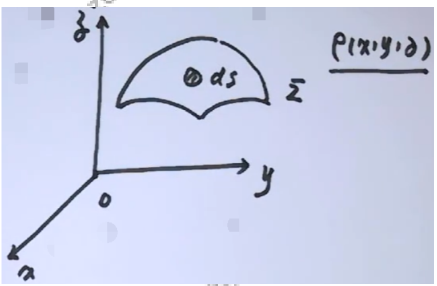
积分域	积分号	例子
线状	\int	\int_a^bf(x)dx \int_L
面状	\iint	\iint_Df(x,y)d\sigma \iint_{\sum}
体状	\iiint	\iiint_{\Omega}f(x,y,z)dV

曲面积分

对面积的曲面积分（第一类曲面积分）

背景: m



$$1.\forall dS \subset \Sigma$$
$$2.dm = \rho(x,y,z)dS$$
$$3.m = \iint_{\Sigma} dm = \iint_{\Sigma} \rho(x,y,z)dS$$

定义

$$\iint_{\Sigma} f(x,y,z)dS$$

性质

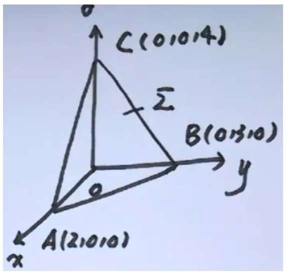
$$1. \iint_{\Sigma} 1dS = A$$

$$2.①\Sigma关于xoy面对称, 上\Sigma_1$$
$$若f(x,y,-z) = -f(x,y,z) \Rightarrow \iint_{\Sigma} f(x,y,z)dS = 0$$
$$若f(x,y,-z) = f(x,y,z) \Rightarrow \iint_{\Sigma} f(x,y,z)dS = 2\iint_{\Sigma_1} f(x,y,z)dS$$

计算方法

特殊法

$$计算I = \iint_{\Sigma}(2x + \frac{4y}{3} + z)dS, 其中\Sigma是平面\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1在第\mathrm{I}卦限的部分.$$

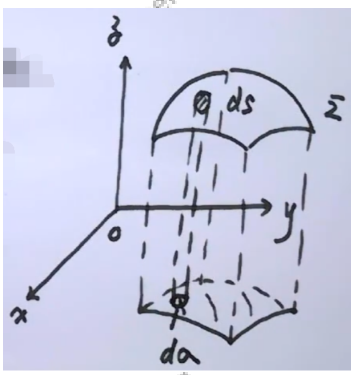


$$\begin{aligned} I =& 4\iint_{\Sigma}(\frac{x}{2}+\frac{y}{3}+\frac{z}{4})dS \\ =& 4\iint_{\Sigma}1dS \end{aligned}$$

$\overrightarrow{AB}=\{-2,3,0\},\overrightarrow{AC}=\{-2,0,4\}$
$\overrightarrow{AB}\times\overrightarrow{AC}=\{12,8,6\}$
$|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{144+64+36}=2\sqrt{61}\Rightarrow S_{\triangle ABC}=\sqrt{61}$
$\therefore$ 原式 $=4\sqrt{61}$

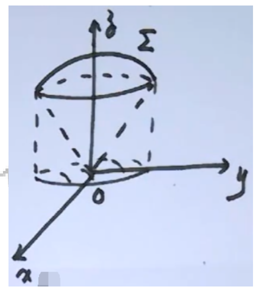
二重积分法

$$I=\iint_{\Sigma}f(x,y,z)dS$$



$1.\Sigma: z=\Phi(x,y),(x,y)\in D_{xy}$
$2.dS=\sqrt{1+(z_x')^2+(z_y')^2}d\sigma$
$3.I=\iint_{D_{xy}}f[x,y,\Phi(x,y)]\cdot\sqrt{1+(z_x')^2+(z_y')^2}d\sigma$

求$I=\iint_{\Sigma}zdS$,其中Σ为$x^2+y^2+z^2=1$被$z=\sqrt{x^2+y^2}$所截的顶部.



$$1.\Sigma : z = \sqrt{1 - x^2 - y^2}$$

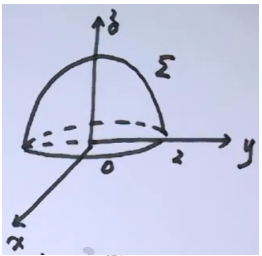
$$D_{xy} : x^2 + y^2 \leq \frac{1}{2}$$

$$2.z'_x = -\frac{x}{\sqrt{1 - x^2 - y^2}}, z'_y = -\frac{y}{\sqrt{1 - x^2 - y^2}}$$

$$dS = \sqrt{1 + \frac{x^2 + y^2}{1 - x^2 - y^2}} d\sigma = \frac{1}{\sqrt{1 - x^2 - y^2}} d\sigma$$

$$3.I = \iint_{D_{xy}} \sqrt{1 - x^2 - y^2} \cdot \frac{1}{\sqrt{1 - x^2 - y^2}} d\sigma = \iint_{D_{xy}} d\sigma = \frac{\pi}{2}$$

$$I = \iint_{\Sigma} (x^2 z + 2xy) dS.$$



$$\iint_{\Sigma} x^2 z dS = \iint_{\Sigma} y^2 z dS$$

$$= \frac{1}{2} \iint_{\Sigma} (x^2 + y^2) z dS$$

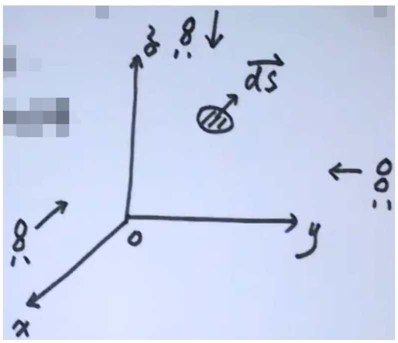
$$1.\Sigma : z = \sqrt{4 - x^2 - y^2}, D_{xy} : x^2 + y^2 \leq 4$$

$$2.z'_x = \frac{-x}{\sqrt{4 - x^2 - y^2}}, z'_y = \frac{-y}{4 - x^2 - y^2}$$

$$dS = \frac{2}{\sqrt{4 - x^2 - y^2}} d\sigma$$

$$3.I = \frac{1}{2} \iint_{D_{xy}} (x^2 + y^2) \cdot 2 d\sigma = 2\pi \int_0^2 r^3 dr = 8\pi$$

对坐标的曲面积分（第二类曲面积分）



Σ有侧曲面

1.从x正半轴将$\overrightarrow{dS}$向yoz面投影, 投影记为$dydz$
若 $\cos\alpha > 0, dydz > 0$
若 $\cos\alpha < 0, dydz < 0$
$dydz = \cos\alpha \cdot dS$

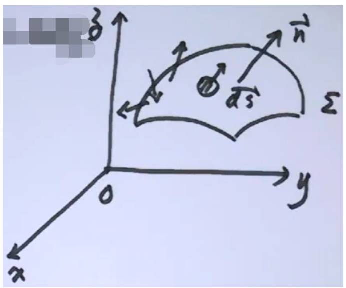
2.从y正半轴将$\overrightarrow{dS}$向xoz面投影, 投影记为$dzdx$
若 $\cos\beta > 0, dzdx > 0$
若 $\cos\beta < 0, dzdx < 0$
$dzdx = \cos\beta \cdot dS$

3.从z正半轴将$\overrightarrow{dS}$向xoy面投影, 投影记为$dxdy$
若 $\cos\gamma > 0, dxdy > 0$
若 $\cos\gamma < 0, dxdy < 0$
$dxdy = \cos\gamma \cdot dS$

$$\begin{cases} ①dydz = \cos\alpha \cdot dS, dzdx = \cos\beta \cdot dS, dxdy = \cos\gamma \cdot dS \\ ②\overrightarrow{dS} = \{dydz, dzdx, dxdy\} \end{cases}$$

背景: 流量

$\vec{v} = \{P, Q, R\}, \Sigma -$ 有侧曲面块
求Φ.



$$1.\forall \overrightarrow{dS} \subset \Sigma, \overrightarrow{dS} = \{dydz, dzdx, dxdy\}$$

$$\begin{aligned} 2.d\Phi &= \vec{v} \cdot \overrightarrow{dS} \\ &= Pdydz + Qdzdx + Rdxdy \end{aligned}$$

$$\begin{aligned} 3.\Phi &= \iint_\Sigma d\Phi \\ &= \iint_\Sigma Pdydz + Qdzdx + Rdxdy \end{aligned}$$

定义

$$\iint_\Sigma Pdydz + Qdzdx + Rdxdy = \iint_\Sigma Pdydz + \iint_\Sigma Qdzdx + \iint_\Sigma Rdxdy$$

性质

$$1. \iint_{\Sigma^-} = -\iint_\Sigma$$

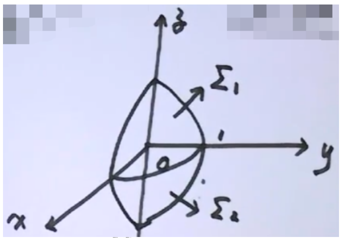
$$2. \iint_\Sigma Pdydz + Qdzdx + Rdxdy = \iint_\Sigma (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS$$

计算方法

二重积分法

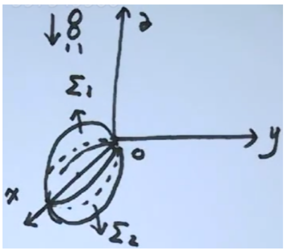
$$① \iint_{\Sigma} P(x,y,z)dydz$$
$$1.\Sigma: x=\Phi(y,z),(y,z)\in D_{yz}$$
$$2.\iint_{\Sigma} Pdydz=\pm\iint_{D_{yz}} P[\Phi(y,z),y,z]dydz$$
$$\cos\alpha>0\text{取}+,\cos\alpha<0\text{取}-$$
$$③ \iint_{\Sigma} R(x,y,z)dxdy$$
$$1.\Sigma: z=\Phi(x,y),(x,y)\in D_{xy}$$
$$2.\iint_{\Sigma} Rdxdy=\pm\iint_{D_{xy}} R[x,y,\Phi(x,y)]dxdy$$
$$\cos\gamma>0\text{取}+,\cos\gamma<0\text{取}-$$

$$I=\iint_{\Sigma} zdxdy.$$



$$1.I=\iint_{\Sigma_1} zdxdy+\iint_{\Sigma_2} zdxdy$$
$$2.①\Sigma_1: z=\sqrt{1-x^2-y^2}$$
$$D_{xy}: x^2+y^2\le 1(x\ge 0,y\ge 0)$$
$$\iint_{\Sigma_1} zdxdy=\iint_{D_{xy}}\sqrt{1-x^2-y^2}dxdy$$
$$②\Sigma_2: z=-\sqrt{1-x^2-y^2}$$
$$D_{xy}: x^2+y^2\le 1(x\ge 0,y\ge 0)$$
$$\iint_{\Sigma_2} zdxdy=-\iint_{D_{xy}}-\sqrt{1-x^2-y^2}dxdy$$
$$3.I=2\iint_{D_{xy}}\sqrt{1-x^2-y^2}dxdy=\pi\int_0^1 r\sqrt{1-r^2}dr$$
$$=-\frac{\pi}{2}\int_0^1(1-r^2)^{\frac{1}{2}}d(1-r^2)$$
$$=-\frac{\pi}{3}(1-r^2)^{\frac{3}{2}}\,|_0^1=\frac{\pi}{3}$$

$$\text{设}\Sigma:(x-1)^2+y^2+z^2=1,\text{取外侧},\text{计算}\iint_{\Sigma} y^2zdxdy.$$



$1.I=\iint_{\Sigma} y^{2} z d x d y=2 \iint_{\Sigma_{1}} y^{2} z d x d y$

$2.\Sigma: z=\sqrt{1-(x-1)^{2}-y^{2}}$

$D_{xy}:(x-1)^{2}+y^{2} \leq 1$

$I=2 \iint_{D_{xy}} y^{2} \sqrt{1-(x-1)^{2}-y^{2}} d(x-1) d y$

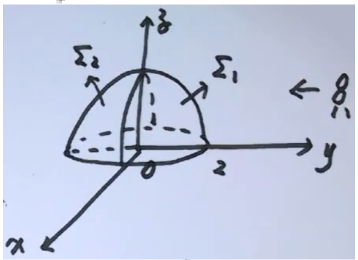
$=2 \iint_{x^{2}+y^{2} \leq 1} y^{2} \sqrt{1-x^{2}-y^{2}} d x d y$

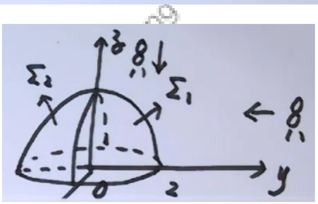
$=\iint_{x^{2}+y^{2} \leq 1}\left(x^{2}+y^{2}\right) \sqrt{1-x^{2}-y^{2}} d x d y$

$=2 \pi \int_{0}^{1} r^{3} \sqrt{1-r^{2}} d r=2 \pi \int_{0}^{\frac{\pi}{2}}\left(\sin ^{3} t-\sin ^{5} t\right) d t, r=\sin t$

$=2 \pi\left(\frac{2}{3}-\frac{4}{5} \cdot \frac{2}{3} \cdot 1\right)=\frac{4}{15} \pi$

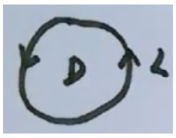
$$I=\iint_{\Sigma} y z d z d x+2 d x d y .$$





$$①I_1=\iint_{\Sigma}yzdzdx$$
$$=2\iint_{\Sigma_1}yzdzx$$
$$\Sigma_1: y=\sqrt{4-x^2-z^2}$$
$$D_{xz}=x^2+z^2\leq 4(z\geq 0)$$
$$I_1=2\iint_{D_{xz}}z\sqrt{4-x^2-z^2}dzdx$$
$$=2\int_0^{\pi}\sin\theta d\theta\int_0^2 r^2\sqrt{4-r^2}dr=4\int_0^2 r^2\sqrt{4-r^2}dr$$
$$=4\int_0^{\frac{\pi}{2}}4\sin^2 t\cdot 4(1-\sin^2 t)dt=64(I_2-I_4),r=2\sin t$$
$$=64(\frac{1}{2}\times\frac{\pi}{2}-\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2})=4\pi$$
$$②I_2=\iint_{\Sigma}2dxdy$$
$$\Sigma: z=\sqrt{4-x^2-y^2},D_{xy}:x^2+y^2\leq 4$$
$$I_2=2\iint_{D_{xy}}1dxdy=8\pi$$
$$\therefore \text{原式}=12\pi$$

三重积分（Gauss 公式）



$$1-dim:F(b)-F(a)=\int_a^b f(x)dx$$
$$2-dim:\oint_L Pdx+Qdy=\iint_D(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})d\sigma$$
$$3-dim:\oiint_{\Sigma}Pdydz+Qdzdx+Rdxdy=\iiint_{\Omega}(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z})dV$$

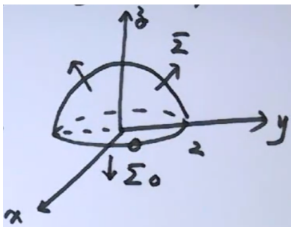
Th.(Gauss)

若①Ω为几何体, Σ为Ω的外表面

②P,Q,R连续可偏导, 则

$$\oiint_{\Sigma}Pdydz+Qdzdx+Rdxdy=\iiint_{\Omega}(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z})dV$$

$$I=\iint_{\Sigma}yzdzdx+2dxdy.$$



$$1.P=0,Q=yz,R=2,\frac{\partial P}{\partial x}=0,\frac{\partial Q}{\partial y}=z,\frac{\partial R}{\partial z}=0$$

$$2.\Sigma_0:z=0(x^2+y^2\leq 4),\text{下}$$

$$I=\oiint_{\Sigma+\Sigma_0}-\iint_{\Sigma_0}$$

$$3.\oiint_{\Sigma+\Sigma_0}=\iiint_{\Omega}zdV=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\Phi\int_0^2 r\cos\Phi\cdot r^2\sin\Phi dr$$

$$=2\pi\times\frac{1}{2}\times 4=4\pi$$

$$\iint_{\Sigma_0}=\iint_{\Sigma_0}2dxdy=-\iint_{D_{xy}}dxdy=-2\times 4\pi=-8pi$$

$$\therefore I=12\pi$$

场论

$$1.\text{梯度}:u=f(x,y,z)$$

$$\text{grad}u=\{\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\}$$

$$2.\text{旋度}:\vec{A}=\{P,Q,R\}$$

$$\text{rot}\vec{A}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ P&Q&R\end{vmatrix}$$

$$\begin{aligned}\vec{A}=&\{yz,x^2+z^2,x+y\}\\ \text{rot}\vec{A}=&(2y-2z)\vec{i}+(y-1)\vec{j}+(2x-z)\vec{k}\\ =&\{2y-2z,y-1,2x-z\}\end{aligned}$$

$$3.\text{散度}:\vec{A}=\{P,Q,R\}$$

$$\text{div}\vec{A}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$$

$$4.\text{通量}:\vec{v}=\{P,Q,R\},\Sigma-\text{有侧}$$

$$\Phi=\iint_{\Sigma}Pdydz+Qdzdx+Rdxdy$$

$$5.\text{环流量}:\vec{v}=\{P,Q,R\},L\text{为有向闭曲线}$$

$$\Phi=\oint_L Pdx+Qdy+Rdz$$