不定积分的积分法

换元法

第一类换元积分法

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{1}{\sqrt{1-(\frac{x}{a})^2}} d(\frac{x}{a}) = \arcsin\frac{x}{a} + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\int \frac{1}{1+(\frac{x}{a})^2} d(\frac{x}{a}) = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\begin{aligned}
&\int \frac{x}{(2x+1)^2}dx \\
=&\frac{1}{4}\int \frac{(2x+1)-1}{(2x+1)^2}d(2x+1) \\
=&\frac{1}{4}[\int \frac{d(2x+1)}{2x+1} - \int \frac{d(2x+1)}{(2x+1)^2}] \\
=&\frac{1}{4}(\ln|2x+1| + \frac{1}{2x+1}) + C
\end{aligned}$$

$$\begin{aligned}
&\int \frac{dx}{x^2+2x+5} \\
=&\int \frac{d(x+1)}{2^2+(x+1)^2} \\
=&\frac{1}{2}\arctan\frac{x+1}{2} + C
\end{aligned}$$

$$\begin{aligned}
&\int \frac{dx}{\sqrt{2x-x^2}} \\
=&\int \frac{d(x-1)}{\sqrt{1-(x-1)^2}} \\
=&\arcsin(x-1) + C
\end{aligned}$$

$$\begin{aligned}
&\int \frac{x+1}{x^2+2x+3}dx \\
\text{原式} =&\frac{1}{2}\int \frac{d(x^2+2x+3)}{x^2+2x+3} \\
=&\frac{1}{2}\ln(x^2+2x+3) + C
\end{aligned}$$

$$\int \frac{x^2}{\sqrt{x^3+1}}dx$$

$$\begin{aligned}原式 &= \frac{1}{3}\int \frac{d(x^3+1)}{\sqrt{x^3+1}} \\ &= \frac{2}{3}\int \frac{d(x^3+1)}{2\sqrt{x^3+1}} \\ &= \frac{2}{3}\sqrt{x^3+1}+C\end{aligned}$$

$$\int \frac{x}{4+x^4}dx$$

$$\begin{aligned}原式 &= \frac{1}{2}\int \frac{d(x^2)}{2^2+(x^2)^2} \\ &= \frac{1}{4}\int \frac{d(x^2)}{2^2+(x^2)^2} \\ &= \frac{1}{4}\arctan\frac{x^2}{2}+C\end{aligned}$$

$$\int \frac{dx}{\sqrt{x}(1+x)}$$

$$\begin{aligned}原式 &= 2\int \frac{dx}{2\sqrt{x}(1+x)} \\ &= 2\int \frac{d\sqrt{x}}{(1+x)} \\ &= 2\arctan\sqrt{x}+C\end{aligned}$$

$$\int \frac{\tan^2\sqrt{x}}{\sqrt{x}}dx$$

$$\begin{aligned}原式 &= 2\int \tan^2\sqrt{x}d\sqrt{x} \\ &= 2\int (\sec^2\sqrt{x}-1)d\sqrt{x} \\ &= 2(\tan\sqrt{x}-\sqrt{x})+C\end{aligned}$$

$$\begin{aligned}x^a &= (\frac{1}{a+1}x^{a+1})' \\ \frac{1}{2\sqrt{x}} &= (\sqrt{x})' \\ e^x &= (e^x)'\end{aligned}$$

$$\begin{aligned}&\int \frac{dx}{\sqrt{x^2+x}} \\ &= 2\int \frac{dx}{2\sqrt{x}\sqrt{x+1}} \\ &= 2\int \frac{d\sqrt{x}}{\sqrt{(\sqrt{x})^2+1}} \\ &= 2\ln(\sqrt{x}+\sqrt{x+1})+C\end{aligned}$$

$$\int \frac{e^x}{4+e^{2x}}dx$$

$$\begin{aligned}\text{原式} &= \int \frac{de^x}{2^2+(e^x)^2} \\ &= \frac{1}{2}\arctan\frac{e^x}{2}+C\end{aligned}$$

$$\int \frac{1}{1+e^x}dx$$

$$\begin{aligned}\text{原式} &= \int \frac{dx}{e^x(e^{-x}+1)} \\ &= \int \frac{e^{-x}}{e^{-x}+1}dx \\ &= -\int \frac{d(e^{-x}+1)}{e^{-x}+1} \\ &= -\ln(e^{-x}+1)+C\end{aligned}$$

$$\text{计算不定积分}\int \frac{\sin x}{\sin x+\cos x}dx.$$

$$\begin{aligned}\text{原式} &= \frac{1}{\sqrt{2}}\int \frac{\sin x}{\cos(x-\frac{\pi}{4})}dx \\ &= \frac{1}{\sqrt{2}}\int \frac{\sin[(x-\frac{\pi}{4})+\frac{\pi}{4}]}{\cos(x-\frac{\pi}{4})}d(x-\frac{\pi}{4}) \\ &= \frac{1}{2}\int \frac{\sin(x-\frac{\pi}{4})+\cos(x-\frac{\pi}{4})}{\cos(x-\frac{\pi}{4})}d(x-\frac{\pi}{4}) \\ &= \frac{1}{2}\int \tan(x-\frac{\pi}{4})d(x-\frac{\pi}{4})+\frac{1}{2}(x-\frac{\pi}{4}) \\ &= -\frac{1}{2}\ln|\cos(x-\frac{\pi}{4})|+\frac{x}{2}+C\end{aligned}$$

$$F'(u)=f(u), u=\Phi(x)\text{可导}$$

$$\int f[\Phi(x)]\Phi'(x)dx=\int f[\Phi(x)]d\Phi(x)=\int f(t)dt$$

$$=F(t)+C=F[\Phi(x)]+C$$

$$\int \frac{dx}{x^2+2x+5}$$

$$\begin{aligned}\text{原式} &= \int \frac{d(x+1)}{2^2+(x+1)^2} \\ &= \frac{1}{2}\arctan\frac{x+1}{2}+C\end{aligned}$$

$$\int \frac{dx}{x^2-x-2}$$

$$
\begin{aligned}
\text{原式} &= \int \frac{dx}{(x-2)(x+1)} \\
&= \frac{1}{3}\int (\frac{1}{x-2} - \frac{1}{x+1})dx \\
&= \frac{1}{3}[\int \frac{d(x-2)}{x-2} - \int \frac{d(x+1)}{x+1}] \\
&= \frac{1}{3}\ln|\frac{x-2}{x+1}| + C
\end{aligned}
$$

$$\int \frac{dx}{\sqrt{x}(4+x)}$$

$$
\begin{aligned}
\text{原式} =& 2\int \frac{d\sqrt{x}}{2^2 + (\sqrt{x})^2} \\
=& \arctan\frac{\sqrt{x}}{2} + C
\end{aligned}
$$

$$\int \frac{dx}{x\ln^2 x}$$

$$
\begin{aligned}
\text{原式} &= \int \frac{d(\ln x)}{\ln^2 x} \\
&= -\frac{1}{\ln x} + C
\end{aligned}
$$

$$
\begin{aligned}
&\int \frac{\sin^2\sqrt{x}}{\sqrt{x}}dx \\
=& 2\int \sin^2\sqrt{x}d(\sqrt{x}), \sqrt{x} = t \\
=& 2\int \sin^2 tdt \\
=& \int (1-\cos 2t)dt = t - \int \cos 2tdt \\
=& t - \frac{1}{2}\sin 2t + C = \sqrt{x} - \frac{1}{2}\sin 2\sqrt{x} + C
\end{aligned}
$$

第二类换元积分法

$$
\begin{aligned}
&\int f(x)dx = \int f[\Phi(t)]\Phi'(t)dt = \int g(t)dt \\
&= G(t) + C = G[\Phi^{-1}(x)] + C
\end{aligned}
$$

无理->有理

$$1.\ \text{无理} \Rightarrow \text{有理}$$

$$
\begin{aligned}
&\int \frac{dx}{\sqrt{x(1-x)}} = 2\int \frac{dx}{2\sqrt{x}\sqrt{1-x}} = 2\frac{d(\sqrt{x})}{\sqrt{1-(\sqrt{x})^2}} \\
&= 2\arcsin\sqrt{x} + C
\end{aligned}
$$

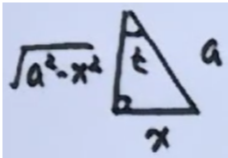
$$\int \frac{dx}{\sqrt{x}(x-2)}$$
$$=2\int \frac{d(\sqrt{x})}{(\sqrt{x})^2-(\sqrt{2})^2}$$
$$=2\frac{1}{2\sqrt{2}}\ln|\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}+\sqrt{2}}|+C$$

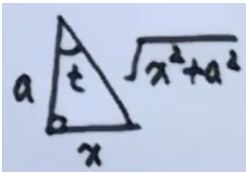
$$\text{求}\int \frac{dx}{1+\sqrt{x}}.$$

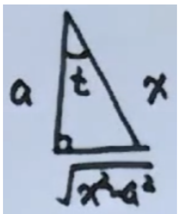
$$t=\sqrt{x}$$

$$\int \frac{dx}{1+\sqrt{x}}=2\int \frac{tdt}{1+t}=2\int(1-\frac{1}{1+t})dt=2(t-\ln|1+t|)+C=2\sqrt{x}-2\ln(1+\sqrt{x})+C$$

平方和 平方差





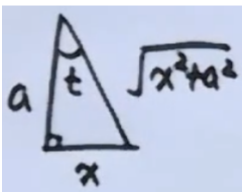


2. 三角代换

①$\sqrt{a^2-x^2}=a\cos t, x=a\sin t$

②$\sqrt{x^2+a^2}=a\sec t, x=a\tan t$

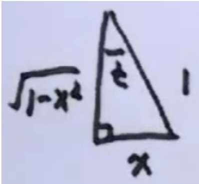
③$\sqrt{x^2-a^2}=a\tan t, x=a\sec t$



$$\int \frac{dx}{\sqrt{x^2+a^2}}=\int \frac{dx}{a\sec t}, x=a\tan t$$
$$=\int \frac{a\sec^2 t}{a\sec t}dx=\int \sec tdt=\ln|\sec t+\tan t|+C$$
$$=\ln\frac{\sqrt{x^2+a^2}+x}{a}+C$$
$$=\ln(x+\sqrt{x^2+a^2})+C$$

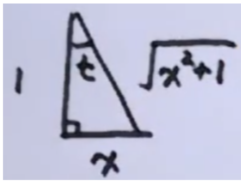
$$求不定积分\int \frac{dx}{x^2\sqrt{1-x^2}}.$$

$$\begin{aligned}原式&=\int \frac{\cos t}{\sin^2 t\cos t}dt, x=\sin t\\&=\int \csc^2 dt\\&=-\cot t+C\\&=-\frac{\sqrt{1-x^2}}{x}+C\end{aligned}$$



$$求\int \frac{1}{x^2\sqrt{x^2+1}}dx$$

$$\begin{aligned}原式&=\int \frac{\sec^2 t}{\tan^2 t\sec t}dt, x=\tan t\\&=\int \frac{\cos t}{\sin^2 t}dt\\&=\frac{d(\sin t)}{\sin^2 t}\\&=-\frac{1}{\sin t}+C\\&=-\frac{\sqrt{x^2+1}}{x}+C\end{aligned}$$



分部积分法

$$\begin{aligned}(uv)'&=u'v+uv'\\uv&=\int u'vdx+\int uv'dx\\&=\int vdu+\int udv\\\int udv&=uv-\int vdu\end{aligned}$$

幂 * 指

$$\int 幂*指dx$$

$$求\int x^2e^x dx.$$

$$\begin{aligned}\int x^2e^xdx=&\int x^2d(e^x)\\=&x^2e^x-2\int xe^xdx\\=&x^2e^x-2\int xd(e^x)\\=&x^2e^x-2(xe^x-\int e^xdx)\\=&x^2e^x-2xe^x+2e^x+C\end{aligned}$$

幂 * 对数

$$\int \text{幂}*\text{对数}dx$$

$$\text{求}\int x\ln^2 xdx.$$

$$\begin{aligned}\int x\ln^2 xdx=&\int \ln^2 xd(\frac{1}{2}x^2)\\=&\frac{x^2}{2}\ln^2 x-\int\frac{1}{2}x^2 2\ln x\frac{1}{x}\\=&\frac{x^2}{2}\ln^2 x-\int \ln xd(\frac{1}{2}x^2)\\=&\frac{x^2}{2}\ln^2 x-(\frac{x^2}{2}\ln^2 x-\int\frac{1}{2}x^2\frac{1}{x}dx)\\=&\frac{x^2}{2}\ln^2 x-\frac{x^2}{2}\ln^2 x+\frac{x^2}{4}+C\end{aligned}$$

幂 * 三角

$$\int \text{幂}*\text{三角}dx$$

$$\text{求}\int x^2\sin 2xdx.$$

$$\begin{aligned}\int x^2\sin 2xdx=&-\frac{1}{2}\int x^2d(\cos 2x)\\=&-\frac{1}{2}(x^2\cos 2x-2\int x\cos 2xdx)\\=&-\frac{x^2}{2}\cos 2x+\frac{1}{2}\int xd(\sin 2x)\\=&-\frac{x^2}{2}\cos 2x+\frac{1}{2}(x\sin 2x-\int\sin 2xdx)\\=&-\frac{x^2}{2}\cos 2x+\frac{x}{2}\sin 2x+\frac{1}{4}\cos 2x+C\end{aligned}$$

$$\int x\tan^2 xdx$$

$$
\begin{aligned}
\int x\tan^2 x dx =& \int x(\sec^2 x - 1)dx \\
=& \int x\sec^2 x dx - \frac{x^2}{2} \\
=& \int x d(\tan x) - \frac{x^2}{2} \\
=& x\tan x - \int \tan x dx - \frac{x^2}{2} \\
=& x\tan x + \ln|\cos x| - \frac{x^2}{2} + C
\end{aligned}
$$

幂 * 反三角

$$
\int 幂 * 反三角 dx
$$

$$
求 \int x\arctan x dx.
$$

$$
\begin{aligned}
\int x\arctan x dx =& \int \arctan x d(\frac{x^2}{2}) \\
=& \frac{x^2}{2}\arctan x - \frac{1}{2}\int x^2 \frac{1}{1+x^2}dx \\
=& \frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x + C
\end{aligned}
$$

$$
\begin{aligned}
\int \arcsin x dx =& x\arcsin x - \int x d(\arcsin x) \\
=& x\arcsin x + \int \frac{-2x}{2\sqrt{1-x^2}}dx \\
=& x\arcsin x + \int \frac{d(1-x^2)}{2\sqrt{1-x^2}} \\
=& x\arcsin x + \sqrt{1-x^2} + C
\end{aligned}
$$

5

$$
\int e^{ax} \begin{cases} \cos bx \\ \sin bx \end{cases} dx
$$

$$
求 \int e^{2x}\cos x dx.
$$

$$
I = \int e^{2x}\cos x dx = \int e^{2x} d(\sin x) = e^{2x}\sin x - 2\int e^{2x}\sin x dx
$$

$$
= e^{2x}\sin x + 2\int e^{2x} d(\cos x) = e^{2x}\sin x + 2(e^{2x}\cos x - 2\int e^{2x}\cos x dx)
$$

$$
= e^{2x}\sin x + 2e^{2x}\cos x - 4I
$$

$$
\Rightarrow 原式 = \frac{1}{5}e^{2x}(\sin x + 2\cos x) + C
$$

6

$$\begin{aligned}\int \sec^4 x dx =& \int (1+\tan^2 x) d(\tan x) \\ =& \tan x + \frac{1}{3}\tan^3 x + C\end{aligned}$$

$$求 \int \sec^3 x dx.$$

$$\begin{aligned}解：I = \int \sec^3 x dx &= \int \sec x d(\tan x) \\ &= \sec x \tan x - \int \tan^2 x \sec x dx \\ &= \sec x \tan x - \int \sec^3 dx + \int \sec x dx \\ &= \sec x \tan x + \ln|\sec x + \tan x| - I \\ \Rightarrow I &= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C\end{aligned}$$