中值定理

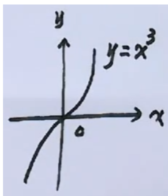
$$y=f(x)(x\in D),x_0\in D$$

$$\begin{aligned}&1.\ 若\exists\delta>0,当0<|x-x_0|<\delta时,\\&\quad f(x)<f(x_0)(左右小,中间大)\\&\quad x=x_0为f(x)的极大点\\&2.\ 若\exists\delta>0,当0<|x-x_0|<\delta时,\\&\quad f(x)>f(x_0)(左右大,中间小)\\&\quad x=x_0为f(x)的极小点\end{aligned}$$

$$f'(a)\begin{cases}>0\\<0\\=0\\不存在\end{cases}$$

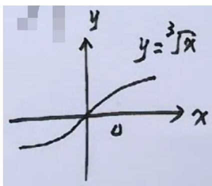
$$\begin{aligned}&1.\ f'(a)>0:f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}>0\\&\quad\exists\delta>0,当0<|x-a|<\delta时,\frac{f(x)-f(a)}{x-a}>0\\&\quad\begin{cases}f(x)<f(a),x\in(a-\delta,a)-左小\\f(x)>f(a),x\in(a,a+\delta)-右大\end{cases}\Rightarrow x=a不是极值点\\&2.\ f'(a)<0:f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}<0\\&\quad\exists\delta>0,当0<|x-a|<\delta时,\frac{f(x)-f(a)}{x-a}<0\\&\quad\begin{cases}f(x)>f(a),x\in(a-\delta,a)-左大\\f(x)<f(a),x\in(a,a+\delta)-右小\end{cases}\Rightarrow x=a不是极值点\end{aligned}$$

$$\begin{aligned}&①\ x=a为f(x)的极值点\Rightarrow(\not\Leftarrow)f'(a)=0或f'(a)不存在\\&②\ f(x)可导且x=a为f(x)的极值点\Rightarrow(\not\Leftarrow)f'(a)=0\end{aligned}$$



$$f(x)=x^3,f'(x)=3x^2,f'(0)=0$$

$$x=0不是极值点$$

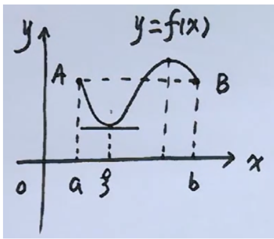


$$f(x)=\sqrt[3]{x}$$

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{\sqrt[3]{x}}{x}=\infty\Rightarrow f'(0)不存在$$

$$x=0不是极值点$$

Rolle



$$\text{若①}f(x)\text{在}[a,b]\text{上连续}$$
$$\text{②}f(x)\text{在}(a,b)\text{内可导}$$
$$\text{③}f(a)=f(b)$$

$$\text{则}\exists\xi\in(a,b),\text{使}f'(\xi)=0$$

$$\begin{aligned}&\text{证}:1.\ f(x)\in C[a,b]\Rightarrow\exists m,M\\&\quad 2.\ \text{①}m=M:f(x)\equiv C_0,\forall\exists\in(a,b),\text{有}f'(\xi)=0\\&\qquad\text{②}m<M:\\&\qquad\quad\because f(a)=f(b),\therefore m,M\text{至少有一个在}(a,b)\text{内取到}\\&\qquad\quad\text{设}\exists\xi\in(a,b),\text{使}f(\xi)=M\\&\qquad\quad\because\xi\text{为极大点},\therefore f'(\xi)=0\text{或}f'(\xi)\text{不存在},\\&\qquad\quad\because f(x)\text{在}(a,b)\text{内可导},\therefore f'(\xi)=0\end{aligned}$$

$$\text{验证函数}f(x)=x^2-2x+4\text{在}[0,2]\text{上满足罗尔定理的条件，并求驻点}\xi.$$

$$f(x)\in C[0,2],f(x)\text{在}(0,2)\text{内可导}$$
$$f(0)=f(2)=4$$
$$\exists\xi\in(0,2),\text{使}f'(\xi)=0$$
$$f'(x)=2x-2,\text{由}2\xi-2=0\Rightarrow\xi=1$$

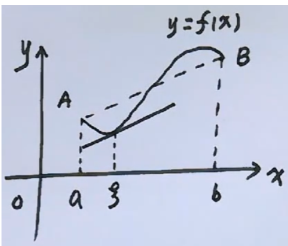
$$\text{设}f(x)\text{在}[0,2]\text{上连续},\text{在}(0,2)\text{内可导},\text{且}f(0)=1,f(1)+f(2)=2,\text{证名}:\text{存在}\xi\in(0,2),\text{使得}f'(\xi)=0.$$

$$\begin{aligned}&\text{证}:1.\ f(x)\in C[1,2]\Rightarrow\exists m,M\\&\qquad 2m\le f(1)+f(2)\le 2M\\&\qquad\because f(1)+f(2)=2,\therefore m\le 1\le M\\&\qquad\therefore\exists c\in[1,2],\text{使}f(c)=1\\&\quad 2.\ \because f(c)=f(0)=1,\therefore\exists\xi\in(0,c)\subset(0,2),\text{使}f'(\xi)=0\end{aligned}$$

$$f(x)\in C[0,2],(0,2)\text{内可导},3f(0)=f(1)+2(f2)$$
$$\text{证}:\exists\xi\in(0,2),\text{使}f'(\xi)=0$$

$$\begin{aligned}&\text{证}:1.\ f(x)\in C[1,2]\Rightarrow\exists m,M\\&\qquad 3m\le f(1)+2f(2)\le 3M\\&\qquad\Rightarrow m\le\frac{f(1)+2f(2)}{3}\le M\\&\qquad\therefore\exists c\in[1,2],\text{使}f(c)=\frac{f(1)+2f(2)}{3}\Rightarrow f(1)+2f(2)=3f(c)\\&\quad 2.\ \because f(0)=f(c),\therefore\exists\xi\in(0,c)\subset(0,2),\text{使}f'(\xi)=0\end{aligned}$$

Lagrange



$$\text{若①} f(x) \text{在} [a,b] \text{上连续}$$
$$\text{②} f(x) \text{在} (a,b) \text{内可导}$$

$$\text{则} \exists \xi \in (a,b), \text{使} f'(\xi) = \frac{f(b)-f(a)}{b-a}$$

$$\text{分析}: L: y = f(x)$$

$$L_{AB}: y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a), \text{即}$$

$$L_{AB}: y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

$$\text{证}: \text{令} \Phi(x) = \text{曲} - \text{直} = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

$\Phi(x) \in C[a,b], (a,b)$内可导

$\because \Phi(a) = \Phi(b) = 0, \therefore \exists \xi \in (a,b)$, 使$\Phi'(\xi) = 0$

而$\Phi'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$

$\therefore f'(\xi) = \dfrac{f(b)-f(a)}{b-a}$

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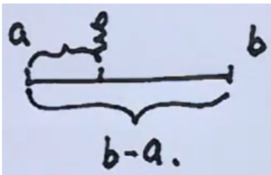
1. 若$f(a) = f(b), L \Rightarrow R$
2. 等价形式,
   $f(b) - f(a) = f'(\xi)(b-a)$
   $f(b) - f(a) = f'[a + \theta(b-a)](b-a) (0 < \theta < 1)$
3. $f(x)$可导
   $f(x) - f(a) = f'(\xi)(x-a) = f'[a + (x-a)\theta](x-a) (0 < \theta < 1)$



设$f(x)$二阶可导, 且$f''(x) > 0$.判断$f'(0), f'(1), f(1) - f(0)$的大小.

1. $f(1) - f(0) = f'(c)(1-0) = f'(c) (0 < c < 1)$
2. $f''(x) > 0 \Rightarrow f'(x) \uparrow$
   $\because 0 < c < 1, \therefore f'(0) < f'(c) < f'(1)$

设函数$f(x)$可导, 且$\lim\limits_{x\to\infty} f'(x) = e$, 求$\lim\limits_{x\to\infty}[f(x+2) - f(x-1)]$.

1. $f(x+2) - f(x-1) = 3f'(\xi) (x-1 < \xi < x+1)$
2. 原式$= 3\lim\limits_{x\to\infty} f'(\xi) = 3e$

$$\lim_{x\to\infty} x^2 (\sin\frac{1}{x-1} - \sin\frac{1}{x+1}) (\infty * 0: \frac{0}{0}, \frac{\infty}{\infty})$$

$$\text{令} f(t) = \sin t, f'(t) = \cos t$$

$$\sin\frac{1}{x-1} - \sin\frac{1}{x+1} = f(\frac{1}{x-1}) - f(\frac{1}{x+1}) = f'(\xi)(\frac{1}{x-1} - \frac{1}{x+1})$$
$$= \frac{2}{x^2-1}\cos\xi (\frac{1}{x+1} < \xi < \frac{1}{x-1})$$
$$\text{原式} = \lim_{x\to\infty} \frac{2x^2}{x^2-1}\cos\xi = 2\cos 0 = 2$$

$$f(x) \text{二阶可导}, \lim_{x\to 0} \frac{f(x)-1}{x} = 1, f(1) = 2, \text{证}: \exists \xi \in (0,1), \text{使} f''(\xi) = 0$$

$$证:\lim_{x\to 0}\frac{f(x)-1}{x}=1\Rightarrow f(0)=1,f'(0)=1$$

$$\exists c\in(0,1),使f'(c)=\frac{f(1)-f(0)}{1-0}=1$$

$$\because f(x)二阶可导,且f'(0)=f'(c)=1$$

$$\therefore \exists\xi\in(0,c)\subset(0,1),使f''(\xi)=0$$

Cauchy

$$\begin{gathered}若①f(x),g(x)在[a,b]上连续\\ ②f(x),g(x)在(a,b)内可导\\ ③g'(x)\neq 0(a<x<b)\end{gathered}$$

$$则\exists\xi\in(a,b),使\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$$

$$\begin{aligned}&1.\ g'(x)\neq 0(a<x<b)\Rightarrow\begin{cases}g'(\xi)\neq 0\\ g(b)-g(a)\neq 0\end{cases}\\ &2.\ 若g(x)=x,C\Rightarrow L\end{aligned}$$

$$分析:L=\Phi(x)=曲-直=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$$

$$C:\Phi(x)=f(x)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]$$

$$证:令\Phi(x)=f(x)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]$$

$$\Phi(x)\in C[a,b],(a,b)内可导,且\Phi(a)=\Phi(b)=0$$

$$\exists\xi\in(a,b),使\Phi'(\xi)=0$$

$$而\Phi'(x)=f'(x)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(x)$$

$$\therefore\frac{f'(\xi)}{g'(\xi)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

$$设f(x)\in C[0,1],(0,1)内可导,f(1)=0,证:\exists\xi\in(0,1),使\xi f'(\xi)+2f(\xi)=0.$$

$$分析:xf'(x)+2f(x)=0\Rightarrow\frac{f'(x)}{f(x)}+\frac{2}{x}=0$$

$$\Rightarrow[\ln f(x)]'+(\ln x^2)'=0\Rightarrow[\ln x^2f(x)]'=0$$

$$证:令\Phi(x)=x^2f(x)$$

$$\because\Phi(0)=\Phi(1)=0$$

$$\therefore\exists\xi\in(0,1),使\Phi'(x)=0$$

$$而\Phi'(x)=2xf(x)+x^2f'(x)$$

$$\therefore 2\xi f(\xi)+\xi^2f'(\xi)=0$$

$$\because\xi\neq 0,\therefore 2f(\xi)+\xi f'(\xi)=0$$

$$f(x)\in C[a,b],(a,b)内可导,f(a)=f(b)=0,证:\exists\xi\in(a,b),使f'(\xi)-f(\xi)=0.$$

$$分析:f'(x)-f(x)=0\Rightarrow\frac{f'(x)}{f(x)}-1=0$$

$$\Rightarrow[\ln f(x)]'+(\ln e^{-x})'=0$$

$$\Rightarrow[\ln e^{-x}f(x)]'=0$$

$$证：令\Phi(x)=e^{-x}f(x)$$
$$\because f(a)=f(b)=0,\therefore \Phi(a)=\Phi(b)=0$$
$$\therefore \exists\xi\in(a,b),使\Phi'(\xi)=0$$
$$而\Phi'(x)=e^{-x}[f'(x)-f(x)]且e^{-x}\neq 0$$
$$\therefore f'(\xi)-f(\xi)=0$$

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①$f(b)-f(a)$或$f(a)\neq f(b)-L$

②$\xi f'(\xi)+2f(\xi)\begin{cases}仅有\xi,无a,b\\2项\\导数差一阶\end{cases}\Rightarrow xf'(x)+2f(x)\Rightarrow\dfrac{f'(x)}{f(x)}+\dfrac{2}{x}=0\Rightarrow[\ln f(x)]'+(\ln x^2)'=0\Rightarrow\Phi(x)=x^2f(x)$

③有ξ,有a,b,ξ与a,b可分开

ξ与a,b分开$\Rightarrow a,b$侧$\begin{cases}\frac{f(b)-f(a)}{b-a}-L\\\frac{f(b)-f(a)}{g(b)-g(a)}-C\end{cases}$

$$设函数f(x)\in C[a,b],在(a,b)内可导(a>0),证明：存在\xi\in(a,b),使得f(b)-f(a)=\xi f'(\xi)\ln\frac{b}{a}.$$
$$\frac{f(b)-f(a)}{\ln b-\ln a}=\xi f'(\xi)$$
$$证：g(x)=\ln x,g'(x)=\frac{1}{x}\neq 0(a<x<b)$$
$$\exists\xi\in(a,b),使\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}\Rightarrow f(b)-f(a)=\xi f'(\xi)\ln\frac{b}{a}$$

Taylor中值定理

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}\neq\lim_{x\to 0}\frac{x-x}{x^3}$$
$$若\sin x=x-\frac{x^3}{6}+o(x^3)\sim\frac{x^3}{6}$$
$$原式=\lim_{x\to 0}\frac{\frac{x^3}{6}-o(x^3)}{x^3}=\frac{1}{6}$$

$$条件：f(x)在x=x_0邻域内有n+1阶导数$$
$$结论：f(x)=P_n(x)+R_n(x)-余项$$
$$P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$
$$R_n(x)=\begin{cases}\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},\xi在x_0与x内\\o((x-x_0)^n)-皮亚诺型余项\end{cases}$$
$$若x_0=0,$$
$$f(x)=f(0)+f'(0)x+\ldots+\frac{f^{(n)}(0)}{n!}x^n+o(x^n)-麦克劳林公式$$

记 $:(x \to 0)$

①$e^x = 1 + x + \ldots + \frac{x^n}{n!} + o(x^n)$

②$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + o(x^{2n+1})$

③$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^n}{(2n)!} x^{2n} + o(x^{2n})$

④$\frac{1}{1-x} = 1 + x + \ldots + x^n + o(x^n)$

⑤$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + o(x^n)$

⑥$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{n-1}}{n} x^n + o(x^n)$

⑦$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!} x^2 + \frac{a(a-1)(a-2)}{3!} x^3 + \ldots$

$$\text{求极限} \lim_{x \to 0} \frac{\ln(1+x) - e^x + 1}{x^2}.$$

$$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$\ln(1+x) - e^x + 1 = -x^2 + o(x^2) \sim -x^2$$

$$\text{原式} = -1$$

$$\text{求} \lim_{x \to 0} \frac{e^{-\frac{x^2}{2}} - 1 + \frac{x^2}{2}}{x^2 - \sin^2 x}.$$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2) \Rightarrow e^{-\frac{x^2}{2}} = 1 - \frac{x^2}{2} + \frac{x^4}{8} + o(x^4)$$

$$e^{-\frac{x^2}{2}} - 1 + \frac{x^2}{2} \sim \frac{x^4}{8}$$

$$\text{原式} = \frac{1}{8} \lim_{x \to 0} \frac{x}{x + \sin x} * \frac{x^3}{x - \sin x} = \frac{1}{16} \lim_{x \to 0} \frac{x^3}{x - \sin x}$$

$$\because \sin x = x - \frac{x^3}{6} + o(x^3), \therefore x - \sin x \sim \frac{1}{6} x^3$$

$$\therefore \text{原式} = \frac{3}{8}$$

$$\lim_{x \to 0} \frac{\sqrt{1+x} + \sqrt{1-x} - 2}{x^2}$$

$$\text{由} (1+x)^a = 1 + ax + \frac{a(a-1)}{2!} x^2 + o(x^2), \text{得}$$

$$\sqrt{1+x} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + o(x^2)$$

$$\sqrt{1-x} = 1 - \frac{1}{2} x - \frac{1}{8} x^2 + o(x^2)$$

$$\Rightarrow \sqrt{1+x} - \sqrt{1-x} - 2 \sim -\frac{1}{4} x^2$$

$$\therefore \text{原式} = -\frac{1}{4}$$