反常积分（广义积分）

$$若f(x)在[a,b]上除有限个第一类间断点外连续 \Rightarrow 可积$$

$$\begin{aligned}反常积分:&f(x)积分区间无限\\&或f(x)在[a,b]上有无穷间断点\end{aligned}$$

积分区间无限

右区间无限

$$\begin{aligned}
&f(x)\in C[a,+\infty):\\
&\int_a^b f(x)dx=F(b)-F(a)\\
&F(b)-F(a)与\int_a^{+\infty} f(x)dx\\
&1.F(b)-F(a)与\int_a^{+\infty} f(x)dx不同\\
&2.\lim_{b\to+\infty}[F(b)-F(a)]与\int_a^{+\infty} f(x)dx同
\end{aligned}$$

$$\begin{aligned}
&若\lim_{b\to\infty}[F(b)-F(a)]=A,称\int_a^{+\infty} f(x)dx收敛于A\Rightarrow\int_a^{+\infty} f(x)dx=A\\
&若\lim_{b\to\infty}[F(b)-F(a)]不存在,称\int_a^{+\infty} f(x)dx发散
\end{aligned}$$

$$\begin{aligned}
&\int_1^{+\infty}\frac{dx}{\sqrt{x}(1+x)}\\
&\forall b>1,\int_1^b\frac{dx}{\sqrt{x}(1+x)}=2\int_1^b\frac{d\sqrt{x}}{1+(\sqrt{x})^2}=2\arctan\sqrt{x}\,|_1^b=2(\arctan\sqrt{b}-\frac{\pi}{4})\\
&\because\lim_{b\to+\infty}2(\arctan\sqrt{b}-\frac{\pi}{4})=\frac{\pi}{2}\\
&\therefore\int_1^{+\infty}\frac{dx}{\sqrt{x}(1+x)}=\frac{\pi}{2}
\end{aligned}$$

判别法

$$若\exists\alpha>1:\lim_{x\to+\infty}x^\alpha f(x)存在\Rightarrow 收敛$$

$$若\exists\alpha\le 1:\lim_{x\to+\infty}x^\alpha f(x)\begin{cases}=A\neq 0\\=\infty\end{cases}\Rightarrow 发散$$

$$\begin{aligned}
&\int_0^{+\infty}\frac{\sqrt{x}dx}{4+x^2}\\
&\because\lim_{x\to\infty}x^{\frac{3}{2}}*\frac{\sqrt{x}}{4+x^2}=1且\alpha=\frac{3}{2}>1\\
&\therefore\int_0^{+\infty}\frac{\sqrt{x}dx}{4+x^2}收敛
\end{aligned}$$

Gamma函数

$$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx$$

$$\int_0^{+\infty}x\sqrt{x}e^{-x}dx=\int_0^{+\infty}x^{\frac{3}{2}}e^{-x}dx=\Gamma(\frac{5}{2})$$

$$\begin{cases}\Gamma(\alpha+1)=\alpha\Gamma(\alpha)\\ \Gamma(n+1)=n!\\ \Gamma(\frac{1}{2})=\sqrt{\pi}\end{cases}$$

$$\int_0^{+\infty}x^7e^{-x^2}dx=\frac{1}{2}\int_0^{+\infty}(x^2)^3e^{-x^2}dx^2=\frac{1}{2}\int_0^{+\infty}x^3e^{-x}dx=\frac{1}{2}\Gamma(4)=3$$
$$\int_0^{+\infty}x\sqrt{x}e^{-x}dx=\int_0^{+\infty}x^{\frac{3}{2}}e^{-x}dx=\Gamma(\frac{3}{2}+1)=\frac{3}{2}\Gamma(\frac{3}{2})=\frac{3}{2}*\frac{1}{2}*\Gamma(\frac{1}{2})=\frac{3\sqrt{\pi}}{4}$$
$$\int_0^{+\infty}x^2e^{-x^2}dx=\int_0^{+\infty}te^{-t}\frac{1}{2\sqrt{t}}dt=\frac{1}{2}\int_0^{+\infty}\sqrt{t}e^{-t}dt=\frac{1}{2}\Gamma(\frac{1}{2}+1)=\frac{\sqrt{\pi}}{4}$$

左区间无限

$$f(x)\in C(-\infty,a],\int_{-\infty}^{a}f(x)dx$$
$$\forall b<a,\int_b^a f(x)dx=F(a)-F(b)$$
$$若\lim_{b\to-\infty}[F(a)-F(b)]=A\Rightarrow\int_{-\infty}^{a}f(x)dx=A$$
$$若\lim_{b\to-\infty}[F(a)-F(b)]不存在\Rightarrow\int_{-\infty}^{a}f(x)dx发散$$

判别法

$$若\exists\alpha>1:\lim_{x\to-\infty}x^\alpha f(x)存在\Rightarrow 收敛$$
$$若\exists\alpha\leq 1:\lim_{x\to-\infty}x^\alpha f(x)\begin{cases}=A\neq 0\\ =\infty\end{cases}\Rightarrow 发散$$

双侧无限

$$f(x)\in C(-\infty,+\infty):\int_{-\infty}^{+\infty}f(x)dx$$

$$\int_{-\infty}^{+\infty}f(x)dx收敛\Leftrightarrow\int_{-\infty}^{a}f(x)dx与\int_a^{+\infty}f(x)dx皆收敛$$

$$问\int_{-\infty}^{+\infty}\frac{x}{1+x^2}dx?(先确定敛散性)$$

对$\int_0^{+\infty} \frac{x}{1+x^2}dx$

$\because \lim_{x\to+\infty} x^1 \cdot \frac{x}{1+x^2} = 1(\neq 0)$且$\alpha = 1 \leq 1$

$\therefore \int_0^{+\infty} \frac{x}{1+x^2}dx$发散

$\therefore \int_{-\infty}^{+\infty} \frac{x}{1+x^2}dx$发散

积分区间有限 函数存在无穷间断点

左端点无穷

$f(x) \in C(a,b]$, 且$f(a+0) = \infty$

$\forall \epsilon > 0, \int_{a+\epsilon}^{b} f(x)dx = F(b) - F(a+\epsilon)$

1.$F(b) - F(a)$与$\int_a^b f(x)dx$不同

2.$\lim_{\epsilon\to0^+}[F(b) - F(a+\epsilon)]$与$\int_a^b f(x)dx$同

若$\lim_{\epsilon\to0^+}[F(b) - F(a+\epsilon)] = A$, 称$\int_a^b f(x)dx$收敛于$A$, 记$\int_a^b f(x)dx = A$

若$\lim_{\epsilon\to0^+}[F(b) - F(a+\epsilon)]$不存在, 称$\int_a^b f(x)dx$发散

$$\int_1^2 \frac{1}{x\sqrt{x-1}}dx$$

解 :$\forall\epsilon > 0$

$\int_{1+\epsilon}^{2} \frac{dx}{x\sqrt{x-1}} = 2\int_{1+\epsilon}^{2} \frac{d(\sqrt{x-1})}{1+(\sqrt{x-1})^2}$

$= 2\arctan\sqrt{x-1} \mid_{1+\epsilon}^{2} = 2(\frac{\pi}{4} - \arctan\sqrt{\epsilon})$

$\because \lim_{\epsilon\to0^+} 2(\frac{\pi}{4} - \arctan\sqrt{\epsilon}) = \frac{\pi}{2}$

$\therefore \int_1^2 \frac{1}{x\sqrt{x-1}}dx = \frac{\pi}{2}$

$\int_0^1 \frac{dx}{\sqrt{x(x+1)}}$

$\lim_{x\to0^+}(x-0)^{\frac{1}{2}} \frac{1}{\sqrt{x(x+1)}} = 1$且$\alpha = \frac{1}{2} < 1, \therefore$ 收敛

$\int_0^1 \frac{dx}{\sqrt{x(x+1)}} = 2\int_0^1 \frac{d\sqrt{x}}{\sqrt{(\sqrt{x})^2+1}} = 2\int_0^1 \frac{dx}{\sqrt{x^2+1}} = 2\ln(x+\sqrt{x^2+1}) \mid_0^1 = 2\ln(1+\sqrt{2})$

判别法

$$若\exists\alpha<1:\lim_{x\to a^+}(x-a)^\alpha f(x)存在\Rightarrow 收敛$$

$$若\exists\alpha\ge 1:\lim_{x\to a^+}(x-a)^\alpha f(x)\begin{cases}=A\ne 0\\=\infty\end{cases}\Rightarrow 发散$$

右端点无穷

$$f(x)\in C[a,b)且f(b-0)=\infty,\int_a^b f(x)dx$$

$$\forall\epsilon>0,\int_a^{b-\epsilon}f(x)dx=F(b-\epsilon)-F(a)$$

$$\lim_{\epsilon\to 0^+}[F(b-\epsilon)-F(a)]=A\Rightarrow\int_a^b f(x)dx=A$$

$$\lim_{\epsilon\to 0^+}[F(b-\epsilon)-F(a)]不存在\Rightarrow\int_a^b f(x)dx发散$$

判别法

$$若\exists\alpha<1:\lim_{x\to b^-}(b-x)^\alpha f(x)存在\Rightarrow 收敛$$

$$若\exists\alpha\ge 1:\lim_{x\to b^-}(b-x)^\alpha f(x)\begin{cases}=A\ne 0\\=\infty\end{cases}\Rightarrow 发散$$

无穷间断点在内部

$$f(x)\in C[a,c)\cup(c,b]且\lim_{x\to c}f(x)=\infty$$

$$\int_a^b f(x)dx收敛\Leftrightarrow\int_a^c f(x)dx与\int_c^b f(x)dx皆收敛$$

$$\int_0^2\frac{dx}{\sqrt{2x-x^2}}$$

$$\begin{aligned}
解:1.\because\lim_{x\to 0^+}(x-0)^{\frac{1}{2}}\cdot\frac{1}{\sqrt{x}\sqrt{2-x}}&=\frac{1}{\sqrt{2}}且\alpha=\frac{1}{2}<1\\
又\because\lim_{x\to 2^-}(2-x)^{\frac{1}{2}}\frac{1}{\sqrt{x}\sqrt{2-x}}&=\frac{1}{\sqrt{2}}且\alpha=\frac{1}{2}<1\\
\therefore 收敛&\\
2.\int_0^2\frac{dx}{\sqrt{2x-x^2}}&=\int_0^2\frac{d(x-1)}{\sqrt{1-(x-1)^2}}=\int_{-1}^1\frac{dx}{\sqrt{1-x^2}}\\
&=2\int_0^1\frac{dx}{\sqrt{1-x^2}}=2\arcsin x\mid_0^1=\pi
\end{aligned}$$

$$\int_0^{+\infty}\frac{dx}{\sqrt{x}(x+1)}$$

解：1. $\because \lim\limits_{x\to 0^+}(x-0)^{\frac{1}{2}}\cdot\dfrac{1}{\sqrt{x}\sqrt{x+1}}=1$且$\alpha=\dfrac{1}{2}<1$

又 $\because \lim\limits_{x\to+\infty}x^{\frac{3}{2}}\dfrac{1}{\sqrt{x}(x+1)}=1$且$\alpha=\dfrac{3}{2}>1$, $\therefore$ 收敛

2. $\displaystyle\int_0^{+\infty}\frac{dx}{\sqrt{x}(x+1)}=2\int_0^{+\infty}\frac{d(\sqrt{x})}{1+(\sqrt{x})^2}$

$=2\arctan\sqrt{x}\ |_0^{+\infty}$

$=\pi$