Modeling Spontaneous Emission with a Density Matrix and a Quantum Trajectories Approach

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Problem

- Quantum computer needs quantum bits (qubits)
- Ion trap model for qubit
- Qubit is a two level system, represented by density matrix:

$$\rho = \sum_i p(i) \left|\psi_i\right\rangle \left\langle\psi_i\right| = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix}$$

- $\left|\psi_i\right\rangle$ is either $\left|e\right\rangle$ (excited state) or $\left|g\right\rangle$ (ground state)
- Following approaches can be extended to quantum systems with n levels
- Ion may spontaneously emit a photon and lose energy, transitioning to a lower energy level

Approach

- Losing energy to the environment is a non-unitary time evolution
- Lindblad equation models time evolution of density matrix interacting with environment:

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H, \rho] + \sum_k [L_k \rho, L_k^\dagger] + [L_k, \rho L_k^\dagger]$$

- H is the system Hamiltonian, and L_k are Lindblad operators modeling interaction with environment (emission)

Approach

- When does the emission happen?
- Rewritten version of Lindblad equation:

$$\frac{\partial |\psi\rangle}{\partial t} = -\frac{i}{\hbar} H |\psi\rangle - L^\dagger L |\psi\rangle$$

- Apply emission operator L at random times
- Take average of $|\psi\rangle \langle\psi|$ over many possible trajectories (emission times) to obtain estimate for ρ
- $|\psi\rangle$ is an n vector, ρ is an $n \times n$ matrix: fewer differential equations to solve

Linear Multi-step Methods

- Initial value problem:

$$y' = f(t, y) \text{ s.t. } y(t_0) = y_0$$

- Linear multi-step method:

$$y_{n+s} = \sum_{k=0}^{s-1} a_k \cdot y_{n+k} + h \cdot \left(\sum_{k=0}^{s} b_k \cdot f(t_{n+k}, y_{n+k}) \right)$$

- Discrete-variable method: provide approximations to exact solutions at discrete points $y(t_i) \approx y_i$
- Coefficients $a_0, \ldots, a_{s-1}$ and $b_0, \ldots, b_s$ completely determine the method

Adams-Moulton Method

- Implicit Adams method for non-stiff system
- Coefficients a_i are fixed:

$$a_{s-1} = -1, \quad a_{s-2}, \ldots, a_0 = 0$$

- Coefficients b_i determined by interpolating polynomial passing through points $p(t_{n+i}) = f(t_{n+i}, y_{n+i}) \; \forall \; i \in \{0, \cdots, s-1\}$ using Lagrange polynomial basis

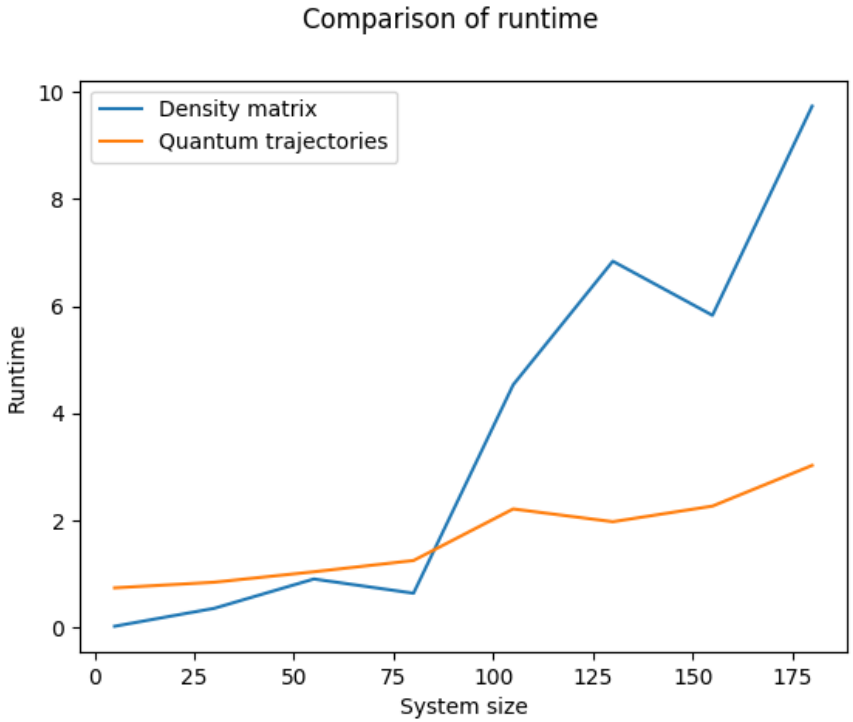
Adams-Moulton Method

- p is a locally good approximation for $y' = f(t, y)$
- Update step:

$$y_{n+s} = y_{n+s-1} + \int_{t_{n+s-1}}^{t_{n+s}} p(t)dt$$

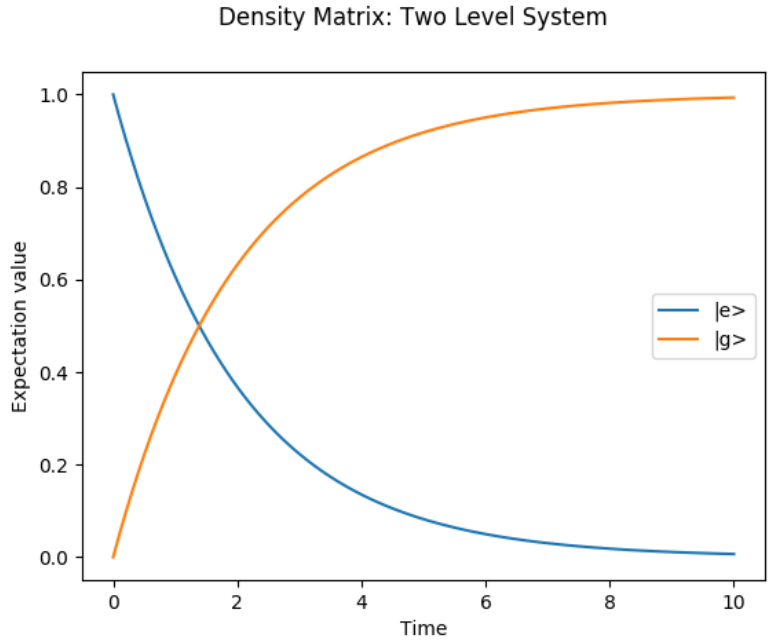
- Replacing $f(t, y)$ by the interpolant p incurs an error of $O(h^{s+1})$
- Implicit Adams method has order $s + 1$

Findings



- Quadratic time complexity for density matrix approach
- Linear for quantum trajectories

Findings


Density Matrix: Two Level System
Expectation value
Time
|e>
|g>
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
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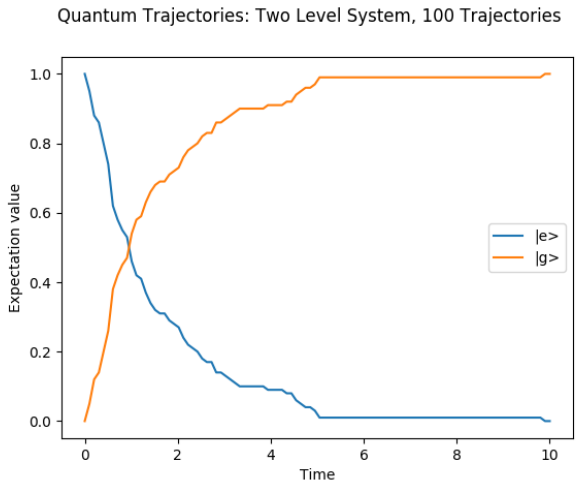
Quantum Trajectories: Two Level System, 100 Trajectories
Expectation value
Time
|e>
|g>
0.0
0.2
0.4
0.6
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1.0
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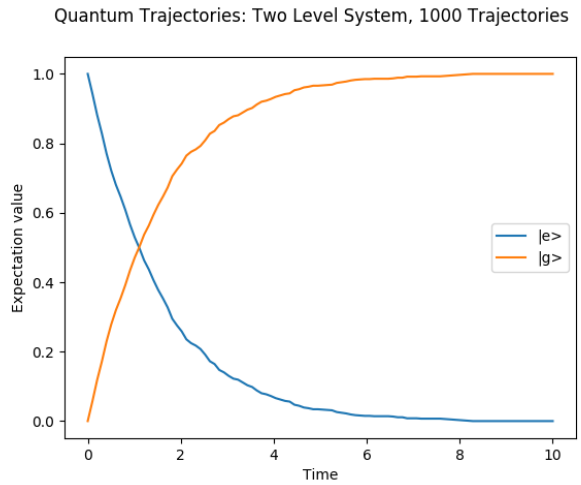
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0.2
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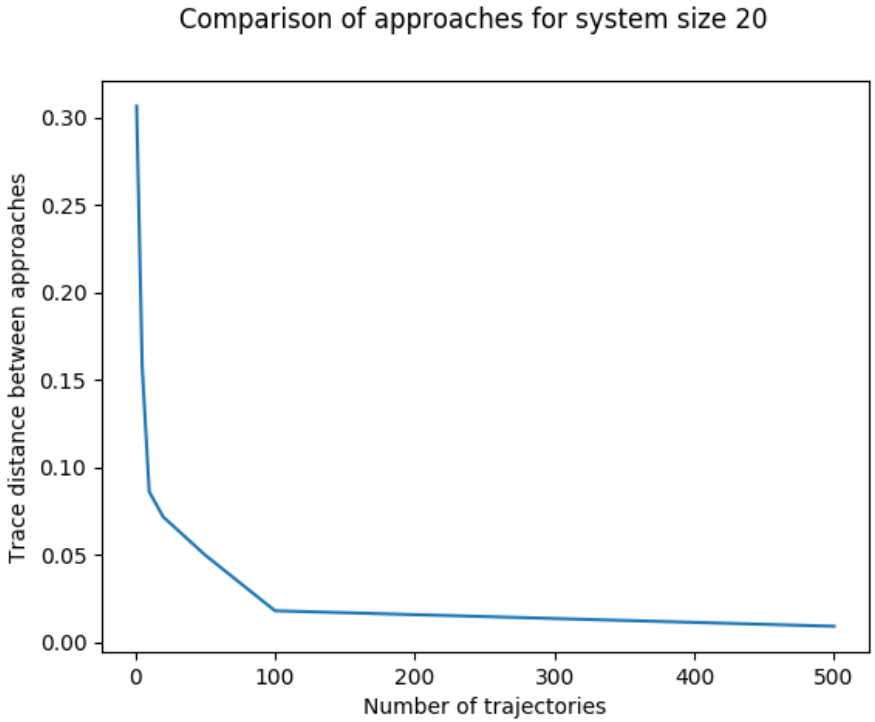
Findings



- Trace distance:

$$T(\rho, \sigma) = \frac{1}{2}\mathrm{Tr}|\rho - \sigma| = \frac{1}{2}\sum_i |\lambda_i|$$

Findings

- Above findings confirm the theoretical equivalence of the two approaches as number of trajectories is increased
- Different Hamiltonians model different physical systems (i.e. qubit, truncated harmonic oscillator, etc.)
- Varying Hamiltonian does not change running time or difference between approaches much (i.e. $H = 0$, $H = constant$, $H = H(t)$)
- Approaches can be used to solve systems which cannot be solved analytically (i.e. perturbed harmonic oscillator)

References

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7. *Quantum toolbox in Python (QuTiP)*
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