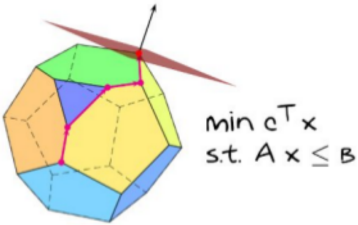
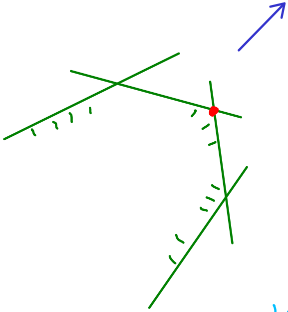
The simplex method

- Adjacent vertices
- Basic idea of simplex algorithm



Recap



$a_1^T \cdot x \le b_1$
$\vdots$
$a_m^T \cdot x \le b_m$

$$\max \ c^T \cdot x$$
$$Ax \le b$$
$$x \in \mathbb{R}^n$$

$\operatorname{rank}(A) = n$

bounded $\Rightarrow \exists$ opt. sol that is also a vertex

$x^* \in P$ vertex $\Leftrightarrow$ subsystem of active constr. $\bar{A}x \le \bar{b}$ satisfies $\operatorname{rank}(\bar{A}) = n$

$\Leftrightarrow \exists B \subseteq \{1, \dots, m\}, |B| = n$ s.th. A_B non-singular and $A_B \cdot x^* = b_B$ $\left(\Leftrightarrow x^* = A_B^{-1} b_B \right)$

$B = \{i_1, \dots, i_n\}$

$$A_B = \begin{pmatrix} a_{i_1}^T \\ \vdots \\ a_{i_n}^T \end{pmatrix}$$

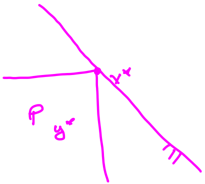
$$b_B = \begin{pmatrix} b_{i_1} \\ \vdots \\ b_{i_n} \end{pmatrix}$$

Adjacent vertices

Two distinct vertices x_1 and x_2 of $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ are *adjacent*, if there exist $n - 1$ *linearly independent inequalities* of $Ax \leqslant b$ active at both x_1 and x_2.

$$\begin{aligned} a_1^T \cdot x &\leq b_1 \\ &\vdots \\ a_k^T x &\leq b_k \end{aligned}$$

are linearly indep. if $a_1, \ldots, a_k$ are lin. indep.

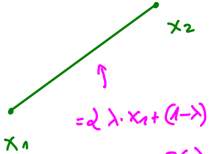


$x^* \in P$ is vertex $\Leftrightarrow$ $\exists$ n lin. indep. inequalities of $Ax \leq b$ that are active at x^*

Adjacent vertices

Theorem

$x_1 \neq x_2 \in P$ are adjacent iff there exists $c \in \mathbb{R}^n$ such that set of optimal solutions of $\max\{c^T x : x \in P\}$ is $\underbrace{\{\lambda x_1 + (1-\lambda)x_2 : \lambda \in \mathbb{R},\ 0 \leqslant \lambda \leqslant 1\}}_{\text{line segment spanned by } x_1 \text{ and } x_2}$.



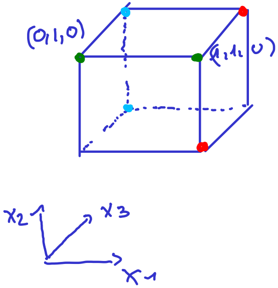
$= \{\lambda \cdot x_1 + (1-\lambda)x_2 : \lambda \in \mathbb{R},\ 0 \leq \lambda \leq 1\}$

Proof: Similar to proof of Vertex and Basic Feasible solution are equivalent concepts.

Quiz

$P = \{x \in \mathbb{R}^3 : \mathbf{0} \leqslant x \leqslant \mathbf{1}\}$

Which pairs of vertices of P are adjacent?



$x_2 \leq 1$

$x_3 \geq 0$

$(0,1,0)$

$(0,0,-1)$

Simplex algorithm

George Dantzig (1914 - 2005)

$\max \ c^T \cdot x$

$Ax \leq b$

$x \in \mathbb{R}^n$

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$

Basic idea:

Start with vertex x^*

`while` x^* is not optimal

Find vertex x' adjacent to x^* with $c^T x' > c^T x^*$
update $x^* := x'$

Or assert that LP is unbounded.