二重积分

定义

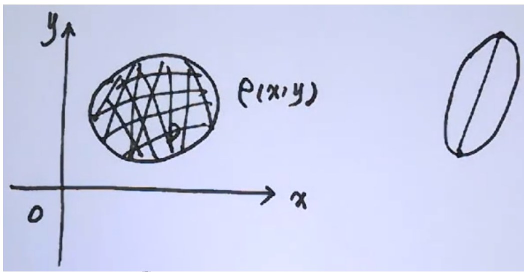
定积分

$$
\begin{gathered}
\text{定积分}, f(x)\text{在}[a,b]\text{上有界} \\
1.\ a = x_0 < x_1 < \ldots < x_n = b \\
2.\ \forall \xi_i \in [x_{i-1}, x_i], \sum_{i=1}^{n} f(\xi_i)\Delta x_i \\
3.\ \lambda = \max\{\Delta x_1, \ldots, \Delta x_n\}
\end{gathered}
$$

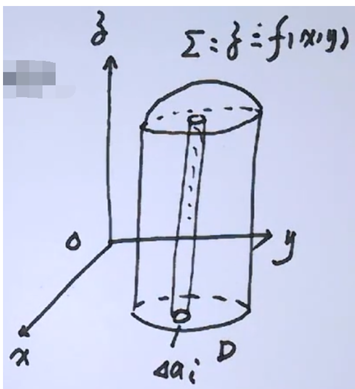
$$
\text{若} \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i \exists, \text{称} f(x)\text{在}[a,b]\text{上可积}
$$

$$
\text{极限值称为} f(x)\text{在}[a,b]\text{上的定积分，记作} \int_a^b f(x)dx
$$

二重积分



$$
\begin{aligned}
&\text{质量}m? \\
&1.D \Rightarrow \Delta\sigma_1, \ldots, \Delta\sigma_n \\
&2.\forall(\xi_i, \eta_i) \in \Delta\sigma_i \\
&\quad m \approx \sum_{i=1}^{n} \rho(\xi_i, \eta_i)\Delta\sigma_i \\
&3.\lambda\text{为}\Delta\sigma_1, \ldots, \Delta\sigma_n\text{直径最大者} \\
&\quad m = \lim_{\lambda \to 0} \sum_{i=1}^{n} \rho(\xi_i, \eta_i)\Delta\sigma_i
\end{aligned}
$$



$$\sum: z=f(x,y)\geq 0,(x,y)\in D$$

体积V?

$1.D\Rightarrow \Delta\sigma_1,\dots,\Delta\sigma_n$

$2.\forall(\xi_i,\eta_i)\in\sigma_i$

$$V\approx\sum_{i=1}^{n}f(\xi_i,\eta_i)\Delta\sigma_i$$

$3.\lambda$为$\Delta\sigma_1,\dots,\Delta\sigma_n$直径最大者

$$V=\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i,\eta_i)\Delta\sigma_i$$

设D为xoy面有界闭区域，$f(x,y)$在D上有界

1. D分成$\Delta\sigma_1,\dots,\Delta\sigma_n$
2. $\forall(\xi_i,\eta_i)\in\Delta\sigma_i(1\leq i\leq n)$, 作$\sum_{i=1}^{n}f(\xi_i,\eta_i)\Delta\sigma_i$
3. λ为$\Delta\sigma_1,\dots,\Delta\sigma_n$直径最大者

$$\text{若}\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i,\eta_i)\Delta\sigma_i\exists\text{, 极限值称}f(x,y)\text{在}D\text{上的二重积分，记作}\iint_D f(x,y)d\sigma$$

$$\text{即}\iint_D f(x,y)d\sigma\triangleq\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i,\eta_i)\Delta\sigma_i$$

定义题型

$$D=\{(x,y)|0\leq x\leq 1,0\leq y\leq 1\}$$

$$1.\lim_{m\to\infty,n\to\infty}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}f(\frac{i}{m},\frac{i}{n})=\iint_D f(x,y)d\sigma$$

$$2.\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}f(\frac{i}{n},\frac{i}{n})=\iint_D f(x,y)d\sigma$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{j}{(n+i)(n^2+j^2)}$$

$$=\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\frac{j}{n}}{(1+\frac{i}{n})[1+(\frac{j}{n})^2]}$$

$$=\int_0^1\frac{1}{1+x}dx\int_0^1\frac{y}{1+y^2}dy$$

$$=\frac{1}{2}\ln 2\ln(1+y^2)\mid_0^1=\frac{1}{2}\ln^2 2$$

性质

1. $D=D_1+D_2, D_1\cap D_2=\emptyset, \iint_D=\iint_{D_1}+\iint_{D_2}$

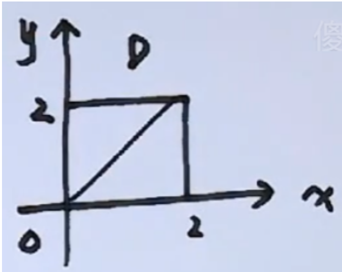
2. $\iint_D 1d\sigma=A$

3. D关于y轴对称，右D_1

$$\begin{cases} f(-x,y)=-f(x,y)\Rightarrow \iint_D f(x,y)d\sigma=0 \\ f(-x,y)=f(x,y)\Rightarrow \iint_D f(x,y)d\sigma=2\iint_{D_1} f(x,y)d\sigma \end{cases}$$

D关于$y=x$对称，则

$$\iint_D f(x,y)d\sigma=\iint_D f(y,x)d\sigma$$



$$f(u)>0, \forall a>0, b>0$$

$$I=\iint_D \frac{af(x)+bf(y)}{f(x)+f(y)}d\sigma=\iint_D \frac{af(y)+bf(x)}{f(y)+f(x)}d\sigma$$

$$2I=(a+b)\iint_D d\sigma=4(a+b), I=2(a+b)$$

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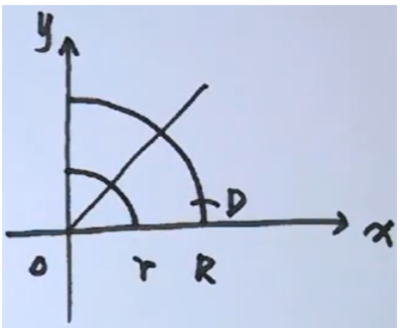
4. $\iint_D 1d\sigma=A$

5.①D关于y轴对称(左右对称),右D_1

$$\begin{cases} 若f(-x,y)=-f(x,y)\Rightarrow \iint_D f(x,y)d\sigma=0 \\ 若f(-x,y)=f(x,y)\Rightarrow \iint_D f(x,y)d\sigma=2\iint_{D_1} f(x,y)d\sigma \end{cases}$$

②D关于$y=x$对称,则

$$\iint_D f(x,y)d\sigma=\iint_D f(y,x)d\sigma$$



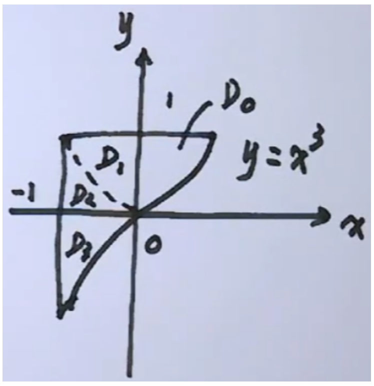
$$f(u)>0\text{连续}, a>0, b>0$$

$$I=\iint_D \frac{a\sqrt{f(x)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}d\sigma$$

$$I=\iint_D \frac{a\sqrt{f(y)}+b\sqrt{f(x)}}{\sqrt{f(x)}+\sqrt{f(y)}}d\sigma$$

$$2I=(a+b)\iint_D 1d\sigma=\frac{\pi}{4}(a+b)(R^2-r^2)$$

$$I=\frac{\pi}{8}(a+b)(R^2-r^2)$$



$$\begin{aligned}I&=\iint_D \sin x\cos y d\sigma\\&=\iint_{D_0+D_1}\sin x\cos y d\sigma+\iint_{D_2+D_3}\sin x\cos y d\sigma\\&=2\iint_{D_2}\sin x\cos y d\sigma\end{aligned}$$

$6.D-$有界闭区域, $f(x,y)\in C(D)$, 则$\exists(\xi,\eta)\in D$, 使

$$\iint_D f(x,y)d\sigma=f(\xi,\eta)A$$

设区域$D=\{(x,y)|x^2+4y^2\le t^2, t>0\}$, 求$\lim\limits_{t\to0^+}\dfrac{\iint_D e^{-x^2}\cos 2ydxdy}{t^2}$.

$$D:\frac{x^2}{t^2}+\frac{y^2}{(\frac{t}{2})^2}\le 1$$

$$\iint_D e^{-x^2}\cos 2ydxdy=e^{-\xi^2}\cdot\cos 2\eta\cdot\pi\cdot t\cdot\frac{t}{2},(\xi,\eta)\in D$$

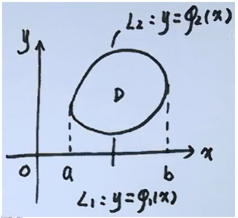
$$\text{原式}=\frac{\pi}{2}\lim_{t\to0}e^{-\xi^2}\cdot\cos 2\eta=\frac{\pi}{2}$$

积分法

直角坐标法

$$\iint_D f(x,y)d\sigma$$

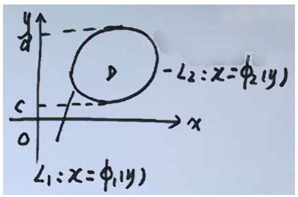
x型区域



$$D=(x,y)|a\le x\le b,\Phi_1(x)\le y\le\Phi_2(x)$$

$$\iint_D f(x,y)d\sigma=\int_a^b dx\int_{\Phi_1(x)}^{\Phi_2(x)}f(x,y)dy$$

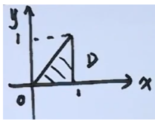
y型区域



$$D=\{(x,y)|c\le y\le d,\phi_1(y)\le x\le\phi_2(y)\}$$

$$\iint_D f(x,y)d\sigma=\int_c^d dy\int_{\phi_1(y)}^{\phi_2(y)}f(x,y)dx$$

计算$\iint_D x^2ydxdy$, 其中D由$y=x,x=1$及x轴所围成.



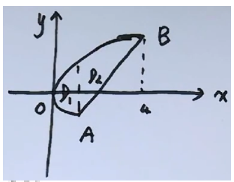
$$D=\{(x,y)|0\le x\le 1,0\le y\le x\}$$

$$\iint_D x^2yd\sigma=\int_0^1x^2dx\int_0^x ydy=\frac{1}{2}\int_0^1x^4dx=\frac{1}{10}$$

$$D=\{(x,y)|0\le y\le 1,y\le x\le 1\}$$

$$\iint_D x^2yd\sigma=\int_0^1ydy\int_y^1x^2dx=\frac{1}{3}\int_0^1(y-y^4)dy=\frac{1}{10}$$

计算$I=\iint_D xdxdy$, 其中D由$x=y^2$与$y=x-2$围成

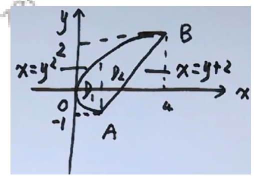


$$由\begin{cases}x=y^2\\y=x-2\end{cases}\Rightarrow A(1,-1),B(4,2)$$

$$D_1=\{(x,y)|0\le x\le 1,-\sqrt{x}\le y\le\sqrt{x}\}$$
$$D_2=\{(x,y)|1\le x\le 4,x-2\le y\le\sqrt{x}\}$$

$$原式=\int_0^1 xdx\int_{-\sqrt{x}}^{\sqrt{x}}1dy+\int_1^4 xdx\int_{x-2}^{\sqrt{x}}1dy$$

$$=2\int_0^1 x^{\frac{3}{2}}dx+\int_1^4(x^{\frac{3}{2}}+x^2-2x)dx$$



$$D=\{(x,y)|y^2\le x\le y+2,-1\le y\le 2\}$$

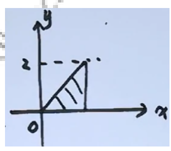
$$原式=\int_{-1}^2 dy\int_{y^2}^{y+2}xdx$$

$$x^{2n}e^{\pm x^2}dx$$

$$e^{\frac{k}{x}}dx$$

$$\cos\frac{k}{x}dx,\sin\frac{k}{x}dx$$

$$计算I=\int_0^2 dy\int_y^2 e^{-x^2}dx$$



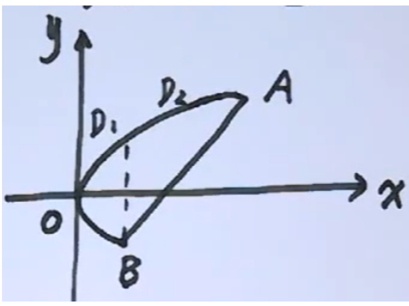
$$I=\int_0^2 dy\int_y^2 e^{-x^2}dx$$

$$=\int_0^2 xe^{-x^2}dx$$

$$=-\frac{1}{2}e^{-x^2}\ |_0^2$$

$$=-\frac{1}{2}(\frac{1}{e^4}-1)$$

$$计算\iint_D(x+y)dxdy,其中D由x=y^2与y=x-2围成.$$



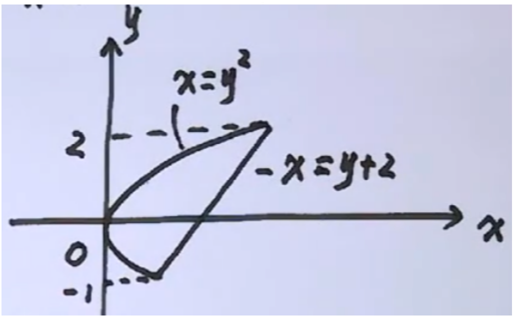
由 $\begin{cases}x=y^2\\y=x-2\end{cases}\Rightarrow\begin{cases}x=1\\y=-1\end{cases},\begin{cases}x=4\\y=2\end{cases}$

法一 $:D_1=\{(x,y)|0\le x\le 1,-\sqrt{x}\le y\le\sqrt{x}$

$D_2=\{(x,y),1\le x\le 4,x-2\le y\le\sqrt{x}\}$

$$原式=\int_0^1 dx\int_{-\sqrt{x}}^{\sqrt{x}}(x+y)dy+\int_1^4 dx\int_{x-2}^{\sqrt{x}}(x+y)dy$$

$$=2\int_0^1 x\sqrt{x}dx+\int_1^4 dx\int_{x-1}^{\sqrt{x}}(x+y)dy$$



$$法二：原式=\int_{-1}^2 dy\int_{y^2}^{y+2}(x+y)dx$$

$$=2\int_0^1 x\sqrt{x}dx+\int_1^4 dx\int_{x-1}^{\sqrt{x}}(x+y)dy$$

计算二重积分时，

$1.x^{2n}e^{\pm x^2}dx$

$2.e^{\frac{k}{x}}dx$

$3.\cos\dfrac{k}{x}dx$

$4.\sin\dfrac{k}{x}dx$

$$计算\int_0^1 dy\int_y^1 e^{x^2}dx.$$

$$原式=\int_0^1 e^{x^2}dx\int_0^x dy$$

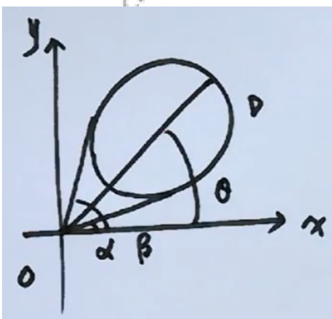
$$=\int_0^1 xe^{x^2}dx=\frac{1}{2}e^{x^2}\ |_0^1=\frac{e-1}{2}$$

极坐标法

特征

$$\begin{gathered}1.D\text{边界曲线含}x^2+y^2\\2.f(x,y)\text{中含}x^2+y^2\end{gathered}$$

变换



$$\begin{cases}x=r\cos\theta\\y=r\sin\theta\end{cases}$$

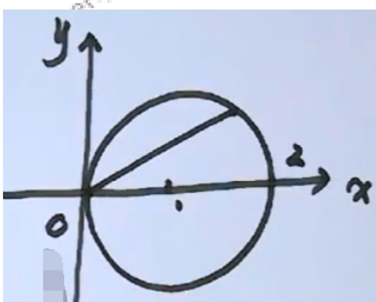
$$\alpha\le\theta\le\beta,r_1(\theta)\le r\le r_2(\theta)$$

面积

$$d\sigma=rdrd\theta$$

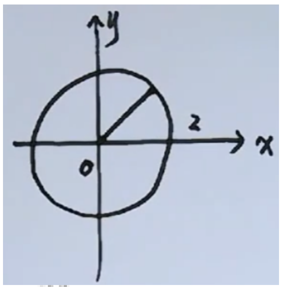
$$\iint_D f(x,y)d\sigma=\int_\alpha^\beta d\theta\int_{r_1(\theta)}^{r_2(\theta)}rf(r\cos\theta,r\sin\theta)dr$$

$$\text{计算}I=\iint_D(x^2+xy)d\sigma,\text{其中}D:x^2+y^2\le 2x.$$



$$\begin{aligned}&D:(x-1)^2+y^2\le 1\\&I=\iint_D x^2d\sigma\\&\text{令}\begin{cases}x=r\cos\theta\\y=r\sin\theta\end{cases}(-\frac{\pi}{2}\le\theta\le\frac{\pi}{2},0\le r\le 2\cos\theta)\end{aligned}$$

$$\begin{aligned}\text{原式}&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta d\theta\int_0^{2\cos\theta}r^3dr\\&=4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^6\theta d\theta=8I_6\\&=8\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{5\pi}{4}\end{aligned}$$

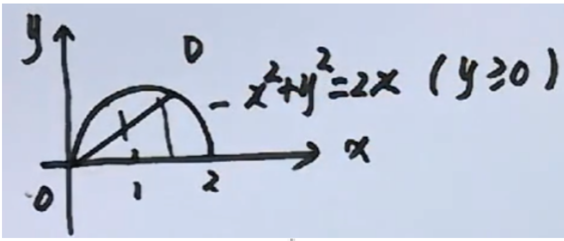


$$计算I=\iint_D(x^2+3xy)d\sigma,其中圆域x^2+y^2\leq 4$$

$$I=\iint_D(x^2+3xy)d\sigma=\iint_D x^2d\sigma=\iint_D y^2d\sigma=\frac{1}{2}\iint_D(x^2+y^2)d\sigma$$

$$令\begin{cases}x=r\cos\theta\\y=r\sin\theta\end{cases}(0\leq\theta\leq 2\pi,0\leq r\leq 2)$$

$$I=\frac{1}{2}\int_0^{2\pi}d\theta\int_0^2 r^3dr=4\pi$$



$$I=\iint_D x^2d\sigma,D由y=\sqrt{2x-x^2}与x轴围成$$

$$令\begin{cases}x=r\cos\theta\\y=r\sin\theta\end{cases}(0\leq\theta\leq\frac{\pi}{2},0\leq r\leq 2\cos\theta)$$

$$\begin{aligned}I&=\int_0^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}r^3\cos^2\theta dr\\&=\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta\int_0^{2\cos\theta}r^3dr\\&=4\int_0^{\frac{\pi}{2}}\cos^6\theta\\&=4*\frac{5}{6}*\frac{3}{4}*\frac{1}{2}*\frac{\pi}{2}\end{aligned}$$