微分方程

可分离变量 微分方程

$$\frac{dy}{dx}=f(x,y)$$
$$f(x,y)=\Phi_1(x)\Phi_2(y)$$
$$\frac{dy}{dx}=\Phi_1(x)\Phi_2(y)\Rightarrow\int\frac{dy}{\Phi_2(y)}=\int\Phi_1(x)dx+C$$

$$\text{求微分方程}\frac{dy}{dx}=2x(1+y^2)\text{的通解.}$$

$$\frac{dy}{1+y^2}=2xdx$$
$$\Rightarrow \arctan y=x^2+C$$
$$y=\tan(x^2+C)$$

$$\frac{dy}{dx}=2xy$$

1. $y=0$为方程的解
2. $y\neq 0$,
   $\frac{dy}{y}=2xdx\Rightarrow \ln|y|=x^2+C_0\Rightarrow|y|=e^{C_0}e^{x^2}$
   $\Rightarrow y=\pm e^{C_0}e^{x^2}$
   令 $\pm e^{C_0}=C(C\neq 0), y=Ce^{x^2}(C\neq 0)$

通解, $y=Ce^{x^2}$(C为任意常数)

齐次 微分方程

$$\frac{dy}{dx}=f(x,y)$$
$$f(x,y)=\Phi(\frac{y}{x})$$
$$\frac{dy}{dx}=\frac{y-x}{y+2x}\Rightarrow\frac{dy}{dx}=\frac{\frac{y}{x}-1}{\frac{y}{x}+2}$$
$$\text{令}\frac{y}{x}=u\Rightarrow y=xu\Rightarrow\frac{dy}{dx}=u+x\frac{du}{dx}$$
$$u+x\frac{du}{dx}=\Phi(u)\Rightarrow\int\frac{du}{\Phi(u)-u}=\int\frac{dx}{x}+C$$

$$\text{求微分方程}\frac{dy}{dx}-\frac{2}{x}y=1\text{的通解.}$$

$$\frac{dy}{dx}=2\frac{y}{x}+1$$

$$令\frac{y}{x}=u,\frac{dy}{dx}=u+x\frac{du}{dx}$$

$$u+x\frac{du}{dx}=2u+1\Rightarrow\frac{du}{u+1}=\frac{dx}{x}$$

$$\Rightarrow\ln(u+1)=\ln x+\ln C$$

$$\Rightarrow u+1=Cx\Rightarrow y=Cx^2-x(C为任意常数)$$

一阶齐次线性 微分方程

$$\frac{dy}{dx}+P(x)y=0$$

$$\frac{dy}{dx}=-P(x)y$$

1. $y=0$为方程的解

2. $y\neq 0,\frac{dy}{y}=-P(x)dx$

$$\Rightarrow\ln|y|=-\int P(x)dx+C_0$$

$$y=\pm e^{C_0}e^{-\int P(x)dx}$$

$$令\pm e^{C_0}=C(C\neq 0),y=Ce^{-\int P(x)dx}(C\neq 0)$$

通解, $y=Ce^{-\int P(x)dx}(C为任意常数)$

求微分方程$y'+2xy=0$的通解.

$$\frac{dy}{dx}+2xy=0$$

$$y=Ce^{-\int 2xdx}=Ce^{-x^2}(C为任意常数)$$

一阶非齐线性 微分方程 (常数变易法)

$$\frac{dy}{dx}+P(x)y=Q(x)$$

$$\frac{dy}{dx}+P(x)y=0\Rightarrow y=Ce^{-\int P(x)dx}(C为任意常数)$$

令$\frac{dy}{dx}+P(x)y=Q(x)$解为$y=C(x)e^{-\int P(x)dx}$, 代入

$$C'(x)e^{-\int P(x)dx}-P(x)C(x)e^{-\int P(x)dx}+P(x)C(x)e^{-\int P(x)dx}=Q(x)$$

$$\Rightarrow C'(x)=Q(x)e^{\int P(x)dx}$$

$$\Rightarrow C(x)=\int Q(x)e^{\int P(x)dx}dx+C$$

通解, $y=[\int Q(x)e^{\int P(x)dx}dx+C]e^{-\int P(x)dx}$

$$\frac{dy}{dx}-\frac{2}{x}y=-1$$

$$P(x)=\frac{2}{x},Q(x)=-1$$

$$y=[\int(-1)e^{-\int\frac{2}{x}dx}dx+C]e^{-\int\frac{2}{x}dx}$$
$$=(\frac{1}{x}+C)x^2=Cx^2+x$$

可降阶的高位 微分方程

$$y^{(n)}=f(x)(n\geq 2)$$

$$设y''=xe^{2x}$$
$$y'=\int xe^{2x}dx+C_1=\frac{1}{2}\int xd(e^{2x})+C_1=\frac{x}{2}e^{2x}-\frac{1}{2}\int e^{2x}dx+C_1=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C_1$$
$$y=\frac{1}{2}\int xe^{2x}dx-\frac{1}{4}\int e^{2x}dx+C_1x+C_2=\frac{1}{4}xe^{2x}-\frac{1}{4}e^{2x}+C_1x+C_2$$

$$f(x,y',y'')=0(缺y)$$
$$令y'=p,y''=\frac{dp}{dx},f(x,p,\frac{dp}{dx})=0$$
$$\Rightarrow p=\Phi(x,C_1),即y'=\Phi(x,C_1)$$
$$y=\int\Phi(x,C_1)dx+C_2$$

$$求微分方程y''+y'=2e^x的通解.$$
$$令y'=p,y''=\frac{dp}{dx}$$
$$\frac{dp}{dx}+p=2e^x$$
$$p=(\int 2e^xe^{\int 1dx}dx+C_1)e^{-\int dx}=C_1e^{-x}+e^x$$
$$y=-C_1e^{-x}+e^x+C_2$$

$$f(y,y',y'')=0(缺x)$$
$$令y'=p,y''=\frac{dp}{dx},f(y,p,\frac{dp}{dx})=0$$
$$y''=\frac{dy}{dx}\frac{dp}{dy}=p\frac{dp}{dy}$$
$$f(y,p,p\frac{dp}{dy})=0$$

$$求微分方程yy''-y'^2=0满足初值条件y(0)=y'(0)=1的特解.$$

$$令y'=p,y''=p\frac{dp}{dy},代入$$

$$yp\frac{dp}{dy}-p^2=0$$

$$\because y'(0)=1,\therefore p\neq 0\Rightarrow \frac{dp}{dy}-\frac{1}{y}p=0$$

$$p=C_1e^{-\int -\frac{1}{y}dy}=C_1y$$

$$即y'=C_1y$$

$$\because y(0)=y'(0)=1,\therefore C_1=1$$

$$\therefore \frac{dy}{dx}-y=0$$

$$y=C_2e^{-\int -dx}=C_2e^x$$

$$\because y(0)=1,\therefore C_2=1,\therefore y=e^x$$

高阶线性 微分方程

二阶齐次线性 微分方程

$$y''+p(x)y'+q(x)y=0(*)$$

二阶非齐线性 微分方程

$$y''+p(x)y'+q(x)y=f(x)(**)$$

$$若f(x)=f_1(x)+f_2(x)$$
$$y''+p(x)y'+q(x)y=f_1(x)(**)'$$
$$y''+p(x)y'+q(x)y=f_2(x)(**)''$$

解的结构

$$\Phi_1(x),...,\Phi_s(x)为(*)解$$
$$\Rightarrow k_1\Phi_1(x)+...+k_s\Phi_s(x)为(*)解$$

$$\Phi_1(x),...,\Phi_s(x)为(**)解$$
$$\Rightarrow k_1\Phi_1(x)+...+k_s\Phi_s(x)为(*)解\Leftrightarrow k_1+...+k_s=0$$
$$\Rightarrow k_1\Phi_1(x)+...+k_s\Phi_s(x)为(**)的解\Leftrightarrow k_1+...+k_s=1$$

$$若\Phi_1(x),\Phi_2(x)为(*)不成比例的解$$
$$(*)通解y=C_1\Phi_1(x)+C_2\Phi_2(x)$$

$$若\Phi_1(x),\Phi_2(x)为(*)不成比例的解,\Phi_0(x)为(**)特解$$
$$则(**)通解y=C_1\Phi_1(x)+C_2\Phi_2(x)+\Phi_0(x)$$

特殊情形下的通解

二阶常系数齐次线性 微分方程

$$y'' + py' + qy = 0(p, q\text{常数})$$
$$\lambda^2 + p\lambda + q = 0$$

$$\begin{aligned}&1.\ \Delta > 0 : \lambda_1 \neq \lambda_2\\&\quad\text{通解}y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}\\&2.\ \Delta = 0 : \lambda_1 = \lambda_2\\&\quad\text{通解}y = (C_1 + C_2x)e^{\lambda_1 x}\\&3.\ \Delta < 0 : \lambda_{1,2} = \alpha \pm i\beta\\&\quad\text{通解}y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)\end{aligned}$$

求微分方程$y'' - y' - 2y = 0$的通解.

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = -1 \neq \lambda_2 = 2$$
$$\text{通解}y = C_1e^{-x} + C_2e^{2x}$$

求微分方程$y'' - 6y' + 9y = 0$的通解.

$$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda_1 = \lambda_2 = 3$$
$$\text{通解}y = (C_1 + C_2x)e^{3x}$$

求微分方程$y'' - 2y' + 5y = 0$的通解.

$$\lambda^2 - 2\lambda + 5 = 0 \Rightarrow \lambda_{1,2} = 1 \pm 2i$$
$$\text{通解}y = e^{\alpha x}(C_1\cos 2x + C_2\sin 2x)$$

二阶常系数非齐线性 微分方程

$$y'' + py' + qy = f(x)(p, q\text{常数})$$

$$f(x) = P(x)e^{kx}$$

求微分方程$y'' - y' - 2y = (2x - 1)e^x$的特解与通解.

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = -1 \neq \lambda_2 = 2$$
$$y'' - y' - 2y = 0 \Rightarrow y = C_1e^{-x} + C_2e^{2x}$$
$$\text{令}y_0(x) = (ax + b)e^x, \text{代入}$$
$$y_0' = (ax + a + b)e^x, y_0'' = (ax + 2a + b)e^x$$
$$ax + 2a + b - ax - a - b - 2ax - 2b = 2x - a$$
$$a = -1, b = 0, \therefore y = C_1e^{-x} + C_2e^{2x} - xe^x$$

求微分方程$y'' + y' - 2y = (2x + 3)e^x$的特解与通解.

$$\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = 1 \neq \lambda_2 = -2$$
$$y'' + y' - 2y = 0 \Rightarrow y = C_1e^x + C_2e^{-2x}$$
$$y_0(x) = x(ax + b)e^x = (ax^2 + bx)e^x, \text{代入}$$

求微分方程$y'' + 2y' + y = (6x-2)e^{-x}$的特解与通解.

$$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1$$
$$y'' + 2y' + y = 0 \Rightarrow y = (C_1 + C_2 x)e^{-x}$$
$$令 y_0(x) = x^2(ax+b)e^{-x} = (ax^3 + bx^2)e^{-x}, 代入$$