APDE: **Charpit:** $F(p,q,u,x,y)=0$ with $u_x=p, u_y=q, \dot{x}=F_p, \dot{y}=F_q$. Then via $F_x, F_y, \&\ p_y=q_x \to p_\tau=-F_x-pF_u,\ q_\tau=-F_y-qF_u, u_\tau=pF_p+qF_q$. Also, $u0_s=p_0x0_s+q_0y0_s; F_0=0$ - last 2 needed to show u defined on Γ. **Riemann:** $\int_D RLu-uL^*R=\int_D \partial_x(Ru_y+auR)+\partial_y(-uR_x+buR)=\int_{\partial D} dy\,(Ru_y+Rau)+dx\,(uR_x-buR)$. Expand over triangle going B-P-A (B at bottom right) $\to$ need $R_x=bR@y=\eta, R_y=aR@x=\xi, R(P)=1, L^*R=0$. Also ensure IVP to get R_y,R_x! **Canonical:** For $au_{xx}+2bu_{xy}+cu_{yy}=f$, we need Cauchy-Kowalevski s.t. first derivs defined: $x':=\frac{dx}{ds}$ s.t. on Γ $p_0'=x_0'u_{xx}+y_0'u_{xy}, q_0'=x_0'u_{xy}+y_0'u_{yy}$. Use these 3, solve det A!=0 s.t. $ay_0'^2-2bx_0'y_0'+cx_0'^2\neq 0$. Solve quadratic s.t. $b^2>ac\to h, b^2<ac\to e, b^2=ac\to p$. H: $\lambda_1,\lambda_2\to\xi,\eta$. E: $\lambda=\lambda_R\pm i\lambda_I;\lambda_R\to\xi,\lambda_I\to\eta$. P: $\lambda_1\to\xi$, choose η independent e.g. xy, x^2. **Green's Fn:** For $u_{xx}+u_{yy}+au_x+bu_y+cu=f$ we have $\int_D GLu-uL^*G=\int_D (u_xG)_x+(u_yG)_y-(uG_x)_x-(uG_y)_y+(auG)_x+(buG)_y=\int_D \boldsymbol{\nabla}\cdot(u_nG-uG_n)+\boldsymbol{\nabla}\cdot\left((a\,b)^T\hat{n}Gu\right)=\int_{\partial D}u_nG-uG_n+(a\,b)^T\hat{n}G$. NB $\hat n=(dy,-dx)$. Also note for quarter plane if we have $G_x(0,y)=0, G(x,0)=0$ then we have same sign at $\xi_1=(-x,y)$, opposite sign at $\xi_2=(x,-y)$, and for the third we reflect ξ_2 across y axis so we have an opposite sign to ξ at $\xi_3=(-x,-y)$. **Types:** Quasi: Coeffs don't depend on highest order derivs Semi: Coeffs depend on x,y. **Causality:** For a n-dim prob, we have n characteristics. Shock intersects $2n$. $\exists\,k$ outgoing, $2n-k$ ingoing. Also have n R-H relations, so $3n-k$ pieces of info. Unknowns are n components of $\vec{u}$ on both sides of shock & slope $\Rightarrow 2n+1$ unknowns. We demand $3n-k=2n+1$ so $k=n-1$ outgoing characterisitcs.

SAM: **Dists:** Need linearity and continuity: $\exists N,C$ s.t. $|(u,\phi)|\le C\sum_{m\le N}\max_{\in[-X,X]}|\phi^{(m)}|$. OR $\lim_{n\to\infty}(u,\phi_n)=(u,\lim_{n\to\infty}\phi_n)$ for a sequence $\phi_n\to 0$ as $n\to\infty$. **Orthog:** $\int_0^\pi \sin(kx)\sin(jx)=\frac{\pi}{2}\delta_{kj}$, same for cos.

NLA: **Cholesky** For matrix $[a_{11},w^*;w,K]=R_1^T\left[I,0;0,K-\frac{ww^*}{a_{11}}\right][\alpha,w^*/\alpha;0,I]$ we have a decomp: for $k=[1,m-1]$: for $j=[k+1,m]$ $R_{j,j:m}=R_{j,j:m}-\frac{R_{kj}}{R_{kk}}R_{k,j:m}$ endfor $R_{k,k:m}=\frac{R_{k,k:m}}{\sqrt{R_{kk}}}$ endfor. $\frac{m^3}{3}$. **Householder** for $k=[1,n]$: $x=A_{k:m,k}; v_k=sgn(x)\|x\|e_k+x; v_k=\frac{v_k}{\|v_k\|}$ for $j=[k,n]$ $A_{k:m,j}=A_{k:m,j}-2v_k\left[v_k^*A_{k:m,j}\right]$ endfor endfor. $\frac{2mn^2}{3}$. **LU** $U=A, L=I$ for $k=[1,m-1]$: for $j=[k+1,m]$ $U_{j,k:m}=U_{j,k:m}-\frac{U_{jk}}{U_{kk}}U_{k,k:m}$ endfor endfor. $\frac{2m^3}{3}$. **MG-S** $V=A$;for $i=[1,n]$: $r_{ii}=\|v_i\|; q_i=\frac{v_i}{r_{ii}}$;for $j=[i+1,n]$ $v_j=v_j-(q_i^Tv_j)q_i; r_{ij}=q_i^Tv_j$ endfor endfor. $2mn^2$. **Givens** $3mn^2$ **SVD:** $=\sum_i^{r:=\min m,n}u_i\sigma_iv_i^T$. **Bounds:** $\|ABB^{-1}\|\ge\|AB\|\|B^{-1}\|\to\|A\|/\|B^{-1}\|\ge\|AB\|$. **Norms:**$\|A\|_F=\sqrt{\sum_i(\sigma_i)^2}$, $\|A\|_\infty=$ max row. **Low-Rank:** For $A\in\mathbb{R}^{m\times n}$ $\min\|A-B\|=\|A-A_r\|$. Proof via $B:=B_1B_2^T$ with $B_1\in\mathbb{R}^{m\times r}$; $\exists W s.t. B_2^TW=0$ with null$(W)\ge n-r$. Then $\exists\, x_V,x_W s.t. V_{r+1}x_V=-Wx_W$. So $\|A-B\|=\|AW\|\ge\|AV_{r+1}x_V\|\ge\sigma_{r+1}$ For reverse $B:=A_r$ **Courant:** $\sigma_i=\max_{dim(S)=i}\{\min_x\|Ax\|/\|x\|\}$. Proof via $V_i=[v_i\dots v_n]$, so dim(S)+dim$(V_i)=n+1$ so $\exists\, w\in S\cap V_i$. Then $\|Aw\|\le\sigma_i$. For reverse take $w=v_i$ when $S=[v_1\dots v_i]$

NPDE: **Hyperbolic:** Implicit: $(A-B,A)=\frac12(\|A\|^2-\|B\|^2)+\frac12\|A-B\|^2$ (time),$(-D_x^+D_x^-U^{m+1}, U^{m+1}-U^m)=(D_x^-U^{m+1}-D_x^-U^m, D_x^-U^{m+1})$ (space). Explicit: 1st rewrite in terms of $D_t^{+-}(\Delta t)^{-2}U_j^m+\frac{c^2(\Delta t)^2}{4}D_x^{+-}((\Delta t)^{-2}U_j^m)-c^2D_x^{+-}\left(U_j^{m+1}+2U_j^m+U_j^{m-1}\right)$. Then use $(D(A-B),A+B)=(DA,A)-(DB,B);(D(A+B),A-B)=(DA,A)-(DB,B)$ by multiplying by $U^{m+1}-U^{m-1}$. Finally WTS $\|V_m\|^2-\frac{c^2(\Delta t)^2}{4}\|D_x^-V^m\|^2\ge0$. Done by noticing: $\|D_x^-V^m\|^2=\sum_i^J\Delta x|D_x^-V_j^m|^2=1/\Delta x\sum_i^J$ $\left(V_j^m-V_{j-1}^m\right)^2\le 2/\Delta x\sum_i^J(V_j^m)^2+(V_{j-1}^m)^2=4/\Delta x^2\sum_i^{J-1}\Delta x\left(V_j^m\right)^2$ **Max Principle:** For $-\Delta u=f\le 0\to\max u\in\partial D$ **P-F Ineq:** $\|V\|_h^2\le c_\star||D_x^-V]|^2$ **Weak Deriv:** w is a weak derivative of u if $\int dx\,wv=(-1)^{|\alpha|}\int dxu(D^\alpha v)$ **Parseval:** $\int dk\,\hat u(k)v(k)=\int dkv(k)\left(\int dx\,u(x)e^{-ixk}\right)=\int dxu(x)\left(\int dk\,v(k)e^{-ixk}\right)=\int dx\,u(x)\hat v(x)$. Now $v(k):=\overline{\hat u(k)}=\overline{F[u(k)]}=\overline{\int dk\,u(k)e^{-ixk}}=\int dk\,\overline{u(k)}e^{ixk}=2\pi F^{-1}\left[\overline{u(k)}\right]\Rightarrow\hat v(x)=2\pi\overline{u(x)}$ **Iterative:** If $U^{j+1}=U^j-\tau\left(AU^j-F\right)\to U-U^j=(I-\tau A)^j\left(U-U^0\right)$ so $\|U-U^j\|\le\|I-\tau A\|^j\|U-U^0\|$. $\|I-\tau A\|=\sigma_1=|\lambda_1|$ as symmetric. If $\lambda\in[\alpha,\beta]$ then $\lambda_1\le\max\{|1-\tau\alpha|,|1-\tau\beta|\}$. Attained when $\tau=2/(\alpha+\beta)\to\lambda_1=\frac{\beta-\alpha}{\beta+\alpha}$. For $-u''+cu=f$ we have $\lambda_k=c+\frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right)$. Lower bound via noting $\sin(y)\ge\frac{2\sqrt2}{\pi}y$ at $y=\frac{\pi}{4}\to\lambda_k\ge c+8$