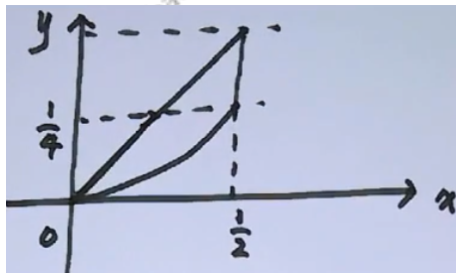


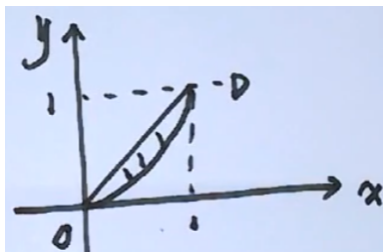
改变积分次序

交换积分次序： $\int_0^{\frac{1}{4}} dy \int_y^{\sqrt{y}} f(x, y) dx + \int_{\frac{1}{4}}^{\frac{1}{2}} dy \int_y^{\frac{1}{2}} f(x, y) dx.$



$$\text{原式} = \int_0^{\frac{1}{2}} dx \int_{x^2}^x f(x, y) dy$$

$$\text{计算 } I = \int_0^1 dy \int_y^{\sqrt{y}} \frac{\sin x}{x} dx.$$



$$\begin{aligned} \text{原式} &= \int_0^1 \frac{\sin x}{x} dx \int_{x^2}^x 1 dy \\ &= \int_0^1 (\sin x - x \sin x) dx = 1 - \cos 1 + \int_0^1 x d(\cos x) \\ &= 1 - \cos 1 + x \cos x \Big|_0^1 - \int_0^1 \cos x dx \\ &= 1 - \sin 1 \end{aligned}$$

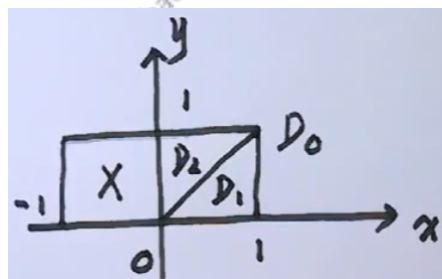
二重积分的计算

直角坐标法

计算 $\iint_D \sqrt{y^2 - xy} dx dy$, 其中 D 是由 $y = x, x = 0, y = 1$ 围成的区域.



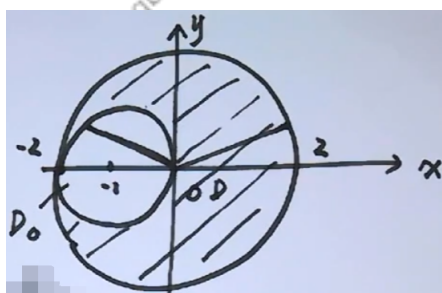
$$\begin{aligned}
 & \int_0^y (y^2 - xy)^{\frac{1}{2}} dx \\
 &= -\frac{1}{y} \int_0^y (y^2 - xy)^{\frac{1}{2}} d(y^2 - xy) \\
 &= -\frac{1}{y} \times \frac{2}{3} (y^2 - xy)^{\frac{3}{2}} \Big|_0^y \\
 &= -\frac{2}{3y} (0 - y^3) = \frac{2}{3} y^2 \\
 \text{原式} &= \int_0^1 dy \int_0^y (y^2 - xy)^{\frac{1}{2}} dx \\
 &= \frac{2}{3} \int_0^1 y^2 dy = \frac{2}{9}
 \end{aligned}$$



$$\begin{aligned}
 I &= \iint_D \sqrt{|y-x|} d\sigma \\
 &= 2 \iint_{D_0} \sqrt{|y-x|} d\sigma = 2I_0 \\
 I_0 &= \iint_{D_1} \sqrt{x-y} d\sigma + \iint_{D_2} \sqrt{y-x} d\sigma \\
 \iint_{D_1} \sqrt{x-y} d\sigma &= -\int_0^1 dx \int_0^x (x-y)^{\frac{1}{2}} d(x-y) \\
 &= -\frac{2}{3} \int_0^1 (x-y)^{\frac{3}{2}} \Big|_0^x dx = -\frac{2}{3} \int_0^1 (0 - x^{\frac{3}{2}}) dx = \frac{4}{15} \\
 \iint_{D_2} \sqrt{y-x} d\sigma &= \int_0^1 dx \int_x^1 (y-x)^{\frac{1}{2}} d(y-x) \\
 &= \frac{2}{3} \int_0^1 (y-x)^{\frac{3}{2}} \Big|_x^1 dx = \frac{2}{3} \int_0^1 (1-x)^{\frac{3}{2}} d(1-x) \\
 &= -\frac{4}{15} (1-x)^{\frac{5}{2}} \Big|_0^1 = -\frac{4}{15} (0-1) = \frac{4}{15} \\
 \therefore I &= \frac{16}{15}
 \end{aligned}$$

极坐标法

求 $\iint_D (\sqrt{x^2+y^2} + y) dx dy$, 其中 D 由 $x^2 + y^2 = 4$ 与 $(x+1)^2 + y^2 = 1$ 围成.



$$\begin{aligned}
I &= \iint_D \sqrt{x^2 + y^2} d\sigma \\
&= \iint_{D+D_0} \sqrt{x^2 + y^2} d\sigma - \iint_{D_0} \sqrt{x^2 + y^2} d\sigma = I_1 - I_2 \\
I_1 &= \int_0^{2\pi} d\theta \int_0^2 r^2 dr = \frac{16}{3} \pi \\
&\text{令 } \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \left(\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}, 0 \leq r \leq -2 \cos \theta \right) \\
I_2 &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\theta \int_0^{-2 \cos \theta} r^2 dr = -\frac{8}{3} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^3 \theta d\theta, \theta - \pi = t \\
&= \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 t dt = \frac{16}{3} \times \frac{2}{3} \times 1 = \frac{32}{9}, \therefore I = \frac{16\pi}{3} - \frac{32}{9}
\end{aligned}$$

计算 $I = \iint_D \frac{dx dy}{\sqrt{x^2 + y^2}}$, 其中 D 由 $x^2 + y^2 \leq 1$ 与 $x + y \geq 1$ 围成.



$$\begin{aligned}
&\text{令 } \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \left(0 \leq \theta \leq \frac{\pi}{2}, \frac{1}{\sin \theta + \cos \theta} \leq r \leq 1 \right) \\
\text{原式} &= \int_0^{\frac{\pi}{2}} d\theta \int_{\frac{1}{\sin \theta + \cos \theta}}^1 1 dr = \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{\sin \theta + \cos \theta} \right) d\theta \\
&= \frac{\pi}{2} - \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \sec \left(\theta - \frac{\pi}{4} \right) d \left(\theta - \frac{\pi}{4} \right) = \frac{\pi}{2} - \sqrt{2} \int_0^{\frac{\pi}{4}} \sec \theta d\theta \\
&= \frac{\pi}{2} - \sqrt{2} \ln | \sec \theta + \tan \theta | \Big|_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{2} - \sqrt{2} \ln(\sqrt{2} + 1)
\end{aligned}$$