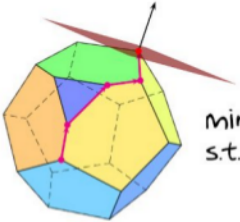
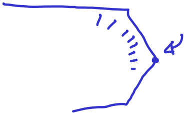
The geometry of linear programming



$$\min c^T x$$
$$\text{s.t. } Ax \leq B$$

The geometry of linear programming

Our plan



CORNERS

- What ARE CORNERS (Vertices)?
- LP is feasible and bounded and if vertices exist,
  ⇒ opt. sol. that is a vertex
- If vertices exist ⇒ only finitely many
- Solve feas. and bounded LP with vertices by enumerating all vertices and picking the best one
- Solving bounded LPs in a similar way

Inefficient!

Polyhedra

A set P of vectors in $\mathbb{R}^n$ is a *polyhedron* if $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ for some matrix A and some vector b.

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$

Max $c^T \cdot x$

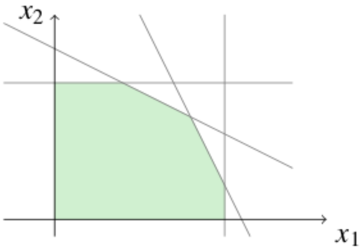
$Ax \leq b$

Set of feasible solutions

Polyhedra

A set P of vectors in $\mathbb{R}^n$ is a *polyhedron* if $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ for some matrix A and some vector b.

Example: Soft-drink production



$$A = \begin{pmatrix} 3 & 6 \\ 8 & 4 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 30 \\ 44 \\ 5 \\ 4 \\ 0 \\ 0 \end{pmatrix}$$

Polyhedra

A set P of vectors in $\mathbb{R}^n$ is a *polyhedron* if $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ for some matrix A and some vector b.

$$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

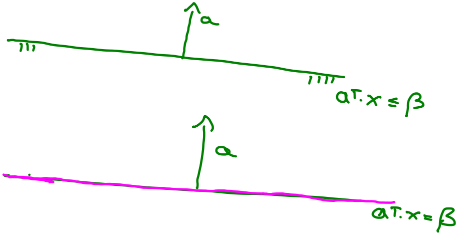
Example:

$P = \emptyset$ $\qquad$ $\{x \in \mathbb{R}^n : \mathbf{0}^T x \leqslant -1\} = \emptyset$

$a \in \mathbb{R}^n \setminus \{0\}$, $\beta \in \mathbb{R}$

$\{x \in \mathbb{R}^n : a^T . x \leq \beta\}$ half space

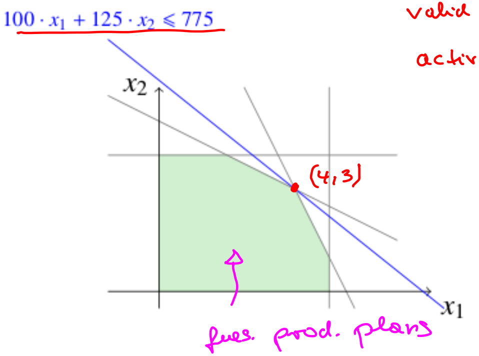
$\{x \in \mathbb{R}^n : a^T . x = \beta\}$ hyperplane



Valid and active inequalities

An inequality $a^T x \leqslant \beta$ is *valid* for a polyhedron P if each $x^* \in P$ satisfies $a^T x^* \leqslant \beta$.
An inequality $a^T x \leqslant \beta$ is *active* at $x^* \in \mathbb{R}^n$ if $a^T x^* = \beta$.

Example: Soft-drink production



valid for P = set of feas. prod. plans

active at $(4,3)$

Quiz

Which of the following inequalities are valid for each polyhedron $P \subseteq \mathbb{R}^n$

- $x_1 \geqslant 0$ not valid for $P = \left\{ \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$
- $\mathbf{0}^T x \leqslant -1$ $\emptyset$
- $\mathbf{0}^T x \leqslant 1$ → valid for each P.

$" \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

Suppose $P \subseteq \mathbb{R}^n$, $n \geqslant 2$ is a non-empty polyhedron containing $\mathbf{1}$ and suppose further that the inequalities $x_i \leqslant 1, \quad i = 1, \ldots, n$ are valid for P.

Which of the following inequalities are valid and active at $\mathbf{1}$:

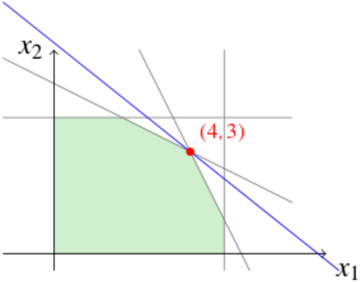
- $\mathbf{1}^T x \leqslant n$ → $\sum_{i=1}^{n} \underbrace{x_i}_{\leq 1} \leq n$ valid for P, active at $\mathbb{1}$
- $\mathbf{1}^T x \leqslant 1$ → violated by $\mathbb{1}$
- $\mathbf{0}^T x \leqslant 1$ → valid for P but not active at $\mathbb{1}$

Vertices

A point $x^* \in P$ is a *vertex* of P if there exists an inequality $a^T x \leqslant \beta$ such that

- $a^T x \leqslant \beta$ is valid for P and
- $a^T x \leqslant \beta$ is active at x^* and <u>not active at any other point</u> in P.

$100 \cdot x_1 + 125 \cdot x_2 \leqslant 775$



$x^* \in P$ is a vertex $\iff \exists\, c \in \mathbb{R}^n$ s.th.

x^* is <u>unique</u> optimal solution of the

linear program $\max\{c^T \cdot x : x \in P\}$

Quiz

Suppose $P \subseteq \mathbb{R}^n$ is a polyhedron such that $x_i \leqslant 1$ are valid for $i = 1, \ldots, n$. Is the following true:

If $x^* = (1, 1, \ldots, 1)^T$ is an element of P, then x^* is a vertex of P.

YES $\cancel{\bigcirc}$ NO $\bigcirc$

$\sum_{i=1}^{n} \underbrace{x_i}_{\leq 1} \leq n$ valid for P, active at $(1, \ldots, 1)^T$

If $y^* \in P$, $y^* \neq x^*$, then $y_i^* \leq 1$, $i = 1, \ldots, n$

$\exists j \; y_j^* < 1$

inactive at y^*

$$\sum_{i=1}^{n} y_i^* = \sum_{\substack{i=1 \\ i \neq j}}^{n} \underbrace{y_i^*}_{\leq 1} + \underbrace{y_j^*}_{< 1} < n$$