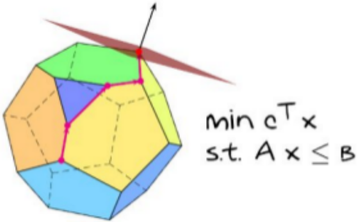
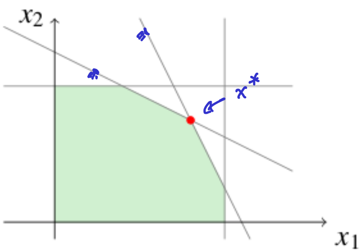
The geometry of linear programming

- Vertices and basic solutions



Alternative characterization of vertices: Intuition



$x^* \in P = \{x \in \mathbb{R}^n : Ax \leq b\}$

$\bar{A}x \leq \bar{b}$ active constraints

x^* is the unique solution

$\bar{A}x = \bar{b}$

$\Updownarrow$

$\operatorname{rank}(\bar{A}) = n$

$\Leftrightarrow$ colums of $\bar{A}$ are linearly independent

Quiz: Linear algebra

Let $A \in \mathbb{R}^{m \times n}$. Which of the following statements are not equivalent to the statement

$m<n$
$m=n$
$m>n$

$$\ker(A) = \{\mathbf{0}\}\,?$$

- The columns of A are linearly independent
- The rows of A are linearly independent $(1 \quad 1)$ $\ker(1,1) = \left\{ \lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$
- A has n linearly independent rows
- A has n linearly independent columns
- The image of A is $\mathbb{R}^m$ $(1,1)$
- The column-rank of A is n
- The row-rank of A is n
- The rank of A is n

Sub-system of active inequalities

$P = \{x \in \mathbb{R}^n : Ax \leqslant b\}, A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$

$Ax \leq b \approx$
$$\begin{aligned} a_1^T x &\leq \beta_1 \\ &\vdots \\ a_m^T x &\leq \beta_m \end{aligned}$$

For $x^* \in \mathbb{R}^n$, $I = \{i : 1 \leqslant i \leqslant m,\ a_i^T x^* = b_i\}$ are *indices of inequalities active at x^**.

Example: $P = \{x \in \mathbb{R}^3 : Ax \leqslant b\}$, $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \end{pmatrix}$, $x^* = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$I = \{1, 4\}$

Sub-system of active inequalities

Write $I = \{i_1, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$ and let $A_I = \begin{pmatrix} a_{i_1}^T \\ \vdots \\ a_{i_k}^T \end{pmatrix}$, $b_I = \begin{pmatrix} b_{i_1} \\ \vdots \\ b_{i_k} \end{pmatrix}$

$A_I x \leqslant b_I$ is *sub-system* of inequalities active at x^*.

Example: $P = \{x \in \mathbb{R}^3 : Ax \leqslant b\}$, $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \end{pmatrix}$, $x^* = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$A_I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ $\quad b_I = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\quad , I = \{1, 4\}$

Basic solutions

not necessarily feasible

Consider polyhedron $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$. A point $x^* \in \mathbb{R}^n$ is a *basic solution* if $\text{rank}(A_I) = n$.

$A_I x \leq b_I$ sub-system of active ineq. at x^*

If $x^* \in P$, then x^* is *basic feasible solution*.

Example: $P = \{x \in \mathbb{R}^3 : Ax \leqslant b\}$, $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \end{pmatrix}$, $x_1^* = \begin{pmatrix} -1/2 \\ 3/2 \\ 5/2 \end{pmatrix}$, $x_2^* = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

$A_1 x \leq b_1$

$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$\text{rank}(A_1) = 3$

x_1^* infeasible

basic sol.

$A_2 x \leq b_2$

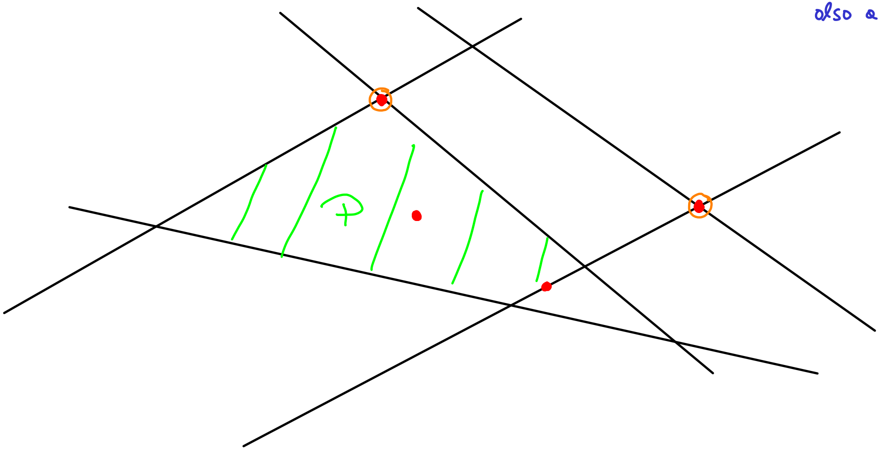
$A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $\text{rank}(A_2) = 3$

x_2^* feas. basic feasible solution

Quiz

Which of the following points are basic solutions?

basic feasible sol is
also a basic sol.



Vertices and basic feasible solutions

Theorem

if and only if

Let $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ and $x^* \in P$. Then x^* is vertex of P iff x^* is basic feasible solution.

$\Rightarrow$" $\underbrace{x^* \in P \text{ vertex}}$, assume not a basic feas. sol.

rank$(A_1) < n \Leftarrow\Rightarrow$ Kernel$(A_1) \overset{?}{\neq} \{0\}$

$Ax \leq b$

$A_1 x \leq b_1$ (active at x^*) $A_1 x^* = b_1$

$A_2 x \leq b_2$ (inactive at x^*) $A_2 x^* < b_2$

$\forall d \in \mathbb{R}^n \; \exists \varepsilon > 0$ s.th. $A_2(x^* \pm \varepsilon \cdot d) < b_2$

Vector
2 CHARACTERS
SMALL CAPS

$A_1(x^* \pm \varepsilon \cdot d)$

$= \underbrace{A_1 x^*}_{= b_1} \pm \varepsilon \cdot \underbrace{A_1 \cdot d}_{= 0}$

Let $d \in \text{Ker}(A_1) \setminus \{0\}$

$A_1 (x^* \pm \varepsilon \cdot d) = \boxed{b_1}$

$\exists c \in \mathbb{R}^n$ s.th. x^* unique opt sol. of the LP $\max\{c^T \cdot x : x \in P\}$

$c^T \cdot x^* > c^T \underbrace{(x^* + \varepsilon \cdot d)}_{\in P}$

$c^T \cdot x^* > c^T \underbrace{(x^* - \varepsilon \cdot d)}_{\in P}$

$\Rightarrow$

$0 > \varepsilon \cdot c^T \cdot d$

$0 > -\varepsilon \, c^T \cdot d$

↯

Vertices and basic feasible solutions

Theorem

Let $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ and $x^* \in P$. Then x^* is vertex of P iff x^* is basic feasible solution.

"$\Leftarrow$" x^* basic feas. sol.

$Ax \leq b$: $A_1 x \leq b_1$ active at x^*; $A_2 x \leq b_2$ inactive at x^*

$\mathrm{rank}(A_1) = n$

$\boxed{x^* \text{ unique sol of } A_1 x = b_1}$

$\exists$ $\underline{c^T \cdot x \leq \delta}$ valid for P, active at x^*

an inactive at any $\underline{y^* \in P,\ y^* \neq x^*}$

$A_1 \cdot y^* \neq b_1$

$A_1 x \leq b_1 \approx$ $a_1^T x \leq \beta_1$, $\vdots$, $a_k^T x \leq \beta_k$ $+$

$A_1 \cdot y^* \leq b_1$

$c^T = (a_1^T + \dots + a_k^T)$, $\delta = (\beta_1 + \dots + \beta_k)$

$c^T \cdot x \leq \delta$ $\underline{\text{valid for } P}$ $\underline{\text{active at } x^*}$

$\exists j$ s.th. $a_j^T \cdot y^* < \beta_j$

$c^T x^* = \sum_{i=1}^{k} a_i^T \cdot x^* = \sum_{i=1}^{k} \beta_i = \delta$

INACTIVE AT ANY OTHER $y^* \in P, y^* \neq x^*$

$c^T \cdot y^* = \sum_{i=1}^{k} a_i^T \cdot y^* = \sum_{\substack{i=1 \\ i \neq j}}^{k} \underbrace{a_i^T \cdot y^*}_{\leq \beta_i} + \underbrace{a_j^T \cdot y^*}_{< \beta_j} < \delta.$

$\Rightarrow x^*$ vertex