

**NLA: Cholesky** For matrix  $[a_{11}, w^*; w, K] = R_1^T \begin{bmatrix} I, 0; 0, K - \frac{ww^*}{a_{11}} \end{bmatrix} [\alpha, w^*/\alpha; 0, I]$  we have a decomp:  
**for**  $k = [1, m-1]$  : **for**  $j = [k+1, m]$   $R_{j,j:m} = R_{j,j:m} - \frac{R_{kj}}{R_{kk}} R_{k,j:m}$  **endfor**  $R_{k,k:m} = \frac{R_{k,k:m}}{\sqrt{R_{kk}}}$  **endfor**  $\frac{m^3}{3}$ . **Householder** **for**  $k = [1, n]$  :  $x = A_{k:m,k}; v_k = \text{sgn}(x)\|x\|e_k + x; v_k = \frac{v_k}{\|v_k\|}$  **for**  $j = [k, n]$   
 $A_{k:m,j} = A_{k:m,j} - 2v_k[v_k^* A_{k:m,j}]$  **endfor** **endfor**.  $2mn^2 - \frac{2n^3}{3}$ . **LU**  $U = A, L = I$  **for**  $k = [1, m-1]$  : **for**  
 $j = [k+1, m]$   $L_{jk} = \frac{U_{jk}}{U_{kk}}; U_{j,k:m} = U_{j,k:m} - (\frac{U_{jk}}{U_{kk}})U_{k,k:m}$  **endfor** **endfor**.  $\frac{2m^3}{3}$ . **MG-S**  $V = A$ ; **for**  $i = [1, n]$  :  
 $r_{ii} = \|v_i\|; q_i = \frac{v_i}{r_{ii}}; \text{for } j = [i+1, n]$   $v_j = v_j - (q_i^T v_j)q_i; r_{ij} = q_i^T v_j$  **endfor** **endfor**.  $2mn^2$ . **G-S**  $V = A$ ; **for**  
 $i = [1, n]$  **for**  $j = [1, i-1]$   $r_{ji} = q_j^T a_i; v_i = v_i - r_{ji}q_j$  **endfor**  $r_{ii} = \|v_i\|; q_i = v_i/r_{ii}$  **endfor**.  $2mn^2$ ?. **Givens**  
 $3mn^2$  **SVD**:  $= \sum_{i=\min(m,n)}^r u_i v_i^T$ . **Bounds**:  $\|ABB^{-1}\| \geq \|AB\|\|B^{-1}\| \rightarrow \|A\|/\|B^{-1}\| \geq \|AB\|$ .  
**Weyls**:  $\sigma_i(A+B) = \sigma_i(A) + [-\|B\|, \|B\|]$  **Norms**:  $\|A\|_F = \sqrt{\sum_i (\sigma_i)^2} = \sqrt{\text{Tr}(AA^T)}$ ,  $\|A\|_\infty =$   
max row sum. **Rev  $\Delta$  Ineq**:  $\|A-B\| \geq \||A\| - \|B\||$  **Low-Rank**: For  $A \in \mathbb{R}^{m \times n}$   $\min \|A-B\| =$   
 $\|A-A_r\|$ . Proof via  $B := B_1 B_2^T$  with  $B_1 \in \mathbb{R}^{m \times r}; \exists W \text{ s.t. } B_2^T W = 0$  with  $\text{null}(W) \geq n-r$ . Then  
 $\exists x_V, x_W \text{ s.t. } V_{r+1} x_V = -W x_W$ . So  $\|A-B\| = \|AW\| \geq \|A V_{r+1} x_V\| \geq \sigma_{r+1}$  For reverse  $B := A_r$   
**Courant**:  $\sigma_i = \max_{\dim(S)=i} \{\min_x \|Ax\|/\|x\|\}$ . Proof via  $V_i = [v_i \dots v_n]$ , so  $\dim(S) + \dim(V_i) = n+1$   
so  $\exists w \in S \cap V_i$ . Then  $\|Aw\| \leq \sigma_i$ . For reverse take  $w = v_i$  when  $S = [v_1 \dots v_i]$  **Schur**: Take  
 $Av_1 = \lambda_1 v_1$ ; construct  $U_1 = [v_1, V_\perp] \rightarrow AU_1 = U_1[e_1, X]$ . Repeat. **Back Subst**: For  $Ux = y$  we  
have  $x_{n-i} = (y_{n-i} - \sum_{n-i+1}^n u_{n-i,j} x_j) / u_{n-i,n-i}; O(i)$  per iteration so  $O(n^2)$  total. **Backwards Sta-**  
**ble**: When  $\hat{f}(x) = f(x + \Delta x)$  with  $\|\Delta x\|/\|x\| \leq O(\varepsilon)$  **Conditioning**  $\kappa_2(A) = \sigma_1/\sigma_n = \|A\|\|A^{-1}\|$   
**Similarity**:  $A \rightarrow P^{-1}AP$ , same  $\lambda$ . **Elementary L**: Define via  $L_i(m) = I - m e_i^T$   
**NPDE: Def'n**: With  $u_{tt} - c^2 u_{xx} = f$  have  $\Delta x = (b-a)/J, \Delta t = T/M, x_j = a + j\Delta x, t = m\Delta t$ .  
I.C:  $U_j^0 = u_0(x_j), U_j^1 = U_j^0 + u_1(x_j)\Delta t, U_0^m = U_j^m = 0$  **Hyp Impl**:  $(A-B, A) = \frac{1}{2}(\|A\|^2 - \|B\|^2) +$   
 $\frac{1}{2}\|A-B\|^2$  with  $A := U^{m+1} - U^m, B := U^m - U^{m-1}$  (T);  $(-D_x^+ D_x^- U^{m+1}, U^{m+1} - U^m) = (D_x^- U^{m+1} -$   
 $D_x^- U^m, D_x^- U^{m+1})$  (X). Then  $\frac{1}{2\Delta t^2}(\|U^{m+1} - U^m\|^2 - \|U^m - U^{m-1}\|^2) + \frac{\Delta t^2}{2\Delta t^2}\|U^{m+1} - 2U^m + U^{m-1}\|^2 +$   
 $\frac{c^2}{2}(\|D_x^- U^{m+1}\|^2 - \|D_x^- U^m\|^2) + \frac{c^2 \Delta t^2}{2\Delta t^2}\|D_x^-(U^{m+1} - U^m)\|^2 = (f, U^{m+1} - U^m)$ . Then  $M^2(U^m) :=$   
 $\left\| \frac{U^m - U^{m-1}}{\Delta t} \right\|^2 + c^2 \|D_x^- U^{m+1}\|^2$ . Write green as  $\leq \|f\| \|U^{m+1} - U^m\| = \sqrt{\Delta t T} \|f\| \sqrt{\frac{\Delta t}{T}} \left\| \frac{U^{m+1} - U^m}{\Delta t} \right\| \leq$   
 $\frac{\Delta t T}{2} \|f\|^2 + \frac{\Delta t}{2T} \left\| \frac{U^{m+1} - U^m}{\Delta t} \right\|^2$ . Then  $(1 - \frac{\Delta t}{T})M^2(U^m) \leq M^2(U^{m-1}) + \Delta t T \|f\|^2 \rightarrow M^2(U^m) \leq (1 +$   
 $\frac{2\Delta t}{T})M^2(U^{m-1}) + 2\Delta t T \|f\|^2$ . Use  $a_m \leq \alpha^m a_0 + \sum_{k=1}^m \alpha^{m-k} b_k$  so  $M^2 \leq e^2 M^2(U^0) + 2e^2 T \sum_{k=1}^m \Delta t \|f\|^2$   
**Hyp Expl**: 1st rewrite in terms of  $D_t^{+-}(\Delta t)^{-2} U_j^m + \frac{c^2(\Delta t)^2}{4} D_x^{+-}((\Delta t)^{-2} D_t^{+-} U_j^m) -$   
 $(c^2/4) D_x^{+-}(U_j^{m+1} + 2U_j^m + U_j^{m-1})$ . Then use  $(D(A-B), A+B) = (DA, A) - (DB, B);$   
 $(D(A+B), A-B) = (DA, A) - (DB, B)$  by multiplying by  $U^{m+1} - U^{m-1}$ . Finally WTS  $\|V_m\|^2 -$   
 $\frac{c^2(\Delta t)^2}{4} \|D_x^- V^m\|^2 \geq 0$ . Done by noticing:  $\|D_x^- V^m\|^2 = \sum_i^J \Delta x |D_x^- V_j^m|^2 = \frac{1}{\Delta x} \sum_i^J (V_j^m - V_{j-1}^m)^2 \leq$   
 $2/\Delta x \sum_i^J (V_j^m)^2 + (V_{j-1}^m)^2 = 4/\Delta x^2 \sum_i^{J-1} \Delta x (V_j^m)^2$ . Eventually show  $N^2(U^m) :=$   
 $\left( \left( I + \frac{c^2 \Delta t^2}{2} D_x^{+-} \right) \frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t} \right) + c^2 \left\| D_x^- \frac{U^{m+1} + U^m}{2} \right\|^2 \rightarrow N^2(U^m) = N^2(U^{m-1}) + (f, U^{m+1} -$   
 $U^m)$  **Max Principle**: For  $-\Delta u = f \leq 0 \rightarrow \max u \in \partial D$ . First show contradiction assuming  
 $LU = f < 0$ , then try some auxillary function  $\psi = U + \alpha (T_{\max}) g(x_i, y_i)$  s.t.  $L\psi < 0$  so  $\max \psi =$   
 $\max_{\partial D} \psi$ . Gets  $\max e_{i,j}$ ; change to  $-\alpha$  for  $\min e_{i,j}$ . **P-F Ineq**:  $\|V\|_h^2 \leq c_* \|D_x^- V\|^2$ . **For 2D**:  $|V_j^m| =$   
 $|\sum_{\alpha=1}^j h(D_x^- V_\alpha^m)|^2 \leq jh \sum_{\alpha=1}^{N-1} h |D_x^- V_\alpha^m|^2 \rightarrow \|V\|_h^2 = \sum_{j=1}^{N-1} h |V_j^m|^2 \leq \sum_{j=1}^{N-1} jh^2 \sum_{\alpha=1}^{N-1} h |D_x^- V_\alpha^m|^2 \leq$   
 $\frac{1}{2} \sum_{j=1}^N h |D_x^- V_j^m|^2$ . Use blue and add for  $x, y$  for  $c_* = 0.25$ . **Weak Deriv**:  $w$  is a weak deriva-  
tive of  $u$  if  $\int dx wv = (-1)^{|\alpha|} \int dx u(D^\alpha v)$  **Parseval**:  $\int dk \hat{u}(k)v(k) = \int dk v(k) (\int dx u(x)e^{-ixk}) =$   
 $\int dx u(x) (\int dk v(k)e^{-ixk}) = \int dx u(x)\hat{v}(x)$ . Now  $v(k) := \hat{u}(k) = \overline{F[u(k)]} = \int dk u(k)e^{-ixk} =$   
 $\int dk \overline{u(k)}e^{ixk} = 2\pi F^{-1}[\overline{u(k)}] \Rightarrow \hat{v}(x) = 2\pi \overline{u(x)}$  **Iterative**: If  $U^{j+1} = U^j - \tau(AU^j - F) \rightarrow U -$   
 $U^j = (I - \tau A)^j (U - U^0)$  so  $\|U - U^j\| \leq \|I - \tau A\|^j \|U - U^0\|$ .  $\|I - \tau A\| = \sigma_1 = |\lambda_1|$  as symmet-  
ric. If  $\lambda \in [\alpha, \beta]$  then  $\lambda_1 \leq \max\{|1 - \tau\alpha|, |1 - \tau\beta|\}$ . Attained when  $\tau = 2/(\alpha + \beta) \rightarrow \lambda_1 = \frac{\beta - \alpha}{\beta + \alpha}$ .  
For  $-u'' + cu = f$  we have  $\lambda_k = c + \frac{4}{h^2} \sin^2(\frac{k\pi h}{2})$ . Lower bound via noting  $\sin(y) \geq \frac{2\sqrt{2}}{\pi} y$  at  
 $y = \frac{\pi}{4} \rightarrow \lambda_k \geq c + 8$  **Errors**:  $(AV, V)_h \geq \|D_x^- V\|_h^2$  & PF Ineq  $\rightarrow (AV, V)_h \geq \|V\|_h^2/c_*$ . Then  
 $(AV, V)_h (1 + c_*) \geq \|V\|_{1,h}^2 \rightarrow (AV, V)_h \geq c_0 \|V\|_{1,h}^2$ . Now  $c_0 \|V\|_{1,h}^2 \leq (AV, V)_h \leq \|f\|_h \|V\|_h \leq$   
 $\|f\|_h \|V\|_{1,h} \rightarrow \|V\|_{1,h} \leq \|f\|_h/c_0$ . (Use  $f := AV \rightarrow \|e\|_{1,h} \leq \|T\|_h/c_0$ ). **Scheme**: For e.g. on  $(0, 1)^2$   
write  $-\Delta u + u = -1$  and  $u|_{\partial D} = b$ , with  $x_j, t_m$ , we have scheme for  $1 \leq j \leq J-1, 1 \leq m \leq M-1$ ,  
and initial conditions  $U_{j,0} = U_{j,N} = b$  for  $0 \leq j \leq N$ , and  $U_{0,m} = U_{J,m} = b$  for  $1 \leq m \leq M-1$ .

Might need to define scheme for  $m = 0$  if e.g.  $\theta$  scheme with no  $U_{i,j-1}$  terms such as  $u_t = u_{xx}$   
**Non Uniform:** We have  $h_{i+1} := x_{i+1} - x_i, h_i := x_i - x_{i-1} \rightarrow \bar{h}_i = \frac{1}{2}(h_{i+1} + h_i)$  so  $D_x^+ D_x^- U_j^m =$   
 $\frac{1}{\bar{h}_i} ([U_{j+1} - U_j]/h_{i+1} - [U_j - U_{j-1}]/h_i)$ . **L<sub>2</sub> F'n:** We approximate  $f_{i,j} \rightarrow \frac{1}{h^2} \int_{K_{i,j}} f$  where  $K_{i,j} =$   
 $[x_i \pm 1/2, y_i \pm 1/2]$ . **For errors:** NB that  $Au - AU = -D_x^+ D_x^- u - D_y^+ D_y^- u + cu - T(\Delta u + cu)$ . NB  $Tu_{xx} =$   
 $D_x^+ \frac{1}{h} \int u(x_i - \frac{h}{2}) dy := D_x^+ \alpha_x$  so  $Ae_{i,j} = D_x^+ \phi_1 + D_x^- \phi_2 + \psi$ , with  $\phi_1 := \alpha_x - D_x^- u, \psi := cu - Tcu$ . Now NB  
 $c_0 \|e\|_{1,h}^2 \leq (Ae, e)$ . Bound  $(D_x^+ \phi_1, e)$  via  $\leq \|\phi_1\|_x \|D_x^- e\|_h$  so  $c_0 \|e\|_{1,h}^2 = (\|\phi_1\|_x^2 + \|\phi_2\|_y^2 + \|\psi\|_h^2) \|e\|_{1,h}^2$   
**L-Bounds:**  $\frac{|f(u)-f(v)|}{|u-v|} \leq |f'|$  **Hyperbolic Signs** For  $u_t + au_x$  when using  $[a]_{\pm}$  we write  $D_t^- U_j^m +$   
 $[a]_+ D_x^- U_j^m + [a]_- D_x^+ U_j^m$ . Eventually get  $U_j^{m+1} = \left(1 - \frac{|a|\Delta t}{\Delta x}\right) U_j^m + \frac{[a]_+ \Delta t}{\Delta x} U_{j-1}^m - \frac{[a]_- \Delta t}{\Delta x} U_{j+1}^m$ . Then via  
 CFL assumption  $\frac{a(\|U^0\|_{\infty})\Delta t}{\Delta x} \leq 1 \rightarrow |a(U)| \leq a(|U|) \leq a(\|U\|_{\infty})$  so  $\|U^{m+1}\|_{\infty} \leq \|U^0\|_{\infty}$  **Summation by**  
**Parts:** We have  $(-D_x^+ D_x^- U, U) = -\sum_{i=1}^{N-1} h(D_x^+ D_x^- U_i) U_i = -\sum_{i=1}^{N-1} \frac{U_{i+1} - U_i}{h} U_i + \sum_{i=1}^{N-1} \frac{U_i - U_{i-1}}{h} U_i =$   
 $-\sum_{i=2}^N \frac{U_i - U_{i-1}}{h} U_{i-1} + \sum_{i=1}^{N-1} \frac{U_i - U_{i-1}}{h} U_{i-1} = -\sum_{i=1}^N \frac{U_i - U_{i-1}}{h} U_{i-1} + \sum_{i=1}^N \frac{U_i - U_{i-1}}{h} U_{i-1} = \sum_{i=1}^N h |D_x^- U_i|^2$   
 $= \|D_x^- U\|_h^2$ , where **blue** from shift of index and **green** from  $U_0 = U_N = 0$ .