常数项级数

$$\sum_{n=1}^{\infty} a_n$$

部分和

$$S_n = a_1 + a_2 + \ldots + a_n$$

$$\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$$

$$\lim_{n\to\infty} S_n = \begin{cases} = S, \sum_{n=1}^{\infty} a_n = S \\ \text{不存在，} \sum_{n=1}^{\infty} a_n \text{发散} \end{cases}$$

常数项级数的性质

1.

$$\sum_{n=1}^{\infty} a_n = A, \sum_{n=1}^{\infty} b_n = B \Rightarrow \sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = a \pm b$$

2.

$$\sum_{n=1}^{\infty} a_n = S \Rightarrow \sum_{n=1}^{\infty} k a_n = k \sum_{n=1}^{\infty} a_n = kS$$

3. 级数前面添加、减少改变有限项，敛散性不变（若收敛，和会改变）
4. 添加括号不降收敛性
5. 若级数收敛，则

$$\lim_{n\to\infty} S_n = S$$

$$a_n = S_n - S_{n-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

$$\sum_{n-1}^{\infty} a_n \text{收敛} \Rightarrow \begin{cases} \lim_{n\to\infty} a_n = 0 \\ \lim_{n\to\infty} S_n \exists \end{cases}$$

6.

两个重要的级数

p-级数

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$p = 1, \sum_{n=1}^{\infty} \frac{1}{n}, \text{调和级数}$$

$$\begin{cases} p > 1, \text{收敛} \\ p \leq 1, \text{发散} \end{cases}$$

几何级数

$$\sum_{n=1}^{\infty} aq^n(a \neq 0)$$

$$\begin{cases} |q| \geq 1, \text{发散} \\ |q| < 1, \text{收敛于} \sum_{n=1}^{\infty} aq^n = \frac{\text{第一项}}{1-q} \end{cases}$$

正项级数及敛散性

$$\sum_{n=1}^{\infty} a_n(a_n \geq 0, n = 1, 2, \dots)$$

$$S_1 \leq S_2 \leq S_3 \leq \dots, \text{即}\{S_n\} \uparrow$$

$$\{S_n\}\text{无界} \Rightarrow \lim_{n\to\infty} S_n = +\infty, \text{即} \sum_{n=1}^{\infty} a_n = +\infty$$

$$\exists M > 0, \text{使}S_n \leq M \Rightarrow \lim_{n\to\infty} S_n \exists \Rightarrow \sum_{n=1}^{\infty} a_n\text{收敛}$$

比较审敛法

$$a_n \geq 0, b_n \geq 0(n = 1, 2, \dots)$$

$$a_n \leq b_n\text{且} \sum_{n=1}^{\infty} b_n\text{收敛} \Rightarrow \sum_{n=1}^{\infty} a_n\text{收敛}$$

$$a_n \geq b_n\text{且} \sum_{n=1}^{\infty} b_n\text{发散} \Rightarrow \sum_{n=1}^{\infty} a_n\text{发散}$$

极限

$$a_n > 0, b_n > 0(n = 1, 2, \dots)$$

$$\lim_{n\to\infty} \frac{b_n}{a_n} = l(0 < l < +\infty) \Rightarrow \sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n\text{敛散性同}$$

比值法

$$a_n > 0(n = 1, 2, \dots)$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \text{收敛}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1 \Rightarrow \text{发散}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \Rightarrow ?$$

根值法

$$a_n > 0(n = 1, 2, \dots)$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$$

$$\rho < 1, \text{收敛}$$

$$\rho > 1, \text{发散}$$

$$\rho = 1, ?$$

交错级数及敛散性

$$a_1 - a_2 + a_3 - a_4 + \ldots = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

$$-a_1 + a_2 - a_3 + a_4 - \ldots = \sum_{n=1}^{\infty}(-1)^{n}a_n$$

$$(a_n > 0, n = 1, 2, \ldots)$$

莱布尼茨法

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n(a_n > 0, n = 1, 2, \ldots)$$

$$\{a_n\} \downarrow 且 \lim_{n\to\infty} a_n = 0 \Rightarrow \sum_{n=1}^{\infty}(-1)^{n-1}a_n收敛, S \leq a_1$$

任意级数

$$若\sum_{n=1}^{\infty}|a_n|收敛, 称\sum_{n=1}^{\infty}a_n绝对收敛$$

$$若\sum_{n=1}^{\infty}a_n收敛, 而\sum_{n=1}^{\infty}|a_n|发散, 称\sum_{n=1}^{\infty}a_n条件收敛$$

$$若\sum_{n=1}^{\infty}a_n绝对收敛 \Rightarrow \sum_{n=1}^{\infty}a_n收敛$$

幂级数

$$\sum_{n=0}^{\infty}a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots$$

收敛域

$$\sum_{n=0}^{\infty}x^n = 1 + x + x^2 + \ldots$$

$$x = \frac{2}{3} : 1 + \frac{2}{3} + (\frac{2}{3})^2 + \ldots = \frac{1}{1-\frac{1}{3}} = 3, x = \frac{2}{3}收敛点$$

$$x = 2 : 1 + 2 + 2^2 + \ldots, x = 2发散点$$

一切收敛点构成的集合

收敛半径

$$\sum_{n=0}^{\infty} a_n x^n, \exists R \geq 0$$

$$当|x| < R或x \in (-R, R):绝对收敛$$

$$当|x| > R或x \in (-\infty, -R) \cup (R, +\infty):发散$$

$$当|x| = R,即x = \pm R :?$$

R称为收敛半径

R及收敛域

$$\sum_{n=0}^{\infty} a_n x^n$$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \rho或\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$$

$$\rho = +\infty \Rightarrow R = 0,唯一收敛域x = 0$$

$$\rho = 0 \Rightarrow R = +\infty,收敛域(-\infty, +\infty)$$

$$0 < \rho < +\infty \Rightarrow R = \frac{1}{\rho}$$

$$\sum_{n=0}^{\infty} a_n x^{2n+1}$$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \rho或\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$$

$$\Rightarrow R = \sqrt{\frac{1}{\rho}}$$

$$对\sum_{n=0}^{\infty} a_n x^n,若\sum_{n=0}^{\infty} a_n x_0^n条件收敛 \Rightarrow |x_0| = R$$

分析性质

$$\sum_{n=0}^{\infty} a_n x^n = S(x), x \in (-R, R)$$

$$S(x) \in C(-R, R)$$

$$x = -R\text{为收敛点，则}\sum_{n=0}^{\infty} a_n (-R)^n = S(-R+0)$$

$$x = R\text{为收敛点，则}\sum_{n=0}^{\infty} a_n R^n = S(R-0)$$

$$(\text{逐项可导性}) x \in (-R, R):$$

$$S'(x) = (\sum_{n=0}^{\infty} a_n x^n)' = \sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\text{且}\sum_{n=1}^{\infty} n a_n x^{n-1}\text{收敛半径}R$$

$$(\text{逐项可积性}) x \in (-R, R):$$

$$\int_0^x S(x) dx = \int_0^x (\sum_{n=0}^{\infty} a_n x^n) dx = \sum_{n=0}^{\infty} \int_0^x a_n x^n dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

$$\text{且}\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}\text{收敛半径}R$$

函数展成幂级数

$$f(x): x = x_0$$

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

直接法

$$f(x)\text{在}x = x_0\text{邻域内有}n+1\text{阶导数}$$

$$\Rightarrow f(x) = P_n(x) + R_n(x)$$

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$R_n(x) = \begin{cases} \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}, \text{拉格朗日型} \\ o((x - x_0)^n), \text{皮亚诺型} \end{cases}$$

$$f(x)\text{在}x = x_0\text{邻域内任意阶可导}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, (?)$$

$$x_0 = 0: f(x) \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \text{麦克劳林级数}, (?)$$

$$e^x=1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!},(-\infty<x<+\infty)$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1},(-\infty<x<+\infty)$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n},(-\infty<x<+\infty)$$

$$\frac{1}{1-x}=1+x+\ldots+x^n+\ldots=\sum_{n=0}^{\infty}x^n,(-1<x<1)$$

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n,(-1<x<1)$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n,(-1<x\leq 1)$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\ln 2$$

$$-\ln(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n},(-1\leq x<1)$$

间接法

$$\begin{cases}1\text{到}7\\ \text{逐项可导, 逐项可积}\end{cases}$$

求S(x)

$$\sum P(n)x^n:4\text{、}5$$

$$\sum\frac{x^n}{P(n)}\begin{cases}6\text{、}7\\ \text{逐项可积性}\end{cases}$$

傅里叶级数

参数计算

$$\text{周期为}2\pi\text{的傅里叶函数}f(x)\text{：}$$

$$f(x)\text{可否分解为}\frac{a_0}{2}+\sum_{n=1}^{+\infty}(a_n\cos nx+b_n\sin nx)$$

$$\frac{a_0}{2},\text{直流成份}$$

$$a_1\cos x+b_1\sin x,\text{一次谐波}$$

$$f(x)\text{与}\frac{a_0}{2}+\sum_{n=1}^{+\infty}(a_n\cos nx+b_n\sin nx)$$

$$(\text{狄利克雷充分条件})\text{设}f(x)\text{以}2\pi\text{为周期，在}[-\pi,\pi)\text{上：}$$

$$f(x)\text{连续或有有限个第一类间断点}$$

$$f(x)\text{有有限个极值点}$$

$$\text{则}f(x)\text{可以展成}\frac{a_0}{2}+\sum_{n=1}^{+\infty}(a_n\cos nx+b_n\sin nx),\text{且}$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

$$n=1,2,\dots$$

$$\text{若}x\text{为}f(x)\text{的连续点，则}\frac{a_0}{2}+\sum_{n=1}^{+\infty}(a_n\cos nx+b_n\sin nx)=f(x)$$

$$\text{若}x\text{为}f(x)\text{的间断点，则}\frac{a_0}{2}+\sum_{n=1}^{+\infty}(a_n\cos nx+b_n\sin nx)=\frac{f(x-0)+f(x+0)}{2}$$

周期延拓

$$f(x)\text{定义域}[-\pi,\pi),f(x)\text{周期延拓：}$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(\dots),(-\pi<x<\pi)$$

奇延拓、展成正弦级数

$$f(x)\text{定义域}[0,\pi)$$

$$f(x)\text{奇延拓或展成正弦级数}$$

1.奇延拓、周期延拓

$$2.a_0 = 0, a_n = 0, b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nxdx$$

$$3.f(x) = \sum_{n=1}^{\infty} b_n \sin nx, (0 \le x < \pi)$$

偶延拓、展成余弦级数

1.偶延拓、周期延拓

$$2.a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx, a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nxdx, b_n = 0$$

$$3.f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, (0 \le x \le \pi)$$

以2l为周期的函数f(x)

1.作图

$$2.a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx, a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx, b_n = \frac{1}{l}\int_{-l}^{l}\sin\frac{n\pi x}{l}dx$$

$$3.x\text{为}f(x)\text{的连续点}, f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}, (-\infty < x < +\infty, x \ne ?)$$

$$4.x = ?, \frac{a_0}{2} + \sum_{n=1}^{\infty}(\dots) = \frac{f(x-0)+f(x+0)}{2}$$

f(x)定义域[-l,l)上

1. 周期延拓
2. $$\begin{cases} a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx \\ a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \\ b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx \end{cases}$$
3. $$f(x) = \frac{a_0}{2} + \dots()$$