MAS342 Applicable Analysis

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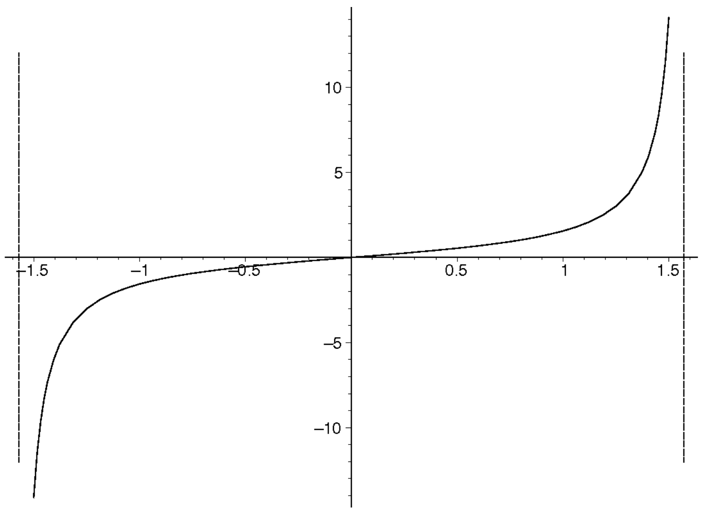
1 Revision

1.1 Some useful inverse functions.

We begin with a reminder of results from the first year which will be used in this course.

Useful Inverse Functions. The graph of $y = \tan x \quad (-\frac{\pi}{2} < x < \frac{\pi}{2})$ is shown below



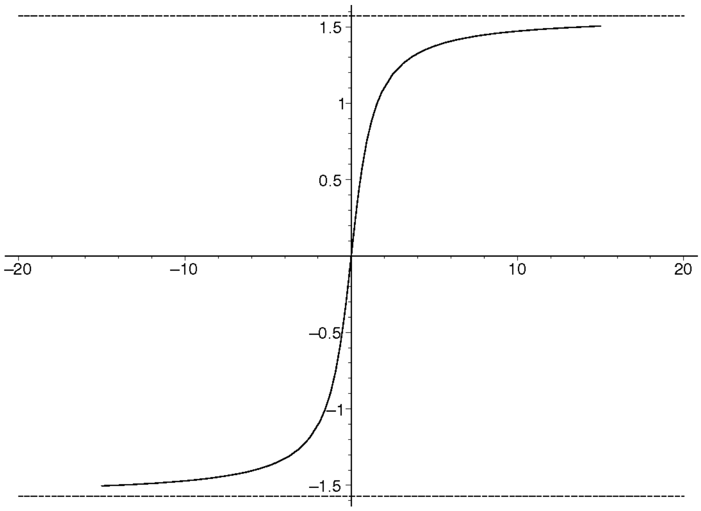
From the graph it is clear that for each real number y there is one and only one real number x such that

$$-\frac{\pi}{2} \; < \; x \; < \; \frac{\pi}{2} \quad \textbf{and} \quad y \; = \; \tan x \, .$$

The inverse function $\tan^{-1}$ is defined on $\mathbb{R}$ by

$$\tan^{-1} y \; = \; x \, , \quad \text{where } x \text{ is } \textbf{the} \text{ real number such that } y \; = \; \tan x \textbf{ and } -\frac{\pi}{2} < x < \frac{\pi}{2} \, .$$

Its graph is shown overleaf .



If $x = \tan^{-1} y$ then $y = \tan x$ and so

$$\frac{dy}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2.$$

Thus for all $y \in \mathbb{R}$,

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{1+y^2} \quad \text{i.e.} \quad \frac{d}{dy}(\tan^{-1} y) = \frac{1}{1+y^2} \quad (y \in \mathbb{R}).$$

Other inverse functions can be defined in a similar way.

For example, $\cos^{-1}$ and $\sin^{-1}$ are defined on $[-1, 1]$ in the following way:

For $-1 \le y \le 1$,

$$\cos^{-1} y = x\,, \quad \text{where } x \text{ is the real number such that } y = \cos x \textbf{ and } 0 \le x \le \pi\,,$$
$$\sin^{-1} y = x\,, \quad \text{where } x \text{ is the real number such that } y = \sin x \textbf{ and } -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2}\,.$$

The inverse function $\sinh^{-1}$ is defined on $\mathbb{R}$ by

$$\sinh^{-1} y = x\,, \quad \text{where } x \text{ is } \textbf{the} \text{ real number such that } y = \sinh x\,.$$

The inverse function $\cosh^{-1}$ is defined on $[1, \infty)$ in the following way. For $y \ge 1$,

$$\cosh^{-1} y = x\,, \quad \text{where } x \text{ is } \textbf{the} \text{ real number such that } y = \cosh x \textbf{ and } x \ge 0\,.$$

Note. Inverse hyperbolic functions can be expressed in terms of logarithms. For example, suppose that $x = \sinh^{-1} y$, then $\sinh x = y$ and

$$\cosh x = \sqrt{1 + \sinh^2 x} = \sqrt{1 + y^2}$$

and

$$\sinh x + \cosh x = \tfrac{1}{2}\left(e^x - e^{-x}\right) + \tfrac{1}{2}\left(e^x + e^{-x}\right) = e^x = y + \sqrt{1 + y^2}$$

giving

$$x \quad = \quad \ln\left(y + \sqrt{1+y^2}\right) \quad \text{i.e.} \;\; \sinh^{-1} y \quad = \quad \ln\left(y + \sqrt{1+y^2}\right).$$

Inverse functions are often useful in evaluating integrals.

1.2 Integration.

The following familiar methods are commonly used to evaluate integrals and will be used extensively in this course:

1. Substitution ,
2. Integration by parts,
3. Partial Fractions.

1. <u>Substitution</u> We will often use the following standard integral.

$$\int_a^b \frac{dx}{1+x^2}$$

To evaluate it we make the substitution $x = \tan t$ so that $\frac{dx}{dt} = \sec^2 t$ and

$$\begin{aligned}\int_a^b \frac{dx}{1+x^2} &= \int_{\tan^{-1}a}^{\tan^{-1}b} \frac{\sec^2 t}{1+\tan^2 t}\,dt = \int_{\tan^{-1}a}^{\tan^{-1}b} \frac{\sec^2 t}{\sec^2 t}\,dt = \int_{\tan^{-1}a}^{\tan^{-1}b} dt \\ &= \left[\, t \,\right]_{\tan^{-1}a}^{\tan^{-1}b} = \tan^{-1} b - \tan^{-1} a\,,\end{aligned}$$

using the relation $1 + \tan^2 t = \sec^2 t$ (which can easily be deduced from $\cos^2 t + \sin^2 t = 1$ by dividing both sides by $\cos^2 t$.)

This can easily be adapted to deal with integrals of the same basic form. For example suppose that $c > 0$. Then

$$\int_a^b \frac{dx}{c^2+x^2}$$

can be found by using the substitution $x \;=\; c\tan t$ or using

$$\int_a^b \frac{dx}{c^2+x^2} = \frac{1}{c}\int_a^b \frac{\frac{1}{c}}{1+(\frac{x}{c})^2}\,dx = \left[\frac{1}{c}\,\tan^{-1}\left(\frac{x}{c}\right)\right]_a^b,$$

using the fact that $\frac{d}{dx}\left(\frac{x}{c}\right) \;=\; \frac{1}{c}$.

Using the same idea, we can show that

$$\int_a^b \frac{dx}{x^2+2x+5} = \int_a^b \frac{dx}{(x+1)^2+4} = \frac{1}{2}\int_a^b \frac{\frac{1}{2}}{\left(\frac{x+1}{2}\right)^2+1}\,dx = \frac{1}{2}\left[\tan^{-1}\left(\frac{x+1}{2}\right)\right]_a^b.$$

Alternatively, we could use the substitution $(x+1) \;=\; 2\tan t$.

The use of substitutions of the form $x = \sin t$ or $x = \cos t$ allows integrals of the form

$$\int_a^b \frac{dx}{\sqrt{1-x^2}} \quad (-1 < x < 1)$$

to be found, giving the answer in terms of $\sin^{-1}$ or $\cos^{-1}$.

When substituting in a definite integral <u>ALWAYS</u> remember to change the limits to the appropriate ones for the new variable.

2. <u>Integration by parts</u> The rule for integration by parts is a consequence of the law for differentiating a product viz.

$$\frac{d}{dx}\,(uv) \quad = \quad u\,\frac{dv}{dx} \;+\; v\,\frac{du}{dx}\,.$$

Thus

$$uv \quad = \quad \int u\,\frac{dv}{dx}\,dx \;+\; \int v\,\frac{du}{dx}\,dx \quad \text{i.e.} \; \int u\,\frac{dv}{dx}\,dx \quad = \quad uv \;-\; \int v\,\frac{du}{dx}\,dx\,.$$

This method can be useful when dealing with integrals involving logarithms as the following example illustrates.

Let $0 < a < b$ evaluate

$$\int_a^b \ln x\,dx\,.$$

To find this integral, use $u = \ln x$ and $\frac{dv}{dx} = 1$ so that $\frac{du}{dx} = \frac{1}{x}$ and $v = x$. Hence

$$\int_a^b \ln x\,dx \;=\; [x\ln x]_a^b \;-\; \int_a^b x\left(\frac{1}{x}\right)\,dx \;=\; [x\ln x \;-\; x]_a^b\,.$$

When finding integrals involving logarithms it is, however, worth looking first of all to see whether the integrand has a factor $\frac{1}{x}$, since $\frac{1}{x}$ is the derivative of $\ln x$ with respect to x. For example, for $0 < a < b$,

$$\int_a^b \frac{2\ln x}{x}\,dx \quad = \quad \int_a^b \frac{d}{dx}\left((\ln x)^2\right)\,dx \quad = \quad \left[(\ln x)^2\right]_a^b\,.$$

1.3 Methods for evaluating $\int_0^\infty e^{-ax} \sin x \, dx$, where $a > 0$.

(i). Integrate by parts twice. Each time integrate the trigonometric function and differentiate the exponential function.

(ii). Integrate by parts twice. Each time integrate the exponential function and differentiate the trigonometric function.

(iii). Use

$$
\begin{aligned}
\int_0^\infty e^{-ax} \sin x \, dx &= \int_0^\infty \mathrm{Im}\left(e^{-ax}\, e^{ix}\right) dx = \int_0^\infty \mathrm{Im}\left(e^{-(a-i)x}\right) dx \\
&= \mathrm{Im}\left[-\frac{1}{a-i}\, e^{-(a-i)x}\right]_0^\infty = \mathrm{Im}\left[-\frac{a+i}{a^2-i^2}\, e^{-ax}\, e^{ix}\right]_0^\infty \\
&= \mathrm{Im}\left[-\frac{a+i}{a^2+1}\, e^{-ax}(\cos x + i \sin x)\right]_0^\infty \\
&= \left[\frac{e^{-ax}}{a^2+1}\,(-\cos x - a \sin x)\right]_0^\infty \\
&= \frac{1}{a^2+1},
\end{aligned}
$$

since

$$\frac{e^{-ax}}{a^2+1}\,(-\cos x - a\sin x) \to 0 \quad \text{as} \quad x \to \infty$$

because $a > 0$.

Note. The same methods will work for $\int_0^\infty e^{-ax} \cos x \, dx$.

3. Partial Fractions Partial fractions can be used to integrate rational functions i.e. quotients of polynomials.

They will be used for this purpose in Chapter 2 on improper integrals. However they have an important role to play in Chapter 4 on Laplace transforms. Their use often allows us to find the inverse Laplace transforms, which we need to solve our problems.

In the past I have found that it has often been necessary to spend some considerable time in lectures on homework feedback on partial fractions. Well this year I have decided that we will look at the partial fractions at the beginning of the course (before the mistakes are made!!) rather than wait until after the mistakes are made and then correct them!!!.

Suppose that

$$f(x) = \frac{p(x)}{q(x)}\,,$$

where p and q are polynomials.

First check the degree of the polynomials p and q.

If the degree of p is greater than or equal to the degree of q use long division to express the quotient in the form

$$\frac{p(x)}{q(x)} = r(x) + \frac{p_1(x)}{q_1(x)}\,,$$

where r , p_1 , q_1 , are polynomials and the degree of p_1 is less than the degree of q_1. Then express $\frac{p_1(x)}{q_1(x)}$ in partial fractions.

This sometimes seems the most difficult part, but there are ways of cutting down on the hard work, as we see below.

Let

$$R(s) = \frac{P(s)}{(s-a)\,Q(s)}\,, \qquad (*)$$

where P, Q are polynomials such that $Q(a) \neq 0$ (i.e. Q(s) does not have a factor $(s-a)$) and

$$\text{degree of } P(s) < \text{degree of } (s-a)\,Q(s)\,.$$

Then the rational function $R(s)$ can be expressed in the form

$$R(s) = \frac{A}{s-a} + \frac{p(s)}{Q(s)}\,, \qquad (**)$$

where

$$\text{degree of } p(s) < \text{degree of } Q(s)$$

and

$$A = \frac{P(a)}{Q(a)}$$

i.e. A is the value we obtain from $R(s)$ when we cover up the factor $(s-a)$ and then substitute $s = a$ in what is left. This is called **the cover-up rule.**

To justify this we note, from $(**)$ that

$$R(s) = \frac{A}{s-a} + \frac{p(s)}{Q(s)} = \frac{A\,Q(s) + (s-a)\,p(s)}{(s-a)\,Q(s)}$$

and so we need to choose A and the polynomial $p(s)$ so that

$$P(s) \;=\; A\,Q(s) \;+\; (s-a)\,p(s)$$

for **all** s. In particular using $s \;=\; a$ gives

$$P(a) \;=\; A\,Q(a) \quad \text{i.e.} \quad A \;=\; \frac{P(a)}{Q(a)}\,.$$

Examples.

1. Using the cover-up rule, we see that for $s \neq \pm 1$,

$$\frac{2}{s^2-1} \;=\; \frac{2}{(s-1)(s+1)} \;=\; \frac{A}{s-1} \;+\; \frac{B}{s+1}\,,$$

where

$$A \;=\; \left(\frac{2}{s+1}\right)_{s=1} \;=\; 1\,, \quad B \;=\; \left(\frac{2}{s-1}\right)_{s=-1} \;=\; -1$$

2. Using the cover-up rule, we see that for $s \neq 1$,

$$\frac{2}{(s-1)(s^2+1)} \;=\; \frac{A}{s-1} \;+\; \frac{Bs+C}{s^2+1}\,,$$

where

$$A \;=\; \left(\frac{2}{s^2+1}\right)_{s=1} \;=\; 1\,.$$

The cover -up rule can **NOT** be used to find $Bs+C$.

We need to find B, C so that

$$A(s^2+1) \;+\; (Bs+C)(s-1) \;=\; 2$$

$$\text{i.e.} \quad (s^2+1) \;+\; (Bs+C)(s-1) \;=\; 2 \tag{1}$$

for **all** s. Substituting $s \;=\; 0$ in (1) gives

$$1 \;-C \;=\; 2 \quad \text{i.e.} \quad C \;=\; -1\,,$$

and equating coefficients of s^2 in (1) gives

$$B \;=\; -1\,.$$

Notes.

1. When using the cover-up rule, the factor you cover up must be of the form $(s-a)$ (i.e. the coefficient of s **in the factor which you cover up must be 1.**)

For example,

$$\frac{1}{(2s-1)(s-2)(s^2+1)} \;=\; \frac{1}{2\left(s-\frac{1}{2}\right)(s-2)(s^2+1)} \;=\; \frac{A}{s-\frac{1}{2}} + \frac{B}{s-2} + \frac{Cs+D}{s^2+1}\,,$$

where

$$A = \left(\frac{1}{2(s-2)(s^2+1)} \right)_{s=\frac{1}{2}} \qquad \text{and} \quad B = \left(\frac{1}{2\left(s-\frac{1}{2}\right)(s^2+1)} \right)_{s=2}$$

using the cover-up rule.

2. Let n be a positive integer with $n \geq 2$. Suppose that P, Q are polynomials such that $Q(a) \neq 0$ (i.e. Q(s) does not have a factor $(s-a)$) and

$$\text{degree of } P(s) < \text{degree of } (s-a)^n Q(s).$$

then we can express the rational function

$$R(s) = \frac{P(s)}{(s-a)^n Q(s)}$$

in partial fractions as

$$R(s) = \frac{p_1(s)}{(s-a)^n} + \frac{p_2(s)}{Q(s)} = \frac{A_n}{(s-a)^n} + \frac{A_{n-1}}{(s-a)^{n-1}} + \cdots + \frac{A_1}{(s-a)} + \frac{p_2(s)}{Q(s)},$$

where p_1, p_2 are polynomials with degree of p_1 $< n$ and degree of p_2 $<$ degree of Q. The cover-up rule can be used to find A_n. Powers of non-zero quadratic factors in the denominator can be dealt with in the same way.

The cover-up rule can be used to find A_n. It gives

$$A_n = \frac{P(a)}{Q(a)}.$$

However it can **NOT** be used to find A_{n-1}, A_{n-2}, $\cdots$ A_1.

Examples. Express

(i) $\dfrac{2s+1}{s^2(s^2-1)}$, (ii) $\dfrac{2s^3+1}{(s^2+1)^2}$

in partial fractions.

Solutions.

(i) For $s \neq 0. \pm 1$,

$$\frac{2s+1}{s^2(s^2-1)} = \frac{2s+1}{s^2(s+1)(s-1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s-1)} + \frac{D}{(s+1)}, \tag{2}$$

where

$$A = -1, \quad C = \tfrac{3}{2}, \quad D = \tfrac{1}{2},$$

using the cover-up rule.

We now need to find B. We note that

$$\frac{2s+1}{s^2(s^2-1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s-1)} + \frac{D}{(s+1)}$$
$$= \frac{A(s^2-1) + Bs(s^2-1) + Cs^2(s+1) + Ds^2(s-1)}{s^2(s^2-1)},$$

and so we need to find B so that

$$A(s^2-1) + Bs(s^2-1) + Cs^2(s+1) + Ds^2(s-1) = 2s+1$$

for **all** s. Comparing coefficients of s^3 gives

$$B + C + D = 0 \quad \text{i.e.} \quad B = -(C+D) = -2\,.$$

Hence

$$\frac{2s+1}{s^2(s^2-1)} = -\frac{1}{s^2} - \frac{2}{s} + \frac{\frac{3}{2}}{(s-1)} + \frac{\frac{1}{2}}{(s+1)}\,.$$

(ii) We see that

$$\frac{2s^3+1}{(s^2+1)^2} = \frac{2s^3 + 2s - 2s + 1}{(s^2+1)^2} = \frac{2s}{(s^2+1)} + \frac{1-2s}{(s^2+1)^2}\,.$$

You will find a list of standard integrals and standard derivatives at the back of the exercise booklet.

1.4 Some useful limits involving exponential functions, powers and logarithms.

1. For all real numbers α,

$$\frac{x^\alpha}{e^x} = x^\alpha e^{-x} \to 0 \quad \text{as} \quad x \to \infty\,.$$

(i.e. the exponential dominates any power.)

2(i) For $\beta > 0$,

$$\frac{\ln y}{y^\beta} \to 0 \quad \text{as} \quad y \to \infty\,.$$

(ii) For $\beta > 0$,

$$t^\beta \ln t \to 0 \quad \text{as} \quad t \to 0+\,.$$

(i.e. t tends to 0 through positive values.)

(Powers dominate logarithms.)

<u>**Proof**</u>.

1. Let α be any real number. Choose an integer q such that $q > \alpha$. Then for all $x > 1$,

$$0 \quad < \quad x^\alpha \quad < \quad x^q \tag{1}$$

and

$$e^x \quad = \quad 1 + x + \frac{x^2}{3!} + \frac{x^3}{3!} + \cdots \quad > \quad \frac{x^{q+1}}{(q+1)!}\,. \tag{2}$$

Hence, for $x > 1$,

$$0 \quad < \quad \frac{x^\alpha}{e^x} \quad < \quad \frac{x^q}{e^x} \quad < \quad \frac{x^q\,(q+1)!}{x^{q+1}} \quad = \quad \frac{(q+1)!}{x}$$

by (1) and (2).

As $x \to \infty\,,\quad \dfrac{(q+1)!}{x} \to 0\,.$ Hence

$$\frac{x^\alpha}{e^x} \to 0 \quad \text{as } x \to \infty\,.$$

2(i). Putting $\alpha = 1$ gives

$$xe^{-x} \to 0 \quad \text{as } x \to \infty\,.$$

Write $x = \beta t$, where $\beta > 0$ to give

$$\beta t e^{-\beta t} \to 0 \quad \text{as } t \to \infty \qquad \text{i.e. } te^{-\beta t} \to 0 \quad \text{as } t \to \infty\,.$$

Now put $e^t = y$ i.e. $t = \ln y$ to give

$$te^{-\beta t} \quad = \quad \frac{t}{e^{\beta t}} \quad = \quad \frac{\ln y}{y^\beta} \to 0 \quad \text{as } y \to \infty\,, \quad \text{when } \beta > 0\,.$$

2(ii). Putting $y = \frac{1}{t}$ $\quad (t > 0)$ in 2(i), we see that

$$\frac{\ln\left(\frac{1}{t}\right)}{\left(\frac{1}{t}\right)^\beta} \to 0 \quad \text{as } t \to 0+$$

(i.e. as t tends to zero through positive values)

Thus

$$t^\beta\,(-\ln t) \to 0 \quad \text{as } t \to 0+ \quad \text{and therefore } t^\beta \ln t \to 0 \quad \text{as } t \to 0+$$

2 Improper Integrals

Riemann integration is initially defined for bounded functions on bounded intervals. However, under certain circumstances, it is possible to define a process of integration in which integrands and intervals of integration may be unbounded. Such integrals will be used throughout much of this course. Informally, they will have been met before, but we now provide a rather more systematic treatment.

2.1 Improper integrals of the first kind

Definition 2.1 If $\int_a^X f(x)\,dx$ exists as an ordinary Riemann integral for every $X > a$, and if $\int_a^X f(x)\,dx$ tends to a finite limit as $X \to \infty$, then this limit is called the *improper integral* of f over $[a, \infty)$ and we write

$$\lim_{X\to\infty} \int_a^X f(x)\,dx = \int_a^\infty f(x)\,dx.$$

If $\int_a^X f(x)\,dx$ does not tend to a finite limit as $X \to \infty$, then we say that *improper integral* of f over $[a, \infty)$ does not exist.

When $\int_a^\infty f(x)\,dx$ exists, we also say that the integral *converges*, and when it does not exist, we say that the integral *diverges*.

Examples 2.2 Decide whether the following integrals exist.

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$. 2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$. 3. $\displaystyle\int_0^\infty \sin x\,dx$.

Solutions.

1. For all $X > 1$, $\displaystyle\int_1^X \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^X = 1 - \frac{1}{X} \to 1$ as $X \to \infty$ and so
$$\int_1^\infty \frac{1}{x^2}\,dx = 1.$$

2. For all $X > 1$, $\displaystyle\int_1^X \frac{1}{x}\,dx = [\ln x]_1^X = \ln X \to \infty$ as $X \to \infty$, and so $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ does not exist.

3. For all $X > 0$, $\int_0^X \sin x\,dx = [-\cos x]_0^X = 1 - \cos X$. This does not tend to a limit as $X \to \infty$, and so $\int_0^\infty \sin x\,dx$ does not exist.

The three simple results in the next theorem follow immediately from the properties of limits.

Theorem 2.3 *1. If $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ exist and k is a constant, then the integrals $\int_a^\infty kf(x)\,dx$ and $\int_a^\infty \{f(x)+g(x)\}dx$ exist, and*

$$\begin{aligned}\int_a^\infty kf(x)\,dx &= k\int_a^\infty f(x)\,dx;\\ \int_a^\infty \{f(x)+g(x)\}dx &= \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx.\end{aligned}$$

2. If $a<b$, then

$$\int_a^b f(x)\,dx + \int_b^\infty f(x)\,dx = \int_a^\infty f(x)\,dx,$$

the existence of either side implying that of the other.

Proof. 1. For all $X>a$,

$$\int_a^X kf(x)\,dx = k\int_a^X f(x)\,dx \to k\int_a^\infty f(x)\,dx \quad \text{as} \quad X\to\infty$$

and

$$\int_a^X \{f(x)+g(x)\}dx = \int_a^X f(x)\,dx + \int_a^X g(x)\,dx \to \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$$

as $X\to\infty$.

2. For all $X>b$,

$$\int_a^b f(x)\,dx + \int_b^X f(x)\,dx = \int_a^X f(x)\,dx\,. \qquad (*)$$

If $\int_b^\infty f(x)\,dx$ exists, then from $(*)$,

$$\int_a^X f(x)\,dx = \int_a^b f(x)\,dx + \int_b^X f(x)\,dx \to \int_a^b f(x)\,dx + \int_b^\infty f(x)\,dx$$

as $X\to\infty$ i.e. $\int_a^\infty f(x)\,dx = \int_a^b f(x)\,dx + \int_b^\infty f(x)\,dx$.

Similarly, if $\int_a^\infty f(x)\,dx$ exists, then from $(*)$,

$$\int_b^X f(x)\,dx = \int_a^X f(x)\,dx - \int_a^b f(x)\,dx \to \int_a^\infty f(x)\,dx - \int_a^b f(x)\,dx$$

as $X\to\infty$ i.e. $\int_a^\infty f(x)\,dx - \int_a^b f(x)\,dx = \int_b^\infty f(x)\,dx$. □

In general, we cannot find convenient explicit expressions for integrals, and so, as in the case of infinite series (with which there is close affinity), we usually rely on convergence tests. For non-negative integrands, the *Comparison Test* is the mainstay of proofs of convergence or divergence of integrals.

Theorem 2.4 (Comparison Test) *Suppose that the functions f, ϕ and ψ are integrable over every interval $[a, X]$.*

1. *If $\phi(x) \geq f(x) \geq 0$ for $x \geq a$ and $\int_a^\infty \phi(x)\,dx$ exists, then $\int_a^\infty f(x)\,dx$ also exists.*
2. *If $f(x) \geq \psi(x) \geq 0$ for $x \geq a$ and $\int_a^\infty \psi(x)\,dx$ does not exist, then $\int_a^\infty f(x)\,dx$ does not exist.*

Proof.

1. Since $f(t) \geq 0$, $F(x) = \int_a^x f(t)\,dt$ increases, and, as $f(t) \leq \phi(t)$,
$$F(x) \leq \int_a^x \phi(t)\,dt \leq \int_a^\infty \phi(t)\,dt.$$
Thus F increases and is bounded above, so that $F(x)$ tends to a finite limit as $x \to \infty$.

2. Since $\int_a^x \psi(t)\,dt$ is increasing and $\int_a^x \psi(t)\,dt$ does not tend to a finite limit as $x \to \infty$, $\int_a^x \psi(t)\,dt \to \infty$ as $x \to \infty$. But $F(x) \geq \int_a^x \psi(t)\,dt$ and so $F(x) \to \infty$ as $x \to \infty$.

$\square$

Example 2.5 Show that $\displaystyle\int_0^\infty \frac{x^3}{x^5+1}\,dx$ exists.

Solution.

For $x \geq 1$,
$$0 < \frac{x^3}{x^5+1} < \frac{1}{x^2}\,.$$
Since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ exists, so does $\displaystyle\int_1^\infty \frac{x^3}{x^5+1}\,dx$ by the Comparison Test. Now $\displaystyle\frac{x^3}{x^5+1}$ is continuous on the closed interval [0,1] and, therefore, $\displaystyle\int_0^1 \frac{x^3}{x^5+1}\,dx$ exists. Hence
$$\int_0^\infty \frac{x^3}{x^5+1}\,dx = \int_0^1 \frac{x^3}{x^5+1}\,dx + \int_1^\infty \frac{x^3}{x^5+1}\,dx$$
exists.

WARNING. In the previous set of examples we showed that $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ exists and so we can use it and the Comparison Test.

However, $\displaystyle\frac{1}{x^2}$ is unbounded at the origin and we will see, in the next section that $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ **diverges** and so $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$ **diverges**. Don't be tempted to use $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$ in conjunction with the Comparison Test for examples like the one above.

This is why we dealt with $\displaystyle\int_0^1$ and $\displaystyle\int_1^\infty$ separately.

When the integrand is of variable sign, the simplest convergence criterion is given by the next theorem.

Theorem 2.6 *If the function f is integrable over the interval $[a, X]$ for every $X > a$, and $\int_a^\infty |f(x)|\,dx$ exists, then $\int_a^\infty f(x)\,dx$ exists.*

Proof. For $x \geq a$,

$$0 \leq f(x) + |f(x)| \ \leq\ 2\,|f(x)|\,.$$

By the Comparison Test $\displaystyle\int_a^\infty [\,f(x) + |f(x)|\,]\,dx$ exists, since $\displaystyle\int_a^\infty |f(x)|\,dx$ exists. Hence

$$\int_a^\infty f(x)\,dx = \int_a^\infty \{\,[\,f(x) + |f(x)|\,] - |f(x)|\,\}\,dx = \int_a^\infty \{f(x) + |f(x)|\}\,dx - \int_a^\infty \{|f(x)|\}\,dx$$

exists.

$\square$

Method for testing for convergence of integrals of the form $\int_a^\infty f(x)\,dx$.

1. If $f(x)$ assumes only non-negative values, try the Comparison Test.

   If $f(x)$ assumes only non-positive values, try applying the Comparison Test to $-f$.

2. If $f(x)$ assumes both positive and negative values, try testing for absolute convergence.

3. If neither of these methods works, then it might be worth trying "Integration by Parts." $\square$

Examples 2.7 1. Show that $\displaystyle\int_1^\infty \frac{\cos x}{x^2}\,dx$ exists.

2. Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ exists, but $\displaystyle\int_0^\infty \left|\frac{\sin x}{x}\right|\,dx$ does not exist.

3. The function $f : [0, \infty) \to \mathbb{R}$ is continuous and $\dfrac{f(x)}{x} \to$ a finite limit as $x \to 0+$. Show that, if $\displaystyle\int_0^\infty f(x)\,dx$ exists, then $\displaystyle\int_0^\infty \frac{f(x)}{x}\,dx$ also exists.

This result will be used in the proof of Corollary 4.18 in Chapter 4 on Laplace Transforms.

Solutions.

1. For $x \geq 1$, $\left|\dfrac{\cos x}{x^2}\right| \leq \dfrac{1}{x^2}$. By the Comparison Test $\displaystyle\int_1^\infty \left|\frac{\cos x}{x^2}\right| dx$ exists since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges. Therefore $\displaystyle\int_1^\infty \frac{\cos x}{x^2}\,dx$ exists.

2. For $X > 1$,
$$\int_1^X \frac{\sin x}{x}\,dx = \left[-\frac{\cos x}{x}\right]_1^X - \int_1^X \frac{\cos x}{x^2}\,dx. \tag{1}$$
Now
$$\left[-\frac{\cos x}{x}\right]_1^X = \cos 1 - \frac{\cos X}{X} \to \cos 1 \text{ as } X \to \infty$$
and $\displaystyle\lim_{X\to\infty}\int_1^X \frac{\cos x}{x^2}\,dx$ exists by Example 1. Hence
$$\int_1^X \frac{\sin x}{x}\,dx \to \cos 1 - \int_1^\infty \frac{\cos x}{x^2}\,dx \quad \text{as} \quad X \to \infty$$
by (1), i.e. $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$ exists.

Also, $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ exists, as $\dfrac{\sin x}{x} \to 1$ as $x \to 0$ and so the integrand, $\dfrac{\sin x}{x}$, is bounded and continuous on the interval $(0, 1]$. Hence
$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx$$
exists.

If r is an **odd integer**, then for $(r-1)\pi \leq x \leq r\pi$, $|\sin x| = \sin x$ and so,
$$\int_{(r-1)\pi}^{r\pi} |\sin x|\,dx = \int_{(r-1)\pi}^{r\pi} \sin x\,dx = [-\cos x]_{(r-1)\pi}^{r\pi} = -\cos r\pi + \cos(r-1)\pi = 2\,.$$
If r is an **even integer**, then for $(r-1)\pi \leq x \leq r\pi$, $|\sin x| = -\sin x$ and so,
$$\int_{(r-1)\pi}^{r\pi} |\sin x|\,dx = -\int_{(r-1)\pi}^{r\pi} \sin x\,dx = [\cos x]_{(r-1)\pi}^{r\pi} = \cos r\pi - \cos(r-1)\pi = 2\,.$$
Thus, for all integers r,
$$\int_{(r-1)\pi}^{r\pi} |\sin x|\,dx = 2\,.$$
For $r = 1, 2, \ldots$,
$$\int_{(r-1)\pi}^{r\pi} \left|\frac{\sin x}{x}\right| dx \geq \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{r\pi}\,dx = \frac{1}{r\pi}\int_{(r-1)\pi}^{r\pi} |\sin x|\,dx = \frac{2}{r\pi}\,.$$
Hence
$$\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{2}{\pi}\sum_{r=1}^{n}\frac{1}{r} = \frac{2}{\pi}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right) \to \infty$$

as $n \to \infty$. This suffices to show that $\displaystyle\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ does not exist.

3. For $x \geq 1$, let

$$F(x) = \int_1^x f(t)\, dt\,. \tag{1}$$

By the Fundamental Theorem of Calculus

$$F'(x) = f(x) \quad (x \geq 1)\,. \tag{2}$$

Using (2) and integration by parts, we see that for $X > 1$,

$$\int_1^X \frac{f(x)}{x}\, dx = \int_1^X \frac{F'(x)}{x}\, dx = \left[\frac{F(x)}{x}\right]_1^X + \int_1^X \frac{F(x)}{x^2}\, dx = \frac{F(X)}{X} + \int_1^X \frac{F(x)}{x^2}\, dx \tag{3}$$

since $F(1) = 0$ by (1).

Now the function F, given by (1), is continuous on $[1, \infty)$, since f is continuous on $[1, \infty)$. In addition, $F(x) \to$ a finite limit as $x \to \infty$ because $\int_1^\infty f(x)\, dx$ exists. Hence F is bounded on $[1, \infty)$ i.e.

$$|F(x)| \quad \leq \quad M \quad (x \geq 1), \tag{4}$$

for some suitable positive real number M. From (4) we see that for all $X \geq 1$,

$$\left|\frac{F(X)}{X}\right| \quad \leq \quad \frac{M}{|X|}$$

and so

$$\frac{F(X)}{X} \to 0 \ \text{ as } \ X \to \infty\,. \tag{5}$$

Moreover,

$$\left|\frac{F(x)}{x^2}\right| \quad \leq \quad \frac{M}{x^2}\,,$$

and $\displaystyle\int_1^\infty \frac{1}{x^2}\, dx$ exists. By the Comparison Test

$$\int_1^\infty \left|\frac{F(x)}{x^2}\right| dx \ \text{ converges}$$

i.e. $\displaystyle\int_1^\infty \frac{F(x)}{x^2}\, dx$ is absolutely convergent and so convergent.

From (3) and (5), we see that

$$\int_1^X \frac{f(x)}{x}\, dx \to \int_1^\infty \frac{F(x)}{x^2}\, dx \ \text{ as } \ X \to \infty.$$

Thus

$$\int_1^\infty \frac{f(x)}{x}\, dx \ \text{ exists.}$$

Now f is continuous on [0,1] and $\dfrac{f(x)}{x}$ tends to a finite limit as $x \to 0+$. Thus $\dfrac{f(x)}{x}$ is bounded and continuous on (0,1]. Hence

$$\int_0^1 \frac{f(x)}{x}\,dx \quad \text{exists.}$$

It follows that

$$\int_0^\infty \frac{f(x)}{x}\,dx = \int_0^1 \frac{f(x)}{x}\,dx + \int_1^\infty \frac{f(x)}{x}\,dx \quad \text{exists .} \qquad \square$$

Definition 2.8 *If the function f is integrable over the interval $[a, X]$ for every $X > a$, and $\int_a^\infty |f(x)|\,dx$ exists, then $\int_a^\infty f(x)$ is said to converge absolutely.*

Theorem 2.6 shows that an absolutely convergent integral also converges. For instance, $\displaystyle\int_1^\infty \frac{\cos x}{x^2}\,dx$ converges absolutely. On the other hand, $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ converges, but it does not converge absolutely. We now actually evaluate this integral.

The working will be fairly straightforward if we first prove some simple results. These results which are contained in the following Lemmas and their Corollaries will already be familiar to some of you.

Lemma 2.9 *For* $0 \le t \le \frac{\pi}{2}$,

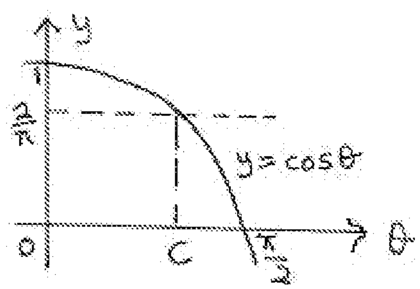
$$\sin t \;\ge\; \frac{2t}{\pi}\ .$$

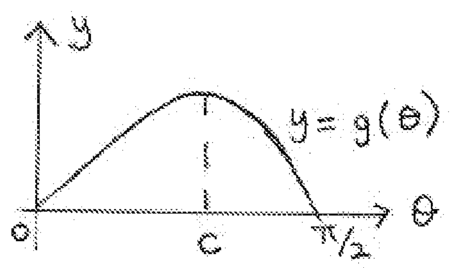
Proof. Let

$$g(t) \;=\; \sin t - \tfrac{2}{\pi}\,t. \tag{1}$$

Then

$$g'(t) \;=\; \cos t - \tfrac{2}{\pi} \tag{2}$$





Now let $c = \cos^{-1}(\frac{2}{\pi})$. Then $0 < c < \frac{\pi}{2}$ and $g'(t) \ge 0$ when $0 \le t \le c$. Thus g is increasing on $[0,c]$ and so $g(t) \geq g(0) = 0$ on $[0,c]$ i.e. $\sin t - \frac{2}{\pi}\, t \geq 0$ for $0 \leq t \leq c$ and so

$$\frac{\sin t}{t} \;\geq\; \frac{2}{\pi} \qquad (0 \,\leq\, t \,\leq\, c)\,.$$

Similarly g is decreasing on $[c, \frac{\pi}{2}]$ because $g'(t) \leq 0$ on $[c, \frac{\pi}{2}]$ and so $g(t) \geq g(\frac{\pi}{2}) = 0$ on $[c, \frac{\pi}{2}]$ giving

$$\frac{\sin t}{t} \;\geq\; \frac{2}{\pi} \qquad (c \,\leq\, t \,\leq\, \tfrac{\pi}{2})\,.$$

Hence $g(t) \geq 0$ on the interval $[0, \frac{\pi}{2}]$,

$$\text{i.e.} \quad \sin t \;\geq\; \frac{2t}{\pi} \qquad (0 \,\leq\, t \,\leq\, \tfrac{\pi}{2})\,.$$

□

Corollary 2.10 *Let $R > 0$, and let*

$$I_R = i \int_0^{\pi} e^{iR(\cos t + i \sin t)}\, dt\,.$$

Then

$$I_R \to 0 \quad \textit{as } R \to \infty.$$

Proof. Since $\sin(\pi - t) = \sin t$, it follows that

$$|I_R| \leq \int_0^{\pi} \left| e^{iR(\cos t + i \sin t)} \right| dt \;=\; \int_0^{\pi} e^{-R \sin t}\, dt = 2 \int_0^{\frac{\pi}{2}} e^{-R \sin t}\, dt\,, \tag{4}$$

using the fact that $|e^{iR\cos t}| = 1$. From the Lemma, it follows that, $-R \sin t \leq -\frac{2tR}{\pi}$ for $0 \leq t \leq \frac{\pi}{2}$, and so

$$|I_R| \leq 2 \int_0^{\frac{\pi}{2}} e^{-\frac{2Rt}{\pi}}\, dt \;=\; \left[-\frac{\pi}{R}\, e^{-\frac{2Rt}{\pi}} \right]_0^{\frac{\pi}{2}} = \frac{\pi}{R}\, \left[1 - e^{-R}\right]$$

and $|I_R| \to 0$ as $R \to \infty$. □

Lemma 2.11 *Let $\epsilon > 0$. Then*

$$\lim_{\epsilon \to 0} \left(i \int_0^{\pi} e^{i\epsilon(\cos t + i \sin t)}\, dt \right) = i\pi\,.$$

Proof. (We note that

$$\lim_{\epsilon \to 0} \left(e^{i\epsilon(\cos t + i \sin t)} \right) = 1 \quad \text{and so} \quad i \int_0^{\pi} \lim_{\epsilon \to 0} \left(e^{i\epsilon(\cos t + i \sin t)} \right) dt = i \int_0^{\pi} dt = i\pi\,.$$

Thus the lemma simply justifies taking the limit through the integral sign.)

Let

$$I_\epsilon = i\int_0^\pi e^{i\epsilon(\cos t + i\sin t)}\,dt\,. \tag{1}$$

Then

$$|I_\epsilon - i\pi| = \left| i\int_0^\pi \left(e^{i\epsilon(\cos t+i\sin t)} - 1\right)\,dt\right| \le \int_0^\pi \left|\,e^{i\epsilon(\cos t + i\sin t)} - 1\right|\,dt\,. \tag{2}$$

Let $z = \epsilon(\cos t + i\sin t)$ and let $0 < \epsilon < 1$. Then

$$\begin{aligned}
\left|e^{i\epsilon(\cos t+i\sin t)} - 1\right| &= |e^{iz} - 1| &&= \left|iz + \tfrac{(iz)^2}{2!} + \tfrac{(iz)^3}{3!} + \cdots\right| \\
&\le |iz| + \left|\tfrac{(iz)^2}{2!}\right| + \left|\tfrac{(iz)^3}{3!}\right| + \cdots &&= \epsilon + \tfrac{\epsilon^2}{2!} + \tfrac{\epsilon^3}{3!} + \cdots \\
&\le \epsilon + \epsilon^2 + \epsilon^3 + \cdots &&= \tfrac{\epsilon}{1-\epsilon}\,. \qquad (3)
\end{aligned}$$

From (2) and (3) we see that

$$|I_\epsilon - i\pi\,| \le \frac{\epsilon\pi}{1-\epsilon} \to 0 \quad \text{as } \epsilon \to 0.$$

i.e. $I_\epsilon - i\pi \to 0$ as $\epsilon \to 0$ giving the required result. □

Theorem 2.12

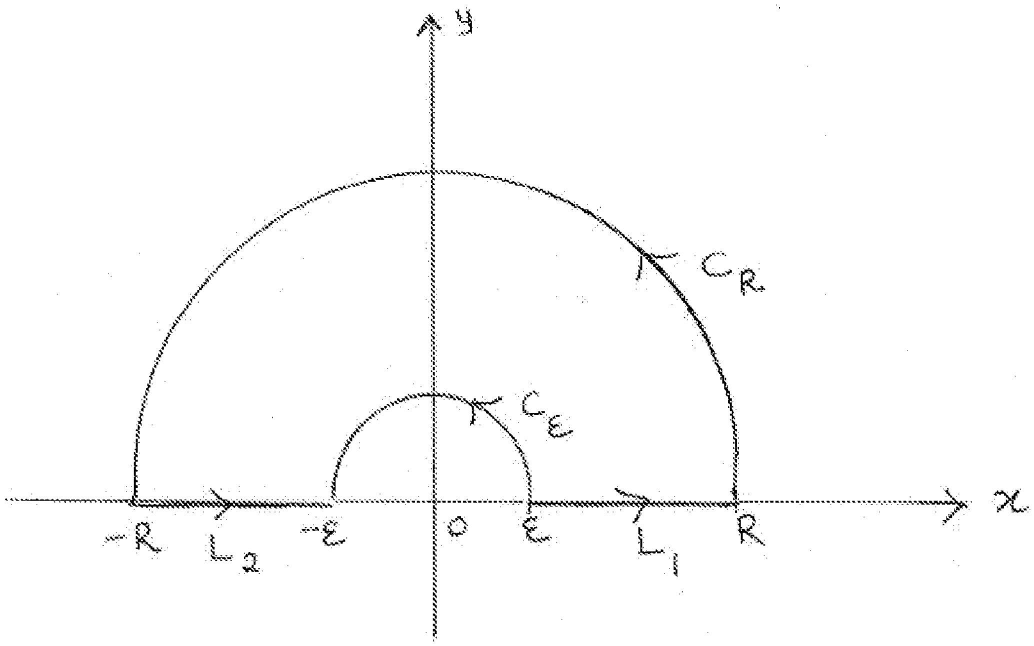
$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Proof. We evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$ using the Residue Theorem.

Let γ be the contour shown below, and let

$$f(z) = \frac{e^{iz}}{z}\,.$$

Then the function f is analytic in $\mathbb{C}$ except for a simple pole at the origin.



Let $0 < \epsilon < R$ and let C_R, C_ϵ, L_1 and L_2 be given by

$$\begin{array}{llll} C_R: \; z = Re^{it} & (0 \le t \le \pi)\,, & C_\epsilon: \; z = \epsilon e^{it} & (0 \le t \le \pi)\,, \\ L_1: \; z = x & (\epsilon \le x \le R)\,, & L_2: z = t & (-R \le t \le -\epsilon) \end{array}$$

and let

$$\gamma = C_R \;+\; L_2 \;-C_\epsilon \;+\; L_1\,.$$

Hence

$$\int_\gamma f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz - \int_{C_\epsilon} f(z)\,dz + \int_{L_1} f(z)\,dz = I_R + I_2 - I_\epsilon + I_1\,. \quad (1)$$

By Cauchy's Residue Theorem

$$\int_\gamma f(z)\,dz = 0.$$

Thus

$$I_R + I_2 - I_\epsilon + I_1 = 0$$

giving

$$I_2 + I_1 = I_\epsilon - I_R\,. \qquad (2)$$

Using the substitution $t = -x$, so that $\frac{dt}{dx} = -1$, gives

$$I_2 = \int_{-R}^{-\epsilon} \frac{e^{it}}{t}\, dt = -\int_{R}^{\epsilon} \frac{e^{-ix}}{-x}\, dx = -\int_{\epsilon}^{R} \frac{e^{-ix}}{x}\, dx\,.$$

Hence

$$I_1 + I_2 = \int_{\epsilon}^{R} \frac{e^{ix}}{x}\, dx - \int_{\epsilon}^{R} \frac{e^{-ix}}{x}\, dx = \int_{\epsilon}^{R} \frac{e^{ix} - e^{-ix}}{x}\, dx = 2i \int_{\epsilon}^{R} \frac{\sin x}{x}\, dx\,. \tag{3}$$

Now

$$I_\epsilon = \int_0^{\pi} \frac{e^{i\epsilon(\cos t + i \sin t)}}{\epsilon e^{it}}\, ie^{it}\, dt = i \int_0^{\pi} e^{i\epsilon(\cos t + i\sin t)}\, dt$$

and so, by Lemma 1.10

$$I_\epsilon \to i\pi \text{ as } \epsilon \to 0+\,. \tag{4}$$

Moreover, by Corollary 1.9,

$$I_R = i \int_0^{\pi} e^{iR(\cos t + i \sin t)}\, dt \to 0 \text{ as } R \to \infty. \tag{5}$$

From (2), (3), (4), and (5), we see that

$$2i \int_{\epsilon}^{R} \frac{\sin x}{x}\, dx \to i\pi \text{ as } \epsilon \to 0 \text{ and } R \to \infty.$$

giving

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}\,.$$

$\square$

Corollary 2.13

$$\int_0^{\infty} \frac{\sin xy}{x}\, dx = \begin{cases} -\frac{\pi}{2} & \text{if } y < 0 \\ 0 & \text{if } y = 0 \\ \frac{\pi}{2} & \text{if } y > 0 \end{cases}$$

Proof. For $y = 0$, the formula is obvious.

When $y > 0$, then using the change of variable $t = xy$ and theorem 2.12, we have

$$\int_0^{X} \frac{\sin xy}{x}\, dx = \int_0^{Xy} \frac{\sin t}{t}\, dt \to \int_0^{\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$

as $X \to \infty$.

When $y < 0$,

$$\int_0^{X} \frac{\sin xy}{x}\, dx = -\int_0^{X} \frac{\sin x(-y)}{x}\, dx \to -\frac{\pi}{2}$$

as $X \to \infty$.

$\square$

Next we evaluate another important non-elementary integral.

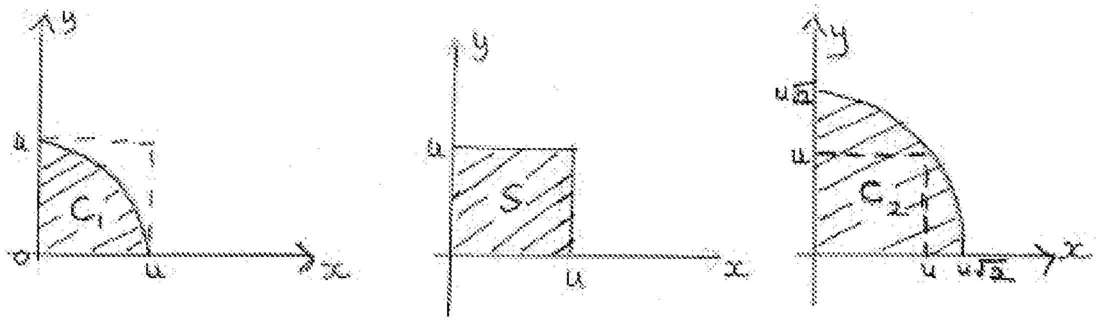
Theorem 2.14

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Proof. For $u > 0$, put $E(u) = \int_0^u e^{-x^2}dx$. Then

$$[E(u)]^2 = \int_0^u e^{-x^2}dx.\int_0^u e^{-y^2}dy = \iint\limits_S e^{-x^2-y^2}dx\,dy,$$

where $S = [0,u]\times[0,u]$ in $\mathbb{R}^2$.



Let C_1, C_2 be the sectors

$$\begin{aligned} C_1 &= \{(x,y)\in\mathbb{R}^2\,|\,x\ge 0, y\ge 0, x^2+y^2\le u^2\},\\ C_2 &= \{(x,y)\in\mathbb{R}^2\,|\,x\ge 0, y\ge 0, x^2+y^2\le 2u^2\},\end{aligned}$$

so that $C_1 \subseteq S \subseteq C_2$. Since $e^{-x^2-y^2} > 0$,

$$\iint\limits_{C_1} e^{-x^2-y^2}dx\,dy \le \iint\limits_{S} e^{-x^2-y^2}dx\,dy \le \iint\limits_{C_2} e^{-x^2-y^2}dx\,dy. \tag{1}$$

Using polar coordinates (r,θ), where $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\iint\limits_{C_1} e^{-x^2-y^2}dx\,dy = \int_0^{\frac{\pi}{2}} d\theta \int_0^u e^{-r^2} r\,dr = \frac{\pi}{2}\left[-\frac{e^{-r^2}}{2}\right]_0^u = \frac{\pi(1-e^{-u^2})}{4}$$

and similarly

$$\iint\limits_{C_2} e^{-x^2-y^2}dx\,dy = \frac{\pi(1-e^{-2u^2})}{4}.$$

Hence, by (1),

$$\frac{\pi(1-e^{-u^2})}{4} \le [E(u)]^2 \le \frac{\pi(1-e^{-2u^2})}{4}$$

and so $[E(u)]^2 \to \frac{\pi}{4}$ as $u\to\infty$. As $E(u) > 0$, this means that $E(u) \to \frac{\sqrt{\pi}}{2}$ as $u \to \infty$. $\square$

A generalisation of this theorem will also be useful.

Corollary 2.15 *For $a \geq 0$,*

$$\int_0^\infty e^{-(x-\frac{a}{x})^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Proof. The case $a = 0$ corresponds to the theorem. Therefore assume that $a > 0$. Then

$$I = \int_0^\infty e^{-\left(x-\frac{a}{x}\right)^2}\,dx = \left(\int_0^{\sqrt{a}} + \int_{\sqrt{a}}^\infty\right) e^{-\left(x-\frac{a}{x}\right)^2}\,dx = I_1 + I_2.$$

In I_1, put $y = \frac{a}{x}$ so that $x = \frac{a}{y}$ and $\frac{dx}{dy} = -\frac{a}{y^2}$ and apply the rule for changing the variable amended in the obvious way,

$$I_1 = \int_0^{\sqrt{a}} e^{-\left(x-\frac{a}{x}\right)^2}\,dx = \int_\infty^{\sqrt{a}} e^{-\left(\frac{a}{y}-y\right)^2}\left(-\frac{a}{y^2}\right)\,dy = \int_{\sqrt{a}}^\infty \frac{a}{y^2}\,e^{-\left(y-\frac{a}{y}\right)^2}\,dy.$$

Hence

$$I = I_1 + I_2 = \int_{\sqrt{a}}^\infty \left(1+\frac{a}{y^2}\right) e^{-\left(y-\frac{a}{y}\right)^2}\,dy.$$

Now put $t = y - \frac{a}{y}$, so that $\frac{dt}{dy} = 1 + \frac{a}{y^2}$. Then

$$I = \int_{\sqrt{a}}^\infty e^{-t^2}\frac{dt}{dy}\,dy = \int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}.$$

$\square$

Integrals of the type $\int_{-\infty}^a f(x)\,dx$ and $\int_{-\infty}^\infty f(x)\,dx$.

The integral

$$\int_{-\infty}^a f(x)\,dx$$

is defined in the obvious way as $\lim\limits_{X\to\infty} \int_{-X}^a f(x)\,dx$, when this limit exists.

We also define

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{\substack{X\to\infty\\ Y\to\infty}} \int_{-X}^Y f(x)\,dx \tag{2}$$

if this limit exists, where X and Y tend to ∞ **independently**.

Evidently, if a is any real number,

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx,$$

the existence of either side implying the existence of the other.

Note that

$$\lim_{X\to\infty}\int_{-X}^X f(x)\,dx$$

may exist without the limit in (2), above, existing, as, for instance, when $f(x) = x$. (In general, of course, if the function f is any odd integrable function, then $\lim\limits_{X\to\infty} \int_{-X}^X f(x)\,dx$

exists and is equal to 0.) If $\int_{-\infty}^{\infty} f(x)\,dx$ does not exist, but $\lim\limits_{X\to\infty} \int_{-X}^{X} f(x)\,dx$ does exist, then this is called the *Cauchy principal value* of the integral, and it is denoted by

$$(P)\int_{-\infty}^{\infty} f(x)\,dx.$$

Example 2.16 Show that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+x+1} = \frac{2\pi}{\sqrt{3}}\,.$$

Solution

$$\begin{aligned}
\int_{-X}^{Y} \frac{dx}{x^2+x+1} &= \int_{-X}^{Y} \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} \\
&= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_{-X}^{Y} \\
&= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2Y+1}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{-2X+1}{\sqrt{3}}\right)\right] \\
&\to \frac{2}{\sqrt{3}}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]
\end{aligned}$$

as X and Y tend to ∞. Hence

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+x+1} = \frac{2\pi}{\sqrt{3}}.$$

Improper integrals of the second kind

In the basic integral of this type, the integrand is unbounded in the neighbourhood of one of the end points of the interval of integration. We use one-sided limits* to define the integral.

Definition 2.17 If the integral $\int_{a+\delta}^{b} f(x)\,dx$ exists as an ordinary Riemann integral whenever $0 < \delta < b-a$, and if $\lim\limits_{\delta\to 0+} \int_{a+\delta}^{b} f(x)\,dx$ exists, then this limit is called the *improper integral* of f over $[a,b]$, and it is written

$$\int_{a+}^{b} f(x)\,dx \qquad \text{or simply} \qquad \int_{a}^{b} f(x)\,dx.$$

There is an analogous definition of $\int_{a}^{b-} f(x)\,dx$; this integral too may simply be written as $\int_a^b f(x)\,dx$.

*The statement that $\delta \to a+$ means δ tends to a through values greater than a.
The statement that $\delta \to b-$ means δ tends to b through values less than b.

If $\int_a^b f(x)\,dx$ exists as an ordinary Riemann integral, then, since indefinite integrals are continuous, $\lim_{\delta\to 0+} \int_{a+\delta}^b f(x)\,dx = \int_a^b f(x)\,dx$. Therefore in this case the definition of $\int_{a+}^b f(x)\,dx$ does not yield anything new. The definition is significant only when f is unbounded near a.

Examples 2.18 Decide whether the following integrals exist

1. $\displaystyle\int_0^1 x^{-1/2}\,dx\,.$ 2. $\displaystyle\int_0^1 \frac{dx}{1-x}\,.$

Solutions.

1. For $0<\delta<1$,
$$\int_\delta^1 x^{-\frac{1}{2}}\,dx = [2x^{\frac{1}{2}}]_\delta^1 = 2(1-\delta^{\frac{1}{2}}) \to 2$$
as $\delta\to 0+$. Hence
$$\int_0^1 x^{-\frac{1}{2}}\,dx = 2.$$

2. For $0<\delta<1$,
$$\int_0^{1-\delta} \frac{dx}{1-x} = [-\ln(1-x)]_0^{1-\delta} = -\ln\delta \to \infty$$
as $\delta\to 0+$. Hence
$$\int_0^1 \frac{dx}{1-x} \text{ does not exist.}$$

Improper integrals of the second kind are handled in much the same way as those of the first kind. In particular, the analogues of Theorems 2.3, 2.4 and 2.6 hold for them.

Example 2.19 1. Show that $\displaystyle\int_0^1 \frac{\cos x}{\sqrt{x}\,(x+1)}\,dx$ converges.

2. Show that $\displaystyle\int_0^1 \frac{e^x}{x\sqrt{x}\,(x+1)}\,dx$ diverges.

Solutions.

1. For $0<x\le 1$,
$$0 \;\le\; \frac{\cos x}{\sqrt{x}\,(x+1)} \;\le\; \frac{1}{\sqrt{x}\,(x+1)} \;\le\; \frac{1}{\sqrt{x}}$$
and
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx \;=\; \int_0^1 x^{-\frac{1}{2}}\,dx \quad \text{converges.}$$
By the Comparison Test
$$\int_0^1 \frac{\cos x}{\sqrt{x}\,(x+1)}\,dx$$
also converges.

2. For $0 < x \le 1$,

$$\frac{e^x}{x\sqrt{x}\,(x+1)} \;\ge\; \frac{1}{x\sqrt{x}\,(x+1)} \;\ge\; \frac{1}{2x\sqrt{x}} \;\ge\; 0$$

and

$$\int_0^1 \frac{1}{2x\sqrt{x}}\,dx \;=\; \tfrac{1}{2}\int_0^1 x^{-\frac{3}{2}}\,dx \;\text{ diverges.}$$

By the Comparison Test

$$\int_0^1 \frac{e^x}{x\sqrt{x}\,(x+1)}\,dx$$

also diverges.

If f is unbounded near a and b, then $\int_a^b f(x)\,dx$ is defined as

$$\lim_{\substack{\delta\to 0+\\ \epsilon\to 0+}} \int_{a+\delta}^{b-\epsilon} f(x)\,dx$$

when this limit exists, where δ and ϵ tend to 0 independently.

Generally, if $a = c_0 < c_1 < \cdots < c_{n-1} < c_n = b$, and f is unbounded near $c_1, \ldots, c_{n-1}$, and possibly near a and b, we define the improper integral $\int_a^b f(x)\,dx$ as the sum $\sum_{r=1}^n \int_{c_{r-1}}^{c_r} f(x)\,dx$, when each integral in the sum exists as one of the improper integrals previously defined.

Finally, the improper integrals of the first and second kinds may occur together. For instance, we may define $\int_a^\infty f(x)\,dx$ as $\lim\limits_{X\to\infty} \int_a^X f(x)\,dx$ when $\int_a^X f(x)\,dx$ exists, for each $X > a$, either an ordinary Riemann integral or as an improper integral of the second kind.

Example 2.20 Show that $\displaystyle\int_0^\infty \frac{\ln x}{(x+2)^2}dx$ exists and find its value.

Solution. When $0 < \delta < X$,

$$\begin{aligned}
\int_\delta^X \frac{\ln x}{(x+2)^2}\,dx &= \left[-\frac{\ln x}{x+2}\right]_\delta^X + \int_\delta^X \frac{dx}{x(x+2)} \\
&= \left[-\frac{\ln x}{x+2}\right]_\delta^X + \int_\delta^X \frac{1}{2}\left(\frac{1}{x} - \frac{1}{(x+2)}\right)\,dx \\
&= \left[-\frac{\ln x}{x+2} + \frac{1}{2}\,(\ln x - \ln(x+2))\right]_\delta^X \\
&= \left(-\frac{\ln X}{X+2} + \frac{\ln X}{2} - \frac{\ln(X+2)}{2}\right) - \left(-\frac{\ln\delta}{\delta+2} + \frac{\ln\delta}{2} - \frac{\ln(\delta+2)}{2}\right) \\
&= -\frac{\ln X}{X+2} - \frac{\ln(1+\frac{2}{X})}{2} - \frac{\delta\ln\delta}{2(\delta+2)} + \frac{\ln(\delta+2)}{2}\,.
\end{aligned}$$

Thus, letting $\delta \to 0+$, gives

$$\int_0^X \frac{\ln x}{(x+2)^2}\,dx = -\frac{\ln X}{X+2} - \frac{\ln(1+\frac{2}{X})}{2} + \frac{\ln 2}{2}$$

using the fact that $\delta \ln \delta \to 0$ as $\delta \to 0+$. Now letting $X \to \infty$ gives

$$\int_0^\infty \frac{\ln x}{(x+2)^2} dx = \frac{\ln 2}{2} \ ,$$

using the standard limit

$$\frac{\ln X}{X} \to 0 \ \text{ as } \ X \to \infty.$$

Functions defined by integrals

A function f on $[a,b] \times [c,d]$† defines a function F on $[c,d]$ by means of the relation

$$F(y) = \int_a^b f(x,y)\, dx \qquad (c \le y \le d) \tag{3}$$

when the integral in (3) exists for every $y \in [c,d]$. The way in which f determines the properties of F was investigated in the second year. The three key results are the following.

Suppose that f is continuous on $[a,b] \times [c,d]$. Then

1. F is continuous on $[c,d]$;
2. $\int_c^d F(y)\, dy = \int_a^b dx \int_c^d f(x,y)\, dy$ (i.e., $\int_c^d dy \int_a^b f(x,y)\, dx = \int_a^b dx \int_c^d f(x,y)\, dy$).
3. If also $\frac{\partial}{\partial y} f(x,y)$ is continuous on $[a,b] \times [c,d]$, then F is differentiable on $[c,d]$ and
$$F'(y) = \int_a^b \frac{\partial}{\partial y} f(x,y)\, dx,$$
(i.e., $\frac{d}{dy} \int_a^b f(x,y)\, dx = \int_a^b \frac{\partial}{\partial y} f(x,y)\, dx$, so that "differentiation under the integral sign" is valid).

Note that the second result may be viewed as "integration under the integral sign" or as changing the order in which two integrations are carried out.

It is important to extend these results to functions defined by improper integrals. For the sake of simplicity, we confine ourselves to improper integrals of the first kind. Thus we begin with a function $f : [a,\infty) \times [c,d] \to \mathbb{R}$ and define the function $F : [c,d] \to \mathbb{R}$ by

$$F(y) = \int_a^\infty f(x,y)\, dx \qquad (c \le y \le d), \tag{4}$$

assuming that the integral in (4) exists for every $y \in [c,d]$.

Before we can formulate results for such integrals we need to introduce a new concept (called majorization‡) which is defined below.

†Let I and J be intervals. The Cartesian product $I \times J$ is the set of all points (x,y) such that $x \in I$, $y \in J$ i.e. $I \times J = \{(x,y) : x \in I,\ y \in J\}$.
Thus, for example, $[a,b] \times [c,d] = \{(x,y) : a \le x \le b \ \text{ and } \ c \le y \le d\ \}$.

‡This concept is also often referred to as *dominated convergence*

Definition 2.21 Let f be a function on $[a, \infty) \times [c, d]$ such that $\int_a^X f(x, y)\, dx$ exists for every $X > a$ and every $y \in [c, d]$. If there exists a function M on $[a, \infty)$ such that, for each $x \in [a, \infty)$,

$$|f(x, y)| \le M(x) \qquad \text{for } c \le y \le d$$

and

$$\int_a^\infty M(x)\, dx \qquad \text{exists,}$$

then $\int_a^\infty f(x, y)\, dx$ is said to be *majorized* by $\int_a^\infty M(x)\, dx$ on $[c, d]$.

Note that, if $\int_a^\infty f(x, y)\, dx$ is majorized on $[c, d]$, then the integral converges absolutely for each $y \in [c, d]$.

Example 2.22 Show that for every $c > 0$, $\displaystyle\int_0^\infty \frac{y \sin xy}{x^2 + y + 1}\, dx$ is majorized on the interval $[0, c]$.

Solution. For $x \ge 0$, $\quad 0 \le y \le c$,

$$\left| \frac{y \sin xy}{x^2 + y + 1} \right| \le \frac{c}{x^2 + 1} = M(x)$$

and

$$\int_0^\infty M(x)\, dx = \int_0^\infty \frac{c}{x^2 + 1} dx$$

exists. Hence

$$\int_0^\infty \frac{y \sin xy}{x^2 + y + 1}\, dx$$

is majorized on the interval $[0, c]$.

Theorem 2.23 *Suppose that the function $f : [a, \infty) \times [c, d] \longrightarrow \mathbb{R}$ is continuous. If $\int_a^\infty f(x, y)\, dx$ is majorized on $[c, d]$, then the function $F : [c, d] \longrightarrow \mathbb{R}$ defined by*

$$F(y) = \int_a^\infty f(x, y)\, dx \qquad (c \le y \le d) \tag{5}$$

is continuous and

$$\int_c^d F(y)\, dy = \int_a^\infty dx \int_c^d f(x, y)\, dy, \tag{6}$$

i.e.,

$$\int_c^d dy \int_a^\infty f(x, y)\, dx = \int_a^\infty dx \int_c^d f(x, y)\, dy.$$

Examples 2.24 1. Prove that, if $0 < a < b$, then

$$\int_a^b dy \int_0^\infty e^{-xy}\,dx = \int_0^\infty dx \int_a^b e^{-xy}\,dy. \tag{7}$$

Hence evaluate

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx.$$

(Note that the integral is not improper at 0, since $\frac{e^{-ax}-e^{-bx}}{x} \to b - a$ as $x \to 0$.)

Solutions.

1. Let

$$f(x, y) \;=\; e^{-xy} \;.$$

Then f is continuous on $[0, \infty) \times [a, b]$. For $x \geq 0$, $a \leq y \leq b$,

$$0 \leq e^{-xy} \leq e^{-ax} = M(x)$$

and

$$\int_0^\infty M(x)\,dx = \int_0^\infty e^{-ax}\,dx$$

exists, since $a > 0$. Thus $\int_0^\infty e^{-xy}\,dx$ is majorized on $[a, b]$ and so (7) holds,

$$\text{i.e.} \quad \int_a^b dy \int_0^\infty e^{-xy}\,dx = \int_0^\infty dx \int_a^b e^{-xy}\,dy. \tag{$*$}$$

Now, for $y > 0$, $X > 0$.

$$\int_0^X e^{-xy}\,dx = \left[-\frac{e^{-xy}}{y}\right]_{x=0}^X = \frac{1}{y}\;\left(1 - e^{-Xy}\right) \to \frac{1}{y} \quad \text{as} \;\; X \to \infty\,.$$

Thus, for all $y > 0$,

$$\int_0^\infty e^{-xy}\,dx \;=\; \frac{1}{y}$$

and so

$$\int_a^b dy \int_0^\infty e^{-xy}\,dx = \int_a^b \frac{dy}{y} = \ln\frac{b}{a}\;.$$

Also, for $x > 0$,

$$\int_a^b e^{-xy}\,dy = \left[-\frac{e^{-xy}}{x}\right]_{y=a}^b = \frac{e^{-ax} - e^{-bx}}{x}.$$

Hence, by $(*)$,

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \ln\frac{b}{a}\;.$$

Theorem 2.25 *Suppose that the function* $f : [a, \infty) \times [c, d] \longrightarrow \mathbb{R}$ *is continuous. If* $\int_a^\infty f(x, y)\, dx$ *exists for* $c \le y \le d$, $\frac{\partial}{\partial y} f(x, y)$ *is continuous on* $[a, \infty) \times [c, d]$, *and* $\int_a^\infty \frac{\partial}{\partial y} f(x, y)\, dx$ *is majorized on* $[c, d]$, *then the function* F *on* $[c, d]$ *defined by Theorem 2.23 is differentiable on* $[c, d]$ *and, for* $c \le y \le d$,

$$F'(y) = \int_a^\infty \frac{\partial}{\partial y} f(x, y)\, dx,$$

i.e.,

$$\frac{d}{dy} \int_a^\infty f(x, y)\, dx = \int_a^\infty \frac{\partial}{\partial y} f(x, y)\, dx.$$

Examples 2.26 1. Prove that, if $0 < c < d$, then

$$\frac{d}{dy} \int_0^\infty \frac{dx}{x^2 + y^2} = -2y \int_0^\infty \frac{dx}{(x^2 + y^2)^2} \quad \text{for } c \le y \le d. \tag{8}$$

Hence evaluate

$$\int_0^\infty \frac{dx}{(x^2 + a^2)^2} \quad \text{for } a > 0.$$

Note that the same method yields $\displaystyle\int_0^\infty \frac{dx}{(x^2 + a^2)^n}$ for any $n > 1$.

Solutions.

1. Let

$$f(x, y) = \frac{1}{x^2 + y^2}\,.$$

Then

$$\frac{\partial f}{\partial y} = -\frac{2y}{(x^2 + y^2)^2}$$

and f and $\frac{\partial f}{\partial y}$ are both continuous on $[0, \infty) \times [c, d]$.

For $X > 0, \quad y > 0$,

$$\int_0^X \frac{dx}{x^2 + y^2} = \left[\frac{1}{y} \tan^{-1}\left(\frac{x}{y}\right)\right]_{x=0}^X = \frac{1}{y} \tan^{-1}\left(\frac{X}{y}\right) \to \frac{\pi}{2y} \quad \text{as } X \to \infty$$

and so, for every $y > 0$,

$$\int_0^\infty \frac{dx}{x^2 + y^2} = \frac{\pi}{2y}\,. \tag{9}$$

Also, for $x \ge 0\,, \quad c \le y \le d$

$$\left|\frac{\partial f}{\partial y}\right| = \left|\frac{\partial}{\partial y}\left(\frac{1}{x^2 + y^2}\right)\right| = \left|\frac{-2y}{(x^2 + y^2)^2}\right| = \frac{2y}{(x^2 + y^2)^2} \le \frac{2d}{(x^2 + c^2)^2} \le \frac{2d}{c^2(x^2 + c^2)}$$

and

$$\int_0^\infty \frac{2d}{c^2(x^2 + c^2)}\, dx$$

exists, since $c > 0$.

Hence $\displaystyle\int_0^\infty \frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right)\,dx$ is majorized on $[c, d]$, and so (8) holds,

$$\text{i.e.}\quad \frac{d}{dy}\int_0^\infty \frac{dx}{x^2+y^2} = -2y\int_0^\infty \frac{dx}{(x^2+y^2)^2} \quad (c \le y \le d)$$

It follows from (8) and (9) that, for $c \le y \le d$,

$$-2y\int_0^\infty \frac{dx}{(x^2+y^2)^2} = \frac{d}{dy}\left(\frac{\pi}{2y}\right) = -\frac{\pi}{2y^2}\ ,$$

i.e., that

$$\int_0^\infty \frac{dx}{(x^2+y^2)^2} = \frac{\pi}{4y^3} \quad \text{for } c \le y \le d. \tag{10}$$

Now take any $a > 0$. Then there are c, d such that $0 < c < a < d$ and therefore, by (10),

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}\ .$$

3 The Gamma and Beta functions

The gamma function is the simplest of the special functions of analysis. It is a generalisation of the factorial function and many naturally occurring definite integrals can be expressed in terms of it. We begin by showing that the integral defining it actually exists.

Theorem 3.1 *For $x > 0$, the integral*

$$\int_0^\infty t^{x-1}e^{-t}\,dt$$

exists.

Proof. We show that $\int_0^1 t^{x-1}e^{-t}\,dt$ and $\int_1^\infty t^{x-1}e^{-t}\,dt$ exist.

(i) $\int_0^1 t^{x-1}e^{-t}\,dt$ exists.

When $x \geq 1$, the integrand is continuous on $[0, 1]$ and so $\int_0^1 t^{x-1}e^{-t}\,dt$ exists.

When $0 < x < 1$, we note that, for $0 < t \leq 1$, $e^{-t} \leq 1$ and so

$$0 < t^{x-1}e^{-t} \leq t^{x-1}.$$

But $\int_0^1 t^{x-1}\,dt$ exists, since $x-1 > -1$. Hence $\int_0^1 t^{x-1}e^{-t}\,dt$ exists by the Comparison Test.

(ii) $\int_1^\infty t^{x-1}e^{-t}\,dt$ exists.

For this, we take any $x > 0$. Then

$$t^{x-1}\,e^{-\frac{t}{2}} \to 0 \quad \text{as} \quad t \to \infty$$

and so there exists T such that $t^{x-1}\,e^{-\frac{t}{2}} \leq 1$, for $t \geq T$. Thus

$$0 < t^{x-1}e^{-t} = t^{x-1}\,e^{-\frac{t}{2}}\,e^{-\frac{t}{2}} \leq e^{-\frac{t}{2}} \quad \text{for } t \geq T.$$

Since $\int_T^\infty e^{-\frac{t}{2}}\,dt$ exists, so does $\int_T^\infty t^{x-1}e^{-t}\,dt$. Hence $\int_1^\infty t^{x-1}e^{-t}\,dt$ also exists. $\square$

Definition 3.2 The *gamma function* Γ on $(0, \infty)$ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

Some properties of the gamma function are collected in the next theorem.

Theorem 3.3 *The Gamma function has the following properties:*

1. $\Gamma(x+1) = x\Gamma(x)$ *for* $x > 0$*;*
2. $\Gamma(n) = (n-1)!$ *for* $n = 1, 2, \ldots$*;*
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$*;*
4. $\Gamma(x) \to \infty$ *as* $x \to 0+$.

Proof.

1. Using integration by parts, gives

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = \left[-t^x e^{-t}\right]_{t=0}^\infty + \int_0^\infty x t^{x-1} e^{-t}\,dt = x\Gamma(x)\,.$$

2. By definition,

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1 = 0!\,.$$

The general formula follows by induction using the previous part.

3. Putting $t = u^2$ in $\int_0^\infty t^{x-1}e^{-t}\,dt$, we have

$$\Gamma(x) = 2\int_0^\infty u^{2x-1} e^{-u^2}\,du. \tag{11}$$

Hence

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}.$$

4. By definition,

$$\begin{aligned}
\Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}\,dt \\
&> \int_0^1 t^{x-1}e^{-t}\,dt \\
&\geq \int_0^1 t^{x-1}e^{-1}\,dt \\
&= \frac{1}{e}\left[\frac{t^x}{x}\right]_{t=0}^1 \\
&= \frac{1}{ex},
\end{aligned}$$

and the right-hand side tends to ∞ as $x \to 0+$. □

We can make Theorem 3.3 (4) more precise (see Example 3.4).

A number of additional properties of the gamma function are needed before we can draw its graph.

It can be shown by means of Theorem 2.25 that $\Gamma'(x)$ is obtained from differentiation under the integral sign:

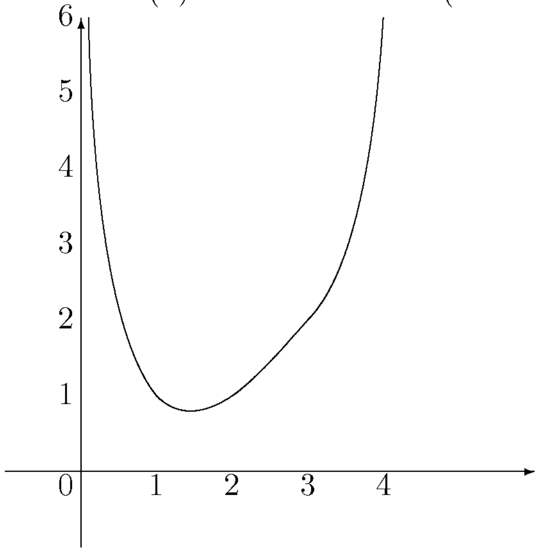
$$\Gamma'(x) = \int_0^\infty (\ln t) t^{x-1} e^{-t}\, dt$$

and similarly

$$\Gamma''(x) = \int_0^\infty (\ln t)^2 t^{x-1} e^{-t}\, dt.$$

Therefore $\Gamma''(x) > 0$, and so $\Gamma'(x)$ strictly increases. Since $\Gamma(1) = \Gamma(2) = 1$, there is a point in between 1 and 2 for which $\Gamma'(x_0) = 0$, and since $\Gamma'(x)$ strictly increases, $\Gamma'(x) < 0$ for $x < x_0$, while $\Gamma'(x) > 0$ for $x > x_0$. Thus x_0 is unique and Γ decreases in the interval $(0, x_0)$ and increases in (x_0, ∞).

By Theorem 3.3(4), $\Gamma(x) \to \infty$ as $x \to 0+$. Also, by Theorem 3.3(2), $\Gamma(n) = (n-1)! \to \infty$ as $n \to \infty$ through integral values. Since Γ increases on (x_0, ∞), it follows that $\Gamma(x) \to \infty$ as $x \to \infty$. (In fact, $\Gamma(x) \to \infty$ very rapidly.)



Note that $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$, about $0.866\ldots$, although this is not the minimum value of Γ.

Example 3.4 Show that for $0 < x < 1$, $\quad \dfrac{1}{x(x+1)} < \Gamma(x) < \dfrac{1}{x}$.

Solution. For $1 < u < 2$, $\Gamma(u) < 1$. Hence, when $0 < x < 1$,

$$\Gamma(x+1) < 1,$$

i.e., $x\Gamma(x) < 1$, so that $\Gamma(x) < \frac{1}{x}$.

For $u > 2$, $\Gamma(u) > 1$. Hence, when $x > 0$,

$$\Gamma(x+2) > 1,$$

i.e., $(x+1)x\Gamma(x) > 1$, and so $\Gamma(x) > \frac{1}{x(x+1)}$.

Note that Example 3.4 implies that

$$0 < \frac{1}{x} - \Gamma(x) < \frac{1}{x} - \frac{1}{x(x+1)} = \frac{1}{x+1} < 1;$$

and also that $x\Gamma(x) \to 1$ as $x \to 0+$ (but the latter also follows from Theorem 3.3(1), the continuity of Γ, and the fact that $\Gamma(1) = 1$).

It is convenient at this stage to introduce another new function which is closely related to the gamma function. But first we must show that the defining integral really exists.

Theorem 3.5 *For $x > 0$, $y > 0$, the integral*

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

exists.

Proof. We just show that $\int_0^{\frac{1}{2}} t^{x-1}(1-t)^{y-1}\,dt$ exists.

When $x \geq 1$, the integrand is continuous on $[0, \frac{1}{2}]$; we therefore assume that $0 < x < 1$. For $0 < t \leq \frac{1}{2}$, $(1-t)^{y-1} < (1-t)^{-1} \leq (\frac{1}{2})^{-1} = 2$, and so

$$0 \leq t^{x-1}(1-t)^{y-1} \leq 2t^{x-1}.$$

But $\int_0^{\frac{1}{2}} 2t^{x-1}\,dt$ exists, since $x - 1 > -1$. Hence $\int_0^{\frac{1}{2}} t^{x-1}(1-t)^{y-1}\,dt$ exists. It is shown similarly that $\int_{\frac{1}{2}}^1 t^{x-1}(1-t)^{y-1}\,dt$ exists; the integral is only improper when $0 < y < 1$. $\square$

Definition 3.6 The *beta function* B on $(0,\infty) \times (0,\infty)$ is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad \text{for } x, y > 0.$$

Note that $B(x,y) = B(y,x)$, since the change of variable $u = 1 - t$ gives

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \int_0^1 (1-u)^{x-1}u^{y-1}\,du.$$

Next we obtain two useful alternative expressions for the beta function.

Theorem 3.7 *For $x, y > 0$,*

1. $B(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta\,d\theta;$
2. $B(x,y) = \displaystyle\int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,du = \int_0^\infty \frac{u^{y-1}}{(1+u)^{x+y}}\,du.$

Proof.

1. Putting $t = \cos^2\theta$ gives

$$\begin{aligned}\int_0^1 t^{x-1}(1-t)^{y-1}\,dt &= \int_{\frac{\pi}{2}}^0 \cos^{2x-2}\theta(1-\cos^2\theta)^{y-1}(-2\cos\theta\sin\theta)\,d\theta \\ &= 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta\sin^{2y-1}\theta\,d\theta.\end{aligned}$$

2. Let $u = \dfrac{t}{1-t}$, so that $t = \dfrac{u}{1+u}$. Then

$$\begin{aligned}\int_0^1 t^{x-1}(1-t)^{y-1}\,dt &= \int_0^\infty \left(\frac{u}{1+u}\right)^{x-1}\left(\frac{1}{1+u}\right)^{y-1}\frac{du}{(1+u)^2} \\ &= \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}du.\end{aligned}$$

□

Corollary 3.8 *For $x, y > 0$,*

1. $B(x+1, y) = \frac{x}{x+y}B(x,y)$;
2. $B(x, y+1) = \frac{y}{x+y}B(x,y)$;
3. $B(x+1, y+1) = \frac{xy}{(x+y+1)(x+y)}B(x,y)$.

Proof.

1. We have, for $x > 0,\ y > 0$,

$$\begin{aligned}B(x+1,y) &= \int_0^\infty \frac{u^x}{(1+u)^{x+y+1}}\,du \\ &= \left[-\frac{1}{x+y}\,\frac{u^x}{(1+u)^{x+y}}\right]_{x=0}^\infty + \frac{x}{x+y}\int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,du \\ &= \frac{x}{x+y}\,B(x,y).\end{aligned}$$

2. Using the above result and the relation $B(x,y) = B(y,x) \quad (x > 0,\ y > 0)$, gives for $x > 0,\ y > 0$,

$$B(x, y+1) = B(y+1, x) = \frac{y}{y+x}\,B(y,x) = \frac{y}{x+y}\,B(x,y)\,.$$

3. Using both the above results , gives for $x > 0,\ y > 0$,

$$\begin{aligned}B(x+1,y+1) &= \frac{x}{x+y+1}\,B(x,y+1) \\ &= \frac{x}{x+y+1}\,\frac{y}{y+x}\,B(x,y) \\ &= \frac{xy}{(x+y+1)(y+x)}\,B(x,y)\,.\end{aligned}$$

□

We are now ready to connect the gamma and beta functions.

Theorem 3.9 *For $x, y > 0$,*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof.[§]

(i) We first assume that $x \geq \frac{1}{2}$ and $y \geq \frac{1}{2}$. Using the substitution $t = u^2$ so that $\frac{dt}{du} = 2u$, we see that

$$\Gamma(x) \;=\; 2\int_0^\infty t^{x-1}e^{-t}\,dt \;=\; 2\int_0^\infty u^{2x-1}e^{-u^2}\,du\,.$$

Let

$$G_R(x) = 2\int_0^R u^{2x-1}e^{-u^2}\,du,$$

so that $G_R(x) \to \Gamma(x)$ as $R \to \infty$.

We now have

$$\begin{aligned} G_R(x)G_R(y) &= 2\int_0^R u^{2x-1}e^{-u^2}\,du\,.\,2\int_0^R v^{2y-1}e^{-v^2}\,dv & (12)\\ &= 4\int_0^R du\int_0^R u^{2x-1}v^{2y-1}e^{-u^2-v^2}\,dv & (13)\\ &= 4\iint_S u^{2x-1}v^{2y-1}e^{-u^2-v^2}\,du\,dv & (14) \end{aligned}$$

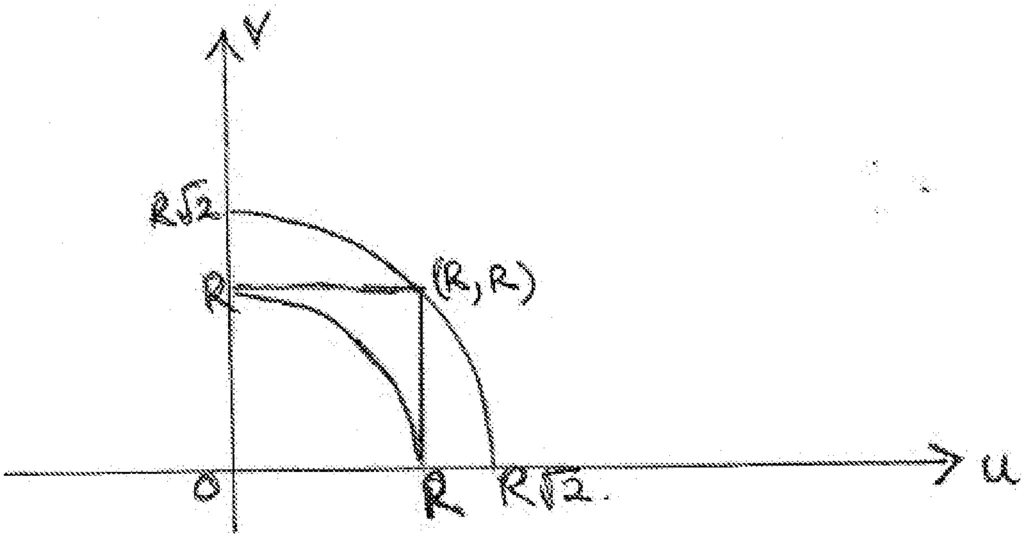
where S is the square $[0,R]\times[0,R]$ in $\mathbb{R}^2$. (Note that the integrand is continuous since $2x-1 \geq 0$ and $2y-1 \geq 0$.)

Let C_1 and C_2 be the sectors

$$\begin{aligned} C_1 &= \{(u,v)\in\mathbb{R}^2 \;:\; u,v\geq 0,\;\; u^2+v^2\leq R^2\}\\ C_2 &= \{(u,v)\in\mathbb{R}^2 \;:\; u,v\geq 0,\;\; u^2+v^2\leq 2R^2\} \end{aligned}$$

[§]The proof is similar to that of the identity $\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$, which is equivalent to the case $x = y = \frac{1}{2}$; for $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-u^2}\,du$, $\Gamma(1) = 1$ and, by Theorem 3.7(1), $B(\frac{1}{2},\frac{1}{2}) = \pi$.

shown in the diagram below



so that $C_1 \subseteq S \subseteq C_2$.
Then, since the integrand in (12) is non-negative,

$$4 \iint_{C_1} u^{2x-1} v^{2y-1} e^{-u^2-v^2} \, du \, dv \leq G_R(x) G_R(y) \leq 4 \iint_{C_2} u^{2x-1} v^{2y-1} e^{-u^2-v^2} \, du \, dv. \tag{15}$$

On the integrals over C_1, C_2, we make the change of variable

$$u = r\cos\theta, \qquad v = r\sin\theta.$$

(This is permitted since the integrand is continuous.) We have

$$\begin{aligned} 4 \iint_{C_1} u^{2x-1} v^{2y-1} e^{-u^2-v^2} \, du \, dv &= 4 \int_0^{\frac{\pi}{2}} d\theta \int_0^R (r\cos\theta)^{2x-1} (r\sin\theta)^{2y-1} e^{-r^2} r \, dr \\ &= 4 \int_0^{\frac{\pi}{2}} d\theta \int_0^R \cos^{2x-1}\theta \sin^{2y-1}\theta r^{2x+2y-1} e^{-r^2} \, dr \\ &= 2 \int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta \, . \, 2 \int_0^R r^{2x+2y-1} e^{-r^2} \, dr \\ &= B(x,y) \, G_R(x+y). \end{aligned}$$

Similarly, the right hand side of (15) is $B(x,y)\, G_{\sqrt{2}R}(x+y)$ and so

$$B(x,y)\, G_R(x+y) \leq G_R(x)\, G_R(y) \leq B(x,y)\, G_{\sqrt{2}R}(x+y).$$

Letting $R \to \infty$, we get

$$B(x,y)\, \Gamma(x+y) \leq \Gamma(x)\, \Gamma(y) \leq B(x,y)\, \Gamma(x+y),$$

i.e.,

$$B(x,y)\, \Gamma(x+y) = \Gamma(x)\, \Gamma(y)$$

as required.

(ii) Assuming only that $x > 0$, $y > 0$, we have $x+1 > 1$ and $y+1 > 1$. Hence, by the calculation we have just done, and by Theorem 3.3(1) and the Corollary to Theorem 3.7, we have

$$\begin{aligned}
B(x,y) &= \frac{(x+y)(x+y+1)}{xy}\, B(x+1,y+1) \\
&= \frac{(x+y)(x+y+1)}{xy}\, \frac{\Gamma(x+1)\,\Gamma(y+1)}{\Gamma(x+y+2)} \\
&= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,.
\end{aligned}$$

$\square$

Examples 3.10 1. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\sin 2x}\, dx$.

2. Let $0 < \alpha < 1$, $\alpha + \beta > 1$. Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{x^\alpha (1+x)^\beta}$.

Solutions.

1. We have,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sqrt{\sin 2x}\, dx &= \sqrt{2} \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}} x \cos^{\frac{1}{2}} x\, dx \\
&= \sqrt{2} \int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}-1} x \cos^{\frac{3}{2}-1} x\, dx \\
&= \frac{\sqrt{2}}{2}\, B(\tfrac{3}{4},\ \tfrac{3}{4}) \\
&= \frac{1}{\sqrt{2}}\, \frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{3}{4})}{\Gamma(\frac{3}{2})} \\
&= \frac{1}{\sqrt{2}}\, \frac{\Gamma(\frac{3}{4})^2}{\frac{\sqrt{\pi}}{2}} \\
&= \sqrt{\tfrac{2}{\pi}}\, [\,\Gamma(\tfrac{3}{4})\,]^2.
\end{aligned}$$

Note that, putting $2x = t$, we have

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin 2x}\, dx = \int_0^{\pi} \sqrt{\sin t}\, \frac{dt}{2} = \int_0^{\frac{\pi}{2}} \sqrt{\sin t}\, dt,$$

as $\sin(\pi - t) = \sin t$, so that $\int_{\frac{\pi}{2}}^{\pi} \sqrt{\sin t}\, dt = \int_0^{\frac{\pi}{2}} \sqrt{\sin t}\, dt$.

2. For $0 < \alpha < 1$, $\alpha + \beta > 1$,

$$\begin{aligned}
\int_0^\infty \frac{dx}{x^\alpha(1+x)^\beta} &= \int_0^\infty \frac{x^{-\alpha}}{(1+x)^\beta} dx \\
&= \int_0^\infty \frac{x^{(1-\alpha)-1}}{(1+x)^{(1-\alpha)+(\alpha+\beta-1)}} dx \\
&= B(1-\alpha, \alpha+\beta-1) \\
&= \frac{\Gamma(1-\alpha)\Gamma(\alpha+\beta-1)}{\Gamma(\beta)} \\
&= \frac{\Gamma(1-\alpha)\Gamma(\alpha+\beta)}{(\alpha+\beta-1)\Gamma(\beta)}.
\end{aligned}$$

We end this chapter with an important and surprising identity.

Theorem 3.11 *For* $0 < \alpha < 1$,

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}.$$

Proof. Let $0 < \alpha < 1$. Then

$$\Gamma(\alpha)\,\Gamma(1-\alpha) = \frac{\Gamma(\alpha)\,\Gamma(1-\alpha)}{\Gamma(1)} = B(\alpha,\, 1-\alpha) = \int_0^\infty \frac{u^{\alpha-1}}{1+u}\, du \, . \tag{1}$$

Let $u = e^x$ so that $x = \ln u$ and $\frac{du}{dx} = e^x$. Then, using (1), we see that

$$\Gamma(\alpha)\,\Gamma(1-\alpha) = \int_0^\infty \frac{u^{\alpha-1}}{1+u}\, du = \int_{-\infty}^\infty \frac{e^{(\alpha-1)x}}{1+e^x}\, e^x\, dx = \int_{-\infty}^\infty \frac{e^{\alpha x}}{1+e^x}\, dx = I\, . \tag{2}$$

We use contour integration to evaluate I.

Let

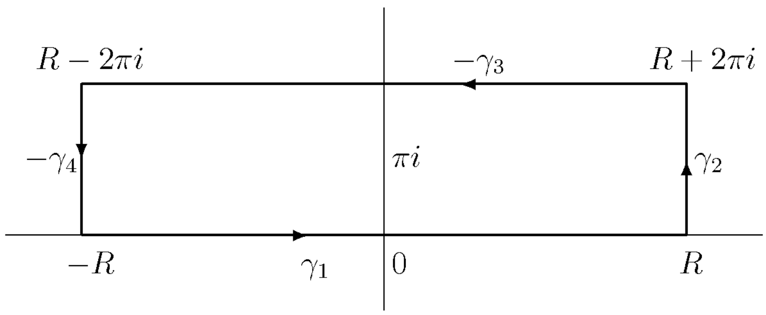
$$f(z) = \frac{e^{\alpha z}}{1+e^z}\, . \tag{3}$$

Then the function f is analytic in $\mathbb{C}$ except for simple poles at the points at which $1 + e^z = 0$ i.e. at the points $\pm\pi i$, $\pm 3\pi i$, $\pm 5\pi i$, $\pm 7\pi i$, $\cdots$.

Let γ be the contour

$$\gamma = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4\, , \tag{4}$$

where $R > 0$, $S > 0$ and γ_r $(r = 1, 2, 3, 4)$ are the paths shown in the diagram below.



Let

$$I_r = \int_{\gamma_r} f(z)\,dz \quad (r = 1, 2, 3, 4) \quad \text{so that} \quad \int_\gamma f(z)\,dz = I_1 + I_2 - I_3 - I_4\,. \tag{5}$$

By Cauchy's Residue Theorem

$$I_1 + I_2 - I_3 - I_4 = \int_\gamma f(z)\,dz = 2\pi i Res\{f; \pi i\} = 2\pi i \left[\frac{e^{\alpha z}}{\frac{d}{dz}(1+e^z)}\right]_{z=\pi i} = -2\pi i e^{\alpha\pi i}\,. \tag{6}$$

Now γ_2 is given by $z = R + it$ $(0 \le t \le 2\pi)$, and so on γ_2,

$$|f(z)| = \left|\frac{e^{\alpha z}}{1+e^z}\right| = \left|\frac{e^{\alpha(R+it)}}{1+e^{R+it}}\right| = \frac{|e^{\alpha(R+it)}|}{|1+e^{R+it}|} \le \frac{e^{\alpha R}}{e^R - 1} = \frac{e^{(\alpha-1)R}}{1-e^{-R}}$$

using the triangle inequalities. Hence

$$|I_2| = \left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{e^{(\alpha-1)R}}{1-e^{-R}} \times \text{ length of } \gamma_2 = 2\pi\,\frac{e^{(\alpha-1)R}}{1-e^{-R}} \;\to 0 \text{ as } R \to \infty,$$

since $\alpha - 1 < 0$ because $0 < \alpha < 1$. Thus

$$I_2 \to 0 \text{ as } R \to \infty. \tag{7}$$

Also γ_4 is given by $z = -S + it$ $(0 \le t \le 2\pi)$, and so on γ_4,

$$|f(z)| = \left|\frac{e^{\alpha z}}{1+e^z}\right| = \left|\frac{e^{\alpha(-S+it)}}{1+e^{-S+it}}\right| = \frac{|e^{\alpha(-S+it)}|}{|1+e^{-S+it}|} \le \frac{e^{-\alpha S}}{1-e^{-S}}$$

using the triangle inequalities. Hence

$$|I_4| = \left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{e^{-S\alpha}}{1-e^{-S}} \times \text{ length of } \gamma_4 = 2\pi\,\frac{e^{-S\alpha}}{1-e^{-S}} \;\to 0 \text{ as } S \to \infty,$$

because $\alpha > 0$. Thus

$$I_4 \to 0 \text{ as } S \to \infty. \tag{8}$$

In addition,

$$\begin{aligned} I_1 - I_3 &= \int_{\gamma_1} f(z)\,dz - \int_{\gamma_3} f(z)\,dz \quad = \int_{-S}^{R} \frac{e^{\alpha x}}{1+e^x}\,dx - \int_{-S}^{R} \frac{e^{\alpha(x+2\pi i)}}{1+e^{(x+2\pi i)}}\,dx \\ &= (1 - e^{2\alpha\pi i}) \int_{-S}^{R} \frac{e^{\alpha x}}{1+e^x}\,dx\,. \end{aligned} \tag{9}$$

From (6) and (9), we see that

$$-2\pi i e^{\alpha\pi i} = \int_\gamma f(z)\,dz = I_1 + I_2 - I_3 - I_4 = (1 - e^{2\alpha\pi i}) \int_{-S}^{R} \frac{e^{\alpha x}}{1+e^x}\,dx + I_2 - I_4\,.$$

Now let $R, S \to \infty$ and use (7) and (8) to obtain

$$\lim_{R,S\to\infty} \int_{-S}^{R} \frac{e^{\alpha x}}{1+e^x}\,dx = \frac{-2\pi i\, e^{\alpha\pi i}}{1 - e^{2\alpha\pi i}} = -\frac{2\pi i}{e^{-\alpha\pi i} - e^{\alpha\pi i}} = \frac{2\pi i}{e^{\alpha\pi i} - e^{-\alpha\pi i}} = \frac{2\pi i}{2i\sin(\pi\alpha)}$$

which gives

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x}\,dx = \frac{\pi}{\sin(\pi\alpha)} \qquad \text{i.e. } \Gamma(\alpha)\,\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}\ ,$$

using (2). $\square$

Examples 3.12 1. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\,d\theta$.

Solution. We have,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\,d\theta &= \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\ \cos^{-\frac{1}{2}}\theta\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}-1}\theta\ \cos^{\frac{1}{2}-1}\theta\,d\theta \\
&= \tfrac{1}{2}B\left(\tfrac{3}{4},\tfrac{1}{4}\right) \\
&= \frac{1}{2}\,\frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{4})}{\Gamma(1)} \\
&= \frac{1}{2}\,\frac{\pi}{\sin\frac{\pi}{4}} \\
&= \frac{\pi}{\sqrt{2}}\ .
\end{aligned}$$

2. Evaluate $\displaystyle\int_0^1 \left(\frac{x-1}{x}\right)^{\frac{1}{3}} dx$.

Solution. We have,

$$\begin{aligned}
\int_0^1 \left(\frac{x-1}{x}\right)^{\frac{1}{3}} dx &= \int_0^1 x^{-\frac{1}{3}}(x-1)^{\frac{1}{3}}dx \\
&= -\int_0^1 x^{\frac{2}{3}-1}(1-x)^{\frac{4}{3}-1}dx \\
&= -B\left(\tfrac{2}{3},\tfrac{4}{3}\right) \\
&= -\frac{\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})}{\Gamma(2)} \\
&= -\Gamma\left(\tfrac{2}{3}\right)\tfrac{1}{3}\Gamma\left(\tfrac{1}{3}\right) \\
&= -\frac{\pi}{3\sin\frac{\pi}{3}} \\
&= -\frac{2\pi}{3\sqrt{3}}.
\end{aligned}$$

3. Prove that

(i) $$\int_0^\infty t^{-1/4} e^{-\sqrt{t}}\, dt = \sqrt{\pi}\,.$$

(ii) $$\int_1^\infty \left(\frac{\ln t}{t^2}\right)^2 dt = \frac{2}{27}\,.$$

Solution. (i) By definition,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad (x > 0).$$

Using the substitution $t = u^2$ we have

$$\begin{aligned}\int_0^\infty t^{-1/4} e^{-\sqrt{t}}\, dt &= 2\int_0^\infty u^{-1/2} e^{-u}\, u\, du = 2\int_0^\infty u^{1/2} e^{-u}\, du = 2\Gamma(\tfrac{3}{2}) \\ &= 2.\tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \sqrt{\pi}\,.\end{aligned}$$

(ii) Put $t = e^u$, so that $\frac{dt}{du} = e^u = t$ and $u = \ln t$. Then we have

$$\begin{aligned}\int_1^\infty \left(\frac{\ln t}{t^2}\right)^2 dt &= \int_0^\infty \left(\frac{u}{e^{2u}}\right)^2 e^u\, du = \int_0^\infty u^2 e^{-3u}\, du \\ &= \int_0^\infty \left(\frac{v}{3}\right)^2 e^{-v}\, \frac{1}{3}\, dv \qquad (v = 3u) \\ &= \frac{1}{27}\int_0^\infty v^2 e^{-v}\, dv = \tfrac{1}{27}\,\Gamma(3) = \tfrac{2}{27}\,.\end{aligned}$$

4. Find the area A of the region enclosed by the curve $|x|^3 + |y|^3 = a^3$ $(a > 0)$.

Solution. Since the curve is symmetrical
about the x-axis and the y-axis,
we see that the area A is four times
the area in the first quadrant.

Thus

$$A = 4\int_0^a y\, dx = 4\int_0^a (a^3 - x^3)^{\frac{1}{3}}\, dx.$$

Put $x^3 = a^3 t$, i.e., $x = a t^{\frac{1}{3}}$, so that $\frac{dx}{dt} = \frac{1}{3} a t^{-\frac{2}{3}}$. Then

$$\begin{aligned} A &= 4\int_0^1 (a^3 - a^3 t)^{\frac{1}{3}} \tfrac{1}{3} a t^{-\frac{2}{3}}\, dt \\ &= \frac{4a^2}{3}\int_0^1 (1-t)^{\frac{1}{3}} t^{-\frac{2}{3}}\, dt \\ &= \frac{4a^2}{3}\int_0^1 t^{\frac{1}{3}-1}(1-t)^{\frac{4}{3}-1}\, dt \\ &= \frac{4a^2}{3} B\left(\tfrac{1}{3}, \tfrac{4}{3}\right) \\ &= \frac{4a^2}{3}\frac{\Gamma(\frac{1}{3})\Gamma(\frac{4}{3})}{\Gamma(\frac{5}{3})} \\ &= \frac{4a^2}{3}\frac{\Gamma(\frac{1}{3})\frac{1}{3}\Gamma(\frac{1}{3})}{\frac{2}{3}\Gamma(\frac{2}{3})} \\ &= \frac{2a^2}{3}\frac{\Gamma(\frac{1}{3})^2}{\Gamma(\frac{2}{3})}\,. \end{aligned}$$

But $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \dfrac{\pi}{\sin\frac{\pi}{3}} = \dfrac{2\pi}{\sqrt{3}}$ and so

$$A = \frac{\Gamma(\frac{1}{3})^3 a^2}{\pi\sqrt{3}}\,.$$

5. Prove that

(i) $$\int_0^{\pi/2} \frac{1}{\sqrt{\cos\theta\,\sin\theta}}\, d\theta = \frac{[\Gamma(\frac{1}{4})]^2}{2\sqrt{\pi}}\,.$$

(ii) $$\int_0^{\infty} \frac{dt}{(1+t^4)^3} = \frac{21\sqrt{2}}{128}\,\pi\,.$$

__Solution.__ (i)

$$\begin{aligned} \int_0^{\pi/2} \frac{1}{\sqrt{\cos\theta\,\sin\theta}}\, d\theta &= \int_0^{\pi/2} \cos^{-1/2}\theta \sin^{-1/2}\theta\, d\theta &&= \tfrac{1}{2}B(\tfrac{1}{4}, \tfrac{1}{4}) \\ &= \frac{1}{2}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{2})} &&= \frac{[\Gamma(\frac{1}{4})]^2}{2\sqrt{\pi}}\,. \end{aligned}$$

(ii) Using the substitution $u = t^4$, we have $t = u^{1/4}$ and $\frac{dt}{du} = \frac{1}{4}\, u^{-3/4}$. Hence

$$\begin{aligned} \int_0^{\infty} \frac{dt}{(1+t^4)^3} &= \int_0^{\infty} \frac{1}{4}\frac{u^{-3/4}}{(1+u)^3}\, du &&= \tfrac{1}{4}B(\tfrac{1}{4}, \tfrac{11}{4}) \\ &= \frac{1}{4}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{11}{4})}{\Gamma(3)} &&= \frac{1}{4}\frac{\Gamma(\frac{1}{4})\frac{7}{4}.\frac{3}{4}\Gamma(\frac{3}{4})}{\Gamma(3)}\,. \end{aligned}$$

Hence

$$\int_0^{\infty} \frac{dt}{(1+t^4)^3} = \frac{21}{64}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{2!} = \frac{21}{128}\frac{\pi}{\sin(\frac{\pi}{4})} = \frac{21\sqrt{2}}{128}\,\pi$$

using the relation

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)} \qquad (0 < \alpha < 1)\,.$$

6. Show that
$$\int_0^\infty \frac{1}{\sqrt{x}(s^2+x^4)}\,dx = \frac{1}{4s^{7/4}}\,\frac{\pi}{\sin\left(\frac{\pi}{8}\right)}\,.$$

Solution. Using the substitution $x^4 = s^2 t$ gives $x = s^{1/2}t^{1/4}$. Hence $\frac{dx}{dt} = \frac{1}{4}\,s^{1/2}\,t^{-3/4}$ and, therefore,

$$\begin{aligned}
\int_0^\infty \frac{1}{\sqrt{x}(s^2+x^4)}\,dx &= \int_0^\infty \frac{\frac{1}{4}\,s^{1/2}\,t^{-3/4}}{s^{1/4}\,t^{1/8}(s^2+s^2t)}\,dt = \frac{1}{4s^{7/4}}\int_0^\infty \frac{t^{-7/8}}{1+t}\,dt \\
&= \frac{1}{4s^{7/4}}\,B(\tfrac{7}{8},\tfrac{1}{8}) \quad . \quad = \frac{1}{4s^{7/4}}\,\frac{\Gamma(\frac{7}{8})\Gamma(\frac{1}{8})}{\Gamma(1)} \\
&= \frac{1}{4s^{7/4}}\,\frac{\pi}{\sin\left(\frac{\pi}{8}\right)}\,.
\end{aligned}$$

4 The Laplace transform

Definition and basic properties

We consider integrals of the from

$$\int_0^\infty e^{-st} f(t)\, dt, \tag{16}$$

where s is real, and the function $f : [0, \infty) \longrightarrow \mathbb{R}$ is continuous. Such integrals may be thought of as integral analogues of power series. For the power series $\sum_{n=0}^{\infty} a_n x^n$ $(x > 0)$ evidently has the integral analogue $\int_0^\infty a(t)x^t\, dt$ or $\int_0^\infty a(t)e^{-st}\, dt$, where $s = -\ln x$.

Every power series has an interval of convergence. The integral (16) has an analogous property which is based on the following result.

Theorem 4.1 *If $\int_0^\infty e^{-st} f(t)\, dt$ converges for $s = s_0$, then the integral also converges for $s > s_0$*[¶].

Proof. Define the function ϕ on $[0, \infty)$ by

$$\phi(x) = \int_0^x e^{-s_0 t} f(t)\, dt \qquad (x \geq 0). \tag{17}$$

Then ϕ is continuous on $[0, \infty)$ and $\lim\limits_{x \to \infty} \phi(x)$ exists. Hence ϕ is bounded, i.e., there exists M such that

$$|\phi(x)| \leq M \text{ for all } x \geq 0.$$

Moreover ϕ is differentiable on $[0, \infty)$ and for $x \geq 0$

$$\phi'(x) = e^{-s_0 x} f(x)$$

by the Fundamental Theorem of Calculus. Now, using integration by parts,

$$\begin{aligned} \int_0^u e^{-st} f(t)\, dt &= \int_0^u e^{-(s-s_0)t} e^{-s_0 t} f(t)\, dt \\ &= \int_0^u e^{-(s-s_0)t}\, \phi'(t)\, dt \\ &= \left[e^{-(s-s_0)t}\phi(t)\right]_{t=0}^u + \int_0^u (s - s_0)e^{-(s-s_0)t}\phi(t)\, dt. \end{aligned} \tag{18}$$

If $s > s_0$, then

$$\left| \left[e^{-(s-s_0)t}\phi(t)\right]_{t=0}^u \right| = |\, e^{-(s-s_0)u}\phi(u)\, | \leq e^{-(s-s_0)u} M \to 0$$

as $u \to \infty$.

Now

$$|e^{-(s-s_0)t}\phi(t)| \leq e^{-(s-s_0)t} M$$

[¶]When $f(t) \geq 0$ for $t \geq 0$, then the result follows at once from the Comparison Test.

and $\displaystyle\int_0^\infty e^{-(s-s_0)t}\,dt$ exists for $s > s_0$. By the Comparison Test

$$\int_0^\infty e^{-(s-s_0)t}\phi(t)\,dt \quad \text{exists}$$

for $s > s_0$. Thus the right hand side of (18) tends to a limit as $u \to \infty$. This means that $\int_0^\infty e^{-st}f(t)\,dt$ exists and, in fact,

$$\int_0^\infty e^{-st}f(t)\,dt = (s-s_0)\int_0^\infty e^{-(s-s_0)t}\phi(t)\,dt. \tag{19}$$

$\square$

Note. In subsequent proofs we will often use the relationship given by equation (19), where the function ϕ is given by (17).

It follows from Theorem 4.1 that there are three possibilities:

1. $\displaystyle\int_0^\infty e^{-st}f(t)\,dt$ converges for all real s;
2. $\displaystyle\int_0^\infty e^{-st}f(t)\,dt$ diverges for all real s;
3. there exists a real number c such that $\displaystyle\int_0^\infty e^{-st}f(t)\,dt$ converges for $s > c$ and diverges for $s < c$. ($c = \inf\{s \in \mathbb{R} \ : \ \int_0^\infty e^{-st}f(t)\,dt \text{ converges}\}$.)

The number c in (3) is called the *abscissa of convergence* of $\int_0^\infty e^{-st}f(t)\,dt$. In cases (1) and (2), we can write $c = -\infty$ and $c = \infty$ respectively.

Examples 4.2 1. Prove that $\int_0^\infty e^{-st}e^{-t^2}\,dt$ converges for all real s.

2. Prove that $\int_0^\infty e^{-st}e^{t^2}\,dt$ diverges for all real s.

3. $\int_0^\infty e^{-st}1\,dt$ converges for $s > 0$ and diverges for $s \leq 0$. Thus $c = 0$.

4. Prove that $\displaystyle\int_0^\infty e^{-st}\frac{dt}{1+t^2}$ converges for $s \geq 0$ and diverges for $s < 0$.

Solutions.

1. Take any $s \in \mathbb{R}$. Then

$$0 \leq e^{-st}e^{-t^2} = e^{-(s+t)t} \leq e^{-t}$$

when $s + t \geq 1$, i.e., when $t > 1 - s$. Since $\int_0^\infty e^{-t}\,dt$ converges, it follows from the Comparison Test that $\int_0^\infty e^{-st}e^{-t^2}\,dt$ converges. This is true for all $s \in \mathbb{R}$ and the given integral converges for all real numbers s i.e. $c = -\infty$.

2. We first note that $\int_0^\infty 1\,dt$ diverges. Now, given any $s \in \mathbb{R}$,

$$e^{-st}e^{t^2} = e^{(-s+t)t} \geq e^0 = 1$$

for $t \geq s$. Hence, by the Comparison Test, $\int_0^\infty e^{-st}e^{t^2}\,dt$ diverges. This is true for all $s \in \mathbb{R}$ and the given integral diverges for all real numbers s i.e. $c = \infty$.

3. $\int_0^\infty e^{-st}1\,dt$ converges for $s > 0$ and diverges for $s \leq 0$. Thus $c = 0$.

4. When $s = 0$,

$$\int_0^\infty e^{-st}\frac{dt}{1+t^2} = \int_0^\infty \frac{dt}{1+t^2}$$

which converges. Thus

$$\int_0^\infty e^{-st}\frac{dt}{1+t^2}$$

converges for $s = 0$ and so it converges for $s > 0$ by Theorem 4.1 (or simply by the Comparison Test). Thus

$$\int_0^\infty e^{-st}\frac{dt}{1+t^2} \quad \text{converges for} \quad s \geq 0\,.$$

Now let $s < 0$, so that $y = -s > 0$. We have

$$\frac{e^{-st}}{1+t^2} = \frac{e^{yt}}{1+t^2} \geq \frac{e^{yt}}{2t^2} \quad (t \geq 1)$$

and $\dfrac{e^{yt}}{2t^2} \to \infty$ as $t \to \infty$. Hence, there is a real number $T \geq 1$ such that $\dfrac{e^{yt}}{2t^2} \geq 1$ for all $t \geq T$. By the Comparison Test, $\displaystyle\int_T^\infty \frac{e^{yt}}{1+t^2}\,dt$ diverges and so $\displaystyle\int_0^\infty \frac{e^{yt}}{1+t^2}\,dt$ diverges. Thus $\displaystyle\int_0^\infty \frac{e^{-st}}{1+t^2}\,dt$ diverges for $s < 0$.

Since $\displaystyle\int_0^\infty \frac{e^{-st}}{1+t^2}\,dt$ converges for $s \geq 0$ and it diverges for $s < 0$ its abscissa of convergence $c = 0$.

Examples 3 and 4 show that $\int_0^\infty e^{-ct}f(t)\,dt$ may converge or diverge when c is the abscissa of convergence of $\int_0^\infty e^{-st}f(t)\,dt$. (Compare the behaviour of a power series at the end points of its interval of convergence.)

Definition 4.3 Suppose that the function $f : [0,\infty) \longrightarrow \mathbb{R}$ is continuous and that the integral

$$\int_0^\infty e^{-st}f(t)\,dt$$

converges for some $s \in \mathbb{R}$, so that it has an abscissa of convergence $c < \infty$. Then the function $F : (c, \infty) \longrightarrow \mathbb{R}$ given by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \qquad (s > c)$$

is called the *Laplace transform* of f and we write

$$F = L(f).$$

Also, f is said to be the *determining function* of F.

Note 4.4 $\displaystyle\int_0^\infty e^{-st} f(t)\, dt$ may converge for $s = c$, so that F may be defined on $[c, \infty)$.

Examples 4.5 1. Let $f(t) = t$ $(t \geq 0)$. Find $L(f) = F$.

2. Let $f(t) = e^{at}$ $(t \geq 0)$. Find $L(f) = F$.

3. Let $f(t) = \sin at$ $(t \geq 0)$. Find $L(f) = F$.

Solutions.

1. Let $f(t) = t$ $(t \geq 0)$. Then, for $s > 0$,

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} t\, dt = \left[-\frac{e^{-st}}{s}\, t\right]_{t=0}^\infty + \int_0^\infty \frac{e^{-st}}{s}\, dt = \left[-\frac{e^{-st}}{s^2}\right]_{t=0}^\infty = \frac{1}{s^2}\,.$$

For $s \leq 0$,

$$\int_0^\infty e^{-st} f(t)\, dt$$

diverges

Thus $L(f) = F$ is given by $F(s) = \dfrac{1}{s^2}$ $(s > 0)$.

2. Let $f(t) = e^{at}$ $(t \geq 0)$. Then, for $s > a$,

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{(-s+a)t}\, dt = \left[\frac{e^{(-s+a)t}}{-s+a}\right]_{t=0}^\infty = \frac{1}{s-a}\,.$$

For $s \leq a$,

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} e^{at}\, dt$$

diverges.

Thus $L(f) = F$, where $F(s) = \dfrac{1}{s-a}$ $(s > a)$. A convenient, though loose, notation is

$$L(e^{at}) = \frac{1}{s-a} \quad (s > a)\,.$$

3. Let $f(t) = \sin at$ $(t \geq 0)$. Then, for $s > 0$,

$$\int_0^\infty e^{-st} e^{iat}\, dt = \int_0^\infty e^{(-s+ia)t}\, dt = \left[\frac{e^{(-s+ia)t}}{-s+ia}\right]_{t=0}^\infty = \frac{1}{s-ai} = \frac{s+ia}{s^2+a^2}\,.$$

and so, for $s > 0$,

$$\int_0^\infty e^{-st} \sin at\, dt = \operatorname{Im}\left(\int_0^\infty e^{-st} e^{iat}\, dt\right) = \operatorname{Im}\left(\frac{s+ai}{s^2+a^2}\right) = \frac{a}{s^2+a^2}\,.$$

For $s \leq 0$,

$$\int_0^\infty e^{-st} \sin at\, dt$$

diverges.

Thus

$$L(\sin at) = \int_0^\infty e^{-st} \sin at\, dt = \frac{a}{s^2+a^2} \quad (s > 0)\,.$$

Sometimes it can be important for the integral defining the Laplace transform to converge absolutely. We therefore obtain the analogue of Theorem 4.1 for absolute convergence.

Theorem 4.6 *If $\int_0^\infty e^{-st} f(t)\, dt$ converges absolutely for $s = s_0$, then the integral converges absolutely for $s > s_0$.*

Proof Let $s > s_0$. Since $\int_0^\infty e^{-s_0 t}|f(t)|\, dt$ converges and

$$e^{-st}|f(t)| \leq e^{-s_0 t}|f(t)| \text{ for } t \geq 0,$$

the Comparison test shows that $\int_0^\infty e^{-st}|f(t)|\, dt$ converges. $\square$

Note. Theorem 4.1 applied to $|f|$ gives the desired result, but the direct proof is particularly simple.

It follows from Theorem 4.6 that to any continuous function $f : [0, \infty) \longrightarrow \mathbb{R}$ there corresponds a c' such that $\int_0^\infty e^{-st}|f(t)|\, dt$ converges for $s > c'$ and diverges for $s < c'$. The number c' (which may be $\pm\infty$) is called the *abscissa of absolute convergence* of $\int_0^\infty e^{-st} f(t)\, dt$. Clearly $c' \geq c$.

Example 4.7 Show that $\displaystyle\int_0^\infty e^{-st} e^t \sin e^t\, dt$ has abscissa of convergence $c = 0$ and abscissa of absolute convergence $c' = 1$.

Solution. Let $f(t) = e^t \sin e^t$ $(t \geq 0)$. The change of variable $u = e^t$ gives

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} e^t \sin e^t\, dt = \int_1^\infty \frac{\sin u}{u^s}\, du,$$

with the existence of one side implying the existence of the other. Now $\int_1^\infty \sin u\, du$ diverges and so the integral on the right diverges when $s = 0$. Thus $\int_0^\infty e^{-st} f(t)\, dt$ diverges for $s = 0$ and so $\int_0^\infty e^{-st} f(t)\, dt$ diverges for all $s \leq 0$.

When $s > 0$, we use integration by parts to show that

$$\int_1^\infty \frac{\sin u}{u^s}\, du$$

exists. Let $T > 1$. Then using integration by parts,

$$\int_1^T \frac{\sin u}{u^s}\, du = \left[-\frac{\cos u}{u^s}\right]_1^T - \int_1^T \frac{s \cos u}{u^{s+1}}\, du = -\frac{\cos T}{T^s} + \cos 1 - \int_1^T \frac{s \cos u}{u^{s+1}}\, du\,. \qquad (*)$$

Since $S > 0$,

$$\frac{\cos T}{T^s} \to 0 \quad \text{as} \quad T \to \infty\,.$$

Now, for $u \geq 1$,

$$\left| \frac{s \cos u}{u^{s+1}} \right| \leq \frac{s}{u^{s+1}} \quad \text{and} \quad \int_0^\infty \frac{s}{u^{s+1}}\, du$$

converges because $s + 1 > 1$. By the Comparison Test

$$\int_1^\infty \frac{s \cos u}{u^{s+1}}\, du$$

is absolutely convergent and so convergent. From $(*)$, we see that

$$\int_1^T \frac{\sin u}{u^s}\, du \to \cos 1 - \int_1^\infty \frac{s \cos u}{u^{s+1}}\, du \quad \text{as} \quad T \to \infty\,.$$

Thus $\int_0^\infty e^{-st} f(t)\, dt$ converges for all $s > 0$. Since it diverges for all $s \leq 0$, the abscissa of convergence $c = 0$.

On the other hand,

$$\int_0^\infty e^{-st} \left|e^t \sin e^t\right|\, dt = \int_1^\infty \frac{|\sin u|}{u^s} du,$$

and the second integral converges for $s > 1$ (since $\frac{|\sin u|}{u^s} \leq \frac{1}{u^s}$). Moreover, we saw in Chapter 1 that $\displaystyle\int_1^\infty \frac{|\sin u|}{u^s} du$ diverges for $s = 1$. Thus $\int_0^\infty e^{-st} |\, f(t)\, |\, dt$ converges for $s > 1$ and it diverges for $s \leq 1$ and so the abscissa of absolute convergence $c' = 1$.

In the next theorem, we show that there is a severe restriction on the type of function which can be a Laplace transform.

Theorem 4.8 *If the function $f : [0, \infty) \longrightarrow \mathbb{R}$ is continuous and the Laplace transform $F = L(f)$ exists on some interval (c, ∞), then $F(s) \to 0$ as $s \to \infty$.*

Proof. Let $s_0 > c$. As in the proof of Theorem 4.1, put

$$\phi(x) = \int_0^x e^{-s_0 t} f(t)\, dt.$$

We have seen that there exists M such that $|\phi(x)| \leq M$ for all $x \geq 0$. Also, by (19), for $s > s_0$,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = (s - s_0) \int_0^\infty e^{-(s-s_0)t} \phi(t)\, dt.$$

Take $\epsilon > 0$. Since ϕ is continuous and $\phi(0) = 0$, there exists $\delta > 0$ such that

$$|\phi(t)| < \epsilon \text{ for } 0 \leq t \leq \delta.$$

We now have

$$\begin{aligned} \left|(s - s_0) \int_0^\delta e^{-(s-s_0)t} \phi(t)\, dt\right| &\leq \int_0^\delta (s - s_0) e^{-(s-s_0)t} \epsilon\, dt \\ &= \epsilon \left[-e^{-(s-s_0)t}\right]_{t=0}^{\delta} \\ &= \epsilon(1 - e^{-(s-s_0)\delta}) \\ &< \epsilon\, ; \end{aligned}$$

and

$$\begin{aligned} \left|(s - s_0) \int_\delta^\infty e^{-(s-s_0)t} \phi(t)\, dt\right| &\leq \int_\delta^\infty (s - s_0) e^{-(s-s_0)t} M\, dt \\ &= M \left[-e^{-(s-s_0)t}\right]_{t=\delta}^{\infty} \\ &= M e^{-(s-s_0)\delta}. \end{aligned}$$

Now $Me^{-(s-s_0)\delta} \to 0$ as $s \to \infty$ and so for sufficiently large s, $Me^{-(s-s_0)\delta} < \epsilon$. Thus, there is a number S such that $Me^{-(s-s_0)\delta} < \epsilon$ for all $s > S$ and so

$$|F(s)| < 2\epsilon$$

for all $s > S$ i.e. for any chosen positive number ϵ, however small, we know that $|F(s)| < 2\epsilon$ for all sufficiently large s. This means that $F(s) \to 0$ as $s \to \infty$. □

The theorem shows, for instance, that a non-zero constant or $\sin s$ cannot be Laplace transforms.

Theorem 4.9 1. *If the functions f_1, $f_2 : [0, \infty) \longrightarrow \mathbb{R}$ are continuous and $L(f_1)$, $L(f_2)$ exists on (c_1, ∞), (c_2, ∞) respectively, then $L(\alpha_1 f_1 + \alpha_2 f_2)$ exists on (c_0, ∞) where $c_0 = \max(c_1, c_2)$ and*

$$L(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 L(f_1) + \alpha_2 L(f_2).$$

2. *If the function* $f : [0,\infty) \longrightarrow \mathbb{R}$ *is continuous and* $L(f) = F$ *exists on* (c,∞)*, then, when* $a > 0$*,*

$$L(f(at)) = \frac{1}{a}\, F\left(\frac{s}{a}\right) \quad (s > ac) \tag{20}$$

or

$$F(as) = L\left(\frac{1}{a}\, f\left(\frac{t}{a}\right)\right) \quad (s > c/a). \tag{21}$$

Also, for any $b \in \mathbb{R}$*,*

$$F(s-b) = L(e^{bt} f(t)) \quad (s > c+b). \tag{22}$$

Proof.

1. Immediate.

2. The change of variable $u = at$ gives

$$\int_0^\infty e^{-st} f(at)\, dt = \int_0^\infty e^{-\frac{su}{a}} f(u)\, \frac{du}{a}\,,$$

which is (20) and (21) is obtained from (20) by replacing a by $\frac{1}{a}$.

To prove (22), we have

$$F(s-b) = \int_0^\infty e^{-(s-b)t} f(t)\, dt = \int_0^\infty e^{-st} e^{bt} f(t)\, dt\,.$$

$\square$

Examples 4.10 1. Find the Laplace Transform of $e^{2t}\sin 3t$.

2. Find the function f continuous $[0,\infty)$ whose Laplace Transform is

$$\frac{s}{s^2+4}\,. \quad (s > 0)$$

3. Find the function f continuous $[0,\infty)$ whose Laplace Transform is

$$\frac{16}{s^2(s^2-4)}\,. \quad (s > 2)$$

4. Find the function f continuous $[0,\infty)$ whose Laplace Transform is

$$\frac{2}{(s^2+2s+2)s}\,. \quad (s > 0)$$

Solutions.

1. Since
$$L(\sin 3t) = \frac{3}{s^2+9} \quad (s>0)\,,$$
we see from equation (22) that
$$L(e^{2t}\sin 3t) = \frac{3}{(s-2)^2+9} = \frac{3}{s^2-4s+13} \quad (s>2)\,.$$

2. The required function is $\cos 2t$, since
$$L(\cos 2t) = \frac{s}{s^2+4} \quad (s>0)\,.$$

3. We use partial fractions. For $s>2$, we have
$$\begin{aligned}\frac{16}{s^2(s^2-4)} &= \frac{4}{s^2-4}-\frac{4}{s^2} &&= \frac{1}{s-2}-\frac{1}{s+2}-\frac{4}{s^2}\\ &= L(e^{2t})-L(e^{-2t})-L(4t) &&= L(e^{2t}-e^{-2t}-4t)\end{aligned}$$
using the table of standard Laplace transforms and equation (22). The required function is, therefore, $e^{2t}-e^{-2t}-4t$.

4. Again we begin by using partial fractions. For $s>0$, we have
$$\begin{aligned}\frac{2}{(s^2+2s+2)s} &= \frac{(s^2+2s+2)-(s^2+2s)}{(s^2+2s+2)s} &&= \frac{1}{s}-\frac{s+2}{s^2+2s+2}\\ &= \frac{1}{s}-\frac{(s+1)+1}{(s+1)^2+1} &&= \frac{1}{s}-\frac{(s+1)}{(s+1)^2+1}-\frac{1}{(s+1)^2+1}\,.\end{aligned}$$
Now
$$\frac{(s+1)}{(s+1)^2+1} = G(s+1)\,, \qquad \text{where} \quad G(s) = \frac{s}{s^2+1} = L(\cos t)\,.$$
By equation (22)
$$G(s+1) = L(e^{-t}\cos t)\,,$$
and similarly
$$\frac{1}{(s+1)^2+1} = L(e^{-t}\sin t)\,.$$
Thus
$$\begin{aligned}\frac{2}{(s^2+2s+2)s} &= \frac{1}{s}-\frac{(s+1)}{(s+1)^2+1}-\frac{1}{(s+1)^2+1}\\ &= L(1)-L(e^{-t}\cos t)-L(e^{-t}\sin t)\\ &= L(1-e^{-t}\cos t-e^{-t}\sin t) = L(1-e^{-t}[\cos t+\sin t])\,.\end{aligned}$$
Thus the required function is $1-e^{-t}(\cos t+\sin t)$.

In view of Theorem 4.9, a linear combination of Laplace transforms is again a Laplace transform. We now prove the remarkable result that the product of Laplace transforms is often also a Laplace transform.

Definition 4.11 Suppose that the functions $f,\ g : [0,\infty) \longrightarrow \mathbb{R}$ are continuous. The function $h : [0,\infty) \longrightarrow \mathbb{R}$ defined by

$$h(t) = \int_0^t f(u)g(t-u)\,du \qquad (t \geq 0) \tag{23}$$

is called the *convolution* of f and g, and we write

$$h = f * g.$$

Often it is convenient to write loosely

$$h(t) = f(t) * g(t).$$

The change of variable $v = t - u$ shows that $f * g = g * f$.

Examples 4.12 1. Find $t * t$.

2. Find $\sin t * \cos t$.

3. Find $1 * f(t)$.

Solutions.

1. For $t \geq 0$,

$$t * t = \int_0^t u(t-u)\,du = \left[\frac{u^2 t}{2} - \frac{u^3}{3}\right]_{u=0}^{t} = \frac{t^3}{2} - \frac{t^3}{3} = \frac{t^3}{6}\,.$$

2. For $t \geq 0$,

$$\begin{aligned}
\sin t * \cos t &= \int_0^t \sin u \cos(t-u)\,du \\
&= \int_0^t \left(\frac{\sin t + \sin(2u-t)}{2}\right) du \\
&= \frac{1}{2}\left[u \sin t - \frac{\cos(2u-t)}{2}\right]_{u=0}^{t} \\
&= \frac{1}{2}\left[t \sin t - \frac{\cos t}{2} + \frac{\cos(-t)}{2}\right] \\
&= \frac{t \sin t}{2}
\end{aligned}$$

3. For $t \geq 0$,

$$1 * f(t) = f(t) * 1 = \int_0^t f(u)\,du\,.$$

Theorem 4.13 *If the functions f, $g : [0, \infty) \longrightarrow \mathbb{R}$ are continuous and the integrals for*

$$F(s) = L(f), \qquad G(s) = L(g)$$

converge absolutely for $s > a$, then

$$H(s) = L(f * g)$$

exists for $s > a$ and

$$H(s) = F(s)G(s),$$

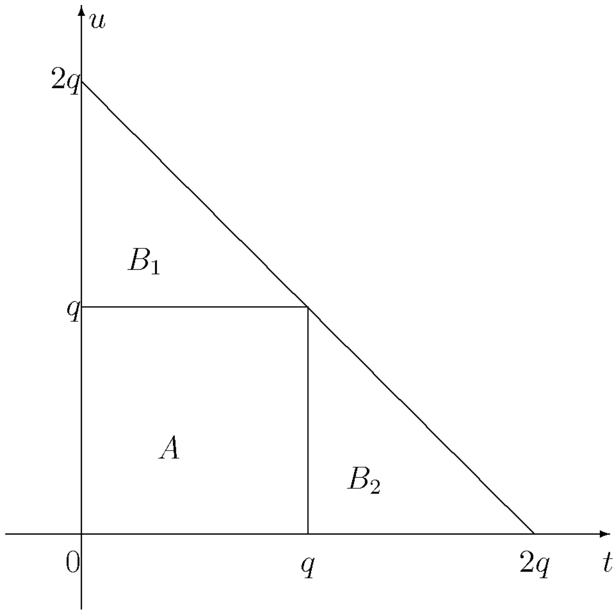
i.e.,

$$L(f * g) = L(f)L(g).$$

Proof. Take any $s > a$. For any set E in the positive quadrant of the (t, u)-plane, put

$$I(E) = \iint_E e^{-s(t+u)} f(t)g(u)\, dt\, du.$$

Let A be the square $[0, q] \times [0, q]$, and let B_1, B_2 be the triangles shown in the diagram.



Then

$$I(A) = \int_0^q e^{-st} f(t)\, dt \int_0^q e^{-su} g(u)\, du,$$

so that

$$I(A) \to F(s)G(s) \text{ as } q \to \infty. \tag{24}$$

Also,

$$\begin{aligned}
|I(B_1)| &= \left| \int_0^q dt \int_q^{2q-t} e^{-s(t+u)} f(t)g(u)\,du \right| \\
&\le \int_0^q dt \int_q^{2q-t} e^{-s(t+u)} |\,f(t)|\,|\,g(u)|\,du \\
&\le \int_0^q dt \int_q^{2q} e^{-s(t+u)} |\,f(t)|\,|\,g(u)|\,du \\
&= \int_0^q e^{-st} |\,f(t)|\,dt . \int_q^{2q} e^{-su} |\,g(u)|\,du \\
&\to \int_0^\infty e^{-st} |\,f(t)|\,dt\,.\,0 \\
&= 0
\end{aligned}$$

as $q \to \infty$. Thus

$$I(B_1) \to 0 \text{ as } q \to \infty \tag{25}$$

and similarly

$$I(B_2) \to 0 \text{ as } q \to \infty. \tag{26}$$

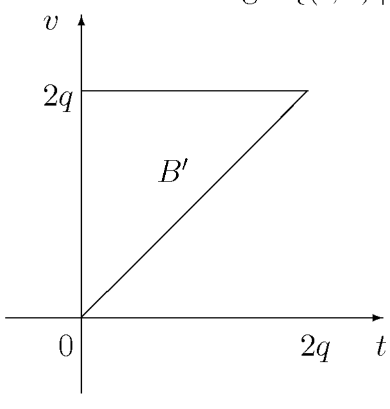
Hence, if $B = A \cup B_1 \cup B_2$, then, by (24), (25) and (26),

$$I(B) = I(A) + I(B_1) + I(B_2) \to F(s)G(s) \text{ as } q \to \infty. \tag{27}$$

But $I(B) = \displaystyle\int_0^{2q} dt \int_0^{2q-t} e^{-s(t+u)} f(t)g(u)\,du$, and putting $v = u + t$ in the inner integral, we have

$$I(B) = \int_0^{2q} dt \int_0^{2q-t} e^{-sv} f(t)g(v-t)\,dv = \iint_{B'} e^{-sv} f(t)g(v-t)\,dt\,dv,$$

where B' is the triangle $\{(t, v)\,|\,0 \le t \le 2q, t \le v \le 2q\}$.



Therefore, by (23),

$$I(B) = \int_0^{2q} dv \int_0^{v} e^{-sv} f(t)g(v-t)\,dt = \int_0^{2q} e^{-sv} h(v)\,dv,$$

where $h = f * g$, and it follows from (27) that $\int_0^\infty e^{-sv} h(v)\,dv$ exists and equals $F(s)G(s)$.

□

Example 4.14

$$L\left(\frac{t^3}{6}\right) = L(t*t) = L(t)L(t) = \frac{1}{s^2}.\frac{1}{s^2} = \frac{1}{s^4},$$

so $L(t^3) = \frac{6}{s^4}$. This result is, of course, quite easily obtained from the definition of the Laplace transform.

We now show that Laplace transforms are very smooth functions.

Theorem 4.15 *If the function $f : [0, \infty) \longrightarrow \mathbb{R}$ is continuous and $F = L(f)$ exists on the interval (c, ∞), then F is differentiable on (c, ∞) and*

$$F'(s) = -\int_0^\infty e^{-st} t f(t)\, dt = -L(tf(t)).$$

Proof.‖ Take an $s_0 > c$ and, as in the proof of Theorem 4.1, put

$$\phi(x) = \int_0^x e^{-s_0 t} f(t)\, dt \qquad (x \geq 0),$$

so that ϕ is continuous and bounded on $[0, \infty)$:

$$|\phi(x)| \leq M \text{ for all } x \geq 0.$$

We saw in (19) that, for $s > s_0$,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = (s - s_0) \int_0^\infty e^{-(s-s_0)t} \phi(t)\, dt. \tag{28}$$

Hence F is differentiable if

$$\psi(s) = \int_0^\infty e^{-(s-s_0)t} \phi(t)\, dt$$

is differentiable. We use theorem 2.25 to show that $\psi'(s)$ can be obtained using differentiation under the integral sign.
Now

$$\frac{\partial}{\partial s}\left[e^{-(s-s_0)t}\phi(t)\right] = -te^{-(s-s_0)t}\phi(t)$$

is continuous on the set $\{(s,t) : s \geq s_0, t \geq 0\}$ and, if $\delta > 0$, then, for $s \geq s_0 + \delta$,

$$\left|-te^{-(s-s_0)t}\phi(t)\right| \leq Mte^{-\delta t}$$

and

$$\int_0^\infty Mte^{-\delta t}\, dt$$

exists. Hence, by Theorem 2.25, $\psi'(s)$ exists and equals $-\displaystyle\int_0^\infty te^{-(s-s_0)t}\phi(t)\, dt$ for $s \geq s_0 + \delta$, and therefore for $s > s_0$. Thus, by (28),

$$F'(s) = \int_0^\infty e^{-(s-s_0)t}\phi(t)\, dt - (s - s_0)\int_0^\infty e^{-(s-s_0)t} t\phi(t)\, dt.$$

‖The result would be obvious if we knew that $F(s) = \int_0^\infty e^{-st} f(t)\, dt$ can be differentiated under the integral sign.

Finally, integrating the second integral by parts, we have

$$\begin{aligned} F'(s) &= \int_0^\infty e^{-(s-s_0)t}\phi(t)\,dt + [e^{-(s-s_0)t} t\,\phi(t)]_{t=0}^\infty - \int_0^\infty e^{-(s-s_0)t}\{\phi(t)+t\phi'(t)\}dt \\ &= -\int_0^\infty e^{-(s-s_0)t} t\phi'(t)\,dt \\ &= -\int_0^\infty e^{-st} t f(t)\,dt, \end{aligned}$$

since $\phi'(t) = e^{-s_0 t} f(t)$.

This has been proved for $s > s_0$. Since s_0 was an arbitrary number greater than c, the formula holds for all $s > c$. $\square$

Corollary 4.16 *If the function $f : [0, \infty) \longrightarrow \mathbb{R}$ is continuous and $F = L(f)$ exists on (c, ∞), then $F = L(f)$ has derivatives of all orders and*

$$F^{(k)}(s) = (-1)^k \int_0^\infty e^{-st} t^k f(t)\,dt = (-1)^k L(t^k f(t)).$$

Proof. The result is proved using mathematical induction.
Clearly the result is true for $k = 1$, by theorem 4.15.
Now suppose that the result is true for $k = r$ for some positive integer r,

$$\text{i.e.} \quad F^{(r)}(s) = (-1)^r \int_0^\infty e^{-st} t^r f(t)\,dt = (-1)^r L(t^r f(t)).$$

Then, using theorem 4.15, gives

$$F^{(r+1)}(s) = \frac{d}{ds}\left(F^{(r)}(s)\right) = -\left[(-1)^r L(t\{t^r f(t)\})\right] = (-1)^{r+1} L(t^{r+1} f(t)).$$

Thus the result is true for $k = r + 1$ whenever it is true for $k = r$. Since the result is true for $k = 1$, it follows by induction that it is true for all positive integers k. $\square$

Theorem 4.15 can also be used to produce new Laplace transforms.

Example 4.17 1. Show that for $s > 0$,

$$L(t \sin at) = \frac{2as}{(s^2 + a^2)^2}$$

and

$$L(t^2 \sin at) = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}\ .$$

2. Find the continuous functions $f : [0, \infty) \longrightarrow \mathbb{R}$ such that

$$\int_0^t f(u) f(t-u)\,du = 2(\sin t - t \cos t)\ . \qquad (I)$$

Solutions.

1. For $s > 0$,

$$L(\sin at) = \frac{a}{s^2 + a^2} \ .$$

Hence, using theorem 4.15,

$$L(t \sin at) = -\frac{d}{ds}\frac{a}{s^2 + a^2} = \frac{2as}{(s^2 + a^2)^2}$$

and, using corollary 4.16,

$$L(t^2 \sin at) = -\frac{d}{ds}\left(\frac{2as}{(s^2 + a^2)^2}\right) = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3} \ .$$

2. Let $h = f * f$. Then equation (I) can be written as

$$h(t) = \int_0^t f(u)f(t-u)\, du = 2(\sin t - t\cos t) \ . \qquad (II)$$

Now,

$$L(2\sin t) = \frac{2}{s^2 + 1} \ ,$$

and using theorem 4.15 gives

$$L(t\cos t) = -\frac{d}{ds}\left(\frac{2s}{s^2 + 1}\right) = -\frac{(s^2+1)2 - (2s)(2s)}{(s^2+1)^2} = -\frac{2}{s^2+1} + \frac{4s^2}{(s^2+1)^2} \ .$$

Hence

$$L(2\sin t - 2t\cos t) = \frac{4}{s^2+1} - \frac{4s^2}{(s^2+1)^2} = \frac{4}{(s^2+1)^2} \ .$$

Taking Laplace transforms in (II) gives

$$L(f * f) = L(h) = L(2\sin t - 2t\cos t) = \frac{4}{(s^2+1)^2} \ . \qquad (III)$$

Let $F = L(f)$. Then (III) gives,

$$L(f * f) = L(f) \ \times \ L(f) = [\, F(s)\,]^2 = \frac{4}{(s^2+1)^2} \ .$$

Hence

$$F(s) = \ \pm\frac{2}{s^2+1}$$

and so

$$f(t) = \ \pm 2\sin t.$$

It is easy to check that this function f satisfies all the required conditions.

Corollary 4.18 *Suppose that the function $f : [0, \infty) \longrightarrow \mathbb{R}$ is continuous and that $F = L(f)$ exists on the interval (c, ∞). If also $\lim_{t\to 0+} \frac{f(t)}{t}$ exists, (i.e. $\frac{f(t)}{t}$ tends to a finite limit as $t \to 0+$,) then*

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(u)\, du \qquad (s > c).$$

Proof. The function $\frac{f(t)}{t}$ may be defined at 0 so as to be continuous at 0 and so on $[0, \infty)$. Also, by an example in chapter 1, the existence of $\int_0^\infty e^{-st} f(t)\, dt$ implies that of $\int_0^\infty e^{-st} \frac{f(t)}{t} dt$. Thus

$$L\left(\frac{f(t)}{t}\right) = \Phi(s),$$

say, exists for $s > c$. Then

$$F(s) = L(f(t)) = L\left(t.\frac{f(t)}{t}\right) = -\Phi'(s)$$

and therefore, if $c_0 > c$,

$$\Phi(s) = -\int_{c_0}^{s} F(u)\, du + K \qquad (s > c).$$

But, by Theorem 4.8, $\Phi(s) \to 0$ as $s \to \infty$. Hence $\int_{c_0}^\infty F(u)\, du$ exists and is equal to K. Thus

$$\Phi(s) = \int_{c_0}^{\infty} F(u)\, du - \int_{c_0}^{s} F(u)\, du = \int_{s}^{\infty} F(u)\, du.$$

□

Example 4.19 Show that for $s > 0$,

$$L\left(\frac{\sin t}{t}\right) = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1} \frac{1}{s}\,.$$

Solution.

1. Let

$$f(t) = \sin t\,, \quad \text{and let} \quad F = L(f).$$

Then, for $s > 0$,

$$F(s) = \frac{1}{s^2 + 1}\,.$$

Now

$$\frac{f(t)}{t} = \frac{\sin t}{t} \to 1 \;\text{ as }\; t \to 0+\;.$$

Hence, by corollary 4.18,

$$L\left(\frac{\sin t}{t}\right) = L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(u)\, du = \int_s^\infty \frac{du}{u^2+1} = [\tan^{-1} u]_{u=s}^{\infty} = \frac{\pi}{2} - \tan^{-1} s.$$

Next we consider the Laplace transforms of the derivatives and integrals of given determining functions.

Theorem 4.20 *Suppose that the function $f : [0, \infty) \longrightarrow \mathbb{R}$ is continuously differentiable, that $F = L(f)$ exists on (c, ∞) and that, for $s > c$,*

$$e^{-st} f(t) \to 0 \text{ as } t \to \infty.$$

Then $L(f')$ exists on (c, ∞), and

$$L(f') = sF(s) - f(0). \tag{29}$$

Proof. Let $s > c$. Then, for $u > 0$,

$$\int_0^u e^{-st} f'(t)\, dt = \left[e^{-st} f(t)\right]_{t=0}^{u} + s \int_0^u e^{-st} f(t)\, dt.$$

As $u \to \infty$,

$$s \int_0^u e^{-st} f(t)\, dt \to sF(s),$$

and

$$\left[e^{-st} f(t)\right]_{t=0}^{u} = e^{-su} f(u) - f(0) \to -f(0).$$

Thus $L(f') = \int_0^\infty e^{-st} f'(t)\, dt$ exists and (29) holds. □

Corollary 4.21 *If the function $f : [0, \infty) \longrightarrow \mathbb{R}$ has a continuous nth derivative, $F = L(f)$ exists on (c, ∞) and, for $s > c$,*

$$e^{-st} f^{(k)}(t) \to 0 \text{ as } t \to \infty \text{ for } k = 0, 1, \ldots, n-1,$$

then $L(f^{(n)})$ exists in (c, ∞) and

$$L(f^{(n)}) = s^n F(s) - \sum_{k=1}^{n} f^{(k-1)}(0) s^{n-k}.$$

Proof. We use mathematical induction to prove the result.

By theorem 4.20 the result is true for $n = 1$.

Now suppose that the result is true for $n = r$, where r is some positive integer,

$$\text{i.e.} \quad L(f^{(r)}) = s^r F(s) - \sum_{k=1}^{r} f^{(k-1)}(0) s^{r-k}.$$

Then, using theorem 4.20,

$$\begin{aligned} L(f^{(r+1)}) = L\left(\frac{d}{dt}\left[f^{(r)}\right]\right) &= s\left[s^r F(s) - \sum_{k=1}^{r} f^{(k-1)}(0) s^{r-k}\right] - f^r(0) \\ &= s\left[s^r F(s) - s^{r-1} f(0) - s^{r-2} f'(0) - \cdots - f^{r-1}(0)\right] - f^r(0) \\ &= s^{r+1} F(s) - s^r f(0) - s^{r-1} f'(0) - \cdots - f^r(0) \\ &= s^{r+1} F(s) - \sum_{k=1}^{r+1} f^{(k-1)}(0) s^{r+1-k}\ . \end{aligned}$$

Thus the result is true for $n = r + 1$ whenever it is true for $n = r$. Since the result is true for $n = 1$, it follows by induction that it is true for all positive integers n. □

Example 4.22 For $s > 0$,

$$L(\cos at) = \frac{s}{s^2 + a^2}$$

and obviously $e^{-st} \cos at \to 0$ as $t \to \infty$. Hence

$$L(-a \sin at) = s\,\frac{s}{s^2 + a^2} - \cos(a0) = \frac{s^2}{s^2 + a^2} - 1 = -\frac{a^2}{s^2 + a^2}\ ,$$

i.e.,

$$L(\sin at) = \frac{a}{s^2 + a^2}.$$

Theorem 4.23 *Suppose that the function $f : [0, \infty) \longrightarrow \mathbb{R}$ is continuous and that $F = L(f)$ exists on (c, ∞). Put $d = \max\{c, 0\}$. Then $L\left(\int_0^t f(u)\,du\right)$ exists on the interval (d, ∞), and*

$$L\left(\int_0^t f(u)\,du\right) = \frac{F(s)}{s} \qquad (s > d).$$

Proof.** Let

$$g(t) = \int_0^t f(u)\,du \qquad (t \geq 0).$$

Then g is continuous, and the proof is largely a matter of showing that, when $s > d$,

$$e^{-st} g(t) \to 0 \text{ as } t \to \infty. \tag{30}$$

Take any $s > d$, and let s_0 be such that $s > s_0 > d$. Put

$$\phi(t) = \int_0^t e^{-s_0 u} f(u)\,du \qquad (t \geq 0)$$

so that, as we have seen before, ϕ is continuous, $\phi(0) = 0$, and there exists M such that

$$|\phi(t)| \leq M \text{ for } t \geq 0$$

and

$$\phi'(t) = e^{-s_0 t} f(t) \quad (t \geq 0).$$

We have, using integration by parts,

$$\begin{aligned} g(t) &= \int_0^t e^{s_0 u}(e^{-s_0 u} f(u))\,du = \int_0^\infty e^{s_0 u} \phi'(u)\,du \\ &= \left[\, e^{s_0 u} \phi(u)\,\right]_{u=0}^t - \int_0^t s_0 e^{s_0 u} \phi(u)\,du \\ &= e^{s_0 t} \phi(t) - \int_0^t s_0 e^{s_0 u} \phi(u)\,du; \end{aligned}$$

**When the integral for $F(s)$ converges absolutely, the result follows at once from Theorem 4.13. For

$$L\left(\int_0^t f(u)\,du\right) = L(1 * f(t)) = L(1)L(f(t)) = \frac{1}{s} F(s).$$

and so

$$\begin{aligned} |\,g(t)\,| &\leq e^{s_0 t}|\,\phi(t)\,| + \int_0^t s_0\, e^{s_0 u}|\phi(u)|\,du \\ &\leq e^{s_0 t}M + \int_0^t s_0\, e^{s_0 u}M\,du \\ &= M(\,e^{s_0 t} + [\,e^{s_0 u}\,]_{u=0}^t\,) \\ &= M(e^{s_0 t} + e^{s_0 t} - 1) \\ &\leq 2Me^{s_0 t}. \end{aligned}$$

Hence

$$|\,e^{-st}g(t)\,| \leq 2Me^{-(s-s_0)t}$$

and so (30) holds.

Now

$$\int_0^u e^{-st}g(t)\,dt = \left[-\frac{1}{s}\,e^{-st}g(t)\right]_{t=0}^u + \frac{1}{s}\int_0^u e^{-st}f(t)\,dt.$$

But $F(s) = \int_0^\infty e^{-st}f(t)\,dt$ exists, $g(0) = 0$ and (30) holds. Therefore $G(s) = \int_0^\infty e^{-st}g(t)\,dt$ exists, and

$$G(s) = \frac{F(s)}{s}.$$

□

Uniqueness of the determining function

Given a continuous function f on $[0,\infty)$, the Laplace transform $F = L(f)$, if it exists, is clearly uniquely determined. It is also true that a given function F cannot be the Laplace transform of more than one continuous function. This fact is important in applications to, for instance, the solution of differential equations. The proof depends on a famous theorem of Weierstrass on the approximation of continuous functions by polynomials. The Uniqueness Theorem and its corollary are stated here. The proofs are given in Appendix 1, which is available to anyone who is interested. The proofs in Appendix 1 are **Not examinable**.

Theorem 4.24 (Uniqueness Theorem) *If the function $f : [0,\infty) \longrightarrow \mathbb{R}$ is continuous, $F = L(f)$ exists on an interval (c,∞) and $F(s) = 0$ for $s > c$, then $f(t) = 0$ for $t \geq 0$.*

Corollary 4.25 *If the functions f, $g : [0,\infty) \longrightarrow \mathbb{R}$ are continuous and if $F = L(f)$ and $G = L(g)$ exist and are equal on an interval (c,∞), then $f(t) = g(t)$ for $t \geq 0$.*

The uniqueness theorem enables us to define the notion of an inverse Laplace transform. If the function $f : [0,\infty) \longrightarrow \mathbb{R}$ is continuous and $F = L(f)$ exists on some interval (c,∞), then we can write

$$f = L^{-1}(F),$$

and f is called the *inverse Laplace transform* of F.

Additional Laplace transforms

So far, we have always stipulated that our determining functions should be continuous on $[0, \infty)$. However, the Laplace transform

$$\int_0^\infty e^{-st} f(t)\, dt$$

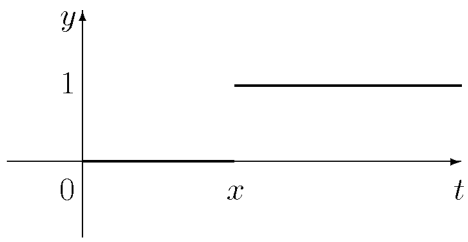
may clearly also be defined when

1. f has a finite number of jump discontinuities[††], and
2. the integral is improper at 0.

Under these circumstances, most of the foregoing theory holds[‡‡], but individual results may be more difficult to prove. Below we obtain Laplace transforms of the extended type. We also introduce a new function which frequently appears in applications of analysis.

1. A simple determining function with a jump discontinuity is ψ_x $(x \geq 0)$ defined on $[0, \infty)$ by

$$\psi_x(t) = \begin{cases} 0 & \text{for } 0 \leq t < x, \\ 1 & \text{for } t \geq x. \end{cases}$$



We have $L(\psi_x) = \Psi_x$, where, for $s > 0$,

$$\Psi_x(s) = \int_0^\infty e^{-st} \psi_x(t)\, dt = \int_x^\infty e^{-st}\, dt = \left[-\frac{1}{s} e^{-st} \right]_{t=x}^\infty = \frac{e^{-sx}}{s}.$$

2. We consider a group of functions on $(0, \infty)$ whose Laplace transforms are integrals improper at 0.

Let

$$f(t) = t^\alpha\,, \quad \text{where} \quad (\alpha > -1)\,.$$

(i) If $\alpha > -1\,,\ \ s > 0$ and $0 < t \leq 1$, then

$$0 \leq f(t) e^{-st} = t^\alpha\, e^{-st} \leq t^\alpha\,.$$

[††] The function f has a *jump discontinuity* at c if the left and right limits $f(c-)$, $f(c+)$ exist, but $f(c-) = f(c) = f(c+)$ does not hold.

[‡‡] Since an integrand may be changed at a finite number of points without any effect on the integral, continuity is clearly needed for the Uniqueness Theorem 4.36 to hold.

Since $\int_0^1 t^\alpha \, dt$ exists for $\alpha > -1$, we see, from the Comparison test , that

$$\int_0^1 e^{-st} f(t)\, dt = \int_0^1 e^{-st} t^\alpha \, dt$$

exists for $\alpha > -1 \,, s > 0$.

(ii) Since

$$t^\alpha e^{-\frac{s}{2}t} \to 0 \ \text{ as } \ t \to \infty \,,$$

there exists $T \geq 1$ such that

$$0 \leq t^\alpha e^{-\frac{s}{2}t} \leq 1 \quad \text{for all} \quad t \geq T \,.$$

Thus, if $s > 0$ and $t \geq T$, then

$$0 \leq f(t)\, e^{-st} = t^\alpha \, e^{-st} = \left(t^\alpha e^{-\frac{s}{2}t}\right) e^{-\frac{s}{2}t} \leq e^{-\frac{s}{2}t} \,.$$

Since

$$\int_T^\infty e^{-\frac{s}{2}t} \, dt$$

exists for $s > 0$, we see, from the Comparison test , that

$$\int_T^\infty e^{-st} f(t)\, dt = \int_T^\infty e^{-st} t^\alpha \, dt$$

exists for $s > 0$. Hence

$$\int_1^\infty e^{-st} f(t)\, dt = \int_1^\infty e^{-st} t^\alpha \, dt$$

exists for $s > 0$.

Thus

$$F(s) = \int_0^\infty e^{-st} t^\alpha \, dt$$

exists for $s > 0$. Putting $st = u$ in the integral we have

$$F(s) = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^\alpha \frac{1}{s} \, du = \frac{1}{s^{\alpha+1}} \int_0^\infty u^\alpha e^{-u} \, du = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

More generally, by Theorem 4.9,

$$L(t^\alpha e^{bt}) = \frac{\Gamma(\alpha+1)}{(s-b)^{\alpha+1}}$$

for $s > b$. Thus, in particular, if $a > 0$,

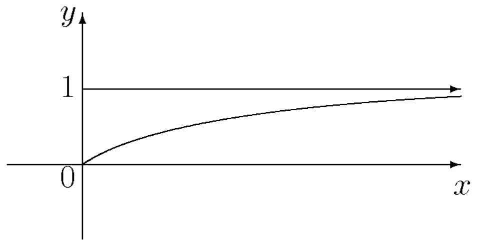
$$L\left(\frac{e^{-at}}{\sqrt{t}}\right) = \frac{\Gamma(\frac{1}{2})}{(s+a)^{\frac{1}{2}}} = \sqrt{\frac{\pi}{s+a}} \tag{31}$$

when $s > -a$.

3. The **error function** $\operatorname{erf} x$ is defined on $[0, \infty)$ by

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \qquad (x \geq 0),$$

so that $\operatorname{erf} 0 = 0$ and $\operatorname{erf} x \to 1$ as $x \to \infty$.



Using Theorem 4.9, we can show that, if $a > 0$,

$$L(\operatorname{erf} a\sqrt{t}) = L(\operatorname{erf} \sqrt{a^2 t}) = \frac{1}{a^2} F\left(\frac{s}{a^2}\right) = \frac{1}{a^2} \frac{1}{\frac{s}{a^2}\sqrt{\frac{s}{a^2} + 1}} = \frac{a}{s\sqrt{s + a^2}} \qquad (s > 0).$$

Since $\dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^\infty e^{-t^2}\, dt = 1$,

$$1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt \qquad (x \geq 0).$$

This function is also called the **complementary error function** and is often denoted by $\operatorname{erfc} x$.

Using Theorem 4.9, we can show that for all $a > 0$,

$$L\left(1 - \operatorname{erf} \frac{a}{\sqrt{t}}\right) = L\left(1 - \operatorname{erf} \frac{1}{\sqrt{\frac{t}{a^2}}}\right) = a^2 F(a^2 s) = a^2 \frac{e^{-2\sqrt{a^2 s}}}{a^2 s} = \frac{e^{-2a\sqrt{s}}}{s} \qquad (s > 0).$$

The proofs of these results are in Appendix 2, which is available to anyone who is interested. The proofs in Appendix 2 are **Not examinable**.

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We are now ready to consider a number of applications of the Laplace transform. As in the derivation of $L(1 - \operatorname{erf} \frac{1}{\sqrt{t}})$, we shall not justify all of the manipulations that have to be carried out, but we shall always draw attention to the assumptions we make.

Evaluation of integrals

Suppose that we wish to evaluate

$$\psi(t) = \int_a^\infty f(x, t)\, dx \qquad (t > 0). \tag{32}$$

If we can calculate $L(\psi)$, then we may be able to obtain ψ. Under suitable conditions

$$L(\psi) = L\left(\int_a^\infty f(x,t)\,dx\right) = \int_a^\infty L(f(x,t))\,dx \tag{33}$$

and the last integral may be much easier to deal with than this integral in (32). Since

$$L\left(\int_a^\infty f(x,t)\,dx\right) = \int_0^\infty e^{-st}\left(\int_a^\infty f(x,t)\,dx\right)dt,$$

the identity (33) is equivalent to

$$\int_0^\infty dt\int_a^\infty e^{-st}f(x,t)\,dx = \int_a^\infty dx\int_0^\infty e^{-st}f(x,t)\,dt.$$

This change in the order of integration is usually hard to justify rigorously, and, in the examples below we shall assume without proof that (33) holds.

Example 4.26 1. Show that $\displaystyle\int_0^\infty \frac{\sin xt}{x}\,dx = \frac{\pi}{2}$ for $t > 0$.

2. Show that $\displaystyle\int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Solutions

1. We assume that

$$L\left(\int_0^\infty \frac{\sin xt}{x}\,dx\right) = \int_0^\infty L\left(\frac{\sin xt}{x}\right)\,dx.$$

Then

$$L\left(\int_0^\infty \frac{\sin xt}{x}\,dx\right) = \int_0^\infty \frac{1}{x}\,\frac{x}{s^2+x^2}\,dx = \int_0^\infty \frac{dx}{s^2+x^2} = \left[\frac{1}{s}\,\tan^{-1}\frac{x}{s}\right]_{x=0}^\infty = \frac{\pi}{2s}.$$

Since $L^{-1}(\frac{1}{s}) = 1$, it follows that

$$\int_0^\infty \frac{\sin xt}{x}\,dx = \frac{\pi}{2} \quad \text{for} \quad t > 0.$$

Note that the integral had previously been evaluated, apparently with much more labour. However, then all the details of the proof were checked, whereas now the principal step is left unjustified.

2. To show that

$$\int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,,$$

we consider

$$\int_0^\infty \sin(x^2 t)\,dx$$

for $t > 0$ and assume that

$$L\left(\int_0^\infty \sin(x^2 t)\,dx\right) = \int_0^\infty L(\sin(x^2 t))\,dx.$$

Then, for $s > 0$,

$$L\left(\int_0^\infty \sin(x^2 t)\,dx\right) = \int_0^\infty \frac{x^2}{s^2 + x^4}\,dx\,.$$

Put $x^4 = s^2 u$, so that $x = s^{1/2}\,u^{1/4}$ and $\dfrac{dx}{du} = s^{1/2}\,\dfrac{1}{4}\,u^{-3/4}$. Then

$$\begin{aligned}
L\left(\int_0^\infty \sin(x^2 t)\,dx\right) &= \int_0^\infty \frac{x^2}{s^2 + x^4}\,dx \\
&= \int_0^\infty \frac{s\,u^{1/2}}{s^2(1+u)}\,\tfrac{1}{4}s^{1/2}\,u^{-3/4}\,du \\
&= \frac{1}{4s^{1/2}}\int_0^\infty \frac{u^{-1/4}}{1+u}\,du \\
&= \frac{1}{4s^{1/2}}\,B\left(\tfrac{3}{4},\tfrac{1}{4}\right) \\
&= \frac{1}{4s^{1/2}}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)} \\
&= \frac{1}{4s^{1/2}}\,\frac{\pi}{\sin\pi/4} = \frac{\pi\sqrt{2}}{4}\,\frac{1}{s^{1/2}}.
\end{aligned}$$

Since $L^{-1}\left(\dfrac{1}{s^{\frac{1}{2}}}\right) = \dfrac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} = \dfrac{t^{-\frac{1}{2}}}{\sqrt{\pi}}$,

$$\int_0^\infty \sin(x^2 t)\,dx = \left(\frac{\pi\sqrt{2}}{4}\right)\left(\frac{t^{-\frac{1}{2}}}{\sqrt{\pi}}\right) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,t^{-\frac{1}{2}}$$

and the desired identity follows on putting $t = 1$.

Solution of ordinary differential equations

Laplace transforms can be very helpful in the solution of ordinary differential equations subject to given initial conditions. Before studying such applications, a short revision section on elementary methods of solving differential equations is included.

An equation connecting $y(t)$, its derivatives and t, such as

$$\frac{dy}{dt} = t + 3\,, \tag{36}$$

$$\frac{d^2y}{dt^2} = t + y\,, \tag{37}$$

$$\left(\frac{d^3y}{dt^3}\right)^5 = t^2 + y\,, \tag{38}$$

is a differential equation.

A function $y(t)$ is **a solution** of a given differential equation if it satisfies the equation i.e. if substituting for $y(t)$ in terms of t satisfies the equation identically. For example $y = \frac{1}{2}t^2 + 3t + c$ satisfies (36) for all values of c.

The **order of a differential equation** is the order of the highest derivative occurring in the differential equation. For example (36) is of order 1, (37) is of order 2, and (38) is of order 3.

In this revision section, we will only consider first order differential equations i.e. ones involving $y(t)$, $\frac{dy}{dt}$ and t, but no higher derivatives. Surprisingly, **this will still enable us to solve some higher order differential equations later using Laplace transforms**. We look at some cases which can be solved analytically i.e. we can find an explicit formula connecting y and t.

1. Separable Variable Case.

These are equations which can be arranged so that they take the form

$$f(y)\,\frac{dy}{dt} = g(t).$$

they can be solved by direct integration to give

$$\int f(y)\,dy = \int g(t)\,dt\,.$$

Examples 4.27 1. Find the general solution of the following differential equations

$$\text{(i)}\quad t^2 \sin y\,\frac{dy}{dt} + 3 = 0\,, \qquad \text{(ii)}\quad (t+1)ty\,\frac{dy}{dt} + 1 = 0\,.$$

Solutions.

1. The differential equation

$$t^2 \sin y\,\frac{dy}{dt} + 3 = 0$$

can be written as

$$\sin y\,\frac{dy}{dt} = -\,\frac{3}{t^2}$$

giving

$$\int \sin y\,dy = -\int \frac{3}{t^2}\,dt + c$$

$$\text{i.e.}\quad -\cos y = \frac{3}{t} + c\,,$$

which can be written as

$$0 = 3 + ct + t\cos y\,.$$

This is the required general solution.

2. The differential equation

$$(t+1)ty\,\frac{dy}{dt}+1=0$$

can be written as

$$y\,\frac{dy}{dt}=-\frac{1}{t(t+1)}=\frac{1}{t+1}-\frac{1}{t}$$

giving

$$\int y\,dy=\int\left(\frac{1}{t+1}-\frac{1}{t}\right)\,dt+c$$

$$\text{i.e.}\quad \tfrac{1}{2}\,y^2=\ln|t+1|-\ln|t|+c\,,$$

and so

$$y^2=2\ln\left|\frac{t+1}{t}\right|+2c=\ln\left[\left(\frac{t+1}{t}\right)^2\right]+2c\,.$$

Hence

$$y=\pm\sqrt{\ln\left[\left(\frac{t+1}{t}\right)^2\right]+2c}\,.$$

This is the required general solution.

2. First Order Linear Differential Equations

An equation of the form

$$h(t)\,\frac{dy}{dt}+f(t)\,y=g(t)\,,$$

where $h(t), f(t), g(t)$ are functions of t only is called a **first order linear differential equation.**

If we divide through by $h(t)$ we can write it in **Standard Form** as

$$\frac{dy}{dt}+p(t)\,y=q(t)\,, \tag{39}$$

where

$$p(t)=\frac{f(t)}{h(t)}\,,\quad q(t)=\frac{g(t)}{h(t)}\,.$$

Multiply (39) by $I(t)$ to obtain

$$I(t)\,\frac{dy}{dt}+p(t)\,I(t)\,y=I(t)\,q(t) \tag{40}$$

and note that

$$\frac{d}{dt}\,[I(t)y]=I(t)\,\frac{dy}{dt}+\frac{dI}{dt}\,y\,. \tag{41}$$

Now the LHS of equation (40) will be $\frac{d}{dt}\,[I(t)\,y]$ if we choose $I(t)$ so that

$$\frac{dI}{dt}=pI\,. \tag{42}$$

So, with this choice of the function $I(t)$, equation (40) becomes

$$\frac{d}{dt}\,[I(t)\,y] = I(t)\,q(t)$$

giving

$$I(t)\,y = \int I(t)\,q(t)\,dt + c$$

and the differential equation is solved.

So all we now need is a formula for $I(t)$. From (42) we have

$$\frac{1}{I(t)}\,\frac{dI}{dt} = p(t)$$

giving

$$\int \frac{1}{I}\,dI = \int p(t)\,dt$$

and so

$$\ln[I(t)] = \int p(t)\,dt$$

giving

$$I(t) = e^{\int p(t)\,dt}\,.$$

Summary of Method for solving First Order Linear Differential Equations

Step 1

Write the differential equation in **Standard Form** as

$$\frac{dy}{dt} + p(t)\,y = q(t)\,. \qquad (*)$$

Step 2

Work out

$$I(t) = e^{\int p(t)\,dt}\,,$$

$I(t)$ **is called the Integrating Factor**, which is commonly abbreviated to **IF**.

Note It may help to simplify the Integrating Factor, in a particular example, if you remember that $y = \ln t$ implies that $t = e^y$ and so $e^{\ln t} = t$. Thus for example,

$$e^{3\ln t} = e^{\ln(t^3)} = t^3, \qquad e^{-\ln t} = e^{\ln(\frac{1}{t})} = \frac{1}{t}\,.$$

Step 3

Check that you have not made a mistake in calculating $I(t)$. If you are correct

$$\frac{d}{dt}\,(I(t)\,y)$$

will be the same as

$$I(t)\,\frac{dy}{dt} + I(t)\,p(t)\,y\,.$$

Step 4

Multiply (*) by $I(t)$ and write the differential equation as

$$\frac{d}{dt}\left(I(t)\, y\right) = I(t)\, q(t)\,.$$

Integrate to give

$$I(t)\, y = \int I(t)\, q(t)\, dt + c\,.$$

Step 5

If a solution is required which satisfies some given initial condition, substitute this in general solution to find the required value of c.

Examples 4.28 1. Find the general solution of the differential equation

$$\frac{dy}{dt} + 2ty + 4e^{-t^2} = 0\,.$$

2. Find the solution of the differential equation

$$t\frac{dy}{dt} - y = 2$$

such that $y = -5$ when $t = 1$.

Solutions.

1. Step 1. The equation is already in the standard form

$$\frac{dy}{dt} + 2t\, y = -4e^{-t^2}\,. \qquad (*)$$

Step 2. The integrating factor

$$I(t) \;=\; e^{\int 2t\, dt} \;=\; e^{t^2}\,.$$

Step 3. Multiply $(*)$ by $I(t)$ to obtain

$$e^{t^2}\frac{dy}{dt} + 2te^{t^2}\, y = -4e^{-t^2}\, e^{t^2} = -4\,. \qquad (1)$$

Now, in this case

$$\frac{d}{dt}\left(I(t)\, y\right) = \frac{d}{dt}\left(e^{t^2}\, y\right) = e^{t^2}\frac{dy}{dt} + 2te^{t^2}\, y$$

and this is the same as the L.H.S. of (1). Hence I.F. is correct.

Step 4. Thus (1) can be written as

$$\frac{d}{dt}\left(e^{t^2}\, y\right) = -4\,.$$

Integration gives

$$e^{t^2}\, y = -4t + c$$

and the required solution is

$$y = -4te^{-t^2} + ce^{-t^2}\,.$$

2. Step 1. In standard form the equation is

$$\frac{dy}{dt} - \frac{1}{t}\,y = \frac{2}{t}\,. \tag{$**$}$$

Step 2. The integrating factor

$$I(t) \;=\; e^{-\int \frac{1}{t}\,dt} \;=\; e^{-\ln t} = e^{\ln\left(\frac{1}{t}\right)} = \frac{1}{t}\,.$$

Step 3. Multiply $(**)$ by $I(t)$ to obtain

$$\frac{1}{t}\,\frac{dy}{dt} - \frac{1}{t^2}\,y = \frac{2}{t^2}\,. \tag{1}$$

Now, in this case

$$\frac{d}{dt}\,(\,I(t)\,y) = \frac{d}{dt}\left(\frac{1}{t}\,y\right) = \frac{1}{t}\,\frac{dy}{dt} - \frac{1}{t^2}\,y$$

and this is the same as the L.H.S. of (1). Hence I.F. is correct.

Step 4. Thus (1) can be written as

$$\frac{d}{dt}\left(\frac{1}{t}\,y\right) = \frac{2}{t^2}\,.$$

Integration gives

$$\frac{1}{t}\,y = -\,\frac{2}{t}\,+c$$

giving

$$y = -2 + c\,t\,. \tag{2}$$

Step 5. We need the solution such that $y = -5$ when $t = 1$. Substituting in (2) gives

$$-5 = -2 + c \quad \text{i.e.} \quad c = -3\,.$$

Thus the required solution is

$$y \;=\; -3t - 2\,.$$

This completes the revision material and we return to the use of Laplace Transforms.

As we remarked earlier, Laplace transforms can be very helpful in the solution of ordinary differential equations subject to given initial conditions. The simplest case is that of the differential equation

$$a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1}y'(t) + a_n y(t) = \phi(t) \qquad (t \geq 0) \tag{43}$$

subject to the conditions

$$y(0) = y_0, \quad y'(0) = y_1, \quad \ldots, \quad y^{(n-1)}(0) = y_{n-1}.$$

If ϕ is continuous, there is known to be a unique solution y, and if there is a constant K such that $|\phi(t)| < e^{Kt}$ for large t, then there is a constant M such that $|y^{(k)}(t)| < e^{Mt}$ for large t $(k = 0, 1, \ldots, n)$. Under these circumstances, $L(\phi) = \Phi(s)$ and $L(y) = Y(s)$ exist for sufficiently large s; moreover, by the corollary after Theorem 4.20, for $m = 1, 2, \ldots, n$,

$$L(y^{(m)}) = s^m Y(s) - \sum_{k=1}^{m} y^{(k-1)}(0) s^{m-k} = s^m Y(s) - \sum_{k=1}^{m} y_{k-1} s^{m-k}.$$

Hence, taking the Laplace transform of both sides of (43), we have

$$a_0 \left(s^n Y(s) - \sum_{k=1}^{n} y_{k-1} s^{n-k} \right) + a_1 \left(s^{n-1} Y(s) - \sum_{k=1}^{n-1} y_{k-1} s^{n--1-k} \right) + \cdots + a_n Y(s) = \Phi(s),$$

i.e.,

$$P(s)Y(s) + Q(s) = \Phi(s),$$

where

$$P(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n$$

and Q is a polynomial of degree at most $n - 1$. Hence

$$Y(s) = \frac{\Phi(s) - Q(s)}{P(s)}$$

and the problem reduces to the evaluation of $L^{-1}(Y(s))$.

Examples 4.29 1. Solve the differential equation

$$y'' - 2y' + 2y = te^t,$$

subject to the conditions $y(0) = 0$, $y'(0) = 2$.

2. Solve the differential equation

$$y''' - y'' + 4y' - 4y = 3$$

subject to the conditions $y(0) = 1$, $y'(0) = 1$, $y''(0) = 0$.

Solutions.

1. We need to solve the differential equation

$$y'' - 2y' + 2y = te^t,$$

subject to the conditions $y(0) = 0$, $y'(0) = 2$.

If $L(y) = Y(s)$, then

$$\begin{aligned} L(y') &= sY(s) - y(0) = sY(s) \\ L(y'') &= s^2Y(s) - (y(0)s + y'(0)) = s^2Y(s) - 2. \end{aligned}$$

Also,

$$L(te^t) = \frac{1}{(s-1)^2}.$$

Hence, taking Laplace transforms in the differential equation gives,

$$(s^2Y(s) - 2) - 2sY(s) + 2Y(s) = \frac{1}{(s-1)^2},$$

i.e.,

$$(s^2 - 2s + 2)Y(s) = \frac{1}{(s-1)^2} + 2 = \frac{2s^2 - 4s + 3}{(s-1)^2}.$$

Thus

$$\begin{aligned} Y(s) &= \frac{2s^2 - 4s + 3}{(s-1)^2(s^2 - 2s + 2)} \\ &= \frac{(s^2 - 2s + 2) + (s^2 - 2s + 1)}{(s-1)^2(s^2 - 2s + 2)} \\ &= \frac{(s^2 - 2s + 2) + (s-1)^2}{(s-1)^2(s^2 - 2s + 2)} \\ &= \frac{1}{(s-1)^2} + \frac{1}{s^2 - 2s + 2} \\ &= \frac{1}{(s-1)^2} + \frac{1}{(s-1)^2 + 1}, \end{aligned}$$

so that

$$y(t) = te^t + e^t \sin t = (t + \sin t)e^t.$$

2. We want to solve the differential equation

$$y''' - y'' + 4y' - 4y = 3$$

subject to the conditions $y(0) = 1$, $y'(0) = 1$, $y''(0) = 0$.

If $L(y) = Y(s)$, then

$$\begin{aligned} L(y') &= sY(s) - y(0) = sY(s) - 1, \\ L(y'') &= s^2Y(s) - (y(0)s + y'(0)) = s^2Y(s) - s - 1, \\ L(y''') &= s^3Y(s) - (y(0)s^2 + y'(0)s + y''(0)) = s^3Y(s) - s^2 - s. \end{aligned}$$

Also,

$$L(3) = \frac{3}{s} \, .$$

Hence, taking Laplace transforms in the differential equation gives,

$$(s^3Y(s) - s^2 - s) - (s^2Y(s) - s - 1) + 4(sY(s) - 1) - 4Y(s) = \frac{3}{s} \, ,$$

i.e.,

$$(s^3 - s^2 + 4s - 4)Y(s) - s^2 - 3 = \frac{3}{s} \, ,$$

i.e.,

$$(s^3 - s^2 + 4s - 4)Y(s) = s^2 + 3 + \frac{3}{s} = \frac{s^3 + 3s + 3}{s} \, .$$

Thus

$$Y(s) = \frac{s^3 + 3s + 3}{s(s^3 - s^2 + 4s - 4)} = \frac{s^3 + 3s + 3}{s(s-1)(s^2 + 4)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs + D}{s^2 + 4} \, .$$

Now, we need constants A, B, C, D such that

$$A(s-1)(s^2 + 4) + Bs(s^2 + 4) + (Cs + D)s(s-1) = s^3 + 3s + 3$$

for all s.

Put $s = 0$: then $-4A = 3$, i.e., $A = -\frac{3}{4}$.

Put $s = 1$: then $5B = 7$, i.e., $B = \frac{7}{5}$.

Compare coefficients of s^3. Then $A + B + C = 1$, i.e., $C = 1 + \frac{3}{4} - \frac{7}{5} = \frac{7}{20}$.

Compare coefficients of s^2. Then $-A - C + D = 0$, i.e., $D = -\frac{3}{4} + \frac{7}{20} = -\frac{2}{5}$.

So

$$Y(s) = \frac{-\frac{3}{4}}{s} + \frac{\frac{7}{5}}{s-1} + \frac{\frac{7}{20}s - \frac{2}{5}}{s^2 + 4}$$

and therefore

$$y(t) = -\frac{3}{4} + \frac{7}{5}\, e^t + \frac{7}{20}\, \cos 2t - \frac{1}{5}\, \sin 2t.$$

The Laplace transform can also be used to solve a differential equation

$$p_0(t)y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \cdots + p_{n-1}(t)y'(t) + p_n(t)y(t) = \phi(t) \qquad (t \geq 0)$$

where $p_0, p_1, \ldots, p_n$ are polynomials. The method depends on the identity

$$L(tf(t)) = -F'(s)$$

of Theorem 4.15.

Examples 4.30 1. Solve the differential equation

$$ty'' + 2y' + ty = -t$$

subject to the conditions $y(0) = 1$, $y'(0) = 0$. (Actually the second condition must be satisfied by every solution of the differential equation.)

2. Solve the differential equation

$$ty'' + (1+2t)y' + 2y = 4(1+t)e^t$$

subject to the conditions $y(0) = 1$, $y'(0) = 2$. (For all solutions of the differential equation, $y'(0) + 2y(0) = 4$.)

Solutions.

1. We need to solve the differential equation

$$ty'' + 2y' + ty = -t$$

subject to the conditions $y(0) = 1$, $y'(0) = 0$.

If $L(y(t)) = Y(s)$, then

$$\begin{aligned}
L(ty(t)) &= -Y'(s), \\
L(y'(t)) &= sY(s) - y(0) = sY(s) - 1, \\
L(y''(t)) &= s^2Y(s) - (y(0)s + y'(0)) = s^2Y(s) - s, \\
L(ty''(t)) &= -\frac{d}{ds}\left(s^2Y(s) - s\right) = -s^2Y'(s) - 2sY(s) + 1.
\end{aligned}$$

Also,

$$L(-t) = -\frac{1}{s^2}\,.$$

Hence, taking Laplace transforms in the differential equation gives,

$$(-s^2Y'(s) - 2sY(s) + 1) + 2(sY(s) - 1) - Y'(s) = -\frac{1}{s^2}$$

i.e.,

$$-(s^2+1)Y'(s) - 1 = -\frac{1}{s^2}$$

so that

$$Y'(s) = -\frac{1}{s^2+1} + \frac{1}{s^2(s^2+1)} = -\frac{1}{s^2+1} + \left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) = -\frac{2}{s^2+1} + \frac{1}{s^2}\,.$$

Thus

$$Y(s) = K - 2\tan^{-1} s - \frac{1}{s}$$

and, since $Y(s) \to 0$ as $s \to \infty$ (by Theorem 4.8), $K = \pi$. Therefore

$$Y(s) = 2\left(\frac{\pi}{2} - \tan^{-1} s\right) - \frac{1}{s}$$

and

$$y(t) = \frac{2\sin t}{t} - 1 \qquad (t > 0)$$

with $y(0)$ defined as 1.

2. We need to solve the differential equation

$$ty'' + (1+2t)y' + 2y = 4(1+t)e^t$$

subject to the conditions $y(0) = 1$, $y'(0) = 2$.

If $L(y(t)) = Y(s)$, then

$$\begin{aligned} L(y'(t)) &= sY(s) - y(0) = sY(s) - 1, \\ L(ty'(t)) &= -\frac{d}{ds}(sY(s) - 1) = -sY'(s) - Y(s), \\ L(y''(t)) &= s^2Y(s) - (y(0)s + y'(0)) = s^2Y(s) - s - 2, \\ L(ty''(t)) &= -\frac{d}{ds}\left(s^2Y(s) - s - 2\right) = -s^2Y'(s) - 2sY(s) + 1. \end{aligned}$$

Also,

$$L(4(e^t + te^t)) = 4\left(\frac{1}{s-1} + \frac{1}{(s-1)^2}\right) = \frac{4s}{(s-1)^2}.$$

Hence, taking Laplace transforms in the differential equation gives,

$$(-s^2Y'(s) - 2sY(s) + 1) + (sY(s) - 1) + 2(-sY'(s) - Y(s)) + 2Y(s) = \frac{4s}{(s-1)^2} \ ,$$

i.e.,

$$-s^2Y'(s) - 2sY'(s) - sY(s) = \frac{4s}{(s-1)^2},$$

i.e.,

$$s(s+2)Y'(s) + sY(s) = \frac{-4s}{(s-1)^2},$$

i.e.,

$$(s+2)Y'(s) + Y(s) = \frac{-4}{(s-1)^2}.$$

Thus

$$\frac{d}{ds}\left((s+2)Y(s)\right) = \frac{-4}{(s-1)^2}$$

i.e.,

$$(s+2)Y(s) = \frac{4}{s-1} + k$$

and so,

$$\begin{aligned} Y(s) &= \frac{4}{(s-1)(s+2)} + \frac{k}{s+2} \\ &= \frac{4}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right) + \frac{k}{s+2} \\ &= \frac{4}{3(s-1)} + \frac{K}{s+2}; \end{aligned}$$

and

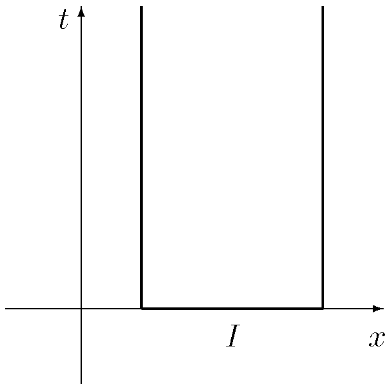
$$y(t) = \frac{4e^t}{3} + Ke^{-2t}.$$

Since $y(0) = 1$, $K = -\frac{1}{3}$, and

$$y(t) = \frac{1}{3}\left(4e^t - e^{-2t}\right).$$

Solution of partial differential equations

In the equations that we shall consider, a function u of two variables x, t is to be found which satisfies, firstly, a given relation involving partial derivatives of u in some infinite rectangle $R = \{(x,t) \in \mathbb{R}^2 : x \in I, t \geq 0\}$, where I is a finite or infinite closed interval, and, secondly, certain additional conditions, usually on the behaviour of u in R or on the boundary of R.



We denote the Laplace transform

$$\int_0^\infty e^{-st} u(x,t)\, dt$$

by $U(x,s)$; our convention is the Laplace transforms are always taken with respect to the second variable t.

It follows from Theorem 4.20 that

$$\begin{aligned} L(u_t(x,t)) &= sU(x,s) - u(x,0), \\ L(u_{tt}(x,t)) &= s^2 U(x,s) - (su(x,0) + u_t(x,0)) \text{ etc.} \end{aligned}$$

Also, assuming that differentiation under the integral sign is permitted, we have

$$L(u_x(x,t)) = \int_0^\infty e^{-st} u_x(x,t)\, dt = \frac{\partial}{\partial x} \int_0^\infty e^{-st} u(x,t)\, dt = U_x(x,s)$$

and

$$L(u_{xx}(x,t)) = U_{xx}(x,s) \text{ etc..}$$

Examples 4.31 1. Solve the partial differential equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \qquad (x \geq 0, t \geq 0)$$

subject to the conditions that

$$u(x,0) = e^{-3x} \text{ for } x \geq 0$$

and $u(x,t)$ is bounded for $x \geq 0$, $t \geq 0$.

2. Solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad (x \geq 0, t \geq 0)$$

subject to the conditions that

$$\begin{aligned} u(x,0) &= 0 \qquad (x \geq 0), \\ u(0,t) &= 1 \qquad (t > 0), \end{aligned}$$

and $u(x,t)$ is bounded for $x \geq 0$, $t \geq 0$.

3. Find the solution of the partial differential equation

$$x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + 3u = 10xe^{-t}\cos t \qquad (x \geq 0, t \geq 0)$$

subject to the conditions that

$$\begin{aligned} u(x,0) &= 0 \qquad (x \geq 0), \\ u(0,t) &= 0 \qquad (t > 0), \end{aligned}$$

and $u(x,t)$ is continuous for $x \geq 0$, $t \geq 0$.

Solutions.

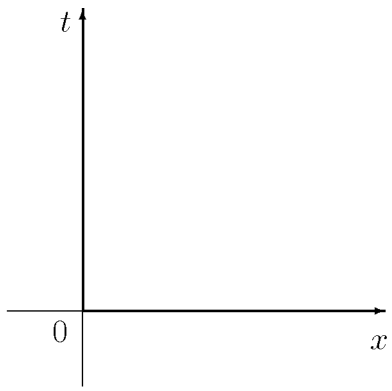
1. We need to solve the partial differential equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \qquad (x \geq 0, t \geq 0) \tag{34}$$

subject to the conditions that

$$u(x,0) = e^{-3x} \text{ for } x \geq 0$$

and $u(x,t)$ is bounded for $x \geq 0$, $t \geq 0$.



The condition that u is bounded ensures that $L(u(x,t)) = U(x,s)$ exists for every $x \geq 0$. We now take the Laplace transform of both sides of (34) noting that

$$L\left(\frac{\partial u}{\partial x}\right) = \frac{\partial U}{\partial x}$$

and

$$L\left(\frac{\partial u}{\partial t}\right) = sU(x,s) - u(x,0) = sU(x,s) - e^{-3x}.$$

Hence

$$\frac{\partial U}{\partial x} = 2(sU - e^{-3x}) + U,$$

i.e.,

$$\frac{\partial U}{\partial x} - (2s+1)U = -2e^{-3x}.$$

For fixed s, this is an ordinary first order differential equation with integrating factor

$$e^{\int -(2s+1)dx} \;=\; e^{-(2s+1)x}.$$

Thus

$$e^{-(2s+1)x}\,\frac{\partial U}{\partial x} - (2s+1)e^{-(2s+1)x}U = -2e^{-(2s+4)x},$$

i.e.,

$$\frac{\partial}{\partial x}(e^{-(2s+1)x}\;U) = -2e^{-(2s+4)x},$$

so that

$$e^{-(2s+1)x}\,U = \frac{e^{-(2s+4)x}}{s+2} + K(s),$$

i.e.,

$$U = \frac{e^{-3x}}{s+2} + K(s)\,e^{(2s+1)x}. \tag{35}$$

Here $K(s)$ is independent of x, but may depend on s. However, we show that, **in this particular example**, $K(s)=0$ for all $s>0$.

If $|u(x,t)| \le M$, say, then for any given $s>0$,

$$|U(x,s)| \;=\; \left|\int_0^\infty e^{-st}u(x,t)\,dt\right| \;\le\; \int_0^\infty e^{-st}M\,dt \;=\; \frac{M}{s} \qquad (x \ge 0). \tag{36}$$

But, if there is an $s>0$ such that $K(s) \ne 0$, then for this particular value of s,

$$|U(x,s)| \to \infty \;\text{ as }\; x \to \infty\,,$$

by (35). This would contradict (36), and so $K(s)=0$ for $s>0$.

Thus

$$U(x,s) = \frac{e^{-3x}}{s+2} \qquad (s>0)$$

and

$$u(x,t) = e^{-3x}e^{-2t} = e^{-(3x+2t)}.$$

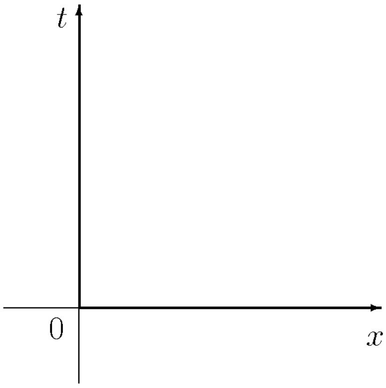
2. We need to solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad (x \ge 0, t \ge 0)$$

subject to the conditions that

$$\begin{aligned} u(x,0) &= 0 \qquad (x \geq 0), \\ u(0,t) &= 1 \qquad (t > 0), \end{aligned}$$

and $u(x,t)$ is bounded for $x \geq 0$, $t \geq 0$.



(The differential equation is an instance of the general *heat equation* $\frac{\partial^2 u}{\partial x^2} = k\frac{\partial u}{\partial t}$ in which $u(x,t)$ is the temperature in a uniform rod at a distance x from a fixed point and at time t. In this case, the rod is semi-infinite, it is initially at temperature 0 and at time $t = 0$ a source of heat is applied to one end so as to keep that end at temperature 1.)

If $$L(u(x,t)) = U(x,s)\,,$$

then
$$L\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial^2 U}{\partial x^2}$$

and
$$L\left(\frac{\partial u}{\partial t}\right) = sU(x,s) - u(x,0) = sU(x,s).$$

Hence
$$\frac{\partial^2 U}{\partial x^2} = sU,$$

$$\text{i.e.} \quad \frac{\partial^2 U}{\partial x^2} - sU = 0$$

and so, when $s > 0$,
$$U(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x}, \tag{37}$$

where $A(s)$, $B(s)$ are independent of x, but may depend on s.

If $|u(x,t)| \leq M$, then, for any $s > 0$,

$$|U(x,s)| = \left|\int_0^\infty e^{-st}u(x,t)\,dt\right| \leq \int_0^\infty e^{-st}M\,dt = \frac{M}{s} \qquad (x \geq 0). \tag{38}$$

But, if there exists $s > 0$ such that $A(s) \neq 0$, then, by (37), $|U(x,s)| \to \infty$ as $x \to \infty$ for this particular value of s. This would contradict (38) and so $A(s) = 0$ for $s > 0$.

Moreover, as $u(0,t) = 1$ for $t > 0$, and so

$$U(0,s) = \int_0^\infty e^{-st}\, u(0,t)\, dt = \int_0^\infty e^{-st}\, u(0,t)\, dt = \frac{1}{s}$$

for $s > 0$ and so, from (37) with $x = 0$, $B(s) = \dfrac{1}{s}$ for $s > 0$. Thus

$$U(x,s) = \frac{1}{s}\, e^{-\sqrt{s}x} \qquad (s > 0)$$

and

$$u(x,t) = 1 - \operatorname{erf} \frac{x}{2\sqrt{t}} = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{t}}}^\infty e^{-y^2}\, dy,$$

with $u(x,0)$ defined as $\lim\limits_{t \to 0+} u(x,t) = 0$.

3. We need to find the solution of the partial differential equation

$$x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + 3u = 10xe^{-t} \cos t \qquad (x \geq 0, t \geq 0)$$

subject to the conditions that

$$\begin{aligned} u(x,0) &= 0 \qquad (x \geq 0), \\ u(0,t) &= 0 \qquad (t > 0), \end{aligned}$$

and $u(x,t)$ is continuous for $x \geq 0$, $t \geq 0$.

Write

$$U(x,s) \quad = \quad \int_0^\infty e^{-st}\, u(x,t)\, dt\ . \tag{39}$$

Then

$$L\left(\frac{\partial u}{\partial t}\right) \quad = \quad sU(x,s) - u(x,0) \quad = \quad sU(x,s)$$

and assuming that differentiation under the integral sign is justified

$$L\left(\frac{\partial u}{\partial x}\right) \quad = \quad \int_0^\infty e^{-st}\, \frac{\partial}{\partial x}(u(x,t))\, dt \quad = \quad \frac{\partial}{\partial x} \int_0^\infty e^{-st}\, u(x,t)\, dt \quad = \quad \frac{\partial U}{\partial x}\ .$$

Taking Laplace transforms in the differential equation gives

$$x\frac{\partial U}{\partial x} + sU + 3U \quad = \quad L(10xe^{-t} \cos t) \quad = \quad \frac{10x(s+1)}{(s+1)^2 + 1} \quad = \quad \frac{10x(s+1)}{s^2 + 2s + 2}$$

giving

$$x\frac{\partial U}{\partial x} + (s+3)U \quad = \quad \frac{10x(s+1)}{s^2 + 2s + 2}\ .$$

For any fixed $s > 0$, this can be considered as a first order linear differential equation for U in terms of x. In standard form, it is

$$\frac{\partial U}{\partial x} + \frac{s+3}{x} U = \frac{10(s+1)}{s^2+2s+2} \tag{40}$$

and the integrating factor

$$I(x) = e^{\int \frac{s+3}{x}\,dx} = e^{(s+3)\ln x} = e^{\ln(x^{s+3})} = x^{s+3}\,. \tag{41}$$

Multiply (40) by $I(x)$ to obtain

$$x^{s+3}\frac{\partial U}{\partial x} + (s+3)\,x^{s+2}\,U = \frac{10x^{s+3}(s+1)}{s^2+2s+2} \tag{42}$$

which can be written as

$$\frac{\partial}{\partial x}\left(x^{s+3}\,U\right) = \frac{10(s+1)}{s^2+2s+2}\,x^{s+3}\,. \tag{43}$$

Thus the LHS of (42) is

$$\frac{\partial}{\partial x}\left(I(x)U\right)$$

and so the I.F. is correct.
Integrate (43) to give

$$x^{s+3}\,U(x,s) = \frac{10(s+1)}{(s^2+2s+2)(s+4)}\,x^{s+4} + k(s)\,, \tag{44}$$

where $k(s)$ is a function of s, but is independent of x.

Now, from (39), with $x=0$, we see that

$$U(0,s) = \int_0^\infty e^{-st}\,u(0,t)\,dt = 0$$

since $u(0,t) = 0 \quad (t \geq 0)$. Hence, putting $x = 0$ in (44), we see that $k(s) = 0$ for all $s > 0$. This gives

$$x^{s+3}\,U(x,s) = \frac{10(s+1)}{(s^2+2s+2)(s+4)}\,x^{s+4}$$

$$\text{i.e.}\quad U(x,s) = \frac{10(s+1)}{(s^2+2s+2)(s+4)}\,x\,.$$

Now

$$\frac{10(s+1)}{(s^2+2s+2)(s+4)} = \frac{Cs+D}{(s^2+2s+2)} + \frac{A}{(s+4)}\,,$$

where A, C, D are chosen so that

$$(Cs+D)(s+4) + A(s^2+2s+2) = 10(s+1)$$

for all s . Putting $s = -4$ gives

$$A(16-8+2) = 10(-3) \qquad \text{i.e.}\quad A = -3\,.$$

Equating coefficients of s^2 gives

$$C + A = 0 \qquad \text{i.e.} \quad C \;=\; 3\,.$$

Equating constants gives

$$4D + 2A = 10. \qquad \text{i.e.} \quad 4D = 10 - 2A = 16 \quad \text{and so} \quad D = 4\,.$$

Thus

$$U(x,s) \;=\; -\frac{3x}{s+4} + \frac{3s+4}{s^2+2s+2}\,x \;=\; -\frac{3x}{s+4} + \frac{3(s+1)}{(s+1)^2+1}\,x + \frac{1}{(s+1)^2+1}\,x$$

and

$$u(x,t) \;=\; L^{-1}U(x,s) \;=\; -\,3xe^{-4t} + 3xe^{-t}\cos t + xe^{-t}\sin t\ .$$

Appendix 1. The Uniqueness Theorem

Uniqueness of the determining function

Given a continuous function f on $[0, \infty)$, the Laplace transform $F = L(f)$, if it exists, is clearly uniquely determined. It is also true that a given function F cannot be the Laplace transform of more than one continuous function. This fact is important in applications to, for instance, the solution of differential equations. The proof depends on a famous theorem of Weierstrass on the approximation of continuous functions by polynomials. This theorem is most simply stated in the notation of metric spaces. We recall that the space $C[a, b]$ of continuous real functions on the interval $[a, b]$, the *usual metric* d is defined by

$$d(f, g) = \max_{a \le x \le b} |f(x) - g(x)|.$$

Theorem 4.32 (Weierstrass Approximation Theorem) *If the function $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, then there is a sequence (P_n) of polynomials such that*

$$d(f, P_n) = \max_{a \le x \le b} |f(x) - P_n(x)| \to 0 \ \textit{as}\ n \to \infty. \tag{45}$$

To prove the theorem, we need a lemma, but first we note that we may take $[a, b]$ to be $[0, 1]$. For suppose that the theorem has been proved for $[0, 1]$, and let f be a continuous function on $[a, b]$. Putting

$$t = \frac{x - a}{b - a} \text{ or } x = a + (b - a)t,$$

so that $0 \le t \le 1$ when $a \le x \le b$, we see that $f(a + (b - a)t)$ is continuous on $[0, 1]$. Hence there exists a sequence (Q_n) of polynomials such that

$$\max_{0 \le t \le 1} |f(a + (b - a)t) - Q_n(t)| \to 0 \text{ as } n \to \infty.$$

If $Q_n(\frac{x-a}{b-a}) = P_n(x)$, then P_n is a polynomial and

$$\max_{a \le x \le b} |f(x) - P_n(x)| \to 0 \text{ as } n \to \infty.$$

Lemma 4.33 *For any positive integer n and for $0 \le r \le n$, let*

$$p_{n,r}(x) = \binom{n}{r} (1 - x)^{n-r} x^r.$$

Then

$$\sum_{r=0}^{n} p_{n,r}(x) = 1 \tag{46}$$

and

$$\sum_{r=0}^{n} p_{n,r}(x) \left(x - \frac{a}{n}\right)^2 = \frac{x(1 - x)}{n}. \tag{47}$$

Proof. By the binomial theorem,

$$\sum_{r=0}^{n} \binom{n}{r} (1-x)^{n-r} y^r = (1-x+y)^n;$$

and, differentiating with respect to y, we have

$$\sum_{r=0}^{n} r \binom{n}{r} (1-x)^{n-r} y^{r-1} = n(1-x+y)^{n-1}$$

and

$$\sum_{r=0}^{n} r(r-1) \binom{n}{r} (1-x)^{n-r} y^{r-2} = n(n-1)(1-x+y)^{n-2}.$$

We now put $y = x$, so that

$$\sum_{r=0}^{n} \binom{n}{r} (1-x)^{n-r} x^r = 1$$

which is (46), and

$$\sum_{r=0}^{n} r \binom{n}{r} (1-x)^{n-r} x^{r-1} = n,$$

i.e.,

$$\sum_{r=0}^{n} r p_{n,r}(x) = nx \tag{48}$$

and also

$$\sum_{r=0}^{n} r(r-1) \binom{n}{r} (1-x)^{n-r} x^{r-2} = n(n-1),$$

i.e.,

$$\sum_{r=0}^{n} r(r-1) p_{n,r}(x) = n(n-1)x^2. \tag{49}$$

Next, we multiply (46), (48) and (49) by n^2x^2, $1-2nx$ and 1 respectively, and add. Since

$$n^2x^2 + (1-2nx)r + r(r-1) = (nx-r)^2$$

and

$$n^2x^2 + (1-2nx)nx + n(n-1) = nx(1-x),$$

we get

$$\sum_{r=0}^{n} p_{n,r}(x)(nx-r)^2 = nx(1-x)$$

which is equivalent to (47). $\square$

Definition 4.34 Given the function $f : [0,1] \longrightarrow \mathbb{R}$, the polynomial

$$B_n(f;x) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right) p_{n,r}(x)$$

is called the *nth Bernstein polynomial* associated with f.

Proof of Theorem 4.32 with $[a, b] = [0, 1]$. We show that (45) holds with $P_n(x) = B_n(f; x)$ by proving that, given $\epsilon > 0$, there exists n_0 such that, when $n > n_0$,

$$|f(x) - B_n(f; x)| \leq \epsilon \text{ for } 0 \leq x \leq 1. \tag{50}$$

Since f is continuous on $[0, 1]$, f is bounded, i.e., there exists $M > 0$ such that

$$|f(x)| \leq M \text{ for } 0 \leq x \leq 1.$$

Also, there exists $\delta > 0$ such that

$$|f(x) - f(u)| < \frac{\epsilon}{2} \text{ whenever } x, u \in [0, 1] \text{ and } |x - u| \leq \delta. \tag{51}$$

(51) describes the property of *uniform continuity* possessed by any continuous function on a bounded, closed interval.

In contrast to (51), if x, $u \in [0, 1]$ and $|x - u| > \delta$, then

$$|f(x) - f(u)| \leq |f(x)| + |f(u)| \leq 2M < \frac{(x-u)^2}{\delta^2} 2M.$$

Hence, for all $x, u \in [0, 1]$,

$$|f(x) - f(u)| < \frac{\epsilon}{2} + \frac{2M(x-u)^2}{\delta^2}. \tag{52}$$

It now follows from (46), (52) and (47) that, for $0 \leq x \leq 1$,

$$\begin{aligned}
|f(x) - B_n(f; x)| &= \left| \sum_{r=0}^{n} f(x) p_{n,r}(x) - \sum_{r=0}^{n} f\left(\frac{r}{n}\right) p_{n,r}(x) \right| \\
&\leq \sum_{r=0}^{n} p_{n,r}(x) \left| f(x) - f\left(\frac{r}{n}\right) \right| \\
&\leq \sum_{r=0}^{n} p_{n,r}(x) \left(\frac{\epsilon}{2} + \frac{2M(x-u)^2}{\delta^2} \right) \\
&= \frac{\epsilon}{2} \sum_{r=0}^{n} p_{n,r}(x) + \frac{2M}{\delta^2} \sum_{r=0}^{n} p_{n,r}(x) \left(x - \frac{r}{n} \right)^2 \\
&= \frac{\epsilon}{2} + \frac{2M}{\delta^2} \frac{x(1-x)}{n} \\
&\leq \frac{\epsilon}{2} + \frac{M}{2n\delta^2}.
\end{aligned}$$

But, for $n > n_0$, say, $\dfrac{M}{2n\delta^2} < \dfrac{\epsilon}{2}$, and so (50) holds. $\square$

The proof of the uniqueness theorem requires a deduction from Theorem 3.9 which is of independent interest.

Lemma 4.35 *If the continuous function* $\psi : [a, b] \longrightarrow \mathbb{R}$ *is such that*

$$\int_a^b \psi(x) x^n \, dx = 0 \textit{ for } n = 0, 1, 2, \ldots$$

then $\psi(x) \equiv 0$ *in* $[a, b]$.

Proof. We first show that

$$\int_a^b \psi^2(x)\,dx = 0.$$

Let M be such that $|\psi(x)| \leq M$ for $a \leq x \leq b$. Also, by Theorem 4.32, given $\epsilon > 0$, there exists a polynomial P such that

$$|\psi(x) - P(x)| < \epsilon \text{ for } a \leq x \leq b;$$

and, by the hypothesis of the theorem,

$$\int_a^b \psi(x)P(x)\,dx = 0.$$

Hence

$$\begin{aligned}
\int_a^b \psi^2(x)\,dx &= \int_a^b \psi(x)\{\psi(x) - P(x)\}dx + \int_a^b \psi(x)P(x)\,dx \\
&= \int_a^b \psi(x)\{\psi(x) - P(x)\}dx \\
&\leq \int_a^b |\psi(x)|.|\psi(x) - P(x)|\,dx \\
&\leq \int_a^b M\epsilon\,dx \\
&= M\epsilon(b-a).
\end{aligned}$$

Since ϵ was arbitrary, it follows that $\displaystyle\int_a^b \psi^2(x)\,dx = 0$.

Now, if $a \leq x \leq b$, then

$$0 \leq \int_a^x \psi^2(t)\,dt \leq \int_a^b \psi^2(t)\,dt = 0.$$

Thus

$$\int_a^x \psi^2(t)\,dt = 0 \text{ for } a \leq x \leq b.$$

Hence

$$\psi^2(x) = \frac{d}{dx}\int_a^x \psi^2(t)\,dt = 0 \text{ for } a \leq x \leq b$$

and so

$$\psi(x) = 0 \text{ for } a \leq x \leq b.$$

□

Theorem 4.36 (Uniqueness Theorem) *If the function $f : [0,\infty) \longrightarrow \mathbb{R}$ is continuous, $F = L(f)$ exists on an interval (c,∞) and $F(s) = 0$ for $s > c$, then $f(t) = 0$ for $t \geq 0$.*

Proof. Let $s_0 > c$ and put

$$\phi(t) = \int_0^t e^{-s_0 u} f(u)\, du,$$

so that ϕ is continuous and, by (19),

$$0 = F(s) = (s - s_0) \int_0^\infty e^{-(s-s_0)t} \phi(t)\, dt \text{ for } s > s_0.$$

Taking $s = s_0 + n$ ($n = 1, 2, \ldots$), we therefore have

$$\int_0^\infty e^{-nt} \phi(t)\, dt = 0 \qquad (n = 1, 2, \ldots).$$

The change of variable $x = e^{-t}$ or $t = -\ln x$ now gives

$$\int_0^1 x^{n-1} \phi(-\ln x)\, dx = 0 \qquad (n = 1, 2, \ldots).$$

Since ϕ is continuous and $\phi(t) \to F(s_0) = 0$ as $t \to \infty$, the function $\psi : [0, 1] \longrightarrow \mathbb{R}$ given by

$$\psi(x) = \begin{cases} 0 & \text{if } x = 0, \\ \phi(-\ln x) & \text{if } 0 < x \leq 1, \end{cases}$$

is continuous on $[0, 1]$. Moreover,

$$\int_0^1 x^{n-1} \psi(x)\, dx = 0 \qquad (n = 1, 2, \ldots).$$

Hence, by Lemma 4.35, $\psi(x) = 0$ for $0 \leq x \leq 1$ and therefore $\psi(t) = 0$ for $t \geq 0$. Thus

$$e^{-s_0 t} f(t) = \phi'(t) = 0 \text{ for } t \geq 0,$$

i.e.,

$$f(t) = 0 \text{ for } t \geq 0.$$

□

Corollary 4.37 *If the functions f, $g : [0, \infty) \longrightarrow \mathbb{R}$ are continuous and if $F = L(f)$ and $G = L(g)$ exist and are equal on an interval (c, ∞), then $f(t) = g(t)$ for $t \geq 0$.*

The uniqueness theorem enables us to define the notion of an inverse Laplace transform. If the function $f : [0, \infty) \longrightarrow \mathbb{R}$ is continuous and $F = L(f)$ exists on some interval (c, ∞), then we can write

$$f = L^{-1}(F),$$

and f is called the *inverse Laplace transform* of F.

Appendix 2. The Error Function

We begin with a reminder of

Theorem 4.23. Suppose that the function $f : [0,\infty) \longrightarrow \mathbb{R}$ is continuous and that $F = L(f)$ exists on (c,∞). Put $d = \max\{c, 0\}$. Then $L\left(\int_0^t f(u)\,du\right)$ exists on the interval (d,∞), and

$$L\left(\int_0^t f(u)\,du\right) = \frac{F(s)}{s} \qquad (s > d). \qquad \square$$

We use this result in the special case $f(t) = \frac{e^{-t}}{\sqrt{t}}$ to prove a result which will be needed later when we consider the **error function**. Using

$$f(t) = \frac{e^{-t}}{\sqrt{t}},$$

we have

$$F(s) = L(f(t)) = \frac{\Gamma(\frac{1}{2})}{(s+1)^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s+1}} \qquad (s > -1).$$

Hence, by Theorem 4.23 above,

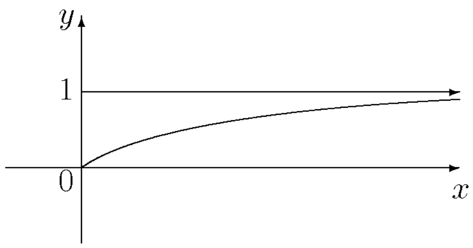
$$L\left(\int_0^t \frac{e^{-u}}{\sqrt{u}}\,du\right) = L\left(\int_0^t f(u)\,du\right) = \frac{F(s)}{s} = \frac{\sqrt{\pi}}{s\sqrt{s+1}} \qquad (s > 0). \qquad (1)$$

— — — — — — — — — — — — — — — — — — — —

The **error function** $\operatorname{erf} x$ is defined on $[0,\infty)$ by

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \qquad (x \geq 0),$$

so that $\operatorname{erf} 0 = 0$ and $\operatorname{erf} x \to 1$ as $x \to \infty$.



Using the substitution $u = t^2$ so that $t = \sqrt{u}$ and $\frac{dt}{du} = \frac{1}{2}\,u^{-1/2}$ we see that

$$\operatorname{erf} x = \frac{1}{\sqrt{\pi}}\int_0^{x^2} \frac{e^{-u}}{\sqrt{u}}\,du.$$

and

$$\operatorname{erf}\sqrt{t} = \frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{-u}}{\sqrt{u}}\,du.$$

Using relation (1) on the previous page, we have,

$$L(\operatorname{erf}\sqrt{t}) = L\left(\frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{-u}}{\sqrt{u}}du\right) = \frac{1}{s\sqrt{\pi}}\sqrt{\frac{\pi}{s+1}} = \frac{1}{s\sqrt{s+1}} \qquad (s > 0).$$

Also, more generally, if $a > 0$, then, by Theorem 4.9,

$$L(\operatorname{erf} a\sqrt{t}) = L(\operatorname{erf}\sqrt{a^2 t}) = \frac{1}{a^2}F\left(\frac{s}{a^2}\right) = \frac{1}{a^2}\frac{1}{\frac{s}{a^2}\sqrt{\frac{s}{a^2}+1}} = \frac{a}{s\sqrt{s+a^2}} \qquad (s > 0).$$

Since $\displaystyle\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt = 1$,

$$1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt \qquad (x \geq 0).$$

This function is also called the *complementary error function* and is often denoted by $\operatorname{erfc} x$.

Now, for $s > 0$,

$$L\left(1 - \operatorname{erf}\frac{1}{\sqrt{t}}\right) = \int_0^\infty e^{-st}\left(\frac{2}{\sqrt{\pi}}\int_{\frac{1}{\sqrt{t}}}^\infty e^{-u^2}\,du\right)dt = \frac{2}{\sqrt{\pi}}\int_0^\infty dt\int_{\frac{1}{\sqrt{t}}}^\infty e^{-u^2}e^{-st}\,du;$$

and, assuming that the order of integration may be reversed, we have

$$\begin{aligned}
L\left(1 - \operatorname{erf}\frac{1}{\sqrt{t}}\right) &= \frac{2}{\sqrt{\pi}}\int_0^\infty du\int_{\frac{1}{u^2}}^\infty e^{-u^2}e^{-st}\,dt \\
&= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}\left[-\frac{1}{s}e^{-st}\right]_{t=\frac{1}{u^2}}^\infty du \\
&= \frac{2}{s\sqrt{\pi}}\int_0^\infty e^{-u^2-\frac{s}{u^2}}\,du \\
&= \frac{2}{s\sqrt{\pi}}\int_0^\infty e^{-(u^2-2\sqrt{s}+\frac{s}{u^2})}e^{-2\sqrt{s}}\,du \\
&= \frac{e^{-2\sqrt{s}}}{s}\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-(u-\frac{\sqrt{s}}{u})^2}\,du \\
&= \frac{e^{-2\sqrt{s}}}{s},
\end{aligned}$$

by the corollary to Theorem 2.14.

More generally, if $a > 0$, then, by Theorem 4.9,

$$L\left(1 - \operatorname{erf}\frac{a}{\sqrt{t}}\right) = L\left(1 - \operatorname{erf}\frac{1}{\sqrt{\frac{t}{a^2}}}\right) = a^2F(a^2 s) = a^2\frac{e^{-2\sqrt{a^2 s}}}{a^2 s} = \frac{e^{-2a\sqrt{s}}}{s} \qquad (s > 0).$$

Appendix 3. Laplace Transforms and Complex Analysis

Those of you who took the Complex Analysis Course MAS332 may be interested to know that the Residue Theorem can be used to invert Laplace transforms. Thus providing us with a method to try when our table of Laplace Transforms is not adequate.

Theorem 4.38 ***(Laplace Inversion Formula)*** *If*

(i) the function F is analytic in some region containing the half plane

$$H = \{s \in \mathbb{C} : \mathrm{Re}\, s \geq a\} ,$$

(ii) and there exist positive constants m, R_0 and k such that

$$|F(s)| \leq \frac{m}{|s|^k}$$

whenever $|s| > R_0$ and $s \in H$.

Then there is a function $f(t)$ whose Laplace Transform is $F(s)$ and this function is given by

$$f(t) = L^{-1}(F(s)) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+\infty} F(s) e^{st}\, dt .$$

Note. The integration is performed along the line Re $s = a$.

The above equation is also called the Bromwich Integral and the Residue Theorem can be used to find it.

Theorem 4.39 *Suppose that*

(i) $F(s)$ is analytic in the s-plane except for a finite number of poles at $s_1, s_2, \cdots, s_n$ which all lie to the left of some vertical line $\mathrm{Re}\, s = a$.

(ii) and there exist positive constants m, R_0 and k such that

$$|F(s)| \leq \frac{m}{|s|^k}$$

for all s such that $|s| > R_0$ and $\mathrm{Re}\, s \leq a$.

Then

$$f(t) = L^{-1} F(s) = \sum_{m=1}^{n} \mathrm{Res}\{G; s_m\} ,$$

where

$$G(s) = F(s)\, e^{st}.$$