# 事件与概率

# 随机试验与随机事件

$$\begin{aligned}&E-\text{试验, 若}\\&\text{①相同条件下可重复}\\&\text{②结果多样且试验前所有可能的结果皆确定}\\&\text{③试验前不确定具体结果}\\&\text{称}E\text{为随机试验}\end{aligned}$$

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$$\begin{aligned}&E-\text{试验}, E\text{的一切可能的基本结果而成的集合}\\&\text{称为样本空间, 记}\Omega\end{aligned}$$

$$\begin{aligned}\Omega&=\{1,2,3,4,5,6\}\\A&=\{2,4,6\}\end{aligned}$$

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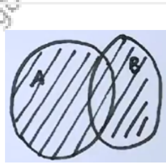
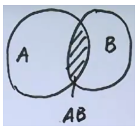
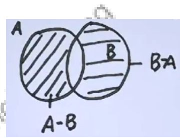
$$\begin{aligned}&\Omega\text{为}E\text{的样本空间}, \Omega\text{的子集称为随机事件}\\&\emptyset\subset\Omega, \emptyset-\text{不可能事件}; \Omega\subset\Omega, \Omega-\text{必然事件}\\\text{如}:&\Omega=\{1,2,3,4,5,6\}\\&A=\{1,2,3\}=\{\text{朝上的数不超过}3\}\end{aligned}$$

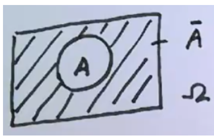
# 事件的运算与关系

## 运算



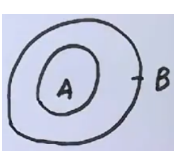



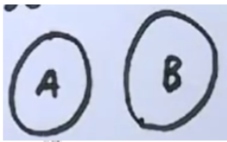



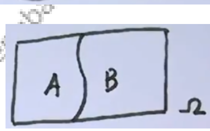




$$\begin{aligned}&1.\text{和}-A+B: A\text{或者}B\text{发生的事件}\\&2.\text{积}-AB: A, B\text{同时发生的事件}\\&3.\text{差}-A-B: A\text{发生且}B\text{不发生的事件}\\&4.\text{补}-\overline{A}: A\text{不发生的事件}\end{aligned}$$

## 关系






1.包含 $-$ 若$A$发生则$B$一定发生, 称$A \subset B$

2.互斥(不相容) $-$ $A, B$不能同时发生, 称$A, B$互斥

$A, B$互斥 $\Leftrightarrow AB = \emptyset$

3.对立 $-$ $A, B$不能同时发生且至少一个发生, 称$A, B$对立

$$A, B\text{对立} \Leftrightarrow \begin{cases} AB = \emptyset \\ A + B = \Omega \end{cases} \Leftrightarrow B = \overline{A}$$

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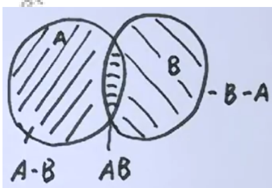


Notes:①$A = (A - B) + AB$且$(A - B)$与$AB$互斥;

②$A + B = (A - B) + AB + (B - A)$且

$A - B, AB, B - A$互斥

# 概率的公理化定义及基本性质

$\Omega$为$E$的样本空间, 在$\Omega$上定义一个函数, 若:

①$\forall A \subset \Omega$, 有$P(A) \geq 0$; (非负性)

②$P(\Omega) = 1$; (归一性)

③设$A_1, A_2, \cdots, A_n, \cdots$两两互斥, 有

$P(A_1 + A_2 + \cdots + A_n + \cdots) = P(A_1) + P(A_2) + \cdots + P(A_n) + \cdots$. (可列可加性)

称$P(A)$为$A$的概率

# 概率的基本性质

1.$P(\emptyset) = 0$

证: 令$A_1 = A_2 = \cdots = A_n = \cdots = \emptyset$

$A_1, A_2, \cdots, A_n, \cdots$两两互斥, 有

$P(A_1 + A_2 + \cdots + A_n + \cdots) = P(A_1) + P(A_2) + \cdots + P(A_n) + \cdots$

即$P(\emptyset) = P(\emptyset) + \cdots + P(\emptyset) + \cdots$

$\because P(\emptyset) \geq 0, \therefore P(\emptyset) = 0$

2.设$A_1, \cdots, A_n$两两互斥, 则

$P(A_1 + \cdots + A_n) = P(A_1) + \cdots + P(A_n)$

证: 取$A_{n+1} = \cdots = \emptyset$

$\because A_1, \cdots, A_n, \cdots$两两互斥

$\therefore P(A_1 + \cdots + A_n + \cdots) = P(A_1) + \cdots + P(A_n) + \cdots$

即$P(A_1 + \cdots + A_n) = P(A_1) + \cdots + P(A_n)$

3.$P(\overline{A}) = 1 - P(A)$

证: $\because A, \overline{A}$互斥, $\therefore P(A + \overline{A}) = P(A) + P(\overline{A})$

又$\because A + \overline{A} = \Omega, \therefore P(A) + P(\overline{A}) = 1$

$\Rightarrow P(\overline{A}) = 1 - P(A)$

# 概率的四大基本公式

1.减法：

$\because A=(A-B)+AB$且$A-B,AB$互斥

$\therefore P(A)=P(A-B)+P(AB)\Rightarrow P(A-B)=P(A)-P(AB)$

$\because A=A\Omega=A(B+\overline{B})=AB+A\overline{B}$

又$\because AB\subset B,A\overline{B}\subset\overline{B}$且$B$与$\overline{B}$互斥

$\therefore AB,A\overline{B}$互斥

$\therefore P(A)=P(AB)+P(A\overline{B})\Rightarrow P(A\overline{B})=P(A)-P(AB)$

$P(A-B)=P(A\overline{B})=P(A)-P(AB)$

2.加法：

$A+B=(A-B)+AB+(B-A)$且$A-B,AB,B-A$互斥

$\therefore P(A+B)=P(A-B)+P(AB)+P(B-A)$

$\because P(A-B)=P(A)-P(AB),P(B-A)=P(B)-P(AB)$

$\therefore$ ①$P(A+B)=P(A)+P(B)-P(AB)$

②$P(A+B+C)=P(A)+P(B)+P(C)-P(AB)-P(AC)-P(BC)+P(ABC)$

3.条件概率公式：$A,B-$事件，且$P(A)>0$

$$P(B|A)=\frac{P(AB)}{P(A)}$$

4.乘法公式：

①$P(AB)=P(A)P(B|A)$

②$P(A_1A_2\cdots A_n)=P(A_1)P(A_2|A_1)P(A_3|A_1A_2)\cdots P(A_n|A_1\cdots A_{n-1})$

# 事件的独立性

$$\text{设}P(A)>0,P(B|A)=\frac{P(AB)}{P(A)}$$

$$\text{若}P(B|A)=P(B),\text{则}\frac{P(AB)}{P(A)}=P(B)\Leftrightarrow P(AB)=P(A)P(B)$$

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1.若$P(AB)=P(A)P(B)$，称$A,B$独立

2.$A,B,C-$三个事件：

$$\text{若}\begin{cases}P(AB)=P(A)P(B)\\P(AC)=P(A)P(C)\\P(BC)=P(B)P(C)\\P(ABC)=P(A)P(B)P(C)\end{cases},\text{称}A,B,C\text{独立}$$

# 独立的性质

Th1.$A, B$、$A, \overline{B}$、$\overline{A}, B$、$\overline{A}, \overline{B}$一对独立其余独立

证：设$A, B$独立，即$P(AB) = P(A)P(B)$

$P(A\overline{B}) = P(A) - P(AB) = P(A)[1 - P(B)] = P(A)P(\overline{B})$，即$A, \overline{B}$独立

同理$\overline{A}, B$独立

$P(\overline{A}\overline{B}) = P(\overline{A+B}) = 1 - P(A+B) = 1 - P(A) - P(B) + P(A)P(B)$

$= [1 - P(A)] \cdot [1 - P(B)] = P(\overline{A})P(\overline{B})$，即$\overline{A}, \overline{B}$独立

Th2.若$P(A) = 0$或$P(A) = 1$，则$A, B$独立

证：设$P(A) = 0$

$\because AB \subset A, \therefore P(AB) \le P(A) = 0$

又$\because P(AB) \ge 0, \therefore P(AB) = 0$

$\therefore P(AB) = P(A)P(B), \therefore A, B$独立

设$P(A) = 1 \Rightarrow P(\overline{A}) = 0 \Rightarrow \overline{A}, B$独立 $\Rightarrow A, B$独立

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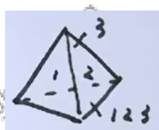


Notes:

①$A, B, C$两两独立 $\not\Rightarrow A, B, C$独立

$A = \{$朝下为1$\}, B = \{$朝下为2$\}, C = \{$朝下为3$\}$

$P(A) = P(B) = P(C) = \frac{1}{2}, P(AB) = P(AC) = P(BC) = \frac{1}{4}, P(ABC) = \frac{1}{4}$

$P(ABC) \neq P(A)P(B)P(C)$

②设$P(A) > 0, P(B) > 0$

若$A, B$独立 $\Rightarrow A, B$不互斥

$P(AB) = P(A)P(B) > 0 \Rightarrow AB \neq \emptyset$

若$A, B$互斥 $\Rightarrow A, B$不独立

$A, B$互斥 $\Rightarrow AB = \emptyset \Rightarrow P(AB) = 0 \neq P(A)P(B) \Rightarrow A, B$不独立

④独立的等价结论：

1.若$P(A) > 0$，则$A, B$独立 $\Leftrightarrow P(B|A) = P(B)$

2.若$0 < P(A) < 1$，则$A, B$独立 $\Leftrightarrow P(B|A) = P(B|\overline{A})$

$$P(B|A) = P(B|\overline{A}) \Leftrightarrow \frac{P(AB)}{P(A)} = \frac{P(B\overline{A})}{P(\overline{A})} = \frac{P(B) - P(AB)}{1 - P(A)}$$