二次型

$$1.\text{二次型}-f=X^TAX\begin{cases}\text{标}\Leftrightarrow A=(\ddots)\\ \text{非标}\Leftrightarrow A^T=A,\text{但}A\neq(\ddots)\end{cases}$$

$$2.\text{标准化}-f=X^TAX\to X=PY,P\text{可逆}\to Y^T(P^TAP)Y$$

$$\text{若}P^TAP=\begin{pmatrix}l_1&&\\&\ddots&\\&&l_n\end{pmatrix},\text{则}f=l_1y_1^2+\ldots+l_ny_n^2$$

$$\text{关键}:\begin{cases}①X=PY,P\text{可逆}\\ ②P^TAP=\begin{pmatrix}l_1&&\\&\ddots&\\&&l_n\end{pmatrix}\end{cases}$$

$$3.\text{合同}-A,B\text{为}n\text{阶阵},\text{若}\exists\text{可逆}P,\text{使}$$
$$P^TAP=B,A\cong B$$

标准化

配方法

$$f(x_1,x_2)=5x_1^2-2x_1x_2+x_2^2,\text{用配方法化为标准型}$$

$$\text{解}:1.A=\begin{pmatrix}5&-1\\-1&1\end{pmatrix},X=\begin{pmatrix}x_1\\x_2\end{pmatrix},f(x_1,x_2)=X^TAX$$

$$2.f=4x_1^2+(x_1-x_2)^2$$

$$3.\text{令}\begin{cases}x_1=y_1\\x_1-x_2=y_2\end{cases}\Rightarrow\begin{cases}x_1=y_1\\x_2=y_1-y_2\end{cases},\text{即}X=PY,P=\begin{pmatrix}1&0\\1&-1\end{pmatrix}\text{可逆}$$

$$4.f=X^TAX\to X=PY\to Y^T(P^TAP)Y=4y_1^2+y_2^2$$

$$P^TAP=\begin{pmatrix}4&0\\0&1\end{pmatrix}$$

$$\text{Q. }f=(2x_1)^2+(x_1-x_2)^2$$

$$\text{令}\begin{cases}2x_1=y_1\\x_1-x_2=y_2\end{cases}\Rightarrow\begin{cases}x_1=\frac{1}{2}y_1\\x_2=\frac{1}{2}y_1-y_2\end{cases},X=PY,P=\begin{pmatrix}\frac{1}{2}&0\\\frac{1}{2}&-1\end{pmatrix}\text{可逆}$$

$$X^TAX\to X=PY\to Y^T(P^TAP)Y=y_1^2+y_2^2$$

$$P^TAP=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$f(x_1,x_2,x_3)=x_1^2+2x_1x_2-4x_2x_3+6x_3^2$$

解：$1.A=\begin{pmatrix}1&1&0\\1&0&-2\\0&-2&6\end{pmatrix},X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},f=X^TAX$

$$\begin{aligned}&2.f=(x_1+x_2)^2-x_2^2-4x_2x_3-4x_3^2+10x_3^2\\&\quad=(x_1+x_2)^2-(x_2+2x_3)^2+10x_3^2\\&3.\begin{cases}x_1+x_2=y_1\\x_2+2x_3=y_2\\x_3=y_3\end{cases}\Rightarrow\begin{cases}x_1=y_1-y_2+2y_3\\x_2=y_2-2y_3\\x_3=y_3\end{cases}\\&\quad X=PY,P=\begin{pmatrix}1&-1&2\\0&1&-2\\0&0&1\end{pmatrix}\text{可逆}\\&4.f=X^TAX\to X=PY\to Y^T(P^TAP)Y=y_1^2-y_2^2+10y_3\end{aligned}$$

正交变换法

$$\begin{aligned}&1.f=X^TAX,A^T=A\\&2.|\lambda E-A|=0\Rightarrow\lambda_1,\ldots,\lambda_n\\&3.\lambda_iE-A\to\ldots:\alpha_1,\ldots,\alpha_n\\&4.\alpha_1\ldots\alpha_n\to\text{正交规范化}\gamma_1\ldots\gamma_n\\&\quad(A\gamma_1=\lambda_1\gamma_1,A\gamma_2=\lambda_2\gamma_2,\ldots,A\gamma_n=\lambda_n\gamma_n)\\&\quad Q=(\gamma_1\ldots\gamma_n),Q^TQ=E,Q^TAQ=\begin{pmatrix}\lambda_1&&\\&\ddots&\\&&\lambda_n\end{pmatrix}\\&5.f=X^TAX\to X=QY\to Y^T(Q^TAQ)Y\\&\quad=\lambda_1y_1^2+\ldots+\lambda_ny_n^2\end{aligned}$$

$$f(x_1,x_2,x_3)=2x_1x_2+2x_1x_3+2x_2x_3$$

解：$1.A=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}, X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, f=X^TAX$

$$2.|\lambda E-A|=\begin{vmatrix}\lambda&-1&-1\\-1&\lambda&-1\\-1&-1&\lambda\end{vmatrix}=0\Rightarrow\lambda_1=\lambda_2=-1,\lambda_3=2$$

$$3.E+A\rightarrow\begin{pmatrix}1&1&1\\&0&\\&&\end{pmatrix},\alpha_1=\begin{pmatrix}-1\\1\\0\end{pmatrix},\alpha_2=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$2E-A=\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}\rightarrow\begin{pmatrix}1&-2&1\\2&-1&-1\\-1&-1&2\end{pmatrix}\rightarrow\begin{pmatrix}1&-2&1\\0&1&-1\\0&0&0\end{pmatrix}$$

$$\rightarrow\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix},\alpha_3=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$4.\beta_1=\alpha_1=\begin{pmatrix}-1\\1\\0\end{pmatrix},\beta_2=\alpha_2-\frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1=\frac{1}{2}\begin{pmatrix}-1\\-1\\2\end{pmatrix},\beta_3=\alpha_3\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\gamma_1=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix},\gamma_2=\frac{1}{\sqrt{6}}\begin{pmatrix}-1\\-1\\2\end{pmatrix},\gamma_3=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$(A\gamma_1=-\gamma_1,A\gamma_2=-\gamma_2,A\gamma_3=2\gamma_3)$

$$令Q=\begin{pmatrix}-\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{3}}\\0&\frac{2}{\sqrt{6}}&\frac{1}{\sqrt{3}}\end{pmatrix},Q^TQ=E,Q^TAQ=\begin{pmatrix}-1&&\\&-1&\\&&2\end{pmatrix}$$

$X^TAX\rightarrow X=QY\rightarrow Y^T(Q^TAQ)Y$

$=-y_1^2-y_2^2+2y_3^2$