APDE: **Charpit:** $F(p,q,u,x,y) = 0$ with $u_x = p, u_y = q, \dot{x} = F_p, \dot{y} = F_q$. Then via $F_x, F_y, \&\ p_y = q_x \to p_\tau = -F_x - pF_u,\ q_\tau = -F_y - qF_u, u_\tau = pF_p + qF_q$. Also, $\frac{du_0}{ds} = p_0\frac{dx_0}{ds} + q_0\frac{dy_0}{ds}; F_0 = 0$ - last 2 needed to show u defined on Γ. **Proof of Charpit:** Write $\frac{dF}{d\tau} = F_p\frac{dp}{d\tau} + F_q\frac{dq}{d\tau} + F_u\frac{du}{d\tau} + F_x\frac{dx}{d\tau} + F_y\frac{dy}{d\tau} = 0$. Then write $\phi := u_s - px_s - qy_s$. Take ϕ_τ to get $\phi_\tau = F_s - \phi F_u + \partial_s\left(u_\tau - pF_p - qF_q\right) \to \phi_\tau = -\phi F_u$. Then we have $u_\tau = x_\tau u_x + y_\tau y_x$, and $u_s = x_s p + y_s q$ which means $p := u_x, q := u_y$. **Max Principle:** For $-\Delta u = f \leq 0 \to \max u \in \partial D$. First show contradiction assuming $LU = f < 0$, then try some auxillary function $\psi = U + \alpha\left(T_{\max}\right) g\left(x_i, y_i\right)$ s.t. $L\psi < 0$ so $\max\psi = \max_{\in\partial D}\psi$. Gets $\max e_{i,j}$; change to $-\alpha$ for $\min e_{i,j}$. **Laplacian:** In $2D: r^{-1}\left(rf_r\right)_r + r^{-2}f_{\theta\theta}$. In $3D: r^{-2}\left(r^2 f_r\right)_r + r^{-2}\sin^{-2}(\theta)f_{\phi\phi} + r^{-2}\sin^{-1}(\theta)\left(\sin(\theta)f_\theta\right)_\theta$ **Green's f'n Circle:** For $G = 0|_{\partial D}$ we have $G = \frac{-1}{4\pi}\left(\frac{1}{|x-\xi|} - \frac{1}{|\xi||x-\xi'|}\right)$ **Riemann:** For $u_{xy} + au_x + bu_y + cu = f$ we have $\int_D RLu - uL^*R = \int_D \partial_x\left(Ru_y + auR\right) + \partial_y\left(-uR_x + buR\right) = \int_{\partial D} dy\left(Ru_y + Rau\right) + dx\left(uR_x - buR\right)$. Expand over triangle going B-P-A (B at bottom right) $\to$ need $R_x = bR@y = \eta, R_y = aR@x = \xi, R(P) = 1, L^*R = 0$. Also ensure IVP on $\int_B^P dyRu_y \to Ru|_B^P - \int_B^P dy\, uR_y$. **Riemann Invariants:** If we have $\frac{d}{dx}[u-v] = -f$ on $y = x + c_1$, and $\frac{d}{dx}[u+v] = f$ on $y = -x + c_2$, then we have: $u - v + \int_{-c_1}^x ds\, f(s, s+c_1) = k_1$, and $u + v - \int_{c_2}^x ds\, f(s, -s+c_2) = k_2$ for constants k_1, k_2. **R-H:** Derived via $P_x\psi + Q_y\psi = R\psi \to \int_D (P\psi)_x + (Q\psi)_y \left(= \int_\Gamma \psi Pdy - \psi Qdx\right) = \int_D P\psi_x + Q\psi_y + R\psi = \int_{D_1+D_2} P\psi_x + Q\psi_y + R\psi$, where $\int_{D_i} = \int_{D_i}(P\psi)_x + (Q\psi)_y + \psi\left(R - P_x - Q_y\right)$. So $\int_\Gamma \psi Pdy - \psi Qdx = \int_{\Gamma+C_1-C_2}\psi Pdy - \psi Qdx$ and so $\int_{C_1+C_2}\psi Pdy - \psi Qdx = 0 \to dy/dx = [Q]_-^+ / [P]_-^+$ **Canonical:** For $au_{xx} + 2bu_{xy} + cu_{yy} = f$, we need Cauchy-Kowalevski s.t. first derivs defined: $x' := \frac{dx}{ds}$ s.t. on Γ $p_0' = x_0'u_{xx} + y_0'u_{xy}, q_0' = x_0'u_{xy} + y_0'u_{yy}$. Use these 3, solve det A!=0 s.t. $ay_0'^2 - 2bx_0'y_0' + cx_0'^2 \neq 0$. Solve quadratic s.t. $b^2 > ac \to h, b^2 < ac \to e, b^2 = ac \to p$. H: $\lambda_1, \lambda_2 \to \xi, \eta$. E: $\lambda = \lambda_R \pm i\lambda_I; \lambda_R \to \xi, \lambda_I \to \eta$. P: $\lambda_1 \to \xi$, choose η independent e.g. xy, x^2. **Canonical Differentials:** $u_x = u_\xi\xi_x + u_\eta\eta_x, u_{xx} = u_{\xi\xi}\xi_x^2 + u_{\eta\eta}\eta_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_\xi\xi_{xx} + u_\eta\eta_{xx}$. Repeat for $\partial_y, \partial_{yy}$ **Green's Fn:** DON'T USE GREENS THM USE NORMALS For $u_{xx} + u_{yy} + au_x + bu_y + cu = f$ we have $\int_D GLu - uL^*G = \int_D \left(u_xG\right)_x + \left(u_yG\right)_y - \left(uG_x\right)_x - \left(uG_y\right)_y + (auG)_x + (buG)_y = \int_D \boldsymbol{\nabla}\cdot\left(u_nG - uG_n\right) + \boldsymbol{\nabla}\cdot\left((a\, b)^T\hat{n}Gu\right) = \int_{\partial D} u_nG - uG_n + (a\, b)^T\hat{n}G$. NB $\hat{n} = (dy, -dx)$. Also note for quarter plane if we have $G_x(0,y) = 0, G(x,0) = 0$ then we have same sign at $\xi_1 = (-x, y)$, opposite sign at $\xi_2 = (x, -y)$, and for the third we reflect ξ_2 across y axis so we have an opposite sign to ξ at $\xi_3 = (-x, -y)$. **Types:** Quasi: Coeffs don't depend on highest order derivs Semi: Coeffs depend on x, y. **Causality:** For a n-dim prob, we have n characteristics. Shock intersects $2n$. $\exists\, k$ outgoing, $2n - k$ ingoing. Also have n R-H relations, so $3n - k$ pieces of info. Unknowns are n components of $\vec{\boldsymbol{u}}$ on both sides of shock & slope $\Rightarrow 2n + 1$ unknowns. We demand $3n - k = 2n + 1$ so $k = n - 1$ outgoing characterisitcs. **d'Alembert:** Consider triangle A-P-B with AB hypoteneuse. Via $\xi = x + t, \eta = x - t$ we get with $R_\eta = 0$ on $\xi = p$, and $R_\xi = 0$ on $\eta = q$, then via riemann f'n $\phi(P) = -\int_D \frac{\hat{f}}{4}$. $|J| = 2$ so $\phi(r,s) = -\int_D \frac{f}{2}dxdt$. Then have triangle ABP with $AP: \eta = q := r - s \to x - t = r - s$, $PB: \xi = p := r + s \to x + t = r + s$, and $AB: y = 0$ so finally $\phi(r,s) = -\frac{1}{2}\int_0^s dt\int_{r-s+t}^{r+s-t} dx f(x,t)$ **Integral Derivs** $\frac{d}{dt}\int_{b(t)}^{a(t)} dx\, f(x,t) = a'(t)f(a,t) - b'(t)f(b,t) + \int_{b(t)}^{a(t)} dt\, f_t(x,t)$

SAM: **Dists:** Need linearity and continuity: $\exists N, C$ s.t. $|(u,\phi)| \leq C\sum_{m\leq N}\max_{\in[-X,X]}|\phi^{(m)}|$. OR $\lim_{n\to\infty}(u,\phi_n) = (u, \lim_{n\to\infty}\phi_n)$ for a sequence $\phi_n \to 0$ as $n \to \infty$. **Orthog:** $\int_0^\pi \sin(kx)\sin(jx) = \frac{\pi}{2}\delta_{kj}$, same for cos. **S-L Operator** For $T := \alpha y'' + \beta y' + \gamma$, multiply by $\exp\left(\int dx\, \beta\right)$ to get T_{SL}