不定积分的基本概念与性质

原函数

$$f(x),F(x)(x\in D),\text{若}\forall x\in D\text{有}$$
$$F'(x)=f(x)$$
$$\text{称}F(x)\text{为}f(x)\text{在}D\text{上的原函数}$$

①若$f(x)\exists$原函数,则一定有无数个原函数,且任两个原函数之间相差常数
$\because F'(x)=f(x)\Rightarrow[F(x)+C]'=f(x)(C\in\mathbb{R})$
②连续函数一定存在原函数,反之不对
③若$F(x)$为$f(x)$的一个原函数,则$F(x)+C$(C为任意数)为$f(x)$的所有原函数
④若$f(x)$有原函数，则任两个原函数相差常数
如$:F'(x)=f(x),G'(x)=g(x)$
$\because[F(x)-G(x)]'=0,\therefore F(x)-G(x)\equiv C_0$
⑤连续函数一定存在原函数,反之不对

举例说明存在第二类间断点,但存在原函数的函数.

$$f(x)=\begin{cases}2x\sin\frac{1}{x}-\cos\frac{1}{x},x\neq0,\\0,x=0.\end{cases},F(x)=\begin{cases}x^2\sin\frac{1}{x},x\neq0,\\0,x=0.\end{cases}$$

$x\neq0:F'(x)=2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}=f(x)$
$x=0:F'(x)=\lim\limits_{x\to0}\dfrac{F(x)-F(0)}{x}=\lim\limits_{x\to0}x\sin\dfrac{1}{x}=0=f(0)$
即$F'(x)=f(x),\therefore F(x)$为$f(x)$的原函数
$\because\lim\limits_{x\to0}f(x)$不存在,$\therefore x=0$为$f(x)$第二类间断点

不定积分

设$F(x)$为$f(x)$的一个原函数,$F(x)+C$即$f(x)$的所有原函数,称为$f(x)$的不定积分,记

$$\int f(x)dx=F(x)+C$$

$$①\frac{d}{dx}\int f(x)dx=f(x)$$
$$②\int f'(x)dx=f(x)+C$$

基本公式

1. $\int kdx = kx + C$

$$2.\begin{cases} ①a \neq -1: \int x^a dx = \frac{1}{a+1}x^{a+1} + C \\ ②a = -1: \int \frac{1}{x}dx = \ln|x| + C \\ \because x < 0\text{时}, [\ln(-x)]' = \frac{1}{-x}x(-1), \therefore \int \frac{1}{x}dx = \ln(-x) + C \\ x > 0\text{时}, (\ln x)' = \frac{1}{x}, \int \frac{1}{x}dx = \ln x + C \\ \therefore \int \frac{1}{x}dx = \ln|x| + C \end{cases}$$

$$3.\begin{cases} ①a \neq e: \int a^x dx = \frac{a^x}{\ln a} + C \\ ②a = e: \int e^x dx = e^x + C \end{cases}$$

4. ① $\int \sin xdx = -\cos x + C$

② $\int \cos xdx = \sin x + C$

③ $\int \tan xdx = -\ln|\cos x| + C$

④ $\int \cot xdx = \ln|\sin x| + C$

⑤ $\int \sec xdx = \ln|\sec x + \tan x| + C$

⑥ $\int \csc xdx = \ln|\csc x - \cot x| + C$

⑦ $\int \sec^2 dx = \tan x + C$

⑧ $\int \csc^2 dx = -\cot x + C$

⑨ $\int \sec x \tan xdx = \sec x + C$

⑩ $\int \csc x \cot xdx = -\csc x + C$

5. 平方和, 平方差 :

$$\begin{cases} ①\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C \\ ②\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{d(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}} = \arcsin\frac{x}{a} + C \end{cases}$$

$$\begin{cases} ③\int \frac{dx}{1+x^2} = \arctan x + C \\ ④\int \frac{dx}{a^2+x^2} = \frac{1}{a}\int \frac{d(\frac{x}{a})}{1+(\frac{x}{a})^2} = \frac{1}{a}\arctan\frac{x}{a} + C \end{cases}$$

$$\begin{cases} ⑤\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2}) + C \\ ⑥\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x + \sqrt{x^2-a^2}| + C \end{cases}$$

⑦ $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}[\int \frac{d(x-a)}{x-a} - \int \frac{d(x+a)}{x+a}] = \frac{1}{2a}\ln|\frac{x-a}{x+a}| + C$

⑧ $\int \sqrt{a^2-x^2}dx = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$