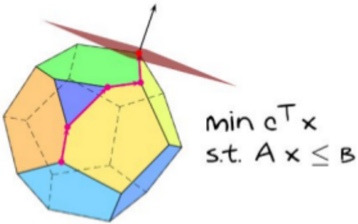
## The simplex method

- Bases and degeneracy
- Moving to a better neighbor



# Bases

A subset $B \subseteq \{1, \ldots, m\}$ of the row-indices with $|B| = n$ and $A_B$ non-singular is called *basis* of the LP.

If in addition $A_B^{-1} b_B$ is feasible, then $B$ is called *feasible basis*.

$x^* \in P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is vertex $\iff$ $\exists\, B \subseteq \{1, \ldots, m\}$

s.th. $|B| = n$, $A_B$ non-singular

and $x^* = A_B^{-1} \cdot b_B$

$$\boxed{\begin{array}{c} \text{MAX } c^T \cdot x \\ x \in P \end{array}}$$

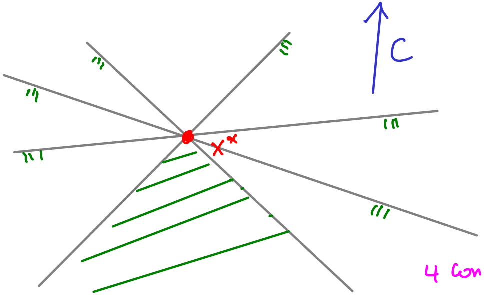
LP

# Vertices and bases

A vertex $x^* \in P$ is represented by a basis $B$.

$x^* = A_B^{-1} \cdot b_B$

A vertex $x^*$ can be represented by several bases.




MAX $c^T . x$
$Ax \le b$
$x \in \mathbb{R}^2$

4 constr.

$\binom{4}{2} = 6$

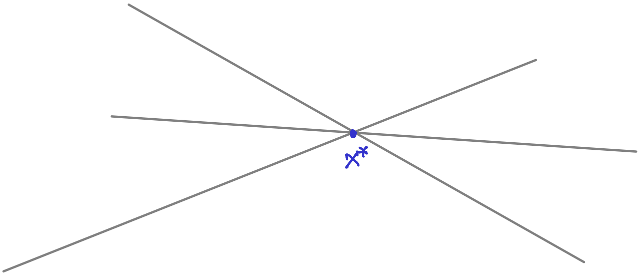
**Quiz:**

How many bases represent $x^*$?

- ○ 2
- ○ 4
- ☒ 6

# Degeneracy

A linear program $\max\{c^T x \colon x \in \mathbb{R}^n,\ Ax \leqslant b\}$ is *degenerate* of there exists an $x^* \in \mathbb{R}^n$ such that there are more than $n$ constraints of $Ax \leqslant b$ that are active at $x^*$.




$n = 2$

3 constraints

active at $x^*$

# Simplex algorithm

George Dantzig (1914 - 2005)

*Basic idea:*

Start with vertex $x^*$

`while` $x^*$ is not optimal

Find vertex $x'$ adjacent to $x^*$ with $c^T x' > c^T x^*$
update $x^* := x'$

Or assert that LP is unbounded.

$$\max \; c^T \cdot x$$
$$Ax \leq b$$
$$x \in \mathbb{R}^n$$

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

# Optimal bases

A basis $B$ is called *optimal* if it is feasible and the unique $\lambda \in \mathbb{R}^m$ with

$$\lambda^T A = c^T \quad \text{and } \lambda_i = 0,\ i \notin B$$

$\lambda_B^T \underline{A_B} = c^T$

$\lambda_B^T = c^T \cdot A_B^{-1}$

satisfies $\lambda \geqslant 0$.

## Theorem

If $B$ is optimal basis, then $x^* = A_B^{-1} b_B$ is optimal solution of LP.

$= \underline{A_B \cdot x^*}$

$a_1^T x \leq b_1 \quad (*\lambda_1)$

$\vdots$

$a_m^T x \leq b_m \quad (*\lambda_m)$

$+$

valid ineq.

$\Rightarrow$

$\underbrace{\lambda^T \cdot A}_{\boxed{c}^T} x \leq \lambda^T \cdot b = \lambda_B^T \cdot b_B = A_B^{-1} \cdot x^*$

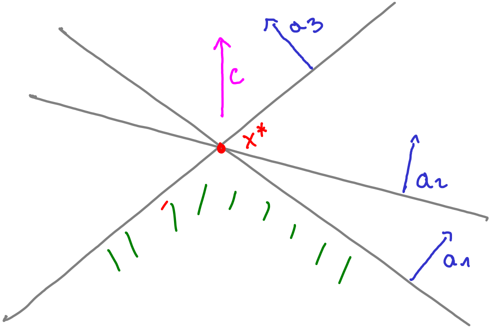
$= \underbrace{\lambda_B^T A_B}_{\boxed{c}^T} \cdot x^*$

$c^T \cdot x \leq c^T \cdot x^*$ valid ineq.

$\Rightarrow x^*$ optimal.

# Quiz

Which bases are optimal?



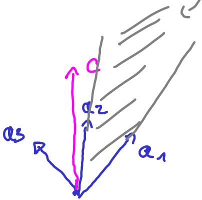

$\{1,2\}$

$\{1,3\}$

$\{2,3\}$

non-neg. linear comb. of $a_2$ and $a_1$

$\Rightarrow$ $c \notin$




$\lambda^T_{\{1,2\}} \cdot A_{\{1,2\}} = c^T$

$\lambda_{\{1,2\}} \ngeq 0$

# The non-degenerate case

max $c^T \cdot x$, $Ax \le b$

$A_B x \le b_B$ active at $x^*$

$A_{\bar{B}} x \le b_{\bar{B}}$ inactive.

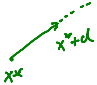
## Theorem

Suppose the LP is non-degenerate and $B$ is a feasible but not optimal basis, then $x^* = A_B^{-1} b_B$ is not an optimal solution.

$\lambda^T A = c^T$, $\lambda_j = 0 \ \forall j \notin B$, $\lambda_i < 0$ for some $i \in B$.

Compute $d \in \mathbb{R}^n$ s.th. $A_{B \setminus \{i\}} \cdot d = 0$, $a_i^T \cdot d = -1$ ($A_B$ non-sing)

$c^T \cdot d = \lambda_B^T \cdot A_B \cdot d = \underbrace{\lambda_i}_{<0} \cdot \underbrace{a_i^T \cdot d}_{=-1}$ $>$ $<$ $0$ $\le$




For $\varepsilon > 0$: $A_B(x^* + \varepsilon \cdot d)$ $\le$ $=$ $b_B$

$b_B + \varepsilon \cdot \begin{pmatrix} 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow i$ $\neq$

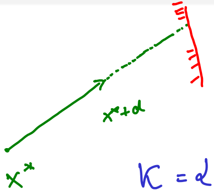
$\exists\, \varepsilon^* > 0$ s.th. $x^* + \varepsilon^* \cdot d$ is feasible.

$c^T(x^* + \varepsilon^* \cdot d) = c^T \cdot x^* + \underbrace{\varepsilon^* \cdot \underbrace{c^T \cdot d}_{>0}}_{>0}$

# Moving to a better neighbor

LP NON-DEG.

- $B$ not an optimal basis
- $x^* = A_B^{-1} b_B$ corresponding basic feasible solution
- $\lambda_i < 0$ for some $i \in B$
- $a_j^T d = 0$ , $j \in B \setminus \{i\}$
  $a_i^T d = -1$
- $c^T d > 0$
- there exists $\varepsilon > 0$ such that $x^* + \varepsilon d$ feasible




$a_1^T x \leq b_1$
$\vdots$
$a_m^T \cdot x \leq b_m$

$K = \{k : 1 \leq k \leq m,\ a_k^T \cdot d > 0\}$

CASE 1: $K = \emptyset$

LP UNBOUNDED

*Question:* How large can $\varepsilon$ be?

# Simplex algorithm

George Dantzig (1914 - 2005)

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$\max\ c^T \cdot x$

$Ax \leq b$

$x \in \mathbb{R}^n$

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Start with vertex $x^*$

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Or assert that LP is unbounded.

$K = \emptyset$

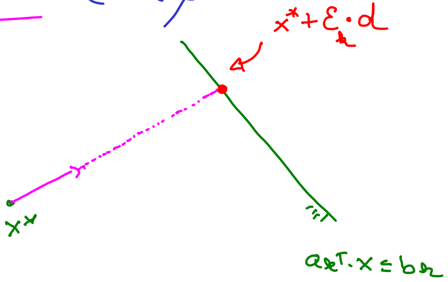
# Moving to a better neighbor

- $B$ not an optimal basis
- $x^* = A_B^{-1} b_B$ corresponding basic feasible solution
- $\lambda_i < 0$ for some $i \in B$
- $a_j^T d = 0$ , $j \in B \setminus \{i\}$
  $a_i^T d = -1$
- $c^T d > 0$
- there exists $\varepsilon > 0$ such that $x^* + \varepsilon d$ feasible

*Question:* How large can $\varepsilon$ be?

CASE 2: $K \neq \emptyset$

$k \in K$




$a_k^T(x^* + \varepsilon_k \cdot d) = b_k$

$\Leftrightarrow \quad \varepsilon_k = \dfrac{b_k - a_k^T \cdot x^*}{\underbrace{a_k^T \cdot d}_{>0}}$

$\varepsilon^* = \min_{k \in K} \varepsilon_k$

$k^* \in K$ Index where min is attained.

$x' = x^* + \varepsilon^* \cdot d$ , $B' = B \setminus \{i\} \cup \{k^*\}$ is BASIS

# Moving to a better neighbor

- $B$ not an optimal basis
- $x^* = A_B^{-1} b_B$ corresponding basic feasible solution
- $\lambda_i < 0$ for some $i \in B$
- $a_j^T d = 0\,, j \in B \setminus \{i\}$
  $a_i^T d = -1$
- $c^T d > 0$
- there exists $\varepsilon > 0$ such that $x^* + \varepsilon d$ feasible

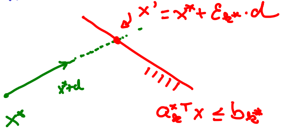
*Question:* How large can $\varepsilon$ be?

$B' = B \setminus \{i\} \cup \{k^*\}$ BASIS.

$d \perp a_j \,, \; j \in B \setminus \{i\}$

$d \not\perp a_{k^*} \,, \; d^T \cdot a_{k^*} > 0$

$\Rightarrow$ $a_{k^*}$ IS NOT A LINEAR COMB. OF THE $a_j,\; j \in B \setminus \{i\}$.




The ineq. $a_j^T \cdot x \leq b_j \,, j \in B'$ are active at $x'$.

$\Rightarrow$ $x'$ IS A VERTEX, ADJACENT TO $X^*$

# Simplex algorithm

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$K \neq \emptyset$

Or assert that LP is unbounded.

$K = \emptyset$