

NLA: Golub for $k = 1 : m, n$: $u_k = (\text{sgn}(b_{k,k})\|b_{k:m,k}\|e_1 + b_{k:m,k})$; $u_k := \hat{u}_k$; $U_k := I - 2u_k u_k^T$;
 $B_{k:m,k:n} := U_k B_{k:m,k:n}$; $U = [I_{k-1,k-1}, 0; 0, U_k]$; **for** $j = 1 : m, n - 1$: $v_k^T := \text{sgn}(b_{k,k+1})\|b_{k,k+1:n}\|e_1 +$
 $b_{k:m,k}$; $V_k := I - 2v_k v_k^T$; $B_{1:m,k+1:n} = B_{1:m,k+1:n} V_k$; $V = [I_{k,k}, 0; 0, V_k]$ **endfor endfor**; $2 \cdot (2mn^2 - 2n^3/3)$
Householder for $k = [1, n]$: $x = A_{k:m,k}$; $v_k = \text{sgn}(x)\|x\|e_k + x$; $v_k = \frac{v_k}{\|v_k\|}$ **for** $j = [k, n]$ $A_{k:m,j} =$
 $A_{k:m,j} - 2v_k [v_k^* A_{k:m,j}]$ **endfor endfor**. $2mn^2 - \frac{2n^3}{3}$. **MG-S** $V = A$; **for** $i = [1, n]$: $r_{ii} = \|v_i\|$; $q_i = \frac{v_i}{r_{ii}}$; **for**
 $j = [i + 1, n]$ $v_j = v_j - (q_i^T v_j) q_i$; $r_{ij} = q_i^T v_j$ **endfor endfor**. $2mn^2$. **Arnoldi**: $q_1 := \hat{b}$; $q_{k+1} h_{k+1,k} =$
 $A q_k - \sum_{i=1}^k q_i h_{ik}$; $h_{ik} = q_i^T (A q_k)$; $h_{k+1,k} := \|v\| \rightarrow A Q_k := Q_k H_k + q_{k+1} [0 \dots h_{k+1,k}]$. **Givens**
 $3mn^2$ **SVD**: $= \sum_{i=\min m,n}^r u_i \sigma_i v_i^T$. **QR Algo**: $A_{k+1} = Q_k^T A_k Q_k \rightarrow A_{k+1} = (Q^{(k)})^T A Q^{(k)}$ & $A^k =$
 $(Q_1 \dots Q_k)(R_k \dots R_1) := Q^{(k)} R^{(k)}$, via induction **GMRES**: $\min \|A Q_k y - b\| \rightarrow \min \|H_k y - \|b\| e_1\|$ **CG**
Convergence: $\|e_k\| = \min_{p(0)=1} \|p_k(A) e_0\| = \min_{p_k(A)} \max |p_k(\lambda)| \|e_0\| \rightarrow \leq 2((\sqrt{k_2} - 1)/(\sqrt{k_2} + 1))^k$;
need $\alpha := 2(\lambda_1 + \lambda_2)$ **Cheb**: $T_k(x) = \frac{1}{2}(z^k + z^{-k})$; $2xT_k = T_{k+1} + T_{k-1}$ **MP**: $\sigma(G) \in [\sqrt{m} - \sqrt{n}, \sqrt{m} +$
 $\sqrt{n}] \rightarrow k_2 = O(1)$ **Sketch**: with $GA\hat{x} = Gb$, and via $C - F \|G[A, b][v, -1]^T\| \leq (s + \sqrt{n+1}) \|R[v, -1]^T\|$,
similar for lower bound via MP $\rightarrow \|A\hat{x} - b\| \leq (\sqrt{s} + \sqrt{n+1})/(\sqrt{s} - \sqrt{n+1}) \|Ax - b\|$ **Blend**: solve
 $\|(A\hat{R}^{-1})y - b\| = 0$ via CG; $k_2(A\hat{R}^{-1}) = O(1)$ with $GA = \hat{Q}\hat{R}$ **PROOF**: $A = QR$; $GA = GQR = \hat{G}R$.
Let $\hat{G} = \hat{Q}\hat{R}$ so $GA = \hat{Q}\hat{R}R \rightarrow \hat{R}^{-1} = R^{-1}\hat{R}^{-1} \rightarrow k_2(A\hat{R}^{-1}) = k_2(\hat{R}^{-1}) = O(1)$ by MP. $O(mn)$ to
solve via normal **Bounds**: $\|ABB^{-1}\| \geq \|AB\| \|B^{-1}\| \rightarrow \|A\|/\|B^{-1}\| \geq \|AB\|$. **Weyls**: $\sigma_i(A + B) =$
 $\sigma_i(A) + [-\|B\|, \|B\|]$ **Rev Δ Ineq**: $\|A - B\| \geq \| \|A\| - \|B\| \|$ **Courant Application**: $\sigma_i([A_1; A_2]) \geq$
 $\max(\sigma_i(A_1), \sigma_i(A_2))$ **Schur**: Take $Av_1 = \lambda_1 v_1$; construct $U_1 = [v_1, V_\perp] \rightarrow AU_1 = U_1[e_1, X]$. Repeat.
Conditioning $\kappa_2(A) = \sigma_1/\sigma_n = \|A\| \|A^{-1}\|$ **Similarity**: $A \rightarrow P^{-1}AP$, same λ .
CO: SD: $\|x_{k+1} - x_*\| \leq (k_2(H) - 1)/(k_2(H) + 1) \|x_k - x_*\|$ with H hessian **bArm**: w/ $\phi(\alpha) = f(x_k +$
 $\alpha_k s_k)$, $\psi(\alpha) = \phi(\alpha) - \phi(0) - \beta \alpha \phi'(0) \leq 0$, show $\psi'(0) = (1 - \beta)\phi'(0) \leq 0 \rightarrow \psi(\alpha) \downarrow$ with α . **BFGS**: To show
 $H_{k+1} \geq 0$ nec. $\gamma^T \delta > 0$. Suff via γ, δ LI \rightarrow use $\|\cdot\|_H \rightarrow \gamma^T \delta > 0$. **Pen. Meth** With $y = -c/\sigma$, $\|\nabla_\sigma \Phi\| \leq$
 $\epsilon^k, \sigma^k \rightarrow 0, x \rightarrow x_*, \nabla c(x_*)$ LI, then $y \rightarrow y_*, x \rightarrow KKT$. **PROOF**: If $y_* := J_*^\dagger \nabla f_* \rightarrow \|y_k - y_*\| =$
 $\|J_k^\dagger \nabla f_k - I y_*\| \leq \|J_k^\dagger\| \|\nabla_\sigma \Phi\| \rightarrow 0$. Also, $\nabla f_* - J_*^T y_* = 0$, and $c_{k \rightarrow *} = -\sigma^{k \rightarrow *} y_{k \rightarrow *} = 0$ so $x_* \rightarrow KKT$
Pen. Meth Newt Have $w = (J\Delta x + c)/\sigma$ so $[\nabla^2 f, J^T; J, -\sigma I][\Delta x, w]^T = -[\nabla f, c]$ **Trust Region**
Radius: $\rho_k := (f(x_k) - f(x_k + s_k))/(f(x_k) - m_k(s_k))$ **TR-Method**: If $\rho \approx 1$ then double radius, update
step $x_{k+1} = x_k + s_k$. If $\rho \geq 0.1$ then same radius, update step. If ρ small shrink radius, don't update
step. **Cauchy**: Want $m_k(s_k) \leq m_k(s_{kc})$, where $s_{kc} := -\alpha_{kc} \nabla f(x_k)$, and $\alpha_{kc} := \arg \min m_k(\alpha \nabla f(x_k))$
subject to $\|\alpha \nabla f\| \leq \Delta$, i.e. $\alpha_{max} := \Delta/\|\nabla f\|$. **Calculation of Cauchy**: We want to prove cauchy
model decrease i.e. $f(x_k) - m_k(s_k) \geq f(x_k) - m_k(s_{kc}) \geq 0.5 \|\nabla f_k\| \min \left\{ \Delta_k, \frac{\|\nabla f_k\|}{\|\nabla^2 f_k\|} \right\}$. First define
 $\Psi(\alpha) := m_k(-\alpha \nabla f)$ s.t. $\Psi := f_k - \alpha \|f_k\|^2 - 0.5 \alpha^2 H_k$, with $H_k := [\nabla f_k]^T [\nabla^2 f_k] [\nabla f_k]$. N.B. that
 $\alpha_{min} := \frac{\|\nabla f_k\|^2}{H_k}$ if $H_k > 0$, from $\Psi'(0) < 0$. Now **A: If $H_k \leq 0$** then we have $\Psi(\alpha) \leq f_k - \alpha \|\nabla f_k\|^2 \rightarrow$
 $\alpha_{kc} = \alpha_{max}$. So, we have $f_k - m_{s_k} \geq f_k - m_{s_{kc}} \geq \|\nabla f_k\| \Delta_k \geq 0.5 \|\nabla f_k\| \min \{\Delta_k\}$. Now **B.i: If**
 $H_k > 0 \rightarrow \alpha_{kc} = \alpha_{min}$. Here $f_k - m_{s_{kc}} = \alpha_{kc} \|\nabla f\|^2 - 0.5 \alpha_{kc}^2 H_k = \frac{\|\nabla f\|^4}{2H_k} \geq \frac{\|\nabla f\|}{2} \min \left\{ \frac{\|\nabla f\|}{\|\nabla^2 f\|} \right\}$ via
C-S. Now **B.ii: If $H_k > 0 \rightarrow \alpha_{kc} = \alpha_{max}$** . Here $\Delta/\|\nabla f\| \leq \|\nabla f\|^2/H_k \rightarrow \alpha_{kc} H_k \leq \|\nabla f\|^2$. So
 $f_k - m_{kc} = -\alpha_{kc} \|\nabla f\|^2 + \frac{\alpha_{kc}^2}{2} H_k \geq \frac{\|\nabla f\|^2}{2} \alpha_{kc} \geq 0.5 \|\nabla f\| \min \{\Delta_k\}$ **TR-Global Convergence**: If
 $m_k(s_k) \leq m_k(s_{kc})$ then either $\exists k \geq 0$ s.t. $\nabla f_k = 0$ or $\lim \|\nabla f\| \rightarrow 0$. Further, require $f \in C^2$, bounded
below and also ∇f L-cont. **PROOF**: Using def of ρ , $f_k - f_{k+1} \geq \frac{0.1}{2} \|\nabla f_k\| \min \{\dots\}$ from above. Let
 $\|\nabla^2 f\| := L$, and assuming $\|\nabla f\| \geq \epsilon$ we have $f_k - f_{k+1} \geq 0.05 \frac{\epsilon}{L} \epsilon^2$ assuming TR has a lower bound $c\epsilon/L$.
Then sum over all successful jumps s.t. $f_0 - f_{lower} \geq \sum_{i \in \mathbb{S}} f_i - f_{i+1} \geq |\mathbb{S}| \frac{0.05 c \epsilon^2}{L}$ **G-N**: $\vec{x}_{k+1} = \vec{x}_k - \frac{\nabla f_k}{J^T J}$
KKT 2nd Order If we have $\min f$ with $c(x) \geq 0$, 2^{nd} order conditions are that $s^T \nabla^2 \mathcal{L} s \geq 0$ for all
 $s \in \mathcal{A}$, with \mathcal{A} defined s.t. **EITHER** $s^T J_i = 0 \forall i$ s.t. $\lambda_i > 0, c_i = 0$, **OR** $s^T J_i \geq 0 \forall i$ s.t. $\lambda_i = 0, c_i = 0$,
for J, c, λ evaluated at x_*