导数的应用

单调性

$$y = f(x)(x \in D)$$

$$\begin{aligned}&1.\ 若\forall x_1, x_2 \in D且x_1 < x_2, 有f(x_1) < f(x_2)\\ &\quad 称f(x)在D上为严格单调递增函数\\ &2. \dots\end{aligned}$$

$$f(x) \in C[a,b], (a,b)内可导$$

$$\begin{aligned}&①f'(x) > 0(a < x < b) \Rightarrow f(x)在[a,b]\uparrow\\ &②f'(x) < 0(a < x < b) \Rightarrow f(x)在[a,b]\downarrow\end{aligned}$$

$$\begin{gathered}证：设f'(x) > 0(a < x < b)\\ \forall x_1, x_2 \in [a,b]且x_1 < x_2\\ f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) > 0(x_1 < \xi < x_2)\\ \Rightarrow f(x_1) < f(x_2) \Rightarrow f(x)在[a,b]上\uparrow\end{gathered}$$

$$\begin{gathered}求单调区间步骤\\ y = f(x)\end{gathered}$$

$$\begin{aligned}&①x \in D\\ &②f'(x)\begin{cases}= 0\\ 不存在\end{cases} \Rightarrow x_1, \dots, x_n\\ &③在每个小区间内判断f'正负, 可得单调区间\end{aligned}$$

极值

$$y = f(x)$$

①$x \in D$

②$f'(x)\begin{cases}=0(\text{驻点})\\ \text{不存在}\end{cases} \Rightarrow x=\ldots$

③判断：

第一充分条件

1. $\begin{cases}x<x_0: f'(x)<0\\ x>x_0: f'(x)>0\end{cases} \Rightarrow x=x_0$为极小点

2. $\begin{cases}x<x_0: f'(x)>0\\ x>x_0: f'(x)<0\end{cases} \Rightarrow x=x_0$为极大点

第二充分条件

设$f'(x_0)=0, f''(x_0)\begin{cases}>0 \Rightarrow x_0\text{为极小点}\\ <0 \Rightarrow x_0\text{为极大点}\end{cases}$

$$f''(x_0)>0: \lim_{x\to x_0}\frac{f'(x)}{x-x_0}>0$$

$$\exists\delta>0, \text{当}0<|x-x_0|<\delta\text{时}, \frac{f'(x)}{x-x_0}>0$$

$$\Rightarrow \begin{cases}f'(x)<0, x\in(x_0-\delta, x_0)\\ f'(x)>0, x\in(x_0, x_0+\delta)\end{cases} \Rightarrow x=x_0\text{为极小点}$$

$$f''(x_0)<0: \lim_{x\to x_0}\frac{f'(x)}{x-x_0}<0$$

$$\exists\delta>0, \text{当}0<|x-x_0|<\delta\text{时}, \frac{f'(x)}{x-x_0}<0$$

$$\Rightarrow \begin{cases}f'(x)>0, x\in(x_0-\delta, x_0)\\ f'(x)<0, x\in(x_0, x_0+\delta)\end{cases} \Rightarrow x=x_0\text{为极大点}$$

求函数$f(x)=x^2e^{-x}$的极值点与极值.

①$x\in(-\infty,+\infty)$

②$f'(x)=(2x-x^2)e^{-x}=0 \Rightarrow x=0, x=2$

③$\begin{cases}x<0: f'(x)<0\\ 0<x<2: f'(x)>0\end{cases} \Rightarrow x=0$为极小点, 极小值$f(0)=0$

$\begin{cases}0<x<2: f'(x)<0\\ x>2: f'(x)<0\end{cases} \Rightarrow x=2$为极大点, 极大值$f(2)=\frac{4}{e^2}$

设$f(x)=\ln x-\frac{x}{e}$, 求函数$f(x)$的极值点与极值.

①$x\in(0,+\infty)$

②$f'(x)=\frac{1}{x}-\frac{1}{e} \Rightarrow x=e$

$f''(x)=-\frac{1}{x^2}$

$\because f''(e)=-\frac{1}{e^2}<0, \therefore x=e$为最大点, $M=f(e)=0$

$$f(x)\text{可导}, f'(1)=0, \lim_{x\to1}\frac{f'(x)}{\sin^3\pi x}=2, x=1?$$

$$\exists\delta>0, \text{当}0<|x-1|<\delta\text{时}, \frac{f'(x)}{\sin^3\pi x}>0$$

$$\begin{cases}f'(x)>0, x\in(1-\delta, 1)\\ f'(x)<0, x\in(1, 1+\delta)\end{cases} \Rightarrow x=1\text{为极大点}$$

$$证:x>0时,\frac{x}{1+x}<\ln(1+x)<x$$

$$证:1.\ f(x)=x-\ln(1+x),f(0)=0$$

$$f'(x)=1-\frac{1}{1+x}>0(x>0)\Rightarrow f(x)在[0,+\infty)\uparrow$$

$$\because\begin{cases}f(0)=0\\f'(x)>0(x>0)\end{cases}\therefore f(x)>0(x>0)$$

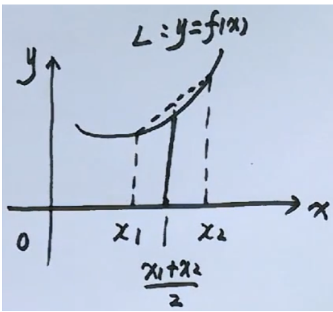
$$2.\ g(x)=\ln(1+x)-\frac{x}{1+x},g(0)=0$$

$$g'(x)=\frac{1}{1+x}-\frac{1}{(1+x)^2}=\frac{x}{(1+x)^2}>0(x>0)$$

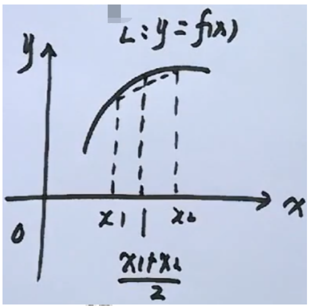
$$\begin{cases}f(0)=0\\g'(x)>0(x>0)\end{cases}\Rightarrow g(x)>0(x>0)$$

凹凸性

$$y=f(x)(x\in D)$$



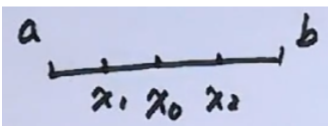
1. 若$\forall x_1,x_2\in D$且$x_1\neq x_2$,有
   $f(\frac{x_1+x_2}{2})<\frac{f(x_1)+f(x_2)}{2}$
   称$f(x)$在D上为下凹函数
2. 若$\forall x_1,x_2\in D$且$x_1\neq x_2$,有
   $f(\frac{x_1+x_2}{2})>\frac{f(x_1)+f(x_2)}{2}$
   称$f(x)$在D上为上凸函数



$$y=f(x)(x\in D),x_0\in D,$$

$$若y=f(x)在x=x_0左、右凹凸性不同,称(x_0,f(x_0))为y=f(x)的拐点$$

$f(x)\in C[a,b],(a,b)$内二阶可导
①若$f''(x)>0(a<x<b)\Rightarrow f(x)$在$[a,b]$上凹
②若$f''(x)<0(a<x<b)\Rightarrow f(x)$在$[a,b]$上凸



$$\text{证}: \forall x_1, x_2 \in [a,b], \text{且} x_1 \neq x_2, \frac{x_1+x_2}{2} \triangleq x_0$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi)}{2!}(x-x_0)^2 (\xi\text{在}x_0\text{与}x\text{之间})$$

$$\because f''(x) > 0 (a < x < b)$$

$$\therefore \frac{f''(\xi)}{2!}(x-x_0)^2 \geq 0$$

$$\Rightarrow f(x) \geq f(x_0) + f'(x_0)(x-x_0)$$

$$= \text{成立} \Leftrightarrow x = x_0$$

$$\because x_1 \neq x_0, x_2 \neq x_0$$

$$\therefore \begin{cases} \frac{1}{2}f(x_1) > \frac{1}{2}f(x_0) + \frac{1}{2}f'(x_0)(x_1-x_0) \\ \frac{1}{2}f(x_2) > \frac{1}{2}f(x_0) + \frac{1}{2}f'(x_0)(x_2-x_0) \end{cases}$$

$$\Rightarrow \frac{f(x_1)+f(x_2)}{2} > f(x_0) + f'(x_0)(\frac{x_1+x_2}{2} - x_0)$$

$$\Rightarrow f(\frac{x_1+x_2}{2}) < \frac{f(x_1)+f(x_2)}{2}$$

$$\therefore f(x)\text{在}[a,b]\text{上凹}$$

$$\text{设曲线}L: y = f(x) = e^{-(x-1)^2}, \text{求曲线的凹凸区间及拐点.}$$

①$x \in (-\infty, +\infty)$

②$f'(x) = -2(x-1)e^{-(x-1)^2}$

$f''(x) = 4e^{-(x-1)^2}[(x-1)^2 - \frac{1}{2}] = 0$

$x = 1 - \frac{1}{\sqrt{2}}, x = 1 + \frac{1}{\sqrt{2}}$

$x \in (-\infty, 1-\frac{1}{\sqrt{2}})$时, $f'' > 0$, $x \in (1-\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}})$时, $f'' < 0$

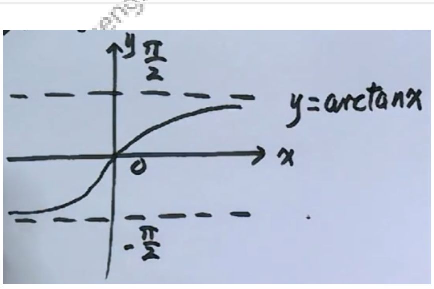
$x \in (1+\frac{1}{\sqrt{2}}, +\infty)$时, $f'' > 0$

$f(1-\frac{1}{\sqrt{2}}) = e^{-\frac{1}{2}}, f(1+\frac{1}{\sqrt{2}}) = e^{-\frac{1}{2}}$

$\therefore (-\infty, 1-\frac{1}{\sqrt{2}}]$与$[1+\frac{1}{\sqrt{2}}, +\infty)$为凹区间, $[1-\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}]$为凸区间

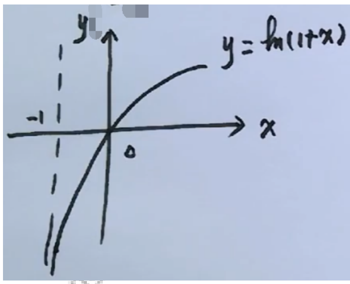
$(1-\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}), (1+\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}})$为拐点

渐近线

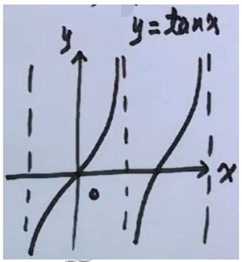


$$\arctan x \to -\frac{\pi}{2} (x \to -\infty)$$

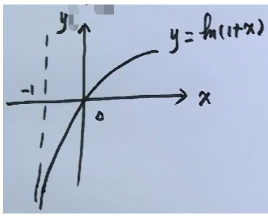
$$\arctan x \to \frac{\pi}{2} (x \to +\infty)$$



$$\ln(1+x) \to -\infty (x \to -1^{+})$$



$$\tan x \to \infty (x \to \frac{\pi}{2})$$



水平渐近线

$$L: y = f(x), \text{若} \lim_{x \to \infty} f(x) = A, y = A\text{称为水平渐近线}$$

铅直渐近线

$$L: y = f(x)$$

$$\text{若}\begin{cases} f(a-0) = \infty \\ f(a+0) = \infty \\ \lim_{x \to a} f(x) = \infty \end{cases}, \text{称}x = a\text{为铅直渐近线}$$

斜渐近线

$$L: y = f(x)$$

$$\text{若} \lim_{x \to \infty} \frac{f(x)}{x} = a(\neq 0, \infty), \lim_{x \to \infty} [f(x) - ax] = b$$

$$y = ax + b\text{为斜渐近线}$$

$$\text{求曲线}y = \frac{x^2 - 3x + 2}{x^2 - 1} e^{\frac{1}{x}}\text{的水平渐近线与铅直渐近线.}$$

$$\because \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-3x+2}{x^2-1} e^{\frac{1}{x}} = 1$$

$$\therefore y = 1\text{为水平渐近线}$$

$$\because \lim_{x\to -1} f(x) = \infty, \therefore x = -1\text{为铅直渐近线}$$

$$\because f(0+0) = -\infty, \therefore x = 0\text{为铅直渐近线}$$

$$\because \lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x-2}{x+1} e^{\frac{1}{x}} = -\frac{e}{2} \neq \infty$$

$$\therefore x = 1\text{不是铅直渐近线}$$

$$\text{求曲线}y = \frac{2x^2-x+3}{x+1}\text{的斜渐近线.}$$

$$\because \lim_{x\to\infty} f(x) = \infty, \therefore \text{无水平渐近线}$$

$$\therefore \lim_{x\to -1} f(x) = \infty, \therefore x = -1\text{为铅直渐近线}$$

$$\because \lim_{x\to\infty} \frac{f(x)}{x} = 2$$

$$\lim_{x\to\infty} [f(x) - 2x] = \lim_{x\to\infty} \frac{-3x+3}{x+1} = -3$$

$$\therefore y = 2x - 3\text{为斜渐近线}$$

$$L: y = f(x) = \sqrt{x^2+4x+8} - x$$

1. $\lim\limits_{x\to -\infty} f(x) = +\infty$

   $$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{4x+8}{\sqrt{x^2+4x+8}+x} = 2$$

   $\Rightarrow y = 2$为水平渐近线

2. $\because y = f(x)$处处连续 $\therefore$ 无铅直渐近线

3. $$\lim_{x\to -\infty} \frac{f(x)}{x} = \lim_{x\to -\infty} \frac{\sqrt{x^2+4x+8}-x}{x} = -2$$

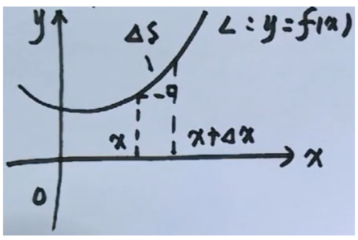
   $$\lim_{x\to -\infty} [f(x)+2x] = \lim_{x\to -\infty} (\sqrt{x^2+4x+8}+x)$$

   $$= \lim_{x\to -\infty} \frac{4x+8}{\sqrt{x^2+4x+8}-x} = -2$$

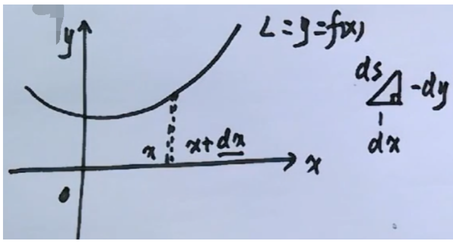
   $\therefore y = -2x - 2$为斜渐近线

曲率、曲率半径

弧微分（弧元素）



$$(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2$$



$$(ds)^2 = (dx)^2 + (dy)^2$$

直角坐标

$$L: y = f(x)$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2} dx$$

$$= \sqrt{1 + f'^2(x)} dx$$

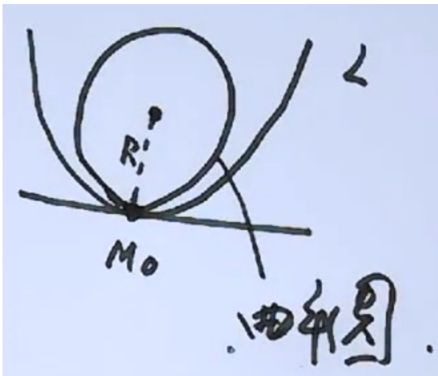
参数形式

$$L: \begin{cases} x = \Phi(t) \\ y = \phi(t) \end{cases}$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

$$= \sqrt{\Phi'^2(t) + \phi^2(t)} dt$$

曲率、曲率半径



$$L: y = f(x), M_0(x_0, y_0) \in L$$

$$k = \frac{|y''|}{(1 + y'^2)^{\frac{3}{2}}}$$

$$R = \frac{1}{k}$$