Charpit: $F(p, q, u, x, y) = 0$ with $u_x = p, u_y = q, \dot{x} = F_p, \dot{y} = F_q$. Then via $F_x, F_y, \& p_y = q_x \rightarrow$ $p_\tau = -F_x - pF_u, q_\tau = -F_y - qF_u, u_\tau = pF_p + qF_q$. Also, $u0_s = p_0 x0_s + q_0 y0_s; F_0 = 0$ - last 2 needed to show u defined on Γ. **Riemann:** $\int_D RLu - uL^*R = \int_D \partial_x (Ru_y + auR) + \partial_y (-uR_x + buR) = \int_{\partial D} dy (Ru_y + Rau) + dx (uR_x - buR)$. Expand over triangle going B-P-A (B at bottom right) $\rightarrow$ need $R_x = bR@y = \eta, R_y = aR@x = \xi, R(P) = 1, L^*R = 0$. Also ensure IVP to get R_y, R_x! **Canonical:** For $au_{xx} + 2bu_{xy} + cu_{yy} = f$, we need Cauchy-Kowalevski s.t. first derivs defined: $x' := \frac{dx}{ds}$ s.t. on Γ $p_0' = x_0' u_{xx} + y_0' u_{xy}, q_0' = x_0' u_{xy} + y_0' u_{yy}$. Use these 3, solve det A!=0 s.t. $ay_0'^2 - 2bx_0'y_0' + cx_0'^2 \neq 0$. Solve quadratic s.t. $b^2 > ac \rightarrow h, b^2 < ac \rightarrow e, b^2 = ac \rightarrow p$. H: $\lambda_1, \lambda_2 \rightarrow \xi, \eta$. E: $\lambda = \lambda_R \pm i\lambda_I; \lambda_R \rightarrow \xi, \lambda_I \rightarrow \eta$. P: $\lambda_1 \rightarrow \xi$, choose η independent e.g. xy, x^2. **Green's Fn:** For $u_{xx} + u_{yy} + au_x + bu_y + cu = f$ we have $\int_D GLu - uL^*G = \int_D (u_xG)_x + (u_yG)_y - (uG_x)_x - (uG_y)_y + (auG)_x + (buG)_y = \int_D \boldsymbol{\nabla} \cdot (u_nG - uG_n) +$ $\boldsymbol{\nabla} \cdot ((a\,b)^T \hat{n}Gu) = \int_{\partial D} u_nG - uG_n + (a\,b)^T \hat{n}G$. NB $\hat{n} = (dy, -dx)$. Also note for quarter plane if we have $G_x(0, y) = 0, G(x, 0) = 0$ then we have same sign at $\xi_1 = (-x, y)$, opposite sign at $\xi_2 = (x, -y)$, and for the third we reflect ξ_2 across y axis so we have an opposite sign to ξ at $\xi_3 = (-x, -y)$. **Types:** Quasi: Coeffs don't depend on highest order derivs Semi: Coeffs depend on x, y. **Causality:** For a n-dim prob, we have n characteristics. Shock intersects $2n$. $\exists k$ outgoing, $2n - k$ ingoing. Also have n R-H relations, so $3n - k$ pieces of info. Unknowns are n components of $\vec{u}$ on both sides of shock & slope $\Rightarrow 2n + 1$ unknowns. We demand $3n - k = 2n + 1$ so $k = n - 1$ outgoing characterisitcs.

SAM: **Dists:** Need linearity and continuity: $\exists N, C$ s.t. $|(u, \phi)| \leq C \sum_{m \leq N} \max_{\in [-X, X]} |\phi^{(m)}|$. OR $\lim_{n \to \infty} (u, \phi_n) = (u, \lim_{n \to \infty} \phi_n)$ for a sequence $\phi_n \to 0$ as $n \to \infty$. **Orthog:** $\int_0^\pi \sin(kx) \sin(jx) = \frac{\pi}{2} \delta_{kj}$, same for cos.

1. NLA: **Cholesky** For matrix $[a_{11}, w^*; w, K] = R_1^T \left[I, 0; 0, K - \frac{ww^*}{a_{11}} \right] [\alpha, w^*/\alpha; 0, I]$ we have a decomp:

2. for $k = [1, m-1]$: for $j = [k+1, m]$ $R_{j,j:m} = R_{j,j:m} - \frac{R_{kj}}{R_{kk}} R_{k,j:m}$ endfor $R_{k,k:m} = \frac{R_{k,k:m}}{\sqrt{R_{kk}}}$ end-

3. for. $\frac{m^3}{3}$. **Householder** for $k = [1, n]$: $x = A_{k:m,k}; v_k = sgn(x)\|x\|e_k + x; v_k = \frac{v_k}{\|v_k\|}$ for $j = [k, n]$

4. $A_{k:m,j} = A_{k:m,j} - 2v_k [v_k^* A_{k:m,j}]$ endfor endfor. $\frac{2mn^2}{3}$. **LU** $U = A, L = I$ for $k = [1, m-1]$: for

5. $j = [k+1, m]$ $U_{j,k:m} = U_{j,k:m} - \frac{U_{jk}}{U_{kk}} U_{k,k:m}$ endfor endfor. $\frac{2m^3}{3}$. **MG-S** $V = A$; for $i = [1, n]$: $r_{ii} =$

6. $\|v_i\|; q_i = \frac{v_i}{r_{ii}}$; for $j = [i+1, n]$ $v_j = v_j - (q_i^T v_j)q_i; r_{ij} = q_i^T v_j$ endfor endfor. $2mn^2$. **Givens** $3mn^2$ **SVD:**

7. $= \sum_i^{r:=\min m,n} u_i \sigma_i v_i^T$. **Bounds:** $\|ABB^{-1}\| \geq \|AB\|\|B^{-1}\| \to \|A\|/\|B^{-1}\| \geq \|AB\|$. **Norms:** $\|A\|_F =$

8. $\sqrt{\sum_i (\sigma_i)^2} = \sqrt{Tr(AA^T)}$, $\|A\|_\infty$ = max row sum. **Low-Rank:** For $A \in \mathbb{R}^{m \times n} \min \|A - B\| =$

9. $\|A - A_r\|$. Proof via $B := B_1 B_2^T$ with $B_1 \in \mathbb{R}^{m \times r}$; $\exists W s.t. B_2^T W = 0$ with null$(W) \geq n - r$. Then

10. $\exists x_V, x_W s.t. V_{r+1} x_V = -W x_W$. So $\|A - B\| = \|AW\| \geq \|AV_{r+1}x_V\| \geq \sigma_{r+1}$ For reverse $B := A_r$

11. **Courant:** $\sigma_i = \max_{dim(S)=i} \{\min_x \|Ax\|/\|x\|\}$. Proof via $V_i = [v_i \dots v_n]$, so $\dim(S) + \dim(V_i) = n + 1$

12. so $\exists w \in S \cap V_i$. Then $\|Aw\| \leq \sigma_i$. For reverse take $w = v_i$ when $S = [v_1 \dots v_i]$ **Schur:** Take

13. $Av_1 = \lambda_1 v_1$; construct $U_1 = [v_1, V_\perp] \to AU_1 = U_1[e_1, X]$. Repeat. **Back Subst:** For $Ux = y$ we have

14. $x_{n-i} = \left(y_{n-i} - \sum_{n-i+1}^n u_{n-i,j} x_j \right) / u_{n-i,n-i}; O(i)$ per iteration so $O(n^2)$ total. **Backwards Stable:**

15. When $\hat{f}(x) = f(x + \Delta x)$ with $\|\Delta x\|/\|x\| \leq O(\varepsilon)$ **Conditioning** $\kappa_2(A) = \sigma_1/\sigma_n = \|A\|\|A^{-1}\|$

16. NPDE: **Hyperbolic:** Implicit: $(A - B, A) = \frac{1}{2}(\|A\|^2 - \|B\|^2) + \frac{1}{2}\|A - B\|^2$ (time), $(-D_x^+ D_x^- U^{m+1},$

17. $U^{m+1} - U^m) = (D_x^- U^{m+1} - D_x^- U^m, D_x^- U^{m+1})$ (space). Explicit: 1st rewrite in terms of $D_t^{+-}(\Delta t)^{-2} U_j^m +$

18. $\frac{c^2(\Delta t)^2}{4} D_x^{+-}((\Delta t)^{-2} U_j^m) - c^2 D_x^{+-}(U_j^{m+1} + 2U_j^m + U_j^{m-1})$. Then use $(D(A-B), A+B) = (DA, A) -$

19. $(DB, B); (D(A+B), A-B) = (DA, A) - (DB, B)$ by multiplying by $U^{m+1} - U^{m-1}$. Finally WTS

20. $\|V_m\|^2 - \frac{c^2(\Delta t)^2}{4}\|D_x^- V^m\|^2 \geq 0$. Done by noticing: $\|D_x^- V^m\|^2 = \sum_i^J \Delta x |D_x^- V_j^m|^2 = 1/\Delta x \sum_i^J$

21. $\left(V_j^m - V_{j-1}^m\right)^2 \leq 2/\Delta x \sum_i^J (V_j^m)^2 + (V_{j-1}^m)^2 = 4/\Delta x^2 \sum_i^{J-1} \Delta x \left(V_j^m\right)^2$ **Max Principle:** For $-\Delta u = f \leq$

22. $0 \to \max u \in \partial D$. First show contradiction assuming $LU = f < 0$, then try some auxillary function

23. $\psi = U + \alpha(T_{\max}) g(x_i, y_i)$ s.t. $L\psi < 0$ so $\max \psi = \max_{\in \partial D} \psi$. Gets max $e_{i,j}$; change to $-\alpha$ for min $e_{i,j}$. **P-**

24. **F Ineq:** $\|V\|_h^2 \leq c_\star \|D_x^- V\|^2$ **Weak Deriv:** w is a weak derivative of u if $\int dx\, wv = (-1)^{|\alpha|} \int dx\, u(D^\alpha v)$

25. **Parseval:** $\int dk\, \hat{u}(k) v(k) = \int dk\, v(k) \left(\int dx\, u(x)e^{-ixk}\right) = \int dx\, u(x) \left(\int dk\, v(k)e^{-ixk}\right) = \int dx\, u(x)\hat{v}(x)$.

26. Now $v(k) := \overline{\hat{u}(k)} = \overline{F[u(k)]} = \overline{\int dk\, u(k)e^{-ixk}} = \int dk\, \overline{u(k)}e^{ixk} = 2\pi F^{-1}\left[\overline{u(k)}\right] \Rightarrow \hat{v}(x) = 2\pi\overline{u(x)}$ **Itera-**

27. **tive:** If $U^{j+1} = U^j - \tau(AU^j - F) \to U - U^j = (I - \tau A)^j (U - U^0)$ so $\|U - U^j\| \leq \|I - \tau A\|^j \|U - U^0\|$.

28. $\|I - \tau A\| = \sigma_1 = |\lambda_1|$ as symmetric. If $\lambda \in [\alpha, \beta]$ then $\lambda_1 \leq \max\{|1 - \tau\alpha|, |1 - \tau\beta|\}$. Attained when

29. $\tau = 2/(\alpha + \beta) \to \lambda_1 = \frac{\beta - \alpha}{\beta + \alpha}$. For $-u'' + cu = f$ we have $\lambda_k = c + \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right)$. Lower bound via noting

30. $\sin(y) \geq \frac{2\sqrt{2}}{\pi} y$ at $y = \frac{\pi}{4} \to \lambda_k \geq c + 8$ **Errors:** $(AV, V)_h \geq \|D_x^- V\|_h^2$ & PF Ineq $\to (AV, V)_h \geq \|V\|_h^2/c_\star$.

31. Then $(AV, V)_h(1 + c_\star) \geq \|V\|_{1,h}^2 \to (AV, V)_h \geq c_0 \|V\|_{1,h}^2$. Now $c_0\|V\|_{1,h}^2 \leq (AV, V)_h \leq \|f\|_h \|V\|_h \leq$

32. $\|f\|_h \|V\|_{1,h} \to \|V\|_{1,h} \leq \|f\|_h/c_0$. (Use $f := AV \to \|e\|_{1,h} \leq \|T\|_h/c_0$)