Does the octonion quasigroup generate a plane of order 17?

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1 Regular planes

By an affine plane we mean a set P (of '*points*') with a collection L of its certain subsets that are regarded as '*lines*', such that the following axioms hold:

- Any two distinct points lie on a unique line.
- Given any line and any point not on that line there is a unique line which contains the point and does not meet the given line.
- (nondegeneracy) There exist four points such that no three of them are on a single line.

An affine plane is finite if the set of its points is finite.
It is well known that finite affine planes and finite projective planes determine each other, so questions concerning finite projective planes naturally translates to finite affine planes.

We can construct an affine plane by starting out from an arbitrary field K and taking $P := K \times K$ and $L = \{\ell_{m,c} : m, c \in K\} \cup \{v_t : t \in K\}$,
where $\ell_{m,c} := \{(x,\, mx + c) \,:\, x \in K\}$ and $v_t := \{(t, x) \,:\, x \in K\}$.
The planes generated this way are called *Desarguian* planes, because geometrically these are exactly the ones that satisfy *Desargue's theorem* besides the axioms above.

If K is a finite field (i.e. $\mathbb{Z}/p\mathbb{Z}$ for a prime p, or any of its finite field extension), we obtain the finite Desarguian planes.

2 A pattern for real / complex / quaternion units

The algebraic condition on K, that it forms a field (has addition, subtraction, multiplication, division, with commutativity, associativity and distributivity constraints), can be somewhat weakened while the above construction still produces an affine plane.

The smallest such example is due to Hall, and it is a *near-field* of 9 elements whose multiplicative reduct is just the 8 element quaternion group $\{\pm 1,\, \pm i,\, \pm j,\, \pm k\}$ plus the zero element 0.
The addition can be given by setting
$1+1=-1, \quad i+1=j, \quad j+1=k.$

Note that the multiplicative reduct of $\mathbb{Z}/3\mathbb{Z}$ is $\{\pm 1,\, 0\}$ (that is, $\{-1,\, 0,\, 1\}$).
The multiplicative reduct of $\mathbb{Z}/5\mathbb{Z}$ is $\{\pm 1,\, \pm i,\, 0\}$ where $1+1=i$
(or, put otherwise, $2 \cdot 2 \;=\; -1$).

This pattern continues by Hall's near-field $\{\pm 1,\, \pm i,\, \pm j,\, \pm k,\;\, 0\}$ which relies on the multiplication of *quaternions*.

Open question. Let H be the quasigroup of base octonions

$$\{\pm 1,\, \pm i,\, \pm j,\, \pm l,\, \pm ij,\, \pm il,\, \pm jl,\, \pm ijl\}\,,$$

and $H^0 := H \cup \{0\}$.
Can we define an addition-like operation on H^0 that produces a non-Desarguian affine plane of order 17?

Alternatively, for $P = H^0 \times H^0$, can we extend the set of 'lines' $\{\ell_{m,0} \,:\, m \in H^0\}$ so that we obtain an affine plane?
Note that, when $c=0$, the above definition $\ell_{m,c}$ does not utilize the addition, as specifically $\ell_{m,0} \;=\; \{(x,\, mx) \,:\, x \in H^0\}$.