$$\sum_{n=1}^{\infty} a_n$$

部分和

$$S_n = a_1 + a_2 + \ldots + a_n$$

$$\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$$

$$\lim_{n\to\infty} S_n = \begin{cases} = S, \sum_{n=1}^{\infty} a_n = S \\ \text{不存在，} \sum_{n=1}^{\infty} a_n \text{发散} \end{cases}$$

$$\text{判断级数} \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \text{的敛散性.}$$

$$S_n = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \ldots + \frac{1}{(2n-1)(2n+1)}$$
$$= \frac{1}{2}(1 - \frac{1}{2n+1})$$
$$\because \lim_{n\to\infty} S_n = \frac{1}{2}, \therefore \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$$

$$\text{Note: 对} \sum_{n=1}^{\infty} (b_{n+1} - b_n) \text{敛散性判断一般用定义.}$$

常数项级数的性质

1. $$\sum_{n=1}^{\infty} a_n = A, \sum_{n=1}^{\infty} b_n = B \Rightarrow \sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = a \pm b$$

2. $$\sum_{n=1}^{\infty} a_n = S \Rightarrow \sum_{n=1}^{\infty} k a_n = k \sum_{n=1}^{\infty} a_n = kS$$

3. 级数前面添加、减少改变有限项，敛散性不变（若收敛，和会改变）

4. 添加括号不降收敛性

$$-1+1-1+1-1+1-\dots$$
$$S_{2n}=0, S_{2n+1}=-1$$
$$\lim_{n\to\infty} S_{2n} \neq \lim_{n\to\infty} S_{2n+1}, \therefore \lim_{n\to\infty} S_n\text{不存在, 即}$$
$$-1+1-1+1-1+1-\dots\text{发散}$$
$$\text{但}(-1+1)+(-1+1)+(-1+1)+\dots=0+0+\dots\text{收敛}$$
$$a_1+a_2+\dots$$
$$(a_1+a_2)+a_3+(a_4+a_5+a_6)+\dots$$
$$\text{如：设}\sum_{n=1}^{\infty} a_n\text{收敛, 问}\sum_{n=1}^{\infty}(a_{2n-1}+a_{2n})?$$
$$\sum_{n=1}^{\infty}(a_{2n-1}+a_{2n})=(a_1+a_2)+(a_3+a_4)+(a_5+a_6)+\dots$$

5. 若级数收敛，则

$$\lim_{n\to\infty} S_n = S$$
$$a_n = S_n - S_{n-1}$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

$$\text{若}\sum_{n=1}^{\infty} a_n\text{收敛} \Rightarrow \begin{cases} ① \lim_{n\to\infty} a_n = 0 \\ ② \lim_{n\to\infty} S_n \exists \end{cases}$$

两个重要的级数

p-级数

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$
$$p=1, \sum_{n=1}^{\infty}\frac{1}{n}, \text{调和级数}$$
$$\begin{cases} p>1, \text{收敛} \\ p\leq 1, \text{发散} \end{cases}$$

几何级数

$$\sum_{n=1}^{\infty} aq^n (a\neq 0)$$
$$\begin{cases} |q|\geq 1, \text{发散} \\ |q|<1, \text{收敛于} \sum_{n=1}^{\infty} aq^n = \frac{\text{第一项}}{1-q} \end{cases}$$

正项级数及敛散性

$$\sum_{n=1}^{\infty} a_n (a_n \geq 0, n=1,2,\dots)\text{称为正项级数}$$
Notes:
$$①S_1\leq S_2\leq S_3\leq\dots, \text{即}\{S_n\}\uparrow$$
$$②\{S_n\}\text{无界} \Rightarrow \lim_{n\to\infty} S_n = +\infty, \text{即}\sum_{n=1}^{\infty} a_n = +\infty$$
$$③\exists M>0, \text{使}S_n\leq M \Rightarrow \lim_{n\to\infty} S_n\exists \Rightarrow \sum_{n=1}^{\infty} a_n\text{收敛}$$

比较审敛法

基本形式

$$
\begin{gathered}
a_n \geq 0, b_n \geq 0(n=1,2,\dots) \\
a_n \leq b_n \text{且} \sum_{n=1}^{\infty} b_n \text{收敛} \Rightarrow \sum_{n=1}^{\infty} a_n \text{收敛} \\
a_n \geq b_n \text{且} \sum_{n=1}^{\infty} b_n \text{发散} \Rightarrow \sum_{n=1}^{\infty} a_n \text{发散}
\end{gathered}
$$

$$
\text{判断级数} \sum_{n=1}^{\infty} \sin\frac{\pi}{2^n} \text{的敛散性.}
$$

$$
\begin{aligned}
&\sin\frac{\pi}{2^n} > 0(n=1,2,\dots) \\
&\because x \geq 0 \text{时}, \sin x \leq x, \therefore 0 < \sin\frac{\pi}{2^n} \leq \frac{\pi}{2^n} \\
&\because \sum_{n=1}^{\infty} \frac{\pi}{2^n} \text{收敛}, \therefore \sum_{n=1}^{\infty} \sin\frac{\pi}{2^n} \text{收敛}
\end{aligned}
$$

$$
\text{设} a_n \leq b_n \leq c_n \text{且级数} \sum_{n=1}^{\infty} a_n \text{与} \sum_{n=1}^{\infty} c_n \text{收敛, 证明级数} \sum_{n=1}^{\infty} b_n \text{收敛.}
$$

$$
\begin{aligned}
&a_n \leq b_n \leq c_n \Rightarrow 0 \leq b_n - a_n \leq c_n - a_n \\
&\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} c_n \text{收敛} \Rightarrow \sum_{n=1}^{\infty} (c_n - a_n) \text{收敛} \\
&\Rightarrow \sum_{n=1}^{\infty} (b_n - a_n) \text{收敛} \\
&\because \sum_{n=1}^{\infty} a_n \text{收敛} \\
&\therefore \sum_{n=1}^{\infty} [(b_n - a_n) + a_n] = \sum_{n=1}^{\infty} b_n \text{收敛}
\end{aligned}
$$

极限形式

$$
\begin{aligned}
&a_n > 0, b_n > 0(n=1,2,\dots) \\
&\lim_{n\to\infty} \frac{b_n}{a_n} = l(0 < l < +\infty) \Rightarrow \sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n \text{敛散性同}
\end{aligned}
$$

比值法

$$
\begin{aligned}
&a_n > 0(n=1,2,\dots) \\
&① \lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \text{收敛} \\
&② \lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1 \Rightarrow \text{发散} \\
&③ \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \Rightarrow ?
\end{aligned}
$$

判断级数$\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}$的敛散性.

$$\because \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n n!}=2\lim_{n\to\infty}(\frac{n}{n+1})^n=$$
$$2\lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^n}=\frac{2}{e}=\rho<1$$

$\therefore$收敛

根值法

$$a_n>0(n=1,2,\dots)$$
$$\lim_{n\to\infty}\sqrt[n]{a_n}=\rho$$

①$\rho<1$, 收敛
②$\rho>1$, 发散
③$\rho=1$, ?

判断级数$\sum_{n=1}^{\infty}(\frac{n}{n+1})^{n^2}$的敛散性.

$$\because \lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}(\frac{n}{n+1})^n=\lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^n}=\frac{1}{e}=\rho<1$$

$\therefore$ 收敛

交错级数及敛散性

$$a_1-a_2+a_3-a_4+\dots=\sum_{n=1}^{\infty}(-1)^{n-1}a_n$$
$$-a_1+a_2-a_3+a_4-\dots=\sum_{n=1}^{\infty}(-1)^n a_n$$
$$(a_n>0,n=1,2,\dots)$$

莱布尼茨法

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n(a_n>0,n=1,2,\dots)$$

①$\{a_n\}\downarrow$
②$\lim_{n\to\infty}a_n=0$
$\Rightarrow\sum_{n=1}^{\infty}(-1)^{n-1}a_n$收敛, $S\le a_1$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$$

$$a_n=\frac{1}{\sqrt{n}}$$

$\because\{\frac{1}{\sqrt{n}}\}\downarrow$且$\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0,\therefore\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$收敛.

$$\sum_{n=1}^{\infty} a_n\text{收敛}, \sum_{n=1}^{\infty} a_n^2\text{不一定收敛}.$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}, \text{而} \sum_{n=1}^{\infty} \frac{1}{n}\text{发散}.$$

$$\sum_{n=1}^{\infty} a_n (a_n \geq 0)\text{收敛} \Rightarrow \sum_{n=1}^{\infty} a_n^2\text{收敛}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n\text{收敛} \Rightarrow \lim_{n\to\infty} a_n = 0 \\
&\text{取}\epsilon = 1, \exists N > 0, \text{当}n > N\text{时}, \\
&|a_n - 0| < 1 \\
&\Rightarrow 0 \leq a_n < 1 \Rightarrow 0 \leq a_n^2 \leq a_n \\
&\because \sum_{n=1}^{\infty} a_n\text{收敛}, \therefore \sum_{n=1}^{\infty} a_n^2\text{收敛}
\end{aligned}$$

任意级数

$$\text{若} \sum_{n=1}^{\infty} |a_n|\text{收敛}, \text{称} \sum_{n=1}^{\infty} a_n\text{绝对收敛}$$

$$\text{若} \sum_{n=1}^{\infty} a_n\text{收敛}, \text{而} \sum_{n=1}^{\infty} |a_n|\text{发散}, \text{称} \sum_{n=1}^{\infty} a_n\text{条件收敛}$$

关系

$$\text{若} \sum_{n=1}^{\infty} a_n\text{绝对收敛} \Rightarrow \sum_{n=1}^{\infty} a_n\text{收敛}$$

$$\begin{aligned}
&\text{证}:\text{已知} \sum_{n=1}^{\infty} |a_n|\text{收敛} \\
&a_n = \frac{|a_n| + a_n}{2} - \frac{|a_n| - a_n}{2} \\
&0 \leq \frac{|a_n| + a_n}{2} \leq |a_n|, 0 \leq \frac{|a_n| - a_n}{2} \leq |a_n| \\
&\because \sum_{n=1}^{\infty} |a_n|\text{收敛}, \therefore \sum_{n=1}^{\infty} \frac{|a_n| + a_n}{2}, \therefore \sum_{n=1}^{\infty} \frac{|a_n| - a_n}{2}\text{皆收敛} \\
&\Rightarrow \sum_{n=1}^{\infty} a_n\text{收敛}
\end{aligned}$$

$$\text{设}f(x)\text{二阶连续可导}, f(0) = 1\text{且} \lim_{x\to 0} \frac{f'(x)}{x} = 2, \text{证明}:\text{级数} \sum_{n=1}^{\infty} [f(\frac{1}{n}) - 1]\text{绝对收敛}.$$

$$\lim_{x\to 0}\frac{f'(x)}{x}=2\Rightarrow f'(0)=0,f''(0)=2$$
$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+o(x^2)$$
$$f(\frac{1}{n})-1=\frac{1}{n^2}+o(\frac{1}{n^2})$$
$$|f(\frac{1}{n})-1|=|\frac{1}{n^2}+o(\frac{1}{n^2})|\sim\frac{1}{n^2}$$
$$\because\sum_{n=1}^{\infty}\frac{1}{n^2}\text{收敛},\therefore\sum_{n=1}^{\infty}|f(\frac{1}{n})-1|\text{收敛}$$
$$\Rightarrow\sum_{n=1}^{\infty}[f(\frac{1}{n})-1]\text{绝对收敛.}$$

判断$\sum_{n=1}^{\infty}(-1)^n(1-\cos\frac{a}{n})(a>0)$的敛散性. 若收敛, 该级数绝对收敛还是条件收敛?

$$0\le|(-1)^n(1-\cos\frac{a}{n})|=|2\sin^2\frac{a}{2n}|\le 2(\frac{a}{2n})^2=\frac{a^2}{2}\cdot\frac{1}{n^2}$$
$$\because\sum_{n=1}^{\infty}\frac{a^2}{2}\cdot\frac{1}{n^2}\text{收敛},\therefore\sum_{n=1}^{\infty}|(-1)^n(1-\cos\frac{a}{n})|\text{收敛}$$

判断级数$\sum_{n=1}^{\infty}\sin\sqrt{n^2+1}\pi$的敛散性, 若收敛, 该级数绝对收敛还是条件收敛?

$$a_n=\sin\sqrt{n^2+1}\pi=\sin[n\pi+(\sqrt{n^2+1}-n)\pi]=(-1)^n\sin\frac{\pi}{\sqrt{n^2+1}+n}$$
$$\because 0<\frac{\pi}{\sqrt{n^2+1}+n}<\frac{\pi}{2},\therefore\sin\frac{\pi}{\sqrt{n^2+1}+n}>0$$
$$\because\{\sin\frac{\pi}{\sqrt{n^2+1}+n}\}\downarrow,\therefore\text{原级数收敛}$$
$$|\sin\sqrt{n^2+1}\pi|\sim\frac{\pi}{\sqrt{n^2+1}+n}\sim\frac{\pi}{2n},\text{而}\sum_{n=1}^{\infty}\frac{\pi}{2n}\text{发散}$$
$$\therefore\sum_{n=1}^{\infty}\sin\sqrt{n^2+1}\pi\text{发散},\therefore\text{原级数条件收敛.}$$

$\sum_{n=1}^{\infty}a_n$收敛, 问$\sum_{n=1}^{\infty}(a_n+a_{n+1})$?

$$S_n=a_1+\ldots+a_n$$
$$\sum_{n=1}^{\infty}a_n\text{收敛}\Rightarrow\sum_{n=1}^{\infty}a_n=0\text{且}\sum_{n=1}^{\infty}S_n\exists$$
$$\text{令}\lim_{n\to\infty}S_n=S$$
$$S_n'=(a_1+a_2)+(a_2+a_3)+\ldots+(a_n+a_{n+1})$$
$$=2S_n-a_1+a_{n+1}$$
$$\because\lim_{n\to\infty}S_n'=2S-a_1$$
$$\therefore\sum_{n=1}^{\infty}(a_n+a_{n+1})\text{收敛}$$

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+n+1}}$$

$$\because \frac{n}{\sqrt{n^3+n+1}} \sim \frac{1}{\sqrt{n}} \text{且} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{发散}, \therefore \text{原级数发散}$$

$$\sum_{n=1}^{\infty} \frac{1}{\int_0^n \sqrt[4]{1+x^4}dx}$$

$$\int_0^n \sqrt[4]{1+x^4}dx \geq \int_0^n xdx = \frac{n^2}{2}$$

$$\Rightarrow 0 < \frac{1}{\int_0^n \sqrt[4]{1+x^4}dx} \leq \frac{2}{n^2}$$

$$\text{而} \sum_{n=1}^{\infty} \frac{2}{n^2} \text{收敛}, \therefore \text{原级数收敛}$$

$$\text{设} a_1 = 2, a_{n+1} = \frac{1}{2}(a_n + \frac{1}{a_n}), \text{证明：}$$

$$(1) \lim_{n\to\infty} a_n \text{存在}$$

$$(2) \sum_{n=1}^{\infty} (\frac{a_n}{a_{n+1}} - 1) \text{收敛}$$

$$(1) a_n > 0$$

$$\because a_n + \frac{1}{a_n} \geq 2, \therefore a_{n+1} \geq 1$$

$$a_{n+1} - a_n = \frac{1}{2}(a_n + \frac{1}{a_n}) - a_n = \frac{1-a_n^2}{2a_n} \leq 0$$

$$\Rightarrow \{a_n\} \downarrow$$

$$\therefore \lim_{n\to\infty} a_n \exists$$

$$(2) \because \{a_n\} \downarrow \text{且} a_n > 0, \therefore \frac{a_n}{a_{n+1}} - 1 \geq 0$$

$$0 \leq \frac{a_n}{a_{n+1}} - 1 = \frac{a_n - a_{n+1}}{a_{n+1}} \leq a_n - a_{n+1}$$

$$\text{对} \sum_{n=1}^{\infty} (a_n - a_{n+1})$$

$$S_n = (a_1 - a_2) + \ldots + (a_n - a_{n+1}) = 2 - a_{n+1}$$

$$\because \lim_{n\to\infty} S_n \exists, \therefore \sum_{n=1}^{\infty} (a_n - a_{n+1}) \text{收敛}$$

$$\therefore \sum_{n=1}^{\infty} (\frac{a_n}{a_{n+1}} - 1) \text{收敛}$$