变积分限函数

$$\text{设}f(x)\text{连续, 且}\lim_{x\to 0}\frac{f(x)}{x}=2, F(x)=\int_0^x t^{n-1}f(x^n-t^n)dt, \text{求}\lim_{x\to 0}\frac{F(x)}{x^{2n}}.$$

$$\begin{aligned}\text{解}: 1.F(x)&=-\frac{1}{n}\int_0^x f(x^n-t^n)d(x^n-t^n)\\&=-\frac{1}{n}\int_{x^n}^0 f(u)du=\frac{1}{n}\int_0^{x^n}f(u)du\\ 2.\text{原式}&=\frac{1}{2n}\lim_{x\to 0}\frac{f(x^n)\cdot x^{n-1}}{x^{2n-1}}=\frac{1}{2n}\lim_{x\to 0}\frac{f(x^n)}{x^n}=\frac{1}{n}\end{aligned}$$

$$f(x)\text{连续}, f(0)=0, f'(0)=2, \text{求}\lim_{x\to 0}\frac{\int_{-x}^x[f(t+x)-f(t-x)]dt}{x^2}.$$

$$\begin{aligned}\text{解}: 1.&\int_{-x}^x[f(t+x)-f(t-x)]dt\\&=\int_{-x}^x f(t+x)d(t+x)-\int_{-x}^x f(t-x)d(t-x)\\&=\int_0^{2x}f(u)du+\int_0^{-2x}f(u)du\\ 2.\text{原式}&=2\lim_{x\to 0}\frac{f(2x)-f(-2x)}{2x}\\&=2[\lim_{x\to 0}\frac{f(2x)-f(0)}{2x}+\lim_{x\to 0}\frac{f(-2x)-f(0)}{-2x}]\\&=4f'(0)=8\end{aligned}$$

常规计算

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\frac{\sin 2x}{1+\sin^4 x}dx\\=&\int_0^{\frac{\pi}{2}}\frac{d(\sin^2 x)}{1+(\sin^2 x)^2}\\=&\arctan\sin^2 x\mid_0^{\frac{\pi}{2}}\\=&\frac{\pi}{4}\end{aligned}$$

$$\begin{aligned}&\int_0^{\ln 2}\sqrt{e^x-1}dx, \sqrt{e^x-1}=t\\=&\int_0^1 t\cdot\frac{2t}{1+t^2}dt=2\int_0^1(1-\frac{1}{1+t^2})dt\\=&2(1-\frac{\pi}{4})\end{aligned}$$

$$
\begin{aligned}
&\int_0^1 x\ln(1+x^2)dx\\
=&\frac{1}{2}\int_0^1 \ln(1+x^2)d(1+x^2)\\
=&\frac{1}{2}\int_1^2 \ln x dx\\
=&\frac{1}{2}(x\ln x\mid_1^2-1)\\
=&\frac{1}{2}(2\ln 2-1)\\
=&\ln 2-\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \frac{\arctan x}{(1+x^2)^2}dx, x=\tan t\\
=&\int_0^{\frac{\pi}{4}} \frac{t}{\sec^4 t}\cdot\sec^2 t\\
=&\int_0^{\frac{\pi}{4}} t\cos^2 t\\
=&\frac{1}{2}\int_0^{\frac{\pi}{4}} t(1+\cos 2t)dt\\
=&\frac{t^2}{4}\mid_0^{\frac{\pi}{4}}+\frac{1}{4}\int_0^{\frac{\pi}{4}} td(\sin 2t)\\
=&\frac{\pi^2}{64}+\frac{1}{4}(t\sin 2t\mid_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\sin 2tdt)\\
=&\frac{\pi^2}{64}+\frac{\pi}{16}-\frac{1}{8}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^2 x^2\sqrt{2x-x^2}dx\\
=&\int_0^2 [1+(x-1)]^2\sqrt{1-(x-1)^2}d(x-1)\\
=&\int_{-1}^1 (1+x)^2\sqrt{1-x^2}dx\\
=&2\int_0^1 (1+x^2)\sqrt{1-x^2}dx, x=\sin t\\
=&2\int_0^{\frac{\pi}{2}} (1+\sin^2 t)(1-\sin^2 t)dt\\
=&2\int_0^{\frac{\pi}{2}} (1-\sin^4 t)dt\\
=&2(\frac{\pi}{2}-\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2})\\
=&\frac{5\pi}{8}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^{\pi^2}\sin^2\sqrt{x}dx,\sqrt{x}=t\\
=&2\int_0^{\pi}t\sin^2 tdt\\
=&\pi\int_0^{\pi}\sin^2 tdt\\
=&2\pi\cdot\frac{1}{2}\cdot\frac{\pi}{2}\\
=&\frac{\pi^2}{2}
\end{aligned}
$$

变积分限函数计算定积分->分部积分

$$f(x)=\int_a^x\Phi(t)dt,\text{求}\int_a^b f(x)dx.$$

$$\text{设}f(x)=\int_1^x e^{-t^2}dt,\text{求}\int_0^1 x^2f(x)dx.$$

$$
\begin{aligned}
&\int_0^1 x^2f(x)dx\\
=&\frac{1}{3}\int_0^1 f(x)d(x^3)\\
=&\frac{x^3}{3}f(x)\mid_0^1-\frac{1}{3}\int_0^1 x^3\cdot e^{-x^2}dx\\
=&-\frac{1}{6}\int_0^1 x^2e^{-x^2}d(x^2)\\
=&-\frac{1}{6}\int_0^1 xe^{-x}dx\\
=&\frac{1}{6}\int_0^1 xd(e^{-x})\\
=&\frac{x}{6}e^{-x}\mid_0^1-\frac{1}{6}\int_0^1 e^{-x}dx\\
=&\frac{1}{6e}+\frac{1}{6}(\frac{1}{e}-1)=\frac{1}{3e}-\frac{1}{6}
\end{aligned}
$$

$$\text{设}f(x)=\int_0^x\frac{\sin t}{\pi-t}dt,\text{求}\int_0^{\pi}f(x)dx$$

$$
\begin{aligned}
&\int_0^{\pi}f(x)dx\\
=&xf(x)\mid_0^{\pi}-\int_0^{\pi}x\cdot\frac{\sin x}{\pi-x}dx\\
=&\pi f(\pi)-\int_0^{\pi}\frac{x\sin x}{\pi-x}dx\\
=&\int_0^{\pi}\frac{\pi\sin x}{\pi-x}dx-\int_0^{\pi}\frac{x\sin x}{\pi-x}dx\\
=&\int_0^{\pi}\sin xdx=2I_1=2
\end{aligned}
$$

证明

情形一

$$f(x)\text{为连续函数}$$

$$\begin{aligned}
&1.\int_{-a}^{0}f(x)dx=\int_{0}^{a},x=-t\\
&2.\int_{a}^{a+b}f(x)dx=\int_{0}^{b},x-a=t\\
&3.\int_{a}^{b}f(x)dx=\int_{a}^{b},x+t=a+b\\
&4.\int_{a}^{b}f(x)dx=\int_{0}^{1},x=a+(b-a)t
\end{aligned}$$

$$\text{设}f(x)\text{连续},\text{证明}:\int_{a}^{b}f(x)dx=\int_{a}^{b}f(a+b-x)dx$$

$$\begin{aligned}
&\int_{a}^{b}f(x)dx,x+t=a+b\\
=&\int_{b}^{a}f(a+b-t)\cdot(-dt)\\
=&\int_{a}^{b}f(a+b-t)dt=\int_{a}^{b}f(a+b-x)dx
\end{aligned}$$

$$\text{设}f(x)\text{连续},\text{证明}:\int_{a}^{b}f(x)dx=(b-a)\int_{0}^{1}f[a+(b-a)x]dx$$

$$\begin{aligned}
&\int_{a}^{b}f(x)dx,x=a+(b-a)t\\
=&\int_{0}^{1}f[a+(b-a)t]\cdot(b-a)dt\\
=&(b-a)\int_{0}^{1}f[a+(b-a)x]dx
\end{aligned}$$

$$\begin{aligned}
&\text{设}f(x)\in C[a,b],\text{且对任意的}x,y\in[a,b]\text{有}|f(x)-f(y)|\leq 2|x-y|,\\
&\text{证明}:|\int_{a}^{b}f(x)dx-f(a)(b-a)|\leq(b-a)^2
\end{aligned}$$

证明：

$$\begin{aligned}
&1.f(a)(b-a)=\int_{a}^{b}f(a)dx\\
&2.|\int_{a}^{b}f(x)dx-f(a)(b-a)|=|\int_{a}^{b}[f(x)-f(a)]dx|\\
&3.|\int_{a}^{b}[f(x)-f(a)]dx|\leq\int_{a}^{b}|f(x)-f(a)|dx\leq\int_{a}^{b}2(x-a)d(x-a)\\
&\quad=(x-a)^2\,|_{a}^{b}=(b-a)^2
\end{aligned}$$

情形二

$f(x)$连续 + 单调

$$设f(x)\in C[a,b]且f(x)单调递增,证明:\int_a^b xf(x)dx \geq \frac{a+b}{2}\int_a^b f(x)dx$$

$$\begin{aligned}
&令\Phi(x)=\int_a^x tf(t)dt-\frac{a+x}{2}\int_a^x f(t)dt,\Phi(a)=0\\
&\Phi'(x)=xf(x)-\frac{1}{2}\int_a^x f(t)dt-\frac{a+x}{2}f(x)\\
&=\frac{x-a}{2}f(x)-\frac{1}{2}\int_a^x f(t)dt\\
&=\frac{x-a}{2}[f(x)-f(\xi)]\geq 0(a<\xi<x<b)\\
\Rightarrow&\Phi(x)\geq 0(a<x\leq b)\\
\Rightarrow&\Phi(b)\geq 0
\end{aligned}$$

$$\begin{aligned}
&设f(x)\in C[0,1]且f(x)单调递减,对任意的\alpha\in(0,1),证明:\\
&\int_0^\alpha f(x)dx\geq\alpha\int_0^1 f(x)dx
\end{aligned}$$

$$\begin{aligned}
法一&\int_0^\alpha f(x)dx=\alpha\int_0^1 f(\alpha t)\cdot dt=\alpha\int_0^1 f(\alpha x)dx,x=\alpha t\\
&\because \alpha x\leq x,\therefore f(\alpha x)\geq f(x)\\
&\therefore\int_0^\alpha f(x)dx\geq\alpha\int_0^1 f(x)dx\\
法二&\int_0^\alpha f(x)dx-\alpha\int_0^1 f(x)dx=(1-\alpha)\int_0^\alpha f(x)dx-\alpha\int_\alpha^1 f(x)dx\\
&=\alpha(1-\alpha)[f(\xi_1)-f(\xi_2)](0\leq\xi_1\leq\alpha,\alpha\leq\xi_2\leq 1)\\
&\because f\downarrow 且\xi_1\leq\xi_2\\
&\therefore\int_0^\alpha f(x)dx-\alpha\int_0^1 f(x)dx\geq 0
\end{aligned}$$

情形三

$f(x)$可导

$$\begin{aligned}
&1.工具\begin{cases}f(x)-f(a)=f'(\xi)(x-a)-L:积分中无导数\\ f(x)-f(a)=\int_a^x f'(t)dt-N-L:积分号中有导数\end{cases}\\
&2.技巧\begin{cases}|\cdot|:|\int_a^b fdx|\leq\int_a^b|f|dx\\ (\ )^2:(\int_a^b fgdx)\leq\int_a^b f^2dx\cdot\int_a^b g^2dx\\ 两边积分\\ \frac{1}{b-a}\int_a^b f(x)dx=f(c)(a\leq c\leq b)\end{cases}
\end{aligned}$$

$$设f'(x)\in C[0,a],f(0)=0,|f'(x)|\leq M,证明:|\int_0^a f(x)dx|\leq\frac{M}{2}a^2$$

证：

$$1. f(x) = f(x) - f(0) = f'(\xi)x (0 < \xi < x)$$

$$2. |\int_0^a f(x)dx| \le \int_0^a |f(x)|dx$$

$$= \int_0^a |f'(\xi)|xdx \le M\int_0^a xdx = \frac{M}{2}a^2$$

设$f'(x) \in C[a,b], f(a) = f(b) = 0, c \in (a,b),$ 证明：

$$|f(c)| \le \frac{1}{2}\int_a^b |f'(x)|dx$$

证：

$$1. \begin{cases} f(c) - f(a) = \int_a^c f'(x)dx \\ f(b) - f(c) = \int_c^b f'(x)dx \end{cases}$$

$$2. \begin{cases} |f(c)| \le \int_a^c |f'(x)|dx \\ |f(c)| \le \int_c^b |f'(x)|dx \end{cases}$$

$$\Rightarrow |f(c)| \le \frac{1}{2}\int_a^b |f'(x)|dx$$

设$f'(x) \in C[a,b], c \in (a,b),$ 证明：

$$|f(c)| \le |\frac{1}{b-a}\int_a^b f(x)dx| + \int_a^b |f'(x)|dx$$

证：

$$1. \frac{1}{b-a}\int_a^b f(x)dx = f(x_0)(a \le x_0 \le b)$$

$$2. f(c) - f(x_0) = \int_{x_0}^c f'(x)dx$$

$$f(c) = f(x_0) + \int_{x_0}^c f'(x)dx$$

$$3. |f(c)| \le |f(x_0)| + |\int_{x_0}^c f'(x)dx|$$

$$\le |\frac{1}{b-a}\int_a^b f(x)dx| + \int_a^b |f'(x)|dx$$