导数与微分

导数

$$y=f(x)(x\in D),a\in D$$
$$\Delta y=f(a+\Delta x)-f(a)$$
$$\text{若}\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\exists,\text{称}f(x)\text{在}x=a\text{处可导},\text{极限值称为}f(x)\text{在}x=a\text{处的导数},\text{记}f'(a),\frac{dy}{dx}\mid_{x=a}$$

$$1.\ f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$
$$2.\ \Delta x\to 0\begin{cases}\Delta x\to 0^-\\ \Delta x\to 0^+\end{cases},x\to a\begin{cases}x\to a^-\\ x\to a^+\end{cases}$$
$$\lim_{\Delta x\to 0^-}\frac{\Delta y}{\Delta x}(=\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a})\triangleq f'_-(a)$$
$$\lim_{\Delta x\to 0^+}\frac{\Delta y}{\Delta x}(=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a})\triangleq f'_+(a)$$
$$f'(a)\exists\Leftrightarrow f'_-(a),f'_+(a)\exists\text{且相等}$$

$$f(x)=\begin{cases}\frac{x*2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}},x\neq 0\\ 0,x=0\end{cases},f'(0)?$$

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}}$$
$$f'_-(0)=0\neq f'_+(0)=1\Rightarrow f'(0)\text{不存在}$$

$$3.\ f(x)\text{在}x=a\text{处可导}\Rightarrow f(x)\text{在}x=a\text{处连续}$$
$$\not\Leftarrow$$

$$\Rightarrow$$

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\exists\Rightarrow\lim_{x\to a}[f(x)-f(a)]=0$$
$$\Rightarrow\lim_{x\to a}f(x)=f(a)$$

$$\not\Leftarrow$$

$$f(x)=|x|\text{在}x=0\text{处连续}$$
$$f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{|x|}{x}=-1$$
$$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{|x|}{x}=1$$
$$\because f'_-(0)\neq f'_+(0),\therefore f(x)\text{在}x=0\text{不可导}$$

$$5.\ f(x)\text{连续},\lim_{x\to a}\frac{f(x)-b}{x-a}=A\Rightarrow f(a)=b,f'(a)=A$$

$$\lim_{x\to a}f(x)=b$$

$$\because f(x)\text{连续},\therefore f(a)=b$$

$$A=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$$

$$\text{设函数}f(x)=\begin{cases}\ln(e+2x),x>0\\1,x=0\\\frac{1}{1+x^2},x<0\end{cases},\text{讨论函数}f(x)\text{在}x=0\text{处的可导性.}$$

$$f'_+(0)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^+}\frac{\ln(e+2x)-\ln e}{x}=\lim_{x\to0^+}\frac{\ln(1+\frac{2x}{e})}{x}=\frac{2}{e}$$

$$f'_-(0)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^-}\frac{\frac{1}{1+x^2}-1}{x}=-\lim_{x\to0^-}\frac{x}{1+x^2}=0$$

$$\Rightarrow f(x)\text{在}x=0\text{不可导}$$

微分

$$y=f(x)(x\in D),a\in D$$

$$\Delta y=f(a+\Delta x)-f(a)(\text{或}\Delta y=f(x)-f(a))$$

$$\text{若}\Delta y=A\Delta x+o(\Delta x)(\text{或}\Delta y=A(x-a)+o(x-a))$$

$$\text{称}f(x)\text{在}x=a\text{可微}$$

$$\text{称}A\Delta x\text{为}y=f(x)\text{在}x=a\text{处的微分, 记}dy\mid_{x=a}=A\Delta x\triangleq Adx$$

$$1.\ f(x)\text{在}x=a\text{可导}\Leftrightarrow f(x)\text{在}x=a\text{可微}$$

$$\Rightarrow$$

$$\lim_{\Delta x\to0}\frac{\Delta y}{\Delta x}=f'(a)\Rightarrow\frac{\Delta y}{\Delta x}=f'(a)+\alpha,\alpha\to0(\Delta x\to0)$$

$$\Rightarrow\Delta y=f'(a)\Delta x+\alpha\Delta x$$

$$\because\lim_{\Delta x\to0}\frac{\alpha\Delta x}{\Delta x}=\lim_{\Delta x\to0}\alpha=0$$

$$\therefore\alpha\Delta x=o(\Delta x)$$

$$\Rightarrow\Delta y=f'(a)\Delta x+o(\Delta x)$$

$$\Leftarrow$$

$$\text{设}\Delta y=A\Delta x+o(\Delta x)$$

$$\Rightarrow\frac{\Delta y}{\Delta x}=A+\frac{o(\Delta x)}{\Delta x}\Rightarrow\lim_{\Delta x\to0}\frac{\Delta y}{\Delta x}=A\Rightarrow f'(a)=A$$

$$2.\ \text{若}\Delta y=A\Delta x+o(\Delta x)\Rightarrow A=f'(a)$$

$$\therefore dy\mid_{x=a}=f'(a)dx$$

$$3.\ \text{设}y=f(x)\text{处处可导}$$

$$dy=df(x)=f'(x)dx$$

$$d(x^3)=3x^2,\cos2xdx=d(\frac{1}{2}\sin2x+C)$$

求导工具

$$初等函数\begin{cases}常数\\基本初等函数\end{cases}加工\begin{cases}四则\\复合\end{cases}而成的式子$$

基本公式

$$
\begin{aligned}
&1.\ (C)'=0\\
&2.\ (x^a)'=ax^{a-1}\begin{cases}(\sqrt{x})'=\frac{1}{2\sqrt{x}}\\(\frac{1}{x})'=-\frac{1}{x^2}\end{cases}\\
&3.\ (a^x)'=a^x\ln a,(e^x)'=e^x\\
&4.\ (\log_a x)'=\frac{1}{x\ln a},(\ln x)'=\frac{1}{x}\\
&5.\ (\sin x)'=\cos x,\\
&\quad(\cos x)'=-\sin x,\\
&\quad(\tan x)'=\sec^2 x,\\
&\quad(\cot x)'=-\csc^2 x,\\
&\quad(\sec x)'=\sec x\tan x,\\
&\quad(\csc x)'=-\csc x\cot x\\
&6.\ (\arcsin x)'=\frac{1}{\sqrt{1-x^2}}\\
&\quad(\arccos x)'=-\frac{1}{\sqrt{1-x^2}}\\
&\quad(\arctan x)'=\frac{1}{1+x^2}\\
&\quad(arccot\ x)'=-\frac{1}{1+x^2}
\end{aligned}
$$

四则

$$
\begin{aligned}
&1.\ (u\pm v)'=u'\pm v'\\
&2.\ (uv)'=u'v+uv'\\
&\quad(ku)'=ku'(k为常数)\\
&3.\ (uvw)'=u'vw+uv'w+uvw'\\
&4.\ (\frac{u}{v})'=\frac{u'v-uv'}{v^2}(v\neq 0)
\end{aligned}
$$

复合求导法则

$$
\begin{gathered}
y=f(u)可导,u=\Phi(x)可导且\Phi'(x)\neq 0\\
\Rightarrow\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=f'(u)\Phi'(x)=f'[\Phi(x)]\Phi'(x)
\end{gathered}
$$

$$
\begin{aligned}
证：&\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}=f'(u)\\
&\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=\Phi'(x)\neq 0\Rightarrow\Delta u=O(\Delta x)\\
&\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}*\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}*\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\\
&\quad=f'(u)\Phi'(x)=f'[\Phi(x)]\Phi'(x)
\end{aligned}
$$

$$y=x^2e^{\sin\frac{1}{x}}$$

$$
\begin{aligned}
y' &= 2xe^{\sin\frac{1}{x}} + x^2(e^{\sin\frac{1}{x}})' \\
&= 2xe^{\sin\frac{1}{x}} + x^2(e^{\sin\frac{1}{x}} * \cos\frac{1}{x} * (-\frac{1}{x^2}))
\end{aligned}
$$

$$y = (1+\sin 2x)^{\ln(1-x)}$$

$$
\begin{aligned}
y &= e^{\ln(1-x)\ln(1+\sin 2x)} \\
y' &= e^{\ln(1-x)\ln(1+\sin 2x)} * [\frac{-1}{1-x}\ln(1+\sin 2x) + \ln(1-x)\frac{2\cos 2x}{1+\sin 2x}]
\end{aligned}
$$

$$y = 2^{\sin^2\frac{1}{x}}, 求y'.$$

$$y' = 2^{\sin^2\frac{1}{x}}\ln 2 * [2\sin\frac{1}{x}\cos\frac{1}{x} * (-\frac{1}{x^2})]$$

反函数导数

$$y = f(x)可导且f'(x) \neq 0, x = \Phi(y)为反函数$$

$$则\Phi'(y) = \frac{1}{f'(x)}$$

$$
\begin{aligned}
证: f'(x) &= \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} \neq 0 \Rightarrow \Delta y = O(\Delta x) \\
\Phi'(y) &= \lim_{\Delta y \to 0}\frac{\Delta x}{\Delta y} = \lim_{\Delta x \to 0}\frac{1}{\frac{\Delta y}{\Delta x}} = \frac{1}{f'(x)}
\end{aligned}
$$

$$设y = \frac{1}{2x+1}, 求y^{(n)}.$$

$$
\begin{aligned}
y &= (2x+1)^{-1} \\
y' &= (-1)(2x+1)^{-2} * 2 \\
y'' &= (-1)(-2)(2x+1)^{-3} * 2^2 \\
&\dots \\
y^{(n)} &= (-1)(-2)\dots(-n)(2x+1)^{-(n+1)}2^n \\
&= \frac{(-1)^n n! 2^n}{(2x+1)^{n+1}}
\end{aligned}
$$