

NLA: Cholesky For matrix $[a_{11}, w^*; w, K] = R_1^T \begin{bmatrix} I, 0; 0, K - \frac{ww^*}{a_{11}} \end{bmatrix} [\alpha, w^*/\alpha; 0, I]$ we have a decomp:
for $k = [1, m-1]$: **for** $j = [k+1, m]$ $R_{j,j:m} = R_{j,j:m} - \frac{R_{kj}}{R_{kk}} R_{k,j:m}$ **endfor** $R_{k,k:m} = \frac{R_{k,k:m}}{\sqrt{R_{kk}}}$ **endfor** $\frac{m^3}{3}$. **Householder** **for** $k = [1, n]$: $x = A_{k:m,k}; v_k = \text{sgn}(x)\|x\|e_k + x; v_k = \frac{v_k}{\|v_k\|}$ **for** $j = [k, n]$
 $A_{k:m,j} = A_{k:m,j} - 2v_k[v_k^* A_{k:m,j}]$ **endfor** **endfor**. $2mn^2 - \frac{2n^3}{3}$. **LU** $U = A, L = I$ **for** $k = [1, m-1]$: **for**
 $j = [k+1, m]$ $L_{jk} = \frac{U_{jk}}{U_{kk}}; U_{j,k:m} = U_{j,k:m} - (\frac{U_{jk}}{U_{kk}})U_{k,k:m}$ **endfor** **endfor**. $\frac{2m^3}{3}$. **MG-S** $V = A; \text{for } i = [1, n]$:
 $r_{ii} = \|v_i\|; q_i = \frac{v_i}{r_{ii}}; \text{for } j = [i+1, n]$ $v_j = v_j - (q_i^T v_j)q_i; r_{ij} = q_i^T v_j$ **endfor** **endfor**. $2mn^2$. **G-S** $V = A; \text{for}$
 $i = [1, n]$ **for** $j = [1, i-1]$ $r_{ji} = q_j^T a_i; v_i = v_i - r_{ji}q_j$ **endfor** $r_{ii} = \|v_i\|; q_i = v_i/r_{ii}$ **endfor**. $2mn^2$. **Givens**
 $3mn^2$ **SVD**: $= \sum_i^{r:=\min m,n} u_i \sigma_i v_i^T$. **Bounds**: $\|ABB^{-1}\| \geq \|AB\|\|B^{-1}\| \rightarrow \|A\|/\|B^{-1}\| \geq \|AB\|$.
Weyls: $\sigma_i(A+B) = \sigma_i(A) + [-\|B\|, \|B\|]$ **Norms**: $\|A\|_F = \sqrt{\sum_i (\sigma_i)^2} = \sqrt{\text{Tr}(AA^T)}$, $\|A\|_\infty =$
max row sum. **Rev Δ Ineq**: $\|A-B\| \geq \||A\| - \|B\||$ **Low-Rank**: For $A \in \mathbb{R}^{m \times n}$ $\min \|A-B\| =$
 $\|A-A_r\|$. Proof via $B := B_1 B_2^T$ with $B_1 \in \mathbb{R}^{m \times r}; \exists W \text{ s.t. } B_2^T W = 0$ with $\text{null}(W) \geq n-r$. Then
 $\exists x_V, x_W \text{ s.t. } V_{r+1} x_V = -W x_W$. So $\|A-B\| = \|AW\| \geq \|A V_{r+1} x_V\| \geq \sigma_{r+1}$ For reverse $B := A_r$
Courant: $\sigma_i = \max_{\dim(S)=i} \{\min_x \|Ax\|/\|x\|\}$. Proof via $V_i = [v_i \dots v_n]$, so $\dim(S) + \dim(V_i) = n+1$
so $\exists w \in S \cap V_i$. Then $\|Aw\| \leq \sigma_i$. For reverse take $w = v_i$ when $S = [v_1 \dots v_i]$ **Schur**: Take
 $Av_1 = \lambda_1 v_1$; construct $U_1 = [v_1, V_\perp] \rightarrow AU_1 = U_1[e_1, X]$. Repeat. **Back Subst**: For $Ux = y$ we
have $x_{n-i} = (y_{n-i} - \sum_{n-i+1}^n u_{n-i,j} x_j) / u_{n-i,n-i}; O(i)$ per iteration so $O(n^2)$ total. **Backwards Sta-**
ble: When $\hat{f}(x) = f(x + \Delta x)$ with $\|\Delta x\|/\|x\| \leq O(\varepsilon)$ **Conditioning** $\kappa_2(A) = \sigma_1/\sigma_n = \|A\|\|A^{-1}\|$
Similarity: $A \rightarrow P^{-1}AP$, same λ . **Elementary L**: Define via $L_i(m) = I - m e_i^T$
NPDE: Def'n: With $u_{tt} - c^2 u_{xx} = f$ have $\Delta x = (b-a)/J, \Delta t = T/M, x_j = a + j\Delta x, t = m\Delta t$.
I.C: $U_j^0 = u_0(x_j), U_j^1 = U_j^0 + u_1(x_j)\Delta t, U_0^m = U_j^m = 0$ **Hyp Impl**: $(A-B, A) = \frac{1}{2}(\|A\|^2 - \|B\|^2) +$
 $\frac{1}{2}\|A-B\|^2$ with $A := U^{m+1} - U^m, B := U^m - U^{m-1}$ (T); $(-D_x^+ D_x^- U^{m+1}, U^{m+1} - U^m) = (D_x^- U^{m+1} -$
 $D_x^- U^m, D_x^- U^{m+1})$ (X). Then $\frac{1}{2\Delta t^2}(\|U^{m+1} - U^m\|^2 - \|U^m - U^{m-1}\|^2) + \frac{\Delta t^2}{2\Delta t^2}\|U^{m+1} - 2U^m + U^{m-1}\|^2 +$
 $\frac{c^2}{2}(\|D_x^- U^{m+1}\|^2 - \|D_x^- U^m\|^2) + \frac{c^2 \Delta t^2}{2\Delta t^2}\|D_x^-(U^{m+1} - U^m)\|^2 = (f, U^{m+1} - U^m)$. Then $M^2(U^m) :=$
 $\left\| \frac{U^m - U^{m-1}}{\Delta t} \right\|^2 + c^2 \|D_x^- U^{m+1}\|^2$. Write green as $\leq \|f\| \|U^{m+1} - U^m\| = \sqrt{\Delta t T} \|f\| \sqrt{\frac{\Delta t}{T}} \left\| \frac{U^{m+1} - U^m}{\Delta t} \right\| \leq$
 $\frac{\Delta t T}{2} \|f\|^2 + \frac{\Delta t}{2T} \left\| \frac{U^{m+1} - U^m}{\Delta t} \right\|^2$. Then $(1 - \frac{\Delta t}{T})M^2(U^m) \leq M^2(U^{m-1}) + \Delta t T \|f\|^2 \rightarrow M^2(U^m) \leq (1 +$
 $\frac{2\Delta t}{T})M^2(U^{m-1}) + 2\Delta t T \|f\|^2$. Use $a_m \leq \alpha^m a_0 + \sum_{k=1}^m \alpha^{m-k} b_k$ so $M^2 \leq e^2 M^2(U^0) + 2e^2 T \sum_{k=1}^m \Delta t \|f\|^2$
Hyp Expl: 1st rewrite in terms of $D_t^{+-}(\Delta t)^{-2} U_j^m + \frac{c^2 (\Delta t)^2}{4} D_x^{+-}((\Delta t)^{-2} D_t^{+-} U_j^m) -$
 $(c^2/4) D_x^{+-}(U_j^{m+1} + 2U_j^m + U_j^{m-1})$. Then use $(D(A-B), A+B) = (DA, A) - (DB, B);$
 $(D(A+B), A-B) = (DA, A) - (DB, B)$ by multiplying by $U^{m+1} - U^{m-1}$. Finally WTS $\|V_m\|^2 -$
 $\frac{c^2 (\Delta t)^2}{4} \|D_x^- V^m\|^2 \geq 0$. Done by noticing: $\|D_x^- V^m\|^2 = \sum_i^J \Delta x |D_x^- V_j^m|^2 = \frac{1}{\Delta x} \sum_i^J (V_j^m - V_{j-1}^m)^2 \leq$
 $2/\Delta x \sum_i^J (V_j^m)^2 + (V_{j-1}^m)^2 = 4/\Delta x^2 \sum_i^{J-1} \Delta x (V_j^m)^2$. Eventually show $N^2(U^m) :=$
 $\left(\left(I + \frac{c^2 \Delta t^2}{2} D_x^{+-} \right) \frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t} \right) + c^2 \left\| D_x^- \frac{U^{m+1} + U^m}{2} \right\|^2 \rightarrow N^2(U^m) = N^2(U^{m-1}) + (f, U^{m+1} -$
 $U^m)$ **Max Principle**: For $-\Delta u = f \leq 0 \rightarrow \max u \in \partial D$. First show contradiction assuming
 $LU = f < 0$, then try some auxillary function $\psi = U + \alpha (T_{\max}) g(x_i, y_i)$ s.t. $L\psi < 0$ so $\max \psi =$
 $\max_{\partial D} \psi$. Gets $\max e_{i,j}$; change to $-\alpha$ for $\min e_{i,j}$. **P-F Ineq**: $\|V\|_h^2 \leq c_* \|D_x^- V\|^2$. **For 2D**: $|V_j^m| =$
 $|\sum_{\alpha=1}^j h(D_x^- V_\alpha^m)|^2 \leq jh \sum_{\alpha=1}^{N-1} h |D_x^- V_\alpha^m|^2 \rightarrow \|V\|_h^2 = \sum_{j=1}^{N-1} h |V_j^m|^2 \leq \sum_{j=1}^{N-1} jh^2 \sum_{\alpha=1}^{N-1} h |D_x^- V_\alpha^m|^2 \leq$
 $\frac{1}{2} \sum_{j=1}^N h |D_x^- V_j^m|^2$. Use blue and add for x, y for $c_* = 0.25$. **Weak Deriv**: w is a weak deriva-
tive of u if $\int dx wv = (-1)^{|\alpha|} \int dx u(D^\alpha v)$ **Parseval**: $\int dk \hat{u}(k)v(k) = \int dk v(k) (\int dx u(x)e^{-ixk}) =$
 $\int dx u(x) (\int dk v(k)e^{-ixk}) = \int dx u(x)\hat{v}(x)$. Now $v(k) := \hat{u}(k) = \overline{F[u(k)]} = \int dk u(k)e^{-ixk} =$
 $\int dk \overline{u(k)} e^{ixk} = 2\pi F^{-1}[\overline{u(k)}] \Rightarrow \hat{v}(x) = 2\pi \overline{u(x)}$ **Iterative**: If $U^{j+1} = U^j - \tau(AU^j - F) \rightarrow U - U^j =$
 $(I - \tau A)^j (U - U^0)$ so $\|U - U^j\| \leq \|I - \tau A\|^j \|U - U^0\|$. $\|I - \tau A\| = \sigma_1 = |\lambda_1|$ as symmetric. If
 $\lambda \in [\alpha, \beta]$ then $\lambda_1 \leq \max\{|1 - \tau\alpha|, |1 - \tau\beta|\}$. Attained when $\tau = 2/(\alpha + \beta) \rightarrow \lambda_1 = \frac{\beta - \alpha}{\beta + \alpha}$. For
 $-u'' + cu = f$ we have $\lambda_k = c + \frac{4}{h^2} \sin^2(\frac{k\pi h}{2})$. Lower bound via noting $\sin(y) \geq \frac{2\sqrt{2}}{\pi} y$ at $y = \frac{\pi}{4} \rightarrow \lambda_k \geq c + 8$
Errors: $(AV, V)_h \geq \|D_x^- V\|_h^2$ & PF Ineq $\rightarrow (AV, V)_h \geq \|V\|_h^2 / c_*$. Then $(AV, V)_h (1 + c_*) \geq \|V\|_{1,h}^2 \rightarrow$
 $(AV, V)_h \geq c_0 \|V\|_{1,h}^2$. Now $c_0 \|V\|_{1,h}^2 \leq (AV, V)_h \leq \|f\|_h \|V\|_h \leq \|f\|_h \|V\|_{1,h} \rightarrow \|V\|_{1,h} \leq \|f\|_h / c_0$. (Use
 $f := AV \rightarrow \|e\|_{1,h} \leq \|T\|_h / c_0$). **Scheme**: For e.g. (on finite domain) $u_t = cu_{xx}$ with x_j, t_m , we have
scheme for $1 \leq j \leq J-1, 0 \leq m \leq M-1$, and initial conditions $\forall j$. **Non Uniform**: We have $h_{i+1} :=$
 $x_{i+1} - x_i, h_i := x_i - x_{i-1} \rightarrow \bar{h}_i = \frac{1}{2}(h_{i+1} + h_i)$ so $D_x^+ D_x^- U_j^m = \frac{1}{\bar{h}_i} ([U_{j+1} - U_j]/h_{i+1} - [U_j - U_{j-1}]/h_i)$.

¹ **L₂ F'n:** We approximate $f_{i,j} \rightarrow \frac{1}{h^2} \int_{K_{i,j}} f$ where $K_{i,j} = [x_i \pm 1/2, y_i \pm 1/2]$. **For errors:** NB that
² $Au - AU = -D_x^+ D_x^- u - D_y^+ D_y^- u + cu - T(\Delta u + cu)$. NB $Tu_{xx} = D_x^+ \frac{1}{h} \int u_x(x_i - \frac{h}{2}) dy := D_x^+ \alpha_x$ so
³ $Ae_{i,j} = D_x^+ \phi_1 + D_x^- \phi_2 + \psi$, with $\phi_1 := \alpha_x - D_x^- u, \psi := cu - Tcu$. Now NB $c_0 \|e\|_{1,h}^2 \leq (Ae, e)$. Bound
⁴ $(D_x^+ \phi_1, e)$ via $\leq \|\phi_1\|_x \|D_x^- e\|_h$ so $c_0 \|e\|_{1,h}^2 = (\|\phi_1\|_x^2 + \|\phi_2\|_y^2 + \|\psi\|_h^2) \|e\|_{1,h}$ **L-Bounds:** $\frac{|f(u)-f(v)|}{|u-v|} \leq |f'|$
