Charpit: $F(p,q,u,x,y) = 0$ with $u_x = p, u_y = q, \dot{x} = F_p, \dot{y} = F_q$. Then via $F_x, F_y,$ & $p_y = q_x \to p_\tau = -F_x - pF_u, \ q_\tau = -F_y - qF_u, u_\tau = pF_p + qF_q$. Also, $u0_s = p_0 x0_s + q_0 y0_s; F_0 = 0$ - last 2 needed to show u defined on Γ. **Laplacian:** In $2D$: $r^{-1}(rf_r)_r + r^{-2}f_{\theta\theta}$. In $3D$: $r^{-2}(r^2 f_r)_r + r^{-2}\sin^{-2}(\theta)f_{\phi\phi} + r^{-2}\sin^{-1}(\theta)(\sin(\theta)f_\theta)_\theta$ **Riemann:** For $u_{xy} + au_x + bu_y + cu = f$ we have $\int_D RLu - uL^*R = \int_D \partial_x(Ru_y + auR) + \partial_y(-uR_x + buR) = \int_{\partial D} dy(Ru_y + Rau) + dx(uR_x - buR)$. Expand over triangle going B-P-A (B at bottom right) $\to$ need $R_x = bR@y = \eta, R_y = aR@x = \xi, R(P) = 1, L^*R = 0$. Also ensure IVP to get R_y, R_x! **R-H:** Derived via $P_x\psi + Q_y\psi = R\psi \to \int_D (P_x\psi)_x + (Q_y\psi)_y (= \int_\Gamma \psi Pdy - \psi Qdx) = \int_D P\psi_x + Q\psi_y + R\psi = \int_{D_1 + D_2} P\psi_x + Q\psi_y + R\psi$, where $\int_{D_i} = \int_{D_i}(P\psi)_x + (Q\psi)_y + \psi(R - P_x - Q_y)$. So $\int_\Gamma \psi Pdy - \psi Qdx = \int_{\Gamma + C_1 - C_2} \psi Pdy - \psi Qdx$ and so $\int_{C_1 + C_2} \psi Pdy - \psi Qdx = 0 \to dy/dx = [Q]_-^+ / [P]_-^+$ **Canonical:** For $au_{xx} + 2bu_{xy} + cu_{yy} = f$, we need Cauchy-Kowalevski s.t. first derivs defined: $x' := \frac{dx}{ds}$ s.t. on Γ $p_0' = x_0' u_{xx} + y_0' u_{xy}, q_0' = x_0' u_{xy} + y_0' u_{yy}$. Use these 3, solve det A!=0 s.t. $ay_0'^2 - 2bx_0'y_0' + cx_0'^2 \neq 0$. Solve quadratic s.t. $b^2 > ac \to h, b^2 < ac \to e, b^2 = ac \to p$. H: $\lambda_1, \lambda_2 \to \xi, \eta$. E: $\lambda = \lambda_R \pm i\lambda_I; \lambda_R \to \xi, \lambda_I \to \eta$. P: $\lambda_1 \to \xi$, choose η independent e.g. xy, x^2. **Green's Fn:** For $u_{xx} + u_{yy} + au_x + bu_y + cu = f$ we have $\int_D GLu - uL^*G = \int_D (u_xG)_x + (u_yG)_y - (uG_x)_x - (uG_y)_y + (auG)_x + (buG)_y = \int_D \boldsymbol{\nabla} \cdot (u_nG - uG_n) + \boldsymbol{\nabla} \cdot ((a\,b)^T \hat{n}Gu) = \int_{\partial D} u_nG - uG_n + (a\,b)^T \hat{n}G$. NB $\hat{n} = (dy, -dx)$. Also note for quarter plane if we have $G_x(0,y) = 0, G(x,0) = 0$ then we have same sign at $\xi_1 = (-x,y)$, opposite sign at $\xi_2 = (x,-y)$, and for the third we reflect ξ_2 across y axis so we have an opposite sign to ξ at $\xi_3 = (-x,-y)$. **Types:** Quasi: Coeffs don't depend on highest order derivs Semi: Coeffs depend on x,y. **Causality:** For a n-dim prob, we have n characteristics. Shock intersects $2n$. $\exists k$ outgoing, $2n - k$ ingoing. Also have n R-H relations, so $3n - k$ pieces of info. Unknowns are n components of $\vec{u}$ on both sides of shock & slope $\Rightarrow 2n + 1$ unknowns. We demand $3n - k = 2n + 1$ so $k = n - 1$ outgoing characterisitcs.

Dists: Need linearity and continuity: $\exists N, C$ s.t. $|(u,\phi)| \leq C\sum_{m \leq N} \max_{\in[-X,X]} |\phi^{(m)}|$. OR $\lim_{n\to\infty}(u,\phi_n) = (u, \lim_{n\to\infty}\phi_n)$ for a sequence $\phi_n \to 0$ as $n \to \infty$. **Orthog:** $\int_0^\pi \sin(kx)\sin(jx) = \frac{\pi}{2}\delta_{kj}$, same for cos.

1. NLA: **Cholesky** For matrix $[a_{11}, w^*; w, K] = R_1^T \left[I, 0; 0, K - \frac{ww^*}{a_{11}} \right] [\alpha, w^*/\alpha; 0, I]$ we have a decomp:

2. for $k = [1, m-1]$: for $j = [k+1, m]$ $R_{j,j:m} = R_{j,j:m} - \frac{R_{kj}}{R_{kk}} R_{k,j:m}$ endfor $R_{k,k:m} = \frac{R_{k,k:m}}{\sqrt{R_{kk}}}$ end-

3. for. $\frac{m^3}{3}$. **Householder** for $k = [1, n]$: $x = A_{k:m,k}; v_k = sgn(x)\|x\|e_k + x; v_k = \frac{v_k}{\|v_k\|}$ for $j = [k, n]$

4. $A_{k:m,j} = A_{k:m,j} - 2v_k [v_k^* A_{k:m,j}]$ endfor endfor. $\frac{2mn^2}{3}$. **LU** $U = A, L = I$ for $k = [1, m-1]$: for

5. $j = [k+1, m]$ $U_{j,k:m} = U_{j,k:m} - \frac{U_{jk}}{U_{kk}} U_{k,k:m}$ endfor endfor. $\frac{2m^3}{3}$. **MG-S** $V = A$; for $i = [1, n]$: $r_{ii} =$

6. $\|v_i\|; q_i = \frac{v_i}{r_{ii}}$; for $j = [i+1, n]$ $v_j = v_j - (q_i^T v_j)q_i; r_{ij} = q_i^T v_j$ endfor endfor. $2mn^2$. **Givens** $3mn^2$ **SVD:**

7. $= \sum_i^{r:=\min m,n} u_i \sigma_i v_i^T$. **Bounds:** $\|ABB^{-1}\| \geq \|AB\|\|B^{-1}\| \to \|A\|/\|B^{-1}\| \geq \|AB\|$. **Norms:** $\|A\|_F =$

8. $\sqrt{\sum_i (\sigma_i)^2} = \sqrt{Tr(AA^T)}, \|A\|_\infty = $ max row sum. **Low-Rank:** For $A \in \mathbb{R}^{m \times n} \min \|A - B\| =$

9. $\|A - A_r\|$. Proof via $B := B_1 B_2^T$ with $B_1 \in \mathbb{R}^{m \times r}$; $\exists W s.t. B_2^T W = 0$ with null$(W) \geq n - r$. Then

10. $\exists x_V, x_W s.t. V_{r+1} x_V = -W x_W$. So $\|A - B\| = \|AW\| \geq \|AV_{r+1} x_V\| \geq \sigma_{r+1}$ For reverse $B := A_r$

11. **Courant:** $\sigma_i = \max_{dim(S)=i} \{\min_x \|Ax\|/\|x\|\}$. Proof via $V_i = [v_i \ldots v_n]$, so $\dim(S) + \dim(V_i) = n + 1$

12. so $\exists w \in S \cap V_i$. Then $\|Aw\| \leq \sigma_i$. For reverse take $w = v_i$ when $S = [v_1 \ldots v_i]$ **Schur:** Take

13. $Av_1 = \lambda_1 v_1$; construct $U_1 = [v_1, V_\perp] \to AU_1 = U_1[e_1, X]$. Repeat. **Back Subst:** For $Ux = y$ we have

14. $x_{n-i} = \left(y_{n-i} - \sum_{n-i+1}^n u_{n-i,j} x_j \right)/u_{n-i,n-i}; O(i)$ per iteration so $O(n^2)$ total. **Backwards Stable:**

15. When $\hat{f}(x) = f(x + \Delta x)$ with $\|\Delta x\|/\|x\| \leq O(\varepsilon)$ **Conditioning** $\kappa_2(A) = \sigma_1/\sigma_n = \|A\|\|A^{-1}\|$

16. NPDE: **Hyp Impl:** $(A - B, A) = \frac{1}{2}(\|A\|^2 - \|B\|^2) + \frac{1}{2}\|A - B\|^2$ with $A := U^{m+1} - U^m, B := U^m - U^{m-1}$

17. (T); $(-D_x^+ D_x^- U^{m+1}, U^{m+1} - U^m) = (D_x^- U^{m+1} - D_x^- U^m, D_x^- U^{m+1})$ (X). Then $\frac{1}{2\Delta t^2}(\|U^{m+1} - U^m\|^2 -$

18. $\|U^m - U^{m-1}\|^2) + \frac{\Delta t^2}{2\Delta t^2}\|U^{m+1} - 2U^m + U^{m-1}\|^2 + \frac{c^2}{2}\left(\|D_x^- U^{m+1}\|^2 - \|D_x^- U^m\|^2\right) +$

19. $\frac{c^2 \Delta t^2}{2\Delta t^2}\|D_x^-(U^{m+1} - U^m)\|^2 = (f, U^{m+1} - U^m)$. Then $M^2(U^m) := \left\|\frac{U^m - U^{m-1}}{\Delta t}\right\|^2 + c^2\|D_x^- U^{m+1}\|^2$. Write

20. green as $\leq \|f\|\|U^{m+1} - U^m\| = \sqrt{\Delta t T}\|f\|\sqrt{\frac{\Delta t}{T}}\left\|\frac{U^{m+1} - U^m}{\Delta t}\right\| \leq \frac{\Delta t T}{2}\|f\|^2 + \frac{\Delta t}{2T}\left\|\frac{U^{m+1} - U^m}{\Delta t}\right\|^2$. Then

21. $(1 - \frac{\Delta t}{T})M^2(U^m) \leq M^2(U^{m-1}) + \Delta t T\|f\|^2 \to M^2(U^m) \leq (1 + \frac{2\Delta t}{T})M^2(U^{m-1}) + 2\Delta t T\|f\|^2$. Use

22. $a_m \leq \alpha^m a_0 + \sum_{k=1}^m \alpha^{m-k} b_k$ so $M^2 \leq e^2 M^2(U^0) + 2e^2 T \sum_{k=1}^m \Delta t\|f\|^2$ **Hyp Expl:** 1st rewrite in terms of

23. $D_t^{+-}(\Delta t)^{-2} U_j^m + \frac{c^2(\Delta t)^2}{4} D_x^{+-}((\Delta t)^{-2} U_j^m) - c^2 D_x^{+-} \left(U_j^{m+1} + 2U_j^m + U_j^{m-1} \right)$. Then use $(D(A - B), A + B)$

24. $= (DA, A) - (DB, B); (D(A + B), A - B) = (DA, A) - (DB, B)$ by multiplying by $U^{m+1} - U^{m-1}$.

25. Finally WTS $\|V_m\|^2 - \frac{c^2(\Delta t)^2}{4}\|D_x^- V^m\|^2 \geq 0$. Done by noticing: $\|D_x^- V^m\|^2 = \sum_i^J \Delta x |D_x^- V_j^m|^2 =$

26. $\frac{1}{\Delta x}\sum_i^J \left(V_j^m - V_{j-1}^m\right)^2 \leq 2/\Delta x \sum_i^J (V_j^m)^2 + (V_{j-1}^m)^2 = 4/\Delta x^2 \sum_i^{J-1} \Delta x \left(V_j^m\right)^2$. Eventually show

27. $N^2(U^m) := \left(\left(I + \frac{c^2 \Delta t^2}{2} D_x^{+-}\right) \frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t}\right) + c^2\left\|D_x^- \frac{U^{m+1} + U^m}{2}\right\|^2 \to N^2(U^m) = N^2(U^{m-1}) +$

28. $(f, U^{m+1} - U^m)$ **Max Principle:** For $-\Delta u = f \leq 0 \to \max u \in \partial D$. First show contradiction as-

29. suming $LU = f < 0$, then try some auxillary function $\psi = U + \alpha (T_{\max}) g(x_i, y_i)$ s.t. $L\psi < 0$

30. so $\max \psi = \max_{\in \partial D} \psi$. Gets $\max e_{i,j}$; change to $-\alpha$ for $\min e_{i,j}$. **P-F Ineq:** $\|V\|_h^2 \leq c_\star \|D_x^- V\|^2$

31. **Weak Deriv:** w is a weak derivative of u if $\int dx\, wv = (-1)^{|\alpha|}\int dx\, u(D^\alpha v)$ **Parseval:** $\int dk\, \hat{u}(k)v(k) =$

32. $\int dk\, v(k)\left(\int dx\, u(x)e^{-ixk}\right) = \int dx\, u(x)\left(\int dk\, v(k)e^{-ixk}\right) = \int dx\, u(x)\hat{v}(x)$. Now $v(k) := \overline{\hat{u}(k)} = \overline{F[u(k)]} =$

33. $\overline{\int dk\, u(k)e^{-ixk}} = \int dk\, \overline{u(k)}e^{ixk} = 2\pi F^{-1}\left[\overline{u(k)}\right] \Rightarrow \hat{v}(x) = 2\pi\overline{u(x)}$ **Iterative:** If $U^{j+1} = U^j - \tau\left(AU^j - F\right) \to$

34. $U - U^j = (I - \tau A)^j\left(U - U^0\right)$ so $\|U - U^j\| \leq \|I - \tau A\|^j\|U - U^0\|$. $\|I - \tau A\| = \sigma_1 = |\lambda_1|$ as symmet-

35. ric. If $\lambda \in [\alpha, \beta]$ then $\lambda_1 \leq \max\{|1 - \tau\alpha|, |1 - \tau\beta|\}$. Attained when $\tau = 2/(\alpha + \beta) \to \lambda_1 = \frac{\beta - \alpha}{\beta + \alpha}$. For

36. $-u'' + cu = f$ we have $\lambda_k = c + \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right)$. Lower bound via noting $\sin(y) \geq \frac{2\sqrt{2}}{\pi}y$ at $y = \frac{\pi}{4} \to \lambda_k \geq c + 8$

37. **Errors:** $(AV, V)_h \geq \|D_x^- V\|_h^2$ & PF Ineq $\to (AV, V)_h \geq \|V\|_h^2/c_\star$. Then $(AV, V)_h (1 + c_\star) \geq \|V\|_{1,h}^2 \to$

38. $(AV, V)_h \geq c_0\|V\|_{1,h}^2$. Now $c_0\|V\|_{1,h}^2 \leq (AV, V)_h \leq \|f\|_h\|V\|_h \leq \|f\|_h\|V\|_{1,h} \to \|V\|_{1,h} \leq \|f\|_h/c_0$. (Use

39. $f := AV \to \|e\|_{1,h} \leq \|T\|_h/c_0$)