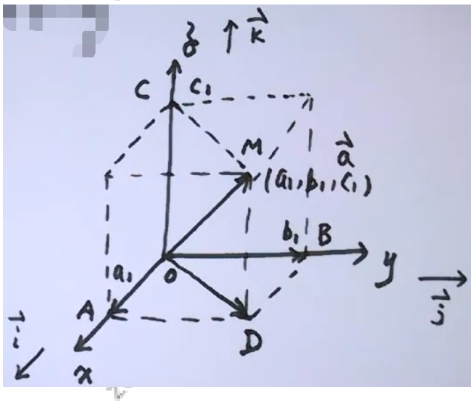
空间解析几何

向量

向量的坐标



$$
\begin{aligned}
&\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB} \\
&\vec{a} = \overrightarrow{OD} + \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} \\
&= a_1\vec{i} + b_1\vec{j} + c_1\vec{k} \\
&\triangleq \{a_1, b_1, c_1\}
\end{aligned}
$$

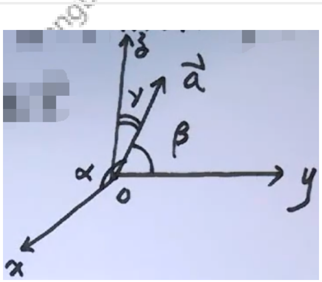
$$
\begin{aligned}
&\text{若} A(x_1, y_1, z_1), B(x_2, y_2, z_2) \\
&\overrightarrow{AB} = \{x_2 - x_1, y_2 - y_1, z_2 - z_1\}
\end{aligned}
$$

若$\vec{a} = \{a_1, b_1, c_1\}$, 则

①$|\vec{a}| = \sqrt{a_1^2 + b_1^2 + c_1^2}$

②$\vec{a}\circ = \dfrac{1}{|\vec{a}|}\vec{a} = \{\dfrac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \dfrac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \dfrac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}\}$

方向角与方向余弦



设$\vec{a} = \{a_1, b_1, c_1\}$

$\vec{a}$与x, y, z轴正方向夹角称为$\vec{a}$的方向角, α, β, γ.

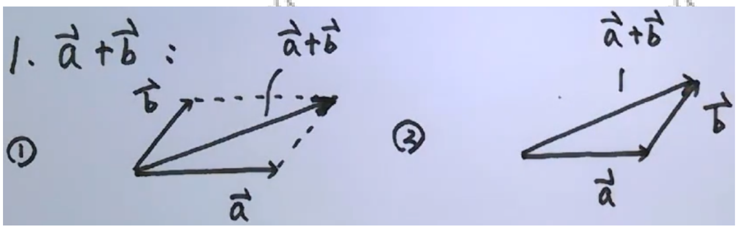
$\cos\alpha = \dfrac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \cos\beta = \dfrac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \cos\gamma = \dfrac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$

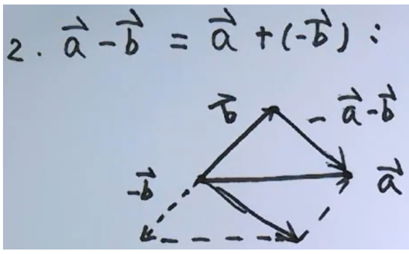
① $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

②$\{\cos\alpha, \cos\beta, \cos\gamma\} = \vec{a}\circ$

向量运算的刻划

几何刻划

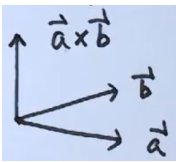




$$3.k\vec{a}\begin{cases}①k>0:k\vec{a}\text{方向与}\vec{a}\text{同, 长为}\vec{a}\text{的}k\text{倍}\\ ②k=0:k\vec{a}=\vec{0}\\ ③k<0:k\vec{a}\text{方向与}\vec{a}\text{反, 长为}\vec{a}\text{的}|\vec{k}|\text{倍}\end{cases}$$

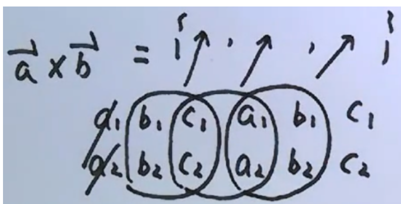
$$4.\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos(\vec{a},\vec{b})$$

$$5.\vec{a}\times\vec{b}:\begin{cases}\text{方向按右手准则}\\ \text{大小}|\vec{a}\times\vec{b}|=|\vec{a}|\cdot|\vec{b}|\cdot\sin(\vec{a},\vec{b})\end{cases}$$

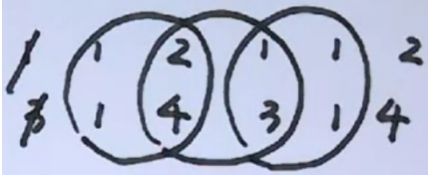


代数刻划

$$\begin{aligned}&\text{设}\vec{a}=\{a_1,b_1,c_1\},\vec{b}=\{a_2,b_2,c_2\}\\ &1.\vec{a}+\vec{b}=\{a_1+a_2,b_1+b_2,c_1+c_2\}\\ &2.\vec{a}-\vec{b}=\{a_1-a_2,b_1-b_2,c_1-c_2\}\\ &3.k\vec{a}=\{ka_1,kb_1,kc_1\}\\ &4.\vec{a}\cdot\vec{b}=\{a_1a_2+b_1b_2+c_1c_2\}\\ &5.\vec{a}\times\vec{b}=\{b_1c_2-b_2c_1,c_1a_2-c_2a_1,a_1b_2-a_2b_1\}\end{aligned}$$



$$\begin{aligned}&\vec{a}=\{1,1,2\},\vec{b}=\{3,1,4\}\\ &\vec{a}\times\vec{b}=\{2,2,-2\}\end{aligned}$$



$1.\vec{a}\cdot\vec{b}:$

①$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

②$\vec{a}\cdot\vec{a}=|\vec{a}|^2$

③$\vec{a}\cdot\vec{b}=0\Leftrightarrow a_1a_2+b_1b_2+c_1c_2=0\Leftrightarrow\vec{a}\perp\vec{b}$

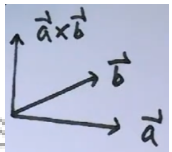
$2.\vec{a}\times\vec{b}:$

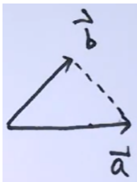
①$\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$

②$\vec{a}\times\vec{b}\perp\vec{a},\vec{a}\times\vec{b}\perp\vec{b}$(法)

③$\vec{a}\times\vec{b}=\vec{0}\Leftrightarrow\vec{a}\parallel\vec{b}\Leftrightarrow\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$

④$|\vec{a}\times\vec{b}|=2S_\triangle$





$A(1,-1,0),B(2,0,1),C(1,1,-1)$,求$S_{\triangle ABC}$.

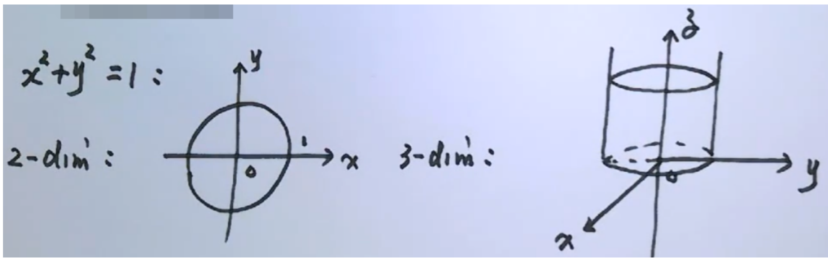
$$
\begin{aligned}
&\overrightarrow{AB}=\{1,1,1\},\overrightarrow{AC}=\{0,2,-1\}\\
&\overrightarrow{AB}\times\overrightarrow{AC}=\{-3,1,2\}\\
&|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{14}\\
&\therefore S_{\triangle ABC}=\frac{\sqrt{14}}{2}
\end{aligned}
$$

向量的应用

空间曲面

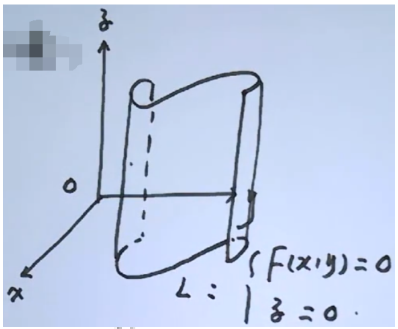
设Σ为空间曲面,$F(x,y,z)=0$为方程.
若Σ上任一点坐标为$F(x,y,z)=0$的解
$F(x,y,z)=0$任一解对应点在Σ上
称$F(x,y,z)=0$为曲面Σ的方程,Σ为$F(x,y,z)=0$的图

$$x^2+y^2=1$$



特殊曲面

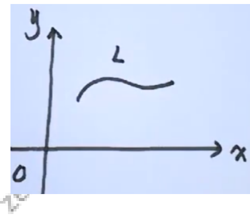
柱面



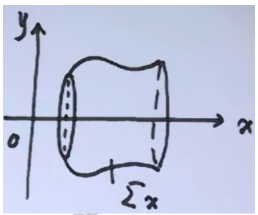
$$\Sigma : F(x,y)=0$$

旋转曲面

2-dim

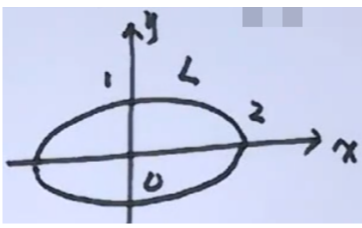


$$L:\begin{cases} f(x,y)=0 \\ z=0 \end{cases}$$



$$①\Sigma_x : f(x, \pm\sqrt{y^2+z^2})=0$$
$$②\Sigma_y : f(\pm\sqrt{x^2+z^2}, y)=0$$

$$L:\begin{cases} \frac{x^2}{4}+y^2=1 \\ z=0 \end{cases}$$

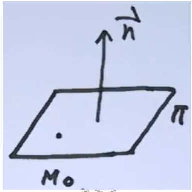


$$\Sigma_x : \frac{x^2}{4} + y^2 + z^2 = 1$$
$$\Sigma_y : \frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1$$

3-dim

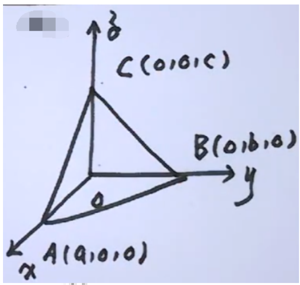
平面(退化)

点法式



$$M_0(x_0, y_0, z_0) \in \pi, \vec{n} = \{A, B, C\} \perp \pi$$
$$\pi : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

截距式



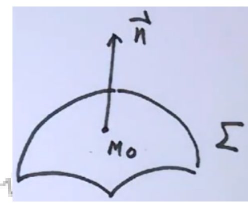
$$\overrightarrow{AB} = \{-a, b, 0\}, \overrightarrow{AC} = \{-a, 0, c\}$$
$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \{bc, ac, ab\}$$
$$\pi : bc(x - a) + ac(y - 0) + ab(z - 0) = 0$$
$$\pi : \frac{x - a}{a} + \frac{y}{b} + \frac{z}{c} = 0$$
$$\therefore \pi : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

一般式

$$\pi : Ax + By + Cz + D = 0$$

切平面和法线

$$空间曲面 \begin{cases} 切平面 \\ 法线 \end{cases}$$

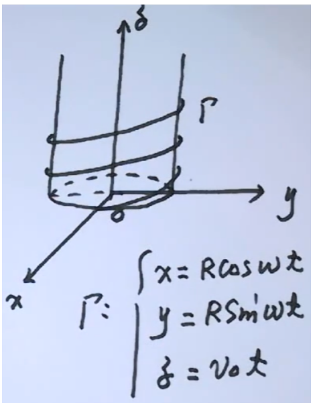


$$\Sigma : F(x, y, z) = 0, M_0 \in \Sigma$$
$$\vec{n} = \{F_x, F_y, F_z\}_{M_0}$$

空间曲线

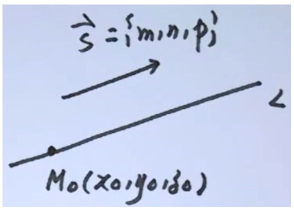
空间曲线的表达形式

$$
\begin{aligned}
&1.\text{一般式：}\\
&\Gamma:\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases}\\
&2.\text{参数式：}\\
&\Gamma:\begin{cases}x=\Phi(t)\\y=\phi(t)\\z=\omega(t)\end{cases}
\end{aligned}
$$



直线(退化)

点向式

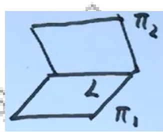


$$
\begin{aligned}
&M_0(x_0,y_0,z_0)\in L,\vec{S}=\{m,n,p\}\parallel L\\
&L:\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}
\end{aligned}
$$

参数式

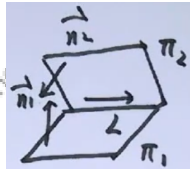
$$
L:\begin{cases}x=x_0+mt\\y=y_0+nt\\z=z_0+pt\end{cases}
$$

一般式



$$
L:\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases}
$$

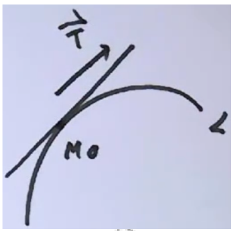
$$
L:\begin{cases}x+y-2z=0\\2x-y+z-2=0\end{cases}\text{化为点向式.}
$$

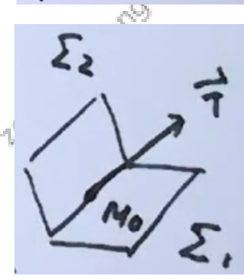


$$
\begin{aligned}
&1.M_0(1,1,1)\in L\\
&2.\vec{S}=\vec{n_1}\times\vec{n_2}=\{1,1,-2\}\times\{2,-1,1\}=\{-1,-5,-3\}\\
&3.L:\frac{x-1}{1}=\frac{y-1}{5}=\frac{z-1}{3}
\end{aligned}
$$

切线和法平面

$$
空间曲线\begin{cases}切线\\法平面\end{cases}
$$





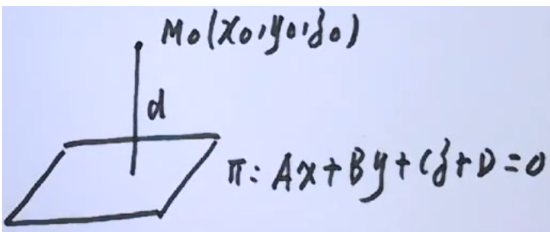
$$
\begin{aligned}
&1.\Gamma:\begin{cases}x=\Phi(t)\\y=\phi(t)\\z=\omega(t)\end{cases},t=t_0\to M_0(x_0,y_0,z_0)\in\Gamma\\
&\quad\vec{T}=\{\Phi'(t_0),\phi'(t_0),\omega'(t_0)\}\\
&2.\Gamma:\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases},M_0(x_0,y_0,z_0)\in\Gamma\\
&\quad\overrightarrow{n_1}=\{F_x,F_y,F_z\}_{M_0},\overrightarrow{n_2}=\{G_x,G_y,G_z\}_{M_0}\\
&\quad\vec{T}=\overrightarrow{n_1}\times\overrightarrow{n_2}
\end{aligned}
$$

距离

两点之距

$$
\begin{aligned}
&A(x_1,y_1,z_1),B(x_2,y_2,z_2)\\
&d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}
\end{aligned}
$$

点到平面之距

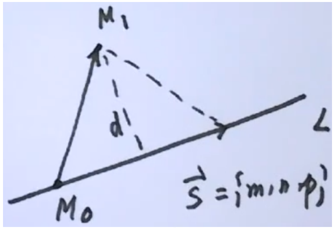


$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

平行平面之距

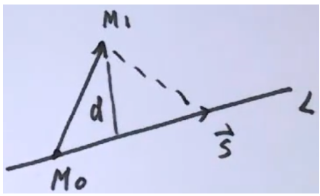
$$\begin{aligned}&\pi_1: Ax+By+Cz+D_1=0\\&\pi_2: Ax+By+Cz+D_2=0\\&d=\frac{|D_1-D_2|}{\sqrt{A^2+B^2+C^2}}\end{aligned}$$

点到直线之距



$$\begin{aligned}&|\overrightarrow{M_0M_1}\times\overrightarrow{S}|=|\overrightarrow{S}|\cdot d\\&\Rightarrow d=\frac{|\overrightarrow{M_0M_1}\times\overrightarrow{S}|}{|\overrightarrow{S}|}\end{aligned}$$

$$M_1(1,2,1), L:\begin{cases}x-y-z-2=0\\2x+y-z-3=0\end{cases}, \text{求}M_1\text{到}L\text{之距}.$$



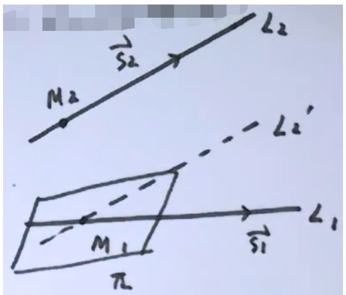
$$\begin{aligned}&M_0(1,0,-1)\in L\\&\vec{S}=\{1,-1,-1\}\times\{2,1,-1\}=\{2,-1,3\}\\&\overrightarrow{M_0M_1}=\{0,2,2\}\\&\overrightarrow{M_0M_1}\times\vec{S}=\{8,4,-4\}, |\overrightarrow{M_0M_1}\times\vec{S}|=4\sqrt{6}\\&|\vec{S}|=\sqrt{14}\\&\text{由}|\vec{S}|\cdot d=4\sqrt{6}\Rightarrow d=\frac{4\sqrt{6}}{\sqrt{14}}\end{aligned}$$

异面直线之距

$$1. L_1, L_2\text{共面}\Leftrightarrow \overrightarrow{S_1}\times\overrightarrow{S_2}\perp\overrightarrow{M_1M_2}\Leftrightarrow(\overrightarrow{S_1}\times\overrightarrow{S_2})\cdot\overrightarrow{M_1M_2}=0$$

$$2. L_1, L_2\text{异面}\Leftrightarrow(\overrightarrow{S_1}\times\overrightarrow{S_2})\cdot\overrightarrow{M_1M_2}\neq 0$$

$$L_1:\frac{x}{1}=\frac{y-1}{-1}=\frac{z}{0}, L_2:\frac{x-1}{2}=\frac{y}{1}=\frac{z-1}{-1}$$



$1.M_1(0,1,0)\in L_1,\overrightarrow{S_1}=\{1,-1,0\}$

$M_2(1,0,1)\in L_2,\overrightarrow{S_2}=\{2,1,-1\}$

$\overrightarrow{S_1}\times\overrightarrow{S_2}=\{1,1,3\},\overrightarrow{M_1M_2}=\{1,-1,1\}$

$\because(\overrightarrow{S_1}\times\overrightarrow{S_2})\cdot\overrightarrow{M_1M_2}=3\neq0,\therefore L_1,L_2$异面

2.过M_1作$L_2'\parallel L_2$

L_1,L_2'成平面π

$\pi:1\times(x-0)+1\times(y-1)+3\times(z-0)$

即$\pi:x+y+3z-1=0$

$$3.d=\frac{|1+0+3-1|}{\sqrt{11}}=\frac{3}{\sqrt{11}}$$