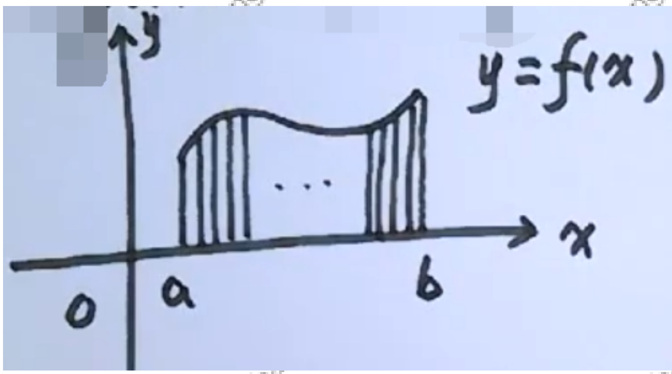
定积分理论

背景

一元不规则量的计算



$1.a=x_0<x_1<\ldots<x_n=b,[a,b]=[x_0,x_1]\cup[x_1,x_2]\cup\ldots\cup[x_{n-1},x_n]$

$2.\forall\xi_i\in[x_{i-1},x_i]$, 作$S\approx\sum\limits_{i=1}^{n}f(\xi_i)\Delta x_i$

$3.\lambda=\max\{\Delta x_1,\Delta x_2,\ldots,\Delta x_n\}$

$$S=\lim_{\lambda\to0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i$$

$$\begin{aligned}&v=v(t),t\in[a,b],S=?\\1.&a=t_0<t_1<\ldots<t_n=b\\&[a,b]=[t_0,t_1]\cup[t_1,t_2]\cup\ldots\cup[t_{n-1},t_n]\\2.&\forall\xi_i\in[t_{i-1},t_i]\\&S\approx\sum_{i=1}^{n}v(\xi_i)\Delta t_i\\3.&\lambda=\max\{\Delta t_1,\Delta t_2,\ldots,\Delta t_n\}\\&S=\lim_{\lambda\to0}\sum_{i=1}^{n}v(\xi_i)\Delta t_i\end{aligned}$$

定积分定义与一般性质

$$设f(x)在[a,b]上有界$$

1.$a=x_0<x_1<\ldots<x_n=b$

2.$\forall\xi_i\in[x_{i-1},x_i]$, 作$\sum_{i=1}^{n}f(\xi_i)\Delta x_i$

3.$\lambda=\max\{\Delta x_1,\Delta x_2,\ldots,\Delta x_n\}$

若$\lim_{\lambda\to0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i\exists$

称$f(x)$在$[a,b]$上可积, 极限值称为$f(x)$在$[a,b]$上的定积分

记$\int_a^b f(x)dx$, 即$\int_a^b f(x)dx\triangleq\lim_{\lambda\to0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i$

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1. $\lim_{\lambda\to0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i$与$\begin{cases}[a,b]\text{分法}\\ \xi_i\text{取法}\end{cases}$无关

2.$\lambda\to0\Rightarrow n\to\infty$

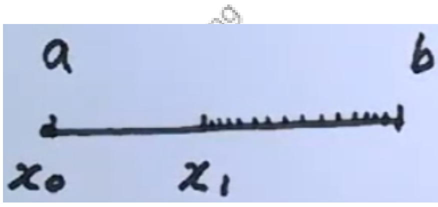
$\lambda\to0\nLeftarrow n\to\infty$

$\Rightarrow$, $b-a=\Delta x_1+\ldots+\Delta x_n\le n\lambda$

$\Rightarrow n\ge\dfrac{b-a}{\lambda}\to\infty(\lambda\to0)$

$\nLeftarrow$, 反例

$n\to\infty$, 但$\lambda=\dfrac{b-a}{2}\nrightarrow0$



3.$f(x)$有界不一定可积

反例：$f(x)=\begin{cases}1, x\in\mathbb{Q}\\ -1, x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$

$f(x)$在$[a,b]$上有界

$\forall\xi_i\in\mathbb{Q}$：

$$\lim_{\lambda\to0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i=\lim_{\lambda\to0}\sum_{i=1}^{n}\Delta x_i=b-a$$

$\forall\xi_i\in\mathbb{R}\setminus\mathbb{Q}$：

$$\lim_{\lambda\to0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i=-\lim_{\lambda\to0}\sum_{i=1}^{n}\Delta x_i=-(b-a)$$

$\Rightarrow\lim\limits_{\lambda\to0}\sum\limits_{i=1}^{n}f(\xi_i)\Delta x_i$不存在

即$f(x)$在$[a,b]$上不可积

4.若$f(x)\in C[a,b]\Rightarrow f(x)$在$[a,b]$上可积

5.设$f(x)$在$[0,1]$上可积,

$$[0,1]=[0,\frac{1}{n}]\cup[\frac{1}{n},\frac{2}{n}]\cup\ldots\cup[\frac{n-1}{n},\frac{n}{n}]$$

$$(\lambda=\frac{1}{n},\lambda\to0\Leftrightarrow n\to\infty)$$

取$\xi_1=\frac{0}{n},\xi_2=\frac{1}{n},\ldots,\xi_n=\frac{n-1}{n}$,即$\xi_i=\frac{i-1}{n}(1\le i\le n)$

或

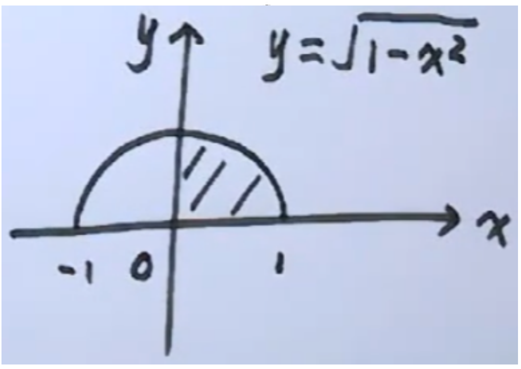
取$\xi_1=\frac{1}{n},\xi_2=\frac{2}{n},\ldots,\xi_n=\frac{n}{n}$,即$\xi_i=\frac{i}{n}(1\le i\le n)$

$$\lim_{\lambda\to0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f(\frac{i-1}{n})=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f(\frac{i}{n})=\int_0^1f(x)dx$$

记 $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}f(\frac{i-1}{n})=\int_0^1f(x)dx$,或

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f(\frac{i}{n})=\int_0^1f(x)dx$$

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n^2}(\sqrt{n^2-1^2}+\ldots+\sqrt{n^2-n^2})\\ =&\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{1-(\frac{i}{n})^2}=\int_0^1\sqrt{1-x^2}dx\\ =&\frac{\pi}{4}\end{aligned}$$



$$\lim_{n\to\infty}(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}})$$
$$\frac{n}{\sqrt{n^2+n}}\leq b_n\leq\frac{n}{\sqrt{n^2+1}}$$
$$原式=1$$

$$\lim_{n\to\infty}(\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}+\ldots+\frac{1}{\sqrt{n^2+n^2}})$$
$$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n^2+i^2}}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{(\frac{i}{n})^2+1}}=\int_0^1\frac{dx}{\sqrt{x^2+1}}=\ln(x+\sqrt{x^2+1})|_0^1=\ln(1+\sqrt{2})$$

$$\lim_{n\to\infty}(\frac{\sqrt{n^4-i^4}}{n^4}+\frac{2\sqrt{n^4-2^4}}{n^4}+\ldots+\frac{n\sqrt{(n^4-n^4)}}{n^4})$$
$$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i\sqrt{n^4-i^4}}{n^4}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}\sqrt{1-(\frac{i}{n})^4}=\int_0^1 x\sqrt{1-x^4}dx$$

$$\lim_{n\to\infty}(\frac{1}{n^2+1^2}+\frac{2}{n^2+2^2}+\ldots+\frac{n}{n^2+n^2})$$
$$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2+i^2}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{i}{n}}{1+(\frac{i}{n})^2}=\int_0^1\frac{x}{1+x^2}dx=\frac{1}{2}\ln 2$$

$$-|f(x)|\leq f(x)\leq|f(x)|$$
$$-\int_a^b|f(x)|dx\leq\int_a^b f(x)dx\leq\int_a^b|f(x)|dx$$
$$-B\leq A\leq B$$
$$|\int_a^b f(x)dx|\leq\int_a^b|f(x)|dx$$

$$积分中值定理$$
$$f(x)\in C[a,b],\exists\xi\in[a,b]$$
$$\int_a^b f(x)dx=f(\xi)(b-a)$$

$$f(x)\in[a,b]\Rightarrow\exists m,M,使m\leq f(x)\leq M$$
$$\int_a^b mdx\leq\int_a^b f(x)dx\leq\int_a^b Mdx$$
$$m(b-a)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq M(b-a)$$
$$\exists\xi\in[a,b],使f(\xi)=\frac{1}{b-a}\int_a^b f(x)dx$$

$$f(x)\in C[0,1],(0,1)可导,f(1)=4\int_0^{\frac{1}{4}}f(x)dx$$

证 :$\exists\xi\in(0,1)$, 使$f'(\xi)=0$

$$f(x)\in C[0,\frac{1}{4}]\Rightarrow\exists c\in[0,\frac{1}{4}],\text{使}$$

$$\int_0^{\frac{1}{4}}f(x)dx=f(c)(\frac{1}{4}-0)\Rightarrow 4\int_0^{\frac{1}{4}}f(x)dx=f(c)$$

$$\Rightarrow f(c)=f(1)$$

$$\exists\xi\in(c,1)\subset(0,1),\text{使}f'(\xi)=0$$

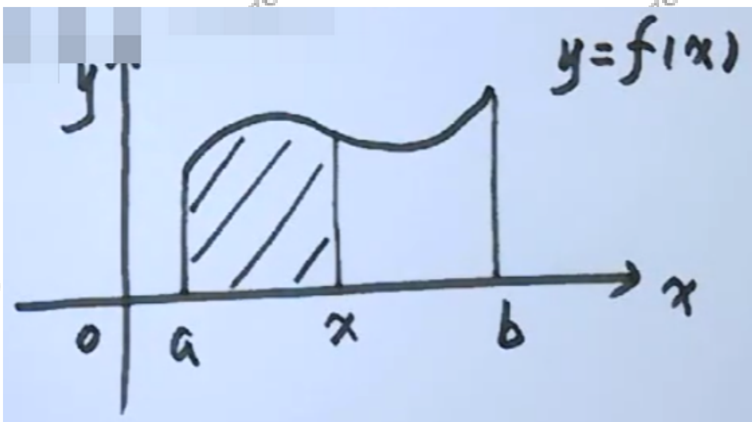
定积分基本定理

$$1.\int x^2dx\neq\int t^2dt$$

$$2.\int_0^1 x^2dx=\int_0^1 t^2dt$$

定积分由上下限和函数关系决定, 与积分变量无关, 即

$$\int_a^b f(x)dx=\int_a^b f(t)dt=\int_a^b f(u)du=\ldots$$



设$f(x)\in C[a,b]$

$$\int_a^x f(x)dx=\int_a^x f(t)dt=\Phi(x)$$

积分上限函数

1. $\int_a^x f(x)dx$表达式x与上限x同否?不同

$$\int_a^x f(x)dx=\int_a^x f(t)dt$$

2. $\int_a^x f(x,t)dt$表达式x与上限x同否?同

定理1

$$\text{设}f(x)\in C[a,b],\Phi(x)=\int_a^x f(t)dt\text{则}$$

$$\Phi'(x)=\frac{d}{dx}\int_a^x f(t)dt=f(x)$$

1.若$f(x)\in C[a,b]\Rightarrow f(x)$∃原函数, $\Phi(x)$即$f(x)$的一个原函数

2.$\dfrac{d}{dx}\displaystyle\int_a^x f(t)dt=f(x)$

$$\frac{d}{dx}\int_a^{\Phi(x)} f(t)dt=f[\Phi(x)]\Phi'(x)$$

$$\frac{d}{dx}\int_{\Phi_1(x)}^{\Phi_2(x)} f(t)dt=f[\Phi_2(x)]\Phi_2'(x)-f[\Phi_1(x)]\Phi_1'(x)$$

证 :$\Delta\Phi=\Phi(x+\Delta x)-\Phi(x)=\displaystyle\int_a^{x+\Delta x} f(t)dt-\int_a^x f(t)dt=\int_x^{x+\Delta x} f(t)dt$

$\because f(x)\in C[a,b],\therefore \Delta\Phi=f(\xi)\Delta x$ (ξ在x与$x+\Delta x$之间)

$\Rightarrow \dfrac{\Delta\Phi}{\Delta x}=f(\xi)\Rightarrow \lim\limits_{\Delta x\to 0}\dfrac{\Delta\Phi}{\Delta x}=\lim\limits_{\Delta x\to 0} f(\xi)$

$\because f(x)$连续, $\therefore \lim\limits_{\Delta x\to 0} f(\xi)=\lim\limits_{\xi\to x} f(\xi)=f(x)$

$\Rightarrow \Phi'(x)=f(x)$

$$f(x)\text{连续}, f(0)=0, f'(0)=\pi, \text{求}\lim_{x\to 0}\frac{\int_0^x f(t)dt}{x-\ln(1+x)}$$

$$\text{原式}=\lim_{x\to 0}\frac{f(x)}{1-\frac{1}{x+1}}=\lim_{x\to 0}\frac{(x+1)f(x)}{x}=\lim_{x\to 0}\frac{f(x)+(x+1)f'(x)}{1}=f(0)+f'(0)=\pi$$

$$\text{设函数}f(x)\text{连续}, \phi(x)=\int_0^x (x-t)f(t)dt, \text{求}\phi''(x)$$

$$\phi(x)=x\int_0^x f(t)dt-\int_0^x tf(t)dt$$

$$\phi'(x)=\int_0^x f(t)dt+xf(x)-xf(x)$$

$$\phi''(x)=f(x)$$

定理2 (N-L)

$$\int_a^b f(x)dx=\lim_{\lambda\to 0}\sum_{i=1}^n f(\xi_i)\Delta x_i$$

牛顿莱布尼茨公式

$f(x)\in C[a,b], F(x)$为$f(x)$的一个原函数, 则

$$\int_a^b f(x)dx=F(b)-F(a)$$

证 :$\Phi(x)=\displaystyle\int_a^x f(t)dt, \Phi'(x)=f(x)$

$\because F'(x)=f(x),\therefore F(x)-\Phi(x)\equiv C_0(a\le x\le b)$

$\Rightarrow F(a)-\Phi(a)=F(b)-\Phi(b)$

$\because \Phi(a)=0,\therefore \Phi(b)=F(b)-F(a)$

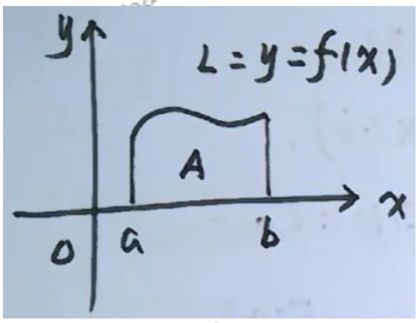
即$\displaystyle\int_a^b f(t)dt=F(b)-F(a)$或$\displaystyle\int_a^b f(x)dx=F(b)-F(a)$

$$\int_0^1 \frac{x}{1+x^4}dx=\frac{1}{2}\int_0^1\frac{d(x^2)}{1+(x^2)^2}=\frac{1}{2}\arctan x^2\ |_0^1=\frac{\pi}{8}$$

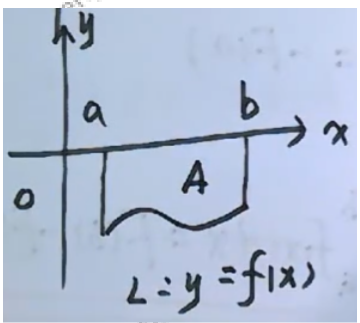
定积分基本性质

一般性质

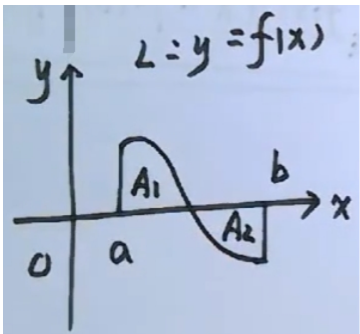
1. $\int_a^b [kf(x) \pm lg(x)]dx = k\int_a^b f(x)dx \pm l\int_a^b g(x)dx (k, l\text{常数})$
2. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
3. $\int_a^b 1dx = b - a$



$$\int_a^b f(x)dx = A$$



$$\int_a^b f(x)dx = -A$$



$$\int_a^b f(x)dx = A_1 - A_2$$

4.①$f(x) \geq 0 (a \leq x \leq b) \Rightarrow \int_a^b f(x)dx \geq 0$

②$f(x) \geq g(x) (a \leq x \leq b) \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$

③$f(x), |f(x)|$在$[a, b]$上可积，则

$$|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$$

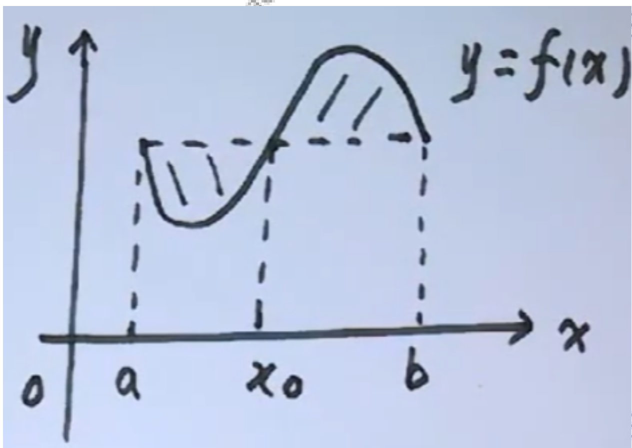
5.$f(x)$在$[a,b]$上可积,且$m \le f(x) \le M$,则

$$\int_a^b m dx \le \int_a^b f(x)dx \le \int_a^b M dx, \text{即}$$

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

6.①(积分中值定理)设$f(x) \in C[a,b] \Rightarrow \exists m, M$

则$\exists \xi \in [a,b]$,使$\int_a^b f(x)dx = f(\xi)(b-a)$



$$\int_a^b f(x)dx = f(\xi)(b-a)$$

$$\xi = a, x_0, b$$

证:$f(x) \in C[a,b] \Rightarrow m, M$

$$m \le f(x) \le M$$

$$\Rightarrow m \le \frac{\int_a^b f(x)dx}{b-a} \le M$$

$\xi \in [a,b]$,使

$$f(\xi) = \frac{\int_a^b f(x)dx}{b-a}$$

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

②(积分中值定理推广)设$f(x) \in C[a,b]$,则$\exists \xi \in (a,b)$,使

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

证:令$F(x) = \int_a^x f(t)dt, F'(x) = f(x)$

$$\int_a^b f(x)dx = F(b) = F(b) - F(a)$$

$$= F'(\xi)(b-a) = f(\xi)(b-a)(a < \xi < b)$$

$f(x) \in C[0,1]$内可导,$f(0) = \int_0^1 f(x)dx$,证:$\exists \xi \in (0,1)$,使$f'(\xi) = 0$

证:令$F(x) = \int_0^x f(t)dt, F'(x) = f(x)$

$$\int_0^1 f(x)dx = F(1) = F(1) - F(0) = F'(c) = f(c)(0 < c < 1)$$

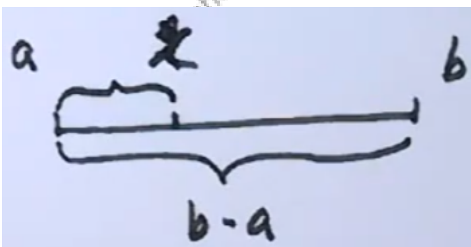
$\because f(0) = f(c), \therefore \exists \xi \in (0,c) \subset (0,1)$,使$f'(\xi) = 0$

特殊性质

对称区间

$$设f(x)\in C[-a,a],则\int_{-a}^{a}f(x)dx=\int_{0}^{a}[f(x)+f(-x)]dx$$

$$
\begin{aligned}
&1.\int_{-a}^{0}f(x)dx=\int_{0}^{a},x=-t\\
&2.\int_{a}^{a+b}f(x)dx=\int_{0}^{b},x-a=t\\
&3.\int_{a}^{b}f(x)dx=\int_{a}^{b},x+t=a+b\\
&4.\int_{a}^{b}f(x)dx=\int_{0}^{1},x=a+(b-a)t
\end{aligned}
$$



$$
\begin{aligned}
&证:左=\int_{-a}^{0}f(x)dx+\int_{0}^{a}f(x)dx\\
&\quad 而\int_{-a}^{0}f(x)dx=\int_{a}^{0}f(-t)(-dt)=\int_{0}^{a}f(-t)dt=\int_{0}^{a}f(-x)dx\\
&\quad 左=\int_{0}^{a}[f(x)+f(-x)]dx\\
&\quad 若f(-x)=-f(x)\Rightarrow\int_{-a}^{a}f(x)dx=0\\
&\quad 若f(-x)=f(x)\Rightarrow\int_{-a}^{a}f(x)dx=2\int_{0}^{a}f(x)dx
\end{aligned}
$$

$$
\begin{aligned}
&\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{e^x+1}dx\\
=&\int_{0}^{\frac{\pi}{4}}(\frac{1}{e^x+1}+\frac{1}{e^{-x}+1})dx\\
=&\int_{0}^{\frac{\pi}{4}}(\frac{1}{e^x+1}+\frac{e^x}{e^x+1})dx\\
=&\frac{\pi}{4}
\end{aligned}
$$

$$
\begin{aligned}
&\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{1-\sin x}dx\\
=&\int_{0}^{\frac{\pi}{4}}(\frac{1}{1-\sin x}+\frac{1}{1+\sin x})dx\\
=&\int_{0}^{\frac{\pi}{4}}\frac{2}{1-\sin^2 x}=2\int_{0}^{\frac{\pi}{4}}\sec^2 xdx\\
=&2\tan x\mid_{0}^{\frac{\pi}{4}}\\
=&2
\end{aligned}
$$

三角函数

$$设f(x)\in C[0,1],则\int_0^{\frac{\pi}{2}} f(\sin x)dx=\int_0^{\frac{\pi}{2}} f(\cos x)dx$$

$$证:x+t=\frac{\pi}{2},d(x+t)=dx+dt=0$$

$$\int_0^{\frac{\pi}{2}} f(\sin x)dx=\int_{\frac{\pi}{2}}^0 f(\cos t)(-dt)=\int_0^{\frac{\pi}{2}} f(\cos t)dt$$

$$\int_0^{\frac{\pi}{2}} f(\sin x)dx=\int_0^{\frac{\pi}{2}} f(\cos x)dx$$

$$求I=\int_0^1\frac{dx}{x+\sqrt{1-x^2}}$$

$$解:令x=\sin t$$

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos t}{\sin t+\cos t}dt=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x+\cos x}dx$$

$$\because I=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\cos x+\sin x}dx$$

$$\therefore 2I=\int_0^{\frac{\pi}{2}}1dx=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$$

重点1

$$记:\int_0^{\frac{\pi}{2}}\sin^n xdx=\int_0^{\frac{\pi}{2}}\cos^n xdx=I_n$$

$$\begin{cases} I_n=\frac{n-1}{n}I_{n-2}\\ I_0=\frac{\pi}{2}\\ I_1=1\end{cases}$$

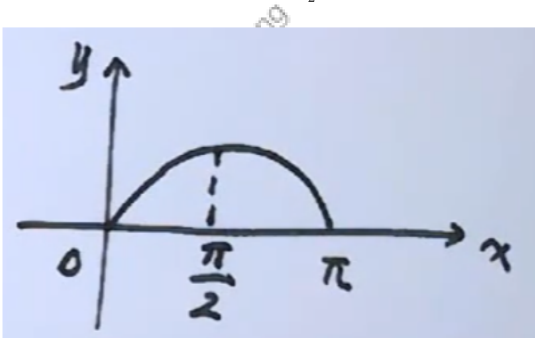
$$\int_0^{\frac{\pi}{2}}\sin^{11}x=I_{11}=\frac{10}{11}*\frac{8}{9}*\frac{6}{7}*\frac{4}{5}*\frac{2}{3}*I_1=\frac{10}{11}*\frac{8}{9}*\frac{6}{7}*\frac{4}{5}*\frac{2}{3}*1$$

$$\int_0^{\frac{\pi}{2}}\cos^{10}x=I_{10}=\frac{9}{10}I_8=\frac{9}{10}*\frac{7}{8}*I_6=\frac{9}{10}*\frac{5}{6}*\frac{3}{4}*\frac{1}{2}*I_0=\frac{9}{10}*\frac{5}{6}*\frac{3}{4}*\frac{1}{2}*\frac{\pi}{2}$$

重点2

$$\int_0^{\pi}f(\sin x)dx=2\int_0^{\frac{\pi}{2}}f(\sin x)dx$$

$$\int_0^{\frac{\pi}{2}}f(\sin x)dx=\int_{\frac{\pi}{2}}^{\pi}f(\sin x)dx$$



$$证:\int_{\frac{\pi}{2}}^{\pi} f(\sin x)dx, x-\frac{\pi}{2}=t$$

$$\int_0^{\frac{\pi}{2}} f(\cos t)dt = \int_0^{\frac{\pi}{2}} f(\cos x)dx$$

$$= \int_0^{\frac{\pi}{2}} f(\sin x)dx$$

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x dx$$

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1dx = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

$$I = \int_{-\pi}^{\pi} \frac{\sin^2 x}{1+e^{-x}} dx$$

$$解:I = \int_0^{\pi} \frac{\sin^2 x}{1+e^{-x}} + \frac{\sin^2 x}{1+e^{x}} dx$$

$$= \int_0^{\pi} (\frac{1}{1+e^{-x}} + \frac{1}{1+e^{x}}) \sin^2 x dx$$

$$= \int_0^{\pi} \sin^2 x dx$$

$$=2 \int_0^{\frac{\pi}{2}} \sin^2 x dx$$

$$=2I_2$$

$$=2 * \frac{1}{2} * \frac{\pi}{2}$$

$$=\frac{\pi}{2}$$

$$f, g \in C[-a, a], f(x) + f(-x) \equiv A, g(-x) = g(x),$$

$$1.证: \int_{-a}^{a} f(x)g(x)dx = A \int_0^a g(x)dx$$

$$2.求I = \int_{-\pi}^{\pi} \arctan e^x \cdot \sin^2 x dx$$

$$1.证: \int_{-a}^{a} f(x)g(x)dx = \int_0^a [f(x)g(x) + f(-x)g(-x)]dx$$

$$= \int_0^a [f(x) + f(-x)]g(x)dx = A \int_0^a g(x)dx$$

$$2.解:I = \int_0^{\pi} (\arctan e^x + \arctan e^{-x}) \cdot \sin^2 x dx$$

$$\because (\arctan e^x + \arctan e^{-x})' = \frac{e^x}{1+e^{2x}} - \frac{e^{-x}}{1+e^{-2x}} = 0$$

$$\therefore \arctan e^x + \arctan e^{-x} \equiv A$$

$$x = 0时, A = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{2} \int_0^{\pi} \sin^2 x dx = \pi I_2 = \frac{\pi}{2} I_0 = \frac{\pi^2}{4}$$

$$\int_0^\pi f(|\cos x|)dx = 2\int_0^{\frac{\pi}{2}} f(\cos x)dx$$
$$\int_0^\pi f(\cos^{2n} x)dx = 2\int_0^{\frac{\pi}{2}} f(\cos^{2n} x)dx$$

$$\begin{aligned}&\int_0^\pi \frac{|\cos x|}{1+\sin^2 x}dx\\ =&2\int_0^{\frac{\pi}{2}} \frac{d(\sin x)}{1+\sin^2 x}\\ =&2\arctan(\sin x)\ |_0^{\frac{\pi}{2}}\\ =&\frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}&\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx\\ \text{证}:I =& \int_0^\pi xf(\sin x)dx, x+t=\pi\\ =&\int_\pi^0 (\pi-t)f(\sin t)\cdot(-dt)\\ =&\int_0^\pi (\pi-t)f(\sin t)dt = \int_0^\pi (\pi-x)f(\sin x)dx\\ =&\pi\int_0^\pi f(\sin x)dx - \int_0^\pi xf(\sin x)dx = \pi\int_0^\pi f(\sin x)dx - I\\ \Rightarrow&\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx\end{aligned}$$

$$\begin{aligned}&\int_0^\pi x\sin^3 xdx\\ =&\frac{\pi}{2}\int_0^\pi \sin^3 xdx\\ =&\pi I_3\\ =&\frac{2\pi}{3}\end{aligned}$$

$$\text{求}I = \int_0^\pi \sin^2\sqrt{x}dx$$

$$\text{解}:\text{令}\sqrt{x}=t, x=t^2$$

$$\begin{aligned}I &= 2\int_0^\pi t\sin^2 tdt\\ &= 2*\frac{\pi}{2}\int_0^\pi \sin^2 tdt\\ &= 2\pi\frac{1}{2}\frac{\pi}{2} = \frac{\pi^2}{2}\end{aligned}$$

平移性质

$$\text{设}f(x)\text{连续且以}T>0\text{为周期, 则}$$
$$1.\int_a^{a+T}f(x)dx=\int_0^T f(x)dx(\text{平移性质})$$
$$2.\int_0^{nT}f(x)dx=n\int_0^T f(x)dx$$

$$\text{证}:\int_a^{a+T}f(x)dx=\int_a^0 f(x)dx+\int_0^T f(x)dx+\int_T^{a+T}f(x)dx$$
$$\text{而}\int_T^{a+T}f(x)dx,x-T=t,\int_0^a f(t)dt=\int_0^a f(x)dx$$
$$\therefore\int_a^{a+T}f(x)dx=\int_0^T f(x)dx$$

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\sin^4 xdx$$
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^4 xdx$$
$$=2\int_0^{\frac{\pi}{2}}\sin^4 xdx$$
$$=2*\frac{3}{4}*\frac{1}{2}*\frac{\pi}{2}=\frac{3}{8}\pi$$

$$I=\int_0^{\pi}|\sin x+\cos x|dx$$

$$\text{解}:(\text{法一}:I=\int_0^{\frac{\pi}{2}})$$
$$\text{法一}:I=\int_0^{\frac{3\pi}{4}}(\sin x+\cos x)dx-\int_{\frac{3\pi}{4}}^{\pi}(\sin x+\cos x)dx$$
$$\text{法二}:I=\sqrt{2}\int_0^{\pi}|\sin(x+\frac{\pi}{4})|d(x+\frac{\pi}{4})$$
$$=\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}|\sin x|dx$$
$$=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|\sin x|dx$$
$$=2\sqrt{2}\int_0^{\frac{\pi}{2}}\sin xdx=2\sqrt{2}$$

补充

$$f(x)\in C[a,b],\Phi(x)=\int_a^x f(t)dt,\text{则}\Phi'(x)=f(x)$$

证：$\Delta\Phi=\Phi(x+\Delta x)-\Phi(x)=\int_a^{x+\Delta x}f(t)dt-\int_a^x f(t)dt$

$=\int_x^{x+\Delta x}f(t)dt=f(\xi)\Delta x$（$\xi$在$x$与$x+\Delta x$之间）

$\Rightarrow\dfrac{\Delta\Phi}{\Delta x}=f(\xi)\Rightarrow\lim\limits_{\Delta x\to 0}\dfrac{\Delta\Phi}{\Delta x}=\lim\limits_{\Delta x\to 0}f(\xi)$

即$\Phi'(x)=f(x)$