型一 n项和的极限

先和后极限

①先和后极限

$$求\lim_{n\to\infty}[\frac{1}{1*3}+\frac{1}{3*5}+...+\frac{1}{(2n-1)(2n+1)}].$$

$$\frac{1}{1*3}+\frac{1}{3*5}+...+\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}(1-\frac{1}{3})+\frac{1}{2}(\frac{1}{3}-\frac{1}{5})+...+\frac{1}{2}(\frac{1}{2n-1}-\frac{1}{2n+1})$$

$$=\frac{1}{2}(1-\frac{1}{2n+1})$$

$$\therefore 原式=\lim_{n\to\infty}\frac{1}{2}(1-\frac{1}{2n+1})=\frac{1}{2}$$

定积分定义

$$②\begin{cases}分子次数齐,分母次数齐\\分母高一次\end{cases}定积分定义$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f(\frac{i-1}{n})=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f(\frac{i}{n})=\int_0^1 f(x)dx$$

$$\lim_{n\to\infty}(\frac{1}{n^2+1^2}+\frac{2}{n^2+2^2}+...+\frac{n}{n^2+n^2})$$

$$原式=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2+i^2}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{i}{n}}{1+(\frac{i}{n})^2}$$

$$=\int_0^1\frac{x}{1+x^2}dx=\frac{1}{2}\int_0^1\frac{d(1+x^2)}{1+x^2}$$

$$=\frac{1}{2}\ln(1+x^2)\mid_0^1=\frac{1}{2}\ln 2$$

$$\lim_{n\to\infty}(\frac{1}{\sqrt{n^2+1^2}}+...+\frac{n}{\sqrt{n^2+n^2}})$$

$$原式=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n^2+i^2}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{1+(\frac{i}{n})^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{x^2+1}}=\ln(x+\sqrt{x^2+1})\mid_0^1=\ln(1+\sqrt{2})$$

夹逼定理

③分子或分母不齐 — 夹逼定理

$$\lim_{n\to\infty}\left(\frac{1^2}{n^3+1}+\frac{2^2}{n^3+2}+\ldots+\frac{n^2}{n^3+n}\right)$$

$$\frac{1^2}{n^3+1}+\frac{2^2}{n^3+2}+\ldots+\frac{n^2}{n^3+n}\triangleq b_n$$

$$\frac{\frac{1}{6}n(n+1)(2n+1)}{n^3+n}\le b_n\le\frac{\frac{1}{6}n(n+1)(2n+1)}{n^3+1}$$

$$\because\lim_{n\to\infty}左=\lim_{n\to\infty}右=\frac{1}{3}，\ \therefore 原式=\frac{1}{3}$$

型二 不定型

$$\frac{0}{0},1^\infty,\frac{\infty}{\infty},\infty\cdot0,\infty-\infty,\infty^0,0^0$$

0/0

$$①\frac{0}{0}$$

①常用方法 $\begin{cases}\text{等价无穷小}\\\text{洛必达法则}\end{cases}$

②习惯动作

$1.u(x)^{v(x)}=e^{v(x)\ln u(x)}$

$2.\ln(\ldots)=\ln(1+\Delta)\sim\Delta,\ldots\to1$

$3.(\ldots)-1=\begin{cases}e^\Delta-1\sim\Delta\\(1+\Delta)^a-1\sim a\Delta\end{cases}(\Delta\to0)$

③阶 $\begin{cases}x,\sin x,\tan x,\arcsin x,\arctan x\text{任两者之差为三阶无穷小}\\x-\ln(1+x)\text{是二阶}\end{cases}$

④误区：

$$\lim_{x\to0}\frac{e^{x^2}+\cos x-2}{x\arctan x}=\lim_{x\to0}\frac{e^{x^2}+\cos x-2}{x^2}=\lim_{x\to0}\frac{(e^{x^2}-1)-(1-\cos x)}{x^2}=\frac{1}{2}$$

$$\lim_{x\to0}\frac{x-\sin x}{x\ln^2(1+2x)}=\frac{1}{4}\lim_{x\to0}\frac{x-\sin x}{x^3}=\frac{1}{4}\lim_{x\to0}\frac{1-\cos x}{3x^2}=\frac{1}{24}$$

$$\begin{aligned}\lim_{x\to0}\frac{(1-\sin2x)^{\ln^2(1+x)}-1}{x^2\arcsin x}&=\lim_{x\to0}\frac{e^{\ln^2(1+x)\cdot\ln(1-\sin2x)}-1}{x^3}\\&=\lim_{x\to0}\frac{\ln^2(1+x)\cdot\ln(1-\sin2x)}{x^3}\\&=-2\end{aligned}$$

$$\begin{aligned}\lim_{x\to0}\frac{(\frac{\cos x+1}{2})^x-1}{x^3}&=\lim_{x\to0}\frac{e^{x\ln\frac{\cos x+1}{2}}-1}{x^3}\\&=\lim_{x\to0}\frac{\ln(1+\frac{\cos x-1}{2})}{x^2}\\&=\frac{1}{2}\lim_{x\to0}\frac{\cos x-1}{x^2}\\&=-\frac{1}{4}\end{aligned}$$

$$
\begin{aligned}
\lim_{x\to0}\frac{\ln\frac{\sin x}{x}}{x^2} &= \lim_{x\to0}\frac{\sin x - x}{x^3}\\
&= \lim_{x\to0}\frac{\cos x - 1}{3x^2}\\
&= -\frac{1}{6}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x\to0}\frac{e^{\tan x}-e^{x}}{x^3} &= \lim_{x\to0} e^{x}\frac{e^{\tan x - x}-1}{x^3}\\
&= \lim_{x\to0}\frac{\tan x - x}{x^3}\\
&= \lim_{x\to0}\frac{\sec^2 x - 1}{3x^2}\\
&= \lim_{x\to0}\frac{\tan^2 x}{3x^2}\\
&= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x\to0}\frac{\sqrt{1+x\cos x}-\sqrt{1+\sin x}}{x^3} &= \frac{1}{2}\lim_{x\to0}\frac{x\cos x - \sin x}{x^3}\\
&= \frac{1}{2}\lim_{x\to0}\frac{\cos x - x\sin x - \cos x}{3x^2}\\
&= -\frac{1}{6}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x\to0}\frac{x^2-\ln^2(1+x)}{x^3} &= \lim_{x\to0}\frac{x+\ln(1+x)}{x}\,\frac{x-\ln(1+x)}{x^2}\\
&= 2\lim_{x\to0}\frac{1-\frac{1}{1+x}}{2x}\\
&= \lim_{x\to0}\frac{1}{1+x}\\
&= 1
\end{aligned}
$$

1^∞

$$
1^{\infty}\begin{cases}凑(1+\Delta)^{\frac{1}{\Delta}}\\ 恒等变形\end{cases}
$$

$$
\begin{aligned}
\lim_{x\to\infty}(\cos\frac{1}{x})^{x^2} &= \lim_{x\to\infty}\{[1+(\cos\frac{1}{x}-1)]^{\frac{1}{\cos\frac{1}{x}-1}}\}^{x^2(\cos\frac{1}{x}-1)}\\
&= e^{\lim_{x\to\infty}\frac{\cos\frac{1}{x}-1}{\frac{1}{x^2}}}\\
&= e^{\lim_{x\to\infty}\frac{\cos t-1}{t^2}},\ \frac{1}{x}=t\\
&= e^{-\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x\to0}(\frac{1+\tan x}{1+\sin x})^{\frac{1}{x^3}} &= \lim_{x\to0}[(1+\frac{\tan x-\sin x}{1+\sin x})^{\frac{1+\sin x}{\tan x-\sin x}}]^{\frac{1}{1+\sin x}\frac{\tan x-\sin x}{x^3}}\\
&= e^{\lim_{x\to\infty}\frac{1}{1+\sin x}\frac{\tan x-\sin x}{x^3}}\\
&= e^{\lim_{x\to\infty}\frac{\tan x}{x}\frac{1-\cos x}{x^2}}\\
&= e^{\frac{1}{2}}
\end{aligned}
$$

∞/∞

$$x \to \infty \begin{cases} \ln^a x \to +\infty \\ x^b \to +\infty \\ c^x \to +\infty \end{cases} \quad (a>0, b>0, c>1)$$

$$\lim_{x\to+\infty} \frac{\ln^{100} x}{\sqrt{x}} = 0, \lim_{x\to+\infty} \frac{x^{50}}{e^x} = 0$$

$$\lim_{x\to+\infty} \frac{a_m x_m + \dots}{b_n x_n + \dots} \begin{cases} = 0, m < n \\ = \frac{a_m}{b_n}, m = n \\ = \infty, m > n \end{cases}$$

$$已知 \lim_{x\to\infty} \frac{(x+1)^3-(x-2)^3}{x^a+1} = b(\neq 0, \infty), 求a, b$$

$$(x+1)^3 = x^3 + 3x^2 + \dots$$

$$(x-2)^3 = x^3 + C_3^1 x^2(-2) + \dots = x^3 - 6x^2 + \dots$$

$$(x+1)^3 - (x-2)^3 = 9x^2 + \dots$$

$$\therefore \begin{cases} a = 2 \\ b = 9 \end{cases}$$

$$\lim_{x\to ?} \frac{f(x)}{g(x)} = \lim_{x\to ?} \frac{f(x)/?}{g(x)/?} = \frac{A}{B}(B \neq 0)$$

$$\lim_{x\to\infty} \frac{3x^2 - 2x\sin 2x}{2x^2 - x + \cos\frac{1}{x}} = \lim_{x\to\infty} \frac{3 - \frac{2}{x}\sin 2x}{2 - \frac{1}{x} + \frac{1}{x^2}\cos\frac{1}{x}}$$

$$= \frac{3}{2}$$

$$\lim_{x\to\infty} \frac{\ln(3x^2+4x+1)}{\ln(6x^4+x^2+3)} = \frac{6x+4}{3x^2+4x+1} / \frac{24x^3+2x}{6x^4+x^2+3}$$

$$= \lim_{x\to\infty} \frac{36x^5 + \dots}{72x^5 + \dots}$$

$$= \frac{1}{2}$$

∞ 0

$$\infty \cdot 0 \begin{cases} \frac{0}{\frac{1}{\infty}} : \frac{0}{0} \\ \frac{\infty}{\frac{1}{0}} : \frac{\infty}{\infty} \\ ? \end{cases}$$

$$\lim_{x\to\infty} (x^2 - x^3 \sin\frac{1}{x})(\infty - \infty)$$

$$= \lim_{x\to\infty} x^3(\frac{1}{x} - sin\frac{1}{x})(\infty \cdot 0)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x} - sin\frac{1}{x}}{\frac{1}{x^3}}(\infty \cdot 0), \frac{1}{x} = t$$

$$= \lim_{t\to\infty} \frac{t - \sin t}{t^3} = \frac{1}{6}$$

∞-∞

$$\infty-\infty\begin{cases}\text{有分母：通分}\\ \text{无分母：分子有理化, 提取}\end{cases}$$

$$\begin{aligned}&\lim_{x\to 0}(\frac{1}{\sin^2 x}-\frac{1}{x^2})\\ =&\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4}\\ =&\lim_{x\to 0}\frac{x+\sin x}{x}\cdot\frac{x-\sin x}{x^3}=\frac{1}{3}\end{aligned}$$

$$\begin{aligned}&\lim_{x\to+\infty}(\sqrt{x^2+4x-1}-\sqrt{x^2-2x+4})\\ =&\lim_{x\to+\infty}\frac{6x-5}{\sqrt{x^2+4x-1}+\sqrt{x^2-2x+4}}=3\\ =&\lim_{x\to+\infty}\frac{6-\frac{5}{x}}{\sqrt{1+\frac{4}{x}-\frac{1}{x^2}}+\sqrt{1-\frac{2}{x}+\frac{4}{x^2}}}=3\end{aligned}$$

$$\begin{aligned}&\lim_{x\to\infty}[x-x^2\ln(1+\frac{1}{x})]\\ =&\lim_{x\to\infty}x^2[\frac{1}{x}-\ln(1+\frac{1}{x})]\\ =&\lim_{x\to\infty}\frac{\frac{1}{x}-\ln(1+\frac{1}{x})}{\frac{1}{x^2}}\\ =&\lim_{t\to 0}\frac{t-\ln(1+t)}{t^2},\frac{1}{x}=t\\ =&\frac{1}{2}\end{aligned}$$

∞^0 0^0

$$\infty^0,0^0:e^{\ln}$$

$$\begin{aligned}\lim_{x\to 0^+}x^{\sin 2x}=&e^{\lim_{x\to 0^+}\sin 2x\cdot\ln x}\\ =&e^{2\lim_{x\to 0^+}\frac{\sin 2x}{2x}\cdot x\ln x}\\ =&e^{2\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}}\\ =&e^{2\lim_{x\to 0^+}(-x)}\\ =&e^0=1\end{aligned}$$

型三 证明数列存在极限

$$\text{证}\lim_{n\to\infty}a_n\exists$$

$\{a_n\}$单调性证明方法
①归纳法
②重要不等式
③$a_{n+1}-a_n$

设$\{a_n\}=\sqrt{2},a_2=\sqrt{2+\sqrt{2}},a_3=\sqrt{2+\sqrt{2+\sqrt{2}}},\ldots$，证明：数列$\{a_n\}$收敛，并求其极限.

$$
\begin{aligned}
&1.\ a_{n+1}=\sqrt{2+a_n}(n=1,2,\ldots)\\
&2.\ a_1<a_2,\text{设}a_k<a_{k+1}\Rightarrow\sqrt{2+a_k}<\sqrt{2+a_{k+1}},\\
&\quad\text{即}a_{k+1}<a_{k+2},\therefore\forall n,a_n<a_{n+1}\Rightarrow\{a_n\}\uparrow\\
&3.\ \text{现证}a_n\le 2\\
&\quad a_1=\sqrt{2}\le 2,\text{设}a_k\le 2,\text{则}\\
&\quad a_{k+1}=\sqrt{2+a_k}\le\sqrt{2+2}=2\\
&\quad\therefore\forall n,\text{有}a_n\le 2\Rightarrow\lim_{n\to\infty}a_n\exists\\
&4.\ \text{令}\lim_{n\to\infty}a_n=A\\
&\quad a_{n+1}=\sqrt{2+a_n}\Rightarrow A=\sqrt{2+A}\\
&\quad\Rightarrow A^2-A-2=0\Rightarrow A=-1(\text{舍}),A=2
\end{aligned}
$$

$$0<a_0<\frac{\pi}{2},a_{n+1}=\sin a_n(n=0,1,2,\ldots),\text{证}\lim_{n\to\infty}a_n\exists,\text{求极限.}$$

$$
\begin{gathered}
\text{证}:a_0\in(0,\frac{\pi}{2})\Rightarrow a_n\in(0,1)(n=1,2,\ldots)\\
\Rightarrow 0<a_n<\frac{\pi}{2}(n=0,1,2,\ldots)\Rightarrow\{a_n\}\text{有界}\\
\because x>0\text{时},\sin x<x\\
\therefore a_{n+1}=\sin a_n<a_n\Rightarrow\{a_n\}\downarrow\\
\therefore\lim_{n\to\infty}a_n\exists\\
\text{令}\lim_{n\to\infty}a_n=A,\text{由}a_{n+1}=\sin a_n\Rightarrow A=\sin A\Rightarrow A=0
\end{gathered}
$$

$$a_1=2,a_{n+1}=\frac{1}{2}(a_n+\frac{1}{a_n}),\text{证}\lim_{n\to\infty}a_n\exists$$

$$
\begin{gathered}
\text{证}:\because a_n>0,\therefore a_n+\frac{1}{a_n}\ge 2\\
\therefore a_{n+1}\ge 1\\
a_{n+1}-a_n=\frac{1}{2}(a_n+\frac{1}{a_n})-a_n=\frac{1-a_n^2}{2a_n}\le 0\Rightarrow\{a_n\}\downarrow\\
\therefore\lim_{n\to\infty}a_n\exists
\end{gathered}
$$

$$0<a_1<2,a_{n+1}=\sqrt{a_n(2-a_n)},\text{证}:\lim_{n\to\infty}a_n\exists$$

$$
\begin{gathered}
\text{证}:a_{n+1}\le\frac{a_n+(2-a_n)}{2}=1\\
a_{n+1}-a_n=\sqrt{a_n(2-a_n)}-a_n=\frac{2a_n(1-a_n)}{\sqrt{a_n(2-a_n)}+a_n}\ge 0\\
\Rightarrow\{a_n\}\uparrow,\therefore\lim_{n\to\infty}a_n\exists
\end{gathered}
$$

型四 连续与间断

$$f(x)\in C[a,b],p>0,q>0,\text{证}:\exists\xi\in[a,b],\text{使}pf(a)+af(b)=(p+q)f(\xi)$$

$$证：f(x)\in C[a,b]\Rightarrow \exists m,M$$
$$(p+q)m\le pf(a)+qf(b)\le (p+q)M$$
$$\Rightarrow m\le \frac{pf(a)+qf(b)}{p+q}\le M$$
$$\exists\xi\in[a,b],使f(\xi)=\frac{pf(a)+qf(b)}{p+q}$$

$$f(x)\in C[a,+\infty),\lim_{x\to+\infty}f(x)=2,证：f(x)在[a,+\infty)上有界$$

$$证：取\epsilon=1,\exists x_0>a,当x>x_0时,有$$
$$|f(x)-2|<1\Rightarrow |f(x)|<3$$
$$\because f(x)\in C[a,x_0],\therefore \exists k>0,使|f(x)|\le k$$
$$取M=\max\{k,3\},则|f(x)|\le M$$

$$f(x)=\frac{\ln|x|}{x^2-1}e^{\frac{1}{x-2}}$$

$$x=-1,0,1,2为f(x)的间断点$$

$$\lim_{x\to-1}f(x)=\lim_{x\to-1}\frac{e^{\frac{1}{x-2}}}{x-1}\frac{\ln(-x)}{x+1}$$
$$=-\frac{1}{2}e^{-\frac{1}{3}}\lim_{x\to-1}\frac{\ln[1-(x+1)]}{x+1}=\frac{1}{2}e^{-\frac{1}{3}}\Rightarrow x=-1为可去间断点$$
$$\because\lim_{x\to0}f(x)=+\infty,\therefore x=0为第二类间断点$$

$$\lim_{x\to1}f(x)=\lim_{x\to1}\frac{e^{\frac{1}{x-2}}}{x-1}\frac{\ln x}{x+1}=\frac{1}{2}e^{-1}\lim_{x\to1}\frac{\ln[1+(x-1)]}{x-1}=\frac{1}{2e}$$
$$\Rightarrow x=1为可去间断点$$
$$f(2-0)=0,f(2+0)=+\infty\Rightarrow x=2为第二类间断点$$