NLA: **Golub** for $k = 1 : m, n$: $u_k = (sgn(b_{k,k})\|b_{k:m,k}\|e_1 + b_{k:m,k})$; $u_k := \hat{u}_k$; $U_k := I - 2u_k u_k^T$; $B_{k:m,k:n} := U_k B_{k:m,k:n}$; $U = [I_{k-1,k-1}, 0; 0, U_k]$; for $j = 1 : m, n-1$: $v_k^T := sgn(b_{k,k+1})\|b_{k,k+1:n}\|e_1 + b_{k:m,k}$; $V_k := I - 2v_k v_k^T$; $B_{1:m,k+1:n} = B_{1:m,k+1:n}V_k$; $V = [I_{k,k}, 0; 0, V_k]$ endfor endfor; $2 \cdot (2mn^2 - 2n^3/3)$ **Householder** for $k = [1, n]$: $x = A_{k:m,k}$; $v_k = sgn(x)\|x\|e_k + x$; $v_k = \frac{v_k}{\|v_k\|}$ for $j = [k, n]$ $A_{k:m,j} = A_{k:m,j} - 2v_k\,[v_k^* A_{k:m,j}]$ endfor endfor. $2mn^2 - \frac{2n^3}{3}$. **MG-S** $V = A$; for $i = [1, n]$: $r_{ii} = \|v_i\|$; $q_i = \frac{v_i}{r_{ii}}$; for $j = [i+1, n]$ $v_j = v_j - (q_i^T v_j)q_i$; $r_{ij} = q_i^T v_j$ endfor endfor. $2mn^2$. **Arnoldi:** $q_1 := \hat{b}$; $q_{k+1}h_{k+1,k} = Aq_k - \sum_{i=1}^{k} q_i h_{ik}$; $h_{ik} = q_i^T(Aq_k)$; $h_{k+1,k} := \|v\| \to AQ_k := Q_k H_k + q_{k+1}[0 \dots h_{k+1,k}]$. **Lanczos:** If $A = A^T$ for arnoldi, $AQ_k = Q_k T_k + q_{k+1}[0 \dots t_{k+1,k}]$. T tridiag via $H_k = Q_k^T A Q_k$. Also have $t_{k+1}q_{k+1} = (A - t_{k,k})q_k - t_{k-1,k}q_{k-1}$. **Givens** $3mn^2$ **SVD:** $= \sum_i^{r:=\min m,n} u_i \sigma_i v_i^T$. **C-F:** $\sigma_i(A)\sigma_n(B) \le \sigma_i(AB) \le \sigma_i(A)\sigma_1(B)$ **QR Algo:** $A_{k+1} = Q_k^T A_k Q_k \to A_{k+1} = \left(Q^{(k)}\right)^T AQ^{(k)}$. Next $A^{k-1} = (Q_1 \dots Q_{k-1})(R_{k-1} \dots R_1)$, so $A_k = Q_k R_k = (Q^{(k-1)})^T AQ^{(k-1)}$ so $Q^{(k-1)}A_k = AQ^{(k-1)}$. So $A^k = (AQ^{(k-1)})R^{(k-1)} = Q^{(k)}R^{(k)}$ as $A_k = Q_k R_k$. **Krylov:** Usually want $x_k - x_0 \in \mathcal{K}_k$. Also note to show CG span properties first show $\{r_k\} \subset \{p_k\}$ etc, then show $\{p_k\}$ LI. **GMRES:** $\min \|AQ_k y - b\|_2 \to \min \|H_k y - \|b\|e_1\|$. Bound $\|r_k\| = \|Mp(\Lambda)M^{-1}r_0\|$ **GMRES Conv:** If $x_k = p_{k-1}(A)b$ have $\min \|Ax_k - b\| = \min_{p(0)=1} \|Ap_{k-1}(A)b - b\| \le k_2(A)\|p(\Lambda)b\|$ with $p(0) = 1$ **CG Bound:** With $c = x - x_0, c_k = x_k - x_0$ s.t. $r_k = A(c - c_k)$ we have $r_k^T v = 0 \;\forall\; v \in \mathcal{K}_k$ so $v^t A(c - c_k) = 0$, s.t. $y = c_k = \arg\min \|c - y\|_A$. WTS $e_k = e_0 p_k(A)$ with $p(0) = 1$, and write $e_0 := \sum \gamma_i v_i$ with $Av_i = \lambda_i v_i \to \|e_k\|_A = \min_{p_k, p(0)=1} \max |p(\lambda_i\|\|e_0\|_A$ **CG Convergence:** $\|e_k\|_A = \min_{p(0)=1} \|p_k(A)e_0\| = \min_{p_k(A)} \max |p_k(\lambda)|\|e_0\| \to \le 2\left((\sqrt{k_2} - 1)/(\sqrt{k_2} + 1)\right)^k$; need $\alpha := 2(\lambda_1 + \lambda_2)$ **Cheb:** $T_k(x) = \frac{1}{2}(z^k + z^{-k})$; $2xT_k = T_{k+1} + T_{k-1}$ **Cheb Shift:** Choose $p(x) = T_k([2x - b - a] \,/\, [b - a])/T_k\left([-b - a] \,/\, [b - a]\right)$ s.t. $p(0) = 1$. Then $p \le 1/|T_k\left([-b - a] \,/\, [b - a]\right)| \le 2\left([\sqrt{\kappa} - 1]/[\sqrt{\kappa} + 1]\right)^k$ **CG Conditions:** To show $r_{k+1}^T r_k = 0$ first show $p_k^T A p_k = p_k^T A r_k$ via β then show $p_k^T r_k = r_k^T r_k$ via $p_{k-1}^T r_k = 0$. **Preconditioning:** For GMRES solve $MAx = Mb$ with MA eigvals clustered far from 0, well conditioned. E.g. if $A = LU$ do $(LU)^{-1}$. For CG, $A = A^T$ difficult so want $M^T M \approx A^{-1}$. So want $M^T AMy = M^T b$, same properties. **MP:** $\sigma(G) \in [\sqrt{m} - \sqrt{n}, \sqrt{m} + \sqrt{n}] \to k_2 = O(1)$ **Sketch:** First $G = \mathbb{R}^{s \times m}$ so with $GA\hat{x} = Gb$, and via $C-F$ $\left\|G[A, b][v, -1]^T\right\| \le (s + \sqrt{n+1})\left\|R[v, -1]^T\right\|$, similar for lower bound via MP $\to \|A\hat{x} - b\| \le (\sqrt{s} + \sqrt{n+1})/(\sqrt{s} - \sqrt{n+1})\|Ax - b\|$. **Blend:** solve $\left\|(A\tilde{R}^{-1})y - b\right\| = 0$ via CG;$k_2(A\tilde{R}^{-1}) = O(1)$ with $GA = \tilde{Q}\tilde{R}$ PROOF: $A = QR; GA = GQR = \hat{G}R$. Let $\hat{G} = \hat{Q}\hat{R}$ so $GA = \hat{Q}\hat{R}R = \hat{Q}(\hat{R}R) \to \tilde{R}^{-1} = R^{-1}\hat{R}^{-1} \to k_2(A\tilde{R}^{-1}) = k_2(\hat{R}^{-1}) = O(1)$ by MP. $O(mn)$ to solve via normal **HMT:** For $X = n \times r$ let $AX = QR$, then if $A = U_r \Sigma_r V_r^T$, span(Q)=span(U_r) so $\hat{A} = QQ^T A$ is a rank r approximant. **HMT Proof:** Goal $\left\|A - \hat{A}\right\| = O(1)\|A - A_r\|$. Have $(I - QQ^T)AX = 0$ so $A - \hat{A} = (I_m - QQ^T)A(I_n - XM^T) = 0 \;\forall\; M^T$. Choose $M^T = (V^T X)^\dagger V^T$, $V \in n \times \hat{r} \le r$. Let $XM^T = P$ s.t. $A(I - P) = A(I - VV^T)(I - P)$. So $\left\|A - \hat{A}\right\| = \left\|(I_m - QQ^T)U_A \Sigma_A [\tilde{V}_{\hat{r}}^T, \tilde{V}_{\hat{r}+1}^T]^T (I - VV^T)(I - P)\right\|$. If $V = \tilde{V}_r$ then $= \left\|(I_m - QQ^T)U_A \Sigma_A [0, \tilde{V}_{\hat{r}+1}]^T (I - P)\right\| \le \|\Sigma_{\hat{r}+1}\|\left\|I_n - XM^T\right\|$. Now note $\left\|I_n - XM^T\right\| = \left\|I_n - X(\tilde{V}_r^T X)^\dagger \tilde{V}_r^T\right\| \le 1 + \|X\|\left\|(\tilde{V}_r^T X)^\dagger\right\|$. Now $\|X\| \le \sqrt{n} + \sqrt{r}$ by MP, and $\left\|(\tilde{V}_r^T X)^\dagger\right\| = \sigma_n(\tilde{V}_r^T X)^{-1} \le (\sqrt{r} - \sqrt{\hat{r}})^{-1}$ by MP. So $\left\|XM^T\right\| \le \frac{\sqrt{n} + \sqrt{r}}{\sqrt{r} - \sqrt{\hat{r}}}$. **Bounds:** $\left\|ABB^{-1}\right\| \ge \|AB\|\left\|B^{-1}\right\| \to \|A\|/\left\|B^{-1}\right\| \ge \|AB\|$. **Weyls:** $\sigma_i(A + B) = \sigma_i(A) + [-\|B\|, \|B\|]$ **Rev Δ Ineq:** $\|A - B\| \ge |\|A\| - \|B\||$ **Courant Application:** $\sigma_i([A_1; A_2]) \ge \max(\sigma_i(A_1), \sigma_i(A_2))$ **Schur:** Take $Av_1 = \lambda_1 v_1$; construct $U_1 = [v_1, V_\perp] \to AU_1 = U_1[e_1, X]$. Repeat. **Conditioning** $\kappa_2(A) = \sigma_1/\sigma_n = \|A\|\left\|A^{-1}\right\|$ **Similarity:** $A \to P^{-1}AP$, same λ. **Pseud-Inv:** $A^\dagger = V\Sigma^{-1}U^T = (A^T A)^{-1}A^T$. Else have $A^\dagger = A^T(AA^T)^{-1}$. $M^\dagger = M^{-1}$ if full rank.

CO: **BFGS:** $\gamma_k = \Delta \nabla f_{k+1,k} = B_{k+1}(x_{k+1} - x_k) = B_{k+1}\alpha_k s_k$ **G-N:** $\vec{x}_{k+1} = \vec{x}_k - \frac{\nabla f_k}{J^T J}$, with $J :=$ Jacobian of $r(x)$ **Linesearch Convergence:** Show $x_{k+1} - x_* = \Psi(x_k) - x_* = \Psi(x_* + e_k) - x_*$ and taylor expand. **SD:** $\|x_{k+1} - x_*\| \le (k_2(H) - 1)\,/\,(k_2(H) + 1)\|x_k - x_*\|$ with H hessian. Also note with EXACT linesearch for quadratic, $H(x - x_*) = -s$. **Rayleigh:** $\frac{s^T H s}{\|s\|^2} \le \|H\|$ **bArm:** To show existence of α, have $\phi(\alpha) = f(x_k + \alpha_k s_k)$, $\psi(\alpha) = \phi(\alpha) - \phi(0) - \beta\alpha\phi'(0) \le 0$, show $\psi'(0) = (1 - \beta)\phi'(0) \le 0 \to \psi(\alpha) \downarrow$ with α. **BFGS:** To show $H_{k+1} \ge 0$ nec. $\gamma^T\delta > 0$. Suff via γ, δ LI $\to$ use $\|\cdot\|_H \to \gamma^T\delta > 0$. **Quad Penalty Meth** With $y = -c/\sigma$, $\|\nabla_\sigma \Phi\| \le \epsilon^k, \sigma^k \to 0, x \to x_*, \nabla c(x_*)$ LI, then $y \to y_*$, $x \to KKT$, if $f, c \in C^1$. Also need J_*^T full row rank. PROOF: If $y_* := J_*^\dagger \nabla f_* \to \|y_k - y_*\| = \left\|J_*^\dagger \nabla f_* - y_k\right\| \le \left\|J_k^\dagger \cancel{\nabla f_k - J_*^\dagger \nabla f_*}\right\| + \left\|J_k^\dagger \nabla f_k - y_k\right\|$. Next $\left\|J_k^\dagger \nabla f_k - y_k\right\| \le \left\|J_k^\dagger\right\|\|\nabla_\sigma \Phi\| \to 0$. Also, $\nabla f_* - J_*^T y_* \to 0$, and $c_{k\to *} = -\sigma^{k\to *}y_{k\to *} = 0$ so $x_* \to KKT$ **Quad Pen. Meth Newt** Have $w = (J\Delta x + c)/\sigma$ so $[\nabla^2 f, J^T; J, -\sigma I][\Delta x, w]^T = -[\nabla f, c]$ **Trust Region Radius:** $\rho_k := (f(x_k) - f(x_k + s_k))/(f(x_k) - m_k(s_k))$ **TR-Method:** If $\rho \ge 0.9$ then double radius, update step $x_{k+1} = x_k + s_k$. If $\rho \ge 0.1$ then same radius, update step. If ρ small shrink radius, don't update step. **Cauchy:** Is the point on gradient which minimises the quadratic model within TR. Want $m_k(s_k) \leq m_k(s_{kc})$, where $s_{kc} := -\alpha_{kc}\nabla f(x_k)$, and $\alpha_{kc} := \arg\min m_k\left(\alpha\nabla f(x_k)\right)$ subject to $\|\alpha\nabla f\| \leq \Delta$, i.e. $\alpha_{max} := \Delta/\|\nabla f\|$. **Calculation of Cauchy:** We want to prove cauchy model decrease i.e. $f(x_k) - m_k(s_k) \geq f(x_k) - m_k(s_{kc}) \geq 0.5\|\nabla f_k\| \min\left\{\Delta_k, \frac{\|\nabla f_k\|}{\|\nabla^2 f_k\|}\right\}$. First define $\Psi(\alpha) := m_k(-\alpha\nabla f)$ s.t. $\Psi := f_k - \alpha\|f_k\|^2 - 0.5\alpha^2 H_k$, with $H_k := \left[\nabla f_k\right]^T\left[\nabla^2 f_k\right]\left[\nabla f_k\right]$. N.B. that $\alpha_{min} := \frac{\|\nabla f_k\|^2}{H_k}$ if $H_k > 0$, from $\Psi'(0) < 0$. Now A: If $H_k \leq 0$ then we have $\Psi(\alpha) \leq f_k - \alpha\|\nabla f_k\|^2 \rightarrow \alpha_{kc} = \alpha_{max}$. So, we have $f_k - m_{s_k} \geq f_k - m_{s_{kc}} \geq \|\nabla f_k\|\Delta_k \geq 0.5\|\nabla f_k\| \min\{\Delta_k\}$. Now B.i: If $H_k > 0 \rightarrow \alpha_{kc} = \alpha_{min}$. Here $f_k - m_{s_{kc}} = \alpha_{kc}\|\nabla f\|^2 - 0.5\alpha_{kc}^2 H_k = \frac{\|\nabla f\|^4}{2H_k} \geq \frac{\|\nabla f\|}{2}\min\left\{\frac{\|\nabla f\|}{\|\nabla^2 f\|}\right\}$ via C-S. Now B.ii: If $H_k > 0 \rightarrow \alpha_{kc} = \alpha_{max}$. Here $\Delta/\|\nabla f\| \leq \|\nabla f\|^2/H_k \rightarrow \alpha_{kc}H_k \leq \|\nabla f\|^2$. So $f_k - m_{kc} = -\alpha_{kc}\|\nabla f\|^2 + \frac{\alpha_{kc}^2}{2}H_k \geq \frac{\|\nabla f\|^2}{2}\alpha_{kc} \geq 0.5\|\nabla f\|\min\{\Delta_k\}$ **TR-Global Convergence:** If $m_k(s_k) \leq m_k(s_{kc})$ then either $\exists k \geq 0$ s.t. $\nabla f_k = 0$ or $\lim\|\nabla f\| \rightarrow 0$. Further, require $f \in C^2$, bounded below and also ∇f L-cont. PROOF: Using def of ρ, $f_k - f_{k+1} \geq \frac{0.1}{2}\|\nabla f_k\|\min\{\dots\}$ from above. Let $\left\|\nabla^2 f\right\| := L$, and assuming $\|\nabla f\| \geq \epsilon$ we have $f_k - f_{k+1} \geq 0.05\frac{c}{L}\epsilon^2$ assuming TR has a lower bound $c\epsilon/L$. Then sum over all successful jumps s.t. $f_0 - f_{lower} \geq \sum_{i\in\mathbb{S}} f_i - f_{i+1} \geq |\mathbb{S}|\frac{0.05c\epsilon^2}{L}$ **Solving TR Prob:** Solve secular $\|s\|^{-1} - \Delta^{-1} = 0$. **KKT Feasibility:** Need $s^T J \geq 0$, $J_E^T s = 0$, and $s^T\nabla f < 0$. **KKT Conditions:** REMEMBER $c \geq 0,\ \lambda \geq 0$! **First Order KKT (Equality):** If we have x_* local min, then let $x = x_* + \alpha s$. Then we have $c_i(x(\alpha)) \rightarrow 0 = c_i(x_*) + \alpha s^T J \rightarrow s^T J = 0$. Further, we have $f(x) = f(x_*) + \alpha s^T\nabla f \rightarrow \alpha s^T\nabla f \geq 0$. Repeat for negative α s.t. $s^T\nabla f = 0$. By Rank-Nullity (assuming $J_E(x_*)$ full rank), we have $\nabla f_* = J_*^T y + s_*$ for some y_*, which then implies (after s^T from LHS) that $\|s_*\| = 0$, so $\nabla f_* = J_*^T y_*$. **KKT 2nd Order** If we have $\min f$ with $c(x) \geq 0$, 2^{nd} order conditions are that $s^T\nabla^2\mathcal{L}s \geq 0$ for all $s \in \mathcal{A}$, with $\mathcal{A}$ defined s.t: EITHER $s^T J_i = 0\ \forall\ i$ s.t. $\lambda_i > 0, c_i = 0$, OR $s^T J_i \geq 0\ \forall\ i$ s.t. $\lambda_i = 0, c_i = 0$, for J, c, λ evaluated at x_* For EQUALITY constraints instead need positive definite $\forall\ s$ s.t. $J^T s = 0$ **Convex Problems** $\hat{x} = KKT \Rightarrow \hat{x} = \arg\min f(x)$. Proof via $f \geq f(\hat{x}) + \nabla f^T(x - \hat{x})$ so $f \geq f(\hat{x}) + \hat{y}^T A(x - \hat{x}) + \sum_{i\in I}\lambda_i J_i^T(x - \hat{x})$. Choose $Ax = b$, and note that c_i concave s.t. $\lambda_i J_i^T(\hat{x})(x - \hat{x}) \geq \lambda_i(c_i(x) - c_i(\hat{x})) = \lambda_i c_i(x) \geq 0 \rightarrow f(x) \geq f(\hat{x})$. **Log-Barrier Global Convergence:** (for $f - \sum\mu\log(c_i)$) With $f \in C^1, \lambda_{ik} = \frac{\mu_k}{c_{ik}}, \|\nabla f_u(x_k)\| \leq \epsilon_k, \mu_k \rightarrow 0, x_k \rightarrow x_*$. Also, $\nabla c(x_*)LI\ \forall\ i \in \mathcal{A}$ (active constraints). Then x_* KKT and $\lambda \rightarrow \lambda_*$. PROOF: Have $J_A^\dagger(x_*) = (J_A(x_*)J_A(x_*)^T)^{-1}J_A(x_*)$. Also, $c_A = 0, c_I > 0$. So $\lambda = \mu/c \rightarrow 0$ so $\lambda_I = 0$ as $c_I > 0$. Next $\|\nabla f_k - J_{Ak}\lambda_{Ak}\| \leq \left\|\nabla f_k - J_k^T\lambda_k\right\| + \|\lambda_I\|M_1 = \|\nabla f_{\mu k}\| \rightarrow 0$. Now $\left\|J_A^\dagger\nabla f_k - \lambda_{kA}\right\| \leq \left\|J_A^\dagger\right\|\left\|\nabla f_k - J_{Ak}^T\lambda_{Ak}\right\| \rightarrow 0$. So with triangle ineq $\left\|\lambda_{kA} - J_{Ak}^\dagger\nabla f_k + J_{Ak}^\dagger\nabla f_k - \lambda_{A*}\right\| \rightarrow 0$, via cont. of ∇f and $J^\dagger$. Thus $\lambda_{Ak} \rightarrow \lambda_{A*} \geq 0$. Combine s.t. $\nabla f_k - J_{Ak}^T\lambda_{AK}$ with $k \rightarrow *$ so get KKT. **Primal-Dual Newton:** Have $\nabla f = J^T\lambda, C(x)\lambda = \mu e$ so $[\nabla^2\mathcal{L}, -J^T; \Lambda J, C][dx, d\lambda]^T = -[\nabla f - J^T\lambda, C\lambda - \mu e]^T$ **Augmented Lagrangian:** Same result as QUAD PEN METH but $x \rightarrow x_*$ if $\sigma \rightarrow 0$ for bounded u_k or u_k to y_* for bounded σ_k. Proof via $\|c_k\| \leq \sigma_k\|y_k - y_*\| + \sigma_k\|u_k - y_*\|$. If u_k bounded then $\rightarrow 0$ as $\sigma \rightarrow 0$, else trivially if $u_k \rightarrow y_*$ then to 0. **GLM Global Convergence:** With $f \in C^1, \nabla f$ L-Cont, and f bdd below, then $\nabla f_l = 0$ or $\lim\min\left\{\frac{|\nabla f^T s|}{\|s\|}, \left|s^T\nabla f\right|\right\}$ to 0. In non-trivial case, via bArmijo, $f_k - f_{k+1} \geq -\beta\alpha_k s_k^T\nabla f_k = \beta\alpha_k\left|s_k^T\nabla f_k\right|$. Sum s.t. $f_0 - f_{k+1} \geq \beta\sum\alpha_k\left|s_k^T\nabla f_k\right|$, so term in sum to 0. For all k successful we then have $\alpha_k\left|s_k^T\nabla f_k\right| \geq \frac{(1-\beta)\tau}{L}\left(\frac{\left|s_k^T\nabla f_k\right|}{\|s_k\|}\right)^2 \geq 0$ so squared term to 0. For unsuccesful steps $\alpha_k \geq \alpha_0$ so no norm term. **Convergence Newton LSearch:** Need $f \in C^2$ then if H_k (hessian) bdd above and below, so $\lambda_n \leq \lambda(H_k) \leq \lambda_1$. So $\left|s_k^T\nabla f_k\right| \geq \lambda_1^{-1}\|\nabla f_k\|^2$. Also $\|s_k\|^2 \leq \lambda_n^{-2}\|\nabla f_k\|^2$. Thus $\lim\min\left\{\lambda_n\lambda_1^{-1}\|\nabla f\|, \lambda_1^{-1}\|\nabla f\|^2\right\} \rightarrow 0$ from GLM Convergence Thm.