二重积分

定义

定积分,f(x)在[a,b]上有界

1.
$$a = x_0 < x_1 < \ldots < x_n = b$$

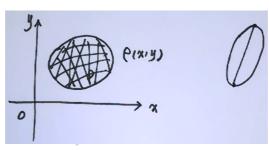
$$2.\ orall \xi_i \in [x_{i-1},x_i], \sum_{i=1}^n f(\xi_i) \Delta x_i$$

$$3.\ \lambda = \max\left\{\Delta x_1,\ldots,\Delta x_n
ight\}$$

若 $\lim_{\lambda o 0} \sum_{i=1}^n f(\xi_i) \Delta x_i$ 习,称f(x)在[a,b]上可积

极限值称为f(x)在[a,b]上可定积分,记作 $\int_a^b f(x)dx$

二重积分



质量m?

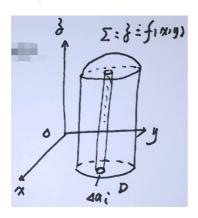
$$1.D\Rightarrow \Delta\sigma_1,\ldots,\Delta\sigma_n$$

$$2.orall (\xi_i,\eta_i)\in \Delta\sigma_i$$

$$mpprox \sum_{i=1}^n
ho(\xi_i,\eta_i)\Delta\sigma_i$$

 $3.\lambda$ 为 $\Delta\sigma_1,\ldots,\Delta\sigma_n$ 直径最大者

$$m = \lim_{\lambda o 0} \sum_{i=1}^n
ho(\xi_i, \eta_i) \Delta \sigma_i$$



$$egin{aligned} \sum_{ ext{}
otin T} : z = f(x,y) &\geq 0, (x,y) \in D \ \text{ \perp} &\downarrow D \Rightarrow \Delta \sigma_1, \dots, \Delta \sigma_n \ 2. &\forall (\xi_i, \eta_i) \in \sigma_i \ V &pprox \sum_{i=1}^n f(\xi_i, \eta_i) \Delta \sigma_i \ 3. &\lambda eta \Delta \sigma_1, \dots, \Delta \sigma_n$$
直径最大者 $V = \lim_{\lambda o 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta \sigma_i \end{aligned}$

设D为xoy面有界闭区域,f(x,y)在D上有界

$$1. D$$
分成 $\Delta \sigma_1, \ldots, \Delta \sigma_n$

$$1.$$
 D) ($\Delta \sigma_1, \ldots, \Delta \sigma_n$ $2.$ $orall (\xi_i, \eta_i) \in \Delta \sigma_i (1 \leq i \leq n), ext{ if } \sum_{i=1}^n f(\xi_i, \eta_i) \Delta \sigma_i$

$$3. \lambda$$
为 $\Delta \sigma_1, \ldots, \Delta \sigma_n$ 直径最大者

若
$$\lim_{\lambda \to 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta \sigma_i$$
习,极限值称 $f(x,y)$ 在 D 上的二重积分,记作 $\iint_D f(x,y) d\sigma$

即
$$\iint_D f(x,y) d\sigma riangleq \lim_{\lambda o 0} \sum_{i=1}^n f(\xi_i,\eta_i) \Delta \sigma_i$$

定义题型

$$D = \{(x, y) | 0 \le x \le 1, 0 \le y \le 1\}$$

$$1.\lim_{m o\infty,n o\infty}rac{1}{mn}\sum_{i=1}^m\sum_{j=1}^nf(rac{i}{m},rac{i}{n})=\iint_Df(x,y)d\sigma$$

$$2.\lim_{n o\infty}rac{1}{n^2}\sum_{i=1}^n\sum_{i=1}^nf(rac{i}{n},rac{i}{n})=\iint_Df(x,y)d\sigma$$

 $egin{aligned} &\lim_{n o\infty}\sum_{i=1}^n\sum_{j=1}^nrac{j}{(n+i)(n^2+j^2)}\ &=\lim_{n o\infty}rac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^nrac{rac{j}{n}}{(1+rac{i}{n})[1+(rac{j}{n})^2]}\ &=\int_0^1rac{1}{1+x}dx\int_0^1rac{y}{1+y^2}dy\ &=rac{1}{2}\ln2\ln(1+y^2)\mid_0^1=rac{1}{2}\ln^22 \end{aligned}$

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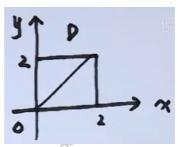
$$1.\ D = D_1 + D_2, D_1 \cap D_2 = \emptyset, \iint_D = \iint_{D_1} + \iint_{D_2}$$

$$2.\, \iint_D 1 d\sigma = A$$

3. D关于y轴对称,右 D_1

$$\begin{cases} f(-x,y) = -f(x,y) \Rightarrow \iint_D f(x,y) d\sigma = 0 \ f(-x,y) = f(x,y) \Rightarrow \iint_D f(x,y) d\sigma = 2 \iint_{D_1} f(x,y) d\sigma \ D$$
关于 $y = x$ 对称,则

$$\iint_D f(x,y)d\sigma = \iint_D f(y,x)d\sigma$$



f(u)>0, orall a>0, b>0

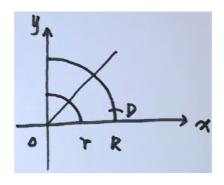
$$I = \iint_D rac{af(x) + bf(y)}{f(x) + f(y)} d\sigma = \iint_D rac{af(y) + bf(x)}{f(y) + f(x)} d\sigma$$

$$2I=(a+b)\iint_{D}d\sigma=4(a+b),I=2(a+b)$$

$$4.\iint_{D}1d\sigma=A$$
 $5.① D 关于 u 轴对$

$$egin{aligned} 4. &\iint_D 1 d\sigma = A \ 5. ① D$$
关于 y 轴对称(左右对称), 右 $D_1 \ &\exists f(-x,y) = -f(x,y) \Rightarrow \iint_D f(x,y) d\sigma = 0 \ &\exists f(-x,y) = f(x,y) \Rightarrow \iint_D f(x,y) d\sigma = 2 \iint_{D_1} f(x,y) d\sigma \end{aligned}$

$$\iint_D f(x,y)d\sigma = \iint_D f(y,x)d\sigma$$



$$egin{aligned} f(u) > 0$$
连续, $a > 0, b > 0 \ I = \iint_D rac{a\sqrt{f(x)} + b\sqrt{f(y)}}{\sqrt{f(x)} + \sqrt{f(y)}} d\sigma \ I = \iint_D rac{a\sqrt{f(y)} + b\sqrt{f(x)}}{\sqrt{f(x)} + \sqrt{f(y)}} d\sigma \ 2I = (a+b) \iint_D 1 d\sigma = rac{\pi}{4}(a+b)(R^2 - r^2) \ I = rac{\pi}{8}(a+b)(R^2 - r^2) \end{aligned}$

-1 Pi y=x3

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$$egin{aligned} I &= \iint_D \sin x \cos y d\sigma \ &= \iint_{D_0 + D_1} \sin x \cos y d\sigma + \iint_{D_2 + D_3} \sin x \cos y d\sigma \ &= 2 \iint_{D_2} \sin x \cos y d\sigma \end{aligned}$$

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$$=2\iint_{D_2}\sin x\cos yd\sigma$$
 $=2\iint_{D_2}\sin x\cos yd\sigma$ $6.D-$ 有界闭区域 $,f(x,y)\in C(D),$ 则 $\exists (\xi,\eta)\in D,$ 使 $\iint_D f(x,y)d\sigma=f(\xi,\eta)A$

设区域
$$D=\{(x,y)|x^2+4y^2\leq t^2, t>0\},$$
求 $\lim_{t o 0^+}rac{\iint_D e^{-x^2}\cos 2y dx dy}{t^2}.$

$$egin{align} D: rac{x^2}{t^2} + rac{y^2}{(rac{t}{2})^2} & \leq 1 \ & \iint_D e^{-x^2} \cos 2y dx dy = e^{-\xi^2} \cdot \cos 2\eta \cdot \pi \cdot t \cdot rac{t}{2}, (\xi, \eta) \in D \ &
otag
Arr = rac{\pi}{2} \lim_{t o 0} e^{-\xi^2} \cdot \cos 2\eta = rac{\pi}{2} \end{split}$$

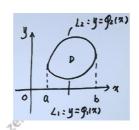
积分法

直角坐标法

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$$\iint_D f(x,y) d\sigma$$

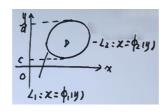
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$$D=(x,y)|a\leq x\leq b, \Phi_1(x)\leq y\leq \Phi_2(x)$$

$$\iint_D f(x,y) d\sigma = \int_a^b dx \int_{\Phi_1(x)}^{\Phi_2(x)} f(x,y) dy$$

y型区域



$$D = \{(x, y) | c \le y \le d, \phi_1(y) \le x \le \phi_2(y) \}$$

$$\iint_D f(x,y) d\sigma = \int_c^d dy \int_{\phi_1(y)}^{\phi_2(y)} f(x,y) dx$$

计算 $\iint_D x^2 y dx dy$,其中D由y=x, x=1及x轴所围成.

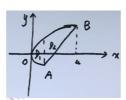


$$D = \{(x, y) | 0 \le x \le 1, 0 \le y \le x\}$$

$$\iint_D x^2 y d\sigma = \int_0^1 x^2 dx \int_0^x y dy = rac{1}{2} \int_0^1 x^4 dx = rac{1}{10}$$
 $D = \{(x,y)|0 \leq y \leq 1, y \leq x \leq 1\}$

$$\iint_D x^2 y d\sigma = \int_0^1 y dy \int_y^1 x^2 dx = rac{1}{3} \int_0^1 (y - y^4) dy = rac{1}{10}$$

计算
$$I=\iint_D x dx dy$$
,其中 D 由 $x=y^2$ 与 $y=x-2$ 围成

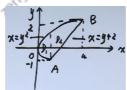


$$\oplus \begin{cases} x = y^2 \\ y = x - 2 \end{cases} \Rightarrow A(1, -1), B(4, 2)$$

$$D_1=\{(x,y)|0\leq x\leq 1, -\sqrt{x}\leq y\leq \sqrt{x}\}$$

$$D_2=\{(x,y)|1\leq x\leq 4, x-2\leq y\leq \sqrt{x}\}$$

原式
$$=\int_0^1 x dx \int_{-\sqrt{x}}^{\sqrt{x}} 1 dy + \int_1^4 x dx \int_{x-2}^{\sqrt{x}} 1 dy$$
 $=2\int_0^1 x^{\frac{3}{2}} dx + \int_1^4 (x^{\frac{3}{2}} + x^2 - 2x) dx$



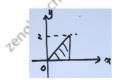
$$D=\{(x,y)|y^2\leq x\leq y+2,-1\leq y\leq 2\}$$
原式 $=\int_{-1}^2 dy\int_{y^2}^{y+2}xdx$

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$$x^{2n}e^{\pm x^2}dx$$
 $e^{rac{k}{x}}dx$ $\cosrac{k}{x}dx,\sinrac{k}{x}dx$

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计算
$$I=\int_0^2 dy \int_y^2 e^{-x^2} dx$$



$$I = \int_0^2 dy \int_y^2 e^{-x^2} dx$$

$$= \int_0^2 x e^{-x^2} dx$$

$$= -\frac{1}{2} e^{-x^2} \mid_0^2$$

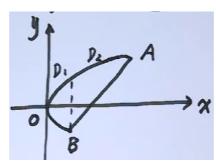
$$= -\frac{1}{2} (\frac{1}{e^4} - 1)$$

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计算
$$\iint_D (x+y) dx dy$$
, 其中 D 由 $x=y^2$ 与 $y=x-2$ 围成.

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NO.

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$$\exists \begin{cases} x = y^2 \\ y = x - 2 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -1 \end{cases}, \begin{cases} x = 4 \\ y = 2 \end{cases}$$
法一: $D_1 = \{(x, y) | 0 \le x \le 1, -\sqrt{x} \le y \le \sqrt{x} \}$

$$D_2 = \{(x, y), 1 \le x \le 4, x - 2 \le y \le \sqrt{x} \}$$
原式 $= \int_0^1 dx \int_{-\sqrt{x}}^{\sqrt{x}} (x + y) dy + \int_1^4 dx \int_{x-2}^{\sqrt{x}} (x + y) dy$

$$= 2 \int_0^1 x \sqrt{x} dx + \int_1^4 dx \int_{x-1}^{\sqrt{x}} (x + y) dy$$

法二:原式 $=\int_{-1}^2 dy \int_{y^2}^{y+2} (x+y) dx$ $=2\int_0^1 x \sqrt{x} dx + \int_1^4 dx \int_{x-1}^{\sqrt{x}} (x+y) dy$

计算二重积分时,

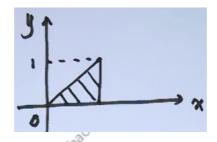
$$1.x^{2n}e^{\pm x^2}dx$$

$$2.e^{\frac{k}{x}}dx$$

$$3.\cos\frac{k}{x}dx$$

$$4.\sin\frac{k}{x}dx$$

计算 $\int_0^1 dy \int_y^1 e^{x^2} dx.$



原式
$$=\int_0^1 e^{x^2} dx \int_0^x dy$$
 $=\int_0^1 x e^{x^2} dx = \frac{1}{2} e^{x^2} \mid_0^1 = \frac{e-1}{2}$

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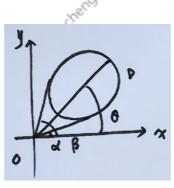
极坐标法

特征

1.
$$D$$
边界曲线含 $x^2 + y^2$
2. $f(x,y)$ 中含 $x^2 + y^2$

变换

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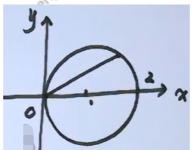


$$egin{cases} x = r\cos{ heta} \ y = r\sin{ heta} \end{cases}$$
 $lpha \leq heta \leq eta, r_1(heta) \leq r \leq r_2(heta)$

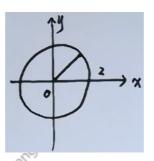
面积

$$d\sigma = r dr d heta \ \iint_D f(x,y) d\sigma = \int_lpha^eta d heta \int_{r_1(heta)}^{r_2(heta)} r f(r\cos heta,r\sin heta) dr$$

计算 $I=\iint_D (x^2+xy)d\sigma$, 其中 $D:x^2+y^2\leq 2x$.



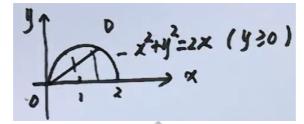
 $D: (x-1)^2 + y^2 \le 1$ $I = \iint_D x^2 d\sigma$ $\Rightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} (-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, 0 \le r \le 2\cos\theta)$ 原式 $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta \int_0^{2\cos\theta} r^3 dr$ $= 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6\theta d\theta = 8I_6$ $= 8 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5\pi}{4}$



计算
$$I=\iint_D (x^2+3xy)d\sigma$$
, 其中圆域 $x^2+y^2\leq 4$

计算
$$I=\iint_D (x^2+3xy)d\sigma$$
, 其中圆域 $x^2+y^2\leq 4$
$$I=\iint_D (x^2+3xy)d\sigma=\iint_D x^2d\sigma=\iint_D y^2d\sigma=\frac{1}{2}\iint_D (x^2+y^2)d\sigma$$
 令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases} (0\leq \theta\leq 2\pi, 0\leq r\leq 2)$

$$I=rac{1}{2}\int_0^{2\pi}d heta\int_0^2r^3dr=4\pi$$



$$I=\iint_D x^2 d\sigma, D$$
由 $y=\sqrt{2x-x^2}$ 与 x 轴围成

$$\Leftrightarrow egin{cases} x = r\cos heta \ y = r\sin heta \end{cases} (0 \le heta \le rac{\pi}{2}, 0 \le r \le 2\cos heta)$$

$$I = \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r^3 \cos^2\theta dr$$

$$= \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta \int_0^{2\cos\theta} r^3 dr$$

$$= 4 \int_0^{\frac{\pi}{2}} \cos^6\theta$$

$$= 4 * \frac{5}{6} * \frac{3}{4} * \frac{1}{2} * \frac{\pi}{2}$$