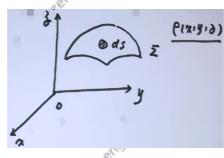
积分域	积分号	例子
线状	\int	\int_a^bf(x)dx \int_L
面状	\iint	\iint_Df(x,y)d\sigma \iint_{\sum}
体状	\iiint	\iiint_{\Omega}f(x,y,z)dV

# 曲面积分

# 对面积的曲面积分 (第一类曲面积分)

# 背景: m



 $egin{aligned} 1. orall dS \subset \Sigma \ 2. dm &= 
ho(x,y,z) dS \end{aligned}$ 

$$3.m = \iint_{\Sigma} dm = \iint_{\Sigma} 
ho(x,y,z) dS$$

# 定义

$$\iint_{\Sigma} f(x, y, z) dS$$

# 性质

$${f f}$$
  $1.\iint_{\Sigma}1dS=A$ 

2.① $\Sigma$ 关于xoy面对称,上 $\Sigma_1$ 

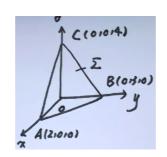
$$egin{aligned} egin{aligned} eta f(x,y,-z) &= -f(x,y,z) \Rightarrow \iint_{\Sigma} f(x,y,z) dS = 0 \ egin{aligned} eta f(x,y,-z) &= f(x,y,z) \Rightarrow \iint_{\Sigma} f(x,y,z) dS = 2 \iint_{\Sigma_1} f(x,y,z) dS \end{aligned}$$

## 计算方法。

## 特殊法。

关
$$ext{计算}I=\iint_{\Sigma}(2x+rac{4y}{3}+z)dS,$$
其中 $\Sigma$ 是平面 $rac{x}{2}+rac{y}{3}+rac{z}{4}=1$ 在第  $I$  卦限的部分.

Lengtagothens



<sub>Lend</sub>baothend

$$egin{aligned} I=&4\iint_{\Sigma}(rac{x}{2}+rac{y}{3}+rac{z}{4})dS\ =&4\iint_{\Sigma}1dS \end{aligned}$$

$$\overrightarrow{AB} = \{-2, 3, 0\}, \overrightarrow{AC} = \{-2, 0, 4\}$$
 $\overrightarrow{AB} imes \overrightarrow{AC} = \{12, 8, 6\}$ 
 $|\overrightarrow{AB} imes \overrightarrow{AC}| = \sqrt{144 + 64 + 36} = 2\sqrt{61} \Rightarrow S_{\triangle ABC} = \sqrt{61}$ 
 $\therefore$  原式  $= 4\sqrt{61}$ 

### 二重积分法

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$$I = \iint_{\Sigma} f(x, y, z) dS$$

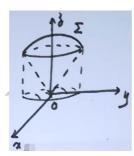
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$$egin{align} 1.\Sigma: z &= \Phi(x,y), (x,y) \in D_{xy} \ 2.dS &= \sqrt{1 + (z_x')^2 + (z_y')^2} d\sigma \ 3.I &= \iint_{D_{xy}} f[x,y,\Phi(x,y)] \cdot \sqrt{1 + (z_x')^2 + (z_y')^2} d\sigma \ \end{cases}$$

求
$$I = \iint_{\Sigma} z dS$$
, 其中 $\Sigma$ 为 $x^2 + y^2 + z^2 = 1$ 被 $z = \sqrt{x^2 + y^2}$ 所截的顶部.

endbachend



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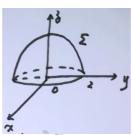
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OZO.

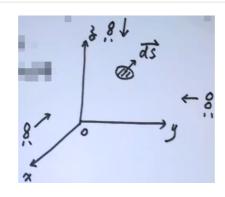
$$egin{align} 1.\Sigma: z &= \sqrt{1-x^2-y^2} \ D_{xy}: x^2+y^2 &\leq rac{1}{2} \ 2.z_x' &= -rac{x}{\sqrt{1-x^2-y^2}}, z_y' &= -rac{y}{\sqrt{1-x^2-y^2}} \ dS &= \sqrt{1+rac{x^2+y^2}{1-x^2-y^2}} d\sigma &= rac{1}{\sqrt{1-x^2-y^2}} d\sigma \ 3.I &= \iint_{D_{xy}} \sqrt{1-x^2-y^2} \cdot rac{1}{\sqrt{1-x^2-y^2}} d\sigma &= \iint_{D_{xy}} d\sigma &= rac{\pi}{2} \ I &= \iint_{\Sigma} (x^2z+2xy) dS. \ .$$

$$I=\iint_{\Sigma}(x^2z+2xy)dS$$



$$egin{aligned} &\iint_{\Sigma} x^2 z dS = \iint_{\Sigma} y^2 z dS \ &= rac{1}{2} \iint_{\Sigma} (x^2 + y^2) z dS \ 1.\Sigma : z &= \sqrt{4 - x^2 - y^2}, D_{xy} : x^2 + y^2 \leq 4 \ 2.z_x' &= rac{-x}{\sqrt{4 - x^2 - y^2}}, z_y' &= rac{-y}{4 - x^2 - y^2} \ dS &= rac{2}{\sqrt{4 - x^2 - y^2}} d\sigma \ 3.I &= rac{1}{2} \iint_{D_{xy}} (x^2 + y^2) \cdot 2 d\sigma = 2\pi \int_0^2 r^3 dr = 8\pi \end{aligned}$$

# 对坐标的曲面积分 (第二类曲面积分)



### Σ有侧曲面

 $\overrightarrow{1.}$ 从x正半轴将 $\overrightarrow{dS}$ 向yoz面投影,投影记为dydz

若 
$$\cos \alpha > 0$$
,  $dydz > 0$ 

若 
$$\cos \alpha < 0, dydz < 0$$

$$dydz = \cos\alpha \cdot dS$$

 $\overrightarrow{2}$ .从y正半轴将dS向xoz面投影,投影记为dzdx

若 
$$\cos \beta > 0$$
,  $dzdx > 0$ 

若 
$$\cos \beta < 0, dz dx < 0$$

$$dzdx = \cos\beta \cdot dS$$

3.从z正半轴将 $d\overset{'}{S}$ 向xoy面投影,投影记为dxdy

若 
$$\cos \gamma > 0$$
,  $dxdy > 0$ 

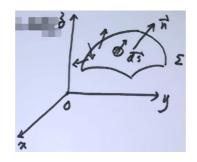
若 
$$\cos \gamma < 0, dxdy < 0$$

$$dxdy = \cos \gamma \cdot dS$$

$$\left\{ egin{aligned} @dydz = \coslpha \cdot dS, dzdx = \coseta \cdot dS, dxdy = \cos\gamma \cdot dS \ & \longrightarrow \ @dS = \{dydz, dzdx, dxdy\} \end{aligned} 
ight.$$

### 背景: 流量

$$\overrightarrow{v} = \{P, Q, R\}, \Sigma -$$
有侧曲面块求 $\Phi$ .



$$1. \forall \overrightarrow{dS} \subset \Sigma, \overrightarrow{dS} = \{dydz, dzdx, dxdy\}$$

$$2.d\Phi = \overrightarrow{v} \cdot \overrightarrow{dS}$$

$$= Pdydz + Qdzdx + Rdxdy$$

$$egin{aligned} 3.\Phi &= \iint_{\Sigma} d\Phi \ &= \iint_{\Sigma} P dy dz + Q dz dx + R dx dy \end{aligned}$$

## 定义

$$\iint_{\Sigma} P dy dz + Q dz dx + R dx dy = \iint_{\Sigma} P dy dz + \iint_{\Sigma} Q dz dx + \iint_{\Sigma} R dx dy$$

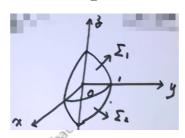
# 性质

$$egin{aligned} 1. \iint_{\Sigma^-} &= -\iint_{\Sigma} \ 2. \iint_{\Sigma} P dy dz + Q dz dx + R dx dy &= \iint_{\Sigma} (P\coslpha + Q\coseta + R\cos\gamma) dS \end{aligned}$$

① 
$$\iint_{\Sigma} P(x,y,z) dy dz$$
 $1.\Sigma: x = \Phi(y,z), (y,z) \in D_{yz}$ 
 $2.\iint_{\Sigma} P dy dz = \pm \iint_{D_{yz}} P[\Phi(y,z),y,z] dy dz$ 
 $\cos \alpha > 0$ 取 $+,\cos \alpha < 0$ 取 $-$ 

$$\cos lpha > 0$$
取 $+,\cos lpha < 0$ 取 $-$ ③ $\iint_{\Sigma} R(x,y,z) dx dy$  $1.\Sigma: z = \Phi(x,y), (x,y) \in D_{xy}$  $2.\iint_{\Sigma} R dx dy = \pm \iint_{D_{xy}} R[x,y,\Phi(x,y)] dx dy$  $\cos \gamma > 0$ 取 $+,\cos \gamma < 0$ 取 $-$ 

 $I = \iint_{\Sigma} z dx dy.$ 



$$egin{align} 1.I &= \iint_{\Sigma_1} z dx dy + \iint_{\Sigma_2} z dx dy \ 2. @\Sigma_1: z &= \sqrt{1-x^2-y^2} \ D_{xy}: x^2+y^2 &\leq 1 (x \geq 0, y \geq 0) \ \end{cases}$$

$$\iint_{\Sigma_1} z dx dy = \iint_{D_{xy}} \sqrt{1-x^2-y^2} dx dy$$

$$2\Sigma_2: z = -\sqrt{1 - x^2 - y^2}$$

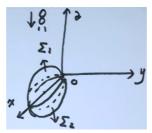
$$D_{xy}:x^2+y^2\leq 1(x\geq 0,y\geq 0)$$

$$egin{align} @\Sigma_2: z = -\sqrt{1-x^2-y^2} \ D_{xy}: x^2+y^2 \leq 1 (x \geq 0, y \geq 0) \ \iint_{\Sigma_2} z dx dy = -\iint_{D_{xy}} -\sqrt{1-x^2-y^2} dx dy \ \end{pmatrix}$$

$$egin{align} 3.I &= 2 \iint_{D_{xy}} \sqrt{1-x^2-y^2} dx dy = \pi \int_0^1 r \sqrt{1-r^2} dr \ &= -rac{\pi}{2} \int_0^1 (1-r^2)^{rac{1}{2}} d(1-r^2) \ &= -rac{\pi}{3} (1-r^2)^{rac{3}{2}} \mid_0^1 = rac{\pi}{3} \end{aligned}$$

设
$$\Sigma:(x-1)^2+y^2+z^2=1,$$
取外侧, 计算 $\iint_{\Sigma}y^2zdxdy.$ 

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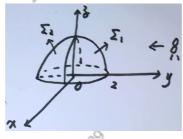


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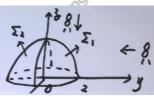
$$\begin{split} 1.I &= \iint_{\Sigma} y^2 z dx dy = 2 \iint_{\Sigma_1} y^2 z dx dy \\ 2.\Sigma &: z = \sqrt{1 - (x - 1)^2 - y^2} \\ D_{xy} &: (x - 1)^2 + y^2 \le 1 \\ I &= 2 \iint_{D_{xy}} y^2 \sqrt{1 - (x - 1)^2 - y^2} d(x - 1) dy \\ &= 2 \iint_{x^2 + y^2 \le 1} y^2 \sqrt{1 - x^2 - y^2} dx dy \\ &= \iint_{x^2 + y^2 \le 1} (x^2 + y^2) \sqrt{1 - x^2 - y^2} dx dy \\ &= 2\pi \int_0^1 r^3 \sqrt{1 - r^2} dr = 2\pi \int_0^{\frac{\pi}{2}} (\sin^3 t - \sin^5 t) dt, r = \sin t \\ &= 2\pi (\frac{2}{3} - \frac{4}{5} \cdot \frac{2}{3} \cdot 1) = \frac{4}{15} \pi \end{split}$$

ndpaothe.

$$I=\iint_{\Sigma}yzdzdx+2dxdy.$$



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$$I_1 = \iint_{\Sigma} yzdzdx$$

$$= 2 \iint_{\Sigma_1} yzdzx$$
 $\Sigma_1 : y = \sqrt{4 - x^2 - z^2}$ 
 $D_{xz} = x^2 + z^2 \le 4(z \ge 0)$ 
 $I_1 = 2 \iint_{D_{xz}} z\sqrt{4 - x^2 - z^2}dzdx$ 

$$= 2 \int_0^{\pi} \sin\theta d\theta \int_0^2 r^2 \sqrt{4 - r^2}dr = 4 \int_0^2 r^2 \sqrt{4 - r^2}dr$$

$$= 4 \int_0^{\frac{\pi}{2}} 4 \sin^2 t \cdot 4(1 - \sin^2 t)dt = 64(I_2 - I_4), r = 2 \sin t$$

$$= 64(\frac{1}{2} \times \frac{\pi}{2} - \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}) = 4\pi$$
② $I_2 = \iint_{\Sigma} 2dxdy$ 

$$\Sigma : z = \sqrt{4 - x^2 - y^2}, D_{xy} : x^2 + y^2 \le 4$$
 $I_2 = 2 \iint_{D_{xy}} 1dxdy = 8\pi$ 

$$\therefore \text{ $\mathbb{R}$} \ \mathbb{R} = 12\pi$$

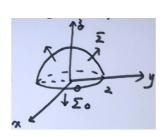
# 三重积分 (Gauss 公式)





$$1 - dim : F(b) - F(a) = \int_{a}^{b} f(x) dx$$
  $2 - dim : \oint_{L} P dx + Q dy = \iint_{D} (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) d\sigma$   $3 - dim : \oiint_{\Sigma} P dy dz + Q dz dx + R dx dy = \iiint_{\Omega} (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}) dV$ 

$$I=\iint_{\Sigma}yzdzdx+2dxdy.$$



$$egin{aligned} 1.P &= 0, Q = yz, R = 2, rac{\partial P}{\partial x} = 0, rac{\partial Q}{\partial y} = z, rac{\partial R}{\partial z} = 0 \ 2.\Sigma_0: z = 0(x^2 + y^2 \leq 4), \, \mathbb{F} \ I &= igoplus_{\Sigma + \Sigma_0} - \iint_{\Sigma_0} \ 3. igoplus_{\Sigma + \Sigma_0} &= \iiint_{\Omega} z dV = \int_0^{2\pi} d heta \int_0^{\frac{\pi}{2}} d\Phi \int_0^2 r \cos \Phi \cdot r^2 \sin \Phi dr \ &= 2\pi imes rac{1}{2} imes 4 = 4\pi \ \iint_{\Sigma_0} &= \iint_{\Sigma_0} 2 dx dy = - \iint_{D_{xy}} dx dy = -2 imes 4\pi = -8pi \ dots : I = 12\pi \end{aligned}$$

场论

$$1.$$
梯度: $u=f(x,y,z)$  
$$\operatorname{grad} u = \{ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \}$$
  $2.$ 旋度: $\vec{A} = \{P,Q,R\}$  
$$\operatorname{rot} \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$egin{aligned} ec{A} = & \{yz, x^2 + z^2, x + y\} \ \mathrm{rot} ec{A} = & (2y - 2z)ec{i} + (y - 1)ec{j} + (2x - z)ec{k} \ = & \{2y - 2z, y - 1, 2x - z\} \end{aligned}$$

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$$egin{aligned} 3.$$
散度: $ec{A} = \{P,Q,R\} \ \mathrm{div}ec{A} &= rac{\partial P}{\partial x} + rac{\partial Q}{\partial y} + rac{\partial R}{\partial z} \ 4.$ 通量: $ec{v} = \{P,Q,R\}, \Sigma - eta$ 例  $\Phi = \iint_{\Sigma} P dy dz + Q dz dx + R dx dy \ 5.$ 环流量: $ec{v} = \{P,Q,R\}, L$ 为有向闭曲线  $\Phi = \oint_{L} P dx + Q dy + R dz$