矩阵对角化过程

(一)$A^T \neq A$:

1.$|\lambda E - A| = 0 \Rightarrow \lambda_1, \ldots, \lambda_n$

2.$\lambda_i E - A \to \ldots : \alpha_1 \ldots \alpha_m \begin{cases} \alpha_1 \ldots \alpha_m \text{线性无关} \\ m \leq n \end{cases}$

3.①$m < n$: A不可相似对角化

②$m = n$: A可相似对角化

$P = (\alpha_1, \ldots, \alpha_n)$, 可逆, $A\alpha_1 = \lambda_1\alpha_1, A\alpha_n = \lambda_n\alpha_n$

$(A\alpha_1, \ldots, A\alpha_n) = (\lambda_1\alpha_1, \ldots, \lambda_n\alpha_n)$

$$AP = P\begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ \ldots & \ldots & \ldots & \\ 0 & 0 & & \lambda_n \end{pmatrix} \Rightarrow P^{-1}AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

向量正交

(一)向量正交

1.若$(\alpha, \beta) = \alpha^T\beta = 0$, 称$\alpha \perp \beta$

2.性质 :

$\alpha_1 \ldots \alpha_n \begin{cases} \text{非零} \\ \text{两两正交} \end{cases} \Rightarrow \alpha_1 \ldots \alpha_n\text{线性无关}, \not\Leftarrow$

$\Rightarrow$, 令$k_1\alpha_1 + \ldots + k_n\alpha_n = 0$

$\because (\alpha_1, k_1\alpha_1 + \ldots + k_n\alpha_n) = 0$

$\therefore k_1(\alpha_1, \alpha_1) = 0$

而$(\alpha_1, \alpha_1) = |\alpha_1|^2 > 0, \therefore k_1 = 0$

$(\alpha_2, k_2\alpha_2 + \ldots + k_n\alpha_n) = 0$

$\Rightarrow k_2(\alpha_2, \alpha_2) = 0$

$\because (\alpha_2, \alpha_2) = |\alpha_2|^2 > 0, \therefore k_2 = 0$

$\ldots$

$k_n\alpha_n = 0$

$\because \alpha_n \neq 0, \therefore k_n = 0, \Rightarrow \alpha_1 \ldots . \alpha_n$线性无关

$\not\Leftarrow, \alpha_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$无关

但$(\alpha_1, \alpha_2) = 5 \neq 0$

3.正交规范化 :

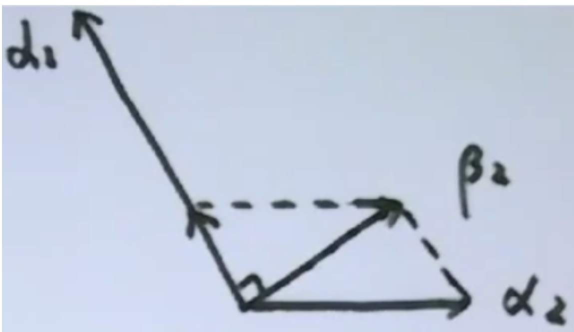
已知$\alpha_1, \alpha_2, \ldots, \alpha_n$线性无关

$\alpha_1, \alpha_2, \alpha_3$线性无关

$\beta_1 = \alpha_1$

$\beta_2 = \alpha_2 + ?\beta_1$

$\beta_3 = \alpha_3 + ?\beta_1 + ?\beta_2$



已知$\alpha_1,\alpha_2,\alpha_3$线性无关

1.正交化：

$\beta_1=\alpha_1$

$$\beta_2=\alpha_2-\frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1$$

$$\beta_3=\alpha_3-\frac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}\beta_1-\frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2$$

β_1,β_2,β_3两两正交

2.规范化：

$$\gamma_1=\frac{1}{|\beta_1|}\beta_1,\gamma_2=\frac{1}{|\beta_2|}\beta_2,\gamma_3=\frac{1}{|\beta_3|}\beta_3$$

$\gamma_1,\gamma_2,\gamma_3$两两正交且规范, 正交规范组

正交矩阵

1.$A_{n\times n}$, 若$A^TA=E$或$AA^T=E$

称A为正交矩阵

2.正交阵性质：

①若$A^TA=E\Leftrightarrow A^T=A^{-1}$

②$A^TA=E\Rightarrow |A|=\pm 1$

证：$A^TA=E\Rightarrow |A^T|\cdot|A|=1\Rightarrow |A|^2=1\Rightarrow |A|=\pm 1$

③$A^TA=E\Rightarrow \lambda=\pm 1$

证：令$AX=\lambda X(X\neq 0)$

$X^TA^T\cdot AX=\lambda X^T\cdot AX\Rightarrow X^TX=\lambda^2X^TX$

$\Rightarrow (\lambda^2-1)X^TX=0$

$\because X^TX=|X|^2>0,\therefore \lambda^2-1=0\Rightarrow \lambda=\pm 1$

3.正交阵等价条件(内在结构)：

Th. $Q_{3\times 3}=(r_1,r_2,r_3)$, 则

$Q^TQ=E\Leftrightarrow r_1,r_2,r_3$两两正交且规范

$$证：Q^TQ=\begin{pmatrix}\gamma_1^T\\ \gamma_2^T\\ \gamma_3^T\end{pmatrix}(\gamma_1,\gamma_2,\gamma_3)=\begin{pmatrix}\gamma_1^T\gamma_1 & \gamma_1^T\gamma_2 & \gamma_1^T\gamma_3\\ \gamma_2^T\gamma_1 & \gamma_2^T\gamma_2 & \gamma_2^T\gamma_3\\ \gamma_3^T\gamma_1 & \gamma_3^T\gamma_2 & \gamma_3^T\gamma_3\end{pmatrix}=E$$

$$\Leftrightarrow\begin{cases}\gamma_1^T\gamma_1=\gamma_2^T\gamma_2=\gamma_3^T\gamma_3=1\\ \gamma_i^T\gamma_j=0(i\neq j)\end{cases}$$

即$\gamma_1,\gamma_2,\gamma_3$正交规范

实对称矩阵正交

$1.|\lambda E-A|=0\Rightarrow\lambda_1,\ldots,\lambda_n$

$2.\lambda_i E-A\rightarrow\ldots:\alpha_1,\ldots,\alpha_n\begin{cases}\alpha_1\ldots\alpha_n\text{线性无关}\\ \text{不同特征值对应的特征向量正交}\end{cases}$

3.①一般要求：找可逆阵P

$P=(\alpha_1,\alpha_2,\ldots,\alpha_n)$

$$P^{-1}AP=\begin{pmatrix}\lambda_1&&\\&\ddots&\\&&\lambda_n\end{pmatrix}$$

②高要求：找正交阵Q

$\alpha_1\ldots\alpha_n\Rightarrow$ 正交规范$\gamma_1\ldots\gamma_n$

$(A\gamma_1=\lambda_1\gamma_1,\ldots,A\gamma_n=\lambda_n\gamma_n)$

$Q=(\gamma_1,\ldots,\gamma_n),Q^TQ=E$

$(A\gamma_1,A\gamma_2,\ldots,A\gamma_n)=(\lambda_1\gamma_1,\ldots,\lambda_n\gamma_n)$

$$AQ=Q\begin{pmatrix}\lambda_1&&&\\&\lambda_2&&\\&&\ddots&\\&&&\lambda_n\end{pmatrix}\Rightarrow Q^TAQ=\begin{pmatrix}\lambda_1&&&\\&\lambda_2&&\\&&\ddots&\\&&&\lambda_n\end{pmatrix}$$

$$A=\begin{pmatrix}1&-1&-1\\-1&1&-1\\-1&-1&1\end{pmatrix},A^T=A$$

$1.\lambda_1=-1,\lambda_2=\lambda_3=2$

$$\alpha_1=\begin{pmatrix}1\\1\\1\end{pmatrix},\alpha_2=\begin{pmatrix}-1\\1\\0\end{pmatrix},\alpha_3=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

2.找可逆阵$P:P=\begin{pmatrix}1&-1&-1\\1&1&0\\1&0&1\end{pmatrix},P^{-1}AP=\begin{pmatrix}-1&&\\&2&\\&&2\end{pmatrix}$

3.找正交阵Q：

$$\beta_1=\alpha_1\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\beta_2=\alpha_2=\begin{pmatrix}-1\\1\\0\end{pmatrix},\beta_3=\alpha_3-\frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2=\begin{pmatrix}-1\\0\\1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-1\\-1\\2\end{pmatrix}$$

$$\gamma_1=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix},\gamma_2=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix},\gamma_3=\frac{1}{\sqrt{6}}\begin{pmatrix}-1\\-1\\2\end{pmatrix}$$

$(A\gamma_1=-\gamma_1,A\gamma_2=2\gamma_2,A\gamma_3=2\gamma_3)$

$$Q=\begin{pmatrix}\frac{1}{\sqrt{3}}&-\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{3}}&0&\frac{2}{\sqrt{6}}\end{pmatrix},Q^TQ=E$$

$$AQ=Q\begin{pmatrix}-1&0&0\\0&2&0\\0&0&2\end{pmatrix}\Rightarrow Q^TAQ=\begin{pmatrix}-1&&\\&2&\\&&2\end{pmatrix}$$