# 线性方程组

$$A_{m\times n}$$
$$AX=0(*)$$
$$AX=b(**)$$
$$\overline{A}=(A\vdots b)-\text{增广矩阵}$$
$$A=(\alpha_1\dots\alpha_n),\overline{A}=(\alpha_1\dots\alpha_n\vdots b)$$

# 解的结构

$1.\xi_1,\dots,\xi_s$为$(*)$解 $\Rightarrow k_1\xi_1+\dots+k_s\xi_s$为$(*)$解

$2.\eta_1,\dots,\eta_s$为$(**)$解

①$k_1\eta_1+\dots+k_s\eta_s$为$(*)$解 $\Leftrightarrow k_1+\dots+k_s=0$

②$k_1\eta_1+\dots+k_s\eta_s$为$(**)$解 $\Leftrightarrow k_1+\dots+k_s=1$

$3.\xi$为$(*)$解,$\eta$为$(**)$解 $\Rightarrow \xi+\eta$为$(**)$解

$4.\eta_1,\eta_2$为$(**)$解 $\Rightarrow \eta_2-\eta_1$为$(*)$的解

# 解的基本定理

Th1. 对$(*)$:

①$(*)$仅有零解 $\Leftrightarrow \alpha_1\dots\alpha_n$线性无关 $\Leftrightarrow r(A)=n$

②$(*)$有非零解 $\Leftrightarrow \alpha_1\dots\alpha_n$线性相关 $\Leftrightarrow r(A)<n$

Th2. $A=(\alpha_1\dots\alpha_n),\overline{A}=(A\vdots b)=(\alpha_1\dots\alpha_n\vdots b)$, 对$(**)$:

①$(**)$有解 $\Leftrightarrow b$可由$\alpha_1\dots\alpha_n$线性表示 $\Leftrightarrow \alpha_1\dots\alpha_n$的秩 $=\alpha_1\dots\alpha_n,b$的秩

$$\Leftrightarrow r(A)=r(\overline{A})\begin{cases}=n,\text{唯一解}\\<n,\text{无数个解}\end{cases}$$

②$(**)$无解 $\Leftrightarrow b$不可由$\alpha_1\dots\alpha_n$线性表示

$\Leftrightarrow \alpha_1\dots\alpha_n,b$秩 $=\alpha_1\dots\alpha_n$的秩 $+1$

$\Leftrightarrow r(\overline{A})=r(A)+1(r(A)\neq r(\overline{A}))$

Q1. ①$AX=0$仅有零解与$AX=b$有唯一解关系?

$r(A)=n\Leftarrow(\not\Rightarrow)r(A)=r(\overline{A})=n$

②$AX=0$有非零解与$AX=b$有无数解关系?

$r(A)<n\Leftarrow(\not\Rightarrow)r(A)=r(\overline{A})<n$

Q2. $A_{m\times n},r(A)=m$, 问$AX=b$是否一定有解? $\Leftrightarrow r(A)=r(\overline{A})$?

$\overline{A}=(A\vdots b)_{m\times(n+1)}$

$r(\overline{A})\geq r(A)=m$

$\because r(\overline{A})\leq m,\therefore r(A)=r(\overline{A})=m$

Th3. $A_{m\times n},B_{n\times s}=(\beta_1,\dots,\beta_s)$, 若$AB=0$

$\beta_1,\beta_2,\dots,\beta_s$为$AX=0$的解.

证: $AB=(A\beta_1,\dots,A\beta_s)$

$\because AB=0,\therefore A\beta_1=0,A\beta_2=0,\dots,A\beta_s=0$

即$\beta_1,\beta_2,\dots,\beta_s$为$AX=0$的解.

# 通解

## 齐次

$$1.\begin{cases}x_1+2x_2-x_3+3x_4-2x_5=0\\2x_1+3x_2+5x_3-2x_4+x_5=0\\3x_1+5x_2+4x_3+x_4-x_5=0\end{cases}$$

$$解:A=\begin{pmatrix}1&2&-1&3&-2\\2&3&5&-2&1\\3&5&4&1&-1\end{pmatrix}\to\begin{pmatrix}1&2&-1&3&-2\\0&-1&7&-8&5\\0&-1&7&-8&5\end{pmatrix}$$

$$\to\begin{pmatrix}1&2&-1&3&-2\\0&-1&7&-8&5\\0&0&0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&13&-13&8\\0&1&-7&8&-5\\0&0&0&0&0\end{pmatrix}$$

$$法一:同解方程组为\begin{cases}x_1=-13x_3+13x_4-8x_5\\x_2=7x_3-8x_4+5x_5\end{cases}$$

$$通解x=\to\begin{pmatrix}-13x_3+13x_4-8x_5\\7x_3-8x_4+5x_5\\x_3\\x_4\\x_5\end{pmatrix}=x_3\begin{pmatrix}-13\\7\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}13\\-8\\0\\1\\0\end{pmatrix}+x_5\begin{pmatrix}-8\\5\\0\\0\\1\end{pmatrix}$$

$$法二:通解x=k_1\begin{pmatrix}-13\\7\\1\\0\\0\end{pmatrix}+k_2\begin{pmatrix}13\\-8\\0\\1\\0\end{pmatrix}+k_3\begin{pmatrix}-8\\5\\0\\0\\1\end{pmatrix}$$

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$$1.\begin{cases}x_1-x_2+2x_3-x_4+3x_5=0\\2x_1-2x_2+3x_3+3x_4-x_5=0\\3x_1-3x_2+5x_3+2x_4+2x_5=0\end{cases}$$

$$解:A=\begin{pmatrix}1&-1&2&-1&3\\2&-2&3&3&-1\\3&-3&5&2&2\end{pmatrix}\to\begin{pmatrix}1&-1&2&-1&3\\0&0&-1&5&-7\\0&0&0&0&0\end{pmatrix}$$

$$\to\begin{pmatrix}1&-1&0&9&-11\\0&0&1&-5&7\\0&0&0&0&0\end{pmatrix}$$

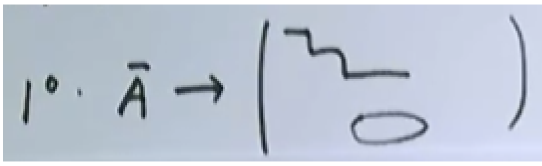
$$通解x=k_1\begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}+k_2\begin{pmatrix}-9\\0\\5\\1\\0\end{pmatrix}+k_3\begin{pmatrix}11\\0\\-7\\0\\1\end{pmatrix}=\xi_1+\xi_2+\xi_3\begin{cases}①\xi_1,\xi_2,\xi_3皆是解\\②线性无关\\③3=5-r(A)\end{cases}$$

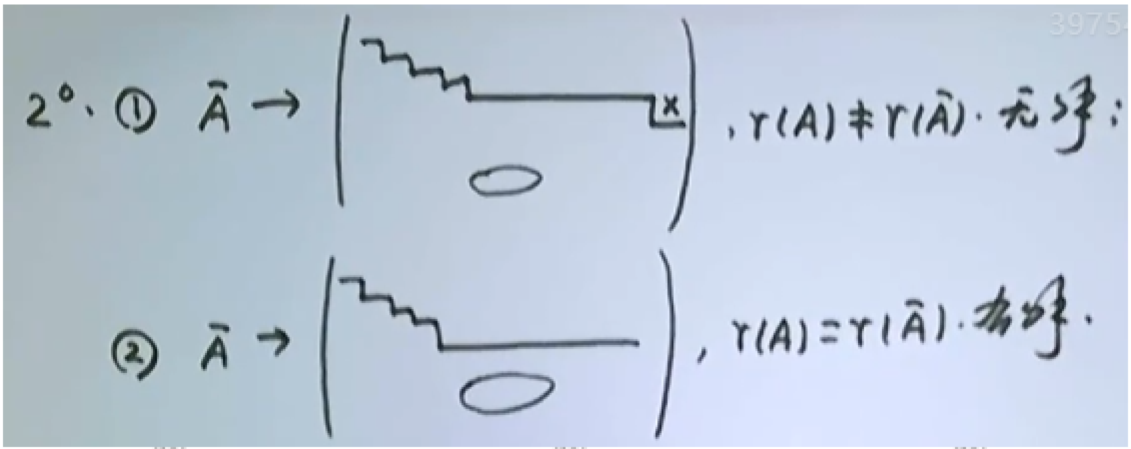
# 基础解系

$$\begin{aligned}&A_{m\times n},r(A)=r<n,若\\&①\xi_1,\ldots,\xi_s为AX=0的解\\&②\xi_1,\ldots,\xi_s线性无关\\&③s=n-r\\&称\xi_1,\ldots,\xi_s为AX=0基础解系\\&通解X=k_1\xi_1+\ldots+k_s\xi_s\end{aligned}$$

## 非齐次

$$AX=b,\overline{A}=(A\vdots b)$$

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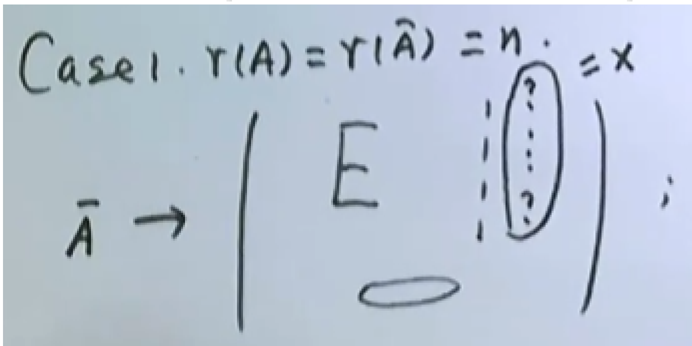
$$\text{Case 1. } r(A)=r(\overline{A})=n$$

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$$\begin{cases}2x_1+x_2=1\\ x_1-x_2=2\\ x_1+2x_2=-1\end{cases}$$

$$\text{解}:\overline{A}=\begin{pmatrix}2&1&1\\1&-1&2\\1&2&-1\end{pmatrix}\to\begin{pmatrix}1&-1&2\\2&1&1\\1&2&-1\end{pmatrix}\to\begin{pmatrix}1&-1&2\\0&3&-3\\0&0&0\end{pmatrix}$$

$r(A)=r(\overline{A})=2$, 唯一解

$$\overline{A}\to\begin{pmatrix}1&-1&2\\0&1&-1\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&1\\0&1&-1\\0&0&0\end{pmatrix},x=\begin{pmatrix}1\\-1\end{pmatrix}$$

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$$\text{Case 2. } r(A)=r(\overline{A})=r<n$$

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判断非齐次线性方程组 $\begin{cases}x_1+x_2+4x_3=4\\ -x_1+4x_2+x_3=16\\ x_1-x_2+2x_3=-4\end{cases}$, 是否有解, 若有无穷多个解, 求出其通解.

$$\overline{A}=\begin{pmatrix}1&1&4&4\\-1&4&1&16\\1&-1&2&-4\end{pmatrix}\rightarrow\begin{pmatrix}1&1&4&4\\0&1&1&4\\0&0&0&0\end{pmatrix}$$

$r(A)=r(\overline{A})=2<3,$ 无数解

$$\overline{A}\rightarrow\begin{pmatrix}1&0&3&0\\0&1&1&4\\0&0&0&0\end{pmatrix}$$

$$x=k\begin{pmatrix}-3\\-1\\1\end{pmatrix}+\begin{pmatrix}0\\4\\0\end{pmatrix}$$

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$$\begin{cases}x_1+2x_2-3x_3+5x_4=3\\2x_1+3x_2+x_3-3x_4=-1\\3x_1+5x_2-2x_3+2x_4=2\end{cases}$$

$$\overline{A}=\begin{pmatrix}1&2&-3&5&3\\2&3&1&-3&-1\\3&5&-2&2&2\end{pmatrix}\rightarrow\begin{pmatrix}1&2&-3&5&3\\0&-1&7&-13&-7\\0&0&0&0&0\end{pmatrix}$$

$r(A)=r(\overline{A})=2<4,$ 有无数个解

$$\overline{A}\rightarrow\begin{pmatrix}1&0&11&-21&-11\\0&1&-7&13&7\\0&0&0&0&0\end{pmatrix}$$

$$X=k_1\begin{pmatrix}-11\\7\\1\\0\end{pmatrix}+k_2\begin{pmatrix}21\\-13\\0\\1\end{pmatrix}+\begin{pmatrix}-11\\7\\0\\0\end{pmatrix}$$