NS: **Inverse 2×2:** For $A := [a, b; c, d]$, $A^{-1} := \frac{1}{ad-bc}[d, -b; -c, a]$ **Adj A:** $\mathrm{Adj}(A)$ is $A^{-1}*\det(A)$ **Radial:** $r\dot{r} = x\dot{x} + y\dot{y}$, $\dot{\theta} = \left[\tan^{-1}(y/x)\right]' = \frac{x\dot{y}-\dot{x}y}{x^2+y^2}$ **Classifications:** Node: $\lambda_i \in \mathbb{R}, \Pi\lambda_i > 0$ Centre: $\lambda_i = \pm ib$ Focus: $\lambda_i = a \pm ib$ Hyperbolic: $\mathrm{Re}(\lambda) \neq 0 \to$ hyperbolic. If all $\lambda < 0$ for $\mathrm{Spec}(Df(x_0))$ then A-Stable **Invariant Set:** $\phi_t(S) \subseteq S\ \forall\ t$ **Lim Pts:** ω pt. if $\lim_{t\to\infty}\phi(x) = p$, i.e. flows *tend to p*. α pt. if $\lim_{t\to-\infty}\phi(x) = p$. **Attracting Set:** A set $A \subseteq S$ if $\exists$ neighbourhood U s.t. $\phi(U) \subseteq U \forall t \geq 0$, and $A = \cap_{t>0}\phi(U)$ **Dense Orbits:** If $\forall\ \epsilon > 0, x \in A$ with A an attracting set, $\exists \tilde{x} \in \Gamma s.t. |x - \tilde{x}| < \epsilon$. I.e. a dense orbit goes as close as needed to *any* point within A **Attractor:** An attracting set with a dense orbit. **Lyapunov Stable:** If $\forall\epsilon > 0, \exists\ \delta > 0 s.t. \forall\ x \in B_\delta, t \geq 0, \phi(t) \in B_\delta$ (i.e. points stay close within region). **Asymptotically Stable:** If L-Stable and $\exists\ \delta > 0 s.t. \phi(x) \to x_0 \forall x \in B_\delta$ **Lyapunov F'n:** $V(x_0) = 0, V(x) > 0 \forall\ x \neq x_0$. Then if $\dot{V} < 0 \to$ A-Stable, or if $\dot{V} \leq 0$ L-Stable. **Stable Manifold:** If spectrum of $Df(x_0)$ has k eigvals with positive real parts, and $n-k$ with negative, then $\exists$ an $n-k$ dim manifold tangent to E^s s.t. for all $t > 0$ $\phi(W^s_{loc}) \subseteq W^s_{loc}$, and $\forall x \in W^s_{loc}\phi(x) \to x_0$ as time increases. Repeat for k-dim unstable manifold but for negative time. Then, define e.g. global stable manifold by $W^s(x_0) := \cup_{t\leq 0}\phi_t(W^s_{loc})$. Note that we search backwards in time for stable, and forwards for unstable! **Centre Manifold:** If x_0 not hyperbolic (0 real part), then E^c is the centre subspace. Then $\exists W^c$ parallel to E^c, of class C^r, and invariant under flow. Want bifurcation at $\mu = 0$, so with change of variables first find eigvecs v_1, v_2. Then, construct $P := [v_1, v_2]$ s.t. $\vec{\boldsymbol{x}} = P\vec{\boldsymbol{\xi}}$. NOTE: first v_i in P is *always* associated with $\mathrm{Re}(\lambda)= 0$. Solve for $\vec{\boldsymbol{\xi}}$ and then expand with $\eta = h(\xi, \tilde{\mu})$ **Alt. Centre Manifold:** If vector $v_1 \sim E^c = [a, b]^T$ then we have $y = bx/a$ (e.g. $[1, 1]^T \to y = x$. If bifurcation at $\mu = \alpha$ then have $\mu = \tilde{\mu} + \alpha$ s.t. bifurcation when $\tilde{\mu} = 0$. Then have $\dot{x}(x, y, \tilde{\mu}) = \ldots$ etc. Next, set up $y = h(x, \tilde{\mu}) = bx/a + b_1\tilde{\mu} + b_2\tilde{\mu}^2 + a_2x^2 + c_2\tilde{\mu}x$ and proceed as usual but at $\tilde{\mu} = 0$, s.t. y is along E^c. **Transcritical Bifurcation:** Always two points, change type at origin. E.g. $\dot{x} = \mu x - x^2$ **Saddle-Node:** E.g. $\dot{x} = \mu - x^2$ Bifurcation begins to exist at origin. **Supercritical:** E.g. $\dot{x} = \mu x - x^3$, where stable$\to 2\times$ stable and one unstable. **Subcritical:** E.g. $\dot{x} = -\mu x + x^3$, where unstable$\to 2\times$ unstable and one stable. **General co-dim 1:** If $\dot{x} = f$ then $\dot{x} = \mu f_u + 0.5x^2 f_{xx} + x\mu f_{x\mu} + 0.5\mu^2 f_{\mu\mu}$. Generally this is a saddle-node but if $f_u = 0$ we have $\dot{x} = x\mu f_{x\mu} + 0.5x^2 f_{xx}$, which is a transcritical. However if flows invariant under $x = -x$ (reflectional symmetry) then $\dot{x} = x(\mu f_{x\mu} + \ldots) + x^3(f_{xxx}/6 + \ldots) \to$ pitchfork. Saddle-node stable under perturbations! **Homoclinic Orbits** Sum of roots of cubic = - coeff. of x^2

FMM: **Point Constraint:** If $G(y, z) = 0$ then $\int_a^b \left(\left[F_y - \frac{d}{dx}(F_{y'})\right]\eta + \left[F_z - \frac{d}{dx}(F_{z'})\right]\xi\right) = 0$. Taylor expand G s.t. $G_y\eta + G_z\xi = 0$, multiply by $\lambda(x)$ s.t. $\int \lambda G_y\eta + \lambda G_z\xi = 0$. Rearrange from before s.t. $F_y - \lambda G_y = \frac{d}{dx}(F_{y'})$, and similar for z, z'. **Integral Constraint** If $J[y] = \int F dx$ with $\int G dx = C$ then $\tilde{J}[y] = \int F - \lambda G dx$ **Hamiltonian:** $H := y'F_{y'} - F \to H' = -F_x$. If $F = F(y, \dot{y})$ then $H = C$ **Hamilton's Eqs:** $p := F_{y'}, q = y$ and so $p' = -H_q, q' = H_p$ **Derive Hamilton Eq:** Have $H = py' - F$, and note $p' = F_y$. EQ1: So $H_{y'} = p + y'p_{y'} - F_{y'} = y'p_{y'}$. Also, $H_{y'} = \cancel{H_q q_{y'}} + H_p p_{y'} = H_p p_{y'}$ as $q = y$. Therefore $y' = q' = H_p$. EQ2: $p' = F_y = (py' - H)_y = y'p_y - H_y$. But $H_y = H_p p_y + H_q q_y = y'p_y + H_q$. So finally $p' = -H_q$. **Free Boundary:** $J[y, b] = \int_a^b F(x, y, y')dx$ where b free. Expand with $y + \epsilon\eta, b + \epsilon\beta \to J = J_0 + \epsilon\left\{\int_a^b \eta F_y + \eta' F_{y'} dx + \beta F(b, y(b), y'(b))\right\}$ If $y(b) = d \to d = y(b + \epsilon\beta) + \epsilon\eta(b + \epsilon\beta) = y(b) + \epsilon(\beta y'(b) + \eta(b))$ so $\eta(b) = -\beta y'(b)$. IVP on integral so $\beta\left[F - y'F_{y'}\right]_{x=b} + \int(\ldots) = 0$ so $F = y'F_{y'}$ at free boundary. **Multiple Ind. Variables:** For $J = \int F(x, \phi, \phi_x, \phi_y)$. $J[\phi + \epsilon\eta] = J_0 + \epsilon\int\int_D \left(\eta F_\phi + \eta_x F_{\phi_x} + \eta_y F_{\phi_y}\right)$. Via Green's $\boldsymbol{\nabla}\cdot(\eta\vec{\boldsymbol{f}}) = \boldsymbol{\nabla}\eta\cdot\vec{\boldsymbol{f}} + \eta\boldsymbol{\nabla}\cdot\vec{\boldsymbol{f}}$, so with $\vec{\boldsymbol{f}} = (F_{\phi_x}, F_{\phi_y})$ we have $\int\int(\eta_x F_{\phi_x} + \eta_y F_{\phi_y}) = -\int\int_D \left(\eta\partial_x(F_{\phi_x}) + \eta\partial_y(F_{\phi_y})\right) + \cancel{\int_{\partial D}\left(\eta\left[F_{\phi_x}\eta_x + F_{\phi_y}\eta_y\right]\right)}$. I.e. we have $F_\phi = \partial_x(F_{\phi_x}) + \partial_y(F_{\phi_y})$ **Control:** Have $\int \xi h_x + \eta h_u dt = 0, \dot{\xi} = \xi f_x + \eta f_u$. Sub for η, IVP s.t. $\frac{d}{dt}\frac{h_u}{f_u} = h_x - f_x\frac{h_u}{f_u}$ and $\dot{x} = f$ **Hamiltonian (Control):** $H := f\frac{h_u}{f_u} - h$ s.t. $\dot{H} = \frac{h_u}{f_u}f_t - h_t \to$ autonomous if $h_t = f_t = 0$. **Fredholm Alt Integ Eqs.** For $y = f + \int K(x, t)y(t)dt$ we have ONE (N) has a unique sol $y = 0$ if $f = 0$, and adjoint has unique sol, or TWO (H) as sols $y_1 \ldots y_r$iff $\forall$ solutions to H^*, z_i, we have $< f, z_i >= 0$. GENERAL CASE: Have $y = f + \lambda AG_1 + \lambda BG_2$. Solve for system $[\alpha_1, \alpha_2; , \alpha_3, \alpha_4][A, B]^T = [\gamma_1, \gamma_2]^T$ with NONUNIQUE sols for $\lambda = \lambda_*$. Now for $\lambda = \lambda_*$, want to solve $L^*w = 0$ and show this is orthogonal to RHS. First solve $[\alpha_1, \alpha_2; , \alpha_3, \alpha_4][A, B]^T = [0, 0]^T$. Then we have $w = \lambda_* A(F(G_1, G_2))$. Check if $\int fw = 0$. If so, return to NONHOM case and solve $[\alpha_1, \alpha_2; , \alpha_3, \alpha_4]_{\lambda_*}[A, B]^T = [\gamma_1, \gamma_2]^T$ to get $B = -\frac{\alpha_1}{\alpha_2}A + \frac{\gamma_1}{\alpha_2}$. Sub this into $y = f + \lambda_* AG_1 + \lambda_* B(A)G_2$. EX: Solve $y = 1 - x^2 + \lambda\int(1 - 5x^2t^2)y(t)\ dt = 1 - x^2 + \lambda A - 5\lambda Bx^2$. Have $A := \int y_N(t) = \int 1 - x^2 + \lambda A + \ldots = \lambda A - \frac{5\lambda}{3} + \frac{2}{3}$. Repeat for B s.t. $[1 - \lambda, 5\lambda/3; -\lambda/3, 1 + \lambda][A, B]^T = [2/3, 2/15]^T$. Unique sols if $\lambda \neq \pm\frac{3}{2} \to$ try when $\lambda_* = \frac{-3}{2}$. Have $L^*w_H = \lambda A - 5\lambda_* Bx^2 \to A := \int \lambda_* A - 5\lambda_* Bx^2$, and $B := \int \ldots$. Both give consistent results $A = B$ so $w_H = \lambda_* A(1 - 5x^2)$. Check that $\int w_H(x)(1 - x^2) = 0$, so we have shown nullspace of adj. orthog. to RHS. Note that we may also find A, B for adjoint quicker via $[1 - \lambda, 5\lambda/3; -\lambda/3, 1 + \lambda]_{\lambda_*}[A, B]^T = [0, 0]^T$. Lastly, return to (N), and having verified λ_* permits a solution, solve $[1-\lambda, 5\lambda/3; -\lambda/3, 1+\lambda]_{\lambda_*}[A,B]^T = [2/3, 2/15]^T \to A-B = 4/15$ when $\lambda = -3/2$. Sub this into $y = 1-x^2\ldots$ for solution. **Fred Diff Eq.** For nonunique sol to exist, need $< Ly, w > = < f, w > \forall\, w s.t. L^* w = 0$ **Trig:** $\int_0^{2\pi} \cos^2 = \int_0^{2\pi} \sin^2 = \pi$ FPDE: **Types:** 1^{st} : $\exists$ scale s.t. solution found, not so for 2^{nd}. **Heat:** $\hat{T} = u(\hat{T}_\infty - \hat{T}_{-\infty}) + \hat{T}_{-\infty}$ **Oil Spread:** Dims: $x = x_f + \varepsilon\xi, t = \tau$ **Ground Spread:** $(1-s)\phi h_t + Q_x = 0; Q \sim -hh_x, 0 < x_s < x_f$. Have $h(x_f) = 0, h_t(x_s) = 0$, and $hh_x|_{x=0,x_f} = 0$ (i.e. no flux at centre and front), and h, hh_x cont. at joint. **Expansions:** Let $\xi = z + \epsilon\eta$ for perturbations **Scale:** Try $x = x_f + \epsilon\xi$ for groundwater **Stefan:** $S_0 = C\left(T_1 - T_m\right)/L$, condition $= \rho L\dot{s} = kT_x|_{s-}^{s+}$ **1ph Stefan:** Bar $= T_h|liq|_s sol|INS$. Use $T = T_m + (T_1 - T_m)u$ s.t. $S_0 u_t = u_{xx}, u = 1$ @ $x = 0, \{\dot{s} = -u_x, u = 0\}$ @ $x = s, s(0) = 0$. Sim. sol is $s = \beta\sqrt{t}, f = f(x/\sqrt{t})$ **2ph Stefan:** (melting) Use $T = T_m + (T_1 - T_m)u$ s.t. $S_0 u_t = u_{xx}$ @ $0 < x < s, (S_0/\kappa)u_t = u_{xx}$ @ $s < x < 1, u = 1$ @ $x = 0, u_x = 0$ @ $x = 1, \left\{\dot{s} = Ku_x|_{s_+} - u_x|_{s_-}, u = 0\right\}$ @ $x = s, \{s = 0, u = -\theta\}$ @ $x = 0$. Here $\theta := (T_m - T_0)/(T_1 - T_m), \kappa := c_1 k_1/(c_2 k_2), K := k_2/k_1$ Sim. sol is $s = \beta\sqrt{t}, f = f(x/\sqrt{t})$ **2-Dim:** Normal velocity $U_n := \hat{n}\cdot u = K(u_2)_n - (u_1)_n$ in Stefan Prob. If $x = f(y,t)$ then $\hat{n} := \boldsymbol{\nabla}(x - f) = [1, -f_y]^T/\sqrt{1 + f_y^2}$. EX: Consider $u_1 := -\lambda_1(x - V_0 t) + \epsilon\tilde{u}_1(x,y,t), u_2 := -\lambda_2(x - V_0 t) + \epsilon\tilde{u}_2(x,y,t)$. If position of boundary $x_b := V_0 t + \epsilon\xi(y,t) = f(y,t)$. So, normal velocity $U_n := \hat{n}\cdot u$, where we have $u = [\dot{x}_b, \dot{y}_b]^T$ so $U_n \sim [1, -f_y][\dot{x}_b, 0]^T = \dot{x}_b = V_0 + \epsilon\xi_t$. On RHS, we have $(u_i)_n = \hat{n}\dot{\boldsymbol{\nabla}} u_i \sim [1, -f_y]^T\boldsymbol{\nabla} u = [1, -\epsilon\xi_y][u_x, u_y]^2 = u_x - \epsilon\xi_y u_y$. For e.g. u_1 we have $(u_1)_n = -\lambda_1 + \epsilon(\tilde{u}_1)_x - \epsilon^2\xi_y(\tilde{u}_1)_y$. So, to $O(\epsilon^0)$ we have $V_0 = -\lambda_2 K + \lambda_1$, and to $O(\epsilon^1)$ $\xi_t = K(\tilde{u}_2)_x - (\tilde{u}_1)_x$. **Welding:** Have $0 < s_2 < s_1$. Have cold $x = a$, no flux $x = 0$. $\theta = 1$ in liquid. In mush $\rho L\theta_t = J^2/\sigma$, CoE $\to \theta\rho L\dot{s} + kT_x|_{s-}^{s+} = 0$. Have θ cont. $(= 0)$ at s_1. I.e. we have $S_0 u_t = u_{xx} + q, u_x = 0$ @ $x = 0, u = -1$ @ $x = 1, \theta = 0$ @ $x = s_1$. Also $\theta_t = q$ in mush. **ERF:** $\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^y e^{-y^2}$ s.t. if $f' = e^{-\frac{\eta^4}{4k}}, f := A\sqrt{\pi k}\operatorname{erf}\frac{\eta}{2k} + B$, $(\operatorname{erf} x)' = \frac{2}{\sqrt{\pi}}e^{-y^2}$.