三重积分

概念

$$\Omega - \text{有界闭区域}, f(x,y,z)\text{在}\Omega\text{上有界}$$

$$\begin{aligned}
&1.\Omega \Rightarrow \Delta v_1, \dots, \Delta v_n \\
&2.\forall(\xi_i, \eta_i, \zeta_i) \in \Delta v_i (1 \le i \le n), \text{作} \\
&\quad \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \cdot \Delta v_i \\
&3.\lambda\text{为}\Delta v_1, \dots, \Delta v_n\text{的直径最大者} \\
&\quad \text{若} \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \Delta v_i \exists, \text{称此极限为} \\
&\quad f(x,y,z)\text{在}\Omega\text{上的三重积分}, \iiint_{\Omega} f(x,y,z) dv, \text{即} \\
&\quad \iiint_{\Omega} f(x,y,z) dv \triangleq \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \Delta v_i
\end{aligned}$$

性质

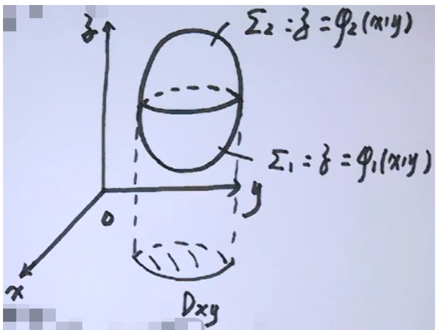
$$\begin{aligned}
&4. \iiint_{\Omega} 1 dV = V \\
&5.\text{①设}\Omega\text{关于}xoy\text{面对称(上下对称), 上}\Omega_1 \\
&\quad \text{若} f(x,y,-z) = -f(x,y,z) \Rightarrow \iiint_{\Omega} f(x,y,z) dV = 0 \\
&\quad \text{若} f(x,y,-z) = f(x,y,z) \Rightarrow \iiint_{\Omega} f(x,y,z) dV = 2 \iiint_{\Omega_1} f(x,y,z) dV
\end{aligned}$$

积分法

直角坐标法

$$\text{对} \iiint_{\Omega} f(x,y,z) dV$$

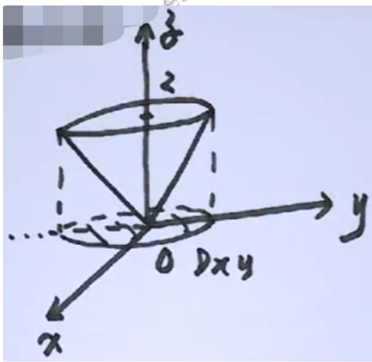
铅直投影法



$$\Omega: \begin{cases} (x,y) \in D_{xy} \\ \phi_1(x,y) \le z \le \phi_2(x,y) \end{cases}$$

$$\iiint_{\Omega} f(x,y,z) dV = \iint_{D_{xy}} dxdy \int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z) dz$$

计算 $\iiint_{\Omega}(z^2+2xy)dV$, 其中$\Omega$为锥面$z=\sqrt{x^2+y^2}$与$z=2$所围成的几何体.

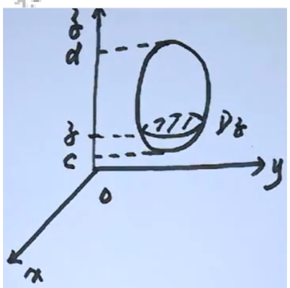


$$
\text{原式}=I=\iiint_{\Omega} z^2 dV
$$

$$
\Omega:\begin{cases}(x,y)\in D_{xy}: x^2+y^2\le 4\\ \sqrt{x^2+y^2}\le z\le 2\end{cases}
$$

$$
\begin{aligned}
I&=\iint_{D_{xy}} dxdy\int_{\sqrt{x^2+y^2}}^{2} z^2 dz\\
&=\frac{1}{3}\iint_{D_{xy}}[8-(x^2+y^2)^{\frac{3}{2}}]dxdy\\
&=\frac{1}{3}\int_0^{2\pi}d\theta\int_0^2(8r-r^4)dr=\frac{2\pi}{3}(16-\frac{32}{5})=\frac{2\pi}{3}\times\frac{48}{5}=\frac{32}{5}\pi
\end{aligned}
$$

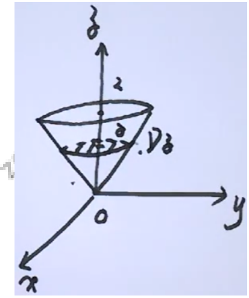
切片法



$$
\Omega:\begin{cases}(x,y)\in D_z\\ c\le z\le d\end{cases}
$$

$$
\iiint_{\Omega} f(x,y,z)dV=\int_c^d dz\iint_{D_z} f(x,y,z)dxdy
$$

计算 $\iiint_{\Omega}(z^2+2xy)dV$, 其中$\Omega$为锥面$z=\sqrt{x^2+y^2}$与$z=2$所围成的几何体.

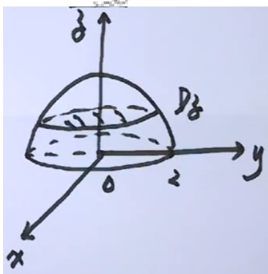


$$原式 = I = \iiint_{\Omega} z^2 dV$$

$$\Omega : \begin{cases} (x,y) \in D_z : x^2+y^2 \le z^2 \\ 0 \le z \le 2 \end{cases}$$

$$原式 = \int_0^2 z^2 dz \iint_{D_z} 1dxdy = \pi \int_0^2 z^4 dz = \frac{32}{5}\pi$$

计算$\iiint_{\Omega}(x^2+y^2-xy+z)dV$,其中$\Omega$为由$z=\sqrt{4-x^2-y^2}$及$xOy$平面围成的几何体.



$$原式 = I = \iiint_{\Omega}(x^2+y^2+z)dV$$

$$法一 :\Omega \begin{cases} (x,y) \in D_{xy} \\ 0 \le z \le \sqrt{4-x^2-y^2} \end{cases}$$

$$\begin{aligned} I &= \iint_{D_{xy}} dxdy \int_0^{\sqrt{4-x^2-y^2}} (x^2+y^2+z)dz \\ &= \iint_{D_{xy}} [(x^2+y^2) \cdot \sqrt{4-x^2-y^2} + \frac{1}{2}(4-x^2-y^2)]dxdy \\ &=2\pi \int_0^2 [r^3\sqrt{4-r^2} + \frac{1}{2}(4r-r^3)]dr \\ &=2\pi[\int_0^{\frac{\pi}{2}} 8\sin^3 t \cdot 4(1-\sin^2 t)dt + \frac{1}{2}(8-4)] \\ &=2\pi[32(\frac{2}{3}\times 1 - \frac{4}{5}\times\frac{2}{3}\times 1)+2] \\ &=\frac{188\pi}{15} \end{aligned}$$

$$法二 :I = \iiint_{\Omega}(x^2+y^2+z)dV$$

$$\Omega \begin{cases} (x,y) \in D_z : x^2+y^2 \le 4-z^2 \\ 0 \le z \le 2 \end{cases}$$

$$\begin{aligned} I &= \int_0^2 dz \iint_{D_z} (x^2+y^2+z)dxdy \\ &=2\pi \int_0^2 dz \int_0^{\sqrt{4-z^2}} (r^3+zr)dr \\ &=2\pi \int_0^2 [\frac{(4-z^2)^2}{4} + z\cdot\frac{4-z^2}{2}]dz \\ &=2\pi \int_0^2 (4+\frac{1}{4}z^4) - 2z^2 + 2z - \frac{z^3}{2}dz = \frac{188\pi}{15} \end{aligned}$$

柱面坐标变换法

1.特征：

①Ω边界曲面含x^2+y^2

②$f(x,y,z)$中含x^2+y^2

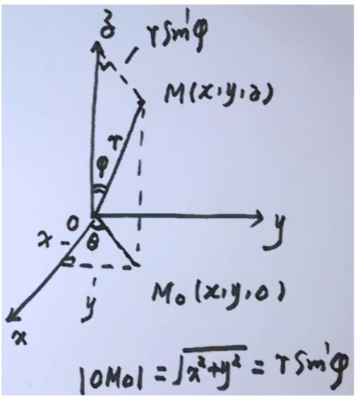
2.步骤：

第一步：$\Omega:\begin{cases}(x,y)\in D_{xy}\\ \phi_1(x,y)\le z\le \phi_2(x,y)\end{cases}$

第二步：令$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\\ z=z\end{cases},\begin{cases}\alpha\le\theta\le\beta\\ r_1(\theta)\le r\le r_2(\theta)\\ \phi_1(\dots)\le z\le\phi_2(\dots)\end{cases}$

第三步：$dV=rdrd\theta dz$

球面坐标变换法



1.特征：

①Ω的边界曲面含$x^2+y^2+z^2$

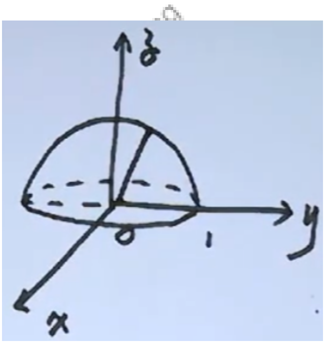
②$f(x,y,z)$中含$x^2+y^2+z^2$

2.变换：

$\begin{cases}x=r\cos\theta\sin\phi\\ y=r\sin\theta\sin\phi\\ z=r\cos\phi\end{cases}$

3.$dV=r^2\sin\phi drd\theta d\phi$

计算$\iiint_\Omega (x^2+y^2)dV$，其中$\Omega=\{(x,y,z)|x^2+y^2+z^2\le 1,z\ge 0\}$.



$$\text{令}\begin{cases}x=r\cos\theta\sin\phi\\y=r\sin\theta\sin\phi\\z=r\cos\phi\end{cases}$$

$$(0\le\theta\le 2\pi,0\le\phi\le\frac{\pi}{2},0\le r\le 1)$$

$$\begin{aligned}\text{原式}=&\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\phi\int_0^1 r^2\sin^2\phi\cdot r^2\sin\phi dr\\=&2\pi\times\frac{2}{3}\times 1\times\frac{1}{5}\\=&\frac{4\pi}{15}\end{aligned}$$

$$f(u)\text{连续},f(0)=0,f'(0)=2,\Omega:x^2+y^2+z^2\le t^2(t>0,z\ge 0)$$

$$\text{求}\lim_{t\to 0}\frac{\iiint_\Omega f(\sqrt{x^2+y^2+z^2})dV}{t^4}$$

$$\begin{aligned}\iiint_\Omega f(\sqrt{x^2+y^2+z^2})dV=&\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\phi\int_0^t f(r)\cdot r^2\sin\phi dr\\=&2\pi\int_0^t r^2f(r)dr\\\text{原式}=&\frac{\pi}{2}\lim_{t\to 0}\frac{t^2f(t)}{t^3}\\=&\frac{\pi}{2}\lim_{t\to 0}\frac{f(t)-f(0)}{t}=\pi\end{aligned}$$

$$I=\iiint_\Omega\sqrt{x^2+y^2}dV$$

$$\text{法一}:\Omega\begin{cases}(x,y)\in D_{xy}:x^2+y^2\le 4\\0\le z\le\sqrt{4-x^2-y^2}\end{cases}$$

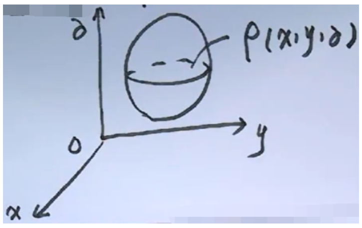
$$\begin{aligned}\text{原式}=&\iint_{D_{xy}}\sqrt{x^2+y^2}dxdy\cdot\int_0^{\sqrt{4-x^2-y^2}}1dz=\iint_{D_{xy}}\sqrt{x^2+y^2}\cdot\sqrt{4-x^2-y^2}dxdy\\=&2\pi\int_0^2 r^2\sqrt{4-r^2}dr\\=&2\pi\int_0^{\frac{\pi}{2}}4\sin^2t\cdot 4(1-\sin^2t)dt,r=2\sin t\\=&32\pi(I_2-I_4)\\=&32\pi(\frac{1}{2}\times\frac{\pi}{2}-\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2})\\=&2\pi^2\end{aligned}$$

$$\text{法二}:\Omega\begin{cases}(x,y)\in D_{xy}:x^2+y^2\le 4-z^2\\0\le z\le 2\end{cases}$$

$$\begin{aligned}\text{原式}=&\int_0^2dz\iint_{D_z}\sqrt{x^2+y^2}dxdy\\=&2\pi\int_0^2dz\int_0^{\sqrt{4-z^2}}r^2dr=\frac{2\pi}{3}\int_0^2(\sqrt{4-z^2})^3dz\\=&\frac{2\pi}{3}\int_0^{\frac{\pi}{2}}8\cos^3t\cdot 2\cos tdt,z=2\sin t\\=&\frac{32\pi}{3}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}=2\pi^2\end{aligned}$$

$$
\text{法三：令}\begin{cases} x = r\cos\theta\sin\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\phi \end{cases}, \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{2} \\ 0 \le r \le 2 \end{cases}
$$

$$
\begin{aligned}
\text{原式} &= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\phi \int_0^2 r\sin\phi \cdot r^2 \sin\phi dr \\
&= 2\pi \int_0^{\frac{\pi}{2}} \sin^2\phi d\phi \int_0^2 r^3 dr \\
&= 2\pi \times \frac{1}{2} \times \frac{\pi}{2} \times 4 = 2\pi^2
\end{aligned}
$$



$$
1. m = \iiint_\Omega \rho(x,y,z) dV
$$

$$
\begin{aligned}
2. \overline{x} &= \iiint_\Omega x\rho dV / \iiint_\Omega \rho dV \\
\overline{y} &= \iiint_\Omega y\rho dV / \iiint_\Omega \rho dV \\
\overline{z} &= \iiint_\Omega z\rho dV / \iiint_\Omega \rho dV
\end{aligned}
$$

$$
\text{若}\rho \equiv C_0, x = \iiint_\Omega x dV / \iiint_\Omega dV
$$