二重积分

定义

$$
\begin{gathered}
\text{定积分}, f(x)\text{在}[a,b]\text{上有界}\\
\begin{aligned}
&1.\ a=x_0<x_1<\ldots<x_n=b\\
&2.\ \forall \xi_i\in[x_{i-1},x_i],\sum_{i=1}^{n}f(\xi_i)\Delta x_i\\
&3.\ \lambda=\max\{\Delta x_1,\ldots,\Delta x_n\}
\end{aligned}\\
\text{若}\lim_{\lambda\to0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i\exists,\text{称}f(x)\text{在}[a,b]\text{上可积}\\
\text{极限值称为}f(x)\text{在}[a,b]\text{上的定积分，记作}\int_a^b f(x)dx
\end{gathered}
$$

$$
\begin{gathered}
\text{设}D\text{为}xoy\text{面有界闭区域，}f(x,y)\text{在}D\text{上有界}\\
\begin{aligned}
&1.\ D\text{分成}\Delta\sigma_1,\ldots,\Delta\sigma_n\\
&2.\ \forall(\xi_i,\eta_i)\in\Delta\sigma_i,\text{作}\sum_{i=1}^{n}f(\xi_i,\eta_i)\Delta\sigma_i\\
&3.\ \lambda\text{为}\Delta\sigma_1,\ldots,\Delta\sigma_n\text{直径最大者}
\end{aligned}\\
\text{若}\lim_{\lambda\to0}\sum_{i=1}^{n}f(\xi_i,\eta_i)\Delta\sigma_i\exists,\text{极限值称}f(x,y)\text{在}D\text{上的二重积分，记作}\iint_D f(x,y)d\sigma\\
\text{即}\iint_D f(x,y)d\sigma\triangleq\lim_{\lambda\to0}\sum_{i=1}^{n}f(\xi_i,\eta_i)\Delta\sigma_i
\end{gathered}
$$

性质

$$
\begin{aligned}
&1.\ D=D_1+D_2,D_1\cap D_2=\emptyset,\iint_D=\iint_{D_1}+\iint_{D_2}\\
&2.\ \iint_D 1d\sigma=A\\
&3.\ D\text{关于}y\text{轴对称，右}D_1\\
&\quad\begin{cases}f(-x,y)=-f(x,y)\Rightarrow\iint_D f(x,y)d\sigma=0\\ f(-x,y)=f(x,y)\Rightarrow\iint_D f(x,y)d\sigma=2\iint_{D_1}f(x,y)d\sigma\end{cases}\\
&\quad D\text{关于}y=x\text{对称，则}\\
&\quad\iint_D f(x,y)d\sigma=\iint_D f(y,x)d\sigma
\end{aligned}
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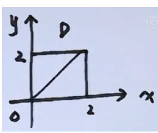


$$f(u) > 0, \forall a > 0, b > 0$$

$$I = \iint_D \frac{af(x) + bf(y)}{f(x) + f(y)} d\sigma = \iint_D \frac{af(y) + bf(x)}{f(y) + f(x)} d\sigma$$

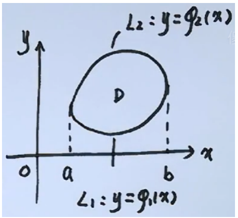
$$2I = (a+b) \iint_D d\sigma = 4(a+b), I = 2(a+b)$$

积分法

直角坐标法

$$\iint_D f(x,y) d\sigma$$

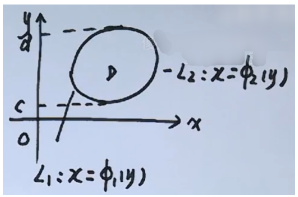
1. x型区域



$$D = (x,y)|a \leq x \leq b, \Phi_1(x) \leq y \leq \Phi_2(x)$$

$$\iint_D f(x,y) d\sigma = \int_a^b dx \int_{\Phi_1(x)}^{\Phi_2(x)} f(x,y) dy$$

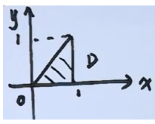
2. y型区域



$$D = \{(x,y)|c \leq x \leq d, \phi_1(y) \leq x \leq \phi_2(y)\}$$

$$\iint_D f(x,y) d\sigma = \int_c^d dy \int_{\phi_1(y)}^{\phi_2(y)} f(x,y) dx$$

$$\text{计算} \iint_D x^2 y dx dy, \text{其中} D \text{由} y = x, x = 1 \text{及} x \text{轴所围成}$$



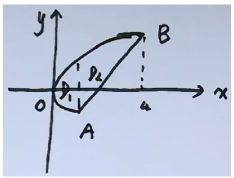
$$D = \{(x,y)|0 \leq x \leq 1, 0 \leq y \leq x\}$$

$$\iint_D x^2 y d\sigma = \int_0^1 x^2 dx \int_0^x y dy = \frac{1}{2} \int_0^1 x^4 dx = \frac{1}{10}$$

$$D = \{(x,y)|0 \leq y \leq 1, y \leq x \leq 1\}$$

$$\iint_D x^2 y d\sigma = \int_0^1 y dy \int_y^1 x^2 dx = \frac{1}{3} \int_0^1 (y - y^4) dy = \frac{1}{10}$$

$$计算I=\iint_D xdxdy,其中D由x=y^2与y=x-2围成$$



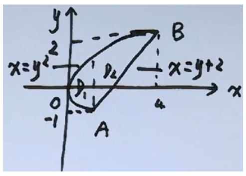
$$由\begin{cases}x=y^2\\y=x-2\end{cases}\Rightarrow A(1,-1),B(4,2)$$

$$D_1=\{(x,y)|0\le x\le 1,-\sqrt{x}\le y\le\sqrt{x}\}$$

$$D_2=\{(x,y)|1\le x\le 4,x-2\le y\le\sqrt{x}\}$$

$$原式=\int_0^1 xdx\int_{-\sqrt{x}}^{\sqrt{x}}1dy+\int_1^4 xdx\int_{x-2}^{\sqrt{x}}1dy$$

$$=2\int_0^1 x^{\frac{3}{2}}dx+\int_1^4(x^{\frac{3}{2}}+x^2-2x)dx$$



$$D=\{(x,y)|y^2\le x\le y+2,-1\le y\le 2\}$$

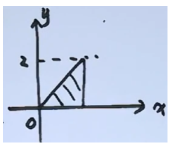
$$原式=\int_{-1}^2 dy\int_{y^2}^{y+2}xdx$$

$$x^{2n}e^{\pm x^2}dx$$

$$e^{\frac{k}{x}}dx$$

$$\cos\frac{k}{x}dx,\sin\frac{k}{x}dx$$

$$计算I=\int_0^2 dy\int_y^2 e^{-x^2}dx$$



$$I=\int_0^2 dy\int_y^2 e^{-x^2}dx$$

$$=\int_0^2 xe^{-x^2}dx$$

$$=-\frac{1}{2}e^{-x^2}\ |_0^2$$

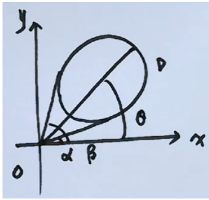
$$=-\frac{1}{2}(\frac{1}{e^4}-1)$$

极坐标法

特征

$$
\begin{array}{c}
1.\ D\text{边界区域含}x^2+y^2 \\
2.\ f(x,y)\text{中含}x^2+y^2
\end{array}
$$

变换

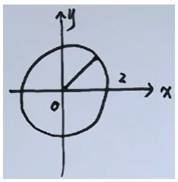


$$
\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}
$$

$$
\alpha \le \theta \le \beta, r_1(\theta) \le r \le r_2(\theta)
$$

面积

$$
d\sigma = r dr d\theta
$$

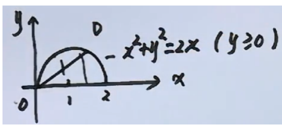


$$
\text{计算}I = \iint_D (x^2+3xy)d\sigma, \text{其中圆域}x^2+y^2 \le 4
$$

$$
I = \iint_D (x^2+3xy)d\sigma = \iint_D x^2 d\sigma = \iint_D y^2 d\sigma = \frac{1}{2}\iint_D (x^2+y^2)d\sigma
$$

$$
\text{令}\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}(0 \le \theta \le 2\pi, 0 \le r \le 2)
$$

$$
I = \frac{1}{2}\int_0^{2\pi} d\theta \int_0^2 r^3 dr = 4\pi
$$



$$
I = \iint_D x^2 d\sigma, D\text{由}y = \sqrt{2x-x^2}\text{与}x\text{轴围成}
$$

$$
\text{令}\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}(0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 2\cos\theta)
$$

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r^3 \cos^2\theta dr \\
&= \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta \int_0^{2\cos\theta} r^3 dr \\
&= 4 \int_0^{\frac{\pi}{2}} \cos^6\theta \\
&= 4 * \frac{5}{6} * \frac{3}{4} * \frac{1}{2} * \frac{\pi}{2}
\end{aligned}
$$