正定二次型

$$1.f(x_1,x_2)=3x_1^2+2x_2^2=X^TAX$$
$$\begin{cases}X^TAX\geq 0\\X^TAX=0\Leftrightarrow X=0\end{cases}\Leftrightarrow \forall x\neq 0,\text{有}X^TAX>0$$
$$2.f(x_1,x_2)=3x_1^2-2x_1x_2+x_2^2=X^TAX$$
$$=2x_1^2+(x_1-x_2)^2$$
$$\begin{cases}X^TAX\geq 0\\X^TAX=0\Leftrightarrow\begin{cases}x_1=0\\x_1-x_2=0\end{cases}\Leftrightarrow X=0\end{cases}\Leftrightarrow \forall x\neq 0,\text{有}X^TAX>0$$

$$f=X^TAX,\text{若}\forall X\neq 0,\text{有}X^TAX>0$$
$$\text{称}X^TAX\text{为正定二次型},A\text{为正定矩阵}$$

判别法

定义法

$$A,B\text{正定阵},\text{证}:A+B\text{正定阵}.$$
$$\text{证}:\forall X\neq 0,X^T(A+B)X=X^TAX+X^TBX$$
$$\because A,B\text{为正定阵},\therefore X^TAX>0,X^TBX>0$$
$$\therefore X^T(A+B)X>0,\therefore A+B\text{为正定阵}$$

$$A_{m\times n},\text{且}r(A)=n,\text{证}:B=A^TA\text{正定阵}.$$
$$\text{证}:B^T=(A^TA)^T=A^TA=B$$
$$\forall X\neq 0,X^TBX=(AX)^T(AX)=\alpha^T\alpha=|\alpha|^2$$
$$\text{其中},\alpha=AX$$
$$\alpha\neq 0,\text{若}\alpha=0\Rightarrow AX=0$$
$$\because r(A)=n,\therefore X=0\text{矛盾},\therefore \alpha\neq 0$$
$$X^TBX=|\alpha|^2>0,\therefore B\text{为正定阵}$$

特征值法

$$\text{Th1. }A^T=A,\text{则}A\text{正定}\Leftrightarrow \lambda_1>0,\ldots,\lambda_n>0$$

$$\text{设}A\text{正定},\text{证}:\text{则}A^{-1}\text{正定}.$$

$$\text{证}:A^T=A,(A^{-1})^T=(A^T)^{-1}=A^{-1}$$
$$\because A\text{正定},\therefore \lambda_1>0,\ldots,\lambda_n>0$$
$$\because A^{-1}\text{特征值}\frac{1}{\lambda_1}>0,\ldots,\frac{1}{\lambda_n}>0,\therefore A^{-1}\text{正定}$$

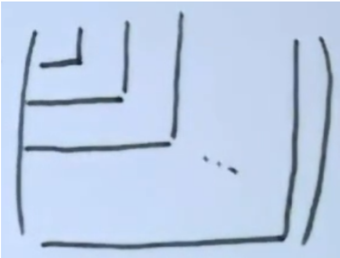
$$A_{n\times n}\text{正定},\text{证}:|A+E|>1.$$

$$\text{证}:A\text{正交}\Rightarrow \lambda_1>0,\ldots,\lambda_n>0$$
$$A+E\text{特征值}\lambda_1+1>1,\ldots,\lambda_{n+1}>1$$
$$\therefore |A+E|=(\lambda_1+1)\ldots(\lambda_n+1)>1$$

顺序主子式法

TH2. $A^T = A$, 则A正定 $\Leftrightarrow$

$$a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \ldots, |A| > 0$$



已知二次型$f(x_1, x_2, x_3) = (1-a)x_1^2 + (1-a)x_2^2 + 2x_3^2 + 2(1+a)x_1x_2$的秩为2.
(1)求常数a;
(2)求正交变换$X = QY$, 把二次型化为标准型;
(3)求方程$f(x_1, x_2, x_3) = 0$的解.

$$\text{解}: A = \begin{pmatrix} 1-a & 1+a & 0 \\ 1+a & 1-a & 0 \\ 0 & 0 & 2 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, f = X^TAX$$

①$r(A) = 2 < 3 \Rightarrow |A| = 0$

$$\text{而}|A| = -8a, \therefore a = 0, A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

②$|\lambda E - A| = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = \lambda_3 = 2$

$$0E - A \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$2E - A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\gamma_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \gamma_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$(A\gamma_1 = 0\gamma_1, A\gamma_2 = 2\gamma_2, A\gamma_3 = 2\gamma_3)$

$$Q = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, Q^TQ = E, Q^TAQ = \begin{pmatrix} 0 & & \\ & 2 & \\ & & 2 \end{pmatrix}$$

$X^TAX \to X = QY \to 2y_2^2 + 2y_3^2$

③$f = x_1^2 + x_2^2 + 2x_3^2 + 2x_1x_2$

$= (x_1 + x_2)^2 + 2x_3^2 = 0$

$$\Leftrightarrow \begin{cases} x_1 + x_2 = 0 \\ x_3 = 0 \end{cases}, \therefore X = K\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

设二次型$f(x_1,x_2,x_3)=x_1^2+x_2^2-x_3^2+4x_1x_3+2ax_2x_3$经过正交变换化为标准形为$f=-3y_1^2+by_2^2+3y_3^2$.

(1)求常数a,b;

(2)求正交矩阵Q, 使得二次型在正交变换$X=QY$下化为标准形.

解 :$A=\begin{pmatrix}1&0&2\\0&1&a\\2&a&-1\end{pmatrix},X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$f=X^TAX$

$\lambda_1=-3,\lambda_2=b,\lambda_3=3$

①$\mathrm{tr}(A)=1=b\Rightarrow b=1$

$|A|=-9,$而$|A|=\begin{vmatrix}1&0&2\\0&1&a\\0&a&-5\end{vmatrix}=-5-a^2$

$\Rightarrow a^2=4,a=\pm2$

$\Rightarrow\begin{cases}a=-2\\b=1\end{cases},\begin{cases}a=2\\b=1\end{cases}$

②case1. $\begin{cases}a=-2\\b=1\end{cases}:A=\begin{pmatrix}1&0&2\\0&1&-2\\2&-2&-1\end{pmatrix}$

$3E+A=\begin{pmatrix}4&0&2\\0&4&-2\\2&-2&2\end{pmatrix}\rightarrow\begin{pmatrix}1&-1&1\\2&0&1\\0&2&-1\end{pmatrix}\rightarrow\begin{pmatrix}1&-1&1\\0&1&-\frac{1}{2}\\0&0&0\end{pmatrix}\rightarrow\begin{pmatrix}1&0&\frac{1}{2}\\0&1&-\frac{1}{2}\\0&0&0\end{pmatrix}$

$\alpha_1=\begin{pmatrix}-1\\1\\2\end{pmatrix}$

$E-A=\begin{pmatrix}0&0&-2\\0&0&2\\-2&2&2\end{pmatrix}\rightarrow\begin{pmatrix}1&-1&0\\0&0&1\\0&0&0\end{pmatrix},\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$

$3E-A=\begin{pmatrix}2&0&-2\\0&2&2\\-2&2&4\end{pmatrix}\rightarrow\begin{pmatrix}1&0&-1\\0&1&1\\0&0&0\end{pmatrix},\alpha_3=\begin{pmatrix}1\\-1\\1\end{pmatrix}$

向量空间

$$1.R^n \triangleq \{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} | x_i \in R, 1 \le i \le n\} - n\text{维向量空间}$$

$$R^3 \triangleq \{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} | a_i \in R, 1 \le i \le 3\}$$

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\forall \beta \in R^3$

$\because \alpha_1, \alpha_2, \alpha_3, \beta$线性相关

$\therefore \beta$可由$\alpha_1, \alpha_2, \alpha_3$唯一线性表示

$\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$

2.基 − 若$\alpha_1, \ldots, \alpha_n$为$n$个$n$维线性无关向量

称$(\alpha_1, \ldots, \alpha_n)$为$R^n$的一组基

设$(\alpha_1, \ldots, \alpha_n)$为$R^n$的一组基

$\forall \beta \in R^n, \exists | k_1, \ldots, k_n$, 使

$\beta = k_1\alpha_1 + \ldots + k_n\alpha_n$

称$(k_1, \ldots, k_n)^T$为β在基$\alpha_1 \ldots \alpha_n$下的坐标

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\text{为}R^3\text{的一组基}$$

$$\text{求}\beta = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}\text{在}\alpha_1, \alpha_2, \alpha_3\text{下的坐标.}$$

解 :设$\beta = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3$

$$= (\alpha_1, \alpha_2, \alpha_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (\alpha_1, \alpha_2, \alpha_3)^{-1}\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

$$\text{由}\begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 & -2 & 5 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ 5 \\ -2 \end{pmatrix}$$

过渡矩阵

$\alpha_1,\alpha_2,\dots,\alpha_n$与$\beta_1,\beta_2,\dots,\beta_n$皆

R^n的两组基

若$\exists Q$使$(\beta_1\dots\beta_n)=(\alpha_1\dots\alpha_n)Q$

称Q为从基$\alpha_1\dots\alpha_n$到基$\beta_1\dots\beta_n$过渡矩阵

$$\begin{cases}\beta_1=l_{11}\alpha_1+\dots+l_{1n}\alpha_n\\ \dots\\ \beta_n=l_{n1}\alpha_1+\dots+l_{nn}\alpha_n\end{cases}$$

$$(\beta_1,\dots,\beta_n)=(\alpha_1,\dots,\alpha_n)\begin{pmatrix}l_{11} & & l_{n1}\\ \vdots & \dots & \vdots\\ l_{1n} & & l_{nn}\end{pmatrix}$$

设$\alpha_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix},\alpha_3=\begin{pmatrix}-2\\3\\1\end{pmatrix}$与$\beta_1=\begin{pmatrix}1\\1\\0\end{pmatrix},\beta_2=\begin{pmatrix}-1\\1\\0\end{pmatrix},\beta_3=\begin{pmatrix}3\\2\\1\end{pmatrix}$

为R^3的两组基,求从$\alpha_1,\alpha_2,\alpha_3$到$\beta_1,\beta_2,\beta_3$的过渡矩阵.

解 :设过渡矩阵为Q

由$(\beta_1,\beta_2,\beta_3)=(\alpha_1,\alpha_2,\alpha_3)Q$

$$\therefore Q=\begin{pmatrix}1&1&-2\\0&1&3\\0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&-1&3\\1&1&2\\0&0&1\end{pmatrix}=?$$