连续与间断

连续

$$f(x)\text{在}x=a\text{的邻域内有定义}$$
$$\text{若}\lim_{x\to a}f(x)=f(a),\text{称}f(x)\text{在}x=a\text{连续}$$

$$\begin{array}{l}1.\ f(x)\text{在}x=a\text{连续}\Leftrightarrow f(a-0)=f(a+0)=f(a)\\2.\ \text{初等函数在定义域内皆连续}\end{array}$$

$$\begin{array}{l}1.\ f(x)\text{在}(a,b)\text{内点点连续}\\2.\ f(a)=f(a+0)(\text{右连续}),f(b)=f(b-0)(\text{左连续})\\\quad\text{称}f(x)\text{在}[a,b]\text{上连续},\text{记}f(x)\in C[a,b]\end{array}$$

间断

$$\text{若}\lim_{x\to x_0}f(x)\neq f(x_0),x_0\text{为}f(x)\text{的间断点}$$

第一类间断点

$$f(x_0-0),f(x_0+0)\exists$$

$$\begin{cases}f(x_0-0)=f(x_0+0)(\neq f(x_0))-x_0\text{为可去间断点}\\f(x_0-0)\neq f(x_0+0)-x_0\text{为跳跃间断点}\end{cases}$$

第二类间断点

$$f(x_0-0),f(x_0+0)\text{至少一个不存在}$$

$$\text{设函数}f(x)=\begin{cases}\frac{\sin ax+e^{2x}-1}{x},x<0\\4,x=0\\\frac{b\arctan x+\ln(1-x)}{x},x>0\end{cases},\text{在}x=0\text{处连续},\text{求常数}a,b\text{的值}.$$

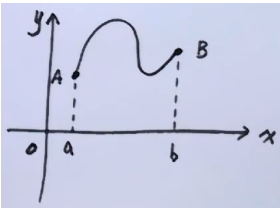
$$f(0-0)=\lim_{x\to0^-}\frac{\sin ax}{x}+\lim_{x\to0^-}\frac{e^{2x}-1}{x}=a+2$$
$$f(0)=4$$
$$f(0+0)=b\lim_{x\to0^+}\frac{\arctan x}{x}+\lim_{x\to0^+}\frac{\ln(1-x)}{x}=b-1$$
$$\because f(x)\text{在}x=0\text{连续},\therefore f(0-0)=f(0)=f(0+0)\Rightarrow a=2,b=5$$

$$\text{设}f(x)=\frac{x^2+x-2}{x^2-1}e^{\frac{1}{x}},\text{求函数}f(x)\text{的间断点并分类}.$$

$$x=-1,0,1\text{为间断点}$$
$$\lim_{x\to-1}f(x)=\infty\Rightarrow x=-1\text{为第二类间断点}$$
$$f(0-0)=\lim_{x\to0^-}\frac{x^2+x-2}{x^2-1}e^{\frac{1}{x}}=0$$
$$f(0+0)=+\infty$$
$$\Rightarrow x=0\text{为第二类间断点}$$
$$\lim_{x\to1}f(x)=\lim_{x\to1}\frac{(x-1)(x+2)}{(x-1)(x+1)}e^{\frac{1}{x}}=\lim_{x\to1}\frac{x+2}{x+1}e^{\frac{1}{x}}=\frac{3e}{2}\Rightarrow x=1\text{为可去间断点}$$

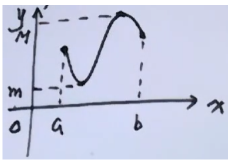
$$\text{设}f(x)=\frac{\ln|x|}{x^2-1},\text{求}f(x)\text{的间断点.}$$
$$x=-1,0,1\text{为间断点}$$
$$\lim_{x\to-1}f(x)=\lim_{x\to-1}\frac{1}{x-1}\frac{\ln(-x)}{x+1}=-\frac{1}{2}\lim_{x\to-1}\frac{\ln[1-(x+1)]}{x+1}=\frac{1}{2}$$
$$\Rightarrow x=-1\text{为可去间断点}$$
$$\lim_{x\to0}f(x)=+\infty\Rightarrow x=0\text{为第二类间断点}$$
$$\lim_{x\to1}f(x)=\lim_{x\to1}\frac{1}{x+1}\frac{\ln x}{x-1}=\lim_{x\to1}\frac{\ln[1+(x-1)]}{x-1}=\frac{1}{2}$$
$$\Rightarrow x=1\text{为可去间断点}$$

闭区间连续性质



$$f(x)\in C[a,b]$$

最值定理

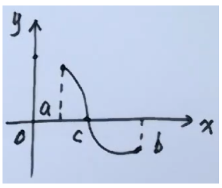


$$(m,M)\text{若}f(x)\in C[a,b]\Rightarrow f(x)\text{在}[a,b]\text{上取到最小值}m\text{和最大值}M$$

有界定理

$$\text{若}f(x)\in C[a,b]\Rightarrow\exists k>0,\text{使}|f(x)|\le k$$
$$\because f(x)\ge m,f(x)\le M,\therefore f(x)\text{在}[a,b]\text{上有界}$$

零点定理

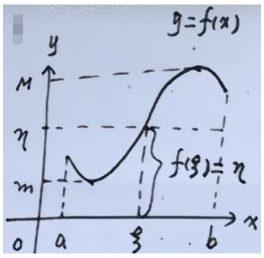


$$\text{若}f(x) \in C[a,b], \text{且}f(a)f(b) < 0$$
$$\Rightarrow \exists c \in (a,b), \text{使}f(c) = 0$$

$$\text{证明}x^5 + 4x - 1 = 0\text{有且仅有一个正根.}$$

$$\text{证：令}f(x) = x^5 + 4x - 1 \in C[0,1]$$
$$f(0) = -1, f(1) = 4$$
$$\because f(0)f(1) < 0, \therefore \exists c \in (0,1), \text{使}f(c) = 0$$
$$\because f'(x) = 5x^4 + 4 > 0 (x > 0)$$
$$\therefore f(x)\text{在}[0, +\infty] \uparrow$$
$$\therefore \text{正根唯一}$$

介值定理



$$[m, M] - \text{值域}$$
$$\text{位于}[m,M]\text{之间的任一个值称为介值}$$
$$\forall \eta \in [m, M], \exists \xi \in [a,b]\text{使}f(\xi) = \eta$$
$$\text{位于}m\text{和}M\text{之间任何值}f(x)\text{皆可取到}$$

$$\text{设}f(x) \in C[a,b], \forall \eta \in [m,M], \exists \xi \in [a,b], \text{使}$$
$$f(\xi) = \eta$$

$$f(x) \in C[0,2], f(0) + 2f(1) + 3f(2) = 6$$
$$\text{证：}\exists \xi \in [0,2], \text{使}f(\xi) = 1$$

$$\text{证：}f(x) \in C[0,2] \Rightarrow \exists m, M$$
$$6m \le f(0) + 2f(1) + 3f(2) \le 6M$$
$$\Rightarrow m \le 1 \le M$$
$$\exists \xi \in [0,2], \text{使}f(\xi) = 1$$

$$\text{设}f(x) \in C[a,b], \text{证}\exists \xi \in [a,b], \text{使}\int_a^b f(x)dx = f(\xi)(b-a)$$

$$\text{证} \because f(x) \in C[a,b], \therefore \exists m, M$$
$$m \le f(x) \le M$$
$$\Rightarrow \int_a^b m dx \le \int_a^b f(x)dx \le \int_a^b M dx$$
$$\Rightarrow m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$$
$$\therefore \exists \xi \in [a,b], \text{使}f(\xi) = \frac{1}{b-a}\int_a^b f(x)dx$$