Advanced Probability tools in modern quantitative financial analysis

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Acknowledgments

I would like to begin this project by respectfully acknowledging that Concordia University is located on unceded Indigenous lands whose custodians are eh Kanien"keh'a:ka Nation.

This project was further inspired by course materials from MAST 671B and MAST 729 (Mathematical & Computational Finance II taught by Professor Cody Hyndman), as well as classical references in option pricing and modern research on stochastic differential equations. I would also like to acknowledge the occasional use of Grammarly to assist in correcting minor grammatical errors throughout the preparation of this report, and scientific computing through libraries in Python for numerical experiments.

Finally, although the numerical experiments in this report are based on synthetic data, they are calibrated to reflect realistic expectations of real financial data. The aim of this project is to work in a controlled environment while remaining faithful to the constraints and behaviours that would be observed in real markets.

Motivation

The central theme of this project is the use of advanced probabilistic tools in modern quantitative finance. To be precise, the practical starting point is a concrete quantitative problem: how to compute the arbitrage-free price of an arithmetic Asian put option when no closed-form solution is available. Asian options are widely used in practice because their payoff depends on an average of the underlying price over time, mitigating sensitivity to short-lived spikes; while geometric Asian options admit closed-form solutions under GBM, the more commonly used arithmetic options do not, requiring numerical methods based on advanced probability

In practice, quantitative finance relies on models for asset prices, interest rates, and risk factors that are formulated in continuous time and driven by stochastic processes. At the heart of these models lies the concept of stochastic differential equations (SDEs) [1]. In their simplest form, an SDE describes a stochastic process X_t driven by Brownian motion, written as

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dB_t \text{ [1]}$$

Where B_t is a Brownian motion, μ is the drift, and σ is the diffusion coefficient. This framework combines randomness (through Brownian motion) with analytic structure (through Ito calculus), forming the probabilistic language in which many asset pricing models are expressed.

A classical example is the modeling of a stock price S_t as a geometric Brownian motion (GBM) [2]. In a simple SDE form, under the real-world (or physical) probability measure, one writes:

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dB_t \text{ [1]}$$

With μ being the drift and $\sigma > 0$ the volatility. Under suitable assumptions and a no-arbitrage argument, one can change probability measure to a risk-neutral measure under which the discounted price process becomes a martingale [2]. In that case, the dynamics become:

$$dS_t = r\, S_t\, dt + \sigma\, S_t\, dW_t^{\mathbb{Q}}$$

with r the risk-free rate and $W_t^{\mathbb{Q}}$ a Brownian motion under the risk-neutral measure. Central to both arbitrage-free pricing and this project, the change of measure and the martingale property of discounted prices provide the essential probabilistic framework for our analysis

One of the most striking consequences of this framework dates to the work of Black, Scholes in the 1970s on option pricing theory [3]. This work gave a closed-form expression for the price of a European option with Maturity T and strike K. In the Black-Scholes model, the time 0 price C_0 of a call option is

$$C_0 = S_0\, \Phi(d_1) \; - \; Ke^{-rT}\, \Phi(d_2)$$

Where $d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$. and Φ denotes the cumulative distribution function of the standard normal distribution. This formula exemplifies the power of the continuous-time probabilistic approach when the payoff depends only on the terminal value S_T and the underlying dynamics are sufficiently simple.

Asian options expose both the strength and the limitations of this classical result. For geometric averaging, the distributional structure is tractable enough that a closed-form pricing formula can be derived under GBM [4]. For arithmetic averaging, which is common in applications, no simple analogue of the Black–Scholes or geometric-Asian formulas is available. The "gap" is therefore clear: the standard closed-form theory can handle only a special subclass of Asian options, while the practically important arithmetic case requires us to return to the definition of the arbitrage-free price as a risk-neutral expectation and approximate that expectation numerically.

This is where our approach comes in. The concrete objective of the project is to price an arithmetic Asian put option under a GBM model using Monte Carlo methods on synthetic but realistic data. This entails: (i) formulating the SDE model and its risk-neutral version, (ii) expressing the Asian option price as an expectation under $\mathbb{Q}$, (iii) constructing and implementing a Monte Carlo estimator based on simulated paths of S_t, and (iv) analyzing the statistical properties of this estimator, including the impact of variance reduction techniques. The problem is nontrivial because one must simultaneously control discretization error from simulating the SDE, statistical error from finite samples, and the additional complexity introduced by path dependence.

Within this framework, the project makes two principal contributions. First, it presents a fully explicit derivation linking core probabilistic concepts such as Brownian motion, SDEs, martingales, and changes of measure, to a practical numerical algorithm for pricing a path-dependent option without a closed-form solution. Second, it provides a quantitative evaluation of Monte Carlo methods in this setting, demonstrating how variance reduction can significantly enhance efficiency even under a simple GBM model. The focus is deliberately on classical probabilistic tools, emphasizing their central role rather than model complexity.

The remainder of the report is organized around the probabilistic building blocks that enable this algorithm. Using the arithmetic Asian put as a test case, the project shows that advanced probability is not abstract formalism: it translates directly into implementable algorithms for realistic options that closed-form Black–Scholes formulas cannot handle.

Brownian motion

Modern continuous-time models in quantitative finance are built on Brownian motion. Historically, the term comes from the irregular motion of microscopic particles suspended in a fluid, first systematically observed by the botanist Robert Brown in 1820s [5]. This physical phenomenon was later given a precise mathematical formulation as a continuous-time stochastic process with independent Gaussian increments. In probability theory, Brownian motion has become a canonical example of a martingale with continuous paths and is the basic building block for stochastic calculus, stochastic differential equations, and arbitrage-free pricing.

Probability space, filtration, and definition

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A filtration $\{\mathcal{F}_t\}_{t\geq 0}$is a non-decreasing family of σ-algebras satisfying $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ whenever $s \leq t$, and is interpreted as the information available up to time t.

A stochastic process $\{B_t\}_{t\geq 0}$ with values in $\mathbb{R}^d$ is called a d-dimensional standard Brownian motion (starting at the origin) if:

1. if $0 \leq \mathrm{t}_1 \leq \mathrm{t}_2 \leq \cdots \leq \mathrm{t}_\mathrm{n}$ then $B_{\mathrm{t}_1} - B_{\mathrm{t}_0}, B_{\mathrm{t}_2} - B_{\mathrm{t}_1}, \ldots,\ B_{\mathrm{t}_\mathrm{n}} - B_{\mathrm{t}_{n-1}}$ are independent
2. if $0 \leq s \leq t$ then $B(t) - B(s) \sim N_d(0,\ (t-s)I_d)$ with I_d the $d \times d$ identity matrix
3. $\mathbb{P}(\{\omega \in \Omega\colon B(\omega, 0) = 0$ and $t \mapsto B(\omega,t)$ is continuous $\}) = 1$

The latter means that the standard Brownian motion starting at the origin is a stochastic process that almost surely starts with $B_0 = 0$, and has continuous paths almost surely.

If we equip the space with the natural filtration $\mathcal{F}_t = \sigma(B_s\colon 0 \leq s \leq t)$, then B is adapted to $\{\mathcal{F}_t\}$ and is called a Brownian motion with respect to this filtration. In what follows, we mainly work with the one-dimensional case $d = 1$and write B_t for a standard Brownian motion.

Fundamental properties

From this definition, several structural properties follow. We record here the ones that will be used repeatedly in the sequel.

I. Marginal distribution:

For each $t \geq 0$, $B_t \sim N(0, t)$. This is an immediate consequence of the increment property: taking $s = 0$, we have $B_t - B_0 \sim N(0, t)$ and $B_0 = 0$ almost surely, hence $B_t \sim N(0, t)$.

II. Martingale property

We now show that $\{B_t\}_{t\geq 0}$ is a martingale with respect to its natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Fix $0 \leq s \leq t$. We can decompose: $B_t = B_s + (B_t - B_s)$.

1. Adaptedness: B_s is $\mathcal{F}_s$-measurable by definition, since $\mathcal{F}_s = \sigma(B_u\colon 0 \leq u \leq s)$.
2. Independence of increments: The increment $B_t - B_s$ depends only on the behavior of B over the interval $[s, t]$and is independent of $\mathcal{F}_s$ by the independent increments' property of Brownian motion. In other words, $B_t - B_s \perp\!\!\!\perp \mathcal{F}_s$
3. Integrability: Since $B_t \sim \mathcal{N}(0, t)$, we have $\mathbb{E}[\mid B_t \mid] = \sqrt{\frac{2t}{\pi}} < \infty$, so the process is integrable, and conditional expectations are well-defined.

Combining these facts, we compute the conditional expectation:

$$\mathbb{E}[B_t \mid \mathcal{F}_s] = \mathbb{E}[B_s \mid \mathcal{F}_s] + \mathbb{E}[B_t - B_s \mid \mathcal{F}_s] = B_s + 0 = B_s.$$

Thus, $\{B_t\}_{t\geq 0}$ satisfies the defining martingale property with respect to $\{\mathcal{F}_t\}$

III. Quadratic variation.

Quadratic variation is one of the features that fundamentally distinguishes Brownian motion from smooth deterministic paths and underlies the need for Itô calculus.

For a standard Brownian motion $B = \{B_t\}_{t\geq 0}$, the quadratic variation on $[0,t]$ is defined as

$$[B,B](t) := \lim_{n\to\infty} \sum_{i=1}^{m_n} \left| B\left(t_i^{(n)}\right) - B\left(t_{i-1}^{(n)}\right) \right|^2 \quad [6]$$

where for each n, $\pi_n = \{0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{m_n}^{(n)} = t\}$ is a partition of $[0,t]$ with mesh $\delta_n := \max_i (t_i^{(n)} - t_{i-1}^{(n)}) \to 0$ as $n \to \infty$.

It is a remarkable fact that, although each partial sum is random, their limit is deterministic:

Theorem (Quadratic variation of Brownian motion)

For a standard Brownian motion B and any sequence of partitions $(\pi_n)_n$ of $[0,t]$ with mesh $\delta_n \to 0$ (under a mild regularity condition on the partitions), we have

$$[B,B](t) = \lim_{n\to\infty} \sum_{i=1}^{m_n} \left| B\left(t_i^{(n)}\right) - B\left(t_{i-1}^{(n)}\right) \right|^2 = t \text{ almost surely.} \quad [6]$$

In particular, Brownian motion has quadratic variation $[B]_t = t$, whereas smooth deterministic functions have zero quadratic variation. This is precisely the phenomenon that forces the extra second-order term to appear in Itô's formula and ultimately justifies the Itô calculus used later in the report. A detailed proof of the theorem is provided in Appendix A.1

Discrete approximation and simulation

For numerical purposes, we cannot simulate Brownian motion at all times, but we can approximate it arbitrarily well on a discrete time grid. This discrete construction underlies both the visualization of Brownian paths and the Monte Carlo simulation of SDE-driven asset prices.

Fix a time horizon $T > 0$and an integer $n \geq 1$. Define a uniform partition

$$t_k = k\,\Delta t, \qquad k = 0,1,\dots,n, \qquad \Delta t = \frac{T}{n}.$$

We generate independent standard normal variables $Z_0, \dots, Z_{n-1} \sim N(0,1)$ and set

$$\Delta B_k := \sqrt{\Delta t}\, Z_k, \qquad B_{t_0} = 0, \qquad B_{t_k} = \sum_{i=0}^{k-1} \Delta B_i\,, \qquad k = 1,\dots,n.$$

By construction:

- the increments $\Delta B_k = B_{t_{k+1}} - B_{t_k}$ are independent and normally distributed with mean 0 and variance Δt,
- the covariance structure satisfies $\mathrm{Cov}(B_{t_k}, B_{t_\ell}) = \min\,(t_k, t_\ell)$

Thus, at the grid points $\{t_k\}$, this discrete process has the same finite-dimensional distributions as Brownian motion.

To obtain a continuous-time approximation on $[0, T]$, we define the piecewise-linear interpolation $B^{(n)}(t)$ by joining the points $\{(t_k, B_{t_k})\}$ linearly on each interval $[t_k, t_{k+1}]$. A classical result (often referred to as Lévy's construction) states that, as $\Delta t \to 0$, the processes $B^{(n)}$converge almost surely and uniformly on compact time intervals to a standard Brownian motion [7] [8] (see Appendix A.2 for a discussion). In practice, this justifies using the discrete scheme above as the basis for simulating Brownian paths.

Simulation procedure

In our numerical experiments, we implement this construction on a finite horizon $[0,10]$ using $n = 200$ time steps. The step size is therefore $\Delta t = 0.05$, and the time grid is $t_k = k\Delta t$, $k = 0, \dots, 200$. The Brownian increments are simulated as

$$\Delta B_i = \sqrt{\Delta t}\, Z_i, \qquad Z_i \sim N(0,1) \text{ iid},$$

and the Brownian motion at the grid points is obtained by cumulative summation:

$$B_{t_0} = 0, \qquad B_{t_k} = \sum_{i=0}^{k-1} \Delta B_i$$

We first generate and plot a single sample path, which illustrates the typical irregular yet continuous behavior of Brownian motion. We then repeat the same sampling procedure with independent sequences $\{Z_i^{(m)}\}$, $m = 1,2, \dots, M$, to obtain multiple independent Brownian paths:

$$B_{t_k}^{(m)} = \sum_{i=0}^{k-1} \sqrt{\Delta t}\, Z_i^{(m)}, m = 1, \dots, M.$$

Visualizing a bundle of such paths over $[0,10]$ highlights both the randomness of individual trajectories and the diffusion of the ensemble as time increases, consistent with the variance growth $\mathrm{Var}(B_t) = t$.

Figure 1 displays a single simulated trajectory of standard Brownian motion on $[0,10]$, generated with the discrete scheme described above. The path is continuous yet highly irregular, with rapid oscillations and no visible deterministic trend, which is consistent with the theoretical properties of Brownian motion.

Figure1 : Single Brownian Motion path

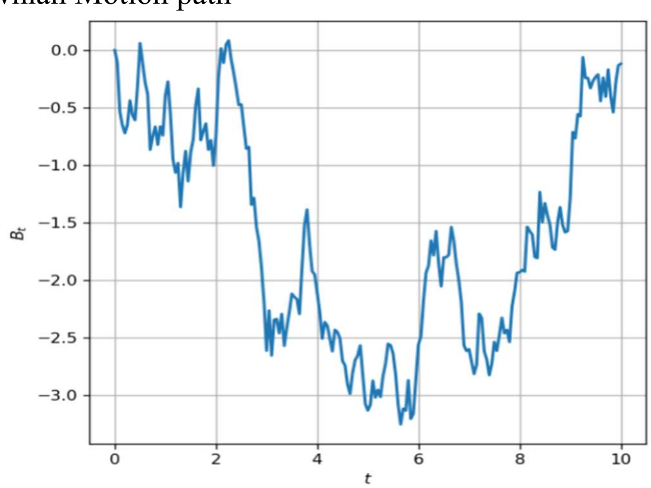
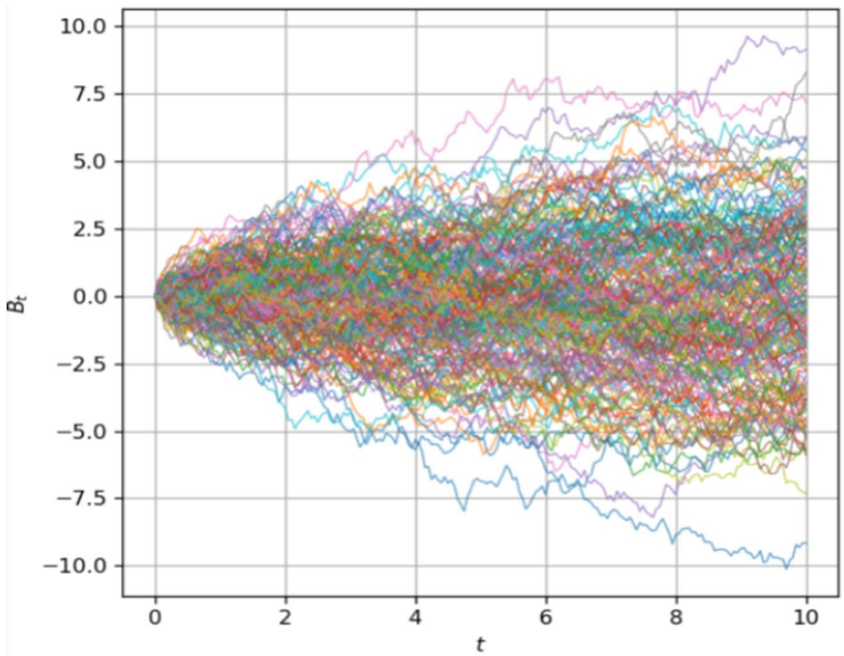


Figure 2 displays 200 independently simulated trajectories of standard Brownian motion on $[0,10]$, generated with the same discretization scheme as in Figure 1. All paths start from the common initial condition $B_0 = 0$ and gradually fan out over time, illustrating the increasing dispersion consistent with $\mathrm{Var}(B_t) = t$and the symmetric fluctuations of Brownian motion around zero.

Figure2 : 200 Brownian Motion paths



Long-time behavior

Beyond finite-horizon simulations, it is useful to recall the long-time behavior of Brownian motion. For standard one-dimensional Brownian motion $(B_t)_{t\geq 0}$, we have the almost sure limit

$$\lim_{t\to\infty} \frac{B_t}{t} = 0 \text{ a.s.}$$

Intuitively, Brownian motion has no drift, so although its variance grows linearly in time, its average rate B_t/t tends to zero almost surely.

A sketch of the argument is as follows. Write Brownian motion as a sum of independent increments over unit intervals. For integer times,

$$B_n = \sum_{k=1}^{n} (B_k - B_{k-1}) = \sum_{k=1}^{n} X_k,$$

where $X_k := B_k - B_{k-1} \sim N(0,1)$ are independent. By the strong law of large numbers [9],

$$\frac{1}{n}\sum_{k=1}^{n} X_k \to 0 \text{ a.s. as } n \to \infty,$$

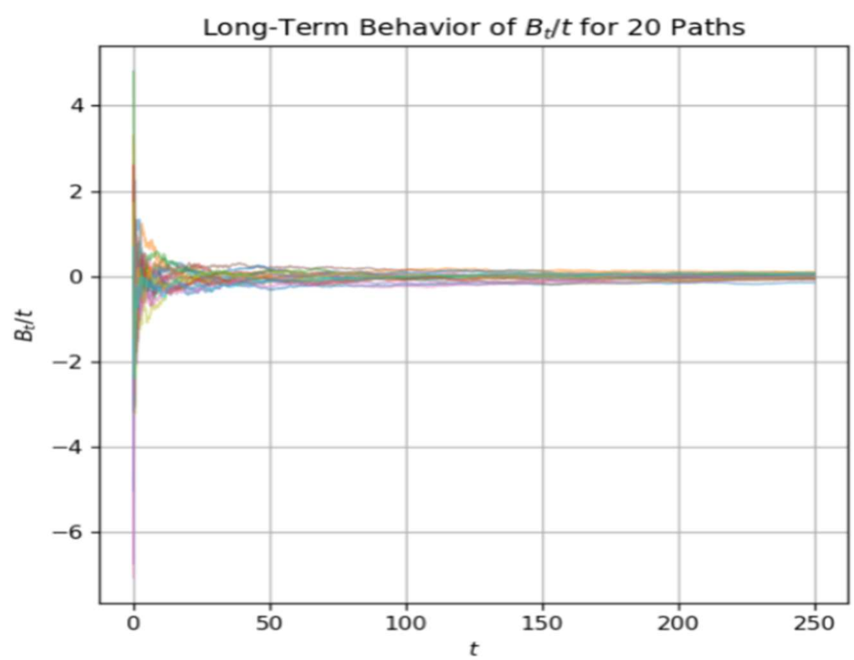
so $B_n/n \to 0$ almost surely along the integers. For non-integer t, write

$$B_t = B_{\lfloor t \rfloor} + (B_t - B_{\lfloor t \rfloor}),$$

and note that the remainder term $(B_t - B_{\lfloor t \rfloor})/t$ is bounded in absolute value by $\mid B_t - B_{\lfloor t \rfloor} \mid / \lfloor t \rfloor$, which tends to zero almost surely because the increment over an interval of length less than 1 is of order 1 ($O(1)$) while $\lfloor t \rfloor \to \infty$. Combining these observations yields $\lim_{t \to \infty} B_t/t = 0$ almost surely. A more refined analysis shows that this convergence is in fact "complete" in the sense of almost sure convergence together with summability of the tail probabilities; we refer to Appendix A.3 for further details.

To illustrate this long-time stabilization, Figure 3 displays 20 independently simulated trajectories of the normalized process $t \mapsto B_t/t$over the horizon $[0,250]$. For very small t, the ratios exhibit large volatility because we divide by a small denominator, but as $t$increases the curves rapidly contract into a narrow band around zero. This visual behavior is fully consistent with the almost sure limit $\lim_{t \to \infty} B_t/t = 0$and with the complete convergence result established in Appendix A.3.

<u>Figure3 :</u> Long term behavior of 10 Brownian Motion paths



In the remainder of the report, this construction plays two roles. First, it provides a practical method to generate sample paths of Brownian motion for illustration and diagnostics. Second, and more importantly, it serves as the core building block for simulating the geometric Brownian motion model under the risk-neutral measure. Those simulated asset paths will then be used to construct Monte Carlo estimators for the price of the arithmetic Asian put option.

Stochastic differential equations (SDEs)

The next ingredient in the project is the stochastic differential equation (SDE) framework. Intuitively, an SDE describes the evolution of a system that has both a deterministic trend and random shocks driven by Brownian motion.

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions and let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion adapted to $\{\mathcal{F}_t\}$.

General form of an SDE and relation to ODEs

A one–dimensional SDE [1] driven by Brownian motion has the form

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dB_t,$$

where

- $\mu(X_t, t)$is the drift (local deterministic trend),
- $\sigma(X_t, t)$is the diffusion coefficient (local volatility),
- $B_t$is standard Brownian motion.

In integral form

$$X_t = X_0 + \int_0^t \mu(X_s, s)\ ds + \int_0^t \sigma(X_s, s)\ dB_s \text{ [1]}$$

This should be compared with the ordinary differential equation (ODE) $\frac{dx(t)}{dt} = \mu(x(t), t)$, which produces a deterministic trajectory once the initial condition is fixed. In an SDE, the Brownian term $\sigma(X_t, t)\, dB_t$ turns the solution into a random process: for a given initial condition, we obtain a distribution over paths rather than a single curve. When $\sigma \equiv 0$, the SDE reduces to the ODE and the randomness disappears. In this sense, SDEs are probabilistic generalizations of ODEs obtained by adding a noise term with non-zero variance.

Under standard regularity assumptions on μ and σ(e.g., global Lipschitz continuity and linear growth), we have existence and pathwise uniqueness of a strong solution to this SDE [1].

Itô's formula

Because the driving noise is Brownian motion, classical calculus rules do not apply directly. Brownian paths have non-zero quadratic variation, and this produces an extra second-order term when we apply a change of variables. This is captured by Itô's formula.

Let $\{Y_t\}_{t\geq 0}$ satisfy $dY_t = a(Y_t, t)\, dt + b(Y_t, t)\, dB_t$,and let $f: \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be $C^{1,2}$(once continuously differentiable in t and twice in y). Then Itô's lemma [1] states that

$$df(Y_t, t) = \left(\frac{\partial f}{\partial t} + a(Y_t, t)\, \frac{\partial f}{\partial y} + \frac{1}{2} b^2(Y_t, t)\, \frac{\partial^2 f}{\partial y^2}\right) dt + b(Y_t, t)\, \frac{\partial f}{\partial y}\, dB_t.$$

The term $\frac{1}{2}b^2\frac{\partial^2 f}{\partial y^2}$ has no analogue in ordinary calculus; it comes from the quadratic variation $[B]_t = t$ of the Brownian motion.

Geometric Brownian motion and its explicit solution

The most fundamental SDE in continuous-time finance models a stock price S_t as a geometric Brownian motion (GBM). As argued for example in Sengupta (2004) [10], this specification is most plausible in a setting where the firm is assumed to remain in operation over the horizon of interest, prices evolve continuously in time without jumps, and the price process can be treated as Markovian, so that the current level $S_t$contains all relevant information for its future evolution. Under these assumptions, proportional price changes over a fixed interval are lognormally distributed, which is equivalent to assuming that continuously compounded returns are Gaussian. Taken together, these hypotheses make GBM both analytically tractable and a reasonable first-order model for liquid equity prices in many applications.

Under the physical measure $\mathbb{P}$, the dynamics are

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t$$

where μ is the real-world drift (expected proportional growth rate) and $\sigma > 0$ is the volatility.

To solve this SDE explicitly, we apply Itô's formula to the logarithm of the process. Set $Y_t = S_t$and $f(y) = \ln\, y$. Then

$$\frac{\partial f}{\partial t} = 0, \qquad \frac{\partial f}{\partial y} = \frac{1}{y}, \qquad \frac{\partial^2 f}{\partial y^2} = -\frac{1}{y^2}.$$

In our SDE, $a(y,t) = \mu y$ and $b(y,t) = \sigma y$. Plugging into Itô's formula gives

$$d(\ln\, S_t) = (\mu - \frac{1}{2}\sigma^2)\, dt + \sigma\, dB_t.$$

Integrating from 0 to t yields $\ln\, S_t - \ln\, S_0 = (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t$, so

$$S_t = S_0 \exp\,((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t).$$

Thus $\ln S_t$ is normal and S_t is lognormal. This explicit solution makes clear how the drift $\mu$and volatility σ shape the distribution of future prices, which is central for pricing and risk management.

Change of measure and risk-neutral dynamics

For derivative pricing, the physical drift μ is not the most convenient parameter. Arbitrage-free valuation is based on a *risk-neutral* measure $\mathbb{Q}$ under which discounted asset prices are martingales. The passage from $\mathbb{P}$ to $\mathbb{Q}$ is a simple special case of Girsanov's theorem [2].

Let $\{B_t\}_{t\geq 0}$be standard Brownian motion under $\mathbb{P}$, and fix a constant $\theta \in \mathbb{R}$. Define

$$M_t := \exp\left(\theta B_t - \frac{1}{2}\theta^2 t\right), \qquad t \geq 0.$$

Using Itô's formula, $\{M_t\}_{t\geq 0}$ is a positive $\mathbb{P}$-martingale with $\mathbb{E}_{\mathbb{P}}[M_t] = 1$for all t; a detailed proof is in Appendix B.1. For each fixed horizon T, we then define a new probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\mid_{\mathcal{F}_T} = M_T.$$

Girsanov's theorem (in this constant-drift setting) states that under $\mathbb{Q}$ the process

$$W_t := B_t - \theta t, \qquad 0 \leq t \leq T,$$

is a standard Brownian motion. In other words, changing from $\mathbb{P}$ to $\mathbb{Q}$ shifts the drift of the Brownian motion but leaves its volatility structure unchanged. Now, we can apply this to the GBM model. Under $\mathbb{P}$,

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t.$$

Let r be the risk-free interest rate and set $\theta = \frac{\mu - r}{\sigma}$. If we define $\mathbb{Q}$ via $M_t = \exp\left(-\theta B_t - \frac{1}{2}\theta^2 t\right)$

Then, by Girsanov, the process $W_t := B_t + \theta t$ is a Brownian motion under $\mathbb{Q}$. Substituting $dB_t = dW_t - \theta\, dt$ into the SDE for S_t gives

$$dS_t = rS_t\, dt + \sigma S_t\, dW_t.$$

From this representation, one can show that the discounted process $\tilde{S}_t := e^{-rt} S_t$ is a martingale under $\mathbb{Q}$, i.e. $\mathbb{E}_{\mathbb{Q}}[\tilde{S}_t \mid \mathcal{F}_s] = \tilde{S}_s$ for $0 \leq s \leq t$.

A detailed verification using Itô's formula and conditional expectations is given in Appendix B.2.

This martingale property is the probabilistic backbone of arbitrage-free pricing: for a payoff $H(S_\cdot)$ at maturity T, the fair time-0 price is

$$\text{Price}_0 = e^{-rT}\, \mathbb{E}_{\mathbb{Q}}[H(S_\cdot)].$$

In this project, that representation is the key link between the SDE model and the Monte Carlo algorithms used later. The GBM SDE and the change-of-measure machinery provide the risk-neutral dynamics for S_t, while the Brownian simulations constructed in the previous section give a practical way to generate sample paths of W_t(and hence S_t). These paths will serve as the raw input for our numerical estimators of the arithmetic Asian put price.

Monte Carlo pricing of arithmetic Asian put options

A path-dependent option is a derivative whose payoff depends on the entire trajectory of the underlying asset, not only on its terminal value S_T. Asian options are a canonical example: their payoff depends on an average of the asset price over a monitoring interval, which dampens the effect of short-lived spikes and typically makes them cheaper than otherwise comparable European options.

Given monitoring dates $0 < t_1 < \cdots < t_n = T$, two standard averaging schemes are:

- **Arithmetic average:** $\bar{S}_A = \frac{1}{n}\sum_{j=1}^{n} S_{t_j}$
- **Geometric average:** $\bar{S}_G = \left(\prod_{j=1}^{n} S_{t_j}\right)^{1/n}$

For a put option with strike K, the arithmetic Asian payoff is $H_T = (K - \bar{S}_A)^+$, where $x^+ = \max(x, 0)$. Under geometric averaging one can exploit log-normality to obtain a closed-form price, but for the arithmetic average no such simple formula exists under GBM. This makes it an ideal test case for Monte Carlo methods grounded in the probabilistic machinery developed earlier.

Risk-neutral valuation and GBM path generation

Under the risk-neutral measure $\mathbb{Q}$, we define a uniform grid $t_k = k\Delta t$ with $\Delta t = T/n$, the exact discretization of GBM under $\mathbb{Q}$ is

$$S_{t_{k+1}} = S_{t_k} \exp\left((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\, Z_k\right), \qquad Z_k \sim N(0,1) \text{ i.i.d.}$$

Our implementation of this scheme is summarized in Algorithm 1 (Path generator for GBM), which takes as input S_0, r, σ, the maturity T, and the number of time steps n, and returns a discrete path $(S_0, S_{t_1}, \ldots, S_{t_n})$suitable for use in the Asian payoff.

Algorithm 1 Path generator for GBM (Asian option setting)

Input: $S_0 = 500$, $r = 0.0175$, $\sigma = 0.25$, $T = 1$, $n = 52$
Output: Price path $(S_0, S_{t_1}, \ldots, S_{t_n})$
Set $\Delta t = T/n$ Initialize an array $S[0..n]$ Set $S[0] = S_0$
for $i = 1, 2, \ldots, n$ **do**
  Draw $Z_i \sim \mathcal{N}(0,1)$; `// standard normal`
  Set $X_i = \left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\, Z_i$ Set $S[i] = S[i-1] \cdot \exp(X_i)$
return $(S[0], S[1], \ldots, S[n])$

Given a simulated path, the arithmetic average $\bar{S}_A = \frac{1}{n}\sum_{j=1}^{n} S_{t_j}$ is straightforward to compute, and the discounted payoff under $\mathbb{Q}$ is discPayoff $= e^{-rT}(K - \bar{S}_A)^+$. By the risk-neutral pricing principle, $P = \mathbb{E}_{\mathbb{Q}}[e^{-r}\ \ (K - \bar{S}_A)^+]$ is the unique arbitrage-free price of the arithmetic Asian put in this GBM setting.

Crude Monte Carlo estimator

To approximate P, we generate N independent GBM paths under $\mathbb{Q}$. For each path $j = 1, \dots, N$ we compute the arithmetic average $\bar{S}_A^{(j)}$ and the corresponding discounted payoff

$$Y_j = e^{-rT}(K - \bar{S}_A^{(j)})^+.$$

The crude Monte Carlo estimator of the option price is $\hat{P}_N = \frac{1}{N}\sum_{j=1}^{N} Y_j$ [11]

Since the Y_j are i.i.d. with finite variance, we have:

- **Unbiasedness:** $\mathbb{E}[\hat{P}_N] = P$.
- **Law of large numbers:** $\hat{P}_N \to P$ almost surely as $N \to \infty$.
- **Central limit theorem** [9]**:** for large N, $\sqrt{N}(\hat{P}_N - P) \xrightarrow{d} N(0, \sigma_Y^2)$, where $\sigma_Y^2 = \mathrm{Var}\,(Y_1)$

In practice we estimate σ_Y^2 by the unbiased sample variance

$$s^2 = \frac{1}{N-1}\sum_{j=1}^{N} Y_j^2 - \frac{N}{N-1}\hat{P}_N^2 \text{ [11]}$$

and form an approximate 95% confidence interval as $\hat{P}_N \pm 1.96\,\frac{s}{\sqrt{N}}$ [11] The full procedure is summarized in Algorithm 2 (Crude Monte Carlo pricing of an arithmetic Asian put), which also tracks the mean squared error estimate and the confidence interval.

Algorithm 2 Crude Monte Carlo pricing of an arithmetic Asian put

Input: $S_0 = 500$, $r = 0.0175$, $\sigma = 0.25$, $T = 1$, $n = 52$, $K = S_0$, $N \in \{10^4, 10^5\}$
Output: Estimated price $\hat{P}_N$, MSE, and 95% confidence interval
Set random seed $= 42$; `// for reproducibility`
Set $\Delta t = T/n$ Set sum $= 0$, sumSq $= 0$
for $i = 1, 2, \dots, N$ **do**
Call **Algorithm 1** to generate a path $(S_0, S_{t_1}, \dots, S_{t_n})$ Compute arithmetic average

$$\bar{S}_A = \frac{1}{n}\sum_{j=1}^{n} S_{t_j}$$

Compute discounted payoff of the Asian put

$$\text{discPayoff} = e^{-rT}\max(K - \bar{S}_A, 0)$$

Update accumulators:

$$\text{sum} \leftarrow \text{sum} + \text{discPayoff}, \qquad \text{sumSq} \leftarrow \text{sumSq} + \text{discPayoff}^2$$

Compute sample mean (price estimator)

$$\hat{P}_N = \frac{\text{sum}}{N}$$

Compute unbiased sample variance

$$s^2 = \frac{\text{sumSq}}{N-1} - \frac{N}{N-1}\hat{P}_N^2$$

Set $s = \sqrt{s^2}$, MSE $= s^2/N$
Compute 95% confidence interval half-width

$$\text{err} = 1.96\,\frac{s}{\sqrt{N}}$$

Set confidence interval

$$\text{CI} = [\hat{P}_N - \text{err},\ \hat{P}_N + \text{err}]$$

return $\hat{P}_N$, *MSE*, *CI*

From a probabilistic standpoint, this estimator is conceptually simple: it directly approximates the risk-neutral expectation defining the price by averaging i.i.d. discounted payoffs. The main practical drawback is variance: because the payoff depends on the entire path, the distribution of Y_j can be quite dispersed, so that large values of N are required to achieve a tight confidence interval.

Variance reduction via antithetic variates

A classical way to improve Monte Carlo efficiency without changing the underlying model is to reduce the variance of the estimator. Among many techniques, antithetic variates [11] are particularly simple to implement and are well suited to GBM under $\mathbb{Q}$.

The key idea can be understood through the inverse transform method. Suppose $U \sim \mathrm{Unif}(0,1)$ and we generate a standard normal variable as $Z = \Phi^{-1}(U)$, where Φ is the cumulative distribution function of $N(0,1)$. For the same U, consider also $Z' = \Phi^{-1}(1-U)$.

For a symmetric distribution such as $N(0,1)$, symmetry implies $\Phi^{-1}(1-u) = -\,\Phi^{-1}(u)$,

so $Z' = -Z$. Moreover, since $u \mapsto \Phi^{-1}(u)$ is increasing, Z and Z' are negatively correlated: when one is large and positive, the other is large and negative. In simulation practice we typically generate $Z \sim N(0,1)$ directly and then set $Z' = -Z$; this is equivalent to starting from the pair $(U, 1-U)$.

Applied to GBM, this yields pairs of "mirror" paths. Given a current value S_{t_k}, the antithetic pair of updates is

$$\begin{aligned} S_{t_{k+1}}^{(+)} &= S_{t_k}^{(+)} \exp\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\, Z_k\right), \\ S_{t_{k+1}}^{(-)} &= S_{t_k}^{(-)} \exp\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t - \sigma\sqrt{\Delta t}\, Z_k\right), \end{aligned}$$

with $Z_k \sim N(0,1)$ i.i.d. and both paths starting from the same S_0. Intuitively, one path tends to move up when the other moves down, though both retain the correct marginal distribution of GBM at each time.

For each pair we obtain two discounted payoffs, say $Y^{(+)}$ and $Y^{(-)}$. The antithetic estimator for that pair is the average $\theta = \frac{1}{2}(Y^{(+)} + Y^{(-)})$.

Because both payoffs have the same expectation, this pairwise averaging preserves unbiasedness:

$$\mathbb{E}[\theta] = \frac{1}{2}\left(\mathbb{E}[Y^{(+)}] + \mathbb{E}[Y^{(-)}]\right) = P.$$

The benefit comes from the variance:

$$\mathrm{Var}\,(\theta) = \frac{1}{4}\left(\mathrm{Var}\,(Y^{(+)}) + \mathrm{Var}\,(Y^{(-)}) + 2\,\mathrm{Cov}\,(Y^{(+)}, Y^{(-)})\right).$$

Since $\mathrm{Var}\,(Y^{(+)}) = \mathrm{Var}\,(Y^{(-)}) = \mathrm{Var}\,(Y)$, this simplifies to

$$\text{Var}\,(\theta) = \frac{1}{2}\text{Var}\,(Y) + \frac{1}{2}\text{Cov}\,(Y^{(+)}, Y^{(-)}).$$

Antithetic construction is designed so that $\text{Cov}\,(Y^{(+)}, Y^{(-)}) < 0$ in typical settings: when one path produces an unusually large payoff, the mirror path tends to produce an unusually small one. Consequently, $\text{Var}\,(\theta) < \text{Var}\,(Y)$, meaning that for the same number of GBM step updates we obtain a lower-variance estimator.

In practice we work with N total GBM paths grouped into $M = N/2$ antithetic pairs. For each pair $j = 1, \ldots, M$, we form the averaged discounted payoff θ_j. The antithetic Monte Carlo estimator of the price is then

$$\hat{P}_N^{\text{anti}} = \frac{1}{M}\sum_{j=1}^{M}\theta_j.$$

Again $\hat{P}_N^{\text{anti}}$is unbiased for P, and its variance is strictly smaller than that of the crude estimator as long as the covariance within pairs is negative. The implementation details, including the computation of the sample variance and a 95% confidence interval, are summarized in:

- Algorithm 3 (Arithmetic averages with antithetic GBM paths), which constructs the pair of arithmetic averages $(\bar{S}_A^{(+)}, \bar{S}_A^{(-)})$for each antithetic pair, and
- Algorithm 4 (Antithetic Monte Carlo pricing of an arithmetic Asian put), which aggregates these averages into the final price estimator and error metrics.

Algorithm 3 Arithmetic averages with antithetic GBM paths

Input: $S_0 = 500$, $r = 0.0175$, $\sigma = 0.25$, $T = 1$, $n = 52$
Output: Arithmetic averages $(\bar{S}_A^{(+)}, \bar{S}_A^{(-)})$ for the two antithetic paths
Set $\Delta t = T/n$ Set $S^{(+)} = S_0$, $S^{(-)} = S_0$ Set $\text{sum}^{(+)} = 0$, $\text{sum}^{(-)} = 0$
for $j = 1, 2, \ldots, n$ **do**
  Draw $Z_j \sim \mathcal{N}(0,1)$ Set $X_j = \left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\,Z_j$ Set $X_j^- = \left(r - \frac{1}{2}\sigma^2\right)\Delta t - \sigma\sqrt{\Delta t}\,Z_j$ Update paths:

$$S^{(+)} \leftarrow S^{(+)}\exp(X_j), \qquad S^{(-)} \leftarrow S^{(-)}\exp(X_j^-)$$

  Accumulate:

$$\text{sum}^{(+)} \leftarrow \text{sum}^{(+)} + S^{(+)}, \qquad \text{sum}^{(-)} \leftarrow \text{sum}^{(-)} + S^{(-)}$$

Set $\bar{S}_A^{(+)} = \text{sum}^{(+)}/n$, $\bar{S}_A^{(-)} = \text{sum}^{(-)}/n$
return $(\bar{S}_A^{(+)}, \bar{S}_A^{(-)})$

Algorithm 4 Antithetic Monte Carlo pricing of an arithmetic Asian put

Input: $S_0 = 500$, $r = 0.0175$, $\sigma = 0.25$, $T = 1$, $n = 52$, $K = S_0$, $N \in \{10^4, 10^5\}$ (with N even)
Output: Estimated price $\hat{P}_N^{\text{anti}}$, MSE, and 95% confidence interval
if N *is odd* **then**
  error: N must be even
Set $M = N/2$; `// number of antithetic pairs`
Set random seed $= 42$; `// for reproducibility`
Set sum $= 0$, sumSq $= 0$
for $i = 1, 2, \ldots, M$ **do**
  Call **Algorithm: Arithmetic averages with antithetic GBM paths** to obtain $(\bar{S}_A^{(+)}, \bar{S}_A^{(-)})$
  Compute discounted payoffs:

$$\text{discPayoff}^{(+)} = e^{-rT} \max(K - \bar{S}_A^{(+)}, 0), \quad \text{discPayoff}^{(-)} = e^{-rT} \max(K - \bar{S}_A^{(-)}, 0)$$

  Average within the pair:

$$\text{discPayoff}_{\text{avg}} = \frac{1}{2}\left(\text{discPayoff}^{(+)} + \text{discPayoff}^{(-)}\right)$$

  Update accumulators:

$$\text{sum} \leftarrow \text{sum} + \text{discPayoff}_{\text{avg}}, \qquad \text{sumSq} \leftarrow \text{sumSq} + \text{discPayoff}_{\text{avg}}^2$$

Compute sample mean (antithetic estimator)

$$\hat{P}_N^{\text{anti}} = \frac{\text{sum}}{M}$$

Compute unbiased sample variance (as in the code)

$$s^2 = \frac{\text{sumSq}}{M-1} - \frac{M}{M-1}\left(\hat{P}_N^{\text{anti}}\right)^2$$

Set $s = \sqrt{s^2}$, $\quad \text{MSE} = s^2/M$
Compute 95% confidence interval half-width

$$\text{err} = 1.96\,\frac{s}{\sqrt{M}}$$

Set confidence interval

$$\text{CI} = \left[\,\hat{P}_N^{\text{anti}} - \text{err},\ \hat{P}_N^{\text{anti}} + \text{err}\,\right]$$

return $\hat{P}_N^{anti}$, *MSE, CI*

Simulation Results

Using the parameter values specified in Algorithms 1–4, we implement both crude and antithetic Monte Carlo estimators for the arithmetic Asian put. The resulting point estimates and 95% confidence intervals can be summarized in Table 1.

Table1: Pricing Asian Arithmetic Option using crude MC and Antithetic estimators

Method	Sample Size (N)	Estimated Price	MSE	95% Confidence Interval
Crude MC	10,000	26.9471	0.139226	(26.2157, 27.6784)
Antithetic MC	10,000	26.4250	0.064996	(25.9253, 26.9247)
Crude MC	100,000	26.4964	0.013459	(26.2690, 26.7238)
Antithetic MC	100,000	26.5803	0.006499	(26.4223, 26.7383)

To quantify the benefit of antithetic variates, we compare the mean-squared error (MSE) of the crude and antithetic estimators at two Monte Carlo budgets, $N = 10^4$and $N = 10^5$. Defining the efficiency gain as $\mathrm{EfficiencyGain} = \frac{\mathrm{MSE}_{\mathrm{crude}}}{\mathrm{MSE}_{\mathrm{anti}}}$, our numerical experiments show gains of approximately $2.14\times$for $N = 10^4$and $2.07\times$ for $N = 10^5$.

In other words, in this GBM–Asian option setting, antithetic variates deliver roughly a factor-of-two improvement in efficiency: to achieve a given accuracy, the antithetic estimator requires about half as many simulated paths as crude Monte Carlo.

This section thus closes the loop between the abstract probabilistic framework and a concrete numerical method: starting from the risk-neutral GBM SDE, we build pathwise simulations, construct unbiased Monte Carlo estimators for a path-dependent payoff, and use variance reduction grounded in dependence structure to significantly enhance practical performance

Conclusion

In summary, this project has used a single, concrete problem – the pricing of an arithmetic Asian put – to showcase how advanced probabilistic tools underpin modern quantitative finance. Starting from Brownian motion and its martingale properties, we built the geometric Brownian motion (GBM) model, passed to the risk-neutral measure via Girsanov's theorem, and expressed the arbitrage-free price as a risk-neutral expectation. This probabilistic representation led naturally to a Monte Carlo approximation based on simulated GBM paths, and to a rigorous analysis of the estimator's variance and convergence. The numerical experiments confirmed the theory: antithetic variates deliver roughly a factor-two reduction in mean squared error for a fixed Monte Carlo budget, demonstrating how variance reduction turns abstract probabilistic ideas into tangible computational gains.

Beyond the specific Asian option considered here, the methodology is directly relevant to a wide range of problems in practice. Path-dependent contracts such as Asian options, lookbacks, barrier options, and many structured products do not admit simple closed-form formulas under standard models. Monte Carlo methods built on SDEs provide a flexible workhorse for pricing and hedging such products, for stress testing and scenario analysis in risk management, and for evaluating the distribution of portfolio P&L under different market regimes. Once the simulation engine is in place, the same framework can be used to approximate sensitivities (the Greeks) via pathwise derivatives or likelihood-ratio methods, which are central to hedging and to model risk assessment. Even though the data used in this project are synthetic, the entire pipeline (calibration, risk-neutral simulation, Monte Carlo pricing, variance analysis) carries over without conceptual change once the model is fitted to historical or implied market data.

At the same time, the project highlights the limitations of the GBM assumption. For GBM to be a reasonable model, one essentially assumes that prices evolve continuously in time, that only the current price matters for future dynamics (Markov property), and that log-returns over fixed intervals are approximately normal with constant volatility and interest rate. These assumptions are often violated in real markets. Earnings announcements, macroeconomic releases, and firm-specific news can generate sudden jumps; volatility clusters and regimes appear; and interest rates or dividend yields can shift in a way that is not captured by a simple constant-parameter GBM. From a risk-management perspective, relying on GBM alone may understate tail risk and misrepresent the distribution of extreme scenarios that are critical for capital allocation and regulatory stress tests.

A natural direction is therefore to enrich the dynamics. One classical extension is the Merton jump–diffusion model [12], in which the stock price follows a diffusion with superimposed price jumps. In differential form, the dynamics can be written as

$$dS_t = S_{t-}[(a - \lambda k)\, dt + \sigma\, dW_t] + S_{t-}(Y - 1)\, dN_t,$$

where $N_t$is a Poisson process with intensity λ, $Y - 1$represents the random jump size so that a jump sends $S_{t-}$to $S_t = YS_{t-}$, and the drift is corrected by $-\lambda k$ with $k = \mathbb{E}[Y - 1]$ to keep the mean behavior consistent. Such models capture large discontinuous moves around earnings or macro events much more realistically than pure GBM. They also push the analysis further into the probabilistic domain: risk-neutral pricing typically requires a combination of jump-diffusion SDE theory and Monte Carlo simulation, sometimes along with more sophisticated variance-reduction or approximation techniques.

Overall, the project demonstrates that advanced probability is not an ornamental layer on top of finance, but the core analytical engine. Concepts such as filtrations, martingales, change of measure, SDE solutions, and variance reduction translate directly into algorithms that produce prices, confidence intervals, and ultimately risk numbers that practitioners rely on. In a broader context, the same machinery supports statistical arbitrage (through model-based backtesting and scenario generation), risk management (through simulation of loss distributions and tail events), and model calibration (through likelihood or moment-matching based on SDE dynamics). By making the full chain from theory to implementation explicit for a nontrivial path-dependent option, this work illustrates how an advanced probability toolkit can be deployed in a disciplined way to address realistic quantitative finance problems, and it lays the groundwork for future extensions to richer models such as stochastic volatility or jump-diffusion

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Appendix

A.1 Proof for the Quadratic variation

Fix $t > 0$ and a sequence $(\pi_n)_n$ as in the statement. For each n, write the increments along the partition as

$$\Delta_i^{(n)}B := B(t_i^{(n)}) - B(t_{i-1}^{(n)}), i = 1, \dots, m_n.$$

By the defining properties of Brownian motion, the random variables $\Delta_i^{(n)}B$ are independent and

$$\Delta_i^{(n)}B \sim N(0, t_i^{(n)} - t_{i-1}^{(n)}).$$

We first compute the expectation:

$$\mathbb{E}[T_n] = \mathbb{E}\left[\sum_{i=1}^{m_n} \left(\Delta_i^{(n)}B\right)^2\right] = \sum_{i=1}^{m_n} \mathbb{E}\left[\left(\Delta_i^{(n)}B\right)^2\right].$$

Since $\Delta_i^{(n)}B \sim N(0, \Delta t_i^{(n)})$ with $\Delta t_i^{(n)} := t_i^{(n)} - t_{i-1}^{(n)}$, we have $\mathbb{E}\left[\left(\Delta_i^{(n)}B\right)^2\right] = \mathrm{Var}\left(\Delta_i^{(n)}B\right) = \Delta t_i^{(n)}$.

Therefore,

$$\mathbb{E}[T_n] = \sum_{i=1}^{m_n} \Delta t_i^{(n)} = t - 0 = t.$$

We next compute Var (T_n). Using independence of the increments,

$$\mathrm{Var}\ (T_n) = \mathrm{Var}\left(\sum_{i=1}^{m_n} \left(\Delta_i^{(n)}B\right)^2\right) = \sum_{i=1}^{m_n} Var\left(\left(\Delta_i^{(n)}B\right)^2\right)$$

If $X \sim N(0, \sigma^2)$, then $\mathbb{E}[X^2] = \sigma^2$ and $\mathbb{E}[X^4] = 3\sigma^4$. Hence

$$\mathrm{Var}\ (X^2) = \mathbb{E}[X^4] - (\mathbb{E}[X^2])^2 = 3\sigma^4 - \sigma^4 = 2\sigma^4.$$

Applying this with $X = \Delta_i^{(n)}B$ and $\sigma^2 = \Delta t_i^{(n)}$, we find $Var\left(\left(\Delta_i^{(n)}B\right)^2\right) = 2\left(\Delta t_i^{(n)}\right)^2$.

Therefore

$$\mathrm{Var}\ (T_n) = \sum_{i=1}^{m_n} 2\left(\Delta t_i^{(n)}\right)^2 \leq 2(\max_i\ \Delta t_i^{(n)}) \sum_{i=1}^{m_n} \Delta t_i^{(n)} = 2t\ \delta_n.$$

Since $\delta_n \to 0$, we obtain

$$\mathrm{Var}\ (T_n) \leq 2t\delta_n \longrightarrow 0 \text{ as } n \to \infty.$$

We have shown that $\mathbb{E}[T_n] = t$ for all n and Var $(T_n) \to 0$. Thus

$$\mathbb{E}[(T_n - t)^2] = \text{Var}\,(T_n) \longrightarrow 0,$$

so $T_n \to t$ in L^2, and in particular in probability.

The assumption $\sum_{n=1}^{\infty} \delta_n < \infty$ implies

$$\sum_{n=1}^{\infty} \text{Var}\,(T_n) \le 2t \sum_{n=1}^{\infty} \delta_n < \infty.$$

Since Var $(T_n) = \mathbb{E}[(T_n - t)^2]$, we have

$$\sum_{n=1}^{\infty} \mathbb{E}[(T_n - t)^2] < \infty.$$

By the monotone convergence theorem, this implies

$$\mathbb{E}\left[\sum_{n=1}^{\infty} (T_n - t)^2\right] < \infty,$$

so $\sum_{n=1}^{\infty} (T_n - t)^2 < \infty$ almost surely. In particular, $(T_n - t)^2 \to 0$ almost surely, hence

$$T_n \longrightarrow t \text{ almost surely as } n \to \infty.$$

This proves the claim for the class of refining partitions with $\delta_n \to 0$and $\sum_n \delta_n < \infty$. It is a standard extension to show that the same conclusion holds for any sequence of successive refinements with mesh $\delta_n \to 0$. In all cases, the quadratic variation of Brownian motion on $[0, t]$ is almost surely equal to t.

A.2 Proof for the convergence of Lévy construction of Brownian motion

Fix $n \in \mathbb{N}$. Set

$$\Delta t_n = \frac{T}{n}, t_k = k\Delta t_n, k = 0, \dots, n.$$

Let $(Z_{n,i})_{i=0}^{n-1}$ be i.i.d. $N(0,1)$on some probability space. Define

$$\Delta B_i^{(n)} := \sqrt{\Delta t_n}\, Z_{n,i}, B_{t_k}^{(n)} := \sum_{i=0}^{k-1} \Delta B_i^{(n)}, B_{t_0}^{(n)} := 0.$$

Then:

- each increment $\Delta B_i^{(n)}$ is $N(0, \Delta t_n)$,
- increments over disjoint intervals are independent.

Thus, the vector $(B_{t_0}^{(n)}, \dots, B_{t_n}^{(n)})$is centered Gaussian with covariance

$$\operatorname{Cov}(B_{t_k}^{(n)}, B_{t_\ell}^{(n)}) = \mathbb{E}[\sum_{i<k} \Delta B_i^{(n)} \sum_{j<\ell} \Delta B_j^{(n)}] = \sum_{i<\mathrm{mi}\quad(k,\ell)} \Delta t_n = \min{(t_k, t_\ell)}.$$

So, for the grid points, $B^{(n)}$already has the same finite-dimensional distributions as Brownian motion restricted to that grid.

Define the continuous-time process $B^{(n)}(t)$ on $[0, T]$ by linear interpolation:

For $t \in [t_k, t_{k+1}]$, $$B^{(n)}(t) = B_{t_k}^{(n)} + \frac{t-t_k}{\Delta t_n}\,(B_{t_{k+1}}^{(n)} - B_{t_k}^{(n)}).$$

Then:

- $B^{(n)}$is continuous on $[0, T]$,
- $B^{(n)}$is a Gaussian process (finite-dimensional distributions are multivariate normal, as linear transforms of Gaussians),
- at the grid points we still have covariance $\min{(t_k, t_\ell)}$.

At this point, for each fixed n, $B^{(n)}$ is some continuous Gaussian process built from Gaussian increments. We have not yet defined the limit process B.

Now we use the standard existence theorem for Brownian motion.

Define a family of finite-dimensional distributions by: for any $0 \leq t_1 < \cdots < t_m \leq T$, the vector $(X_{t_1}, \ldots, X_{t_m})$ is centered Gaussian with $\operatorname{Cov}(X_{t_i}, X_{t_j}) = \min{(t_i, t_j)}$.

By the Kolmogorov extension theorem, there exists a probability space and a process $X = (X_t)_{t\in[0,T]}$with these finite-dimensional distributions.

By Kolmogorov's continuity theorem, because the covariance structure implies suitable moment bounds (e.g. $\mathbb{E}[(X_t - X_s)^4] = C \mid t - s \mid^2$), there exists a continuous modification of X. Call this continuous version B.

Then B is, by definition, a standard Brownian motion on $[0, T]$:

- Gaussian increments,
- covariance structure $\min{(s, t)}$,
- continuous paths almost surely,
- $B_0 = 0$.

So, at this stage we *have* a standard Brownian motion B.

To talk about almost sure convergence, we need $B^{(n)}$and Bon the same probability space.

One clean way (common in probability theory) is:

- Work on the canonical space $(C[0, T], \mathcal{B}(C[0, T]), \mathbb{W})$, where $C[0, T]$is the space of continuous real-valued functions, $\mathcal{B}$its Borel sigma-algebra, and $\mathbb{W}$the Wiener measure.
- Let $B_t(\omega) = \omega(t)$be the coordinate process. Under $\mathbb{W}$, $B$is standard Brownian motion.

Now define $B^{(n)}$as a functional of B on this same space:

$$B^{(n)}(t,\omega) := \text{piecewise linear interpolation of } \{B(t_k,\omega)\}_{k=0}^{n}.$$

That is, we are no longer building $B^{(n)}$directly from independent Z_i; instead, we are interpolating the actual Brownian path.

But since for fixed $n$the increments $B(t_{k+1}) - B(t_k)$ have the same law as the $\sqrt{\Delta t_n} Z_{n,k}$, the process $B^{(n)}$has the same law as the discrete construction using $Z_{n,k}$. So, we are not cheating: this is just a coupling.

The advantage: now both $B^{(n)}$and $B$are built on the same space, so we can talk about almost sure convergence.

Fix ω such that the path $t \mapsto B(t,\omega)$ is continuous on $[0,T]$. This happens on a set of probability 1, since Brownian paths are almost surely continuous.

On a compact interval, every continuous function is uniformly continuous. So, for this ω:

For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|t-s| < \delta \Rightarrow |B(t,\omega) - B(s,\omega)| < \varepsilon$.

Now consider a partition with mesh $|\pi_n| = \Delta t_n < \delta$. On each interval $[t_k, t_{k+1}]$,

$$B^{(n)}(t,\omega) = B(t_k,\omega) + \frac{t-t_k}{\Delta t_n}(B(t_{k+1},\omega) - B(t_k,\omega)),$$

so the maximum error on that interval is bounded by the maximum oscillation of $B(\cdot,\omega)$on $[t_k, t_{k+1}]$:

$$\sup_{t\in[t_k,t_{k+1}]} \left|B^{(n)}(t,\omega) - B(t,\omega)\right| \le \sup_{t\in[t_k,t_{k+1}]} \max\left(|B(t,\omega) - B(t_k,\omega)|,\ |B(t,\omega) - B(t_{k+1},\omega)|\right) < \varepsilon,$$

because both $|t-t_k|$ and $|t-t_{k+1}|$ are at most $\Delta t_n < \delta$. Taking the supremum over all k yields

$$\sup_{t\in[0,T]} \left|B^{(n)}(t,\omega) - B(t,\omega)\right| \le \varepsilon,$$

for all sufficiently large n (i.e. all partitions with mesh smaller than δ).

Thus, for every ω in a probability-1 set,

$$\sup_{t\in[0,T]} |\ B^{(n)}(t,\omega) - B(t,\omega)\ | \underset{n\to\infty}{\rightarrow} 0.$$

Equivalently,

$$\sup_{t\in[0,T]} |\ B^{(n)}(t) - B(t)\ | \overset{\text{a.s.}}{\underset{n\to\infty}{\rightarrow}} 0.$$

This is exactly the almost sure uniform convergence on $[0,T]$.

A.3 Long-time Complete Convergence proof

Define $X_n = B_n/n$. We first prove that $X_n \to 0$ completely.

Fix $\varepsilon > 0$. Since $B_n \sim N(0,n)$, we can write

$$\frac{B_n}{\sqrt{n}} = dZ$$

with $Z \sim N(0,1)$. Then

$$\mathbb{P}\left(\frac{|B_n|}{n} > \varepsilon\right) = \mathbb{P}(|B_n| > \varepsilon n) = \mathbb{P}\left(\left|\frac{B_n}{\sqrt{n}}\right| > \varepsilon\sqrt{n}\right) = \mathbb{P}(|\mathrm{Z}| > \varepsilon\sqrt{n}).$$

Using a standard Gaussian tail bound, there exists a constant $c > 0$(for instance $c = 1/2$) such that for all $x > 0$, $\mathbb{P}(| Z |> x) \leq 2e^{-cx^2}$.

Applying this with $x = \varepsilon\sqrt{n}$ gives

$$\mathbb{P}\left(\frac{|B_n|}{n} > \varepsilon\right) \leq 2e^{-c\varepsilon^2 n}.$$

Therefore

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{|B_n|}{n} > \varepsilon\right) \leq \sum_{n=1}^{\infty} 2e^{-c\varepsilon^2 n} < \infty,$$

since the right-hand side is a convergent geometric series. By the definition of complete convergence, this shows

$$\frac{B_n}{n} \longrightarrow 0 \text{ completely as } n \to \infty.$$

In particular, by Borel–Cantelli, we recover the almost sure limit

$$\frac{B_n}{n} \longrightarrow 0 \text{ a.s.}$$

Notice how independence and Gaussian tails enter crucially here: the exact law of B_n as a sum of independent increments allows us to compute (and then summably bound) the tail probabilities.

We now extend the result from integer times to all $t \geq 0$. There are different ways to phrase “complete convergence” in continuous time; a natural one, consistent with the discrete definition, is:

For $\varepsilon > 0$, consider the events

$$A_n(\varepsilon) := \left\{ \sup_{t\in[n,n+1]} \frac{|B_\mathrm{t}|}{\mathrm{t}} > \varepsilon \right\}$$

We say that $B_t/t \to 0$completely if, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n(\varepsilon)) < \infty.$$

This implies that $\sup_{t\in[n,n+1]} | B_t |/t \to 0$almost surely, and hence $B_t/t \to 0$almost surely as $t \to \infty$.

We now show that the series of $\mathbb{P}(A_n(\varepsilon))$is finite.

Fix $\varepsilon > 0$and $n \geq 1$. For $t \in [n, n+1]$, we have $t \geq n$, hence

$$\sup_{t\in[n,n+1]} \frac{|B_\mathrm{t}|}{\mathrm{t}} \leq \sup_{t\in[n,n+1]} \frac{|B_\mathrm{t}|}{\mathrm{n}}.$$

Using the decomposition $B_t = B_n + (B_t - B_n)$, we obtain

$$\sup_{t\in[n,n+1]} \frac{|B_\mathrm{t}|}{\mathrm{n}} \leq \frac{|B_n|}{n} + \frac{1}{n} \sup_{t\in[n,n+1]} |B_t - B_n|.$$

Therefore

$$A_n(\varepsilon) \subseteq \left\{\frac{|B_n|}{n} > \frac{\varepsilon}{2}\right\} \cup \left\{\frac{1}{n} \sup_{t\in[n,n+1]} |B_t - B_n| > \frac{\varepsilon}{2}\right\}.$$

Taking probabilities and using the union bound yields

$$\mathbb{P}(A_n(\varepsilon)) \leq \mathbb{P}\left(\frac{|B_n|}{n} > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(\sup_{t\in[n,n+1]} |B_t - B_n| > \frac{\varepsilon n}{2}\right).$$

We already controlled the first term in the previous subsection:

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{|B_n|}{n} > \frac{\varepsilon}{2}\right) < \infty.$$

It remains to show that the series of the second term is also finite. For this, we use stationarity and scaling of increments, plus a Gaussian tail bound on the supremum over an interval.

By stationary independent increments, the process $(B_{n+s} - B_n)_{0\leq s\leq 1}$has the same law as $(B_s)_{0\leq s\leq 1}$. Hence

$$\sup_{t\in[n,n+1]} |B_t - B_n| = d \sup_{0\leq s\leq 1} |B_\mathrm{s}|.$$

Thus $\mathbb{P}\left(\sup_{t\in[n,n+1]} |B_t - B_n| > \frac{\varepsilon n}{2}\right) = \mathbb{P}\left(\sup_{0\leq s\leq 1} |B_\mathrm{s}| > \frac{\varepsilon n}{2}\right)$

A standard reflection principle argument shows that $\mathbb{P}(\sup_{0\leq s\leq 1} B_s > x) = 2\,\mathbb{P}(B_1 > x)$,

and similarly one can bound $\mathbb{P}(\sup_{0\leq s\leq 1} \mid B_s \mid > x) \leq 4\,\mathbb{P}(B_1 > x)$.

Using again the Gaussian tail bound for $B_1 \sim N(0,1)$, there exist constants $C, c > 0$such that

$$\mathbb{P}(\sup_{0\leq s\leq 1} \mid B_s \mid > x) \leq Ce^{-cx^2}, \text{for all } x > 0.$$

Applying this with $x = \varepsilon n/2$gives

$$\mathbb{P}\left(\sup_{t\in[n,n+1]}|B_t - B_n| > \frac{\varepsilon n}{2}\right) \leq C\exp\left(-c\left(\frac{\varepsilon n}{2}\right)^2\right) = C\exp(-c'\varepsilon^2 n^2),$$

for some constant $c' > 0$. Therefore

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\sup_{t\in[n,n+1]}|B_t - B_n| > \frac{\varepsilon n}{2}\right) \leq \sum_{n=1}^{\infty} Ce^{-c'\varepsilon^2 n^2} < \infty,$$

since the series has super-exponentially decaying terms.

Combining everything, we obtain

$$\sum_{n=1}^{\infty}\mathbb{P}(A_n(\varepsilon)) \leq \sum_{n=1}^{\infty}\mathbb{P}\left(\frac{|B_n|}{n} > \frac{\varepsilon}{2}\right) + \sum_{n=1}^{\infty}\mathbb{P}\left(\sup_{t\in[n,n+1]}|B_t - B_n| > \frac{\varepsilon n}{2}\right) < \infty$$

for every $\varepsilon > 0$. By definition, this is exactly complete convergence of B_t/t to 0 as $t \to \infty$.

Thus, in a strong sense (complete convergence), Brownian motion does not accumulate a linear drift: the ratio B_t/t tends to 0 almost surely, with tail probabilities decaying fast enough to be summable.

B.1 Proof of $\mathrm{M_t}$ being a positive martingale

For every $\omega \in \Omega$and $t \geq 0$, $M_t(\omega) = \exp(\theta B_t(\omega) - \frac{1}{2}\theta^2 t) > 0$, so the process is strictly positive.

Fix $t \geq 0$. Since $B_t \sim \mathrm{N}(0,t)$, the random variable θB_t is Gaussian with mean 0and variance $\theta^2 t$. The moment generating function of a centered Gaussian is finite for all real arguments; in particular,

$$\mathbb{E}_{\mathbb{P}}[e^{\theta B_t}] = \exp\left(\frac{1}{2}\theta^2 t\right) < \infty.$$

Therefore

$$\mathbb{E}_{\mathbb{P}}[M_t] = \mathbb{E}_{\mathbb{P}}\left[e^{\theta B_t - \frac{1}{2}\theta^2 t}\right] = e^{-\frac{1}{2}\theta^2 t}\,\mathbb{E}_{\mathbb{P}}[e^{\theta B_t}] = e^{-\frac{1}{2}\theta^2 t}\,e^{\frac{1}{2}\theta^2 t} = 1.$$

In particular, $M_t \in L^1(\mathbb{P})$for every t, so the process is integrable at all times.

Define the function $f: \mathbb{R} \times [0,\infty) \to \mathbb{R}$by

$$f(x,t) := \exp(\theta x - \frac{1}{2}\theta^2 t).$$

Then $M_t = f(B_t, t)$. We compute the partial derivatives:

$$\frac{\partial f}{\partial x}(x,t) = \theta\exp(\theta x - \frac{1}{2}\theta^2 t) = \theta f(x,t),$$
$$\frac{\partial^2 f}{\partial x^2}(x,t) = \theta^2\exp(\theta x - \frac{1}{2}\theta^2 t) = \theta^2 f(x,t),$$

$$\frac{\partial f}{\partial t}(x,t) = -\frac{1}{2}\theta^2 \exp\left(\theta x - \frac{1}{2}\theta^2 t\right) = -\frac{1}{2}\theta^2 f(x,t).$$

Now apply Itô's formula to the Itô process $Y_t = B_t$(with dynamics $dB_t = 1 \cdot dB_t$, zero drift) and function f:

$$dM_t = df(B_t,t) = \frac{\partial f}{\partial x}(B_t,t)\, dB_t + \frac{\partial f}{\partial t}(B_t,t)\, dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(B_t,t)\, dt.$$

$$dM_t = \theta f(B_t,t)\, dB_t + (-\frac{1}{2}\theta^2 f(B_t,t))\, dt + \frac{1}{2}\theta^2 f(B_t,t)\, dt.$$

Hence,

$$dM_t = \theta f(B_t,t)\, dB_t = \theta M_t\, dB_t.$$

Thus $(M_t)_{t\geq 0}$is a continuous local martingale given by the stochastic integral

$$M_t = M_0 + \int_0^t \theta M_s\ dB_s = 1 + \int_0^t \theta M_s\ dB_s,$$

since $M_0 = \exp\left(\theta B_0 - \frac{1}{2}\theta^2 \cdot 0\right) = 1.$

The integrand $\theta M_s$is adapted and, as we saw above, square-integrable on finite intervals (because $M_s \in L^2$for each fixed s). Hence the stochastic integral is well-defined, and $M$is a continuous local martingale. At this stage, we still need to show it is a true martingale (not just local).

Let $0 \leq s \leq t$. Using the independent increments of Brownian motion, write

$$B_t = B_s + (B_t - B_s).$$

Then $M_t = \exp\left(\theta B_t - \frac{1}{2}\theta^2 t\right) = \exp\left(\theta B_s - \frac{1}{2}\theta^2 s\right)\exp\left(\theta(B_t - B_s) - \frac{1}{2}\theta^2 (t-s)\right).$

Define $M_s := \exp\left(\theta B_s - \frac{1}{2}\theta^2 s\right), Z_{s,t} := \exp\left(\theta(B_t - B_s) - \frac{1}{2}\theta^2 (t-s)\right),$

so that $M_t = M_s \cdot Z_{s,t}$.

Now note:

1. $M_s$is $\mathcal{F}_s$-measurable by construction.
2. $B_t - B_s$is independent of $\mathcal{F}_s$and distributed as $\mathcal{N}(0, t-s)$by the definition of Brownian motion. Hence $Z_{s,t}$is independent of $\mathcal{F}_s$.
3. By the same Gaussian moment generating function computation as in Step 1,
   $\mathbb{E}_{\mathbb{P}}[Z_{s,t}] = \mathbb{E}_{\mathbb{P}}[e^{\theta(B_t - B_s) - \frac{1}{2}\theta^2 (t-s)}] = e^{-\frac{1}{2}\theta^2 (t-s)}\ \mathbb{E}[e^{\theta(B_t - B_s)}] = e^{-\frac{1}{2}\theta^2 (t-s)} e^{\frac{1}{2}\theta^2 (t-s)} = 1.$

We can now compute the conditional expectation:

$$\mathbb{E}_{\mathbb{P}}[M_t \mid \mathcal{F}_s] = \mathbb{E}_{\mathbb{P}}[M_s Z_{s,t} \mid \mathcal{F}_s].$$

Since $M_s$is $\mathcal{F}_s$-measurable and $Z_{s,t}$is independent of $\mathcal{F}_s$, we can pull M_s outside and replace $Z_{s,t}$ by its unconditional expectation:

$$\mathbb{E}_{\mathbb{P}}[M_t \mid \mathcal{F}_s] = M_s\, \mathbb{E}_{\mathbb{P}}[Z_{s,t} \mid \mathcal{F}_s] = M_s\, \mathbb{E}_{\mathbb{P}}[Z_{s,t}] = M_s \cdot 1 = M_s.$$

Thus, for all $0 \le s \le t$, $\mathbb{E}_{\mathbb{P}}[M_t \mid \mathcal{F}_s] = M_s$ almost surely,

so $(M_t)_{t\ge0}$is a martingale with respect to $(\mathcal{F}_t)$.

Note that this also implies

$$\mathbb{E}_{\mathbb{P}}[M_t] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[M_t \mid \mathcal{F}_0]] = \mathbb{E}_{\mathbb{P}}[M_0] = 1,$$

We have shown:

- $M_t > 0$almost surely for all $t \ge 0$.
- $M_t \in L^1(\mathbb{P})$for all t, with $\mathbb{E}_{\mathbb{P}}[M_t] = 1$.
- $(M_t)_{t\ge0}$is a continuous local martingale (via its SDE $dM_t = \theta M_t\, dB_t$).
- Using independent increments and conditional expectations, we verified the martingale property: $\mathbb{E}_{\mathbb{P}}[M_t \mid \mathcal{F}_s] = M_s$for all $0 \le s \le t$.

Hence $(M_t)_{t\ge0}$is a strictly positive $\mathbb{P}$-martingale with unit expectation, as claimed.

B.2 Proof of Martingale property of Discounted Asset process

We first use Itô's formula to compute the dynamics of $\tilde{S}_t$.

Consider the function $f: [0,\infty) \times (0,\infty) \to \mathbb{R}$ given by $f(t,x) = e^{-rt}x$.

Then $\tilde{S}_t = f(t, S_t)$. We compute the partial derivatives:

- Derivatives in t: $\frac{\partial f}{\partial t}(t,x) = -re^{-rt}x = -rf(t,x)$,
- Derivatives in x: $\frac{\partial f}{\partial x}(t,x) = e^{-rt}$, $\frac{\partial^2 f}{\partial x^2}(t,x) = 0$.

Under $\mathbb{Q}$, the Itô process S_t satisfies

$$dS_t = rS_t\, dt + \sigma S_t\, dW_t.$$

By Itô's formula for a function $f(t,x)$of a one-dimensional Itô process X_t,

$$df(t,X_t) = \frac{\partial f}{\partial t}(t,X_t)\, dt + \frac{\partial f}{\partial x}(t,X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\, (dX_t)^2.$$

Applying this to $X_t = S_t$, we get

$$d\tilde{S}_t = \frac{\partial f}{\partial t}(t,S_t)\, dt + \frac{\partial f}{\partial x}(t,S_t)\, dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,S_t)\, (dS_t)^2.$$

Substitute the derivatives and the SDE for S_t:

- $\frac{\partial f}{\partial t}(t,S_t) = -re^{-rt}S_t = -r\tilde{S}_t$,
- $\frac{\partial f}{\partial x}(t,S_t) = e^{-rt}$,
- $\frac{\partial^2 f}{\partial x^2}(t,S_t) = 0$,

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t.$$

Thus

$$\begin{aligned} d\tilde{S}_t \quad &= (-r\tilde{S}_t)\,dt + e^{-rt}(rS_t\,dt + \sigma S_t\,dW_t) + \frac{1}{2}\cdot 0\cdot (dS_t)^2 \\ &= (-re^{-rt}S_t)\,dt + e^{-r}\ rS_t\,dt + e^{-rt}\sigma S_t\,dW_t \\ &= 0\cdot dt + \sigma e^{-r}\ S_t\,dW_t \\ &= \sigma\tilde{S}_t\,dW_t. \end{aligned}$$

So $\tilde{S}$ satisfies the SDE

$$d\tilde{S}_t = \sigma\tilde{S}_t\,dW_t, \qquad \tilde{S}_0 = S_0.$$

In particular,

$$\tilde{S}_t = \tilde{S}_0 + \int_0^t \sigma\tilde{S}_u\ dW_u = S_0 + \int_0^t \sigma\tilde{S}_u\ dW_u.$$

This shows that $\tilde{S}$is a continuous local martingale (being a stochastic integral with respect to Brownian motion). To conclude it is a true martingale, we need to verify appropriate integrability and the martingale property via conditional expectations.

We can also solve the SDE for $S_t$explicitly under $\mathbb{Q}$. The equation

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t$$

is a GBM SDE. Using the same Itô trick as before (apply Itô to $\ln S_t$), one obtains

$$S_t = S_0\exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t\right).$$

Therefore, the discounted process is

$$\tilde{S}_t = e^{-rt}S_t = S_0\exp\left(-rt + (r - \frac{1}{2}\sigma^2)t + \sigma W_t\right) = S_0\exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right).$$

Define $N_t := \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)$.

Then $\tilde{S}_t = S_0 N_t$.

By the exponential martingale result you proved earlier (with $\theta = \sigma$), the process $(N_t)_{t\geq 0}$is a strictly positive $\mathbb{Q}$–martingale, satisfies

$$\mathbb{E}_{\mathbb{Q}}[N_t] = 1, t \geq 0,$$

and has finite moments of all orders. Hence $\tilde{S}_t$is integrable for each t, with

$$\mathbb{E}_{\mathbb{Q}}[\mid \tilde{S}_t \mid] = S_0\mathbb{E}_{\mathbb{Q}}[N_t] = S_0 < \infty.$$

So $(\tilde{S}_t)$is not only a local martingale but also integrable at every time t.

To verify the martingale property, fix $0 \leq s \leq t$. Using the explicit representation,

$$\tilde{S}_t = S_0 \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right).$$

Write

$$W_t = W_s + (W_t - W_s),$$

where $W_t - W_s$is independent of $\mathcal{F}_s$and distributed as $\mathcal{N}(0, t-s)$under $\mathbb{Q}$. Then

$$\tilde{S}_t = S_0 exp\left(\sigma W_s + \sigma(W_t - W_s) - \frac{1}{2}\sigma^2 s - \frac{1}{2}\sigma^2(t-s)\right)$$

$$\tilde{S}_t = \left[S_0 \exp\left(\sigma W_s - \frac{1}{2}\sigma^2 s\right)\right] \cdot \exp\left(\sigma(W_t - W_s) - \frac{1}{2}\sigma^2(t-s)\right)$$

Recognize the first factor as $\tilde{S}_s$: $\tilde{S}_s = S_0 \exp\left(\sigma W_s - \frac{1}{2}\sigma^2 s\right)$,

and define

$$Z_{s,t} := \exp\left(\sigma(W_t - W_s) - \frac{1}{2}\sigma^2(t-s)\right).$$

Thus

$$\tilde{S}_t = \tilde{S}_s \cdot Z_{s,t}.$$

Now:

1. $\tilde{S}_s$is $\mathcal{F}_s$–measurable (it is a measurable function of $(W_u)_{0 \leq u \leq s}$).
2. $W_t - W_s$is independent of $\mathcal{F}_s$and has law $\mathcal{N}(0, t-s)$; hence $Z_{s,t}$ is independent of $\mathcal{F}_s$.
3. As before, by the Gaussian moment generating function,

$$\mathbb{E}_{\mathbb{Q}}[Z_{s,t}] = \mathbb{E}_{\mathbb{Q}}[e^{\sigma(W_t - W_s) - \frac{1}{2}\sigma^2(t-s)}] = e^{-\frac{1}{2}\sigma^2(t-s)}\, \mathbb{E}_{\mathbb{Q}}[e^{\sigma(W_t - W_s)}] = 1.$$

We now compute the conditional expectation:

$$\mathbb{E}_{\mathbb{Q}}[\tilde{S}_t \mid \mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[\tilde{S}_s Z_{s,t} \mid \mathcal{F}_s].$$

Since $\tilde{S}_s$is $\mathcal{F}_s$–measurable and $Z_{s,t}$is independent of $\mathcal{F}_s$, we may pull $\tilde{S}_s$out of the conditional expectation and replace $Z_{s,t}$by its unconditional expectation:

$$\mathbb{E}_{\mathbb{Q}}\left[\tilde{S}_t \mid \mathcal{F}_s\right] = \tilde{S}_s\, \mathbb{E}_{\mathbb{Q}}\left[Z_{s,t} \mid \mathcal{F}_s\right] = \tilde{S}_s\, \mathbb{E}_{\mathbb{Q}}\left[Z_{s,t}\right] = \tilde{S}_s \cdot 1 = \tilde{S}_s.$$

This holds almost surely for every $0 \leq s \leq t$. Together with the integrability established, it shows that $(\tilde{S}_t)_{t \geq 0}$ is a $\mathbb{Q}$–martingale.