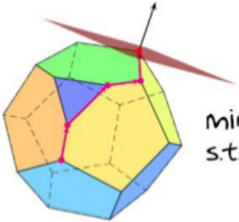
The geometry of linear programming

- A first (inefficient) algorithm for linear programming



$$\min c^T x$$
$$\text{s.t. } A x \leq B$$

Algorithm for bounded LPs with vertices

!! INEFFICIENT

Solve $\max\{c^T x \colon x \in \mathbb{R}^n,\ Ax \leqslant b\}$

- BOUNDED
- rank(A) = n

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$S := \emptyset$
for each $B \in \binom{[m]}{n}$
        if $A_B$ is invertible and $x_B = A_B^{-1} b_B$ feasible
            $S := S \cup \{x_B\}$
if $S = \emptyset$
    LP not feasible
else
    return $x \in S$ with largest obj. value $c^T x$
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Why inefficient?

$$\#\{B : B \subseteq \{1,\dots,n\},\ |B| = n\} = \binom{m}{n} = \frac{m!}{n!\,(m-n)!}$$

$\underline{m=2n}$:

$$\binom{m}{n} = \overset{\geq 2}{\boxed{\frac{2n}{n}}}\ \overset{\geq 2}{\boxed{\frac{(2n-1)}{(n-1)}}}\cdots\overset{\geq 2}{\boxed{\frac{(n+1)}{(1)}}}$$

$$\geq 2^{n}$$

$m = 200$ CONSTR.

$n = 100$ VARIABLES.

Alg generates $\geq 2^{100}$ Bs

2^{25} Bs $\geq$ 1 sec

$\leq 2^{50}$ Bs in one year.

1 year $\leq 2^{25}$ sec

years to finish $\geq \dfrac{2^{100}}{2^{50}} = 2^{50}$

$\geq$ Age of the universe.

Existence of optimal solutions

Theorem

A feasible and bounded linear program $\max\{c^T x \colon x \in \mathbb{R}^n,\ Ax \leqslant b\}$ has an optimal solution.

proof:

$$\begin{array}{l} \max\ c^T.x \\ Ax \leq b \\ x \in \mathbb{R}^n \end{array} \quad \approx \quad \max\ c^T(z-y) \quad \boxed{\begin{array}{l} A(z-y) \leq b \\ z \geq 0 \\ y \geq 0 \end{array}} \quad \begin{array}{l} z \in \mathbb{R}^n \\ y \in \mathbb{R}^n \end{array} \qquad A'\binom{z}{y} \leq b'$$

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} n$$

$$A' = \begin{pmatrix} A & -I \\ -I & -A \end{pmatrix} \bigcirc ,$$

$$A' = \begin{pmatrix} A & -A \\ -I & 0 \\ 0 & -I \end{pmatrix} \quad \cancel{\bigcirc}$$

$\mathrm{rank}(A') = 2n$

An inefficient algorithm for linear programming

- Goal: Solve *bounded* linear program

$$\max\{c^T x : x \in \mathbb{R}^n,\ Ax \leqslant b\}.$$

- Transform into equivalent linear program

$$\max\{c^T(x_1 - x_2) : x_1, x_2 \in \mathbb{R}^n,\ A(x_1 - x_2) \leqslant b,\ x_1 \geqslant 0,\ x_2 \geqslant 0\}.$$

Inefficient!

- Enumerate all basic solutions.
- If all basic solutions are infeasible, assert LP infeasible.
- Otherwise, output feasible basic solution with largest objective value.