导数的应用

单调性

$$y = f(x)(x \in D)$$

$$\begin{aligned} &1.\ \text{若}\forall x_1, x_2 \in D\text{且}x_1 < x_2,\text{有}f(x_1) < f(x_2) \\ &\quad\text{称}f(x)\text{在}D\text{上为严格单调递增函数} \\ &2.\ \ldots \end{aligned}$$

$$f(x) \in C[a,b], (a,b)\text{内可导}$$

$$\begin{gathered} ①f'(x) > 0(a < x < b) \Rightarrow f(x)\text{在}[a,b]\uparrow \\ ②f'(x) < 0(a < x < b) \Rightarrow f(x)\text{在}[a,b]\downarrow \end{gathered}$$

$$\begin{gathered} \text{证：设}f'(x) > 0(a < x < b) \\ \forall x_1, x_2 \in [a,b]\text{且}x_1 < x_2 \\ f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) > 0(x_1 < \xi < x_2) \\ \Rightarrow f(x_1) < f(x_2) \Rightarrow f(x)\text{在}[a,b]\text{上}\uparrow \end{gathered}$$

$$\text{求单调区间步骤}$$

$$y = f(x)$$

$$\begin{aligned} &①x \in D \\ &②f'(x)\begin{cases} = 0 \\ \text{不存在} \end{cases} \Rightarrow x_1, \ldots, x_n \\ &③\text{在每个小区间内判断}f'\text{正负,可得单调区间} \end{aligned}$$

极值

$$y = f(x)$$

①$x \in D$

②$f'(x)\begin{cases} =0(\text{驻点}) \\ \text{不存在} \end{cases} \Rightarrow x = \ldots$

③判断：

第一充分条件

1. $\begin{cases} x<x_0: f'(x)<0 \\ x>x_0: f'(x)>0 \end{cases} \Rightarrow x=x_0$为极小点

2. $\begin{cases} x<x_0: f'(x)>0 \\ x>x_0: f'(x)<0 \end{cases} \Rightarrow x=x_0$为极大点

第二充分条件

设$f'(x_0)=0, f''(x_0)\begin{cases} >0 \Rightarrow x_0\text{为极小点} \\ <0 \Rightarrow x_0\text{为极大点} \end{cases}$

$$f''(x_0)>0: \lim_{x\to x_0}\frac{f'(x)}{x-x_0}>0$$

$$\exists\delta>0,\text{当}0<|x-x_0|<\delta\text{时},\frac{f'(x)}{x-x_0}>0$$

$$\Rightarrow \begin{cases} f'(x)<0, x\in(x_0-\delta,x_0) \\ f'(x)>0, x\in(x_0,x_0+\delta) \end{cases} \Rightarrow x=x_0\text{为极小点}$$

$$f''(x_0)<0: \lim_{x\to x_0}\frac{f'(x)}{x-x_0}<0$$

$$\exists\delta>0,\text{当}0<|x-x_0|<\delta\text{时},\frac{f'(x)}{x-x_0}<0$$

$$\Rightarrow \begin{cases} f'(x)>0, x\in(x_0-\delta,x_0) \\ f'(x)<0, x\in(x_0,x_0+\delta) \end{cases} \Rightarrow x=x_0\text{为极大点}$$

求函数$f(x)=x^2e^{-x}$的极值点与极值.

①$x\in(-\infty,+\infty)$

②$f'(x)=(2x-x^2)e^{-x}=0 \Rightarrow x=0, x=2$

③$\begin{cases} x<0: f'(x)<0 \\ 0<x<2: f'(x)>0 \end{cases} \Rightarrow x=0$为极小点, 极小值$f(0)=0$

$\begin{cases} 0<x<2: f'(x)<0 \\ x>2: f'(x)<0 \end{cases} \Rightarrow x=2$为极大点, 极大值$f(2)=\dfrac{4}{e^2}$

设$f(x)=\ln x-\dfrac{x}{e}$, 求函数$f(x)$的极值点与极值.

①$x\in(0,+\infty)$

②$f'(x)=\dfrac{1}{x}-\dfrac{1}{e} \Rightarrow x=e$

$$f''(x)=-\frac{1}{x^2}$$

$$\because f''(e)=-\frac{1}{e^2}<0, \therefore x=e\text{为最大点}, M=f(e)=0$$

$$f(x)\text{可导}, f'(1)=0, \lim_{x\to 1}\frac{f'(x)}{\sin^3\pi x}=2, x=1?$$

$$\exists\delta>0,\text{当}0<|x-1|<\delta\text{时},\frac{f'(x)}{\sin^3\pi x}>0$$

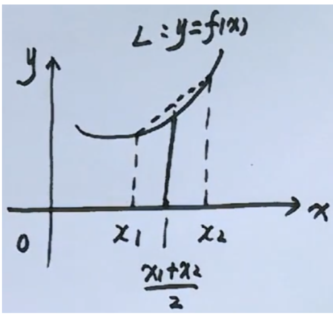
$$\begin{cases}f'(x)>0,x\in(1-\delta,1)\\f'(x)<0,x\in(1,1+\delta)\end{cases}\Rightarrow x=1\text{为极大点}$$

$$\text{证}:x>0\text{时},\frac{x}{1+x}<\ln(1+x)<x$$

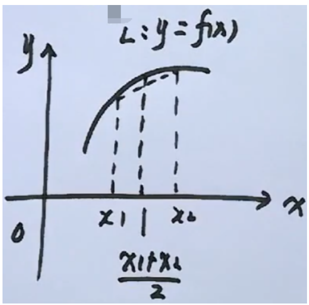
$$\begin{aligned}&\text{证}:1.\ f(x)=x-\ln(1+x),f(0)=0\\&\quad f'(x)=1-\frac{1}{1+x}>0(x>0)\Rightarrow f(x)\text{在}[0,+\infty)\uparrow\\&\quad\because\begin{cases}f(0)=0\\f'(x)>0(x>0)\end{cases}\therefore f(x)>0(x>0)\\&\ 2.\ g(x)=\ln(1+x)-\frac{x}{1+x},g(0)=0\\&\quad g'(x)=\frac{1}{1+x}-\frac{1}{(1+x)^2}=\frac{x}{(1+x)^2}>0(x>0)\\&\quad\begin{cases}f(0)=0\\g'(x)>0(x>0)\end{cases}\Rightarrow g(x)>0(x>0)\end{aligned}$$

凹凸性

$$y=f(x)(x\in D)$$



1. 若$\forall x_1,x_2\in D$且$x_1\neq x_2$,有
   $$f(\frac{x_1+x_2}{2})<\frac{f(x_1)+f(x_2)}{2}$$
   称$f(x)$在D上为下凹函数
2. 若$\forall x_1,x_2\in D$且$x_1\neq x_2$,有
   $$f(\frac{x_1+x_2}{2})>\frac{f(x_1)+f(x_2)}{2}$$
   称$f(x)$在D上为上凸函数



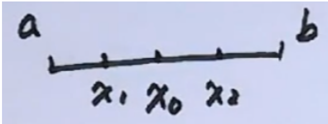
$$y=f(x)(x\in D),x_0\in D,$$

若$y=f(x)$在$x=x_0$左、右凹凸性不同,称$(x_0,f(x_0))$为$y=f(x)$的拐点

$f(x) \in C[a,b], (a,b)$内二阶可导

①若$f''(x) > 0 (a < x < b) \Rightarrow f(x)$在$[a,b]$上凹

②若$f''(x) < 0 (a < x < b) \Rightarrow f(x)$在$[a,b]$上凸



证：$\forall x_1, x_2 \in [a,b]$, 且$x_1 \neq x_2, \dfrac{x_1+x_2}{2} \triangleq x_0$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi)}{2!}(x-x_0)^2 (\xi\text{在}x_0\text{与}x\text{之间})$$

$$\because f''(x) > 0 (a < x < b)$$

$$\therefore \frac{f''(\xi)}{2!}(x-x_0)^2 \geq 0$$

$$\Rightarrow f(x) \geq f(x_0) + f'(x_0)(x-x_0)$$

$$= \text{成立} \Leftrightarrow x = x_0$$

$$\because x_1 \neq x_0, x_2 \neq x_0$$

$$\therefore \begin{cases} \frac{1}{2}f(x_1) > \frac{1}{2}f(x_0) + \frac{1}{2}f'(x_0)(x_1-x_0) \\ \frac{1}{2}f(x_2) > \frac{1}{2}f(x_0) + \frac{1}{2}f'(x_0)(x_2-x_0) \end{cases}$$

$$\Rightarrow \frac{f(x_1)+f(x_2)}{2} > f(x_0) + f'(x_0)(\frac{x_1+x_2}{2} - x_0)$$

$$\Rightarrow f(\frac{x_1+x_2}{2}) < \frac{f(x_1)+f(x_2)}{2}$$

$$\therefore f(x)\text{在}[a,b]\text{上凹}$$

设曲线$L: y = f(x) = e^{-(x-1)^2}$, 求曲线的凹凸区间及拐点.

①$x \in (-\infty, +\infty)$

②$f'(x) = -2(x-1)e^{-(x-1)^2}$

$f''(x) = 4e^{-(x-1)^2}[(x-1)^2 - \dfrac{1}{2}] = 0$

$x = 1 - \dfrac{1}{\sqrt{2}}, x = 1 + \dfrac{1}{\sqrt{2}}$

$x \in (-\infty, 1 - \dfrac{1}{\sqrt{2}})$时, $f'' > 0, x \in (1 - \dfrac{1}{\sqrt{2}}, 1 + \dfrac{1}{\sqrt{2}})$时, $f'' < 0$

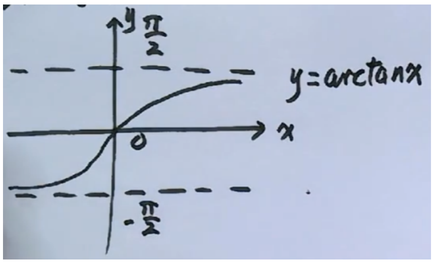
$x \in (1 + \dfrac{1}{\sqrt{2}}, +\infty)$时, $f'' > 0$

$f(1 - \dfrac{1}{\sqrt{2}}) = e^{-\frac{1}{2}}, f(1 + \dfrac{1}{\sqrt{2}}) = e^{-\frac{1}{2}}$

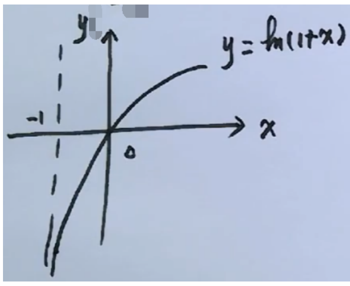
$\therefore (-\infty, 1 - \dfrac{1}{\sqrt{2}}]$与$[1 + \dfrac{1}{\sqrt{2}}, +\infty)$为凹区间, $[1 - \dfrac{1}{\sqrt{2}}, 1 + \dfrac{1}{\sqrt{2}}]$为凸区间

$(1 - \dfrac{1}{\sqrt{2}}, e^{-\frac{1}{2}}), (1 + \dfrac{1}{\sqrt{2}}, e^{-\frac{1}{2}})$为拐点

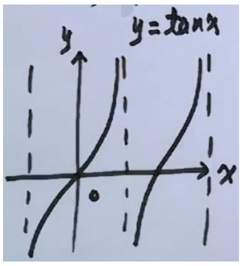
渐近线



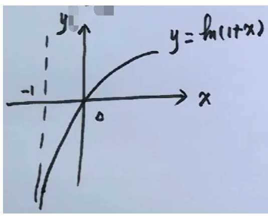
$$\arctan x \to -\frac{\pi}{2}(x \to -\infty)$$
$$\arctan x \to \frac{\pi}{2}(x \to +\infty)$$



$$\ln(1+x) \to -\infty(x \to -1^{+})$$



$$\tan x \to \infty(x \to \frac{\pi}{2})$$



水平渐近线

$$L: y = f(x), \text{若} \lim_{x\to\infty} f(x) = A, y = A\text{称为水平渐近线}$$

铅直渐近线

$$L: y = f(x)$$

$$\text{若} \begin{cases} f(a-0) = \infty \\ f(a+0) = \infty \\ \lim_{x\to a} f(x) = \infty \end{cases}, \text{称} x = a \text{为铅直渐近线}$$

斜渐近线

$$L: y = f(x)$$

$$\text{若} \lim_{x\to\infty} \frac{f(x)}{x} = a(\neq 0, \infty), \lim_{x\to\infty}[f(x) - ax] = b$$

$$y = ax + b\text{为斜渐近线}$$

$$\text{求曲线} y = \frac{x^2 - 3x + 2}{x^2 - 1} e^{\frac{1}{x}} \text{的水平渐近线与铅直渐近线.}$$

$$\because \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2 - 3x + 2}{x^2 - 1} e^{\frac{1}{x}} = 1$$

$$\therefore y = 1\text{为水平渐近线}$$

$$\because \lim_{x\to -1} f(x) = \infty, \therefore x = -1\text{为铅直渐近线}$$

$$\because f(0+0) = -\infty, \therefore x = 0\text{为铅直渐近线}$$

$$\because \lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x-2}{x+1} e^{\frac{1}{x}} = -\frac{e}{2} \neq \infty$$

$$\therefore x = 1\text{不是铅直渐近线}$$

$$\text{求曲线} y = \frac{2x^2 - x + 3}{x + 1} \text{的斜渐近线.}$$

$$\because \lim_{x\to\infty} f(x) = \infty, \therefore \text{无水平渐近线}$$

$$\therefore \lim_{x\to -1} f(x) = \infty, \therefore x = -1\text{为铅直渐近线}$$

$$\because \lim_{x\to\infty} \frac{f(x)}{x} = 2$$

$$\lim_{x\to\infty}[f(x) - 2x] = \lim_{x\to\infty} \frac{-3x + 3}{x + 1} = -3$$

$$\therefore y = 2x - 3\text{为斜渐近线}$$

$$L: y = f(x) = \sqrt{x^2 + 4x + 8} - x$$

1. $\lim\limits_{x\to -\infty} f(x) = +\infty$

   $\lim\limits_{x\to +\infty} f(x) = \lim\limits_{x\to +\infty} \dfrac{4x + 8}{\sqrt{x^2 + 4x + 8} + x} = 2$

   $\Rightarrow y = 2$为水平渐近线
2. $\because y = f(x)$处处连续 $\therefore$ 无铅直渐近线
3. $\lim\limits_{x\to -\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to -\infty} \dfrac{\sqrt{x^2 + 4x + 8} - x}{x} = -2$

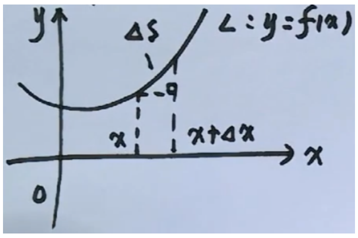
   $\lim\limits_{x\to -\infty}[f(x) + 2x] = \lim\limits_{x\to -\infty}(\sqrt{x^2 + 4x + 8} + x)$

   $= \lim\limits_{x\to -\infty} \dfrac{4x + 8}{\sqrt{x^2 + 4x + 8} - x} = -2$

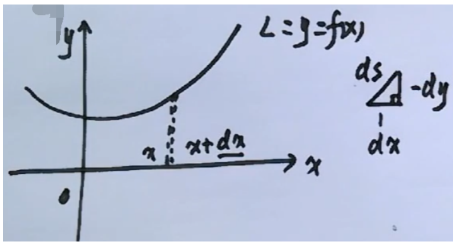
   $\therefore y = -2x - 2$为斜渐近线

曲率、曲率半径

弧微分（弧元素）



$$(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2$$



$$(ds)^2 = (dx)^2 + (dy)^2$$

直角坐标

$$L: y = f(x)$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2}dx$$

$$= \sqrt{1 + f'^2(x)}dx$$

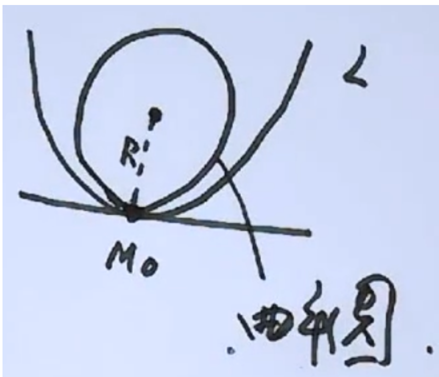
参数形式

$$L: \begin{cases} x = \Phi(t) \\ y = \phi(t) \end{cases}$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$$

$$= \sqrt{\Phi'^2(t) + \phi^2(t)}dt$$

曲率、曲率半径



$$L : y = f(x), M_0(x_0, y_0) \in L$$

$$k = \frac{|y''|}{(1 + y'^2)^{\frac{3}{2}}}$$

$$R = \frac{1}{k}$$