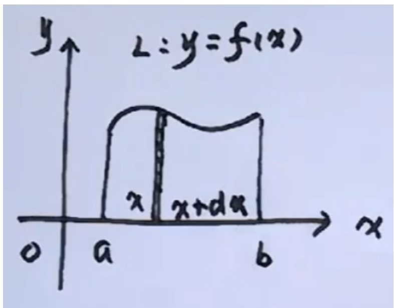
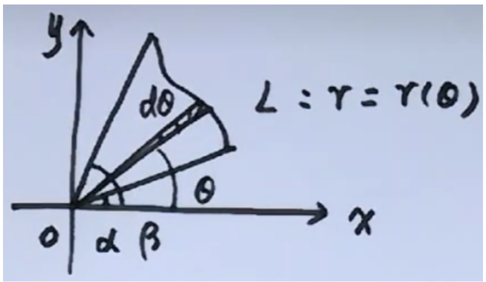
定积分的几何应用

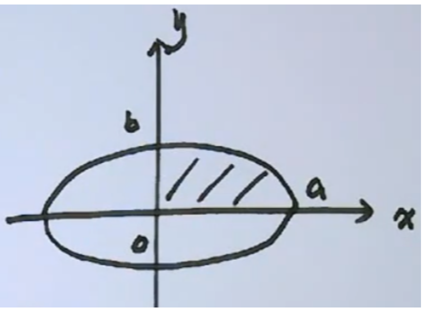
面积



$$
\begin{aligned}
&1.\forall[x,x+dx]\subset[a,b]\\
&2.dA=f(x)dx\\
&3.A=\int_a^b f(x)dx
\end{aligned}
$$



$$
\begin{aligned}
&1.\forall[\theta,\theta+d\theta]\subset[\alpha,\beta]\\
&2.dA=\frac{1}{2}r^2(\theta)d\theta\\
&3.A=\frac{1}{2}\int_\alpha^\beta r^2(\theta)d\theta
\end{aligned}
$$



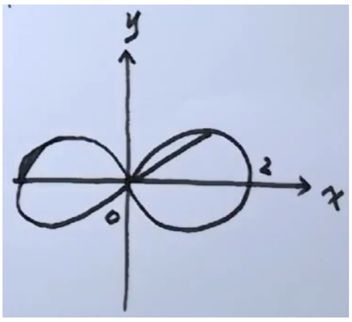
求椭圆$L:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>0,b>0)$所围成的区域面积.

解 :法一

$$
\begin{aligned}
&L_1:y=\frac{b}{a}\sqrt{a^2-x^2}(0\le x\le a)\\
&A_1=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}dx=\frac{b}{a}*\frac{\pi}{4}a^2=\frac{\pi}{4}ab\\
&A=4A_1=\pi ab
\end{aligned}
$$

法二

$$
\begin{aligned}
&令\begin{cases}x=r\cos\theta\\y=r\sin\theta\end{cases}\\
&L:r^2=\frac{a^2b^2}{b^2\cos^2\theta+a^2\sin^2\theta}\\
&A_1=\frac{a^2b^2}{2}\int_0^{\frac{\pi}{2}}\frac{d\theta}{b^2\cos^2\theta+a^2\sin^2\theta}=\frac{ab^2}{2}\int_0^{\frac{\pi}{2}}\frac{d(a\tan\theta)}{b^2+(a\tan\theta)^2}=\frac{ab^2}{2}*\frac{1}{b}\arctan\frac{a\tan\theta}{b}\Big|_0^{\frac{\pi}{2}}=\frac{ab}{2}*\frac{\pi}{2}=\frac{\pi ab}{4}\\
&A=4A_1=\pi ab
\end{aligned}
$$



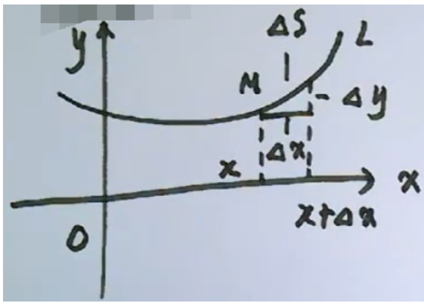
求$L:(x^2+y^2)^2=4(x^2-y^2)$围成面积.

解:$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

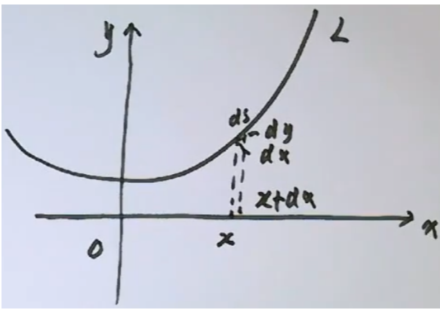
$L:r^2=4\cos 2\theta$

$A_1=\frac{1}{2}\int_0^{\frac{\pi}{4}}4\cos 2\theta d\theta=\int_0^{\frac{\pi}{4}}\cos 2\theta d(2\theta)=\int_0^{\frac{\pi}{4}}\cos 2\theta d(2\theta)=\int_0^{\frac{\pi}{2}}\cos\theta d\theta$

$\Rightarrow A=4A_1=4$



$$(\Delta s)^2\approx(\Delta x)^2+(\Delta y)^2$$



$$(ds)^2=(dx)^2+(dy)^2$$

$1.L:y=f(x)$

$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{1+(\frac{dy}{dx})^2}dx$

$=\sqrt{1+f'^2(x)}dx$

$2.L:\begin{cases} x=\Phi(t) \\ y=\phi(t) \end{cases}$

$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{\Phi'^2(t)+\phi'^2(t)}dt$

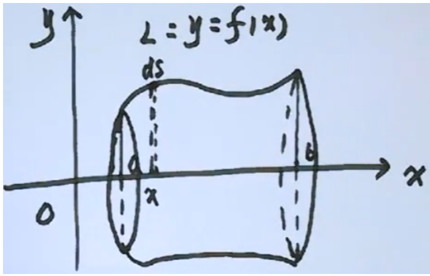
3.

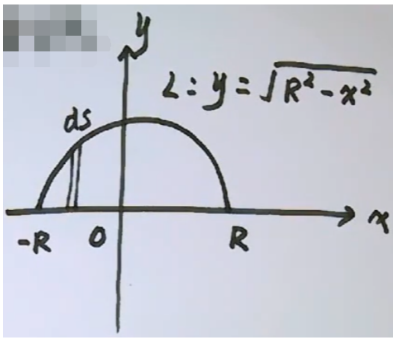
①. 取$[x,x+dx]\subset[a,b]$

②. $dA=2\pi|f(x)|\cdot dS$

$=2\pi|y|\cdot\sqrt{1+y'^2}dx$

③. $A_x=2\pi\int_a^b|y|\cdot\sqrt{1+y'^2}dx$

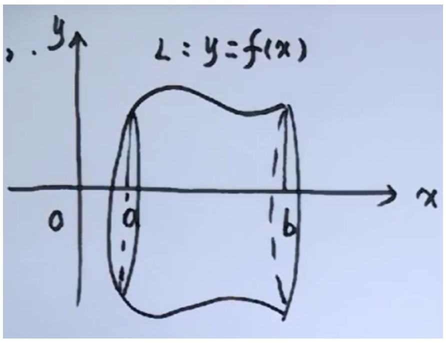




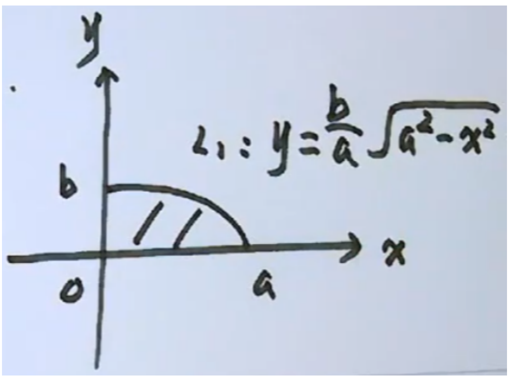
求半径为R的球的表面积

$$\begin{aligned}
\text{解：}&1.\text{取}[x,x+dx]\subset[-R,R]\\
&2.dA=2\pi y\cdot ds\\
&\quad=2\pi\cdot\sqrt{R^2-x^2}\cdot\sqrt{1+\frac{x^2}{R^2-x^2}}dx=2\pi Rdx\\
&3.A=\int_{-R}^{R}2\pi Rdx=4\pi R^2
\end{aligned}$$

体积

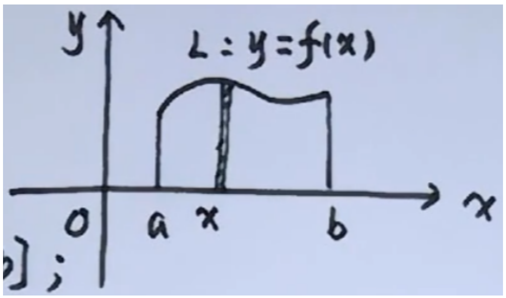


$$\begin{aligned}
&1.\text{取}[x,x+dx]\subset[a,b]\\
&2.dV_x=\pi f^2(x)dx\\
&3.V_x=\pi\int_a^b f^2(x)dx
\end{aligned}$$

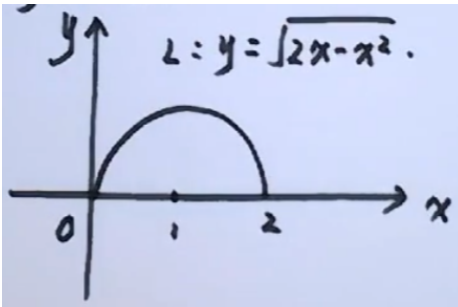


$$L:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\text{求}V_x$$

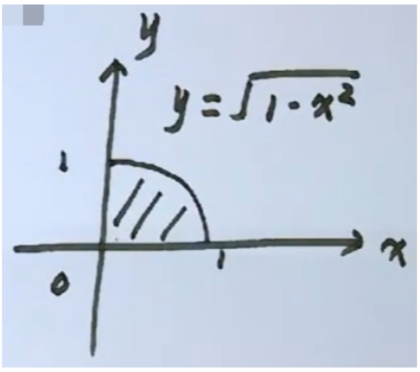
$$
\begin{aligned}
解:V_x &= 2\int_0^a \frac{b^2}{a^2}(a^2-x^2)dx \\
&= \frac{2b^2\pi}{a^2}(a^3-\frac{a^3}{3}) = \frac{4}{3}\pi ab^2
\end{aligned}
$$

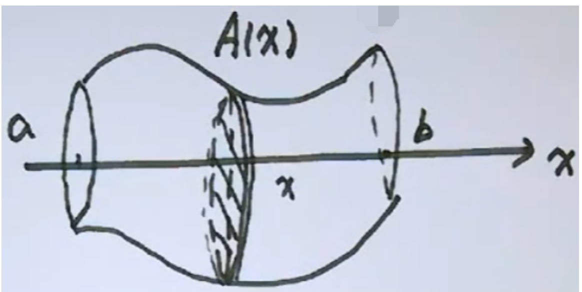


1.取$[x, x+dx] \subset [a,b]$
2.$dV_y = 2\pi|x| \cdot |y|dx$
3.$V_y = 2\pi\int_a^b |x| \cdot |y|dx$

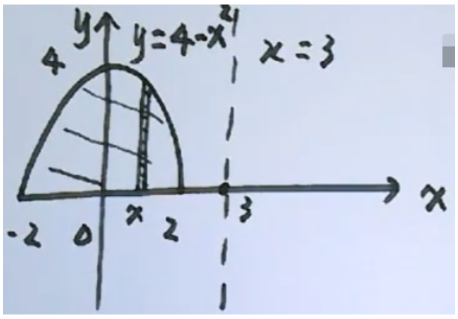


1.$V_x = \pi\int_0^2 f^2(x)dx = \pi\int_0^2 y^2dx = \pi\int_0^2(2x-x^2)dx = \pi(4-\frac{8}{3}) = \frac{4}{3}\pi$
2.取$[x, x+dx] \subset [0,2]$
$dV_y = 2\pi x\sqrt{2x-x^2}dx$
$V_y = 2\pi\int_0^2 x\sqrt{2x-x^2}dx = 2\pi\int_0^2[1+(x-1)]\sqrt{1-(x-1)^2}d(x-1) = 2\pi\int_{-1}^1(1+x)\sqrt{1-x^2}dx = 4\pi\int_0^1\sqrt{1-x^2}dx = \pi^2$





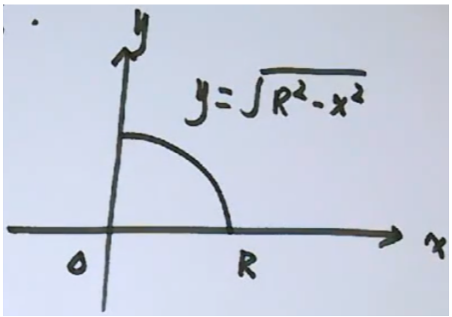
1.取$[x, x+dx] \subset [a,b]$
2.$dV = A(x)dx$
3.$V = \int_a^b A(x)dx$



$$1.\text{取}[x,x+dx]\subset[-2,2]$$
$$2.dV=2\pi(3-x)\cdot y\cdot dx=2\pi(3-x)(4-x^2)dx$$
$$3.V=2\pi\int_{-2}^{2}(3-x)(4-x^2)dx=12\pi\int_{0}^{2}(4-x^2)dx$$
$$=12\pi(8-\frac{8}{3})=12\pi\cdot\frac{16}{3}=64\pi$$

弧长

$$1.L:y=f(x)(a\leq x\leq b)$$
$$①\forall[x,x+dx]\subset[a,b]$$
$$②ds=\sqrt{1+f'^2(x)}dx$$
$$③l=\int_a^b\sqrt{1+f'^2(x)}dx$$
$$2.L:\begin{cases}x=\Phi(t)\\y=\phi(t)\end{cases}(\alpha\leq t\leq\beta)$$
$$①\text{取}[t,t+dt]\subset[\alpha,\beta]$$
$$②ds=\sqrt{\Phi'^2(t)+\phi'^2(t)}dt$$
$$③l=\int_\alpha^\beta\sqrt{\Phi'^2(t)+\phi'^2(t)}dt$$

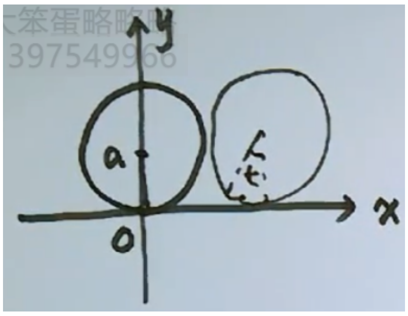


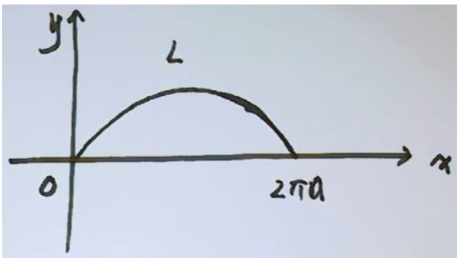
求半径为R的圆周长

$$\text{解}:l=4\int_0^R\sqrt{1+y'^2}dx$$
$$=4\int_0^R\sqrt{1+\frac{R}{R^2-x^2}}dx$$
$$=4R\int_0^R\frac{dx}{\sqrt{R^2-x^2}}=4R\arcsin\frac{x}{R}\ |_0^R=2\pi R$$

特殊曲线

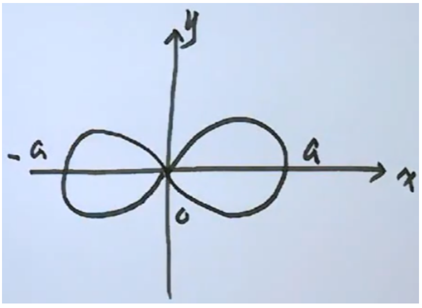
摆线





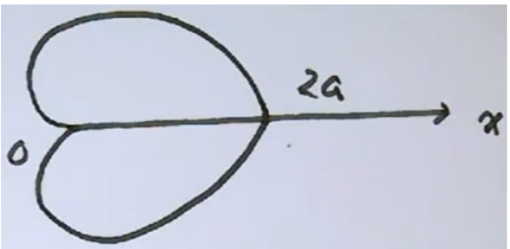
$$L:\begin{cases}x=a(t-\sin t)\\y=a(1-\cos t)\end{cases}(0\le t\le 2\pi)$$

双纽线



$$L:(x^2+y^2)^2=a^2(x^2-y^2)$$
$$L:r^2=a^2\cos 2\theta$$

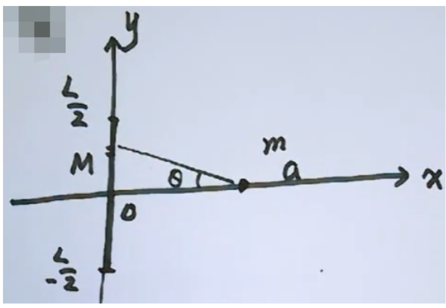
心脏线



$$L:r=a(1+\cos\theta)$$

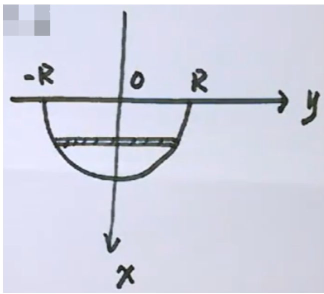
定积分的物理应用

引力



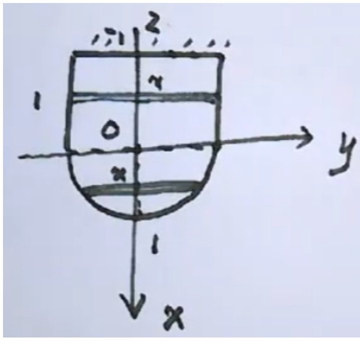
$$
\begin{aligned}
&解:1.取[y,y+dy]\subset[-\frac{L}{2},\frac{L}{2}]\\
&\quad 2.dF=K\cdot\frac{m\cdot\frac{M}{L}dy}{a^2+y^2},dF_x=dF\cdot\frac{a}{\sqrt{a^2+y^2}}\\
&\quad\quad dF_x=\frac{KamM}{L}\cdot\frac{dy}{(a^2+y^2)^{\frac{3}{2}}}\\
&\quad 3.F_x=\int_{-\frac{L}{2}}^{\frac{L}{2}}dF_x=\frac{2KamM}{L}\int_0^{\frac{L}{2}}\frac{dy}{(y^2+a^2)^{\frac{3}{2}}}\\
&\quad\quad =\frac{2KamM}{L}\cdot\frac{y}{\sqrt{y^2+a^2}}\ |_0^{\frac{L}{2}}
\end{aligned}
$$

压力



圆柱形水桶盛一半的水,底面半径为R,将圆柱水平放置,求水对地面的压力.

$$
\begin{aligned}
&解:1.取[x,x+dx]\subset[0,R]\\
&\quad 2.dF=\rho gx\cdot dA\\
&\quad\quad =\rho gx\cdot(y_2-y_1)dx\\
&\quad\quad \because x^2+y^2=R^2,\therefore y_1=-\sqrt{R^2-x^2},y_2=\sqrt{R^2-x^2}\\
&\quad\quad \therefore dF=2\rho gx\sqrt{R^2-x^2}dx\\
&\quad 3.F=2\rho g\int_0^R x\sqrt{R^2-x^2}dx\\
&\quad\quad =-\rho g\int_0^R(R^2-x^2)^{\frac{1}{2}}d(R^2-x^2)\\
&\quad\quad =-\frac{2}{3}\rho g(R^2-x^2)^{\frac{3}{2}}\ |_0^R=\frac{2}{3}\rho gR^3
\end{aligned}
$$



解：
①1.取$[x,x+dx]\subset[0,-1]$

$$2.dF_1=\rho g\cdot(x+1)\cdot 2dx$$
$$=\rho gx\cdot(y_2-y_1)dx$$
$$3.F_1=2\rho g\int_{-1}^{0}(x+1)dx=2\rho g(-\frac{1}{2}+1)=\rho g$$

②1.取$[x,x+dx]\subset[0,1]$

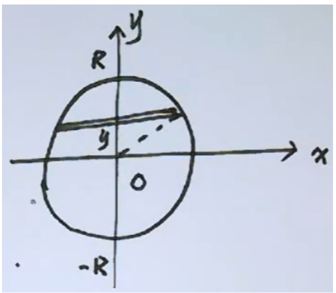
$$2.dF_2=\rho g\cdot(x+1)\cdot(y_2-y_1)dx$$
$$=2\rho g(x+1)\sqrt{1-x^2}dx$$
$$3.F_2=2\rho g\int_{0}^{1}(x+1)\sqrt{1-x^2}dx$$
$$=-\rho g\int_{0}^{1}(1-x^2)^{\frac{1}{2}}d(1-x^2)+2\rho g\frac{\pi}{4}$$
$$=\frac{2}{3}\rho g+\frac{\pi}{2}\rho g$$
$$\therefore F=(\frac{5}{3}+\frac{\pi}{2})\rho g$$

功

有一电荷量为q_1, 且带正电的固定质点位于原点, 在距离原点a处有一电荷量为q_2且带正电的活动质点, 若固定质点, 将活动质点从距离a处排斥到距离b处, 求排斥力所做的功.

解：1.取$[x,x+dx]\subset[a,b]$

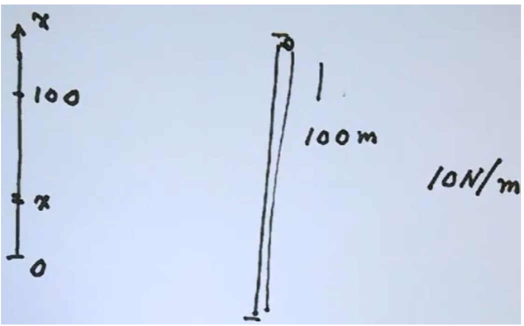
$$2.dW=k\cdot\frac{q_1q_2}{x^2}dx$$
$$3.W=\int_{a}^{b}dW=-\frac{kq_1q_2}{x}\mid_{a}^{b}$$
$$=kq_1q_2(\frac{1}{a}-\frac{1}{b})$$



半径为R的球体充满水, 将水从顶部抽出, 求W.

解：1.取$[y,y+dy]\subset[-R,R]$

$$2.dW=\rho gdV\cdot(R-y)$$
$$dV=\pi x^2\cdot dy=\pi(R^2-y^2)dy$$
$$dW=\pi\rho g(R-y)(R^2-y^2)dy$$
$$3.W=\pi\rho g\int_{-R}^{R}(R-y)(R^2-y^2)dy$$
$$=2\pi\rho gR\int_{0}^{R}(R^2-y^2)dy$$
$$=\frac{4\pi}{3}\rho gR^4$$



解：1.取$[x, x+dx] \subset [0,100]$

2.$dW = 10(100-x) \cdot dx$

3.$W = \int_0^{100} dW = 10\int_0^{100}(100-x)dx$

$= 10 * \frac{1}{2} * 100 * 100 = 50000(J)$