不定积分的基本概念与性质

原函数

$$f(x),F(x)(x\in D),\text{若}\forall x\in D\text{有}$$
$$F'(x)=f(x)$$
$$\text{称}F(x)\text{为}f(x)\text{在}D\text{上的原函数}$$

①若$f(x)\exists$原函数, 则一定有无数个原函数, 且任两个原函数之间相差常数

$\quad\because F'(x)=f(x)\Rightarrow[F(x)+C]'=f(x)(C\in\mathbb{R})$

②连续函数一定存在原函数, 反之不对

③若$F(x)$为$f(x)$的一个原函数, 则$F(x)+C$(C为任意数)为$f(x)$的所有原函数

④若$f(x)$有原函数，则任两个原函数相差常数

$\quad$如 : $F'(x)=f(x),G'(x)=g(x)$

$\quad\because[F(x)-G(x)]'=0,\therefore F(x)-G(x)\equiv C_0$

⑤连续函数一定存在原函数, 反之不对

举例说明存在第二类间断点, 但存在原函数的函数.

$$f(x)=\begin{cases}2x\sin\frac{1}{x}-\cos\frac{1}{x},x\neq0,\\0,x=0.\end{cases},F(x)=\begin{cases}x^2\sin\frac{1}{x},x\neq0,\\0,x=0.\end{cases}$$

$$\begin{aligned}&x\neq0:F'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}=f(x)\\&x=0:F'(x)=\lim_{x\to0}\frac{F(x)-F(0)}{x}=\lim_{x\to0}x\sin\frac{1}{x}=0=f(0)\\&\text{即}F'(x)=f(x),\therefore F(x)\text{为}f(x)\text{的原函数}\\&\because\lim_{x\to0}f(x)\text{不存在},\therefore x=0\text{为}f(x)\text{第二类间断点}\end{aligned}$$

不定积分

设$F(x)$为$f(x)$的一个原函数, $F(x)+C$即$f(x)$的所有原函数, 称为$f(x)$的不定积分, 记

$$\int f(x)dx=F(x)+C$$

$$①\frac{d}{dx}\int f(x)dx=f(x)$$
$$②\int f'(x)dx=f(x)+C$$

基本公式

1. $\int kdx = kx + C$

2. $\begin{cases} ①a \neq -1: \int x^a dx = \frac{1}{a+1} x^{a+1} + C \\ ②a = -1: \int \frac{1}{x} dx = \ln|x| + C \\ \because x < 0时, [\ln(-x)]' = \frac{1}{-x} x(-1), \therefore \int \frac{1}{x} dx = \ln(-x) + C \\ x > 0时, (\ln x)' = \frac{1}{x}, \int \frac{1}{x} dx = \ln x + C \\ \therefore \int \frac{1}{x} dx = \ln|x| + C \end{cases}$

3. $\begin{cases} ①a \neq e: \int a^x dx = \frac{a^x}{\ln a} + C \\ ②a = e: \int e^x dx = e^x + C \end{cases}$

4. ① $\int \sin x dx = -\cos x + C$

② $\int \cos x dx = \sin x + C$

③ $\int \tan x dx = -\ln|\cos x| + C$

④ $\int \cot x dx = \ln|\sin x| + C$

⑤ $\int \sec x dx = \ln|\sec x + \tan x| + C$

⑥ $\int \csc x dx = \ln|\csc x - \cot x| + C$

⑦ $\int \sec^2 dx = \tan x + C$

⑧ $\int \csc^2 dx = -\cot x + C$

⑨ $\int \sec x \tan x dx = \sec x + C$

⑩ $\int \csc x \cot x dx = -\csc x + C$

5. 平方和,平方差:

$\begin{cases} ① \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C \\ ② \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{d(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}} = \arcsin \frac{x}{a} + C \end{cases}$

$\begin{cases} ③ \int \frac{dx}{1+x^2} = \arctan x + C \\ ④ \int \frac{dx}{a^2+x^2} = \frac{1}{a} \int \frac{d(\frac{x}{a})}{1+(\frac{x}{a})^2} = \frac{1}{a} \arctan \frac{x}{a} + C \end{cases}$

$\begin{cases} ⑤ \int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2}) + C \\ ⑥ \int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x + \sqrt{x^2-a^2}| + C \end{cases}$

⑦ $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}[\int \frac{d(x-a)}{x-a} - \int \frac{d(x+a)}{x+a}] = \frac{1}{2a} \ln|\frac{x-a}{x+a}| + C$

⑧ $\int \sqrt{a^2-x^2} dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2-x^2} + C$