Time Series Filtering

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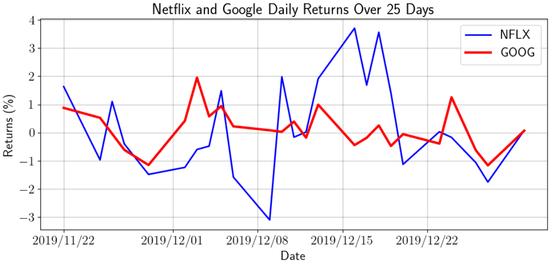
Introduction



- Problem: We wish to extract some measure of "similarity" between stock prices, but they are noisy. Can we seperate noise and signal?
  - Useful in e.g. pairs trading.
- Solution: Singular Spectrum Analysis (SSA).

Consider[1] a time series $Z_T = (z_1, \dots z_T)$. With fixed *window length* L and *rank* r, and with $K := T - L + 1$:

1. Construct the (Hankel) trajectory matrix:

$$\mathbf{X} := \begin{bmatrix} z_1 & z_2 & z_3 & \cdots & z_K \\ z_2 & z_3 & z_4 & \cdots & z_{K+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_L & z_{L+1} & z_{L+2} & \cdots & z_T \end{bmatrix} \tag{1}$$

[1] Hassani, Mahmoudvand, et al. 2011.

Outline of SSA: Taking a Low-Rank Approximation

2. Compute SVD (via HMT algorithm) of $\mathbf{X}$:

$$\mathbf{X} = U\Sigma V^T = \sum_{i=1}^{n} u_i v_i^T \sigma_i$$

3. Truncate the SVD to r rank-1 matrices, with r chosen s.t. $r \leq n$:

$$\mathbf{X} \approx \mathcal{X} = \sum_{i=1}^{r} u_i v_i^T \sigma_i$$

4. $\mathcal{X}$ is not necessarily Hankel, so re-diagonalise on the off-diagonals to reconstruct a de-noised series $\bar{Z}_T = (\bar{z}_1, \ldots \bar{z}_T)$ from the averaged Hankel matrix

$$\bar{\mathbf{X}} := \begin{bmatrix} \bar{z}_1 & \bar{z}_2 & \bar{z}_3 & \ldots & \bar{z}_K \\ \bar{z}_2 & \bar{z}_3 & \bar{z}_4 & \ldots & \bar{z}_{K+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{z}_L & \bar{z}_{L+1} & \bar{z}_{L+2} & \ldots & \bar{z}_T \end{bmatrix} \tag{2}$$

1. To set L, we first define the *weighted inner product*

$$(Z_T, Y_T)_w := \sum_{i=1}^{T} \min\{i, L, T-i+1\} z_i y_i \tag{3}$$

with associated norm

$$\|Z_T\|_w^2 := (Z_T, Z_T)_w \,.$$

2. We then construct the $w-$correlation

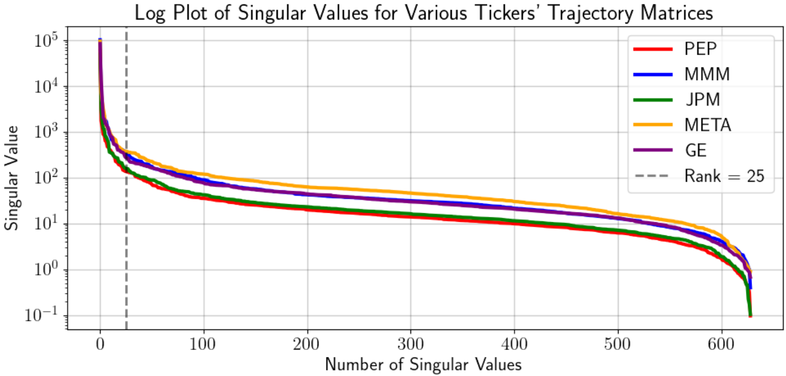
$$\rho_w(Z_T, Y_T) := \frac{(Z_T, Y_T)_w}{\|Z_T\|_w \|Y_T\|_w}$$

which we use as a measure of seperability.

It can be shown[2] that a choice of $L = \frac{T+1}{2}$ minimises ρ_w.

[2] Hassani, Kalantari, et al. 2017.

To choose r, examine the scree plot showing a knee at approximately 25 singular values.



We measure similarity of two time (de-noised) time series using the Time Warped Edit Distance[3] (TWED). First define

$$f(x_i, y_j) = |x_i - y_j|$$

for two time series $X_T = (x_1, \dots x_T)$ and $Y_T = (y_1, \dots y_T)$ as a "difference measurement".

[3] Marteau 2008.

TWED: Initialising the Grid

Next, we initialise a grid $D_{i,j}$ s.t.

$$\begin{aligned} D_{0,0} &= 0, \\ D_{0,j} &= \infty \quad j = 1, \ldots T, \\ D_{i,0} &= \infty \quad i = 1, \ldots T. \end{aligned}$$

	1	∞			
Y_T	1	∞			
	9	∞			
		0	∞	∞	∞
			5	-3	1
				X_T	

Figure 1: Initialised TWED grid.

TWED: Populating the Grid

Then define the *TWED-Closeness* by $D_{T,T}$, where we construct

$$D_{i,j} = \min\left\{D_{i-1,j-1} + \Gamma_{X,Y}\ , D_{i-1,j} + \Gamma_X , D_{i,j-1} + \Gamma_Y\right\} \quad (4)$$

for $1 \leq i, j \leq T$, where

$$\Gamma_{X,Y} = f\left(x_i, y_j\right) + f\left(x_{i-1}, y_{j-1}\right) + 2\nu|i-j|, \quad (5)$$

$$\Gamma_X = f\left(x_i, x_{i-1}\right) + \nu + \lambda, \quad (6)$$

$$\Gamma_Y = f\left(y_j, y_{j-1}\right) + \nu + \lambda, \quad (7)$$

with

- λ: deletion penalty
- ν: stiffness parameter

TWED: Graphical Example

	1	∞	18	17	16
Y_T	1	∞	13	12	13
	9	∞	4	13	14
		0	∞	∞	∞
			5	-3	1
				X_T	

Figure 2: Populated TWED grid, with $\nu = \lambda = 0.5$. $D_{T,T} = 16$.

Current Results: SSA Reconstruction

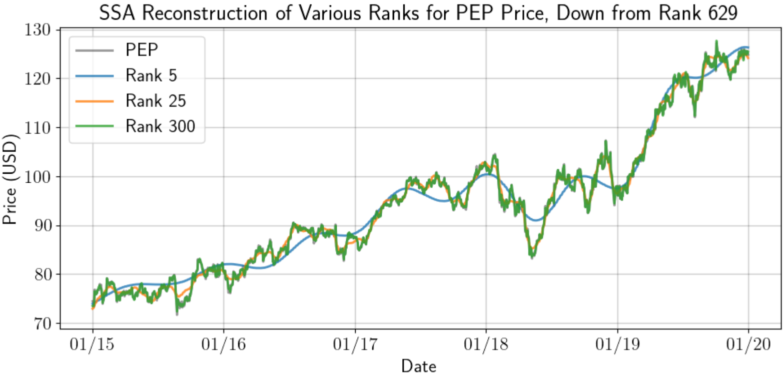


Figure 3: Different rank SSA reconstructions. Note underfitting at $r = 5$, and overfitting at $r = 300$.

Current Results: TWED-Closeness Analysis

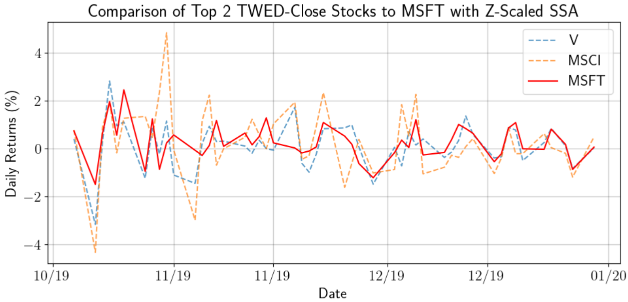


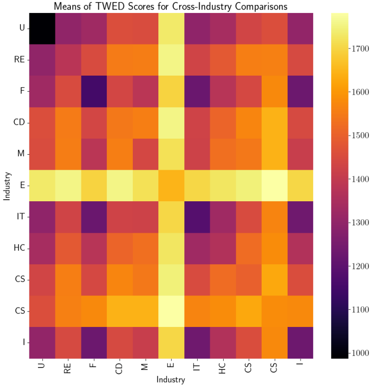
Figure 4: Returns over 50 days for the top 2 most "similar" stocks to MSFT. Note how when returns diverge, they eventually reconverge.

With n time series of length m, TWED-scoring complexity is $O(m^2n^2) \sim$ 12 hours for 500 stocks over 5 years.

Current Results: Inter-Industry Similarity (Mean)

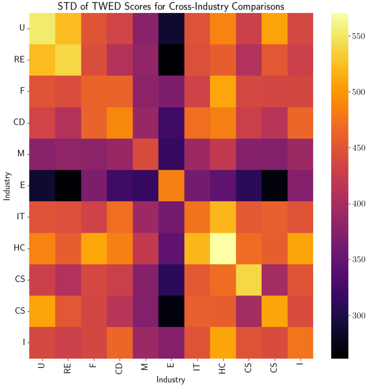
Key takeaways:

- Energy, Consumer Staples sector dissimilar to other sectors.
- Utilities, Finance, IT show strong inter-similarity.

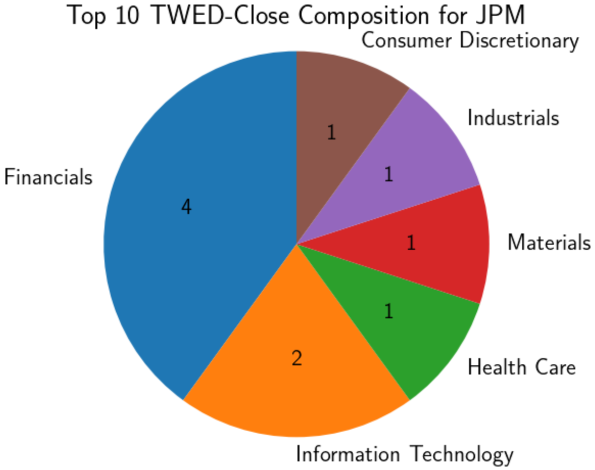


Key takeaways:

- Energy sector conclusions strong.
- Utilities, Health Care conclusion very weak.



Current Results: Similarity Composition for JPM



Avenues for Future Work

- Neural-network based approaches
  - There is promising work being done on *Siamese Neural Networks*[4], which use a pair of Recurrent Neural Networks that share weights to classify time series.
- Fine-tuning the choice of SSA parameters to better classify data
- Forecasting?

[4] Hou et al. 2019.

(1) Dey, S.; Dutta, A.; Toledo, J. I.; Ghosh, S. K.; Llados, J.; Pal, U. SigNet: Convolutional Siamese Network for Writer Independent Offline Signature Verification, Number: arXiv:1707.02131, 2017.

(2) Ghodsi, M.; Hassani, H.; Rahmani, D.; Silva, E. S. *Journal of Applied Statistics* **2018**, *45*, Publisher: Taylor & Francis _eprint: https://doi.org/10.1080/02664763.2017.1401050, 1872–1899.

(3) Hassani, H.; Kalantari, M.; Yarmohammadi, M. *Comptes Rendus Mathematique* **2017**, *355*, 1026–1036.

(4) Hassani, H.; Mahmoudvand, R.; Zokaei, M. *Comptes Rendus Mathematique* **2011**, *349*, 987–990.

(5) Hou, L.; Jin, X.; Zhao, Z. In *2019 12th International Congress on Image and Signal Processing, BioMedical Engineering and Informatics (CISP-BMEI)*, 2019 12th International Congress on Image and Signal Processing, BioMedical Engineering and Informatics (CISP-BMEI), 2019, pp 1–6.

(6) Marteau, P.-F. Time Warp Edit Distance, Number: arXiv:0802.3522, 2008.

(7) Serrà, J.; Arcos, J. L. *Knowledge-Based Systems* **2014**, *67*, 305–314.

Siamese Neural Networks

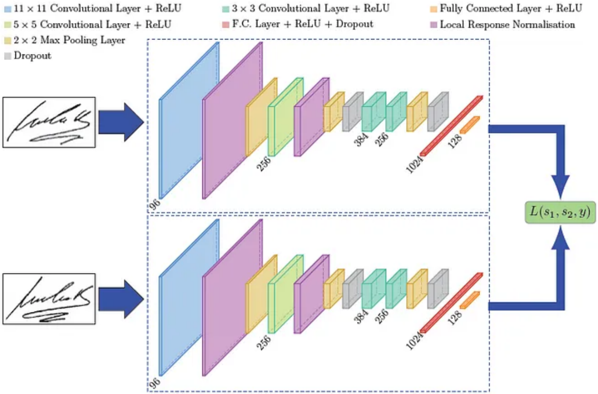


Figure 5: Overview of an SNN, as used in SigNet[5].

[5] Dey et al. 2017.

There exist two different types of SSA forecasting: recurrent, and vector. We go over them in turn:

1. Recurrent forecasting[6]: Consider the left singular vectors $u_1, u_2, \ldots u_r$. Take their L^{th} components, denoted π_i, and define

$$v^2 := \sum_{i=1}^{r} \pi_i^2. \tag{8}$$

   Denote by $\hat{u}_i$ the $L-1 \times 1$ vector which is u_i with the final component removed.

[6] Ghodsi et al. 2018.

Then define

$$A = (\alpha_{L-1}, \dots \alpha_1)^T = \frac{1}{1 - v^2} \sum_{i=1}^{r} \pi_i \hat{u}_i,$$

and thus construct

$$z_t = \begin{cases} \bar{z}_t & t = 1, \dots T, \\ \displaystyle\sum_{i=1}^{L-1} \alpha_i z_{t-i} & t = T+1, \dots T+h, \end{cases}$$

for a forecast to h steps ahead.

2. Vector forecasting[7]: First define

$$\hat{\mathbf{U}} = [\hat{u}_1, \dots \hat{u}_r]\,,$$

and construct

$$\mathbf{\Pi} = \hat{\mathbf{U}}\hat{\mathbf{U}}^T + \left(1 - v^2\right) AA^T.$$

Finally, construct the operator Θ s.t.

$$\Theta V := \begin{bmatrix} \mathbf{\Pi}\hat{V} \\ A^T\hat{V} \end{bmatrix},$$

where $\hat{V}$ denotes the vector V with the last element removed.

[7] Ghodsi et al. 2018.

Define now

$$\Xi_i = \begin{cases} \mathcal{X}_i & i = 1, \dots K, \\ \Theta \Xi_{i-1} & i = K+1, \dots K+h+L-1, \end{cases}$$

where $\mathcal{X}_i$ are the columns of $\mathcal{X}$. Next construct

$$\boldsymbol{\Xi} = \left[\Xi_1, \dots \Xi_{K+h+L-1} \right],$$

and hankelise to get the matrix $\bar{\boldsymbol{\Xi}}$ from which we recover an "extended" time series containing forecasted values.