多元函数微分学

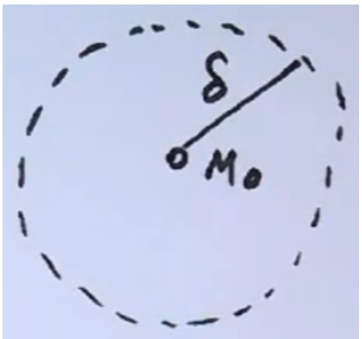
$$M_0(x_0,y_0),\delta>0$$
$$\{(x,y)|\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta\}$$
$$\{(x,y)|0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta\}$$

极限

一元

$$f(x)\text{在}x=a\text{去心邻域内有定义,若}\forall\epsilon>0,\exists\delta>0,\text{当}0<|x-a|<\delta\text{时,}$$
$$|f(x)-A|<\epsilon$$
$$\lim_{x\to a}f(x)=A$$
$$\lim_{x\to a}f(x)\text{存在}\Longleftrightarrow f(a-0),f(a+0)\text{存在且相等}$$

二元



$$f(x,y)\text{在}M_0(\text{若}\forall\epsilon>0,\exists\delta>0,\text{当}0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta\text{时,}$$
$$|f(x,y)-A|<\epsilon$$
$$\lim_{x\to x_0,y\to y_0}f(x,y)=A$$

$$\text{设}f(x,y)=(\frac{\sin xy}{xy})^{\frac{1}{x^2}},\text{求}\lim_{x\to0,y\to2}f(x,y).$$

$$\begin{aligned}\text{原式}=&\lim_{x\to0,y\to2}[(1+\frac{\sin xy-xy}{xy})^{\frac{xy}{\sin xy-xy}}]^{\frac{\sin xy-xy}{(xy)^3}\cdot y^2}\\=&e^{\lim_{x\to0,y\to2}\frac{\sin xy-xy}{(xy)^3}\cdot y^2}\\=&e^{4\lim_{t\to0}\frac{\sin t-t}{t^3}}\\=&e^{-\frac{2}{3}}\end{aligned}$$

$$\text{设}f(x,y)=\begin{cases}\frac{xy}{x^2+y^2},(x,y)\neq(0,0)\\0,(x,y)=(0,0)\end{cases}\text{讨论}\lim_{x\to0,y\to0}f(x,y)\text{是否存在.}$$

$$\begin{aligned}&\because\lim_{x\to0,y=x}f(x,y)=\frac{1}{2}\neq\lim_{x\to0,y=-x}f(x,y)=-\frac{1}{2}\\&\therefore\lim_{x\to0,y\to0}f(x,y)\text{不存在.}\end{aligned}$$

$$\lim_{x\to 0,y\to 2}\frac{\sqrt{1+xy}-\sqrt{1-xy}}{\sin x}$$

$$\begin{aligned}
\text{原式} &= \lim_{x\to 0,y\to 2}\frac{\sqrt{1+xy}-\sqrt{1-xy}}{xy}y \\
&= 2\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t} \\
&= 2\lim_{t\to 0}\frac{1}{\sqrt{1+t}+\sqrt{1-t}}*2 \\
&= 2
\end{aligned}$$

$$f(x,y)=\begin{cases}\frac{x+y}{|x|+|y|},(x,y)\neq(0,0),\\ 0,(x,y)=(0,0),\end{cases}\text{ 讨论 }\lim_{x\to 0,y\to 0}f(x,y)\text{是否存在}$$

$$\lim_{x\to 0,y=0}f(x,y)=\lim_{x\to 0}\frac{x}{|x|}\text{不存在}\Rightarrow\lim_{x\to 0,y\to 0}\frac{x+y}{|x|+|y|}\text{不存在}$$

偏导数

一元导数

$$y=f(x)\text{在}x=a\text{邻域内有定义},y=f(x)(x\in D),a\in D$$
$$\text{若}\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\text{存在(或}\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\text{存在)，称}f(x)\text{在}x=a\text{可导}$$
$$f'(a),\frac{dy}{dx}\mid_{x=a}$$

$$\begin{aligned}
&1.f(x)\text{在}x=a\text{可导}\Rightarrow f(x)\text{在}x=a\text{连续}\\
&\ \ f(x)\text{在}x=a\text{可导}\nLeftarrow f(x)\text{在}x=a\text{连续}\\
&2.f'(a)\exists\Leftrightarrow f'_-(a),f'_+(a)\exists\text{且等}\\
&3.f(x)\text{在}x=a\text{可导}\Leftrightarrow f(x)\text{在}x=a\text{可微}
\end{aligned}$$

二元

$$f(x,y)\text{在}(x_0,y_0)\text{邻域内有定义},z=f(x,y)((x,y)\in D),(x_0,y_0)\in D$$

$$\begin{aligned}
\Delta z_x &\triangleq f(x_0+\Delta x,y_0)-f(x_0,y_0)(=f(x,y_0)-f(x_0,y_0))\\
\Delta z_y &\triangleq f(x_0,y_0+\Delta y)-f(x_0,y_0)(=f(x_0,y)-f(x_0,y_0))\\
\Delta z &\triangleq f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)(=f(x,y)-f(x_0,y_0))
\end{aligned}$$

$$\text{若}\lim_{\Delta x\to 0}\frac{\Delta z_x}{\Delta x}\text{存在(或}\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}\text{存在),称}f(x,y)\text{在}(x_0,y_0)\text{对}x\text{可偏导}$$
$$\text{极限值即对}x\text{的偏导数,记}f_x(x_0,y_0),\frac{\partial z}{\partial x}\mid_{x_0,y_0}$$
$$\text{若}\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}\text{存在(或}\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}\text{存在),称}f(x,y)\text{在}(x_0,y_0)\text{对}y\text{可偏导}$$
$$\text{极限值即对}y\text{的偏导数,记}f_y(x_0,y_0),\frac{\partial z}{\partial y}\mid_{x_0,y_0}$$

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}\triangleq f_x(x,y)$$

$$
\text{设}z=f(x,y)\text{的偏导数}f_x(x,y),f_y(x,y)\text{仍可偏导}
$$

$$
\frac{\partial^2 z}{\partial x^2}=f_{xx}(x,y),\frac{\partial^2 z}{\partial y^2}=f_{yy}(x,y)
$$

$$
\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}(\frac{\partial z}{\partial x})\triangleq f_{xy}(x,y)
$$

$$
\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial x}(\frac{\partial z}{\partial y})\triangleq f_{yx}(x,y)
$$

$$
\begin{aligned}
&z=f(x,y)=x^3-x^2y+2xy^2-y^3\\
&\frac{\partial z}{\partial x}=3x^2-2xy+2y^2,\frac{\partial z}{\partial y}=-x^2+4xy-3y^2\\
&\frac{\partial^2 z}{\partial x^2}=6x-2y,\frac{\partial^2 z}{\partial y^2}=4x-6y\\
&\frac{\partial^2 z}{\partial x\partial y}=-2x+4y=\frac{\partial^2 z}{\partial y\partial x}=-2x+4y
\end{aligned}
$$

$$
z=x^2e^{\sin y}
$$

$$
\frac{\partial z}{\partial x}=2xe^{\sin y},\frac{\partial z}{\partial y}=x^2e^{\sin y}\cos y
$$

$$
\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial z}{\partial x})=2e^{\sin y},\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}(\frac{\partial z}{\partial y})=x^2e^{\sin y}\cos^2 y-x^2e^{\sin y}\sin y
$$

$$
\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}(\frac{\partial z}{\partial x})=2xe^{\sin y}\cos y,\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial x}(\frac{\partial z}{\partial y})=2xe^{\sin y}\cos y
$$

连续

一元

$$
f(x)\text{在}x=a\text{邻域内有定义}
$$

$$
\text{若}\lim_{x\to a}f(x)=f(a),\text{称}f(x)\text{在}x=a\text{连续}
$$

$$
f(x)\text{在}x=a\text{连续}\Leftrightarrow f(a-0)=f(a+0)=f(a)
$$

二元

$$
f(x,y)\text{在}(x_0,y_0)\text{邻域内有定义,}
$$

$$
\text{若}\lim_{x\to x_0,y\to y_0}f(x,y)=f(x_0,y_0),\text{称}f(x,y)\text{在}(x_0,y_0)\text{处连续}
$$

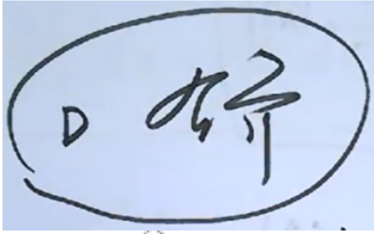
连续的性质

一元

$$
f(x)\text{在有界闭区域上，}f(x)\in C[a,b]
$$

1. $\exists m,M$
2. $\exists k>0,|f(x)|\leq k$
3. 若$f(a)f(b)<0\Rightarrow\exists c\in(a,b)$,使$f(c)=0$
4. $\forall\eta\in[m,M],\exists\xi\in[a,b]$,使$f(\xi)=\eta$

二元



若$\exists R>0$,使D在$x^2+y^2\le R^2$内

二元函数在有界闭区域上的连续性质

设$f(x,y)$在有界闭区域D上连续,

$D-xoy$面上有界闭区域, $f(x,y)\in C(D)$

1. $\exists m, M$
2. $\exists k>0$, 使$|f(x,y)|\le k$
3. $\forall\delta\in[m,M], \exists(\xi,\eta)\in D$, 使$f(\xi,\eta)=\delta$
4. 若$z=f(x,y)$二阶连续可偏导 $\Rightarrow \dfrac{\partial^2 z}{\partial y\partial x}=\dfrac{\partial^2 z}{\partial x\partial y}$

可(全)微

一元

$$y=f(x)\text{在}x=a\text{邻域内有定义}, y=f(x)(x\in D), a\in D$$

$$\Delta y=f(a+\Delta x)-f(a)(\text{或}\Delta y=f(x)-f(a))$$

$$\text{若}\Delta y=A\Delta x+o(\Delta x), \text{称}y=f(x)\text{在}x=a\text{可微}$$

$$A\Delta x\triangleq\frac{dy}{dx}\mid_{x=a}, A\Delta x=Adx$$

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1. $f(x)$在$x=a$可导 $\Leftrightarrow f(x)$在$x=a$可微
2. 若$\Delta y=A\Delta x+o(\Delta x)$, 则$A=f'(a)$
3. 设$y=f(x)$可导, $dy=df(x)=f'(x)dx$

二元

$$z=f(x,y)\text{在}(x_0,y_0)\text{邻域内有定义}, z=f(x,y)((x,y)\in D), (x_0,y_0)\in D$$

$$\Delta z=f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)=f(x,y)-f(x_0,y_0)$$

$$\begin{cases}\Delta x=x-x_0\\ \Delta y=y=y_0\end{cases}$$

$$\text{令}\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}(\rho=\sqrt{(x-x_0)^2+(y-y_0)^2})$$

$$\text{若}\Delta z=A\Delta x+B\Delta y+o(\rho), \text{称}z=f(x,y)\text{在}(x_0,y_0)\text{处可全微(可微)}$$

$$A\Delta x+B\Delta y\triangleq dz\mid_{M_0}=Adx+Bdy$$

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1. 若$\Delta z = A\Delta x + B\Delta y + o(\rho)$

$$\Rightarrow A = f_x(x_0, y_0) = \frac{\partial z}{\partial x}\ |_{M_0}, B = f_y(x_0, y_0) = \frac{\partial z}{\partial y}\ |_{M_0}$$

证：$\Delta z = A\Delta x + B\Delta y + o(\rho)$

取$\Delta y = 0 \Rightarrow \Delta z_x = A\Delta x + o(\Delta x)$

$$\Rightarrow \frac{\Delta z_x}{\Delta x} = A + \frac{o(\Delta x)}{\Delta x} \Rightarrow \lim_{\Delta x \to 0} \frac{\Delta z_x}{\Delta x} = A$$

即$f_x(x_0, y_0) = A$，同理$f_y(x_0, y_0) = B$

即可微 $\Rightarrow$ 可偏导，且$A = f_x(x_0, y_0), B = f_y(x_0, y_0)$

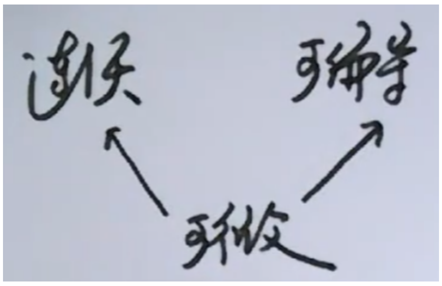
可微 $\not\Leftarrow$ 可偏导

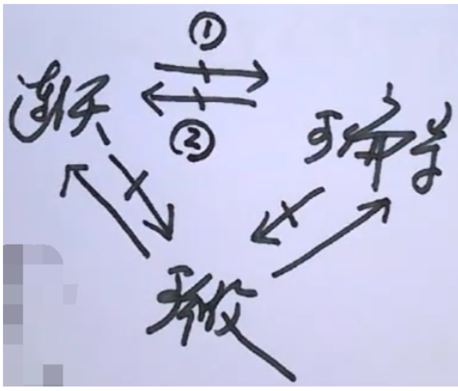
2. 若$z = f(x, y)$处处可微，

则$dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$

$$z = x^2 \ln(1 + \tan y)$$
$$\frac{\partial z}{\partial x} = 2x\ln(1 + \tan y), \frac{\partial z}{\partial y} = \frac{x^2 \sec^2 y}{1 + \tan y}$$
$$dz = 2x\ln(1 + \tan y)dx + \frac{x^2 \sec^2 y}{1 + \tan y}dy$$

连续 可偏导 可微 关系





可微 $\Rightarrow$ 连续

$$\Delta z = f(x, y) - f(x_0, y_0)$$
$$= A(x - x_0) + B(y - y_0) + o(\sqrt{(x - x_0)^2 + (y - y_0)^2})$$
$$\because \lim_{x \to x_0, y \to y_0} \Delta z = 0, \therefore \lim_{x \to x_0, y \to y_0} f(x, y) = f(x_0, y_0)$$

可微 $\Rightarrow$ 可偏导

$$
\begin{aligned}
\Delta z &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) \\
&= A\Delta x + B\Delta y + o(\rho)
\end{aligned}
$$

取$\Delta y = 0 \Rightarrow \Delta z_x = A\Delta x + o(\Delta x)$

$\Rightarrow \dfrac{\Delta z_x}{\Delta x} = A + \dfrac{o(\Delta x)}{\Delta x} \Rightarrow \lim\limits_{\Delta x\to 0} \dfrac{\Delta z_x}{\Delta x} = A$, 且$\dfrac{\partial z}{\partial x}\,|_{M_0} = A$

同理$\dfrac{\partial z}{\partial y}\,|_{M_0} = B$

连续 $\nRightarrow$ 可偏导

$z = f(x,y) = |x| + |y|$

$\lim\limits_{x\to 0, y\to 0} f(x,y) = 0 = f(0,0) \Rightarrow f(x,y)$在$(0,0)$连续

$\lim\limits_{x\to 0} \dfrac{f(x,0) - f(0,0)}{x-0} = \lim\limits_{x\to 0} \dfrac{|x|}{x}$不存在

$\Rightarrow f(x,y)$在$(0,0)$对x不可偏导

同理$f(x,y)$在$(0,0)$对y不可偏导

可偏导 $\nRightarrow$ 连续

设$z = f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0) \end{cases}$讨论$f(x,y)$在$(0,0)$处的连续性与可偏导性.

$\lim\limits_{x\to 0} \dfrac{f(x,0) - f(0,0)}{x-0} = \lim\limits_{x\to 0} \dfrac{0}{x^3} = 0 \Rightarrow f_x(0,0) = 0$

同理$f_y(0,0) = 0$

$\lim\limits_{x\to 0, y=x} f(x,y) = \dfrac{1}{2} \neq \lim\limits_{x\to 0, y=-x} f(x,y) = -\dfrac{1}{2}$

$\therefore \lim\limits_{x\to 0, y\to 0} f(x,y)$不存在, 而$f(0,0) = 0$

$\because \lim\limits_{x\to 0, y\to 0} f(x,y) \neq f(0,0), \therefore f(x,y)$在$(0,0)$不连续