Value-at-Risk Using Delta-Normal Approximation

The code uses the Black-Scholes model to calculate the price of a European call option and applies the Delta Normal Approximation to estimate the VAR for a given position. The simulation incorporates **Importance Sampling**, a technique used to shift the probability distribution of the asset's returns, focusing on the tails of the distribution to estimate the probability of extreme events (such as large VAR breaks).

Case Scenario

Let us consider the problem of calculating the **5-day Value at Risk (VaR)** for a **long European call option** using the **Delta-Normal approximation**. The asset's price evolves over time according to the **Black-Scholes model**, where the asset's price follows a geometric Brownian motion under the risk-neutral measure $\mathbb{Q}$,

The stochastic differential equation (SDE) for the asset price S_t^1 is:

$$dS_t^1 \; = \; \mu \, S_t^1 \, dt \; + \; \sigma \, S_t^1 \, dB_t.$$

With:

- μ: drift term of the underlying asset,
- σ_i: volatility of the asset,
- dB_t: increment of a Brownian motion under the risk-neutral measure.

The bank account follows $dS_t^0 \; = \; r \, S_t^0 \, dt \; ; S_0^0 = 1$

Now, the European call option expiring at time T with strike price K on this forward contract satisfies the following partial differential equation (PDE):

$$-rC(t,F) + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 C(t,F)}{\partial F^2} = \frac{\partial C(t,F)}{\partial t}$$

where F is the forward price at time t, and the terminal condition is given by:

$$C(T, F(T,T)) \; = \; (F(T,T) - K)^+$$

The goal of our project is to estimate the **Value at Risk (VaR)** for a **long position** in this European call option over a 5-day period.

Project Highlights

1. Black-Scholes Formula for Call Option Price

The **Black-Scholes model** provides the price of a European call option as:

$$C(t, St) = St \cdot N(d1) - K \cdot e^{-r(T-t)} \cdot N(d2)$$

2. Delta of the Call Option

The **Delta** of the option is the Sensitivity of the option to changes in the asset price, it can be calculated using Monte Carlo simulations. For a call option:

$$\Delta = \frac{\partial}{\partial S_0^1} E^{\mathbb{Q}}[e^{-rT} * max(0, S_t^1 - k)] = e^{-rT} * E^{\mathbb{Q}}\left[1_{\{S_t^1 > K\}} \frac{S_T^1}{S_0^1}\right]$$

In the **Delta-Normal approximation** for VaR, the change in the value of the financial instrument (here the European call option) is approximated by the change in the underlying asset price:

$$\Delta \approx V_0(S_0)\Delta S$$

Where:

- $V_0(S_0)$ is the initial value of the option,
- Δs is the change in the underlying asset price.

Since S (the asset price) is assumed to follow a normal distribution $N(0, \sigma^2)$, the change in the value of the option V will also follow a normal distribution with mean 0 and variance $\sigma^2 \Delta S^2$:

$$V \sim N(0, \Delta S^2)$$

Thus, the VaR based on the Delta-Normal approximation can be calculated as:

$$VaR = -Z_\alpha * \Delta * \sigma\sqrt{\Delta t}$$

Code

```
# Delta Normal Approximation to VAR for a Long Option Position using Importan
ce Sampling


# Parameters

s0 <- 100

K <- 100

r <- 0.01

sigma <- 0.5

T <- 1

Dt <- 5/365  # Calculating 5-day VAR


# Function to calculate delta_t

delta_t <- function(s0, K, r, sigma, T, t) {

  d1 <- (log(s0/K) + (r + 0.5*sigma^2)*(T - t)) / (sigma * sqrt(T - t))
```

```r
  d2 <- d1 - sigma * sqrt(T - t)
  N1 <- pnorm(d1)
  N2 <- pnorm(d2)
  Delta_t = s0 * dnorm(d1) / (sigma * sqrt(T - t)) + s0 * N1 - K * exp(-r * (
T - t)) * dnorm(d2) / (sigma * sqrt(T - t))
  return(Delta_t)
}

# Calculating delta_t at t = 0
t = 0
Delta_t = delta_t(s0, K, r, sigma, T, t)

# VAR for a long position in a European call option under Black-Scholes Model
alpha = 0.05
z_alpha = qnorm(1 - alpha)

VAR_L = -z_alpha * sigma * sqrt(Dt) * Delta_t

# Black-Scholes formula to calculate option prices
BScall <- function(t, T, S, K, r, sigma) {
  d1 <- (log(S / K) + (r + 0.5 * sigma^2) * (T - t)) / (sigma * sqrt(T - t))
  d2 <- d1 - sigma * sqrt(T - t)
  N1 <- pnorm(d1)
  N2 <- pnorm(d2)
  y <- S * N1 - K * exp(-r * (T - t)) * N2
  return(y)
}

# Importance Sampling: Change of measure
# Modify the drift of the process to focus on the tail (Increase volatility t
o capture extreme movements)

# For importance sampling, we'll shift the drift upwards so that extreme move
s are more probable
shift_factor = 2  # This can be adjusted based on your needs
```

```r
mu_is <- r - 0.5 * sigma^2 + shift_factor * sigma  # Adjusted drift for impor
tance sampling


# Simulate the number of VAR breaks using Importance Sampling
N <- 10^6
Breaks <- 0

set.seed(461)

for (i in 1:N) {
  # Simulating using the importance sampling distribution
  S_is <- s0 * exp((mu_is - 0.5 * sigma^2) * Dt + sigma * sqrt(Dt) * rnorm(1)
)  # Importance sampling path
  V0 <- BScall(0, T, s0, K, r, sigma)
  V <- BScall(Dt, T, S_is, K, r, sigma)
  dV <- V - V0

  # Re-weighting the probability based on the ratio of the original and the m
odified distribution
  weight <- exp(-(mu_is - r) * Dt)  # Importance sampling weight
  if (dV < VAR_L) {
    Breaks <- Breaks + weight
  }
}

# Estimating the alpha (probability of VAR break)
alpha_est_is <- Breaks / N

cat("5-days VaR with the Delta-Normal Approximation:", VAR_L, "\n")
```

```
## 5-days VaR with the Delta-Normal Approximation: -5.837263
```

```r
# Output the estimated alpha from importance sampling
cat("Estimated probability of a VAR break using Importance Sampling:", alpha_
est_is, "\n")
```

```
## Estimated probability of a VAR break using Importance Sampling: 0.02256787
```

Results

In the code provided above, we observe that The 5-day VaR calculated using the Delta-Normal Approximation is -5.837263 , which indicates that our 95% VaR should be expected to be exceeded 5% of the time, meaning that -5.837263 is the minimum loss incurred in the 5% worst scenarios

Furthermore, using **Importance Sampling**, we estimated the probability of a **VAR break** to be 0.02256787 meaning there was a 2.26% overall percentage of our VaR exceeding our found value in the simulated runs, highlighting the importance of considering alternative risk measures and tail risk.

Project improvement

- Real-Time Data Integration: To enhance the accuracy of VaR estimates, we can integrate live market data using sources like Yahoo Finance (yfinance). This would allow us to update the calculations dynamically, reflecting real-time market conditions.
- Model Refinements: While the Delta-Normal Approximation is useful, it assumes that the underlying asset returns follow a normal distribution, which may not capture extreme market events. Exploring non-parametric methods or stochastic models (such as GARCH or Monte Carlo simulations) could improve the robustness of the risk assessment.
- Dynamic Risk Management: Considering the use of dynamic hedging strategies or risk limits based on real-time VaR calculations could further optimize portfolio management and reduce the likelihood of large, unexpected losses.