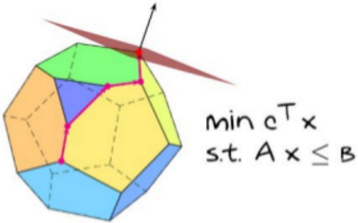
The geometry of linear programming

- Optimal vertices



Optimality of vertices

Theorem

If a linear program $\max\{c^T x : x \in \mathbb{R}^n, Ax \leqslant b\}$ is feasible and bounded and if $\operatorname{rank}(A) = n$, then the LP has an optimal solution that is a vertex.

proof: $x^* \in P$, $Ax \leq b$ → $A_1 x \leq b_1$ active at x^*; → $A_2 x \leq b_2$ inactive at x^*

x^* not a vertex, then rank $(A_1) < \boxed{n}$

Idea: construct $y^* \in P$ s.th 1.) $c^T \cdot y^* \geq c^T \cdot x^*$

2.) $Ax \leq b$ → $\bar{A}_1 x \leq \bar{b}_1$ active at y^*; → $\bar{A}_2 x \leq \bar{b}_2$ in-active at y^*

rank$(\bar{A}_1) \geq$ rank$(A_1) + 1$

$\boxed{\text{This procedure can not be repeated more than } n \text{ times}}$

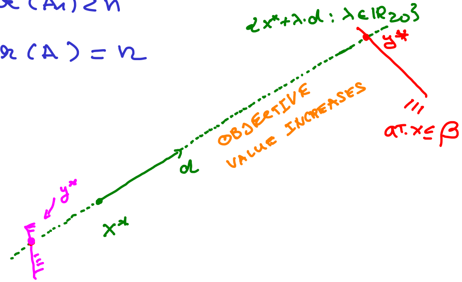
For any $x^* \in P$ $\exists$ vertex $z^* \in P$ s.th $c^T \cdot z^* \geq c^T \cdot x^*$

$\Rightarrow$ $\max\{c^T \cdot x : x \in \mathbb{R}^n, Ax \leq b\}$ exists and it is attained at a vertex

Optimality of vertices

$x^* \in P$ $\quad Ax \le b \to$ $A_1 x \le b_1$ (active) $\quad$ rank$(A_1) < n$

$\qquad\qquad\qquad\quad\to$ $A_2 x \le b_2$ (inactive) $\quad$ rank$(A) = n$



Let $d \in \ker(A_1) \setminus \{0\}$, w.l.o.g. $c^T\cdot d \ge 0$

CASE 1: $c^T\cdot d > 0$

VECTOR
2 CHARACTERS
SMALL CAPS

$$A_1 y^* = \boxed{b_1}$$

$y^* \in P$

$$a^T y^* = \boxed{\beta}$$

Greek letter
small caps

$$\text{rank}\begin{pmatrix} A_1 \\ a^T \end{pmatrix} = \text{rank}(A_1) + \boxed{1}$$

Assume $a^T = z^T A_1$
$\qquad\qquad\uparrow$ vector

$c^T\cdot y^* > c^T\cdot x^*$

$$\beta = a^T\cdot y^* = a^T(x^* + \lambda\cdot d) = a^T\cdot x^* + \lambda\cdot a^T\cdot d$$

$$= \underbrace{a^T\cdot x^*}_{<\beta} + \lambda\cdot z^T\cdot \underbrace{A_1\cdot d}_{=0} \quad ↯$$

CASE 2: $c^T\cdot d = 0$ $\qquad$ rank$(A) = n$

$\qquad\qquad\qquad\qquad\qquad$ $\ker(A) = \{0\}$

Similar argument!

∎

Consequence: Restrict to vertices

MAX $c^T \cdot x$

$Ax \leq b$

$x \in \mathbb{R}^n$

Bounded

$\text{Rank}(A) = n$

$A \in \mathbb{R}^{m \times n}$

Important consequence:

CAN BE SOLVED BY enumerating all vertices and by picking the best one.

$\Rightarrow \exists$ vertex that is also opt. sol.

x^* is vertex $\Rightarrow \exists\ B \subseteq \{1, \dots, m\}$, s.th. , $|B| = n$

x^* is unique solution of $A_B \cdot x = b_B$

$$\binom{\boxed{m}}{\boxed{n}} = \frac{m!}{n!\,(m-n)!}$$

- Enumerate all $B \subseteq \{1, \dots, m\}$, $|B| = n$
- If A_B is non-singular, $\underbrace{A_B^{-1} \cdot b_B}_{}$ feasible $\rightarrow$ store it.

Quiz

Consider

$$\begin{aligned} \max \quad & x_1 + x_2 & & \\ & x_1 + x_2 & \leqslant & \ 1 \\ & x_1 & \leqslant & \ 1 \\ & x_2 & \leqslant & \ 1 \end{aligned}$$

Which of the following statements are true?

- Each optimal solution is a vertex.
- There exists an optimal solution that is a vertex.
- There are infinitely many optimal solutions.