反常积分（广义积分）

$$若f(x)在[a,b]上除有限个第一类间断点外连续 \Rightarrow 可积$$

$$\begin{aligned}反常积分:&f(x)积分区间无限\\&或f(x)在[a,b]上有无穷间断点\end{aligned}$$

积分区间无限

右区间无限

$$\begin{aligned}&f(x)\in C[a,+\infty):\\&\int_a^b f(x)dx=F(b)-F(a)\\&F(b)-F(a)与\int_a^{+\infty} f(x)dx\\&1.F(b)-F(a)与\int_a^{+\infty} f(x)dx不同\\&2.\lim_{b\to+\infty}[F(b)-F(a)]与\int_a^{+\infty} f(x)dx同\end{aligned}$$

$$\begin{aligned}&若\lim_{b\to\infty}[F(b)-F(a)]=A,称\int_a^{+\infty} f(x)dx收敛于A\Rightarrow\int_a^{+\infty} f(x)dx=A\\&若\lim_{b\to\infty}[F(b)-F(a)]不存在,称\int_a^{+\infty} f(x)dx发散\end{aligned}$$

$$\begin{aligned}&\int_1^{+\infty}\frac{dx}{\sqrt{x}(1+x)}\\&\forall b>1,\int_1^b\frac{dx}{\sqrt{x}(1+x)}=2\int_1^b\frac{d\sqrt{x}}{1+(\sqrt{x})^2}=2\arctan\sqrt{x}\mid_1^b=2(\arctan\sqrt{b}-\frac{\pi}{4})\\&\because\lim_{b\to+\infty}2(\arctan\sqrt{b}-\frac{\pi}{4})=\frac{\pi}{2}\\&\therefore\int_1^{+\infty}\frac{dx}{\sqrt{x}(1+x)}=\frac{\pi}{2}\end{aligned}$$

判别法

$$若\exists\alpha>1:\lim_{x\to+\infty}x^\alpha f(x)存在\Rightarrow 收敛$$

$$若\exists\alpha\le1:\lim_{x\to+\infty}x^\alpha f(x)\begin{cases}=A\ne0\\=\infty\end{cases}\Rightarrow 发散$$

$$\begin{aligned}&\int_0^{+\infty}\frac{\sqrt{x}dx}{4+x^2}\\&\because\lim_{x\to\infty}x^{\frac{3}{2}}*\frac{\sqrt{x}}{4+x^2}=1且\alpha=\frac{3}{2}>1\\&\therefore\int_0^{+\infty}\frac{\sqrt{x}dx}{4+x^2}收敛\end{aligned}$$

Gamma函数

$$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx$$

$$\int_0^{+\infty}x\sqrt{x}e^{-x}dx=\int_0^{+\infty}x^{\frac{3}{2}}e^{-x}dx=\Gamma(\frac{5}{2})$$

$$\begin{cases}\Gamma(\alpha+1)=\alpha\Gamma(\alpha)\\ \Gamma(n+1)=n!\\ \Gamma(\frac{1}{2})=\sqrt{\pi}\end{cases}$$

$$\int_0^{+\infty}x^7e^{-x^2}dx=\frac{1}{2}\int_0^{+\infty}(x^2)^3e^{-x^2}dx^2=\frac{1}{2}\int_0^{+\infty}x^3e^{-x}dx=\frac{1}{2}\Gamma(4)=3$$
$$\int_0^{+\infty}x\sqrt{x}e^{-x}dx=\int_0^{+\infty}x^{\frac{3}{2}}e^{-x}dx=\Gamma(\frac{3}{2}+1)=\frac{3}{2}\Gamma(\frac{3}{2})=\frac{3}{2}*\frac{1}{2}*\Gamma(\frac{1}{2})=\frac{3\sqrt{\pi}}{4}$$
$$\int_0^{+\infty}x^2e^{-x^2}dx=\int_0^{+\infty}te^{-t}\frac{1}{2\sqrt{t}}dt=\frac{1}{2}\int_0^{+\infty}\sqrt{t}e^{-t}dt=\frac{1}{2}\Gamma(\frac{1}{2}+1)=\frac{\sqrt{\pi}}{4}$$

左区间无限

$$f(x)\in C(-\infty,a],\int_{-\infty}^{a}f(x)dx$$
$$\forall b<a,\int_b^a f(x)dx=F(a)-F(b)$$
$$\text{若}\lim_{b\to-\infty}[F(a)-F(b)]=A\Rightarrow\int_{-\infty}^{a}f(x)dx=A$$
$$\text{若}\lim_{b\to-\infty}[F(a)-F(b)]\text{不存在}\Rightarrow\int_{-\infty}^{a}f(x)dx\text{发散}$$

判别法

$$\text{若}\exists\alpha>1:\lim_{x\to-\infty}x^{\alpha}f(x)\text{存在}\Rightarrow\text{收敛}$$
$$\text{若}\exists\alpha\le 1:\lim_{x\to-\infty}x^{\alpha}f(x)\begin{cases}=A\neq 0\\ =\infty\end{cases}\Rightarrow\text{发散}$$

双侧无限

$$f(x)\in C(-\infty,+\infty):\int_{-\infty}^{+\infty}f(x)dx$$

$$\int_{-\infty}^{+\infty}f(x)dx\text{收敛}\Leftrightarrow\int_{-\infty}^{a}f(x)dx\text{与}\int_a^{+\infty}f(x)dx\text{皆收敛}$$

$$\text{问}\int_{-\infty}^{+\infty}\frac{x}{1+x^2}dx?(\text{先确定敛散性})$$

对$\int_0^{+\infty}\frac{x}{1+x^2}dx$

$\because \lim\limits_{x\to+\infty}x^1\cdot\frac{x}{1+x^2}=1(\neq 0)$且$\alpha=1\leq 1$

$\therefore \int_0^{+\infty}\frac{x}{1+x^2}dx$发散

$\therefore \int_{-\infty}^{+\infty}\frac{x}{1+x^2}dx$发散

积分区间有限 函数存在无穷间断点

左端点无穷

$f(x)\in C(a,b]$, 且$f(a+0)=\infty$

$\forall\epsilon>0, \int_{a+\epsilon}^{b}f(x)dx=F(b)-F(a+\epsilon)$

1.$F(b)-F(a)$与$\int_a^b f(x)dx$不同

2.$\lim\limits_{\epsilon\to0^+}[F(b)-F(a+\epsilon)]$与$\int_a^b f(x)dx$同

若$\lim\limits_{\epsilon\to0^+}[F(b)-F(a+\epsilon)]=A$, 称$\int_a^b f(x)dx$收敛于$A$, 记$\int_a^b f(x)dx=A$

若$\lim\limits_{\epsilon\to0^+}[F(b)-F(a+\epsilon)]$不存在, 称$\int_a^b f(x)dx$发散

$$\int_1^2\frac{1}{x\sqrt{x-1}}dx$$

解 :$\forall\epsilon>0$

$\int_{1+\epsilon}^{2}\frac{dx}{x\sqrt{x-1}}=2\int_{1+\epsilon}^{2}\frac{d(\sqrt{x-1})}{1+(\sqrt{x-1})^2}$

$=2\arctan\sqrt{x-1}\mid_{1+\epsilon}^{2}=2(\frac{\pi}{4}-\arctan\sqrt{\epsilon})$

$\because \lim\limits_{\epsilon\to0^+}2(\frac{\pi}{4}-\arctan\sqrt{\epsilon})=\frac{\pi}{2}$

$\therefore \int_1^2\frac{1}{x\sqrt{x-1}}dx=\frac{\pi}{2}$

$\int_0^1\frac{dx}{\sqrt{x(x+1)}}$

$\lim\limits_{x\to0^+}(x-0)^{\frac{1}{2}}\frac{1}{\sqrt{x(x+1)}}=1$且$\alpha=\frac{1}{2}<1, \therefore$ 收敛

$\int_0^1\frac{dx}{\sqrt{x(x+1)}}=2\int_0^1\frac{d\sqrt{x}}{\sqrt{(\sqrt{x})^2+1}}=2\int_0^1\frac{dx}{\sqrt{x^2+1}}=2\ln(x+\sqrt{x^2+1})\mid_0^1=2\ln(1+\sqrt{2})$

判别法

若$\exists\alpha<1:\lim\limits_{x\to a^+}(x-a)^\alpha f(x)$存在 $\Rightarrow$ 收敛

若$\exists\alpha\geq1:\lim\limits_{x\to a^+}(x-a)^\alpha f(x)\begin{cases}=A\neq0\\=\infty\end{cases}\Rightarrow$ 发散

右端点无穷

$$
\begin{aligned}
& f(x) \in C[a,b)\text{且}f(b-0)=\infty, \int_a^b f(x)dx \\
& \forall \epsilon > 0, \int_a^{b-\epsilon} f(x)dx = F(b-\epsilon)-F(a) \\
& \lim_{\epsilon \to 0^+}[F(b-\epsilon)-F(a)] = A \Rightarrow \int_a^b f(x)dx = A \\
& \lim_{\epsilon \to 0^+}[F(b-\epsilon)-F(a)]\text{不存在} \Rightarrow \int_a^b f(x)dx\text{发散}
\end{aligned}
$$

判别法

$$
\begin{aligned}
& \text{若}\exists\alpha<1: \lim_{x\to b^-}(b-x)^\alpha f(x)\text{存在} \Rightarrow \text{收敛} \\
& \text{若}\exists\alpha\geq 1: \lim_{x\to b^-}(b-x)^\alpha f(x) \begin{cases} = A \neq 0 \\ = \infty \end{cases} \Rightarrow \text{发散}
\end{aligned}
$$

无穷间断点在内部

$$
f(x) \in C[a,c)\cup(c,b]\text{且}\lim_{x\to c} f(x) = \infty
$$

$$
\int_a^b f(x)dx\text{收敛} \Leftrightarrow \int_a^c f(x)dx\text{与}\int_c^b f(x)dx\text{皆收敛}
$$

$$
\int_0^2 \frac{dx}{\sqrt{2x-x^2}}
$$

$$
\begin{aligned}
\text{解}: 1. & \because \lim_{x\to 0^+}(x-0)^{\frac{1}{2}} \cdot \frac{1}{\sqrt{x}\sqrt{2-x}} = \frac{1}{\sqrt{2}}\text{且}\alpha = \frac{1}{2} < 1 \\
& \text{又}\because \lim_{x\to 2^-}(2-x)^{\frac{1}{2}} \frac{1}{\sqrt{x}\sqrt{2-x}} = \frac{1}{\sqrt{2}}\text{且}\alpha = \frac{1}{2} < 1 \\
& \therefore \text{收敛} \\
2. & \int_0^2 \frac{dx}{\sqrt{2x-x^2}} = \int_0^2 \frac{d(x-1)}{\sqrt{1-(x-1)^2}} = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} \\
& = 2\int_0^1 \frac{dx}{\sqrt{1-x^2}} = 2\arcsin x \mid_0^1 = \pi
\end{aligned}
$$

$$
\int_0^{+\infty} \frac{dx}{\sqrt{x}(x+1)}
$$

$$
\begin{aligned}
\text{解}: 1. & \because \lim_{x\to 0^+}(x-0)^{\frac{1}{2}} \cdot \frac{1}{\sqrt{x}\sqrt{x+1}} = 1\text{且}\alpha = \frac{1}{2} < 1 \\
& \text{又}\because \lim_{x\to +\infty} x^{\frac{3}{2}} \frac{1}{\sqrt{x}(x+1)} = 1\text{且}\alpha = \frac{3}{2} > 1, \therefore \text{收敛} \\
2. & \int_0^{+\infty} \frac{dx}{\sqrt{x}(x+1)} = 2\int_0^{+\infty} \frac{d(\sqrt{x})}{1+(\sqrt{x})^2} \\
& = 2\arctan\sqrt{x} \mid_0^{+\infty} \\
& = \pi
\end{aligned}
$$