逆阵(矩阵理论一)

$$\begin{cases} ①什么叫逆阵? \\ ②可逆否? \\ ③若可逆,求A^{-1}? \end{cases}$$

$A_{n\times n}$, 若$\exists B_{n\times n}$, 使$BA=E$(或$AB=E$)
称A可逆, $B\triangleq A^{-1}$

Nots:用定义法求A^{-1}方法为
$A\times(?)=E$

设A为n阶矩阵, 若$A^2+A-2E=0$, 求A^{-1}.

$$A(A+E)=2E\Rightarrow A\cdot\frac{1}{2}(A+E)=E$$
$$\Rightarrow A^{-1}=\frac{1}{2}(A+E)$$

设A为n阶矩阵, 若$A^2+A-4E=O$, 求$(A-E)^{-1}$.

$$(A-E)\cdot(A+2E)-2E=0$$
$$\Rightarrow (A-E)\cdot\frac{1}{2}(A+2E)=E\Rightarrow (A-E)^{-1}=\frac{1}{2}(A+2E)$$

设$A\neq 0$, 但$A^2=0$, 求$(E-A)^{-1}$.

$$(E-A)(E+A)=E-A^2=E$$
$$\Rightarrow (E-A)^{-1}=E+A$$

判别

Notes :

1.①$|A^T|=|A|$;②$|A^{-1}|=\dfrac{1}{|A|}$

③$|A^*|=|A|^{n-1}$;④$A_{n\times n}$, $|kA|=k^n|A|$

⑤$A_{n\times n}, B_{n\times n}$, $|AB|=|A|\cdot|B|$(Laplace法则)

2.①$(A^{-1})^{-1}$;②$(A^T)^{-1}=(A^{-1})^T$

③$(A^{-1})^m=(A^m)^{-1}$;④$(kA)^{-1}=\dfrac{1}{k}A^{-1}$

⑤$(AB)^{-1}=B^{-1}A^{-1}$

⑥$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^{-1}=\begin{pmatrix} A^{-1} & \\ & B^{-1} \end{pmatrix}, \begin{pmatrix} & A \\ B & \end{pmatrix}^{-1}=\begin{pmatrix} & B^{-1} \\ A^{-1} & \end{pmatrix}$

$\text{Th.} A_{n\times n}$可逆 $\Leftrightarrow |A| \neq 0$

证 $:\Rightarrow \exists B,$ 使 $BA = E \Rightarrow |B||A| = E \Rightarrow |A| \neq 0$

$\Leftarrow \because AA^* = |A|E,$ 而 $|A| \neq 0$

$\therefore A \cdot \dfrac{1}{|A|} A^* = E$

$\therefore A$可逆且$A^{-1} = \dfrac{1}{|A|} A^* (n \leq 3)$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ 3 & 4 & 3 \end{pmatrix}, \text{问}A\text{可逆否?若可逆}, A^{-1} =?$$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ 3 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0 \Rightarrow A\text{可逆}$$

$$\begin{cases} M_{11} = 13, A_{11} = 13 \\ M_{12} = 9, A_{12} = -9 \\ M_{13} = -1, A_{13} = -1 \end{cases}, \begin{cases} M_{21} = -5, A_{21} = 5 \\ M_{22} = -3, A_{22} = -3, \\ M_{23} = 1, A_{23} = -1 \end{cases} \begin{cases} M_{31} = -7, A_{31} = -7 \\ M_{32} = -5, A_{32} = 5 \\ M_{33} = 1, A_{33} = 1 \end{cases}$$

$$A^* = \begin{pmatrix} 13 & 5 & -7 \\ -9 & -3 & 5 \\ -1 & -1 & 1 \end{pmatrix}, \therefore A^{-1} = \frac{1}{|A|} A^* = \frac{1}{2} \begin{pmatrix} 13 & 5 & -7 \\ -9 & -3 & 5 \\ -1 & -1 & 1 \end{pmatrix}$$

Note:若A可逆, 则$A^* = |A| A^{-1}$

设A为n阶可逆矩阵, 且$k \neq 0$, 求$(kA)^*$.

$$(kA)^* = |kA| \cdot (kA)^{-1} = k^n |A| \cdot \frac{1}{k} A^{-1} = k^{n-1} A^*$$

A, B为n阶可逆阵, $(AB)^*$?

$$(AB)^* = |AB| \cdot (AB)^{-1} = |A| \cdot |B| \cdot B^{-1} A^{-1}$$
$$= |B| B^{-1} \cdot |A| A^{-1} = B^* A^*$$

$$A, B\text{可逆}, \begin{pmatrix} A & \\ & B \end{pmatrix}^* ?$$

$$\begin{pmatrix} A & \\ & B \end{pmatrix}^* = \begin{vmatrix} A & \\ & B \end{vmatrix} \cdot \begin{pmatrix} A & \\ & B \end{pmatrix}^{-1} = |A| \cdot |B| \cdot \begin{pmatrix} A^{-1} & \\ & B^{-1} \end{pmatrix}$$

$$\begin{pmatrix} |B| A^* & \\ & |A| B^* \end{pmatrix}$$

逆阵求法

方法一：伴随矩阵法：$A^{-1}=\dfrac{1}{|A|}A^*(n\le 3)$

方法二：初等变换法思想体系

1.方程组的同解变形：$\begin{cases}3x_1-2x_2+x_3+2x_4=0\\ x_1-x_2+2x_3-x_4=0\\ \dots\end{cases}\Leftrightarrow\begin{cases}x_1-x_2+2x_3-x_4=0\\ 3x_1-2x_2+x_3+2x_4=0\\ \dots\end{cases}$

①对调两个方程

②一方程 $\times C\neq 0$

③一方程 $\times k$加到另一方程

①到③称为同解变形

$$\begin{cases}3x_1-2x_2+x_3+2x_4=0\\ x_1-x_2+2x_3-x_4=0\end{cases}\Leftrightarrow\begin{pmatrix}3&-2&1&2\\1&-1&2&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=0,\text{一方程一行}$$

矩阵的三种初等行变换

①对调两行

②一行 $\times C\neq 0$

③一行 $\times k$加到另一行

称①到③为矩阵的初等行变换

Notes:

① $\begin{cases}\text{对调两列}\\ \text{一列}\times C\neq 0\\ \text{一列}\times k\text{加到另一列}\end{cases}$ 称为矩阵的初等列变换

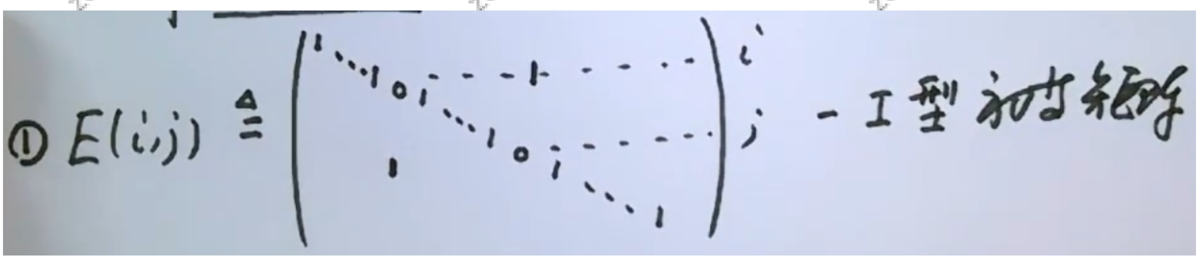
②解方程组禁止使用初等列变换

三种初等矩阵

I

$$E=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\triangleq E(2,3)$$

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{pmatrix}=\begin{pmatrix}a_1&b_1&c_1\\a_3&b_3&c_3\\a_2&b_2&c_2\end{pmatrix}$$

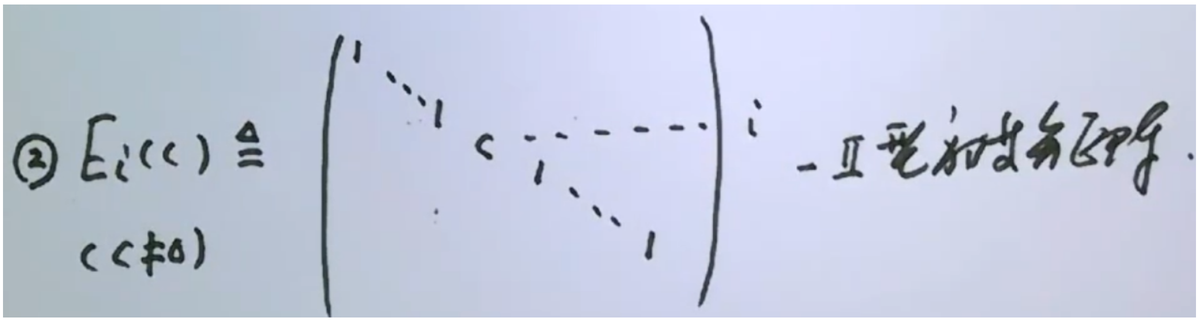


$$\begin{cases} E(i,j)A - A\text{的}i,j\text{行对调} \\ AE(i,j) - A\text{的}i,j\text{列对调} \\ |E(i,j)| = -1 \\ E^{-1}(i,j) = E(i,j), E(i,j)\cdot E(i,j) = E \Leftrightarrow E^2(i,j) = E \end{cases}$$

Ⅱ

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \triangleq E_2(3)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ 3a_2 & 3b_2 & 3c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

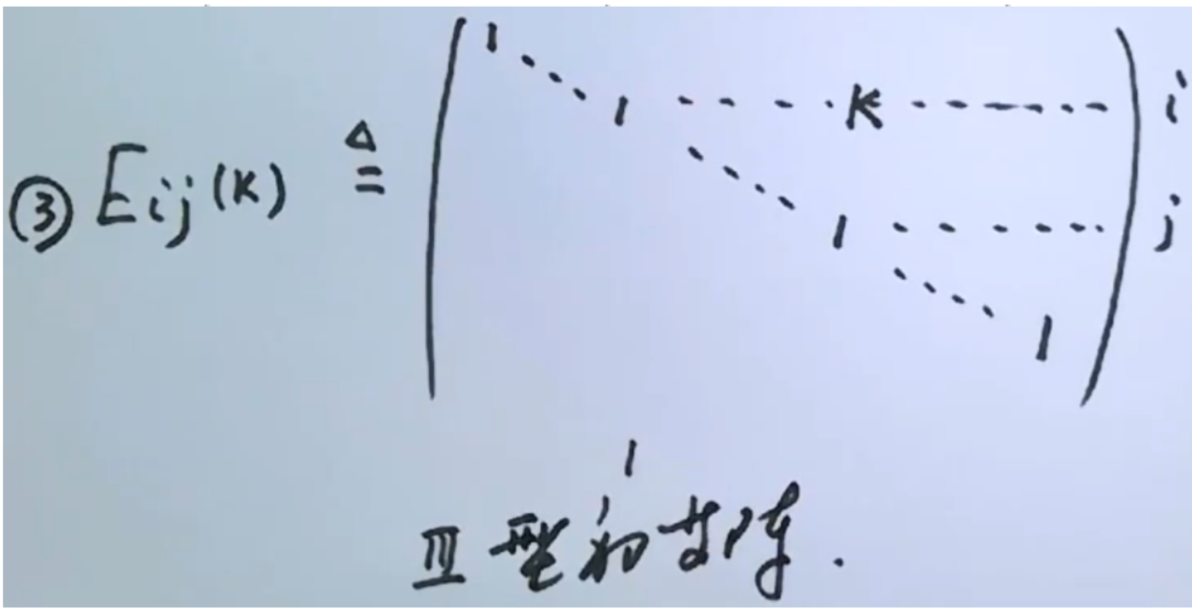


$$E_i(C): \begin{cases} E_i(C)A - A\text{的}i\text{行} \times C \neq 0 \\ AE_i(C) - A\text{的}i\text{列} \times C \neq 0 \\ |E_i(C)| = C \neq 0 \\ E_i^{-1}(C) = E_i(\frac{1}{C}) \end{cases}$$

Ⅲ

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \triangleq E_{31}(2)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 2a_1 + a_3 & 2b_1 + b_3 & 2c_1 + c_3 \end{pmatrix}$$



$$E_{ij}(k):\begin{cases}E_{ij}(k)A-A\text{的}j\text{行}k\text{倍加到}i\text{行}\\AE_{ij}(k)-A\text{的}i\text{列}k\text{倍加到}j\text{列}\\|E_{ij}(k)|=1\neq 0\\E_{ij}^{-1}(k)=E_{ij}(-k)\end{cases}$$

Notes:
①A三种初等行变换 $\Leftrightarrow$ A左乘$E(i,j),E_i(C),E_{ij}(k)$
A的三种初等列变换 $\Leftrightarrow$ A右乘$E(i,j),E_i(C),E_{ij}(k)$
②方程组同解变形 $\Leftrightarrow$ 初等行变换 $\Leftrightarrow$ 左乘$E(i,j),E_i(C),E_{ij}(k)$
③$E^{-1}(i,j)=E(i,j),E_i^{-1}(C)=E_i(\frac{1}{C}),E_{ij}^{-1}(k)=E_{ij}(-k)$
初等矩阵逆阵仍为初等阵且型不改