NLA: **Cholesky** For matrix $[a_{11}, w^*; w, K] = R_1^T \left[I, 0; 0, K - \frac{ww^*}{a_{11}}\right][\alpha, w^*/\alpha; 0, I]$ we have a decomp: for $k = [1, m-1]$: for $j = [k+1, m]$ $R_{j,j:m} = R_{j,j:m} - \frac{R_{kj}}{R_{kk}} R_{k,j:m}$ endfor $R_{k,k:m} = \frac{R_{k,k:m}}{\sqrt{R_{kk}}}$ endfor. $\frac{m^3}{3}$. **Householder** for $k = [1, n]$: $x = A_{k:m,k}; v_k = sgn(x)\|x\|e_k + x; v_k = \frac{v_k}{\|v_k\|}$ for $j = [k, n]$ $A_{k:m,j} = A_{k:m,j} - 2v_k[v_k^* A_{k:m,j}]$ endfor endfor. $2mn^2 - \frac{2n^3}{3}$. **LU** $U = A, L = I$ for $k = [1, m-1]$: for $j = [k+1, m]$ $L_{jk} = \frac{U_{jk}}{U_{kk}}; U_{j,k:m} = U_{j,k:m} - (\frac{U_{jk}}{U_{kk}})U_{k,k:m}$ endfor endfor. $\frac{2m^3}{3}$. **MG-S** $V = A$;for $i = [1, n]$: $r_{ii} = \|v_i\|; q_i = \frac{v_i}{r_{ii}}$;for $j = [i+1, n]$ $v_j = v_j - (q_i^T v_j)q_i; r_{ij} = q_i^T v_j$ endfor endfor. $2mn^2$. **G-S** $V = A$;for $i = [1, n]$for $j = [1, i-1]$ $r_{ji} = q_j^T a_i; v_i = v_i - r_{ji}q_j$ endfor $r_{ii} = \|v_i\|; q_i = v_i/r_{ii}$ endfor. $2mn^2$?. **Givens** $3mn^2$ **SVD:** $= \sum_i^{r:=\min m,n} u_i \sigma_i v_i^T$. **Construct SVD:** Write $A^TA = V\Gamma V^T \to (AV)^T(AV) = \Sigma^2$, and $AV\Sigma^{-1} := U$ **Bounds:** $\|ABB^{-1}\| \ge \|AB\|\|B^{-1}\| \to \|A\|/\|B^{-1}\| \ge \|AB\|$. **Weyls:** $\sigma_i(A+B) = \sigma_i(A) + [-\|B\|, \|B\|]$ **Norms:**$\|A\|_F = \sqrt{\sum_i (\sigma_i)^2} = \sqrt{Tr(AA^T)}$, $\|A\|_\infty =$ max row sum. **Rev Δ Ineq:** $\|A-B\| \ge |\|A\| - \|B\||$ **Low-Rank:** For $A \in \mathbb{R}^{m\times n}$ $\min\|A-B\| = \|A - A_r\|$. Proof via $B := B_1B_2^T$ with $B_1 \in \mathbb{R}^{m\times r}$; $\exists W s.t. B_2^TW = 0$ with null(W)$\ge n-r$. Then $\exists\, x_V, x_W s.t. V_{r+1}x_V = -Wx_W$. So $\|A-B\| = \|AW\| \ge \|AV_{r+1}x_V\| \ge \sigma_{r+1}$ For reverse $B := A_r$ **Courant:** $\sigma_i = \max_{dim(S)=i}\{\min_x \|Ax\|/\|x\|\}$. Proof via $V_i = [v_i \dots v_n]$, so dim(S)+dim(V_i) $= n+1$ so $\exists\, w \in S \cap V_i$. Then $\|Aw\| \le \sigma_i$. For reverse take $w = v_i$ when $S = [v_1 \dots v_i]$ **Courant Application:** $\sigma_i([A_1; A_2]) \ge \max(\sigma_i(A_1), \sigma_i(A_2))$ **Schur:** Take $Av_1 = \lambda_1 v_1$; construct $U_1 = [v_1, V_\perp] \to AU_1 = U_1[e_1, X]$. Repeat. **Back Subst:** For $Ux = y$ we have $x_{n-i} = \left(y_{n-i} - \sum_{n-i+1}^n u_{n-i,j}x_j\right)/u_{n-i,n-i}; O(i)$ per iteration so $O(n^2)$ total. **Backwards Stable:** When $\hat{f}(x) = f(x + \Delta x)$ with $\|\Delta x\|/\|x\| \le O(\varepsilon)$ **Conditioning** $\kappa_2(A) = \sigma_1/\sigma_n = \|A\|\|A^{-1}\|$ **Similarity:** $A \to P^{-1}AP$, same λ. **Elementary L:** Define via $L_i(m) = I - me_i^T$

NPDE: **Def'n:** With $u_{tt} - c^2u_{xx} = f$ have $\Delta x = (b-a)/J, \Delta t = T/M, x_j = a + j\Delta x, t = m\Delta t$. I.C: $U_j^0 = u_0(x_j), U_j^1 = U_j^0 + u_1(x_j)\Delta t, U_0^m = U_J^m = 0$ **Hyp Impl:** $(A-B, A) = \frac{1}{2}(\|A\|^2 - \|B\|^2) + \frac{1}{2}\|A-B\|^2$ with $A := U^{m+1} - U^m, B := U^m - U^{m-1}$ (T);$(-D_x^+D_x^-U^{m+1}, U^{m+1} - U^m) = (D_x^-U^{m+1} - D_x^-U^m, D_x^-U^{m+1})$ (X). Then $\frac{1}{2\Delta t^2}(\|U^{m+1} - U^m\|^2 - \|U^m - U^{m-1}\|^2) + \frac{\Delta t^2}{2\Delta t^2}\|U^{m+1} - 2U^m + U^{m-1}\|^2 + \frac{c^2}{2}\left(\|D_x^-U^{m+1}\|^2 - \|D_x^-U^m\|^2\right) + \frac{c^2\Delta t^2}{2\Delta t^2}\|D_x^-(U^{m+1} - U^m)\|^2 = (f, U^{m+1} - U^m)$. Then $M^2(U^m) := \left\|\frac{U^m - U^{m-1}}{\Delta t}\right\|^2 + c^2\|D_x^-U^{m+1}\|^2$. Write green as $\le \|f\|\|U^{m+1} - U^m\| = \sqrt{\Delta t T}\|f\|\sqrt{\frac{\Delta t}{T}}\left\|\frac{U^{m+1} - U^m}{\Delta t}\right\| \le \frac{\Delta t T}{2}\|f\|^2 + \frac{\Delta t}{2T}\left\|\frac{U^{m+1} - U^m}{\Delta t}\right\|^2$. Then $(1 - \frac{\Delta t}{T})M^2(U^m) \le M^2(U^{m-1}) + \Delta t T\|f\|^2 \to M^2(U^m) \le (1 + \frac{2\Delta t}{T})M^2(U^{m-1}) + 2\Delta t T\|f\|^2$. Use $a_m \le \alpha^m a_0 + \sum_{k=1}^m \alpha^{m-k}b_k$ so $M^2 \le e^2M^2(U^0) + 2e^2T\sum_{k=1}^m \Delta t\|f\|^2$ **Hyp Expl:** 1st rewrite in terms of $D_t^{+-}(\Delta t)^{-2}U_j^m + \frac{c^2(\Delta t)^2}{4}D_x^{+-}((\Delta t)^{-2}D_t^{+-}U_j^m) - (c^2/4)D_x^{+-}\left(U_j^{m+1} + 2U_j^m + U_j^{m-1}\right)$. Then use $(D(A-B), A+B) = (DA, A) - (DB, B)$; $(D(A+B), A-B) = (DA, A) - (DB, B)$ by multiplying by $U^{m+1} - U^{m-1}$. Finally WTS $\|V_m\|^2 - \frac{c^2(\Delta t)^2}{4}\|D_x^-V^m\|^2 \ge 0$. Done by noticing: $\|D_x^-V^m\|^2 = \sum_i^J \Delta x|D_x^-V_j^m|^2 = \frac{1}{\Delta x}\sum_i^J (V_j^m - V_{j-1}^m)^2 \le 2/\Delta x\sum_i^J (V_j^m)^2 + (V_{j-1}^m)^2 = 4/\Delta x^2\sum_i^{J-1}\Delta x\,(V_j^m)^2$. Eventually show $N^2(U^m) := \left(\left(I + \frac{c^2\Delta t^2}{2}D_x^{+-}\right)\frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t}\right) + c^2\left\|D_x^-\frac{U^{m+1} + U^m}{2}\right\|^2 \to N^2(U^m) = N^2(U^{m-1}) + (f, U^{m+1} - U^m)$ **Max Principle:** For $-\Delta u = f \le 0 \to \max u \in \partial D$. First show contradiction assuming $LU = f < 0$, then try some auxillary function $\psi = U + \alpha(T_{\max})g(x_i, y_i)$ s.t. $L\psi < 0$ so $\max\psi = \max_{\in\partial D}\psi$. Gets $\max e_{i,j}$; change to $-\alpha$ for $\min e_{i,j}$. **P-F Ineq:** $\|V\|_h^2 \le c_\star\|D_x^-V\|^2$. For 2D: $|V_j^m| = |\sum_{\alpha=1}^j h(D_x^-V_\alpha^m)|^2 \le jh\sum_{\alpha=1}^{N-1}h|D_x^-V_\alpha^m|^2 \to \|V\|_h^2 = \sum_{j=1}^{N-1}h|V_j^m|^2 \le \sum_{j=1}^{N-1}jh^2\sum_{\alpha=1}^{N-1}h|D_x^-V_\alpha^m|^2 \le \frac{1}{2}\sum_{j=1}^N h|D_x^-V_j^m|^2$. Use blue and add for x, y for $c_\star = 0.25$. **Weak Deriv:** w is a weak derivative of u if $\int dx\, wv = (-1)^{|\alpha|}\int dx\, u(D^\alpha v)$ **Parseval:** $\int dk\, \hat{u}(k)\overline{v(k)} = \int dk\, \overline{v(k)}\left(\int dx\, u(x)e^{-ixk}\right) = \int dx\, u(x)\overline{\left(\int dk\, v(k)e^{-ixk}\right)} = \int dx\, u(x)\overline{\hat{v}(x)}$. Now $v(k) := \overline{\hat{u}(k)} = \overline{F[u(k)]} = \overline{\int dk\, u(k)e^{-ixk}} = \int dk\, \overline{u(k)}e^{ixk} = 2\pi F^{-1}\left[\overline{u(k)}\right] \Rightarrow \hat{v}(x) = 2\pi\overline{u(x)}$ **Iterative:** If $U^{j+1} = U^j - \tau(AU^j - F) \to U - U^j = (I - \tau A)^j(U - U^0)$ so $\|U - U^j\| \le \|I - \tau A\|^j\|U - U^0\|$. $\|I - \tau A\| = \sigma_1 = |\lambda_1|$ as symmetric. If $\lambda \in [\alpha, \beta]$ then $\lambda_1 \le \max\{|1 - \tau\alpha|, |1 - \tau\beta|\}$. Attained when $\tau = 2/(\alpha + \beta) \to \lambda_1 = \frac{\beta - \alpha}{\beta + \alpha}$. For $-u'' + cu = f$ we have $\lambda_k = c + \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right)$. Lower bound via noting $\sin(y) \ge \frac{2\sqrt{2}}{\pi}y$ at $y = \frac{\pi}{4} \to \lambda_k \ge c + 8$ **Errors:** $(AV, V)_h \ge \|D_x^-V\|_h^2$ & PF Ineq $\to (AV, V)_h \ge \|V\|_h^2/c_\star$. Then $(AV, V)_h(1 + c_\star) \ge \|V\|_{1,h}^2 \to (AV, V)_h \ge c_0\|V\|_{1,h}^2$. Now $c_0\|V\|_{1,h}^2 \le (AV, V)_h \le \|f\|_h\|V\|_h \le \|f\|_h\|V\|_{1,h} \to \|V\|_{1,h} \le \|f\|_h/c_0$. (Use $f := AV \to \|e\|_{1,h} \le \|T\|_h/c_0$). **Scheme:** For e.g. on $(0,1)^2$

write $-\Delta u + u = -1$ and $u|_{\partial D} = b$, with x_j, t_m, we have scheme for $1 \leq j \leq J-1, 1 \leq m \leq M-1$, and initial conditions $U_{j,0} = U_{j,N} = b$ for $0 \leq j \leq N$, and $U_{0,m} = U_{J,m} = b$ for $1 \leq m \leq M-1$. Might need to define scheme for $m = 0$ if e.g. θ scheme with no $U_{i,j-1}$ terms such as $u_t = u_{xx}$ **Non Uniform:** We have $h_{i+1} := x_{i+1} - x_i, h_i := x_i - x_{i-1} \to \hbar_i = \frac{1}{2}(h_{i+1} + h_i)$ so $D_x^+ D_x^- U_j^m = \frac{1}{\hbar_i}([U_{j+1} - U_j]/h_{i+1} - [U_j - U_{j-1}]/h_i)$. $\mathbf{L}_2$ **F'n:** We approximate $f_{i,j} \to \frac{1}{h^2}\int_{K_{i,j}} f$ where $K_{i,j} = [x_i \pm 1/2, y_i \pm 1/2]$. For errors: NB that $Au - AU = -D_x^+ D_x^- u - D_y^+ D_y^- u + cu - T(\Delta u + cu)$. NB $Tu_{xx} = D_x^+ \frac{1}{h} \int u_x(x_i - \frac{h}{2})dy := D_x^+ \alpha_x$ so $Ae_{i,j} = D_x^+ \phi_1 + D_x^- \phi_2 + \psi$, with $\phi_1 := \alpha_x - D_x^- u, \psi := cu - Tcu$. Now NB $c_0 \|e\|_{1,h}^2 \leq (Ae, e)$. Bound $(D_x^+ \phi_1, e)$ via $\leq ||\phi_1]|_x ||D_x^- e]|_h$ so $c_0 \|e\|_{1,h}^2 = (||\phi_1]|_x^2 + ||\phi_2]|_y^2 + \|\psi\|_h^2)\|e\|_{1,h}$ **L-Bounds:** $\frac{|f(u)-f(v)|}{|u-v|} \leq |f'|$ **Hyperbolic Signs** For $u_t + au_x$ when using $[a]_\pm$ we write $D_t^- U_j^m + [a]_+ D_x^- U_j^m + [a]_- D_x^+ U_j^m$. Eventually get $U_j^{m+1} = \left(1 - \frac{|a|\Delta T}{\Delta x}\right) U_j^m + \frac{[a]_+ \Delta t}{\Delta x} U_{j-1}^m - \frac{[a]_- \Delta t}{\Delta x} U_{j+1}^m$. Then via CFL assumption $\frac{a(\|U^0\|_\infty)\Delta t}{\Delta x} \leq 1 \to |a(U)| \leq a(|U|) \leq a(\|U\|_\infty)$ so $\left\|U^{m+1}\right\|_\infty \leq \left\|U^0\right\|_\infty$ **Summation by Parts:** We have $(-D_x^+ D_x^- U, U) = -\sum_{i=1}^{N-1} h(D_x^+ D_x^- U_i)U_i = -\sum_{i=1}^{N-1} \frac{U_{i+1} - U_i}{h} U_i + \sum_{i=1}^{N-1} \frac{U_i - U_{i-1}}{h} U_i = -\sum_{i=2}^{N} \frac{U_i - U_{i-1}}{h} U_{i-1} + \sum_{i=1}^{N-1} \frac{U_i - U_{i-1}}{h} U_{i-1} = -\sum_{i=1}^{N} \frac{U_i - U_{i-1}}{h} U_{i-1} + \sum_{i=1}^{N} \frac{U_i - U_{i-1}}{h} U_{i-1} = \sum_{i=1}^{N} h|D_x^- U_i|^2 = ||D_x^- U]|_h^2$, where blue from shift of index and green from $U_0 = U_N = 0$.