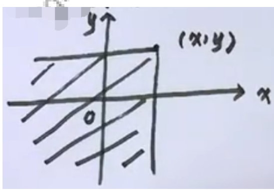
二维随机变量及其分布

$$\text{def1. } X-\text{随机变量}, Y-\text{随机变量}, \text{称}(X,Y)\text{为二维随机变量}$$
$$\text{def2. 联合分布函数}-(X,Y)\text{为二维随机变量}$$
$$F(x,y)=P\{X\le x, Y\le y\}-\text{联合分布函数}$$



$$F_X(x)=P\{X\le x\}-X\text{的边缘分布函数}$$
$$F_Y(y)=P\{Y\le y\}-Y\text{的边缘分布函数}$$

F(x,y) 联合分布函数性质

$$1. 0\le F(x,y)\le 1$$
$$2. F(x,y)\text{关于}x,y\text{不减}$$
$$3. F(x,y)\text{关于}x,y\text{右连续}$$
$$4. F(-\infty,-\infty)=F(-\infty,+\infty)=F(+\infty,-\infty)=0, F(+\infty,+\infty)=1$$

$$\text{反之, 若}F(x,y)\text{满足1到4, 则}F(x,y)\text{也为联合分布函数}$$

二维离散型随机变量及分布

联合分布律

$$\text{称}P\{X=x_i,Y=y_j\}=P_{ij}(i=1,2,\cdots,m;j=1,2,\cdots,n)\text{或}$$

X \ Y	y_1	y_2	$\cdots$	y_n
x_1	p_{11}	p_{12}	$\cdots$	p_{1n}
x_2	p_{21}	p_{22}	$\cdots$	p_{2n}
$\vdots$		$\cdots$		
x_m	p_{m1}	p_{m2}	$\cdots$	p_{mn}

$$\text{为}(X,Y)\text{的联合分布律}$$

Notes:
$$1. p_{ij}\ge 0(1\le i\le m, 1\le j\le n)$$
$$2. \sum_{i=1}^{m}\sum_{j=1}^{n}p_{ij}=1$$

边缘分布律

$$A_1=\{Y=y_1\},\cdots,A_n=\{Y=y_n\}$$
$$A_1\cdots A_n\text{完备组}$$
$$B=\{X=x_1\}$$

$$
\begin{aligned}
&P\{X=x_1\}=P(B)\\
=&P(\Omega B)=P(A_1B+\cdots+A_nB)\\
=&P(A_1B)+\cdots+P(A_nB)\\
=&P\{X=x_1,Y=y_1\}+\cdots+P\{X=x_1,Y=y_n\}\\
=&p_{11}+p_{12}+\cdots+p_{1n}=p_{1\cdot}
\end{aligned}
$$

X	x_1	x_2	$\cdots$	x_m
P	$p_{1\cdot}$	$p_{2\cdot}$	$\cdots$	$p_{m\cdot}$

$(p_{i\cdot}=p_{i1}+p_{i2}+\cdots+p_{in})$

X的边缘分布律.

Y	y_1	y_2	$\cdots$	y_n
P	$p_{\cdot 1}$	$p_{\cdot 2}$	$\cdots$	$p_{\cdot n}$

$(p_{\cdot j}=p_{1j}+p_{2j}+\cdots+p_{mj})$

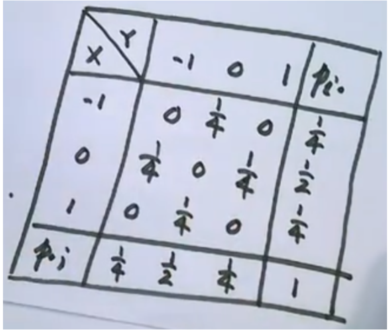
Y的边缘分布律

合并：

$X \backslash Y$	y_1 $\cdots$ y_n	$p_{i\cdot}$
x_1	p_{11} $\cdots$ p_{1n}	$p_{1\cdot}$
$\vdots$	$\cdots$	$\vdots$
x_m	p_{m1} $\cdots$ p_{mn}	$p_{m\cdot}$
$p_{\cdot j}$	$p_{\cdot 1}$ $\cdots$ $p_{\cdot n}$	1

设X,Y同分布,其中$X\sim\begin{pmatrix}-1&0&1\\ \frac{1}{4}&\frac{1}{2}&\frac{1}{4}\end{pmatrix}$,且$P\{XY=0\}=1$,求$(X,Y)$的联合分布律与边缘分布律.

$$
\begin{aligned}
&P\{XY=0\}=1\Rightarrow P\{XY\neq 0\}=0\\
&\Rightarrow P\{X=-1,Y=-1\}=P\{X=-1,Y=1\}\\
&=P\{X=1,Y=-1\}=P\{X=1,Y=1\}=0\\
&P\{X=-1\}=\frac{1}{4}\Rightarrow P\{X=-1,Y=0\}=\frac{1}{4}\\
&P\{X=1\}=\frac{1}{4}\Rightarrow P\{X=1,Y=0\}=\frac{1}{4}\\
&P\{Y=-1\}=\frac{1}{4}\Rightarrow P\{X=0,Y=-1\}=\frac{1}{4}\\
&P\{Y=0\}=\frac{1}{2}\Rightarrow P\{X=0,Y=0\}=0\\
&P\{Y=1\}=\frac{1}{4}\Rightarrow P\{X=0,Y=1\}=\frac{1}{4}
\end{aligned}
$$



X \ Y	-1	0	1	$p_{i\cdot}$
-1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
$p_{\cdot j}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

二维连续型随机变量

联合密度函数 (X,Y)-2-dim r.v.

$$
\begin{aligned}
&F(x,y)=P\{X\le x, Y\le y\}\\
&\text{若}\exists f(x,y)\ge 0,\text{使}\int_{-\infty}^{x}du\int_{-\infty}^{y}f(u,v)dv=F(x,y)\\
&\text{称}(X,Y)\text{为}2-\text{dim}\text{连续型随机变量}\\
&f(x,y)\text{称}(X,Y)\text{的联合密度函数}
\end{aligned}
$$

边缘密度函数

$$
\begin{aligned}
f_X(x)&=\int_{-\infty}^{+\infty}f(x,y)dy-X\text{的边缘密度函数}\\
f_Y(y)&=\int_{-\infty}^{+\infty}f(x,y)dx-Y\text{的边缘密度函数}
\end{aligned}
$$

$$
\begin{aligned}
&\text{Notes:}\\
&1.\begin{cases}F(x,y)=\int_{-\infty}^{x}du\int_{-\infty}^{y}f(u,v)dv\\ f(x,y)=\begin{cases}\frac{\partial^2F}{\partial x\partial y},F(x,y)\text{在}(x,y)\text{处二阶可偏导}\\ 0,F(x,y)\text{在}(x,y)\text{处二阶不可偏导}\end{cases}\end{cases}\\
&2.f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy\\
&f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx\\
&F_X(x)=\int_{-\infty}^{x}f_X(x)dx=\int_{-\infty}^{x}dx\int_{-\infty}^{+\infty}f(x,y)dy=F(x,+\infty)\\
&F_Y(y)=\int_{-\infty}^{y}f_Y(y)dy=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{y}f(x,y)dy=F(+\infty,y)
\end{aligned}
$$

$$
\begin{aligned}
&(X,Y)-2-\text{dim r.v.}\\
&F(x,y)=P\{X\le x,Y\le y\}\\
&F_X(x)=P\{X\le x\}=F(x,+\infty)\\
&F_Y(y)=P\{Y\le y\}=F(+\infty,y)
\end{aligned}
$$

$$(X,Y)-2-\text{dim离散型r.v.}$$

X \ Y	$y_1 \cdots y_n$	$p_{i\cdot}$
x_1	$p_{11} \cdots p_{1n}$	$p_{1\cdot}$
$\vdots$	$\cdots$	$\vdots$
x_m	$p_{m1} \cdots p_{mn}$	$p_{m\cdot}$
$p_{\cdot j}$	$p_{\cdot 1} \cdots p_{\cdot n}$	1

$(X,Y)-2-\mathrm{dim}$连续型r.v.

$F(x,y)=P\{X\le x, Y\le y\}$

若$\exists f(x,y)\ge 0: \int_{-\infty}^{x}dx\int_{-\infty}^{y}f(x,y)dy=F(x,y)$

$f(x,y)-$联合密度函数

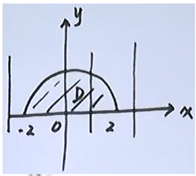
$f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy$

$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx$

设二维连续型随机变量(X,Y)的联合概率密度函数为

$$f(x,y)=\begin{cases}A(x^2+y^2), x^2+y^2\le 4且y\ge 0\\0,其他\end{cases}$$

(1)求常数A; (2)求X,Y的边缘概率密度函数.



①$1=A\pi\int_0^2 r^3dr=4\pi A$

$A=\frac{1}{4\pi}$

②$f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy$

$x\le -2$或$x\ge 2$时, $f_X(x)=0$

$-2<x<2$时, $f_X(x)=\frac{1}{4\pi}\int_0^{\sqrt{4-x^2}}(x^2+y^2)dy$

$=\frac{1}{4\pi}[x^2\sqrt{4-x^2}+\frac{1}{3}(4-x^2)\sqrt{4-x^2}]=\frac{1}{6\pi}(x^2+2)\sqrt{4-x^2}$

$$f_X(x)=\begin{cases}\frac{1}{6\pi}(x^2+2)\sqrt{4-x^2}, -2<x<2\\0,其他\end{cases}$$

$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx$

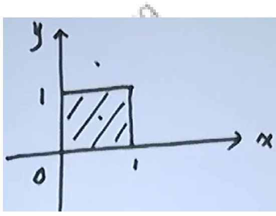
$y\le 0$或$y\ge 2$时, $f_Y(y)=0$

$0<y<2$时, $f_Y(y)=\frac{1}{4\pi}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}(x^2+y^2)dx=\frac{1}{3\pi}(y^2+2)\sqrt{4-y^2}$

设(X,Y)的联合概率密度函数为

$$f(x,y)=\begin{cases}2-x-y, 0<x<1, 0<y<1\\ 0, 其他\end{cases}$$

(1)求(X,Y)的联合分布函数$F(x,y)$; (2)求X,Y的边缘概率密度函数.



①$F(x,y)=\int_{-\infty}^{x}dx\int_{-\infty}^{y}f(x,y)dy$

$x<0$或$y<0$时, $F(x,y)=0$

$0\le x<1$且$0\le y<1$时, $F(x,y)=\int_{0}^{x}dx\int_{0}^{y}(2-x-y)dy$

$=\int_{0}^{x}(2y-xy-\frac{y^2}{2})dx=2xy-\frac{x^2y}{2}-\frac{xy^2}{2}$

$0\le x<1$且$y\ge 1$时, $F(x,y)=\int_{0}^{x}dx\int_{0}^{1}(2-x-y)dy$

$=\int_{0}^{x}(\frac{3}{2}-x)dx=\frac{3}{2}x-\frac{x^2}{2}$

$x\ge 1$且$0\le y<1$时, $F(x,y)=\frac{3}{2}y-\frac{y^2}{2}$

$x\ge 1$且$y\ge 1$时, $F(x,y)=1$

②$f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy$

$x\le 0$或$x\ge 1$时, $f_X(x)=0$

$0<x<1$时, $f_X(x)=\int_{0}^{1}(2-x-y)dy=\frac{3}{2}-x$

$$f_X(x)=\begin{cases}\frac{3}{2}-x, 0<x<1\\ 0, 其他\end{cases}$$

常见的2-dim随机变量及分布

均匀分布

D为xoy面上有界区域，面积为A

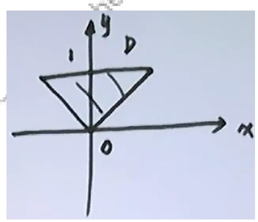
$(X,Y)-2-\dim$ r.v., 若(X,Y)联合密度函数为

$$f(x,y)=\begin{cases}\frac{1}{A}, (x,y)\in D\\ 0, (x,y)\notin D\end{cases}$$

称(X,Y)在D上服从均匀分布. 记$(X,Y)\sim U(D)$

设区域D由$y=|x|$与$y=1$围成, 二维随机变量(X,Y)在区域D内服从均匀分布.

(1)求(X,Y)的联合概率密度函数; (2)求X,Y的边缘分布函数.



$$①f(x,y)=\begin{cases}1,(x,y)\in D\\0,(x,y)\notin D\end{cases}$$

$$②f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy$$

$$|x|\geq 1时,f_X(x)=0$$

$$-1<x\leq 0时,f_X(x)=\int_{-x}^{1}dy=1+x$$

$$0<x<1时,f_X(x)=\int_{x}^{1}1dy=1-x$$

$$\therefore f_X(x)=\begin{cases}1-|x|,|x|<1\\0,|x|\geq 1\end{cases}$$

$$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx$$

$$y\leq 0或y\geq 1时,f_Y(y)=0$$

$$0<y<1时,f_Y(y)=\int_{-y}^{y}1dx=2y$$

$$f_Y(y)=\begin{cases}2y,0<y<1\\0,其他\end{cases}$$

二维正态分布

$(X,Y)-2-\dim$ r.v.若(X,Y)的

联合密度函数为

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\cdot e^{-\frac{1}{2(1-\rho^2)}\{\frac{(x-\mu_1)^2}{\sigma_1^2}-2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2}\}}$$

$(\mu_1,\mu_2,\sigma_1>0,\sigma_2>0,\rho-常数,-\infty<x,y<+\infty)$

称(X,Y)服从二维正态分布. 记$(X,Y)\sim N(\mu_1,\mu_2;\sigma_1^2,\sigma_2^2;\rho)$

Notes:

$1.X\sim N(\mu,\sigma^2):$

$$①P\{X\leq\mu\}=P\{X>\mu\}=\frac{1}{2}$$

$$②\frac{X-\mu}{\sigma}\sim N(0,1)$$

$$③F(x)=\Phi(\frac{x-\mu}{\sigma})$$

$2.(X,Y)\sim N(\mu_1,\mu_2;\sigma_1^2,\sigma_2^2;\rho)$

$①|\rho|<1$

$②X\sim N(\mu_1,\sigma_1^2),Y\sim N(\mu_2,\sigma_2^2)$

2-dim r.v. 条件分布

$X \backslash Y$	$y_1 \cdots y_n$	$p_{i\cdot}$
x_1	$p_{11} \cdots p_{1n}$	$p_{1\cdot}$
$\vdots$	$\cdots$	$\vdots$
x_m	$p_{m1} \cdots p_{mn}$	$p_{m\cdot}$
$p_{\cdot j}$	$p_{\cdot 1} \cdots p_{\cdot n}$	1

若$P(A)>0$,则$P(B|A)=\dfrac{P(AB)}{P(A)}$

(一)离散型：

1.在$X=x_i$下,Y的条件分布律：

$$P\{Y=y_j|X=x_i\}=\frac{p_{ij}}{p_{i\cdot}}(j=1,2,\cdots,n)$$

2.在$Y=y_j$下,X的条件分布律：

$$P\{X=x_i|Y=y_j\}=\frac{p_{ij}}{p_{\cdot i}}(i=1,2,\cdots,m)$$

(二)连续型：

$(X,Y)\sim f(x,y)$

1.在$X=x$下,Y的条件密度函数为

$$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}$$

2.在$Y=y$下,X的条件密度函数为：

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}$$

设随机变量X,Y同分布,且$X\sim\begin{pmatrix}0 & 1\\ \frac{2}{3} & \frac{1}{3}\end{pmatrix}$,$P\{XY=0\}=\dfrac{5}{6}$.

(1)求(X,Y)的联合分布律和边缘分布律;

(2)求在$X=0$条件下Y的条件分布律.

X \ Y	0	1	$p_{i\cdot}$
0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}$
1	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
$p_{\cdot j}$	$\frac{2}{3}$	$\frac{1}{3}$	1

①$P\{XY=0\}=\dfrac{5}{6}$

$$\Rightarrow P\{XY\neq0\}=\frac{1}{6}\Rightarrow P\{X=1,Y=1\}=\frac{1}{6}$$

$$P\{X=1,Y=0\}=\frac{1}{3}-\frac{1}{6}=\frac{1}{6}$$

$$P\{X=0,Y=0\}=\frac{2}{3}-\frac{1}{6}=\frac{1}{2}$$

$$P\{X=0,Y=1\}=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$$

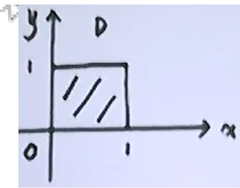
②$P\{X=0\}=\dfrac{2}{3}$

$$P\{Y=0|X=0\}=\frac{1}{2}/\frac{2}{3}=\frac{3}{4}$$

$$P\{Y=1|X=0\}=\frac{1}{6}/\frac{2}{3}=\frac{1}{4}$$

设(X,Y)的联合概率密度函数为$f(x,y)=\begin{cases}4xy,0<x<1,0<y<1,\\0,其他.\end{cases}$

(1)求X,Y的边缘概率密度函数;(2)求Y的条件概率密度函数.



$①x \leq 0或x \geq 1时, f_X(x) = 0$

$0 < x < 1时, f_X(x) = \int_0^1 4xydy = 2x$

$\therefore f_X(x) = \begin{cases} 2x, 0 < x < 1 \\ 0, 其他 \end{cases}$

$同理 f_Y(y) = \begin{cases} 2y, 0 < y < 1 \\ 0, 其他 \end{cases}$

$②在X = x下(0 < x < 1), Y的条件密度为$

$f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)} = \begin{cases} 2y, 0 < y < 1 \\ 0, 其他 \end{cases}$

独立性

随机事件：

$若P(AB) = P(A)P(B), 称A, B独立$

$① \begin{cases} A, B \\ A, \overline{B} \\ \overline{A}, B \\ \overline{A}, \overline{B} \end{cases} 一对独立, 其余独立$

$②若P(A) = 0或P(A) = 1 \Rightarrow A, B独立$

$③若P(A) > 0, 则A, B独立 \Leftrightarrow P(B|A) = P(B)$

$若0 < P(A) < 1, 则A, B独立 \Leftrightarrow P(B|A) = P(B|\overline{A})$

$(一)(X, Y)为2-\dim \text{r.v.}若$

$F(x, y) = F_X(x)F_Y(y)$

$称X, Y独立$

(二)等价判别法：

$1.离散型：P\{X = x_i, Y = y_i\} = p_{ij}(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$

$则X, Y独立 \Leftrightarrow p_{ij} = p_{i\cdot} \times p_{\cdot j}$

$(1 \leq i \leq m, 1 \leq j \leq n)$

$2.连续型：(X, Y) \sim f(x, y)$

$则X, Y独立 \Leftrightarrow f(x, y) = f_X(x)f_Y(y)$

$\text{Notes}：设(X, Y)为2-\dim连续型\text{r.v.}一般需要f(x, y)$

考试时有如下三种情形：

$①f(x, y)已知;$

$②X的边缘分布已知, Y的条件分布已知$

$f(x, y) = f_X(x) \cdot f_{Y|X}(y|x)$

$③X, Y边缘分布已知, 且X, Y独立$

$则f(x, y) = f_X(x)f_Y(y)$

Z=\phi(X,Y)的分布

(X,Y)分布已知, $Z=\phi(X,Y)$的分布

(二维随机变量函数的分布)

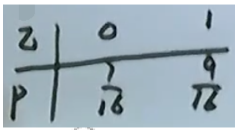
(一)(X,Y)离散, $Z=\phi(X,Y)$离散

(二)(X,Y)连续型, Z离散

(三)(X,Y)连续型, $Z=\phi(X,Y)$连续型

(四)X离散, Y连续, $Z=\phi(X,Y)$, 求$F_Z(z)$

设X,Y独立同分布, 且$X\sim\begin{pmatrix}0 & 1\\ \frac{1}{4} & \frac{3}{4}\end{pmatrix}$, 令$Z=\min\{X,Y\}$, 求$Z$的分布律.



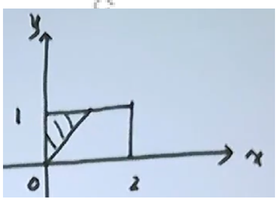
Z	0	1
P	7/16	9/16

1.Z的可能取值0, 1

2.$P\{Z=1\}=P\{X=1,Y=1\}=P\{X=1\}P\{Y=1\}=\dfrac{1}{16}$

$P\{Z=0\}=\dfrac{7}{16}$

设随机变量(X,Y)在$D=\{(x,y)|0\le x\le 2,0\le y\le 1\}$上服从均匀分布, 当$(x,y)\in D$时, Z $=\begin{cases}0,Y<X\\1,Y\ge X\end{cases}$, 求$Z$的分布律.



$$1.f(x,y)=\begin{cases}\frac{1}{2},(x,y)\in D\\0,(x,y)\notin D\end{cases}$$

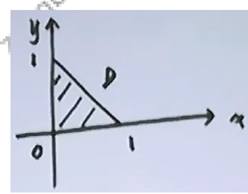
$$2.P\{Z=1\}=P\{Y\ge X\}=\iint_{y\ge x}f(x,y)d\sigma$$

$$=\int_0^1 dy\int_0^y\frac{1}{2}dx=\frac{1}{2}\int_0^1 ydy=\frac{1}{4}$$

$$P\{Z=0\}=\frac{3}{4}$$

Z	0	1
P	3/4	1/4

设区域D由$x+y=1$, x轴及y轴围成, 随机变量(X,Y)在D上服从均匀分布, 求$Z=X+Y$的概率密度函数.



$1.f(x,y)=\begin{cases}2,(x,y)\in D\\0,(x,y)\notin D\end{cases}$

$2.Z=X+Y$的分布函数为

$$F_Z(z)=P\{Z\le z\}=P\{X+Y\le z\}=\iint_{x+y\le z}f(x,y)d\sigma$$

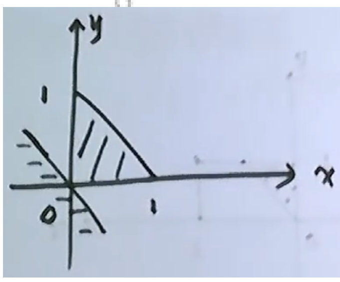
$$x+y\le z\Leftrightarrow y\le -(x-z),y\le -x$$

$$z<0:F_Z(z)=0;z\ge 1:F_Z(z)=1$$

$$0\le z<1:F_Z(z)=\iint_{D_z}2d\sigma=2\times\frac{1}{2}z^2=z^2$$

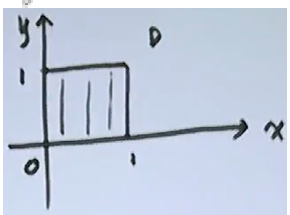
$$F_Z(z)=\begin{cases}z^2,0\le z<1\\0,其他\end{cases}$$

$3.f_Z(z)=\begin{cases}2z,0<z<1\\0,其他\end{cases}$



设随机变量(X,Y)的联合概率密度函数为$f(x,y)=\begin{cases}2x,0<x<1,0<y<1\\0,其他\end{cases}$

求$Z=X+Y$的概率密度函数.



$1.F_Z(z)=P\{x+y\le z\}=\iint_{x+y\le z}f(x,y)d\sigma$

$$z<0:F_Z(z)=0;z\ge 2:F_Z(z)=1$$

$$0\le z<1:F_Z(z)=\int_0^z dx\int_0^{z-x}2xdy=\int_0^z(2zx-2x^2)dx=\frac{z^3}{3}$$

$2.1\le z<2:F_Z(z)=1-\int_{z-1}^1 2xdx\int_{z-x}^1 dy$

$$=1-\int_{z-1}^1(2x-2zx-2x^2)dx=-\frac{1}{3}+z^2-\frac{z^3}{3}$$

$$F_Z(z)=\begin{cases}0,z<0\\\frac{1}{3}z^3,0\le z<1\\-\frac{1}{3}+z^2-\frac{1}{3}z^3,1\le z<2\\1,z\ge 1\end{cases}$$

$$\therefore f_Z(z)=\begin{cases}z^2,0<z<1\\2z-z^2,1\le z<2\\0,其他\end{cases}$$

$X\sim E(2),Y\sim E(1),X,Y$独立.

①$Z=\min\{X,Y\}$;②$Z=\max\{X,Y\}$,求$f_Z(z)$.

$$X \sim E(2) \Rightarrow F_X(x) = \begin{cases} 1-e^{-2x}, x \geq 0 \\ 0, x < 0 \end{cases}$$

$$Y \sim E(1) \Rightarrow F_Y(y) = \begin{cases} 1-e^{-y}, y \geq 0 \\ 0, y < 0 \end{cases}$$

$$①F_Z(z) = P\{Z \leq z\} = 1 - P\{Z > z\} = 1 - P\{X > z\}P\{Y > z\}$$

$$= 1 - [1 - F_X(z)][1 - F_Y(z)] = \begin{cases} 0, z < 0 \\ 1 - e^{-3z}, z \geq 0 \end{cases}$$

$$f_Z(z) = \begin{cases} 3e^{-3z}, z > 0 \\ 0, z \leq 0 \end{cases}, \text{即} Z \sim E(3)$$

$$②F_Z(z) = P\{Z \leq z\} = P\{X \leq z\}P\{Y \leq z\}$$

$$= F_X(z)F_Y(z) = \begin{cases} 0, z < 0 \\ 1 - e^{-z} - e^{-2z} + e^{-3z}, z \geq 0 \end{cases}$$

$$f_Z(z) = \begin{cases} e^{-z} + 2e^{-2z} - 3e^{-3z}, z > 0 \\ 0, z \leq 0 \end{cases}$$

设$X \sim \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, Y \sim U(0,2)$且$(X,Y)$相互独立,求$Z = X + Y$的分布函数.

$$Y \sim U(0,2) \Rightarrow f_Y(y) = \begin{cases} \frac{1}{2}, 0 < y < 2 \\ 0, \text{其他} \end{cases}$$

$$F_Z(z) = P\{Z \leq z\} = P\{X + Y \leq z\}$$

$$z < 0 : F_Z(z) = 0; z \geq 3 : F_Z(z) = 1$$

$$0 \leq z < 1 : F_Z(z) = P\{X = 0\}P\{X + Y \leq z | X = 0\} = \frac{1}{2}P\{Y \leq z\}$$

$$= \frac{1}{2}\int_0^z \frac{1}{2}dy = \frac{3}{4}$$

$$1 \leq z < 2 : F_Z(z) = P\{X = 0\}P\{X + Y \leq z | X = 0\} + P\{X = 1\}P\{X + Y \leq z | X = 1\}$$

$$= \frac{1}{2}P\{Y \leq z\} + \frac{1}{2}P\{Y \leq z - 1\} = \frac{1}{2}\int_0^z \frac{1}{2}dy + \frac{1}{2}\int_0^{z-1} \frac{1}{2}dz = \frac{z}{2} - \frac{1}{4}$$

$$2 \leq z < 3 : F_Z(z) = \frac{1}{2} + \frac{1}{2}\int_0^{z-1} \frac{1}{2}dy = \frac{1}{2} + \frac{z-1}{4} = \frac{z+1}{4}$$

型一

设A, B为随机事件,令$X = \begin{cases} 1, \text{当}A\text{发生} \\ 0, \text{当}\overline{A}\text{发生时} \end{cases} Y = \begin{cases} 1, \text{当}B\text{发生} \\ 0, \text{当}\overline{B}\text{发生时} \end{cases}$

$P(A) = \frac{1}{3}, P(B|A) = \frac{1}{4}, P(A|B) = \frac{1}{2}$,求$(X,Y)$的联合分布律.

X \ Y	0	1
0	$\frac{7}{12}$	$\frac{1}{6}$
1	$\frac{1}{6}$	$\frac{1}{12}$

$$1.P(B|A)=\frac{1}{4}\Rightarrow P(AB)=\frac{1}{12}$$

$$P(A|B)=\frac{1}{2}\Rightarrow P(B)=\frac{1}{6}$$

$$2.(X,Y):(0,0),(0,1),(1,0),(1,1)$$

$$P\{X=1,Y=1\}=P(AB)=\frac{1}{12}$$

$$P\{X=1,Y=0\}=P(A\overline{B})=\frac{1}{6}$$

$$P\{X=0,Y=1\}=P(\overline{A}B)=\frac{1}{6}$$

$$P\{X=0,Y=0\}=\frac{7}{12}$$

设随机变量X,Y独立同分布,且随机变量X的分布律为$X\sim\begin{pmatrix}0&1\\ \frac{2}{3}&\frac{1}{3}\end{pmatrix}$,令

$U=\max\{X,Y\},V=\min\{X,Y\}$

(1)求(U,V)的联合分布律;

(2)求Cov(U,V).

$$①(U,V):(1,1),(1,2),(2,1),(2,2)$$

$$P\{U=1,V=1\}=P\{X=1\}P\{Y=1\}=\frac{4}{9}$$

$$P\{U=1,V=2\}=0$$

$$P\{U=2,V=2\}=P\{X=2\}P\{Y=2\}=\frac{1}{9}$$

$$P\{U=2,V=1\}=\frac{4}{9}$$

U \ V	1	2
1	$\frac{4}{9}$	0
2	$\frac{4}{9}$	$\frac{1}{9}$

袋中有10个大小相同的球,其中6个红球4个白球,随机抽取2个，每次抽取1个,定义如下两个随机变量：

$X=\begin{cases}1,\text{第1次抽到红球},\\0,\text{第1次抽到白球},\end{cases}Y=\begin{cases}1,\text{第2次抽到红球},\\0,\text{第2次抽到白球},\end{cases}$

就下列两种情况,求(X,Y)的联合分布律：

(1)每次抽取后放回;(2)每次抽取后不放回.

$$①(X,Y):(0,0),(0,1),(1,0),(1,1)$$

$$P\{X=0,Y=0\}=\frac{4}{10}\times\frac{4}{10}=\frac{4}{25}$$

$$P\{X=0,Y=1\}=\frac{4}{10}\times\frac{6}{10}=\frac{6}{25}$$

$$P\{X=1,Y=0\}=\frac{6}{10}\times\frac{4}{10}=\frac{6}{25}$$

$$P\{X=1,Y=1\}=\frac{6}{10}\times\frac{6}{10}=\frac{9}{25}$$

X\Y	0	1
0	$\frac{4}{25}$	$\frac{6}{25}$
1	$\frac{6}{25}$	$\frac{9}{25}$

$$
\begin{aligned}
&②(X,Y):(0,0),(0,1),(1,0),(1,1)\\
&\quad P\{X=0,Y=0\}=\frac{4}{10}\times\frac{3}{9}=\frac{2}{15}\\
&\quad P\{X=0,Y=1\}=\frac{4}{10}\times\frac{6}{9}=\frac{4}{15}\\
&\quad P\{X=1,Y=0\}=\frac{6}{10}\times\frac{4}{9}=\frac{4}{15}\\
&\quad P\{X=1,Y=1\}=\frac{6}{10}\times\frac{5}{9}=\frac{5}{15}
\end{aligned}
$$

型二

$$f(x,y),f_X(x),f_Y(y)$$

设二维随机变量(X,Y)的联合概率密度为

$f(x,y)=\begin{cases}axe^{-x(y+1)},x>0,y>0,\\0,\text{其他}\end{cases}(a>0).$

(1)求常数a;

(2)求随机变量X,Y的边缘概率密度函数.

$$
\begin{aligned}
&①1=a\int_0^{+\infty}dx\int_0^{+\infty}e^{-x(y+1)}d[x(y+1)]\\
&\quad =a\int_0^{+\infty}x^0e^{-x}dx=a\Rightarrow a=1\\
&②f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy\\
&\quad x\le 0:f_X(x)=0\\
&\quad x>0:f_X(x)=\int_0^{+\infty}xe^{-x(y+1)}dy=e^{-x}\\
&\quad \therefore f_X(x)=\begin{cases}e^{-x},x>0\\0,x\le 0\end{cases}\\
&\quad f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx\\
&\quad y\le 0:f_Y(y)=0\\
&\quad y>0:f_Y(y)=\int_0^{+\infty}xe^{-x(y+1)}dx\\
&\quad =\frac{1}{(y+1)^2}\int_0^{+\infty}x(y+1)e^{-x(y+1)}dx(y+1)\\
&\quad =\frac{1}{(y+1)^2}\int_0^{+\infty}te^{-t}dt=\frac{1}{(y+1)^2}\\
&\quad f_Y(y)=\begin{cases}\frac{1}{(y+1)^2},y>0\\0,y\le 0\end{cases}
\end{aligned}
$$