

APDE: Charpit: $F(p, q, u, x, y) = 0$ with $u_x = p, u_y = q, \dot{x} = F_p, \dot{y} = F_q$. Then via F_x, F_y , & $p_y = q_x \rightarrow p_\tau = -F_x - pF_u, q_\tau = -F_y - qF_u, u_\tau = pF_p + qF_q$. Also, $u0_s = p_0x0_s + q_0y0_s; F_0 = 0$ - last 2 needed to show u defined on Γ . **Laplacian:** In $2D : r^{-1}(rf_r)_r + r^{-2}f_{\theta\theta}$. In $3D : r^{-2}(r^2f_r)_r + r^{-2}\sin^{-2}(\theta)f_{\phi\phi} + r^{-2}\sin^{-1}(\theta)(\sin(\theta)f_\theta)_\theta$ **Riemann:** For $u_{xy} + au_x + bu_y + cu = f$ we have $\int_D RLu - uL^*R = \int_D \partial_x(Ru_y + auR) + \partial_y(-uR_x + buR) = \int_{\partial D} dy(Ru_y + Rau) + dx(uR_x - buR)$. Expand over triangle going B-P-A (B at bottom right) \rightarrow need $R_x = bR@y = \eta, R_y = aR@x = \xi, R(P) = 1, L^*R = 0$. Also ensure IVP to get $R_y, R_x!$ **R-H:** Derived via $P_x\psi + Q_y\psi = R\psi \rightarrow \int_D (P_x\psi)_x + (Q_y\psi)_y (= \int_\Gamma \psi Pdy - \psi Qdx) = \int_D P\psi_x + Q\psi_y + R\psi = \int_{D_1+D_2} P\psi_x + Q\psi_y + R\psi$, where $\int_{D_i} = \int_{D_i} (P\psi)_x + (Q\psi)_y + \psi(R - P_x - Q_y)$. So $\int_\Gamma \psi Pdy - \psi Qdx = \int_{\Gamma+C_1-C_2} \psi Pdy - \psi Qdx$ and so $\int_{C_1+C_2} \psi Pdy - \psi Qdx = 0 \rightarrow dy/dx = [Q]_-^+ / [P]_-^+$
Canonical: For $au_{xx} + 2bu_{xy} + cu_{yy} = f$, we need **Cauchy-Kowalevski** s.t. first derivs defined: $x' := \frac{dx}{ds}$ s.t. on Γ $p'_0 = x'_0u_{xx} + y'_0u_{xy}, q'_0 = x'_0u_{xy} + y'_0u_{yy}$. Use these 3, solve $\det A \neq 0$ s.t. $ay_0'^2 - 2bx_0'y_0' + cx_0'^2 \neq 0$. Solve quadratic s.t. $b^2 > ac \rightarrow h, b^2 < ac \rightarrow e, b^2 = ac \rightarrow p$. **H:** $\lambda_1, \lambda_2 \rightarrow \xi, \eta$. **E:** $\lambda = \lambda_R \pm i\lambda_I; \lambda_R \rightarrow \xi, \lambda_I \rightarrow \eta$. **P:** $\lambda_1 \rightarrow \xi$, choose η independent e.g. xy, x^2 . **Green's Fn:** For $u_{xx} + u_{yy} + au_x + bu_y + cu = f$ we have $\int_D GLu - uL^*G = \int_D (u_xG)_x + (u_yG)_y - (uG_x)_x - (uG_y)_y + (auG)_x + (buG)_y = \int_D \nabla \cdot (u_nG - uG_n) + \nabla \cdot ((ab)^T \hat{n}Gu) = \int_{\partial D} u_nG - uG_n + (ab)^T \hat{n}G$. NB $\hat{n} = (dy, -dx)$. **Also note for quarter plane** if we have $G_x(0, y) = 0, G(x, 0) = 0$ then we have same sign at $\xi_1 = (-x, y)$, opposite sign at $\xi_2 = (x, -y)$, and for the third we reflect ξ_2 across y axis so we have an opposite sign to ξ at $\xi_3 = (-x, -y)$. **Types: Quasi:** Coeffs don't depend on highest order derivs **Semi:** Coeffs depend on x, y . **Causality:** For a n -dim prob, we have n characteristics. Shock intersects $2n$. $\exists k$ outgoing, $2n - k$ ingoing. Also have n R-H relations, so $3n - k$ pieces of info. Unknowns are n components of \vec{u} on both sides of shock & slope $\Rightarrow 2n + 1$ unknowns. We demand $3n - k = 2n + 1$ so $k = n - 1$ outgoing characteristics.
SAM: Dists: Need linearity and continuity: $\exists N, C$ s.t. $|(u, \phi)| \leq C \sum_{m \leq N} \max_{x \in [-X, X]} |\phi^{(m)}|$. OR $\lim_{n \rightarrow \infty} (u, \phi_n) = (u, \lim_{n \rightarrow \infty} \phi_n)$ for a sequence $\phi_n \rightarrow 0$ as $n \rightarrow \infty$. **Orthog:** $\int_0^\pi \sin(kx) \sin(jx) = \frac{\pi}{2} \delta_{kj}$, same for cos.

NLA: Cholesky For matrix $[a_{11}, w^*; w, K] = R_1^T \begin{bmatrix} I, 0; 0, K - \frac{ww^*}{a_{11}} \end{bmatrix} [\alpha, w^*/\alpha; 0, I]$ we have a decomp:
for $k = [1, m-1]$: **for** $j = [k+1, m]$ $R_{j,j:m} = R_{j,j:m} - \frac{R_{kj}}{R_{kk}} R_{k,j:m}$ **endfor** $R_{k,k:m} = \frac{R_{k,k:m}}{\sqrt{R_{kk}}}$ **endfor**. $\frac{m^3}{3}$. **Householder** **for** $k = [1, n]$: $x = A_{k:m,k}; v_k = \text{sgn}(x)\|x\|e_k + x; v_k = \frac{v_k}{\|v_k\|}$ **for** $j = [k, n]$
 $A_{k:m,j} = A_{k:m,j} - 2v_k[v_k^* A_{k:m,j}]$ **endfor** **endfor**. $\frac{2mn^2}{3}$. **LU** $U = A, L = I$ **for** $k = [1, m-1]$: **for**
 $j = [k+1, m]$ $U_{j,k:m} = U_{j,k:m} - \frac{U_{jk}}{U_{kk}} U_{k,k:m}$ **endfor** **endfor**. $\frac{2m^3}{3}$. **MG-S** $V = A$; **for** $i = [1, n]$: $r_{ii} =$
 $\|v_i\|; q_i = \frac{v_i}{r_{ii}}; \text{for } j = [i+1, n]$ $v_j = v_j - (q_i^T v_j)q_i; r_{ij} = q_i^T v_j$ **endfor** **endfor**. $2mn^2$. **Givens** $3mn^2$ **SVD**:
 $= \sum_i^{r:=\min m,n} u_i \sigma_i v_i^T$. **Bounds**: $\|ABB^{-1}\| \geq \|AB\| \|B^{-1}\| \rightarrow \|A\|/\|B^{-1}\| \geq \|AB\|$. **Norms**: $\|A\|_F =$
 $\sqrt{\sum_i (\sigma_i)^2} = \sqrt{\text{Tr}(AA^T)}$, $\|A\|_\infty = \max \text{row sum}$. **Low-Rank**: For $A \in \mathbb{R}^{m \times n}$ $\min \|A - B\| =$
 $\|A - A_r\|$. Proof via $B := B_1 B_2^T$ with $B_1 \in \mathbb{R}^{m \times r}; \exists W \text{ s.t. } B_2^T W = 0$ with $\text{null}(W) \geq n - r$. Then
 $\exists x_V, x_W \text{ s.t. } V_{r+1} x_V = -W x_W$. So $\|A - B\| = \|AW\| \geq \|A V_{r+1} x_V\| \geq \sigma_{r+1}$ For reverse $B := A_r$
Courant: $\sigma_i = \max_{\dim(S)=i} \{\min_x \|Ax\|/\|x\|\}$. Proof via $V_i = [v_i \dots v_n]$, so $\dim(S) + \dim(V_i) = n + 1$
so $\exists w \in S \cap V_i$. Then $\|Aw\| \leq \sigma_i$. For reverse take $w = v_i$ when $S = [v_1 \dots v_i]$ **Schur**: Take
 $Av_1 = \lambda_1 v_1$; construct $U_1 = [v_1, V_\perp] \rightarrow AU_1 = U_1[e_1, X]$. Repeat. **Back Subst**: For $Ux = y$ we have
 $x_{n-i} = (y_{n-i} - \sum_{n-i+1}^n u_{n-i,j} x_j) / u_{n-i,n-i}; O(i)$ per iteration so $O(n^2)$ total. **Backwards Stable**:
When $\hat{f}(x) = f(x + \Delta x)$ with $\|\Delta x\|/\|x\| \leq O(\varepsilon)$ **Conditioning** $\kappa_2(A) = \sigma_1/\sigma_n = \|A\| \|A^{-1}\|$
NPDE: Def'n: With $u_{tt} - c^2 u_{xx} = f$ have $\Delta x = (b-a)/J, \Delta t = T/M, x_j = a + j\Delta x, t = m\Delta t$.
I.C: $U_j^0 = u_0(x_j), U_j^1 = U_j^0 + u_1(x_j)\Delta t, U_0^m = U_j^m = 0$ **Hyp Impl**: $(A - B, A) = \frac{1}{2}(\|A\|^2 - \|B\|^2) +$
 $\frac{1}{2}\|A - B\|^2$ with $A := U^{m+1} - U^m, B := U^m - U^{m-1}$ (T); $(-D_x^+ D_x^- U^{m+1}, U^{m+1} - U^m) = (D_x^- U^{m+1} -$
 $D_x^- U^m, D_x^- U^{m+1})$ (X). Then $\frac{1}{2\Delta t^2}(\|U^{m+1} - U^m\|^2 - \|U^m - U^{m-1}\|^2) + \frac{\Delta t^2}{2\Delta t^2} \|U^{m+1} - 2U^m + U^{m-1}\|^2 +$
 $\frac{c^2}{2} (\|D_x^- U^{m+1}\|^2 - \|D_x^- U^m\|^2) + \frac{c^2 \Delta t^2}{2\Delta t^2} \|D_x^- (U^{m+1} - U^m)\|^2 = (f, U^{m+1} - U^m)$. Then $M^2(U^m) :=$
 $\left\| \frac{U^m - U^{m-1}}{\Delta t} \right\|^2 + c^2 \|D_x^- U^{m+1}\|^2$. Write green as $\leq \|f\| \|U^{m+1} - U^m\| = \sqrt{\Delta t T} \|f\| \sqrt{\frac{\Delta t}{T}} \left\| \frac{U^{m+1} - U^m}{\Delta t} \right\| \leq$
 $\frac{\Delta t T}{2} \|f\|^2 + \frac{\Delta t}{2T} \left\| \frac{U^{m+1} - U^m}{\Delta t} \right\|^2$. Then $(1 - \frac{\Delta t}{T}) M^2(U^m) \leq M^2(U^{m-1}) + \Delta t T \|f\|^2 \rightarrow M^2(U^m) \leq (1 +$
 $\frac{2\Delta t}{T}) M^2(U^{m-1}) + 2\Delta t T \|f\|^2$. Use $a_m \leq \alpha^m a_0 + \sum_{k=1}^m \alpha^{m-k} b_k$ so $M^2 \leq e^2 M^2(U^0) + 2e^2 T \sum_{k=1}^m \Delta t \|f\|^2$
Hyp Expl: 1st rewrite in terms of $D_t^{+-}(\Delta t)^{-2} U_j^m + \frac{c^2(\Delta t)^2}{4} D_x^{+-}((\Delta t)^{-2} D_t^{+-} U_j^m) -$
 $(c^2/4) D_x^{+-} (U_j^{m+1} + 2U_j^m + U_j^{m-1})$. Then use $(D(A - B), A + B) = (DA, A) - (DB, B);$
 $(D(A + B), A - B) = (DA, A) - (DB, B)$ by multiplying by $U^{m+1} - U^{m-1}$. Finally WTS $\|V_m\|^2 -$
 $\frac{c^2(\Delta t)^2}{4} \|D_x^- V^m\|^2 \geq 0$. Done by noticing: $\|D_x^- V^m\|^2 = \sum_i^J \Delta x |D_x^- V_j^m|^2 = \frac{1}{\Delta x} \sum_i^J (V_j^m - V_{j-1}^m)^2 \leq$
 $2/\Delta x \sum_i^J (V_j^m)^2 + (V_{j-1}^m)^2 = 4/\Delta x^2 \sum_i^{J-1} \Delta x (V_j^m)^2$. Eventually show $N^2(U^m) :=$
 $\left(\left(I + \frac{c^2 \Delta t^2}{2} D_x^{+-} \right) \frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t} \right) + c^2 \left\| D_x^- \frac{U^{m+1} + U^m}{2} \right\|^2 \rightarrow N^2(U^m) = N^2(U^{m-1}) + (f, U^{m+1} -$
 $U^m)$ **Max Principle**: For $-\Delta u = f \leq 0 \rightarrow \max u \in \partial D$. First show contradiction assuming
 $LU = f < 0$, then try some auxillary function $\psi = U + \alpha(T_{\max})g(x_i, y_i)$ s.t. $L\psi < 0$ so $\max \psi =$
 $\max_{\partial D} \psi$. Gets $\max e_{i,j}$; change to $-\alpha$ for $\min e_{i,j}$. **P-F Ineq**: $\|V\|_h^2 \leq c_\star \|D_x^- V\|^2$. For 2D: $|V_j^m| =$
 $|\sum_{\alpha=1}^j h(D_x^- V_\alpha^m)|^2 \leq jh \sum_{\alpha=1}^{N-1} h |D_x^- V_\alpha^m|^2 \rightarrow \|V\|_h^2 = \sum_{j=1}^{N-1} h |V_j^m|^2 \leq \sum_{j=1}^{N-1} jh^2 \sum_{\alpha=1}^{N-1} h |D_x^- V_\alpha^m|^2 \leq$
 $\frac{1}{2} \sum_{j=1}^N h |D_x^- V_j^m|^2$. Use blue and add for x, y for $c_\star = 0.25$. **Weak Deriv**: w is a weak deriva-
tive of u if $\int dx wv = (-1)^{|\alpha|} \int dx u(D^\alpha v)$ **Parseval**: $\int dk \hat{u}(k)v(k) = \int dk v(k) (\int dx u(x)e^{-ixk}) =$
 $\int dx u(x) (\int dk v(k)e^{-ixk}) = \int dx u(x)\hat{v}(x)$. Now $v(k) := \hat{u}(k) = \overline{F[u(k)]} = \overline{\int dk u(k)e^{-ixk}} =$
 $\int dk \overline{u(k)}e^{ixk} = 2\pi F^{-1}[\overline{u(k)}] \Rightarrow \hat{v}(x) = 2\pi \overline{u(x)}$ **Iterative**: If $U^{j+1} = U^j - \tau(AU^j - F) \rightarrow U - U^j =$
 $(I - \tau A)^j (U - U^0)$ so $\|U - U^j\| \leq \|I - \tau A\|^j \|U - U^0\|$. $\|I - \tau A\| = \sigma_1 = |\lambda_1|$ as symmetric. If
 $\lambda \in [\alpha, \beta]$ then $\lambda_1 \leq \max\{|1 - \tau\alpha|, |1 - \tau\beta|\}$. Attained when $\tau = 2/(\alpha + \beta) \rightarrow \lambda_1 = \frac{\beta - \alpha}{\beta + \alpha}$. For
 $-u'' + cu = f$ we have $\lambda_k = c + \frac{4}{h^2} \sin^2(\frac{k\pi h}{2})$. Lower bound via noting $\sin(y) \geq \frac{2\sqrt{2}}{\pi} y$ at $y = \frac{\pi}{4} \rightarrow \lambda_k \geq c + 8$
Errors: $(AV, V)_h \geq \|D_x^- V\|_h^2$ & PF Ineq $\rightarrow (AV, V)_h \geq \|V\|_h^2 / c_\star$. Then $(AV, V)_h (1 + c_\star) \geq \|V\|_{1,h}^2 \rightarrow$
 $(AV, V)_h \geq c_0 \|V\|_{1,h}^2$. Now $c_0 \|V\|_{1,h}^2 \leq (AV, V)_h \leq \|f\|_h \|V\|_h \leq \|f\|_h \|V\|_{1,h} \rightarrow \|V\|_{1,h} \leq \|f\|_h / c_0$. (Use
 $f := AV \rightarrow \|e\|_{1,h} \leq \|T\|_h / c_0$).