向量理论（二）

$$\text{向量理论(二)}\begin{cases}\text{向量组等价}\\ \text{极大线性无关组与向量组的秩}\end{cases}$$

向量组等价

1.向量组等价

Ⅰ：$\alpha_1, \alpha_2, \ldots, \alpha_m$

Ⅱ：$\beta_1, \beta_2, \ldots, \beta_n$

$$\text{若}\begin{cases}\alpha_1 = k_{11}\beta_1 + \ldots + k_{1n}\beta_n\\ \ldots\\ \alpha_m = k_{m1}\beta_1 + \ldots + k_{mn}\beta_n\end{cases} \quad ①$$

称Ⅰ可由Ⅱ线性表示

$$\text{若}\begin{cases}\beta_1 = l_{11}\alpha_1 + \ldots + l_{1m}\alpha_m\\ \ldots\\ \beta_n = l_{n1}\alpha_1 + \ldots + l_{nm}\alpha_m\end{cases} \quad ②$$

称Ⅱ可由Ⅰ线性表示, 若①②成立, 称Ⅰ Ⅱ等价

极大线性无关组与向量组的秩

2.极大线性无关组与向量组的秩

若①$\exists r$个向量线性无关

②$\forall r+1$个向量线性相关或不存在$r+1$个向量

称r个线性无关的向量为极大线性无关组

极大组所含向量的个数称为向量组的秩

Notes:

①极大组不一定唯一

如：$\alpha_1=\begin{pmatrix}1\\2\end{pmatrix},\alpha_2=\begin{pmatrix}1\\-1\end{pmatrix},\alpha_3=\begin{pmatrix}3\\1\end{pmatrix}$线性相关

$\alpha_1\alpha_2,\alpha_1\alpha_3,\alpha_2\alpha_3$皆为极大组

②向量组与极大组等价

证：设α_1,α_2为$\alpha_1,\alpha_2,\alpha_3$的极大组

$\because\begin{cases}\alpha_1=1\alpha_1+0\alpha_2+0\alpha_3\\\alpha_2=0\alpha_1+1\alpha_2+0\alpha_3\end{cases},\therefore\alpha_1,\alpha_2$可由$\alpha_1,\alpha_2,\alpha_3$线性表示

又$\because\begin{cases}\alpha_1=1\alpha_1+0\alpha_2\\\alpha_2=0\alpha_1+1\alpha_2\\\alpha_3=k_1\alpha_1+k_2\alpha_2\end{cases},\therefore\alpha_1,\alpha_2,\alpha_3$可由$\alpha_1,\alpha_2$线性表示

$\therefore\alpha_1,\alpha_2$与$\alpha_1,\alpha_2,\alpha_3$等价

③Ⅰ：$\alpha_1,\dots,\alpha_n$

case 1. $\alpha_1\dots\alpha_n$线性无关$\Leftrightarrow$ Ⅰ的秩$=n$

case 2. $\alpha_1\dots\alpha_n$线性相关$\Leftrightarrow$ Ⅰ的秩$<n$

④Ⅰ：$\alpha_1,\dots,\alpha_n$; Ⅱ：$\alpha_1,\dots,\alpha_n,b$

case 1. Ⅰ秩$=$ Ⅱ秩$\Leftrightarrow b$可由$\alpha_1\dots\alpha_n$线性表示

case 2. Ⅱ秩$=$ Ⅰ秩$+1\Leftrightarrow b$不可由$\alpha_1\dots\alpha_n$线性表示

⑤$A_{m\times n}=\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_m\end{pmatrix}=(\beta_1,\beta_2,\dots,\beta_n)$

$\alpha_1,\alpha_2,\dots,\alpha_m-A$的行向量组，$\alpha_1\dots\alpha_m$的秩称为$A$的行秩

$\beta_1,\dots,\beta_n-A$的列向量组，$\beta_1\dots\beta_n$的秩称为A的列秩

⑥$\begin{cases}AB=A(\beta_1\dots\beta_s)=(A\beta_1,\dots,A\beta_s)\\\text{如：}A=(\alpha_1,\alpha_2),B=\begin{pmatrix}2&1\\1&-3\end{pmatrix},AB=(2\alpha_1+\alpha_2,\alpha_1-3\alpha_2)\\(\alpha_1+\alpha_2,2\alpha_2-\alpha_3,\alpha_1+4\alpha_3)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&0&1\\1&2&0\\0&-1&4\end{pmatrix}\end{cases}$

性质

1.矩阵的秩，矩阵的行秩，矩阵的列秩相等

Notes: 设$\alpha_1\dots\alpha_n$为向量组，$A=(\alpha_1,\dots,\alpha_n)$

①$\alpha_1\dots\alpha_n$线性无关$\Leftrightarrow r(A)=n$

②$\alpha_1\dots\alpha_n$线性相关$\Leftrightarrow r(A)<n$

$\alpha_1,\alpha_2,\alpha_3$线性无关，$\beta_1=\alpha_1+\alpha_2,\beta_2=\alpha_2+\alpha_3,\beta_3=\alpha_3+\alpha_1$

证：β_1,β_2,β_3线性无关.

证：$A=(\alpha_1,\alpha_2,\alpha_3),r(A)=3$

$$B=(\beta_1,\beta_2,\beta_3)=A\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}$$

$$\because\begin{vmatrix}1&0&1\\1&1&0\\0&1&1\end{vmatrix}=2\neq0,\therefore\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}\text{可逆}$$

$\therefore r(B)=r(A)=3,\therefore\beta_1,\beta_2,\beta_3$线性无关

α_1,α_2线性无关,α_3不可由α_1,α_2线性表示,α_4可由α_1,α_2线性表示,

证 : $\alpha_1,\alpha_2,\alpha_3-\alpha_4$线性无关.

证 :法一,令$\alpha_4=l_1\alpha_1+l_2\alpha_2$

设$k_1\alpha_1+k_2\alpha_2+k_3(\alpha_3-\alpha_4)=0$

$\Rightarrow (k_1-l_1k_3)\alpha_1+(k_2-l_2k_3)\alpha_2+k_3\alpha_3=0$

$\because \alpha_1,\alpha_2$无关,$\alpha_3$不可由$\alpha_1,\alpha_2$表示,$\therefore \alpha_1,\alpha_2,\alpha_3$无关

$$\therefore \begin{cases} k_1-l_1k_3=0 \\ k_2-l_2k_3=0 \Rightarrow k_1=0,k_2=0,k_3=0 \Rightarrow \alpha_1,\alpha_2,\alpha_3-\alpha_4\text{线性无关} \\ k_3=0 \end{cases}$$

法二,$\because \alpha_1,\alpha_2$线性无关

又$\because \alpha_3-\alpha_4$不可由$\alpha_1,\alpha_2$线性表示

$\therefore \alpha_1,\alpha_2,\alpha_3-\alpha_4$线性无关

法三,$\because \alpha_1,\alpha_2$无关,$\alpha_3$不可由$\alpha_1,\alpha_2$表示,$\therefore \alpha_1,\alpha_2,\alpha_3$无关,令$A=(\alpha_1,\alpha_2,\alpha_3),r(A)=3$

$\alpha_4=l_1\alpha_1+l_2\alpha_2$

$B=(\alpha_1,\alpha_2,\alpha_3-\alpha_4)=(\alpha_1,\alpha_2,-l_1\alpha_1-l_2\alpha_2+\alpha_3)$

$$=A\begin{pmatrix}1&0&-l_1\\0&1&-l_2\\0&0&1\end{pmatrix}$$

$$\because \begin{pmatrix}1&0&-l_1\\0&1&-l_2\\0&0&1\end{pmatrix}\text{可逆},\therefore r(B)=r(A)=3\Rightarrow \alpha_1,\alpha_2,\alpha_3-\alpha_4\text{无关}$$

Notes:

①研究向量组相关性,可以研究矩阵的秩

$\alpha_1,\ldots,\alpha_n,A=(\alpha_1\ldots\alpha_n)$

②已知矩阵,问行向量和列向量相关性

$A_{m\times n}\neq 0,B_{n\times s}\neq 0,AB=0(D)$

$(A)A$行相关,B行相关;$(B)A$列相关,B列相关

$(C)A$行相关,B列相关;$(D)A$列相关,B行相关

$A\neq 0,B\neq 0\Rightarrow r(A)\geq 1,r(B)\geq 1$

$AB=0\Rightarrow r(A)+r(B)\leq n\Rightarrow r(A)<n,r(B)<n$

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2. Ⅰ : $\alpha_1\ldots\alpha_m$; Ⅱ : $\beta_1\ldots\beta_n$

若Ⅰ可由Ⅱ线性表示$\Rightarrow$ Ⅰ秩$\leq$ Ⅱ秩

证 :令$A=(\alpha_1\ldots\alpha_m),B=(\beta_1\ldots\beta_n)$

$$\because \begin{cases} \alpha_1=k_{11}\beta_1+\ldots+k_{1n}\beta_n \\ \ldots \\ \alpha_m=k_{m1}\beta_1+\ldots+k_{mn}\beta_n \end{cases}$$

$$\therefore A=B\begin{pmatrix}k_{11}&&k_{m1}\\ \vdots&\ldots&\vdots\\ k_{1n}&&k_{mn}\end{pmatrix}=BK$$

而$r(A)=r(BK)\leq r(B)$

$\therefore$ Ⅰ秩$\leq$ Ⅱ秩

Ⅰ：$\alpha_1,\dots,\alpha_r$；Ⅱ：$\beta_1\dots\beta_s$

Ⅰ可由Ⅱ线性表示，且$r>s$

证：Ⅰ可由Ⅱ线性表示 $\Rightarrow$ Ⅰ秩 $\leq$ Ⅱ秩

而Ⅱ秩 $\leq s$

$\therefore$ Ⅰ秩 $\leq s<r \Rightarrow$ Ⅰ秩 $<r \Rightarrow$ Ⅰ线性相关

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3. ⅠⅡ等价 $\Rightarrow$ Ⅰ秩 $=$ Ⅱ秩，$\not\Leftarrow$

$$\not\Leftarrow,\ \text{Ⅰ}:\alpha_1=\begin{pmatrix}1\\2\\0\\0\end{pmatrix},\alpha_2=\begin{pmatrix}1\\-1\\0\\0\end{pmatrix},\ \text{Ⅰ秩}=2$$

$$\text{Ⅱ}:\beta_1=\begin{pmatrix}0\\0\\2\\3\end{pmatrix},\beta_2=\begin{pmatrix}0\\0\\3\\-1\end{pmatrix},\ \text{Ⅱ秩}=2$$

Ⅰ不可由Ⅱ线性表示，Ⅱ也不可由Ⅰ线性表示，ⅠⅡ不等价