

APDE: Charpit: $F(p, q, u, x, y) = 0$ with $u_x = p, u_y = q, \dot{x} = F_p, \dot{y} = F_q$. Then via F_x, F_y , & $p_y = q_x \rightarrow p_\tau = -F_x - pF_u, q_\tau = -F_y - qF_u, u_\tau = pF_p + qF_q$. Also, $u0_s = p_0x0_s + q_0y0_s; F_0 = 0$ - last 2 needed to show u defined on Γ . **Riemann:** $\int_D RLu - uL^*R = \int_D \partial_x (Ru_y + auR) + \partial_y (-uR_x + buR) = \int_{\partial D} dy (Ru_y + Rau) + dx (uR_x - buR)$. Expand over triangle going B-P-A (B at bottom right) \rightarrow need $R_x = bR@y = \eta, R_y = aR@x = \xi, R(P) = 1, L^*R = 0$. Also ensure IVP to get R_y, R_x ! **Canonical:** For $au_{xx} + 2bu_{xy} + cu_{yy} = f$, we need **Cauchy-Kowalevski** s.t. first derivs defined: $x' := \frac{dx}{ds}$ s.t. on Γ $p'_0 = x'_0 u_{xx} + y'_0 u_{xy}, q'_0 = x'_0 u_{xy} + y'_0 u_{yy}$. Use these 3, solve $\det A! = 0$ s.t. $ay_0'^2 - 2bx_0'y'_0 + cx_0'^2 \neq 0$. Solve quadratic s.t. $b^2 > ac \rightarrow h, b^2 < ac \rightarrow e, b^2 = ac \rightarrow p$. **H:** $\lambda_1, \lambda_2 \rightarrow \xi, \eta$. **E:** $\lambda = \lambda_R \pm i\lambda_I; \lambda_R \rightarrow \xi, \lambda_I \rightarrow \eta$. **P:** $\lambda_1 \rightarrow \xi$, choose η independent e.g. xy, x^2 . **Green's Fn:** For $u_{xx} + u_{yy} + au_x + bu_y + cu = f$ we have $\int_D GLu - uL^*G = \int_D (u_x G)_x + (u_y G)_y - (uG_x)_x - (uG_y)_y + (auG)_x + (buG)_y = \int_D \nabla \cdot (u_n G - uG_n) + \nabla \cdot ((ab)^T \hat{n} Gu) = \int_{\partial D} u_n G - uG_n + (ab)^T \hat{n} G$. NB $\hat{n} = (dy, -dx)$. **Also note for quarter plane** if we have $G_x(0, y) = 0, G(x, 0) = 0$ then we have same sign at $\xi_1 = (-x, y)$, opposite sign at $\xi_2 = (x, -y)$, and for the third we reflect ξ_2 across y axis so we have an opposite sign to ξ at $\xi_3 = (-x, -y)$. **Types: Quasi:** Coeffs don't depend on highest order derivs **Semi:** Coeffs depend on x, y . **Causality:** For a n -dim prob, we have n characteristics. Shock intersects $2n$. $\exists k$ outgoing, $2n - k$ ingoing. Also have n R-H relations, so $3n - k$ pieces of info. Unknowns are n components of \vec{u} on both sides of shock & slope $\Rightarrow 2n + 1$ unknowns. We demand $3n - k = 2n + 1$ so $k = n - 1$ outgoing characteristics. **SAM: Dists:** Need linearity and continuity: $\exists N, C$ s.t. $|(u, \phi)| \leq C \sum_{m \leq N} \max_{\epsilon \in [-X, X]} |\phi^{(m)}|$. OR $\lim_{n \rightarrow \infty} (u, \phi_n) = (u, \lim_{n \rightarrow \infty} \phi_n)$ for a sequence $\phi_n \rightarrow 0$ as $n \rightarrow \infty$. **Orthog:** $\int_0^\pi \sin(kx) \sin(jx) = \frac{\pi}{2} \delta_{kj}$, same for cos.

NLA: Cholesky For matrix $[a_{11}, w^*; w, K] = R_1^T \begin{bmatrix} I, 0; 0, K - \frac{ww^*}{a_{11}} \end{bmatrix} [\alpha, w^*/\alpha; 0, I]$ we have a decomp: **for**
 $k = [1, m-1] : \text{for } j = [k+1, m] \ R_{j,j:m} = R_{j,j:m} - \frac{R_{kj}}{R_{kk}} R_{k,j:m} \text{endfor } R_{k,k:m} = \frac{R_{k,k:m}}{\sqrt{R_{kk}}} \text{endfor. } \frac{m^3}{3}.$
Householder **for** $k = [1, n] : x = A_{k:m,k}; v_k = \text{sgn}(x) \|x\| e_k + x; v_k = \frac{v_k}{\|v_k\|} \text{for } j = [k, n] \ A_{k:m,j} =$
 $A_{k:m,j} - 2v_k [v_k^* A_{k:m,j}] \text{endfor endfor. } \frac{2mn^2}{3}.$ **LU** $U = A, L = I \text{for } k = [1, m-1] : \text{for } j = [k+1, m]$
 $U_{j,k:m} = U_{j,k:m} - \frac{U_{jk}}{U_{kk}} U_{k,k:m} \text{endfor endfor. } \frac{2m^3}{3}.$ **MG-S** $V = A; \text{for } i = [1, n] : r_{ii} = \|v_i\|; q_i = \frac{v_i}{r_{ii}}; \text{for } j =$
 $[i+1, n] \ v_j = v_j - (q_i^T v_j) q_i; r_{ij} = q_i^T v_j \text{endfor endfor. } 2mn^2.$ **Givens** $3mn^2$ **SVD**: $= \sum_i^{r:=\min m,n} u_i \sigma_i v_i^T.$
Bounds: $\|ABB^{-1}\| \geq \|AB\| \|B^{-1}\| \rightarrow \|A\|/\|B^{-1}\| \geq \|AB\|.$ **Norms**: $\|A\|_F = \sqrt{\sum_i (\sigma_i)^2}, \|A\|_\infty =$
 $\max \text{row.}$ **Low-Rank**: For $A \in \mathbb{R}^{m \times n} \min \|A - B\| = \|A - A_r\|.$ Proof via $B := B_1 B_2^T$ with $B_1 \in \mathbb{R}^{m \times r};$
 $\exists W \text{ s.t. } B_2^T W = 0$ with $\text{null}(W) \geq n - r.$ Then $\exists x_V, x_W \text{ s.t. } V_{r+1} x_V = -W x_W.$ So $\|A - B\| = \|AW\| \geq$
 $\|AV_{r+1} x_V\| \geq \sigma_{r+1}$ For reverse $B := A_r$ **Courant**: $\sigma_i = \max_{\dim(S)=i} \{\min_x \|Ax\|/\|x\|\}.$ Proof via
 $V_i = [v_i \dots v_n],$ so $\dim(S) + \dim(V_i) = n + 1$ so $\exists w \in S \cap V_i.$ Then $\|Aw\| \leq \sigma_i.$ For reverse take $w = v_i$
when $S = [v_1 \dots v_i]$
NPDE: Hyperbolic: Implicit: $(A - B, A) = \frac{1}{2}(\|A\|^2 - \|B\|^2) + \frac{1}{2}\|A - B\|^2$ (time), $(-D_x^+ D_x^- U^{m+1},$
 $U^{m+1} - U^m) = (D_x^- U^{m+1} - D_x^- U^m, D_x^- U^{m+1})$ (space). **Explicit**: 1st rewrite in terms of $D_t^{+-}(\Delta t)^{-2} U_j^m +$
 $\frac{c^2(\Delta t)^2}{4} D_x^{+-}((\Delta t)^{-2} U_j^m) - c^2 D_x^{+-}(U_j^{m+1} + 2U_j^m + U_j^{m-1}).$ Then use $(D(A - B), A + B) = (DA, A) -$
 $(DB, B); (D(A + B), A - B) = (DA, A) - (DB, B)$ by multiplying by $U^{m+1} - U^{m-1}.$ Finally WTS
 $\|V_m\|^2 - \frac{c^2(\Delta t)^2}{4} \|D_x^- V^m\|^2 \geq 0.$ Done by noticing: $\|D_x^- V^m\|^2 = \sum_i^J \Delta x |D_x^- V_j^m|^2 = 1/\Delta x \sum_i^J$
 $(V_j^m - V_{j-1}^m)^2 \leq 2/\Delta x \sum_i^J (V_j^m)^2 + (V_{j-1}^m)^2 = 4/\Delta x^2 \sum_i^{J-1} \Delta x (V_j^m)^2$ **Max Principle**: For $-\Delta u = f \leq$
 $0 \rightarrow \max u \in \partial D.$ First show contradiction assuming $LU = f < 0,$ then try some auxillary function
 $\psi = U + \alpha (T_{\max}) g(x_i, y_i)$ s.t. $L\psi < 0$ so $\max \psi = \max_{\partial D} \psi.$ Gets max $e_{i,j};$ change to $-\alpha$ for min $e_{i,j}.$ **P-**
F Ineq: $\|V\|_h^2 \leq c_* \|D_x^- V\|^2$ **Weak Deriv**: w is a weak derivative of u if $\int dx wv = (-1)^{|\alpha|} \int dx u(D^\alpha v)$
Parseval: $\int dk \hat{u}(k) v(k) = \int dk v(k) (\int dx u(x) e^{-ixk}) = \int dx u(x) (\int dk v(k) e^{-ixk}) = \int dx u(x) \hat{v}(x).$
Now $v(k) := \overline{\hat{u}(k)} = \overline{F[u(k)]} = \overline{\int dk u(k) e^{-ixk}} = \int dk \overline{u(k)} e^{ixk} = 2\pi F^{-1} [\overline{u(k)}] \Rightarrow \hat{v}(x) = 2\pi \overline{u(x)}$ **Itera-**
tive: If $U^{j+1} = U^j - \tau (AU^j - F) \rightarrow U - U^j = (I - \tau A)^j (U - U^0)$ so $\|U - U^j\| \leq \|I - \tau A\|^j \|U - U^0\|.$
 $\|I - \tau A\| = \sigma_1 = |\lambda_1|$ as symmetric. If $\lambda \in [\alpha, \beta]$ then $\lambda_1 \leq \max\{|1 - \tau\alpha|, |1 - \tau\beta|\}.$ Attained when
 $\tau = 2/(\alpha + \beta) \rightarrow \lambda_1 = \frac{\beta - \alpha}{\beta + \alpha}.$ For $-u'' + cu = f$ we have $\lambda_k = c + \frac{4}{h^2} \sin^2(\frac{k\pi h}{2}).$ Lower bound via noting
 $\sin(y) \geq \frac{2\sqrt{2}}{\pi} y$ at $y = \frac{\pi}{4} \rightarrow \lambda_k \geq c + 8$ **Errors**: $(AV, V)_h \geq \|D_x^- V\|_h^2$ & PF Ineq $\rightarrow (AV, V)_h \geq \|V\|_h^2 / c_*.$
Then $(AV, V)_h (1 + c_*) \geq \|V\|_{1,h}^2 \rightarrow (AV, V)_h \geq c_0 \|V\|_{1,h}^2.$ Now $c_0 \|V\|_{1,h}^2 \leq (AV, V)_h \leq \|f\|_h \|V\|_h \leq$
 $\|f\|_h \|V\|_{1,h} \rightarrow \|V\|_{1,h} \leq \|f\|_h / c_0.$ (Use $f := AV \rightarrow \|e\|_{1,h} \leq \|T\|_h / c_0$)