NLA: **Golub** for $k = 1 : m,n$: $u_k = (sgn(b_{k,k})\|b_{k:m,k}\|e_1 + b_{k:m,k})$; $u_k := \hat{u}_k$; $U_k := I - 2u_k u_k^T$; $B_{k:m,k:n} := U_k B_{k:m,k:n}$; $U = [I_{k-1,k-1}, 0; 0, U_k]$;for $j = 1 : m, n-1$: $v_k^T := sgn(b_{k,k+1})\|b_{k,k+1:n}\|e_1 + b_{k:m,k}$;$V_k := I - 2v_k v_k^T$; $B_{1:m,k+1:n} = B_{1:m,k+1:n}V_k$; $V = [I_{k,k}, 0; 0, V_k]$endfor endfor; $2 \cdot (2mn^2 - 2n^3/3)$ **Householder** for $k = [1,n]$: $x = A_{k:m,k}$; $v_k = sgn(x)\|x\|e_k + x$; $v_k = \frac{v_k}{\|v_k\|}$ for $j = [k,n]$ $A_{k:m,j} = A_{k:m,j} - 2v_k\,[v_k^* A_{k:m,j}]$ endfor endfor. $2mn^2 - \frac{2n^3}{3}$. **MG-S** $V = A$;for $i = [1,n]$: $r_{ii} = \|v_i\|$; $q_i = \frac{v_i}{r_{ii}}$;for $j = [i+1,n]$ $v_j = v_j - (q_i^T v_j)q_i$; $r_{ij} = q_i^T v_j$ endfor endfor. $2mn^2$. **Arnoldi:** $q_1 := \hat{b}$; $q_{k+1}h_{k+1,k} = Aq_k - \sum_{i=1}^k q_i h_{ik}$; $h_{ik} = q_i^T(Aq_k)$; $h_{k+1,k} := \|v\| \to AQ_k := Q_k H_k + q_{k+1}[0 \dots h_{k+1,k}]$. **Givens** $3mn^2$ **SVD:** $= \sum_i^{r:=\min m,n} u_i \sigma_i v_i^T$. **QR Algo:** $A_{k+1} = Q_k^T A_k Q_k \to A_{k+1} = \left(Q^{(k)}\right)^T AQ^{(k)} \& A^k = (Q_1 \dots Q_k)(R_k \dots R_1) := Q^{(k)}R^{(k)}$, via induction **GMRES:** $\min \|AQ_k y - b\| \to \min \|H_k y - \|b\|e_1\|$ **CG Convergence:** $\|e_k\| = \min_{p(0)=1} \|p_k(A)e_0\| = \min_{p_k(A)} \max |p_k(\lambda)| \|e_0\| \to \leq 2\left((\sqrt{k_2} - 1)/(\sqrt{k_2} + 1)\right)^k$; need $\alpha := 2(\lambda_1 + \lambda_2)$ **Cheb:** $T_k(x) = \frac{1}{2}(z^k + z^{-k})$; $2xT_k = T_{k+1} + T_{k-1}$ **MP:** $\sigma(G) \in [\sqrt{m} - \sqrt{n}, \sqrt{m} + \sqrt{n}] \to k_2 = O(1)$ **Sketch:** with $GA\hat{x} = Gb$, and via $C-F$ $\left\|G[A,b][v,-1]^T\right\| \leq (s + \sqrt{n+1})\left\|R[v,-1]^T\right\|$, similar for lower bound via MP $\to \|A\hat{x} - b\| \leq (\sqrt{s} + \sqrt{n+1})/(\sqrt{s} - \sqrt{n+1})\|Ax - b\|$ **Blend:** solve $\left\|(A\hat{R}^{-1})y - b\right\| = 0$ via CG;$k_2(A\hat{R}^{-1}) = O(1)$ with $GA = \hat{Q}\hat{R}$ PROOF: $A = QR; GA = GQR = \hat{G}R$. Let $\hat{G} = \hat{Q}\hat{R}$ so $GA = \hat{Q}\hat{R}R \to \tilde{R}^{-1} = R^{-1}\hat{R}^{-1} \to k_2(A\tilde{R}^{-1}) = k_2(\hat{R}^{-1}) = O(1)$ by MP. $O(mn)$ to solve via normal **Bounds:** $\left\|ABB^{-1}\right\| \geq \|AB\|\left\|B^{-1}\right\| \to \|A\|/\left\|B^{-1}\right\| \geq \|AB\|$. **Weyls:** $\sigma_i(A + B) = \sigma_i(A) + [-\|B\|, \|B\|]$ **Rev Δ Ineq:** $\|A - B\| \geq |\|A\| - \|B\||$ **Courant Application:** $\sigma_i([A_1; A_2]) \geq \max(\sigma_i(A_1), \sigma_i(A_2))$ **Schur:** Take $Av_1 = \lambda_1 v_1$; construct $U_1 = [v_1, V_\perp] \to AU_1 = U_1[e_1, X]$. Repeat. **Conditioning** $\kappa_2(A) = \sigma_1/\sigma_n = \|A\|\left\|A^{-1}\right\|$ **Similarity:** $A \to P^{-1}AP$, same λ.

CO: **SD:** $\|x_{k+1} - x_*\| \leq (k_2(H) - 1)/(k_2(H) + 1)\|x_k - x_*\|$ with H hessian **bArm:** w/ $\phi(\alpha) = f(x_k + \alpha_k s_k)$, $\psi(\alpha) = \phi(\alpha) - \phi(0) - \beta\alpha\phi'(0) \leq 0$, show $\psi'(0) = (1-\beta)\phi'(0) \leq 0 \to \psi(\alpha) \downarrow$ with α. **BFGS:** To show $H_{k+1} \geq 0$ nec. $\gamma^T\delta > 0$. Suff via γ, δ LI $\to$ use $\|\cdot\|_H \to \gamma^T\delta > 0$. **Pen. Meth** With $y = -c/\sigma$, $\|\nabla_\sigma \Phi\| \leq \epsilon^k$, $\sigma^k \to 0$, $x \to x_*$, $\nabla c(x_*)$ LI, then $y \to y_*$, $x \to KKT$. PROOF: If $y_* := J_*^\dagger \nabla f_* \to \|y_k - y_*\| = \left\|J_k^\dagger \nabla f_k - Iy_*\right\| \leq \left\|J_k^\dagger\right\|\|\nabla_\sigma \Phi\| \to 0$. Also, $\nabla f_* - J_*^T y_* = 0$, and $c_{k\to*} = -\sigma^{k\to*}y_{k\to*} = 0$ so $x_* \to KKT$ **Pen. Meth Newt** Have $w = (J\Delta x + c)/\sigma$ so $[\nabla^2 f, J^T; J, -\sigma I][\Delta x, w]^T = -[\nabla f, c]$ **Trust Region Radius:** $\rho_k := (f(x_k) - f(x_k + s_k))/(f(x_k) - m_k(s_k))$ **TR-Method:** If $\rho \approx 1$ then double radius, update step $x_{k+1} = x_k + s_k$. If $\rho \geq 0.1$ then same radius, update step. If ρ small shrink radius, don't update step. **Cauchy:** Want $m_k(s_k) \leq m_k(s_{kc})$, where $s_{kc} := -\alpha_{kc}\nabla f(x_k)$, and $\alpha_{kc} := \arg\min m_k(\alpha \nabla f(x_k))$ subject to $\|\alpha \nabla f\| \leq \Delta$, i.e. $\alpha_{max} := \Delta/\|\nabla f\|$. **Calculation of Cauchy:** We want to prove cauchy model decrease i.e. $f(x_k) - m_k(s_k) \geq f(x_k) - m_k(s_{kc}) \geq 0.5\|\nabla f_k\| \min\left\{\Delta_k, \frac{\|\nabla f_k\|}{\|\nabla^2 f_k\|}\right\}$. First define $\Psi(\alpha) := m_k(-\alpha \nabla f)$ s.t. $\Psi := f_k - \alpha\|f_k\|^2 - 0.5\alpha^2 H_k$, with $H_k := [\nabla f_k]^T [\nabla^2 f_k] [\nabla f_k]$. N.B. that $\alpha_{min} := \frac{\|\nabla f_k\|^2}{H_k}$ if $H_k > 0$, from $\Psi'(0) < 0$. Now A: If $H_k \leq 0$ then we have $\Psi(\alpha) \leq f_k - \alpha\|\nabla f_k\|^2 \to \alpha_{kc} = \alpha_{max}$. So, we have $f_k - m_{s_k} \geq f_k - m_{s_{kc}} \geq \|\nabla f_k\|\Delta_k \geq 0.5\|\nabla f_k\| \min\{\Delta_k\}$. Now B.i: If $H_k > 0 \to \alpha_{kc} = \alpha_{min}$. Here $f_k - m_{s_{kc}} = \alpha_{kc}\|\nabla f\|^2 - 0.5\alpha_{kc}^2 H_k = \frac{\|\nabla f\|^4}{2H_k} \geq \frac{\|\nabla f\|}{2} \min\left\{\frac{\|\nabla f\|}{\|\nabla^2 f\|}\right\}$ via C-S. Now B.ii: If $H_k > 0 \to \alpha_{kc} = \alpha_{max}$. Here $\Delta/\|\nabla f\| \leq \|\nabla f\|^2/H_k \to \alpha_{kc}H_k \leq \|\nabla f\|^2$. So $f_k - m_{kc} = -\alpha_{kc}\|\nabla f\|^2 + \frac{\alpha_{kc}^2}{2}H_k \geq \frac{\|\nabla f\|^2}{2}\alpha_{kc} \geq 0.5\|\nabla f\| \min\{\Delta_k\}$ **TR-Global Convergence:** If $m_k(s_k) \leq m_k(s_{kc})$ then either $\exists k \geq 0$ s.t. $\nabla f_k = 0$ or $\lim \|\nabla f\| \to 0$ sd