极限存在准则与重要极限

夹逼定理

$$
\text{若} a_n \le b_n \le c_n \\
\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = A \Rightarrow \lim_{n\to\infty} b_n = A
$$

$$
\text{证}: \forall\epsilon > 0, \exists N_1 > 0, \text{当} n > N_1 \text{时}, \\
|a_n - A| < \epsilon \Leftrightarrow A - \epsilon < a_n < A + \epsilon (*) \\
\exists N_2 > 0, \text{当} n > N_2 \text{时}, \\
|c_n - A| < \epsilon \Leftrightarrow A - \epsilon < c_n < A + \epsilon (**) \\
\text{取} N = \max\{N_1, N_2\}, \text{当} n > N \text{时}, (*)(**) \text{成立}, \\
\text{当} n > N \text{时} \\
A - \epsilon < a_n \le b_n \le c_n < A + \epsilon \\
\Rightarrow A - \epsilon < b_n < A + \epsilon \Leftrightarrow |b_n - A| < \epsilon \\
\therefore \lim_{n\to\infty} b_n = A
$$

$$
\text{若} f(x) \le g(x) \le h(x) \\
\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = A \\
\Rightarrow \lim_{x\to a} g(x) = A
$$

$$
\text{证}: \forall\epsilon > 0 \\
\exists\delta_1 > 0, \text{当} 0 < |x - a| < \delta_1 \text{时}, \\
|f(x) - A| < \epsilon \Leftrightarrow A - \epsilon < f(x) < A + \epsilon (*) \\
\exists\delta_2 > 0, \text{当} 0 < |x - a| < \delta_2 \text{时}, \\
|h(x) - A| < \epsilon \Leftrightarrow A - \epsilon < h(x) < A + \epsilon (**) \\
\text{取} \delta = \min\{\delta_1, \delta_2\}, \text{当} 0 < |x - a| < \delta \text{时}, (*)(**) \text{皆对} \\
A - \epsilon < f(x) \le g(x) \le h(x) < A + \epsilon \\
\Rightarrow A - \epsilon < g(x) < A + \epsilon \Leftrightarrow |g(x) - A| < \epsilon \\
\lim_{x\to a} g(x) = A
$$

$$
\text{求极限} \lim_{n\to\infty} \sqrt[n]{2^n + 3^n}.
$$

$$
3^n \le 2^n + 3^n \le 2 * 3^n \Rightarrow 3 \le \sqrt[n]{2^n + 3^n} \le 2^{\frac{1}{n}} * 3 \\
\because \lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1, \therefore \lim_{n\to\infty} 3 = 3 = \lim_{n\to\infty} 2^{\frac{1}{n}} * 3 = 3 \\
\therefore \text{原式} = 3
$$

$$
\lim_{n\to\infty} (2^n + 3^n + 5^n)^{\frac{1}{n}}
$$

$$\because 5^n \le 2^n+3^n+5^n \le 3\cdot 5^n$$
$$\therefore 5 \le (2^n+3^n+5^n)^{\frac{1}{n}} \le 3^{\frac{1}{n}}\cdot 5$$
$$\because \lim_{n\to\infty} 3^{\frac{1}{n}} = 3^0 = 1, \therefore \lim_{n\to\infty} 左 = \lim_{n\to\infty} 右 = 5$$
$$\therefore 原式 = 5$$

$$a>0, b>0, c>0, 则$$
$$\lim_{n\to\infty} \sqrt[n]{a^n+b^n+c^n} = \max\{a,b,c\}$$

$$x>0$$
$$求 \lim_{n\to\infty} \sqrt[n]{x^n+x^{2n}}.$$
$$\lim_{n\to\infty} \sqrt[n]{x^n+x^{2n}} = \lim_{n\to\infty} \sqrt[n]{x^n+(x^2)^n}$$
$$= \max\{x,x^2\} = \begin{cases} x, 0<x<1 \\ x^2, x\ge 1 \end{cases}$$

$$求极限 \lim_{n\to\infty}\left(\frac{1}{2n^2+1}+\frac{2}{2n^2+2}\ldots+\frac{n}{2n^2+n}\right).$$
$$b_n = \frac{1}{2n^2+1}+\frac{2}{2n^2+2}\ldots+\frac{n}{2n^2+n}$$
$$\because \frac{i}{2n^2+n} \le \frac{i}{2n^2+i} \le \frac{i}{2n^2+1} (1\le i\le n)$$
$$\therefore \frac{1+2+\ldots+n}{2n^2+n} \le \frac{1}{2n^2+1}+\frac{2}{2n^2+2}+\ldots+\frac{n}{2n^2+n} \le \frac{1+2+\ldots+n}{2n^2+1}$$
$$即 \frac{1}{2}\frac{n(n+1)}{2n^2+n} \le b_n \le \frac{1}{2}\frac{n(n+1)}{2n^2+1}$$
$$\because \lim_{n\to\infty} 左 = \frac{1}{2}\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2+\frac{1}{n}} = \frac{1}{4}$$
$$\lim_{n\to\infty} 右 = \frac{1}{2}\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2+\frac{1}{n^2}} = \frac{1}{4}$$
$$\therefore 原式 = \frac{1}{4}$$

$$\lim_{n\to\infty}\frac{n^2}{2^n}$$
$$n充分大时, 2^n = (1+1)^n = C_n^0+C_n^1+C_n^2+C_n^3+\ldots+C_n^n$$
$$\ge C_n^3 = \frac{n(n-1)(n-2)}{6}$$
$$0 \le \frac{1}{2^n} \le \frac{6}{n(n-1)(n-2)}$$
$$\Rightarrow 0 \le \frac{n^2}{2^n} \le \frac{6n^2}{n(n-1)(n-2)}$$
$$\because \lim_{n\to\infty} 右 = 6\lim_{n\to\infty}\frac{n}{(n-1)(n-2)} = 0, \therefore \lim_{n\to\infty}\frac{n^2}{2^n} = 0$$

$$f(x)=\lim_{n\to\infty}\sqrt[n]{x^n+x^{2n}}(x>0),\text{求}f(x).$$

$$\text{当}0<x\leq 1\text{时},$$

$$x^n\leq x^n+x^{2n}\leq 2x^n\Rightarrow x\leq\sqrt[n]{x^n+x^{2n}}\leq 2^{\frac{1}{n}}x$$

$$\because\lim_{n\to\infty}2^{\frac{1}{n}}=1,\therefore\lim_{n\to\infty}\text{左}=\lim_{n\to\infty}\text{右}=x\Rightarrow f(x)=x$$

$$\text{当}x>1\text{时},$$

$$x^{2n}\leq x^n+x^{2n}\leq 2x^{2n}\Rightarrow x^2\leq\sqrt[n]{x^n+x^{2n}}\leq 2^{\frac{1}{n}}x^2$$

$$\because\lim_{n\to\infty}\text{左}=\lim_{n\to\infty}\text{右}=x^2\Rightarrow f(x)=x^2$$

$$\therefore f(x)\begin{cases}x,0<x\leq 1\\ x^2,x>1\end{cases}$$

$$\lim_{n\to\infty}(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}})$$

$$b_n\triangleq\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}}$$

$$\frac{n}{\sqrt{n^2+n}}\leq b_n\leq\frac{n}{\sqrt{n^2+1}}$$

$$\lim_{n\to\infty}\text{左}=\lim_{n\to\infty}\text{右}=1$$

$$\therefore\text{原式}=1$$

单调、有界数列必有极限

$$\{a_n\}\uparrow\begin{cases}\text{无上界}\Rightarrow\lim_{n\to\infty}a_n=+\infty\\ a_n\leq M\Rightarrow\lim_{n\to\infty}a_n\exists\end{cases}$$

$$\{a_n\}\downarrow\begin{cases}\text{无下界}\Rightarrow\lim_{n\to\infty}a_n=-\infty\\ a_n\geq M\Rightarrow\lim_{n\to\infty}a_n\exists\end{cases}$$

设$\{a_n\}=\sqrt{2},a_2=\sqrt{2+\sqrt{2}},a_3=\sqrt{2+\sqrt{2+\sqrt{2}}},\ldots,$证明：数列$\{a_n\}$收敛，并求其极限.

1. $a_{n+1}=\sqrt{2+a_n}(n=1,2,\ldots)$
2. $\{a_n\}\uparrow$
3. 现证$a_n\leq 2$

   $a_1=\sqrt{2}\leq 2,$设$a_k\leq 2,$则

   $a_{k+1}=\sqrt{2+a_k}\leq\sqrt{2+2}=2$

   $\therefore\forall n,$有$a_n\leq 2\Rightarrow\lim\limits_{n\to\infty}a_n\exists$
4. 令$\lim\limits_{n\to\infty}a_n=A$

   $a_{n+1}=\sqrt{2+a_n}\Rightarrow A=\sqrt{2+A}$

   $\Rightarrow A^2-A-2=0\Rightarrow A=-1(\text{舍}),A=2$

$$a_1=2,a_{n+1}=\frac{1}{2}(a_n+\frac{1}{a_n}),\text{证：}\lim_{n\to\infty}a_n\exists$$

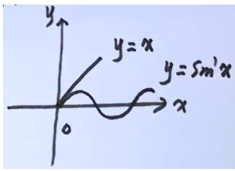
1. $\because a_n + \dfrac{1}{a_n} \geq 2$

$\therefore a_{n+1} \geq 1$

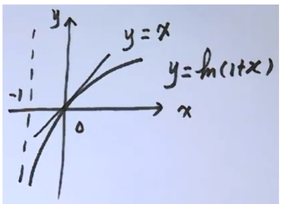
而$a_1 = 2 \geq 1, \therefore a_n \geq 1$

2. $a_{n+1} - a_n = \dfrac{1}{2}(a_n + \dfrac{1}{a_n}) - a_n = \dfrac{1-a_n^2}{2a_n} \leq 0$

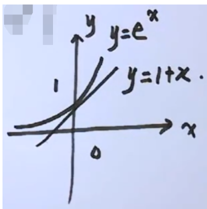
$\Rightarrow \{a_n\} \downarrow \Rightarrow \lim\limits_{n\to\infty} a_n \exists$



$x \geq 0, \sin x \leq x$

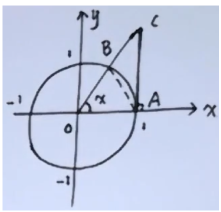


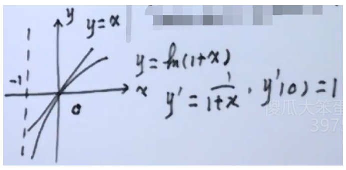
$x > -1, \ln(1+x) \leq x$

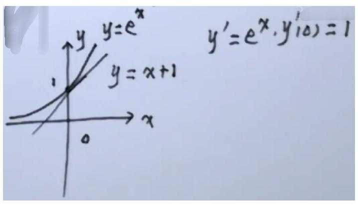


$e^x \geq (1+x)$

重要极限







$$1.\ 0<x<\frac{\pi}{2}\text{时},0<\sin x<x<\tan x$$

$$S_{\Delta AOB}=\frac{1}{2}\sin x$$

$$S_{\text{扇}AOB}=\frac{1}{2}x$$

$$S_{Rt\Delta AOC}=\frac{1}{2}\tan x$$

$$2.\ x>-1\text{时},\ln(1+x)\leq x$$

$$3.\ x\in(-\infty,+\infty),e^x\geq 1+x$$

$$\lim_{\Delta\to 0}=\frac{\sin\Delta}{\Delta}=1$$

$$\lim_{\Delta\to 0}(1+\Delta)^{\frac{1}{\Delta}}=e$$

$$\lim_{x\to 0}\frac{1-\cos^3 x}{x\ln(1+2x)}$$

$$\begin{aligned}\text{原式}=&\lim_{x\to 0}\frac{1-\cos^2 x}{2x^2}\\ =&\frac{1}{2}\lim_{x\to 0}(1+\cos x)\frac{1-\cos x}{x^2}\\ =&\frac{1}{2}*2*\frac{1}{2}\\ =&\frac{1}{2}\end{aligned}$$

$$\lim_{x\to 0}\frac{e^{\tan x}-e^{\sin x}}{x\arcsin^2 x}$$

$$\begin{aligned}\text{原式}=&\lim_{x\to 0}\frac{e^{\tan x}-e^{\sin x}}{x^3}\\ =&\lim_{x\to 0}e^{\sin x}\frac{e^{\tan x-\sin x}-1}{x^3}\\ =&1*\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}\\ =&\lim_{x\to 0}\frac{\tan x}{x}*\frac{1-\cos x}{x^2}\\ =&\frac{1}{2}\end{aligned}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x\cos x}-\sqrt{1+x}}{x^3}$$

$$\begin{aligned}\text{原式}=&\lim_{x\to 0}\frac{1}{\sqrt{1+x\cos x}+\sqrt{1+x}}\frac{x\cos x-x}{x^3}\\ =&\frac{1}{2}\lim_{x\to 0}\frac{\cos x-1}{x^2}\\ =&-\frac{1}{4}\end{aligned}$$

$$\lim_{x\to 0}(1-\sin 2x^2)^{\frac{1}{x^2}}$$

$$\begin{aligned}
\text{原式} &= \lim_{x\to 0}\left[(1-\sin 2x^2)^{-\frac{1}{\sin 2x^2}}\right]^{\frac{1}{x^2}*(-\sin 2x^2)} \\
&= e^{-\lim_{x\to 0}\frac{\sin 2x^2}{x^2}} \\
&= e^{-2}
\end{aligned}$$

$$\lim_{x\to 0}(\cos x)^{\frac{1}{x\ln(1-x)}}$$

$$\begin{aligned}
\text{原式} &= \lim_{x\to 0}\left\{[1+(\cos x-1)]^{\frac{1}{\cos x-1}}\right\}^{\frac{\cos x-1}{x\ln(1-x)}} \\
&= e^{\lim_{x\to 0}\frac{\cos x-1}{x\ln(1-x)}} \\
&= e^{\lim_{x\to 0}\frac{-\frac{1}{2}x^2}{-x^2}} \\
&= e^{\frac{1}{2}}
\end{aligned}$$

$$\lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\frac{1}{x^3}}$$

$$\begin{aligned}
\text{原式} &= \lim_{x\to 0}\left\{\left[1+\left(\frac{1+\tan x}{1+\sin x}-1\right)\right]^{\frac{1}{\frac{1+\tan x}{1+\sin x}-1}}\right\}^{\frac{1}{x^3}\left(\frac{1+\tan x}{1+\sin x}-1\right)} \\
&= e^{\lim_{x\to 0}\frac{1}{x^3}\left(\frac{\tan x-\sin x}{1+\sin x}\right)} \\
&= e^{\lim_{x\to 0}\frac{1}{1+\sin x}\frac{\tan x-\sin x}{x^3}} \\
&= e^{\lim_{x\to 0}\frac{\tan x}{x}\frac{1-\cos x}{x^2}} \\
&= e^{\frac{1}{2}}
\end{aligned}$$

$$\lim_{x\to\infty}\left(\frac{2x^2+3x+1}{x-1}-2x\right)$$

$$\begin{aligned}
\text{原式} &= \lim_{x\to\infty}\frac{2x^2+3x+1-2x(x-1)}{x-1} \\
&= \lim_{x\to\infty}\frac{5x+1}{x-1} \\
&= \lim_{x\to\infty}\frac{5+\frac{1}{x}}{1-\frac{1}{x}} \\
&= 5
\end{aligned}$$

$$\lim_{x\to\infty}\left(\sqrt{x^2+4x+1}-x\right)$$

$$\begin{aligned}
\text{原式} &= \lim_{x\to\infty}\frac{4x+1}{\sqrt{x^2+4x+1}+x} \\
&= \lim_{x\to\infty}\frac{4+\frac{1}{x}}{\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}+1} \\
&= 2
\end{aligned}$$