数字特征

数学期望（均值）

X	x_1	$\cdots$	x_n
P	p_1	$\cdots$	p_n

1.(1-dim离散)

①$EX \triangleq \sum_{i=1}^{n} x_i p_i$

②$Y=\phi(X)$, 则

$EY \triangleq \sum_{i=1}^{n} \phi(x_i) p_i$

2.(1-dim连续)设$X \sim f(x)$

①$EX \triangleq \int_{-\infty}^{+\infty} x f(x) dx$

②$Y=\phi(X)$, 则

$EY \triangleq \int_{-\infty}^{+\infty} \phi(x) f(x) dx$

3.(2-dim离散)设$P\{X=x_i, Y=y_j\}=p_{ij}(1 \leq i \leq m, 1 \leq j \leq n)$

若$Z=\phi(X, Y)$

$EZ \triangleq \sum_{i=1}^{m} \sum_{j=1}^{n} \phi(x_i, y_j) \cdot p_{ij}$

4.(2-dim连续)设$(X, Y) \sim f(x, y)$

$Z=\phi(X, Y)$

$EZ=\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \phi(x, y) f(x, y) dy$

C	C
P	1

1.$E(C)=C$

2.$E(aX+bY)=aEX+bEY$

3.若X, Y独立, 则$EXY=EX \cdot EY$

方差

$$
\begin{aligned}
&(一)X-\text{r.v.}, DX \triangleq E(X-EX)^2\\
&(二)计算公式, DX=E(X^2-2EX\cdot X+(EX)^2)\\
&\quad\quad =EX^2-2(EX)^2+(EX)^2\\
&\quad\quad =EX^2-(EX)^2\\
&\quad\ DX=EX^2-(EX)^2\\
&(三)方差的性质:\\
&\ \ 1.D(C)=0\\
&\ \ 2.D(aX+b)=D(aX)=a^2DX\\
&\ \ 3.X,Y独立时, D(aX+bY)=a^2DX+b^2DY
\end{aligned}
$$

Notes:

$1.X\sim B(n,p):$

$P\{X=k\}=C_n^kp^k(1-p)^{n-k}(0<p<1,k=0,1,2,\cdots,n)$

$EX=np,DX=np(1-p)$

$2.X\sim P(\lambda)(\lambda>0):$

①$P\{X=k\}=\dfrac{\lambda^k}{k!}e^{-\lambda}(k=0,1,2,\cdots)$

$$EX=\sum_{k=0}^{\infty}k\cdot\frac{\lambda^k}{k!}e^{-\lambda}=e^{-\lambda}\sum_{n=1}^{\infty}n\cdot\frac{\lambda^n}{n!}=\lambda e^{-\lambda}\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!}=\lambda e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}=\lambda$$

$$EX^2=e^{-\lambda}\sum_{n=1}^{\infty}n^2\cdot\frac{\lambda^n}{n!}=e^{-\lambda}\sum_{n=1}^{\infty}\frac{1+(n-1)}{(n-1)!}\lambda^n=\lambda e^{-\lambda}\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!}+\lambda^2e^{-\lambda}\sum_{n=2}^{\infty}\frac{\lambda^{n-2}}{(n-2)!}$$

$=\lambda+\lambda^2,DX=EX^2-(EX)^2=\lambda$

②$EX=\lambda,DX=\lambda$

$3.X\sim U(a,b):$

①$f(x)=\begin{cases}\frac{1}{b-a},a<x<b\\0,其他\end{cases},F(x)=\begin{cases}0,x<a\\\frac{x-a}{b-a},a\le x<b\\1,x\ge b\end{cases}$

$$EX=\int_a^b x\cdot\frac{1}{b-a}=\frac{a+b}{2}$$

②$EX=\dfrac{a+b}{2},DX=\dfrac{(b-a)^2}{12}$

$4.X\sim E(\lambda)(\lambda>0):$

①$f(x)=\begin{cases}\lambda e^{-\lambda x},x>0\\0,x\le 0\end{cases};F(x)=\begin{cases}1-e^{-\lambda x},x\ge 0\\0,x<0\end{cases}$

$$EX=\int_0^{+\infty}\lambda xe^{-\lambda x}dx=\frac{1}{\lambda}$$

$$EX^2=\int_0^{+\infty}x^2e^{-\lambda x}d(\lambda x)=\frac{1}{\lambda^2}\int_0^{+\infty}t^2e^{-t}dt=\frac{2}{\lambda^2}$$

$$DX=EX^2-(EX)^2=\frac{1}{\lambda^2}$$

②$EX=\dfrac{1}{\lambda},DX=\dfrac{1}{\lambda^2}$

如：$X\sim E(\lambda),P\{X>\sqrt{DX}\}$

$$DX=\frac{1}{\lambda^2},F(x)=\begin{cases}1-e^{-\lambda x},x\geq 0\\0,x<0\end{cases}$$

$$原式=P\{X>\frac{1}{\lambda}\}$$

$$=1-P\{X\leq\frac{1}{\lambda}\}=1-F(\frac{1}{\lambda})=\frac{1}{e}$$

$$设随机变量X\sim N(\mu,\sigma^2),求E(X),D(X).$$

$$X\sim N(\mu,\sigma^2)\Rightarrow f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$EX=\int_{-\infty}^{+\infty}x\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}d(\frac{x-\mu}{\sigma})=\int_{-\infty}^{+\infty}(\mu+\sigma t)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt,\frac{x-\mu}{\sigma}=t$$

$$=\mu\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt=\mu$$

$$EX^2=\int_{-\infty}^{+\infty}(\mu^2+2\mu\sigma t+\sigma^2t^2)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$$

$$=\mu^2+\frac{2\sigma^2}{\sqrt{2\pi}}\int_0^{+\infty}x^2e^{-\frac{x^2}{2}}dx=\mu^2+\frac{2\sigma^2}{\sqrt{2\pi}}\int_0^{+\infty}2t\cdot e^{-t}\cdot\sqrt{2}\cdot\frac{dt}{2\sqrt{t}},\frac{x^2}{2}=t$$

$$=\mu^2+\frac{2\sigma^2}{\sqrt{\pi}}\int_0^{+\infty}t^{\frac{1}{2}}e^{-t}dt$$

$$=\mu^2+\frac{2\sigma^2}{\sqrt{\pi}}\Gamma(\frac{1}{2}+1)=\mu^2+\sigma^2$$

$$DX=EX^2-(EX)^2=\sigma^2$$

协方差

$$DX\triangleq E(X-EX)^2=E(X-EX)(X-EX)$$

$$1.协方差-COV(X,Y)\triangleq E(X-EX)(Y-EY)(COV(X,X)=DX)$$
$$2.计算公式:COV(X,Y)=EXY-EX\cdot EY$$
$$3.性质:$$
$$①COV(X,X)=DX$$
$$②COV(X,Y)=COV(Y,X)$$
$$③COV(X,k_1Y_1+\cdots+k_nY_n)=k_1COV(X,Y_1)+\cdots+k_nCOV(X,Y_n)$$
$$④X,Y独立\Rightarrow EXY=EX\cdot EY\Leftrightarrow COV(X,Y)=0$$
$$⑤D(aX+bY)=COV(aX+bY,aX+bY)$$
$$=a^2DX+b^2DY+2abCOV(X,Y)$$

相关系数

$$1.\rho_{XY}=\frac{COV(X,Y)}{\sqrt{DX}\cdot\sqrt{DY}}$$
$$2.性质:$$
$$①|\rho_{XY}|\leq 1$$
$$②若\rho_{XY}=0,称X,Y不相关$$
$$\rho_{XY}=0\Leftrightarrow COV(X,Y)=0\Leftrightarrow EXY=EX\cdot EY$$
$$③若\rho_{XY}=-1,称X,Y负相关$$
$$\rho_{XY}=-1\Leftrightarrow P\{Y=aX+b\}=1(a<0)$$
$$④若\rho_{XY}=1,称X,Y正相关$$
$$\rho_{XY}=1\Leftrightarrow P\{Y=aX+b\}=1(a>0)$$

Notes:

①X,Y独立 $\Rightarrow X,Y$不相关, $\not\Leftarrow$

$\Rightarrow X,Y$独立 $\Rightarrow EXY=EX\cdot EY \Rightarrow COV(X,Y)=0 \Rightarrow \rho_{XY}=0$

$\not\Leftarrow$ 设$X\sim U(-1,1), Y=X^2$

$EX=0, EXY=EX^3=\int_{-1}^{1}x^3\cdot\frac{1}{2}dx=0$

$COV(X,Y)=EXY-EXEY=0\Rightarrow \rho_{XY}=0$

设(X,Y)的联合分布函数为$F(x,y)$

$F(\frac{1}{2},\frac{1}{4})=P\{X\le\frac{1}{2},X^2\le\frac{1}{4}\}=P\{-\frac{1}{2}\le X\le\frac{1}{2}\}=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{2}dx=\frac{1}{2}$

$F_X(\frac{1}{2})=P\{X\le\frac{1}{2}\}=\int_{-1}^{\frac{1}{2}}\frac{1}{2}dx=\frac{3}{4}$

$F_Y(\frac{1}{4})=P\{X^2\le\frac{1}{4}\}=\frac{1}{2}$

$\because F(\frac{1}{2},\frac{1}{4})\neq F_X(\frac{1}{2})\cdot F_Y(\frac{1}{4}), \therefore X,Y$不独立

②设$(X,Y)\sim N(\mu_1,\mu_2;\sigma_1^2,\sigma_2^2;\rho)$

$\Rightarrow X\sim N(\mu_1,\sigma_1^2), Y\sim N(\mu_2,\sigma_2^2)$

X,Y独立 $\Leftrightarrow X,Y$不相关 $\Leftrightarrow \rho=0$

设$(X,Y)\sim N(0,1;1,4;0)$, 求$P\{XY<Y\}$

$X\sim N(0,1), Y\sim N(1,4)$

$\because \rho=0, \therefore X,Y$独立

$P\{XY<X\}=P\{X(Y-1)<0\}=P\{X<0\}P\{Y>1\}+P\{X>0\}P\{Y<1\}=\frac{1}{2}$

投币n次, 正反分别为XY次, 求ρ_{XY}

法一 : $\because P\{X+Y=n\}=1$

$\therefore P\{Y=-X+n\}=1, \therefore \rho_{XY}=-1$

法二 :$Y=-X+n$

$COV(X,Y)=COV(X,-X+n)=-DX$

$DY=D(-X+n)=D(-X)=DX$

$\rho_{XY}=\frac{COV(X,Y)}{\sqrt{DX}\sqrt{DY}}=\frac{-DX}{DX}=-1$

型一 一维离散型随机变量的数学期望和方差

设随机变量X的分布函数为

$$F(x)=\begin{cases}0, x<0\\ \theta, 0\le x<1\\ 2\theta, 1\le x<2\\ 1, x\ge 2\end{cases}$$

(1)求$E(X), D(X)$; (2)$E(X^2+2X)$.

$X=0,1,2$

$P\{X=0\}=F(0)-F(0-0)=\theta$

$P\{X=1\}=F(1)-F(1-0)=\theta$

$P\{X=2\}=F(2)-F(2-0)=1-2\theta$

X	0	1	2
P	θ	θ	$1-2\theta$.

设试验成功的概率为$\dfrac{3}{4}$,失败的概率为$\dfrac{1}{4}$,独立重复该试验直到成功两次为止,求试验次数的数学期望.

$$1.P\{X=k\}=C_{k-1}^{1}\cdot\frac{3}{4}\cdot(\frac{1}{4})^{k-2}\cdot\frac{3}{4}=\frac{9}{16}(k-1)(\frac{1}{4})^{k-2}(k=2,3,\cdots)$$

$$2.EX=\sum_{n=2}^{\infty}n\cdot\frac{9}{16}\cdot(n-1)(\frac{1}{4})^{n-2}=\frac{9}{16}\sum_{n=2}^{\infty}n(n-1)(\frac{1}{4})^{n-2}$$

令$S(x)=\sum\limits_{n=2}^{\infty}n(n-1)x^{n-2}(-1<x<1)$

$$=(\sum_{n=2}^{\infty}x^{n})''=(\frac{x^2}{1-x})''=\frac{2}{(1-x)^3}$$

$$\therefore EX=\frac{9}{16}\times S(\frac{1}{4})=\frac{8}{3}$$

$$X\sim f(x)=\begin{cases}\frac{1}{2}\cos\frac{x}{2},0<x<\pi\\0,\text{其他}\end{cases}$$

对X观察4次,Y表示4次$\{X>\dfrac{\pi}{3}\}$次数,求EY^2

$1.Y\sim B(4,p)$

$$p=P\{X>\frac{\pi}{3}\}=\int_{\frac{\pi}{3}}^{\pi}\frac{1}{2}\cos\frac{x}{2}dx=\frac{1}{2}$$

$$\therefore Y\sim B(4,\frac{1}{2})$$

$2.EY=2,DY=1$

$\because DY=EY^2-(EY)^2$

$\therefore EY^2=DY+(EY)^2=5$

型二 一维连续型随机变量的数学期望和方差

设随机变量X的概率密度函数为$f(x)=\dfrac{1}{2}e^{-|x-\theta|}$,求$E(X),D(X)$.

$$EX=\frac{1}{2}\int_{-\infty}^{+\infty}[(x-\theta)+\theta]e^{-|x-\theta|}d(x-\theta)$$

$$=\frac{1}{2}\int_{-\infty}^{+\infty}(x+\theta)e^{-|x|}dx=\theta\int_{0}^{+\infty}x^0e^{-x}dx=\theta$$

$$EX^2=\frac{1}{2}\int_{-\infty}^{+\infty}[(x-\theta)+\theta]^2e^{-|x-\theta|}d(x-\theta)$$

$$=\frac{1}{2}\int_{-\infty}^{+\infty}(x+\theta)^2e^{-|x|}dx$$

$$=\int_{0}^{+\infty}(x^2+\theta^2)e^{-x}dx=2+\theta^2$$

$$DX=EX^2-(EX)^2=2$$

设随机变量X的分布函数为$F(x)=0.4\phi(\frac{x-1}{2})+0.6\phi(3x+1)$,其中$\phi(x)$是服从标准正态分布的随机变量的分布函数,求$E(X)$.

1.X的密度函数为

$$f(x)=0.2\phi(\frac{x-1}{2})+1.8\phi(3x+1)$$

$$2.EX=0.2\int_{-\infty}^{+\infty}x\phi(\frac{x-1}{2})dx+1.8\int_{-\infty}^{+\infty}x\phi(3x+1)dx$$

$$=\int_{-\infty}^{+\infty}(\frac{x-1}{2}+\frac{1}{2})\phi(\frac{x-1}{2})d(\frac{x-1}{2})+\int_{-\infty}^{+\infty}[(3x+1)-1]\phi(3x+1)d(3x+1)$$

$$=0.8\int_{-\infty}^{+\infty}(x+\frac{1}{2})\phi(x)dx+0.2\int_{-\infty}^{+\infty}(x-1)\phi(x)dx$$

$$=0.4-0.2=0.2$$

型三 二维离散型随机变量的数字特征

设X,Y独立同分布,且$X\sim$,$U=\max\{X,Y\},V=\min\{X,Y\}$,求:
(1)$E(U),D(U),E(V),D(V)$;
(2)ρ_{UV}.

(U,V)的分布律为

U \ V	1	2	$p_{i\cdot}$
1	$\frac{1}{9}$	0	$\frac{1}{9}$
2	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{8}{9}$
$p_{\cdot j}$	$\frac{5}{9}$	$\frac{4}{9}$	1

U	1	2
P	$\frac{1}{9}$	$\frac{8}{9}$

V	1	2
P	$\frac{5}{9}$	$\frac{4}{9}$

UV	1	2	4
P	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{4}{9}$

$$EU=\frac{17}{9},EU^2=\frac{33}{9}$$

$$DU=\frac{33}{9}-\frac{17^2}{81}=\frac{8}{81}$$

$$EV=\frac{13}{9}\cdot EV^2=\frac{21}{9},DV=EV^2-(EV)^2=\frac{20}{81}$$

$$EUV=\frac{25}{9},COV(U,V)=\frac{25}{9}-\frac{17}{9}\times\frac{13}{9}=\frac{4}{81}$$

$$\rho_{UV}=\frac{4}{81}/(\frac{2\sqrt{2}}{9}\cdot\frac{2\sqrt{5}}{9})=\frac{\sqrt{10}}{10}$$

型四 二维连续型随机变量的数字特征

$$X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu_2, \sigma_2^2), X, Y\text{独立}$$
$$\Rightarrow aX + bY \sim N(a\mu_1 + b\mu_2, a^2 a_1^2 + b^2 a_2^2)$$

$$X \sim N(1,1), Y \sim N(0,1), X, Y\text{独立}$$
$$P\{X+Y<1\} = \frac{1}{2}$$
$$\because X, Y\text{独立}, \therefore X+Y \sim N(1,2)$$
$$\therefore P\{X+Y<1\} = \frac{1}{2}$$

设$X \sim N(0,1), Y \sim N(0,1)$, 且$X, Y$相互独立.

(1)求$E(|X-Y|), D(|X-Y|)$;

(2)设$U = \max\{X, Y\}, V = \min\{X, Y\}$, 求$E(U), E(V)$及$E(UV)$.

令$Z = X - Y, Z \sim N(0,2) \Rightarrow \frac{Z}{\sqrt{2}} \sim N(0,1)$

①$E(|X-Y|) = E|Z| = \sqrt{2}E|\frac{Z}{\sqrt{2}}|$

$$= \sqrt{2}\int_{-\infty}^{+\infty} |u| \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \frac{u^2}{2} e^{-\frac{u^2}{2}} d(\frac{u^2}{2}) = \frac{2}{\sqrt{\pi}}$$

$$D|X-Y| = E(X-Y)^2 - (E|X-Y|)^2$$
$$E(X-Y)^2 = 2E(\frac{X-Y}{\sqrt{2}})^2 = 2E(\frac{Z}{\sqrt{2}})^2$$
$$\because \frac{Z}{\sqrt{2}} \sim N(0,1), \therefore E(\frac{Z}{\sqrt{2}}) = 0, D(\frac{Z}{\sqrt{2}}) = 1$$
$$\therefore E(\frac{Z}{\sqrt{2}})^2 = D(\frac{Z}{\sqrt{2}}) + (E(\frac{Z}{\sqrt{2}}))^2 = 1$$
$$\therefore D|X-Y| = 2 - \frac{4}{\pi}$$