Compartmental models of infectious disease dynamics

Sansao Pedro

Universidade Eduardo Mondlane
Department of Mathematics and Computer Sciences

sansaopedro@gmail.com/sansao.pedro@uem.ac.mz

November 21, 2023

Overview

Facts on infecious diseases

- Disease outbreaks of acquired immunodeficiency syndrome, severe acute respiratory syndrome, pandemic H1N1, H7N9, H5N1, Ebola, Zika, Middle East respiratory syndrome, and recently COVID-19 have raised the attention of the public over the past half-century;

Facts on infecious diseases

- Disease outbreaks of acquired immunodeficiency syndrome, severe acute respiratory syndrome, pandemic H1N1, H7N9, H5N1, Ebola, Zika, Middle East respiratory syndrome, and recently COVID-19 have raised the attention of the public over the past half-century;
- The emergence and reemergence of neglected tropical diseases continue to pose significant challenges to humanities and societies;

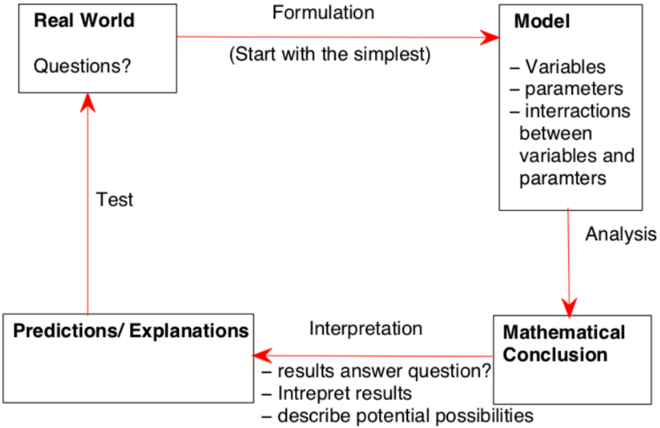
Facts on infecious diseases

- Disease outbreaks of acquired immunodeficiency syndrome, severe acute respiratory syndrome, pandemic H1N1, H7N9, H5N1, Ebola, Zika, Middle East respiratory syndrome, and recently COVID-19 have raised the attention of the public over the past half-century;
- The emergence and reemergence of neglected tropical diseases continue to pose significant challenges to humanities and societies;
- Revealing the characteristics and epidemic trends are important parts of disease control;

Facts on infecious diseases

- Disease outbreaks of acquired immunodeficiency syndrome, severe acute respiratory syndrome, pandemic H1N1, H7N9, H5N1, Ebola, Zika, Middle East respiratory syndrome, and recently COVID-19 have raised the attention of the public over the past half-century;
- The emergence and reemergence of neglected tropical diseases continue to pose significant challenges to humanities and societies;
- Revealing the characteristics and epidemic trends are important parts of disease control;
- The biological scenarios including transmission characteristics can be constructed and translated into mathematical models, which can help to predict and gain a deeper understanding of diseases.

The interface between infectious diseases and mathematical modelling



What is our interest?

- dynamics of infectious diseases;

What is our interest?

- dynamics of infectious diseases;
- how a process that happens at an individual level affects a given population;

What is our interest?

- dynamics of infectious diseases;
- how a process that happens at an individual level affects a given population;
- The risk of getting infected given that there is a possibility of getting an infection;

What is our interest?

- dynamics of infectious diseases;
- how a process that happens at an individual level affects a given population;
- The risk of getting infected given that there is a possibility of getting an infection;
- The analysis of transmission patterns in various populations;

What is our interest?

- dynamics of infectious diseases;
- how a process that happens at an individual level affects a given population;
- The risk of getting infected given that there is a possibility of getting an infection;
- The analysis of transmission patterns in various populations;
- Methods to assess the effectiveness of control strategies;

What is our interest?

- dynamics of infectious diseases;
- how a process that happens at an individual level affects a given population;
- The risk of getting infected given that there is a possibility of getting an infection;
- The analysis of transmission patterns in various populations;
- Methods to assess the effectiveness of control strategies;
- Translating reality into an abstract system that can be analyzed and the results translated back to reality;

What is our interest?

- dynamics of infectious diseases;
- how a process that happens at an individual level affects a given population;
- The risk of getting infected given that there is a possibility of getting an infection;
- The analysis of transmission patterns in various populations;
- Methods to assess the effectiveness of control strategies;
- Translating reality into an abstract system that can be analyzed and the results translated back to reality;

Summary

understand the general principles, assumptions and basic techniques used in mathematical models for infectious diseases, appreciate the value and limits of mathematical models and explore the behavior of different structures.

Mathematical models

Mathematical model

- a conceptual tool that uses the language of mathematics to produce a more refined and precise description of a system;

Mathematical models

Mathematical model

- a conceptual tool that uses the language of mathematics to produce a more refined and precise description of a system;
- a set of equations describing the structure and interaction of individuals in an area or region;

Mathematical model

- a conceptual tool that uses the language of mathematics to produce a more refined and precise description of a system;
- a set of equations describing the structure and interaction of individuals in an area or region;
- Eykhoff (1974) defined a mathematical model as 'a representation of the essential aspects of an existing system (or a system to be constructed) which presents knowledge of that system in usable form.

Mathematical models

Mathematical model

- a conceptual tool that uses the language of mathematics to produce a more refined and precise description of a system;
- a set of equations describing the structure and interaction of individuals in an area or region;
- Eykhoff (1974) defined a mathematical model as 'a representation of the essential aspects of an existing system (or a system to be constructed) which presents knowledge of that system in usable form.

- Expressing ideas mathematically clarifies thinking;

Mathematical models

Mathematical model

- a conceptual tool that uses the language of mathematics to produce a more refined and precise description of a system;
- a set of equations describing the structure and interaction of individuals in an area or region;
- Eykhoff (1974) defined a mathematical model as 'a representation of the essential aspects of an existing system (or a system to be constructed) which presents knowledge of that system in usable form.

- Expressing ideas mathematically clarifies thinking;
- J. M. Smith: Describing complex, poorly understood reality with a complex, poorly understood model is not progress at all;

Mathematical models

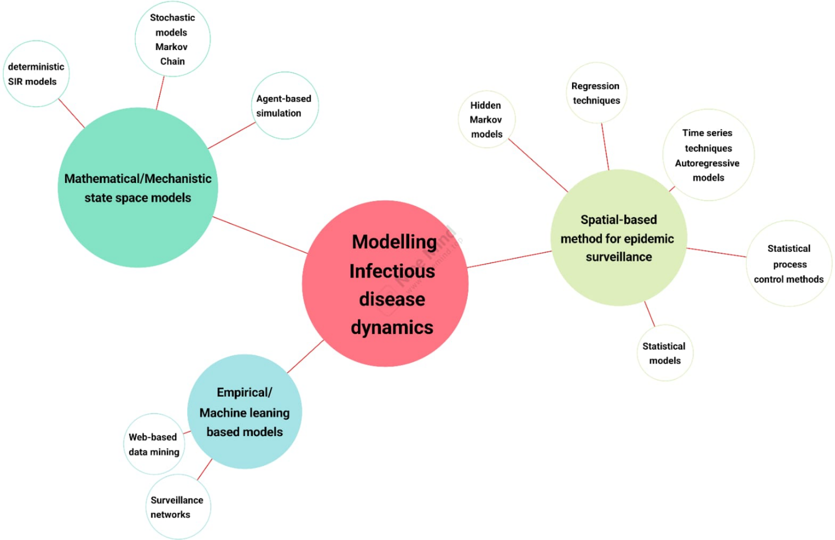
Mathematical model

- a conceptual tool that uses the language of mathematics to produce a more refined and precise description of a system;
- a set of equations describing the structure and interaction of individuals in an area or region;
- Eykhoff (1974) defined a mathematical model as 'a representation of the essential aspects of an existing system (or a system to be constructed) which presents knowledge of that system in usable form.

- Expressing ideas mathematically clarifies thinking;
- J. M. Smith: Describing complex, poorly understood reality with a complex, poorly understood model is not progress at all;
- A. Einstein: Everything should be made as simple as possible, but not simpler;

Model types

Influenced by the scientific question of concern - *research question*



What models can do?

Prediction

- Guiding difficult policy decisions where a trade-off between alternative control strategies exists;
- Predict epidemic thresholds;
- Estimate critical parameters;

What models can do?

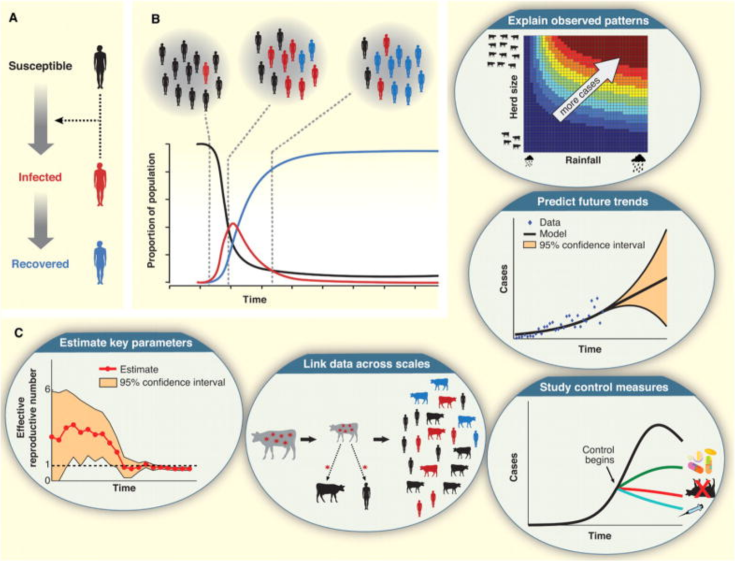
Prediction

- Guiding difficult policy decisions where a trade-off between alternative control strategies exists;
- Predict epidemic thresholds;
- Estimate critical parameters;

Understanding

- How various complexity affect the dynamics;
- The risks associated with the global spread of infectious diseases.
- A framework for examining disease features in a fairly robust and generic way;
- Understanding gained can help to build more sophisticated predictive models;

What models can do? . . .



Lyod-Smith et al (2009)

Modelling process

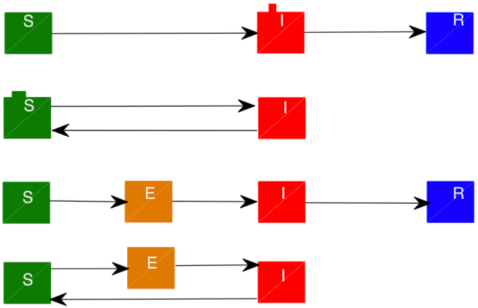
- Models are designed to address scientific questions - **Rule of thumb**
  - Choose the simplest model that explains observed phenomena (data)
  - Choose a model that is able to answer the question of interest
  - Model development can be an iterative process based on new observations or omissions

Formulating an model

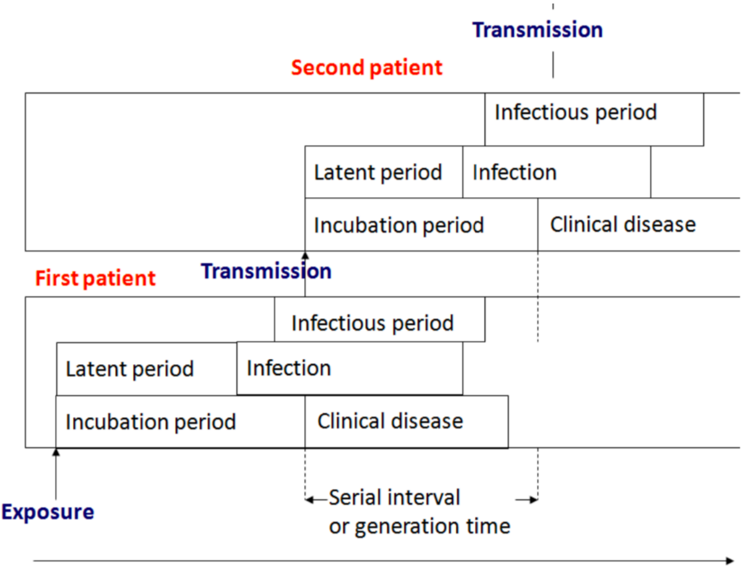
- Specify the *state variables*;
- Specify the *processes affecting* the state variables;
- Specify the *process rates* of the state variables;
- Produce the *dynamic equation* specifying the 'state variables' change over time.

Specifying the state variables

- epidemiological compartmental structure:
  - SI, SIS, SEI, SEIS, SIR, SIRS, SEIR, SEIRS;
  - More classes can be added: P (passively immune), A (asymptomatic infectives), C (carriers), V (vaccinated), SS (Super Spreaders);



Stages of infection according to time



State variables and processes

Specifying the **state variables**

- **Susceptible** - no pathogen present;
- **Exposed** - host encounters infected individuals, infected with the pathogen which grows over time but no obvious signs of infection - low pathogen load;
- **Infectious** - level of parasite grows and individual has high potential to transmit infection to other individuals;
- **Recovered** - immune system clears the parasite;

Specifying the **processes affecting** state variables

- Transmission processes - horizontal, vertical and environmental;

State variables and processes

Specifying the state variables

- **Susceptible** - no pathogen present;
- **Exposed** - host encounters infected individuals, infected with the pathogen which grows over time but no obvious signs of infection - low pathogen load;
- **Infectious** - level of parasite grows and individual has high potential to transmit infection to other individuals;
- **Recovered** - immune system clears the parasite;

Specifying the processes affecting state variables

- Transmission processes - horizontal, vertical and environmental;
- Demographics - birth, death, migration;

State variables and processes

Specifying the **state variables**

- **Susceptible** - no pathogen present;
- **Exposed** - host encounters infected individuals, infected with the pathogen which grows over time but no obvious signs of infection - low pathogen load;
- **Infectious** - level of parasite grows and individual has high potential to transmit infection to other individuals;
- **Recovered** - immune system clears the parasite;

Specifying the **processes affecting** state variables

- Transmission processes - horizontal, vertical and environmental;
- Demographics - birth, death, migration;
- Waiting times in compartment;

State variables and processes

Specifying the **state variables**

- **Susceptible** - no pathogen present;
- **Exposed** - host encounters infected individuals, infected with the pathogen which grows over time but no obvious signs of infection - low pathogen load;
- **Infectious** - level of parasite grows and individual has high potential to transmit infection to other individuals;
- **Recovered** - immune system clears the parasite;

Specifying the **processes affecting** state variables

- Transmission processes - horizontal, vertical and environmental;
- Demographics - birth, death, migration;
- Waiting times in compartment;
- recovery from infection - immune/non-immune;

Building simple basic outbreak models

Let us consider a simple case where a single case give rise to 3 new cases every day:

Let us consider a simple case where a single case give rise to 3 new cases every day:

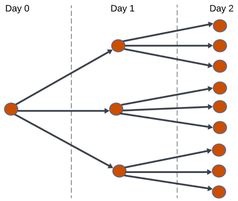
Sequence of cases

$$1,\ 3,\ 9, \ldots\ ??$$

Let us consider a simple case where a single case give rise to 3 new cases every day:

Sequence of cases

$$1,\ 3,\ 9, \ldots\ ??$$

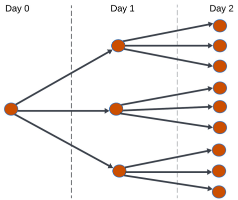


Building simple basic outbreak models

Let us consider a simple case where a single case give rise to 3 new cases every day:

Sequence of cases

$$1,\ 3,\ 9, \ldots\ ??$$



Geometric progression model

$$I_t = 3^t$$

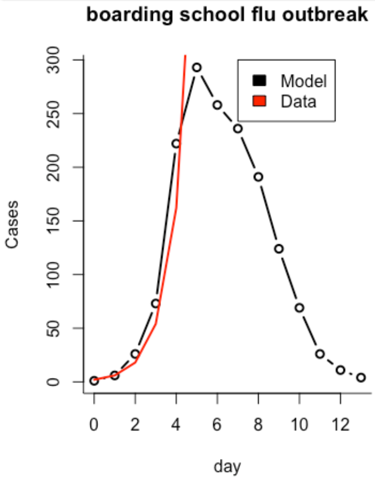
Building simple basic outbreak models

$I_t = 3^t$

Is this a good model? Can it be used to represent a real disease outbreak?

$I_t = 3^t$

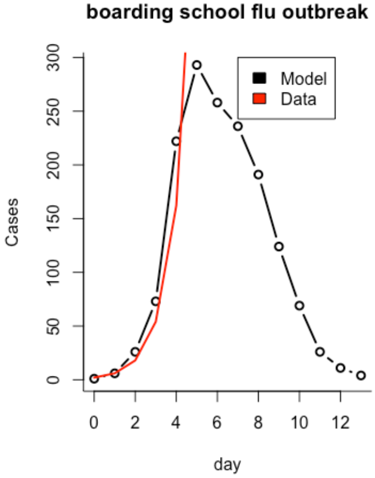
Is this a good model? Can it be used to represent a real disease outbreak?



Building simple basic outbreak models

$I_t = 3^t$

Is this a good model? Can it be used to represent a real disease outbreak?

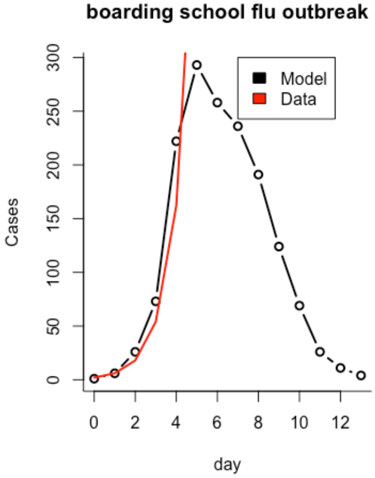


Glory

It predicts the initial phase of the outbreak.

$I_t = 3^t$

Is this a good model? Can it be used to represent a real disease outbreak?



Glory

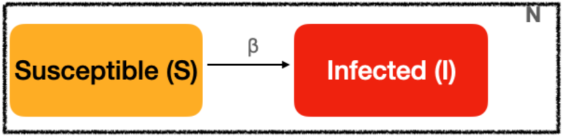
It predicts the initial phase of the outbreak.

Misery

- The model predicts exponential growth infinitely;
- It fails to capture the turn down;
- It missed the disease biology (what exactly?);

Building simple compartmental model

An *SI* model

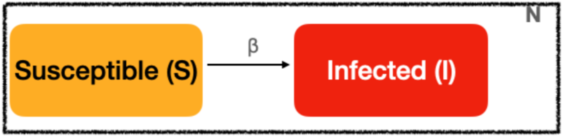


$$\frac{ds}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI$$

Building simple compartmental model

An *SI* model



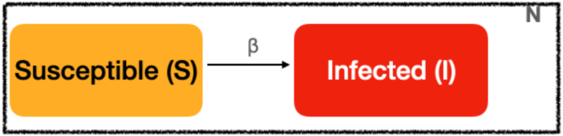
$$\frac{ds}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI$$

- Separate total population in groups;

Building simple compartmental model

An *SI* model



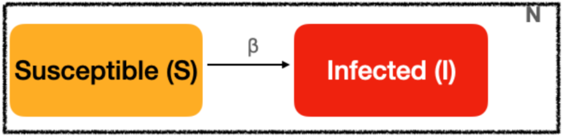
$$\frac{ds}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI$$

- Separate total population in groups;
- Keep track of each individual in each group;

Building simple compartmental model

An *SI* model



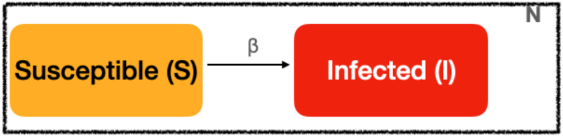
$$\frac{ds}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI$$

- Separate total population in groups;
- Keep track of each individual in each group;
- Include epidemiological processes;

Building simple compartmental model

An SI model



$$\frac{ds}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI$$

- Separate total population in groups;
- Keep track of each individual in each group;
- Include epidemiological processes;
- Overall rate of new infections depend on S and I;

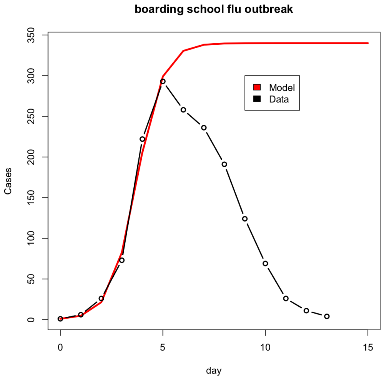
Building simple *SI* outbreak model

SI model

Is this a good model? Can it be used to represent a real disease outbreak?

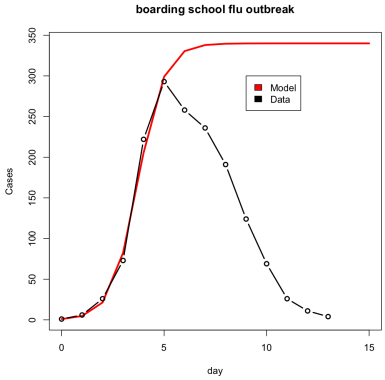
SI model

Is this a good model? Can it be used to represent a real disease outbreak?



SI model

Is this a good model? Can it be used to represent a real disease outbreak?



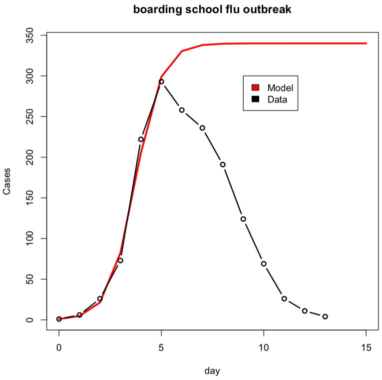
Glory

- It predicts the initial phase of the outbreak;
- Epidemic no longer grows without bounds;

Building simple *SI* outbreak model

SI model

Is this a good model? Can it be used to represent a real disease outbreak?



Glory

- It predicts the initial phase of the outbreak;
- Epidemic no longer grows without bounds;

Misery

- It fails to capture the turn down;
- It misses some disease biology (what exactly?);

Building a simple *SIR* model

- Some model assumptions and simplifications;
  - The populations is well-mixed.
  - It is spatially homogeneous.
- This defines implicitly time and space scales;
- Then, we classify individuals in three compartments:
  - S susceptibles;
  - I infectious (we will use 'infected' interchangeably);
  - R recovered (including immune and dead);
- From population biology point of view, we have a structured population;

Building a simple SIR model: **time scales**

- Our focus is on the dynamics the disease not that of the population itself;

Building a simple *SIR* model: **time scales**

- Our focus is on the dynamics the disease not that of the population itself;
- Study the qualitative behaviours of the epidemic, the conditions for its occurrence, its prevalence, and if it will come to an end or not;

Building a simple *SIR* model: **time scales**

- Our focus is on the dynamics the disease not that of the population itself;
- Study the qualitative behaviours of the epidemic, the conditions for its occurrence, its prevalence, and if it will come to an end or not;
- One would separate the dynamics of the population and that of the epidemic if the typical time scale associated with the disease is much shorter than the time scale for changes in the population;

Building a simple *SIR* model: **time scales**

- Our focus is on the dynamics the disease not that of the population itself;
- Study the qualitative behaviours of the epidemic, the conditions for its occurrence, its prevalence, and if it will come to an end or not;
- One would separate the dynamics of the population and that of the epidemic if the typical time scale associated with the disease is much shorter than the time scale for changes in the population;
- That is, we assume that the time scale of disease spread is sufficiently fast so as not to be affected by population births and deaths;

Building a simple *SIR* model: **time scales**

- Our focus is on the dynamics the disease not that of the population itself;
- Study the qualitative behaviours of the epidemic, the conditions for its occurrence, its prevalence, and if it will come to an end or not;
- One would separate the dynamics of the population and that of the epidemic if the typical time scale associated with the disease is much shorter than the time scale for changes in the population;
- That is, we assume that the time scale of disease spread is sufficiently fast so as not to be affected by population births and deaths;
- If this is the case, we can take the population as a constant, N, (note that this case is valid for a wide range of both communicable and non-communicable diseases).

The Kermack & McKendrick (1927) model

The Kermack & McKendrick (1927) model

Rate of change of susceptibles $$\frac{dS}{dt} = -rSI \tag{1}$$

Rate of change of infected $$\frac{dI}{dt} = rSI - \gamma I \tag{2}$$

Rate of change of recovered $$\frac{dR}{dt} = \gamma I \tag{3}$$

Model description

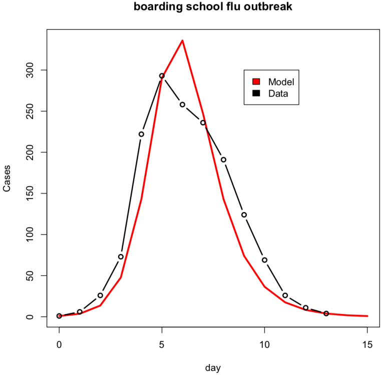
- The per capita rate of change of susceptibles is proportional to the number of infected, where r is the infection rate.
- The per capita rate of change of the infected is proportional to the number of susceptibles minus a factor accounting for the removal.
- The rate of change of the recovered is proportional to the number of infected.

SI model

Is this a good model? Can it be used to represent a real disease outbreak?

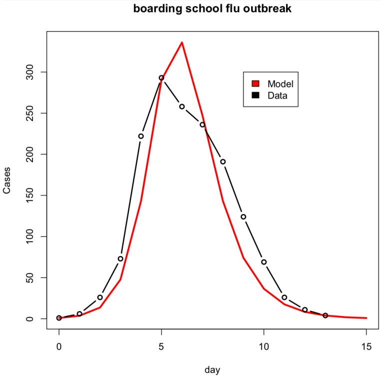
SI model

Is this a good model? Can it be used to represent a real disease outbreak?



SI model

Is this a good model? Can it be used to represent a real disease outbreak?



Glory

- It predicts the initial phase of the outbreak;
- It predicts the entire course of the outbreak with just parameters;
- Including recoveries allows to capture the overall "shape";

SIR model extensions

$$S \xrightarrow{\beta SI} I \xrightarrow{\gamma I} R$$

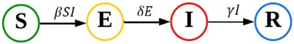
Coronavirus case

If you live with others and you are the first in the household to have symptoms of coronavirus, then you must stay home for **7 days**, but all other household members who remain well must stay home and not leave the house for **14 days**. why?

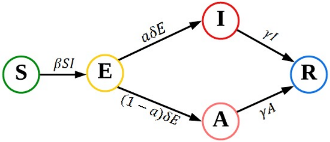
SIR model extensions

Coronavirus case

If you live with others and you are the first in the household to have symptoms of coronavirus, then you must stay home for **7 days**, but all other household members who remain well must stay home and not leave the house for **14 days**. why?



- delay between infection and symptoms;
- delay between infections and infectiousness;



Problem - tutorial sessions

Consider an *SIR* epidemic model and extend it by assuming that there a treatment for infection once a person has been infected. Model this by supposing that a fraction γ per unit time of infectives is selected for treatment, and that treatment reduces infectivity by a fraction δ. Suppose that the rate of removal from the treated class is η.

Problem - tutorial sessions

Formulate a model to describe the course of an epidemic when control measures are begun under the assumptions:

1. Exposed members may be infective with infectivity reduced by a factor ϵ_E, $0 \leq \epsilon_E < 1$.
2. Exposed members who are not isolated become infective at rate κ_1.
3. We introduce a class Q of quarantined members and a class J of isolated members.
4. Exposed members are quarantined at a proportional rate γ_1 in unit time (in practice, a quarantine will also be applied to many susceptibles, but we ignore this in the model). Quarantine is not perfect, but reduces the contact rate by a factor ϵ_Q. The effect of this assumption is that some susceptibles make fewer contacts than the model assumes.
5. There may be transmission of disease by isolated members, with an in- fectivity factor of ϵ_J .

Problems - tutorial session ...

1. Infectives are diagnosed at a proportional rate γ_2 per unit time and isolated. In addition, quarantined members are monitored and when they develop symptoms at rate κ_2 they are isolated immediately.
2. Infectives leave the infective class at rate α_1 and a fraction f_1 of these recover, and isolated members leave the isolated class at rate α_2 with a fraction f_2 recovering.

End Lecturer One!

References

M.J. Keeling and P. Rohani (2007)
Modeling Infectious Diseases in Humans and Animals
Princeton.

J.D. Murray (2002)
Mathematical Biology I
Springer.

R.M. Anderson and (1982)
Population Dynamics of Infectious Diseases: Theory and Applications
Springer.

G.F. Raggett: Modeling the Eyam plague. IMA J., 18, 221–226 (1982)

References

O. Diekmann, J.A.P. Heesterbeek, J.A.J. Metz (1990),
On the definition and the computation of the basic reprodcution ration R_0 in models for infectious diseases in heterogeneous populations,
J. Math. Biol. 28: 365-382.

P. VAn den Driessche, J. Watmough (2002),
Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission,
Mathematical Biosciences, 180: 29-48.

Pedro SA, Ndjomatchoua FT, Jentsch P, Tchuenche JM, Anand M and Bauch CT (2020) Conditions for a Second Wave of COVID-19 Due to Interactions Between Disease Dynamics and Social Processes. Front. Phys. 8:574514. doi: 10.3389/fphy.2020.574514

R. M. Anderson and R. M. May (1992),
Infectious Disease of Humans.,
Oxiford University Press

The End