Compartmental models of infectious disease dynamics

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Overview

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    - That is, looking for conditions for $I(t) > I_0$ for some time t;

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- Also, for $dS/dt < 0$, $S < S_0$ for all t.
- Thus, if $S_0 < \gamma/r$ then $S(t) < \gamma/r$ for all t

$$\left[\frac{dI}{dt}\right] = rSI - \gamma I = I(rS - \gamma) < 0,$$

  and thus $I(t) < I_0$ and will be no epidemic.
- If $S_0 > \gamma/r$ there will be an epidemic (as $\left[\frac{dI}{dt}\right]_0 > 0$).Is $I(t)$ a monotonic function?

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- Alternatively, we can interpret γ/r, the relative removal rate, to be small enough to permit the disease to spread;
- The inverse of the relative removal rate is the well known threshold parameter called the basic reproductive ratio (R_0), for this case $R_0 = \frac{rS_0}{\gamma}$; if $R_0 < 1$ no epidemic other we have an outbreak.

Phase portrait of the SIR model

- Let us look at the dynamics in the phase space.
- We have three variables, but as $S + I + R = N$, we can eliminate one of them. Say, R. So we now have a two-dimensional phase space S e I.

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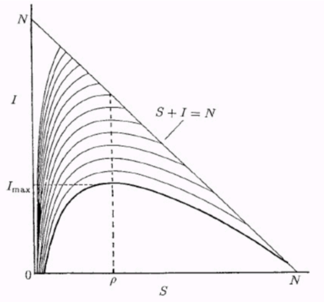

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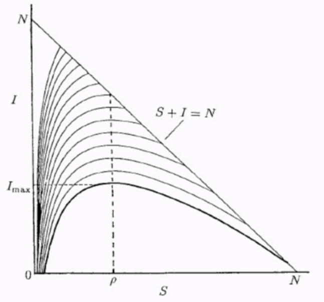

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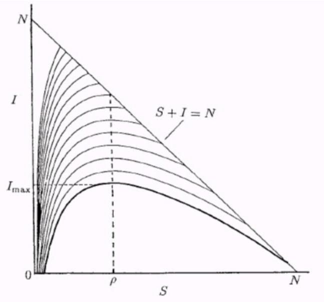

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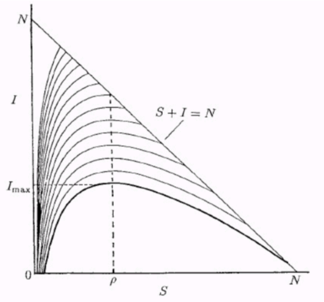

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- This will be further extended through final epidemic size analysis.

Problem - tutorial sessions

Find the curves governing the (S, I) phase plane. Determine when the maximum value of I on each of these orbits is attained. Give the expression of I_{max}.

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- If we assume that $S_0 \approx N$, then $S_0 < \gamma/r$ implies $R_0 = \frac{rN}{\gamma} < 1$ while $S_0 > \gamma/r$ implies $R_0 = \frac{rN}{\gamma} > 1$.
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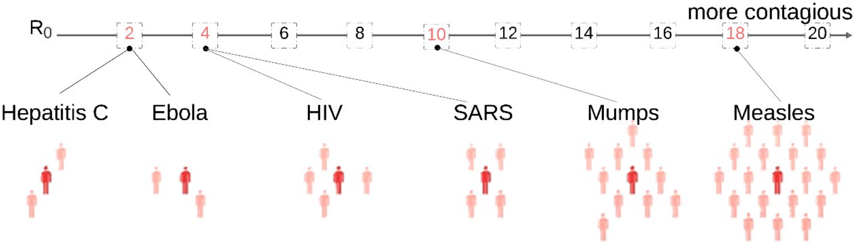

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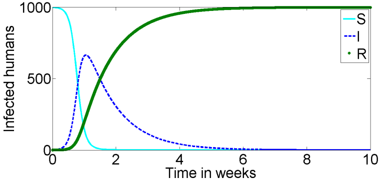

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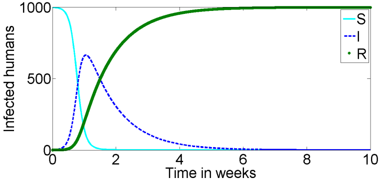

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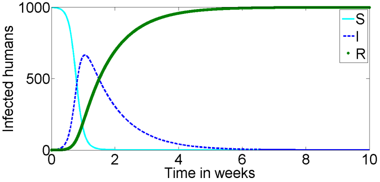

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Illustrating the qualitative behaviour of *SIR* model

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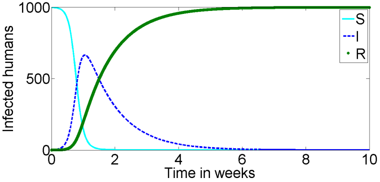

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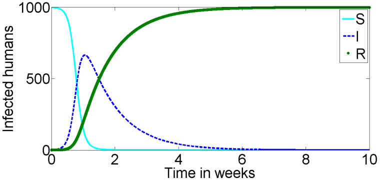

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- such a typical model was found to fit data from a flu epidemic in a boarding school in the UK.
- we can use the model as a starting point then extend it for more realistic scenarios.

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Remark

The chain of transmission eventually breaks due to the decline in infectives, not due to a complete lack of susceptibles.

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- So, there is a zero, which is the solution.

Herd immunity and Social distancing

Herd immunity

The interruption or resistance to onward infections that results if a sufficient high proportion of individuals are immune.

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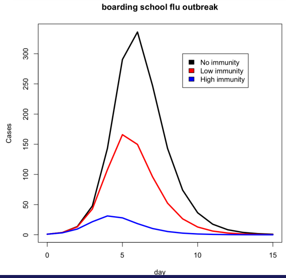

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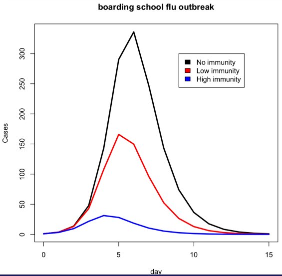

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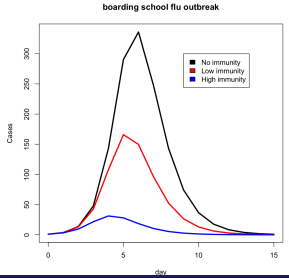

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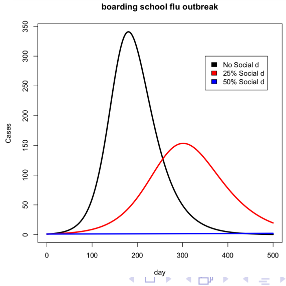

frontiers in Physics

ORIGINAL RESE
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Conditions for a Second Wave of COVID-19 Due to Interactions Between Disease Dynamics and Social Processes

*Sansao A. Pedro[1], Frank T. Ndjomatchoua[2], Peter Jentsch[3,4], Jean M. Tchuenche[5], Madhur Anand[3] and Chris T. Bauch[4]**

[1] *Departamento de Matemática e Informática, Universidade Eduardo Mondlane, Maputo, Mozambique,* [2] *Sustainable Platform, Geospatial Science and Modelling Cluster, International Rice Research Institute, Metro Manila, Philippines,* [3] *of Environmental Sciences, University of Guelph, Guelph, ON, Canada,* [4] *Department of Applied Mathematics, Univers*

Summary: general properties of epidemics

- For an infectious disease with an average infectious period given by $1/\gamma$ and a transmission rate $r = \beta$, its basic reproductive ratio R_0 is determined by β/γ.
- In a closed population, an infectious disease with a specified R_0 can invade only if there is a threshold fraction of susceptibles greater than $1/R_0$.
- Vaccination can be used to reduce the proportion of susceptibles below $1/R_0$ and hence eradicate the disease;
- Social distancing can be used to delay the peak and buy time for control;
- The chain of transmission eventually breaks due to the decline in infectives, not due to a complete lack of susceptibles.

Problem - tutorial sessions

- In many infectious diseases there is an exposed period after the transmission of infection from susceptibles to potentially infective members but before these potential infectives can transmit infection. Establish the final size relation for the extended model.
- Consider an *SIR* epidemic model and extend it by assuming that there a treatment for infection once a person has been infected. Model this by supposing that a fraction γ per unit time of infectives is selected for treatment, and that treatment reduces infectivity by a fraction δ. Suppose that the rate of removal from the treated class is η. Give the corresponding flow chart and equations of the model.

End Lecturer Two!

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