Efficient energy preserving Galerkin–Legendre spectral methods for fractional nonlinear Schrödinger equation with wave operator

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In this paper, three energy preserving numerical methods are proposed, including the Crank–Nicolson Galerkin–Legendre spectral (CN–GLS) method, the SAV Galerkin–Legendre spectral (SAV–GLS) method, and the ESAV Galerkin–Legendre spectral (ESAV–GLS) method, for the space fractional nonlinear Schrödinger equation with wave operator. In theoretical analyses, we take the CN–GLS method as an example to analyze the boundness of numerical solution and the unconditional spectral–accuracy convergence in L^2 and L^∞ norms. The effective numerical implementations of the proposed spectral Galerkin methods are discussed in detail. Numerical comparisons are reported to illustrate that the theoretical results are reasonable and the proposed spectral Galerkin methods have high efficiency for energy preservation in long–time computations.

$|u|^2$

1. Introduction

In this paper, we numerically investigate the following fractional nonlinear Schrödinger wave (FNSW) equation

$$u_{tt} + (-\Delta)^{\frac{\alpha}{2}} u + \mathrm{i}\kappa u_t + \beta |u|^2 u = 0 \quad \text{in} \quad \Omega \times (0, T], \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \tag{1.2}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0, T], \tag{1.3}$$

where $1 < \alpha < 2$, $\Omega = (a, b)$, κ and β are given positive constants, $u_0(x)$ and $u_1(x)$ are smooth functions, and $\mathrm{i}^2 = -1$. The Riesz fractional derivative is represented by [34]

$$-(-\Delta)^{\frac{\alpha}{2}} u = -\frac{1}{2\cos(\frac{\alpha\pi}{2})} \left({}_{a}^{RL}D_x^\alpha + {}_{x}^{RL}D_b^\alpha \right) u,$$

where the left and right Riemann-Liouville fractional derivatives are defined as

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$$ {}_{a}^{RL}D_{x}^{\alpha}u=\frac{1}{\Gamma(2-\alpha)}\frac{\partial^{2}}{\partial x^{2}}\int_{a}^{x}\frac{u(\xi,t)d\xi}{(x-\xi)^{\alpha-1}},\quad {}_{x}^{RL}D_{b}^{\alpha}u=\frac{1}{\Gamma(2-\alpha)}\frac{\partial^{2}}{\partial x^{2}}\int_{x}^{b}\frac{u(\xi,t)d\xi}{(\xi-x)^{\alpha-1}}, $$

in which $\Gamma(z)=\int_0^{\infty}x^{z-1}e^{-x}dx$.

It is worth noticing that the solutions of FNSW equation (1.1) conserve the total energy, i.e.,

$$ \mathcal{E}(t)=\int_{\Omega}(u_t)^2+\bar{u}(-\Delta)^{\frac{\alpha}{2}}u+\frac{\beta}{2}|u|^4dx\equiv\mathcal{E}(0), $$

where $\bar{u}$ is the complex conjugate of u.

The nonlinear Schrödinger (NLS) equation is famous for its real applications in many fields of physics, involving quantum physics, plasma physics and nonlinear optics [36,51]. It is derived by using Feynman path integrals over Brownian process. Recently, Laskin [29] confirms that the Lévy-like process offers possibilities for developing the fractional quantum mechanics which is a generalized quantum mechanics. In virtue of such an attempt, the fractional nonlinear Schrödinger equation is proposed, which involves the fractional derivative with the Lévy index α instead of the classical one. AS for $\alpha=2$, the FNSW equation reduces to the classical cubic NLS equation. In the special case of $\kappa=0$, it collapses to the fractional nonlinear wave equation [30,37]. Nowadays, the FNSW equation has drawn much attention in propagation dynamics [53], water wave dynamic [27] and biopolymer [28]. Other studies on the Cauchy problem, the existence of solitary wave solutions of the FNSW equation (1.1) have been studied, readers can refer to [6,10,18,20] for more details.

Indeed, it is hard for researchers to discuss the analytical solution for the FNSW equation due to the nonlocality of fractional derivative and nonlinearity of the fractional model. Designing an efficient numerical method for the model (1.1) that is important to recognize the solitary wave propagation of the FNSW equation. Moreover, the key of numerical methods is the approximation of fractional derivative. In recent years, a large amount of literature on numerical approaches for the fractional derivative have been studied, such as the finite difference methods [12,13,24,26,45,55], finite element methods [8,32,50] and spectral methods [11,43,47,52,54]. Also, some novel approximations are worth mentioning, for instance, the finite volume methods [15], meshless methods [1] and fractional iteration methods [2,3] and so forth. While for the FNSW equation, there exist some works on numerical methods, for example, we refer readers to the finite difference methods [38,40], finite element methods [9,31] and references therein. However, the previous methods have low-order accuracy in space, it is important and challenging to construct high-order spatial scheme for the FNSW equation. The spectral Galerkin methods have been widely used in numerical simulations of partial differential equations with the fractional derivative due to their spectral accuracy [43,52]. To our knowledge, there is no work on spectral Galerkin method for the FNSW equation. This motivates us to develop a high-order spectral Galerkin scheme for the FNSW equation. In this way, it is easy to achieve the desired numerical accuracy by equipping few space storage.

Numerous theoretical and experimental results indicate that the structure-preserving algorithms can preserve the intrinsic physical invariants of nonlinear conservative systems and often have excellent numerical properties, such as the linear error growth, long-time behavior and smaller amplitude [21]. One of the most important components of the structure-preserving scheme is exactly the time-stepping approach. The well-known invariant-preserving integrators for conservative systems are about the averaged vector field (AVF) method [39], discrete variational derivative (DVD) method [16], Hamiltonian boundary value methods (HBVMs) [7] and so on. In recent years, there are some energy stable methods for gradient flow systems which have been applied to construct the energy preserving schemes for the conservative systems, for examples, we refer readers to the invariant energy quadratization (IEQ) method [48], scalar auxiliary variable (SAV) method [42] and exponential scalar auxiliary variable (ESAV) [35] method.

The objective of this paper is to propose three energy preserving spectral Galerkin methods for FNSW equation (1.1). After spatial discretization, we first present the Crank–Nicolson energy preserving spectral Galerkin method for the FNSW equation. Second, by introducing a scalar auxiliary variable [42], we reformulate the original system (1.1) into an equivalent system, then we construct an energy quadratization SAV spectral Galerkin method. Third, we transform the system (1.1) into another equivalent system by introducing an exponential scalar auxiliary variable [35], we propose a linearly implicit energy preserving ESAV spectral Galerkin method based on the extrapolation strategy. One of the significance of this paper is that we establish the spectral–accuracy convergence and stability in L^2 and L^{∞} norms without grid-ratio condition for the CN–GLS method. The other significance of this paper is that our proposed spectral Galerkin methods are more accurate than the existed finite difference methods, only small algebraic systems need to be solved in our numerical implementations. This fact shows the great superiority of our methods in long-time computations. Extensive numerical results demonstrate that the proposed spectral Galerkin methods perform more effectively than the compared numerical schemes.

The rest of this paper is organized as follows. In Section 2, we introduce some fractional Sobolev spaces and lemmas. In Section 3, we construct the CN–GLS method and discuss the associated theoretical analyses. Then, we establish the SAV–GLS method and ESAV–GLS method in Section 4 and Section 5, respectively. In Section 6, we give a brief implementation to illustrate the numerical schemes. Finally, we report some numerical results for comparisons in Section 7 and draw some conclusions in Section 8.

2. Preliminaries

In this section, we introduce some essential definitions and lemmas of the spectral Galerkin method.

Definition 2.1. Define the inner product, L^p norm $(1 \le p < \infty)$ and L^∞ norm as

$$(u,v) := \int_\Omega \bar{v} u dx, \quad \left\| u \right\| := \sqrt{\left(u,u\right)}, \quad \left\| u \right\|_p := \Big(\int_\Omega |u|^p dx \Big)^{1/p}, \quad \left\| u \right\|_\infty := \operatorname*{ess\,sup}_{x\in\Omega} \Big\{ |u(x)| \Big\}.$$

Definition 2.2 *(Left fractional derivative space [14]).* For $\mu > 0$, define the semi-norm and norm of the left fractional derivative space on Ω as

$$\left| u \right|_{J_L^\mu(\Omega)} := \left\| {}_a^{RL}D_x^\mu u \right\|, \quad \left\| u \right\|_{J_L^\mu(\Omega)} := \Big(\left\| u \right\|^2 + \left| u \right|^2_{J_L^\mu(\Omega)} \Big)^{1/2}.$$

$J_L^\mu(\Omega)$ (or $J_{L,0}^\mu(\Omega)$) denotes the closure of $C^\infty(\Omega)$ (or $C_0^\infty(\Omega)$) with respect to $\left\| \cdot \right\|_{J_L^\mu(\Omega)}$.

Definition 2.3 *(Right fractional derivative space [14]).* For $\mu > 0$, define the semi-norm and the norm of the right fractional derivative space on Ω as

$$\left| u \right|_{J_R^\mu(\Omega)} := \left\| {}_x^{RL}D_b^\mu u \right\|, \quad \left\| u \right\|_{J_R^\mu(\Omega)} := \Big(\left\| u \right\|^2 + \left| u \right|^2_{J_R^\mu(\Omega)} \Big)^{1/2}.$$

$J_R^\mu(\Omega)$ (or $J_{R,0}^\mu(\Omega)$) denotes the closure of $C^\infty(\Omega)$ (or $C_0^\infty(\Omega)$) with respect to $\left\| \cdot \right\|_{J_R^\mu(\Omega)}$.

Definition 2.4 *(Symmetric fractional derivative space [14]).* For $\mu > 0$, where $\mu \ne n - 1/2, n \in \mathbb{N}$, define the semi-norm and the norm as

$$\left| u \right|_{J_S^\mu(\Omega)} := \left| \Big({}_a^{RL}D_x^\mu u, {}_x^{RL}D_b^\mu u \Big) \right|^{1/2}, \quad \left\| u \right\|_{J_S^\mu(\Omega)} := \Big(\left\| u \right\|^2 + \left| u \right|^2_{J_S^\mu(\Omega)} \Big)^{1/2}.$$

$J_S^\mu(\Omega)$ (or $J_{S,0}^\mu(\Omega)$) denotes the closure of $C^\infty(\Omega)$ (or $C_0^\infty(\Omega)$) with respect to $\left\| \cdot \right\|_{J_S^\mu(\Omega)}$.

Definition 2.5 *(Fractional Sobolev space [14]).* For $\mu > 0$, define the semi-norm and the norm as

$$\left| u \right|_{H^\mu(\Omega)} := \left\| |\xi|^\mu \mathcal{F}[u,\xi] \right\|, \quad \left\| u \right\|_{H^\mu(\Omega)} := \Big(\left\| u \right\|^2 + \left| u \right|^2_{H^\mu(\Omega)} \Big)^{1/2}.$$

$H^\mu(\Omega)$ (or $H_0^\mu(\Omega)$) denotes the closure of $C^\infty(\Omega)$ (or $C_0^\infty(\Omega)$) with respect to $\left\| \cdot \right\|_{H^\mu(\Omega)}$, and $\mathcal{F}$ represents the Fourier transform.

Lemma 2.1 *([14]). Suppose $\mu > 0$ and $\mu \ne n - 1/2, n \in \mathbb{N}$. Then the spaces $J_L^\mu(\Omega)$, $J_R^\mu(\Omega)$, $J_S^\mu(\Omega)$ and $H^\mu(\Omega)$ are equal with equivalent semi-norms and norms, and the spaces $J_{L,0}^\mu(\Omega)$, $J_{R,0}^\mu(\Omega)$, $J_{S,0}^\mu(\Omega)$ and $H_0^\mu(\Omega)$ are equal with equivalent semi-norms and norms.*

Lemma 2.2 *([14]). Suppose $1 < \mu \le 2$. For any $u \in H_0^\mu(\Omega)$ and $v \in H_0^{\mu/2}(\Omega)$, we have*

$$\Big({}_a^{RL}D_x^\mu u, v \Big) = \Big({}_a^{RL}D_x^{\mu/2} u, {}_x^{RL}D_b^{\mu/2} v \Big), \quad \Big({}_x^{RL}D_b^\mu u, v \Big) = \Big({}_x^{RL}D_b^{\mu/2} u, {}_a^{RL}D_x^{\mu/2} v \Big).$$

Lemma 2.3 *(Fractional Poincaré-Friedrichs inequality [14]). For $u \in J_{L,0}^\mu(\Omega)$ and $0 < \lambda \le \mu$, it holds that*

$$\left\| u \right\| \le C \left| u \right|_{J_L^\mu(\Omega)}, \quad \left\| u \right\|_{J_L^\lambda(\Omega)} \le C \left| u \right|_{J_L^\mu(\Omega)}.$$

Besides, for $u \in J_{R,0}^\mu(\Omega)$ and $0 < \lambda \le \mu$, we have

$$\left\| u \right\| \le C \left| u \right|_{J_R^\mu(\Omega)}, \quad \left\| u \right\|_{J_R^\lambda(\Omega)} \le C \left| u \right|_{J_R^\mu(\Omega)}.$$

Similar conclusion can be deduced for $u \in H_0^\mu(\Omega)$ and $\mu \ne n - 1/2, n \in \mathbb{N}$.

3. Crank–Nicolson Galerkin–Legendre spectral method

Define the following bilinear form

$$\mathcal{A}(u,w)=\frac{1}{2\cos(\frac{\alpha\pi}{2})}\Big[\Big({}_a^{RL}D_x^{\alpha/2}u,{}_x^{RL}D_b^{\alpha/2}w\Big)+\Big({}_x^{RL}D_b^{\alpha/2}u,{}_a^{RL}D_x^{\alpha/2}w\Big)\Big],\quad \forall u,w\in H_0^{\alpha/2}(\Omega).$$

For simplicity, we introduce the semi-norm and the norm as

$$\Big|u\Big|_{\alpha/2}:=\sqrt{\mathcal{A}\Big(u,u\Big)},\quad \Big\|u\Big\|_{\alpha/2}:=\Big(\Big\|u\Big\|^2+\Big|u\Big|_{\alpha/2}^2\Big)^{1/2}.$$

We observe from Lemma 2.1 that $|\cdot|_{\alpha/2}$ and $\|\cdot\|_{\alpha/2}$ are equivalent with the semi-norms and norms of $J_L^{\mu}(\Omega)$, $J_R^{\mu}(\Omega)$ and $J_S^{\mu}(\Omega)$. From Ref. [14], the bilinear form $\mathcal{A}(u,w)$ has the continuous and coercive properties, i.e., there exist positive constants C_1 and C_2, such that

$$\Big|\mathcal{A}(u,w)\Big|\le C_1\Big\|u\Big\|_{\alpha/2}\Big\|w\Big\|_{\alpha/2},\quad \Big|\mathcal{A}(u,u)\Big|\ge C_2\Big\|u\Big\|_{\alpha/2}^2.$$

Based on Lemma 2.4 and the definition of Riesz fractional derivative, the weak formulation of (1.1) reads: finding $u\in H_0^{\alpha/2}(\Omega)$, such that

$$(u_{tt},w)+\mathcal{A}(u,w)+\mathrm{i}\kappa(u_t,w)+\beta(|u|^2u,w)=0,\quad \forall w\in H_0^{\alpha/2}(\Omega), \tag{3.1}$$

with the initial conditions given by

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x).$$

3.1. The CN–GLS method

Introduce $v=u_t$ to reformulate (3.1) into an equivalent system as follows, i.e., finding $u,v\in H_0^{\alpha/2}(\Omega)$, for $w\in H_0^{\alpha/2}(\Omega)$, such that

$$(u_t,w)=(v,w), \tag{3.2}$$

$$(v_t,w)+\mathcal{A}(u,w)+\mathrm{i}\kappa(u_t,w)+\beta(|u|^2u,w)=0. \tag{3.3}$$

For simplicity, introducing the following notations

$$u^n=u(x,t_n),\quad \delta_t u^{n+\frac12}=\frac{u^{n+1}-u^n}{\tau},\quad u^{n+\frac12}=\frac{u^{n+1}+u^n}{2},\quad \widetilde{u}^{n+\frac12}=\frac{3u^n-u^{n-1}}{2},$$

where $t_n=n\tau$, $n=0,1,\cdots,N_t$ and $\tau=T/N_t$ is the time step size. We denote u^n and U_N^n as the exact solution and numerical approximation at $t=t_n$, respectively. For theoretical analysis, assume that

$$\max_{0\le t\le T}\Big\{\Big\|u\Big\|,\Big|u\Big|_{\alpha/2},\Big\|u\Big\|_\infty,\Big\|v\Big\|\Big\}\le\mathcal{C}. \tag{3.4}$$

Without loss of generality, let $\mathcal{C}$ be a general positive constant which may vary in different circumstance. Denote $P_N(\Omega)$ to be the polynomial space with the degree no more than N in interval Ω. The approximation space $X_N^0(\Omega)$ is defined as

$$X_N^0(\Omega)=P_N(\Omega)\cap H_0^{\alpha/2}(\Omega).$$

It is clear that $X_N^0(\Omega)$ is a subspace of $H_0^{\alpha/2}(\Omega)$. By using the modified Crank–Nicolson scheme for temporal derivative in (3.2)–(3.3), the Crank–Nicolson scheme with truncation errors reads: finding $u^n,v^n\in H_0^{\alpha/2}(\Omega)$, for $w\in H_0^{\alpha/2}(\Omega)$, such that

$$(\delta_t u^{n+\frac12},w)=(v^{n+\frac12},w)+(\mathcal{R}_1^n,w),\quad n=0,1,\cdots,N_t-1, \tag{3.5}$$

$$(\delta_t v^{n+\frac12},w)+\mathcal{A}(u^{n+\frac12},w)+\mathrm{i}\kappa(\delta_t u^{n+\frac12},w)+\frac{\beta}{2}\Big((|u^n|^2+|u^{n+1}|^2)u^{n+\frac12},w\Big)=(\mathcal{R}_2^n,w). \tag{3.6}$$

Omitting the residual terms $(\mathcal{R}_1^n,w)$, $(\mathcal{R}_2^n,w)$ in (3.5)-(3.6) and replacing the exact solutions u^n and v^n with the numerical solutions U_N^n and V_N^n, we propose the fully discrete CN–GLS scheme as follows: finding $U_N^n,V_N^n\in X_N^0(\Omega)$, for $w_N\in X_N^0(\Omega)$, such that

$$(\delta_t U_N^{n+\frac12},w_N)=(V_N^{n+\frac12},w_N),\quad n=0,1,\cdots,N_t-1, \tag{3.7}$$

$$(\delta_t V_N^{n+\frac12},w_N)+\mathcal{A}(U_N^{n+\frac12},w_N)+\mathrm{i}\kappa(\delta_t U_N^{n+\frac12},w_N)+\frac{\beta}{2}\Big((|U_N^n|^2+|U_N^{n+1}|^2)U_N^{n+\frac12},w_N\Big)=0, \tag{3.8}$$

with the initial conditions

$$U_N^0 = \Pi_N^{\alpha/2,0} u_0(x), \quad V_N^0 = \Pi_N^{\alpha/2,0} u_1(x),$$

where the orthogonal projection operator $\Pi_N^{\alpha/2,0}$: $H_0^{\alpha/2}(\Omega) \to X_N^0(\Omega)$ is defined as

$$\mathcal{A}(u - \Pi_N^{\alpha/2,0} u, w_N) = 0, \quad \forall w_N \in X_N^0(\Omega).$$

The orthogonal projection operator $\Pi_N^{\alpha/2,0}$ has the following properties.

Lemma 3.1 *([46,52]). Let μ and r be arbitrary real numbers satisfying $\frac{1}{2} < \mu \le 1 < r$. Then there exists a positive constant $\mathcal{C}$ independent of N, such that, for any function $u \in H^r(\Omega) \cap H_0^\mu(\Omega)$, the following estimate holds:*

$$\left\| u - \Pi_N^{\mu,0} u \right\| \le \mathcal{C} N^{-r} \left\| u \right\|_{H^r(\Omega)},$$
$$\left| u - \Pi_N^{\mu,0} u \right|_{H^\mu} \le \mathcal{C} N^{\mu - r} \left\| u \right\|_{H^r(\Omega)}.$$

Theorem 3.1 *(CN–GLS: Conservative law). The numerical solution of the CN–GLS scheme (3.7)-(3.8) is conservative in the sense that*

$$\mathcal{E}^{n+1} = \mathcal{E}^n = \cdots = \mathcal{E}^0,$$

where

$$\mathcal{E}^n = \left\| V_N^n \right\|^2 + \left| U_N^n \right|_{\alpha/2}^2 + \frac{\beta}{2} \left\| U_N^n \right\|_4^4, \quad 0 \le n \le N_t.$$

Proof. Choosing $w_N = \delta_t U_N^{n+\frac{1}{2}}$ in (3.8) and taking the real part we arrive at

$$\mathrm{Re}\Big\{ (\delta_t V_N^{n+\frac{1}{2}}, \delta_t U_N^{n+\frac{1}{2}}) + \mathcal{A}(U_N^{n+\frac{1}{2}}, \delta_t U_N^{n+\frac{1}{2}}) + \frac{\beta}{2} \Big((|U_N^n|^2 + |U_N^{n+1}|^2) U_N^{n+\frac{1}{2}}, \delta_t U_N^{n+\frac{1}{2}} \Big) \Big\} = 0, \tag{3.9}$$

where “Re” represents the real part of a complex number.

Notice that

$$\mathrm{Re}(\delta_t V_N^{n+\frac{1}{2}}, \delta_t U_N^{n+\frac{1}{2}}) = \mathrm{Re}(\delta_t V_N^{n+\frac{1}{2}}, V_N^{n+\frac{1}{2}}) = \frac{\left\| V_N^{n+1} \right\|^2 - \left\| V_N^n \right\|^2}{2\tau}, \tag{3.10}$$

$$\mathrm{Re}\Big\{ \mathcal{A}(U_N^{n+\frac{1}{2}}, \delta_t U_N^{n+\frac{1}{2}}) \Big\} = \frac{\left| U_N^{n+1} \right|_{\alpha/2}^2 - \left| U_N^n \right|_{\alpha/2}^2}{2\tau}, \tag{3.11}$$

and

$$\mathrm{Re}\Big((|U_N^n|^2 + |U_N^{n+1}|^2) U_N^{n+\frac{1}{2}}, \delta_t U_N^{n+\frac{1}{2}} \Big) = \frac{\left\| U_N^{n+1} \right\|_4^4 - \left\| U_N^n \right\|_4^4}{2\tau}. \tag{3.12}$$

Substituting (3.10)-(3.12) into (3.9), one has

$$\left\| V_N^{n+1} \right\|^2 + \left| U_N^{n+1} \right|_{\alpha/2}^2 + \frac{\beta}{2} \left\| U_N^{n+1} \right\|_4^4 = \left\| V_N^n \right\|^2 + \left| U_N^n \right|_{\alpha/2}^2 + \frac{\beta}{2} \left\| U_N^n \right\|_4^4.$$

This ends the proof. $\square$

3.2. The prior bound

Lemma 3.2 *(Fractional Gagliardo-Nirenberg inequality [22,46]). For $1 < \alpha < 2$, there exists a positive constant $\mathcal{C}$, such that*

$$\left\| u \right\|_4^4 \le \mathcal{C} \left| u \right|_{\alpha/2}^{2/\alpha} \left\| u \right\|^{4-2/\alpha}.$$

Lemma 3.3 *(Sobolev inequality [28]). For $\frac{1}{2} < \mu \le 1$, then there exists a positive constant $\mathcal{C}$, such that*

$$\left\| u \right\|_\infty \le \mathcal{C} \left\| u \right\|_{H^\mu(\Omega)}, \quad \forall u \in H_0^\mu(\Omega).$$

Theorem 3.2. *The numerical solution of the scheme (3.7)-(3.8) is long-time bounded in the following sense*

$$\left\|U_N^n\right\| \leq \mathcal{C}, \quad \left|U_N^n\right|_{\alpha/2} \leq \mathcal{C}, \quad \left\|U_N^n\right\|_\infty \leq \mathcal{C}, \quad 0 \leq n \leq N_t.$$

Proof. It follows from the triangle inequality that

$$\left\|\Pi_N^{\alpha/2,0} u_0\right\| - \left\|u_0 - \Pi_N^{\alpha/2,0} u_0\right\| \leq \left\|u_0\right\|. \tag{3.13}$$

By admitting the assumptions on exact solution (3.4) and Lemma 3.1, for sufficiently small N^{-1} and smooth exact solution, we get

$$\left\|U_N^0\right\| = \left\|\Pi_N^{\alpha/2,0} u_0\right\| \leq \left\|u_0\right\| + \left\|u_0 - \Pi_N^{\alpha/2,0} u_0\right\| \leq \mathcal{C}_0 + \mathcal{C}_0 N^{-r} \leq \mathcal{C}, \tag{3.14}$$

where $\mathcal{C}_0$ is a positive constant which is independent of τ and N.

By employing the similar procedures and Lemma 3.3, one has

$$\left\|V_N^0\right\| \leq \mathcal{C}, \quad \left|U_N^0\right|_{\alpha/2} \leq \mathcal{C}, \quad \left\|U_N^0\right\|_\infty \leq \mathcal{C}\sqrt{\left\|U_N^0\right\|^2 + \left|U_N^0\right|_{\alpha/2}^2} \leq \mathcal{C}. \tag{3.15}$$

Thanks to (3.14), (3.15) and Lemma 3.2, we obtain

$$\left\|U_N^0\right\|_4^4 \leq \mathcal{C}\left|U_N^0\right|_{\alpha/2}^{2/\alpha}\left\|U_N^0\right\|^{4-2/\alpha} \leq \mathcal{C}.$$

According to Theorem 3.1, we derive

$$\left\|V_N^n\right\|^2 + \left|U_N^n\right|_{\alpha/2}^2 \leq \mathcal{E}^n = \mathcal{E}^0 = \left\|V_N^0\right\|^2 + \left|U_N^0\right|_{\alpha/2}^2 + \frac{\beta}{2}\left\|U_N^0\right\|_4^4 \leq \mathcal{C},$$

which implies

$$\left\|V_N^n\right\| \leq \mathcal{C}, \quad \left|U_N^n\right|_{\alpha/2} \leq \mathcal{C}, \quad n = 0, 1, \cdots, N_t.$$

It follows from (3.7) that

$$(\delta_t U_N^{n+\frac{1}{2}} - V_N^{n+\frac{1}{2}}, w_N) = 0, \quad \forall w_N \in X_N^0(\Omega).$$

Setting $w_N = \delta_t U_N^{n+\frac{1}{2}} - V_N^{n+\frac{1}{2}}$ in above equation, we deduce

$$\delta_t U_N^{n+\frac{1}{2}} = V_N^{n+\frac{1}{2}}. \tag{3.16}$$

Taking L^2-norm of both sides of (3.16), one obtains

$$\left\|U_N^{n+1}\right\| \leq \left\|U_N^n\right\| + \mathcal{C}\tau\left(\left\|V_N^{n+1}\right\| + \left\|V_N^n\right\|\right) = \left\|U_N^0\right\| + \mathcal{C}\tau\sum_{k=0}^{n}\left(\left\|V_N^{k+1}\right\| + \left\|V_N^k\right\|\right) \leq \left\|U_N^0\right\| + \mathcal{C} \leq \mathcal{C}.$$

In virtue of the Sobolev inequality (Lemma 3.3), we conclude

$$\left\|U_N^n\right\|_\infty \leq \mathcal{C}\sqrt{\left\|U_N^n\right\|^2 + \left|U_N^n\right|_{\alpha/2}^2} \leq \mathcal{C}, \quad n = 0, 1, \cdots, N_t.$$

This ends the proof. □

3.3. Unique solvability

Lemma 3.4 *(Browder fixed point theorem [4]). Let $(\mathcal{H}, (\cdot,\cdot))$ be a finite dimensional inner product space, $||\cdot||$ the associated norm, and $\mathscr{F}: \mathcal{H} \to \mathcal{H}$ be continuous, such that*

$$\exists\, \delta > 0,\ \forall\, z \in \mathcal{H},\ ||z|| = \delta,\ s.t.\ Re(\mathscr{F}(z), z) \geq 0,$$

there exists $z^* \in \mathcal{H}$, $||z^*|| \leq \delta$ such that $\mathscr{F}(z^*) = 0$.

Lemma 3.5 *([44]). For any complex functions W_1, W_2, Z_1, and Z_2, the following inequality holds:*

$$\Big||W_1|^2W_2-|Z_1|^2Z_2\Big|\le\Big(\max\Big\{|W_1|,|W_2|,|Z_1|,|Z_2|\Big\}\Big)^2\Big(2|W_1-Z_1|+|W_2-Z_2|\Big).$$

Theorem 3.3. *The solution of numerical scheme (3.7)–(3.8) uniquely exists.*

Proof. For the sake of readability, we remove the proof of this theorem to the Appendix A. □

3.4. Convergence and stability analyses for the CN–GLS method

Lemma 3.6. *Suppose that the exact solution $u(\cdot,t)\in C^4([0,T])$. Then we have*

$$\max_{0\le n\le N_t-1}\Big\{\Big|\mathcal{R}_1^n\Big|,\Big|\delta_t\mathcal{R}_1^{n+1/2}\Big|,\Big|\mathcal{R}_2^n\Big|\Big\}\le\mathcal{C}\tau^2,$$

where $\mathcal{C}$ is a positive constant independent of N and τ.

Proof. Using the Taylor's expansion for u at $t=t_{n+1/2}$, one has

$$\mathcal{R}_1^n=u_t(\cdot,t_{n+1/2})-\delta_tu^{n+1/2}+v^{n+1/2}-v(\cdot,t_{n+1/2})=\mathcal{I}_1+\mathcal{I}_2,$$

where

$$\mathcal{I}_1=u_t(\cdot,t_{n+1/2})-\delta_tu^{n+1/2}=-\frac{\tau^2}{16}\int_0^1\Big[u_{ttt}(\cdot,t_{n+1/2}+\frac{\tau}{2}s)+u_{ttt}(\cdot,t_{n+1/2}-\frac{\tau}{2}s)\Big](1-s)^2ds,$$

and

$$\begin{aligned}\mathcal{I}_2=v^{n+1/2}-v(\cdot,t_{n+1/2})&=\frac{\tau^2}{8}\int_0^1\Big[v_{tt}(\cdot,t_{n+1/2}+\frac{\tau}{2}s)+v_{tt}(\cdot,t_{n+1/2}-\frac{\tau}{2}s)\Big](1-s)ds\\&=\frac{\tau^2}{8}\int_0^1\Big[u_{ttt}(\cdot,t_{n+1/2}+\frac{\tau}{2}s)+u_{ttt}(\cdot,t_{n+1/2}-\frac{\tau}{2}s)\Big](1-s)ds.\end{aligned}$$

By assuming $u(\cdot,t)\in C^3([0,T])$, we deduce

$$\Big|\mathcal{R}_1^n\Big|\le\mathcal{C}\tau^2,\quad n=0,1,\cdots,N_t-1.$$

Thus,

$$\begin{aligned}\delta_t\mathcal{R}_1^{n+\frac12}=&-\frac{\tau^2}{16}\int_0^1\Big[\frac{u_{ttt}(\cdot,t_{n+\frac32}+\frac{\tau}{2}s)-u_{ttt}(\cdot,t_{n+\frac12}+\frac{\tau}{2}s)}{\tau}+\frac{u_{ttt}(\cdot,t_{n+\frac32}-\frac{\tau}{2}s)-u_{ttt}(\cdot,t_{n+\frac12}-\frac{\tau}{2}s)}{\tau}\Big](1-s)^2ds\\&+\frac{\tau^2}{8}\int_0^1\Big[\frac{u_{ttt}(\cdot,t_{n+\frac32}+\frac{\tau}{2}s)-u_{ttt}(\cdot,t_{n+\frac12}+\frac{\tau}{2}s)}{\tau}+\frac{u_{ttt}(\cdot,t_{n+\frac32}-\frac{\tau}{2}s)-u_{ttt}(\cdot,t_{n+\frac12}-\frac{\tau}{2}s)}{\tau}\Big](1-s)ds.\end{aligned}$$

Employing the Lagrange's mean value theorem, we obtain

$$u_{ttt}(\cdot,t_{n+3/2}+\frac{\tau}{2}s)-u_{ttt}(\cdot,t_{n+1/2}+\frac{\tau}{2}s)=\tau u^{(4)}(\cdot,\xi_1),\quad t_{n+1/2}+\frac{\tau}{2}s<\xi_1<t_{n+3/2}+\frac{\tau}{2}s,$$

and

$$u_{ttt}(\cdot,t_{n+3/2}-\frac{\tau}{2}s)-u_{ttt}(\cdot,t_{n+1/2}-\frac{\tau}{2}s)=\tau u^{(4)}(\cdot,\xi_2),\quad t_{n+1/2}-\frac{\tau}{2}s<\xi_2<t_{n+3/2}-\frac{\tau}{2}s.$$

Suppose that $u(\cdot,t)\in C^4([0,T])$. We deduce

$$\Big|\delta_t\mathcal{R}_1^{n+1/2}\Big|\le\mathcal{C}\tau^2,\quad n=0,1,\cdots,N_t-1.$$

Similar proof to Ref. [26], we derive

$$\left|\mathcal{R}_2^n\right| \le \mathcal{C}\tau^2, \quad n = 0, 1, \cdots, N_t - 1.$$

The proof is completed. $\square$

Lemma 3.7 *([23]). For time sequences $w = \{w^0, w^1, \cdots, w^n, w^{n+1}\}$ and $g = \{g^0, g^1, \cdots, g^n, g^{n+1}\}$, there is*

$$\mid 2\tau\sum_{k=0}^{n} g^k \delta_t w^{k+1/2} \mid \le \tau\sum_{k=1}^{n} \mid w^k \mid^2 + \tau\sum_{k=0}^{n-1} \mid \delta_t g^{k+1/2} \mid^2 + \frac{1}{2} \mid w^{n+1} \mid^2 + 2 \mid g^n \mid^2 + \mid w^0 \mid^2 + \mid g^0 \mid^2 .$$

Lemma 3.8 *(Gronwall inequality I [56]). Suppose that the discrete grid function $\{w^n \mid n = 0, 1, 2, \cdots, N_t; N_t\tau = T\}$ satisfies the following inequality*

$$w^n - w^{n-1} \le A\tau w^n + B\tau w^{n-1} + C_n\tau,$$

where A, B and C_n are non-negative constants, then

$$\max_{1\le n\le N_t} \mid w^n \mid \le \left(w^0 + \tau\sum_{k=1}^{N_t} C_k\right)e^{2(A+B)T},$$

where τ is sufficiently small, such that $(A + B)\tau \le \frac{N_t-1}{2N_t} < \frac{1}{2}$, $(N_t > 1)$.

Lemma 3.9 *(Gronwall inequality II [56]). Suppose that the discrete grid function $\{w^n \mid n = 0, 1, 2, \cdots, N_t; N_t\tau = T\}$ satisfies the following inequality*

$$w^n \le A + \tau\sum_{k=1}^{n} B_k w^k,$$

where A and B_k $(k = 0, 1, 2, \cdots, N_t)$ are non-negative constants, then

$$\max_{1\le n\le N_t} \mid w^n \mid \le A\exp(2\tau\sum_{k=1}^{N_t} B_k),$$

where τ is sufficiently small, such that $\tau \max_{1\le k\le N_t} B_k \le 1/2$.

Theorem 3.4 *(Convergence). Suppose $u \in C^4(0, T; H^r(\Omega) \cap H_0^{\alpha/2}(\Omega))$ $(r > 1)$ is the solution of (1.1) and U_N^n is the solution of (3.7)–(3.8). For sufficiently small τ and N^{-1}, the CN–GLS scheme is unconditionally convergent in the sense that*

$$\left\|u^n - U_N^n\right\| \le \mathcal{C}(\tau^2 + N^{-r}), \qquad \left\|u^n - U_N^n\right\|_\infty \le \mathcal{C}(\tau^2 + N^{\alpha/2-r}), \quad 0 \le n \le N_t.$$

Proof. Denote the error functions

$$\begin{aligned}
&\varepsilon_u^n = u^n - U_N^n = (u^n - \Pi_N^{\alpha/2,0}u^n) + (\Pi_N^{\alpha/2,0}u^n - U_N^n) = \eta_u^n + \xi_u^n,\\
&\varepsilon_v^n = v^n - V_N^n = (v^n - \Pi_N^{\alpha/2,0}v^n) + (\Pi_N^{\alpha/2,0}v^n - V_N^n) = \eta_v^n + \xi_v^n,\\
&\varepsilon_u^0 = u_0 - \Pi_N^{\alpha/2,0}u_0, \qquad \varepsilon_v^0 = u_1 - \Pi_N^{\alpha/2,0}u_1.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\xi_u^0 = \Pi_N^{\alpha/2,0}u_0 - U_N^0 = \Pi_N^{\alpha/2,0}u_0 - \Pi_N^{\alpha/2,0}u_0 = 0,\\
&\xi_v^0 = \Pi_N^{\alpha/2,0}u_1 - V_N^0 = \Pi_N^{\alpha/2,0}u_1 - \Pi_N^{\alpha/2,0}u_1 = 0.
\end{aligned}$$

Subtract (3.7)-(3.8) from (3.5)-(3.6). By virtue of the definition of orthogonal projection operator $\Pi_N^{\alpha/2,0}$, for $w_N \in X_N^0(\Omega)$, we obtain the error equations

$$(\delta_t\xi_u^{n+\frac{1}{2}}, w_N) = (\xi_v^{n+\frac{1}{2}}, w_N) + (\widehat{\mathcal{R}}_1^n, w_N), \tag{3.17}$$

$$(\delta_t\xi_v^{n+\frac{1}{2}}, w_N) + \mathcal{A}(\xi_u^{n+\frac{1}{2}}, w_N) + \mathrm{i}\kappa(\delta_t\xi_u^{n+\frac{1}{2}}, w_N) + \frac{\beta}{2}\left(\mathcal{N}^{n+1/2}, w_N\right) = (\widehat{\mathcal{R}}_2^n, w_N), \tag{3.18}$$

in which

$$\begin{aligned}\mathcal{N}^{n+1/2} &= (|u^n|^2+|u^{n+1}|^2)u^{n+\frac{1}{2}} - (|U_N^n|^2+|U_N^{n+1}|^2)U_N^{n+\frac{1}{2}}\\ &= (|U_N^n|^2+|U_N^{n+1}|^2)\varepsilon_u^{n+\frac{1}{2}} + \Big[(|u^n|^2+|u^{n+1}|^2) - (|U_N^n|^2+|U_N^{n+1}|^2)\Big]u^{n+\frac{1}{2}}\\ &= \mathcal{N}_1^{n+1/2}+\mathcal{N}_2^{n+1/2},\end{aligned}$$

and

$$\widehat{\mathcal{R}}_1^n = \mathcal{R}_1^n + \eta_v^{n+\frac{1}{2}} - \delta_t\eta_u^{n+\frac{1}{2}}, \quad \widehat{\mathcal{R}}_2^n = \mathcal{R}_2^n - \delta_t\eta_v^{n+\frac{1}{2}} - \mathrm{i}\kappa\,\delta_t\eta_u^{n+1/2}.$$

According to Taylor's expansion and Lemma 3.1, we derive

$$\begin{aligned}\left\|\eta_u^{n+\frac{1}{2}}\right\| &\le \left\|\eta_u^{n+\frac{1}{2}} - \eta_u(\cdot,t_{n+1/2})\right\| + \left\|\eta_u(\cdot,t_{n+1/2})\right\| \le \mathcal{C}(\tau^2+N^{-r}),\\ \left\|\delta_t\eta_u^{n+\frac{1}{2}}\right\| &\le \left\|\delta_t\eta_u^{n+\frac{1}{2}} - \partial_t\eta_u(\cdot,t_{n+1/2})\right\| + \left\|\partial_t\eta_u(\cdot,t_{n+1/2})\right\| \le \mathcal{C}(\tau^2+N^{-r}).\end{aligned}$$

Similarly, one has

$$\left\|\delta_t\eta_v^{n+\frac{1}{2}}\right\| \le \mathcal{C}(\tau^2+N^{-r}).$$

Therefore, combining the above relations and Lemma 3.6, we have

$$\left\|\widehat{\mathcal{R}}_1^n\right\| \le \mathcal{C}(\tau^2+N^{-r}), \quad \left\|\delta_t\widehat{\mathcal{R}}_1^{n+\frac{1}{2}}\right\| \le \mathcal{C}(\tau^2+N^{-r}), \quad \left\|\widehat{\mathcal{R}}_2^n\right\| \le \mathcal{C}(\tau^2+N^{-r}).$$

By employing Theorem 3.2 and the assumption on exact solution, we deduce

$$\left\|\mathcal{N}_1^{n+1/2}\right\|^2 = \left\|(|U_N^n|^2+|U_N^{n+1}|^2)\varepsilon_u^{n+\frac{1}{2}}\right\|^2 \le \mathcal{C}\Big(\left\|\varepsilon_u^{n+1}\right\|^2+\left\|\varepsilon_u^n\right\|^2\Big),$$

and

$$\begin{aligned}\left\|\mathcal{N}_2^{n+1/2}\right\|^2 &= \left\|\Big[2\mathrm{Re}(\bar{\varepsilon}_u^{n+1}u^{n+1}) - \bar{\varepsilon}_u^{n+1}(u^{n+1}-U_N^{n+1}) + 2\mathrm{Re}(\bar{\varepsilon}_u^n u^n) - \bar{\varepsilon}_u^n(u^n-U_N^n)\Big]u^{n+\frac{1}{2}}\right\|^2\\ &\le \mathcal{C}\Big(\left\|\varepsilon_u^{n+1}\right\|^2+\left\|\varepsilon_u^n\right\|^2\Big).\end{aligned}$$

Further, by admitting Lemma 3.1, we derive

$$\left\|\mathcal{N}^{n+1/2}\right\|^2 \le \mathcal{C}\Big(\left\|\mathcal{N}_1^{n+1/2}\right\|^2+\left\|\mathcal{N}_2^{n+1/2}\right\|^2\Big) \le \mathcal{C}\Big(\left\|\varepsilon_u^{n+1}\right\|^2+\left\|\varepsilon_u^n\right\|^2\Big) \le \mathcal{C}\Big(\left\|\xi_u^{n+1}\right\|^2+\left\|\xi_u^n\right\|^2+N^{-2r}\Big).$$

Choosing $w_N = \xi_u^{n+1/2}$ in (3.17) and taking the real part, by using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\frac{\left\|\xi_u^{n+1}\right\|^2-\left\|\xi_u^n\right\|^2}{2\tau} &= \mathrm{Re}\Big\{(\xi_v^{n+\frac{1}{2}},\xi_u^{n+1/2}) + (\widehat{\mathcal{R}}_1^n,\xi_u^{n+1/2})\Big\}\\ &\le \mathcal{C}\Big(\left\|\xi_u^{n+1}\right\|^2+\left\|\xi_u^n\right\|^2+\left\|\xi_v^{n+1}\right\|^2+\left\|\xi_v^n\right\|^2+\tau^4+N^{-2r}\Big).\end{aligned}$$

Thus,

$$\left\|\xi_u^{n+1}\right\|^2 \le \left\|\xi_u^n\right\|^2 + \mathcal{C}\tau\Big(\left\|\xi_u^{n+1}\right\|^2+\left\|\xi_u^n\right\|^2+\left\|\xi_v^{n+1}\right\|^2+\left\|\xi_v^n\right\|^2+\tau^4+N^{-2r}\Big). \tag{3.19}$$

Applying the Gronwall inequality I (Lemma 3.8) for (3.19), we derive

$$\left\|\xi_u^{n+1}\right\|^2 \le \mathcal{C}\tau\sum_{k=0}^{n}\left\|\xi_v^{k+1}\right\|^2 + \mathcal{C}(\tau^4+N^{-2r}), \tag{3.20}$$

which implies

$$\begin{aligned}\tau\sum_{k=0}^{n}\left\|\xi_u^{k+1}\right\|^2 &\le \mathcal{C}\tau^2\sum_{k=0}^{n}\sum_{j=0}^{k}\left\|\xi_v^{j+1}\right\|^2+\mathcal{C}(\tau^4+N^{-2r})\le \mathcal{C}\tau^2\sum_{k=0}^{n}\sum_{j=0}^{n}\left\|\xi_v^{j+1}\right\|^2+\mathcal{C}(\tau^4+N^{-2r})\\ &\le \mathcal{C}\tau\sum_{k=0}^{n}\left\|\xi_v^{k+1}\right\|^2+\mathcal{C}(\tau^4+N^{-2r}).\end{aligned}\tag{3.21}$$

Setting $w_N=\delta_t\xi_u^{n+1/2}$ in (3.18) and taking the real part, one obtains

$$\mathrm{Re}\Big(\delta_t\xi_v^{n+1/2},\delta_t\xi_u^{n+1/2}\Big)=\mathrm{Re}\Big\{(\widehat{\mathcal{R}}_2^{n+\frac12},\delta_t\xi_u^{n+1/2})-\mathcal{A}(\xi_u^{n+\frac12},\delta_t\xi_u^{n+1/2})-\frac{\beta}{2}\Big(\mathcal{N}^{n+1/2},\delta_t\xi_u^{n+1/2}\Big)\Big\}.\tag{3.22}$$

Noticing (3.17), there has

$$\begin{aligned}\mathrm{Re}\Big(\delta_t\xi_v^{n+1/2},\delta_t\xi_u^{n+1/2}\Big)&=\mathrm{Re}(\xi_v^{n+\frac12},\delta_t\xi_v^{n+1/2})+\mathrm{Re}(\widehat{\mathcal{R}}_1^n,\delta_t\xi_v^{n+1/2})\\ &=\frac{\left\|\xi_v^{n+1}\right\|^2-\left\|\xi_v^{n}\right\|^2}{2\tau}+\mathrm{Re}(\widehat{\mathcal{R}}_1^n,\delta_t\xi_v^{n+1/2}).\end{aligned}\tag{3.23}$$

Taking into account of the definitions of ξ_u^n, ξ_v^n and $\delta_t U_N^{n+\frac12}=V_N^{n+\frac12}$, we deduce

$$(\xi_v^{n+1/2},w_N)=(\delta_t\xi_u^{n+1/2}+\Pi_N^{\alpha/2,0}v^{n+1/2}-\Pi_N^{\alpha/2,0}\delta_t u^{n+1/2},w_N),\quad \forall w_N\in X_N^0(\Omega),$$

which implies

$$\delta_t\xi_u^{n+1/2}=\xi_v^{n+1/2}+\Pi_N^{\alpha/2,0}\delta_t u^{n+1/2}-\Pi_N^{\alpha/2,0}v^{n+1/2}.$$

In virtue of Lemma 3.1, Lemma 3.6 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\mathrm{Re}\Big\{(\widehat{\mathcal{R}}_2^n,\delta_t\xi_u^{n+1/2})\Big\}\le\frac12\Big(\left\|\widehat{\mathcal{R}}_2^n\right\|^2+\left\|\xi_v^{n+1/2}+\Pi_N^{\alpha/2,0}\delta_t u^{n+1/2}-\Pi_N^{\alpha/2,0}v^{n+1/2}\right\|^2\Big)\\ &=\frac12\Big(\left\|\widehat{\mathcal{R}}_2^n\right\|^2+\left\|\xi_v^{n+1/2}+\Pi_N^{\alpha/2,0}\delta_t u^{n+1/2}-\delta_t u^{n+1/2}+\delta_t u^{n+1/2}-v^{n+1/2}+v^{n+1/2}-\Pi_N^{\alpha/2,0}v^{n+1/2}\right\|^2\Big)\\ &\le\mathcal{C}\Big(\left\|\xi_v^{n}\right\|^2+\left\|\xi_v^{n+1}\right\|^2+\tau^4+N^{-2r}\Big),\\ &\mathrm{Re}\Big\{\mathcal{A}(\xi_u^{n+\frac12},\delta_t\xi_u^{n+1/2})\Big\}=\frac{\left|\xi_u^{n+1}\right|_{\alpha/2}^2-\left|\xi_u^{n}\right|_{\alpha/2}^2}{2\tau},\end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}\Big(\mathcal{N}^{n+1/2},\delta_t\xi_u^{n+1/2}\Big)&\le\frac12\Big(\left\|\mathcal{N}^{n+1/2}\right\|^2+\left\|\xi_v^{n+1/2}+\Pi_N^{\alpha/2,0}\delta_t u^{n+1/2}-\Pi_N^{\alpha/2,0}v^{n+1/2}\right\|^2\Big)\\ &\le\mathcal{C}\Big(\left\|\xi_u^{n+1}\right\|^2+\left\|\xi_u^{n}\right\|^2+\left\|\xi_v^{n+1}\right\|^2+\left\|\xi_v^{n}\right\|^2+\tau^4+N^{-2r}\Big).\end{aligned}\tag{3.24}$$

Substituting (3.23)-(3.24) into (3.22), we obtain

$$\begin{aligned}\left|\xi_u^{n+1}\right|_{\alpha/2}^2+\left\|\xi_v^{n+1}\right\|^2&\le\left|\xi_u^{n}\right|_{\alpha/2}^2+\left\|\xi_v^{n}\right\|^2-2\tau\mathrm{Re}(\widehat{\mathcal{R}}_1^n,\delta_t\xi_v^{n+1/2})\\ &\quad+\mathcal{C}\tau\Big(\left\|\xi_u^{n+1}\right\|^2+\left\|\xi_u^{n}\right\|^2+\left\|\xi_v^{n+1}\right\|^2+\left\|\xi_v^{n}\right\|^2+\tau^4+N^{-2r}\Big)\\ &\le\left|\xi_u^{0}\right|_{\alpha/2}^2+\left\|\xi_v^{0}\right\|^2-2\tau\mathrm{Re}\sum_{k=0}^{n}(\widehat{\mathcal{R}}_1^k,\delta_t\xi_v^{k+1/2})\\ &\quad+\mathcal{C}\tau\sum_{k=0}^{n}\Big(\left\|\xi_u^{k+1}\right\|^2+\left\|\xi_v^{k+1}\right\|^2\Big)+\mathcal{C}(\tau^4+N^{-2r}).\end{aligned}\tag{3.25}$$

It follows from (3.25), (3.21) and Lemma 3.7 that

$$
\begin{aligned}
\left|\xi_u^{n+1}\right|_{\alpha/2}^2+\left\|\xi_v^{n+1}\right\|^2 &\le \mathcal{C}\tau\sum_{k=0}^{n}\left(\left\|\xi_u^{k+1}\right\|^2+\left\|\xi_v^{k+1}\right\|^2\right)+\mathcal{C}(\tau^4+N^{-2r})-2\tau\,\mathrm{Re}\sum_{k=0}^{n}(\widehat{\mathcal{R}}_1^k,\delta_t\xi_v^{k+1/2})\\
&\le \mathcal{C}\tau\sum_{k=0}^{n}\left\|\xi_v^{k+1}\right\|^2+\mathcal{C}(\tau^4+N^{-2r})-2\tau\,\mathrm{Re}\sum_{k=0}^{n}(\widehat{\mathcal{R}}_1^k,\delta_t\xi_v^{k+1/2})\\
&\le \mathcal{C}\tau\sum_{k=0}^{n}\left\|\xi_v^{k+1}\right\|^2+\frac{1}{2}\left\|\xi_v^{n+1}\right\|^2+\mathcal{C}(\tau^4+N^{-2r}).
\end{aligned}\tag{3.26}
$$

Further, we rewrite (3.26) as

$$
\begin{aligned}
\left|\xi_u^{n+1}\right|_{\alpha/2}^2+\frac{1}{2}\left\|\xi_v^{n+1}\right\|^2 &\le \mathcal{C}\tau\sum_{k=0}^{n}\frac{1}{2}\left\|\xi_v^{k+1}\right\|^2+\mathcal{C}(\tau^4+N^{-2r})\\
&\le \mathcal{C}\tau\sum_{k=0}^{n}\left(\left|\xi_u^{k+1}\right|_{\alpha/2}^2+\frac{1}{2}\left\|\xi_v^{n+1}\right\|^2\right)+\mathcal{C}(\tau^4+N^{-2r}).
\end{aligned}
$$

Applying the Gronwall inequality II (Lemma 3.9) for above equation, we get

$$
\left\|\xi_v^{n+1}\right\|^2\le\mathcal{C}(\tau^4+N^{-2r}),\quad \left|\xi_u^{n+1}\right|_{\alpha/2}^2\le\mathcal{C}(\tau^4+N^{-2r}).
$$

By admitting the relation (3.20), we arrive at

$$
\left\|\xi_u^{n+1}\right\|^2\le\mathcal{C}\tau\sum_{k=0}^{n}\left\|\xi_v^{k+1}\right\|^2+\mathcal{C}(\tau^4+N^{-2r})\le\mathcal{C}(\tau^4+N^{-2r}).
$$

Thanks to Lemma 3.1, one has the following estimates

$$
\left\|u^n-U_N^n\right\|^2\le\mathcal{C}\left(\left\|u^n-\Pi_N^{\alpha/2,0}u_N^n\right\|^2+\left\|\xi_u^n\right\|^2\right)\le\mathcal{C}(\tau^4+N^{-2r}),
$$

and

$$
\left|u^n-U_N^n\right|_{\alpha/2}^2\le\mathcal{C}\left(\left|u^n-\Pi_N^{\alpha/2,0}u_N^n\right|_{\alpha/2}^2+\left|\xi_u^n\right|_{\alpha/2}^2\right)\le\mathcal{C}(\tau^4+N^{\alpha-2r}).
$$

Therefore, we obtain the L^∞ norm error estimate

$$
\left\|u^n-U_N^n\right\|_\infty\le\mathcal{C}\sqrt{\left\|u^n-U_N^n\right\|^2+\left|u^n-U_N^n\right|_{\alpha/2}^2}\le\mathcal{C}(\tau^2+N^{\alpha/2-r}).
$$

This completes the proof. □

Corollary 3.1. *Under the conditions of Theorem 3.4, the numerical solution U_N^n of the CN–GLS method (3.7)-(3.8) is unconditionally stable with respect to initial values in L^2 and L^∞ norms.*

4. The energy quadratization SAV–GLS method

The SAV formulation of the FNSW equation (1.1) introduces a scalar auxiliary variable

$$
p(t)=\sqrt{\int_\Omega\frac{1}{2}|u|^4dx+C_0}\quad\text{with}\quad F(u)=\frac{|u|^2}{\sqrt{\int_\Omega\frac{1}{2}|u|^4dx+C_0}},
$$

with a positive C_0, and reformulate (1.1)–(1.3) into an equivalent system

$$
u_t=v\quad\text{in}\quad\Omega\times(0,T],\tag{4.1}
$$

$$
v_t+(-\Delta)^{\frac{\alpha}{2}}u+\mathrm{i}\kappa u_t+\beta pF(u)u=0\quad\text{in}\quad\Omega\times(0,T],\tag{4.2}
$$

$$
p_t=\mathrm{Re}(F(u)u,u_t)\quad\text{in}\quad\Omega\times(0,T],\tag{4.3}
$$

$$
u=0\qquad\text{on}\quad\partial\Omega\times(0,T],\tag{4.4}
$$

$$
u=u_0,\quad v=u_1,\quad p=p_0\quad\text{in}\quad\Omega\times\{0\},\tag{4.5}
$$

where $p_0 = \sqrt{\int_\Omega \frac{1}{2}|u_0|^4 dx + C_0}$. The energy conservation in the SAV formulation is

$$\frac{d}{dt}\left(\left\|v\right\|^2 + \left|u\right|^2_{\alpha/2} + \beta p^2 - \beta C_0\right) = 0.$$

By using the implicit midpoint method in time, and the spectral Galerkin method is applied in space for (4.1)-(4.3), the fully discrete SAV–GLS scheme is constructed, i.e., finding $U_N^n, V_N^n \in X_N^0(\Omega)$, for $w_N \in X_N^0(\Omega)$, such that

$$(\delta_t U_N^{n+1/2}, w_N) = (V_N^{n+1/2}, w_N), \tag{4.6}$$

$$(\delta_t V_N^{n+1/2}, w_N) + \mathcal{A}(U_N^{n+1/2}, w_N) + \mathrm{i}\kappa(\delta_t U_N^{n+1/2}, w_N) + \beta P^{n+1/2}(F(U_N^{n+1/2})U_N^{n+1/2}, w_N) = 0, \tag{4.7}$$

$$\delta_t P^{n+1/2} = \mathrm{Re}(F(U_N^{n+1/2})U_N^{n+1/2}, \delta_t U_N^{n+1/2}). \tag{4.8}$$

Theorem 4.1 *(SAV–GLS: Conservative law). The numerical solution of the SAV-GLS scheme (4.6)-(4.8) is conservative in the sense that*

$$\mathcal{E}^{n+1} = \mathcal{E}^n = \cdots = \mathcal{E}^0,$$

where

$$\mathcal{E}^n = \left\|V_N^n\right\|^2 + \left|U_N^n\right|^2_{\alpha/2} + \beta(P^n)^2 - \beta C_0, \quad 0 \le n \le N_t.$$

Proof. Choosing $w_N = \delta_t U_N^{n+\frac{1}{2}}$ in (4.7) and taking the real part, the energy conservation is immediate. $\square$

Theorem 4.2 *(SAV–GLS: Boundness). The numerical solution of the scheme (4.6)-(4.8) is long-time bounded in the following sense*

$$\left\|U_N^n\right\| \le \mathcal{C}, \quad \left\|U_N^n\right\|_\infty \le \mathcal{C}, \quad 0 \le n \le N_t.$$

Proof. In view of the energy formulation, the proof is same as Theorem 3.2. Here we omit the proof for brevity. $\square$

Remark 4.1. It is worth mentioning that there are some works focusing on the linearly implicit SAV methods for the gradient flow system [42] and the conservative system [25] by the extrapolation technique. We noticed that the extrapolation technique could cause the implementation of SAV-GLS scheme to become more complex. In the current paper, we just consider the implicit-midpoint SAV-GLS method with quadratic energy.

Remark 4.2. By admitting the L^∞ norm boundness of the numerical solutions of SAV–GLS method, it is trivial to check that the nonlinear term $F(u)$ satisfying the global Lipschitz conditions. This fact offers possibilities for the SAV–GLS method to analyze the unconditional convergence and stability in L^2 and L^∞ norms.

5. The linearly implicit ESAV–GLS method

The ESAV formulation of the FNSW equation (1.1) introduces an exponential scalar auxiliary variable

$$q(t) = \exp\left(\int_\Omega \frac{1}{2}|u|^4 dx\right) \quad \text{with} \quad \mathcal{F}(u) = \frac{|u|^2}{\exp\left(\int_\Omega \frac{1}{2}|u|^4 dx\right)}.$$

$\frac{|u|^2}{q(t)}$.

The system (1.1)–(1.3) can be reformulated as

$$u_t = v \quad \text{in} \quad \Omega \times (0, T], \tag{5.1}$$

$$v_t + (-\Delta)^{\frac{\alpha}{2}} u + \mathrm{i}\kappa u_t + \beta q \mathcal{F}(u) u = 0 \quad \text{in} \quad \Omega \times (0, T], \tag{5.2}$$

$$\frac{d}{dt}\ln(q) = 2\mathrm{Re}(q\mathcal{F}(u)u, u_t) \quad \text{in} \quad \Omega \times (0, T], \tag{5.3}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, T], \tag{5.4}$$

$$u = u_0, \quad v = u_1, \quad q = q_0 \quad \text{in} \quad \Omega \times \{0\}, \tag{5.5}$$

$u^2 \to$ $q(t).F(u)$

where $q_0 = \exp\left(\int_\Omega \frac{1}{2}|u_0|^4 dx\right)$. The energy conservation in the ESAV formulation is

$\int_a^b |\psi|^2 \phi \, dx.$

$$\frac{d}{dt}\Big(\big\|v\big\|^2+\big|u\big|^2_{\alpha/2}+\beta\ln(q)\Big)=0.$$

The ESAV–GLS scheme is constructed by combining the implicit midpoint method and spectral Galerkin method for (5.1)-(5.3), i.e., finding $U_N^n, V_N^n \in X_N^0(\Omega)$, for $w_N \in X_N^0(\Omega)$, such that

$$(\delta_t U_N^{n+1/2}, w_N) = (V_N^{n+1/2}, w_N), \tag{5.6}$$

$$(\delta_t V_N^{n+1/2}, w_N) + \mathcal{A}(U_N^{n+1/2}, w_N) + \mathrm{i}\kappa(\delta_t U_N^{n+1/2}, w_N) + \beta \widetilde{Q}^{n+1/2}(F(\widetilde{U}_N^{n+1/2})\widetilde{U}_N^{n+1/2}, w_N) = 0, \tag{5.7}$$

$$\delta_t \ln(Q^{n+1/2}) = 2\widetilde{Q}^{n+1/2}\mathrm{Re}(F(\widetilde{U}_N^{n+1/2})\widetilde{U}_N^{n+1/2}, \delta_t U_N^{n+1/2}), \tag{5.8}$$

with the initial-start scheme

$$(\delta_t U_N^{1/2}, w_N) = (V_N^{1/2}, w_N), \tag{5.9}$$

$$(\delta_t V_N^{1/2}, w_N) + \mathcal{A}(U_N^{1/2}, w_N) + \mathrm{i}\kappa(\delta_t U_N^{1/2}, w_N) + \beta Q^{1/2}(F(U_N^{1/2})U_N^{1/2}, w_N) = 0, \tag{5.10}$$

$$\delta_t \ln(Q^{1/2}) = 2Q^{1/2}\mathrm{Re}(F(U_N^{1/2})U_N^{1/2}, \delta_t U_N^{1/2}). \tag{5.11}$$

In a similar proof to SAV-GLS, we have the following energy conservation.

Theorem 5.1 *(ESAV–GLS: Conservative law). The numerical solution of the SAV–GLS scheme (5.6)-(5.8) is conservative in the sense that*

$$\mathcal{E}^{n+1} = \mathcal{E}^n = \cdots = \mathcal{E}^0,$$

where

$$\mathcal{E}^n = \big\|V_N^n\big\|^2 + \big|U_N^n\big|^2_{\alpha/2} + \beta\ln(Q^n), \quad 0 \le n \le N_t.$$

6. Implementation

In this section, we take the CN–GLS method (3.7)-(3.8) as an example to illustrate the implementations. The approximation polynomial space $X_N^0(\Omega)$ can be expressed as

$$X_N^0(\Omega) = \mathrm{span}\{\phi_k(x) : k = 0, 1, \cdots, N-2\},$$

in which $\phi_k(x)$ is determined by the following recurrence relation

$$\phi_k(x) = L_k(\widehat{x}) - L_{k+2}(\widehat{x}), \quad \widehat{x} \in [-1, 1], \quad x = \frac{(b-a)\widehat{x} + (a+b)}{2} \in [a, b],$$

where $L_k(\hat{x})$ is the Legendre polynomial which satisfies the three-term recurrence relation [41]

$$L_0(\widehat{x}) = 1, \quad L_1(\widehat{x}) = \widehat{x},$$
$$(k+1)L_{k+1}(\widehat{x}) = (2k+1)\widehat{x}L_k(\widehat{x}) - kL_{k-1}(\widehat{x}), \quad k \ge 1.$$

Denote the numerical solutions U_N^n and V_N^n as

$$U_N^n = \sum_{k=0}^{N-2} \widehat{U}_k^n \phi_k(x), \qquad V_N^n = \sum_{k=0}^{N-2} \widehat{V}_k^n \phi_k(x).$$

Then we give the mass and stiff matrices of the CN–GLS scheme as follows

$$M_{l,k}^x = (\phi_k(x), \phi_l(x)), \quad S_{l,k}^x = \frac{1}{2\cos(\frac{\alpha\pi}{2})}\Big[\Big({}_a^{RL}D_x^{\alpha/2}\phi_k(x), {}_x^{RL}D_b^{\alpha/2}\phi_l(x)\Big) + \Big({}_x^{RL}D_b^{\alpha/2}\phi_k(x), {}_a^{RL}D_x^{\alpha/2}\phi_l(x)\Big)\Big],$$

where M^x, and S^x are all symmetric, the computations of the entries of the mass and stiff matrices can be found in [52]. The linearized iterative algorithm of CN–GLS scheme can be expressed as

$$\Big(M^x + \frac{\tau^2}{4}S^x + \frac{\mathrm{i}\kappa\tau}{2}M^x\Big)\widehat{U}^{n+1,s+1} = \Big(M^x - \frac{\tau^2}{4}S^x + \frac{\mathrm{i}\kappa\tau}{2}M^x\Big)\widehat{U}^n + \tau M^x\widehat{V}^n - \frac{\beta\tau^2}{8}\mathcal{N}^{n+1,s}, \; s = 0, 1, 2, \cdots,$$

with

Table 1
Abbreviations of the numerical schemes.

• CN-GLS:	Spectral Galerkin method (3.7)-(3.8) with an accuracy of $\mathcal{O}(\tau^2 + N^{(\alpha/2)-r})$.
• SAV-GLS:	Spectral Galerkin method (4.6)-(4.8) with an accuracy of $\mathcal{O}(\tau^2 + N^{(\alpha/2)-r})$.
• ESAV-GLS:	Spectral Galerkin method (5.6)-(5.8) with an accuracy of $\mathcal{O}(\tau^2 + N^{(\alpha/2)-r})$.
• LF-FDM:	Finite difference method [40] with an accuracy of $\mathcal{O}(\tau^2 + N^{-2})$.

$$\widehat{U}^{n+1,0} = \begin{cases} \widehat{U}^0, & n = 0, \\ 2\widehat{U}^n - \widehat{U}^{n-1}, & n \ge 1, \end{cases}$$

where

$$(\mathcal{N}^{n+1,s})_l = \Big((|U_N^n|^2 + |U_N^{n+1,s}|^2)(U_N^n + U_N^{n+1,s}), \phi_l \Big), \quad l = 0, 1, \cdots, N-2.$$

Then $U_N^{n+1,s+1}$ numerically converges to U_N^{n+1} if there satisfies $\left\| U_N^{n+1,s+1} - U_N^{n+1,s} \right\| \le \epsilon$ for the given stopping criterion ϵ [5].

7. Numerical experiments

In this section, we intend to report the numerical results of the CN–GLS method, SAV–GLS method and ESAV–GLS method by using Matlab R2018a software on a computer with *Intel Core i7 and 16 GB RAM*. Without special instructions, we always take the stopping criterion $\epsilon = 10^{-14}$ and the constant $C_0 = 1$.

Compute the L^2 norm error as

$$\left\| u^{N_t} - U_N^{N_t} \right\| = \sqrt{\frac{b-a}{2} \sum_{j=0}^{M} \Big(u(x_j, T) - U_N^{N_t}(x_j) \Big)^2 w_j}, \quad \text{with} \quad x_j = \frac{(b-a)\widehat{x}_j + (a+b)}{2},$$

where $\{\widehat{x}_j\}$ and $\{w_j\}$ are points and weights of the Legendre–Gauss–Lobatto quadrature [41], respectively, and $M = \mu N$ ($\mu > 1, \mu \in \mathbb{N}_+$).

The L^∞ norm error is computed by

$$\left\| u^{N_t} - U_N^{N_t} \right\|_\infty = \max_{0 \le j \le M} \left| u(x_j, T) - U_N^{N_t}(x_j) \right|.$$

The relative energy deviation is defined as

$$\mathrm{RE}^n = \frac{|\mathcal{E}^n - \mathcal{E}^0|}{\mathcal{E}^0}.$$

It is worth noting that, for the case which the exact solution is unknown, we compute the reference "exact solutions" with large $N = 600$ and $\tau = 10^{-5}$. To illustrate the numerical efficiency of the proposed scheme, we present some schemes in Table 1 for comparisons.

Example 7.1 *(Numerical accuracy).* Consider the following FNSW equation with a source term

$$u_{tt} + (-\Delta)^{\frac{\alpha}{2}} u + 2\mathrm{i} u_t + |u|^2 u = f(x,t), \quad (x,t) \in (0,1) \times (0,1].$$

The initial data are determined by the exact solution

$$u(x,t) = \mathrm{i}\exp(-t) x^s (1-x)^s, \quad s \in \mathbb{N}_+.$$

The explicit expression of $f(x,t)$ can be found in Ref. [12].

For this linear example, we apply our proposed spectral Galerkin methods (CN–GLS method, SAV–GLS method and ESAV–GLS method) and the existing finite difference method to solve Example 7.1. For the fixed polynomial degree $N = 400$ and $s = 2$, the L^2 and L^∞ norm errors as well as the corresponding temporal convergence orders are listed in Tables 2–4. It is obvious to find that the compared numerical schemes uniformly possess the second order accuracy in time direction. Taking the temporal step $\tau = 10^{-4}$ and various polynomial degrees N for $s = 2, 4, 6$, the numerical results are shown in Tables 5–7. The spatial observation orders confirm that the spectral accuracy could be observed for the sufficiently large s and small time step τ. Numerical results of Tables 5–7 also illustrate that the spectral Galerkin methods enjoy the more superior numerical accuracy than finite difference method. In the other word, the spectral method can quickly reach the machine accuracy with a small N, but the finite difference method can not achieve.

Table 2
Temporal accuracy of the numerical schemes for Example 7.1 with $N=400$, $s=2$ and $\alpha=1.4$.

Schemes	Errors	Steps			
		$\tau=\frac{1}{10}$	$\tau=\frac{1}{20}$	$\tau=\frac{1}{40}$	$\tau=\frac{1}{80}$
CN–GLS	$\Vert u^{N_t}-U_N^{N_t}\Vert$	5.0687e-05	1.2682e-05	3.1713e-06	7.9287e-07
	Conv.rate	–	1.9988	1.9997	1.9999
	$\Vert u^{N_t}-U_N^{N_t}\Vert_\infty$	6.9832e-05	1.7233e-05	4.2934e-06	1.0724e-06
	Conv.rate	–	2.0187	2.0050	2.0013
SAV–GLS	$\Vert u^{N_t}-U_N^{N_t}\Vert$	5.0655e-05	1.2674e-05	3.1693e-06	7.9237e-07
	Conv.rate	–	1.9988	1.9997	1.9999
	$\Vert u^{N_t}-U_N^{N_t}\Vert_\infty$	6.9793e-05	1.7224e-05	4.2909e-06	1.0718e-06
	Conv.rate	–	2.0187	2.0050	2.0013
ESAV–GLS	$\Vert u^{N_t}-U_N^{N_t}\Vert$	5.0512e-05	1.2633e-05	3.1581e-06	7.8947e-07
	Conv.rate	–	1.9995	2.0000	2.0001
	$\Vert u^{N_t}-U_N^{N_t}\Vert_\infty$	6.9599e-05	1.7171e-05	4.2771e-06	1.0682e-06
	Conv.rate	–	2.0191	2.0053	2.0014
LF–FDM [40]	$\Vert u^{N_t}-U_N^{N_t}\Vert$	6.9750e-05	1.9606e-05	5.2819e-06	1.4715e-06
	Conv.rate	–	1.8309	1.8922	1.8438
	$\Vert u^{N_t}-U_N^{N_t}\Vert_\infty$	1.0195e-04	2.7380e-05	7.1959e-06	2.0072e-06
	Conv.rate	–	1.8967	1.9279	1.8420

Table 3
Temporal accuracy of the numerical schemes for Example 7.1 with $N=400$, $s=2$ and $\alpha=1.6$.

Schemes	Errors	Steps			
		$\tau=\frac{1}{10}$	$\tau=\frac{1}{20}$	$\tau=\frac{1}{40}$	$\tau=\frac{1}{80}$
CN–GLS	$\Vert u^{N_t}-U_N^{N_t}\Vert$	5.4240e-05	1.3554e-05	3.3877e-06	8.4689e-07
	Conv.rate	–	2.0007	2.0003	2.0001
	$\Vert u^{N_t}-U_N^{N_t}\Vert_\infty$	6.6599e-05	1.7028e-05	4.2651e-06	1.0653e-06
	Conv.rate	–	1.9676	1.9972	2.0013
SAV–GLS	$\Vert u^{N_t}-U_N^{N_t}\Vert$	5.4213e-05	1.3547e-05	3.3861e-06	8.4646e-07
	Conv.rate	–	2.0007	2.0003	2.0001
	$\Vert u^{N_t}-U_N^{N_t}\Vert_\infty$	6.6568e-05	1.7020e-05	4.2631e-06	1.0648e-06
	Conv.rate	–	1.9676	1.9973	2.0013
ESAV–GLS	$\Vert u^{N_t}-U_N^{N_t}\Vert$	5.4084e-05	1.3510e-05	3.3764e-06	8.4399e-07
	Conv.rate	–	2.0011	2.0005	2.0002
	$\Vert u^{N_t}-U_N^{N_t}\Vert_\infty$	6.6416e-05	1.6977e-05	4.2517e-06	1.0619e-06
	Conv.rate	–	1.9680	1.9975	2.0014
LF–FDM [40]	$\Vert u^{N_t}-U_N^{N_t}\Vert$	5.7982e-05	1.6326e-05	4.497e-06	1.3755e-06
	Conv.rate	–	1.8284	1.8601	1.7090
	$\Vert u^{N_t}-U_N^{N_t}\Vert_\infty$	8.6524e-05	2.2784e-05	5.8879e-06	1.8948e-06
	Conv.rate	–	1.9251	1.9522	1.6357

Example 7.2. We consider the FNSW equation (1.1) in domain $(-20,20)\times(0,T]$, the initial data are given as follows

$$u(x,0)=(1+\mathrm{i})x\exp\big(-10(1-x)^2\big),\quad u_t(x,0)=0,\quad x\in(-20,20).$$

In this simulation, we carry out utilizing our proposed spectral methods for Example 7.2 with $\kappa=\beta=1$. Taking the parameters $N=400$, $\tau=0.1$ and $T=10$, the profiles of the numerical solutions of the CN–GLS method are plotted in Fig. 1, which are in great agreement with the existing references [31,40]. For the fixed polynomial degree $N=400$, the CPU time costs of the CN–GLS method, the SAV–GLS method and the ESAV–GLS method with various time steps τ ($\tau=1/20, 1/40, 1/80, 1/160, 1/320$) at final time $T=1$ are shown in Fig. 2 and Fig. 3. The numerical results verify that the linearly implicit ESAV–GLS method enjoys a more superior computation efficiency than the CN–GLS method and the SAV–GLS method, also, the CN–GLS method performs more effectively than the SAV–GLS method. For testing the conservative properties of the proposed numerical schemes, we apply our spectral methods and the finite difference method to solve Example 7.2 with $N=100$, $\tau=0.1$ and various fractional orders for comparisons, and the time evolution of relative energy deviations are plotted in Fig. 4. The relative energy errors uniformly reach the machine accuracy verifying that our proposed

Table 4
Temporal accuracy of the numerical schemes for Example 7.1 with $N = 400$, $s = 2$ and $\alpha = 1.8$.

Schemes	Errors	Steps			
		$\tau = \frac{1}{10}$	$\tau = \frac{1}{20}$	$\tau = \frac{1}{40}$	$\tau = \frac{1}{80}$
CN–GLS	$\|u^{N_t} - U_N^{N_t}\|$	5.4681e-05	1.3581e-05	3.3901e-06	8.472e-07
	Conv.rate	–	2.0095	2.0022	2.0005
	$\|u^{N_t} - U_N^{N_t}\|_\infty$	7.2078e-05	1.9661e-05	4.9326e-06	1.2304e-06
	Conv.rate	–	1.8742	1.9949	2.0032
SAV–GLS	$\|u^{N_t} - U_N^{N_t}\|$	5.4660e-05	1.3575e-05	3.3888e-06	8.4688e-07
	Conv.rate	–	2.0095	2.0022	2.0005
	$\|u^{N_t} - U_N^{N_t}\|_\infty$	7.2051e-05	1.9654e-05	4.9308e-06	1.2300e-06
	Conv.rate	–	1.8742	1.9949	2.0032
ESAV–GLS	$\|u^{N_t} - U_N^{N_t}\|$	5.4554e-05	1.3547e-05	3.3814e-06	8.4502e-07
	Conv.rate	–	2.0097	2.0023	2.0006
	$\|u^{N_t} - U_N^{N_t}\|_\infty$	7.1916e-05	1.9616e-05	4.9208e-06	1.2274e-06
	Conv.rate	–	1.8743	1.9951	2.0033
LF–FDM [40]	$\|u^{N_t} - U_N^{N_t}\|$	4.6025e-05	1.2717e-05	3.6312e-06	1.3500e-06
	Conv.rate	–	1.8557	1.8083	1.4275
	$\|u^{N_t} - U_N^{N_t}\|_\infty$	5.9557e-05	1.6485e-05	5.0681e-06	2.0064e-06
	Conv.rate	–	1.8531	1.7016	1.3368

Table 5
Spatial accuracy of the numerical schemes for Example 7.1 with $\tau = 10^{-4}$ and $s = 2$.

α	N	CN–GLS				LF–FDM [40]			
		$\|u^{N_t} - U_N^{N_t}\|$	Conv.rate	$\|u^{N_t} - U_N^{N_t}\|_\infty$	Conv.rate	$\|u^{N_t} - U_N^{N_t}\|$	Conv.rate	$\|u^{N_t} - U_N^{N_t}\|_\infty$	Conv.rate
1.4	2^3	7.8251e-07	–	1.4387e-06	–	1.0625e-03	–	1.5240e-03	–
	2^4	3.2646e-08	$N^{-4.5831}$	9.5681e-08	$N^{-3.9103}$	2.3044e-04	$N^{-2.2049}$	3.4017e-04	$N^{-2.1636}$
	2^5	8.8670e-10	$N^{-5.2023}$	2.9528e-09	$N^{-5.0181}$	5.0747e-05	$N^{-2.1830}$	7.7553e-05	$N^{-2.1330}$
	2^6	6.6681e-11	$N^{-3.7331}$	1.8638e-10	$N^{-3.9858}$	1.1409e-05	$N^{-2.1531}$	1.7949e-05	$N^{-2.1113}$
1.6	2^3	9.0825e-07	–	1.5756e-06	–	1.6408e-03	–	2.2045e-03	–
	2^4	4.5634e-08	$N^{-4.3149}$	9.5053e-08	$N^{-4.0510}$	3.7188e-04	$N^{-2.1415}$	4.6951e-04	$N^{-2.2312}$
	2^5	1.7700e-09	$N^{-4.6883}$	4.3493e-09	$N^{-4.4499}$	8.4706e-05	$N^{-2.1343}$	1.1534e-04	$N^{-2.0253}$
	2^6	4.0768e-11	$N^{-5.4402}$	2.5453e-10	$N^{-4.0949}$	1.9453e-05	$N^{-2.1225}$	2.8469e-05	$N^{-2.0184}$
1.8	2^3	8.8544e-07	–	1.1569e-06	–	2.3621e-03	–	3.2931e-03	–
	2^4	5.4509e-08	$N^{-4.0218}$	8.3552e-08	$N^{-3.7914}$	5.6108e-04	$N^{-2.0738}$	7.9435e-04	$N^{-2.0516}$
	2^5	2.5355e-09	$N^{-4.4261}$	4.7136e-09	$N^{-4.1478}$	1.3312e-04	$N^{-2.0755}$	1.9501e-04	$N^{-2.0262}$
	2^6	2.2045e-10	$N^{-3.5238}$	3.6016e-10	$N^{-3.7101}$	3.1589e-05	$N^{-2.0752}$	4.6206e-05	$N^{-2.0774}$

Table 6
Spatial accuracy of the numerical schemes for Example 7.1 with $\tau = 10^{-4}$ and $s = 4$.

α	N	CN–GLS				LF–FDM [40]			
		$\|u^{N_t} - U_N^{N_t}\|$	Conv.rate	$\|u^{N_t} - U_N^{N_t}\|_\infty$	Conv.rate	$\|u^{N_t} - U_N^{N_t}\|$	Conv.rate	$\|u^{N_t} - U_N^{N_t}\|_\infty$	Conv.rate
1.4	2^3	2.6210e-10	–	5.2541e-10	–	3.4780e-05	–	5.3906e-05	–
	2^4	3.5097e-12	$N^{-6.2227}$	4.5208e-12	$N^{-6.8607}$	8.4337e-06	$N^{-2.0440}$	1.2631e-05	$N^{-2.0935}$
1.6	2^3	2.3398e-10	–	3.7366e-10	–	6.4787e-05	–	9.0932e-05	–
	2^4	1.2636e-12	$N^{-7.5327}$	2.1298e-12	$N^{-7.4549}$	1.5777e-05	$N^{-2.0379}$	2.4341e-05	$N^{-1.9014}$
1.8	2^3	2.0096e-10	–	2.6384e-10	–	3.4745e-05	–	4.8634e-05	–
	2^4	6.5536e-12	$N^{-4.9385}$	8.9557e-12	$N^{-4.8807}$	6.7183e-06	$N^{-2.3706}$	8.6281e-06	$N^{-2.4949}$

numerical schemes preserve the energy well. This fact exactly confirms the correctness of Theorem 3.1, Theorem 4.1 and Theorem 5.1.

8. Conclusion

In this work, we have presented three Galerkin–Legendre spectral methods, involving the CN–GLS method, the SAV–GLS method and the ESAV–GLS method, for solving the space-fractional nonlinear Schrödinger equation with wave operator.

Table 7
Spatial accuracy of the numerical schemes for Example 7.1 with $\tau = 10^{-4}$ and $s = 6$.

α	N	CN–GLS				LF–FDM [40]			
		$\|u^{N_t} - U_N^{N_t}\|$	Conv.rate	$\|u^{N_t} - U_N^{N_t}\|_\infty$	Conv.rate	$\|u^{N_t} - U_N^{N_t}\|$	Conv.rate	$\|u^{N_t} - U_N^{N_t}\|_\infty$	Conv.rate
1.4	2^3	6.0943e-07	–	8.6411e-07	–	3.4293e-06	–	5.1708e-06	–
	2^4	9.6911e-14	$N^{-22.5840}$	1.4380e-13	$N^{-22.5190}$	7.2950e-07	$N^{-2.2329}$	1.1791e-06	$N^{-2.1328}$
1.6	2^3	5.1591e-07	–	1.0640e-06	–	4.7316e-06	–	7.8430e-06	–
	2^4	1.4686e-13	$N^{-21.7440}$	2.2672e-13	$N^{-22.1620}$	1.3482e-06	$N^{-1.8113}$	2.4203e-06	$N^{-1.6962}$
1.8	2^3	4.8355e-07	–	7.1443e-07	–	4.1374e-06	–	6.8622e-06	–
	2^4	2.5897e-13	$N^{-20.8320}$	3.4652e-13	$N^{-20.9750}$	6.0925e-07	$N^{-2.7636}$	9.9311e-07	$N^{-2.7886}$

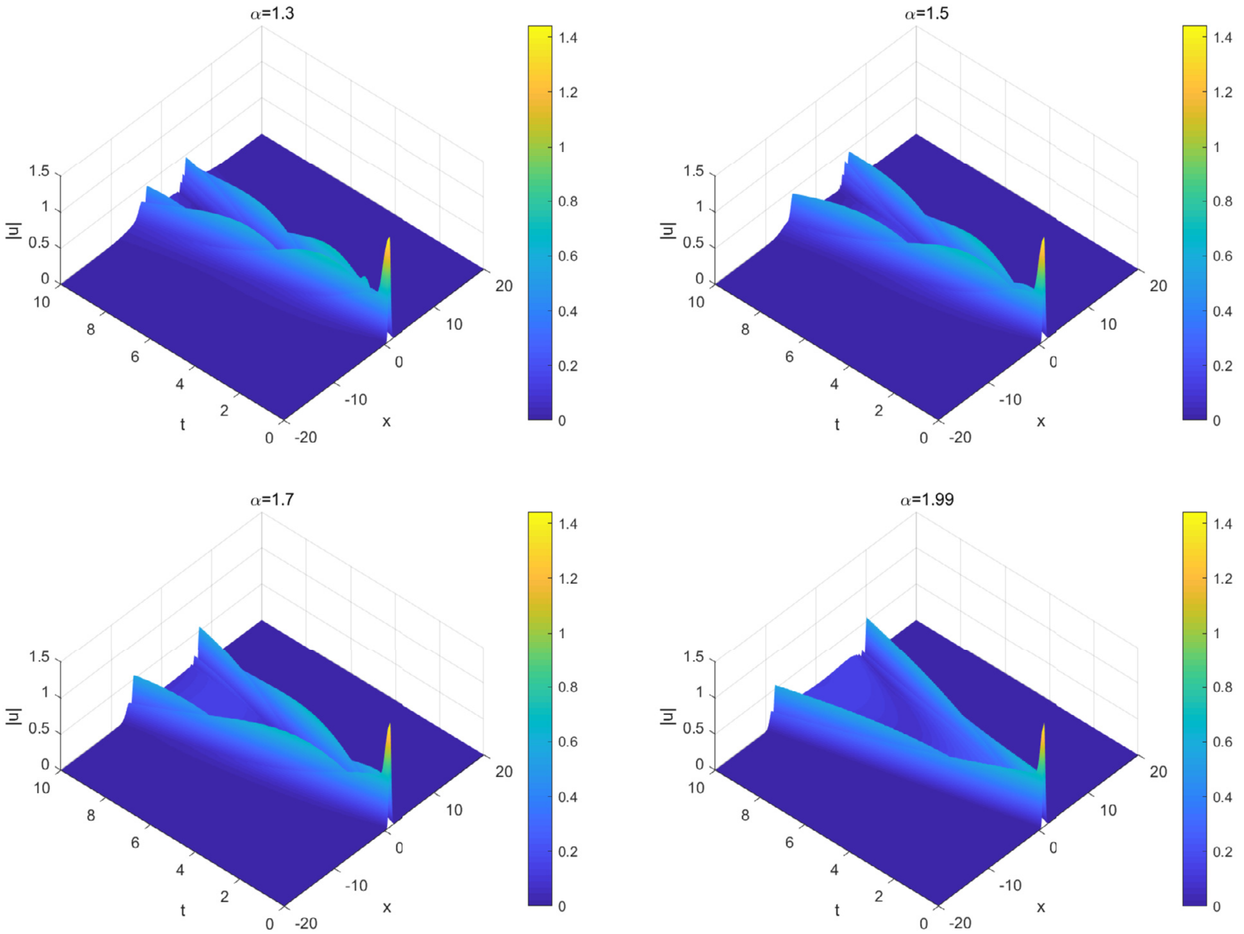

Fig. 1. Time evolution of numerical solutions for Example 7.2 with $\beta = 1$ and various α. (For interpretation of the colors in the figures, the reader is referred to the web version of this article.)

All the methods preserve the corresponding energy as continuous models. Moreover, we take the CN–GLS method as an example to illustrate that the proposed methods are unconditionally convergent and stable in L^∞ norm. Extensive numerical comparisons show that the theoretical analyses are reasonable and correct. In theoretical aspect, the CN–GLS method and the SAV–GLS method both enjoy the L^∞ norm boundness, but the ESAV–GLS method does not. This fact offers some possibilities for the CN–GLS method and the SAV–GLS method to analyze the unconditional convergence. In simulations, the linearly implicit ESAV–GLS method enjoys a more superior computation efficiency than the CN–GLS method and the SAV–GLS method. In recent years, there are some conservative schemes for the two-dimensional classical Schrödinger-type equations that have also drawn much attention [17,19,33,49]. In future studies, we try to extend our proposed spectral Galerkin methods and the corresponding theoretical analyses to the two-dimensional fractional models.

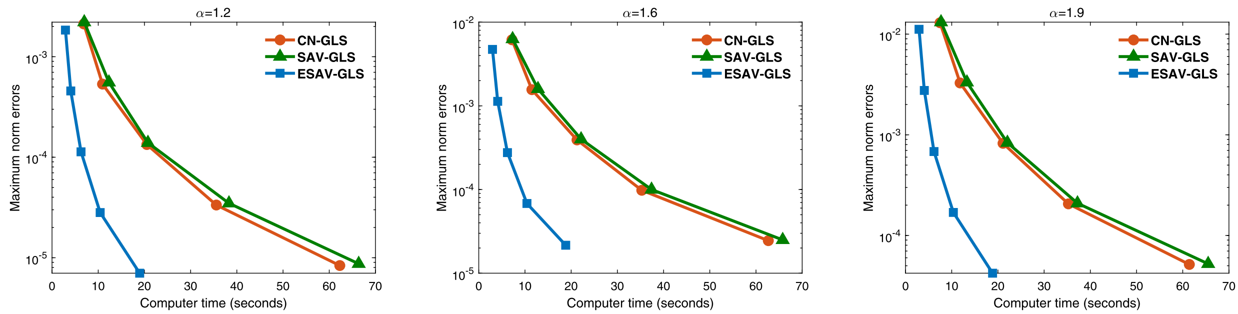

Fig. 2. Maximum norm errors and the corresponding CPU time costs for Example 7.2 with $\beta = 1$.

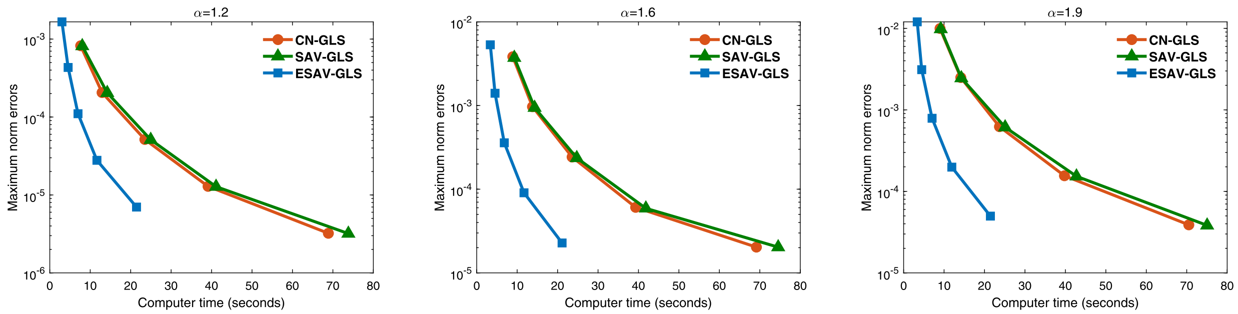

Fig. 3. Maximum norm errors and the corresponding CPU time costs for Example 7.2 with $\beta = -1$.

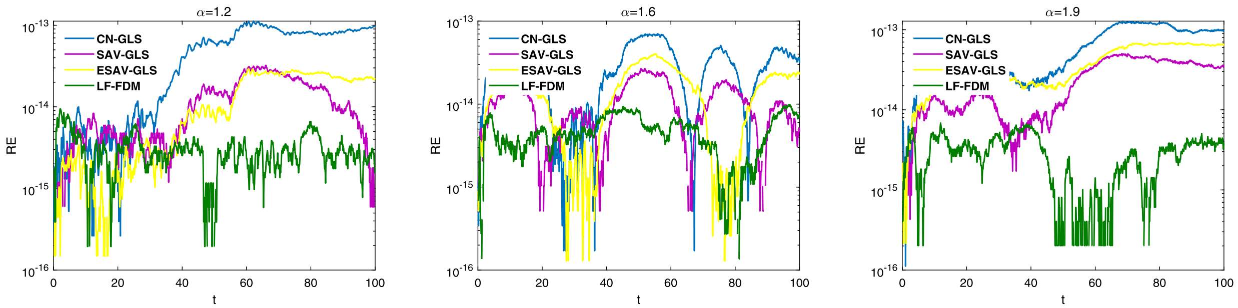

Fig. 4. Time evolution of the relative energy deviation for Example 7.2 with $\beta = 1$.

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Appendix A. Proof of Theorem 3.3

The proof of Theorem 3.3 is divided into two parts, including the existence and uniqueness.

(I) **Existence**:

Proof. Noticing that $U_N^{n+1} = 2U_N^{n+1/2} - U_N^n$ and $\delta_t U_N^{n+1/2} = V_N^{n+1/2}$. We reformulate the spectral Galerkin scheme (3.7)–(3.8) into the following form

$$\begin{aligned}\frac{\kappa\tau}{2}(U_N^{n+1/2} - U_N^n, w_N) - \mathrm{i}(U_N^{n+1/2} - U_N^n - \frac{\tau}{2}V_N^n, w_N) - \frac{\tau^2}{4}\mathrm{i}\mathcal{A}(U_N^{n+\frac{1}{2}}, w_N)\\ - \frac{\beta\tau^2}{8}\mathrm{i}\Big((|U_N^n|^2 + |2U_N^{n+1/2} - U_N^n|^2)U_N^{n+\frac{1}{2}}, w_N\Big) = 0, \quad \forall w_N \in X_N^0(\Omega).\end{aligned} \tag{A.1}$$

We carry out proving the existence of the numerical solution $U_N^{n+1/2}$. For convenience, we define the map $\mathscr{F}\colon X_N^0(\Omega)\to X_N^0(\Omega)$, such that

$$(\mathscr{F}(\Theta), w_N) = \frac{\kappa\tau}{2}(\Theta - U_N^n, w_N) - \mathrm{i}(\Theta - U_N^n - \frac{\tau}{2}V_N^n, w_N) - \frac{\tau^2}{4}\mathrm{i}\mathcal{A}(\Theta, w_N) \\ - \frac{\beta\tau^2}{8}\mathrm{i}\Big((|U_N^n|^2 + |2\Theta - U_N^n|^2)\Theta, w_N\Big), \quad \forall w_N \in X_N^0(\Omega). \tag{A.2}$$

Setting $w_N = \Theta$ in (A.2) and taking the real part, we have

$$\mathrm{Re}(\mathscr{F}(\Theta), \Theta) = \frac{\kappa\tau}{2}\left\|\Theta\right\|^2 - \frac{\kappa\tau}{2}\mathrm{Re}(U_N^n, \Theta) - \mathrm{Re}(U_N^n + \frac{\tau}{2}V_N^n, \Theta).$$

By employing the Young's inequality, we obtain

$$\frac{\kappa\tau}{2}\mathrm{Re}(U_N^n, \Theta) \le \frac{\kappa\tau}{2}\left\|U_N^n\right\|^2 + \frac{\kappa\tau}{8}\left\|\Theta\right\|^2,$$

and

$$\mathrm{Re}(U_N^n + \frac{\tau}{2}V_N^n, \Theta) \le \left\|U_N^n + \frac{\tau}{2}V_N^n\right\|\left\|\Theta\right\| \le \frac{2}{\kappa\tau}\left\|U_N^n + \frac{\tau}{2}V_N^n\right\|^2 + \frac{\kappa\tau}{8}\left\|\Theta\right\|^2.$$

Hence, one has

$$\begin{aligned}\mathrm{Re}(\mathscr{F}(\Theta), \Theta) &\ge \frac{\kappa\tau}{4}\left\|\Theta\right\|^2 - \frac{\kappa\tau}{2}\left\|U_N^n\right\|^2 - \frac{2}{\kappa\tau}\left\|U_N^n + \frac{\tau}{2}V_N^n\right\|^2 \\ &= \frac{\kappa\tau}{4}\Big(\left\|\Theta\right\|^2 - 2\left\|U_N^n\right\|^2 - \frac{8}{\kappa^2\tau^2}\left\|U_N^n + \frac{\tau}{2}V_N^n\right\|^2\Big) \\ &\ge \frac{\kappa\tau}{4}\Big(\left\|\Theta\right\|^2 - \left\|\sqrt{2}|U_N^n| + \frac{2\sqrt{2}}{\kappa\tau}|U_N^n + \frac{\tau}{2}V_N^n|\right\|^2\Big),\end{aligned}$$

where $|a|^2 + |b|^2 \le (|a| + |b|)^2$ is used in the last inequality. Taking $\delta = \left\|\sqrt{2}|U_N^n| + \frac{2\sqrt{2}}{\kappa\tau}|U_N^n + \frac{\tau}{2}V_N^n|\right\|$, which satisfies the condition of Lemma 3.4, we derive

$$\mathrm{Re}(\mathscr{F}(\Theta), \Theta) \ge 0, \quad \forall\Theta: \ \|\Theta\| = \delta.$$

Therefore, we prove that the solution of (3.7)–(3.8) exists. This completes the proof of existence. □

(II) **Uniqueness**:

Proof. The uniqueness is proven by induction. Notice that the (U_N^0, V_N^0) exists and is unique. Suppose that (U_N^n, V_N^n) for $n = 0, 1, \cdots, N_t - 1$. Assume that there exist two numerical solutions $X^{n+1/2}$ and $Y^{n+1/2}$ for (3.7)–(3.8), which satisfy (A.1)

$$\frac{\kappa\tau}{2}(X^{n+1/2} - U_N^n, w_N) - \mathrm{i}(X^{n+1/2} - U_N^n - \frac{\tau}{2}V_N^n, w_N) - \frac{\tau^2}{4}\mathrm{i}\mathcal{A}(X^{n+1/2}, w_N) \\ - \frac{\beta\tau^2}{8}\mathrm{i}\Big((|U_N^n|^2 + |2X^{n+1/2} - U_N^n|^2)X^{n+1/2}, w_N\Big) = 0, \tag{A.3}$$

$$\frac{\kappa\tau}{2}(Y^{n+1/2} - U_N^n, w_N) - \mathrm{i}(Y^{n+1/2} - U_N^n - \frac{\tau}{2}V_N^n, w_N) - \frac{\tau^2}{4}\mathrm{i}\mathcal{A}(Y^{n+1/2}, w_N) \\ - \frac{\beta\tau^2}{8}\mathrm{i}\Big((|U_N^n|^2 + |2Y^{n+1/2} - U_N^n|^2)Y^{n+1/2}, w_N\Big) = 0. \tag{A.4}$$

Subtracting (A.4) from (A.3), we have

$$\frac{\kappa\tau}{2}(X^{n+1/2} - Y^{n+1/2}, w_N) - \mathrm{i}(X^{n+1/2} - Y^{n+1/2}, w_N) - \frac{\tau^2}{4}\mathrm{i}\mathcal{A}(X^{n+1/2} - Y^{n+1/2}, w_N) \\ - \frac{\beta\tau^2}{8}\mathrm{i}\Big((|U_N^n|^2 + |2X^{n+1/2} - U_N^n|^2)X^{n+1/2} - (|U_N^n|^2 + |2Y^{n+1/2} - U_N^n|^2)Y^{n+1/2}, w_N\Big) = 0. \tag{A.5}$$

Setting $w_N = X^{n+1/2} - Y^{n+1/2}$ and taking the real part, by the Cauchy-Schwarz inequality and Lemma 3.5, we obtain

$$\begin{aligned}\frac{\kappa\tau}{2}\left\|X^{n+1/2}-Y^{n+1/2}\right\|^2 &= -\frac{\beta\tau^2}{8}\mathrm{Im}\Big((|U_N^n|^2+|2X^{n+1/2}-U_N^n|^2)X^{n+1/2}\\ &\quad -(|U_N^n|^2+|2Y^{n+1/2}-U_N^n|^2)Y^{n+1/2}, X^{n+1/2}-Y^{n+1/2}\Big)\\ &\le \mathcal{C}\tau^2\left\|X^{n+1/2}-Y^{n+1/2}\right\|^2.\end{aligned} \tag{A.6}$$

Therefore, we have

$$\left\|X^{n+1/2}-Y^{n+1/2}\right\|^2 \le \frac{2\mathcal{C}\tau}{\kappa}\left\|X^{n+1/2}-Y^{n+1/2}\right\|^2. \tag{A.7}$$

By assuming $\frac{2\mathcal{C}\tau}{\kappa}<1$, it leads to $\left\|X^{n+1/2}-Y^{n+1/2}\right\|=0$, which implies $X^{n+1/2}=Y^{n+1/2}$. This ends the proof of uniqueness. □

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