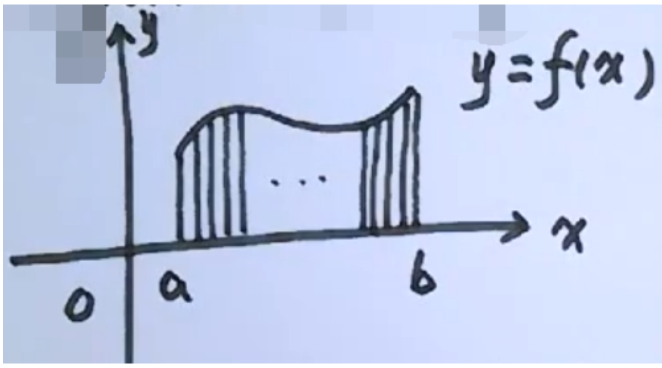
# 定积分理论

## 背景

一元不规则量的计算




$1.a=x_0<x_1<\ldots<x_n=b,[a,b]=[x_0,x_1]\cup[x_1,x_2]\cup\ldots\cup[x_{n-1},x_n]$

$2.\forall\xi_i\in[x_{i-1},x_i],$作$S\approx\sum_{i=1}^{n}f(\xi_i)\Delta x_i$

$3.\lambda=\max\{\Delta x_1,\Delta x_2,\ldots,\Delta x_n\}$

$$S=\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i$$

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$$\begin{aligned}&v=v(t),t\in[a,b],S=?\\1.&a=t_0<t_1<\ldots<t_n=b\\&[a,b]=[t_0,t_1]\cup[t_1,t_2]\cup\ldots\cup[t_{n-1},t_n]\\2.&\forall\xi_i\in[t_{i-1},t_i]\\&S\approx\sum_{i=1}^{n}v(\xi_i)\Delta t_i\\3.&\lambda=\max\{\Delta t_1,\Delta t_2,\ldots,\Delta t_n\}\\&S=\lim_{\lambda\to 0}\sum_{i=1}^{n}v(\xi_i)\Delta t_i\end{aligned}$$

## 定积分定义与一般性质

$$设f(x)在[a,b]上有界$$

$1.a = x_0 < x_1 < \ldots < x_n = b$

$2.\forall \xi_i \in [x_{i-1}, x_i]$, 作 $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$

$3.\lambda = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$

若 $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i \exists$

称$f(x)$在$[a,b]$上可积, 极限值称为$f(x)$在$[a,b]$上的定积分

记 $\int_a^b f(x)dx$, 即 $\int_a^b f(x)dx \triangleq \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$

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1. $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$与$\begin{cases}[a,b]\text{分法} \\ \xi_i\text{取法}\end{cases}$无关

$2.\lambda \to 0 \Rightarrow n \to \infty$

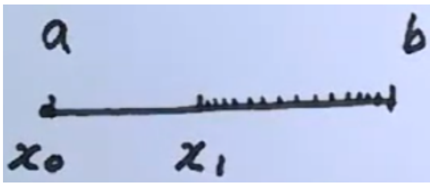
$\lambda \to 0 \not\Leftarrow n \to \infty$

$\Rightarrow, b - a = \Delta x_1 + \ldots + \Delta x_n \le n\lambda$

$\Rightarrow n \ge \frac{b-a}{\lambda} \to \infty (\lambda \to 0)$

$\not\Leftarrow$, 反例

$n \to \infty$, 但$\lambda = \frac{b-a}{2} \nrightarrow 0$



3.$f(x)$有界不一定可积

反例：$f(x)=\begin{cases}1, x\in\mathbb{Q}\\ -1, x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$

$f(x)$在$[a,b]$上有界

$\forall \xi_i\in\mathbb{Q}$：

$$\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i=\lim_{\lambda\to 0}\sum_{i=1}^{n}\Delta x_i=b-a$$

$\forall \xi_i\in\mathbb{R}\setminus\mathbb{Q}$：

$$\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i=-\lim_{\lambda\to 0}\sum_{i=1}^{n}\Delta x_i=-(b-a)$$

$\Rightarrow \lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i$不存在

即$f(x)$在$[a,b]$上不可积

4.若$f(x)\in C[a,b]\Rightarrow f(x)$在$[a,b]$上可积

5.设$f(x)$在$[0,1]$上可积,

$[0,1]=[0,\frac{1}{n}]\cup[\frac{1}{n},\frac{2}{n}]\cup\ldots\cup[\frac{n-1}{n},\frac{n}{n}]$

$(\lambda=\frac{1}{n},\lambda\to 0\Leftrightarrow n\to\infty)$

取$\xi_1=\frac{0}{n},\xi_2=\frac{1}{n},\ldots,\xi_n=\frac{n-1}{n}$,即$\xi_i=\frac{i-1}{n}(1\le i\le n)$

或

取$\xi_1=\frac{1}{n},\xi_2=\frac{2}{n},\ldots,\xi_n=\frac{n}{n}$,即$\xi_i=\frac{i}{n}(1\le i\le n)$

$$\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(\frac{i-1}{n})=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(\frac{i}{n})=\int_0^1 f(x)dx$$

记 $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n} f(\frac{i-1}{n})=\int_0^1 f(x)dx$,或

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(\frac{i}{n})=\int_0^1 f(x)dx$$

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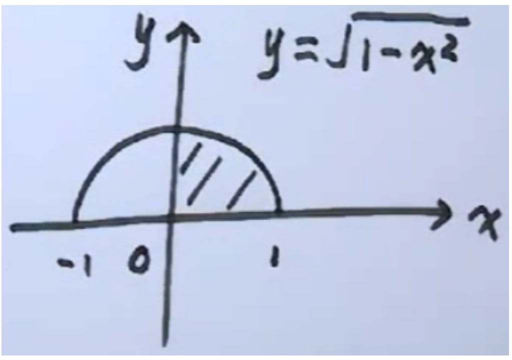
$$\lim_{n\to\infty}\frac{1}{n^2}(\sqrt{n^2-1^2}+\ldots+\sqrt{n^2-n^2})$$
$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{1-(\frac{i}{n})^2}=\int_0^1\sqrt{1-x^2}dx$$
$$=\frac{\pi}{4}$$

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$$\lim_{n\to\infty}(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}})$$

$$\frac{n}{\sqrt{n^2+n}}\le b_n\le\frac{n}{\sqrt{n^2+1}}$$

$$\text{原式}=1$$

---

$$\lim_{n\to\infty}(\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}+\ldots+\frac{1}{\sqrt{n^2+n^2}})$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n^2+i^2}}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{(\frac{i}{n})^2+1}}=\int_0^1\frac{dx}{\sqrt{x^2+1}}=\ln(x+\sqrt{x^2+1})|_0^1=\ln(1+\sqrt{2})$$

---

$$\lim_{n\to\infty}(\frac{\sqrt{n^4-i^4}}{n^4}+\frac{2\sqrt{n^4-2^4}}{n^4}+\ldots+\frac{n\sqrt{(n^4-n^4)}}{n^4})$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i\sqrt{n^4-i^4}}{n^4}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}\sqrt{1-(\frac{i}{n})^4}=\int_0^1x\sqrt{1-x^4}dx$$

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$$\lim_{n\to\infty}(\frac{1}{n^2+1^2}+\frac{2}{n^2+2^2}+\ldots+\frac{n}{n^2+n^2})$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2+i^2}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{i}{n}}{1+(\frac{i}{n})^2}=\int_0^1\frac{x}{1+x^2}dx=\frac{1}{2}\ln 2$$

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$$-|f(x)|\le f(x)\le|f(x)|$$

$$-\int_a^b|f(x)|dx\le\int_a^b f(x)dx\le\int_a^b|f(x)|dx$$

$$-B\le A\le B$$

$$|\int_a^b f(x)dx|\le\int_a^b|f(x)|dx$$

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积分中值定理

$$f(x)\in C[a,b],\exists\xi\in[a,b]$$

$$\int_a^b f(x)dx=f(\xi)(b-a)$$

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$$f(x)\in[a,b]\Rightarrow\exists m,M,\text{使}m\le f(x)\le M$$

$$\int_a^b mdx\le\int_a^b f(x)dx\le\int_a^b Mdx$$

$$m(b-a)\le\frac{1}{b-a}\int_a^b f(x)dx\le M(b-a)$$

$$\exists\xi\in[a,b],\text{使}f(\xi)=\frac{1}{b-a}\int_a^b f(x)dx$$

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$$f(x)\in C[0,1],(0,1)\text{可导},f(1)=4\int_0^{\frac{1}{4}}f(x)dx$$

证 $:\exists\xi\in(0,1)$, 使$f'(\xi)=0$

$$f(x)\in C[0,\frac{1}{4}]\Rightarrow\exists c\in[0,\frac{1}{4}], \text{使}$$

$$\int_0^{\frac{1}{4}}f(x)dx=f(c)(\frac{1}{4}-0)\Rightarrow 4\int_0^{\frac{1}{4}}f(x)dx=f(c)$$

$$\Rightarrow f(c)=f(1)$$

$$\exists\xi\in(c,1)\subset(0,1), \text{使}f'(\xi)=0$$

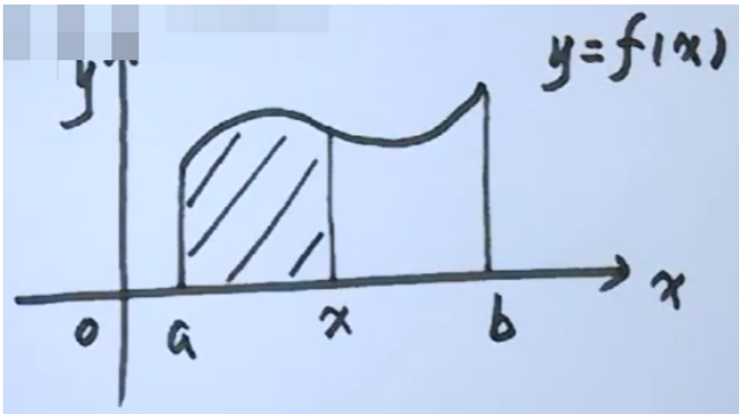
# 定积分基本定理

$$1.\int x^2dx\neq\int t^2dt$$

$$2.\int_0^1x^2dx=\int_0^1t^2dt$$

定积分由上下限和函数关系决定, 与积分变量无关, 即

$$\int_a^bf(x)dx=\int_a^bf(t)dt=\int_a^bf(u)du=\dots$$




设$f(x)\in C[a,b]$

$$\int_a^xf(x)dx=\int_a^xf(t)dt=\Phi(x)$$

积分上限函数

$1.\int_a^xf(x)dx$表达式$x$与上限$x$同否?不同

$$\int_a^xf(x)dx=\int_a^xf(t)dt$$

$2.\int_a^xf(x,t)dt$表达式$x$与上限$x$同否?同

## 定理1

$$\text{设}f(x)\in C[a,b],\Phi(x)=\int_a^xf(t)dt\text{则}$$

$$\Phi'(x)=\frac{d}{dx}\int_a^xf(t)dt=f(x)$$

$$
\begin{aligned}
&1.\text{若} f(x)\in C[a,b]\Rightarrow f(x)\exists\text{原函数},\Phi(x)\text{即}f(x)\text{的一个原函数}\\
&2.\frac{d}{dx}\int_a^x f(t)dt=f(x)\\
&\quad\frac{d}{dx}\int_a^{\Phi(x)} f(t)dt=f[\Phi(x)]\Phi'(x)\\
&\quad\frac{d}{dx}\int_{\Phi_1(x)}^{\Phi_2(x)} f(t)dt=f[\Phi_2(x)]\Phi_2'(x)-f[\Phi_1(x)]\Phi_1'(x)
\end{aligned}
$$

$$
\begin{aligned}
\text{证 }:&\Delta\Phi=\Phi(x+\Delta x)-\Phi(x)=\int_a^{x+\Delta x}f(t)dt-\int_a^x f(t)dt=\int_x^{x+\Delta x}f(t)dt\\
&\because f(x)\in C[a,b],\therefore\Delta\Phi=f(\xi)\Delta x\ (\xi\text{在}x\text{与}x+\Delta x\text{之间})\\
&\Rightarrow\frac{\Delta\Phi}{\Delta x}=f(\xi)\Rightarrow\lim_{\Delta x\to0}\frac{\Delta\Phi}{\Delta x}=\lim_{\Delta x\to0}f(\xi)\\
&\because f(x)\text{连续},\therefore\lim_{\Delta x\to0}f(\xi)=\lim_{\xi\to x}f(\xi)=f(x)\\
&\Rightarrow\Phi'(x)=f(x)
\end{aligned}
$$

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$$
f(x)\text{连续},f(0)=0,f'(0)=\pi,\text{求}\lim_{x\to0}\frac{\int_0^x f(t)dt}{x-\ln(1+x)}
$$

$$
\text{原式}=\lim_{x\to0}\frac{f(x)}{1-\frac{1}{x+1}}=\lim_{x\to0}\frac{(x+1)f(x)}{x}=\lim_{x\to0}\frac{f(x)+(x+1)f'(x)}{1}=f(0)+f'(0)=\pi
$$

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$$
\text{设函数}f(x)\text{连续},\phi(x)=\int_0^x(x-t)f(t)dt,\text{求}\phi''(x)
$$

$$
\begin{aligned}
\phi(x)&=x\int_0^x f(t)dt-\int_0^x tf(t)dt\\
\phi'(x)&=\int_0^x f(t)dt+xf(x)-xf(x)\\
\phi''(x)&=f(x)
\end{aligned}
$$

## 定理2 (N-L)

$$
\int_a^b f(x)dx=\lim_{\lambda\to0}\sum_{i=1}^n f(\xi_i)\Delta x_i
$$

$$
\begin{aligned}
&\text{牛顿莱布尼茨公式}\\
&f(x)\in C[a,b],F(x)\text{为}f(x)\text{的一个原函数},\text{则}\\
&\int_a^b f(x)dx=F(b)-F(a)
\end{aligned}
$$

$$
\begin{aligned}
\text{证 }:&\Phi(x)=\int_a^x f(t)dt,\Phi'(x)=f(x)\\
&\because F'(x)=f(x),\therefore F(x)-\Phi(x)\equiv C_0(a\le x\le b)\\
&\Rightarrow F(a)-\Phi(a)=F(b)-\Phi(b)\\
&\because\Phi(a)=0,\therefore\Phi(b)=F(b)-F(a)\\
&\text{即}\int_a^b f(t)dt=F(b)-F(a)\text{或}\int_a^b f(x)dx=F(b)-F(a)
\end{aligned}
$$

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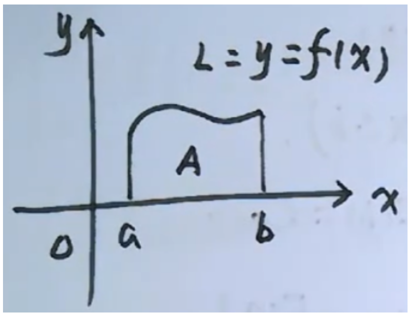
$$
\int_0^1\frac{x}{1+x^4}dx=\frac12\int_0^1\frac{d(x^2)}{1+(x^2)^2}=\frac12\arctan x^2\ |_0^1=\frac{\pi}{8}
$$

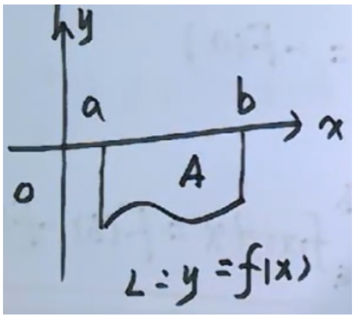
# 定积分基本性质

## 一般性质

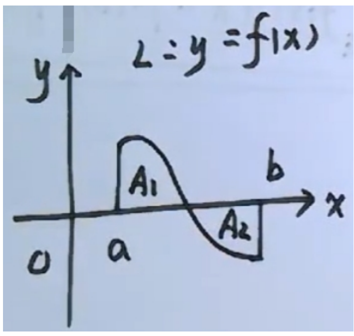
$$1. \int_a^b [kf(x) \pm lg(x)]dx = k\int_a^b f(x)dx \pm l\int_a^b g(x)dx(k, l\text{常数})$$
$$2. \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
$$3. \int_a^b 1dx = b - a$$




$$\int_a^b f(x)dx = A$$




$$\int_a^b f(x)dx = -A$$




$$\int_a^b f(x)dx = A_1 - A_2$$

$$4.①f(x) \geq 0(a \leq x \leq b) \Rightarrow \int_a^b f(x)dx \geq 0$$
$$②f(x) \geq g(x)(a \leq x \leq b) \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$$
③$f(x), |f(x)|$在$[a, b]$上可积, 则
$$|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$$

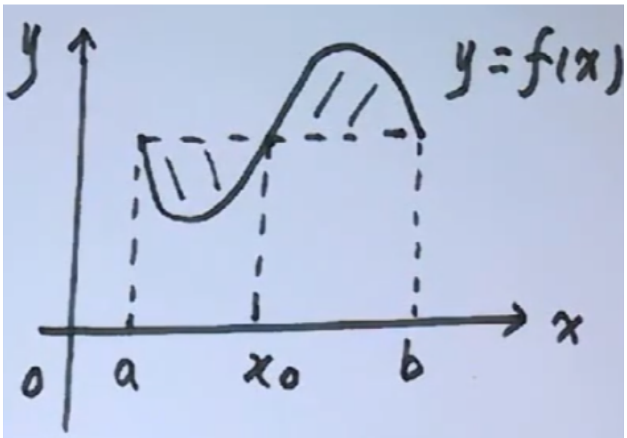
5.$f(x)$在$[a,b]$上可积,且$m \leq f(x) \leq M$,则

$$\int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx, \text{即}$$

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

6.①(积分中值定理)设$f(x) \in C[a,b] \Rightarrow \exists m, M$

则$\exists \xi \in [a,b]$,使$\int_a^b f(x)dx = f(\xi)(b-a)$




$$\int_a^b f(x)dx = f(\xi)(b-a)$$

$$\xi = a, x_0, b$$

证:$f(x) \in C[a,b] \Rightarrow m, M$

$$m \leq f(x) \leq M$$

$$\Rightarrow m \leq \frac{\int_a^b f(x)dx}{b-a} \leq M$$

$\xi \in [a,b]$,使

$$f(\xi) = \frac{\int_a^b f(x)dx}{b-a}$$

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

②(积分中值定理推广)设$f(x) \in C[a,b]$,则$\exists \xi \in (a,b)$,使

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

证:令$F(x) = \int_a^x f(t)dt, F'(x) = f(x)$

$$\int_a^b f(x)dx = F(b) = F(b) - F(a)$$

$$= F'(\xi)(b-a) = f(\xi)(b-a)(a < \xi < b)$$

$f(x) \in C[0,1]$内可导,$f(0) = \int_0^1 f(x)dx$,证:$\exists \xi \in (0,1)$,使$f'(\xi) = 0$

证:令$F(x) = \int_0^x f(t)dt, F'(x) = f(x)$

$$\int_0^1 f(x)dx = F(1) = F(1) - F(0) = F'(c) = f(c)(0 < c < 1)$$

$\because f(0) = f(c), \therefore \exists \xi \in (0,c) \subset (0,1)$,使$f'(\xi) = 0$

# 特殊性质

## 对称区间

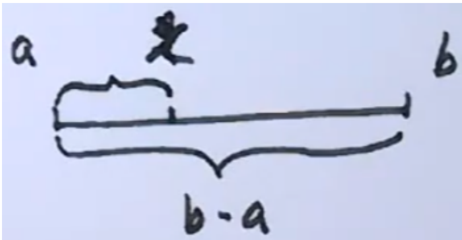
$$设f(x) \in C[-a,a], 则\int_{-a}^{a} f(x)dx = \int_0^a [f(x)+f(-x)]dx$$

$$1. \int_{-a}^{0} f(x)dx = \int_0^a, x=-t$$

$$2. \int_{a}^{a+b} f(x)dx = \int_0^b, x-a=t$$

$$3. \int_{a}^{b} f(x)dx = \int_a^b, x+t=a+b$$

$$4. \int_{a}^{b} f(x)dx = \int_0^1, x=a+(b-a)t$$




$$证:左 = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$$

$$而 \int_{-a}^{0} f(x)dx = \int_a^0 f(-t)(-dt) = \int_0^a f(-t)dt = \int_0^a f(-x)dx$$

$$左 = \int_0^a [f(x)+f(-x)]dx$$

$$若f(-x)=-f(x) \Rightarrow \int_{-a}^{a} f(x)dx = 0$$

$$若f(-x)=f(x) \Rightarrow \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

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$$\begin{aligned}
&\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{e^x+1}dx \\
=&\int_0^{\frac{\pi}{4}} (\frac{1}{e^x+1} + \frac{1}{e^{-x}+1})dx \\
=&\int_0^{\frac{\pi}{4}} (\frac{1}{e^x+1} + \frac{e^x}{e^x+1})dx \\
=&\frac{\pi}{4}
\end{aligned}$$

---

$$
\begin{aligned}
&\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1-\sin x} dx \\
=&\int_{0}^{\frac{\pi}{4}} (\frac{1}{1-\sin x}+\frac{1}{1+\sin x}) dx \\
=&\int_{0}^{\frac{\pi}{4}} \frac{2}{1-\sin^2 x} = 2\int_{0}^{\frac{\pi}{4}} \sec^2 x dx \\
=&2\tan x \mid_{0}^{\frac{\pi}{4}} \\
=&2
\end{aligned}
$$

## 三角函数

$$
设f(x) \in C[0,1],则 \int_{0}^{\frac{\pi}{2}} f(\sin x) dx = \int_{0}^{\frac{\pi}{2}} f(\cos x) dx
$$

$$
\begin{aligned}
&证: x+t=\frac{\pi}{2}, d(x+t)=dx+dt=0 \\
&\quad \int_{0}^{\frac{\pi}{2}} f(\sin x) dx = \int_{\frac{\pi}{2}}^{0} f(\cos t)(-dt) = \int_{0}^{\frac{\pi}{2}} f(\cos t) dt \\
&\quad \int_{0}^{\frac{\pi}{2}} f(\sin x) dx = \int_{0}^{\frac{\pi}{2}} f(\cos x) dx
\end{aligned}
$$

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$$
求 I = \int_{0}^{1} \frac{dx}{x+\sqrt{1-x^2}}
$$

$$
\begin{aligned}
&解: 令 x=\sin t \\
&\quad I = \int_{0}^{\frac{\pi}{2}} \frac{\cos t}{\sin t+\cos t} dt = \int_{0}^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x} dx \\
&\quad \because I = \int_{0}^{\frac{\pi}{2}} \frac{\sin x}{\cos x+\sin x} dx \\
&\quad \therefore 2I = \int_{0}^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}
\end{aligned}
$$

**重点1**

$$
记: \int_{0}^{\frac{\pi}{2}} \sin^n x dx = \int_{0}^{\frac{\pi}{2}} \cos^n x dx = I_n
$$

$$
\begin{cases}
I_n = \frac{n-1}{n} I_{n-2} \\
I_0 = \frac{\pi}{2} \\
I_1 = 1
\end{cases}
$$

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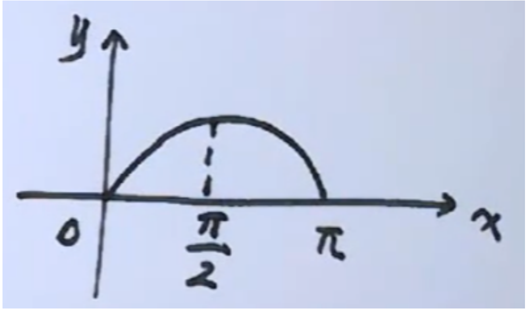
$$
\begin{aligned}
&\int_{0}^{\frac{\pi}{2}} \sin^{11} x = I_{11} = \frac{10}{11} * \frac{8}{9} * \frac{6}{7} * \frac{4}{5} * \frac{2}{3} * I_1 = \frac{10}{11} * \frac{8}{9} * \frac{6}{7} * \frac{4}{5} * \frac{2}{3} * 1 \\
&\int_{0}^{\frac{\pi}{2}} \cos^{10} x = I_{10} = \frac{9}{10} I_8 = \frac{9}{10} * \frac{7}{8} * I_6 = \frac{9}{10} * \frac{5}{6} * \frac{3}{4} * \frac{1}{2} * I_0 = \frac{9}{10} * \frac{5}{6} * \frac{3}{4} * \frac{1}{2} * \frac{\pi}{2}
\end{aligned}
$$

**重点2**

$$
\begin{aligned}
&\int_{0}^{\pi} f(\sin x) dx = 2\int_{0}^{\frac{\pi}{2}} f(\sin x) dx \\
&\int_{0}^{\frac{\pi}{2}} f(\sin x) dx = \int_{\frac{\pi}{2}}^{\pi} f(\sin x) dx
\end{aligned}
$$




$$证:\int_{\frac{\pi}{2}}^{\pi} f(\sin x)dx, x-\frac{\pi}{2}=t$$
$$\int_{0}^{\frac{\pi}{2}} f(\cos t)dt=\int_{0}^{\frac{\pi}{2}} f(\cos x)dx$$
$$=\int_{0}^{\frac{\pi}{2}} f(\sin x)dx$$

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$$I=\int_{0}^{\frac{\pi}{2}} \sin^2 xdx$$
$$I=\int_{0}^{\frac{\pi}{2}} \cos^2 xdx$$
$$2I=\int_{0}^{\frac{\pi}{2}} 1dx=\frac{\pi}{2}$$
$$I=\frac{\pi}{4}$$

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$$I=\int_{-\pi}^{\pi} \frac{\sin^2 x}{1+e^{-x}}dx$$

$$解:I=\int_{0}^{\pi} \frac{\sin^2 x}{1+e^{-x}}+\frac{\sin^2 x}{1+e^{x}}dx$$
$$=\int_{0}^{\pi} (\frac{1}{1+e^{-x}}+\frac{1}{1+e^{x}})\sin^2 xdx$$
$$=\int_{0}^{\pi} \sin^2 xdx$$
$$=2\int_{0}^{\frac{\pi}{2}} \sin^2 xdx$$
$$=2I_2$$
$$=2*\frac{1}{2}*\frac{\pi}{2}$$
$$=\frac{\pi}{2}$$

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$$f,g\in C[-a,a], f(x)+f(-x)\equiv A, g(-x)=g(x),$$
$$1.证:\int_{-a}^{a} f(x)g(x)dx=A\int_{0}^{a} g(x)dx$$
$$2.求 I=\int_{-\pi}^{\pi} \arctan e^x\cdot\sin^2 xdx$$

$$1.\text{证}:\int_{-a}^{a} f(x)g(x)dx=\int_0^a [f(x)g(x)+f(-x)g(-x)]dx$$
$$=\int_0^a [f(x)+f(-x)]g(x)dx=A\int_0^a g(x)dx$$
$$2.\text{解}:I=\int_0^\pi (\arctan e^x+\arctan e^{-x})\cdot \sin^2 xdx$$
$$\because (\arctan e^x+\arctan e^{-x})'=\frac{e^x}{1+e^{2x}}-\frac{e^{-x}}{1+e^{-2x}}=0$$
$$\therefore \arctan e^x+\arctan e^{-x}\equiv A$$
$$x=0\text{时},A=\frac{\pi}{2}$$
$$\therefore I=\frac{\pi}{2}\int_0^\pi \sin^2 xdx=\pi I_2=\frac{\pi}{2}I_0=\frac{\pi^2}{4}$$

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$$\int_0^\pi f(|\cos x|)dx=2\int_0^{\frac{\pi}{2}} f(\cos x)dx$$
$$\int_0^\pi f(\cos^{2n} x)dx=2\int_0^{\frac{\pi}{2}} f(\cos^{2n} x)dx$$

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$$\int_0^\pi \frac{|\cos x|}{1+\sin^2 x}dx$$
$$=2\int_0^{\frac{\pi}{2}} \frac{d(\sin x)}{1+\sin^2 x}$$
$$=2\arctan(\sin x)\Big|_0^{\frac{\pi}{2}}$$
$$=\frac{\pi}{2}$$

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$$\int_0^\pi xf(\sin x)dx=\frac{\pi}{2}\int_0^\pi f(\sin x)dx$$
$$\text{证}:I=\int_0^\pi xf(\sin x)dx,x+t=\pi$$
$$=\int_\pi^0 (\pi-t)f(\sin t)\cdot(-dt)$$
$$=\int_0^\pi (\pi-t)f(\sin t)dt=\int_0^\pi (\pi-x)f(\sin x)dx$$
$$=\pi\int_0^\pi f(\sin x)dx-\int_0^\pi xf(\sin x)dx=\pi\int_0^\pi f(\sin x)dx-I$$
$$\Rightarrow \int_0^\pi xf(\sin x)dx=\frac{\pi}{2}\int_0^\pi f(\sin x)dx$$

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$$\int_0^\pi x\sin^3 xdx$$
$$=\frac{\pi}{2}\int_0^\pi \sin^3 xdx$$
$$=\pi I_3$$
$$=\frac{2\pi}{3}$$

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$$求I=\int_0^{\pi}\sin^2\sqrt{x}dx$$

$$\begin{aligned}&解:令\sqrt{x}=t,x=t^2\\&I=2\int_0^{\pi}t\sin^2 tdt\\&=2*\frac{\pi}{2}\int_0^{\pi}\sin^2 tdt\\&=2\pi\frac{1}{2}\frac{\pi}{2}=\frac{\pi^2}{2}\end{aligned}$$

## 平移性质

$$\begin{aligned}&设f(x)连续且以T>0为周期,则\\&1.\int_a^{a+T}f(x)dx=\int_0^T f(x)dx(平移性质)\\&2.\int_0^{nT}f(x)dx=n\int_0^T f(x)dx\end{aligned}$$

$$\begin{aligned}&证:\int_a^{a+T}f(x)dx=\int_a^0 f(x)dx+\int_0^T f(x)dx+\int_T^{a+T}f(x)dx\\&而\int_T^{a+T}f(x)dx,x-T=t,\int_0^a f(t)dt=\int_0^a f(x)dx\\&\therefore\int_a^{a+T}f(x)dx=\int_0^T f(x)dx\end{aligned}$$

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$$\begin{aligned}&\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\sin^4 xdx\\&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^4 xdx\\&=2\int_0^{\frac{\pi}{2}}\sin^4 xdx\\&=2*\frac{3}{4}*\frac{1}{2}*\frac{\pi}{2}=\frac{3}{8}\pi\end{aligned}$$

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$$I=\int_0^{\pi}|\sin x+\cos x|dx$$

$$\begin{aligned}&解:(法一:I=\int_0^{\frac{\pi}{2}})\\&法一:I=\int_0^{\frac{3\pi}{4}}(\sin x+\cos x)dx-\int_{\frac{3\pi}{4}}^{\pi}(\sin x+\cos x)dx\\&法二:I=\sqrt{2}\int_0^{\pi}|\sin(x+\frac{\pi}{4})|d(x+\frac{\pi}{4})\\&=\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}|\sin x|dx\\&=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|\sin x|dx\\&=2\sqrt{2}\int_0^{\frac{\pi}{2}}\sin xdx=2\sqrt{2}\end{aligned}$$

# 补充

$$f(x)\in C[a,b],\Phi(x)=\int_a^x f(t)dt,\text{则}\Phi'(x)=f(x)$$

$$\text{证}:\Delta\Phi=\Phi(x+\Delta x)-\Phi(x)=\int_a^{x+\Delta x} f(t)dt-\int_a^x f(t)dt$$

$$=\int_x^{x+\Delta x} f(t)dt=f(\xi)\Delta x(\xi\text{在}x\text{与}x+\Delta x\text{之间})$$

$$\Rightarrow\frac{\Delta\Phi}{\Delta x}=f(\xi)\Rightarrow\lim_{\Delta x\to 0}\frac{\Delta\Phi}{\Delta x}=\lim_{\Delta x\to 0}f(\xi)$$

$$\text{即}\Phi'(x)=f(x)$$