微分方程

微分方程

$$
\text{微分方程}-\text{含导数或微分的方程}\\
\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-3y=0\\
2ydx+(2x+1)dy=x^2+y^2
$$

阶

$$
\text{阶}-\text{导数或微分的最高阶数, 称为微分方程的阶}
$$

微分方程的解

$$
\text{解}-\text{使微分方程成立的函数, 称为微分方程的解}
$$

$$
\begin{aligned}
&y'+2xy=0\\
&y=e^{-x^2},\text{代入,}\\
&-2xe^{-x^2}+2xe^{-x^2}\equiv 0, y=e^{-x^2}\text{为}y'+2xy=0\text{的一个解}
\end{aligned}
$$

一阶微分方程

可分离变量的微分方程

$$
\frac{dy}{dx}=f(x,y)
$$

解法

$$
\begin{aligned}
&f(x,y)=\Phi_1(x)\Phi_2(y)\\
&\frac{dy}{dx}=\Phi_1(x)\Phi_2(y)\Rightarrow\int\frac{dy}{\Phi_2(y)}=\int\Phi_1(x)dx+C
\end{aligned}
$$

例题

$$
\text{求}\frac{dy}{dx}=1+x+y^2+xy^2\text{的通解.}
$$

$$
\begin{aligned}
&\frac{dy}{dx}=(1+x)(1+y^2)\\
&\Rightarrow\frac{dy}{1+y^2}=(1+x)dx\Rightarrow\arctan y=x+\frac{x^2}{2}+C
\end{aligned}
$$

$$
\text{求微分方程}\frac{dy}{dx}=2x(1+y^2)\text{的通解.}
$$

$$
\begin{aligned}
&\frac{dy}{1+y^2}=2xdx\\
&\Rightarrow\arctan y=x^2+C\\
&y=\tan(x^2+C)
\end{aligned}
$$

求$\dfrac{dy}{dx}=2xy$的通解.

1. $y=0$为方程的解
2. $y\neq 0$,

$$\frac{dy}{y}=2xdx\Rightarrow\ln|y|=x^2+C_0\Rightarrow|y|=e^{C_0}e^{x^2}$$

$$\Rightarrow y=\pm e^{C_0}e^{x^2}$$

令$\pm e^{C_0}=C(C\neq 0),y=Ce^{x^2}(C\neq 0)$

通解,$y=Ce^{x^2}$(C为任意常数)

齐次微分方程

$\dfrac{dy}{dx}=f(x,y)$,若$f(x,y)=\Phi(\dfrac{y}{x})$,称该微分方程方程为齐次微分方程

$$\frac{dy}{dx}=\frac{y+2x}{y-x}\Rightarrow\frac{dy}{dx}=\frac{\frac{y}{x}+2}{\frac{y}{x}-1}$$

解法

$$\frac{dy}{dx}=\Phi(\frac{y}{x})$$

令$u=\dfrac{y}{x},y=xu,\dfrac{dy}{dx}=u+x\dfrac{du}{dx}$,代入

$$u+x\frac{du}{dx}=\Phi(u)$$

$$\Rightarrow\int\frac{du}{\Phi(u)-u}=\int\frac{dx}{x}+C$$

例题

求微分方程$\dfrac{dy}{dx}-\dfrac{2}{x}y=1$的通解.

$$\frac{dy}{dx}=2\frac{y}{x}+1$$

令$\dfrac{y}{x}=u,\dfrac{dy}{dx}=u+x\dfrac{du}{dx}$

$$u+x\frac{du}{dx}=2u+1\Rightarrow\frac{du}{u+1}=\frac{dx}{x}$$

$$\Rightarrow\ln(u+1)=\ln x+\ln C$$

$\Rightarrow u+1=Cx\Rightarrow y=Cx^2-x$($C$为任意常数)

求微分方程$x\dfrac{dy}{dx}=y+\sqrt{x^2+y^2}(x>0)$满足$y(1)=0$的特解.

$$\frac{dy}{dx}=\frac{y}{x}+\sqrt{1+(\frac{y}{x})^2}(x>0)$$
$$令u=\frac{y}{x},代入,u+x\frac{du}{dx}=u+\sqrt{u^2+1}$$
$$\Rightarrow\frac{du}{\sqrt{u^2+1}}=\frac{dx}{x}\Rightarrow\ln(u+\sqrt{u^2+1})=\ln x+\ln C$$
$$u+\sqrt{u^2+1}=Cx$$
$$\because y(1)=0,\therefore u(1)=0\Rightarrow C=1$$
$$u+\sqrt{u^2+1}=x$$
$$\because -u+\sqrt{u^2+1}=\frac{1}{x},\therefore u=\frac{1}{2}(x-\frac{1}{x})$$
$$\therefore y=\frac{1}{2}(x^2-1)$$

一阶齐次线性微分方程

$$\frac{dy}{dx}+P(x)y=0,称该微分方程为一阶齐次线性微分方程$$

解法

$$\frac{dy}{dx}+P(x)y=0\Rightarrow\frac{dy}{dx}=-P(x)y$$
$$1.\ y=0为方程的解$$
$$2.\ y\neq0,\frac{dy}{y}=-P(x)dx$$
$$\Rightarrow\ln|y|=-\int P(x)dx+C_0$$
$$y=\pm e^{C_0}e^{-\int P(x)dx}$$
$$令\pm e^{C_0}=C(C\neq0),y=Ce^{-\int P(x)dx}(C\neq0)$$
$$通解,y=Ce^{-\int P(x)dx}(C为任意常数)$$

例题

$$求微分方程y'+2xy=0的通解.$$

$$\frac{dy}{dx}+2xy=0$$
$$y=Ce^{-\int 2xdx}=Ce^{-x^2}(C为任意常数)$$

一阶非齐线性微分方程(常数变易法)

$$\frac{dy}{dx}+P(x)y=Q(x),称该方程为一阶非齐线性微分方程$$

解法

$$\frac{dy}{dx}+P(x)y=0(*)$$
$$\frac{dy}{dx}+P(x)y=Q(x)(**)$$
$$(*)\text{通解}y=Ce^{-\int P(x)dx}(C\text{为任意常数})$$
$$\text{令}(**)\text{通解为}y=C(x)e^{-\int P(x)dx}(C\text{为任意常数}),\text{代入}(**)$$
$$C'(x)e^{-\int P(x)dx}-P(x)C(x)e^{-\int P(x)dx}+P(x)C(x)e^{-\int P(x)dx}=Q(x)$$
$$\Rightarrow C'(x)=Q(x)e^{\int P(x)dx}$$
$$\Rightarrow C(x)=\int Q(x)e^{\int P(x)dx}dx+C$$
$$\therefore (**)\text{通解},y=[\int Q(x)e^{\int P(x)dx}dx+C]e^{-\int P(x)dx}$$

例题

$$\frac{dy}{dx}-\frac{2}{x}y=-1$$
$$\begin{aligned}P(x)&=-\frac{2}{x},Q(x)=-1\\ y&=[\int Q(x)e^{\int P(x)dx}dx+C]e^{-\int P(x)dx}\\ &=[\int(-1)e^{-\int\frac{2}{x}dx}dx+C]e^{-\int\frac{2}{x}dx}\\ &=(\frac{1}{x}+C)x^2=Cx^2+x\end{aligned}$$

求$y'+y\tan x=\cos x$的通解.

$$\begin{aligned}y&=(\int\cos xe^{\int\tan xdx}dx+C)e^{-\int\tan xdx}\\ &=(x+C)\cos x\end{aligned}$$

求微分方程$ydx-(x-4y)dy=0(y>0)$的通解.

$$\begin{aligned}&\frac{dx}{dy}-\frac{1}{y}x=-4\\ &x=(\int Q(y)e^{\int P(y)dy}dy+C)e^{-\int P(y)dy}\\ &=(-4\int e^{-\int\frac{1}{y}dy}dy+C)e^{-\int-\frac{1}{y}dy}=(-4\ln y+C)y\end{aligned}$$

可降阶高位微分方程

型一

$$y^{(n)}=f(x)(n\geq 2)$$

例题

$$
\begin{aligned}
y'' &= \cos 2x + x \\
y' &= \frac{1}{2}\sin 2x + \frac{1}{2}x^2 + C_1 \\
y &= -\frac{1}{4}\cos 2x + \frac{1}{6}x^3 + C_1 x + C_2
\end{aligned}
$$

$$
\begin{aligned}
y'' &= xe^{2x} \\
y' &= \int xe^{2x}dx + C_1 = \frac{1}{2}\int xd(e^{2x}) + C_1 = \frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}dx + C_1 = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C_1 \\
y &= \frac{1}{2}\int xe^{2x}dx - \frac{1}{4}\int e^{2x}dx + C_1 x + C_2 = \frac{1}{4}xe^{2x} - \frac{1}{4}e^{2x} + C_1 x + C_2
\end{aligned}
$$

型二

$$f(x, y', y'') = 0(\text{缺}y)$$

解法

$$
\begin{aligned}
&\text{令}y' = p, y'' = \frac{dp}{dx}, f(x, p, \frac{dp}{dx}) = 0 \\
&\Rightarrow p = \Phi(x, C_1), \text{即}y' = \Phi(x, C_1) \\
&y = \int \Phi(x, C_1)dx + C_2
\end{aligned}
$$

例题

$$y'' + y' = x$$

$$
\begin{aligned}
&\text{令}y' = p, y'' = \frac{dp}{dx} \\
&\frac{dp}{dx} + p = x \\
&p = (\int xe^{\int dx}dx + C_1)e^{-\int dx} = [(x-1)e^x + C_1]e^{-x} = C_1 e^{-x} + x - 1 \\
&\text{即}y' = C_1 e^{-x} + x - 1 \\
&y = -C_1 e^{-x} + \frac{x^2}{2} - x + C_2
\end{aligned}
$$

$$xy'' + 2y' = 0$$

$$
\begin{aligned}
&\text{法一}:\text{令}y' = p, y'' = \frac{dp}{dx} \\
&\quad \frac{dP}{dx} + \frac{2}{x}p = 0 \\
&\quad p = C_1 e^{-\int \frac{2}{x}dx} = \frac{C_1}{x^2}, \text{即}y' = \frac{C_1}{x^2} \Rightarrow y = -\frac{C_1}{x} + C_2 \\
&\text{法二}: xy'' + 2y' = 0 \Rightarrow x^2 y'' + 2xy' = 0 \\
&\quad \Rightarrow (x^2 y')' = 0 \Rightarrow x^2 y' = C_1 \\
&\quad y' = \frac{C_1}{x^2}, \therefore = -\frac{C_1}{x} + C_2
\end{aligned}
$$

求微分方程$y'' + y' = 2e^x$的通解.

$$
\begin{aligned}
&令 y'=p, y''=\frac{dp}{dx}\\
&\frac{dp}{dx}+p=2e^x\\
&p=(\int 2e^x e^{\int 1dx}dx+C_1)e^{-\int dx}=C_1e^{-x}+e^x\\
&y=-C_1e^{-x}+e^x+C_2
\end{aligned}
$$

型三

$$f(y,y',y'')=0(缺x)$$

解法

$$
\begin{aligned}
&令 y'=p, y''=\frac{dp}{dx}\Rightarrow f(y,p,\frac{dp}{dx})=0\\
&y''=\frac{dy}{dx}\frac{dp}{dy}=p\frac{dp}{dy}\\
&f(y,p,p\frac{dp}{dy})=0\\
&p=\Phi(y,C_1)\Rightarrow\int\frac{dy}{\Phi(y,C_1)}=\int dx+C_2
\end{aligned}
$$

例题

求微分方程$yy''=y'^2$满足初值条件$y(0)=y'(0)=1$的特解.

$$
\begin{aligned}
&令 y'=p, y''=p\frac{dp}{dy},代入\\
&yp\frac{dp}{dy}-p^2=0\\
&\because y'(0)=1,\therefore p\neq 0\Rightarrow\frac{dp}{dy}-\frac{1}{y}p=0\\
&p=C_1e^{-\int-\frac{1}{y}dy}=C_1y,即 y'=C_1y\\
&\because y(0)=y'(0)=1,\therefore C_1=1\\
&\Rightarrow\frac{dy}{dx}-y=0\\
&y=C_2e^{-\int-dx}=C_2e^x\\
&\because y(0)=1,\therefore C_2=1\\
&\therefore y=e^x
\end{aligned}
$$

高阶线性微分方程

$$
\begin{aligned}
&y^{(n)}+a_1(x)y^{(n-1)}+\ldots+a_{n-1}y'+a_n(x)y=0(*)-n阶齐次线性微分方程\\
&y^{(n)}+a_1(x)y^{(n-1)}+\ldots+a_{n-1}y'+a_n(x)y=f(x)(**)-n阶非齐次线性微分方程
\end{aligned}
$$

$$
\begin{aligned}
&若 f(x)=f_1(x)+f_2(x)\\
&y^{(n)}+a_1(x)y^{(n-1)}+\ldots+a_{n-1}y'+a_n(x)y=f_1(x)(**)'\\
&y^{(n)}+a_1(x)y^{(n-1)}+\ldots+a_{n-1}y'+a_n(x)y=f_2(x)(**)''
\end{aligned}
$$

解的结构

$$
\begin{aligned}
&\Phi_1(x),\ldots,\Phi_s(x)为(*)解\\
&\Rightarrow k_1\Phi_1(x)+\ldots+k_s\Phi_s(x)为(*)解
\end{aligned}
$$

$\Phi_1(x),\ldots,\Phi_s(x)$为$(**)$的解,则
①$k_1\Phi_1(x)+\ldots+k_s\Phi_s(x)$为$(*)$解 $\Leftrightarrow k_1+\ldots+k_s=0$
②$k_1\Phi_1(x)+\ldots+k_s\Phi_s(x)$为$(**)$的解 $\Leftrightarrow k_1+\ldots+k_s=1$

$\Phi_1(x),\Phi_2(x)$为$(*),(**)$的解 $\Rightarrow \Phi_1(x)+\Phi_2(x)$为$(**)$的解

$\Phi_1(x),\Phi_2(x)$为$(**)$的解 $\Rightarrow \Phi_1(x)-\Phi_2(x)$为$(*)$的解

$\Phi_1(x),\Phi_2(x)$为$(**)',(**)''$的解 $\Rightarrow \Phi_1(x)+\Phi_2(x)$为$(**)$的解

$y_1=x+e^x, y_2=x-e^x$为$y'+a(x)y=b(x)$的特解,求$a(x),b(x)$.
令$y'+a(x)y=0(*)$
$y'+a(x)y=b(x)(**)$
$y_1-y_2=2e^x$代入$(*):2e^x+a(x)2e^x=0\Rightarrow a(x)=-1$
$y'-y=b(x)(**)$
$y_0=\dfrac{1}{2}y_1+\dfrac{1}{2}y_2=x$,代入$(**):b(x)=1-x$

若$\Phi_1(x),\Phi_2(x)$为$(*)$不成比例的解
$(*)$通解$y=C_1\Phi_1(x)+C_2\Phi_2(x)$

若$\Phi_1(x),\Phi_2(x)$为$(*)$不成比例的解,$\Phi_0(x)$为$(**)$特解
则$(**)$通解$y=C_1\Phi_1(x)+C_2\Phi_2(x)+\Phi_0(x)$

二阶常系数齐次线性微分方程

1.$y''+py'+qy=0(*)$
where p,q are constants.
2.$\lambda^2+p\lambda+q=0$称为$(*)$的特征方程

解法

1.$\Delta>0:\lambda_1\neq\lambda_2$
通解$y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$
2.$\Delta=0:\lambda_1=\lambda_2$
通解$y=(C_1+C_2x)e^{\lambda_1 x}$
3.$\Delta<0:\lambda_{1,2}=\alpha\pm i\beta$
通解$y=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$

例题

求微分方程$y''-y'-2y=0$的通解.

$\lambda^2-\lambda-2=0\Rightarrow\lambda_1=-1\neq\lambda_2=2$
通解$y=C_1e^{-x}+C_2e^{2x}$

求微分方程$y''-6y'+9y=0$的通解.

$\lambda^2-6\lambda+9=0\Rightarrow\lambda_1=\lambda_2=3$
通解$y=(C_1+C_2x)e^{3x}$

求微分方程$y''-2y'+5y=0$的通解.

$$\lambda^2-2\lambda+5=0\Rightarrow\lambda_{1,2}=1\pm 2i$$
$$\text{通解}y=e^{\alpha x}(C_1\cos 2x+C_2\sin 2x)$$

二阶常系数非齐线性微分方程（特解）

$$y''+py'+qy=f(x)(**)$$
where p,q are constants.

$$\text{若}f(x)=f_1(x)+f_2(x)$$
$$y''+p(x)y'+q(x)y=f_1(x)(**)'$$
$$y''+p(x)y'+q(x)y=f_2(x)(**)''$$

型一

$$f(x)=P_n(x)e^{kx}$$

解法

1.照$f(x)$的形式
2.①$k\neq\lambda_1,k\neq\lambda_2:y_0(x)=(a_nx^n+\ldots+a_1x+a_0)e^{kx}$
②$k=\lambda_1,k\neq\lambda_2:y_0(x)=x(a_nx^n+\ldots+a_1x+a_0)e^{kx}$
③$k=\lambda_1=\lambda_2:y_0(x)=x^2(a_nx^n+\ldots+a_1x+a_0)e^{kx}$

例题

$$y''+y'-2y=(2x+1)e^{2x}$$

$$\lambda^2+\lambda-2=0\Rightarrow\lambda_1=1,\lambda_2=-2$$
$$y''+y'-2y=0\Rightarrow y=C_1e^x+C_2e^{-2x}$$
$$\text{令}y_0(x)=(ax+b)e^{2x},\text{代入},a=\frac{1}{2},b=-\frac{3}{8}$$
$$\therefore\text{通解}y=C_1e^x+C_2e^{-2x}+(\frac{x}{2}-\frac{3}{8})e^{2x}$$

$$y''+y'-2y=(4x-1)e^x$$

$$\lambda^2+\lambda-2=0\Rightarrow\lambda_1=1,\lambda_2=-2$$
$$y''+y'-2y=0\Rightarrow y=C_1e^x+C_2e^{-2x}$$
$$\text{令}y_0(x)=(ax^2+bx)e^x=(ax^2+bx)e^x,\text{代入},a=\frac{2}{3},b=-\frac{7}{9}$$
$$\therefore\text{通解}y=C_1e^x+C_2e^{-2x}+(\frac{2}{3}x^2-\frac{7}{9}x)e^x$$

$$y''-2y'+y=(4x+3)e^x$$

$$\lambda^2-2\lambda+1=0\Rightarrow\lambda_1=\lambda_2=1$$
$$y''-2y'+y=0\Rightarrow y=(C_1+C_2x)e^x$$
$$\text{令}y_0(x)=(ax^3+bx^2)e^x,\text{代入},a=\frac{2}{3},b=\frac{3}{2}$$
$$\text{通解}y=(C_1+C_2x)e^x+(\frac{2}{3}x^3+\frac{3}{2}x^2)e^x$$

贝努利方程

$$\frac{dy}{dx}+P(x)y=Q(x)y^n(n\neq 0,1)-\text{称为贝努利方程}$$

解法

$$\begin{aligned}&\text{令}u=y^{1-n},\text{则}\\&\frac{du}{dx}+(1-n)P(x)u=(1-n)Q(x)\end{aligned}$$

例题

$$x^2y'+xy=y^2,y(1)=1$$

$$\begin{aligned}&\frac{dy}{dx}+\frac{1}{x}y=\frac{1}{x^2}\cdot y^2\\&\text{令}u=y^{1-2}=y^{-1},\text{则}\\&\frac{du}{dx}-\frac{1}{x}u=-\frac{1}{x^2}\\&u=(\int-\frac{1}{x^2}e^{\int-\frac{1}{x}dx}dx+C)\cdot e^{-\int-\frac{1}{x}dx}=(\frac{1}{2x^2}+C)x\\&=Cx+\frac{1}{2x}\\&\because y(1)=1,\therefore u(1)-1\\&\therefore 1=C+\frac{1}{2}\Rightarrow C=\frac{1}{2}\Rightarrow u=\frac{1+x^2}{2x}\\&\therefore y=\frac{2x}{1+x^2}\end{aligned}$$

欧拉方程

$$x^ny^{(n)}+a^1x^{n-1}y^{(n-1)}+\ldots+a^{n-1}xy'+a_ny=f(x)-\text{称为欧拉方程}$$

解法

$$\begin{aligned}&\text{令}x=e^t,\frac{d}{dt}\triangleq D,\frac{d^2}{dt^2}\triangleq D^2\\&xy'=Dy=\frac{dy}{dt}\\&x^2y''=D(D-1)y=\frac{d^2y}{dt^2}-\frac{dy}{dt}\\&x^3y'''=D(D-1)(D-2)y\end{aligned}$$

例题

$$x^2y''+2xy'-2y=3x-2(x>0)$$

令$x=e^t, xy'=Dy=\frac{dy}{dt}, x^2y''=D(D-1)y=\frac{d^2y}{dt^2}-\frac{dy}{dt}$,代入

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}-2y=3e^t-2$$

$$\lambda^2+\lambda-2=0 \Rightarrow \lambda_1=1, \lambda_2=-2$$

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}-2y=0 \Rightarrow y=C_1e^t+C_2e^{-2t}$$

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}-2y=3e^t(**)'$$

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}-2y=-2(**)''$$

$y_1(t)=ate^t$,代入$(**)', a=1, y_1(t)=te^t$

$y_2(t)=1$

$\therefore y_0(t)=te^t+1$

$\therefore$ 通解$y=C_1e^t+C_2e^{-2t}+te^t+1$

$\therefore$ 原方程通解$y=C_1x+\frac{C_2}{x^2}+x\ln x+1$