极限存在准则与重要极限

夹逼定理

$$
\begin{gathered}
若 a_n \le b_n \le c_n \\
\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = A \Rightarrow \lim_{n\to\infty} b_n = A
\end{gathered}
$$

$$
\begin{gathered}
证: \forall \epsilon > 0, \exists N_1 > 0, 当 n > N_1 时, \\
|a_n - A| < \epsilon \Leftrightarrow A - \epsilon < a_n < A + \epsilon (*) \\
\exists N_2 > 0, 当 n > N_2 时, \\
|c_n - A| < \epsilon \Leftrightarrow A - \epsilon < c_n < A + \epsilon (**) \\
取 N = \max\{N_1, N_2\}, 当 n > N 时, (*)(**) 成立, \\
当 n > N 时 \\
A - \epsilon < a_n \le b_n \le c_n < A + \epsilon \\
\Rightarrow A - \epsilon < b_n < A + \epsilon \Leftrightarrow |b_n - A| < \epsilon \\
\therefore \lim_{n\to\infty} b_n = A
\end{gathered}
$$

$$
\begin{gathered}
若 f(x) \le g(x) \le h(x) \\
\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = A \\
\Rightarrow \lim_{x\to a} g(x) = A
\end{gathered}
$$

$$
\begin{gathered}
证: \forall \epsilon > 0 \\
\exists \delta_1 > 0, 当 0 < |x - a| < \delta_1 时, \\
|f(x) - A| < \epsilon \Leftrightarrow A - \epsilon < f(x) < A + \epsilon (*) \\
\exists \delta_2 > 0, 当 0 < |x - a| < \delta_2 时, \\
|h(x) - A| < \epsilon \Leftrightarrow A - \epsilon < h(x) < A + \epsilon (**) \\
取 \delta = \min\{\delta_1, \delta_2\}, 当 0 < |x - a| < \delta 时, (*)(**) 皆对 \\
A - \epsilon < f(x) \le g(x) \le h(x) < A + \epsilon \\
\Rightarrow A - \epsilon < g(x) < A + \epsilon \Leftrightarrow |g(x) - A| < \epsilon \\
\lim_{x\to a} g(x) = A
\end{gathered}
$$

$$
求极限 \lim_{n\to\infty} \sqrt[n]{2^n + 3^n}.
$$

$$
\begin{gathered}
3^n \le 2^n + 3^n \le 2 * 3^n \Rightarrow 3 \le \sqrt[n]{2^n + 3^n} \le 2^{\frac{1}{n}} * 3 \\
\because \lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1, \therefore \lim_{n\to\infty} 3 = 3 = \lim_{n\to\infty} 2^{\frac{1}{n}} * 3 = 3 \\
\therefore 原式 = 3
\end{gathered}
$$

$$
\lim_{n\to\infty} (2^n + 3^n + 5^n)^{\frac{1}{n}}
$$

$$\because 5^n \le 2^n + 3^n + 5^n \le 3 \cdot 5^n$$

$$\therefore 5 \le (2^n + 3^n + 5^n)^{\frac{1}{n}} \le 3^{\frac{1}{n}} \cdot 5$$

$$\because \lim_{n\to\infty} 3^{\frac{1}{n}} = 3^0 = 1, \therefore \lim_{n\to\infty} 左 = \lim_{n\to\infty} 右 = 5$$

$$\therefore 原式 = 5$$

$$a > 0, b > 0, c > 0, 则$$

$$\lim_{n\to\infty} \sqrt[n]{a^n + b^n + c^n} = \max\{a, b, c\}$$

$$x > 0$$

$$求 \lim_{n\to\infty} \sqrt[n]{x^n + x^{2n}}.$$

$$\lim_{n\to\infty} \sqrt[n]{x^n + x^{2n}} = \lim_{n\to\infty} \sqrt[n]{x^n + (x^2)^n}$$

$$= \max\{x, x^2\} = \begin{cases} x, 0 < x < 1 \\ x^2, x \ge 1 \end{cases}$$

$$求极限 \lim_{n\to\infty} (\frac{1}{2n^2+1} + \frac{2}{2n^2+2} \ldots + \frac{n}{2n^2+n}).$$

$$b_n = \frac{1}{2n^2+1} + \frac{2}{2n^2+2} \ldots + \frac{n}{2n^2+n}$$

$$\because \frac{i}{2n^2+n} \le \frac{i}{2n^2+i} \le \frac{i}{2n^2+1} (1 \le i \le n)$$

$$\therefore \frac{1+2+\ldots+n}{2n^2+n} \le \frac{1}{2n^2+1} + \frac{2}{2n^2+2} + \ldots + \frac{n}{2n^2+n} \le \frac{1+2+\ldots+n}{2n^2+1}$$

$$即 \frac{1}{2} \frac{n(n+1)}{2n^2+n} \le b_n \le \frac{1}{2} \frac{n(n+1)}{2n^2+1}$$

$$\because \lim_{n\to\infty} 左 = \frac{1}{2} \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{1}{n}} = \frac{1}{4}$$

$$\lim_{n\to\infty} 右 = \frac{1}{2} \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{1}{n^2}} = \frac{1}{4}$$

$$\therefore 原式 = \frac{1}{4}$$

$$\lim_{n\to\infty} \frac{n^2}{2^n}$$

$$n充分大时, 2^n = (1+1)^n = C_n^0 + C_n^1 + C_n^2 + C_n^3 + \ldots + C_n^n$$

$$\ge C_n^3 = \frac{n(n-1)(n-2)}{6}$$

$$0 \le \frac{1}{2^n} \le \frac{6}{n(n-1)(n-2)}$$

$$\Rightarrow 0 \le \frac{n^2}{2^n} \le \frac{6n^2}{n(n-1)(n-2)}$$

$$\because \lim_{n\to\infty} 右 = 6 \lim_{n\to\infty} \frac{n}{(n-1)(n-2)} = 0, \therefore \lim_{n\to\infty} \frac{n^2}{2^n} = 0$$

$$f(x)=\lim_{n\to\infty}\sqrt[n]{x^n+x^{2n}}(x>0),求f(x).$$

$$当0<x\leq 1时,$$

$$x^n\leq x^n+x^{2n}\leq 2x^n\Rightarrow x\leq\sqrt[n]{x^n+x^{2n}}\leq 2^{\frac{1}{n}}x$$

$$\because\lim_{n\to\infty}2^{\frac{1}{n}}=1,\therefore\lim_{n\to\infty}左=\lim_{n\to\infty}右=x\Rightarrow f(x)=x$$

$$当x>1时,$$

$$x^{2n}\leq x^n+x^{2n}\leq 2x^{2n}\Rightarrow x^2\leq\sqrt[n]{x^n+x^{2n}}\leq 2^{\frac{1}{n}}x^2$$

$$\because\lim_{n\to\infty}左=\lim_{n\to\infty}右=x^2\Rightarrow f(x)=x^2$$

$$\therefore f(x)\begin{cases}x,0<x\leq 1\\x^2,x>1\end{cases}$$

$$\lim_{n\to\infty}(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}})$$

$$b_n\triangleq\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}}$$

$$\frac{n}{\sqrt{n^2+n}}\leq b_n\leq\frac{n}{\sqrt{n^2+1}}$$

$$\lim_{n\to\infty}左=\lim_{n\to\infty}右=1$$

$$\therefore原式=1$$

单调、有界数列必有极限

$$\{a_n\}\uparrow\begin{cases}无上界\Rightarrow\lim_{n\to\infty}a_n=+\infty\\a_n\leq M\Rightarrow\lim_{n\to\infty}a_n\exists\end{cases}$$

$$\{a_n\}\downarrow\begin{cases}无下界\Rightarrow\lim_{n\to\infty}a_n=-\infty\\a_n\geq M\Rightarrow\lim_{n\to\infty}a_n\exists\end{cases}$$

$$设\{a_n\}=\sqrt{2},a_2=\sqrt{2+\sqrt{2}},a_3=\sqrt{2+\sqrt{2+\sqrt{2}}},\ldots,证明：数列\{a_n\}收敛，并求其极限.$$

1. $a_{n+1}=\sqrt{2+a_n}(n=1,2,\ldots)$
2. $\{a_n\}\uparrow$
3. 现证$a_n\leq 2$

   $a_1=\sqrt{2}\leq 2,设a_k\leq 2,则$

   $a_{k+1}=\sqrt{2+a_k}\leq\sqrt{2+2}=2$

   $\therefore\forall n,有a_n\leq 2\Rightarrow\lim\limits_{n\to\infty}a_n\exists$
4. 令$\lim\limits_{n\to\infty}a_n=A$

   $a_{n+1}=\sqrt{2+a_n}\Rightarrow A=\sqrt{2+A}$

   $\Rightarrow A^2-A-2=0\Rightarrow A=-1(舍),A=2$

$$a_1=2,a_{n+1}=\frac{1}{2}(a_n+\frac{1}{a_n}),证：\lim_{n\to\infty}a_n\exists$$

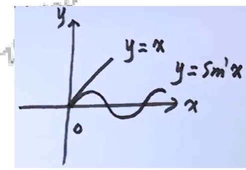
1. $\because a_n + \dfrac{1}{a_n} \geq 2$

$\therefore a_{n+1} \geq 1$

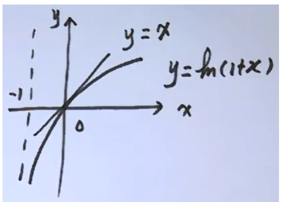
而$a_1 = 2 \geq 1, \therefore a_n \geq 1$

2. $a_{n+1} - a_n = \dfrac{1}{2}(a_n + \dfrac{1}{a_n}) - a_n = \dfrac{1 - a_n^2}{2a_n} \leq 0$

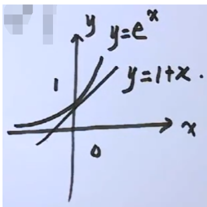
$\Rightarrow \{a_n\} \downarrow \Rightarrow \lim\limits_{n\to\infty} a_n \exists$



$x \geq 0, \sin x \leq x$

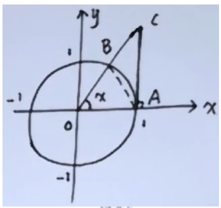


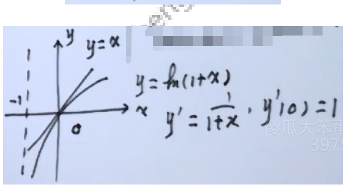
$x > -1, \ln(1+x) \leq x$

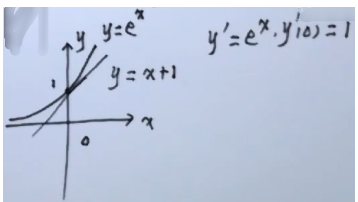


$e^x \geq (1+x)$

重要极限







1. $0<x<\frac{\pi}{2}$时, $0<\sin x<x<\tan x$

$$S_{\Delta AOB}=\frac{1}{2}\sin x$$

$$S_{扇AOB}=\frac{1}{2}x$$

$$S_{Rt\Delta AOC}=\frac{1}{2}\tan x$$

2. $x>-1$时, $\ln(1+x)\leq x$

3. $x\in(-\infty,+\infty), e^x\geq 1+x$

$$\lim_{\Delta\to 0}=\frac{\sin\Delta}{\Delta}=1$$

$$\lim_{\Delta\to 0}(1+\Delta)^{\frac{1}{\Delta}}=e$$

$$\lim_{x\to 0}\frac{1-\cos^3 x}{x\ln(1+2x)}$$

$$\begin{aligned}
原式 &=\lim_{x\to 0}\frac{1-\cos^2 x}{2x^2}\\
&=\frac{1}{2}\lim_{x\to 0}(1+\cos x)\frac{1-\cos x}{x^2}\\
&=\frac{1}{2}*2*\frac{1}{2}\\
&=\frac{1}{2}
\end{aligned}$$

$$\lim_{x\to 0}\frac{e^{\tan x}-e^{\sin x}}{x\arcsin^2 x}$$

$$\begin{aligned}
原式 &=\lim_{x\to 0}\frac{e^{\tan x}-e^{\sin x}}{x^3}\\
&=\lim_{x\to 0}e^{\sin x}\frac{e^{\tan x-\sin x}-1}{x^3}\\
&=1*\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}\\
&=\lim_{x\to 0}\frac{\tan x}{x}*\frac{1-\cos x}{x^2}\\
&=\frac{1}{2}
\end{aligned}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x\cos x}-\sqrt{1+x}}{x^3}$$

$$\begin{aligned}
原式 &=\lim_{x\to 0}\frac{1}{\sqrt{1+x\cos x}+\sqrt{1+x}}\frac{x\cos x-x}{x^3}\\
&=\frac{1}{2}\lim_{x\to 0}\frac{\cos x-1}{x^2}\\
&=-\frac{1}{4}
\end{aligned}$$

$$\lim_{x\to 0}(1-\sin 2x^2)^{\frac{1}{x^2}}$$

$$原式=\lim_{x\to 0}[(1-\sin 2x^2)^{-\frac{1}{\sin 2x^2}}]^{\frac{1}{x^2}*(-\sin 2x^2)}$$
$$=e^{-\lim_{x\to 0}\frac{\sin 2x^2}{x^2}}$$
$$=e^{-2}$$

$$\lim_{x\to 0}(\cos x)^{\frac{1}{x\ln(1-x)}}$$

$$原式=\lim_{x\to 0}\{[1+(\cos x-1)]^{\frac{1}{\cos x-1}}\}^{\frac{\cos x-1}{x\ln(1-x)}}$$
$$=e^{\lim_{x\to 0}\frac{\cos x-1}{x\ln(1-x)}}$$
$$=e^{\lim_{x\to 0}\frac{-\frac{1}{2}x^2}{-x^2}}$$
$$=e^{\frac{1}{2}}$$

$$\lim_{x\to 0}(\frac{1+\tan x}{1+\sin x})^{\frac{1}{x^3}}$$

$$原式=\lim_{x\to 0}\{[1+(\frac{1+\tan x}{1+\sin x}-1)]^{\frac{1}{\frac{1+\tan x}{1+\sin x}-1}}\}^{\frac{1}{x^3}(\frac{1+\tan x}{1+\sin x}-1)}$$
$$=e^{\lim_{x\to 0}\frac{1}{x^3}(\frac{\tan x-\sin x}{1+\sin x})}$$
$$=e^{\lim_{x\to 0}\frac{1}{1+\sin x}\frac{\tan x-\sin x}{x^3}}$$
$$=e^{\lim_{x\to 0}\frac{\tan x}{x}\frac{1-\cos x}{x^2}}$$
$$=e^{\frac{1}{2}}$$

$$\lim_{x\to\infty}(\frac{2x^2+3x+1}{x-1}-2x)$$

$$原式=\lim_{x\to\infty}\frac{2x^2+3x+1-2x(x-1)}{x-1}$$
$$=\lim_{x\to\infty}\frac{5x+1}{x-1}$$
$$=\lim_{x\to\infty}\frac{5+\frac{1}{x}}{1-\frac{1}{x}}$$
$$=5$$

$$\lim_{x\to\infty}(\sqrt{x^2+4x+1}-x)$$

$$原式=\lim_{x\to\infty}\frac{4x+1}{\sqrt{x^2+4x+1}+x}$$
$$=\lim_{x\to\infty}\frac{4+\frac{1}{x}}{\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}+1}$$
$$=2$$