Digital Signal Processing

Module 7: Stochastic Signal Processing and Quantization

Module Overview:

- Module 7.1: Stochastic signals
- Module 7.2: Quantization
- Module 7.2: A/D and D/A conversion

Digital Signal Processing

Module 7.1: Stochastic signal processing

Overview:

- A simple random signal
- Power spectral density
- Filtering a stochastic signal
- Noise

Overview:

- A simple random signal
- Power spectral density
- Filtering a stochastic signal
- Noise

Overview:

- A simple random signal
- Power spectral density
- Filtering a stochastic signal
- Noise

Overview:

- A simple random signal
- Power spectral density
- Filtering a stochastic signal
- Noise

Deterministic vs. stochastic

- deterministic signals are known in advance: $x[n] = \sin(0.2\,n)$
- interesting signals are *not* known in advance: $s[n] =$ what I'm going to say next
- we usually know something, though: $s[n]$ is a speech signal
- stochastic signals can be described probabilistically
- can we do signal processing with random signals? Yes!
- will not develop stochastic signal processing rigorously but give enough intuition to deal with things such as "noise"

Deterministic vs. stochastic

- deterministic signals are known in advance: $x[n] = \sin(0.2\,n)$
- interesting signals are *not* known in advance: $s[n] =$ what I'm going to say next
- we usually know something, though: $s[n]$ is a speech signal
- stochastic signals can be described probabilistically
- can we do signal processing with random signals? Yes!
- will not develop stochastic signal processing rigorously but give enough intuition to deal with things such as "noise"

Deterministic vs. stochastic

- deterministic signals are known in advance: $x[n] = \sin(0.2\,n)$
- interesting signals are *not* known in advance: $s[n] =$ what I'm going to say next
- we usually know something, though: $s[n]$ is a speech signal
- stochastic signals can be described probabilistically
- can we do signal processing with random signals? Yes!
- will not develop stochastic signal processing rigorously but give enough intuition to deal with things such as "noise"

Deterministic vs. stochastic

- deterministic signals are known in advance: $x[n] = \sin(0.2\, n)$
- interesting signals are *not* known in advance: $s[n]$ = what I'm going to say next
- we usually know something, though: $s[n]$ is a speech signal
- stochastic signals can be described probabilistically
- can we do signal processing with random signals? Yes!
- will not develop stochastic signal processing rigorously but give enough intuition to deal with things such as "noise"

Deterministic vs. stochastic

- deterministic signals are known in advance: $x[n] = \sin(0.2\, n)$
- interesting signals are *not* known in advance: $s[n] =$ what I'm going to say next
- we usually know something, though: $s[n]$ is a speech signal
- stochastic signals can be described probabilistically
- can we do signal processing with random signals? Yes!
- will not develop stochastic signal processing rigorously but give enough intuition to deal with things such as "noise"

Deterministic vs. stochastic

- deterministic signals are known in advance: $x[n] = \sin(0.2\,n)$
- interesting signals are *not* known in advance: $s[n] =$ what I'm going to say next
- we usually know something, though: $s[n]$ is a speech signal
- stochastic signals can be described probabilistically
- can we do signal processing with random signals? Yes!
- will not develop stochastic signal processing rigorously but give enough intuition to deal with things such as "noise"

A simple discrete-time random signal generator

For each new sample, toss a fair coin:

$$x[n] = \begin{cases} +1 & \text{if the outcome of the } n\text{-th toss is head} \\ -1 & \text{if the outcome of the } n\text{-th toss is tail} \end{cases}$$

- each sample is independent from all others
- each sample value has a 50% probability

A simple discrete-time random signal generator

For each new sample, toss a fair coin:

$$x[n] = \begin{cases} +1 & \text{if the outcome of the } n\text{-th toss is head} \\ -1 & \text{if the outcome of the } n\text{-th toss is tail} \end{cases}$$

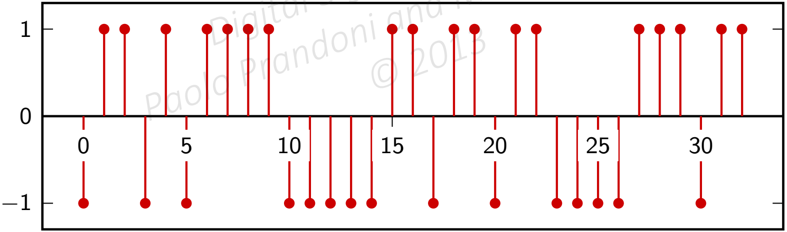
- each sample is independent from all others
- each sample value has a 50% probability

A simple discrete-time random signal generator

- every time we turn on the generator we obtain a different *realization* of the signal
- we know the "mechanism" behind each instance
- but how can we analyze a random signal?

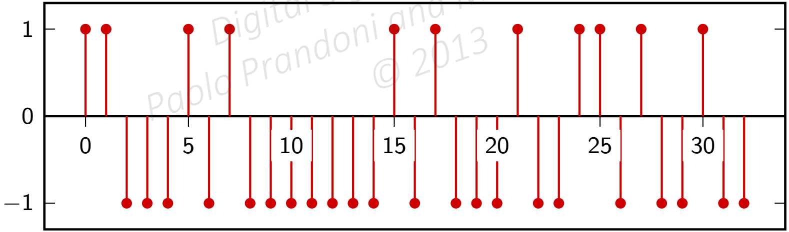
A simple discrete-time random signal generator

- every time we turn on the generator we obtain a different *realization* of the signal
- we know the "mechanism" behind each instance
- but how can we analyze a random signal?



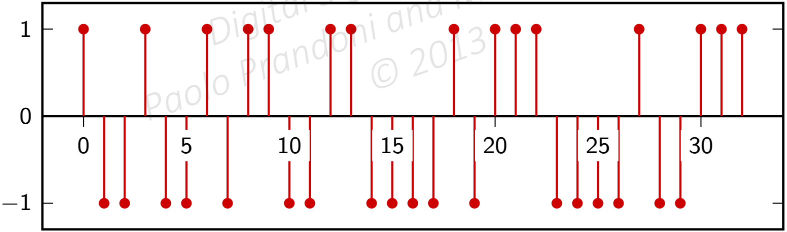
A simple discrete-time random signal generator

- every time we turn on the generator we obtain a different *realization* of the signal
- we know the "mechanism" behind each instance
- but how can we analyze a random signal?



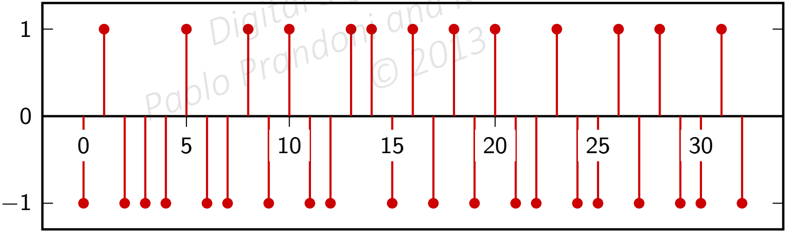
A simple discrete-time random signal generator

- every time we turn on the generator we obtain a different *realization* of the signal
- we know the "mechanism" behind each instance
- but how can we analyze a random signal?



A simple discrete-time random signal generator

- every time we turn on the generator we obtain a different *realization* of the signal
- we know the "mechanism" behind each instance
- but how can we analyze a random signal?

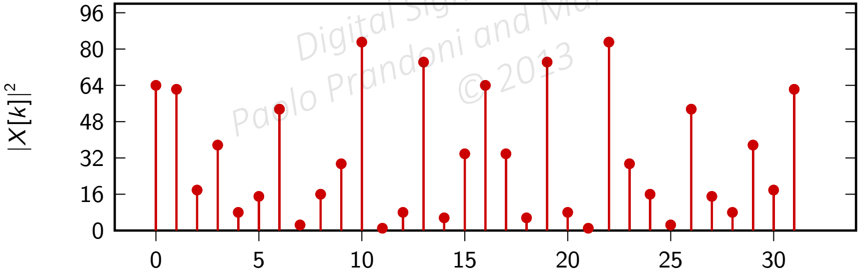


Spectral properties?

- let's try with the DFT of a finite set of random samples
- every time it's different; maybe with more data?
- no clear pattern... we need a new strategy

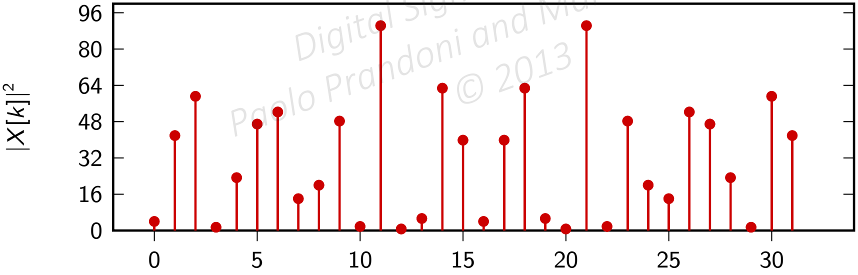
Spectral properties?

- let's try with the DFT of a finite set of random samples
- every time it's different; maybe with more data?
- no clear pattern... we need a new strategy



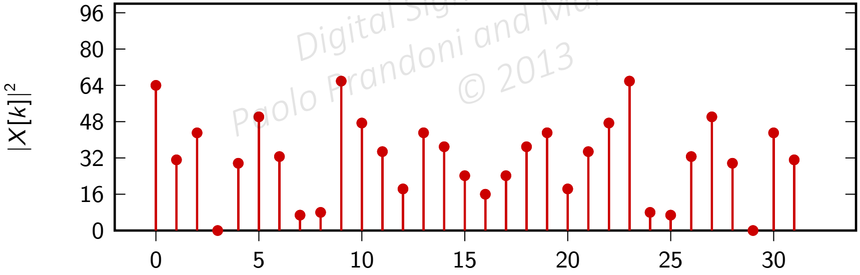
Spectral properties?

- let's try with the DFT of a finite set of random samples
- every time it's different; maybe with more data?
- no clear pattern... we need a new strategy



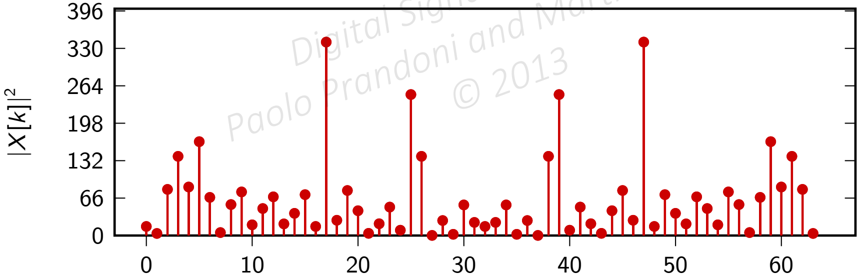
Spectral properties?

- let's try with the DFT of a finite set of random samples
- every time it's different; maybe with more data?
- no clear pattern... we need a new strategy



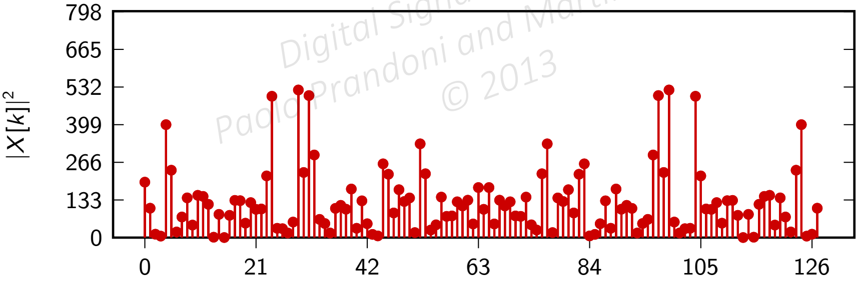
Spectral properties?

- let's try with the DFT of a finite set of random samples
- every time it's different; maybe with more data?
- no clear pattern... we need a new strategy



Spectral properties?

- let's try with the DFT of a finite set of random samples
- every time it's different; maybe with more data?
- no clear pattern... we need a new strategy



Averaging

- when faced with random data an intuitive response is to take "averages"
- in probability theory the average is across realizations and it's called *expectation*
- for the coin-toss signal:

$$\mathrm{E}\left[x[n]\right] = -1 \cdot P[\text{n-th toss is tail}] + 1 \cdot P[\text{n-th toss is head}] = 0$$

- so the average value for each sample is zero...

Averaging

- when faced with random data an intuitive response is to take "averages"
- in probability theory the average is across realizations and it's called *expectation*
- for the coin-toss signal:

$$\mathrm{E}[x[n]] = -1 \cdot P[\text{n-th toss is tail}] + 1 \cdot P[\text{n-th toss is head}] = 0$$

- so the average value for each sample is zero...

Averaging

- when faced with random data an intuitive response is to take "averages"
- in probability theory the average is across realizations and it's called *expectation*
- for the coin-toss signal:

$$\mathrm{E}\left[x[n]\right] = -1 \cdot P[\text{n-th toss is tail}] + 1 \cdot P[\text{n-th toss is head}] = 0$$

- so the average value for each sample is zero...

Averaging

- when faced with random data an intuitive response is to take “averages”
- in probability theory the average is across realizations and it's called *expectation*
- for the coin-toss signal:

$$\mathrm{E}\left[x[n]\right] = -1 \cdot P[\text{n-th toss is tail}] + 1 \cdot P[\text{n-th toss is head}] = 0$$

- so the average value for each sample is zero...

Averaging the DFT

- … as a consequence, averaging the DFT will not work
- $E[X[k]] = 0$
- however the signal "moves", so its energy or power must be nonzero

Averaging the DFT

- … as a consequence, averaging the DFT will not work
- $\mathrm{E}[X[k]] = 0$
- however the signal "moves", so its energy or power must be nonzero

Averaging the DFT

- … as a consequence, averaging the DFT will not work
- $\mathrm{E}[X[k]] = 0$
- however the signal "moves", so its energy or power must be nonzero

Energy and power

- the coin-toss signal has infinite energy (see Module 2.1):

$$E_x = \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} (2N+1) = \infty$$

- however it has finite power over any interval:

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = 1$$

Averaging

let's try to average the DFT's square magnitude, normalized:

- pick an interval length N
- pick a number of iterations M
- run the signal generator M times and obtain M N-point realizations
- compute the DFT of each realization
- average their square magnitude divided by N

Averaging

let's try to average the DFT's square magnitude, normalized:

- pick an interval length N
- pick a number of iterations M
- run the signal generator M times and obtain M N-point realizations
- compute the DFT of each realization
- average their square magnitude divided by N

let's try to average the DFT's square magnitude, normalized:

- pick an interval length N
- pick a number of iterations M
- run the signal generator M times and obtain M N-point realizations
- compute the DFT of each realization
- average their square magnitude divided by N

Averaging

let's try to average the DFT's square magnitude, normalized:

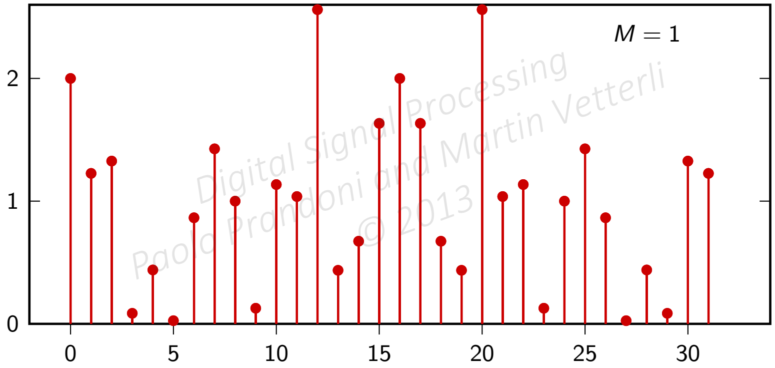
- pick an interval length N
- pick a number of iterations M
- run the signal generator M times and obtain M N-point realizations
- compute the DFT of each realization
- average their square magnitude divided by N

Averaging

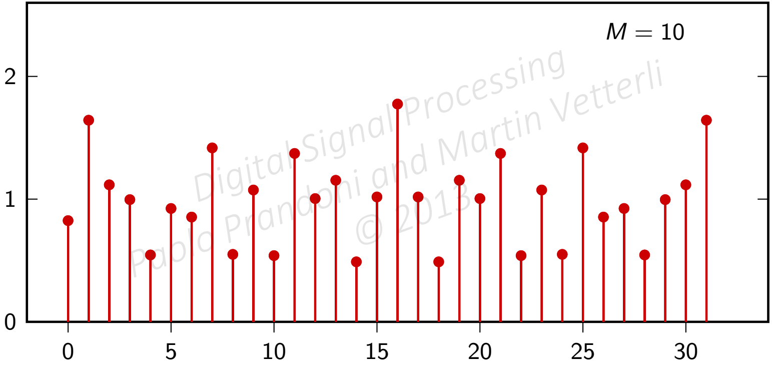
let's try to average the DFT's square magnitude, normalized:

- pick an interval length N
- pick a number of iterations M
- run the signal generator M times and obtain M N-point realizations
- compute the DFT of each realization
- average their square magnitude divided by N

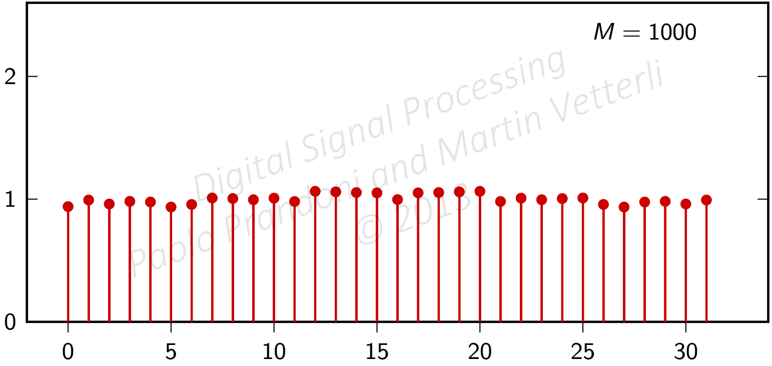
Averaged DFT square magnitude



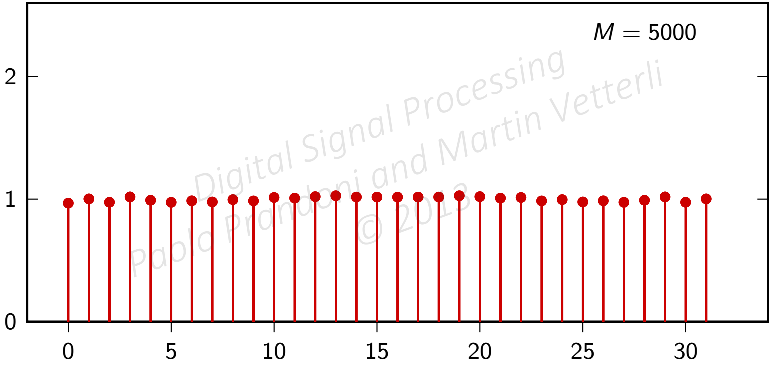
Averaged DFT square magnitude



Averaged DFT square magnitude



Averaged DFT square magnitude



Power spectral density

$$P[k] = \mathrm{E}\left[|X_N[k]|^2/N\right]$$

- it looks very much as if $P[k] = 1$
- if $|X_N[k]|^2$ tends to the *energy* distribution in frequency...
- ...$|X_N[k]|^2/N$ tends to the *power* distribution (aka *density*) in frequency
- the frequency-domain representation for stochastic processes is the power spectral density

$$P[k] = \mathrm{E}\left[|X_N[k]|^2/N\right]$$

- it looks very much as if $P[k] = 1$
- if $|X_N[k]|^2$ tends to the *energy* distribution in frequency...
- ...$|X_N[k]|^2/N$ tends to the *power* distribution (aka *density*) in frequency
- the frequency-domain representation for stochastic processes is the power spectral density

Power spectral density

$$P[k] = \mathrm{E}\left[|X_N[k]|^2/N\right]$$

- it looks very much as if $P[k] = 1$
- if $|X_N[k]|^2$ tends to the *energy* distribution in frequency...
- ...$|X_N[k]|^2/N$ tends to the *power* distribution (aka *density*) in frequency
- the frequency-domain representation for stochastic processes is the power spectral density

Power spectral density

$$P[k] = \mathrm{E}\left[|X_N[k]|^2/N\right]$$

- it looks very much as if $P[k] = 1$
- if $|X_N[k]|^2$ tends to the *energy* distribution in frequency...
- ...$|X_N[k]|^2/N$ tends to the *power* distribution (aka *density*) in frequency
- the frequency-domain representation for stochastic processes is the power spectral density

Power spectral density: intuition

- $P[k] = 1$ means that the power is equally distributed over all frequencies
- i.e., we cannot predict if the signal moves "slowly" or "super-fast"
- this is because each sample is independent of each other: we could have a realization of all ones or a realization in which the sign changes every other sample or anything in between

Power spectral density: intuition

- $P[k] = 1$ means that the power is equally distributed over all frequencies
- i.e., we cannot predict if the signal moves "slowly" or "super-fast"
- this is because each sample is independent of each other: we could have a realization of all ones or a realization in which the sign changes every other sample or anything in between

Power spectral density: intuition

- $P[k] = 1$ means that the power is equally distributed over all frequencies
- i.e., we cannot predict if the signal moves "slowly" or "super-fast"
- this is because each sample is independent of each other: we could have a realization of all ones or a realization in which the sign changes every other sample or anything in between

Filtering a random process

- let's filter the random process with a 2-point Moving Average filter
- $y[n] = (x[n] + x[n-1])/2$
- what is the power spectral density?

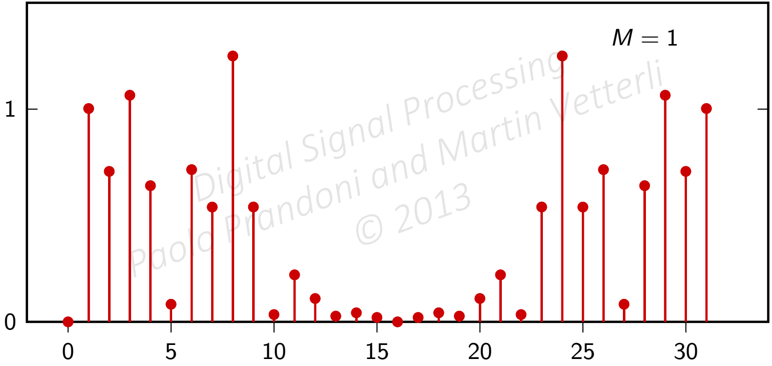
Filtering a random process

- let's filter the random process with a 2-point Moving Average filter
- $y[n] = (x[n] + x[n-1])/2$
- what is the power spectral density?

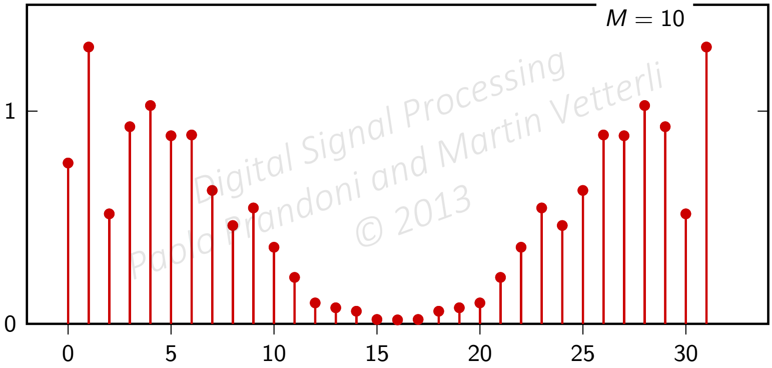
Filtering a random process

- let's filter the random process with a 2-point Moving Average filter
- $y[n] = (x[n] + x[n-1])/2$
- what is the power spectral density?

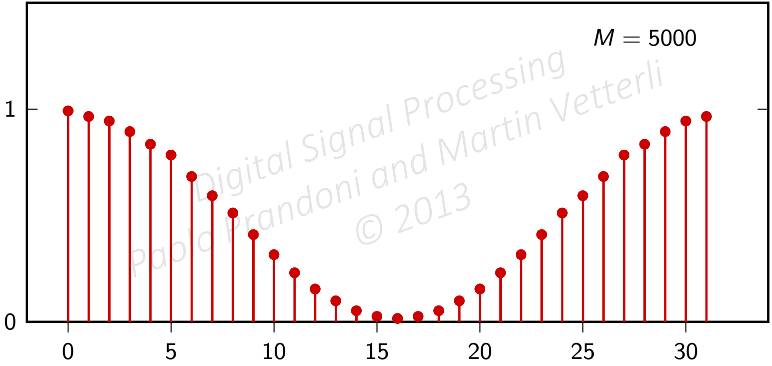
Averaged DFT magnitude of filtered process



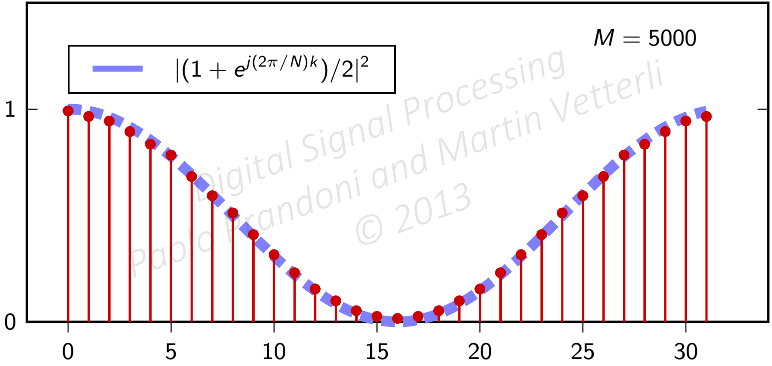
Averaged DFT magnitude of filtered process



Averaged DFT magnitude of filtered process



Averaged DFT magnitude of filtered process



Filtering a random process

- it looks like $P_y[k] = P_x[k]\,|H[k]|^2$, where $H[k] = \text{DFT}\{h[n]\}$
- can we generalize these results beyond a finite set of samples?

Filtering a random process

- it looks like $P_y[k] = P_x[k]\,|H[k]|^2$, where $H[k] = \text{DFT}\{h[n]\}$
- can we generalize these results beyond a finite set of samples?

Stochastic signal processing

- a stochastic process is characterized by its power spectral density (PSD)
- it can be shown (see the textbook) that the PSD is

$$P_x(e^{j\omega}) = \text{DTFT}\{r_x[n]\}$$

where $r_x[n] = \mathrm{E}[x[k]\,x[n+k]]$ is the autocorrelation of the process.

- for a filtered stochastic process $y[n] = \mathcal{H}\{x[n]\}$, it is:

$$P_y(e^{j\omega}) = |H(e^{j\omega})|^2 P_x(e^{j\omega})$$

Stochastic signal processing

- a stochastic process is characterized by its power spectral density (PSD)
- it can be shown (see the textbook) that the PSD is

$$P_x(e^{j\omega}) = \text{DTFT}\{r_x[n]\}$$

where $r_x[n] = \mathrm{E}[x[k]\,x[n+k]]$ is the autocorrelation of the process.

- for a filtered stochastic process $y[n] = \mathcal{H}\{x[n]\}$, it is:

$$P_y(e^{j\omega}) = |H(e^{j\omega})|^2 P_x(e^{j\omega})$$

- a stochastic process is characterized by its power spectral density (PSD)
- it can be shown (see the textbook) that the PSD is

$$P_x(e^{j\omega}) = \mathrm{DTFT}\{r_x[n]\}$$

where $r_x[n] = \mathrm{E}[x[k]\,x[n+k]]$ is the autocorrelation of the process.

- for a filtered stochastic process $y[n] = \mathcal{H}\{x[n]\}$, it is:

$$P_y(e^{j\omega}) = |H(e^{j\omega})|^2 \, P_x(e^{j\omega})$$

key points:

- filters designed for deterministic signals still work (in magnitude) in the stochastic case
- we lose the concept of phase since we don't know the shape of a realization in advance

key points:

- filters designed for deterministic signals still work (in magnitude) in the stochastic case
- we lose the concept of phase since we don't know the shape of a realization in advance

Noise

- noise is everywhere:
  - thermal noise
  - sum of extraneous interferences
  - quantization and numerical errors
  - ...
- we can model noise as a stochastic signal
- the most important noise is white noise

Noise

- noise is everywhere:
  - thermal noise
  - sum of extraneous interferences
  - quantization and numerical errors
  - ...
- we can model noise as a stochastic signal
- the most important noise is white noise

Noise

- noise is everywhere:
  - thermal noise
  - sum of extraneous interferences
  - quantization and numerical errors
  - ...
- we can model noise as a stochastic signal
- the most important noise is white noise

Noise

- noise is everywhere:
  - thermal noise
  - sum of extraneous interferences
  - quantization and numerical errors
  - ...
- we can model noise as a stochastic signal
- the most important noise is white noise

Noise

- noise is everywhere:
  - thermal noise
  - sum of extraneous interferences
  - quantization and numerical errors
  - ...
- we can model noise as a stochastic signal
- the most important noise is white noise

Noise

- noise is everywhere:
  - thermal noise
  - sum of extraneous interferences
  - quantization and numerical errors
  - ...
- we can model noise as a stochastic signal
- the most important noise is white noise

Noise

- noise is everywhere:
  - thermal noise
  - sum of extraneous interferences
  - quantization and numerical errors
  - ...
- we can model noise as a stochastic signal
- the most important noise is white noise

White noise

- "white" indicates uncorrelated samples
- $r_w[n] = \sigma^2\delta[n]$
- $P_w(e^{j\omega}) = \sigma^2$

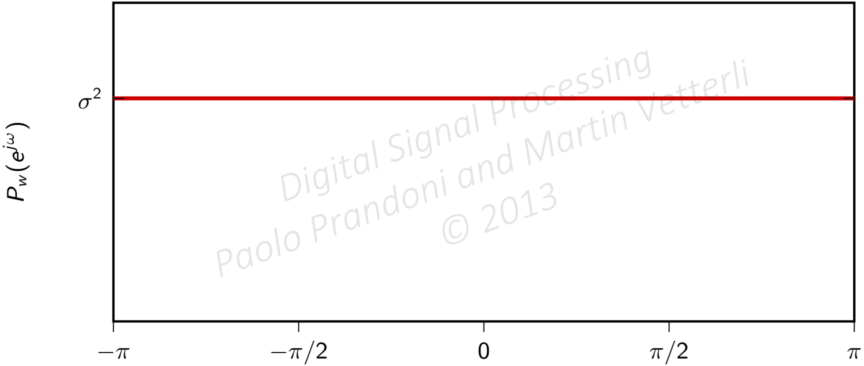
White noise

- "white" indicates uncorrelated samples
- $r_w[n] = \sigma^2 \delta[n]$
- $P_w(e^{j\omega}) = \sigma^2$

White noise

- "white" indicates uncorrelated samples
- $r_w[n] = \sigma^2 \delta[n]$
- $P_w(e^{j\omega}) = \sigma^2$

White noise



White noise

- the PSD is independent of the probability distribution of the single samples (depends only on the variance)
- distribution is important to estimate bounds for the signal
- very often a Gaussian distribution models the experimental data the best
- AWGN: additive white Gaussian noise

White noise

- the PSD is independent of the probability distribution of the single samples (depends only on the variance)
- distribution is important to estimate bounds for the signal
- very often a Gaussian distribution models the experimental data the best
- AWGN: additive white Gaussian noise

White noise

- the PSD is independent of the probability distribution of the single samples (depends only on the variance)
- distribution is important to estimate bounds for the signal
- very often a Gaussian distribution models the experimental data the best
- AWGN: additive white Gaussian noise

White noise

- the PSD is independent of the probability distribution of the single samples (depends only on the variance)
- distribution is important to estimate bounds for the signal
- very often a Gaussian distribution models the experimental data the best
- AWGN: additive white Gaussian noise

END OF MODULE 7.1

Digital Signal Processing

Module 7.2: Quantization

Overview:

- Quantization
- Uniform quantization and error analysis
- Clipping, saturation, companding

Overview:

- Quantization
- Uniform quantization and error analysis
- Clipping, saturation, companding

Overview:

- Quantization
- Uniform quantization and error analysis
- Clipping, saturation, companding

Quantization

- digital devices can only deal with integers (b bits per sample)
- we need to map the range of a signal onto a finite set of values
- irreversible loss of information $\rightarrow$ quantization noise

Quantization

- digital devices can only deal with integers (b bits per sample)
- we need to map the range of a signal onto a finite set of values
- irreversible loss of information $\rightarrow$ quantization noise

Quantization

- digital devices can only deal with integers (b bits per sample)
- we need to map the range of a signal onto a finite set of values
- irreversible loss of information $\rightarrow$ quantization noise

Quantization schemes

$$x[n] \longrightarrow \boxed{\mathcal{Q}\{\cdot\}} \longrightarrow \hat{x}[n]$$

Several factors at play:

- storage budget (bits per sample)
- storage scheme (fixed point, floating point)
- properties of the input
  - range
  - probability distribution

Quantization schemes

$$x[n] \longrightarrow \boxed{\mathcal{Q}\{\cdot\}} \longrightarrow \hat{x}[n]$$

Several factors at play:

- storage budget (bits per sample)
- storage scheme (fixed point, floating point)
- properties of the input
  - range
  - probability distribution

Quantization schemes

$$x[n] \longrightarrow \boxed{\mathcal{Q}\{\cdot\}} \longrightarrow \hat{x}[n]$$

Several factors at play:

- storage budget (bits per sample)
- storage scheme (fixed point, floating point)
- properties of the input
  - range
  - probability distribution

Quantization schemes

$$x[n] \longrightarrow \boxed{\mathcal{Q}\{\cdot\}} \longrightarrow \hat{x}[n]$$

Several factors at play:

- storage budget (bits per sample)
- storage scheme (fixed point, floating point)
- properties of the input
  - range
  - probability distribution

Quantization schemes

$$x[n] \longrightarrow \boxed{\mathcal{Q}\{\cdot\}} \longrightarrow \hat{x}[n]$$

Several factors at play:

- storage budget (bits per sample)
- storage scheme (fixed point, floating point)
- properties of the input
  - range
  - probability distribution

Scalar quantization

$$x[n] \longrightarrow \boxed{\mathcal{Q}\{\cdot\}} \longrightarrow \hat{x}[n]$$

The simplest quantizer:

- each sample is encoded individually (hence *scalar*)
- each sample is quantized independently (memoryless quantization)
- each sample is encoded using R bits

Scalar quantization

$$x[n] \longrightarrow \boxed{\mathcal{Q}\{\cdot\}} \longrightarrow \hat{x}[n]$$

The simplest quantizer:

- each sample is encoded individually (hence *scalar*)
- each sample is quantized independently (memoryless quantization)
- each sample is encoded using R bits

Scalar quantization

$$x[n] \longrightarrow \boxed{\mathcal{Q}\{\cdot\}} \longrightarrow \hat{x}[n]$$

R bps

The simplest quantizer:

- each sample is encoded individually (hence *scalar*)
- each sample is quantized independently (memoryless quantization)
- each sample is encoded using R bits

Scalar quantization

Assume input signal bounded: $A \leq x[n] \leq B$ for all n:

- each sample quantized over 2^R possible values $\Rightarrow 2^R$ intervals.
- each interval associated to a quantization value

Scalar quantization

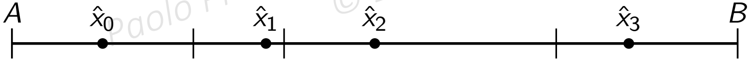
Assume input signal bounded: $A \leq x[n] \leq B$ for all n:

- each sample quantized over 2^R possible values $\Rightarrow 2^R$ intervals.
- each interval associated to a quantization value

Scalar quantization

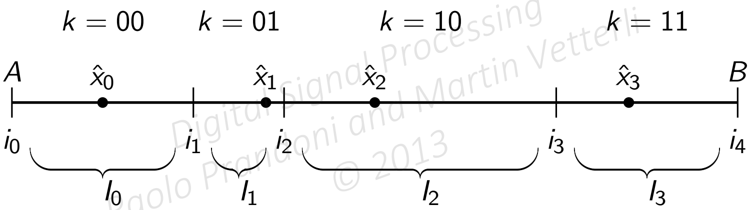
Assume input signal bounded: $A \leq x[n] \leq B$ for all n:

- each sample quantized over 2^R possible values $\Rightarrow$ 2^R intervals.
- each interval associated to a quantization value



Scalar quantization

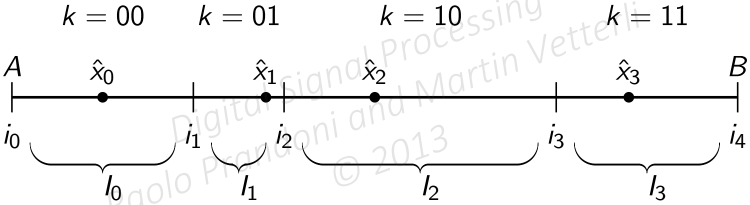
Example for $R = 2$:



- what are the optimal interval boundaries i_k?
- what are the optimal quantization values $\hat{x}_k$?

Scalar quantization

Example for $R = 2$:



- what are the optimal interval boundaries i_k?
- what are the optimal quantization values $\hat{x}_k$?

$$e[n] = \mathcal{Q}\{x[n]\} - x[n] = \hat{x}[n] - x[n]$$

- model $x[n]$ as a stochastic process
- model error as a white noise sequence:
  - error samples are uncorrelated
  - all error samples have the same distribution
- we need statistics of the input to study the error

Quantization Error

$$e[n] = \mathcal{Q}\{x[n]\} - x[n] = \hat{x}[n] - x[n]$$

- model $x[n]$ as a stochastic process
- model error as a white noise sequence:
  - error samples are uncorrelated
  - all error samples have the same distribution
- we need statistics of the input to study the error

Quantization Error

$$e[n] = \mathcal{Q}\{x[n]\} - x[n] = \hat{x}[n] - x[n]$$

- model $x[n]$ as a stochastic process
- model error as a white noise sequence:
  - error samples are uncorrelated
  - all error samples have the same distribution
- we need statistics of the input to study the error

Quantization Error

$$e[n] = \mathcal{Q}\{x[n]\} - x[n] = \hat{x}[n] - x[n]$$

- model $x[n]$ as a stochastic process
- model error as a white noise sequence:
  - error samples are uncorrelated
  - all error samples have the same distribution
- we need statistics of the input to study the error

Quantization Error

$$e[n] = \mathcal{Q}\{x[n]\} - x[n] = \hat{x}[n] - x[n]$$

- model $x[n]$ as a stochastic process
- model error as a white noise sequence:
  - error samples are uncorrelated
  - all error samples have the same distribution
- we need statistics of the input to study the error

Uniform quantization

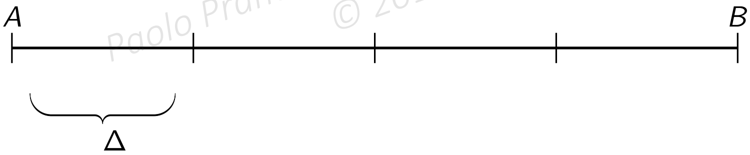
- simple but very general case
- range is split into 2^R *equal* intervals of width $\Delta = (B - A)2^{-R}$

Uniform quantization

- simple but very general case
- range is split into 2^R *equal* intervals of width $\Delta = (B - A)2^{-R}$

Uniform quantization

- simple but very general case
- range is split into 2^R *equal* intervals of width $\Delta = (B - A)2^{-R}$



Uniform quantization

Mean Square Error is the variance of the error signal:

$$\sigma_e^2 = \mathrm{E}\left[\left| \mathcal{Q}\{x[n]\} - x[n] \right|^2 \right]$$

$$= \int_A^B f_x(\tau)\left(\mathcal{Q}\{\tau\} - \tau\right)^2 d\tau$$

$$= \sum_{k=0}^{2^R-1} \int_{I_k} f_x(\tau)(\hat{x}_k - \tau)^2 d\tau$$

error depends on the probability distribution of the input

Uniform quantization

Mean Square Error is the variance of the error signal:

$$\sigma_e^2 = \mathrm{E}\left[|\mathcal{Q}\{x[n]\} - x[n]|^2\right]$$

$$= \int_A^B f_x(\tau)(\mathcal{Q}\{\tau\} - \tau)^2\, d\tau$$

$$= \sum_{k=0}^{2^R-1} \int_{I_k} f_x(\tau)(\hat{x}_k - \tau)^2\, d\tau$$

error depends on the probability distribution of the input

Mean Square Error is the variance of the error signal:

$$
\begin{aligned}
\sigma_e^2 &= \mathrm{E}\left[|\mathcal{Q}\{x[n]\} - x[n]|^2\right] \\
&= \int_A^B f_x(\tau)(\mathcal{Q}\{\tau\} - \tau)^2 \, d\tau \\
&= \sum_{k=0}^{2^R-1} \int_{I_k} f_x(\tau)(\hat{x}_k - \tau)^2 \, d\tau
\end{aligned}
$$

error depends on the probability distribution of the input

Uniform quantization

Mean Square Error is the variance of the error signal:

$$
\begin{aligned}
\sigma_e^2 &= \mathrm{E}\left[|\mathcal{Q}\{x[n]\} - x[n]|^2\right] \\
&= \int_A^B f_x(\tau)(\mathcal{Q}\{\tau\} - \tau)^2 \, d\tau \\
&= \sum_{k=0}^{2^R-1} \int_{I_k} f_x(\tau)(\hat{x}_k - \tau)^2 \, d\tau
\end{aligned}
$$

error depends on the probability distribution of the input

Uniform quantization of uniform input

Uniform-input hypothesis:

$$f_x(\tau) = \frac{1}{B-A}$$

$$\sigma_e^2 = \sum_{k=0}^{2^R-1} \int_{I_k} \frac{(\hat{x}_k - \tau)^2}{B-A}\, d\tau$$

Let's find the optimal quantization point by minimizing the error

$$\frac{\partial \sigma_e^2}{\partial \hat{x}_m} = \frac{\partial}{\partial \hat{x}_m} \sum_{k=0}^{2^R-1} \int_{I_k} \frac{(\hat{x}_k - \tau)^2}{B - A} \, d\tau$$

$$= \int_{I_m} \frac{2(\hat{x}_m - \tau)}{B - A} \, d\tau$$

$$= \left. \frac{(\hat{x}_m - \tau)^2}{B - A} \right|_{A+m\Delta}^{A+m\Delta+\Delta}$$

Uniform quantization of uniform input

Let's find the optimal quantization point by minimizing the error

$$
\begin{aligned}
\frac{\partial \sigma_e^2}{\partial \hat{x}_m} &= \frac{\partial}{\partial \hat{x}_m} \sum_{k=0}^{2^R-1} \int_{I_k} \frac{(\hat{x}_k - \tau)^2}{B-A}\, d\tau \\
&= \int_{I_m} \frac{2(\hat{x}_m - \tau)}{B-A}\, d\tau \\
&= \left. \frac{(\hat{x}_m - \tau)^2}{B-A} \right|_{A+m\Delta}^{A+m\Delta+\Delta}
\end{aligned}
$$

Let's find the optimal quantization point by minimizing the error

$$
\begin{aligned}
\frac{\partial \sigma_e^2}{\partial \hat{x}_m} &= \frac{\partial}{\partial \hat{x}_m} \sum_{k=0}^{2^R-1} \int_{I_k} \frac{(\hat{x}_k - \tau)^2}{B - A} \, d\tau \\
&= \int_{I_m} \frac{2(\hat{x}_m - \tau)}{B - A} \, d\tau \\
&= \left. \frac{(\hat{x}_m - \tau)^2}{B - A} \right|_{A+m\Delta}^{A+m\Delta+\Delta}
\end{aligned}
$$

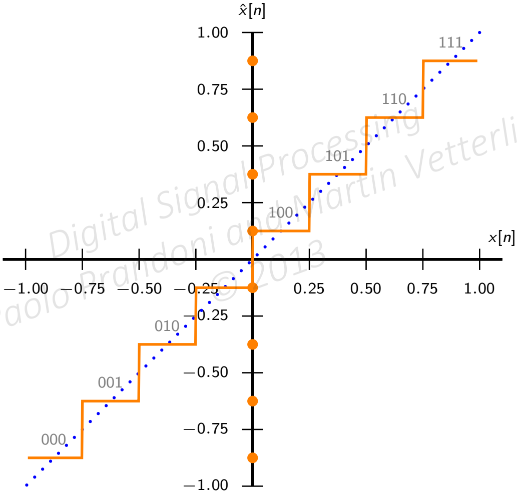
Uniform quantization of uniform input

Minimizing the error:

$$\frac{\partial \sigma_e^2}{\partial \hat{x}_m} = 0 \quad \text{for } \hat{x}_m = A + m\Delta + \frac{\Delta}{2}$$

optimal quantization point is the interval's midpoint, for all intervals

Uniform 3-Bit quantization function



Quantizer's mean square error:

$$\sigma_e^2 = \sum_{k=0}^{2^R-1} \int_{A+k\Delta}^{A+k\Delta+\Delta} \frac{(A+k\Delta+\Delta/2-\tau)^2}{B-A}\, d\tau$$

$$= 2^R \int_0^\Delta \frac{(\Delta/2-\tau)^2}{B-A}\, d\tau$$

$$= \frac{\Delta^2}{12}$$

Uniform quantization of uniform input

Quantizer's mean square error:

$$\sigma_e^2 = \sum_{k=0}^{2^R-1} \int_{A+k\Delta}^{A+k\Delta+\Delta} \frac{(A + k\Delta + \Delta/2 - \tau)^2}{B - A}\, d\tau$$

$$= 2^R \int_0^{\Delta} \frac{(\Delta/2 - \tau)^2}{B - A}\, d\tau$$

$$= \frac{\Delta^2}{12}$$

Quantizer's mean square error:

$$\sigma_e^2 = \sum_{k=0}^{2^R-1} \int_{A+k\Delta}^{A+k\Delta+\Delta} \frac{(A + k\Delta + \Delta/2 - \tau)^2}{B - A}\, d\tau$$

$$= 2^R \int_0^{\Delta} \frac{(\Delta/2 - \tau)^2}{B - A}\, d\tau$$

$$= \frac{\Delta^2}{12}$$

Error analysis

- error energy

$$\sigma_e^2 = \Delta^2/12, \qquad \Delta = (B - A)/2^R$$

- signal energy

$$\sigma_x^2 = (B - A)^2/12$$

- signal to noise ratio

$$\mathrm{SNR} = 2^{2R}$$

- in dB

$$\mathrm{SNR}_{\mathrm{dB}} = 10 \log_{10} 2^{2R} \approx 6R \text{ dB}$$

Error analysis

- error energy

$$\sigma_e^2 = \Delta^2/12, \qquad \Delta = (B - A)/2^R$$

- signal energy

$$\sigma_x^2 = (B - A)^2/12$$

- signal to noise ratio

$$\text{SNR} = 2^{2R}$$

- in dB

$$\text{SNR}_{\text{dB}} = 10 \log_{10} 2^{2R} \approx 6R \text{ dB}$$

Error analysis

- error energy

$$\sigma_e^2 = \Delta^2/12, \qquad \Delta = (B - A)/2^R$$

- signal energy

$$\sigma_x^2 = (B - A)^2/12$$

- signal to noise ratio

$$\mathrm{SNR} = 2^{2R}$$

- in dB

$$\mathrm{SNR_{dB}} = 10 \log_{10} 2^{2R} \approx 6R \text{ dB}$$

Error analysis

- error energy

$$\sigma_e^2 = \Delta^2/12, \qquad \Delta = (B - A)/2^R$$

- signal energy

$$\sigma_x^2 = (B - A)^2/12$$

- signal to noise ratio

$$\mathrm{SNR} = 2^{2R}$$

- in dB

$$\mathrm{SNR}_{\mathrm{dB}} = 10 \log_{10} 2^{2R} \approx 6R \text{ dB}$$

The "6dB/bit" rule of thumb

- a compact disk has 16 bits/sample:

$$\max \text{SNR} = 96\text{dB}$$

- a DVD has 24 bits/sample:

$$\max \text{SNR} = 144\text{dB}$$

The "6dB/bit" rule of thumb

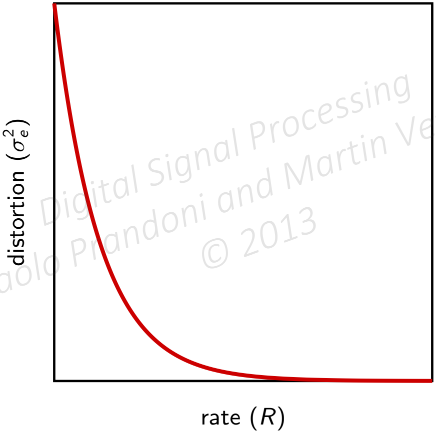
- a compact disk has 16 bits/sample:

$$\max \text{SNR} = 96\text{dB}$$

- a DVD has 24 bits/sample:

$$\max \text{SNR} = 144\text{dB}$$

Rate/Distortion Curve



Other quantization errors

If input is not bounded to $[A, B]$:

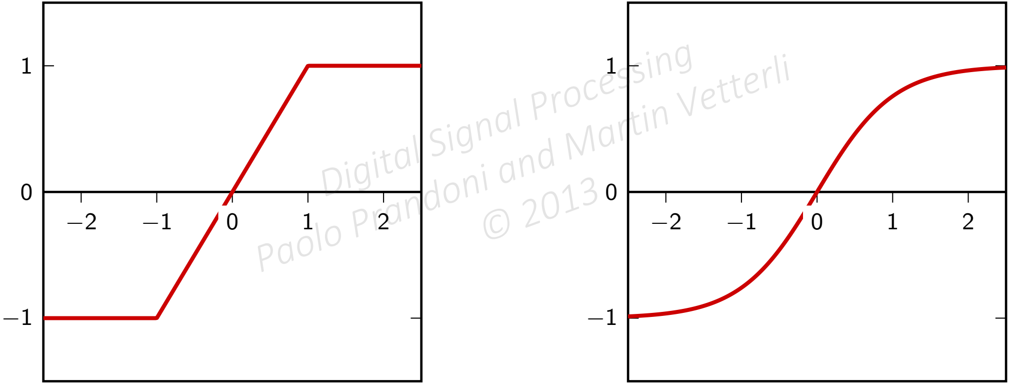
- clip samples to $[A, B]$: linear distortion (can be put to good use in guitar effects!)
- smoothly saturate input: this simulates the saturation curves of analog electronics

Other quantization errors

If input is not bounded to $[A, B]$:

- clip samples to $[A, B]$: linear distortion (can be put to good use in guitar effects!)
- smoothly saturate input: this simulates the saturation curves of analog electronics

Clipping vs saturation



Other quantization errors

If input is not uniform:

- use uniform quantizer and accept increased error.
  For instance, if input is Gaussian:

$$\sigma_e^2 = \frac{\sqrt{3}\pi}{2}\sigma^2\,\Delta^2$$

- design optimal quantizer for input distribution, if known (Lloyd-Max algorithm)
- use "companders"

Other quantization errors

If input is not uniform:

- use uniform quantizer and accept increased error.
  For instance, if input is Gaussian:

$$\sigma_e^2 = \frac{\sqrt{3}\pi}{2}\sigma^2\,\Delta^2$$

- design optimal quantizer for input distribution, if known (Lloyd-Max algorithm)
- use "companders"

Other quantization errors

If input is not uniform:

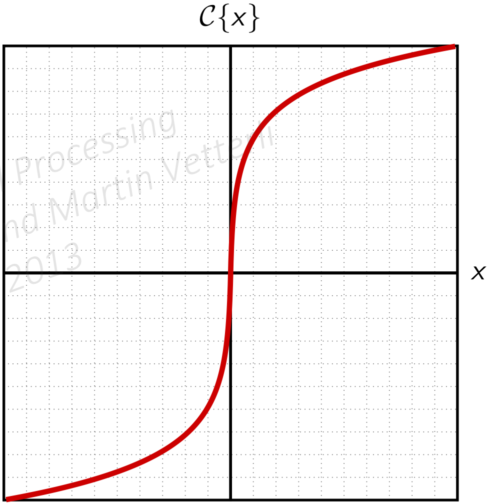
- use uniform quantizer and accept increased error.
  For instance, if input is Gaussian:

$$\sigma_e^2 = \frac{\sqrt{3}\pi}{2}\sigma^2\,\Delta^2$$

- design optimal quantizer for input distribution, if known (Lloyd-Max algorithm)
- use "companders"

μ-law compander

$$\mathcal{C}\{x[n]\} = \mathrm{sgn}(x[n])\frac{\ln(1+\mu|x[n]|)}{\ln(1+\mu)}$$



END OF MODULE 7.2

Digital Signal Processing

Module 7.3: A/D and D/A Conversion

Overview:

- Analog-to-digital (A/D) conversion
- Digital-to-analog (D/A) conversion

Overview:

- Analog-to-digital (A/D) conversion
- Digital-to-analog (D/A) conversion

From analog to digital

- sampling discretizes time
- quantization discretized amplitude
- how is it done in practice?

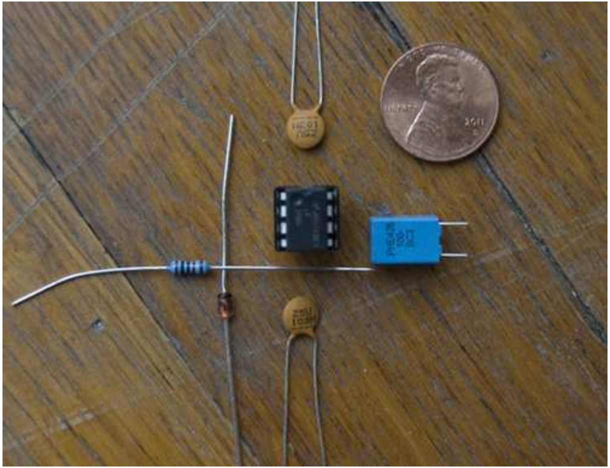
From analog to digital

- sampling discretizes time
- quantization discretized amplitude
- how is it done in practice?

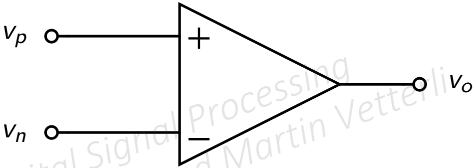
From analog to digital

- sampling discretizes time
- quantization discretized amplitude
- how is it done in practice?

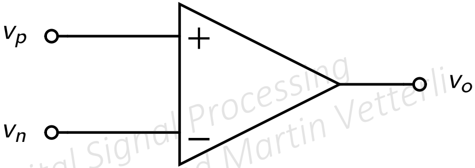
From analog to digital



A tiny bit of electronics: the op-amp



$$v_o = G(v_p - v_n)$$



$$v_o = G(v_p - v_n)$$

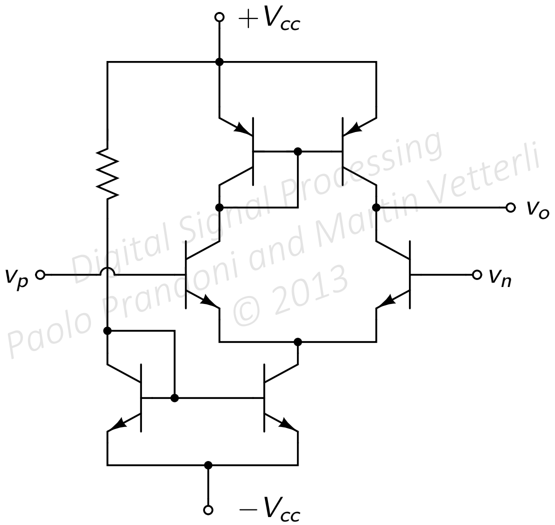
The two key properties

- infinite input gain ($G \approx \infty$)
- zero input current

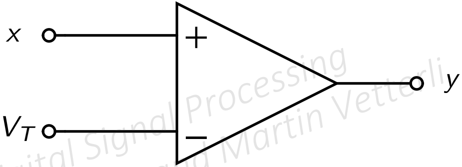
The two key properties

- infinite input gain ($G \approx \infty$)
- zero input current

Inside the box

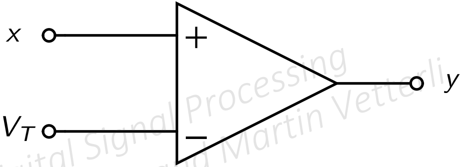


The op-amp in open loop: comparator



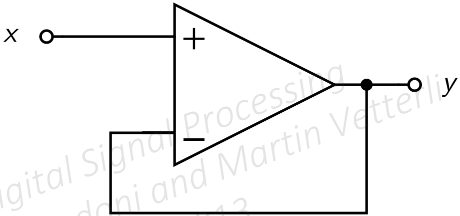
$$y = \begin{cases} +V_{cc} & \text{if } x > V_T \\ -V_{cc} & \text{if } x < V_T \end{cases}$$

The op-amp in open loop: comparator



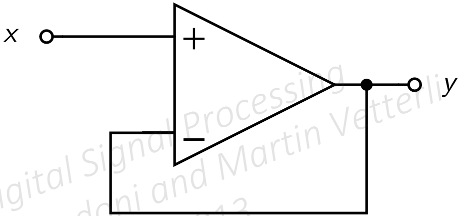
$$y = \begin{cases} +V_{cc} & \text{if } x > V_T \\ -V_{cc} & \text{if } x < V_T \end{cases}$$

The op-amp in closed loop: buffer



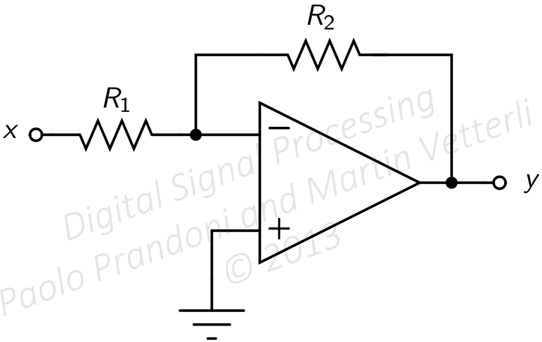
$$y = x$$

The op-amp in closed loop: buffer

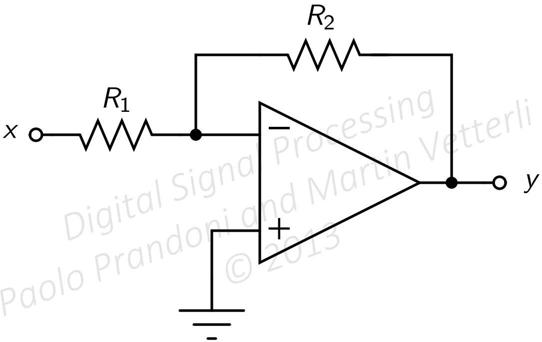


$$y = x$$

The op-amp in closed loop: inverting amplifier

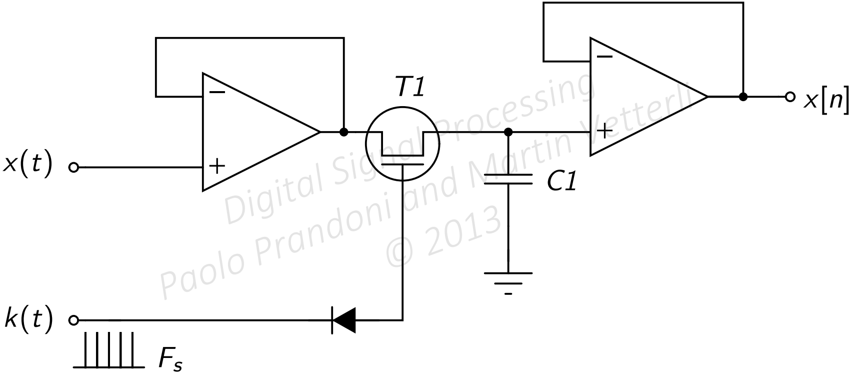


$$y = -(R_2/R_1)x$$

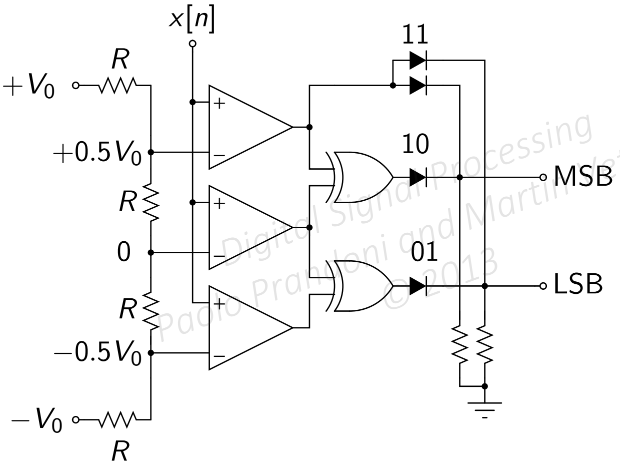


$$y = -(R_2/R_1)x$$

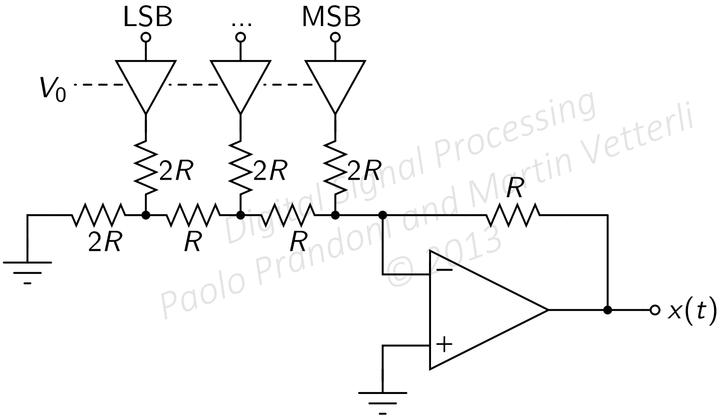
A/D Converter: Sample & Hold



A/D Converter: 2-Bit Quantizer



D/A Converter



END OF MODULE 7.3

END OF MODULE 7