无穷小与无穷大

无穷小

$$\alpha(x)\text{在}x=a\text{的去心邻域内有定义}$$

$$\text{若}\lim_{x\to a}\alpha(x)=0,\text{称}\alpha(x)\text{当}x\to a\text{时为无穷小}$$

1. 0为与自变量趋向无关的无穷小
2. 非零函数是否为无穷小与自变量趋向有关

无穷小的比较

$$\text{设}\alpha\to 0,\beta\to 0$$

$$1.\ \text{若}\lim\frac{\beta}{\alpha}=0,\beta=o(\alpha)$$

$$2.\ \text{若}\lim\frac{\beta}{\alpha}=k(\neq 0,\infty),\beta=O(\alpha)$$

$$\text{若}\lim\frac{\beta}{\alpha}=1,\alpha\sim\beta$$

无穷大

$$\alpha(x)\text{在}x=a\text{的去心邻域内有定义}$$

$$\text{若}\forall M>0,\exists\delta>0,\text{当}0<|x-a|<\delta\text{时},$$
$$|f(x)|\geq M$$
$$\text{称}f(x)\text{当}x\to a\text{时为无穷大},\text{记}\lim_{x\to a}\alpha(x)=\infty$$

$$\text{若}\lim_{x\to a}\frac{1}{\alpha(x)}=0,\text{称}\alpha(x)\text{当}x\to a\text{时为无穷大}$$

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1. 无界 $*$ 无界 $\neq$ 无界
2. 无穷大 $*$ 无穷大 $=$ 无穷大

$$\text{无穷大与无穷大之和为无穷大}(\times)$$
$$a_n=2n+1,b_n=-2n,a_n+b_n=1$$

$$\text{无穷大与有界函数之积为无穷大}(\times)$$

$$a_n=n,b_n=\sin\frac{1}{n^2}$$

$$\lim_{n\to\infty}a_nb_n=\lim_{n\to\infty}n\frac{1}{n^2}\frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}}=0$$

$$\text{无界量与无界量之积是无界量}(\times)$$
$$a_n=1,0,3,0,5,\ldots$$
$$b_n=0,2,0,4,0,\ldots$$
$$\{a_n\}\{b_n\}\text{无界},a_nb_n\equiv 0$$

无穷小的性质

一般性质

$$
\begin{aligned}
&1.\ \alpha \to 0, \beta \to 0 \Rightarrow \begin{cases} \alpha \pm \beta \to 0 \\ \alpha\beta \to 0 \\ k\alpha \to 0(k\text{常数}) \end{cases} \\
&2.\ |\alpha| \le M, \beta \to 0 \Rightarrow \alpha\beta \to 0 \\
&\quad \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0 \\
&3.\ \lim_{x\to a} f(x) = A \Leftrightarrow f(x) = A + \alpha, \alpha \to 0(x \to a)
\end{aligned}
$$

$$
\begin{gathered}
\Rightarrow, \forall \epsilon > 0, \exists \delta > 0, \text{当} 0 < |x-a| < \delta\text{时}, \\
|f(x) - A| < \epsilon \\
f(x) = A + f(x) - A = A + \alpha, \alpha = f(x) - A \\
\because \forall \epsilon > 0, \exists \delta > 0, \text{当} 0 < |x-a| < \delta\text{时}, \\
|\alpha(x) - 0| < \epsilon \\
\therefore \lim_{x\to a} \alpha = 0
\end{gathered}
$$

$$
\begin{gathered}
\Leftarrow, \text{设} f(x) = A + \alpha, \alpha \to 0(x \to a) \\
\Rightarrow \lim_{x\to a} f(x) = A
\end{gathered}
$$

$$
\text{设} \lim f(x) = A, \lim g(x) = B, \text{证明}: \lim[f(x) \pm g(x)] = A \pm B.
$$

$$
\begin{gathered}
\text{证}: \\
\lim f(x) = A \Leftrightarrow f(x) = A + \alpha, \alpha \to 0 \\
\lim g(x) = B \Leftrightarrow g(x) = B + \beta, \beta \to 0 \\
f(x) \pm g(x) = (A \pm B) + (\alpha \pm \beta) \\
\because \lim(\alpha \pm \beta) = 0, \therefore \lim[f(x) \pm g(x)] = A \pm B
\end{gathered}
$$

$$
\text{设} \lim f(x) = A, \lim g(x) = B, \text{证明}: \lim f(x)g(x) = AB.
$$

$$
\begin{gathered}
\text{证}: \\
f(x)g(x) = AB + (A\beta + B\alpha + \alpha\beta) \\
\because \lim(A\beta + B\alpha + \alpha\beta) = 0 \\
\therefore \lim f(x)g(x) = AB
\end{gathered}
$$

等价性质

$$
\alpha \sim \beta \begin{cases} \alpha \to 0, \beta \to 0 \\ \frac{\beta}{\alpha} = 1 \end{cases}
$$

1. $\alpha \to 0, \beta \to 0, \gamma \to 0$

①$\alpha \sim \alpha$

②若$\alpha \sim \beta \Rightarrow \beta \sim \alpha$

③若$\alpha \sim \beta, \beta \sim \gamma \Rightarrow \alpha \sim \gamma$

证：$\dfrac{\gamma}{\alpha} = \dfrac{\beta}{\alpha}\dfrac{\gamma}{\beta}$

$\because \alpha \sim \beta, \beta \sim \gamma,$

$\therefore \dfrac{\beta}{\alpha} \to 1, \dfrac{\gamma}{\beta} \to 1 \Rightarrow \dfrac{\gamma}{\alpha} \to 1,$

$\therefore \alpha \sim \gamma$

2. $\alpha \sim \alpha_1, \beta \sim \beta_1$, 且$\lim \dfrac{\beta_1}{\alpha_1} = A \Rightarrow \lim \dfrac{\beta}{\alpha} = A$

证：$\dfrac{\beta}{\alpha} = \dfrac{\alpha_1}{\alpha}\dfrac{\beta_1}{\alpha_1}\dfrac{\beta}{\beta_1}$

$\because \alpha \sim \alpha_1, \beta \sim \beta_1, \therefore \dfrac{\alpha_1}{\alpha} \to 1, \dfrac{\beta}{\beta_1} \to 1$

$\therefore \lim \dfrac{\beta}{\alpha} = \lim \dfrac{\beta_1}{\alpha_1} = A$

$x \to 0$时

①$x \sim \sin x \sim \tan x \sim \arctan x \sim e^x - 1 \sim \ln(1+x)$

②$1 - \cos x \sim \dfrac{1}{2}x^2$

③$(1+x)^a - 1 \sim ax$