幂级数

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$
$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + \dots$$

收敛域

收敛域：一切收敛点构成的集合

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$
$$x = \frac{2}{3} : 1 + \frac{2}{3} + (\frac{2}{3})^2 + \dots = \frac{1}{1-\frac{1}{3}} = 3, x = \frac{2}{3}\text{收敛点}$$
$$x = 2 : 1 + 2 + 2^2 + \dots, x = 2\text{发散点}$$

收敛半径

收敛半径：R

$$\sum_{n=0}^{\infty} a_n x^n : (\text{Abel})\exists R \geq 0$$
$$①\text{当}|x| < R\text{或}x \in (-R, R)：\text{绝对收敛}$$
$$②\text{当}|x| > R\text{或}x \in (-\infty, -R) \cup (R, +\infty)：\text{发散}$$
$$③\text{当}|x| = R, \text{即}x = \pm R :?$$

R及收敛域

$$\sum_{n=0}^{\infty} a_n x^n$$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \rho\text{或}\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$$
$$①\rho = +\infty \Rightarrow R = 0, \text{唯一收敛域}x = 0$$
$$②\rho = 0 \Rightarrow R = +\infty, \text{收敛域}(-\infty, +\infty)$$
$$③0 < \rho < +\infty \Rightarrow R = \frac{1}{\rho}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

$$\because \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1, \therefore R = 1$$
$$\text{当}x = \pm 1\text{时}, |\frac{(\pm 1)^n}{n^2}| = \frac{1}{n^2}, \text{级数绝对收敛.}$$
$$\text{收敛域}[-1, 1]$$

$$\sum_{n=0}^{\infty} n! x^n$$

$$\because \lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=1, \therefore R=1$$

当$x=-1$时, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$收敛; 当$x=1$时, $\sum_{n=1}^{\infty}\frac{1}{n}$发散.

$\therefore$ 收敛域$[-1,1)$

Notes :

①如$\sum_{n=0}^{\infty}a_nx^{2n+1}$, 若$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=\rho$或$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\rho$

$$\Rightarrow R=\sqrt{\frac{1}{\rho}}$$

②对$\sum_{n=0}^{\infty}a_nx^n$, 若$\sum_{n=1}^{\infty}a_nx_0^n$条件收敛 $\Rightarrow |x_0|=R$

分析性质

$$\sum_{n=0}^{\infty}a_nx^n=S(x), x\in(-R,R)$$

$$S(x)\in C(-R,R)$$

$x=-R$为收敛点，则$\sum_{n=0}^{\infty}a_n(-R)^n=S(-R+0)$

$x=R$为收敛点，则$\sum_{n=0}^{\infty}a_nR^n=S(R-0)$

(逐项可导性)$x\in(-R,R)$:

$$S'(x)=(\sum_{n=0}^{\infty}a_nx^n)'=\sum_{n=0}^{\infty}(a_nx^n)'=\sum_{n=1}^{\infty}na_nx^{n-1}$$

且$\sum_{n=1}^{\infty}na_nx^{n-1}$收敛半径$R$

(逐项可积性)$x\in(-R,R)$:

$$\int_0^x S(x)dx=\int_0^x(\sum_{n=0}^{\infty}a_nx^n)dx=\sum_{n=0}^{\infty}\int_0^x a_nx^ndx=\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$$

且$\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$收敛半径$R$

函数展成幂级数

$$f(x): x=x_0$$

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

直接法

$f(x)$在$x=x_0$邻域内有$n+1$阶导数

$\Rightarrow f(x)=P_n(x)+R_n(x)$

$$P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$R_n(x)=\begin{cases}\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}, \text{拉格朗日型}\\ o((x-x_0)^n), \text{皮亚诺型}\end{cases}$$

$$f(x)\text{在}x=x_0\text{邻域内任意阶可导}$$
$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,(?)$$
$$x_0=0:f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n,\text{麦克劳林级数},(?)$$

$$①e^x=1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!},(-\infty<x<+\infty)$$
$$②\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1},(-\infty<x<+\infty)$$
$$③\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n},(-\infty<x<+\infty)$$
$$④\frac{1}{1-x}=1+x+\ldots+x^n+\ldots=\sum_{n=0}^{\infty}x^n,(-1<x<1)$$
$$⑤\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n,(-1<x<1)$$
$$⑥\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n,(-1<x\le 1)$$
$$\text{Note: }1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\ln 2$$
$$⑦-\ln(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n},(-1\le x<1)$$

间接法

$$\begin{cases}\text{①到⑦}\\ \text{逐项可导, 逐项可积}\end{cases}$$

$$\text{将}f(x)=\frac{1}{x^2-3x+2}\text{展开成}x-4\text{的幂级数.}$$

$$f(x)=\frac{1}{x-2}-\frac{1}{x-1}=\frac{1}{2+(x-4)}-\frac{1}{3+(x-4)}$$
$$\frac{1}{2+(x-4)}=\frac{1}{2}\cdot\frac{1}{1+\frac{x-4}{2}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-4)^n(2<x<6)$$
$$\frac{1}{3+(x-4)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-4)^n(1<x<7)$$
$$\Rightarrow f(x)=\sum_{n=0}^{\infty}(-1)^n(\frac{1}{2^{n+1}}-\frac{1}{3^{n+1}})(x-4)^n(2<x<6)$$

$$\text{将}f(x)=\frac{5x-1}{x^2-x-2}\text{展开成}x-1\text{的幂级数.}$$

$$f(x)=\frac{5x-1}{(x+1)(x-2)}=\frac{A}{x-2}+\frac{B}{x+1}$$

$$\text{由}A(x+1)+B(x-2)=5x-1\Rightarrow\begin{cases}A=3\\B=2\end{cases}$$

$$f(x)=\frac{3}{x-2}+\frac{2}{x+1}$$

$$=\frac{2}{2+(x-1)}-\frac{3}{1-(x-1)}=\frac{1}{1+\frac{x-1}{2}}-3\frac{1}{1-(x-1)}$$

$$\frac{1}{1+\frac{x-1}{2}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}(x-1)^n(-1<x<3)$$

$$\frac{1}{1-(x-1)}=\sum_{n=0}^{\infty}(x-1)^n(0<x<2)$$

$$\therefore f(x)=\sum_{n=0}^{\infty}[\frac{(-1)^n}{2^n}-3](x-1)^n(0<x<2)$$

将$f(x)=\arctan x$展开成x的幂级数.

$$f(0)=0$$

$$f'(x)=\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}(-1<x<1)$$

$$f(x)=f(x)-f(0)=\int_0^x f'(x)dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}(-1\le x\le 1)$$

求S(x)

$$\sum P(n)x^n:④⑤$$

$$\sum\frac{x^n}{P(n)}\begin{cases}⑥⑦\\ \text{逐项可积性}\end{cases}$$

求$\sum_{n=0}^{\infty}n^2x^n$的$S(x)$.

$$1.\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=1\Rightarrow R=1$$

$$x=\pm1\text{时},n^2\cdot(\pm1)^n\nrightarrow 0(n\to\infty),\text{收敛域为}(-1,1)$$

$$2.S(x)=\sum_{n=1}^{\infty}n^2x^n=\sum_{n=1}^{\infty}[n(n-1)+n]x^n$$

$$=x^2\sum_{n=2}^{\infty}n(n-1)x^{n-2}+x\sum_{n=1}^{\infty}nx^{n-1}$$

$$=x^2(\sum_{n=2}^{\infty}x^n)''+x(\sum_{n=1}^{\infty}x^n)'$$

$$=x^2(\frac{x^2}{1-x})''+x(\frac{x}{1-x})'$$

$\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$,求$S(x)$.

1. $\lim\limits_{n\to\infty}|\dfrac{a_{n+1}}{a_n}|=1\Rightarrow R=1$

$x=\pm1$时,$|\dfrac{(\pm1)^n}{n(n+1)}|\sim\dfrac{1}{n^2}$且$\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$收敛

$\Rightarrow$ 收敛域$[-1,1]$

2.$S(x)=\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n(n+1)}=\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n}-\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n+1}$

①$S(0)=0$

②$x\neq0$:

$$S(x)=-\ln(1-x)-\frac{1}{x}\sum_{n=2}^{\infty}\frac{x^n}{n}$$

$$=-\ln(1-x)-\frac{1}{x}(\sum_{n=1}^{\infty}\frac{x^n}{n}-x)$$

$$=(\frac{1}{x}-1)\ln(1-x)+1(-1\leq x<1\text{且}x\neq0)$$

③$S(1)=\sum\limits_{n=1}^{\infty}(\dfrac{1}{n}-\dfrac{1}{n+1})$

$S_n=(1-\dfrac{1}{2})+\ldots+(\dfrac{1}{n}-\dfrac{1}{n+1})=1-\dfrac{1}{n+1}$

$\because\lim\limits_{n\to\infty}S_n=1,\therefore S(1)=1$

$$\therefore S(x)=\begin{cases}0,x=0\\1,x=1\\(\frac{1}{x}-1)\ln(1-x)+1,-1\leq x<1\text{且}x\neq0\end{cases}$$