中值定理

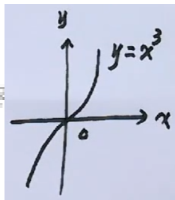
$$y=f(x)(x\in D),x_0\in D$$

$$\begin{aligned}&1.\ 若\exists\delta>0,当0<|x-x_0|<\delta时,\\&\quad f(x)<f(x_0)(左右小,中间大)\\&\quad x=x_0为f(x)的极大点\\&2.\ 若\exists\delta>0,当0<|x-x_0|<\delta时,\\&\quad f(x)>f(x_0)(左右大,中间小)\\&\quad x=x_0为f(x)的极小点\end{aligned}$$

$$f'(a)\begin{cases}>0\\<0\\=0\\不存在\end{cases}$$

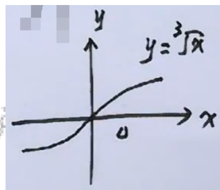
$$\begin{aligned}&1.\ f'(a)>0:f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}>0\\&\quad\exists\delta>0,当0<|x-a|<\delta时,\frac{f(x)-f(a)}{x-a}>0\\&\quad\begin{cases}f(x)<f(a),x\in(a-\delta,a)-左小\\f(x)>f(a),x\in(a,a+\delta)-右大\end{cases}\Rightarrow x=a不是极值点\\&2.\ f'(a)<0:f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}<0\\&\quad\exists\delta>0,当0<|x-a|<\delta时,\frac{f(x)-f(a)}{x-a}<0\\&\quad\begin{cases}f(x)>f(a),x\in(a-\delta,a)-左大\\f(x)<f(a),x\in(a,a+\delta)-右小\end{cases}\Rightarrow x=a不是极值点\end{aligned}$$

$$\begin{aligned}&①\ x=a为f(x)的极值点\Rightarrow(\not\Leftarrow)f'(a)=0或f'(a)不存在\\&②\ f(x)可导且x=a为f(x)的极值点\Rightarrow(\not\Leftarrow)f'(a)=0\end{aligned}$$



$$f(x)=x^3,f'(x)=3x^2,f'(0)=0$$

$$x=0不是极值点$$

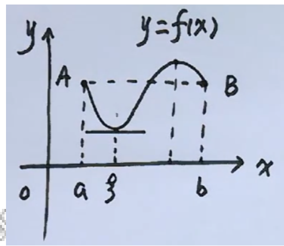


$$f(x)=\sqrt[3]{x}$$

$$\lim_{x\to0}\frac{f(x)-f(0)}{x}=\lim_{x\to0}\frac{\sqrt[3]{x}}{x}=\infty\Rightarrow f'(0)不存在$$

$$x=0不是极值点$$

Rolle



$$\text{若①}f(x)\text{在}[a,b]\text{上连续}$$
$$\text{②}f(x)\text{在}(a,b)\text{内可导}$$
$$\text{③}f(a)=f(b)$$

$$\text{则}\exists\xi\in(a,b),\text{使}f'(\xi)=0$$

$$\begin{aligned}
\text{证}:&1.\ f(x)\in C[a,b]\Rightarrow\exists m,M\\
&2.\ \text{①}m=M:f(x)\equiv C_0,\forall\exists\in(a,b),\text{有}f'(\xi)=0\\
&\quad\text{②}m<M:\\
&\qquad\because f(a)=f(b),\therefore m,M\text{至少有一个在}(a,b)\text{内取到}\\
&\qquad\text{设}\exists\xi\in(a,b),\text{使}f(\xi)=M\\
&\qquad\because \xi\text{为极大点},\therefore f'(\xi)=0\text{或}f'(\xi)\text{不存在},\\
&\qquad\because f(x)\text{在}(a,b)\text{内可导},\therefore f'(\xi)=0
\end{aligned}$$

验证函数$f(x)=x^2-2x+4$在$[0,2]$上满足罗尔定理的条件，并求驻点ξ.

$$\begin{gathered}
f(x)\in C[0,2],f(x)\text{在}(0,2)\text{内可导}\\
f(0)=f(2)=4\\
\exists\xi\in(0,2),\text{使}f'(\xi)=0\\
f'(x)=2x-2,\text{由}2\xi-2=0\Rightarrow\xi=1
\end{gathered}$$

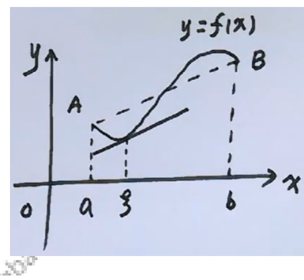
设$f(x)$在$[0,2]$上连续,在$(0,2)$内可导,且$f(0)=1,f(1)+f(2)=2$,证名:存在$\xi\in(0,2)$,使得$f'(\xi)=0$.

$$\begin{aligned}
\text{证}:&1.\ f(x)\in C[1,2]\Rightarrow\exists m,M\\
&\quad 2m\le f(1)+f(2)\le 2M\\
&\quad \because f(1)+f(2)=2,\therefore m\le 1\le M\\
&\quad \therefore\exists c\in[1,2],\text{使}f(c)=1\\
&2.\ \because f(c)=f(0)=1,\therefore\exists\xi\in(0,c)\subset(0,2),\text{使}f'(\xi)=0
\end{aligned}$$

$$f(x)\in C[0,2],(0,2)\text{内可导},3f(0)=f(1)+2(f2)$$
$$\text{证}:\exists\xi\in(0,2),\text{使}f'(\xi)=0$$

$$\begin{aligned}
\text{证}:&1.\ f(x)\in C[1,2]\Rightarrow\exists m,M\\
&\quad 3m\le f(1)+2f(2)\le 3M\\
&\quad \Rightarrow m\le\frac{f(1)+2f(2)}{3}\le M\\
&\quad \therefore\exists c\in[1,2],\text{使}f(c)=\frac{f(1)+2f(2)}{3}\Rightarrow f(1)+2f(2)=3f(c)\\
&2.\ \because f(0)=f(c),\therefore\exists\xi\in(0,c)\subset(0,2),\text{使}f'(\xi)=0
\end{aligned}$$

Lagrange



若①$f(x)$在$[a,b]$上连续

②$f(x)$在(a,b)内可导

$$则\exists\xi\in(a,b),使f'(\xi)=\frac{f(b)-f(a)}{b-a}$$

$$分析:L:y=f(x)$$

$$L_{AB}:y-f(a)=\frac{f(b)-f(a)}{b-a}(x-a),即$$

$$L_{AB}:y=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$$

$$证:令\Phi(x)=曲-直=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$$

$\Phi(x)\in C[a,b],(a,b)$内可导

$\because\Phi(a)=\Phi(b)=0,\therefore\exists\xi\in(a,b),使\Phi'(\xi)=0$

$$而\Phi'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$$

$$\therefore f'(\xi)=\frac{f(b)-f(a)}{b-a}$$

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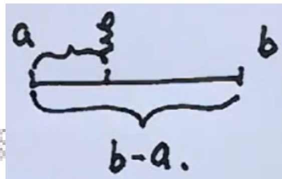
1. 若$f(a)=f(b),L\Rightarrow R$
2. 等价形式,
   $f(b)-f(a)=f'(\xi)(b-a)$
   $f(b)-f(a)=f'[a+\theta(b-a)](b-a)(0<\theta<1)$
3. $f(x)$可导
   $f(x)-f(a)=f'(\xi)(x-a)=f'[a+(x-a)\theta](x-a)(0<\theta<1)$



设$f(x)$二阶可导,且$f''(x)>0$.判断$f'(0),f'(1),f(1)-f(0)$的大小.

1. $f(1)-f(0)=f'(c)(1-0)=f'(c)(0<c<1)$
2. $f''(x)>0\Rightarrow f'(x)\uparrow$
   $\because 0<c<1,\therefore f'(0)<f'(c)<f'(1)$

设函数$f(x)$可导,且$\lim\limits_{x\to\infty}f'(x)=e$,求$\lim\limits_{x\to\infty}[f(x+2)-f(x-1)]$.

1. $f(x+2)-f(x-1)=3f'(\xi)(x-1<\xi<x+1)$
2. 原式$=3\lim\limits_{x\to\infty}f'(\xi)=3e$

$$\lim_{x\to\infty}x^2(\sin\frac{1}{x-1}-\sin\frac{1}{x+1})(\infty*0:\frac{0}{0},\frac{\infty}{\infty})$$

$$令f(t)=\sin t,f'(t)=\cos t$$

$$\sin\frac{1}{x-1}-\sin\frac{1}{x+1}=f(\frac{1}{x-1})-f(\frac{1}{x+1})=f'(\xi)(\frac{1}{x-1}-\frac{1}{x+1})$$

$$=\frac{2}{x^2-1}\cos\xi(\frac{1}{x+1}<\xi<\frac{1}{x-1})$$

$$原式=\lim_{x\to\infty}\frac{2x^2}{x^2-1}\cos\xi=2\cos 0=2$$

$$f(x)二阶可导,\lim_{x\to 0}\frac{f(x)-1}{x}=1,f(1)=2,证:\exists\xi\in(0,1),使f''(\xi)=0$$

$$证：\lim_{x\to0}\frac{f(x)-1}{x}=1\Rightarrow f(0)=1,f'(0)=1$$

$$\exists c\in(0,1),使f'(c)=\frac{f(1)-f(0)}{1-0}=1$$

$$\because f(x)二阶可导,且f'(0)=f'(c)=1$$

$$\therefore \exists\xi\in(0,c)\subset(0,1),使f''(\xi)=0$$

Cauchy

$$若①f(x),g(x)在[a,b]上连续$$

$$②f(x),g(x)在(a,b)内可导$$

$$③g'(x)\neq0(a<x<b)$$

$$则\exists\xi\in(a,b),使\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$$

$$1.\ g'(x)\neq0(a<x<b)\Rightarrow\begin{cases}g'(\xi)\neq0\\g(b)-g(a)\neq0\end{cases}$$

$$2.\ 若g(x)=x,C\Rightarrow L$$

$$分析：L=\Phi(x)=曲-直=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$$

$$C:\Phi(x)=f(x)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]$$

$$证：令\Phi(x)=f(x)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]$$

$$\Phi(x)\in C[a,b],(a,b)内可导,且\Phi(a)=\Phi(b)=0$$

$$\exists\xi\in(a,b),使\Phi'(\xi)=0$$

$$而\Phi'(x)=f'(x)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(x)$$

$$\therefore\frac{f'(\xi)}{g'(\xi)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

$$设f(x)\in C[0,1],(0,1)内可导,f(1)=0,证：\exists\xi\in(0,1),使\xi f'(\xi)+2f(\xi)=0.$$

$$分析：xf'(x)+2f(x)=0\Rightarrow\frac{f'(x)}{f(x)}+\frac{2}{x}=0$$

$$\Rightarrow[\ln f(x)]'+(\ln x^2)'=0\Rightarrow[\ln x^2f(x)]'=0$$

$$证：令\Phi(x)=x^2f(x)$$

$$\because\Phi(0)=\Phi(1)=0$$

$$\therefore\exists\xi\in(0,1),使\Phi'(x)=0$$

$$而\Phi'(x)=2xf(x)+x^2f'(x)$$

$$\therefore 2\xi f(\xi)+\xi^2f'(\xi)=0$$

$$\because\xi\neq0,\therefore 2f(\xi)+\xi f'(\xi)=0$$

$$f(x)\in C[a,b],(a,b)内可导,f(a)=f(b)=0,证：\exists\xi\in(a,b),使f'(\xi)-f(\xi)=0.$$

$$分析：f'(x)-f(x)=0\Rightarrow\frac{f'(x)}{f(x)}-1=0$$

$$\Rightarrow[\ln f(x)]'+(\ln e^{-x})'=0$$

$$\Rightarrow[\ln e^{-x}f(x)]'=0$$

$$证：令\Phi(x)=e^{-x}f(x)$$
$$\because f(a)=f(b)=0,\therefore \Phi(a)=\Phi(b)=0$$
$$\therefore \exists\xi\in(a,b),使\Phi'(\xi)=0$$
$$而\Phi'(x)=e^{-x}[f'(x)-f(x)]且e^{-x}\neq 0$$
$$\therefore f'(\xi)-f(\xi)=0$$

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①$f(b)-f(a)$或$f(a)\neq f(b)-L$

②$\xi f'(\xi)+2f(\xi)\begin{cases}仅有\xi,无a,b\\2项\\导数差一阶\end{cases}\Rightarrow xf'(x)+2f(x)\Rightarrow\frac{f'(x)}{f(x)}+\frac{2}{x}=0\Rightarrow[\ln f(x)]'+(\ln x^2)'=0\Rightarrow\Phi(x)=x^2f(x)$

③有ξ,有a,b,ξ与a,b可分开

ξ与a,b分开 $\Rightarrow a,b$侧$\begin{cases}\frac{f(b)-f(a)}{b-a}-L\\\frac{f(b)-f(a)}{g(b)-g(a)}-C\end{cases}$

$$设函数f(x)\in C[a,b],在(a,b)内可导(a>0),证明：存在\xi\in(a,b),使得f(b)-f(a)=\xi f'(\xi)\ln\frac{b}{a}.$$

$$\frac{f(b)-f(a)}{\ln b-\ln a}=\xi f'(\xi)$$

$$证：g(x)=\ln x,g'(x)=\frac{1}{x}\neq 0(a<x<b)$$
$$\exists\xi\in(a,b),使\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}\Rightarrow f(b)-f(a)=\xi f'(\xi)\ln\frac{b}{a}$$

Taylor中值定理

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}\neq\lim_{x\to 0}\frac{x-x}{x^3}$$
$$若\sin x=x-\frac{x^3}{6}+o(x^3)\sim\frac{x^3}{6}$$
$$原式=\lim_{x\to 0}\frac{\frac{x^3}{6}-o(x^3)}{x^3}=\frac{1}{6}$$

$$条件：f(x)在x=x_0邻域内有n+1阶导数$$

$$结论：f(x)=P_n(x)+R_n(x)-余项$$

$$P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$R_n(x)=\begin{cases}\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},\xi在x_0与x内\\o((x-x_0)^n)-皮亚诺型余项\end{cases}$$

$$若x_0=0,$$
$$f(x)=f(0)+f'(0)x+\ldots+\frac{f^{(n)}(0)}{n!}x^n+o(x^n)-麦克劳林公式$$

记 :$(x \to 0)$

①$e^x = 1 + x + \ldots + \frac{x^n}{n!} + o(x^n)$

②$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + o(x^{2n+1})$

③$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^n}{(2n)!} x^{2n} + o(x^{2n})$

④$\frac{1}{1-x} = 1 + x + \ldots + x^n + o(x^n)$

⑤$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + o(x^n)$

⑥$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{n-1}}{n} x^n + o(x^n)$

⑦$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!} x^2 + \frac{a(a-1)(a-2)}{3!} x^3 + \ldots$

$$\text{求极限} \lim_{x \to 0} \frac{\ln(1+x) - e^x + 1}{x^2}.$$

$$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$\ln(1+x) - e^x + 1 = -x^2 + o(x^2) \sim -x^2$$

$$\text{原式} = -1$$

$$\text{求} \lim_{x \to 0} \frac{e^{-\frac{x^2}{2}} - 1 + \frac{x^2}{2}}{x^2 - \sin^2 x}.$$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2) \Rightarrow e^{-\frac{x^2}{2}} = 1 - \frac{x^2}{2} + \frac{x^4}{8} + o(x^4)$$

$$e^{-\frac{x^2}{2}} - 1 + \frac{x^2}{2} \sim \frac{x^4}{8}$$

$$\text{原式} = \frac{1}{8} \lim_{x \to 0} \frac{x}{x + \sin x} * \frac{x^3}{x - \sin x} = \frac{1}{16} \lim_{x \to 0} \frac{x^3}{x - \sin x}$$

$$\because \sin x = x - \frac{x^3}{6} + o(x^3), \therefore x - \sin x \sim \frac{1}{6} x^3$$

$$\therefore \text{原式} = \frac{3}{8}$$

$$\lim_{x \to 0} \frac{\sqrt{1+x} + \sqrt{1-x} - 2}{x^2}$$

$$\text{由} (1+x)^a = 1 + ax + \frac{a(a-1)}{2!} x^2 + o(x^2), \text{得}$$

$$\sqrt{1+x} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + o(x^2)$$

$$\sqrt{1-x} = 1 - \frac{1}{2} x - \frac{1}{8} x^2 + o(x^2)$$

$$\Rightarrow \sqrt{1+x} - \sqrt{1-x} - 2 \sim -\frac{1}{4} x^2$$

$$\therefore \text{原式} = -\frac{1}{4}$$