问题一 性质与结构

$$\begin{aligned}&\text{方程组的解取决于秩}\\&\text{齐}r(A)?\\&\text{非齐}r(A)=r(\overline{A})?\end{aligned}$$

设A为四阶矩阵, $r(A)=3$, 且A的每行元素之和为0，求方程组$AX=0$的通解.

$$\begin{aligned}&1.r(A)=3<4\\&2.\because A\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=0,\therefore\text{通解}X=k\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\end{aligned}$$

设A为四阶矩阵, $r(A)<4$, 且$A_{21}\neq 0$, 求方程组$AX=0$的通解.

$$\begin{aligned}&1.r(A)<4\Rightarrow r(A^*)=0\text{或}1\\&\quad\because A_{21}\neq 0,\therefore A^*\neq 0\Rightarrow r(A^*)=1\Rightarrow r(A)=3<4\\&2.\because AA^*=|A|E=0,\therefore A^*\text{列为}AX=0\text{的解}.\\&\quad\because A^*=\begin{pmatrix}A_{11}&A_{21}&\cdots\\A_{12}&A_{22}&\cdots\\A_{13}&A_{23}&\cdots\\A_{14}&A_{24}&\cdots\end{pmatrix}\text{且}A_{21}\neq 0,\therefore\text{通解}X=k\begin{pmatrix}A_{21}\\A_{22}\\A_{23}\\A_{24}\end{pmatrix}\end{aligned}$$

设$\eta_1=\begin{pmatrix}1\\-2\\1\\2\end{pmatrix},\eta_2=\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},\eta_3=\begin{pmatrix}2\\1\\3\\-2\end{pmatrix}$为方程组$\begin{cases}x_1+a_2x_2+4x_3-x_4=d_1\\b_1x_1+x_2+b_3x_3+b_4x_4=d_2\\2x_1+c_2x_2+c_3x_3+x_4=d_3\end{cases}$的三个解, 求该方程的通解.

$$\begin{aligned}&1.\overline{A}=\begin{pmatrix}1&a_2&4&-1&d_1\\b_1&1&b_3&b_4&d_2\\2&c_2&c_3&1&d_3\end{pmatrix}\\&\quad r(A)=r(\overline{A})<4\\&\quad\because r(A)\geq 2,\therefore 2\leq r(A)=r(\overline{A})\leq 3\\&\quad\xi_1=\eta_2-\eta_1=\begin{pmatrix}-1\\3\\-1\\-3\end{pmatrix},\xi_2=\eta_3-\eta_1=\begin{pmatrix}1\\3\\2\\-4\end{pmatrix}\\&\quad\because\xi_1,\xi_2\text{为}AX=0\text{线性无关解}\\&\quad\therefore 4-r(A)\geq 2\Rightarrow r(A)\leq 2\Rightarrow r(A)=r(\overline{A})=2<4\\&2.\text{通解}X=k_1\xi_1+k_2\xi_2+\eta_1\end{aligned}$$

设$A=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$为四阶矩阵, 方程组$AX=0$的通解为$X=k(1,0,-4,0)^T$, 下列向量组中是$A^*X=0$的基础解系的为$(C)$.
$(A)\alpha_1,\alpha_2,\alpha_3$
$(B)\alpha_1,\alpha_2$
$(C)\alpha_1,\alpha_2,\alpha_4$
$(D)\alpha_1,\alpha_3,\alpha_4$

$$\begin{aligned}&1.r(A)=3<4\Rightarrow r(A^*)=1<4\\&\quad\Rightarrow A^*X=0\text{基础解系含3个线性无关列向量}\\&2.\because A\begin{pmatrix}1\\0\\-4\\0\end{pmatrix}=0,\therefore\alpha_1-4\alpha_3=0\Rightarrow\alpha_1=4\alpha_3\\&\quad\because r(A)=3,\therefore\alpha_1,\alpha_2,\alpha_4\text{或}\alpha_2,\alpha_3,\alpha_4\text{线性无关}\\&3.\because A^*A=|A|E=0,\therefore\alpha_1\sim\alpha_4\text{为}A^*X=0\text{的解}\\&\quad\therefore\alpha_1,\alpha_2,\alpha_3\text{或}\alpha_2,\alpha_3,\alpha_4\text{为}A^*X=0\text{的基础解系}\end{aligned}$$

$$A=\begin{pmatrix}2 & b & c\\ & & \\ & & \end{pmatrix}_{3\times3},B=\begin{pmatrix}1&2&3\\2&4&6\\3&6&k\end{pmatrix},\text{且}AB=0,\text{求}AX=0\text{通解}.$$

$\because A\neq 0,\therefore r(A)\geq 1$

又 $\because AB=0,\therefore r(A)+r(B)\leq 3$

Case 1. $k\neq 9: r(B)=2\Rightarrow r(A)=1<3$

$$\because AB=0,\therefore X=C_1\begin{pmatrix}1\\2\\3\end{pmatrix}+C_2\begin{pmatrix}3\\6\\k\end{pmatrix}$$

Case 2. $k=9: r(B)=1\Rightarrow 1\leq r(A)\leq 2$

①$r(A)=2<3$

$$\because AB=0,\therefore X=C\begin{pmatrix}1\\2\\3\end{pmatrix}$$

②$r(A)=1<3:$

$$A\to\begin{pmatrix}2&b&c\\&0&\end{pmatrix}\to\begin{pmatrix}1&\frac{b}{2}&\frac{c}{2}\\&0&\end{pmatrix},X=C_1\begin{pmatrix}-\frac{b}{2}\\1\\0\end{pmatrix}+C_2\begin{pmatrix}-\frac{c}{2}\\0\\1\end{pmatrix}$$

$A=(\alpha_1,\alpha_2,\alpha_3,\alpha_4),\alpha_1,\alpha_3$无关$,\alpha_2=2\alpha_1+\alpha_3$

$\alpha_4=\alpha_1+4\alpha_2-\alpha_3,$又$b=\alpha_1-\alpha_2+\alpha_3,$求$AX=b$通解.

$AX=b\Leftrightarrow x_1\alpha_1+x_2\alpha_2+x_3\alpha_3+x_4\alpha_4=b$

1. $\because b=\alpha_1-\alpha_2+\alpha_3,\therefore r(A)=r(\overline{A})$

又 $\because r(A)=2<4,\therefore r(A)=r(\overline{A})=2<4$

$$2.\because\begin{cases}2\alpha_1-\alpha_2+\alpha_3+0\alpha_4=0\\ \alpha_1+4\alpha_2-\alpha_3-\alpha_4=0\\ \alpha_1-\alpha_2+\alpha_3+0\alpha_4=b\end{cases},\therefore X=k_1\begin{pmatrix}2\\-1\\1\\0\end{pmatrix}+k_2\begin{pmatrix}1\\4\\-1\\-1\end{pmatrix}+\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}$$

问题二 含参方程组解的讨论

$$\text{设}A=\begin{pmatrix}1&2&1&2\\0&1&t&t\\1&t&0&1\end{pmatrix},\text{且}AX=0\text{的基础解系含有两个线性无关的解向量},\text{求}AX=0\text{的通解}.$$

1. $\because 4-r(A)=2,\therefore r(A)=2<4$

$$2.A\to\begin{pmatrix}1&2&1&2\\0&1&t&t\\0&t-2&-1&-1\end{pmatrix}$$

$$\because r(A)=2,\therefore\frac{1}{t-2}=\frac{t}{-1}=\frac{t}{-1}\Rightarrow t=1$$

$$3.A\to\begin{pmatrix}1&2&1&2\\0&1&1&1\\0&0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&-1&0\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

$$\therefore X=k_1\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}+k_2\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}$$

$$\text{已知齐次线性方程组}\begin{cases}x_1-x_2+2x_3=0\\2x_1+(a-3)x_2+5x_3=0\\-x_1+x_2+(4-a)x_3=0\\ax_1-ax_2+(2a+3)x_3=0\end{cases}\text{有非零解, 求常数}a\text{，并求该方程组的通解}.$$

$$A=\begin{pmatrix}1&-1&2\\2&a-3&5\\-1&1&4-a\\a&-a&2a+3\end{pmatrix}=r(A)<3$$

$$A\to\begin{pmatrix}1&-1&2\\0&a-1&1\\0&0&6-a\\0&0&3\end{pmatrix}\to\begin{pmatrix}1&-1&2\\0&a-1&1\\0&0&1\\0&0&0\end{pmatrix}$$

若$a\neq 1\Rightarrow r(A)=3$,矛盾.$\therefore a=1$

$$A\to\begin{pmatrix}1&-1&2\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&-1&0\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

$$X=k\begin{pmatrix}1\\1\\0\end{pmatrix}$$

设方程组 $\begin{pmatrix}1&2&1\\2&3&a+2\\1&a&-2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\3\\0\end{pmatrix}$,讨论$a$的取值,使得方程组有唯一解,无解,有无穷多个解,当方程组有无穷多解时,求出其通解.

$$\overline{A}=\begin{pmatrix}1&2&1&1\\2&3&a+2&3\\1&a&-2&0\end{pmatrix}\to\begin{pmatrix}1&2&1&1\\0&-1&a&1\\0&a-2&-3&-1\end{pmatrix}$$

$$\to\begin{pmatrix}1&2&1&1\\0&-1&a&1\\0&0&(a+1)(a-3)&a-3\end{pmatrix}$$

①$a\neq -1$且$a\neq 3:r(A)=r(\overline{A})=3$,唯一解

$$\overline{A}\to\begin{pmatrix}1&2&1&1\\0&1&-a&-1\\0&0&1&\frac{1}{a+1}\end{pmatrix}\to\begin{pmatrix}1&2&0&\frac{a}{a+1}\\0&1&0&-\frac{1}{a+1}\\0&0&1&\frac{1}{a+1}\end{pmatrix}$$

$$\to\begin{pmatrix}1&0&0&\frac{a+2}{a+1}\\0&1&0&-\frac{1}{a+1}\\0&0&1&\frac{1}{a+1}\end{pmatrix},X=\begin{pmatrix}\frac{a+2}{a+1}\\-\frac{1}{a+1}\\\frac{1}{a+1}\end{pmatrix}$$

②$a=-1:\overline{A}\to\begin{pmatrix}1&2&1&1\\0&-1&-1&1\\0&0&0&-4\end{pmatrix}$

$r(A)=2\neq r(\overline{A})=3$,无解

③$a=3:\overline{A}\to\begin{pmatrix}1&2&1&1\\0&-1&3&1\\0&0&0&0\end{pmatrix}$

$r(A)=r(\overline{A})=2<3$

$$\overline{A}\to\begin{pmatrix}1&2&1&1\\0&1&-3&-1\\0&0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&7&3\\0&1&-3&-1\\0&0&0&0\end{pmatrix}$$

$$\therefore X=k\begin{pmatrix}-7\\3\\1\end{pmatrix}+\begin{pmatrix}3\\-1\\0\end{pmatrix}$$

设$A=\begin{pmatrix}1&a&0&0\\0&1&a&0\\0&0&1&a\\a&0&0&1\end{pmatrix},\beta=\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$

(1)计算$|A|$;

(2)讨论当a为何值时,方程组$AX=\beta$有无数个解,并求出其通解.

$$①|A| = \begin{vmatrix} 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & a \\ a & 0 & 0 & 1 \end{vmatrix} = A_{11} + aA_{12} = M_{11} - aM_{12}$$

$$= 1 - a \begin{vmatrix} 0 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 1 - a \times aA_{12} = 1 + a^2 M_{12} = 1 - a^4$$

$②r(A) = r(\overline{A}) < 4 \Rightarrow (\nLeftarrow) r(A) < 4 \Rightarrow |A| = 0 \Rightarrow a = \pm 1$

Case 1. $a = 1:$

$$\overline{A} \to \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

$r(A) = 3 \neq r(\overline{A}) = 4,$无解$, a = 1$舍

Case 2. $a = -1:$

$$\overline{A} \to \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$r(A) = r(\overline{A}) = 3 < 4,$无数解$, \therefore a = -1$

$$\overline{A} \to \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \therefore X = k \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$