极限

epsilon-N 数列极限

$$\begin{gathered}
\text{若}(\exists A),\text{对}\forall\epsilon>0,\exists N>0,\text{当}n>N\text{时},\\
|a_n-A|<\epsilon\\
\lim_{n\to\infty}a_n=A\text{或}a_n\to A(n\to\infty)
\end{gathered}$$

$$a_n=\frac{n+1}{2n},\lim_{n\to\infty}a_n=\frac{1}{2},\frac{n+1}{2n}\neq\frac{1}{2}$$

$$\text{证明：若}\lim_{n\to\infty}a_n=A,\text{则}\lim_{n\to\infty}|a_n|=|A|,\text{反之不对}.$$

$$\begin{gathered}
\Rightarrow\\
\because\lim_{n\to\infty}a_n=A\\
\therefore\forall\epsilon>0,\exists N>0,\text{当}n>N\text{时},\\
|a_n-A|<\epsilon\\
\because||a_n|-|A||\leq|a_n-A|\\
\therefore\forall\epsilon>0,\exists N>0,\text{当}n>N\text{时}\\
||a_n|-|A||<\epsilon\\
\therefore\lim_{n\to\infty}|a_n|=|A|
\end{gathered}$$

$$\begin{gathered}
\nLeftarrow\\
a_n=(-1)^n,\lim_{n\to\infty}a_n\text{不存在},\text{但}\lim_{n\to\infty}|a_n|=1
\end{gathered}$$

epsilon-delta

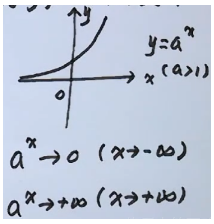
$$\begin{gathered}
f(x)\text{在}x=a\text{的去心邻域内有定义}\\
\text{若}\forall\epsilon>0,\exists\delta>0,\text{当}0<|x-a|<\delta,\text{有}\\
|f(x)-A|<\epsilon\\
\lim_{x\to a}f(x)=A\text{或}f(x)\to A(x\to a)
\end{gathered}$$

$$\begin{aligned}
&1.\quad x\to a:\begin{cases}x\neq a\\x\to a^-,x\to a^+\end{cases}\\
&2.\quad\lim_{x\to a}f(x)\text{与}f(a)\text{无关}
\end{aligned}$$

$$\begin{gathered}
f(x)=\frac{x^2+x-2}{x^2-1}(x\neq\pm1)\\
\lim_{x\to1}f(x)=\lim_{x\to1}\frac{(x-1)(x+2)}{(x-1)(x+1)}=\lim_{x\to1}\frac{x+2}{x+1}=\frac{3}{2},\text{而}f(1)\text{不存在}
\end{gathered}$$

$$
\begin{aligned}
&3.\ 若\forall\epsilon>0,\exists\delta_1>0,当x\in(a-\delta_1,a)时\\
&\quad|f(x)-A|<\epsilon\\
&\quad\lim_{x\to a^-}f(x)=A或f(a-0)=A\\
&\quad若\forall\epsilon>0,\exists\delta_2>0,当x\in(a,a+\delta_2)时\\
&\quad|f(x)-B|<\epsilon\\
&\quad\lim_{x\to a^+}f(x)=B或f(a+0)=B\\
&\quad\lim_{x\to a}f(x)\exists\Leftrightarrow f(a-0),f(a+0)\exists且等
\end{aligned}
$$

$$
f(x)含\begin{cases}a^{\frac{?}{x-b}}\\a^{\frac{?}{b-x}}\end{cases}当x\to b时分左右极限
$$



$$
设f(x)=\frac{1-2^{\frac{1}{x-1}}}{1+2^{\frac{1}{x-1}}},判断\lim_{x\to1}f(x)是否存在.
$$

$$
x\to1^-\Rightarrow\frac{1}{x-1}\to-\infty\Rightarrow2^{\frac{1}{x-1}}\to0
$$

$$
f(1-0)=1
$$

$$
x\to1^+\Rightarrow\frac{1}{x-1}\to+\infty\Rightarrow2^{\frac{1}{x-1}}\to+\infty
$$

$$
f(1+0)=-1
$$

$$
\because f(1-0)\neq f(1+0),\therefore f(x)不存在
$$

epsilon-x

case1

$$
若\forall\epsilon>0,\exists X>0,当x>X时,
$$

$$
|f(x)-A|<\epsilon
$$

$$
\lim_{x\to+\infty}f(x)=A
$$

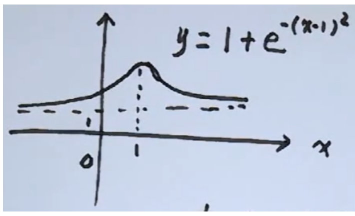
case2

$$
若\forall\epsilon>0,\exists X>0,当x<-X时,
$$

$$
|f(x)-A|<\epsilon
$$

$$
\lim_{x\to-\infty}f(x)=A
$$

case3



$$若\forall\epsilon>0,\exists X>0,当|x|>X时,$$
$$|f(x)-A|<\epsilon$$
$$\lim_{x\to\infty}f(x)=A$$

一般性质

唯一性

$$极限存在必唯一$$
$$证:设\lim_{x\to a}f(x)=A,\lim_{x\to a}f(x)=B$$
$$若A>B:取\epsilon=\frac{A-B}{2}>0,$$
$$\because\lim_{x\to a}f(x)=A,\therefore\exists\delta_1>0,当0<|x-a|<\delta_1时,$$
$$|f(x)-A|<\frac{A-B}{2}$$
$$即\frac{A+B}{2}<f(x)<\frac{3A-B}{2}(*)$$
$$\because\lim_{x\to a}f(x)=B,\therefore\exists\delta_2>0,当0<|x-a|<\delta_2时,$$
$$|f(x)-B|<\frac{A-B}{2}$$
$$即\frac{3B-A}{2}<f(x)<\frac{A+B}{2}(**)$$
$$取\delta=\min\{\delta_1,\delta_2\},当0<|x-a|<\delta时,(*)(**)皆对矛盾$$
$$\therefore A>B不对$$
$$若A<B:同理不对$$
$$\therefore A=B$$

有界性

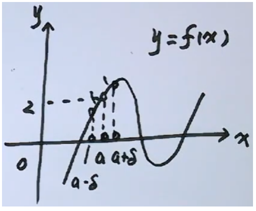
$$若\lim_{n\to\infty}a_n=A\Rightarrow\exists M>0,\forall n,|a_n|\le M$$
$$\not\Leftarrow$$
$$证:\Rightarrow 取\epsilon=1$$
$$\because\lim_{n\to\infty}a_n=A,\therefore\lim_{n\to\infty}|a_n|=|A|$$
$$\therefore\exists N>0,当n>N时,有|a_n-A|<1$$
$$||a_n|-|A||<1\Rightarrow|a_n|<1+|A|$$
$$取M=\max\{|a_1|,|a_2|,\ldots,|a_N|,1+|A|\}$$
$$\therefore\forall n,|a_n|\le M$$
$$\not\Leftarrow a_n=2+(-1)^n,|a_n|\le3,但\lim_{n\to\infty}a_n不存在$$

保号性

$$若\lim_{x\to a} f(x) = A \begin{cases} > 0 \\ < 0 \end{cases}, 则$$

$$\exists \delta > 0, 当0 < |x-a| < \delta时，$$

$$f(x) \begin{cases} > 0 \\ < 0 \end{cases}$$



$$\lim_{x\to a} f(x) = 2 > 0$$

$$当0 < |x-a| < \delta时$$

$$f(x) > 0$$

$$A > 0$$

$$取\epsilon = \frac{A}{2}$$

$$\because \lim_{x\to a} f(x) = A$$

$$\therefore \exists \delta > 0, 当0 < |x-a| < \delta时$$

$$|f(x) - A| < \frac{A}{2}$$

$$\Rightarrow f(x) > \frac{A}{2} > 0$$

$$A < 0$$

$$取\epsilon = -\frac{A}{2}$$

$$\because \lim_{x\to a} f(x) = A$$

$$\therefore \exists \delta > 0, 当0 < |x-a| < \delta时$$

$$|f(x) - A| < -\frac{A}{2}$$

$$\Rightarrow f(x) < \frac{A}{2} < 0$$

$$设\lim_{x\to 1} \frac{f(x)-3}{\ln^2 x} = -2且f(1) = 3, 讨论x = 1是否为函数f(x)的极值点.$$

$$\because \lim_{x\to 1} \frac{f(x)-3}{\ln^2 x} = -2 < 0$$

$$\therefore \exists \delta > 0, 当0 < |x-1| < \delta时, \frac{f(x)-3}{\ln^2 x} < 0$$

$$\because \ln^2 x > 0, \therefore f(x) - 3 < 0$$

$$即当0 < |x-1| < \delta时, f(x) < f(1)$$

$$\therefore x = 1为f(x)的极大点$$

$$设f'(0)=0,又\lim_{x\to 0}\frac{f'(x)}{x^3}=-2,讨论x=0是否为f(x)的极值点.$$

$$\because \lim_{x\to 0}\frac{f'(x)}{x^3}=-2<0,$$

$$\therefore \exists\delta>0,当0<|x|<\delta时,\frac{f'(x)}{x^3}<0$$

$$\begin{cases} f'(x)>0, x\in(-\delta,0) \\ f'(x)<0, x\in(0,\delta) \end{cases} \Rightarrow x=0为f(x)的极大点$$

$$f(1)=2,\lim_{x\to 1}\frac{f(x)-2}{(x-1)^2}=3, x=1?$$

$$\exists\delta>0,当0<|x-1|<\delta时,$$

$$\frac{f(x)-2}{(x-1)^2}>0\Rightarrow f(x)-2>0$$

$$\Rightarrow f(x)>f(1)\Rightarrow x=1为f(x)极小点$$

$$f'(1)=0,\lim_{x\to 1}\frac{f'(x)}{(x-1)^3}=-2, x=1?$$

$$\exists\delta>0,当0<|x-1|<\delta时,$$

$$\frac{f'(x)}{(x-1)^3}<0$$

$$\begin{cases} f'(x)>0, x\in(1-\delta,1) \\ f'(x)<0, x\in(1,1+\delta) \end{cases} \Rightarrow x=1为f(x)的极大点$$

运算性质

四则

1. 若$\lim f(x),\lim g(x)\exists\Rightarrow\lim[f(x)\pm g(x)]=\lim f(x)\pm\lim g(x)$
2. 若$\lim f(x),\lim g(x)\exists\Rightarrow\lim f(x)g(x)=\lim f(x)\lim g(x)$
3. 若$\lim f(x)\exists,\lim g(x)\exists\neq 0\Rightarrow\lim\frac{f(x)}{g(x)}=\frac{\lim f(x)}{\lim g(x)}$

复合

$$\begin{aligned} &1.\ \lim_{u\to a}f(u)=A,\lim_{x\to x_0}\Phi(x)=a\Rightarrow \\ &\quad\lim_{x\to x_0}f[\Phi(x)]=A \\ &2.\ \lim_{u\to a}f(u)=f(a),\lim_{x\to x_0}\Phi(x)=a\Rightarrow \\ &\quad\lim_{x\to x_0}f[\Phi(x)]=f[\lim_{x\to x_0}\Phi(x)]=f(a) \end{aligned}$$