变积分限函数

$$设f(x)连续, 且\lim_{x\to 0}\frac{f(x)}{x}=2, F(x)=\int_0^x t^{n-1}f(x^n-t^n)dt, 求\lim_{x\to 0}\frac{F(x)}{x^{2n}}.$$

$$
\begin{aligned}
&解:1.F(x)=-\frac{1}{n}\int_0^x f(x^n-t^n)d(x^n-t^n)\\
&=-\frac{1}{n}\int_{x^n}^0 f(u)du=\frac{1}{n}\int_0^{x^n}f(u)du\\
&2.原式=\frac{1}{2n}\lim_{x\to 0}\frac{f(x^n)\cdot x^{n-1}}{x^{2n-1}}=\frac{1}{2n}\lim_{x\to 0}\frac{f(x^n)}{x^n}=\frac{1}{n}
\end{aligned}
$$

$$f(x)连续, f(0)=0, f'(0)=2, 求\lim_{x\to 0}\frac{\int_{-x}^x[f(t+x)-f(t-x)]dt}{x^2}.$$

$$
\begin{aligned}
&解:1.\int_{-x}^x[f(t+x)-f(t-x)]dt\\
&=\int_{-x}^x f(t+x)d(t+x)-\int_{-x}^x f(t-x)d(t-x)\\
&=\int_0^{2x}f(u)du+\int_0^{-2x}f(u)du\\
&2.原式=2\lim_{x\to 0}\frac{f(2x)-f(-2x)}{2x}\\
&=2[\lim_{x\to 0}\frac{f(2x)-f(0)}{2x}+\lim_{x\to 0}\frac{f(-2x)-f(0)}{-2x}]\\
&=4f'(0)=8
\end{aligned}
$$

常规计算

$$
\begin{aligned}
&\int_0^{\frac{\pi}{2}}\frac{\sin 2x}{1+\sin^4 x}dx\\
&=\int_0^{\frac{\pi}{2}}\frac{d(\sin^2 x)}{1+(\sin^2 x)^2}\\
&=\arctan\sin^2 x\ |_0^{\frac{\pi}{2}}\\
&=\frac{\pi}{4}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^{\ln 2}\sqrt{e^x-1}dx, \sqrt{e^x-1}=t\\
&=\int_0^1 t\cdot\frac{2t}{1+t^2}dt=2\int_0^1(1-\frac{1}{1+t^2})dt\\
&=2(1-\frac{\pi}{4})
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 x\ln(1+x^2)dx \\
=&\frac{1}{2}\int_0^1 \ln(1+x^2)d(1+x^2) \\
=&\frac{1}{2}\int_1^2 \ln x dx \\
=&\frac{1}{2}(x\ln x\mid_1^2 -1) \\
=&\frac{1}{2}(2\ln 2-1) \\
=&\ln 2-\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \frac{\arctan x}{(1+x^2)^2}dx, x=\tan t \\
=&\int_0^{\frac{\pi}{4}} \frac{t}{\sec^4 t}\cdot\sec^2 t \\
=&\int_0^{\frac{\pi}{4}} t\cos^2 t \\
=&\frac{1}{2}\int_0^{\frac{\pi}{4}} t(1+\cos 2t)dt \\
=&\frac{t^2}{4}\mid_0^{\frac{\pi}{4}} +\frac{1}{4}\int_0^{\frac{\pi}{4}} td(\sin 2t) \\
=&\frac{\pi^2}{64}+\frac{1}{4}(t\sin 2t\mid_0^{\frac{\pi}{4}} -\int_0^{\frac{\pi}{4}}\sin 2tdt) \\
=&\frac{\pi^2}{64}+\frac{\pi}{16}-\frac{1}{8}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^2 x^2\sqrt{2x-x^2}dx \\
=&\int_0^2 [1+(x-1)]^2\sqrt{1-(x-1)^2}d(x-1) \\
=&\int_{-1}^1 (1+x)^2\sqrt{1-x^2}dx \\
=&2\int_0^1 (1+x^2)\sqrt{1-x^2}dx, x=\sin t \\
=&2\int_0^{\frac{\pi}{2}} (1+\sin^2 t)(1-\sin^2 t)dt \\
=&2\int_0^{\frac{\pi}{2}} (1-\sin^4 t)dt \\
=&2(\frac{\pi}{2}-\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}) \\
=&\frac{5\pi}{8}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^{\pi^2}\sin^2\sqrt{x}dx,\sqrt{x}=t\\
=&2\int_0^{\pi}t\sin^2 tdt\\
=&\pi\int_0^{\pi}\sin^2 tdt\\
=&2\pi\cdot\frac{1}{2}\cdot\frac{\pi}{2}\\
=&\frac{\pi^2}{2}
\end{aligned}
$$

变积分限函数计算定积分->分部积分

$$f(x)=\int_a^x\Phi(t)dt,\text{求}\int_a^b f(x)dx.$$

$$\text{设}f(x)=\int_1^x e^{-t^2}dt,\text{求}\int_0^1 x^2f(x)dx.$$

$$
\begin{aligned}
&\int_0^1 x^2f(x)dx\\
=&\frac{1}{3}\int_0^1 f(x)d(x^3)\\
=&\frac{x^3}{3}f(x)\mid_0^1-\frac{1}{3}\int_0^1 x^3\cdot e^{-x^2}dx\\
=&-\frac{1}{6}\int_0^1 x^2e^{-x^2}d(x^2)\\
=&-\frac{1}{6}\int_0^1 xe^{-x}dx\\
=&\frac{1}{6}\int_0^1 xd(e^{-x})\\
=&\frac{x}{6}e^{-x}\mid_0^1-\frac{1}{6}\int_0^1 e^{-x}dx\\
=&\frac{1}{6e}+\frac{1}{6}(\frac{1}{e}-1)=\frac{1}{3e}-\frac{1}{6}
\end{aligned}
$$

$$\text{设}f(x)=\int_0^x\frac{\sin t}{\pi-t}dt,\text{求}\int_0^{\pi}f(x)dx$$

$$
\begin{aligned}
&\int_0^{\pi}f(x)dx\\
=&xf(x)\mid_0^{\pi}-\int_0^{\pi}x\cdot\frac{\sin x}{\pi-x}dx\\
=&\pi f(\pi)-\int_0^{\pi}\frac{x\sin x}{\pi-x}dx\\
=&\int_0^{\pi}\frac{\pi\sin x}{\pi-x}dx-\int_0^{\pi}\frac{x\sin x}{\pi-x}dx\\
=&\int_0^{\pi}\sin xdx=2I_1=2
\end{aligned}
$$

证明

情形一

$$f(x)\text{为连续函数}$$

$$\begin{aligned}
&1.\int_{-a}^{0} f(x)dx = \int_{0}^{a}, x = -t \\
&2.\int_{a}^{a+b} f(x)dx = \int_{0}^{b}, x - a = t \\
&3.\int_{a}^{b} f(x)dx = \int_{a}^{b}, x + t = a + b \\
&4.\int_{a}^{b} f(x)dx = \int_{0}^{1}, x = a + (b-a)t
\end{aligned}$$

$$\text{设}f(x)\text{连续}, \text{证明}: \int_{a}^{b} f(x)dx = \int_{a}^{b} f(a+b-x)dx$$

$$\begin{aligned}
&\int_{a}^{b} f(x)dx, x + t = a + b \\
=&\int_{b}^{a} f(a+b-t)\cdot(-dt) \\
=&\int_{a}^{b} f(a+b-t)dt = \int_{a}^{b} f(a+b-x)dx
\end{aligned}$$

$$\text{设}f(x)\text{连续}, \text{证明}: \int_{a}^{b} f(x)dx = (b-a)\int_{0}^{1} f[a+(b-a)x]dx$$

$$\begin{aligned}
&\int_{a}^{b} f(x)dx, x = a + (b-a)t \\
=&\int_{0}^{1} f[a+(b-a)t]\cdot(b-a)dt \\
=&(b-a)\int_{0}^{1} f[a+(b-a)x]dx
\end{aligned}$$

$$\begin{aligned}
&\text{设}f(x)\in C[a,b], \text{且对任意的}x,y\in[a,b]\text{有}|f(x)-f(y)|\le 2|x-y|, \\
&\text{证明}: |\int_{a}^{b} f(x)dx - f(a)(b-a)| \le (b-a)^2
\end{aligned}$$

证明：

$$\begin{aligned}
&1.f(a)(b-a) = \int_{a}^{b} f(a)dx \\
&2.|\int_{a}^{b} f(x)dx - f(a)(b-a)| = |\int_{a}^{b}[f(x)-f(a)]dx| \\
&3.|\int_{a}^{b}[f(x)-f(a)]dx| \le \int_{a}^{b}|f(x)-f(a)|dx \le \int_{a}^{b} 2(x-a)d(x-a) \\
&\quad = (x-a)^2\ |_{a}^{b} = (b-a)^2
\end{aligned}$$

情形二

$$f(x)\text{连续}+\text{单调}$$

$$\text{设}f(x)\in C[a,b]\text{且}f(x)\text{单调递增,证明}:\int_a^b xf(x)dx\geq\frac{a+b}{2}\int_a^b f(x)dx$$

$$\begin{aligned}&\text{令}\Phi(x)=\int_a^x tf(t)dt-\frac{a+x}{2}\int_a^x f(t)dt,\Phi(a)=0\\&\Phi'(x)=xf(x)-\frac{1}{2}\int_a^x f(t)dt-\frac{a+x}{2}f(x)\\&=\frac{x-a}{2}f(x)-\frac{1}{2}\int_a^x f(t)dt\\&=\frac{x-a}{2}[f(x)-f(\xi)]\geq0(a<\xi<x<b)\\&\Rightarrow\Phi(x)\geq0(a<x\leq b)\\&\Rightarrow\Phi(b)\geq0\end{aligned}$$

$$\begin{aligned}&\text{设}f(x)\in C[0,1]\text{且}f(x)\text{单调递减,对任意的}\alpha\in(0,1),\text{证明}:\\&\int_0^\alpha f(x)dx\geq\alpha\int_0^1 f(x)dx\end{aligned}$$

$$\begin{aligned}&\text{法一}\int_0^\alpha f(x)dx=\alpha\int_0^1 f(\alpha t)\cdot dt=\alpha\int_0^1 f(\alpha x)dx,x=\alpha t\\&\because\alpha x\leq x,\therefore f(\alpha x)\geq f(x)\\&\therefore\int_0^\alpha f(x)dx\geq\alpha\int_0^1 f(x)dx\\&\text{法二}\int_0^\alpha f(x)dx-\alpha\int_0^1 f(x)dx=(1-\alpha)\int_0^\alpha f(x)dx-\alpha\int_\alpha^1 f(x)dx\\&=\alpha(1-\alpha)[f(\xi_1)-f(\xi_2)](0\leq\xi_1\leq\alpha,\alpha\leq\xi_2\leq1)\\&\because f\downarrow\text{且}\xi_1\leq\xi_2\\&\therefore\int_0^\alpha f(x)dx-\alpha\int_0^1 f(x)dx\geq0\end{aligned}$$

情形三

$$f(x)\text{可导}$$

$$1.\text{工具}\begin{cases}f(x)-f(a)=f'(\xi)(x-a)-L:\text{积分中无导数}\\f(x)-f(a)=\int_a^x f'(t)dt-N-L:\text{积分号中有导数}\end{cases}$$

$$2.\text{技巧}\begin{cases}|\cdot|:|\int_a^b fdx|\leq\int_a^b|f|dx\\(\ )^2:(\int_a^b fgdx)\leq\int_a^b f^2dx\cdot\int_a^b g^2dx\\\text{两边积分}\\\frac{1}{b-a}\int_a^b f(x)dx=f(c)(a\leq c\leq b)\end{cases}$$

$$\text{设}f'(x)\in C[0,a],f(0)=0,|f'(x)|\leq M,\text{证明}:|\int_0^a f(x)dx|\leq\frac{M}{2}a^2$$

证：

$$1.f(x)=f(x)-f(0)=f'(\xi)x(0<\xi<x)$$

$$2.|\int_0^a f(x)dx|\leq\int_0^a |f(x)|dx$$

$$=\int_0^a |f'(\xi)|xdx\leq M\int_0^a xdx=\frac{M}{2}a^2$$

设$f'(x)\in C[a,b],f(a)=f(b)=0,c\in(a,b),$证明：

$$|f(c)|\leq\frac{1}{2}\int_a^b |f'(x)|dx$$

证：

$$1.\begin{cases}f(c)-f(a)=\int_a^c f'(x)dx\\ f(b)-f(c)=\int_c^b f'(x)dx\end{cases}$$

$$2.\begin{cases}|f(c)|\leq\int_a^c |f'(x)|dx\\ |f(c)|\leq\int_c^b |f'(x)|dx\end{cases}$$

$$\Rightarrow|f(c)|\leq\frac{1}{2}\int_a^b |f'(x)|dx$$

设$f'(x)\in C[a,b],c\in(a,b),$证明：

$$|f(c)|\leq|\frac{1}{b-a}\int_a^b f(x)dx|+\int_a^b |f'(x)|dx$$

证：

$$1.\frac{1}{b-a}\int_a^b f(x)dx=f(x_0)(a\leq x_0\leq b)$$

$$2.f(c)-f(x_0)=\int_{x_0}^c f'(x)dx$$

$$f(c)=f(x_0)+\int_{x_0}^c f'(x)dx$$

$$3.|f(c)|\leq|f(x_0)|+|\int_{x_0}^c f'(x)dx|$$

$$\leq|\frac{1}{b-a}\int_a^b f(x)dx|+\int_a^b |f'(x)|dx$$