型一 n项和的极限

先和后极限

①先和后极限

$$求\lim_{n\to\infty}[\frac{1}{1*3}+\frac{1}{3*5}+\dots+\frac{1}{(2n-1)(2n+1)}].$$

$$\frac{1}{1*3}+\frac{1}{3*5}+\dots+\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}(1-\frac{1}{3})+\frac{1}{2}(\frac{1}{3}-\frac{1}{5})+\dots+\frac{1}{2}(\frac{1}{2n-1}-\frac{1}{2n+1})$$

$$=\frac{1}{2}(1-\frac{1}{2n+1})$$

$$\therefore 原式=\lim_{n\to\infty}\frac{1}{2}(1-\frac{1}{2n+1})=\frac{1}{2}$$

定积分定义

$$② \begin{cases} 分子次数齐,分母次数齐 \\ 分母高一次 \end{cases} 定积分定义$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f(\frac{i-1}{n})=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f(\frac{i}{n})=\int_0^1 f(x)dx$$

$$\lim_{n\to\infty}(\frac{1}{n^2+1^2}+\frac{2}{n^2+2^2}+\dots+\frac{n}{n^2+n^2})$$

$$原式=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2+i^2}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{i}{n}}{1+(\frac{i}{n})^2}$$

$$=\int_0^1\frac{x}{1+x^2}dx=\frac{1}{2}\int_0^1\frac{d(1+x^2)}{1+x^2}$$

$$=\frac{1}{2}\ln(1+x^2)\mid_0^1=\frac{1}{2}\ln 2$$

$$\lim_{n\to\infty}(\frac{1}{\sqrt{n^2+1^2}}+\dots+\frac{n}{\sqrt{n^2+n^2}})$$

$$原式=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n^2+i^2}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{1+(\frac{i}{n})^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{x^2+1}}=\ln(x+\sqrt{x^2+1})\mid_0^1=\ln(1+\sqrt{2})$$

夹逼定理

③分子或分母不齐 — 夹逼定理

$$\lim_{n\to\infty}(\frac{1^2}{n^3+1}+\frac{2^2}{n^3+2}+\ldots+\frac{n^2}{n^3+n})$$

$$\frac{1^2}{n^3+1}+\frac{2^2}{n^3+2}+\ldots+\frac{n^2}{n^3+n}\triangleq b_n$$

$$\frac{\frac{1}{6}n(n+1)(2n+1)}{n^3+n}\le b_n\le\frac{\frac{1}{6}n(n+1)(2n+1)}{n^3+1}$$

$$\because \lim_{n\to\infty}\text{左}=\lim_{n\to\infty}\text{右}=\frac{1}{3}，\therefore\text{原式}=\frac{1}{3}$$

型二 不定型

$$\frac{0}{0},1^{\infty},\frac{\infty}{\infty},\infty\cdot0,\infty-\infty,\infty^0,0^0$$

0/0

$$①\frac{0}{0}$$

$$\begin{array}{l}
①\text{常用方法}\begin{cases}\text{等价无穷小}\\ \text{洛必达法则}\end{cases}\\
②\text{习惯动作}\\
\quad 1.u(x)^{v(x)}=e^{v(x)\ln u(x)}\\
\quad 2.\ln(\ldots)=\ln(1+\Delta)\sim\Delta,\ldots\to1\\
\quad 3.(\ldots)-1=\begin{cases}e^{\Delta}-1\sim\Delta\\ (1+\Delta)^a-1\sim a\Delta\end{cases}(\Delta\to0)\\
③\text{阶}\begin{cases}x,\sin x,\tan x,\arcsin x,\arctan x\text{任两者之差为三阶无穷小}\\ x-\ln(1+x)\text{是二阶}\end{cases}\\
④\text{误区：}\\
\quad \lim_{x\to0}\frac{e^{x^2}+\cos x-2}{x\arctan x}=\lim_{x\to0}\frac{e^{x^2}+\cos x-2}{x^2}=\lim_{x\to0}\frac{(e^{x^2}-1)-(1-\cos x)}{x^2}=\frac{1}{2}\\
\quad \lim_{x\to0}\frac{x-\sin x}{x\ln^2(1+2x)}=\frac{1}{4}\lim_{x\to0}\frac{x-\sin x}{x^3}=\frac{1}{4}\lim_{x\to0}\frac{1-\cos x}{3x^2}=\frac{1}{24}
\end{array}$$

$$\begin{aligned}
\lim_{x\to0}\frac{(1-\sin2x)^{\ln^2(1+x)}-1}{x^2\arcsin x}&=\lim_{x\to0}\frac{e^{\ln^2(1+x)\cdot\ln(1-\sin2x)}-1}{x^3}\\
&=\lim_{x\to0}\frac{\ln^2(1+x)\cdot\ln(1-\sin2x)}{x^3}\\
&=-2
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to0}\frac{(\frac{\cos x+1}{2})^x-1}{x^3}&=\lim_{x\to0}\frac{e^{x\ln\frac{\cos x+1}{2}}-1}{x^3}\\
&=\lim_{x\to0}\frac{\ln(1+\frac{\cos x-1}{2})}{x^2}\\
&=\frac{1}{2}\lim_{x\to0}\frac{\cos x-1}{x^2}\\
&=-\frac{1}{4}
\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0}\frac{\ln\frac{\sin x}{x}}{x^2} &= \lim_{x\to 0}\frac{\sin x - x}{x^3}\\ &= \lim_{x\to 0}\frac{\cos x - 1}{3x^2}\\ &= -\frac{1}{6}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0}\frac{e^{\tan x}-e^x}{x^3} &= \lim_{x\to 0} e^x\frac{e^{\tan x - x}-1}{x^3}\\ &= \lim_{x\to 0}\frac{\tan x - x}{x^3}\\ &= \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}\\ &= \lim_{x\to 0}\frac{\tan^2 x}{3x^2}\\ &= \frac{1}{3}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{1+x\cos x}-\sqrt{1+\sin x}}{x^3} &= \frac{1}{2}\lim_{x\to 0}\frac{x\cos x - \sin x}{x^3}\\ &= \frac{1}{2}\lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{3x^2}\\ &= -\frac{1}{6}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0}\frac{x^2-\ln^2(1+x)}{x^3} &= \lim_{x\to 0}\frac{x+\ln(1+x)}{x}\,\frac{x-\ln(1+x)}{x^2}\\ &= 2\lim_{x\to 0}\frac{1-\frac{1}{1+x}}{2x}\\ &= \lim_{x\to 0}\frac{1}{1+x}\\ &= 1\end{aligned}$$

1^∞

$$1^{\infty}\begin{cases}\text{凑}(1+\Delta)^{\frac{1}{\Delta}}\\ \text{恒等变形}\end{cases}$$

$$\begin{aligned}\lim_{x\to\infty}(\cos\frac{1}{x})^{x^2} &= \lim_{x\to\infty}\{[1+(\cos\frac{1}{x}-1)]^{\frac{1}{\cos\frac{1}{x}-1}}\}^{x^2(\cos\frac{1}{x}-1)}\\ &= e^{\lim_{x\to\infty}\frac{\cos\frac{1}{x}-1}{\frac{1}{x^2}}}\\ &= e^{\lim_{x\to\infty}\frac{\cos t-1}{t^2}},\ \frac{1}{x}=t\\ &= e^{-\frac{1}{2}}\end{aligned}$$

$$\lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\frac{1}{x^3}}=\lim_{x\to 0}\left[\left(1+\frac{\tan x-\sin x}{1+\sin x}\right)^{\frac{1+\sin x}{\tan x-\sin x}}\right]^{\frac{1}{1+\sin x}\frac{\tan x-\sin x}{x^3}}$$
$$=e^{\lim_{x\to\infty}\frac{1}{1+\sin x}\frac{\tan x-\sin x}{x^3}}$$
$$=e^{\lim_{x\to\infty}\frac{\tan x}{x}\frac{1-\cos x}{x^2}}$$
$$=e^{\frac{1}{2}}$$

∞/∞

$$x\to\infty\begin{cases}\ln^a x\to+\infty\\ x^b\to+\infty\\ c^x\to+\infty\end{cases}\quad(a>0,b>0,c>1)$$
$$\lim_{x\to+\infty}\frac{\ln^{100}x}{\sqrt{x}}=0,\ \lim_{x\to+\infty}\frac{x^{50}}{e^x}=0$$

$$\lim_{x\to+\infty}\frac{a_m x_m+\ldots}{b_n x_n+\ldots}\begin{cases}=0,m<n\\ =\frac{a_m}{b_n},m=n\\ =\infty,m>n\end{cases}$$

$$\text{已知}\lim_{x\to\infty}\frac{(x+1)^3-(x-2)^3}{x^a+1}=b(\neq 0,\infty),\text{求}a,b$$
$$(x+1)^3=x^3+3x^2+\ldots$$
$$(x-2)^3=x^3+C_3^1x^2(-2)+\ldots=x^3-6x^2+\ldots$$
$$(x+1)^3-(x-2)^3=9x^2+\ldots$$
$$\therefore\begin{cases}a=2\\ b=9\end{cases}$$

$$\lim_{x\to ?}\frac{f(x)}{g(x)}=\lim_{x\to ?}\frac{f(x)/?}{g(x)/?}=\frac{A}{B}(B\neq 0)$$

$$\lim_{x\to\infty}\frac{3x^2-2x\sin 2x}{2x^2-x+\cos\frac{1}{x}}=\lim_{x\to\infty}\frac{3-\frac{2}{x}\sin 2x}{2-\frac{1}{x}+\frac{1}{x^2}\cos\frac{1}{x}}$$
$$=\frac{3}{2}$$

$$\lim_{x\to\infty}\frac{\ln(3x^2+4x+1)}{\ln(6x^4+x^2+3)}=\frac{6x+4}{3x^2+4x+1}\Big/\frac{24x^3+2x}{6x^4+x^2+3}$$
$$=\lim_{x\to\infty}\frac{36x^5+\ldots}{72x^5+\ldots}$$
$$=\frac{1}{2}$$

∞ 0

$$\infty\cdot 0\begin{cases}\frac{0}{\frac{1}{\infty}}:\frac{0}{0}\\ \frac{\infty}{\frac{1}{0}}:\frac{\infty}{\infty}\\ ?\end{cases}$$

$$
\begin{aligned}
&\lim_{x\to\infty}(x^2-x^3\sin\frac{1}{x})(\infty-\infty)\\
&=\lim_{x\to\infty}x^3(\frac{1}{x}-sin\frac{1}{x})(\infty\cdot 0)\\
&=\lim_{x\to\infty}\frac{\frac{1}{x}-sin\frac{1}{x}}{\frac{1}{x^3}}(\infty\cdot 0),\frac{1}{x}=t\\
&=\lim_{t\to\infty}\frac{t-\sin t}{t^3}=\frac{1}{6}
\end{aligned}
$$

∞-∞

$$
\infty-\infty\begin{cases}有分母：通分\\ 无分母：分子有理化,提取\end{cases}
$$

$$
\begin{aligned}
&\lim_{x\to 0}(\frac{1}{\sin^2 x}-\frac{1}{x^2})\\
&=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4}\\
&=\lim_{x\to 0}\frac{x+\sin x}{x}\cdot\frac{x-\sin x}{x^3}=\frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{x\to+\infty}(\sqrt{x^2+4x-1}-\sqrt{x^2-2x+4})\\
&=\lim_{x\to+\infty}\frac{6x-5}{\sqrt{x^2+4x-1}+\sqrt{x^2-2x+4}}=3\\
&=\lim_{x\to+\infty}\frac{6-\frac{5}{x}}{\sqrt{1+\frac{4}{x}-\frac{1}{x^2}}+\sqrt{1-\frac{2}{x}+\frac{4}{x^2}}}=3
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{x\to\infty}[x-x^2\ln(1+\frac{1}{x})]\\
&=\lim_{x\to\infty}x^2[\frac{1}{x}-\ln(1+\frac{1}{x})]\\
&=\lim_{x\to\infty}\frac{\frac{1}{x}-\ln(1+\frac{1}{x})}{\frac{1}{x^2}}\\
&=\lim_{t\to 0}\frac{t-\ln(1+t)}{t^2},\frac{1}{x}=t\\
&=\frac{1}{2}
\end{aligned}
$$

∞^0 0^0

$$
\infty^0,0^0:e^{\ln}
$$

$$
\begin{aligned}
\lim_{x\to 0^+}x^{\sin 2x}&=e^{\lim_{x\to 0^+}\sin 2x\cdot\ln x}\\
&=e^{2\lim_{x\to 0^+}\frac{\sin 2x}{2x}\cdot x\ln x}\\
&=e^{2\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}}\\
&=e^{2\lim_{x\to 0^+}(-x)}\\
&=e^0=1
\end{aligned}
$$

型三 证明数列存在极限

$$\text{证} \lim_{n\to\infty} a_n \exists$$

$\{a_n\}$单调性证明方法
①归纳法
②重要不等式
③$a_{n+1}-a_n$

设$\{a_n\}=\sqrt{2}, a_2=\sqrt{2+\sqrt{2}}, a_3=\sqrt{2+\sqrt{2+\sqrt{2}}},\dots$，证明：数列$\{a_n\}$收敛，并求其极限.

1. $a_{n+1}=\sqrt{2+a_n}(n=1,2,\dots)$
2. $a_1<a_2$，设$a_k<a_{k+1}\Rightarrow\sqrt{2+a_k}<\sqrt{2+a_{k+1}}$，
   即$a_{k+1}<a_{k+2}$，$\therefore\forall n, a_n<a_{n+1}\Rightarrow\{a_n\}\uparrow$
3. 现证$a_n\le 2$
   $a_1=\sqrt{2}\le 2$，设$a_k\le 2$，则
   $a_{k+1}=\sqrt{2+a_k}\le\sqrt{2+2}=2$
   $\therefore\forall n$，有$a_n\le 2\Rightarrow\lim\limits_{n\to\infty}a_n\exists$
4. 令$\lim\limits_{n\to\infty}a_n=A$
   $a_{n+1}=\sqrt{2+a_n}\Rightarrow A=\sqrt{2+A}$
   $\Rightarrow A^2-A-2=0\Rightarrow A=-1(\text{舍}), A=2$

$$0<a_0<\frac{\pi}{2}, a_{n+1}=\sin a_n(n=0,1,2,\dots), \text{证} \lim_{n\to\infty}a_n\exists, \text{求极限}.$$

$$\text{证：}a_0\in(0,\frac{\pi}{2})\Rightarrow a_n\in(0,1)(n=1,2,\dots)$$

$$\Rightarrow 0<a_n<\frac{\pi}{2}(n=0,1,2,\dots)\Rightarrow\{a_n\}\text{有界}$$

$$\because x>0\text{时}, \sin x<x$$

$$\therefore a_{n+1}=\sin a_n<a_n\Rightarrow\{a_n\}\downarrow$$

$$\therefore\lim_{n\to\infty}a_n\exists$$

$$\text{令}\lim_{n\to\infty}a_n=A, \text{由}a_{n+1}=\sin a_n\Rightarrow A=\sin A\Rightarrow A=0$$

$$a_1=2, a_{n+1}=\frac{1}{2}(a_n+\frac{1}{a_n}), \text{证}\lim_{n\to\infty}a_n\exists$$

$$\text{证}\because a_n>0, \therefore a_n+\frac{1}{a_n}\ge 2$$

$$\therefore a_{n+1}\ge 1$$

$$a_{n+1}-a_n=\frac{1}{2}(a_n+\frac{1}{a_n})-a_n=\frac{1-a_n^2}{2a_n}\le 0\Rightarrow\{a_n\}\downarrow$$

$$\therefore\lim_{n\to\infty}a_n\exists$$

$$0<a_1<2, a_{n+1}=\sqrt{a_n(2-a_n)}, \text{证：}\lim_{n\to\infty}a_n\exists$$

$$证:a_{n+1} \leq \frac{a_n+(2-a_n)}{2}=1$$

$$a_{n+1}-a_n=\sqrt{a_n(2-a_n)}-a_n=\frac{2a_n(1-a_n)}{\sqrt{a_n(2-a_n)}+a_n} \geq 0$$

$$\Rightarrow \{a_n\}\uparrow, \therefore \lim_{n\to\infty} a_n \exists$$

型四 连续与间断

$$f(x)\in C[a,b], p>0, q>0, 证:\exists \xi \in [a,b], 使pf(a)+af(b)=(p+q)f(\xi)$$

$$证:f(x)\in C[a,b] \Rightarrow \exists m, M$$

$$(p+q)m \leq pf(a)+qf(b) \leq (p+q)M$$

$$\Rightarrow m \leq \frac{pf(a)+qf(b)}{p+q} \leq M$$

$$\exists \xi \in [a,b], 使f(\xi)=\frac{pf(a)+qf(b)}{p+q}$$

$$f(x)\in C[a,+\infty), \lim_{x\to+\infty} f(x)=2, 证:f(x)在[a,+\infty)上有界$$

$$证:取\epsilon=1, \exists x_0>a, 当x>x_0时, 有$$

$$|f(x)-2|<1 \Rightarrow |f(x)|<3$$

$$\because f(x)\in C[a,x_0], \therefore \exists k>0, 使|f(x)| \leq k$$

$$取M=\max\{k,3\}, 则|f(x)| \leq M$$

$$f(x)=\frac{\ln|x|}{x^2-1}e^{\frac{1}{x-2}}$$

$$x=-1,0,1,2为f(x)的间断点$$

$$\lim_{x\to-1} f(x)=\lim_{x\to-1}\frac{e^{\frac{1}{x-2}}}{x-1}\frac{\ln(-x)}{x+1}$$

$$=-\frac{1}{2}e^{-\frac{1}{3}}\lim_{x\to-1}\frac{\ln[1-(x+1)]}{x+1}=\frac{1}{2}e^{-\frac{1}{3}} \Rightarrow x=-1为可去间断点$$

$$\because \lim_{x\to0} f(x)=+\infty, \therefore x=0为第二类间断点$$

$$\lim_{x\to1} f(x)=\lim_{x\to1}\frac{e^{\frac{1}{x-2}}}{x-1}\frac{\ln x}{x+1}=\frac{1}{2}e^{-1}\lim_{x\to1}\frac{\ln[1+(x-1)]}{x-1}=\frac{1}{2e}$$

$$\Rightarrow x=1为可去间断点$$

$$f(2-0)=0, f(2+0)=+\infty \Rightarrow x=2为第二类间断点$$