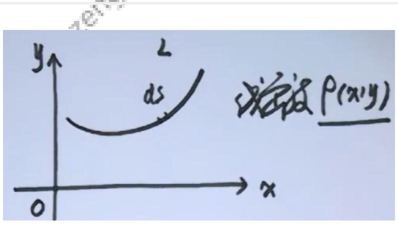
积分域	积分号	例子
线状	\int	\int_a^bf(x)dx \int_L
面状	\iint	\iint_Df(x,y)d\sigma \iint_{\sum}
体状	\iiint	\iiint_{\Omega}f(x,y,z)dV

曲线积分

对弧长的曲线积分（第一类曲线积分）

背景: m



$$
\begin{aligned}
&1.\forall ds \subset L \\
&2.dm = \rho(x,y)ds \\
&3.m = \int_L dm = \int_L \rho(x,y)ds
\end{aligned}
$$

定义

$$\int_L f(x,y)dx$$

$$f(x,y)\text{在曲线段}L\text{上对弧长的曲线积分}$$

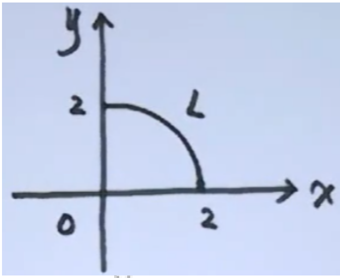
性质

$$
\begin{aligned}
&1.\int_L 1ds = l \\
&2.\text{①}L\text{左右对称, 右}L1, \\
&\quad\text{若}f(-x,y) = -f(x,y) \Rightarrow \int_L f(x,y)ds = 0 \\
&\quad\text{若}f(-x,y) = f(x,y) \Rightarrow \int_L f(x,y)ds = 2\int_{L_1} f(x,y)ds \\
&\quad\text{②}L\text{关于}y = x\text{对称, 则} \\
&\quad\int_L f(x,y)ds = \int_L f(y,x)ds
\end{aligned}
$$

计算方法

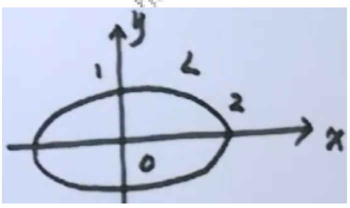
特殊法

$$I=\int_L x^2 ds.$$



$$\begin{aligned}&I=\int_L y^2 ds\\&\Rightarrow 2I=\int_L (x^2+y^2)ds=4\int_L 1ds=4\times\frac{1}{4}\times 2\pi\times 2=4\pi\\&\Rightarrow I=2\pi\end{aligned}$$

$$L:\frac{x^2}{4}+y^2=1, L\text{长为}a,\text{求}I=\int_L (x-2y)^2 ds.$$



$$\begin{aligned}I=&\int_L (x^2+4y^2)ds\\=&4\int_L (\frac{x^2}{4}+y^2)ds=4\int_L 1ds=4a\end{aligned}$$

定积分法

$$I=\int_L f(x,y)ds$$

$$\begin{aligned}&\text{case1.}L:y=\phi(x)(a\le x\le b)\\&\qquad I=\int_a^b f[x,\phi(x)]\cdot\sqrt{1+\phi'^2(x)}dx\\&\text{case2.}L:\begin{cases}x=\Phi(t)\\y=\phi(t)\end{cases}(\alpha\le t\le\beta)\\&\qquad I=\int_\alpha^\beta f[\Phi(t),\phi(t)]\cdot\sqrt{\Phi'^2(t)+\phi'^2(t)}dt\end{aligned}$$

$$\text{计算}\int_L xe^{\sqrt{x^2+y^2}}ds,\text{其中}L\text{为第一象限中由}x\text{轴},y=x\text{及}x^2+y^2=4\text{围成的曲线段}.$$

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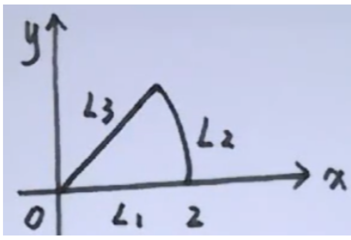


$1.L_1: y=0(0\leq x\leq 2)$

$$\int_{L_1} xe^{\sqrt{x^2+y^2}}ds=\int_0^2 xe^x dx$$

$$=(x-1)e^x\ |_0^2=e^2-(-1)=e^2+1$$

$2.L_2:\begin{cases}x=2\cos t\\ y=2\sin t\end{cases}(0\leq t\leq\frac{\pi}{4})$

$$\int_{L_2} xe^{\sqrt{x^2+y^2}}ds=e^2\int_0^{\frac{\pi}{4}}2\cos t\cdot\sqrt{4\sin^2 t+4\cos^2 t}tdt$$

$$=4e^2\sin t\ |_0^{\frac{\pi}{4}}=2\sqrt{2}e^2$$

$3.L_3: y=x(0\leq x\leq\sqrt{2})$

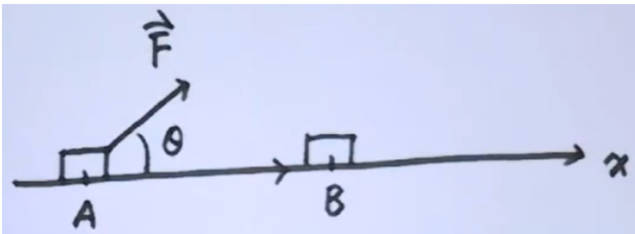
$$\int_{L_3} xe^{\sqrt{x^2+y^2}}ds=\frac{1}{\sqrt{2}}\int_0^{\sqrt{2}}\sqrt{2}xe^{\sqrt{2}x}d(\sqrt{2}x)=\frac{1}{\sqrt{2}}\int_0^2 xe^x dx=\frac{e^2+1}{\sqrt{2}}$$

$$\therefore 原式=(1+\frac{1}{\sqrt{2}})(e^2+1)+2\sqrt{2}e^2$$

对坐标的曲线积分（第二类曲线积分）

背景: 做功

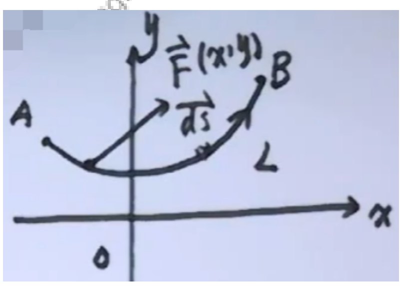
双理想



$$\text{case1.双理想}:$$

$$\begin{aligned}W=&|\overrightarrow{F}|\cos\theta\cdot|\overrightarrow{AB}|\\ =&|\overrightarrow{F}|\cdot|\overrightarrow{AB}|\cdot\cos(\overrightarrow{F},\overrightarrow{AB})\triangleq\overrightarrow{F}\cdot\overrightarrow{AB}\end{aligned}$$

2-dim 双不理想



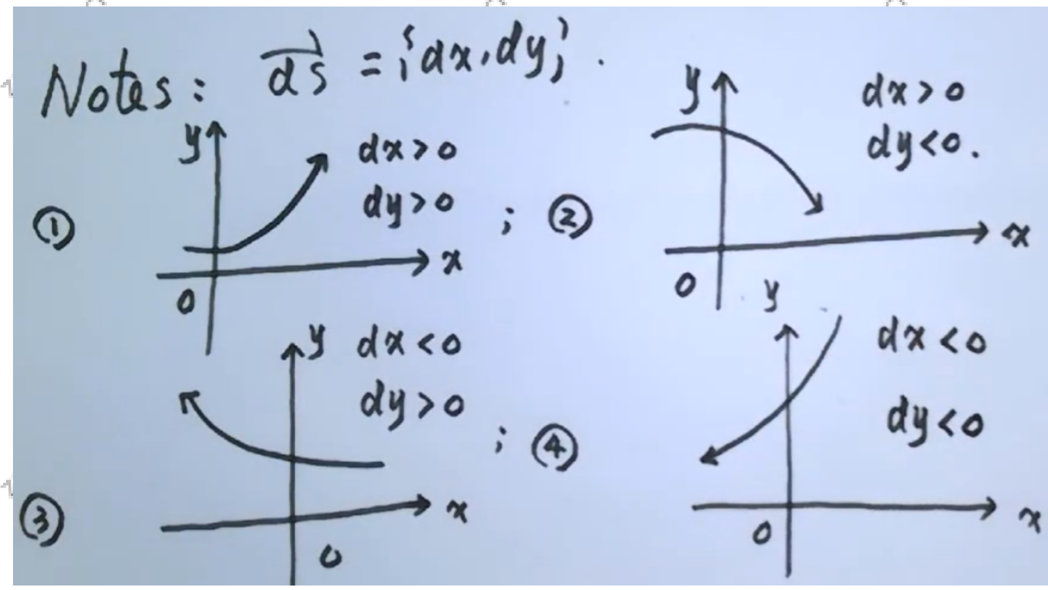
case2.(2-dim)双不理想

$$\overrightarrow{F}=\{P(x,y),Q(x,y)\}$$

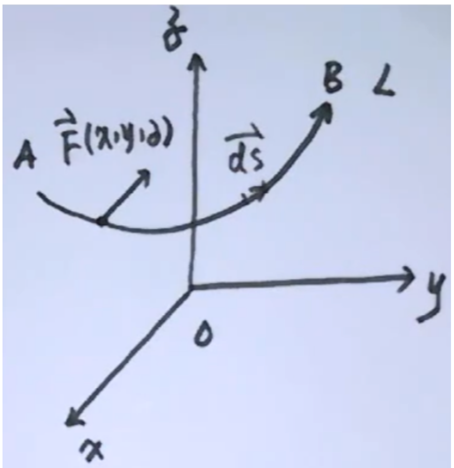
$$1.\forall \overrightarrow{ds}\subset L,\overrightarrow{ds}=\{dx,dy\}$$

$$2.dW=\overrightarrow{F}\cdot\overrightarrow{ds}=P(x,y)dx+Q(x,y)dy$$

$$3.W=\int_L dW=\int_L P(x,y)dx+Q(x,y)dy$$



3-dim 双不理想



case3.(3-dim)双不理想

$$\overrightarrow{F}=\{P,Q,R\}$$

$$1.\forall \overrightarrow{ds}\subset L,\overrightarrow{ds}=\{dx,dy,dz\}$$

$$2.dW=\overrightarrow{F}\cdot\overrightarrow{ds}=Pdx+Qdy+Rdz$$

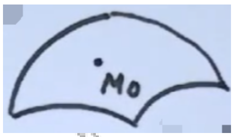
$$3.W=\int_L dW=\int_L Pdx+Qdy+Rdz$$

定义

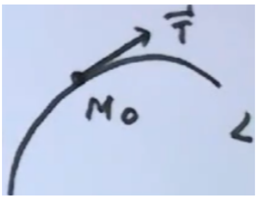
$$1.(\text{2-dim}):\int_L Pdx+Qdy=\int_L Pdx+\int_L Qdy$$

$$2.(\text{3-dim}):\int_L Pdx+Qdy+Rdz=\int_L Pdx+\int_L Qdy+\int_L Rdz$$

线切/面法



$\Sigma : F(x,y,z)=0$
法向量为$\vec{n}=\{F_x,F_y,F_z\}_{M_0}$



case1
$$L:\begin{cases}x=\Phi(t)\\ y=\phi(t)\\ z=w(t)\end{cases},M_0\leftrightarrow t_0$$
$$\vec{\tau}=\{\Phi'(t_0),\phi'(t_0),w'(t_0)\}$$
case2
$$L:\begin{cases}F(x,y,z)=0\\ G(x,y,z)=0\end{cases},M_0\in L$$
$$\overrightarrow{n_1}=\{F_x,F_y,F_z\}_{M_0},\overrightarrow{n_2}=\{G_x,G_y,G_z\}_{M_0}$$
$$\vec{\tau}=\overrightarrow{n_1}\times\overrightarrow{n_2}$$

性质

$$1.\int_{L^-}=-\int_L$$
$$2.①\int_L Pdx+Qdy$$
$$=\int_L(P\cos\alpha+Q\cos\beta)ds$$
$\cos\alpha,\cos\beta$为L切向量的方向余弦
$$②\int_L Pdx+Qdy+Rdz$$
$$=\int_L(P\cos\alpha+Q\cos\beta+R\cos\gamma)ds$$
$\cos\alpha,\cos\beta,\cos\gamma$为$L$的切向量的方向余弦

计算方法

2-dim 情形

$$\int_L Pdx+Qdy$$

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定积分法

$$①L: y=\Phi(x)(\text{起}x=a,\text{终}x=b)$$

$$\int_L Pdx+Qdy=\int_a^b\{P[x,\Phi(x)]+Q[x,\Phi(x)]\Phi'(x)\}dx$$

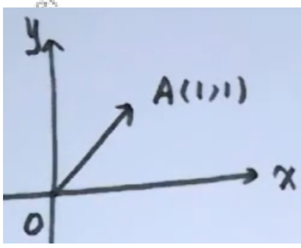
$$②L:\begin{cases}x=\Phi(t)\\ y=\phi(t)\end{cases}(\text{起}t=\alpha,\text{终}t=\beta)$$

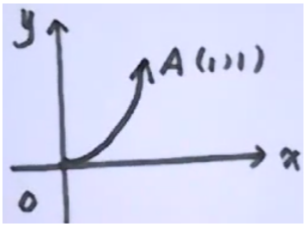
$$\int_L Pdx+Qdy=\int_\alpha^\beta\{P[\Phi(t),\phi(t)]\Phi'(t)+Q[\Phi(t),\phi(t)]\phi'(t)\}dt$$

求以下情况下的曲线积分 $\int_L(y+1)dx+(2x-1)dy:$

(1) L是从点$O(0,0)$经$y=x$到点$A(1,1)$的有向曲线段;

(2) L是从点$O(0,0)$经$y=x^2$到点$A(1,1)$的有向曲线段.





$1.L: y=x(\text{起}x=0,\text{终}x=1)$

$$I=\int_0^1(x+1)dx+(2x-1)dx=\int_0^1 3xdx=\frac{3}{2}$$

$2.L: y=x^2(\text{起}x=0,\text{终}x=1)$

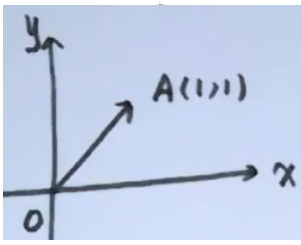
$$I=\int_0^1(x^2+1)dx+(2x-1)\cdot 2xdx$$

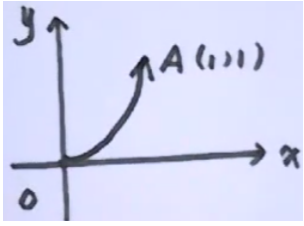
$$=\int_0^1(5x^2-2x+1)dx=\frac{5}{3}$$

求以下情况下的曲线积分 $\int_L(2y+1)dx+(2x-3)dy:$

(1) L是从点$O(0,0)$经$y=x$到点$A(1,1)$的有向曲线段;

(2) L是从点$O(0,0)$经$y=x^2$到点$A(1,1)$的有向曲线段.

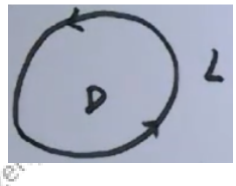




$$1.L:y=x(\text{起}x=0,\text{终}x=1)$$
$$I=\int_0^1(2x+1)dx+(2x-3)dx=\int_0^1(4x-2)dx=0$$
$$2.L:y=x^2(\text{起}x=0,\text{终}x=1)$$
$$I=\int_0^1(2x^2+1)dx+(2x-3)\cdot 2xdx$$
$$=\int_0^1(6x^2-6x+1)dx=2-3+1=0$$

二重积分法（Green 公式）

单连通区域

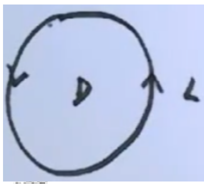


$$\text{单连通区域}$$
$$L\text{正方向：逆时针}$$

多连通区域

$$\text{多连通区域}$$
$$L=L_1+L_2\text{正方向：外逆内顺}$$

格林公式



$$\text{1-dim}:F(b)-F(a)=\int_a^b f(x)dx$$
$$\text{2-dim}:L-\text{边界}:\int_L,D-\text{区域}\iint_D$$

Th.①D为连通区域, L为D的正向边界

②$P(x,y),Q(x,y)$在D上连续可偏导, 则

$$\oint_L Pdx+Qdy=\iint_D(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})d\sigma$$

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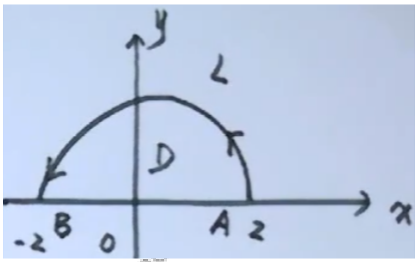
$$I=\int_L 3ydx-(x+1)dy$$



法一.$L:\begin{cases}x=2\cos t\\ y=2\sin t\end{cases}$(起$t=0$,终$t=\pi$)

$$\begin{aligned}I&=\int_0^\pi 6\sin t\cdot(-2\sin t)dt-(2\cos t+1)\cdot 2\cos tdt\\&=-\int_0^\pi(12\sin^2t+4\cos^2t+2\cos t)dt\\&=-\int_0^\pi(4+8\sin^2t+2\cos t)dt=-4\pi-16\times\frac{1}{2}\times\frac{\pi}{2}\\&=-8\pi\end{aligned}$$

法二.

1.$P=3y,Q=-(x+1),\dfrac{\partial P}{\partial y}=3,\dfrac{\partial Q}{\partial x}=-1$

2.$I=\oint_{L+\overline{BA}}+\int_{\overline{AB}}$

3.$\oint_{L+\overline{BA}}=\iint_D(-4)d\sigma=-4\times 2\pi=-8\pi$

4.$\int_{\overline{AB}}=\int_2^{-2}0dx=0$

$\therefore$原式$=-8\pi$

求$I=\oint_L\dfrac{ydx-xdy}{x^2+y^2}$,$L$为不经过$o$的正向闭曲线.

$$1.P=\frac{y}{x^2+y^2},Q=\frac{-x}{x^2+y^2}$$

$$\frac{\partial Q}{\partial x}=-\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y}=\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}((x,y)\neq(0,0))$$

$$2.①O(0,0)\notin D$$

$$I=\iint_D(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})d\sigma=0$$

$$②O(0,0)\in D$$

$$L_0:x^2+y^2=r^2$$

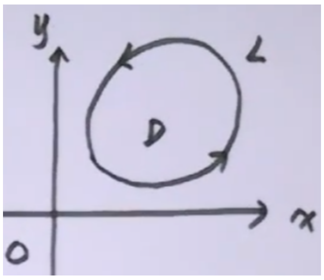
$$(r>0,L_0\text{在}L\text{内},L_0\text{逆时针})$$

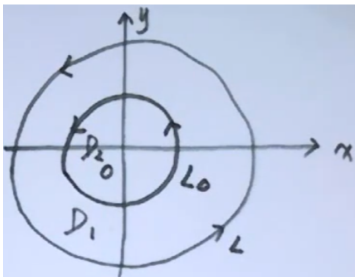
$$\oint_{L+L_0^-}=\iint_{D_1}0d\sigma=0$$

$$\Rightarrow\oint_L-\oint_{L_0}=0\Rightarrow I=\oint_L=\oint_{L_0}\frac{ydx-xdy}{x^2+y^2}$$

$$=\frac{1}{r^2}\oint_{L_0}ydx-xdy=\frac{1}{r^2}\iint_{D_2}(-2)d\sigma$$

$$=\frac{-2}{r^2}\times\pi r^2=-2\pi$$





曲线积分与路径无关问题

Th. $D-$单连通区域$,P,Q$在D上连续可偏导

以下命题等价：

1. $\int_L Pdx+Qdy$与路径无关
2. 任取闭曲线$C\subset D$,有$\oint_C Pdx+Qdy=0$
3. $\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}$(C.-R.)
4. $\exists u(x,y)$, 使$Pdx+Qdy=du$

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1.若$\dfrac{\partial Q}{\partial x}\equiv\dfrac{\partial P}{\partial y}$,则

$$\int_L Pdx+Qdy=\int_{(x_0,y_0)}^{(x_1,y_1)}Pdx+Qdy$$
$$=\int_{x_0}^{x_1}P(x,y_0)dx+\int_{y_0}^{y_1}Q(x_1,y)dy$$

2.若$\dfrac{\partial Q}{\partial x}\equiv\dfrac{\partial P}{\partial y}$,且$Pdx+Qdy=du(x,y)$,则

$$\int_L Pdx+Qdy=\int_{(x_0,y_0)}^{(x_1,y_1)}du(x,y)$$
$$=u(x_1,y_1)-u(x_0,y_0)$$

3.若$\dfrac{\partial Q}{\partial x}\equiv\dfrac{\partial P}{\partial y}$,则

$$u(x,y)=\int_{(x_0,y_0)}^{(x,y)}Pdx+Qdy$$
$$=\int_{x_0}^{x}P(x,y_0)dx+\int_{y_0}^{y}P(x_0,y)dy$$

证$:x>0$内$\dfrac{xdy-ydx}{4x^2+y^2}$为某二元函数$u(x,y)$的全微分,求$u(x,y)$.

$$P=-\frac{y}{4x^2+y^2},Q=\frac{x}{4x^2+y^2}$$
$$\because\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}=\frac{y^2-4x^2}{(4x^2+y^2)^2}$$
$$\therefore\exists u(x,y),\text{使}du=\frac{xdy-ydx}{4x^2+y^2}$$
$$u(x,y)=\int_{(1,0)}^{(x,y)}\frac{xdy-ydx}{4x^2+y^2}$$
$$=\int_1^x\frac{-0dx}{4x^2+0^2}+\int_0^y\frac{xdy}{4x^2+y^2}$$
$$=x\int_0^y\frac{dy}{(2x)^2+y^2}=\frac{1}{2}\arctan\frac{y}{2x}=\frac{1}{2}\arctan\frac{y}{2x}$$

4.全微分方程$-P(x,y)dx+Q(x,y)dy=0$

若$\dfrac{\partial Q}{\partial x}\equiv\dfrac{\partial P}{\partial y}$,称该微分方程为全微分方程

解法$:\because\dfrac{\partial Q}{\partial x}\equiv\dfrac{\partial P}{\partial y},\therefore\exists u(x,y)$,使$Pdx+Qdy=du$

$Pdx+Qdy=0\Leftrightarrow du=0$

$\therefore$通解$u(x,y)=C$

求$(2xy^2+x)dx+(2x^2y+2y)dy=0$通解.

$P=2xy^2+x, Q=2x^2y+2y$

$\because \dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}=4xy, \therefore$ 该方程为全微分方程

$$法一.u(x,y)=\int_{(0,0)}^{(x,y)}(2xy^2+x)dx+(2x^2y+2y)dy$$

$$=\int_0^x xdx+\int_0^y(2x^2y+2y)dy=\frac{x^2}{2}+x^2y^2+y^2$$

$\therefore$ 通解 $\dfrac{x^2}{2}+x^2y^2+y^2=C$

$$法二.(2xy^2dx+2x^2ydy)+xdx+2ydy=0$$

$$\Rightarrow d(x^2y^2+\frac{x^2}{2}+y^2)=0$$

$$\therefore x^2y^2+\frac{x^2}{2}+y^2=C$$

设曲线积分 $\int_L xy^2dx+y\Phi(x)dy$ 与路径无关, 其中 Φ 连续可导, 且 $\Phi(0)=0$, 计算

$$\int_{(0,0)}^{(1,1)}xy^2dx+y\Phi(x)dy.$$

$P=xy^2, Q=y\Phi(x)$

由 $\dfrac{\partial Q}{\partial x}=\dfrac{\partial P}{\partial y}\Rightarrow y\Phi'(x)=2xy\Rightarrow\Phi'(x)=2x$

$\Phi(x)=x^2+C$

由 $\Phi(0)=0\Rightarrow C=0\Rightarrow\Phi(x)=x^2$

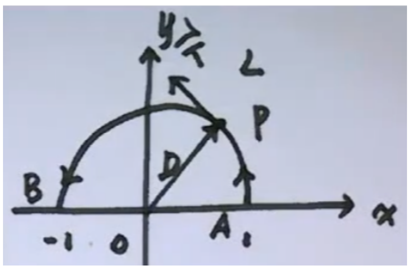
$$法一.原式=\int_0^1 x\cdot 0^2dx+\int_0^1 ydy=\frac{1}{2}$$

$$法二.原式=\int_{(0,0)}^{(1,1)}d(\frac{1}{2}x^2y)$$

$$=\frac{1}{2}x^2y^2|_{(0,0)}^{(1,1)}=\frac{1}{2}$$

坐标转化对弧长积分（第二类转第一类）

$$I=\int_L xdy-(y+1)dx$$



法一.令$L:\begin{cases}x=\cos t\\y=\sin t\end{cases}$(起$t=0$,终$t=\pi$)

$$I=\int_0^{\pi}[\cos^2 t+(\sin t+1)\sin t]dt=\int_0^{\pi}(1+\sin t)dt=\pi+2$$

法二.$P=-(y+1),Q=x,\dfrac{\partial Q}{\partial x}=1,\dfrac{\partial P}{\partial y}=-1$

$$I=\oint_{L+\overline{BA}}+\oint_{\overline{AB}}$$

$$\oint_{L+\overline{BA}}=2\iint_D d\sigma=\pi$$

$$\int_{\overline{AB}}=\int_1^{-1}-dx=2,\therefore I=\pi+2$$

法三.$\forall P(x,y)\in L,\overrightarrow{OP}=\{x,y\}$

$$\vec{\tau}=\{-y,x\},\cos\alpha=\frac{-y}{\sqrt{x^2+y^2}}=-y,\cos\beta=\frac{x}{\sqrt{x^2+y^2}}=x$$

$$I=\int_L(y^2+y+x^2)ds=\int_L(1+y)dS$$

$$L:\begin{cases}x=\cos t\\y=\sin t\end{cases}(0\le t\le\pi)$$

$$I=\int_0^{\pi}(1+\sin t)\sqrt{\sin^2 t+\cos^2 t}dt=\pi+2$$

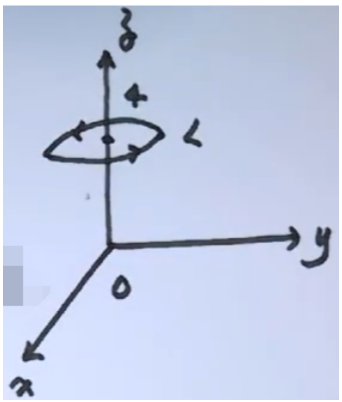
3-dim 情形

定积分法

$$L:\begin{cases}x=\Phi(t)\\y=\phi(t)\\z=w(t)\end{cases}(起t=\alpha,终t=\beta)$$

$$I=\int_{\alpha}^{\beta}\dots dt$$

$$I=\int_L ydx-(x+1)dy+2dz$$



$$1.L:\begin{cases}x=\cos t\\y=\sin t\\z=4\end{cases}(起t=0,终t=2\pi)$$

$$2.I=\int_0^{2\pi}[-\sin^2 t-(\cos t+1)\cos t]dt=-\int_0^{2\pi}(1+\cos t)dt=-2\pi$$

$$I=\int_L -ydx+xdy+zdz$$

$$L:\begin{cases}x^2+y^2=1\\ x+y-z=0\end{cases}, \text{从}z\text{轴正向看逆时针}$$

$$1.L:\begin{cases}x=\cos t\\ y=\sin t\\ z=\sin t+\cos t\end{cases}$$

$$(\text{起}t=0, \text{终}t=2\pi)$$

$$2.I=\int_0^{2\pi}\ldots dt$$

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