极限

epsilon-N 数列极限

$$
\begin{gathered}
若(\exists A), 对\forall \epsilon > 0, \exists N > 0, 当n > N时,\\
|a_n - A| < \epsilon\\
\lim_{n\to\infty} a_n = A或a_n \to A(n \to \infty)
\end{gathered}
$$

$$
a_n = \frac{n+1}{2n}, \lim_{n\to\infty} a_n = \frac{1}{2}, \frac{n+1}{2n} \neq \frac{1}{2}
$$

$$
\begin{gathered}
证明：若\lim_{n\to\infty} a_n = A, 则\lim_{n\to\infty} |a_n| = |A|, 反之不对.\\
\Rightarrow\\
\because \lim_{n\to\infty} a_n = A\\
\therefore \forall \epsilon > 0, \exists N > 0, 当n > N时,\\
|a_n - A| < \epsilon\\
\because ||a_n| - |A|| \leq |a_n - A|\\
\therefore \forall \epsilon > 0, \exists N > 0, 当n > N时\\
||a_n| - |A|| < \epsilon\\
\therefore \lim_{n\to\infty} |a_n| = |A|\\
\nLeftarrow\\
a_n = (-1)^n, \lim_{n\to\infty} a_n不存在, 但\lim_{n\to\infty} |a_n| = 1
\end{gathered}
$$

epsilon-delta

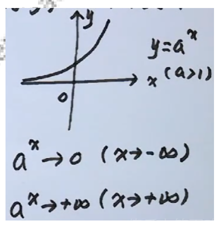
$$
\begin{gathered}
f(x)在x = a的去心邻域内有定义\\
若\forall \epsilon > 0, \exists \delta > 0, 当0 < |x - a| < \delta, 有\\
|f(x) - A| < \epsilon\\
\lim_{x\to a} f(x) = A或f(x) \to A(x \to a)
\end{gathered}
$$

$$
\begin{aligned}
&1.\quad x \to a : \begin{cases} x \neq a \\ x \to a^-, x \to a^+ \end{cases}\\
&2.\quad \lim_{x\to a} f(x)与f(a)无关
\end{aligned}
$$

$$
\begin{gathered}
f(x) = \frac{x^2 + x - 2}{x^2 - 1}(x \neq \pm 1)\\
\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x+2}{x+1} = \frac{3}{2}, 而f(1)不存在
\end{gathered}
$$

$$
\begin{aligned}
&3.\ \text{若}\forall\epsilon>0,\exists\delta_1>0,\text{当}x\in(a-\delta_1,a)\text{时}\\
&|f(x)-A|<\epsilon\\
&\lim_{x\to a^-}f(x)=A\text{或}f(a-0)=A\\
&\text{若}\forall\epsilon>0,\exists\delta_2>0,\text{当}x\in(a,a+\delta_2)\text{时}\\
&|f(x)-B|<\epsilon\\
&\lim_{x\to a^+}f(x)=B\text{或}f(a+0)=B\\
&\lim_{x\to a}f(x)\exists\Leftrightarrow f(a-0),f(a+0)\exists\text{且等}
\end{aligned}
$$

$$
f(x)\text{含}\begin{cases}a^{\frac{?}{x-b}}\\ a^{\frac{?}{b-x}}\end{cases}\text{当}x\to b\text{时分左右极限}
$$



$$
\text{设}f(x)=\frac{1-2^{\frac{1}{x-1}}}{1+2^{\frac{1}{x-1}}},\text{判断}\lim_{x\to1}f(x)\text{是否存在.}
$$

$$
x\to1^-\Rightarrow\frac{1}{x-1}\to-\infty\Rightarrow2^{\frac{1}{x-1}}\to0
$$

$$
f(1-0)=1
$$

$$
x\to1^+\Rightarrow\frac{1}{x-1}\to+\infty\Rightarrow2^{\frac{1}{x-1}}\to+\infty
$$

$$
f(1+0)=-1
$$

$$
\because f(1-0)\neq f(1+0),\therefore f(x)\text{不存在}
$$

epsilon-x

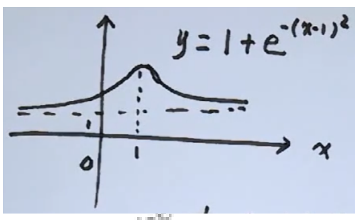
case1

$$
\begin{gathered}
\text{若}\forall\epsilon>0,\exists X>0,\text{当}x>X\text{时},\\
|f(x)-A|<\epsilon\\
\lim_{x\to+\infty}f(x)=A
\end{gathered}
$$

case2

$$
\begin{gathered}
\text{若}\forall\epsilon>0,\exists X>0,\text{当}x<-X\text{时},\\
|f(x)-A|<\epsilon\\
\lim_{x\to-\infty}f(x)=A
\end{gathered}
$$

case3



$$\text{若}\forall\epsilon>0,\exists X>0,\text{当}|x|>X\text{时},$$
$$|f(x)-A|<\epsilon$$
$$\lim_{x\to\infty}f(x)=A$$

一般性质

唯一性

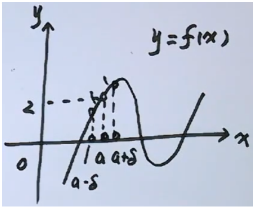
$$\text{极限存在必唯一}$$
$$\text{证}:\text{设}\lim_{x\to a}f(x)=A,\lim_{x\to a}f(x)=B$$
$$\text{若}A>B:\text{取}\epsilon=\frac{A-B}{2}>0,$$
$$\because\lim_{x\to a}f(x)=A,\therefore\exists\delta_1>0,\text{当}0<|x-a|<\delta_1\text{时},$$
$$|f(x)-A|<\frac{A-B}{2}$$
$$\text{即}\frac{A+B}{2}<f(x)<\frac{3A-B}{2}(*)$$
$$\because\lim_{x\to a}f(x)=B,\therefore\exists\delta_2>0,\text{当}0<|x-a|<\delta_2\text{时},$$
$$|f(x)-B|<\frac{A-B}{2}$$
$$\text{即}\frac{3B-A}{2}<f(x)<\frac{A+B}{2}(**)$$
$$\text{取}\delta=\min\{\delta_1,\delta_2\},\text{当}0<|x-a|<\delta\text{时},(*)(**)\text{皆对矛盾}$$
$$\therefore A>B\text{不对}$$
$$\text{若}A<B:\text{同理不对}$$
$$\therefore A=B$$

有界性

$$\text{若}\lim_{n\to\infty}a_n=A\Rightarrow\exists M>0,\forall n,|a_n|\leq M$$
$$\not\Leftarrow$$
$$\text{证}:\Rightarrow\ \text{取}\epsilon=1$$
$$\because\lim_{n\to\infty}a_n=A,\therefore\lim_{n\to\infty}|a_n|=|A|$$
$$\therefore\exists N>0,\text{当}n>N\text{时},\text{有}|a_n-A|<1$$
$$||a_n|-|A||<1\Rightarrow|a_n|<1+|A|$$
$$\text{取}M=\max\{|a_1|,|a_2|,\ldots,|a_N|,1+|A|\}$$
$$\therefore\forall n,|a_n|\leq M$$
$$\not\Leftarrow a_n=2+(-1)^n,|a_n|\leq3,\text{但}\lim_{n\to\infty}a_n\text{不存在}$$

保号性

$$\text{若}\lim_{x\to a}f(x)=A\begin{cases}>0\\<0\end{cases},\text{则}$$
$$\exists\delta>0,\text{当}0<|x-a|<\delta\text{时，}$$
$$f(x)\begin{cases}>0\\<0\end{cases}$$



$$\lim_{x\to a} f(x) = 2 > 0$$
$$\text{当}0 < |x-a| < \delta\text{时}$$
$$f(x) > 0$$

$$A > 0$$
$$\text{取}\epsilon = \frac{A}{2}$$
$$\because \lim_{x\to a} f(x) = A$$
$$\therefore \exists\delta > 0, \text{当}0 < |x-a| < \delta\text{时}$$
$$|f(x) - A| < \frac{A}{2}$$
$$\Rightarrow f(x) > \frac{A}{2} > 0$$
$$A < 0$$
$$\text{取}\epsilon = -\frac{A}{2}$$
$$\because \lim_{x\to a} f(x) = A$$
$$\therefore \exists\delta > 0, \text{当}0 < |x-a| < \delta\text{时}$$
$$|f(x) - A| < -\frac{A}{2}$$
$$\Rightarrow f(x) < \frac{A}{2} < 0$$

设$\lim\limits_{x\to 1}\dfrac{f(x)-3}{\ln^2 x} = -2$且$f(1)=3$, 讨论$x=1$是否为函数$f(x)$的极值点.

$$\because \lim_{x\to 1}\frac{f(x)-3}{\ln^2 x} = -2 < 0$$
$$\therefore \exists\delta > 0, \text{当}0 < |x-1| < \delta\text{时}, \frac{f(x)-3}{\ln^2 x} < 0$$
$$\because \ln^2 x > 0, \therefore f(x) - 3 < 0$$
$$\text{即当}0 < |x-1| < \delta\text{时}, f(x) < f(1)$$
$$\therefore x = 1\text{为}f(x)\text{的极大点}$$

设$f'(0)=0$, 又$\lim\limits_{x\to 0}\dfrac{f'(x)}{x^3} = -2$, 讨论$x=0$是否为$f(x)$的极值点.

$$\because \lim_{x\to 0}\frac{f'(x)}{x^3} = -2 < 0,$$
$$\therefore \exists\delta > 0, \text{当}0 < |x| < \delta\text{时}, \frac{f'(x)}{x^3} < 0$$
$$\begin{cases} f'(x) > 0, x \in (-\delta, 0) \\ f'(x) < 0, x \in (0, \delta) \end{cases} \Rightarrow x = 0\text{为}f(x)\text{的极大点}$$

$$f(1)=2,\lim_{x\to 1}\frac{f(x)-2}{(x-1)^2}=3,x=1?$$

$$\exists\delta>0,\text{当}0<|x-1|<\delta\text{时},$$
$$\frac{f(x)-2}{(x-1)^2}>0\Rightarrow f(x)-2>0$$
$$\Rightarrow f(x)>f(1)\Rightarrow x=1\text{为}f(x)\text{极小点}$$

$$f'(1)=0,\lim_{x\to 1}\frac{f'(x)}{(x-1)^3}=-2,x=1?$$

$$\exists\delta>0,\text{当}0<|x-1|<\delta\text{时},$$
$$\frac{f'(x)}{(x-1)^3}<0$$
$$\begin{cases}f'(x)>0,x\in(1-\delta,1)\\ f'(x)<0,x\in(1,1+\delta)\end{cases}\Rightarrow x=1\text{为}f(x)\text{的极大点}$$

运算性质

四则

1. 若 $\lim f(x),\lim g(x)\exists\Rightarrow\lim[f(x)\pm g(x)]=\lim f(x)\pm\lim g(x)$
2. 若 $\lim f(x),\lim g(x)\exists\Rightarrow\lim f(x)g(x)=\lim f(x)\lim g(x)$
3. 若 $\lim f(x)\exists,\lim g(x)\exists\neq 0\Rightarrow\lim\frac{f(x)}{g(x)}=\frac{\lim f(x)}{\lim g(x)}$

复合

1. $\lim\limits_{u\to a}f(u)=A,\lim\limits_{x\to x_0}\Phi(x)=a\Rightarrow$
   $\lim\limits_{x\to x_0}f[\Phi(x)]=A$
2. $\lim\limits_{u\to a}f(u)=f(a),\lim\limits_{x\to x_0}\Phi(x)=a\Rightarrow$
   $\lim\limits_{x\to x_0}f[\Phi(x)]=f[\lim\limits_{x\to x_0}\Phi(x)]=f(a)$