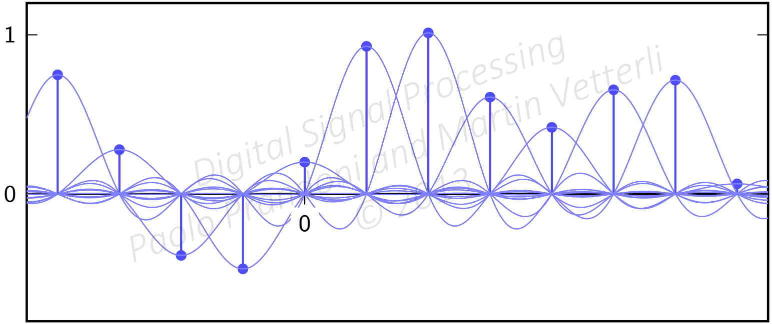


- $Y_c(j\Omega) = e^{-j\Omega d} X_c(j\Omega)$
- $y_c(t) = x_c(t - d)$
- $y[n]$ is the sampled interpolation of $x[n]$ delayed by d

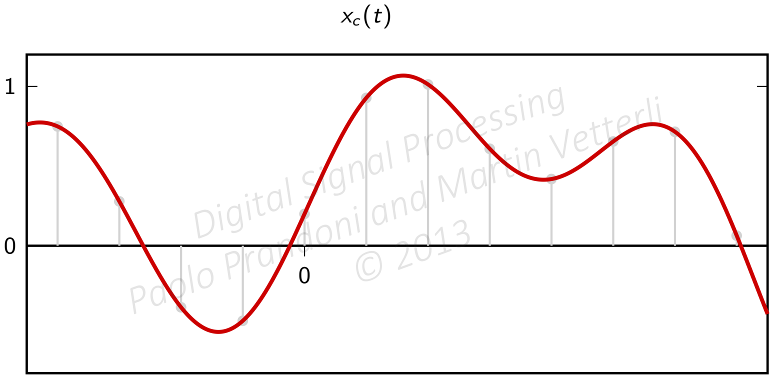
Example: fractional delay



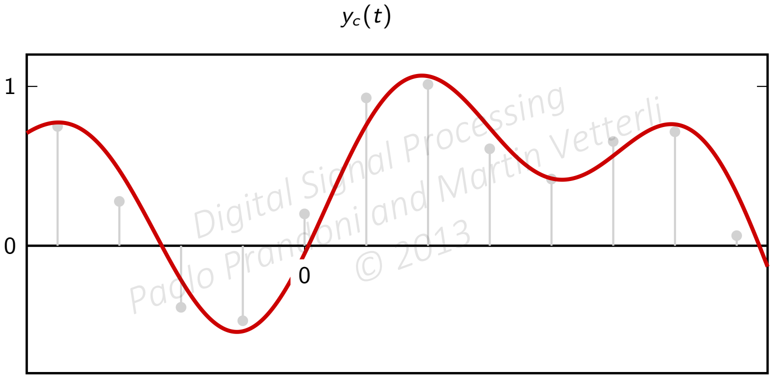
Example: fractional delay



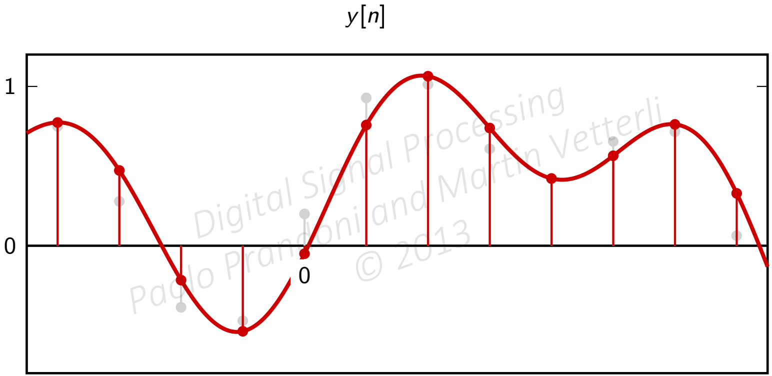
Example: fractional delay



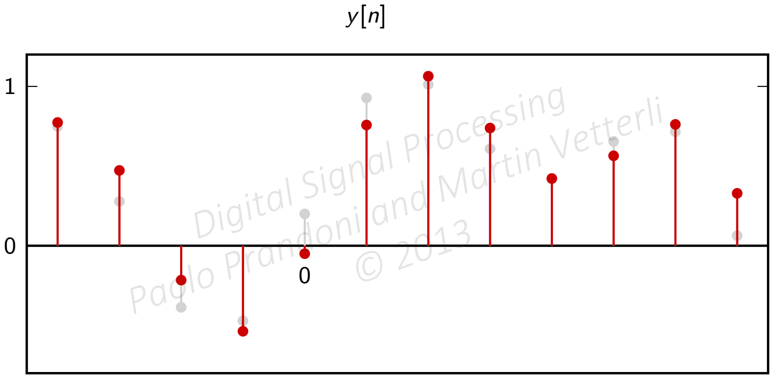
Example: fractional delay



Example: fractional delay



Example: fractional delay



Example: fractional delay

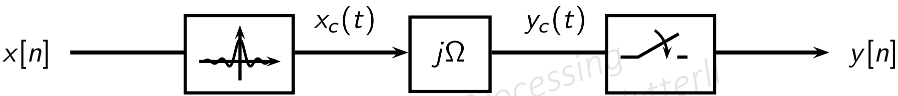
- to delay a discrete-time signal by a fraction of a sample we need an ideal filter!
- efficient time-variant approximations exist (see Module 11)

Example: differentiator

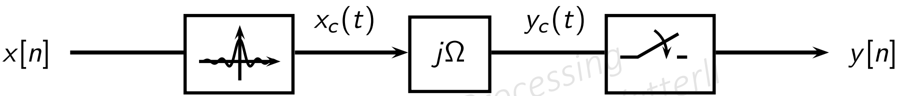
$$H(e^{j\omega}) = j\omega$$

- in continuous time we know that $\mathrm{FT}\{x_c'(t)\} = j\Omega\, X_c(j\Omega)$
- in discrete time...

By duality



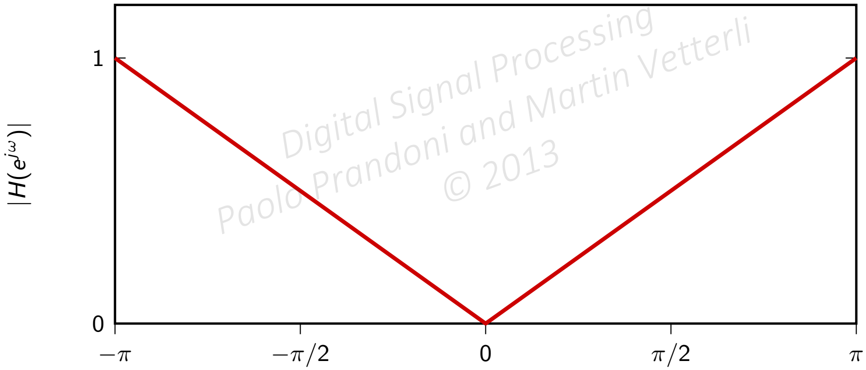
By duality



- $Y_c(j\Omega) = j\Omega X_c(j\Omega)$
- $y_c(t) = x_c'(t)$
- $y[n]$ is the sampled interpolation of $x[n]$, differentiated

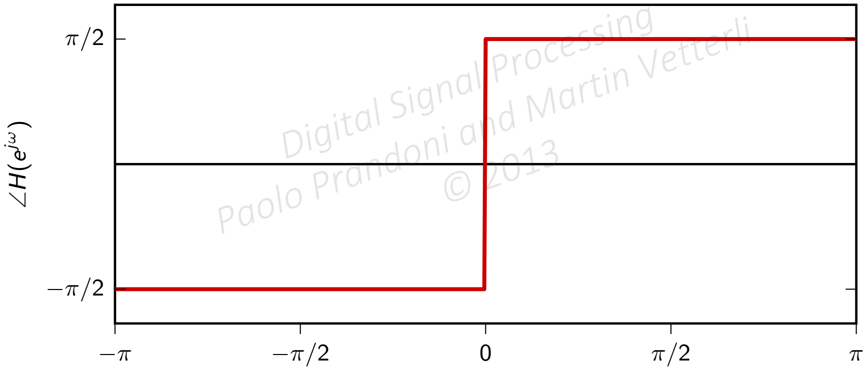
Digital differentiator, magnitude response

$$H(e^{j\omega}) = j\omega$$



Digital differentiator, phase response

$$H(e^{j\omega}) = j\omega$$



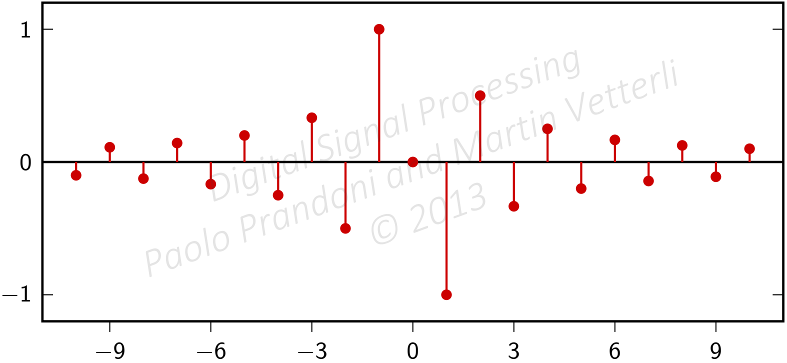
Digital differentiator, impulse response

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega e^{j\omega n} d\omega$$

$$= \ldots \textit{(integration by parts)} \ldots$$

$$= \begin{cases} 0 & n = 0 \\ \dfrac{(-1)^n}{n} & n \neq 0 \end{cases}$$

Digital differentiator, impulse response



Digital differentiator

- the digital differentiator is again an ideal filter!
- many approximations exist, with different properties

Wrap up

- Continuous-time processing of discrete-time sequences
- Discrete-time processing of continuous-time signals
- Jumping back and forth using sampling and interpolation
- In practice: Many applications of processing continuous-time signals in discrete time!

END OF MODULE 6.6

END OF MODULE 6