有理函数不定积分

$$1.\ R(x)=\frac{P(x)}{Q(x)},\text{其中}P(x),Q(x)\text{为多项式},R(x)\text{为有理函数}$$

$$2.\ \text{分类}:\begin{cases}\deg P(x)<\deg Q(x),R(x)-\text{真分式}\\ \deg P(x)\geq \deg Q(x),R(x)-\text{假分式}\end{cases}$$

$$\int R(x)dx\,?$$

步骤

$$\text{①若}R(x)\text{为假分式}:R(x)=\text{多项式}+\text{真分式}$$

$$\frac{3x^4+x^2-3}{x^2+x-1}=(3x^2-3x+7)+\frac{-10x+4}{x^2+x-1}$$

$$\text{②若}R(x)\text{为真分式}:\frac{\text{分子不变}}{\text{分母因式分解}}\Rightarrow\text{部分和}$$

$$1.\ \frac{3x+2}{(x+1)(2x-1)}=\frac{A}{x+1}+\frac{B}{2x-1}$$

$$A(2x+1)+B(x+1)=3x+2\Rightarrow\begin{cases}2A+B=3\\ -A+B=2\end{cases}$$

$$2.\ \frac{x^2-3x+1}{(x+1)^2(2x-1)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{2x-1}$$

$$3.\ \frac{2x^2-1}{(x-1)^2(x^2+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+1}$$

$$\int\frac{3-x-2x^2}{2x+1}dx$$

$$\begin{aligned}\text{原式}&=-\frac{x^2}{2}+\frac{3}{2}\int\frac{d(2x+1)}{2x+1}\\&=-\frac{x^2}{2}+\frac{3}{2}\ln|2x+1|+C\end{aligned}$$

$$\int\frac{5x-8}{2x^2-x-1}dx$$

$$\begin{aligned}\text{原式}&=-\int\frac{d(x-1)}{x-1}+\frac{7}{2}\int\frac{d(2x+1)}{2x+1}\\&=-\ln|x-1|+\frac{7}{2}\ln|2x+1|+C\end{aligned}$$

$$\int\frac{2x^2-x+3}{(x-1)(x^2+1)}dx$$

$$\begin{aligned}\text{原式}=&2\int\frac{d(x-1)}{x-1}-\int\frac{dx}{1+x^2}\\=&2\ln|x-1|-\arctan x+C\end{aligned}$$

$$\int \frac{dx}{x^2+x+1} = \int \frac{d(x+\frac{1}{2})}{(\frac{\sqrt{3}}{2})^2+(x+\frac{1}{2})^2} = \frac{2}{\sqrt{3}}\arctan\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}+C$$

$$\int \frac{x-2}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x-4}{x^2+x+1}dx = \frac{1}{2}\int \frac{(2x+1)-5}{x^2+x+1}dx$$

$$= \frac{1}{2}\int \frac{d(x^2+x+1)}{x^2+x+1} - \frac{5}{2}\int \frac{d(x+\frac{1}{2})}{(\frac{\sqrt{3}}{2})^2+(x+\frac{1}{2})^2} = \frac{1}{2}\ln(x^2+x+1) - \frac{5}{2}\frac{2}{\sqrt{3}}\arctan\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}+C$$

$$\int \frac{dx}{x(x^2+4)}$$

$$\frac{1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\int \frac{dx}{x(x^2+4)} = \frac{1}{2}\int \frac{dx^2}{x^2(x^2+4)}$$

$$= \frac{1}{8}\int (\frac{1}{x^2} - \frac{1}{x^2+4})d(x^2) = \frac{1}{8}\ln\frac{x^2}{x^2+4}+C$$

$$\int \frac{dx}{x(x^4+2)} = \frac{1}{4}\int \frac{d(x^4)}{x^4(x^4+2)}, x^4 = t$$

$$= \frac{1}{4}\int \frac{dt}{t(t+2)}$$

$$= \frac{1}{8}\int (\frac{1}{t} - \frac{1}{t+2})dt$$

$$= \frac{1}{8}\ln|\frac{t}{t+2}|+C = \frac{1}{8}\ln\frac{x^4}{x^4+2}+C$$

$$\int \frac{x^2+1}{x^4+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$= \int \frac{d(x-\frac{1}{x})}{(\sqrt{2})^2+(x-\frac{1}{x})^2}$$

$$= \frac{1}{\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}}+C$$

$$\int \frac{x^2-1}{x^4+1}dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$= \int \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-(\sqrt{2})^2}$$

$$= \frac{1}{2\sqrt{2}}\ln|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}|+C$$