APDE: **Charpit:** $F(p,q,u,x,y)=0$ with $u_x=p, u_y=q, \dot{x}=F_p, \dot{y}=F_q$. Then via $F_x, F_y, \&\ p_y=q_x \rightarrow p_\tau=-F_x-pF_u,\ q_\tau=-F_y-qF_u, u_\tau=pF_p+qF_q$. Also, $u0_s=p_0x0_s+q_0y0_s; F_0=0$ - last 2 needed to show u defined on Γ. **Laplacian:** In $2D: r^{-1}\left(rf_r\right)_r+r^{-2}f_{\theta\theta}$. In $3D: r^{-2}\left(r^2f_r\right)_r+r^{-2}\sin^{-2}(\theta)f_{\phi\phi}+r^{-2}\sin^{-1}(\theta)\left(\sin(\theta)f_\theta\right)_\theta$ **Riemann:** For $u_{xy}+au_x+bu_y+cu=f$ we have $\int_D RLu-uL^*R = \int_D \partial_x\left(Ru_y+auR\right)+\partial_y\left(-uR_x+buR\right)=\int_{\partial D} dy\left(Ru_y+Rau\right)+dx\left(uR_x-buR\right)$. Expand over triangle going B-P-A (B at bottom right) $\rightarrow$ need $R_x=bR@y=\eta, R_y=aR@x=\xi, R(P)=1, L^*R=0$. Also ensure IVP to get R_y, R_x! **R-H:** Derived via $P_x\psi+Q_y\psi=R\psi \rightarrow \int_D\left(P_x\psi\right)_x+\left(Q_y\psi\right)_y\left(=\int_\Gamma \psi Pdy-\psi Qdx\right)=\int_D P\psi_x+Q\psi_y+R\psi=\int_{D_1+D_2}P\psi_x+Q\psi_y+R\psi$, where $\int_{D_i}=\int_{D_i}\left(P\psi\right)_x+\left(Q\psi\right)_y+\psi\left(R-P_x-Q_y\right)$. So $\int_\Gamma \psi Pdy-\psi Qdx=\int_{\Gamma+C_1-C_2}\psi Pdy-\psi Qdx$ and so $\int_{C_1+C_2}\psi Pdy-\psi Qdx=0 \rightarrow dy/dx=[Q]_-^+/[P]_-^+$
Canonical: For $au_{xx}+2bu_{xy}+cu_{yy}=f$, we need Cauchy-Kowalevski s.t. first derivs defined: $x':=\frac{dx}{ds}$ s.t. on Γ $p_0'=x_0'u_{xx}+y_0'u_{xy}, q_0'=x_0'u_{xy}+y_0'u_{yy}$. Use these 3, solve det A!=0 s.t. $ay_0'^2-2bx_0'y_0'+cx_0'^2\neq 0$. Solve quadratic s.t. $b^2>ac\rightarrow h, b^2<ac\rightarrow e, b^2=ac\rightarrow p$. H: $\lambda_1,\lambda_2\rightarrow\xi,\eta$. E: $\lambda=\lambda_R\pm i\lambda_I;\lambda_R\rightarrow\xi,\lambda_I\rightarrow\eta$. P: $\lambda_1\rightarrow\xi$, choose η independent e.g. xy, x^2. **Green's Fn:** For $u_{xx}+u_{yy}+au_x+bu_y+cu=f$ we have $\int_D GLu-uL^*G=\int_D\left(u_xG\right)_x+\left(u_yG\right)_y-\left(uG_x\right)_x-\left(uG_y\right)_y+\left(auG\right)_x+\left(buG\right)_y=\int_D \boldsymbol{\nabla}\cdot\left(u_nG-uG_n\right)+\boldsymbol{\nabla}\cdot\left((a\,b)^T\hat{n}Gu\right)=\int_{\partial D}u_nG-uG_n+(a\,b)^T\hat{n}G$. NB $\hat{n}=(dy,-dx)$. Also note for quarter plane if we have $G_x(0,y)=0, G(x,0)=0$ then we have same sign at $\xi_1=(-x,y)$, opposite sign at $\xi_2=(x,-y)$, and for the third we reflect ξ_2 across y axis so we have an opposite sign to ξ at $\xi_3=(-x,-y)$. **Types:** Quasi: Coeffs don't depend on highest order derivs Semi: Coeffs depend on x,y. **Causality:** For a n-dim prob, we have n characteristics. Shock intersects $2n$. $\exists\, k$ outgoing, $2n-k$ ingoing. Also have n R-H relations, so $3n-k$ pieces of info. Unknowns are n components of $\vec{\boldsymbol{u}}$ on both sides of shock & slope $\Rightarrow 2n+1$ unknowns. We demand $3n-k=2n+1$ so $k=n-1$ outgoing characterisitcs.
SAM: **Dists:** Need linearity and continuity: $\exists N, C$ s.t. $|(u,\phi)|\leq C\sum_{m\leq N}\max_{\in[-X,X]}|\phi^{(m)}|$. OR $\lim_{n\rightarrow\infty}(u,\phi_n)=(u,\lim_{n\rightarrow\infty}\phi_n)$ for a sequence $\phi_n\rightarrow 0$ as $n\rightarrow\infty$. **Orthog:** $\int_0^\pi \sin(kx)\sin(jx)=\frac{\pi}{2}\delta_{kj}$, same for cos.

NLA: **Cholesky** For matrix $[a_{11}, w^*; w, K] = R_1^T \left[I, 0; 0, K - \frac{ww^*}{a_{11}}\right][\alpha, w^*/\alpha; 0, I]$ we have a decomp: for $k=[1,m-1]$: for $j=[k+1,m]$ $R_{j,j:m} = R_{j,j:m} - \frac{R_{kj}}{R_{kk}} R_{k,j:m}$ endfor $R_{k,k:m} = \frac{R_{k,k:m}}{\sqrt{R_{kk}}}$ endfor. $\frac{m^3}{3}$. **Householder** for $k=[1,n]$: $x = A_{k:m,k}; v_k = sgn(x)\|x\|e_k + x; v_k = \frac{v_k}{\|v_k\|}$ for $j=[k,n]$ $A_{k:m,j} = A_{k:m,j} - 2v_k[v_k^* A_{k:m,j}]$ endfor endfor. $\frac{2mn^2}{3}$. **LU** $U=A, L=I$ for $k=[1,m-1]$: for $j=[k+1,m]$ $U_{j,k:m} = U_{j,k:m} - \frac{U_{jk}}{U_{kk}}U_{k,k:m}$ endfor endfor. $\frac{2m^3}{3}$. **MG-S** $V=A$;for $i=[1,n]$: $r_{ii} = \|v_i\|; q_i = \frac{v_i}{r_{ii}}$;for $j=[i+1,n]$ $v_j = v_j - (q_i^T v_j) q_i; r_{ij} = q_i^T v_j$ endfor endfor. $2mn^2$. **Givens** $3mn^2$ **SVD:** $= \sum_i^{r:=\min m,n} u_i\sigma_i v_i^T$. **Bounds:** $\|ABB^{-1}\| \ge \|AB\|\|B^{-1}\| \to \|A\|/\|B^{-1}\| \ge \|AB\|$. **Norms:**$\|A\|_F = \sqrt{\sum_i(\sigma_i)^2} = \sqrt{Tr(AA^T)}$, $\|A\|_\infty$ = max row sum. **Low-Rank:** For $A\in\mathbb{R}^{m\times n}$ $\min\|A-B\| = \|A-A_r\|$. Proof via $B := B_1B_2^T$ with $B_1\in\mathbb{R}^{m\times r}$; $\exists W s.t. B_2^T W = 0$ with null(W)$\ge n-r$. Then $\exists\, x_V, x_W s.t. V_{r+1}x_V = -Wx_W$. So $\|A-B\| = \|AW\| \ge \|AV_{r+1}x_V\| \ge \sigma_{r+1}$ For reverse $B := A_r$ **Courant:** $\sigma_i = \max_{dim(S)=i}\{\min_x \|Ax\|/\|x\|\}$. Proof via $V_i = [v_i \dots v_n]$, so dim(S)+dim(V_i) $= n+1$ so $\exists\, w\in S\cap V_i$. Then $\|Aw\| \le \sigma_i$. For reverse take $w=v_i$ when $S=[v_1\dots v_i]$ **Schur:** Take $Av_1 = \lambda_1 v_1$; construct $U_1 = [v_1, V_\perp] \to AU_1 = U_1[e_1, X]$. Repeat. **Back Subst:** For $Ux=y$ we have $x_{n-i} = \left(y_{n-i} - \sum_{n-i+1}^n u_{n-i,j}x_j\right)/u_{n-i,n-i}; O(i)$ per iteration so $O(n^2)$ total. **Backwards Stable:** When $\hat f(x) = f(x+\Delta x)$ with $\|\Delta x\|/\|x\| \le O(\varepsilon)$ **Conditioning** $\kappa_2(A) = \sigma_1/\sigma_n = \|A\|\|A^{-1}\|$

NPDE: **Hyperbolic:** Implicit: $(A-B,A) = \frac12(\|A\|^2 - \|B\|^2) + \frac12\|A-B\|^2$ (time),$(-D_x^+D_x^-U^{m+1}, U^{m+1}-U^m) = (D_x^-U^{m+1} - D_x^-U^m, D_x^-U^{m+1})$ (space). Explicit: 1st rewrite in terms of $D_t^{+-}(\Delta t)^{-2}U_j^m + \frac{c^2(\Delta t)^2}{4}D_x^{+-}((\Delta t)^{-2}U_j^m) - c^2D_x^{+-}\left(U_j^{m+1} + 2U_j^m + U_j^{m-1}\right)$. Then use $(D(A-B), A+B) = (DA,A) - (DB,B); (D(A+B), A-B) = (DA,A)-(DB,B)$ by multiplying by $U^{m+1}-U^{m-1}$. Finally WTS $\|V_m\|^2 - \frac{c^2(\Delta t)^2}{4}\|D_x^-V^m\|^2 \ge 0$. Done by noticing: $\|D_x^-V^m\|^2 = \sum_i^J \Delta x|D_x^-V_j^m|^2 = 1/\Delta x\sum_i^J \left(V_j^m - V_{j-1}^m\right)^2 \le 2/\Delta x \sum_i^J (V_j^m)^2 + (V_{j-1}^m)^2 = 4/\Delta x^2 \sum_i^{J-1}\Delta x\left(V_j^m\right)^2$ **Max Principle:** For $-\Delta u = f \le 0 \to \max u \in \partial D$. First show contradiction assuming $LU = f < 0$, then try some auxillary function $\psi = U + \alpha(T_{\max})g(x_i,y_i)$ s.t. $L\psi < 0$ so $\max\psi = \max_{\in\partial D}\psi$. Gets max $e_{i,j}$; change to $-\alpha$ for min $e_{i,j}$. **P-F Ineq:** $\|V\|_h^2 \le c_\star||D_x^-V]|^2$ **Weak Deriv:** w is a weak derivative of u if $\int dx\, wv = (-1)^{|\alpha|}\int dx\, u(D^\alpha v)$ **Parseval:** $\int dk\,\hat u(k)v(k) = \int dk\, v(k)\left(\int dx\, u(x)e^{-ixk}\right) = \int dx\, u(x)\left(\int dk\, v(k)e^{-ixk}\right) = \int dx\, u(x)\hat v(x)$. Now $v(k) := \overline{\hat u(k)} = \overline{F[u(k)]} = \overline{\int dk\, u(k)e^{-ixk}} = \int dk\,\overline{u(k)}e^{ixk} = 2\pi F^{-1}\left[\overline{u(k)}\right] \Rightarrow \hat v(x) = 2\pi\overline{u(x)}$ **Iterative:** If $U^{j+1} = U^j - \tau(AU^j - F) \to U - U^j = (I-\tau A)^j(U-U^0)$ so $\|U-U^j\| \le \|I-\tau A\|^j\|U-U^0\|$. $\|I-\tau A\| = \sigma_1 = |\lambda_1|$ as symmetric. If $\lambda\in[\alpha,\beta]$ then $\lambda_1 \le \max\{|1-\tau\alpha|, |1-\tau\beta|\}$. Attained when $\tau = 2/(\alpha+\beta) \to \lambda_1 = \frac{\beta-\alpha}{\beta+\alpha}$. For $-u''+cu=f$ we have $\lambda_k = c + \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right)$. Lower bound via noting $\sin(y) \ge \frac{2\sqrt2}{\pi}y$ at $y=\frac{\pi}{4} \to \lambda_k \ge c+8$ **Errors:** $(AV,V)_h \ge ||D_x^-V]|_h^2$ & PF Ineq $\to (AV,V)_h \ge \|V\|_h^2/c_\star$. Then $(AV,V)_h(1+c_\star) \ge \|V\|_{1,h}^2 \to (AV,V)_h \ge c_0\|V\|_{1,h}^2$. Now $c_0\|V\|_{1,h}^2 \le (AV,V)_h \le \|f\|_h\|V\|_h \le \|f\|_h\|V\|_{1,h} \to \|V\|_{1,h} \le \|f\|_h/c_0$. (Use $f := AV \to \|e\|_{1,h} \le \|T\|_h/c_0$)