

傅里叶级数

周期为 2π 的函数 $f(x)$:

$$f(x) \text{ 可否分解为 } \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$$

$$\frac{a_0}{2}, \text{ 直流成份}$$

$$a_1 \cos x + b_1 \sin x, \text{ 一次谐波}$$

$$f(x) \text{ 与 } \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx) \text{ 的关系?}$$

(狄利克雷充分条件) 设 $f(x)$ 以 2π 为周期, 在 $[-\pi, \pi)$ 上 :

① $f(x)$ 连续或有有限个第一类间断点

② $f(x)$ 有有限个极值点

则 $f(x)$ 可以展成 $\frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$, 且

$$\begin{cases} a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, n = 1, 2, \dots \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \end{cases}$$

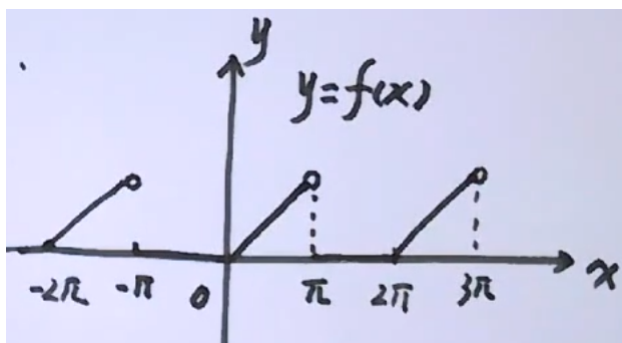
若 x 为 $f(x)$ 的连续点, 则 $\frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx) = f(x)$

若 x 为 $f(x)$ 的间断点, 则 $\frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx) = \frac{f(x-0) + f(x+0)}{2}$

$f(x)$ 以 2π 为周期, 当 $x \in [-\pi, \pi)$ 时,

$$f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ x, & 0 \leq x < \pi \end{cases}$$

将 $f(x)$ 展成Fourier级数.



1. 作 $y = f(x)$ 图

$x = (2k+1)\pi (k \in \mathbb{Z})$ 为间断点

$$2. a_0 = \frac{1}{\pi} \int_0^{\pi} x dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} x \cos nx dx = \frac{1}{n\pi} \int_0^{\pi} x d(\sin nx)$$

$$= -\frac{1}{n\pi} \int_0^{\pi} \sin nx dx = \frac{(-1)^n - 1}{n^2\pi}$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x \sin nx dx = -\frac{1}{n\pi} \int_0^{\pi} x d(\cos nx)$$

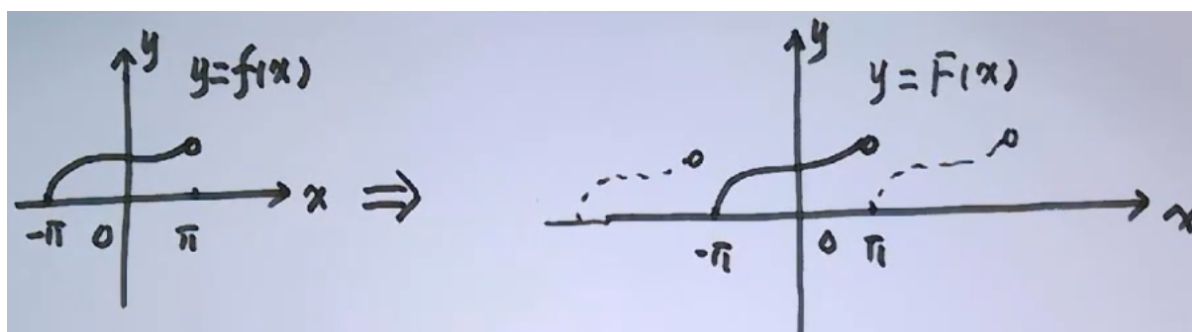
$$3. \textcircled{1} f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2\pi} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right]$$

$(-\infty < x < +\infty \text{ 且 } x \neq (2k+1)\pi (k \in \mathbb{Z}))$

$\textcircled{2}$ 当 $x = (2k+1)\pi$ 时 $(k \in \mathbb{Z})$

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2\pi} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right] = \frac{\pi}{2}$$

周期延拓



1. $f(x)$ 定义域 $[-\pi, \pi)$, $f(x)$ 周期延拓:

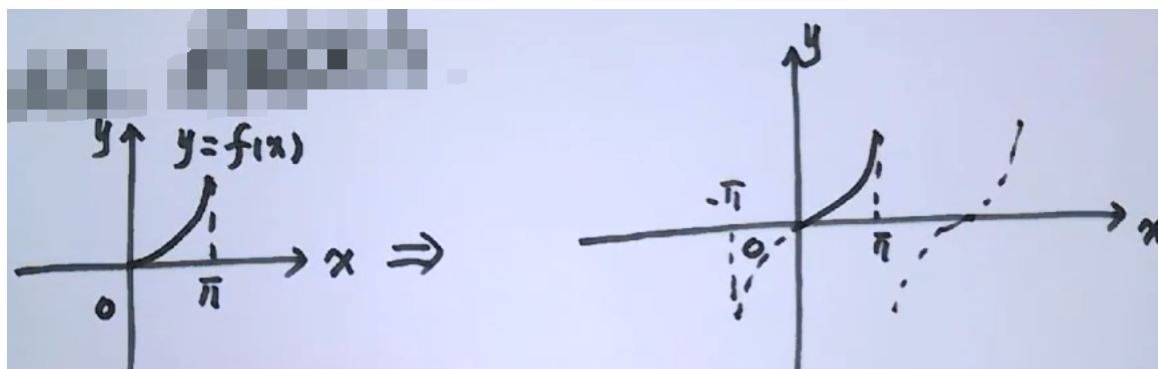
$$2. \begin{cases} a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \end{cases}$$

$$3. f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (\dots), (-\pi < x < \pi)$$

奇延拓 展成正弦级数

$f(x)$ 定义域 $[0, \pi)$

$f(x)$ 奇延拓或展成正弦级数



1. 奇延拓、周期延拓

$$2. a_0 = 0, a_n = 0, b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

$$3. f(x) = \sum_{n=1}^{\infty} b_n \sin nx, (0 \leq x < \pi)$$

偶延拓 展成余弦级数

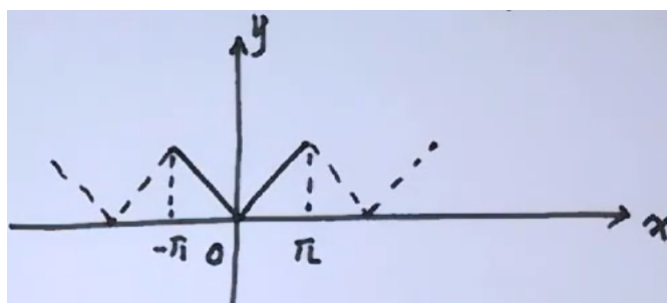


1. 偶延拓、周期延拓

$$2. a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx, a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx, b_n = 0$$

$$3. f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, (0 \leq x \leq \pi)$$

$$f(x) = |x| (-\pi \leq x < \pi) \text{ 展开 } F \text{ 级数, 求 } \sum_{n=1}^{\infty} \frac{1}{n^2} =$$



1. 周期延拓

$$2. a_0 = \frac{2}{\pi} \int_0^{\pi} x dx = \pi$$

$$\begin{aligned} a_n &= \frac{2}{n\pi} \int_0^{\pi} x d(\sin x) = -\frac{2}{n\pi} \int_0^{\pi} \sin nx dx \\ &= \frac{2}{n^2\pi} \cos nx \Big|_0^{\pi} = \frac{2[(-1)^n - 1]}{n^2\pi} = \begin{cases} -\frac{4}{n^2\pi}, n = 1, 3, 5, \dots \\ 0, n = 2, 4, 6, \dots \end{cases} \end{aligned}$$

$$b_n = 0 (n = 1, 2, \dots)$$

$$3. |x| = \frac{\pi}{2} - \frac{4}{\pi} \left(\cos x + \frac{1}{3^2} \cos 3x + \dots \right) (-\pi \leq x \leq \pi)$$

$$4. x = 0 \text{ 时}, \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

$$\text{令 } S = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \left(\frac{1}{1^2} + \frac{1}{1^2} + \dots \right) + \left(\frac{1}{1^2} + \frac{1}{1^2} + \frac{1}{1^2} + \dots \right)$$

$$= \frac{\pi^2}{8} + \frac{1}{4} S$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

以2l为周期的函数f(x)

1. 作图

$$2. \begin{cases} a_0 = \frac{1}{l} \int_{-l}^l f(x) dx \\ a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx \\ b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx \end{cases}$$

$$3. x \text{ 为 } f(x) \text{ 的连续点}, f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}, (-\infty < x < +\infty, x \neq ?)$$

$$4. x = ?, \frac{a_0}{2} + \sum_{n=1}^{\infty} (\dots) = \frac{f(x-0) + f(x+0)}{2}$$

f(x)定义域[-l,l]上

1. 周期延拓

$$2. \begin{cases} a_0 = \frac{1}{l} \int_{-l}^l f(x) dx \\ a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx \\ b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx \end{cases}$$

$$3. f(x) = \frac{a_0}{2} + \dots ()$$