THEORY BEHIND CALIBRATION OF IMPLIED VOLATILITY TREE

This document is but the summary to help with already seen materials on the topic

DEFINITONS

After calibrations, we Put

$$u(n,j) = \frac{S(n+1,j+1)}{S(n,j)};\ d(n,j) = \frac{S(n+1,j)}{S(n,j)}$$
$$\tilde{p}(n,j) = \frac{1 + r(n,j) - d(n,j)}{u(n,j) - d(n,j)}$$

ARROW-DEBREW SECURTITES:

- $\lambda(n,j)$ value at t_0 of a derivative $\begin{cases} 1\$ & \text{at node } (n,j) \\ 0 & \text{else.} \end{cases}$
- AD prices follow a recursive relationship called Jamshidian's forward induction formula.
- in binomial model, we have:

$$V(n,j) = \frac{1}{R(n,j)}\left[\tilde{p}(n,j)V(n+1,j+1) + \tilde{q}(n,j)\,V(n+1,j)\right]$$

also:

$$V(0,0) = \sum_{j=0}^{N} \lambda(N,j)V(N,j)$$

Setting $\begin{cases} \lambda(0,0) = 1 \\ \lambda(n,j) = 0, \ \ j < 0 \text{ or } j > n. \end{cases}$

The price at time to of AD that pays 1$ at (n,j) should be the same as the security having the following payoff at n-1:

for $1 \leqslant j \leqslant n-1$:

$$V(n-1,R) = \begin{cases} 0, & k > j \\ \dfrac{1 - \hat{p}(n-1,j)}{R(n-1,j)}, & k = j \\ \dfrac{\hat{p}(n-1,j-1)}{R(n-1,j-1)}, & k = j-1 \\ 0, & k < j-1 \end{cases}$$

Therefore, for $1 \leqslant j \leqslant n-1$:

$$\lambda(n,j) = \frac{1-\hat{p}(n-1,j)}{R(n-1,j)}\lambda(n-1,j) + \frac{\hat{p}(n-1,j-1)}{R(n-1,j-1)}\lambda(n-1,j-1)$$

For j = 0:

$$V(n-1,R) = \begin{cases} 0, & k > 0 \\ \dfrac{1-\hat{p}(n-1,0)}{R(n-1,0)}, & k = 0 \end{cases}$$

Thus:

$$\lambda(n,j) = \begin{cases} \dfrac{1-\hat{p}(n-1,j)}{R(n-1,j)}\lambda(n-1,j), & j = 0 \\ \dfrac{1-\hat{p}(n-1,j)}{R(n-1,j)}\lambda(n-1,j) + \dfrac{\hat{p}(n-1,j-1)}{R(n-1,j-1)}\lambda(n-1,j-1), & 1 \leq j \leq n \\ \dfrac{\hat{p}(n-1,j-1)}{R(n-1,j-1)}\lambda(n-1,j-1), & j = n \end{cases}$$

CASE 1

S(n-1, j) and S(n, j+1) are known , S(n, j) is unknown

Let K = S(n-1, j)

$$V^{put}(n-1,j) = \frac{1}{R(n-1,j)}[(1-\hat{\rho}(n-1,j))(K-S(n,j))]$$

By the definition of $\hat{\rho}$, we have:

$$1-\hat{\rho}(n-1,j) = \frac{S(n,j+1) - R(n-1,j)S(n-1,j)}{S(n,j+1) - S(n,j)}$$

$$1-\hat{\rho}(n-1,j) = \frac{S(n,j+1) - R(n-1,j)K}{S(n,j+1) - S(n,j)}$$

Substituting (3) into $V^{put}(n-1,j)$ and solving for $S(n,j)$, we get:

$$S(n,j) = \frac{V^{put}(n-1,j)S(n,j+1) + K\left(K - \dfrac{S(n,j+1)}{R(n-1,j)}\right)}{V^{put}(n-1,j) + K - \dfrac{S(n,j+1)}{R(n-1,j)}}$$

$$S(n,j) = \frac{V^{put}(n-1,j)S(n,j+1) + S(n-1,j)\left(S(n-1,j) - \dfrac{S(n,j+1)}{R(n-1,j)}\right)}{V^{put}(n-1,j) + S(n-1,j) - \dfrac{S(n,j+1)}{R(n-1,j)}}$$

CASE 2

S(n-1, j) and S(n, j) are known , S(n, j+1) is unknown

Let K = S(n-1, j) . Then,

$$V^{call}(n-1,j) = \frac{1}{R(n-1,j)}\hat{\rho}(n-1,j)[S(n,j+1)-K]$$

We use:

$$\hat{\rho}(n-1,j) = \frac{R(n-1,j)K - S(n,j)}{S(n,j+1)-S(n,j)}$$

to find:

$$S(n,j+1) = \frac{V^{call}(n-1,j)S(n,j) + K\left[\frac{S(n,j)}{R(n-1,j)} - K\right]}{V^{call}(n-1,j) + \frac{S(n,j)}{R(n-1,j)} - K}$$

Therefore:

$$S(n,j+1) = \frac{V^{call}(n-1,j)S(n,j) + S(n-1,j)\left[\frac{S(n,j)}{R(n-1,j)} - S(n-1,j)\right]}{V^{call}(n-1,j) + \frac{S(n,j)}{R(n-1,j)} - S(n-1,j)}$$

CASE 3

Let K = S(n-1, j)

We have:

$S(n, j+1) = K\, u(n-1, j)$
$S(n, j) = K\, d(n-1, j)$
$u(n-1, j) * d(n-1, j) = 1$

$$\begin{aligned} V^{put} &= \frac{1}{R(n-1,j)}\left[(1-\hat{\rho}(n-1,j))(K - S(n,j))\right] \\ &= \frac{1}{R(n-1,j)}\left[\frac{u(n-1,j) - R(n-1,j)}{u(n-1,j) - d(n-1,j)}\right][1 - d(n-1,j)]K \\ &= \frac{1}{R(n-1,j)}\left[\frac{u(n-1,j) - R(n-1,j)}{u(n-1,j) - \frac{1}{u(n-1,j)}}\right]\left[1 - \frac{1}{u(n-1,j)}\right]K \\ V^{put} &= \frac{1}{R(n-1,j)}\frac{u(n-1,j) - R(n-1,j)}{u(n-1,j) + 1}K \end{aligned}$$

Which can be solved for $u(n-1, j)$:

$$u(n-1,j) = \frac{S(n-1,j) + V^{put}(n-1,j)}{\frac{S(n-1,j)}{R(n-1,j)} - V^{put}(n-1,j)}$$

Thus:

$$S(n, j+1) = S(n-1,j)u(n-1,j)$$
$$S(n,j) = \frac{S(n-1,j)}{u(n-1,j)}$$

CALIBRATION OF IMPLIED VOLATILITY: INPUTS

- $P^{(n,k)}(0,0)$: price at t_o of European put.
- $C^{(n,k)}(0,0)$: price at t_o of european call.
- need to calculate $V^{\text{put}}(n-1,j)$ and $V^{\text{call}}(n-1,j)$ with $K = S(n-1,j)$
  We HAVE

$$p^{(n,k)}(n-1,k) = \frac{[\tilde{\rho}(n-1,k)(K-S(n,k+1))^+ + [1-\tilde{p}(n-1,k)](K-S(n,k))^+]}{R(n-1,k)}$$

$$P^{(n,k)}(n-1,k) = \frac{1}{R(n-1,k)}[K-S(n-1,k)R(n-1,k)]$$

We have $P^{(n,k)}(n-1,k) = 0$ if $k > j$

$$P^{(n,k)}(n-1,j) = V^{put}(n-1,j)$$

Using Arrow Debrew:

$$\begin{aligned} P^{(n,k)}(0,0) &= \sum_{k=0}^{j} \lambda(n-1,k)P^{(n,k)}(n-1,k) \\ &= \sum_{k=0}^{j-1} \lambda(n-1,k)P^{(n,k)}(n-1,k) + \lambda(n-1,j)V^{put}(n-1,j) \\ &= \textstyle\sum_p (n-1,j) + \lambda(n-1,j)V^{put}(n-1,j) \end{aligned}$$

where $\sum_p (n-1,j) = \sum_{k=0}^{j-1} \lambda(n-1,k)p^{(n,k)}(n-1,k)$

which we could solve for $V^{put}(n-1,j)$

SIMILARLY

$$V^{call}(n-1,j) = \frac{C^{(n,K)}(0,0) - \sum_c \ (n-1,j)}{\lambda(n-1,j)}$$

where $\sum_c (n-1),) = \sum_{k=j+1}^{n-1} \lambda(n-1,k)\ C^{(n,k)}(n-1,k)$

$$k = S(n-1,j)$$

At even stage check.

$$S(n,j) < S(n-1,j)R(n-1,j) < S(n,j+1)$$

CASE:

$$\begin{cases} S(n,j+1) & \text{Known} \\ S(n,j) > & S(n-1,j)R(n-1,j) \end{cases}$$

REPLACE:

$$S(n,j) \rightarrow S'(n,j) = \frac{S(n,j+1)S(n-1,j)}{S(n-1,j+1)}$$

CASE:

$$\begin{cases} S(n,j) & \text{Known} \\ S(n,j+1) < & S(n-1,j)R(n-1,j) \end{cases}$$

REPLACE:

$$S(n,j+1) \rightarrow S'(n,j+1) = \frac{S(n,j)S(n-1,j)}{S(n-1,j+1)}$$

Sources

- Shreve, S. E. (2003). *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*. Springer.
- Van der Hoek, J., & Elliott, R. J. (2006). *Binomial Models in Finance*. Springer.
- Derman, E., & Kani, I. (1994). The volatility smile and its implied tree. *Goldman Sachs Quantitative Strategies Research Notes*.
- Hindman, C. *MACF 402 (Mathematical and Computational Finance II): Implied Volatility Trees*. [Slides].