

常数项级数

$$\sum_{n=1}^{\infty} a_n$$

部分和

$$S_n = a_1 + a_2 + \dots + a_n$$

$$\lim_{n \rightarrow \infty} S_n = \sum_{n=1}^{\infty} a_n$$

$$\lim_{n \rightarrow \infty} S_n = \begin{cases} = S, \sum_{n=1}^{\infty} a_n = S \\ \text{不存在, } \sum_{n=1}^{\infty} a_n \text{ 发散} \end{cases}$$

判断级数 $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ 的敛散性.

$$\begin{aligned} S_n &= \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \dots + \frac{1}{(2n-1)(2n+1)} \\ &= \frac{1}{2} \left(1 - \frac{1}{2n+1} \right) \\ \therefore \lim_{n \rightarrow \infty} S_n &= \frac{1}{2}, \therefore \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2} \end{aligned}$$

Note: 对 $\sum_{n=1}^{\infty} (b_{n+1} - b_n)$ 敛散性判断一般用定义.

常数项级数的性质

1.

$$\sum_{n=1}^{\infty} a_n = A, \sum_{n=1}^{\infty} b_n = B \Rightarrow \sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = A \pm B$$

2.

$$\sum_{n=1}^{\infty} a_n = S \Rightarrow \sum_{n=1}^{\infty} k a_n = k \sum_{n=1}^{\infty} a_n = kS$$

3. 级数前面添加、减少改变有限项, 敛散性不变 (若收敛, 和会改变)

4. 添加括号不降收敛性

$$-1+1-1+1-1+1-\dots$$

$$S_{2n}=0, S_{2n+1}=-1$$

$$\lim_{n \rightarrow \infty} S_{2n} \neq \lim_{n \rightarrow \infty} S_{2n+1}, \therefore \lim_{n \rightarrow \infty} S_n \text{ 不存在, 即}$$

$$-1+1-1+1-1+1-\dots \text{ 发散}$$

$$\text{但 } (-1+1)+(-1+1)+(-1+1)+\dots=0+0+\dots \text{ 收敛}$$

$$a_1+a_2+\dots$$

$$(a_1+a_2)+a_3+(a_4+a_5+a_6)+\dots$$

$$\text{如: 设 } \sum_{n=1}^{\infty} a_n \text{ 收敛, 问 } \sum_{n=1}^{\infty} (a_{2n-1}+a_{2n})?$$

$$\sum_{n=1}^{\infty} (a_{2n-1}+a_{2n}) = (a_1+a_2) + (a_3+a_4) + (a_5+a_6) + \dots$$

5. 若级数收敛, 则

$$\lim_{n \rightarrow \infty} S_n = S$$

$$a_n = S_n - S_{n-1}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} = S - S = 0$$

$$\text{若 } \sum_{n=1}^{\infty} a_n \text{ 收敛} \Rightarrow \begin{cases} \text{① } \lim_{n \rightarrow \infty} a_n = 0 \\ \text{② } \lim_{n \rightarrow \infty} S_n \exists \end{cases}$$

两个重要的级数

p-级数

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$p=1, \sum_{n=1}^{\infty} \frac{1}{n}, \text{ 调和级数}$$

$$\begin{cases} p > 1, \text{ 收敛} \\ p \leq 1, \text{ 发散} \end{cases}$$

几何级数

$$\sum_{n=1}^{\infty} aq^n (a \neq 0)$$

$$\begin{cases} |q| \geq 1, \text{ 发散} \\ |q| < 1, \text{ 收敛于 } \sum_{n=1}^{\infty} aq^n = \frac{\text{第一项}}{1-q} \end{cases}$$

正项级数及敛散性

$$\sum_{n=1}^{\infty} a_n (a_n \geq 0, n=1, 2, \dots) \text{ 称为正项级数}$$

Notes:

$$\text{① } S_1 \leq S_2 \leq S_3 \leq \dots, \text{ 即 } \{S_n\} \uparrow$$

$$\text{② } \{S_n\} \text{ 无界} \Rightarrow \lim_{n \rightarrow \infty} S_n = +\infty, \text{ 即 } \sum_{n=1}^{\infty} a_n = +\infty$$

$$\text{③ } \exists M > 0, \text{ 使 } S_n \leq M \Rightarrow \lim_{n \rightarrow \infty} S_n \exists \Rightarrow \sum_{n=1}^{\infty} a_n \text{ 收敛}$$

比较审敛法

基本形式

$$a_n \geq 0, b_n \geq 0 (n = 1, 2, \dots)$$

$$a_n \leq b_n \text{ 且 } \sum_{n=1}^{\infty} b_n \text{ 收敛} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ 收敛}$$

$$a_n \geq b_n \text{ 且 } \sum_{n=1}^{\infty} b_n \text{ 发散} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ 发散}$$

判断级数 $\sum_{n=1}^{\infty} \sin \frac{\pi}{2^n}$ 的敛散性.

$$\sin \frac{\pi}{2^n} > 0 (n = 1, 2, \dots)$$

$$\because x \geq 0 \text{ 时, } \sin x \leq x, \therefore 0 < \sin \frac{\pi}{2^n} \leq \frac{\pi}{2^n}$$

$$\because \sum_{n=1}^{\infty} \frac{\pi}{2^n} \text{ 收敛, } \therefore \sum_{n=1}^{\infty} \sin \frac{\pi}{2^n} \text{ 收敛}$$

设 $a_n \leq b_n \leq c_n$ 且级数 $\sum_{n=1}^{\infty} a_n$ 与 $\sum_{n=1}^{\infty} c_n$ 收敛, 证明级数 $\sum_{n=1}^{\infty} b_n$ 收敛.

$$a_n \leq b_n \leq c_n \Rightarrow 0 \leq b_n - a_n \leq c_n - a_n$$

$$\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} c_n \text{ 收敛} \Rightarrow \sum_{n=1}^{\infty} (c_n - a_n) \text{ 收敛}$$

$$\Rightarrow \sum_{n=1}^{\infty} (b_n - a_n) \text{ 收敛}$$

$$\because \sum_{n=1}^{\infty} a_n \text{ 收敛}$$

$$\therefore \sum_{n=1}^{\infty} [(b_n - a_n) + a_n] = \sum_{n=1}^{\infty} b_n \text{ 收敛}$$

极限形式

$$a_n > 0, b_n > 0 (n = 1, 2, \dots)$$

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = l (0 < l < +\infty) \Rightarrow \sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n \text{ 敛散性同}$$

比值法

$$a_n > 0 (n = 1, 2, \dots)$$

$$\textcircled{1} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \text{收敛}$$

$$\textcircled{2} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 1 \Rightarrow \text{发散}$$

$$\textcircled{3} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1 \Rightarrow ?$$

判断级数 $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$ 的敛散性.

$$\begin{aligned}\therefore \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} = 2 \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n = \\ &= 2 \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{2}{e} = \rho < 1 \\ \therefore &\text{收敛}\end{aligned}$$

根值法

$$a_n > 0 (n = 1, 2, \dots)$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \rho$$

① $\rho < 1$, 收敛

② $\rho > 1$, 发散

③ $\rho = 1$, ?

判断级数 $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ 的敛散性.

$$\begin{aligned}\therefore \lim_{n \rightarrow \infty} \sqrt[n]{a_n} &= \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} = \rho < 1 \\ \therefore &\text{收敛}\end{aligned}$$

交错级数及敛散性

$$a_1 - a_2 + a_3 - a_4 + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

$$-a_1 + a_2 - a_3 + a_4 - \dots = \sum_{n=1}^{\infty} (-1)^n a_n$$

$$(a_n > 0, n = 1, 2, \dots)$$

莱布尼茨法

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n (a_n > 0, n = 1, 2, \dots)$$

① $\{a_n\} \downarrow$

② $\lim_{n \rightarrow \infty} a_n = 0$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} a_n \text{ 收敛, } S \leq a_1$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$$

$$a_n = \frac{1}{\sqrt{n}}$$

$$\therefore \left\{ \frac{1}{\sqrt{n}} \right\} \downarrow \text{ 且 } \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0, \therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \text{ 收敛.}$$

$\sum_{n=1}^{\infty} a_n$ 收敛, $\sum_{n=1}^{\infty} a_n^2$ 不一定收敛.

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$, 而 $\sum_{n=1}^{\infty} \frac{1}{n}$ 发散.

$\sum_{n=1}^{\infty} a_n (a_n \geq 0)$ 收敛 $\Rightarrow \sum_{n=1}^{\infty} a_n^2$ 收敛

$\sum_{n=1}^{\infty} a_n$ 收敛 $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$

取 $\epsilon = 1, \exists N > 0$, 当 $n > N$ 时,

$$|a_n - 0| < 1$$

$$\Rightarrow 0 \leq a_n < 1 \Rightarrow 0 \leq a_n^2 \leq a_n$$

$$\therefore \sum_{n=1}^{\infty} a_n \text{ 收敛}, \therefore \sum_{n=1}^{\infty} a_n^2 \text{ 收敛}$$

任意级数

若 $\sum_{n=1}^{\infty} |a_n|$ 收敛, 称 $\sum_{n=1}^{\infty} a_n$ 绝对收敛

若 $\sum_{n=1}^{\infty} a_n$ 收敛, 而 $\sum_{n=1}^{\infty} |a_n|$ 发散, 称 $\sum_{n=1}^{\infty} a_n$ 条件收敛

关系

若 $\sum_{n=1}^{\infty} a_n$ 绝对收敛 $\Rightarrow \sum_{n=1}^{\infty} a_n$ 收敛

证: 已知 $\sum_{n=1}^{\infty} |a_n|$ 收敛

$$a_n = \frac{|a_n| + a_n}{2} - \frac{|a_n| - a_n}{2}$$

$$0 \leq \frac{|a_n| + a_n}{2} \leq |a_n|, 0 \leq \frac{|a_n| - a_n}{2} \leq |a_n|$$

$$\therefore \sum_{n=1}^{\infty} |a_n| \text{ 收敛}, \therefore \sum_{n=1}^{\infty} \frac{|a_n| + a_n}{2}, \therefore \sum_{n=1}^{\infty} \frac{|a_n| - a_n}{2} \text{ 皆收敛}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ 收敛}$$

设 $f(x)$ 二阶连续可导, $f(0) = 1$ 且 $\lim_{x \rightarrow 0} \frac{f'(x)}{x} = 2$, 证明: 级数 $\sum_{n=1}^{\infty} [f(\frac{1}{n}) - 1]$ 绝对收敛.

$$\lim_{x \rightarrow 0} \frac{f'(x)}{x} = 2 \Rightarrow f'(0) = 0, f''(0) = 2$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + o(x^2)$$

$$f\left(\frac{1}{n}\right) - 1 = \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$$

$$\left|f\left(\frac{1}{n}\right) - 1\right| = \left|\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right| \sim \frac{1}{n^2}$$

$$\because \sum_{n=1}^{\infty} \frac{1}{n^2} \text{收敛}, \therefore \sum_{n=1}^{\infty} \left|f\left(\frac{1}{n}\right) - 1\right| \text{收敛}$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[f\left(\frac{1}{n}\right) - 1\right] \text{绝对收敛.}$$

判断 $\sum_{n=1}^{\infty} (-1)^n (1 - \cos \frac{a}{n}) (a > 0)$ 的敛散性. 若收敛, 该级数绝对收敛还是条件收敛?

$$0 \leq |(-1)^n (1 - \cos \frac{a}{n})| = |2 \sin^2 \frac{a}{2n}| \leq 2 \left(\frac{a}{2n}\right)^2 = \frac{a^2}{2} \cdot \frac{1}{n^2}$$

$$\because \sum_{n=1}^{\infty} \frac{a^2}{2} \cdot \frac{1}{n^2} \text{收敛}, \therefore \sum_{n=1}^{\infty} |(-1)^n (1 - \cos \frac{a}{n})| \text{收敛}$$

判断级数 $\sum_{n=1}^{\infty} \sin \sqrt{n^2 + 1} \pi$ 的敛散性, 若收敛, 该级数绝对收敛还是条件收敛?

$$a_n = \sin \sqrt{n^2 + 1} \pi = \sin[n\pi + (\sqrt{n^2 + 1} - n)\pi] = (-1)^n \sin \frac{\pi}{\sqrt{n^2 + 1} + n}$$

$$\because 0 < \frac{\pi}{\sqrt{n^2 + 1} + n} < \frac{\pi}{2}, \therefore \sin \frac{\pi}{\sqrt{n^2 + 1} + n} > 0$$

$$\because \left\{ \sin \frac{\pi}{\sqrt{n^2 + 1} + n} \right\} \downarrow, \therefore \text{原级数收敛}$$

$$|\sin \sqrt{n^2 + 1} \pi| \sim \frac{\pi}{\sqrt{n^2 + 1} + n} \sim \frac{\pi}{2n}, \text{而 } \sum_{n=1}^{\infty} \frac{\pi}{2n} \text{发散}$$

$$\therefore \sum_{n=1}^{\infty} \sin \sqrt{n^2 + 1} \pi \text{发散}, \therefore \text{原级数条件收敛.}$$

$$\sum_{n=1}^{\infty} a_n \text{收敛, 问 } \sum_{n=1}^{\infty} (a_n + a_{n+1})?$$

$$S_n = a_1 + \dots + a_n$$

$$\sum_{n=1}^{\infty} a_n \text{收敛} \Rightarrow \sum_{n=1}^{\infty} a_n = 0 \text{ 且 } \sum_{n=1}^{\infty} S_n \exists$$

$$\text{令 } \lim_{n \rightarrow \infty} S_n = S$$

$$S'_n = (a_1 + a_2) + (a_2 + a_3) + \dots + (a_n + a_{n+1})$$

$$= 2S_n - a_1 + a_{n+1}$$

$$\therefore \lim_{n \rightarrow \infty} S'_n = 2S - a_1$$

$$\therefore \sum_{n=1}^{\infty} (a_n + a_{n+1}) \text{收敛}$$

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 + n + 1}}$$

$\therefore \frac{n}{\sqrt{n^3 + n + 1}} \sim \frac{1}{\sqrt{n}}$ 且 $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ 发散, \therefore 原级数发散

$$\sum_{n=1}^{\infty} \frac{1}{\int_0^n \sqrt[4]{1+x^4} dx}$$

$$\int_0^n \sqrt[4]{1+x^4} dx \geq \int_0^n x dx = \frac{n^2}{2}$$

$$\Rightarrow 0 < \frac{1}{\int_0^n \sqrt[4]{1+x^4} dx} \leq \frac{2}{n^2}$$

而 $\sum_{n=1}^{\infty} \frac{2}{n^2}$ 收敛, \therefore 原级数收敛

设 $a_1 = 2, a_{n+1} = \frac{1}{2}(a_n + \frac{1}{a_n})$, 证明 :

(1) $\lim_{n \rightarrow \infty} a_n$ 存在

(2) $\sum_{n=1}^{\infty} (\frac{a_n}{a_{n+1}} - 1)$ 收敛

(1) $a_n > 0$

$$\therefore a_n + \frac{1}{a_n} \geq 2, \therefore a_{n+1} \geq 1$$

$$a_{n+1} - a_n = \frac{1}{2}(a_n + \frac{1}{a_n}) - a_n = \frac{1 - a_n^2}{2a_n} \leq 0$$

$$\Rightarrow \{a_n\} \downarrow$$

$$\therefore \lim_{n \rightarrow \infty} a_n \exists$$

(2) $\because \{a_n\} \downarrow$ 且 $a_n > 0, \therefore \frac{a_n}{a_{n+1}} - 1 \geq 0$

$$0 \leq \frac{a_n}{a_{n+1}} - 1 = \frac{a_n - a_{n+1}}{a_{n+1}} \leq a_n - a_{n+1}$$

$$\text{对 } \sum_{n=1}^{\infty} (a_n - a_{n+1})$$

$$S_n = (a_1 - a_2) + \dots + (a_n - a_{n+1}) = 2 - a_{n+1}$$

$\therefore \lim_{n \rightarrow \infty} S_n \exists, \therefore \sum_{n=1}^{\infty} (a_n - a_{n+1})$ 收敛

$\therefore \sum_{n=1}^{\infty} (\frac{a_n}{a_{n+1}} - 1)$ 收敛