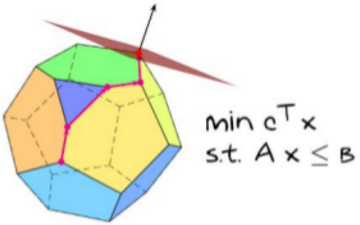
The simplex method

- The simplex algorithm in basis notation
- Example



Simplex algorithm in basis notation

$\lambda^T \cdot A = c^T$ and $\lambda_j = 0 \ \forall j \notin B$

Start with feasible basis B

`while` B is not optimal

Let $i \in B$ be index with $\lambda_i < 0$

Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0,\ j \in B \setminus \{i\}$ and $a_i^T d = -1$

Determine $K = \{k \colon 1 \leqslant k \leqslant m,\ a_k^T d > 0\}$

`if` $K = \emptyset$

assert LP unbounded

`else`

Let $k \in K$ index where $\min\limits_{k \in K}(b_k - a_k^T x^*)/a_k^T d$ is attained

update $B := B \setminus \{i\} \cup \{k\}$

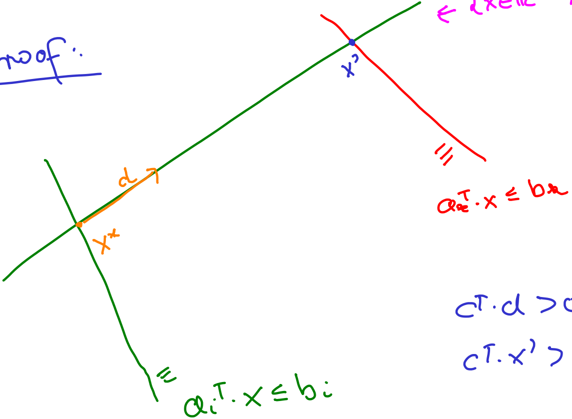
The non-degenerate case

$B \to B' \to B'' \to B''' \cdots\cdots B^{(i)} \quad B^{(j)}$

Theorem

If the linear program is non-degenerate, then the simplex algorithm terminates.

Proof:



$x^* = A_B^{-1} \cdot b_B$

$B' = B \setminus \{i\} \cup \{k\}$

$x' = A_{B'}^{-1} \cdot b_{B'}$

$c^T \cdot d > 0$

$c^T \cdot x' > c^T \cdot x^*$

Each iteration:

Progress!

$\boxed{\equiv}$

Example

LP defined by

$$c = \begin{pmatrix} 6 \\ 14 \\ 13 \end{pmatrix} A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} b = \begin{pmatrix} 10 \\ 14 \\ 11 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- Starting basis: $B = \{1, 2, 3\}$
- $A_B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, b_B = \begin{pmatrix} 10 \\ 14 \\ 11 \end{pmatrix},$

  $x = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$
- Objective value: 63.0
- $\lambda_B^T = \begin{pmatrix} 36/5 & -4/5 & 1/5 \end{pmatrix}$

$\lambda^T \cdot A = c^T$

$\lambda_B^T \cdot A_B = c^T$

$\lambda_B^T = c^T \cdot A_B^{-1}$

while B is not optimal

Let $i \in B$ be index with $\lambda_i < 0$

Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0,\ j \in B \setminus \{i\}$ and $a_i^T d = -1$

Determine $K = \{k\colon 1 \leqslant k \leqslant m,\ a_k^T d > 0\}$

if $K = \emptyset$

assert LP unbounded

else

Let $k \in K$ index where $\min\limits_{k \in K}(b_k - a_k^T x^*)/a_k^T d$ is attained

update $B := B \setminus \{i\} \cup \{k\}$

- Which index leaves B? $\boxed{2}$

Example

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- *2* leaves B

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- Let $i \in B$ be index with $\lambda_i < 0$
- Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0,\ j \in B \setminus \{i\}$ and $a_i^T d = -1$
- Determine $K = \{k\colon 1 \leqslant k \leqslant m,\ a_k^T d > 0\}$
- `if` $K = \emptyset$
  - *assert LP unbounded*
- `else`
  - Let $k \in K$ index where $\min_{k \in K}(b_k - a_k^T x^*)/a_k^T d$ is attained
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- 2 leaves B
- $d = \begin{pmatrix} -2/5 \\ 3/5 \\ -2/5 \end{pmatrix}$, $A_B \cdot d = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$
- $A \cdot d = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2/5 \\ -3/5 \\ 2/5 \end{pmatrix}$ What is K? incr. no separator $\boxed{K=\{4,6\}}$

Example

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- *2* leaves B
- $K = \{4, 6\}$
- Which index enters B? $\boxed{6}$

$4/(2/5) = 10$

$3/(2/5) = 7.5$

Example

```
while B is not optimal
    Let i ∈ B be index with λ_i < 0
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    if K = ∅
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    else
        Let k ∈ K index where min_{k∈K}(b_k − a_k^T x*)/a_k^T d is attained
        update B := B \ {i} ∪ {k}
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- *2* leaves B
- $K = \{4, 6\}$
- *6* enters B
- *New basis*: $B = \{1, 6, 3\}$

Example (cont.)

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$$c = \begin{pmatrix} 6 \\ 14 \\ 13 \end{pmatrix} A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} b = \begin{pmatrix} 10 \\ 14 \\ 11 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- Current basis: $B = \{1, 6, 3\}$
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  $x = \begin{pmatrix} 1 \\ 9/2 \\ 0 \end{pmatrix}$
- Objective value: 69.0
- $\lambda_B^T = \begin{pmatrix} 8 & 2 & -1 \end{pmatrix}$

`while` B is not optimal
  Let $i \in B$ be index with $\lambda_i < 0$
  Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0,\ j \in B \setminus \{i\}$ and $a_i^T d = -1$
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  `if` $K = \emptyset$
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  `else`
    Let $k \in K$ index where $\min_{k \in K} (b_k - a_k^T x^*)/a_k^T d$ is attained
    update $B := B \setminus \{i\} \cup \{k\}$

- Which index leaves B? $\boxed{3}$

Example (cont.)

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- *3* leaves B

Example (cont.)

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- *3* leaves B
- $d = \begin{pmatrix} -1 \\ 1/2 \\ 0 \end{pmatrix}$, $A_B \cdot d = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$, $A \cdot d = \begin{pmatrix} 0 \\ -3/2 \\ -1 \\ 1 \\ -1/2 \\ 0 \end{pmatrix}$

What is K? incr. no separator

$\boxed{\{4\}}$

Example (cont.)

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- Which index enters B? $\boxed{4}$

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- Objective value: 69.0
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- *3* leaves B
- $K = \{4\}$
- *4* enters B
- *New basis*: $B = \{1, 6, 4\}$

Example (cont.)

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- Current basis: $B = \{1, 6, 4\}$
- $A_B = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$, $b_B = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$, $x = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$

  Objective value: 70.0
- $\lambda_B^T = \begin{pmatrix} 7 & 1 & 1 \end{pmatrix}$

optimal Basis !

- $B = \{1, 6, 4\}$ is *optimal basis*
- $x = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$ is optimal solution