多元函数微分学

$$M_0(x_0,y_0),\delta>0$$
$$\{(x,y)|\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta\}$$
$$\{(x,y)|0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta\}$$

极限

$$若\forall\epsilon>0,\exists\delta>0,当0<|x-a|<\delta时,$$
$$|f(x)-A|<\epsilon$$
$$\lim_{x\to a}f(x)=A$$
$$\lim_{x\to a}f(x)存在\iff f(a-0),f(a+0)存在且相等$$

$$若\forall\epsilon>0,\exists\delta>0,当0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta时,$$
$$|f(x,y)-A|<\epsilon$$
$$\lim_{x\to x_0,y\to y_0}f(x,y)=A$$

$$\lim_{x\to 0,y\to 2}\frac{\sqrt{1+xy}-\sqrt{1-xy}}{\sin x}$$
$$\begin{aligned}原式&=\lim_{x\to 0,y\to 2}\frac{\sqrt{1+xy}-\sqrt{1-xy}}{xy}y\\&=2\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}\\&=2\lim_{t\to 0}\frac{1}{\sqrt{1+t}+\sqrt{1-t}}*2\\&=2\end{aligned}$$

$$f(x,y)=\begin{cases}\frac{x+y}{|x|+|y|},(x,y)\neq(0,0),\\0,(x,y)=(0,0),\end{cases}讨论\lim_{x\to 0,y\to 0}f(x,y)是否存在$$
$$\lim_{x\to 0,y=0}f(x,y)=\lim_{x\to 0}\frac{x}{|x|}不存在\Rightarrow\lim_{x\to 0,y\to 0}\frac{x+y}{|x|+|y|}不存在$$

连续

$$f(x)在x=a连续\Leftrightarrow f(a-0)=f(a+0)=f(a)$$
$$若\lim_{x\to x_0,y\to y_0}f(x,y)=f(x_0,y_0)$$

偏导数

$$y=f(x)(x\in D),a\in D$$
$$若\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}存在(或\lim_{x\to a}\frac{f(x)-f(a)}{x-a}存在)，称f(x)在x=a可导$$
$$f'(a),\frac{dy}{dx}|_{x=a}$$
$$f'(a)存在\Rightarrow f(x)在x=a连续$$
$$z=f(x,y)((x,y)\in D),(x_0,y_0)\in D$$
$$f(x_0+\Delta x,y_0)-f(x_0,y_0)\triangleq\Delta z_x$$
$$f(x_0,y_0+\Delta y)-f(x_0,y_0)\triangleq\Delta z_y$$
$$f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)\triangleq\Delta z$$
$$若\lim_{\Delta x\to 0}\frac{\Delta z_x}{\Delta x}存在(或\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}存在),称f(x,y)在(x_0,y_0)对x可偏导$$
$$记f_x(x_0,y_0),\frac{\partial z}{\partial x}|_{x_0,y_0}$$

$$z=x^2e^{\sin y}$$
$$\frac{\partial z}{\partial x}=2xe^{\sin y},\frac{\partial z}{\partial y}=x^2e^{\sin y}\cos y$$
$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial z}{\partial x})=2e^{\sin y},\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}(\frac{\partial z}{\partial y})=x^2e^{\sin y}\cos^2 y-x^2e^{\sin y}\sin y$$
$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}(\frac{\partial z}{\partial x})=2xe^{\sin y}\cos y,\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial x}(\frac{\partial z}{\partial y})=2xe^{\sin y}\cos y$$

连续的性质

$$f(x)\text{在有界闭区域上，}f(x)\in C[a,b]$$

1. $\exists m,M$
2. $\exists k>0,|f(x)|\le k$
3. 若$f(a)f(b)<0\Rightarrow\exists c\in(a,b)$,使$f(c)=0$
4. $\forall\eta\in[m,M],\exists\xi\in[a,b]$,使$f(\xi)=\eta$

$$D-xoy\text{面上有界闭区域},f(x,y)\in C(D)$$

1. $\exists m,M$
2. $\exists k>0$,使$|f(x,y)|\le k$
3. $\forall\delta\in[m,M],\exists(\xi,\eta)\in D$,使$f(\xi,\eta)=\delta$
4. 若$z=f(x,y)$二阶连续可偏导 $\Rightarrow\dfrac{\partial^2 z}{\partial y\partial x}=\dfrac{\partial^2 z}{\partial x\partial y}$

全微分

$$y=f(x)(x\in D),a\in D,\Delta y=f(a+\Delta x)-f(a)(\text{或}\Delta y=f(x)-f(a))$$
$$\text{若}\Delta y=A\Delta x+o(\Delta x),\text{称}y=f(x)\text{在}x=a\text{可微}$$
$$A\Delta x\triangleq\frac{dy}{dx}|_{x=a},A\Delta x=Adx$$

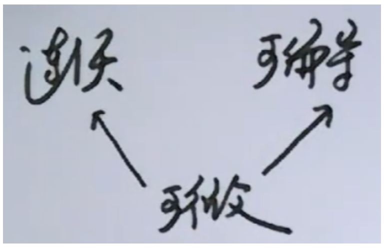
1. $f(x)$在$x=a$可导 $\Leftrightarrow f(x)$在$x=a$可微
2. 若$\Delta y=A\Delta x+o(\Delta x)$,则$A=f'(a)$
3. 设$y=f(x)$可导,$dy=df(x)=f'(x)dx$

$$z=f(x,y)((x,y)\in D),(x_0,y_0)\in D$$
$$\Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=f(x,y)-f(x_0,y_0)$$
$$\text{令}\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$$
$$\text{若}\Delta z=A\Delta x+B\Delta y+o(\rho),\text{称}z=f(x,y)\text{在}(x_0,y_0)\text{处可全微(可微)}$$
$$A\Delta x+B\Delta y\triangleq dz|_{M_0}=Adx+Bdy$$

1. 若$\Delta z=A\Delta x+B\Delta y+o(\rho)\Rightarrow A=\dfrac{\partial z}{\partial x}|_{M_0},B=\dfrac{\partial z}{\partial y}|_{M_0}$
2. 若$z=f(x,y)$可微,则$dz=\dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy$

$$z=x^2\ln(1+\tan y)$$
$$\frac{\partial z}{\partial x}=2x\ln(1+\tan y),\frac{\partial z}{\partial y}=\frac{x^2\sec^2 y}{1+\tan y}$$
$$dz=2x\ln(1+\tan y)dx+\frac{x^2\sec^2 y}{1+\tan y}dy$$

连续 可偏导 可微 关系



可微 $\Rightarrow$ 连续

$$\Delta z = f(x,y) - f(x_0,y_0) = A(x-x_0) + B(y-y_0) + o(\sqrt{(x-x_0)^2+(y-y_0)^2})$$
$$\Rightarrow \lim_{x\to x_0, y\to y_0} \Delta z = 0 \Rightarrow \lim_{x\to x_0, y\to y_0} f(x,y) = f(x_0,y_0)$$

可微 $\Rightarrow$ 可偏导

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) = A\Delta x + B\Delta y + o(\rho)$$
$$\text{取}\Delta y = 0 : \Delta z_x = A\Delta x + o(\Delta x)$$
$$\Rightarrow \frac{\Delta z_x}{\Delta x} = A + \frac{o(\Delta x)}{\Delta x} \Rightarrow \lim_{\Delta x\to 0}\frac{\Delta z_x}{\Delta x} = A, \text{且}\frac{\partial z}{\partial x}\mid_{M_0} = A$$
$$\text{同理}\frac{\partial z}{\partial x}\mid_{M_0} = B$$

连续 $\not\Rightarrow$ 可偏导

$$z = f(x,y) = |x| + |y|$$
$$\lim_{x\to 0, y\to 0} f(x,y) = 0 = f(0,0)$$
$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0}\frac{|x|}{x}\text{不存在}$$
$$\Rightarrow f(x,y)\text{在}(0,0)\text{对}x\text{不可偏导}$$

可偏导 $\not\Rightarrow$ 连续

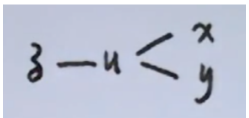
$$z = f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0) \end{cases}$$
$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0}\frac{0}{x^3} = 0 \Rightarrow f_x(0,0) = 0$$
$$\text{同理}f_y(0,0) = 0$$
$$\lim_{x\to 0, y=x} f(x,y) = \frac{1}{2} \neq \lim_{x\to 0, y=-x} f(x,y) = -\frac{1}{2}$$
$$\Rightarrow \lim_{x\to 0, y\to 0} f(x,y)\text{不存在}, \text{而}f(0,0) = 0$$
$$\therefore f(x,y)\text{在}(0,0)\text{不连续}$$

显函数求偏导

$$z = \arctan\frac{x+y}{1-xy}$$
$$\frac{\partial z}{\partial x} = \frac{1}{1+(\frac{x+y}{1-xy})^2}\frac{(1-xy)-(x+y)(-y)}{(1-xy)^2}$$

复合函数求偏导

$$1.\ z = f(x^2+y^2) : z = f(u), u = x^2+y^2$$
$$2.\ z = f(t^2, \sin t) : z = f(u,v), \begin{cases} u = t^2 \\ v = \sin t \end{cases}$$
$$3.\ z = f(x^2+y^2, xy) : z = f(u,v), \begin{cases} u = x^2+y^2 \\ v = xy \end{cases}$$



$$z=f(x^2\sin y),\text{求}\frac{\partial^2 z}{\partial x\partial y}$$

$$\frac{\partial z}{\partial x}=f'(x^2\sin y)2x\sin y$$

$$\frac{\partial^2 z}{\partial x\partial y}=f''(x^2\sin y)x^2\cos y2x\sin y+f'(x^2\sin y)2x\cos y$$

$$z=f(t^2,\sin t),\text{求}\frac{dz}{dt}$$

$$\frac{dz}{dt}=2tf_1+\cos tf_2$$

$$\frac{d^2z}{dt^2}=2f_1+2t(2tf_{11}+\cos tf_{12})-\sin tf_2+\cos t(2tf_{21}+\cos tf_{22})=2f_1-\sin tf_2+4t^2f_{11}+4t\cos tf_{12}+\cos^2 tf_{22}$$

$$z=f(x+y,xy),\text{求}\frac{\partial^2 z}{\partial x\partial y}$$

$$\frac{\partial z}{\partial x}=f_1+yf_2$$

$$\frac{\partial^2 z}{\partial x\partial y}=f_{11}+xf_{12}+f_2+y(f_{21}+xf_{22})=f_{11}+(x+y)f_{12}+f_2+xyf_{22}$$

隐函数(组)求偏导

$$\begin{aligned}&1.\ F(x,y)=0\Rightarrow y=\Phi(x)\\&2.\ F(x,y,z)=0\Rightarrow z=\Phi(x,y)\\&3.\ \begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases}\Rightarrow\begin{cases}y=y(x)\\z=z(x)\end{cases}\end{aligned}$$

$$\tan(x+y+z)=x^2+y^2+z,z=z(x,y),\text{求}\frac{\partial z}{\partial x}$$

$$\sec^2(x+y+z)(1+\frac{\partial z}{\partial x})=2x+\frac{\partial z}{\partial x}\Rightarrow\frac{\partial z}{\partial x}=\frac{2x-\sec^2(x+y+z)}{\tan^2(x+y+z)}$$

$$\begin{cases}x-y+2z=1\\x^2+y^2+4z^2=4\end{cases},\text{求}\frac{dz}{dx}$$

$$\begin{cases}x-y+2z=1\\x^2+y^2+4z^2=4\end{cases}\Rightarrow\begin{cases}y=y(x)\\z=z(x)\end{cases}$$

$$\begin{cases}1-\frac{dy}{dx}+2\frac{dz}{dx}=0\\2x+2y\frac{dy}{dx}+8z\frac{dz}{dx}=0\end{cases}$$

$$\begin{cases}xu+yv=1\\xv-y^2u=e^{x+y}\end{cases},u=u(x,y),v=v(x,y),\text{求}\frac{\partial u}{\partial x},\frac{\partial v}{\partial y}$$

$$\begin{cases}xu+yv=1\\xv-y^2u=e^{x+y}\end{cases}\Rightarrow\begin{cases}u=u(x,y)\\v=v(x,y)\end{cases}$$

$$\begin{cases}u+x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial x}=0\\v+x\frac{\partial v}{\partial x}-y^2\frac{\partial u}{\partial x}=e^{x+y}\end{cases}\Rightarrow\begin{cases}\frac{\partial u}{\partial x}\\\frac{\partial v}{\partial x}\end{cases}$$

$$\begin{cases}x\frac{\partial u}{\partial y}+v+y\frac{\partial v}{\partial y}=0\\x\frac{\partial v}{\partial y}-2yu-y^2\frac{\partial u}{\partial y}=e^{x+y}\end{cases}\Rightarrow\begin{cases}\frac{\partial u}{\partial y}\\\frac{\partial v}{\partial y}\end{cases}$$

多元函数极值

$$y=f(x)$$

$$\begin{aligned}&1.\ x\in D\\&2.\ f'(x)\begin{cases}=0\\ \text{不存在}\end{cases}\\&3.\ \text{判别法}\\&Th1.\begin{cases}x<x_0:f'<0\\x>x_0:f'>0\end{cases},\text{极小值}\\&\qquad\ \begin{cases}x<x_0:f'>0\\x>x_0:f'<0\end{cases},\text{极大值}\\&Th2.f'(x_0)=0,f''(x_0)\begin{cases}<0,\text{极大值}\\>0,\text{极小值}\end{cases}\end{aligned}$$

$$z=f(x,y)((x,y)\in D),(x_0,y_0)\in D$$

$$\text{若}\exists\delta>0,\text{当}0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta\text{时}$$

$$f(x,y)<f(x_0,y_0)$$

$$(x_0,y_0)\text{为极大点},f(x_0,y_0)\text{为极大值}$$

无条件极值

$$z=f(x,y),(x,y)\in D(\text{开区域})$$

$$\begin{aligned}&1.\ \begin{cases}\frac{\partial z}{\partial x}=0\\\frac{\partial z}{\partial y}=0\end{cases}\Rightarrow\begin{cases}x\\y\end{cases}\\&2.\ \text{设}(x,y)=(x_0,y_0)\\&\quad A=\frac{\partial^2 z}{\partial x^2}\mid_{(x_0,y_0)},B=\frac{\partial^2 z}{\partial x\partial y}\mid_{(x_0,y_0)},C=\frac{\partial^2 z}{\partial y^2}\mid_{(x_0,y_0)}\\&3.\ AC-B^2\begin{cases}<0,\times\\>0,\surd\begin{cases}A>0,\text{极小值}\\A<0,\text{极大值}\end{cases}\end{cases}\end{aligned}$$

$$\text{求}z=f(x,y)=x^3-3x+y^2+2y+2\text{的极值点和极值}$$

$$\begin{aligned}&1.\ \begin{cases}\frac{\partial z}{\partial x}=3x^2-3=0\\\frac{\partial z}{\partial y}=2y+2=0\end{cases}\Rightarrow\begin{cases}x=-1\\y=-1\end{cases},\begin{cases}x=1\\y=-1\end{cases}\\&2.\ \text{设}(x,y)=(x_0,y_0)\\&\quad A=\frac{\partial^2 z}{\partial x^2}\mid_{(x_0,y_0)}=6x_0,B=\frac{\partial^2 z}{\partial x\partial y}\mid_{(x_0,y_0)}=0,C=\frac{\partial^2 z}{\partial y^2}\mid_{(x_0,y_0)}=2\\&3.\ (x_0,y_0)=(-1,-1),\\&\quad AC-B^2<0\Rightarrow(-1,-1)\text{不是极值点}\\&\quad (x_0,y_0)=(1,-1),\\&\quad AC-B^2>0\text{且}A>0\Rightarrow(1,-1)\text{是极小点}\end{aligned}$$

条件极值

$$\text{如}z=f(x,y),s.t.\Phi(x,y)=0$$

$$\begin{aligned}&1.\ F=f(x,y)+\lambda\Phi(x,y)\\&2.\ \begin{cases}F_x=f_x+\lambda\Phi_x=0\\F_y=f_y+\lambda\Phi_y=0\\F_\lambda=\Phi(x,y)\end{cases}\Rightarrow\begin{cases}x\\y\end{cases}\end{aligned}$$