APDE: **Charpit:** $F(p,q,u,x,y)=0$ with $u_x=p, u_y=q, \dot{x}=F_p, \dot{y}=F_q$. Then via $F_x, F_y, \& \, p_y=q_x \to p_\tau=-F_x-pF_u,\ q_\tau=-F_y-qF_u, u_\tau=pF_p+qF_q$. Also, $u0_s=p_0x0_s+q_0y0_s; F_0=0$ - last 2 needed to show u defined on Γ. **Riemann:** $\int_D RLu-uL^*R=\int_D \partial_x(Ru_y+auR)+\partial_y(-uR_x+buR)=\int_{\partial D} dy\,(Ru_y+Rau)+dx\,(uR_x-buR)$. Expand over triangle going B-P-A (B at bottom right) $\to$ need $R_x=bR@y=\eta, R_y=aR@x=\xi, R(P)=1, L^*R=0$. Also ensure IVP to get R_y, R_x! **Canonical:** For $au_{xx}+2bu_{xy}+cu_{yy}=f$, we need Cauchy-Kowalevski s.t. first derivs defined: $x':=\frac{dx}{ds}$ s.t. on Γ $p_0'=x_0'u_{xx}+y_0'u_{xy}, q_0'=x_0'u_{xy}+y_0'u_{yy}$. Use these 3, solve det A!=0 s.t. $ay_0'^2-2bx_0'y_0'+cx_0'^2\neq 0$. Solve quadratic s.t. $b^2>ac\to h, b^2<ac\to e, b^2=ac\to p$. H: $\lambda_1,\lambda_2\to\xi,\eta$. E: $\lambda=\lambda_R\pm i\lambda_I; \lambda_R\to\xi, \lambda_I\to\eta$. P: $\lambda_1\to\xi$, choose η independent e.g. xy, x^2. **Green's Fn:** For $u_{xx}+u_{yy}+au_x+bu_y+cu=f$ we have $\int_D GLu-uL^*G=\int_D (u_xG)_x+(u_yG)_y-(uG_x)_x-(uG_y)_y+(auG)_x+(buG)_y=\int_D \boldsymbol{\nabla}\cdot(u_nG-uG_n)+\boldsymbol{\nabla}\cdot\left((a\,b)^T\hat{n}Gu\right)=\int_{\partial D} u_nG-uG_n+(a\,b)^T\hat{n}G$. NB $\hat{n}=(dy,-dx)$. Also note for quarter plane if we have $G_x(0,y)=0, G(x,0)=0$ then we have same sign at $\xi_1=(-x,y)$, opposite sign at $\xi_2=(x,-y)$, and for the third we reflect ξ_2 across y axis so we have an opposite sign to ξ at $\xi_3=(-x,-y)$. **Types:** Quasi: Coeffs don't depend on highest order derivs Semi: Coeffs depend on x,y. **Causality:** For a n-dim prob, we have n characteristics. Shock intersects $2n$. $\exists\, k$ outgoing, $2n-k$ ingoing. Also have n R-H relations, so $3n-k$ pieces of info. Unknowns are n components of $\vec{\boldsymbol{u}}$ on both sides of shock & slope $\Rightarrow 2n+1$ unknowns. We demand $3n-k=2n+1$ so $k=n-1$ outgoing characterisitcs.
SAM: **Dists:** Need linearity and continuity: $\exists N, C$ s.t. $|(u,\phi)|\leq C\sum_{m\leq N}\max_{\in[-X,X]}|\phi^{(m)}|$. OR $\lim_{n\to\infty}(u,\phi_n)=(u,\lim_{n\to\infty}\phi_n)$ for a sequence $\phi_n\to 0$ as $n\to\infty$. **Orthog:** $\int_0^\pi \sin(kx)\sin(jx)=\frac{\pi}{2}\delta_{kj}$, same for cos.

NLA: **Cholesky** For matrix $[a_{11}, w^*; w, K] = R_1^T \left[I, 0; 0, K - \frac{ww^*}{a_{11}}\right][\alpha, w^*/\alpha; 0, I]$ we have a decomp: for $k = [1, m-1]$: for $j = [k+1, m]$ $R_{j,j:m} = R_{j,j:m} - \frac{R_{kj}}{R_{kk}} R_{k,j:m}$ endfor $R_{k,k:m} = \frac{R_{k,k:m}}{\sqrt{R_{kk}}}$ endfor. $\frac{m^3}{3}$. **Householder** for $k = [1, n]$: $x = A_{k:m,k}; v_k = sgn(x)\|x\| e_k + x; v_k = \frac{v_k}{\|v_k\|}$ for $j = [k, n]$ $A_{k:m,j} = A_{k:m,j} - 2v_k\,[v_k^* A_{k:m,j}]$ endfor endfor. $\frac{2mn^2}{3}$. **LU** $U = A, L = I$ for $k = [1, m-1]$: for $j = [k+1, m]$ $U_{j,k:m} = U_{j,k:m} - \frac{U_{jk}}{U_{kk}} U_{k,k:m}$ endfor endfor. $\frac{2m^3}{3}$. **MG-S** $V = A$;for $i = [1, n]$: $r_{ii} = \|v_i\|; q_i = \frac{v_i}{r_{ii}}$;for $j = [i+1, n]$ $v_j = v_j - (q_i^T v_j) q_i; r_{ij} = q_i^T v_j$ endfor endfor. $2mn^2$. **Givens** $3mn^2$ **SVD:** $= \sum_i^{r:=\min m,n} u_i \sigma_i v_i^T$. **Bounds:** $\|ABB^{-1}\| \geq \|AB\|\|B^{-1}\| \to \|A\|/\|B^{-1}\| \geq \|AB\|$. **Norms:**$\|A\|_F = \sqrt{\sum_i (\sigma_i)^2}$, $\|A\|_\infty =$ max row. **Low-Rank:** For $A \in \mathbb{R}^{m\times n}$ $\min \|A - B\| = \|A - A_r\|$. Proof via $B := B_1 B_2^T$ with $B_1 \in \mathbb{R}^{m\times r}$; $\exists W s.t. B_2^T W = 0$ with null$(W) \geq n - r$. Then $\exists\, x_V, x_W s.t. V_{r+1} x_V = -W x_W$. So $\|A - B\| = \|AW\| \geq \|AV_{r+1} x_V\| \geq \sigma_{r+1}$ For reverse $B := A_r$ **Courant:** $\sigma_i = \max_{dim(S)=i}\{\min_x \|Ax\|/\|x\|\}$. Proof via $V_i = [v_i \dots v_n]$, so dim(S)+dim$(V_i) = n+1$ so $\exists\, w \in S \cap V_i$. Then $\|Aw\| \leq \sigma_i$. For reverse take $w = v_i$ when $S = [v_1 \dots v_i]$

NPDE: **Hyperbolic:** Implicit: $(A - B, A) = \frac{1}{2}(\|A\|^2 - \|B\|^2) + \frac{1}{2}\|A - B\|^2$ (time),$(-D_x^+ D_x^- U^{m+1}, U^{m+1} - U^m) = (D_x^- U^{m+1} - D_x^- U^m, D_x^- U^{m+1})$ (space). Explicit: 1st rewrite in terms of $D_t^{+-}(\Delta t)^{-2} U_j^m + \frac{c^2(\Delta t)^2}{4} D_x^{+-}((\Delta t)^{-2} U_j^m) - c^2 D_x^{+-}\left(U_j^{m+1} + 2U_j^m + U_j^{m-1}\right)$. Then use $(D(A-B), A+B) = (DA, A) - (DB, B); (D(A+B), A-B) = (DA, A) - (DB, B)$ by multiplying by $U^{m+1} - U^{m-1}$. Finally WTS $\|V_m\|^2 - \frac{c^2(\Delta t)^2}{4}\|D_x^- V^m\|^2 \geq 0$. Done by noticing: $\|D_x^- V^m\|^2 = \sum_i^J \Delta x |D_x^- V_j^m|^2 = 1/\Delta x \sum_i^J \left(V_j^m - V_{j-1}^m\right)^2 \leq 2/\Delta x \sum_i^J (V_j^m)^2 + (V_{j-1}^m)^2 = 4/\Delta x^2 \sum_i^{J-1} \Delta x \left(V_j^m\right)^2$ **Max Principle:** For $-\Delta u = f \leq 0 \to \max u \in \partial D$. **P-F Ineq:** $\|V\|_h^2 \leq c_\star ||D_x^- V]|^2$ **Weak Deriv:** w is a weak derivative of u if $\int dx\, wv = (-1)^{|\alpha|}\int dx u(D^\alpha v)$ **Parseval:** $\int dk\ \hat{u}(k) v(k) = \int dk v(k)\left(\int dx\ u(x) e^{-ixk}\right) = \int dx u(x)\left(\int dk\ v(k) e^{-ixk}\right) = \int dx\ u(x)\hat{v}(x)$. Now $v(k) := \overline{\hat{u}(k)} = \overline{F[u(k)]} = \overline{\int dk\, u(k) e^{-ixk}} = \int dk\ \overline{u(k)} e^{ixk} = 2\pi F^{-1}\left[\overline{u(k)}\right] \Rightarrow \hat{v}(x) = 2\pi\overline{u(x)}$ **Iterative:** If $U^{j+1} = U^j - \tau\left(AU^j - F\right) \to U - U^j = (I - \tau A)^j\left(U - U^0\right)$ so $\left\|U - U^j\right\| \leq \|I - \tau A\|^j \left\|U - U^0\right\|$. $\|I - \tau A\| = \sigma_1 = |\lambda_1|$ as symmetric. If $\lambda \in [\alpha, \beta]$ then $\lambda_1 \leq \max\{|1 - \tau\alpha|, |1 - \tau\beta|\}$. Attained when $\tau = 2/(\alpha + \beta) \to \lambda_1 = \frac{\beta - \alpha}{\beta + \alpha}$. For $-u'' + cu = f$ we have $\lambda_k = c + \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right)$. Lower bound via noting $\sin(y) \geq \frac{2\sqrt{2}}{\pi} y$ at $y = \frac{\pi}{4} \to \lambda_k \geq c + 8$ **Errors:** $(AV, V)_h \geq ||D_x^- V]|_h^2$ & PF Ineq $\to (AV, V)_h \geq \|V\|_h^2 / c_\star$. Then $(AV, V)_h\,(1 + c_\star) \geq \|V\|_{1,h}^2 \to (AV, V)_h \geq c_0\|V\|_{1,h}^2$. Now $c_0\|V\|_{1,h}^2 \leq (AV, V)_h \leq \|f\|_h \|V\|_h \leq \|f\|_h \|V\|_{1,h} \to \|V\|_{1,h} \leq \|f\|_h / c_0$. (Use $f := AV \to \|e\|_{1,h} \leq \|T\|_h / c_0$)