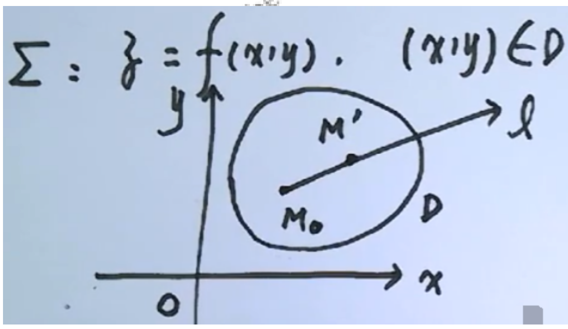
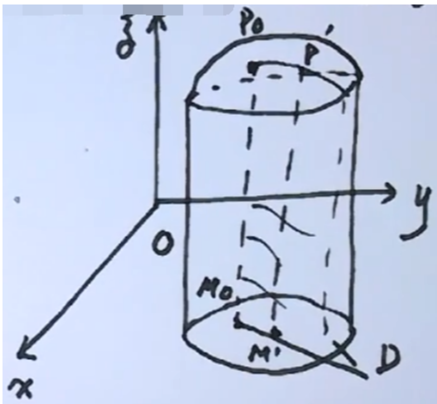
方向导数与梯度

$$\Sigma: z = f(x,y), (x,y) \in D, M_0(x_0,y_0) \in D$$

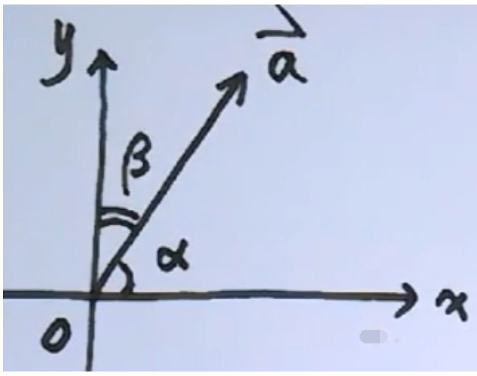




$$\begin{aligned}
&\text{在}l\text{上取}M'(x_0+\Delta x, y_0+\Delta y) \in l \\
&\rho = |M_0M'| = \sqrt{(\Delta x)^2 + (\Delta y)^2} \\
&\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) \\
&\frac{\Delta z}{\rho}
\end{aligned}$$

方向导数

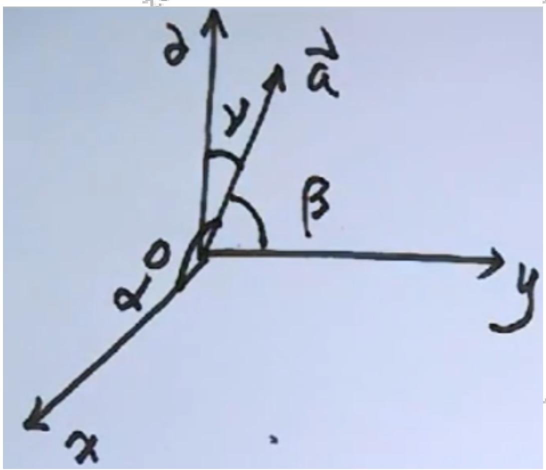
$$\begin{aligned}
&1.z = f(x,y)((x,y) \in D), M_0(x_0,y_0) \in D, \text{在}xoy\text{面内由}M_0\text{作射线}l, \\
&\quad M'(x_0+\Delta x, y_0+\Delta y) \in l, \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2} \\
&\quad \Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) \\
&\quad \text{若}\lim_{\rho \to 0} \frac{\Delta z}{\rho}\exists, \text{称此极限为}z = f(x,y)\text{在}M_0\text{处沿射线} \\
&\quad l\text{的方向导数, 记}\frac{\partial z}{\partial l}\mid_{M_0} \\
&2.u = f(x,y,z)((x,y,z) \in \Omega), M_0 \in \Omega \\
&\quad \text{过}M_0\text{作射线}l, M'(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z) \in l \\
&\quad \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \\
&\quad \Delta u = f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z) - f(x_0,y_0,z_0) \\
&\quad \text{若}\lim_{\rho \to 0} \frac{\Delta u}{\rho}\exists, \text{称此极限为}f(x,y,z)\text{在}M_0\text{处沿射线}l \\
&\quad \text{的方向导数, 记}\frac{\partial u}{\partial l}\mid_{M_0}
\end{aligned}$$



$1.\alpha,\beta$为$\vec{a}$的方向角

$\cos\alpha,\cos\beta$为$\vec{a}$的方向余弦

设$\vec{a}=\{a_1,b_1\},\vec{a}^o=\{\frac{a_1}{\sqrt{a_1^2+b_1^2}},\frac{b_1}{\sqrt{a_1^2+b_1^2}}\}=\{\cos\alpha,\cos\beta\}$



$$2.\vec{a}=\{a_1,b_1,c_1\},|\vec{a}|=\sqrt{a_1^2+b_1^2+c_1^2}$$
$$\vec{a}^o=\{\frac{a_1}{|\vec{a}|},\frac{b_1}{|\vec{a}|},\frac{c_1}{|\vec{a}|}\}=\{\cos\alpha,\cos\beta,\cos\gamma\}$$
$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

如何计算方向导数?

二元

$z=f(x,y),M_0(x_0,y_0)\in D,$过$M_0$作射线$l$

方向角为α,β:

$$\frac{\partial z}{\partial l}|_{M_0}=\frac{\partial z}{\partial x}|_{M_0}\cdot\cos\alpha+\frac{\partial z}{\partial y}|_{M_0}\cdot\cos\beta$$

$z=(x^2+y^2)^{xy}$在$(1,1)$处从$(1,1)$指向$(0,2)$的方向导数.

解 $:z=e^{xy\ln(x^2+y^2)}, \dfrac{\partial z}{\partial x}=(x^2+y^2)^{xy}\cdot[y\ln(x^2+y^2)+\dfrac{2x^2y}{x^2+y^2}]$

$$\frac{\partial z}{\partial y}=(x^2+y^2)^{xy}\cdot[x\ln(x^2+y^2)+\frac{2xy^2}{x^2+y^2}]$$

$$\frac{\partial z}{\partial x}|_{(1,1)}=2(\ln 2+1),\frac{\partial z}{\partial y}|_{(1,1)}=2(\ln 2+1)$$

从$(1,1)$指向$(0,2)$的向量为$\{-1,1\}$

$$\cos\alpha=-\frac{1}{\sqrt{2}},\cos\beta=\frac{1}{\sqrt{2}}$$

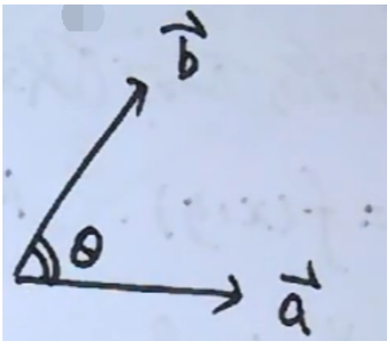
$$\frac{\partial z}{\partial l}|_{(1,1)}=2(\ln 2+1)\cdot(-\frac{1}{\sqrt{2}})+2(\ln 2+1)\cdot\frac{1}{\sqrt{2}}=0$$

三元

$$u=f(x,y,z),M_0\in\Omega,\text{过}M_0\text{作射线}l,\text{方向余弦为}\cos\alpha,\cos\beta,\cos\gamma$$

$$\frac{\partial u}{\partial l}|_{M_0}=\frac{\partial u}{\partial x}|_{M_0}\cdot\cos\alpha+\frac{\partial u}{\partial y}|_{M_0}\cdot\cos\beta+\frac{\partial u}{\partial z}|_{M_0}\cdot\cos\gamma$$

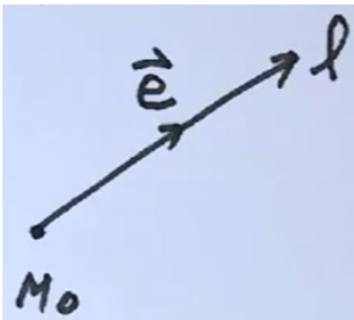
梯度



$$\vec{a}=\{a_1,b_1,c_1\},\vec{b}=\{a_2,b_2,c_2\}$$

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\theta$$

$$\vec{a}\cdot\vec{b}=a_1a_2+b_1b_2+c_1c_2$$



$u=f(x,y,z), M_0\in\Omega$, 过M_0作射线l, 方向余弦 $\cos\alpha, \cos\beta, \cos\gamma$

$\{\cos\alpha, \cos\beta, \cos\gamma\}=\vec{e}$

$$\frac{\partial u}{\partial l}\mid_{M_0}=\frac{\partial u}{\partial x}\mid_{M_0}\cdot\cos\alpha+\frac{\partial u}{\partial y}\mid_{M_0}\cdot\cos\beta+\frac{\partial u}{\partial z}\mid_{M_0}\cdot\cos\gamma$$

$$=\{\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\}_{M_0}\cdot\{\cos\alpha,\cos\beta,\cos\gamma\}$$

$$=\{\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\}_{M_0}\cdot\vec{e}$$

$$=\sqrt{(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2+(\frac{\partial u}{\partial z})^2}\mid_{M_0}\cdot\cos\theta$$

当 $\cos\theta=1$, 或$\theta=0$, 即l的方向与$\{\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\}_{M_0}$同向时,

$\frac{\partial u}{\partial l}\mid_{M_0}$ 取最大值$\sqrt{(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2+(\frac{\partial u}{\partial z})^2}\mid_{M_0}$

记$\{\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\}_{M_0}\triangleq \mathrm{grad}u\mid_{M_0}$

$u=f(x,y,z)$在M_0处的梯度

$$\begin{aligned}\frac{\partial u}{\partial l}\mid_{M_0}&=\mathrm{grad}u\mid_{M_0}\cdot\vec{e}\\&=\sqrt{(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2+(\frac{\partial u}{\partial z})^2}\mid_{M_0}\cdot\cos\theta\end{aligned}$$

当$\theta=0$, 即l的方向与梯度同向, 函数增长速度最快