相关性与线性表示(向量理论一)

方程组三种形式

1.一般形式

$$\begin{cases}a_{11}x_1+\ldots+a_{1n}x_n=0\\ \cdots\\ a_{m1}x_1+\ldots+a_{mn}x_n=0\end{cases}(*)$$

$$\begin{cases}a_{11}x_1+\ldots+a_{1n}x_n=b_1\\ \cdots\\ a_{m1}x_1+\ldots+a_{mn}x_n=b_n\end{cases}(**)$$

2.矩阵形式

$$A=\begin{pmatrix}a_{11}&\ldots&a_{1n}\\ \cdots\\ a_{m1}&\ldots&a_{mn}\end{pmatrix},X=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix},b=\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}$$

$$AX=0(*)$$

$$AX=b(**)$$

3.向量形式

$$\alpha_1=\begin{pmatrix}a_{11}\\ a_{21}\\ \vdots\\ a_{m1}\end{pmatrix},\alpha_2=\begin{pmatrix}a_{12}\\ a_{22}\\ \vdots\\ a_{m2}\end{pmatrix},\cdots,\alpha_n=\begin{pmatrix}a_{1n}\\ a_{2n}\\ \vdots\\ a_{mn}\end{pmatrix}$$

$$x_1\alpha_1+x_2\alpha_2+\cdots+x_n\alpha_n=0(*)$$

$$x_1\alpha_1+x_2\alpha_2+\cdots+x_n\alpha_n=b(**)$$

齐次与非齐次解的情形

$$1.\begin{cases}x_1-2x_2=0\\ 2x_1+x_2=0\end{cases}\Rightarrow\begin{cases}x_1=0\\ x_2=0\end{cases};(\text{无自由变量})$$

$$A=\begin{pmatrix}1&-2\\ 2&1\end{pmatrix},|A|=5\neq0\Rightarrow r(A)=2$$

$$2.\begin{cases}x_1+x_2-2x_3=0\\ 2x_1+3x_2+x_3=0\end{cases}\Rightarrow\begin{cases}x_1=0\\ x_2=0,\\ x_3=0\end{cases}\begin{cases}x_1=7\\ x_2=-5,\\ x_3=1\end{cases}\begin{cases}x_1=-7\\ x_2=5\\ x_3=-1\end{cases},\cdots(\text{有自由变量})$$

$$A=\begin{pmatrix}1&1&-2\\ 2&3&1\end{pmatrix}\to\begin{pmatrix}1&1&-2\\ 0&1&5\end{pmatrix},r(A)=2<3$$

$$(*)\begin{cases}\text{仅有零解(变量与约束条件个数等)}\\ \text{除零解外有无数个非零解(变量多于约束条件的个数)}\end{cases}$$

$$1.\begin{cases}2x_1+x_2=1\\ x_1-x_2=2\end{cases}\Rightarrow\begin{cases}x_1=1\\ x_2=-1\end{cases}(\text{约束条件等于未知数个数})$$

$$2.\begin{cases}x_1+2x_3=-1\\ 2x_1-x_2=1\end{cases}\Rightarrow\begin{cases}x_1=1\\ x_2=1\\ x_3=-1\end{cases},\begin{cases}x_1=0\\ x_2=-1\\ x_3=-\frac{1}{2}\end{cases},\cdots(\text{约束条件少于未知数个数})$$

$$3.\begin{cases}x_1+x_2=1\\ 2x_1+2x_2=4\end{cases}\text{无解}$$

$$(**)\begin{cases}\text{有解}\begin{cases}\text{唯一解}\\ \text{无数解}\end{cases}\\ \text{无解}\end{cases}$$

相关性

$$\alpha_1,\alpha_2,\ldots,\alpha_n:$$
$$x_1\alpha_1+x_2\alpha_2+\ldots+x_n\alpha_n=0(*)$$
case1. $(*)$仅有零解, 称$\alpha_1,\alpha_2,\ldots,\alpha_n$线性无关
case2. $(*)$有非零解, 称$\alpha_1,\alpha_2,\ldots,\alpha_n$线性相关

线性表示

$$\alpha_1,\alpha_2,\ldots,\alpha_n,b$$
$$x_1\alpha_1+\ldots+x_n\alpha_n=b(**)$$
case1. $(**)$有解, 称b可由$\alpha_1,\ldots,\alpha_n$线性表示
case2. $(**)$无解, 称b不可由$\alpha_1,\ldots,\alpha_n$线性表示

性质

1.$\alpha_1,\ldots,\alpha_n$线性相关 $\Leftrightarrow$ 至少有一个向量可由其余向量线性表示

证 : $\Rightarrow$, $\exists$不全为0的$k_1,k_2,\ldots,k_n$, 使

$$k_1\alpha_1+\ldots+k_n\alpha_n=0$$

设$k_1\neq 0\Rightarrow \alpha_1=-\dfrac{k_2}{k_1}\alpha_2-\ldots-\dfrac{k_n}{k_1}\alpha_n$

$\Leftarrow$, 设$\alpha_k=l_1\alpha_1+\ldots+l_{n-1}\alpha_{n-1}+l_{k+1}\alpha_{k+1}+\ldots+l_n\alpha_n$

$\Rightarrow l_1\alpha_1+\ldots+l_{n-1}\alpha_{n-1}+(-1)l_k+l_{k+1}\alpha_{k+1}+\ldots+l_n\alpha_n=0$

$\therefore \alpha_1\ldots\alpha_n$线性相关

Notes:

①含零向量的向量组一定相关

证 : 设$\alpha_1=0,\alpha_2\neq 0,\alpha_3\neq 0$

法一 : $\because 2\alpha_1+0\alpha_2+0\alpha_3=0$

$\therefore \alpha_1,\alpha_2,\alpha_3$线性相关

法二 : $\because \alpha_1=0\alpha_2+0\alpha_3$

$\therefore \alpha_1,\alpha_2,\alpha_3$线性相关

②α,β线性相关 $\Leftrightarrow$ α,β成比例

证 : $\Rightarrow$ $\exists$不全为0的k_1,k_2, 使

$$k_1\alpha+k_2\beta=0$$

设$k_1\neq 0\Rightarrow \alpha=-\dfrac{k_2}{k_1}\beta$

$\Leftarrow$ 设$\beta=k\alpha\Rightarrow k\alpha+(-1)\beta=0$

$\therefore \alpha,\beta$线性相关

2.设$\alpha_1,...,\alpha_n$线性无关

①$\alpha_1,...,\alpha_n,\beta$线性无关$\Leftrightarrow\beta$不可由$\alpha_1,...,\alpha_n$线性表示

②$\alpha_1,...,\alpha_n,\beta$线性相关$\Rightarrow\beta$可由$\alpha_1,...,\alpha_n$唯一线性表示

②证明$:\because\alpha_1,...,\alpha_n,\beta$线性相关

$\therefore\exists$不全为0的$k_1,...,k_n,k_0$,使

$k_1\alpha_1+...+k_n\alpha_n+k_0\beta=0$

$k_0\neq 0$,若$k_0=0\Rightarrow k_1\alpha_1+...+k_n\alpha_n=0$

$\because\alpha_1...\alpha_n$线性无关,$\therefore k_1=...=k_n=0$,矛盾,$\therefore k_0\neq 0$

$\Rightarrow\beta=-\frac{k_1}{k_0}\alpha_1-...-\frac{k_n}{k_0}\alpha_n$

(反)设$\beta=l_1\alpha_1+...+l_n\alpha_n$

$\beta=t_1\alpha_1+...+t_n\alpha_n$

$\Rightarrow(l_1-t_1)\alpha_1+...+(l_n-t_n)\alpha_n=0$

$\because\alpha_1...\alpha_n$线性无关,$\therefore l_i=t_i(1\leq i\leq n)$

3.全组无关$\Rightarrow$ 部分组无关

4.部分组相关$\Rightarrow$ 全组相关

$\alpha_1,\alpha_2,\alpha_3$线性无关,$\alpha_2,\alpha_3,\alpha_4$线性相关,

问α_4可否由$\alpha_1,\alpha_2,\alpha_3$线性表示?

$\alpha_1,\alpha_2,\alpha_3$线性无关$\Rightarrow\alpha_2,\alpha_3$线性无关

$\alpha_2,\alpha_3,\alpha_4$线性相关$\Rightarrow\alpha_4=k_2\alpha_2+k_3\alpha_3$

$\alpha_4=0\alpha_1+k_2\alpha_2+k_3\alpha_3$

α_1,α_2线性无关,β_1不可由α_1,α_2线性表示,β_2可由α_1,α_2线性表示

①$\alpha_1,\alpha_2,k\beta_1+\beta_2$②$\beta_1+k\beta_2$

①若$k=0:\alpha_1,\alpha_2,k\beta_1+\beta_2$线性相关

若$k\neq 0:\alpha_1,\alpha_2,k\beta_1+\beta_2$线性无关

②$\alpha_1,\alpha_2,\beta_1+k\beta_2$线性无关

5.$\alpha_1,\alpha_2,...,\alpha_n$为$n$个$n$维向量

①$\alpha_1,\alpha_2,...,\alpha_n$线性无关$\Leftrightarrow|\alpha_1...\alpha_n|\neq 0$

②$\alpha_1,\alpha_2,...,\alpha_n$线性相关$\Leftrightarrow|\alpha_1...\alpha_n|=0$

证$:$令$A=(\alpha_1...\alpha_n)$

①$\alpha_1...\alpha_n$线性无关$\Leftrightarrow Ax=0$仅有零解

$\Leftrightarrow r(A)=n\Leftrightarrow|A|\neq 0$

②$\alpha_1...\alpha_n$线性相关$\Leftrightarrow Ax=0$有非零解

$\Leftrightarrow r(A)<n\Leftrightarrow|A|=0$

$$\alpha_1=\begin{pmatrix}1\\-1\\1\end{pmatrix},\alpha_2=\begin{pmatrix}1\\2\\4\end{pmatrix},\alpha_3=\begin{pmatrix}1\\a\\a^2\end{pmatrix}$$

$$|\alpha_1,\alpha_2,\alpha_3|=\begin{vmatrix}1&1&1\\-1&2&a\\1&4&a^2\end{vmatrix}=3(a+1)(a-2)$$

①$a\neq-1$且$a\neq-2:\alpha_1,\alpha_2,\alpha_3$线性无关

①$a=-1$或$a=-2:\alpha_1,\alpha_2,\alpha_3$线性相关

$$\begin{cases}x_1+3x_2-2x_3=0\\x_1+x_2+4x_3=0\end{cases}$$

$$x_1\begin{pmatrix}1\\1\end{pmatrix}+x_2\begin{pmatrix}3\\1\end{pmatrix}+x_3\begin{pmatrix}-2\\4\end{pmatrix}=0,\text{一维一个方程}$$

6.$\alpha_1,\alpha_2,\dots,\alpha_n$为$n$个$m$维向量且$m<n$

$\Rightarrow \alpha_1\dots\alpha_n$线性相关

证 :令$A=(\alpha_1\dots\alpha_n)_{m\times n}$

$x_1\alpha_1+\dots+x_n\alpha_n=0\Leftrightarrow Ax=0$

$\because m<n,\therefore r(A)\leq m<n$

$\Rightarrow Ax=0$有非零解$\Leftrightarrow \alpha_1\dots\alpha_n$线性相关

设$\alpha_1,\alpha_2,\alpha_3$为3个3维线性无关向量

$\forall\beta$, 证 : β可由$\alpha_1,\alpha_2,\alpha_3$唯一线性表示

证 : $\because \alpha_1,\alpha_2,\alpha_3$线性无关

又 $\because \alpha_1,\alpha_2,\alpha_3,\beta$线性相关

$\therefore \beta$可由$\alpha_1,\alpha_2,\alpha_3$唯一线性表示

7.①加个数提交相关性

②加维数提升无关性

$$\alpha_1=\begin{pmatrix}2\\3\\1\\0\\0\end{pmatrix},\alpha_2=\begin{pmatrix}1\\-1\\0\\1\\0\end{pmatrix},\alpha_3=\begin{pmatrix}3\\-2\\0\\0\\1\end{pmatrix}$$

$$\because\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}=1\neq0,\therefore\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\text{线性无关}$$

$\Rightarrow \alpha_1,\alpha_2,\alpha_3$线性无关

8.$\alpha_1,\alpha_2,\dots,\alpha_n$非零且两两正交$\Rightarrow \alpha_1,\alpha_2,\dots,\alpha_n$线性无关

$\not\Leftarrow$

证 :$\Rightarrow$, 令$k_1\alpha_1+\dots+k_n\alpha_n=0$

$(\alpha_1,k_1\alpha_1+k_2\alpha_2+\dots+k_n\alpha_n)=0$

$k_1(\alpha_1,\alpha_1)=0$

$\because(\alpha_1,\alpha_1)=\alpha_1^T\alpha_1=|\alpha_1|^2>0,\therefore k_1=0$

$(\alpha_2,k_2\alpha_2+\dots+k_n\alpha_n)=0$

$k_2(\alpha_2,\alpha_2)=0$

$\because(\alpha_2,\alpha_2)=|\alpha_2|^2>0,\therefore k_2=0$

$k_n\alpha_n=0$

$\because\alpha_n\neq0,\therefore k_n=0\Rightarrow\alpha_1\dots\alpha_n$线性无关

$$\not\Leftarrow,\alpha_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix},\alpha_3=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\because\begin{vmatrix}1&1&1\\0&1&1\\0&0&1\end{vmatrix}=1\neq0,\therefore\alpha_1,\alpha_2,\alpha_3\text{线性无关}$$

$(\alpha_1,\alpha_2)=1\neq0,(\alpha_1,\alpha_3)=1\neq0,(\alpha_2,\alpha_3)=2\neq0$

相关性证明

$$\text{相关性证明}\begin{cases}\text{性质}\\\text{定义法}\end{cases}$$

$1.\alpha_1,\alpha_2,\alpha_3$线性无关, $\beta_1=\alpha_1+\alpha_2,\beta_2=\alpha_2+\alpha_3,\beta_3=\alpha_3+\alpha_1$

β_1,β_2,β_3

解：令$k_1\beta_1+k_2\beta_2+k_3\beta_3=0$

$\Rightarrow (k_1+k_3)\alpha_1+(k_1+k_2)\alpha_2+(k_2+k_3)\alpha_3=0$

$\because \alpha_1,\alpha_2,\alpha_3$线性无关

$$\therefore\begin{cases}k_1+k_3=0\\k_1+k_2=0\ (*)\\k_2+k_3=0\end{cases}$$

$$\because |A|=\begin{vmatrix}1&0&1\\1&1&0\\0&1&1\end{vmatrix}=2\neq 0,\therefore r(A)=3$$

$\Rightarrow k_1=k_2=k_3=0\Rightarrow \beta_1,\beta_2,\beta_3$线性无关

$2.\alpha_1\sim\alpha_4$线性无关, $\beta_1=\alpha_1+\alpha_2,\beta_2=\alpha_2+\alpha_3,\beta_3=\alpha_3+\alpha_4,\beta_4=\alpha_4+\alpha_1$

问$\beta_1\sim\beta_4$?

$k_1\beta_1+k_2\beta_2+k_3\beta_3+k_4\beta_4=0$

$\Rightarrow (k_1+k_4)\alpha_1+(k_1+k_2)\alpha_2+(k_2+k_3)\alpha_3+(k_3+k_4)\alpha_4=0$

$\because \alpha_1\sim\alpha_4$线性无关

$$\therefore\begin{cases}k_1+k_4=0\\k_1+k_2=0\\k_2+k_3=0\\k_4+k_4=0\end{cases}(*)$$

$$|A|=\begin{vmatrix}1&0&0&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{vmatrix}=0\Rightarrow r(A)<4$$

$\Rightarrow (*)$有非零解 $\Rightarrow \beta_1\sim\beta_4$线性相关

$\beta_1-\beta_2=\alpha_1-\alpha_3$

$\beta_3-\beta_4=\alpha_3-\alpha_1$

$\Rightarrow\beta_1-\beta_2+\beta_3-\beta_4=0\Rightarrow\beta_1\sim\beta_4$线性相关

$3.\alpha_1,\alpha_2$线性无关, $\beta\neq 0$与α_1,α_2正交, 证：α_1,α_2,β线性无关

令$k_1\alpha_1+k_2\alpha_2+k_0\beta=0$

由$(\beta,k_1\alpha_1+k_2\alpha_2+k_0\beta=0$

$\because \beta$与α_1,α_2正交, $\therefore k_0(\beta,\beta)=0$

而$(\beta,\beta)=|\beta|^2>0,\therefore k_0=0$

$\Rightarrow k_1\alpha_1+k_2\alpha_2=0$

$\because \alpha_1,\alpha_2$线性无关, $\therefore k_1=0,k_2=0$

$\therefore \alpha_1,\alpha_2,\beta$线性无关