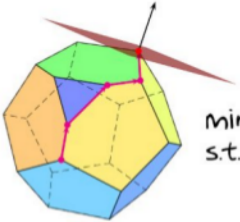
Fourier-Motzkin elimination



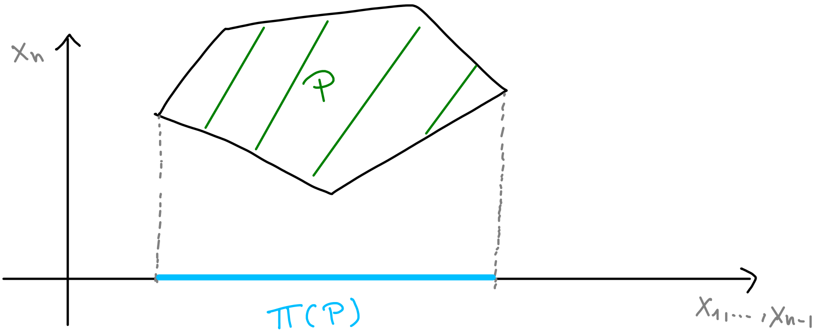
$$\min c^T x$$
$$\text{s.t. } Ax \le b$$

The projection mapping

The *projection mapping* is the function $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ with

$$\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}).$$

For $S \subseteq \mathbb{R}^n$ the *projection* of S is the set $\pi(S) = \{\pi(x) : x \in S\}$.



Quiz

Suppose $P = \{x \in \mathbb{R}^3 : Ax \leqslant b\}$ with

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 1 & -2 \\ 1 & 4 & 2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 3 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$2 \cdot x_1 + 3x_2 + 3x_3 \le 6$

$3x_3 \le 6 - 2 - 3$

$= 1$

$x_3 \le 1/3$

$-2x_3 \le -1$

$x_3 \ge \frac{1}{2}$

Is $(1, 1) \in \pi(P)$?

YES NO

Completing a point in the projection

- Suppose we want to know whether $(x_1^*, \ldots, x_{n-1}^*)$ is in $\pi(P)$ where $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$.
- Re-write each constraint $\sum_{j=1}^n a_{ij} x_j \leqslant b_i$ as

$I_> = \{i : a_{in} > 0\}$

$J_< = \{j : a_{jn} < 0\}$

$$a_{in} x_n \leqslant -\sum_{j=1}^{n-1} a_{ij} x_j + b_i \qquad \left(* \frac{1}{a_{in}} \right)$$

- If $a_{in} \neq 0$ divide both sides by a_{in}. With $\overline{x} = (x_1, \ldots, x_{n-1})$ we obtain an equivalent representation of P

$$\begin{array}{rlr} x_n^* \leqslant & d_i + f_i^T \overline{x}^* & i \in I_> \\ x_n \geqslant & d_j + f_j^T \overline{x}^* & j \in J_< \\ 0 \leqslant & d_k + f_k^T \overline{x}^* & j \in K \end{array}$$

MIN

MAX

x_n^*

$(x_1^*, \ldots, x_{n-1}^*)$ can be completed

$\Leftrightarrow \max_{j \in J_<} d_j + f_j^T \cdot \overline{x^*}$

$\leq \min_{i \in I_>} d_i + f_i^T \cdot \overline{x^*}$

And $0 \leq d_k + f_k^T \overline{x^*} \quad \forall k \in K$

The projection of a polyhedron

If $P \subseteq \mathbb{R}^n$ is represented by

$$\begin{aligned} x_n &\leqslant \underline{d_i + f_i^T \overline{x}} & i &\in I_> \\ x_n &\geqslant \underline{d_j + f_j^T \overline{x}} & j &\in J_< \\ \rightarrow\ \underline{0 \leqslant}&\ \underline{d_k + f_k^T \overline{x}} & k &\in K \end{aligned}$$

then $\pi(P)$ is represented by

$$\left.\begin{aligned} \underline{d_j + f_j^T \overline{x}} &\leqslant \underline{d_i + f_i^T \overline{x}} & i &\in I_>, j \in J_< \\ \rightarrow\ \underline{0} &\underline{\leqslant d_k + f_k^T \overline{x}} & k &\in K \end{aligned}\right\} (1)$$

Describes Polyhedron $\subseteq \mathbb{R}^{n-1}$

VARS: $(x_1, \dots, x_{n-1}) = \overline{x}$

$$\boxed{\max_{j \in J_<} d_j + f_j^T \cdot \overline{x^*} \leq \min_{i \in I_>} d_i + f_i^T \cdot \overline{x^*}}$$

Clearly holds.

<u>Proof:</u> "$\subseteq$" $x^* = (x_1^*, \dots, x_n^*) \in P$

to show $\overline{x^*} = (x_1^*, \dots, x_{n-1}^*)$ sat. (1)

$$d_j + f_j^T \cdot \overline{x^*} \leq x_n^* = x_n^* \leq d_i + f_i^T \cdot \overline{x^*}$$

"$\supseteq$" $\overline{x^*} = (x_1^*, \dots, x_{n-1}^*)$ sat (1).

<u>To show:</u> $\exists\ \underline{x_n^*}$ s.th. $(x_1^*, \dots, x_{n-1}^*, x_n^*) \in P$

The projection of a polyhedron (cont.)

Corollary

If $P \subseteq \mathbb{R}^n$ is a polyhedron, then $\pi(P)$ is a polyhedron.

Solving linear programming with Fourier-Motzkin elimination

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- Compute

$$\pi(Q), \pi(\pi(Q)), \ldots, \pi^n(Q)$$

and the corresponding inequality representations

$$A_1 x^{(1)} \leqslant b_1, \ldots, A^{(n)} x^{(n)} \leqslant b^{(n)}, \text{ where } x^{(i)} = \begin{pmatrix} y \\ x_1 \\ \vdots \\ x_{n-i} \end{pmatrix} \in \mathbb{R}^{n+1-i}.$$

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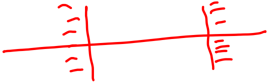

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- If $A^{(n)} x^{(n)} \leqslant b^{(n)}$ is infeasible, then LP is infeasible.

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Inefficient

and the corresponding inequality representations

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- If $A^{(n)} x^{(n)} \leqslant b^{(n)}$ is infeasible, then LP is infeasible.
- Otherwise determine largest $x^{(n)^*} = y^*$ and from there complement to $x^{(n-1)^*}, \ldots, x^{(0)^*}$.

$(x_1^*, \ldots, x_n^*, y^*)$