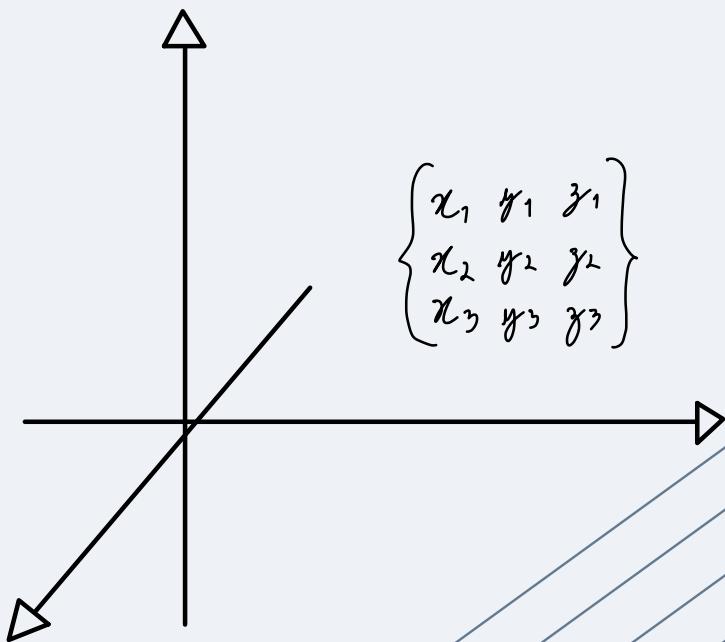


Linear Algebra



LINEAR ALGEBRA

- INTRODUCTIONS
- VECTORS
 - NOTATION
 - PROPERTIES
 - OPERATIONS
 - PYTHON
- MATRICES
 - NOTATION
 - PROPERTIES
 - OPERATIONS
 - PYTHON
- EIGENVALUES & EIGENVECTORS

INTRODUCTIONS

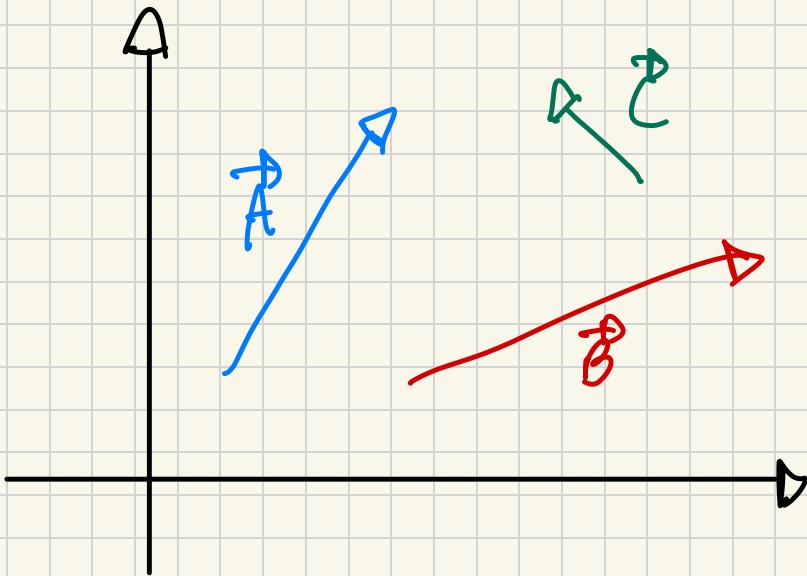
- INSTRUCTOR
- CLASS STRUCTURE
- ICE BREAKER
- CONTENT
- WHY LINEAR ALGEBRA?

VECTORS

VECTORS

- NOTATIONS
- OPERATIONS: ADD & SUBST
- PARALLELOGRAM LAW
- OPERATIONS: SCALAR MULTIPLICAT
- DECOMPOSITION
- NORM
- DISTANCE
- OPERATIONS: DOT PRODUCT
- ANGLE
- PROJECTION
- OPERATIONS: CROSS PRODUCT
- LINEAR INDEPENDENCE & ORTHOGONALITY
- PYTHON APPS

VECTORS NOTATION



VECTORS HAVE :

- MAGNITUDE

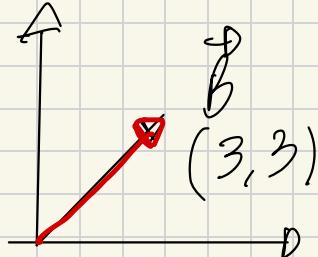
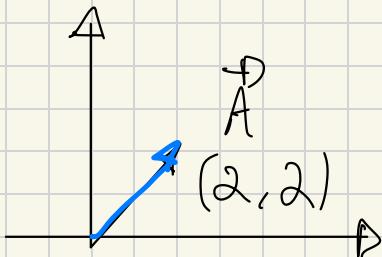
- DIRECTION

EXAMPLES:

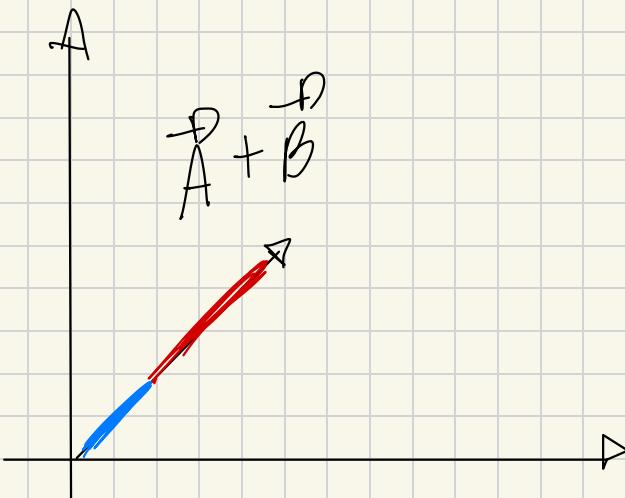
- VELOCITY & FORCE

SIMPLE OPERATIONS

ADDITION & SUBTRACTION

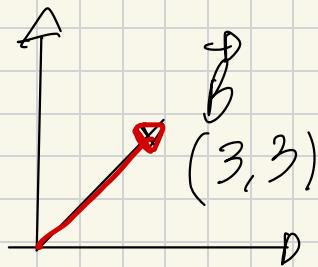
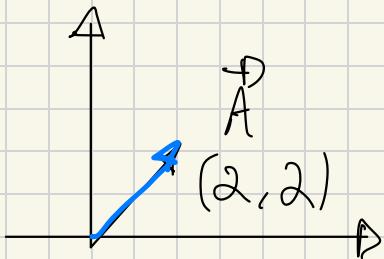


BECAUSE \vec{A} & \vec{B} HAVE THE SAME DIRECTION, THE MAGNITUDE OF $\vec{A} + \vec{B}$ IS THE SUM OF BOTH MAGNITUDES:

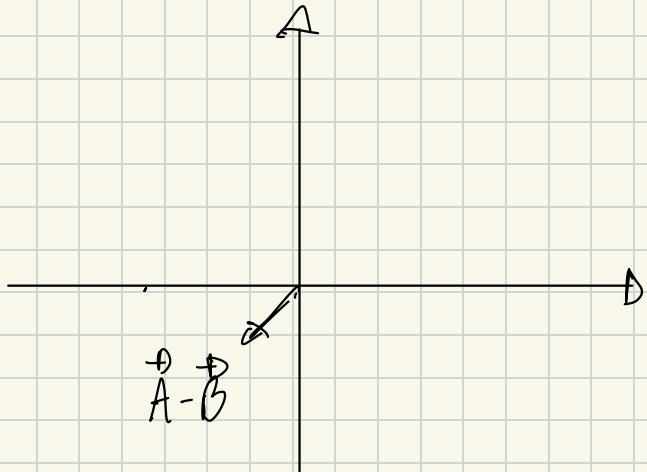


SIMPLE OPERATIONS

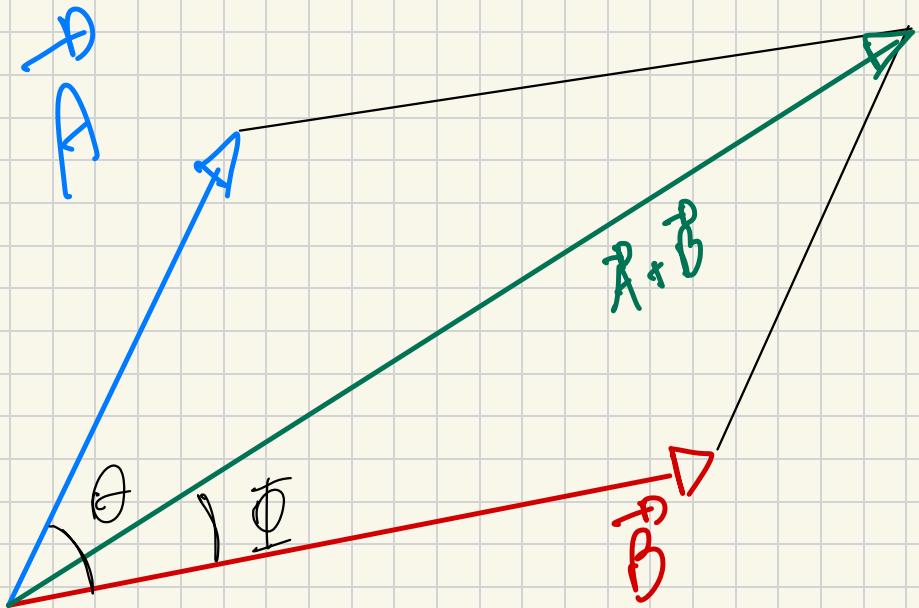
ADDITION & SUBTRACTION



BECAUSE \vec{A} & \vec{B} HAVE THE SAME DIRECTION, THE MAGNITUDE OF $\vec{A} - \vec{B}$ IS THE SUBS OF BOTH MAGNITUDES:



PARALLELOGRAM LAW

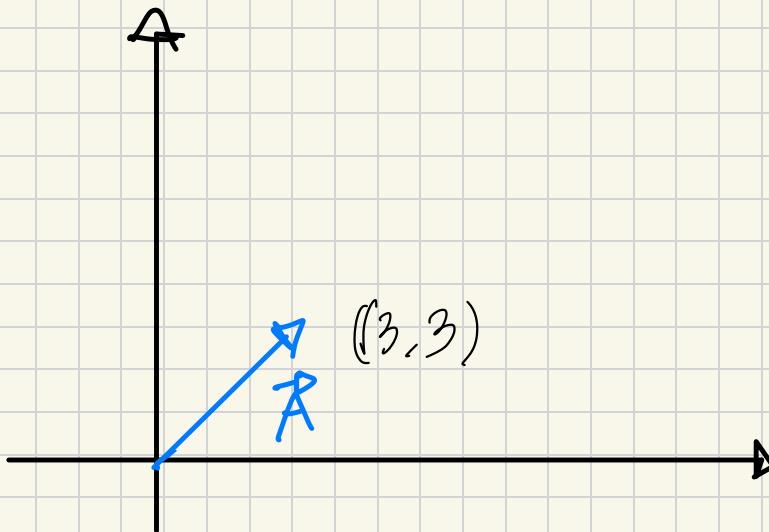


$$|\vec{A} + \vec{B}| = \sqrt{|\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos(\theta)}$$

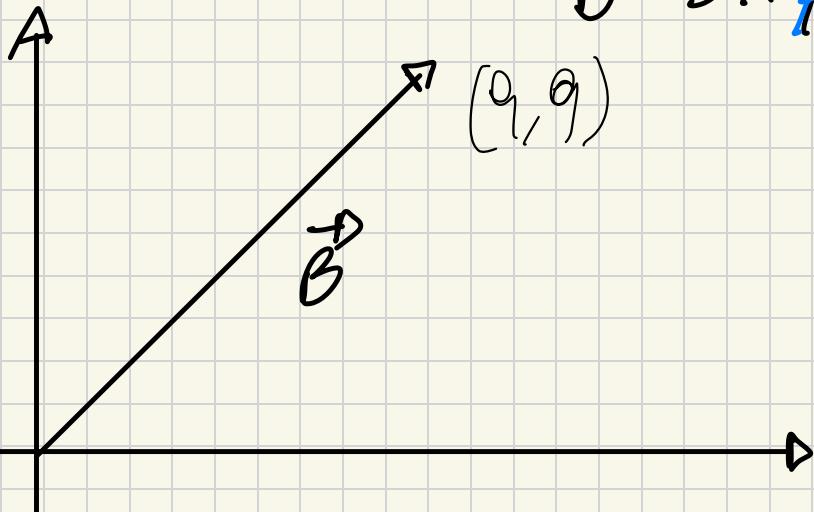
$$\theta = \tan^{-1} \left(\frac{|\vec{A}| \sin(\theta)}{|\vec{B}| + |\vec{A}| \cos(\theta)} \right)$$

SCALAR MULTIPLICATION

ONLY THE MAGNITUDE CHANGES:

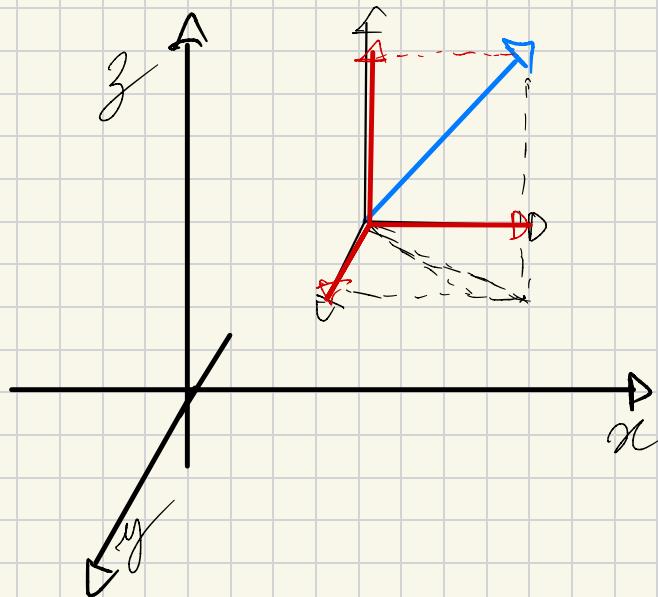
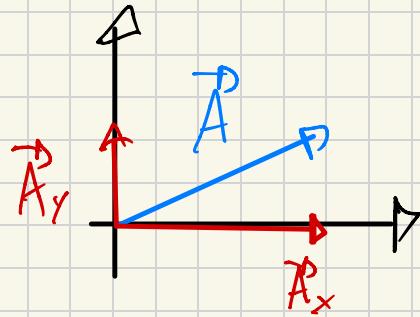


$$\vec{B} = 3 \times \vec{A}$$



DECOMPOSITION OF VECTORS

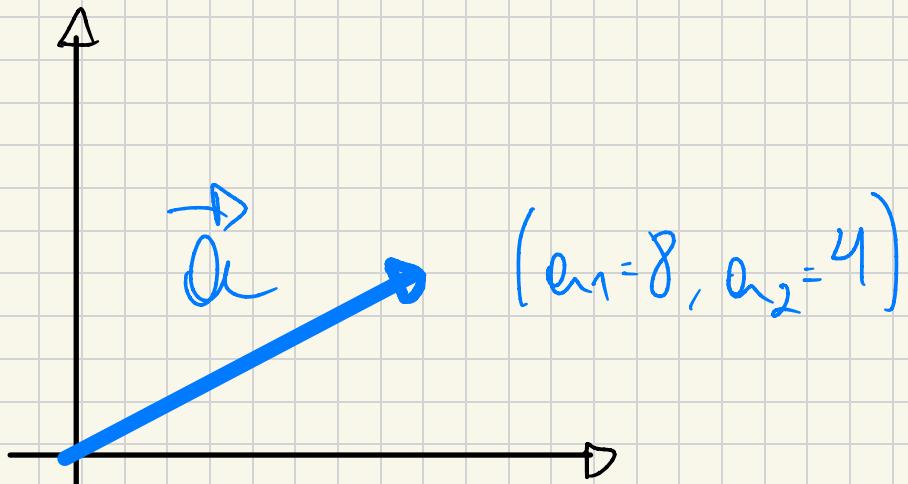
BREAKING ONE VECTOR INTO N SCALARS CAN HELP US CALCULATE MORE OPERATIONS:



NORM

NORM OF A VECTOR REPRESENTS THE LENGTH OF A VECTOR:

$$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \dots + a_n^2}$$

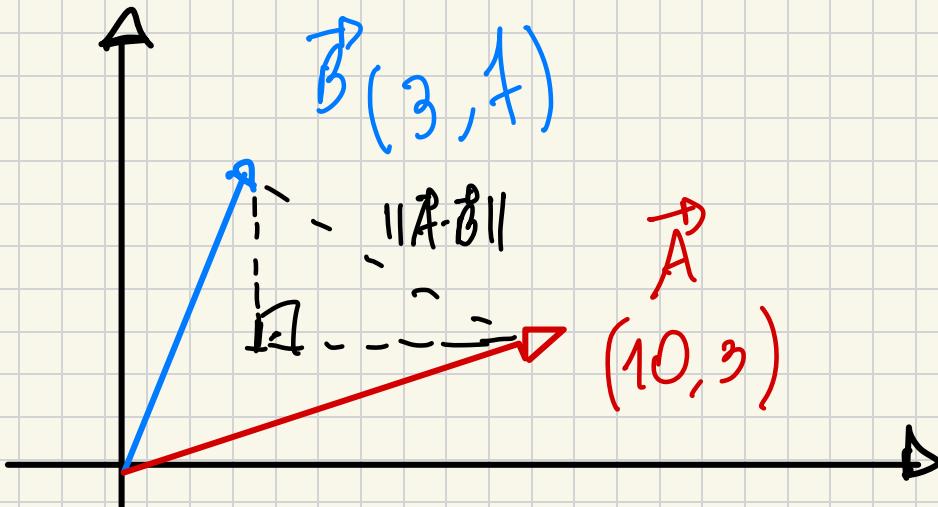


$$\|a\| = \sqrt{8^2 + 4^2} \approx 8.94$$

`np.linalg.norm()`

DISTANCE

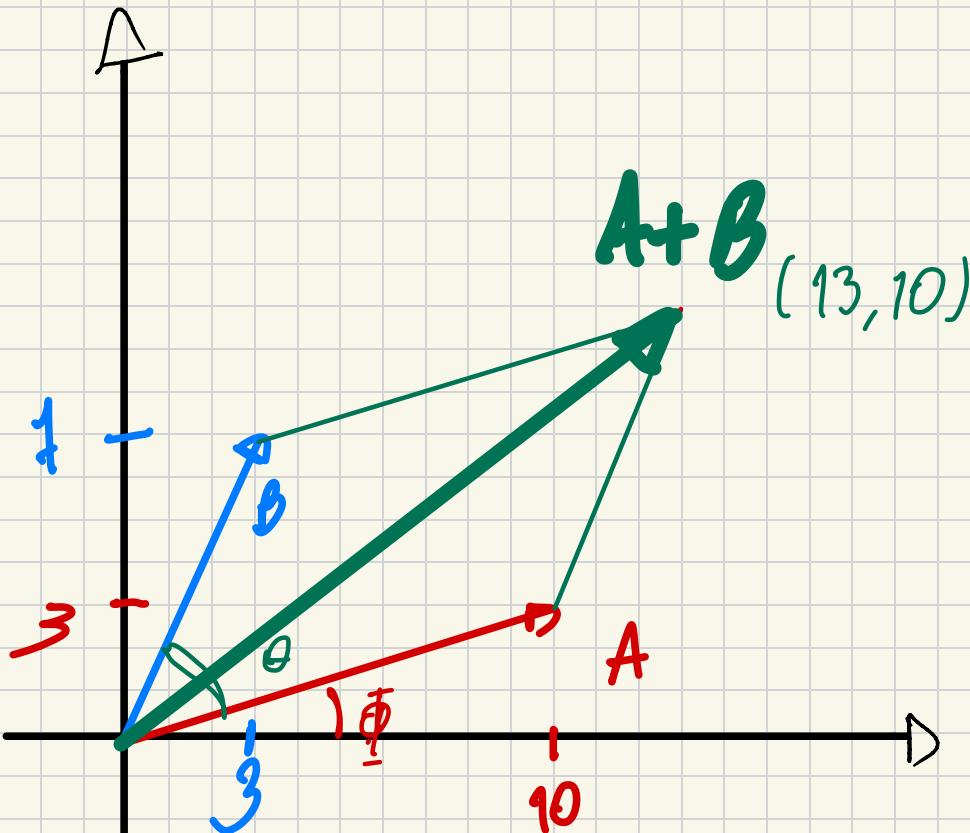
DISTANCE IS A VERY USED CALCULATION TO UNDERSTAND HOW CLOSE TWO VECTORS ARE:



$$d(\vec{A}, \vec{B}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$
$$= \sqrt{(10 - 3)^2 + (3 - 1)^2} \approx 8.06$$

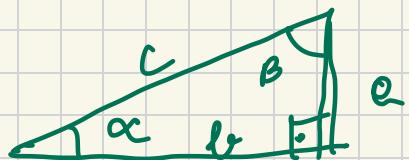
$$d(u, v) = ||u - v|| = \sqrt{(u_1 - v_1)^2 + \dots + (u_n - v_n)^2}$$

SCRATCH (IGNORE it)



$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = c$$

$$\alpha = \arcsin\left(\frac{c}{a}\right)$$



Dot Product (Scalar Product)

IF DECOMPOSITIONS ARE AVAILABLE:

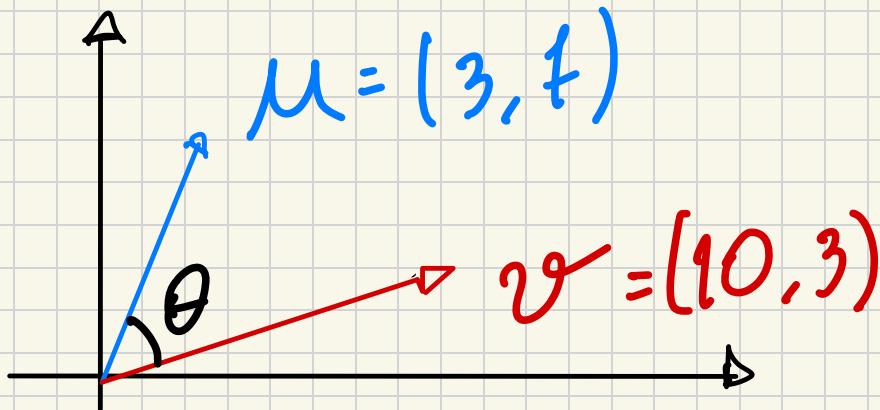
$$\begin{aligned} \mu &= (\mu_1, \mu_2, \dots, \mu_n) \\ v &= (v_1, v_2, \dots, v_n) \\ \mu \cdot v &= \mu_1 \cdot v_1 + \mu_2 \cdot v_2 + \dots + \mu_n \cdot v_n \\ &= \mu^T \cdot v \end{aligned}$$

IF MAGNITUDES AND ANGLE:

$$\mu \cdot v = |\mu| \cdot |v| \cdot \cos(\theta)$$

np.dot(u, v)

Dot Product (Scalar Product)



$$\underline{\underline{\mu \cdot v = 3 \cdot 10 + t \cdot 3 = 51}}$$

OR:

$$|\mu| = \sqrt{3^2 + t^2} \approx 4.62$$

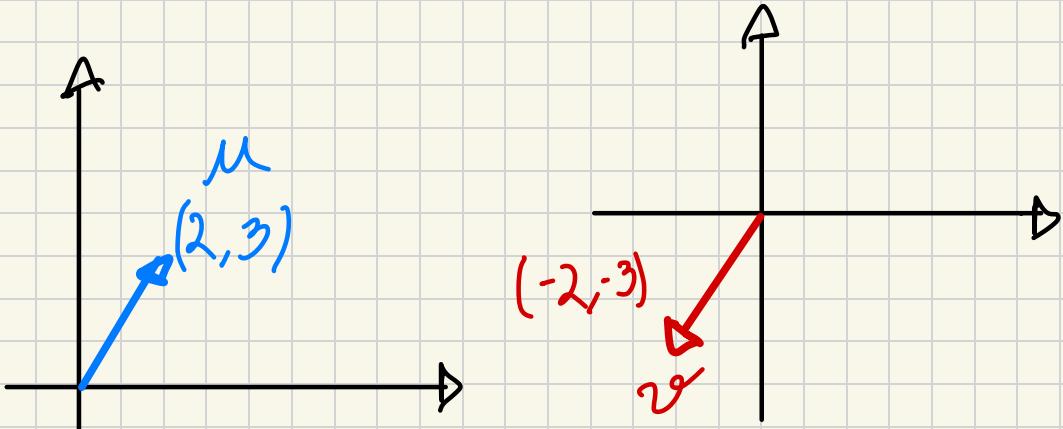
$$|v| = \sqrt{10^2 + 3^2} \approx 10.44$$

USING
 $\rightarrow \sin(\theta)$
from ▲

$$\theta = 0.87 \text{ RAD}$$
$$\cos(\theta) = 0.64$$

$$\mu \cdot v = |\mu| \cdot |v| \cdot \cos(\theta) = 51$$

Dot Product (Scalar Product)



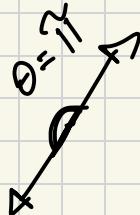
$$u \cdot v = 2 \cdot (-2) + 3 \cdot (-3) = -13$$

$$u \cdot v = |u| \cdot |v| \cdot \cos(\theta)$$

$$|u| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

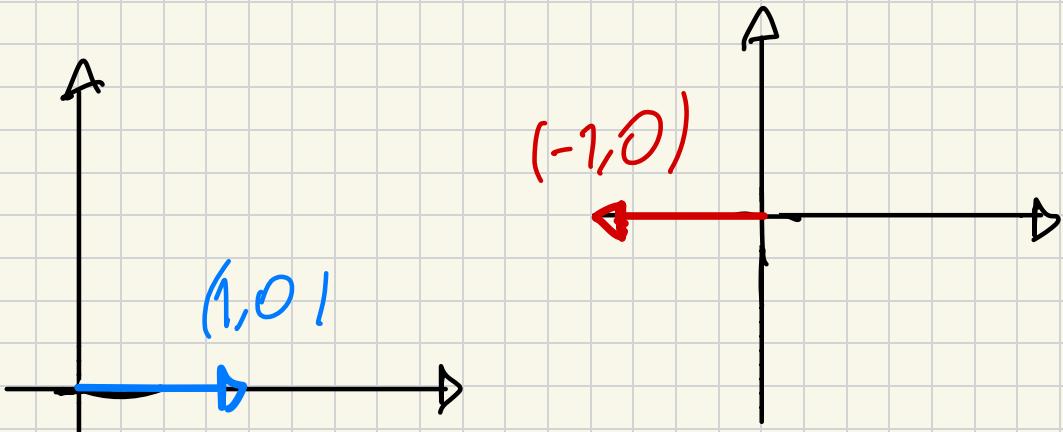
$$|v| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$$

$$\cos(\theta) = -1$$



$$u \cdot v = \sqrt{13} \cdot \sqrt{13} \cdot (-1) = -13$$

Dot Product (Scalar Product)



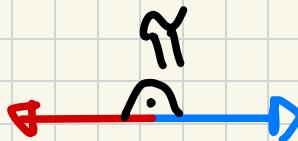
$$u \cdot v = 1 \cdot (-1) + 0 \cdot 0 = -1$$

$$u \cdot v = |u| \cdot |v| \cdot \cos(\theta)$$

$$|u| = 1$$

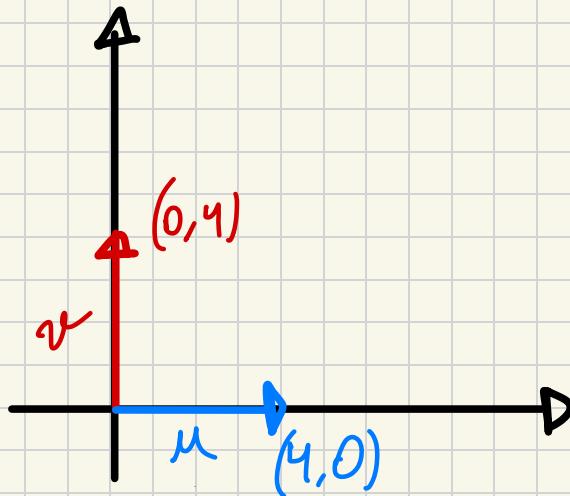
$$|v| = 1$$

$$\cos(\theta) = -1$$



$$u \cdot v = 1 \cdot 1 \cdot (-1) = -1$$

Dot Product (Scalar Product)



$$u \cdot v = 0 \cdot 4 + 4 \cdot 0 = 0$$

$$u \cdot v = |u| \cdot |v| \cdot \cos(\theta)$$

$$|u| = 4$$

$$|v| = 4$$

$$\theta = \frac{\pi}{2} \Rightarrow \cos(\theta) = 0$$

$$u \cdot v = 4 \cdot 4 \cdot 0 = 0$$

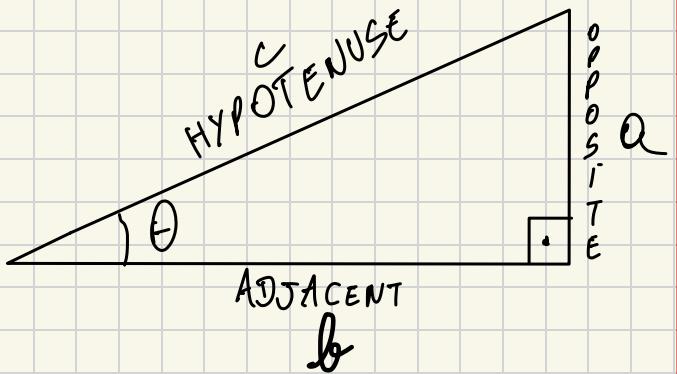
ANGLE

$$\angle(a, b) = \arccos\left(\frac{a^T \cdot b}{\|a\| \cdot \|b\|}\right)$$
$$= \arccos\left(\frac{\text{DOT}(a, b)}{\text{NORM}(a) \cdot \text{NORM}(b)}\right)$$

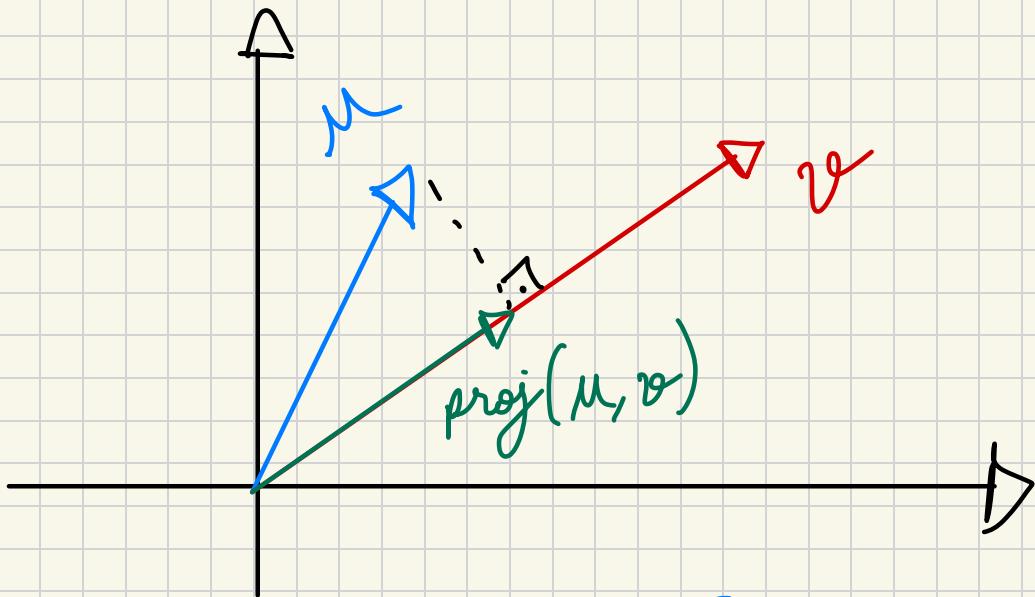
$$\sin(\theta) = \frac{a}{c}$$

$$\cos(\theta) = \frac{b}{c}$$

$$\tan(\theta) = \frac{a}{b}$$



PROJECTION



$$|\text{proj}(u, v)| = \frac{u^T \cdot v}{|v|}$$

$$\text{proj}(u, v) = \frac{|\text{proj}(u, v)|}{|v|} \cdot \vec{v}$$

PROJECTION

$$u = (1, -2, 3), \quad v = (2, 4, 5)$$

$$\text{proj}(u, v) = ?$$

$$|u| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$$

$$|v| = \sqrt{2^2 + 4^2 + 5^2} = \sqrt{45}$$

$$u \cdot v = 2 - 8 + 15 = 9$$

$$|\text{proj}(u, v)| = \frac{9}{\sqrt{45}}$$

$$\text{proj}(u, v) = \frac{|\text{proj}(u, v)|}{|v|} \cdot (2, 4, 5) =$$

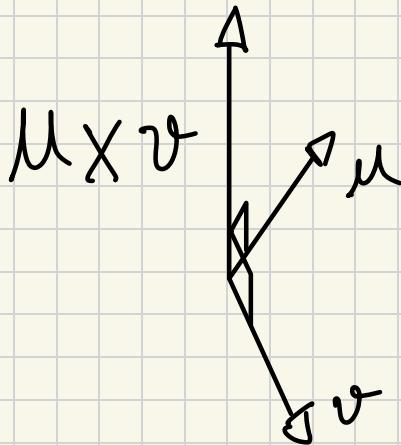
$$= \frac{9}{\sqrt{45}} \cdot \frac{1}{\sqrt{45}} (2, 4, 5) = \frac{9}{45} (2, 4, 5) = \frac{1}{5} (2, 4, 5) =$$

$$= (0.4, 0.8, 1)$$

CROSS PRODUCT

A CROSS PRODUCT OF μ AND ν IS A VECTOR (DIFFERENTLY THEN THE DOT PRODUCT)

THE DIRECTION WOULD BE PERPENDICULAR TO THE PLANE OF μ AND ν .



THE MAGNITUDE OF $\mu \times \nu$ IS:

$$|\mu \times \nu| = |\mu| \cdot |\nu| \cdot \sin(\theta)$$

θ = ANGLE BETWEEN μ & ν

CROSS PRODUCT

WE CAN ALSO REPRESENT THE CROSS PRODUCT AS A MATRIX:

IF i, j AND k ARE UNIT VECTORS $(1,0,0), (0,1,0)$ AND $(0,0,1)$, WE CAN DECOMPOSE ANY 3D VECTOR IN:

$$\begin{aligned}\vec{a} &= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k} \\ \vec{b} &= b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}\end{aligned}\quad \left| \begin{array}{l} \vec{c} = \vec{a} \times \vec{b} \end{array} \right.$$

$$\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\vec{c} = i |a_2 b_3 - a_3 b_2| - j |a_1 b_3 - a_3 b_1| + k |a_1 b_2 - a_2 b_1|$$

CROSS PRODUCT

$$\vec{C} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\vec{C} = i |a_2 b_3 - a_3 b_2| - j |a_1 b_3 - a_3 b_1| + k |a_1 b_2 - a_2 b_1|$$

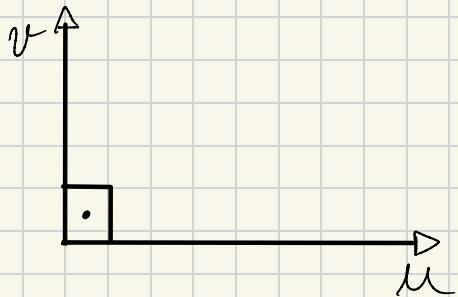
$$u = (1, 2, 5) \quad v = (2, -4, 5) \quad \boxed{u \times v = ?}$$

$$u \times v = (2 \cdot 5 + 5 \cdot 4, -(1 \cdot 5 - 5 \cdot 2), 1 \cdot (-4) - 2 \cdot 2) \\ = (30, 5, -8)$$

$$|u \times v| = \sqrt{30^2 + 5^2 + (-8)^2} \approx 31.45$$

$$\theta_{u \times v, u} = \arccos \left(\frac{30 \cdot 1 + 5 \cdot 2 + (-8) \cdot 5}{31.45 \cdot \sqrt{1^2 + 2^2 + 5^2}} \right) = 90^\circ$$

LINEAR INDEPENDENCE



WE CALL 2 VECTORS ORTHOGONAL
IF THE ANGLE BETWEEN THEM IS 90° .

$$u = (2, 0) \quad v = (0, 5)$$

v CAN'T BE WRITTEN AS
LINEAR COMBINATION OF ANY VECTOR
IN THE SAME DIMENSION OF u .

VECTORS APPLICATIONS

- CROSS PRODUCT: MAGNETIC FIELD
- DISTANCE ; - K-MEANS CLUSTER
 - IMAGE RECOGNITION
- WORD EMBEDDINGS

MATRICES

MATRICES

- CLASSIC NOTATION
- MATRIX AS LINEAR EQUATIONS
- MATRIX VS VECTOR
- PLOTTING A MATRIX AS VECTORS
- OPERATIONS
 - ADDITION
 - SCALAR MULTIPLICATION
 - MATRIX MULTIPLICATION
- LINEAR TRANSFORMATIONS
- DETERMINANT
- RANK OF A MATRIX
- ROW ECHOLON FORM
- SINGULAR
- TRANSPPOSE
- INVERSE
- EIGEN VECTORS & EIGEN VALUES,

MATRICES

MATRICES ARE A RECTANGULAR
REPRESENTATIONS OF LINEAR TRANSFORMATIONS
OF A N-DIMENSION VECTOR TO A M-DIMENSION
ONE:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

MATRICES

CONSIDER 3 3D LINEAR EQUATIONS

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

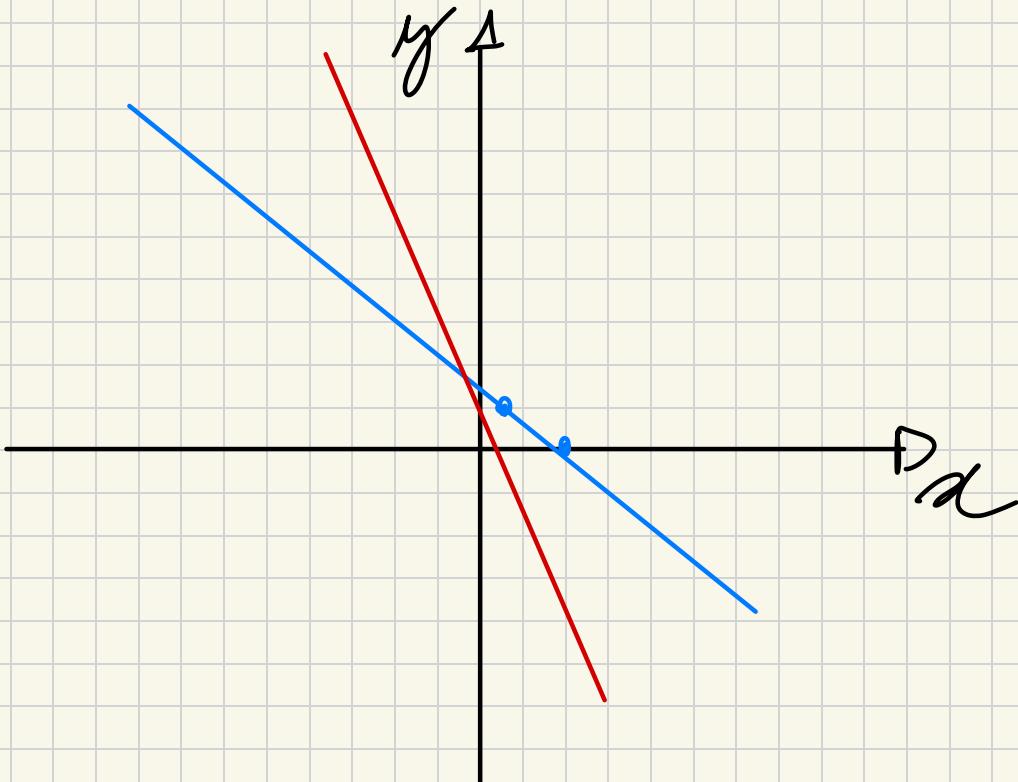
WE COULD REPRESENT it AS:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

MATRICES

$$\begin{aligned}2x + 3y &= 4 \\6x + 2y &= 1\end{aligned}$$

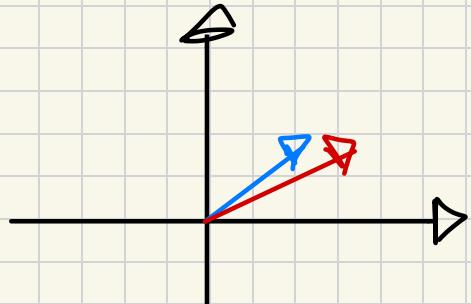
$$\begin{bmatrix} 2 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$



MATRICES

WE CAN ALSO PLOT A MATRIX AS VECTOR:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$



MATRIX PROBLEM 1

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad A(BA) = ?$$

$$BA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 4+1 \\ 1+4 & 2+2 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$$

$$A(BA) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 4+10 & 5+8 \\ 8+5 & 10+4 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$$

MATRIX PROBLEM 2

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \end{bmatrix}_{2 \times 3} \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}_{3 \times 2} \quad AB = ?$$

$$AB = \begin{bmatrix} 2-6+1 & 1-4+1 \\ 4+3+3 & 2+2+3 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 10 & 7 \end{bmatrix}$$

MATRIX OPERATIONS

ADDITION:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{bmatrix}$$

SCALAR MULTIPLICATION:

$$3 \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 \cdot a_{11} & 3 \cdot a_{12} & 3 \cdot a_{13} \\ 3 \cdot a_{21} & 3 \cdot a_{22} & 3 \cdot a_{23} \\ 3 \cdot a_{31} & 3 \cdot a_{32} & 3 \cdot a_{33} \end{bmatrix}$$

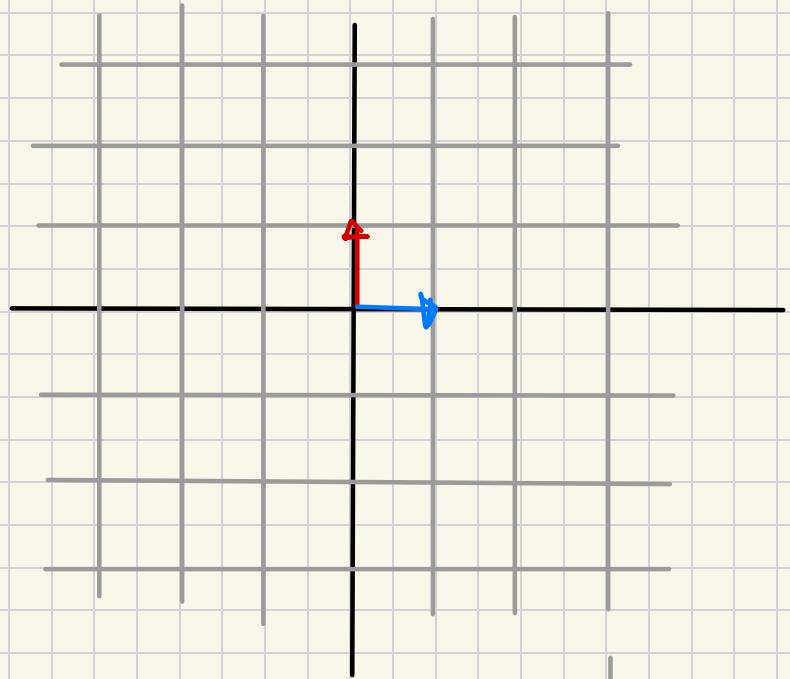
MATRIX MULTIPLICATION

$$C = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2 \times 3} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3 \times 2}$$

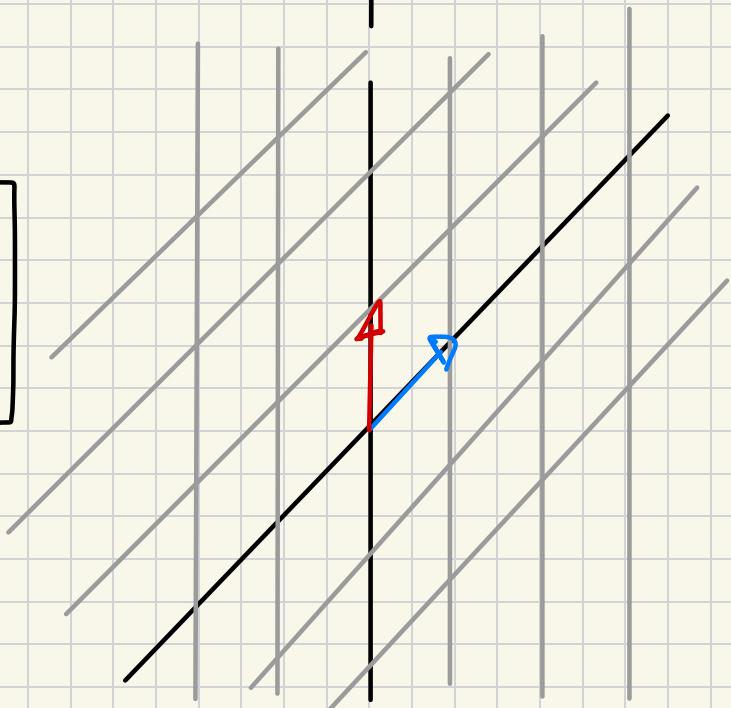
$$= \begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} \end{bmatrix}$$

LINEAR TRANSFORMATIONS

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



DETERMINANT

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}$$

$$\text{DET}(A) = ?$$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}$$

$$2 \cdot (0 \cdot 5 - (-1) \cdot 4) = 2 \cdot (0 + 4) = 8$$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}$$

$$-3 \cdot (2 \cdot 5 - (-1) \cdot 1) = -3 \cdot 11 = -33$$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}$$

$$1 \cdot (2 \cdot 4 - 0 \cdot 1) = 8$$

$$\text{DET}(A) = 8 - (-33) + 8 = 49$$

DETERMINANT

For: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\text{DET}(A) = 1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

(x, y) UNCHANGED

For: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\text{DET}(A) = 4$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

(x, y) CHANGED
SCALE

For: $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

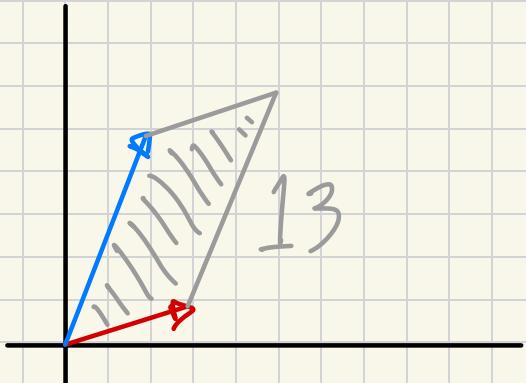
$\text{DET}(A) = 0$

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ 2 \cdot (2x + y) \end{bmatrix}$$

$x \& y$ ARG
NOW 1D

DETERMINANT

FOR: $A = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$



$$\text{DET}(A) = 15 - 2 = 13$$

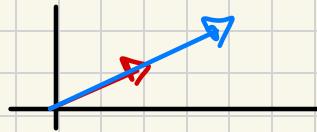
$$\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + y \\ 2x + 5y \end{bmatrix}$$

(x, y) CHANGED SCALE

RANK OF A MATRIX

FOR:

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$



EVEN THOUGH A IS A 2D MATRIX,
THE 2 VECTORS ARE ALIGNED IN
1 DIMENSION.

$$\text{RANK}(A) = 1$$

Row Echelon Form

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \\ 2 & 4 & 2 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 2R_1$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \\ 2 & 4 & 2 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \\ 2 & 4 & 2 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_1$$

$$R_4 = R_3 + R_2$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

RANK = 2

DET = ?

Row Echelon Form

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \xrightarrow{R_2 \rightarrow 2R_1 - R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 4 & 1 & 8 \end{bmatrix}$$

$R_3 \rightarrow 4R_1 - R_3$

$$\xrightarrow{R_3 \rightarrow R_2 + R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_2 + R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{RANK}(A) = 3$$

SINGULAR MATRIX

A SQUARE MATRIX is SINGULAR

IF $\text{DET}(A) = 0$, HENCE IT IS
NOT INVERTIBLE.

$$\text{RANK}(A) \leq \text{ORDER}(A)$$

SINGULAR MATRIX HAVE LINEAR
DEPENDENT ROWS/COLUMNS.

TRANSPOSE

$$A = \begin{bmatrix} 2 & -9 & 3 \\ 13 & 11 & -11 \\ 3 & 6 & 15 \\ 4 & 13 & 1 \end{bmatrix} \quad 4 \times 3$$

$$A^T = \begin{bmatrix} 2 & 13 & 3 & 4 \\ -9 & 11 & 6 & 13 \\ 3 & -11 & 15 & 1 \end{bmatrix} \quad 3 \times 4$$

TRANSPOSE PROPERTIES

1)

$$(A^T)^T = A$$

2)

$$(A + B)^T = A^T + B^T$$

3)

$$(kA)^T = k(A^T)$$

4)

$$(AB)^T = B^T \cdot A^T$$

INVERSE

A^{-1} = INVERSE OF A

$$A \cdot A^{-1} = I$$

I = IDENTITY

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{m \times n}$$

INVERSE

$$A^{-1} = \text{adj}(A) / \det(A)$$

$$\text{adj}(A) = C^T$$

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

M_{ij} = MATRIX WITHOUT
i ROW AND j COLUMN

INVERSE

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \text{adj}(A) / \det(A)$$

$$\text{adj}(A) = C^T$$

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

M_{ij} = MATRIX WITHOUT
i ROW AND j COLUMN

$$C = \begin{bmatrix} +d & -c \\ -b & +a \end{bmatrix}$$

$$C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det(A) = a \cdot d - b \cdot c$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{24 - 14} \begin{bmatrix} 6 & -1 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.1 \\ -0.2 & 0.4 \end{bmatrix}$$

INVERSE

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1-4 & -(2-(-2)) & 4-(1) \\ -(2-(-2)) & 1-1 & -(2-(-2)) \\ 4-(1) & -(2-(-2)) & 1-4 \end{bmatrix} = \begin{bmatrix} -3 & -4 & 5 \\ -4 & 0 & -4 \\ 5 & -4 & -3 \end{bmatrix}$$

$$C^T = \begin{bmatrix} -3 & -4 & 5 \\ -4 & 0 & -4 \\ 5 & -4 & -3 \end{bmatrix}$$

$$A^{-1} = \text{adj}(A) / \det(A)$$

$$\text{adj}(A) = C^T$$

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

M_{ij} = MATRIX WITHOUT
i ROW AND j COLUMN

CONTINUE...

INVERSE

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

$$C^T = \begin{bmatrix} -3 & -4 & 5 \\ -4 & 0 & -4 \\ 5 & -4 & -3 \end{bmatrix}$$

$$\det(A) = 1 \cdot (1-4) - 2 \cdot (2-(-2)) + (-1)(4-(-1))$$

$$= -3 - 8 - 5 = -16$$

$$A^{-1} = \begin{bmatrix} \frac{3}{16} & \frac{1}{4} & \frac{5}{16} \\ \frac{1}{4} & 0 & \frac{1}{4} \\ -\frac{5}{16} & \frac{1}{4} & \frac{3}{16} \end{bmatrix}$$

$$A^{-1} = \text{adj}(A) / \det(A)$$

$$\text{adj}(A) = C^T$$

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

M_{ij} = MATRIX WITHOUT
i ROW AND j COLUMN

INVERSE (PROPERTIES)

1) If A is NONSINGULAR $(A^{-1})^{-1} = A$

2) If A & B ARE NONSINGULAR

$$AB^{-1} = B^{-1}A^{-1}$$

3) $(A^T)^{-1} = (A^{-1})^T$

4) $A \cdot A^{-1} = A^{-1} \cdot A = I$

EIGEN VECTORS & EIGEN VALUES

EIGENVECTOR OF A MATRIX IS A VECTOR THAT DOESN'T CHANGE ITS DIRECTION WHEN MULTIPLIED BY THE MATRIX:

$$A \vec{x} = \lambda \vec{x}$$

Diagram illustrating the equation $A \vec{x} = \lambda \vec{x}$:

- The matrix A is highlighted in yellow.
- The vector \vec{x} is highlighted in yellow.
- The scalar λ is highlighted in pink.
- The resulting vector \vec{x} is highlighted in yellow.
- An arrow points from the scalar λ to the right, labeled "EIGEN VALUE".
- An arrow points from the vector \vec{x} to the left, labeled "EIGEN VECTOR".

2x2 EXAMPLE:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

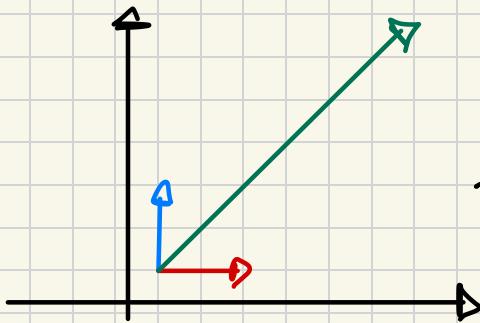
EIGEN VECTORS & EIGENVALUES

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

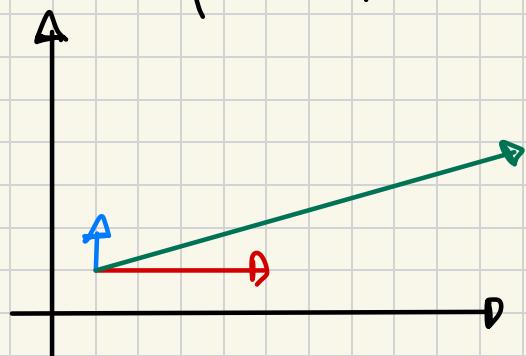
$$\mu = (1, 0)$$

$$v = (0, 1)$$

$$w = (3, 3)$$



A-vector



$$A\mu = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\vec{\pi}_1; \lambda_1 = 2$$

$$Av = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

$$\vec{\pi}_2; \lambda_2 = 0.5$$

$$Aw = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1.5 \end{bmatrix}$$

NOT EIGEN!!!

PROPERTIES OF EIGENVALUES

- $A^+ = -A$ MATRICES HAVE IMAGINARY OR ZERO λ
- SINGULAR MATRICES HAVE 0λ 's
- SUM OF $\lambda_1, \lambda_2, \dots, \lambda_n$ IS THE TRACE.
- EIGENVECTORS ARE LINEARLY INDEPENDENT
- IF A SQUARE, $\lambda=0$ IS NOT A EIGENVALUE.
- $a \cdot \lambda$ IS A EIGENVALUE OF $a \cdot A$
- IF λ IS EIGENVALUE OF A , λ IS ALSO EIGENVALUE OF A^T

CALCULATION OF \vec{v} & λ

1) FIND λ 'S SOLVING:

$$\text{DET}(A - \lambda I) = 0$$

2) FIND \vec{v}_i SOLVING:

$$A \cdot \vec{v}_i = \lambda_i \cdot \vec{v}_i$$

CALCULATION OF \vec{v} & λ

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \quad A - \lambda I = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$$

$$(5-\lambda)(2-\lambda) - 4 = 0 \Rightarrow 10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 6)(\lambda - 1) = 0$$

$$\lambda = 6 \quad \text{OR} \quad \lambda = 1$$

$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \Rightarrow -v_{11} = 2v_{12}$$
$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 6v_{21} \\ 6v_{22} \end{bmatrix} \Rightarrow v_{21} = 4v_{22}$$

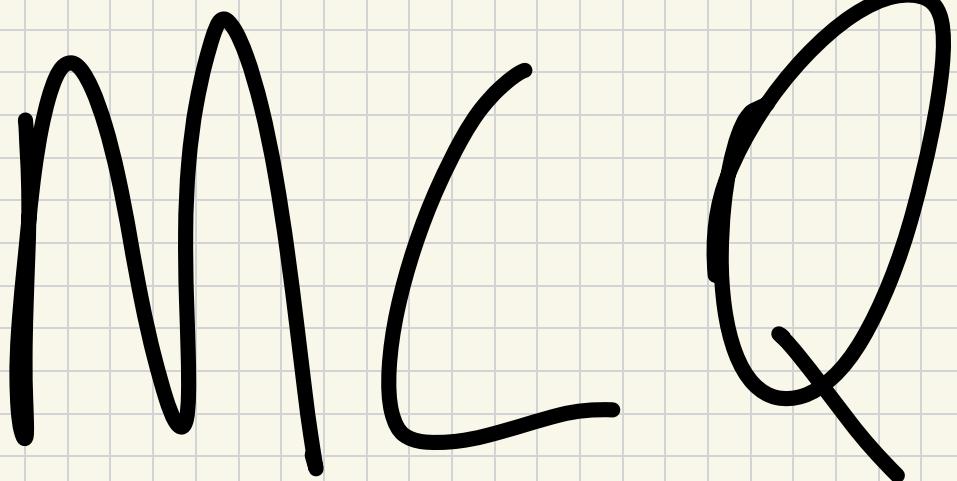
$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 6v_{21} \\ 6v_{22} \end{bmatrix} \Rightarrow v_{21} = 4v_{22}$$

f

i

m





Q1

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\text{RANK}(A) = ?$$

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{array} \right] \xrightarrow{\begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{array}} \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\text{RANK}(A) = 1$$

Q2

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{DET}(B) = ?$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$1 \cdot (5 \cdot 9 - 6 \cdot 8) = 45 - 48 = -3$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$2 \cdot (4 \cdot 9 - 6 \cdot 7) = 2 \cdot (36 - 42) = -12$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

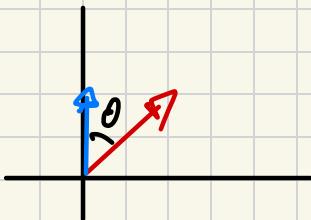
$$3 \cdot (4 \cdot 8 - 5 \cdot 7) = 3 \cdot (32 - 35) = -9$$

$$\text{DET}(B) = -3 + 12 - 9 = \emptyset$$

Q3

$$v_1 = (1, 0) \quad v_2 = (1, 1)$$

$$\angle(v_1, v_2) = ?$$



$$\theta = 45^\circ$$

$$\angle(v_1, v_2) = \cos^{-1} \left(\frac{v_1^T \cdot v_2}{|v_1| |v_2|} \right) = \Theta$$

$$v_1^T \cdot v_2 = 1 \cdot 1 + 0 \cdot 1 = 1$$

$$|v_1| = \sqrt{1^2 + 0^2} = 1 \quad |v_2| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\Theta = \cos^{-1} \left(\frac{1}{\sqrt{2}} \right); \quad \cos(\Theta) = \frac{1}{\sqrt{2}}$$

$$\Theta = 45^\circ$$

Q4

LINEAR TRANSFORMATION PROPERTIES

A: LT PRESERVES ORIGIN

B: LT PRESERVES DISTANCE

C: LT PRESERVES ANGLE

D: LT MAPS STRAIGHT LINES.
TO STRAIGHT LINES

WHICH ONE IS FALSE?

Q4

EXAMPLES OF LINEAR TRANSFORMATIONS:

IDENTITY: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

REFLECTION: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

SCALING: $\begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$

ROTATION: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

PROJECTION: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

LINEAR TRANSFORMATIONS:

$$T(a\vec{v}_1 + b\vec{v}_2) = aT(\vec{v}_1) + b \cdot T(\vec{v}_2)$$

Q4

LINEAR TRANSFORMATION PROPERTIES

A: LT PRESERVES ORIGIN



B: LT PRESERVES DISTANCE



C: LT PRESERVES ANGLE

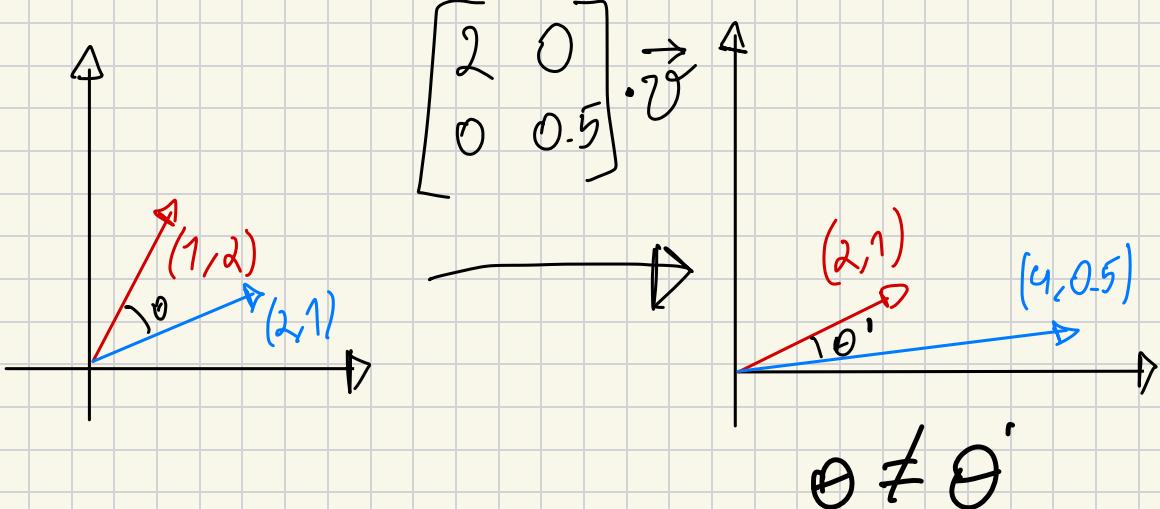


D: LT MAPS STRAIGHT LINES
TO STRAIGHT LINES



EXAMPLE:

$$\begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$



Q5

WHICH OF THE FOLLOWING IS NOT
A VALID MATRIX OPERATION?

- A) ADDITION ✓
- B) SUBTRACTION ✓
- C) DIVISION ✗
- D) MULTIPLICATION ✓

Q6

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$C \vec{x} = \lambda \vec{x}$$

$$\text{DET}(C - \lambda I) = 0$$

$$C - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

$$\text{DET}(C - \lambda I) = (2-\lambda)^2 - 1$$

$$(2-\lambda)^2 = 1$$

$$2-\lambda = \pm 1$$

$$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 3 \end{cases}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ 3y_2 \end{bmatrix}$$

$$\begin{cases} 2x_2 + y_2 = 3x_2 \\ x_2 + 2y_2 = 3y_2 \end{cases}$$

$$\begin{cases} x_2 - y_2 = 0 \\ y_2 - x_2 = 0 \end{cases}$$

$$x_2 = y_2$$

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Q1

WHICH IS TRUE ABOUT ORTHOGONAL MATRIX:

A) $\det(A) = -1$

X

(M1)

B) $A^{-1} = A^T$

✓

C) $\underbrace{A \cdot B}_{\text{ORTHOGONAL}} = \underbrace{C}_{\text{ORTHOGONAL}}$

✓

(M3)

D) λ OF A IS ALWAYS REAL

X

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

M2

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

M3

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

M4

Q8

$$v_1 = (1, 2) \quad v_2 = (2, 1)$$

$$\text{proj}(v_1, v_2) = ?$$

$$|\text{proj}(v_1, v_2)| = \frac{v_1 \cdot v_2}{|v_2|} = \frac{1 \cdot 2 + 2 \cdot 1}{\sqrt{2^2 + 1^2}} = \frac{4}{\sqrt{5}}$$

$$|v_2| = \sqrt{5}$$

$$\text{proj}(v_1, v_2) = \frac{4}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} (2, 1)$$

$$= \left(\frac{4}{5} \cdot 2, \frac{4}{5} \right)$$

$$= (1.6, 0.8)$$

Q9

Which is NOT TRUE ?

- A) $\text{DET}(A) = \emptyset$ if A^{-1} exists X
- B) $\text{DET}(A) = -\text{DET}(B)$ if B is A HAS
ONLY TWO ROWS INTERCHANGED ✓

C) $\text{DET}(A) = \lambda_1 \cdot \lambda_2 \cdot \dots \lambda_n$ ✓

D) $\text{DET}(A) = \sum_{i=1}^n a_{ii}$ X

Q10

Which is NOT INVERTIBLE?

A) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\text{DET}(A) = 1$

B) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\text{DET}(B) = 1$

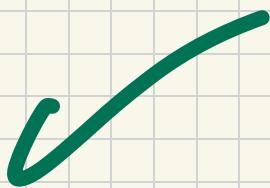
C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\text{DET}(C) = -1$

D) $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ $\text{DET}(D) = \emptyset$

Q11

WHICH IS THE DETERMINANT OF A
IDENTITY MATRIX:

- A) 0
- B) 1
- C) -1
- D) UNDEFINED



Q12

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}$$

$$B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}_{3 \times 2}$$

$$AB = ?$$

$$AB = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 \end{bmatrix}$$

$$= \begin{bmatrix} 7 + 18 + 33 & 8 + 20 + 36 \\ 28 + 45 + 66 & 32 + 50 + 72 \end{bmatrix}$$

$$= \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Q13

$$M = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \quad \lambda = ?$$

$$M \vec{x} = \lambda \vec{x}$$

$$\det(M - \lambda I) = 0$$

$$\begin{vmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 4 = 0$$

$$-6 - 3\lambda + 2\lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$\begin{aligned} a &= 1 \\ b &= -1 \\ c &= -2 \end{aligned}$$

$$\Delta^2 - 4ac = (-1)^2 - 4 \cdot 1 \cdot (-2) = 9$$

$$\frac{-b \pm \sqrt{\Delta}}{2a} = \frac{1 \pm 3}{2}$$

$2 = \lambda_1$
 $-1 = \lambda_2$

