导数与微分

导数

$$y=f(x)(x\in D),a\in D$$
$$\Delta y=f(a+\Delta x)-f(a)$$
$$若\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\exists,称f(x)在x=a处可导,极限值称为f(x)在x=a处的导数,记f'(a),\frac{dy}{dx}|_{x=a}$$

$$1.\ f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$
$$2.\ \Delta x\to 0\begin{cases}\Delta x\to 0^-\\ \Delta x\to 0^+\end{cases},x\to a\begin{cases}x\to a^-\\ x\to a^+\end{cases}$$
$$\lim_{\Delta x\to 0^-}\frac{\Delta y}{\Delta x}(=\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a})\triangleq f'_-(a)$$
$$\lim_{\Delta x\to 0^+}\frac{\Delta y}{\Delta x}(=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a})\triangleq f'_+(a)$$
$$f'(a)\exists\Leftrightarrow f'_-(a),f'_+(a)\exists且相等$$

$$f(x)=\begin{cases}\frac{x*2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}},x\neq 0\\ 0,x=0\end{cases},f'(0)?$$
$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}}$$
$$f'_-(0)=0\neq f'_+(0)=1\Rightarrow f'(0)不存在$$

$$3.\ f(x)在x=a处可导\Rightarrow f(x)在x=a处连续$$
$$\not\Leftarrow$$
$$\Rightarrow$$
$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\exists\Rightarrow\lim_{x\to a}[f(x)-f(a)]=0$$
$$\Rightarrow\lim_{x\to a}f(x)=f(a)$$
$$\not\Leftarrow$$
$$f(x)=|x|在x=0处连续$$
$$f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{|x|}{x}=-1$$
$$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{|x|}{x}=1$$
$$\because f'_-(0)\neq f'_+(0),\therefore f(x)在x=0不可导$$

$$5.\ f(x)连续,\lim_{x\to a}\frac{f(x)-b}{x-a}=A\Rightarrow f(a)=b,f'(a)=A$$
$$\lim_{x\to a}f(x)=b$$
$$\because f(x)连续,\therefore f(a)=b$$
$$A=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$$

$$设函数f(x)=\begin{cases}\ln(e+2x),x>0\\1,x=0\\\frac{1}{1+x^2},x<0\end{cases},讨论函数f(x)在x=0处的可导性.$$

$$f'_+(0)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^+}\frac{\ln(e+2x)-\ln e}{x}=\lim_{x\to0^+}\frac{\ln(1+\frac{2x}{e})}{x}=\frac{2}{e}$$

$$f'_-(0)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^-}\frac{\frac{1}{1+x^2}-1}{x}=-\lim_{x\to0^-}\frac{x}{1+x^2}=0$$

$$\Rightarrow f(x)在x=0不可导$$

微分

$$y=f(x)(x\in D),a\in D$$

$$\Delta y=f(a+\Delta x)-f(a)(或\Delta y=f(x)-f(a))$$

$$若\Delta y=A\Delta x+o(\Delta x)(或\Delta y=A(x-a)+o(x-a))$$

$$称f(x)在x=a可微$$

$$称A\Delta x为y=f(x)在x=a处的微分,记dy\mid_{x=a}=A\Delta x\triangleq Adx$$

$$1.\ f(x)在x=a可导\Leftrightarrow f(x)在x=a可微$$

$$\Rightarrow$$

$$\lim_{\Delta x\to0}\frac{\Delta y}{\Delta x}=f'(a)\Rightarrow\frac{\Delta y}{\Delta x}=f'(a)+\alpha,\alpha\to0(\Delta x\to0)$$

$$\Rightarrow\Delta y=f'(a)\Delta x+\alpha\Delta x$$

$$\because\lim_{\Delta x\to0}\frac{\alpha\Delta x}{\Delta x}=\lim_{\Delta x\to0}\alpha=0$$

$$\therefore\alpha\Delta x=o(\Delta x)$$

$$\Rightarrow\Delta y=f'(a)\Delta x+o(\Delta x)$$

$$\Leftarrow$$

$$设\Delta y=A\Delta x+o(\Delta x)$$

$$\Rightarrow\frac{\Delta y}{\Delta x}=A+\frac{o(\Delta x)}{\Delta x}\Rightarrow\lim_{\Delta x\to0}\frac{\Delta y}{\Delta x}=A\Rightarrow f'(a)=A$$

$$\begin{aligned}&2.\ 若\Delta y=A\Delta x+o(\Delta x)\Rightarrow A=f'(a)\\&\quad\therefore dy\mid_{x=a}=f'(a)dx\\&3.\ 设y=f(x)处处可导\\&\quad dy=df(x)=f'(x)dx\\&\quad d(x^3)=3x^2,\cos2xdx=d(\frac{1}{2}\sin2x+C)\end{aligned}$$

求导工具

$$初等函数\begin{cases}常数\\基本初等函数\end{cases}加工\begin{cases}四则\\复合\end{cases}而成的式子$$

基本公式

$$
\begin{aligned}
&1. (C)' = 0 \\
&2. (x^a)' = ax^{a-1} \begin{cases} (\sqrt{x})' = \frac{1}{2\sqrt{x}} \\ (\frac{1}{x})' = -\frac{1}{x^2} \end{cases} \\
&3. (a^x)' = a^x \ln a, (e^x)' = e^x \\
&4. (\log_a x)' = \frac{1}{x \ln a}, (\ln x)' = \frac{1}{x} \\
&5. (\sin x)' = \cos x, \\
&\quad (\cos x)' = -\sin x, \\
&\quad (\tan x)' = \sec^2 x, \\
&\quad (\cot x)' = -\csc^2 x, \\
&\quad (\sec x)' = \sec x \tan x, \\
&\quad (\csc x)' = -\csc x \cot x \\
&6. (\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \\
&\quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \\
&\quad (\arctan x)' = \frac{1}{1+x^2} \\
&\quad (arccot\, x)' = -\frac{1}{1+x^2}
\end{aligned}
$$

四则

$$
\begin{aligned}
&1. (u \pm v)' = u' \pm v' \\
&2. (uv)' = u'v + uv' \\
&\quad (ku)' = ku' (k\text{为常数}) \\
&3. (uvw)' = u'vw + uv'w + uvw' \\
&4. (\frac{u}{v})' = \frac{u'v - uv'}{v^2} (v \neq 0)
\end{aligned}
$$

复合求导法则

$$
y = f(u)\text{可导}, u = \Phi(x)\text{可导且}\Phi'(x) \neq 0
$$
$$
\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'(u)\Phi'(x) = f'[\Phi(x)]\Phi'(x)
$$

$$
\begin{aligned}
\text{证：} &\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = f'(u) \\
&\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \Phi'(x) \neq 0 \Rightarrow \Delta u = O(\Delta x) \\
&\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} * \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} * \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&\quad = f'(u)\Phi'(x) = f'[\Phi(x)]\Phi'(x)
\end{aligned}
$$

$$
y = x^2 e^{\sin \frac{1}{x}}
$$

$$
\begin{aligned}
y' =& 2xe^{\sin \frac{1}{x}} + x^2 (e^{\sin \frac{1}{x}})' \\
=& 2xe^{\sin \frac{1}{x}} + x^2 (e^{\sin \frac{1}{x}} * \cos \frac{1}{x} * (-\frac{1}{x^2}))
\end{aligned}
$$

$$
y = (1 + \sin 2x)^{\ln(1-x)}
$$

$$
\begin{aligned}
y &= e^{\ln(1-x)\ln(1+\sin 2x)} \\
y' &= e^{\ln(1-x)\ln(1+\sin 2x)} * [\frac{-1}{1-x}\ln(1+\sin 2x) + \ln(1-x)\frac{2\cos 2x}{1+\sin 2x}]
\end{aligned}
$$

$$y = 2^{\sin^2 \frac{1}{x}}, \text{求} y'.$$

$$y' = 2^{\sin^2 \frac{1}{x}} \ln 2 * [2\sin\frac{1}{x}\cos\frac{1}{x} * (-\frac{1}{x^2})]$$

反函数导数

$$y = f(x)\text{可导且}f'(x) \neq 0, x = \Phi(y)\text{为反函数}$$

$$\text{则}\Phi'(y) = \frac{1}{f'(x)}$$

$$
\begin{aligned}
\text{证}: f'(x) &= \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} \neq 0 \Rightarrow \Delta y = O(\Delta x) \\
\Phi'(y) &= \lim_{\Delta y \to 0}\frac{\Delta x}{\Delta y} = \lim_{\Delta x \to 0}\frac{1}{\frac{\Delta y}{\Delta x}} = \frac{1}{f'(x)}
\end{aligned}
$$

$$\text{设} y = \frac{1}{2x+1}, \text{求} y^{(n)}.$$

$$
\begin{aligned}
y &= (2x+1)^{-1} \\
y' &= (-1)(2x+1)^{-2} * 2 \\
y'' &= (-1)(-2)(2x+1)^{-3} * 2^2 \\
&\dots \\
y^{(n)} &= (-1)(-2)\dots(-n)(2x+1)^{-(n+1)}2^n \\
&= \frac{(-1)^n n! 2^n}{(2x+1)^{n+1}}
\end{aligned}
$$