相关性与线性表示(向量理论一)

方程组三种形式

1.一般形式

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = 0 \\ \cdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = 0 \end{cases} (*)$$

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \cdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_n \end{cases} (**)$$

2.矩阵形式

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \cdots & & \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$AX = 0(*)$$

$$AX = b(**)$$

3.向量形式

$$\alpha_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \alpha_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, \alpha_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = 0(*)$$

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = b(**)$$

齐次与非齐次解的情形

$$1.\begin{cases} x_1 - 2x_2 = 0 \\ 2x_1 + x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}; (\text{无自由变量})$$

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, |A| = 5 \neq 0 \Rightarrow r(A) = 2$$

$$2.\begin{cases} x_1 + x_2 - 2x_3 = 0 \\ 2x_1 + 3x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0, \\ x_3 = 0 \end{cases} \begin{cases} x_1 = 7 \\ x_2 = -5, \\ x_3 = 1 \end{cases} \begin{cases} x_1 = -7 \\ x_2 = 5 \\ x_3 = -1 \end{cases}, \cdots (\text{有自由变量})$$

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 5 \end{pmatrix}, r(A) = 2 < 3$$

$$(*) \begin{cases} \text{仅有零解(变量与约束条件个数等)} \\ \text{除零解外有无数个非零解(变量多于约束条件的个数)} \end{cases}$$

$$1.\begin{cases} 2x_1 + x_2 = 1 \\ x_1 - x_2 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = -1 \end{cases} (\text{约束条件等于未知数个数})$$

$$2.\begin{cases} x_1 + 2x_3 = -1 \\ 2x_1 - x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = -1 \end{cases}, \begin{cases} x_1 = 0 \\ x_2 = -1 \\ x_3 = -\frac{1}{2} \end{cases}, \cdots (\text{约束条件少于未知数个数})$$

$$3.\begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 4 \end{cases} \text{无解}$$

$$(**) \begin{cases} \text{有解} \begin{cases} \text{唯一解} \\ \text{无数解} \end{cases} \\ \text{无解} \end{cases}$$

相关性

$$\alpha_1,\alpha_2,\ldots,\alpha_n:$$
$$x_1\alpha_1+x_2\alpha_2+\ldots+x_n\alpha_n=0(*)$$
$$\text{case1. }(*)\text{仅有零解,称}\alpha_1,\alpha_2,\ldots,\alpha_n\text{线性无关}$$
$$\text{case2. }(*)\text{有非零解,称}\alpha_1,\alpha_2,\ldots,\alpha_n\text{线性相关}$$

线性表示

$$\alpha_1,\alpha_2,\ldots,\alpha_n,b$$
$$x_1\alpha_1+\ldots+x_n\alpha_n=b(**)$$
$$\text{case1. }(**)\text{有解,称}b\text{可由}\alpha_1,\ldots,\alpha_n\text{线性表示}$$
$$\text{case2. }(**)\text{无解,称}b\text{不可由}\alpha_1,\ldots,\alpha_n\text{线性表示}$$

性质

$$1.\alpha_1,\ldots,\alpha_n\text{线性相关}\Leftrightarrow\text{至少有一个向量可由其余向量线性表示}$$
$$\text{证}:\Rightarrow,\exists\text{不全为0的}k_1,k_2,\ldots,k_n,\text{使}$$
$$k_1\alpha_1+\ldots+k_n\alpha_n=0$$
$$\text{设}k_1\neq0\Rightarrow\alpha_1=-\frac{k_2}{k_1}\alpha_2-\ldots-\frac{k_n}{k_1}\alpha_n$$
$$\Leftarrow,\text{设}\alpha_k=l_1\alpha_1+\ldots+l_{n-1}\alpha_{n-1}+l_{k+1}\alpha_{k+1}+\ldots+l_n\alpha_n$$
$$\Rightarrow l_1\alpha_1+\ldots+l_{n-1}\alpha_{n-1}+(-1)l_k+l_{k+1}\alpha_{k+1}+\ldots+l_n\alpha_n=0$$
$$\therefore\alpha_1\ldots\alpha_n\text{线性相关}$$

$$\text{Notes:}$$
$$\text{①含零向量的向量组一定相关}$$
$$\text{证}:\text{设}\alpha_1=0,\alpha_2\neq0,\alpha_3\neq0$$
$$\text{法一}:\because 2\alpha_1+0\alpha_2+0\alpha_3=0$$
$$\therefore\alpha_1,\alpha_2,\alpha_3\text{线性相关}$$
$$\text{法二}:\because\alpha_1=0\alpha_2+0\alpha_3$$
$$\therefore\alpha_1,\alpha_2,\alpha_3\text{线性相关}$$
$$\text{②}\alpha,\beta\text{线性相关}\Leftrightarrow\alpha,\beta\text{成比例}$$
$$\text{证}:\Rightarrow\exists\text{不全为0的}k_1,k_2,\text{使}$$
$$k_1\alpha+k_2\beta=0$$
$$\text{设}k_1\neq0\Rightarrow\alpha=-\frac{k_2}{k_1}\beta$$
$$\Leftarrow\text{设}\beta=k\alpha\Rightarrow k\alpha+(-1)\beta=0$$
$$\therefore\alpha,\beta\text{线性相关}$$

$2.$设$\alpha_1,...,\alpha_n$线性无关

①$\alpha_1,...,\alpha_n,\beta$线性无关 $\Leftrightarrow \beta$不可由$\alpha_1,...,\alpha_n$线性表示

②$\alpha_1,...,\alpha_n,\beta$线性相关 $\Rightarrow \beta$可由$\alpha_1,...,\alpha_n$唯一线性表示

②证明 $:\because \alpha_1,...,\alpha_n,\beta$线性相关

$\therefore \exists$不全为0的$k_1,...,k_n,k_0,$使

$k_1\alpha_1+...+k_n\alpha_n+k_0\beta=0$

$k_0\neq 0,$若$k_0=0\Rightarrow k_1\alpha_1+...+k_n\alpha_n=0$

$\because \alpha_1...\alpha_n$线性无关$,\therefore k_1=...=k_n=0,$矛盾$,\therefore k_0\neq 0$

$$\Rightarrow \beta=-\frac{k_1}{k_0}\alpha_1-...-\frac{k_n}{k_0}\alpha_n$$

(反)设$\beta=l_1\alpha_1+...+l_n\alpha_n$

$\beta=t_1\alpha_1+...+t_n\alpha_n$

$\Rightarrow (l_1-t_1)\alpha_1+...+(l_n-t_n)\alpha_n=0$

$\because \alpha_1...\alpha_n$线性无关$,\therefore l_i=t_i(1\leq i\leq n)$

$3.$全组无关 $\Rightarrow$ 部分组无关

$4.$部分组相关 $\Rightarrow$ 全组相关

$\alpha_1,\alpha_2,\alpha_3$线性无关$,\alpha_2,\alpha_3,\alpha_4$线性相关$,$

问α_4可否由$\alpha_1,\alpha_2,\alpha_3$线性表示?

$\alpha_1,\alpha_2,\alpha_3$线性无关 $\Rightarrow \alpha_2,\alpha_3$线性无关

$\alpha_2,\alpha_3,\alpha_4$线性相关 $\Rightarrow \alpha_4=k_2\alpha_2+k_3\alpha_3$

$\alpha_4=0\alpha_1+k_2\alpha_2+k_3\alpha_3$

α_1,α_2线性无关$,\beta_1$不可由α_1,α_2线性表示$,\beta_2$可由α_1,α_2线性表示

①$\alpha_1,\alpha_2,k\beta_1+\beta_2$②$\beta_1+k\beta_2$

①若$k=0:\alpha_1,\alpha_2,k\beta_1+\beta_2$线性相关

若$k\neq 0:\alpha_1,\alpha_2,k\beta_1+\beta_2$线性无关

②$\alpha_1,\alpha_2,\beta_1+k\beta_2$线性无关

$5.\alpha_1,\alpha_2,...,\alpha_n$为$n$个$n$维向量

①$\alpha_1,\alpha_2,...,\alpha_n$线性无关 $\Leftrightarrow |\alpha_1...\alpha_n|\neq 0$

②$\alpha_1,\alpha_2,...,\alpha_n$线性相关 $\Leftrightarrow |\alpha_1...\alpha_n|=0$

证 :令$A=(\alpha_1...\alpha_n)$

①$\alpha_1...\alpha_n$线性无关 $\Leftrightarrow Ax=0$仅有零解

$\Leftrightarrow r(A)=n\Leftrightarrow |A|\neq 0$

②$\alpha_1...\alpha_n$线性相关 $\Leftrightarrow Ax=0$有非零解

$\Leftrightarrow r(A)<n\Leftrightarrow |A|=0$

$$\alpha_1=\begin{pmatrix}1\\-1\\1\end{pmatrix},\alpha_2=\begin{pmatrix}1\\2\\4\end{pmatrix},\alpha_3=\begin{pmatrix}1\\a\\a^2\end{pmatrix}$$

$$|\alpha_1,\alpha_2,\alpha_3|=\begin{vmatrix}1&1&1\\-1&2&a\\1&4&a^2\end{vmatrix}=3(a+1)(a-2)$$

①$a\neq -1$且$a\neq -2:\alpha_1,\alpha_2,\alpha_3$线性无关

①$a=-1$或$a=-2:\alpha_1,\alpha_2,\alpha_3$线性相关

$$\begin{cases}x_1+3x_2-2x_3=0\\x_1+x_2+4x_3=0\end{cases}$$

$$x_1\begin{pmatrix}1\\1\end{pmatrix}+x_2\begin{pmatrix}3\\1\end{pmatrix}+x_3\begin{pmatrix}-2\\4\end{pmatrix}=0,\text{一维一个方程}$$

6.$\alpha_1,\alpha_2,\ldots,\alpha_n$为$n$个$m$维向量且$m<n$

$\Rightarrow \alpha_1\ldots\alpha_n$线性相关

证 :令$A=(\alpha_1\ldots\alpha_n)_{m\times n}$

$x_1\alpha_1+\ldots+x_n\alpha_n=0\Leftrightarrow Ax=0$

$\because m<n,\therefore r(A)\le m<n$

$\Rightarrow Ax=0$有非零解$\Leftrightarrow \alpha_1\ldots\alpha_n$线性相关

设$\alpha_1,\alpha_2,\alpha_3$为3个3维线性无关向量

$\forall\beta$, 证 : β可由$\alpha_1,\alpha_2,\alpha_3$唯一线性表示

证 : $\because \alpha_1,\alpha_2,\alpha_3$线性无关

又 $\because \alpha_1,\alpha_2,\alpha_3,\beta$线性相关

$\therefore \beta$可由$\alpha_1,\alpha_2,\alpha_3$唯一线性表示

7.①加个数提交相关性

②加维数提升无关性

$$\alpha_1=\begin{pmatrix}2\\3\\1\\0\\0\end{pmatrix},\alpha_2=\begin{pmatrix}1\\-1\\0\\1\\0\end{pmatrix},\alpha_3=\begin{pmatrix}3\\-2\\0\\0\\1\end{pmatrix}$$

$$\because \begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}=1\ne 0,\therefore \begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\text{线性无关}$$

$\Rightarrow \alpha_1,\alpha_2,\alpha_3$线性无关

8.$\alpha_1,\alpha_2,\ldots,\alpha_n$非零且两两正交 $\Rightarrow \alpha_1,\alpha_2,\ldots,\alpha_n$线性无关

$\not\Leftarrow$

证 :$\Rightarrow$, 令$k_1\alpha_1+\ldots+k_n\alpha_n=0$

$(\alpha_1,k_1\alpha_1+k_2\alpha_2+\ldots+k_n\alpha_n)=0$

$k_1(\alpha_1,\alpha_1)=0$

$\because (\alpha_1,\alpha_1)=\alpha_1^T\alpha_1=|\alpha_1|^2>0,\therefore k_1=0$

$(\alpha_2,k_2\alpha_2+\ldots+k_n\alpha_n)=0$

$k_2(\alpha_2,\alpha_2)=0$

$\because (\alpha_2,\alpha_2)=|\alpha_2|^2>0,\therefore k_2=0$

$k_n\alpha_n=0$

$\because \alpha_n\ne 0,\therefore k_n=0\Rightarrow \alpha_1\ldots\alpha_n$线性无关

$$\not\Leftarrow,\alpha_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix},\alpha_3=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\because \begin{vmatrix}1&1&1\\0&1&1\\0&0&1\end{vmatrix}=1\ne 0,\therefore \alpha_1,\alpha_2,\alpha_3\text{线性无关}$$

$(\alpha_1,\alpha_2)=1\ne 0,(\alpha_1,\alpha_3)=1\ne 0,(\alpha_2,\alpha_3)=2\ne 0$

相关性证明

$$相关性证明\begin{cases}性质\\定义法\end{cases}$$

$1.\alpha_1,\alpha_2,\alpha_3$线性无关$,\beta_1=\alpha_1+\alpha_2,\beta_2=\alpha_2+\alpha_3,\beta_3=\alpha_3+\alpha_1$

β_1,β_2,β_3

解：令$k_1\beta_1+k_2\beta_2+k_3\beta_3=0$

$\Rightarrow(k_1+k_3)\alpha_1+(k_1+k_2)\alpha_2+(k_2+k_3)\alpha_3=0$

$\because\alpha_1,\alpha_2,\alpha_3$线性无关

$$\therefore\begin{cases}k_1+k_3=0\\k_1+k_2=0\ (*)\\k_2+k_3=0\end{cases}$$

$$\because|A|=\begin{vmatrix}1&0&1\\1&1&0\\0&1&1\end{vmatrix}=2\neq0,\therefore r(A)=3$$

$\Rightarrow k_1=k_2=k_3=0\Rightarrow\beta_1,\beta_2,\beta_3$线性无关

$2.\alpha_1\sim\alpha_4$线性无关$,\beta_1=\alpha_1+\alpha_2,\beta_2=\alpha_2+\alpha_3,\beta_3=\alpha_3+\alpha_4,\beta_4=\alpha_4+\alpha_1$

问$\beta_1\sim\beta_4$?

$k_1\beta_1+k_2\beta_2+k_3\beta_3+k_4\beta_4=0$

$\Rightarrow(k_1+k_4)\alpha_1+(k_1+k_2)\alpha_2+(k_2+k_3)\alpha_3+(k_3+k_4)\alpha_4=0$

$\because\alpha_1\sim\alpha_4$线性无关

$$\therefore\begin{cases}k_1+k_4=0\\k_1+k_2=0\\k_2+k_3=0\\k_4+k_4=0\end{cases}(*)$$

$$|A|=\begin{vmatrix}1&0&0&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{vmatrix}=0\Rightarrow r(A)<4$$

$\Rightarrow(*)$有非零解$\Rightarrow\beta_1\sim\beta_4$线性相关

$\beta_1-\beta_2=\alpha_1-\alpha_3$

$\beta_3-\beta_4=\alpha_3-\alpha_1$

$\Rightarrow\beta_1-\beta_2+\beta_3-\beta_4=0\Rightarrow\beta_1\sim\beta_4$线性相关

$3.\alpha_1,\alpha_2$线性无关$,\beta\neq0$与α_1,α_2正交, 证：α_1,α_2,β线性无关

令$k_1\alpha_1+k_2\alpha_2+k_0\beta=0$

由$(\beta,k_1\alpha_1+k_2\alpha_2+k_0\beta=0$

$\because\beta$与α_1,α_2正交$,\therefore k_0(\beta,\beta)=0$

而$(\beta,\beta)=|\beta|^2>0,\therefore k_0=0$

$\Rightarrow k_1\alpha_1+k_2\alpha_2=0$

$\because\alpha_1,\alpha_2$线性无关$,\therefore k_1=0,k_2=0$

$\therefore\alpha_1,\alpha_2,\beta$线性无关