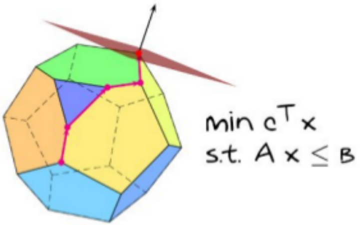
The simplex method

- The degenerate case
- Avoid cycling: Bland's pivot rule



Simplex algorithm: Bland's rule (Bland 1977)

LP DEGENERATE

Start with feasible basis B

B, B', B''' WITHOUT PROGRESS.

`while` B is not optimal

B_1

Cycling.

Let $i \in B$ be *smallest* index with $\lambda_i < 0$

Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0,\ j \in B \setminus \{i\}$ and $a_i^T d = -1$

Determine $K = \{k \colon 1 \leqslant k \leqslant m,\ a_k^T d > 0\}$

`if` $K = \emptyset$

assert LP unbounded

`else`

Let $k \in K$ be *smallest* index where $\min\limits_{k \in K} (b_k - a_k^T x^*)/a_k^T d$ is attained

update $B := B \setminus \{i\} \cup \{k\}$

Pivoting rule.

Bland's rule avoids cycles

$\lambda, \quad \lambda^T A = c^T$

$d \quad c^T d > 0$

Theorem

If Bland's rule is applied, the simplex algorithm terminates.

Proof: Assume basis is re-visited

$j \in \{1, \ldots, m\}$ largest index removed/added

$u \; v$

$B_0 \quad B_1 \quad B_2 \quad \ldots\ldots \quad B_{k-1} \quad B_k$

" $B \setminus \{u\} \cup \{v\}$ "

$\underbrace{B_0 \quad B_1 \quad B_2 \quad \ldots\ldots}_{j} = \underbrace{\ldots \quad B_{k-1} \quad B_k}$

$B_p \quad B_q$

$0 \le p < k$

$0 \le q < k$

$\lambda^{(p)T} \cdot A = c^T \quad , \quad c^T \cdot d^{(q)} > 0$

$\Rightarrow \underline{(\lambda^{(p)})^T (A \cdot d^{(q)}) > 0}$

Bland's rule avoids cycles

$B_p \to j$ $\qquad$ $B_q \to j$

$\Rightarrow \quad \exists\, i \in \{1,\dots,m\} \quad (\underbrace{\lambda_i^{(p)}}_{\neq 0}) \cdot (a_i^T \cdot d^{(q)}) > 0$

$i \in B_p$

CASE 1: $i > j$: $\quad i \in B_q$, $a_i^T \cdot d^{(q)} = 0$ ruled out.

CASE 2: $i < j$: $\quad \lambda_i^{(p)} > 0$, $\quad a_i^T \cdot d^{(q)} \not> 0$, $i \in K$

≤ 0

ruled out!

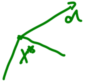
$a_i^T \cdot x \leq b_i$ is active at current vertex throughout iterations $0,\dots,k-1$

CASE 3: $i = j$

$\lambda_i^{(p)} < 0$, $\quad a_i^T \cdot d^{(q)} > 0$

ruled out!

$\lightning$



$\blacksquare$