性质

一般性质

$$
\begin{aligned}
&1.A_{n\times n},\lambda_1\neq\lambda_2\\
&\quad\lambda_1E-A\to\cdots:\alpha_1,\ldots,\alpha_s\\
&\quad\alpha_1\ldots\alpha_s\text{线性无关},\text{且}A\alpha_1=\lambda_1\alpha_1,\ldots,A\alpha_s=\lambda_1\alpha_s\\
&\quad\lambda_2E-A\to\cdots:\beta_1,\ldots,\beta_t\\
&\quad\beta_1\ldots\beta_t\text{线性无关},\text{且}A\beta_1=\lambda_2\beta_1,\ldots,A\beta_t=\lambda_2\beta_t\\
&\quad\Rightarrow\alpha_1,\ldots,\alpha_s,\beta_1,\ldots,\beta_t\text{线性无关}
\end{aligned}
$$

$$
\begin{aligned}
&\text{证}:\text{令}(k_1\alpha_1+\ldots+k_s\alpha_s)+(l_1\beta_1+\ldots+l_t\beta_t)=0(*)\\
&\because A\alpha_1=\lambda_1\alpha_1,\ldots,A\alpha_s=\lambda_1\alpha_s\\
&\quad A\beta_1=\lambda_2\beta_1,\ldots,A\beta_t=\lambda_2\beta_t\\
&\therefore A\times(*):\\
&\quad\lambda_1(k_1\alpha_1+\ldots+k_s\alpha_s)+\lambda_2(l_1\beta_1+\ldots+l_t\beta_t)=0(**)\\
&\quad(*)\times\lambda_2-(**)\\
&\quad(\lambda_2-\lambda_1)(k_1\alpha_1+\ldots+k_s\alpha_s)=0\\
&\because\lambda_1\neq\lambda_2,\therefore\begin{cases}k_1\alpha_1+\ldots+k_s\alpha_s=0\\ l_1\beta_1+\ldots+l_t\beta_t=0\end{cases}\\
&\because\alpha_1\ldots\alpha_s\text{及}\beta_1\ldots\beta_t\text{线性无关}\\
&\therefore k_1=\ldots=k_s=0,l_1=\ldots=l_t=0\\
&\therefore\alpha_1\ldots\alpha_s,\beta_1\ldots\beta_t\text{线性无关}
\end{aligned}
$$

$$
\begin{aligned}
&A=\begin{pmatrix}1&-1&-1\\-1&1&-1\\-1&-1&1\end{pmatrix}\\
&|\lambda E-A|=0\Rightarrow\lambda_1=-1,\lambda_2=\lambda_3=2\\
&\lambda_1=-1:E+A\to:\alpha_1=\begin{pmatrix}1\\1\\1\end{pmatrix}\\
&\lambda_2=\lambda_3=2:2E-A\to:\alpha_2=\begin{pmatrix}-1\\1\\0\end{pmatrix},\alpha_3=\begin{pmatrix}-1\\0\\1\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&1.\alpha_1,\alpha_2,\alpha_3\text{线性无关}\\
&2.A\alpha_1=-\alpha_1,A\alpha_2=2\alpha_2,A\alpha_3=2\alpha_3\\
&3.(A\alpha_1,A\alpha_2,A\alpha_3)=(-\alpha_1,2\alpha_2,2\alpha_3)\\
&\quad A(\alpha_1,\alpha_2,\alpha_3)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}-1&0&0\\0&2&0\\0&0&2\end{pmatrix}\\
&\quad\text{令}P=(\alpha_1,\alpha_2,\alpha_3)=\begin{pmatrix}1&-1&-1\\1&1&0\\1&0&1\end{pmatrix}\text{可逆}\\
&\quad AP=P\begin{pmatrix}-1&0&0\\0&2&0\\0&0&2\end{pmatrix}\Rightarrow P^{-1}AP=\begin{pmatrix}-1&0&0\\0&2&0\\0&0&2\end{pmatrix}
\end{aligned}
$$

$2.A_{n\times n},\lambda_1\neq\lambda_2,A\alpha=\lambda_1\alpha,A\beta=\lambda_2\beta(\alpha\neq 0,\beta\neq 0)$

$\Rightarrow\alpha+\beta$一定不是特征向量

证：(反)设$A(\alpha+\beta)=\lambda_3(\alpha+\beta)$

$\Rightarrow\lambda_1\alpha+\lambda_2\beta=\lambda_3\alpha+\lambda_3\beta$

$\Rightarrow(\lambda_1-\lambda_3)\alpha+(\lambda_2-\lambda_3)\beta=0$

$\because\alpha,\beta$线性无关$,\therefore\lambda_1=\lambda_2=\lambda_3,$矛盾

$\therefore\alpha+\beta$一定不是特征向量

$3.A\alpha=\lambda_0\alpha(\alpha\neq 0)$

①$f(A)\alpha=f(\lambda_0)\alpha$

②若A可逆，则$A^{-1}\alpha=\dfrac{1}{\lambda_0}\alpha,A^*\alpha=\dfrac{|A|}{\lambda_0}\alpha$

证：①$A\alpha=\lambda_0\alpha\Rightarrow A^2\alpha=A\cdot A\alpha=\lambda_0A\alpha=\lambda_0^2\alpha$

$A^3\alpha=\lambda_0^3\alpha,\dots$

若$f(A)=a_nA^n+\dots+a_1A+a_0E$

$f(A)\alpha=a_nA^n\alpha+\dots+a_1A\alpha+a_0\alpha$

$=(a_0\lambda_0^n+\dots+a_1\lambda_0+a_0)\alpha=f(\lambda_0)\alpha$

② $\because A$可逆$,\therefore\lambda_0\neq 0$

$A\alpha=\lambda_0\alpha\Rightarrow\alpha=\lambda_0A^{-1}\alpha\Rightarrow A^{-1}\alpha=\dfrac{1}{\lambda_0}\alpha$

$A^*\alpha=|A|A^{-1}\alpha=\dfrac{|A|}{\lambda_0}\alpha$

Notes: A可逆时$,A,A^{-1},A^*$特征向量相同

A可逆$,A\alpha=\lambda_0\alpha$

$((A^*)^2+E)=[(\dfrac{|A|}{\lambda_0})^2+1]\alpha$

解：$A\alpha=\lambda_0\alpha\Rightarrow A^*\alpha=\dfrac{|A|}{\lambda_0}\alpha$

$((A^*)^2+E)\alpha=[(\dfrac{|A|}{\lambda_0})^2+1]\alpha$

$A\sim B$为4阶阵$,A$特征值$\dfrac{1}{2},\dfrac{1}{3},\dfrac{1}{4},\dfrac{1}{5}$

求$|B^{-1}-E|$?

解：$A\sim B\Rightarrow A^{-1}\sim B^{-1}$

A^{-1}特征值$2,3,4,5\Rightarrow B^{-1}$特征值$2,3,4,5$

$\Rightarrow B^{-1}-E$特征值$1,2,3,4$

$\therefore|B^{-1}-E|=24$

$4.A_{n\times n},A$可相似对角化$\Leftrightarrow A$有n个线性无关特征向量

实对称阵

$A^T = A$:

1.若$A^T = A \Rightarrow \lambda_1 \in R, \ldots, \lambda_n \in R$

2.若$A^T = A, \lambda_1 \neq \lambda_2, A\alpha = \lambda_1\alpha, A\beta = \lambda_2\beta$, 则 $\alpha \perp \beta$

证 : $\alpha^T A^T = \lambda_1 \alpha^T \Rightarrow \alpha^T A = \lambda_1 \alpha^T$

$\Rightarrow \alpha^T A\beta = \lambda_1 \alpha^T \beta \Rightarrow (\lambda_2 - \lambda_1)\alpha^T \beta = 0$

$\because \lambda_1 \neq \lambda_2, \therefore \alpha^T \beta = 0$, 即$\alpha \perp \beta$

3.若$A^T = A$, 则A一定可相似对角化