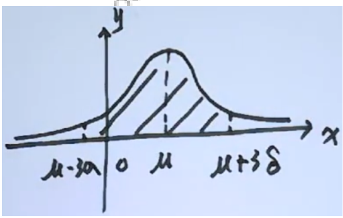
大数定律与中心极限定理

切比雪夫不等式



$$X \sim N(\mu, \sigma^2)$$
$$\frac{X-\mu}{\sigma} \sim N(0,1)$$
$$P\{\mu - 3\sigma < X < \mu + 3\sigma\} = P\{-3 < \frac{X-\mu}{\sigma} \leq 3\}$$
$$= P\{\frac{X-\mu}{\sigma} \leq 3\} - P\{\frac{X-\mu}{\sigma} \leq -3\} = \Phi(3) - \Phi(-3) = 2\Phi(3) - 1$$

$$\text{Th.}X - \text{r.v.}\exists EX, DX, \forall \epsilon > 0, \text{有}$$
$$P\{|X - EX| < \epsilon\} \geq 1 - \frac{DX}{\epsilon^2}$$
$$\Leftrightarrow P\{|X - EX| \geq \epsilon\} \leq \frac{DX}{\epsilon^2}$$

$$\text{设}X \sim N(1,4), Y \sim E(1), \text{且}X, Y\text{相互独立, 用切比雪夫不等式估计}$$
$$P\{-3 < X + Y < 7\}.$$

$$\text{令}Z = X + Y$$
$$EZ = 1 + 1 = 2, DZ = DX + DY = 5$$
$$P\{-3 < X + Y < 7\} = P\{|Z - EZ| < 5\} \geq 1 - \frac{DZ}{25} = 0.8$$

大数定律

$$\{X_n\}\text{为r.v.序列}, a\text{常数, 若}\forall \epsilon > 0$$
$$\lim_{n\to\infty} P\{|X_n - a| < \epsilon\} = 1$$
$$\text{称}X_n\text{以概率}P\text{收敛于}a(n \to \infty)$$

$$X_n \xrightarrow{P} a \ (n \to \infty)$$

$$\text{Th1(车氏)若①}x_1, \cdots, x_n, \cdots \text{独立}$$
$$\text{②}\exists C > 0, \text{使}DX_i \leq C(i = 1, 2, \cdots), \text{则}$$
$$\forall \epsilon > 0, \text{有} \lim_{n\to\infty} P\{|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} EX_i| < \epsilon\} = 1$$
$$\text{即}\frac{1}{n}\sum_{i=1}^{n} X_i\text{以概率}P\text{收敛于}\frac{1}{n}\sum_{i=1}^{n} EX_i$$

$$\frac{1}{n}\sum_{i=1}^{n}X_i \xrightarrow{P} \frac{1}{n}\sum_{i=1}^{n}EX_i$$

Th2(独立同分布)若①$x_1,\cdots,x_n,\cdots$独立同分布

②$\exists EX_i=\mu, DX_i=\sigma^2(i=1,2,\cdots)$，则

$$\forall\epsilon>0,\text{有}\lim_{n\to\infty}P\{|\frac{1}{n}\sum_{i=1}^{n}X_i-\mu|<\epsilon\}=1$$

$$\text{即}\frac{1}{n}\sum_{i=1}^{n}X_i\text{以概率}P\text{收敛于}\mu$$

$$\frac{1}{n}\sum_{i=1}^{n}X_i \xrightarrow{P} \mu$$

Th3(辛钦)若①$x_1,\cdots,x_n,\cdots$独立同分布

②$\exists EX_i=\mu(i=1,2,\cdots)$，则

$$\forall\epsilon>0,\text{有}\lim_{n\to\infty}P\{|\frac{1}{n}\sum_{i=1}^{n}X_i-\mu|<\epsilon\}=1$$

$$\text{即}\frac{1}{n}\sum_{i=1}^{n}X_i\text{以概率}P\text{收敛于}\mu$$

$$\frac{1}{n}\sum_{i=1}^{n}X_i \xrightarrow{P} \mu$$

中心极限定理

Th1(Levy-Lindberg)若①$x_1,\cdots,x_n,\cdots$独立同分布

②$\exists EX_i=\mu, DX_i=\sigma^2$，则

$$\sum_{i=1}^{n}X_i\dot{\sim}N(n\mu,n\sigma^2)$$

$$\Rightarrow\frac{\sum_{i=1}^{n}X_i-n\mu}{\sqrt{n}\sigma}\dot{\sim}N(0,1)$$

$$\forall x\in R, P\{\frac{\sum_{i=1}^{n}X_i-n\mu}{\sqrt{n}\sigma}\le x\}\approx\Phi(x)$$

设$X_1,X_2,\cdots,X_{30}$独立同分布于$U(0,2)$，用中心极限定理估计$P\{\sum_{i=1}^{30}X_i\le 35\}$.

$$EX_i=1, DX_i=\frac{4}{12}=\frac{1}{3}(1\le i\le 30)$$

$$\sum_{i=1}^{30}X_i\dot{\sim}N(30,10)\Rightarrow\frac{\sum_{i=1}^{30}X_i-30}{\sqrt{10}}\dot{\sim}N(0,1)$$

$$P\{\sum_{i=1}^{30}X_i\le 35\}=P\{\frac{\sum_{i=1}^{30}X_i-30}{\sqrt{10}}\le\frac{5}{\sqrt{10}}\}\approx\Phi(\frac{\sqrt{10}}{2})$$

$$\text{Th2(Laplace)}X_n\sim B(n,p)$$

$$\Rightarrow X_n\dot{\sim}N(np,np(1-p))$$

$$\frac{X_n-np}{\sqrt{np(1-p)}}\dot{\sim}N(0,1)$$

$$\forall x\in R, P\{\frac{X_n-np}{\sqrt{np(1-p)}}\le x\}\approx\Phi(x)$$

$$X\sim B(100,\frac{1}{10}),\text{用中心极限定理估计}P\{X\le 16\}.$$

$$X\dot{\sim}N(10,9)\Rightarrow\frac{X-10}{3}\dot{\sim}N(0,1)$$

$$P\{X\le 16\}=P\{\frac{X-10}{3}\le 2\}\approx\Phi(2)$$