问题一

$$\lambda ?$$

Notes:
①公式法：$|\lambda E-A|=0$
②定义法：$AX=\lambda X(X\neq 0)\Rightarrow\lambda=?$

设$A=\begin{pmatrix}1&-1&0\\2&0&1\\1&a&0\end{pmatrix}$, 且存在非零向量$\alpha$, 使得$A\alpha=2\alpha$, 求$a$.

$\because A\alpha=2\alpha$且$\alpha\neq 0,\therefore\lambda=2$为特征值,
$\therefore|2E-A|=0$

$$|2E-A|=\begin{pmatrix}1&1&0\\-2&2&-1\\-1&-a&2\end{pmatrix}=\begin{pmatrix}1&1&0\\0&4&-1\\0&1-a&2\end{pmatrix}=9-a=0\Rightarrow a=9$$

设$A=\begin{pmatrix}1&-1&2\\4&a&0\\-3&1&b\end{pmatrix}$, 且$\alpha=\begin{pmatrix}1\\1\\1\end{pmatrix}$为矩阵$A$的一个特征向量, 求$a,b$及$\alpha$所对应的特征值.

令$A\alpha=\lambda\alpha$, 即$\begin{pmatrix}1&-1&2\\4&a&0\\-3&1&b\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow$

$$\begin{cases}2=\lambda\\a+4=\lambda\Rightarrow\lambda=2,a=-2,b=4\\b-2=\lambda\end{cases}$$

设A为三阶降秩矩阵, α,β线性无关, 且$A\alpha=\beta,A\beta=\alpha$, 求$|(A+2E)^*-E|$.

1.$A\alpha=\beta,A\beta=\alpha\Rightarrow\begin{cases}A(\alpha+\beta)=\alpha+\beta\\A(\alpha-\beta)=-(\alpha-\beta)\end{cases}$
$\Rightarrow\lambda_1=1,\lambda_2=-1$
$\because r(A)<3,\therefore|A|=0\Rightarrow\lambda_3=0$
2.$A+2E$特征值$3,1,2,|A+2E|=6$
$(A+2E)^*$特征值$2,6,3$
$(A+2E)^*-E$特征值$1,5,2$
3.$|(A+2E)^*-E|=10$

$A_{3\times 3},A^2+A-2E=0,|A|=-2$, 求$A$特征值.

解：令$AX=\lambda X(X\neq 0)$
$(A^2+A-2E)X=(\lambda^2+\lambda-2)X=0$
$\because X\neq 0,\therefore\lambda^2+\lambda-2=0\Rightarrow\lambda=1$或$-2$
$\because|A|=-2,\therefore\lambda_1=\lambda_2=1,\lambda_3=-2$

$A_{3\times3}$, 每行元素之和为2

$$A\begin{pmatrix}-1&-1\\1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\-1&0\\0&0\end{pmatrix}, 求A.$$

$$解:A\begin{pmatrix}1\\1\\1\end{pmatrix}=2\begin{pmatrix}1\\1\\1\end{pmatrix}\Rightarrow\lambda_1=2,\alpha_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$A\begin{pmatrix}-1\\1\\0\end{pmatrix}=-\begin{pmatrix}-1\\1\\0\end{pmatrix}\Rightarrow\lambda_2=-1,\alpha_2=\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$A\begin{pmatrix}-1\\0\\1\end{pmatrix}=0\begin{pmatrix}-1\\0\\1\end{pmatrix}\Rightarrow\lambda_3=0,\alpha_3=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$P=\begin{pmatrix}1&-1&-1\\1&1&0\\1&0&1\end{pmatrix}可逆$$

$$P^{-1}AP=\begin{pmatrix}2&&\\&-1&\\&&0\end{pmatrix}$$

$$\Rightarrow A=P\begin{pmatrix}2&&\\&-1&\\&&0\end{pmatrix}P^{-1}$$

$$③关联法:\begin{cases}A可逆:A,A^{-1},A^*\\P^{-1}AP=B:A\sim B\end{cases}$$

$A_{n\times n},\alpha_1,\ldots,\alpha_n$无关

$1.P=(\alpha_1\ldots\alpha_n)$

$2.AP=\ldots=PB\Rightarrow P^{-1}AP=B$

$A_{3\times3},\alpha_1,\alpha_2,\alpha_3$线性无关

$A\alpha_1=-\alpha_2,A\alpha_2=\alpha_1+2\alpha_2,A\alpha_3=3\alpha_1+2\alpha_2-\alpha_3$

求A的λ.

解:$1.P=(\alpha_1,\alpha_2,\alpha_3)$可逆

$2.AP=(A\alpha_1,A\alpha_2,A\alpha_3)=(-\alpha_2,\alpha_1+2\alpha_2,3\alpha_1+2\alpha_2-\alpha_3)$

$$=P\begin{pmatrix}0&1&3\\-1&2&2\\0&0&-1\end{pmatrix}$$

$$3.P^{-1}AP=\begin{pmatrix}0&1&3\\-1&2&2\\0&0&-1\end{pmatrix}\triangleq B,A\sim B$$

$$|\lambda E-B|=0\Rightarrow\lambda_1=\lambda_2=1,\lambda_3=-1$$

问题二

矩阵对角化的判断

$A_{n\times n}$, 若$\exists$可逆P, 使$P^{-1}AP=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$

称A可相似对角化

过程：

case1. $A^T\neq A$：

1.$\lambda_1\dots\lambda_n$

2.$\alpha_1,\dots,\alpha_m\begin{cases}线性无关\\ m\leq n\end{cases}$

3.①$m<n$：A不可相似对角化

②$m=n$：A可相似对角化

$P=(\alpha_1\dots\alpha_n), A\alpha_1=\lambda_1\alpha_1,\dots,A\alpha_n=\lambda_n\alpha_n$

$$AP=P\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}\Rightarrow P^{-1}AP=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$$

case2. $A^T=A$：

1.$\lambda_1,\dots,\lambda_n$

2.$\alpha_1,\dots,\alpha_n\begin{cases}线性无关\\ 不同特征值之间：正交\end{cases}$

3.①一般要求：找可逆阵P

$$P=(\alpha_1,\dots,\alpha_n), P^{-1}AP=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$$

②特殊要求：找正交阵Q

$\alpha_1\dots\alpha_n\Rightarrow$ 正交规范化, $\gamma_1\dots\gamma_n$

$(A\gamma_1=\lambda_1\gamma_1,\dots,A\gamma_n=\lambda_n\gamma_n)$

$Q=(\gamma_1\dots\gamma_n), Q^TQ=E$

$$AQ=Q\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}\Rightarrow Q^TAQ=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$$

判别：

1.若$A^T=A$, 则A可对角化

2.若$A^T\neq A$：

case1. 找出B, 使$A\sim B$

case2. 找不到B, 使$A\sim B\Rightarrow\lambda_1,\dots,\lambda_n$

$\begin{cases}①\lambda_1,\dots,\lambda_n皆单值\Rightarrow A可相似对角化\\ ②重根：重数与无关特征向量个数一致\end{cases}$

$$1.\alpha=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix},\beta=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\text{单位正交},A=\alpha\beta^T+\beta\alpha^T.$$

①证：$\alpha+\beta,\alpha-\beta$为$A$特征向量

②证：A可相似对角化

证：①$A(\alpha+\beta)=(\alpha\beta^T+\beta\alpha^T)(\alpha+\beta)=\alpha+\beta$

$\Rightarrow\alpha+\beta$为$\lambda_1=1$的特征向量

$A(\alpha-\beta)=(\alpha\beta^T+\beta\alpha^T)(\alpha-\beta)=-\alpha+\beta=-(\alpha-\beta)$

$\Rightarrow\alpha-\beta$为$\lambda_2=-1$的特征向量

②法 $\because A^T=\beta\alpha^T+\alpha\beta^T=A$

$\therefore A$可相似对角化

$$2.A=\begin{pmatrix}0&2&3\\-2&4&1\\0&0&1\end{pmatrix}$$

解：$|\lambda E-A|=\begin{vmatrix}\lambda&-2&-3\\2&\lambda-4&-1\\0&0&\lambda-1\end{vmatrix}=0\Rightarrow\lambda_1=1,\lambda_2=\lambda_3=2$

$$2E-A=\begin{pmatrix}2&-2&-3\\2&-2&-1\\0&0&1\end{pmatrix}\to\begin{pmatrix}2&-2&-3\\0&0&1\\0&0&0\end{pmatrix}$$

$\because r(2E-A)=2<3,\therefore A$不可相似对角化

设$A=\begin{pmatrix}0&0&1\\x&1&y\\1&0&0\end{pmatrix}$有三个线性无关的特征向量，求$x,y$所满足的条件.

$$|\lambda E-A|=\begin{vmatrix}\lambda&0&-1\\-x&\lambda-1&-y\\-1&0&\lambda\end{vmatrix}=(\lambda-1)A_{22}=(\lambda-1)M_{22}=(\lambda+1)(\lambda-1)^2=0$$

$\Rightarrow\lambda_1=-1,\lambda_2=\lambda_3=1$

$\because A$可相似对角化，$\therefore r(E-A)=1<3$

而$E-A=\begin{pmatrix}1&0&-1\\-x&0&-y\\-1&0&1\end{pmatrix}\to\begin{pmatrix}1&0&-1\\0&0&-x-y\\0&0&0\end{pmatrix},\therefore x+y=0$

设$A=\begin{pmatrix}1&0&-1\\-x&0&-y\\-1&0&1\end{pmatrix}$相似于对角矩阵，求常数$a$，并求可逆矩阵$P$使得$P^{-1}AP$为对角矩阵.

$$|\lambda E-A|=\begin{vmatrix}\lambda-2 & -2 & 0\\ -8 & \lambda-2 & -a\\ 0 & 0 & \lambda-6\end{vmatrix}=0\Rightarrow\lambda_1=-2,\lambda_2=\lambda_3=6$$

$\because A$可相似对角化$,\therefore r(6E-A)=1<3$

而$6E-A=\begin{pmatrix}4 & -2 & 0\\ -8 & 4 & -a\\ 0 & 0 & 0\end{pmatrix}$,则$a=0$

$$2E+A=\begin{pmatrix}4 & 2 & 0\\ 8 & 4 & 0\\ 0 & 0 & 8\end{pmatrix}\to\begin{pmatrix}1 & \frac{1}{2} & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$

$\lambda_1=-2$对应线性无关特征向量为$\alpha_1=\begin{pmatrix}-1\\ 2\\ 0\end{pmatrix}$

$$6E-A\to\begin{pmatrix}1 & \frac{1}{2} & 0\\ & 0 & \end{pmatrix}$$

$\lambda_2=\lambda_3=6$对应线性无关特征向量为$\alpha_2=\begin{pmatrix}1\\ 2\\ 0\end{pmatrix},\alpha_3=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$

$$P=\begin{pmatrix}-1 & 1 & 0\\ 2 & 2 & 0\\ 0 & 0 & 1\end{pmatrix},P^{-1}AP=\begin{pmatrix}-2 & & \\ & 6 & \\ & & 6\end{pmatrix}$$

$A_{3\times3},\alpha_1,\alpha_2,\alpha_3$线性无关$,A\alpha_1=-\alpha_2,A\alpha_2=\alpha_1+2\alpha_2,$
$A\alpha_3=3\alpha_1+2\alpha_2+\alpha_3,$问$A$可否对角化?

解$:P=(\alpha_1,\alpha_2,\alpha_3)$,可逆

$$AP=(A\alpha_1,A\alpha_2,A\alpha_3)=(-\alpha_2,\alpha_1+2\alpha_2,3\alpha_1+2\alpha_2+\alpha_3)$$

$$=P\begin{pmatrix}0 & 1 & 3\\ -1 & 2 & 2\\ 0 & 0 & 1\end{pmatrix}\Rightarrow P^{-1}AP=\begin{pmatrix}0 & 1 & 3\\ -1 & 2 & 2\\ 0 & 0 & 1\end{pmatrix}\triangleq B,A\sim B$$

$$|\lambda E-B|=0\Rightarrow\lambda_1=\lambda_2=\lambda_3=1$$

$$E-B=\begin{pmatrix}1 & -1 & -3\\ 1 & -1 & -2\\ 0 & 0 & 0\end{pmatrix}$$

$\because r(E-B)=2<3,\therefore B$不可相似对角化$,\therefore A$不可相似对角化

问题三

$$A\sim B?$$

若$\exists$可逆P,使$P^{-1}AP=B,A\sim B$

必要条件$:A\sim B\Rightarrow|\lambda E-A|=|\lambda E-B|\Rightarrow$特征值同$,\nLeftarrow$

$1.|\lambda E-A|=|\lambda E-B|$(必)

2.①A可对角化,B可对角化 $\Rightarrow A\sim B$

证 : A,B特征值$\lambda_1,\dots,\lambda_n$

$\lambda_i E-A\rightarrow\dots:\alpha_1,\dots,\alpha_n$

令$P_1=(\alpha_1\dots\alpha_n),P_1^{-1}AP_1=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$

$\lambda_i E-B\rightarrow\dots:\beta_1\dots\beta_n$

令$P_2=(\beta_1\dots\beta_n),P_2^{-1}BP_2=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$

$P_1^{-1}AP_1=P_2^{-1}BP_2\Rightarrow P_2P_1^{-1}AP_1P_2^{-1}=B$

$\Rightarrow(P_1P_2^{-1})^{-1}A(P_1P_2^{-1})=B$,令$P=P_1P_2^{-1},P^{-1}AP=B$,即$A\sim B$

②A可相似对角化,B不可相似对角化 $\Rightarrow A\nsim B$

$$A=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix},B=\begin{pmatrix}0&0&1\\0&0&2\\0&0&3\end{pmatrix},\text{证}A\sim B$$

证 :$1.|\lambda E-A|=|\lambda E-B|=\lambda^2(\lambda-3)=0\Rightarrow\lambda_1=\lambda_2=0,\lambda_3=3$

$2.\because A^T=A,\therefore A$可相似对角化

$r(0E-B)=r(B)=1<3\Rightarrow B$可相似对角化 $\Rightarrow A\sim B$

$$A=\begin{pmatrix}2&-1&3\\1&0&-1\\0&0&1\end{pmatrix},B=\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \end{pmatrix},A\sim B?$$

$1.A,B:-1,1,1$

$2.E-A=\begin{pmatrix}-1&1&3\\-1&1&1\\0&0&0\end{pmatrix}$

$\because r(E-A)=2<3,\therefore A$不可相似对角化

$\because B^T=B,\therefore B$可相似对角化,$\therefore A\nsim B$