NS: **Inverse 2 × 2:** For $A := [a,b;c,d], A^{-1} := \frac{1}{ad-bc}[d,-b;-c,a]$ **Radial:** $r\dot{r} = x\dot{x} + y\dot{y}$, $\dot{\theta} = \left[\tan^{-1}(y/x)\right]' = \frac{x\dot{y}-\dot{x}y}{x^2+y^2}$ **Classifications:** Node: $\lambda_i \in \mathbb{R}, \Pi\lambda_i > 0$ Centre: $\lambda_i = \pm ib$ Focus: $\lambda_i = a \pm ib$ Hyperbolic: $\mathrm{Re}(\lambda) \neq 0 \to$ hyperbolic. If all $\lambda < 0$ for $\mathrm{Spec}(Df(x_0))$ then A-Stable **Invariant Set:** $\phi_t(S) \subseteq S$ **Lim Pts:** ω pt. if $\lim_{t\to\infty}\phi(x) = p$, i.e. flows *tend to p*. α pt. if $\lim_{t\to-\infty}\phi(x) = p$. **Attracting Set:** A set $A \subseteq S$ if $\exists$ neighbourhood U s.t. $\phi(U) \subseteq U \forall t \geq 0$, and $A = \cap_{t>0}\phi(U)$ **Dense Orbits:** If $\forall\, \epsilon > 0, x \in A$ with A an attracting set, $\exists \tilde{x} \in \Gamma s.t. |x - \tilde{x}| < \epsilon$. I.e. a dense orbit goes as close as needed to *any* point within A **Attractor:** An attracting set with a dense orbit. **Lyapunov Stable:** If $\forall \epsilon > 0, \exists\, \delta > 0 s.t. \forall\, x \in B_\delta, t \geq 0, \phi(t) \in B_\delta$ (i.e. points stay close within region). **Asymptotically Stable:** If L-Stable and $\exists\, \delta > 0 s.t. \phi(x) \to x_0 \forall x \in B_\delta$ **Lyapunov F'n:** $V(x_0) = 0, V(x) > 0 \forall\, x \neq x_0$. Then if $\dot{V} < 0 \to$ A-Stable, or if $\dot{V} \leq 0$ L-Stable. **Stable Manifold:** If spectrum of $Df(x_0)$ has k eigvals with positive real parts, and $n-k$ with negative, then $\exists$ an $n-k$ dim manifold tangent to E^s s.t. for all $t > 0$ $\phi(W^s_{loc}) \subseteq W^s_{loc}$, and $\forall x \in W^s_{loc} \phi(x) \to x_0$ as time increases. Repeat for k-dim unstable manifold but for negative time. Then, define e.g. global stable manifold by $W^s(x_0) := \cup_{t\leq 0}\phi_t(W^s_{loc})$. Note that we search backwards in time for stable, and forwards for unstable! **Centre Manifold:** If x_0 not hyperbolic (0 real part), then E^c is the centre subspace. Then $\exists W^c$ parallel to E^c, of class C^r, and invariant under flow. Want bifurcation at $\mu = 0$, so with change of variables first find eigvecs v_1, v_2. Then, construct $P := [v_1, v_2]$ s.t. $\vec{\boldsymbol{x}} = P\vec{\boldsymbol{\xi}}$. NOTE: first v_i in P is *always* associated with Re(λ)= 0. Solve for $\vec{\boldsymbol{\xi}}$ and then expand with $\eta = h(\xi, \tilde{\mu})$ **Alt. Centre Manifold:** If vector $v_1 \sim E^c = [a,b]^T$ then we have $y = bx/a$ (e.g. $[1,1]^T \to y = x$. If bifurcation at $\mu = \alpha$ then have $\mu = \tilde{\mu} + \alpha$ s.t. bifurcation when $\tilde{\mu} = 0$. Then have $\dot{x}(x, y, \tilde{\mu}) = \ldots$ etc. Next, set up $y = h(x, \tilde{\mu}) = bx/a + b_1\tilde{\mu} + b_2\tilde{\mu}^2 + a_2x^2 + c_2\tilde{\mu}x$ and proceed as usual, s.t. y is along E^c. **Transcritical Bifurcation:** Always two points, change type at origin. E.g. $\dot{x} = \mu x - x^2$ **Saddle-Node:** E.g. $\dot{x} = \mu - x^2$ Bifurcation begins to exist at origin. **Supercritical:** E.g. $\dot{x} = \mu x - x^3$, where stable$\to$ 2× stable and one unstable. **Subcritical:** E.g. $\dot{x} = -\mu x + x^3$, where unstable$\to$ 2× unstable and one stable. **General co-dim 1:** If $\dot{x} = f$ then $\dot{x} = \mu f_u + 0.5x^2 f_{xx} + x\mu f_{x\mu} + 0.5\mu^2 f_{\mu\mu}$. Generally this is a saddle-node but if $f_u = 0$ we have $\dot{x} = x\mu f_{x\mu} + 0.5x^2 f_{xx}$, which is a transcritical. However if $x = -x$ then $\dot{x} = x(\mu f_{x\mu} + \ldots) + x^3(f_{xxx}/6 + \ldots) \to$ pitchfork. Saddle-node stable under perturbations! **Homoclinic Orbits** Sum of roots of cubic = - coeff. of x^2

FPDE: **Types:** 1^{st} : $\exists$ scale s.t. solution found, not so for 2^{nd}. **Heat:** $\hat{T} = u(\hat{T}_\infty - \hat{T}_{-\infty}) + \hat{T}_{-\infty}$ **Oil Spread:** Dims: $x = x_f + \varepsilon\xi, t = \tau$ **Ground Spread:** $(1-s)\phi h_t + Q_x = 0; Q \sim -hh_x, 0 < x_s < x_f$. Have $h(x_f) = 0, h_t(x_s) = 0$, and $hh_x|_{x=0,x_f} = 0$ (i.e. no flux at centre and front), and h, hh_x cont. at joint. **Expansions:** Let $\xi = z + \epsilon\eta$ for perturbations **Scale:** Try $x = x_f + \epsilon\xi$ for groundwater **Stefan:** $S_0 = C(T_1 - T_m)/L$, condition $= \rho L\dot{s} = kT_x|^{s+}_{s-}$ **1ph Stefan:** Bar $= T_h|liq|_s sol|INS$. Use $T = T_m + (T_1 - T_m)u$ s.t. $S_0 u_t = u_{xx}, u = 1$ @ $x = 0, \{\dot{s} = -u_x, u = 0\}$ @ $x = s, s(0) = 0$. Sim. sol is $s = \beta\sqrt{t}, f = f(x/\sqrt{t})$ **2ph Stefan:** Use $T = T_m + (T_1 - T_m)u$ s.t. $S_0 u_t = u_{xx}$ @ $0 < x < s, (S_0/\kappa)u_t = u_{xx}$ @ $s < x < 1, u = 1$ @ $x = 0, u_x = 0$ @ $x = 1, \left\{\dot{s} = Ku_x|_{s_+} - u_x|_{s_-}, u = 0\right\}$ @ $x = s, \{s = 0, u = -\theta\}$ @ $x = 0$. Here $\theta := (T_m - T_0)/(T_1 - T_m), \kappa := c_1k_1/(c_2k_2), K := k_2/k_1$ Sim. sol is $s = \beta\sqrt{t}, f = f(x/\sqrt{t})$ **2-Dim:** $U_n = \hat{n}\cdot u = K(u_2)_n - (u_1)_n$. If $x = f(y,t)$ then $\hat{n} := \boldsymbol{\nabla}(x - f) = [1, -f_y]^T/\sqrt{1 + f_y^2}$ **Welding:** Have $0 < s_2 < s_1$. Have cold $x = a$, no flux $x = 0$. $\theta = 1$ in liquid. In mush $\rho L\theta_t = J^2/\sigma$, CoE $\to \theta\rho L\dot{s} + kT_x|^{s+}_{s-} = 0$. Have θ cont. $(= 0)$ at s_1. I.e. we have $S_0 u_t = u_{xx} + q, u_x = 0$ @ $x = 0, u = -1$ @ $x = 1, \theta = 0$ @ $x = s_1$. Also $\theta_t = q$ in mush.

FMM: **Integral Constraint** If $J[y] = \int F dx$ with $\int G dx = C$ then $\tilde{J}[y] = \int F - \lambda G dx$ **Hamiltonian:** $H := y'F_{y'} - F \to H' = -F_x$. If $F = F(y, \dot{y})$ then $H = C$ **Hamilton's Eqs:** $p := F_{y'}, q = y$ and so $p' = -H_q, q' = H_p$ **Free Boundary:** $J[y,b] = \int_a^b F(x,y,y')dx$ where b free. Expand with $y + \epsilon\eta, b + \epsilon\beta \to J = J_0 + \epsilon\left\{\int_a^b \eta F_y + \eta' F_{y'} dx + \beta F(b, y(b), y'(b))\right\}$ If $y(b) = d \to d = y(b + \epsilon\beta) + \epsilon\eta(b + \epsilon\beta) = y(b) + \epsilon(\beta y'(b) + \eta(b))$ so $\eta(b) = -\beta y'(b)$. IVP on integral so $\beta\left[F - y'F_{y'}\right]_{x=b} + \int(\ldots) = 0$ so $F = y'F_{y'}$ at free boundary. **Control:** Have $\int \xi h_x + \eta h_u dt = 0, \dot{\xi} = \xi f_x + \eta f_u$. Sub for η, IVP s.t. $\frac{d}{dt}\frac{h_u}{f_u} = h_x - f_x\frac{h_u}{f_u}$ and $\dot{x} = f$ **Hamiltonian (Control):** $H := f\frac{h_u}{f_u} - h$ s.t. $\dot{H} = \frac{h_u}{f_u}f_t - h_t \to$ autonomous if $h_t = f_t = 0$. **Fredholm Alt Integ Eqs.** For $y = f + \int K(x,t)y(t)dt$ we have ONE (N) has a unique sol $y = 0$ if $f = 0$, and adjoint has unique sol, or TWO (H) as sols $y_1 \ldots y_r$iff $\forall$ solutions to H^*, z_i, we have $< f, z_i >= 0$. EX: Solve $y = f + \lambda\int\sin(x+t)y(t)dt$. Unique sol iff (H) has trivial sol $\to X_1 = \int y\cos(t) = \int\cos(t)y_H(t) \to$ solve $[1, -\lambda\pi; -\lambda\pi, 1][X_1, X_2]^T = [0,0]^T \to$ unique sol if $\lambda \neq \pm 1/\pi$. In this case $X_1 = \int\cos(x)y_N(x) = \lambda\pi X_2 + \int f(x)\cos(x)$, and similar for X_2. Invert matrix and solve. If non-unique sol, then find sols to (H) first. If $\lambda = 1/\pi$ then $X_1 = X_2 = X$ so $Ly = y - \pi^{-1}(\sin(x) + \cos(x))\int\cos(x)y(x)dx$ with sols $y = c_1(\sin(x) + \cos(x))$ by inspection. Problem self adjoint so $Ly = 0 = L^*w$ so need $\int f(x)w(x) = 0$ i.e. $\int f(x)(\sin(x) + \cos(x)) = 0$, repeat for $\lambda = -1/\pi$. Then $y = y_p(x) + \sum_i y_{h,i}(x)$ **Fred Diff Eq.** For nonunique sol to exist, need $< Ly, w >=< f, w > \forall\, w s.t. L^*w = 0$