中值定理

型一

$$f^{(n)}(\xi)=0$$

$$1.\ n=1: f(a)=f(b)$$

$$\begin{aligned}&f(x)\in C[0,2],(0,2)\text{内可导，}f(0)=1,f(1)+2f(2)=3\\&\text{证}:\exists\xi\in(0,2),\text{使}f'(\xi)=0\end{aligned}$$

$$\begin{aligned}\text{证}:&f(x)\in C[1,2]\Rightarrow\exists m,M\\&\because 3m\le f(1)+2f(2)\le 3M,\therefore m\le 1\le M\\&\therefore\exists c\in[1,2],\text{使}f(c)=1\\&\because f(0)=f(c)=1,\therefore\exists\xi\in(0,c)\subset(0,2),\text{使}f'(\xi)=0\end{aligned}$$

$$\begin{aligned}&f(x)\in C[0,1],(0,1)\text{内可导，}f(0)=-1,f(\frac{1}{2})=1,f(1)=\frac{1}{2}\\&\text{证}:\exists\xi\in(0,1),\text{使}f'(\xi)=0\end{aligned}$$

$$\begin{aligned}\text{证}:&\text{令}h(x)=f(x)-\frac{1}{2}\in C[0,\frac{1}{2}]\\&h(0)=-\frac{3}{2},h(\frac{1}{2})=\frac{1}{2}\\&\because h(0)h(\frac{1}{2})<0,\therefore c\in(0,\frac{1}{2}),\text{使}h(c)=0\\&\Rightarrow f(c)=\frac{1}{2}\\&\because f(c)=f(1)=\frac{1}{2}\\&\therefore\exists\xi\in(c,1)\subset(0,1),\text{使}f'(\xi)=0\end{aligned}$$

$$2.\ n=2\begin{cases}f(a),f(c),f(b)\\f'(\xi_1)=f'(\xi_2)\end{cases}$$

$$\begin{aligned}&f(x)\in C[0,5],(0,5)\text{内二阶可导}\\&3f(0)=f(1)+2f(2)=f(3)+f(4)+f(5)\\&\text{证}:\exists\xi\in(0,5),\text{使}f''(\xi)=0\end{aligned}$$

$$
\begin{aligned}
&证: 1.\ f(x)\in C[1,2]\Rightarrow \exists m, M\\
&\quad \because m\le \frac{f(1)+2f(2)}{3}\le M\\
&\quad \therefore \exists x_1\in[1,2],使f(1)+2f(2)=3f(x_1)\\
&\ \ 2.\ f(x)\in C[3,5]\Rightarrow \exists m, M\\
&\quad m\le \frac{f(3)+f(4)+f(5)}{3}\le M\\
&\quad \exists x_2\in[3,5],使f(3)+f(4)+f(5)=3f(x_2)\\
&\ \ 3.\ \because f(0)=f(x_1)=f(x_2)\\
&\quad \therefore \exists \xi_1\in(0,x_1),\xi_2\in(x_1,x_2),使\\
&\quad f'(\xi_1)=f'(\xi_2)=0\\
&\ \ 4.\ \exists \xi\in(\xi_1,\xi_2)\subset(0,5),使\\
&\quad f''(\xi)=0
\end{aligned}
$$

$$
\begin{aligned}
&f(x)二阶可导,\lim_{x\to0}\frac{f(x)-1}{x}=1,f(1)=2\\
&证:\exists\xi\in(0,1),使f''(\xi)=0
\end{aligned}
$$

$$
\begin{aligned}
&证:\lim_{x\to0}\frac{f(x)-1}{x}=1\Rightarrow f(0)=1,f'(0)=1\\
&\quad \exists c\in(0,1),使f'(c)=\frac{f(1)-f(0)}{1-0}=1\\
&\quad \because f'(0)=f'(c)=1\\
&\quad \exists\xi\in(0,c)\subset(0,1),使\\
&\quad f''(\xi)=0
\end{aligned}
$$

型二

仅有ξ, 无其他字母

$$
1.\ \begin{cases}两项\\导数差一阶\end{cases}-还原法
$$

$$
工具:\frac{f'}{f}=(\ln f)',\frac{f''}{f'}=(\ln f')'
$$

$$
\begin{aligned}
&f(x)\in C[0,1],(0,1)内可导,f(1)=0\\
&证:\exists\xi\in(0,1),使\xi f'(\xi)+3f(\xi)=0
\end{aligned}
$$

$$
\begin{aligned}
&分析: xf'(x)+3f(x)=0\Rightarrow\frac{f'(x)}{f(x)}+\frac{3}{x}=0\\
&\quad\Rightarrow[\ln f(x)]'+(\ln x^3)'=0\\
&证:令\Phi(c)=x^3f(x)\\
&\quad\because f(1)=0,\therefore\Phi(0)=\Phi(1)=0\\
&\quad\therefore\exists\xi\in(0,1),使\Phi'(\xi)=0\\
&\quad而\Phi'(x)=3x^2f(x)+x^3f'(x)\\
&\quad\therefore 3\xi^2f(\xi)+\xi^3f'(\xi)=0\\
&\quad\because\xi\ne0,\therefore\xi f'(\xi)+3f(\xi)=0
\end{aligned}
$$

$$
\begin{aligned}
&f(x)\in C[a,b],(a,b)内可导,f(a)=f(b)=0\\
&证:\exists\xi\in(a,b),使f'(\xi)=2f(\xi)
\end{aligned}
$$

$$
\begin{aligned}
&\text{分析}: f'(x)-2f(x)=0 \Rightarrow \frac{f'(x)}{f(x)}-2=0\\
&\quad \Rightarrow [\ln f(x)]'+(\ln e^{-2x})'=0\\
&\text{证}: \text{令}\Phi(x)=e^{-2x}f(x)\\
&\quad \because f(a)=f(b)=0, \therefore \Phi(a)=\Phi(b)=0\\
&\quad \therefore \exists \xi \in (a,b), \text{使}\Phi'(\xi)=0\\
&\quad \text{而}\Phi'(x)=e^{-2x}[f'(x)-2f(x)]\text{且}e^{-2x}\neq 0\\
&\quad \therefore f'(\xi)=2f(\xi)
\end{aligned}
$$

$$
\begin{aligned}
&f(x)\text{二阶可导}, f(0)=f(1), \text{证}: \exists \xi \in (0,1), \text{使}\\
&f''(\xi)=\frac{4}{1-\xi}f'(\xi)
\end{aligned}
$$

$$
\begin{aligned}
&\text{分析}: \frac{f''(x)}{f'(x)}+\frac{4}{x-1}=0 \Rightarrow [\ln f'(x)]'+[\ln(x-1)^4]'=0\\
&\text{证}: \text{令}\Phi(x)=(x-1)^4 f'(x)\\
&\quad \Phi(1)=0\\
&\quad \because f(0)=f(1), \therefore \exists c \in (0,1), \text{使}f'(c)=0\\
&\quad \therefore \Phi(c)=0\\
&\quad \Phi(c)=\Phi(1)=0, \therefore \exists \xi \in (c,1) \subset (0,1), \text{使}\Phi'(\xi)=0\\
&\quad \text{而}\Phi'(x)=4(x-1)^3 f'(x)+(x-1)^4 f''(x)\\
&\quad \therefore 4(\xi-1)^3 f'(\xi)+(\xi-1)^4 f''(\xi)\\
&\quad \text{而}\xi \neq 1, \therefore f''(\xi)+\frac{4}{\xi-1}f'(\xi)=0
\end{aligned}
$$

$$
2. \begin{cases} \text{多余两项或} \\ \text{导数差不是一阶} \end{cases} - \text{分组法}
$$

$$
\begin{aligned}
&f(x) \in C[0,1], (0,1)\text{内可导}, f(0)=0, f(\frac{1}{2})=1, f(1)=\frac{1}{2}\\
&\text{证}: \text{①}\exists c \in (0,1), \text{使}f(c)=c\\
&\quad \text{②}\exists \xi \in (0,1), \text{使}f'(\xi)+2f(\xi)=1+2\xi
\end{aligned}
$$

$$
\begin{aligned}
&\text{证}: \text{①令}h(x)=f(x)-x\\
&\quad h(\frac{1}{2})=\frac{1}{2}, h(1)=-\frac{1}{2} \ \because h(\frac{1}{2})h(1)<0 \ \therefore \exists c \in (\frac{1}{2},1) \subset (0,1), \text{使}\\
&\quad h(c)=0 \Rightarrow f(c)=c\\
&\text{分析}: \text{②}f'(x)-1+2f(x)-2x=0\\
&\quad [f(x)-x]'+2[f(x)-x]=0, \text{即}h'+2h=0\\
&\quad \frac{h'}{h}+2=0 \Rightarrow [\ln h(x)]'+[\ln e^{2x}]'=0\\
&\text{证}: \text{②令}\Phi(x)=e^{2x}[f(x)-x]\\
&\quad \because f(0)=0, f(c)=c\\
&\quad \therefore \Phi(0)=\Phi(c)=0, \therefore \exists \xi \in (0,c) \subset (0,1), \text{使}\Phi'(\xi)=0\\
&\quad \text{而}\Phi'(x)=e^{2x}[f'(x)-1+2f(x)-2x]\text{且}e^{2x}\neq 0\\
&\quad \therefore f'(\xi)+2f(\xi)=1+2\xi
\end{aligned}
$$

型三

$$有\xi, 有a, b$$

$$1.\ \xi与a, b可分离$$

$$1.\ \xi与a, b分离$$

$$2.\ 法一：a, b侧\begin{cases}\frac{f(b)-f(a)}{b-a}-L\\ \frac{f(b)-f(a)}{g(b)-g(a)-C}\end{cases}$$

$$法二：\xi侧\begin{cases}(\ )'-L\\ \frac{(\ )'}{(\ )'}-C\end{cases}$$

$$0<a<b, 证：\exists\xi\in(a,b), 使ae^b-be^a=(a-b)(1-\xi)e^\xi$$

$$分析：\frac{ae^b-be^a}{a-b}=(1-\xi)e^\xi$$

$$法一：\frac{ae^b-be^a}{a-b}=\frac{\frac{e^b}{b}-\frac{e^a}{a}}{\frac{1}{b}-\frac{1}{a}}$$

$$f(x)=\frac{e^x}{x}, g(x)=\frac{1}{x}$$

$$法二：(1-x)e^x=\frac{\frac{(x-1)e^x}{x^2}}{-\frac{1}{x^2}}=\frac{(\frac{e^x}{x})'}{(\frac{1}{x})'}$$

$$f(x)=\frac{e^x}{x}, g(x)=\frac{1}{x}$$

$$证：令f(x)=\frac{e^x}{x}, g(x)=\frac{1}{x}, g'(x)=-\frac{1}{x^2}\neq 0(a<x<b)$$

$$\exists\xi\in(a,b), 使\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$$

$$f(x)\in C[1,2], (1,2)内可导, 证：\exists\xi\in(1,2), 使f(2)-2f(1)=\xi f'(\xi)-f(\xi)$$

$$分析：法一：f(2)-2f(1)=\frac{f(2)-2f(1)}{1}=\frac{\frac{f(2)}{2}-\frac{f(1)}{1}}{-\frac{1}{2}-(-\frac{1}{1})}$$

$$\frac{f(x)}{x}, -\frac{1}{x}$$

$$法二：xf'(x)-f(x)=\frac{xf'(x)-f(x)}{1}$$

$$=\frac{\frac{xf'(x)-f(x)}{x^2}}{\frac{1}{x^2}}=\frac{[\frac{f(x)}{x}]'}{(-\frac{1}{x})'}$$

$$证：令F(x)=\frac{f(x)}{x}, G(x)=-\frac{1}{x}, G'(x)=\frac{1}{x^2}\neq 0(1<x<2)$$

$$\exists\xi\in(1,2), 使\frac{F(2)-F(1)}{G(2)-G(1)}=\frac{F'(\xi)}{G'(\xi)}$$

$$\Rightarrow\frac{\frac{f(2)}{2}-\frac{f(1)}{1}}{\frac{1}{2}}=\frac{\xi f'(\xi)-f(\xi)}{\xi^2}/\frac{1}{\xi^2}$$

$$\Rightarrow f(2)-2f(1)=\xi f'(\xi)-f(\xi)$$

$$2.\ \xi\text{与}a,b\text{不可分}$$

$$\begin{cases}\xi\to x\\ \text{去分母, 移项}\end{cases}\Rightarrow \text{式子}=0\Rightarrow(\Phi(x))'=0$$

$$\text{如}:f''g+f'g'=(f'g)'$$

$$f,g\in C[a,b],(a,b)\text{内可导},g'(x)\neq 0(a<x<b)$$
$$\text{证}:\exists\xi\in(a,b),\text{使}\frac{f(\xi)-f(a)}{g(b)-g(\xi)}=\frac{f'(\xi)}{g'(\xi)}$$

$$\begin{aligned}\text{分析}:&f(x)g'(x)-f(a)g'(x)-f'(x)g(b)+f'(x)g(x)=0\\&[f(x)g(x)-f(a)g(x)-f(x)g(b)]'=0\\ \text{证}:&\text{令}\Phi(x)=f(x)g(x)-f(a)g(x)-f(x)g(b)\\&\Phi(a)=-f(a)g(b),\Phi(b)=-f(a)g(b)\\&\because\Phi(a)=\Phi(b)\ \therefore\xi\in(a,b),\text{使}\Phi'(\xi)=0\\&\Rightarrow[f(\xi)-f(a)]g'(\xi)-f'(\xi)[g(b)-g(\xi)]=0\\&\because g'(\xi)\neq0,g(b)-g(\xi)\neq0\end{aligned}$$

型四

$$\text{有}\xi,\eta$$

$$1.\ \text{仅有}f'(\xi),f'(\eta)\begin{cases}\text{找三点}\\2L\end{cases}$$

$$f(x)\in C[0,1],(0,1)\text{内可导},f(0)=0,f(1)=1$$
$$\text{证}:①\exists c\in(0,1),\text{使}f(c)=\frac{1}{2}$$
$$②\exists\xi,\eta\in(0,1),\text{使}\frac{1}{f'(\xi)}+\frac{1}{f'(\eta)}=2$$

$$\begin{aligned}\text{证}:&①\text{令}h(x)=f(x)-\frac{1}{2}\\&h(0)=-\frac{1}{2},h(1)=\frac{1}{2},\because h(0)h(1)<0,\therefore\exists c\in(0,1),\text{使}\\&h(c)=0\Rightarrow f(c)=\frac{1}{2}\\&②\exists\xi\in(0,c),\eta\in(c,1),\text{使}\\&f'(\xi)=\frac{f(c)-f(0)}{c-0}=\frac{1}{2c}\\&f'(\eta)=\frac{f(1)-f(c)}{1-c}=\frac{1}{2(1-c)}\\&\Rightarrow\frac{1}{f'(\xi)}=2c,\frac{1}{f'(\eta)}=2(1-c)\end{aligned}$$

$$2.\ \xi,\eta\text{对应的项复杂度不同}$$

$$\text{留复杂中值项}\Rightarrow\begin{cases}(\ )'-L\\ \frac{(\ )'}{(\ )'}-C\end{cases}$$

$$\text{如}:e^{2\xi}[f'(\xi)+2f(\xi)]=[e^{2x}f(x)]',e^{\xi}f'(\xi)=\frac{f'(\xi)}{e^{-\xi}},\frac{f(x)}{-e^{-x}}$$

$$f(x) \in C[a,b], (a,b)\text{内可导}(a>0)$$
$$\text{证}:\exists \xi,\eta \in (a,b),\text{使}f'(\xi)=(a+b)\frac{f'(\eta)}{2\eta}$$

$$\text{分析}:\frac{f'(\eta)}{2\eta},\frac{f(x)}{x^2}$$

$$\begin{aligned}\text{证}:&\text{令}g(x)=x^2,g'(x)=2x\neq 0(a<x<b)\\ &\exists \eta\in(a,b),\text{使}\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\eta)}{g'(\eta)}\\ &\Rightarrow \frac{f(b)-f(a)}{b^2-a^2}=\frac{f'(\eta)}{2\eta}\Rightarrow\frac{f(b)-f(a)}{b-a}=(a+b)\frac{f(\eta)}{2\eta}\\ &\exists \xi\in(a,b),\text{使}f'(\xi)=\frac{f(b)-f(a)}{b-a}\end{aligned}$$

$$\begin{aligned}&0<a<b,\text{证}:\exists\xi,\eta\in(a,b),\text{使}\\ &abf'(\xi)=\eta^2f'(\eta)\end{aligned}$$

$$\text{分析}:\eta^2f'(\eta)=\frac{f'(\eta)}{\frac{1}{\eta^2}},\frac{f(x)}{-\frac{1}{x}}$$

$$\begin{aligned}&\text{证}:\text{令}g(x)=-\frac{1}{x},g'(x)=\frac{1}{x^2}\neq 0(a<x<b)\\ &\exists\eta\in(a,b),\text{使}\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\eta)}{g'(\eta)}\\ &\Rightarrow\frac{f(b)-f(a)}{\frac{1}{a}-\frac{1}{b}}=\frac{f'(\eta)}{\frac{1}{\eta^2}}\Rightarrow ab\frac{f(b)-f(a)}{b-a}=\eta^2f'(\eta)\\ &\exists\xi\in(a,b),\text{使}f'(\xi)=\frac{f(b)-f(a)}{b-a}\end{aligned}$$

型五

$$\begin{aligned}&①f(b)-f(a),\frac{f(b)-f(a)}{b-a},f(a)\neq f(b)-L\\ &②f(a),f(c),f(b)\text{或}f'(a),f'(c),f'(b)-2L\end{aligned}$$

$$\lim_{x\to\infty}f'(x)=e,\lim_{x\to\infty}[f(x+2)-f(x)]=\lim_{x\to\infty}(\frac{x+a}{x-a})^x$$
$$\text{求}a.$$

$$\begin{aligned}\text{解}:&f(x+2)-f(x)=2f'(\xi)(x<\xi<x+2)\\ &\text{左}=2\lim_{x\to\infty}f'(\xi)=2e\\ &\text{右}=\lim_{x\to\infty}[(1+\frac{2a}{x-a})^{\frac{x-a}{2a}}]^{x\cdot\frac{2a}{x-a}}=e^{2x}\\ &\Rightarrow e^{2a}=2e\Rightarrow 2a=1+\ln 2\Rightarrow a=\frac{1+\ln 2}{2}\end{aligned}$$

$$f''(x)>0,\text{问}f'(0),f'(1),f(1)-f(0)\text{大小?}$$

$$\begin{aligned}\text{解}:&1.\ f(1)-f(0)=f'(c)(0<c<1)\\ &2.\ f''(x)>0\Rightarrow f'(x)\uparrow\\ &\quad\because 0<c<1,\therefore f'(0)<f'(c)<f'(1)\end{aligned}$$

$$求\lim_{x\to\infty}x^2(e^{\frac{1}{2x-1}}-e^{\frac{1}{2x+1}})$$

$$\begin{aligned}&解:令f(t)=e^t,f'(t)=e^t\\&\quad e^{\frac{1}{2x-1}}-e^{\frac{1}{2x+1}}=f(\frac{1}{2x-1})-f(\frac{1}{2x+1})=f'(\xi)(\frac{1}{2x-1}-\frac{1}{2x+1})\\&\quad=\frac{2}{4x^2-1}e^{\xi}(\xi在\frac{1}{2x-1}与\frac{1}{2x+1}之间)\\&\quad原式=\lim_{x\to\infty}\frac{2x^2}{4x^2-1}e^{\xi}=\frac{1}{2}\end{aligned}$$

$$f''(x)>0,f(0)=0,证:2f(1)<f(2)$$

$$\begin{aligned}证:&f(1)-f(0)=f'(\xi_1),0<\xi_1<1\\&f(2)-f(1)=f'(\xi_2),1<\xi_2<2\\&\because f''(x)>0,\therefore f'(x)\uparrow\\&又\because\xi_1,\xi_2,\therefore f'(\xi_1)<f'(\xi_2)\\&\Rightarrow f(1)<f(2)-f(1)\end{aligned}$$

$$\begin{aligned}&f(x)\in c[0,2],(0,2)上可导,|f'(x)|\le M\\&f(x)在(0,2)内至少一个零点,证:\\&|f(0)|+|f(2)|\le 2M\end{aligned}$$

$$\begin{aligned}证:&\exists c\in(0,2),使f(c)=0\\&f(c)-f(0)=f'(\xi_1)c,0<\xi_1<c\\&f(2)-f(c)=f'(\xi_2)(2-c),c<\xi_2<2\\&\Rightarrow\begin{cases}|f(0)|\le MC\\|f(2)|\le M(2-C)\end{cases}\\&\Rightarrow|f(0)|+|f(2)|\le 2M\end{aligned}$$

极值、渐近线

型一 极值点判断

$$\begin{aligned}&y=f(x):\\&1.\ x\in D\\&2.\ f'(x)\begin{cases}=0\\不存在\end{cases}\\&3.\ 法一:\\&\quad ①\begin{cases}f'<0,x<x_0\\f'>0,x>x_0\end{cases}\Rightarrow x_0为极小点\\&\quad ②\begin{cases}f'>0,x<x_0\\f'<0,x>x_0\end{cases}\Rightarrow x_0为极大点\\&\quad 法二:f'(x_0)=0,f''(x_0)\begin{cases}>0:x_0为极小点\\<0:x_0为极大点\end{cases}\end{aligned}$$

$$f'(1)=0,\lim_{x\to1}\frac{f'(x)}{\sin\pi x}=-2,x=1?$$

$$\text{解}:\text{法一},\exists\delta>0,\text{当}0<|x-1|<\delta\text{时},\frac{f'(x)}{\sin\pi x}<0$$

$$\begin{cases}f'(x)<0,x\in(1-\delta,1)\\f'(x)>0,x\in(1,1+\delta)\end{cases}\Rightarrow x=1\text{为极小点}$$

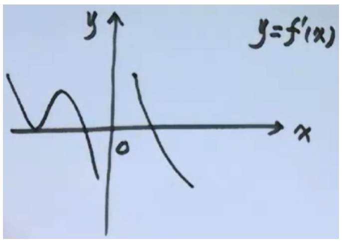
$$\text{法二},f'(1)=0$$

$$-2=\lim_{x\to1}\frac{f'(x)}{\sin[\pi+\pi(x-1)]}=-\lim_{x\to1}\frac{f'(x)}{\pi(x-1)}$$

$$=-\frac{1}{\pi}\lim_{x\to1}\frac{f'(x)-f'(1)}{x-1}=-\frac{1}{\pi}f''(1)\Rightarrow f''(1)=2\pi>0$$

$$\therefore x=1\text{为极小点}$$

$$f(x)\in C(-\infty,+\infty),\text{求}f(x)\text{的极值点个数}$$



$$\text{解}:1.\ x\in(-\infty,+\infty)$$

$$2.\ f'(x)\begin{cases}=0\\\text{无}\end{cases}\Rightarrow x=x_1,x_2,0,x_3$$

$$3.\ \begin{cases}f'>0,x<x_1\\f'>0,x>x_1\end{cases}$$

$$\begin{cases}f'>0,x<x_2\\f'<0,x>x_2\end{cases}\Rightarrow x_2\text{为极大点}$$

$$\begin{cases}f'<0,x<0\\f'>0,x>0\end{cases}\Rightarrow x=0\text{为极小点}$$

$$\begin{cases}f'>0,x<x_3\\f'<0,x>x_3\end{cases}\Rightarrow x_3\text{为极大点}$$

$$f(x):xf''(x)+3x^2f'(x)=1-e^{-2x}$$
$$x=a\text{为}f(x)\text{的极值点},\text{问极大还是极小}.$$

$$\text{解}:1.\ f'(a)=0$$

$$2.\ af''(a)=1-e^{-2a}\Rightarrow f''(a)=\frac{1-e^{-2a}}{a}$$

$$3.\ ①a<0:-2a>0\Rightarrow e^{-2a}>1$$
$$f''(a)>0$$
$$②a>0:-2a<0\Rightarrow e^{-2a}<1$$
$$f''(a)>0$$
$$\therefore x=a\text{为极小点}$$

型二 函数的零点或方程的解

①零点定理

$$\text{证}:x^5+4x-1=0\text{有且仅有一个正根}$$

$$
\begin{aligned}
&\text{证}:\text{令} f(x)=x^5+4x-1\\
&\quad 1.\ f(0)=-1, f(1)=4\\
&\qquad \because f(0)f(1)<0, \therefore \exists c\in(0,1), \text{使} f(c)=0\\
&\quad 2.\ \because f'(x)=5x^4+4>0(x>0)\\
&\qquad \therefore f(x)\text{在}[0,+\infty)\uparrow\\
&\qquad f(x)\text{仅有一个正零点}
\end{aligned}
$$

$$
\text{②} Rolle: f(x), \text{找} F(x), F'(x)=f(x)
$$
$$
\text{若} F(a)=F(b)\Rightarrow \exists c\in(a,b), \text{使} F'(c)=0\Rightarrow f(c)=0
$$

$$
\begin{aligned}
&\text{已知} a_0+\frac{a_1}{2}+\ldots+\frac{a_n}{n+1}=0\\
&\text{证}:\text{方程} a_0+a_1x+\ldots+a_nx^n=0\text{至少有一个正根}
\end{aligned}
$$

$$
\begin{aligned}
&\text{证}:\text{令} f(x)=a_0+a_1x+\ldots+a_nx^n\\
&\quad F(x)=a_0x+\frac{a_1}{2}x^2+\ldots+\frac{a_n}{n+1}x^{n+1}, F'(x)=f(x)\\
&\quad \because F(0)=F(1)=0\\
&\quad \therefore \exists c\in(0,1), \text{使} F'(c)=0\Rightarrow f(c)=0
\end{aligned}
$$

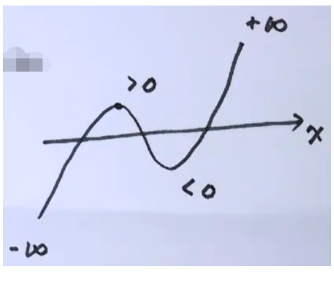
---

③单调法

$$
\begin{aligned}
&1.\ f(x)(x\in D)\\
&2.\ f'(x)\begin{cases}=0\\ \text{不存在}\end{cases}\Rightarrow \text{极大点, 极值}\\
&3.\ \text{研究两侧的变化趋势作草图}
\end{aligned}
$$

$$
f(x)=x^3-3x^2-9x+2
$$

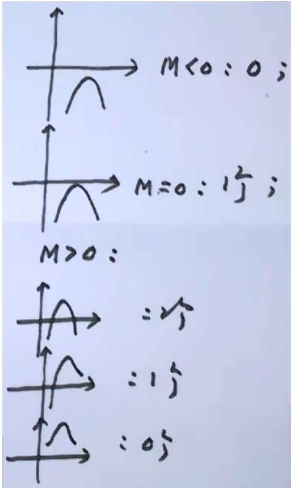
$$
\begin{aligned}
&\text{解}: 1.\ x\in(-\infty,+\infty)\\
&\quad 2.\ f'(x)=3(x^2-2x-3)=3(x+1)(x-3)=0\\
&\qquad \Rightarrow x=-1, x=3\\
&\qquad \begin{cases}f'>0, x<-1\\ f'<0, x>-1\end{cases}\Rightarrow x=-1\text{为极大点}, f(-1)=7\\
&\qquad \begin{cases}f'<0, x<3\\ f'>0, x>3\end{cases}\Rightarrow x=3\text{为极小点}, f(3)=-25\\
&\quad 3.\ f(-\infty)=-\infty, f(+\infty)=+\infty
\end{aligned}
$$



$$
\therefore f(x)\text{有且仅有3个零点}
$$

$$
\text{讨论方程} \ln x=\frac{x}{e}-2\text{几个根.}
$$

$$
\begin{aligned}
&\text{解}: \text{令} f(x)=\ln x-\frac{x}{e}+2(x>0) \\
&\quad \text{令} f'(x)=\frac{1}{x}-\frac{1}{e}=0 \Rightarrow x=e \\
&\quad \because f''(e)=-\frac{1}{e^2}<0 \\
&\quad \therefore x=e \text{为} f(x) \text{的最大点}, M=f(e)=2>0
\end{aligned}
$$



$$
\begin{aligned}
&\because f(0+0)=-\infty, f(+\infty)=-\infty \\
&\therefore \ln x=\frac{x}{e}-2 \text{有且仅有} 2 \text{个根}
\end{aligned}
$$

$$x=ae^{x}(a>0)\text{几个根.}$$

$$
\begin{aligned}
\text{解}: &1.\ x=ae^x \Leftrightarrow xe^{-x}-a=0 \\
&\quad \text{令} f(x)=xe^{-x}-a(x>0) \\
&2.\ f'(x)=(1-x)e^{-x}=0 \\
&\quad \Rightarrow x=1 \\
&\quad 0<x<1\text{时}, f'(x)>0, x>1\text{时}, f'(x)<0 \\
&\quad \Rightarrow x=1\text{为} f(x)\text{的最大点}, M=f(1)=\frac{1}{e}-a \\
&3.\ ①M<0, \text{即} a>\frac{1}{e}, \text{方程无解} \\
&\quad ②M=0, \text{即} a=\frac{1}{e}, \text{方程唯一解} x=1 \\
&\quad ③M>0, \text{即} 0<a<\frac{1}{e} \\
&\quad f(0)=-a<0, f(+\infty)=-a<0 \\
&\quad \therefore \text{方程有} 2 \text{个根}
\end{aligned}
$$

型三 不等式证明

$$
\begin{aligned}
&①\frac{f(b)-f(a)}{b-a}, \frac{f(b)-f(a)}{g(b)-g(a)} - \text{中值定理} \\
&②\text{单调法}
\end{aligned}
$$

$$0<a<b, \text{证}: \arctan b-\arctan a<b-a$$

$$\text{证}:\text{令}f(x)=\arctan x, f'(x)=\frac{1}{1+x^2}$$

$$\arctan b-\arctan a=f(b)-f(a)=f'(\xi)(b-a)$$

$$=\frac{1}{1+\xi^2}(b-a)(a<\xi<b)$$

$$\because \frac{1}{1+\xi^2}<1, \therefore \arctan b-\arctan a<b-a$$

$$0<a<b, \text{证}:\frac{\ln b-\ln a}{b-a}<\frac{2a}{a^2+b^2}$$

$$\text{证}:\text{令}f(x)=\ln x, f'(x)=\frac{1}{x}$$

$$\text{左}=\frac{f(b)-f(a)}{b-a}=f'(\xi)=\frac{1}{\xi}(a<\xi<b)$$

$$\frac{1}{\xi}>\frac{1}{b}>\frac{2a}{a^2+b^2}$$

$$\text{证}:x>0\text{时}, \frac{x}{1+x}<\ln(1+x)<x$$

$$\text{法一}:f(t)=\ln(1+t), f'(t)=\frac{1}{1+t}$$

$$\ln(1+x)=\ln(1+x)-\ln(1+0)=f(x)-f(0)$$

$$=f'(\xi)x=\frac{x}{1+\xi}(0<\xi<x)$$

$$\because \frac{x}{1+x}<\frac{x}{1+\xi}<x$$

$$\therefore \frac{x}{1+x}<\ln(1+x)<x$$

$$\text{法二}:\text{令}f(x)=x-\ln(1+x), f(0)=0$$

$$f'(x)=1-\frac{1}{1+x}>0(x>0)$$

$$\begin{cases} f(0)=0 \\ f'(x)>0(x>0) \end{cases} \Rightarrow f(x)>0(x>0), \text{即}\ln(1+x)<x(x>0)$$

$$\text{令}g(x)=\ln(1+x)-\frac{x}{1+x}, g(0)=0$$

$$g'(x)=\frac{1}{1+x}-\frac{1}{(1+x)^2}>0(x>0)$$

$$\begin{cases} g(0)=0 \\ g'(x)>0(x>0) \end{cases} \Rightarrow g(x)>0(x>0)$$

$$\text{即}\frac{x}{1+x}<\ln(1+x)(x>0)$$

$$e<a<b, \text{证}:a^b>b^a$$

$$\text{证}:a^b>b^a \Leftrightarrow b\ln a-a\ln b>0$$

$$\text{令}\Phi(x)=x\ln a-a\ln x, \Phi(a)=0$$

$$\Phi'(x)=\ln a-\frac{a}{x}>0(x>a)$$

$$\begin{cases} \Phi(a)=0 \\ \Phi'(x)>0(x>a) \end{cases} \Rightarrow \Phi(x)>0(x>a)$$

$$\because b>a \ \therefore \Phi(b)>0$$

$$0<a<b,\text{证}:\ln\frac{b}{a}>\frac{2(b-a)}{a+b}$$

$$\begin{aligned}
\text{证}:&\ln\frac{b}{a}>\frac{2(b-a)}{a+b}\Leftrightarrow(a+b)(\ln b-\ln a)-2(b-a)>0\\
&\text{令}\Phi(x)=(a+x)(\ln x-\ln a)-2(x-a),\Phi(a)=0\\
&\Phi'(x)=\ln x-\ln a+\frac{a}{x}-1,\Phi'(a)=0\\
&\Phi''(x)=\frac{1}{x}-\frac{a}{x^2}=\frac{x-a}{x^2}>0(x>a)\\
&\because\begin{cases}\Phi'(a)=0\\\Phi''(x)>0(x>a)\end{cases}\therefore\Phi'(x)>0(x>a)\\
&\because\begin{cases}\Phi(a)=0\\\Phi'(x)>0(x>a)\end{cases}\therefore\Phi(x)>0(x>a)\\
&\because b>a\ \therefore\Phi(b)>0
\end{aligned}$$

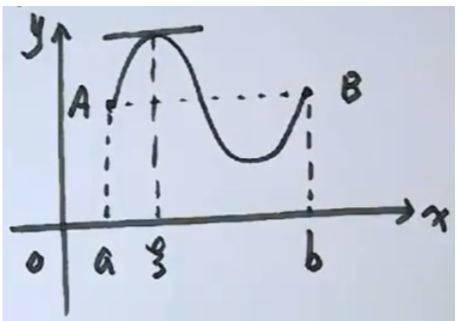
$$\text{证}:x>0\text{时},x^2e^x>(e^x-1)^2.$$

$$\begin{aligned}
\text{证}:&f(x)=x^2e^x-(e^x-1)^2,f(0)=0\\
&f'(x)=2xe^x+x^2e^x-2e^x(e^x-1)\\
&\ =e^x[2x+x^2-2(e^x-1)],e^x>0\\
&h(x)=2x+x^2-2(e^x-1),h(0)=0\\
&h'(x)=2+2x-2e^x=2(1+x-e^x)\\
&\because x>0\text{时},e^x>1+x\ \therefore h'(x)<0(x>0)\\
&\because\begin{cases}h(0)=0\\h'(x)<0(x>0)\end{cases}\Rightarrow h(x)<0(x>0)\Rightarrow f'(x)<0(x>0)\\
&\because\begin{cases}f(0)=0\\f'(x)<0(x>0)\end{cases}\Rightarrow f(x)<0(x>0)\Rightarrow x^2e^x<(e^x-1)^2(x>0)\\
&\because b>a\ \therefore\Phi(b)>0
\end{aligned}$$

罗尔定理 Rolle

$$f(x)\in C[a,b]$$
$$f(x)\text{在}(a,b)\text{可导}$$
$$f(a)=f(b)$$

$$\exists\xi\in(a,b),\text{使}f'(\xi)=0$$



$$f(x)\in C[a,b]\Rightarrow m,M$$

1. m=M

$$f(x)\equiv C_0,\text{则}\forall\xi\in(a,b),\text{有}f'(\xi)=0$$

2. m<M

$$\because f(a)=f(b)\therefore m,M\text{至少一个在}(a,b)\text{内取到}$$
$$\text{设}\exists\xi\in(a,b),\text{使}f(\xi)=M\Rightarrow\xi\text{为极大值}\Rightarrow f'(\xi)=0\text{或不存在}$$
$$\because f(x)\text{可导}\therefore f'(\xi)=0$$

$$f(0)=1,f(1)=2,f(2)=-1$$
$$h(x)=f(x)-1$$
$$h(0)=0,h(1)=1,h(2)=-2$$
$$\exists c\in(1,2),h(c)=h(0)=0$$
$$\exists\xi\in(0,c)\subset(0,2),h'(\xi)=0=f'(\xi)$$

$$f(0)+2f(1)=2f(2)+f(3)$$
$$m\leqslant\frac{f(0)+2f(1)}{3}=\frac{2f(2)+f(3)}{3}\leqslant M$$
$$\exists\xi_1\in(0,1),\xi_2\in(2,3),\text{使}f(\xi_1)=\frac{f(0)+2f(1)}{3}=f(\xi_2)=\frac{2f(2)+f(3)}{3}$$
$$\exists\xi\in(\xi_1,\xi_2)\subset(0,3),\text{使}f'(\xi)=0$$

$$f(x)\text{在}[0,1]\text{上二阶可导}$$
$$\text{连接端点}A(0,f(0)),B(1,f(1))\text{的线段交曲线}y=f(x)\text{于点}C(c,f(c))(0<c<1)$$
$$\exists\xi_1\in(0,c),\xi_2\in(c,1),\text{使}f'(\xi_1)=\frac{f(c)-c(0)}{c-0}=f'(\xi_2)=\frac{f(1)-c(c)}{1-c}$$
$$\exists\xi\in(\xi_1,\xi_2)\subset(0,1),\text{使}f''(\xi)=0$$

$$f'(\xi)=-f(\xi)\cot\xi$$
$$f(x)\cos x+f'(\xi)\sin x=(f(x)\sin x)'$$

$$f'(\xi)+2f(\xi)g'(\xi)=0$$
$$(f(x)e^{2g(x)})'=f'(x)e^{2g(x)}+e^{2g(x)}2g'(x)f(x)=e^{2g(x)}(f'(x)+2g'(x)f(x))$$

$$f'(\xi)+2\xi f(\xi)=0$$
$$(f(x)e^{x^2})'=f'(x)e^{x^2}+e^{x^2}2xf(x)=e^{x^2}(f'(x)+2xf(x))$$

$$\frac{f(a)-f(\xi)}{g(\xi)-g(b)}=\frac{f'(\xi)}{g'(\xi)}$$
$$F'(x)=f'(\xi)[g(\xi)-g(b)]-g'(\xi)[f(a)-f(\xi)]=[f'(\xi)g(\xi)+g'(\xi)f(\xi)]-[g(b)f'(\xi)+f(a)g'(\xi)]$$
$$F(x)=f(x)g(x)-[f(x)g(b)+f(a)g(x)]F(a)=F(b)=-f(a)g(b)$$
$$\exists\xi\in(a,b),\text{使}F'(\xi)=0$$

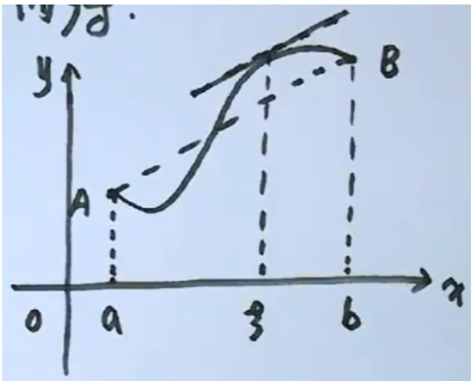
$$f(a)+f(b)<f(a+b)$$
$$f(a)-f(0)=f'(\xi_1)a$$
$$f(a+b)-f(b)=f'(\xi_2)a$$
$$f'(\xi_1)a<f'(\xi_2)a$$
$$f(a)+f(b)<f(a+b)$$

$$|f(0)|+|f(2)|\leqslant 2M$$
$$\exists c\in(0,2),f(c)=0$$
$$f(c)-f(0)=f'(\xi_1)c$$
$$f(2)-f(c)=f'(\xi_2)(2-c)$$
$$|f(0)|=|f'(\xi_1)c|\leqslant cM$$
$$|f(2)|=|f'(\xi_2)(2-c)|\leqslant(2-c)M$$
$$|f(0)|+|f(2)|\leqslant 2M$$

$$|f'(a)|+|f'(b)| \leqslant M(b-a)$$
$$\exists c \in (a,b), 使 f'(c)=0$$
$$f'(c)-f'(a)=f''(\xi_1)(c-a)$$
$$f'(b)-f'(c)=f''(\xi_2)(b-c)$$
$$|f'(a)|=|f''(\xi_1)(c-a)| \leqslant M|c-a|$$
$$|f'(b)|=|f''(\xi_2)(b-c)| \leqslant M|b-c|$$
$$|f'(a)|+|f'(b)| \leqslant M[|c-a|+|b-c|] \leqslant M|b-a|$$

拉格朗日中值定理 Lagrange

$$f(x) \in C[a,b]$$
$$f(x) 在 (a,b) 可导$$
$$\exists \xi \in (a,b), 使 f'(\xi)=\frac{f(b)-f(a)}{b-a}$$



$$L: y=f(x)$$
$$L_{AB}: y-f(a)=\frac{f(b)-f(a)}{b-a}(x-a)$$
$$即 L_{AB}: y=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$$
$$令 \psi(x)=L-L_{AB}=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$$
$$\psi \in C[a,b], 在 (a,b) 内可导$$
$$又 \psi(a)=\psi(b)=0, \therefore \exists \xi \in (a,b), 使 \psi'(\xi)=0$$
$$而 \psi'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$$
$$\therefore f'(\xi)=\frac{f(b)-f(a)}{b-a}$$

$$\exists \xi(0,\frac{1}{2}), \eta(\frac{1}{2},1)$$
$$f'(\xi)=\frac{f(\frac{1}{2})-f(0)}{\frac{1}{2}-0}=2[f(\frac{1}{2})-f(0)]$$
$$f'(\eta)=\frac{f(1)-f(\frac{1}{2})}{1-\frac{1}{2}}=2[f(1)-f(\frac{1}{2})]$$
$$f'(\xi)+f'(\eta)=2[f(1)-f(0)]=0$$

柯西中值定理 Cauchy

$$f(x),g(x)\in C[a,b]$$
$$f(x),g(x)在(a,b)可导$$
$$g'(x)\neq 0\ (a<x<b)$$
$$\exists\xi\in(a,b),使\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$$
$$\psi(x)=L-L_{AB}=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$$
$$令\psi(x)=f(x)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]$$
$$\psi\in C[a,b],(a,b)内可导$$
$$\psi(a)=\psi(b)=0$$
$$\exists\xi\in(a,b),使\psi'(\xi)=0$$
$$而\psi'(x)=f'(x)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(x)$$
$$\therefore f'(\xi)=\frac{f(b)-f(a)}{g(b)-g(a)}g'(x)$$
$$\because g'(\xi)\neq 0,\therefore\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$$
$$f(b)-f(a)=(1+\xi)f'(\xi)\ln\frac{1+b}{1+a}$$
$$\frac{f(b)-f(a)}{\ln(1+b)-\ln(1+a)}=\frac{f'(\xi)}{\frac{1}{1+\xi}}$$
$$g(x)=\ln(1+x)$$
$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$$
$$ae^b-be^a=(a-b)(1-\xi)e^\xi$$
$$\frac{e^b-e^a}{\frac{1}{b}-\frac{1}{a}}=\frac{e^\xi}{\frac{1}{1-\xi}}$$
$$f(x)=e^x,g(x)=lnx$$
$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$$
$$f(2)=\xi f'(\xi)-f(\xi)$$
$$F(x)=\frac{f(x)}{x},G(x)=-\frac{1}{x}$$
$$\frac{F(2)-F(1)}{G(2)-G(1)}=\frac{F'(\xi)}{G'(\xi)}$$
$$\frac{\frac{f(2)}{2}}{-\frac{1}{2}-(-1)}=\frac{\frac{\xi f'(\xi)-f(\xi)}{\xi^2}}{\frac{1}{\xi^2}}$$

$$4f(2)=\xi^2 f(\xi)+\xi^3$$
$$F(x)=xf(x),G(x)=-\frac{1}{x}$$
$$\frac{F(2)-F(1)}{G(2)-G(1)}=\frac{F'(\xi)}{G'(\xi)}$$

$$f'(\xi)=\frac{a+b}{2\eta}f'(\eta)$$
$$F(x)=x^2,F'(x)=2x$$
$$\frac{f(b)-f(a)}{F(b)-F(a)}=\frac{f'(\eta)}{F'(\eta)}$$
$$\frac{f(b)-f(a)}{b-a}=\frac{a+b}{2\eta}f'(\eta)$$
$$f'(\xi)=\frac{a+b}{2\eta}f'(\eta)$$

$$\frac{f'(\zeta)}{f'(\xi)}=\frac{\xi}{\eta}$$
$$F(x)=\ln x,F'(x)=\frac{1}{x}$$
$$\frac{f(2)-f(1)}{F(2)-F(1)}=\frac{f'(\xi)}{F'(\xi)}\Rightarrow\frac{f(2)-f(1)}{\ln 2-\ln 1}=\frac{f'(\xi)}{\frac{1}{\xi}}=\xi f'(\xi)$$
$$\ln 2-\ln 1=\frac{1}{\eta}*(2-1)=\frac{1}{\eta}$$
$$f(2)-f(1)=\frac{\xi}{\eta}f'(\xi)$$
$$f(2)-f(1)=f'(\zeta)(2-1)=f'(\zeta)$$
$$\frac{f'(\zeta)}{f'(\xi)}=\frac{\xi}{\eta}$$

泰勒中值定理 Taylor

$$f(x)\text{在}x=x_0\text{领域内}n+1\text{阶可导}$$
$$f(x)=P_n(x)+R_n(x)$$
$$P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$
$$R_n(x)=\begin{cases}\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} & \xi\text{介于}x_0\text{与}x\text{之间}\quad\text{拉格朗日型}\\ o((x-x_0)^n) & \text{皮亚诺型}\end{cases}$$

$$x_0=0$$
$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+R_n(x)$$
$$R_n(x)=\begin{cases}\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1} & \xi\text{介于}0\text{与}x\text{之间}\\ o(x^n)\end{cases}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2}$$

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2$$

$$a(x)=\sqrt{1+x}$$

$$a'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}}$$

$$a''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}}$$

$$a(x)=1+\frac{1}{2}x-\frac{1}{4*2}x^2$$

$$b(x)=\sqrt{1-x}$$

$$b'(x)=-\frac{1}{2}(1-x)^{-\frac{1}{2}}$$

$$b''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}}$$

$$b(x)=1-\frac{1}{2}x-\frac{1}{4*2}x^2$$

$$\text{原式}=-\frac{1}{4}$$

$$\lim_{x\to 0}\frac{e^{-\frac{x^2}{2}}-1+\frac{x^2}{2}}{x^3\arcsin x}$$

$$\lim_{x\to 0}\frac{e^{-\frac{x^2}{2}}-1+\frac{x^2}{2}}{x^4}$$

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2$$

$$a(x)=e^x$$

$$a'(x)=e^x$$

$$a''(x)=e^x$$

$$a(x)=1+x+\frac{1}{2}x^2$$

$$a(-\frac{x^2}{2})=1-\frac{x^2}{2}+\frac{1}{2}*\frac{x^4}{4}$$

$$\text{原式}=\frac{1}{8}$$

$$|f'(x)|\leqslant\frac{M}{2}$$

$$f(0)=f(x)+f'(x)(0-x)+\frac{f''(\xi)}{2!}(0-x)^2,\xi\in(0,x)$$

$$f(1)=f(x)+f'(x)(1-x)+\frac{f''(\eta)}{2!}(1-x)^2,\eta\in(x,1)$$

$$f(x)+f'(x)(0-x)+\frac{f''(\xi)}{2!}(0-x)^2=f(x)+f'(x)(1-x)+\frac{f''(\eta)}{2!}(1-x)^2$$

$$f'(x)=\frac{f''(\xi)}{2!}(0-x)^2-\frac{f''(\eta)}{2!}(1-x)^2$$

$$|f'(x)|=\frac{1}{2}|f''(\xi)x^2-f''(\eta)(1-x)^2|\leqslant\frac{M}{2}|2x-1|\leqslant\frac{M}{2}$$

$$f(x)\text{在}(a,b)\text{内为凹函数}$$

$$x_0=\frac{x_1+x_2}{2}$$

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2!}(x-x_0)^2,\xi\in(x_0,x)$$

$$f''(x)>0,\therefore f(x)\geqslant f(x_0)+f'(x_0)(x-x_0),\text{当且仅当}x=x_0\text{等式成立}$$

$$\begin{cases}\frac{1}{2}f(x_1)>\frac{1}{2}f(x_0)+\frac{1}{2}f'(x_0)(x_1-x_0)\\\frac{1}{2}f(x_2)>\frac{1}{2}f(x_0)+\frac{1}{2}f'(x_0)(x_2-x_0)\end{cases}$$

$$\frac{f(x_1)+f(x_2)}{2}>f(x_0)$$

$$\frac{f(x_1)+f(x_2)}{2}>f(\frac{x_1+x_2}{2})$$

$$\text{由凹函数定义得},f(x)\text{在}(a,b)\text{内为凹函数}$$