参数估计

$$X \Rightarrow (X_1, \cdots, X_n) \Rightarrow (x_1, \cdots, x_n)$$

参数估计的思想

$$\text{总体}X\text{分布已知, 但含未知参数}\theta\text{, 对}\theta\text{进行估计}$$

$$\text{方式一:点估计}\hat{\theta} = \varphi(X_1, \cdots, X_n) - \text{估计量}$$

$$\hat{\theta} = \varphi(x_1, \cdots, x_n) - \text{估计值}$$

$$\text{方式二:区间估计}, \forall \alpha, \underline{\theta} = \varphi_1(X_1, \cdots, X_n), \overline{\theta} = \varphi(X_1, \cdots, X_n)$$

$$P\{\underline{\theta} < \theta < \overline{\theta}\} = 1 - \alpha$$

$$(\underline{\theta}, \overline{\theta})\text{为}\theta\text{的置信度为}1-\alpha\text{的置信区间}$$

参数的点估计

矩估计法

$$\text{(一)矩估计法}$$

$$\text{1.理论基础} - \text{大数定律}$$

$$X \Rightarrow (X_1, \cdots, X_n)$$

$$A_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k\text{依概率收敛于}\frac{1}{n}\sum_{i=1}^{n} EX_i^k = EX^k$$

$$A_K = \frac{1}{n}\sum_{i=1}^{n} X_i^K \xrightarrow{P} \frac{1}{n}\sum_{i=1}^{n} EX_i^K = EX^K$$

$$\text{2.矩估计法步骤}: (X\text{分布已知}, \begin{cases} X\text{离散} \\ X\text{连续} \end{cases})$$

case1.含θ:

$$\text{若}EX = ?\text{含}\theta\text{, 令}\overline{X} = EX \Rightarrow \hat{\theta} = ?$$

$$\text{若}EX = ?\text{不含}\theta, EX^2 = ?\text{令}EX^2 = A_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \Rightarrow \hat{\theta} = ?$$

$$(\frac{1}{n}\sum_{i=1}^{n} X_i^2\text{依概率收敛于}EX^2)$$

case2.含θ_1, θ_2:

$$\text{1.求}EX = ?EX^2 = ?$$

$$\text{2.令}\begin{cases} EX = \overline{X} \\ EX^2 = A_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \end{cases} \Rightarrow \hat{\theta}_1 = ?, \hat{\theta}_2 = ?$$

$$\begin{cases} X\text{离散}: \begin{array}{c|ccc} X & x_1 & \cdots & x_n \\ \hline P & p_1 & \cdots & p_n \end{array}, \\ X\text{连续}: f(x) \end{cases}$$

设总体$X \sim \begin{pmatrix} 0 & 1 & 2 \\ \theta & 2\theta & 1-3\theta \end{pmatrix}$, $(X_1, X_2, \cdots, X_n)$为来自总体X的简单随机样本,
求参数θ的矩估计量.

$$EX = 2 - 4\theta$$
$$令EX = \overline{X} \Rightarrow \hat{\theta} = \frac{2 - \overline{X}}{4}$$

设总体$X \sim N(\mu, \sigma^2)$, $(X_1, X_2, \cdots, X_n)$为来自总体X的简单随机样本,
(1)设$\mu = 2$, 求参数σ^2的矩估计量; (2)设μ未知, 求参数σ^2的矩估计量.

$$①EX = 2, EX^2 = DX + (EX)^2 = \sigma^2 + 4$$
$$令EX^2 = A_2 \Rightarrow \widehat{\sigma}^2 = A_2 - 4$$
$$②EX = \mu, EX^2 = \sigma^2 + \mu^2$$
$$令\begin{cases} EX = \overline{X} \\ EX^2 = A_2 \end{cases} \Rightarrow \widehat{\mu} = \overline{X}, \widehat{\sigma}^2 = A_2 - \mu^2$$
$$\therefore \widehat{\sigma}^2 = A_2 - \overline{X}^2$$

最大似然估计法

X总体分布已知

$(X_1, \cdots, X_n) \Rightarrow (x_1, \cdots, x_n)$

1.总体X离散型:

1°分布律

$2^\circ L = P\{X_1 = x_1\} \cdots P\{X_n = x_n\} = P\{X = X_1\} \cdots P\{X = X_n\}$

$3^\circ \ln L = \cdots$

4°case1. 含θ:

$$令\frac{d}{d\theta}\ln L = \cdots = 0 \Rightarrow \hat{\theta} = ?$$

case2. 含θ_1, θ_2:

$$令\begin{cases} \frac{\partial}{\partial\theta_1}\ln L = \cdots = 0 \\ \frac{\partial}{\partial\theta_2}\ln L = \cdots = 0 \end{cases} \Rightarrow \hat{\theta}_1 = ?\hat{\theta}_2 = ?$$

2.$X \sim f(x)$:

$1^\circ L = f(x_1) \cdots f(x_n)$

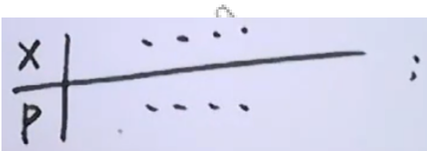
$2^\circ \ln L = \cdots$

3°case1.含θ:

$$令\frac{d}{d\theta}\ln L = \cdots = 0 \Rightarrow \hat{\theta} = ?$$

case2.含θ_1, θ_2:

$$令\begin{cases} \frac{\partial}{\partial\theta_1}\ln L = \cdots = 0 \\ \frac{\partial}{\partial\theta_2}\ln L = \cdots = 0 \end{cases} \Rightarrow \hat{\theta}_1 = ?, \hat{\theta}_2 = ?$$



设总体$X \sim \begin{pmatrix} -1 & 0 & 2 \\ \theta & \theta & 1-2\theta \end{pmatrix}$, (X_1, X_2, X_3, X_4)为来自总体的简单随机样本,
其观察值为$(0, 2, -1, 0)$, 求参数θ的最大似然估计值.

X	-1	0	2
P	θ	θ	$1-2\theta$

1.$L=P\{X=0\}P\{X=2\}P\{X=-1\}P\{X=0\}=\theta^3(1-2\theta)$

2.$\ln L=3\ln\theta+\ln(1-2\theta)$

3.令$\dfrac{d}{d\theta}\ln L=\dfrac{3}{\theta}-\dfrac{2}{1-2\theta}=0\Rightarrow 3-8\theta=0$

$\Rightarrow\hat{\theta}=\dfrac{3}{8}$

设总体$X\sim N(\mu,\sigma^2),(X_1,X_2,\cdots,X_n)$为来自总体$X$的简单随机样本.

(1)若$\mu=1$,求参数σ^2的最大似然估计量;

(2)若μ未知,求参数μ,σ^2的最大似然估计量.

①$X\sim f(x)=\dfrac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-1)^2}{2\sigma^2}}$

1.$L=f(x_1)\cdots f(x_n)=(2\pi)^{-\frac{n}{2}}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-1)^2}$

2.$\ln L=-\dfrac{n}{2}\ln 2\pi-\dfrac{n}{2}\ln\sigma^2-\dfrac{1}{2\sigma^2}\sum\limits_{i=1}^{n}(x_i-1)^2$

3.$\dfrac{d}{d\sigma^2}\ln L=-\dfrac{n}{2\sigma^2}+\dfrac{1}{2\sigma^4}\sum\limits_{i=1}^{n}(x_i-1)^2=0$

$\Rightarrow\widehat{\sigma}^2=\dfrac{1}{n}\sum\limits_{i=1}^{n}(x_i-1)^2$(最大似然估计法)

$\therefore\sigma^2$的最大似然估计量为

$\widehat{\sigma}^2=\dfrac{1}{n}\sum\limits_{i=1}^{n}(X_i-1)^2$

②$L=f(x_1)\cdots f(x_n)=(2\pi)^{-\frac{n}{2}}(\sigma^2)^{-\frac{n}{2}}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$

$\ln L=-\dfrac{n}{2}\ln 2\pi-\dfrac{n}{2}\ln\sigma^2-\dfrac{1}{2\sigma^2}\sum\limits_{i=1}^{n}(x_i-\mu)^2$

由$\begin{cases}\frac{\partial}{\partial\mu}\ln L=-\frac{-2}{2\sigma^2}\cdot\sum_{i=1}^{n}(x_i-\mu)=0\\ \frac{\partial}{\partial\sigma^2}\ln L=-\frac{n}{2\sigma^2}+\frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2=0\end{cases}$

$\Rightarrow\widehat{\mu}=\overline{x},\widehat{\sigma}^2=\dfrac{1}{n}\sum\limits_{i=1}^{n}(x_i-\overline{x})^2$(估计法)

$\therefore\mu,\sigma^2$的最大似然估计量为

$\widehat{\mu}=\overline{X},\widehat{\sigma}^2=\dfrac{1}{n}\sum\limits_{i=1}^{n}(X_i-\overline{X})^2$

设总体X的概率密度为

$f(x)=\begin{cases}\theta x^{\theta-1},0<x<1,\\ 0,其他\end{cases}$

$(X_1,X_2,\cdots,X_n)$为来自总体X的简单随机样本.

(1)求参数θ的矩估计量;(2)求参数θ的最大似然估计量.

$$①EX=\int_0^1 \theta x^\theta dx=\frac{\theta}{\theta+1}$$

$$令EX=\overline{X}\Rightarrow 1+\frac{1}{\theta}=\frac{1}{\overline{X}}\Rightarrow\hat{\theta}=\frac{\overline{X}}{1-\overline{X}}$$

$$②L=f(x_1)\cdots f(x_n)=\theta^n(x_1\cdots x_n)^{\theta-1}(0<x_i<1,1\le i\le n)$$

$$\ln L=n\ln\theta+(\theta-1)\sum_{i=1}^n\ln x_i$$

$$令\frac{d}{d\theta}\ln L=\frac{n}{\theta}+\sum_{i=1}^n\ln x_i=0\Rightarrow\hat{\theta}=-\frac{n}{\sum_{i=1}^n\ln x_i}$$

$\therefore\theta$的最大似然估计量为

$$\hat{\theta}=-\frac{n}{\sum_{i=1}^n\ln x_i}$$

$$总体X\sim f(x)=\begin{cases}2e^{-2(x-\theta)},x>\theta\\0,x\le\theta\end{cases}$$

$$(X_1,X_2,X_3)\Rightarrow(x_1,x_2,x_3)$$

①θ的矩估计量$\hat{\theta}$;②θ的最大似然估计量$\hat{\theta}$;

③②中$\hat{\theta}$,求$E\hat{\theta}$.

$$①EX=\int_\theta^{+\infty}2xe^{-2(x-\theta)}dx=\frac{1}{2}\int_\theta^{+\infty}[2(x-\theta)+2\theta]e^{-2(x-\theta)}d[2(x-\theta)]$$

$$=\frac{1}{2}\int_0^{+\infty}(x+2\theta)e^{-x}dx=\frac{1}{2}(1+2\theta)=\theta+\frac{1}{2}$$

$$令EX=\overline{X}\Rightarrow\hat{\theta}=\overline{X}-\frac{1}{2}$$

$$②L=f(x_1)f(x_2)f(x_3)=2^3\cdot e^{-2(x_1+x_2+x_3)+6\theta}$$

$$(\theta<x_i,i=1,2,3)$$

$$\ln L=3\ln 2-2(x_1+x_2+x_3)+6\theta$$

$$令\frac{d}{d\theta}\ln L=6>0\Rightarrow L(\theta)\uparrow$$

$$\Rightarrow\hat{\theta}=\min\{x_1,x_2,x_3\}$$

θ的最大似然估计量为

$$\hat{\theta}=\min\{x_1,x_2,x_3\}$$

$$③1.总体X的分布函数F(x)=\int_{-\infty}^x f(t)dt$$

$$x<\theta:F(x)=0$$

$$x\ge\theta:F(x)=\int_\theta^x 2e^{-2(t-\theta)}dt=1-e^{-2(x-\theta)}$$

$$F(x)=\begin{cases}1-e^{-2(x-\theta)},x\ge\theta\\0,x<\theta\end{cases}$$

$$2.F_{\hat{\theta}}(x)=P\{\hat{\theta}\le x\}=1-P\{\hat{\theta}<x\}$$

$$=1-P\{x_1>x\}P\{x_2>x\}P\{x_3>x\}=1-P\{X>x\}$$

$$=1-[1-F(x)]^3=\begin{cases}1-e^{-6(x-\theta)},x\ge\theta\\0,x<\theta\end{cases}$$

$$3.f_{\hat{\theta}}(x)=\begin{cases}6e^{-6(x-\theta)},x>\theta\\0,x\le\theta\end{cases}$$

$$4.E\hat{\theta}=\int_\theta^{+\infty}6xe^{-6(x-\theta)}dx=\frac{1}{6}\int_\theta^{+\infty}[6(x-\theta)+6\theta]e^{-6(x-\theta)}d[6(x-\theta)]$$

$$=\frac{1}{6}\int_0^{+\infty}(x+6\theta)e^{-x}dx=\frac{1}{6}(1+6\theta)=\theta+\frac{1}{6}$$

估计量的评价标准

1.无偏性

$X \Rightarrow (X_1, \cdots, X_n)$(含$\theta$)

$\hat{\theta} = \varphi(X_1, \cdots, X_n)$为$\theta$的估计量

若$E\hat{\theta} = \theta$, 称$\hat{\theta} = \varphi(X_1, \cdots, X_n)$为$\theta$的无偏估计量.