求偏导

显函数

$$设z=\arctan\frac{x-y}{x+y},求\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$$

$$解:\frac{\partial z}{\partial x}=\frac{1}{1+(\frac{x-y}{x+y})^2}\times\frac{(x+y)-(x-y)}{(x+y)^2}$$

$$设z=(x^2+y^2)^{xy},求\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}.$$

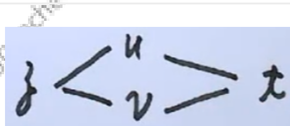
$$解:z=e^{xy\ln(x^2+y^2)}$$

$$\frac{\partial z}{\partial x}=(x^2+y^2)^{xy}\cdot[y\ln(x^2+y^2)+xy\cdot\frac{2x}{x^2+y^2}]$$

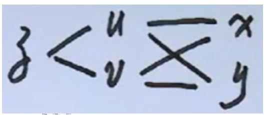
$$z=\arctan\frac{x+y}{1-xy}$$

$$\frac{\partial z}{\partial x}=\frac{1}{1+(\frac{x+y}{1-xy})^2}\frac{(1-xy)-(x+y)(-y)}{(1-xy)^2}$$

复合函数



$$1.z=f(u,v)\begin{cases}u=\Phi(t)\\v=\phi(t)\end{cases}$$



$$2.z=f(u,v)\begin{cases}u=u(x,y)\\v=v(x,y)\end{cases}$$

$$1.z=f(x^2+y^2):z为x,y的二元函数,f一元,z=f(u),u=x^2+y^2$$
$$2.z=f(e^t,t^2):z一元,f二元$$
$$3.z=f(x+y,xy):z二元,f二元$$
$$4.z=f(x^3,x+y,\frac{y}{x}):z二元,f三元$$

$$5.z=f(u,v),\begin{cases}u=\dots\\v=\dots\end{cases}$$

$$\frac{\partial f}{\partial u}\triangleq f_1,f_u,f_1(u,v);\frac{\partial f}{\partial v}\triangleq f_2,f_v,f_2(u,v)$$

$$\frac{\partial^2 f}{\partial u^2}\triangleq f_{11};\frac{\partial^2 f}{\partial u\partial v}\triangleq f_{12};\dots$$

$$设f(u,v)二阶连续可偏导,且z=f(t,\sin t),求\frac{d^2z}{dt^2}$$

$$解:\frac{\partial z}{\partial t}=f_1+\cos tf_2$$

$$\frac{\partial^2 z}{\partial t^2}=f_{11}+\cos t\cdot f_{12}-\sin tf_2+\cos t\cdot(f_{21}+\cos t\cdot f_{22})$$

$$=f_{11}+2\cos t\cdot f_{12}-\sin tf_2+\cos^2 t\cdot f_{22}$$

$$z=f(xy,x+y),\text{求}\frac{\partial^2 z}{\partial x\partial y}.$$

$$\begin{aligned}\frac{\partial z}{\partial x}&=yf_1+f_2\\ \frac{\partial^2 z}{\partial x\partial y}&=f_1+y(xf_{11}+f_{12})+xf_{21}+f_{22}\\ &=f_1+xyf_{11}+(x+y)f_{12}+f_{22}\end{aligned}$$

$$\text{设}z=f(x+y,xy,2x),\text{其中}f\text{二阶连续可偏导,求}\frac{\partial^2 z}{\partial x\partial y}.$$

$$\begin{aligned}\frac{\partial z}{\partial x}&=f_1+yf_2+2f_3\\ \frac{\partial^2 z}{\partial x\partial y}&=(f_{11}+xf_{12})+[f_2+y(f_{21}+xf_{22})]+2(f_{31}+xf_{32})\\ &=f_{11}+(x+y)f_{12}+f_2+xyf_{22}+2f_{31}+2xf_{32}\end{aligned}$$

$$z=f(x^2\sin y),\text{求}\frac{\partial^2 z}{\partial x\partial y}.$$

$$\begin{aligned}\frac{\partial z}{\partial x}&=f'(x^2\sin y)2x\sin y\\ \frac{\partial^2 z}{\partial x\partial y}&=f''(x^2\sin y)x^2\cos y2x\sin y+f'(x^2\sin y)2x\cos y\end{aligned}$$

$$z=f(t^2,\sin t),\text{求}\frac{dz}{dt}.$$

$$\begin{aligned}\frac{dz}{dt}&=2tf_1+\cos tf_2\\ \frac{d^2z}{dt^2}&=2f_1+2t(2tf_{11}+\cos tf_{12})-\sin tf_2+\cos t(2tf_{21}+\cos tf_{22})=2f_1-\sin tf_2+4t^2f_{11}+4t\cos tf_{12}+\cos^2 tf_{22}\end{aligned}$$

隐函数(组)

$$\begin{aligned}&1.F(x,y)=0:\text{一个一元}\\ &\quad\text{若}F_x\neq 0,\text{则由}F(x,y)=0\Rightarrow x=\Phi(x)\\ &\quad\text{若}F_y\neq 0,\text{则由}F(x,y)=0\Rightarrow y=\phi(x)\\ &2.F(x,y,z)=0:\text{一个二元}\\ &\quad\text{若}F_z\neq 0,\text{由}F(x,y,z)=0\Rightarrow z=\Phi(x,y)\\ &3.\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases}:\text{两个一元}\Rightarrow\begin{cases}y=y(x)\\z=z(x)\end{cases}\end{aligned}$$

$$\text{设}z=z(x,y)\text{由}\ln\sqrt{x^2+y^2+z^2}=xyz+1\text{确定,求}\frac{\partial z}{\partial x}.$$

$$\begin{aligned}&\text{解}:\ln\sqrt{x^2+y^2+z^2}=xyz+1\Rightarrow z=z(x,y)\\ &\frac{1}{\sqrt{x^2+y^2+z^2}}\times\frac{2x+2z\cdot\frac{\partial z}{\partial x}}{2\sqrt{x^2+y^2+z^2}}=y(z+x\frac{\partial z}{\partial x})\\ &x+z\frac{\partial z}{\partial x}=yz(x^2+y^2+z^2)+xy(x^2+y^2+z^2)\frac{\partial z}{\partial x}\end{aligned}$$

$$\text{设}\begin{cases}x^2+2y^2+3z^2=21\\x-3y+2z=5\end{cases}\text{求}\frac{dy}{dx},\frac{dz}{dx}.$$

$$1.\begin{cases}x^2+2y^2+3z^2=21\\x-3y+2z=5\end{cases}\Rightarrow\begin{cases}y=y(x)\\z=z(x)\end{cases}$$

$$2.\begin{cases}2x+4y\cdot\frac{dy}{dx}+6z\cdot\frac{dz}{dx}=0\\1-3\frac{dy}{dx}+2\frac{dz}{dx}=5\end{cases}$$

$$\Rightarrow\begin{cases}2y\cdot\frac{dy}{dx}+3z\cdot\frac{dz}{dx}=-x\\3\frac{dy}{dx}-2\frac{dz}{dx}=1\end{cases}$$

$$D=\begin{vmatrix}2y & 3z\\3 & -2\end{vmatrix}=-(4y+9z)$$

$$D_1=\begin{vmatrix}-x & 3z\\3 & -2\end{vmatrix}=2x-3z,D_2=\begin{vmatrix}2y & -x\\3 & 1\end{vmatrix}=3x+2y$$

$$\frac{dy}{dx}=-\frac{2x-3z}{4y+9z},\frac{dz}{dx}=-\frac{3x+2y}{4y+9z}$$

$$F\text{连续可偏导},F(y+z,x+z,x+y)=0$$

$$\text{求}\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}.$$

$$1.F(y+z,x+z,x+y)=0\Rightarrow z=z(x,y)$$

$$2.\begin{cases}F_1\cdot\frac{\partial z}{\partial x}+F_2\cdot(1+\frac{\partial z}{\partial x})+F_3=0\\F_1\cdot(1+\frac{\partial z}{\partial y})+F_2\cdot\frac{\partial z}{\partial y}+F_3=0\end{cases}$$

$$t=F(x,y),y=f(x,t),f,F\text{连续可偏导},\text{求}\frac{dt}{dx}.$$

$$\text{解}:1.\begin{cases}t=F(x,y)\\y=f(x,t)\end{cases}\Rightarrow\begin{cases}y=y(x)\\t=t(x)\end{cases}$$

$$2.\begin{cases}\frac{dt}{dx}=F_1+F_2\cdot\frac{dy}{dx}\\\frac{dy}{dx}=f_1+f_2\cdot\frac{dt}{dx}\end{cases}$$

$$\tan(x+y+z)=x^2+y^2+z,z=z(x,y),\text{求}\frac{\partial z}{\partial x}.$$

$$\sec^2(x+y+z)(1+\frac{\partial z}{\partial x})=2x+\frac{\partial z}{\partial x}\Rightarrow\frac{\partial z}{\partial x}=\frac{2x-\sec^2(x+y+z)}{\tan^2(x+y+z)}$$

$$\begin{cases}x-y+2z=1\\x^2+y^2+4z^2=4\end{cases},\text{求}\frac{dz}{dx}.$$

$$\begin{cases}x-y+2z=1\\x^2+y^2+4z^2=4\end{cases}\Rightarrow\begin{cases}y=y(x)\\z=z(x)\end{cases}$$

$$\begin{cases}1-\frac{dy}{dx}+2\frac{dz}{dx}=0\\2x+2y\frac{dy}{dx}+8z\frac{dz}{dx}=0\end{cases}$$

$$\begin{cases}xu+yv=1\\xv-y^2u=e^{x+y}\end{cases},u=u(x,y),v=v(x,y),\text{求}\frac{\partial u}{\partial x},\frac{\partial v}{\partial y}.$$

$$\begin{cases}xu+yv=1\\xv-y^2u=e^{x+y}\end{cases}\Rightarrow\begin{cases}u=u(x,y)\\v=v(x,y)\end{cases}$$

$$\begin{cases}u+x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial x}=0\\v+x\frac{\partial v}{\partial x}-y^2\frac{\partial u}{\partial x}=e^{x+y}\end{cases}\Rightarrow\begin{cases}\frac{\partial u}{\partial x}\\\frac{\partial v}{\partial x}\end{cases}$$

$$\begin{cases}x\frac{\partial u}{\partial y}+v+y\frac{\partial v}{\partial y}=0\\x\frac{\partial v}{\partial y}-2yu-y^2\frac{\partial u}{\partial y}=e^{x+y}\end{cases}\Rightarrow\begin{cases}\frac{\partial u}{\partial y}\\\frac{\partial v}{\partial y}\end{cases}$$

多元函数极值

一元

$$y=f(x)$$

$$\begin{aligned}&1.x\in D\\&2.f'(x)\begin{cases}=0\\\text{不存在}\end{cases}\\&3.\text{判别法}\\&\quad ①\begin{cases}x<x_0:f'<0\\x>x_0:f'>0\end{cases}\Rightarrow x=x_0\text{为极小点}\\&\quad\begin{cases}x<x_0:f'>0\\x>x_0:f'<0\end{cases},x=x_0\text{为极大点}\\&\quad ②f'(x_0)=0,f''(x_0)\begin{cases}<0,x=x_0\text{为极大点}\\>0,x=x_0\text{为极小点}\end{cases}\end{aligned}$$

二元

$$z=f(x,y)((x,y)\in D),(x_0,y_0)\in D$$

$$\text{若}\exists\delta>0,\text{当}0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta\text{时}$$
$$f(x,y)<f(x_0,y_0)$$
$$(x_0,y_0)\text{为极大点},f(x_0,y_0)\text{为极大值}$$

无条件极值

$$z=f(x,y),(x,y)\in D(\text{开区域})$$
$$\text{求}z=f(x,y)\text{在}D\text{内极值,称为无条件极值}$$

$$\begin{aligned}&1.\ \begin{cases}\frac{\partial z}{\partial x}=0\\\frac{\partial z}{\partial y}=0\end{cases}\Rightarrow\begin{cases}x\\y\end{cases}\\&2.\ \text{设}(x,y)=(x_0,y_0)\\&\quad A=\frac{\partial^2 z}{\partial x^2}\mid_{(x_0,y_0)},B=\frac{\partial^2 z}{\partial x\partial y}\mid_{(x_0,y_0)},C=\frac{\partial^2 z}{\partial y^2}\mid_{(x_0,y_0)}\\&3.\ AC-B^2\begin{cases}<0,\times\\>0,\surd\begin{cases}A>0,(x_0,y_0)\text{为极小点}\\A<0,(x_0,y_0)\text{为极大点}\end{cases}\end{cases}\end{aligned}$$

$$\text{求}z=f(x,y)=x^3-3x^2-9x+y^2+2y+2\text{的极值.}$$

$$\begin{aligned}&1.\ \begin{cases}\frac{\partial z}{\partial x}=3x^2-6x-9=0\\\frac{\partial z}{\partial y}=2y+2=0\end{cases}\Rightarrow\begin{cases}x=-1\\y=-1\end{cases},\begin{cases}x=3\\y=-1\end{cases}\\&2.\ A=\frac{\partial^2 z}{\partial x^2}=6x-6,B=\frac{\partial^2 z}{\partial x\partial y}=0,C=\frac{\partial^2 z}{\partial y^2}=2\\&3.\ (x,y)=(-1,-1),A=-12,B=0,C=2\\&\quad\because AC-B^2<0\Rightarrow(-1,-1)\text{不是极值点}\\&\quad(x,y)=(3,-1),A=12,B=0,C=2\\&\quad AC-B^2>0\text{且}A>0\Rightarrow(3,-1)\text{是极小点}\\&\quad\text{极小值为}f(3,-1)\end{aligned}$$

$$\text{求}z=f(x,y)=x^3-3x+y^2+2y+2\text{的极值点和极值.}$$

1. $\begin{cases}\frac{\partial z}{\partial x}=3x^2-3=0\\ \frac{\partial z}{\partial y}=2y+2=0\end{cases}\Rightarrow\begin{cases}x=-1\\y=-1\end{cases},\begin{cases}x=1\\y=-1\end{cases}$

2. 设$(x,y)=(x_0,y_0)$

$A=\dfrac{\partial^2 z}{\partial x^2}\mid_{(x_0,y_0)}=6x_0,B=\dfrac{\partial^2 z}{\partial x\partial y}\mid_{(x_0,y_0)}=0,C=\dfrac{\partial^2 z}{\partial y^2}\mid_{(x_0,y_0)}=2$

3. $(x_0,y_0)=(-1,-1)$,

$AC-B^2<0\Rightarrow(-1,-1)$不是极值点

$(x_0,y_0)=(1,-1)$,

$AC-B^2>0$且$A>0\Rightarrow(1,-1)$是极小点

条件极值

$$z=f(x,y)\text{在s.t.}\Phi(x,y)=0\text{下的极值.}$$

1. $F=f(x,y)+\lambda\Phi(x,y)$
2. 由$\begin{cases}F_x=f_x+\lambda\Phi_x=0\\F_y=f_y+\lambda\Phi_y=0\\F_\lambda=\Phi(x,y)=0\end{cases}\Rightarrow\begin{cases}x\\y\end{cases}$

求函数$z=f(x,y)=x^2-y^2+2$在$x^2+4y^2\leq4$上的m,M.

解：1.$x^2+4y^2<4$时

由$\begin{cases}z'_x=2x=0\\z'_y=-2y=0\end{cases}\Rightarrow\begin{cases}x=0\\y=0\end{cases},f(0,0)=2$

2.$x^2+4y^2-4=0$时

令$F=x^2-y^2+2+\lambda(x^2+4y^2-4)$

由$\begin{cases}F_x=2x+2\lambda x=0\\F_y=-2y+4\lambda y=0\\F_\lambda=x^2+4y^2-4=0\end{cases}\Rightarrow\begin{cases}x(\lambda+1)=0\\y(2\lambda-1)=0\\x^2+4y^2=4\end{cases}$

若$x=0\Rightarrow\begin{cases}x=0\\y=\pm1\end{cases}$

若$\lambda=-1\Rightarrow y=0\Rightarrow\begin{cases}x=\pm2\\y=0\end{cases}$

$f(0,\pm1)=1,f(\pm2,0)=6$

$\therefore m=1,M=6$