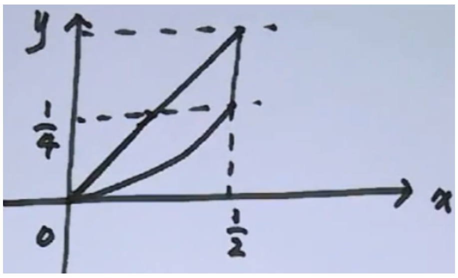
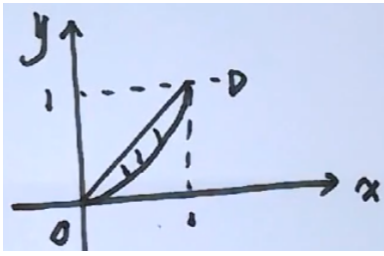
改变积分次序

$$\text{交换积分次序：}\int_0^{\frac{1}{4}} dy \int_y^{\sqrt{y}} f(x,y)dx + \int_{\frac{1}{4}}^{\frac{1}{2}} dy \int_y^{\frac{1}{2}} f(x,y)dx.$$



$$\text{原式} = \int_0^{\frac{1}{2}} dx \int_{x^2}^{x} f(x,y)dy$$

$$\text{计算}I = \int_0^1 dy \int_y^{\sqrt{y}} \frac{\sin x}{x} dx.$$



$$\begin{aligned}
\text{原式} &= \int_0^1 \frac{\sin x}{x} dx \int_{x^2}^{x} 1dy \\
&= \int_0^1 (\sin x - x\sin x)dx = 1 - \cos 1 + \int_0^1 xd(\cos x) \\
&=1 - \cos 1 + x\cos x \mid_0^1 - \int_0^1 \cos x dx \\
&=1 - \sin 1
\end{aligned}$$

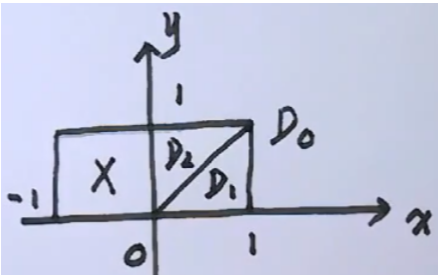
二重积分的计算

直角坐标法

$$\text{计算} \iint_D \sqrt{y^2 - xy}dxdy, \text{其中}D\text{是由}y = x, x = 0, y = 1\text{围成的区域.}$$

$$
\begin{aligned}
&\int_0^y (y^2-xy)^{\frac{1}{2}}dx\\
&=-\frac{1}{y}\int_0^y (y^2-xy)^{\frac{1}{2}}d(y^2-xy)\\
&=-\frac{1}{y}\times\frac{2}{3}(y^2-xy)^{\frac{3}{2}}\ |_0^y\\
&=-\frac{2}{3y}(0-y^3)=\frac{2}{3}y^2
\end{aligned}
$$

$$
\begin{aligned}
\text{原式}&=\int_0^1 dy\int_0^y (y^2-xy)^{\frac{1}{2}}dx\\
&=\frac{2}{3}\int_0^1 y^2dy=\frac{2}{9}
\end{aligned}
$$



$$
\begin{aligned}
I&=\iint_D \sqrt{|y-|x||}d\sigma\\
&=2\iint_{D_0}\sqrt{|y-x|}d\sigma=2I_0
\end{aligned}
$$

$$
I_0=\iint_{D_1}\sqrt{x-y}d\sigma+\iint_{D_2}\sqrt{y-x}d\sigma
$$

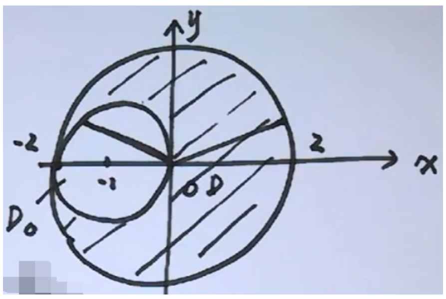
$$
\begin{aligned}
\iint_{D_1}\sqrt{x-y}d\sigma&=-\int_0^1 dx\int_0^x (x-y)^{\frac{1}{2}}d(x-y)\\
&=-\frac{2}{3}\int_0^1 (x-y)^{\frac{3}{2}}\ |_0^x\ dx=-\frac{2}{3}\int_0^1 (0-x^{\frac{3}{2}})dx=\frac{4}{15}
\end{aligned}
$$

$$
\begin{aligned}
\iint_{D_2}\sqrt{y-x}d\sigma&=\int_0^1 dx\int_x^1 (y-x)^{\frac{1}{2}}d(y-x)\\
&=\frac{2}{3}\int_0^1 (y-x)^{\frac{3}{2}}\ |_x^1\ dx=-\frac{2}{3}\int_0^1 (1-x)^{\frac{3}{2}}d(1-x)\\
&=-\frac{4}{15}(1-x)^{\frac{5}{2}}\ |_0^1=-\frac{4}{15}(0-1)=\frac{4}{15}
\end{aligned}
$$

$$
\therefore I=\frac{16}{15}
$$

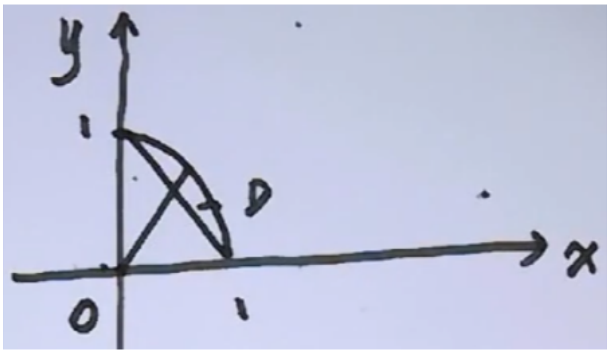
极坐标法

求$\iint_D(\sqrt{x^2+y^2}+y)dxdy$, 其中$D$由$x^2+y^2=4$与$(x+1)^2+y^2=1$围成.



$$
\begin{aligned}
I &= \iint_D \sqrt{x^2+y^2}d\sigma \\
&= \iint_{D+D_0} \sqrt{x^2+y^2}d\sigma - \iint_{D_0} \sqrt{x^2+y^2}d\sigma = I_1 - I_2 \\
I_1 &= \int_0^{2\pi} d\theta \int_0^2 r^2 dr = \frac{16}{3}\pi \\
&\text{令}\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}(\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}, 0 \le r \le -2\cos\theta) \\
I_2 &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\theta \int_0^{-2\cos\theta} r^2 dr = -\frac{8}{3}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^3\theta d\theta, \theta - \pi = t \\
&= \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 t dt = \frac{16}{3} \times \frac{2}{3} \times 1 = \frac{32}{9}, \therefore I = \frac{16\pi}{3} - \frac{32}{9}
\end{aligned}
$$

计算$I = \iint_D \frac{dxdy}{\sqrt{x^2+y^2}}$, 其中$D$由$x^2+y^2 \le 1$与$x+y \ge 1$围成.



$$
\begin{aligned}
&\text{令}\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}(0 \le \theta \le \frac{\pi}{2}, \frac{1}{\sin\theta + \cos\theta} \le r \le 1) \\
\text{原式} &= \int_0^{\frac{\pi}{2}} d\theta \int_{\frac{1}{\sin\theta+\cos\theta}}^1 1 dr = \int_0^{\frac{\pi}{2}} (1 - \frac{1}{\sin\theta + \cos\theta}) d\theta \\
&= \frac{\pi}{2} - \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \sec(\theta - \frac{\pi}{4}) d(\theta - \frac{\pi}{4}) = \frac{\pi}{2} - \sqrt{2}\int_0^{\frac{\pi}{4}} \sec\theta d\theta \\
&= \frac{\pi}{2} - \sqrt{2}\ln|\sec\theta + \tan\theta| \,|_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{2} - \sqrt{2}\ln(\sqrt{2}+1)
\end{aligned}
$$