Apparent Retrograde Motion of Mars Using Geometric Algebra

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Abstract

A planet orbiting the Sun outside the Earths orbit moves with a different frequency. When observed at Earth, the movement of the planet does not simply just rise and fall similar to the movement of the Sun and the Moon. Eventually, the superior planet would move backwards, or in retrograde, as seen from Earth. In this paper, we will show this apparent retrograde movement using Geometric Algebra. © 2013 Samahang Pisika ng Pilipinas

Keywords: orbital mechanics, geometric algebra, vector analysis

1. Introduction

In the Copernican Heliocentric Model, the retrograde motion of a planet superior (has an orbit outside of Earth's) to Earth can be described as the apparent backwards movement of the superior planet as it travels in the background of fixed stars. The path of a superior planet is already known, using projective/geometric analysis, using multiple straight lines from Earth to the planet, and projecting it to a plane, representing the background. This is shown in [2], using Jupiter, and in [3], using Mars. The latter is the main focus of this paper, though we can show the motion of other superior planets such as Jupiter by modifying the frequency and radius of the other planet.

In this paper, we propose to present a more direct algebraic approach in the description of the orbits about any arbitrary axis. For this, we shall use Geometric Algebra, a vector analysis method combining dot and cross products. What makes this method unique from ordinary vector analysis is that it unifies these two products into a single geometric product that contains the imaginary number. This product is similar to the Pauli Identity used in Quantum Mechanics, which allows us to express vector rotations in a more compact and direct manner using exponentials and imaginary vectors. This method of expressing rotations is much clearer than the standard rotation matrices and Euler Angles.

2. Geometric Algebra

Using three spatial unit vectors, Clifford, or Geometric Algebra is generated where these three unit vectors satisfy the orthonormality axiom

$$\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = 2\delta_{jk}, \tag{1}$$

where δ_{jk} is the Kronecker delta function where it is either 1 if $j = k$ or 0 if jk. Once we apply this to Eq (1), we obtain the following

$$\mathbf{e}_j^2 = 1, \tag{2a}$$

$$\mathbf{e}_j\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_j. \tag{2b}$$

Here, we see the rule that makes Geometric Algebra unique we can multiply vectors without a dot or a cross, by simply juxtaposing them. Using two arbitrary vectors **a** and **b**, we can use the distributive property, and Eq. (2) in order to write

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + i(\mathbf{a}\times\mathbf{b}), \tag{3}$$

where we define i as

$$i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3. \tag{4}$$

It can be easily shown that $i^2 = 1$, which makes it by definition an imaginary number, and it commutes with both scalars and vectors.

We can also express the vector $\mathbf{a}$ as the perpendicular and parallel vector projections onto $\mathbf{b}$. With this, we can obtain

$$\mathbf{a}_{\parallel}\mathbf{b} = \mathbf{b}\mathbf{a}_{\parallel}, \tag{5a}$$

$$\mathbf{a}_{\perp}\mathbf{b} = -\mathbf{b}\mathbf{a}_{\perp}. \tag{5b}$$

With the existence of the imaginary number in Geometric Algebra, we can also make use the Euler Identity. We can expand it by using a vector as the argument of the exponential

$$e^{i\mathbf{b}} = \cos|\mathbf{b}| + i\frac{\mathbf{b}}{|\mathbf{b}|}\sin|\mathbf{b}|. \tag{6}$$

This identity is useful in describing vector rotations. We begin with an arbitrary vector $\mathbf{a}$ and multiply this with the exponential above. Here, the vector $\mathbf{b}$ defines the axis of rotation, as well as the magnitude of the rotation, and the direction, as given by the right-hand rule. Assuming that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, we can write the rotation as

$$\mathbf{a}' = \mathbf{a}e^{i\mathbf{b}} = \mathbf{a}\cos b + \mathbf{a}i\mathbf{b}\sin b = \mathbf{a}\cos b + i(\mathbf{a}\cdot\mathbf{b})\sin b - \mathbf{a}\times\mathbf{b}\sin b, \tag{7}$$

where we derived the last equality using Eq. (3). The general case of the rotation, that is when $\mathbf{a}$ and $\mathbf{b}$ are no perpendicular, as well as an in-depth discussion of the derivation of the rotation formula, can be found in [5].

3. Retrograde Motion

For this paper, we will use the Ecliptic Coordinate System. An in-depth discussion of this coordinate system can be found in [4]. We will implement this in the Spherical Coordinate system, with the Sun as the point of origin and the plane of Earths orbit as the reference plane. However, we will use ϕ and θ as the longitude and latitude, respectively, instead of λ and β [1, 4]. We will first have to define the location of the center of the Earth with respect to the Sun.

For simplicity, we will assume that the planets orbit is circular. We will define the position of Earth using the vector $\mathbf{r}_E$ in polar spherical coordinates. However, since the Earth moves in a plane around the Sun, the azimuthal angle is irrelevant. Earth revolves around the Sun with frequency ω_E. We will write this as

$$\mathbf{r}_E = R_E\mathbf{e}_1 e^{i\mathbf{e}_3\omega_E t}, \tag{8}$$

where R_E is the mean radius of Earths orbit, measured in Astronomical Units and t is the time elapsed from a reference point (to be defined later).

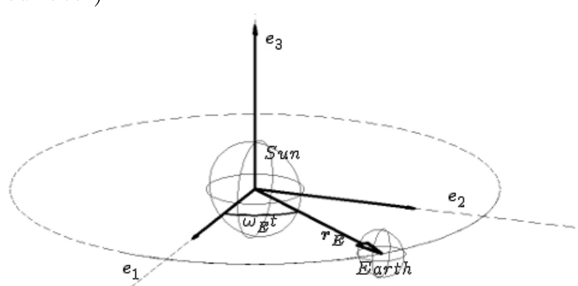

Figure 1: Starting from the reference $\mathbf{e}_1$, the Earth rotates with an angle $\omega_E t$ on the plane defined by a normal vector $\mathbf{e}_3$.

Similar to how we described Earths orbit, we will also use a rotating vector, $\mathbf{r}_M$, on a plane to describe the orbit of Mars. In order to take into account the tilt of Mars, we have to rotate the normal vector of the plane by an angle α_M the inclination of Mars with respect to the Ecliptic. Using geometry, we can say that these two rotations are equal. We first define an equation for the normal vector of Mars orbit,

$$\hat{\mathbf{n}_M} = \mathbf{e}_3 e^{i\mathbf{e}_1\alpha_M} \tag{9}$$

Note that we used the vector $\mathbf{e}_1$ as the axis of rotation. This is where the plane of Mars orbit and the Ecliptic will intersect, and we will later define this point as our point of reference. We can now write the position vector of Mars as

$$\mathbf{r}_M = R_M \mathbf{e}_1 e^{i\hat{\mathbf{n}}_M \omega_M t} \tag{10}$$

where R_M is the radius of Mars orbit, measured in AU, rotating with frequency ω_M.

We will set our reference point to be where Mars is at opposition where the Sun, the Earth, and Mars, form a straight line in that order, and when the distance between Earth and Mars is a minimum where the line that the bodies lies on the Ecliptic, or where the plane of Mars orbit intersects with the Ecliptic. We will set this to be $t = 0$, where time is measured in Earth-years.

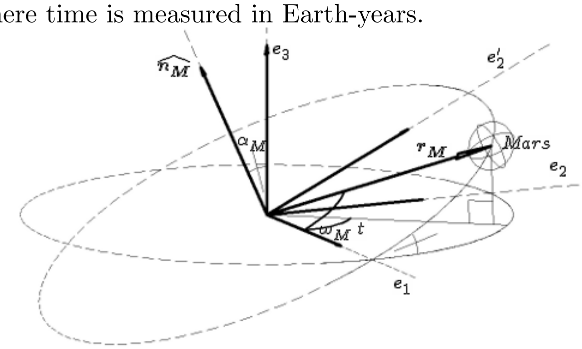

Figure 2: From our reference point, Mars orbits with a frequency ω_M on the plane defined by its normal vector, $\mathbf{e}_3$ rotated by the inclination angle.

With all these data in place, we can form the vector relating the positions of Earth and Mars. We simply connect these two planets with a vector starting from Earth, ending at Mars. With this, we form a triangle, and we can obtain this vector relation:

$$\mathbf{r}_E + \mathbf{r} + \mathbf{r}_M = \mathbf{0}, \tag{11}$$

or,

$$\mathbf{r} = \mathbf{r}_M - \mathbf{r}_E. \tag{12}$$

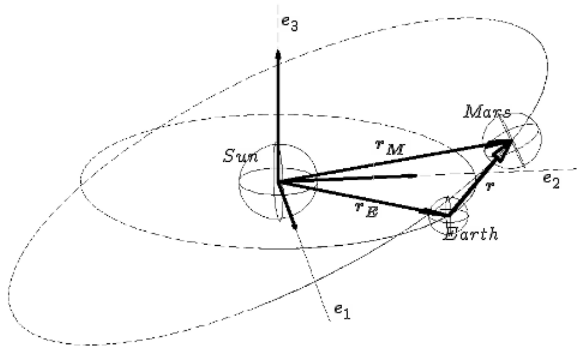

Figure 3: The position of Mars relative to Earth is given by the difference of the position vector of Mars and the position vector of Earth.

We can now plug in Eq. (8) and Eq. (10) in Eq. (12) to obtain

$$\mathbf{r} = R_M \mathbf{e}_1 e^{i\hat{\mathbf{n}}_M \omega_M t} - R_E \mathbf{e}_1 e^{i\mathbf{e}_3 \omega_E t}. \tag{13}$$

Using the rules of Geometric Algebra in (6), we can expand the exponentials in (13) and express $\mathbf{r}$ in terms of the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

$$\mathbf{r} = (R_M \cos\omega_M t - R_E \cos\omega_E t)\mathbf{e}_1 + (R_M \cos\alpha_M \sin\omega_M t - R_E \sin\omega_E t)\mathbf{e}_2 + R_M \sin\alpha_M \sin\omega_M t \mathbf{e}_3 \tag{14}$$

Note that this is of the form $x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. We can then write $\mathbf{r}$ as a set of three parametric equations:

$$x = R_M \cos\omega_M t - R_E \cos\omega_E t, \tag{15a}$$

$$y = R_M \cos\alpha_M \sin\omega_M t - R_E \sin\omega_E t, \tag{15b}$$

$$z = R_M \sin\alpha_M \sin\omega_M t. \tag{15c}$$

We can convert these equations from a Rectangular Coordinate System to a Spherical System using the following transformations:

$$\rho = \sqrt{x^2 + y^2 + z^2} \tag{16a}$$

$$\phi = \arccos \frac{x}{\rho} \tag{16b}$$

$$\theta = \arctan \frac{y}{x} \tag{16c}$$

With these equations, we can plot the graph of the movement of Mars with respect to the center of the Earth (Fig. 4).

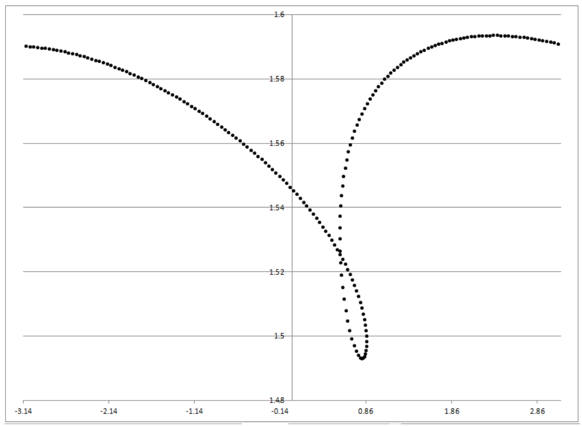

Figure 4: Mars, as seen at the center of the Earth, at t = 1.33 to 3.26 years. Angles are in radians. The hoop in the center represents the retrograde motion of Mars. In this time interval, Mars travels backwards for 63.87 days. Other time intervals would produce other values. This pattern repeats in 47 years; this graph is identical at t = 48.33 to 50.26 years.

4. Conclusion

In this paper, we have shown using Geometric Algebra a simplified method of computing the orbit of Mars with respect to Earth as compared to standard methods.

This paper can be expanded by considering the observer on the surface of Earth. We can also use this method to compute for the rise and set times of the planets, which may be difficult to compute using vector analysis alone.

Acknowledgments

This work was supported by Manila Observatory and the Department of Physics of Ateneo de Manila University.

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