Fractals, Vol. 30, No. 6 (2022) 2250108 (7 pages)

DOI: 10.1142/S0218348X22501080

FRACTAL NETWORKS MODELED BY SOME FRACTAL CARPET

QIN WANG*
College of Big Data and Software Engineering
Zhejiang Wanli University
Ningbo 315100, P. R. China
qinwang@126.com

KEQIN CUI,† WENJIA MA‡ and QINGCHENG ZENG§
School of Mathematics and Statistics
Ningbo University
Ningbo 315211, P. R. China
†*ckq0613@163.com*
‡*2429510040@163.com*
§*1062670874@qq.com*

Received February 25, 2022
Accepted April 1, 2022
Published June 6, 2022

Abstract

We construct a family of evolving networks modeled by some fractal carpet which is not symmetric. We investigate the scale-free effect and small-word effect of our fractal networks, including the power-law of the cumulative degree distribution, the average clustering coefficient and the average geodesic distance.

Keywords: Fractal Network; Self-Similarity; Scale-Free Effect; Small-World Effect.

*Corresponding author.

1. INTRODUCTION

Recently complex networks have attracted more and more attention of scholars. There are some interesting properties revealed in complex networks, such as the small-world effect[1] and scale-free effect,[2] also see Ref. 3.

Mandelbrot[4] introduced notions of the fractal and the self-similarity, and then Hutchinson[5] characterized deterministic self-similar sets in terms of the iterated function system (IFS) and calculated the Box and Hausdorff dimensions of self-similar sets under the open set condition. Song *et al.*[6–8] discovered the self-similarity and fractal dimension on many real-world networks, for example, the WWW, the human brain, metabolic network, protein interaction network and so on. On the other hand, complex networks can be constructed modeled by self-similar fractals, for example, Sierpinski networks (by Zhang *et al.*[9–11]), Koch networks (by Dai *et al.*[12,13]) and Vicsek networks (by Zhang *et al.*[14] and by Deng *et al.*[15]).

In particular, Xi *et al.* proposed a new approach to construct fractal networks based on the symbolic system and the IFS,[16,17] and studied average geodesic distances on networks[18–20]; From then on, using the approach of the symbolic system and the IFS, Xue *et al.*,[21–24] Niu and Shao[25] and Huang and Peng[26] researched fractal networks modeled by self-similar fractals. In their models, fractals have *symmetric* structures.

In this paper, we will construct evolving networks based on some *non-symmetric* fractal carpet and investigate the small-world effect and scale-free effect of our networks.

Let us consider an initial solid square $Q = [0,1]^2$ and an IFS $\{S_i(x) = x/3 + a_i\}_{i=1}^5$ of some planar carpet, where

$$a_1 = (0,0), \quad a_2 = (1/3,0), \quad a_3 = (2/3,0),$$
$$a_4 = (2/3,2/3), \quad a_5 = (0,2/3),$$

then there is a unique invariant nonempty compact set

$$E = \bigcup_{i=1}^5 S_i(E).$$

We call E a fractal carpet. Given a word $\sigma = i_1 i_2 \cdots i_k \in \{1,\ldots,5\}^k$, we write its length $|\sigma| = k$ and

$$S_\sigma = S_{i_1 i_2 \cdots i_k} = S_{i_1} \circ S_{i_2} \cdots \circ S_{i_k}$$

and denote by $Q_\sigma = S_\sigma(Q)$ a basic (solid) square of rank k with side length 3^{-k}. Note that $E = \bigcap_{k=1}^\infty E_k$ with $E_k = \bigcup_{|\sigma|=k} S_\sigma(Q)$, where E_k is said to be the kth construction of the fractal carpet. See Fig. 1 for the first four constructions of the fractal carpet.

By the iteration, the above IFS provides a natural encoding approach for the fractal carpet. We use the word σ to encode (or represent) the solid square Q_σ, see Fig. 2.

Suppose $\sigma = i_1 i_2 \cdots i_k$ and $\tau = i_1 i_2 \cdots i_k i_{k+1} \cdots i_{|\tau|}$ with $\sigma \neq \tau$, we call σ a prefix of τ, denoted by $\sigma \prec \tau$. In particular, we call σ the father of τ and call τ the child of σ, if $\sigma \prec \tau$ and $|\tau| = |\sigma| + 1$. Then any word has five children, and any word $\sigma = i_1 i_2 \cdots i_{k-1} i_k$ has a unique father $\sigma^- = i_1 i_2 \cdots i_{k-1}$. For any integer $k \geq 1$, we let Δ_k denote the set of all words of length k. When $k = 0$, let $\Delta_0 = \{\emptyset\}$, where $\emptyset$ is the empty word with $|\emptyset| = 0$.

We take the above finite words including the empty word as the node set of the networks. Fix an integer $t \geq 0$, we construct G_t with node set $V_t = \bigcup_{k=0}^t \Delta_k$. We also define the edge set of G_t as follows, for two distinct words $\sigma, \tau \in V_t$, denoted by $\sigma \sim \tau$, if and only if the intersection of ∂Q_σ and ∂Q_τ contains at least a non-degenerate line segment, where ∂A is the boundary of the set A. Then we obtain $\{G_t\}$. See Fig. 3 for G_3.

Let $\sharp V_t$ be the number of words in the set V_t. Then we have

$$\sharp V_t = 1 + 5 + 5^2 + \cdots + 5^t = (5^{t+1} - 1)/4. \quad (1.1)$$

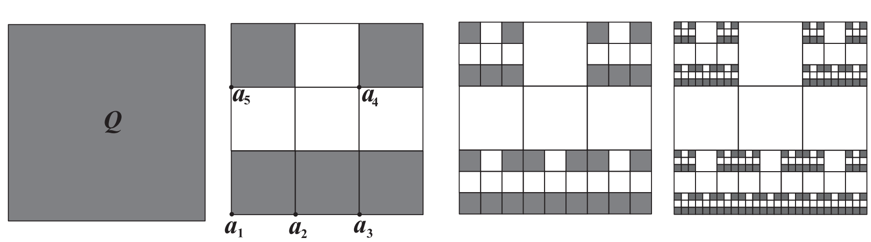

Fig. 1 The first four constructions of the fractal carpet.

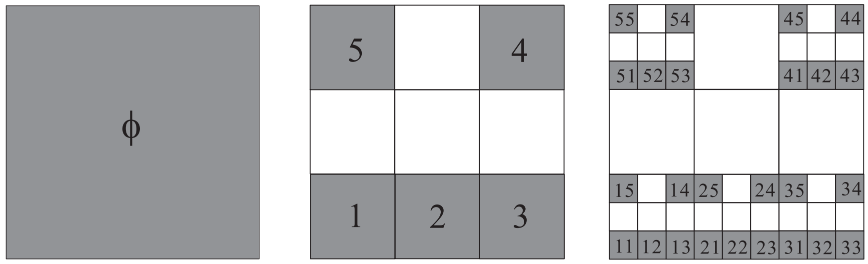

Fig. 2 Encoding basic squares of rank $0, 1, 2$.

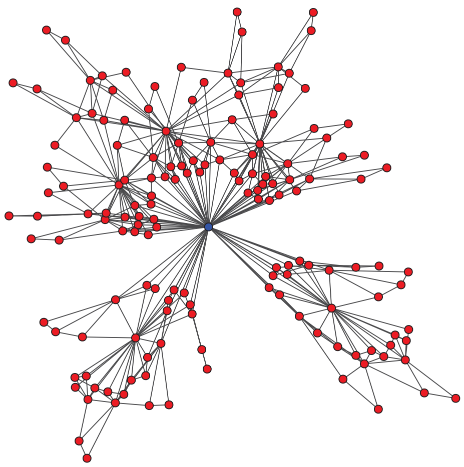

Fig. 3 G_3.

We notice that if $\sigma = i_1 i_2 \cdots i_k \in \{1,2,3\}^k$ then Q_σ intersects Q with the bottom line of $Q = [0,1]^2$. In the same way, we have the following remark.

Remark 1. If $\tau = \sigma\beta$, then $\sigma \sim \tau$ if and only if

$$\beta \in \{1,2,3\}^{|\beta|}, \{3,4\}^{|\beta|}, \{4,5\}^{|\beta|} \text{ or } \{5,1\}^{|\beta|}.$$

For any word $\sigma = i_1 i_2 \cdots i_k \neq \emptyset$, we can find a unique shortest word $f(\sigma)$ such that

$$f(\sigma) \prec \sigma \quad \text{and} \quad f(\sigma) \sim \sigma.$$

For a word $\sigma = \tau\tau'$, where τ' is the maximal suffix of σ such that $\tau' \in \{1,2,3\}^{|\tau'|}$, $\{3,4\}^{|\tau'|}$, $\{4,5\}^{|\tau'|}$ or $\{5,1\}^{|\tau'|}$. Iterating f again and again, we obtain a sequence from σ to $\emptyset$:

$$\sigma \sim f(\sigma) \sim f^2(\sigma) \sim \cdots \sim f^n(\sigma) = \emptyset.$$

Then we let $\omega(\sigma) = n$.

Example 1. For $\sigma = 1245123 = (12)(45)(123)$, we have

$$f(\sigma) = (12)(45),$$
$$f^2(\sigma) = (12),$$
$$f^3(\sigma) = \emptyset,$$

and $\omega(\sigma) = 3$.

2. CUMULATIVE DEGREE DISTRIBUTION

Let $\deg(\sigma)$ be the numbers of neighbors of σ, and $P_{\text{cum}}(u)$ the cumulative degree distribution of G_t, i.e.

$$P_{\text{cum}}(u) = \frac{\sharp\{\sigma : \deg(\sigma) \geq u\}}{\sharp V_t}.$$

Theorem 1. *Letting $t \geq 10$ and $u \geq 3^{t/2+1}$, we have*

$$P_{\text{cum}}(u) \propto u^{-\log_3 5}.$$

See Fig. 4 for the log–log graphs of cumulative degree distributions with $t = 4, 5, 6$. To prove this theorem, we first claim the following facts.

Claim 1. *Suppose $\sigma \in V_t$.*

(1) *For $i > |\sigma|$, we have*

$$3^{i-|\sigma|} \leq \sharp\{\tau \in \Delta_i : \tau \sim \sigma\} \leq 3^{i-|\sigma|} + 3 \cdot 2^{i-|\sigma|}.$$

(2) *For $i \leq |\sigma|$, we get*

$$\sharp\{\tau \in \Delta_i : \tau \sim \sigma\} \leq 2.$$

Lemma 1. *Suppose $t \geq 4$, $k \leq \frac{t}{2}$ and $t - k \geq 5$, we have*

$$\{\sigma : \deg(\sigma) \geq 3^{t-k+1}\} = \{\sigma : |\sigma| \leq k\}.$$

Proof. If $|\sigma| < k$, by (1) of Claim 1 we have

$$\deg(\sigma) \geq \sum_{i>|\sigma|} 3^{i-|\sigma|} = \frac{3}{2}(3^{t-k+1} - 1) \geq 3^{t-k+1}$$

since $(3^{t-k+1} - 1) \geq \frac{2}{3} \cdot 3^{t-k+1}$ due to $t - k \geq 2$.

If $k \leq |\sigma| \leq t$, by Claim 1 we have

$$\deg(\sigma) \leq \sum_{i>|\sigma|} 3^{i-|\sigma|} + 3\sum_{i>|\sigma|} 2^{i-|\sigma|} + 2\sum_{i\leq|\sigma|} 1$$
$$= \frac{3}{2}(3^{t-|\sigma|} - 1) + 6(2^{t-|\sigma|} - 1) + 2|\sigma|$$

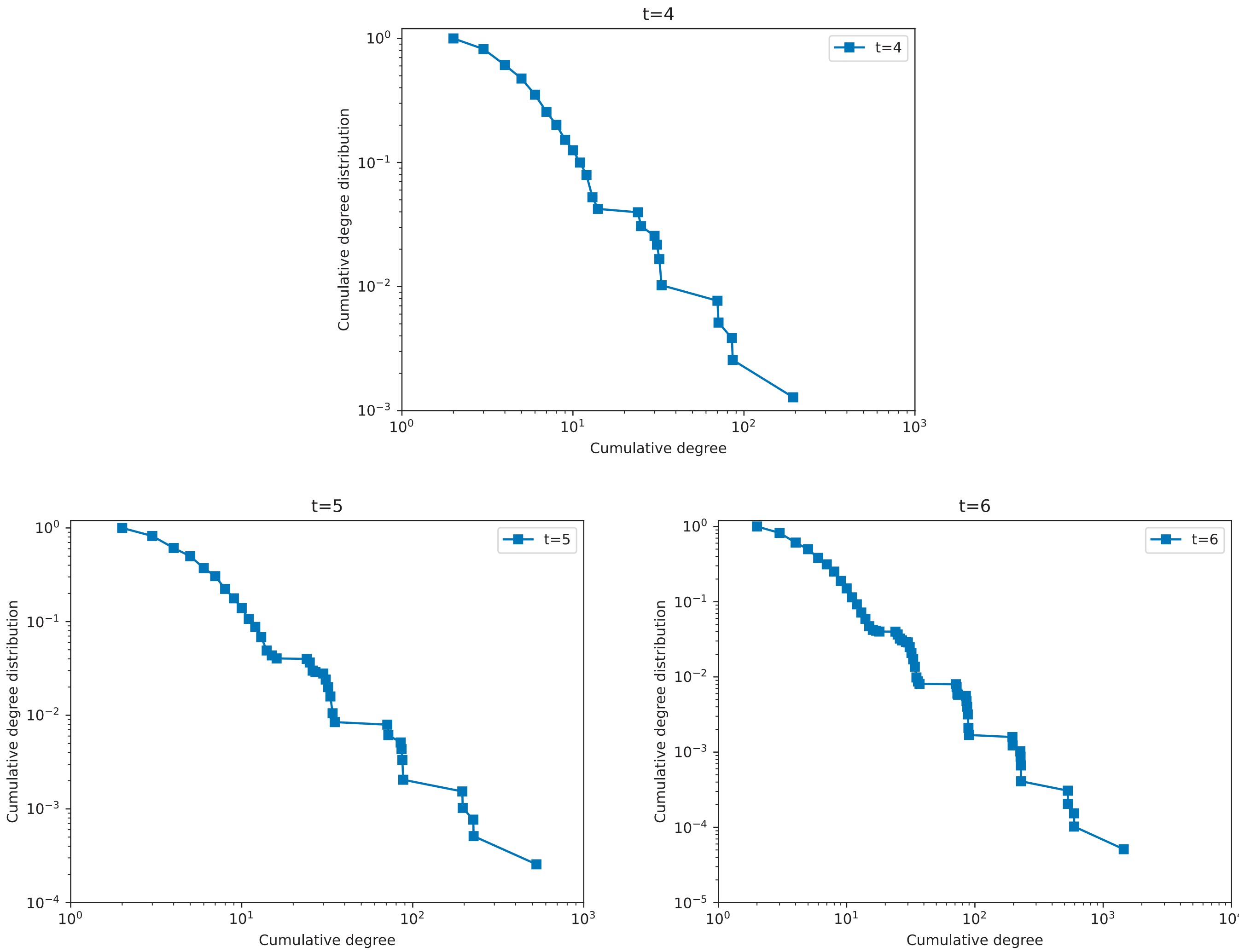

Fig. 4 The log–log graphs of cumulative degree distributions for $t = 4, 5, 6$.

$$\leq \frac{3}{2} \cdot 3^{t-k} + 6 \cdot 2^{t-k} + 2t$$

$$< \frac{3}{2} \cdot 3^{t-k+1}$$

since $6 \cdot 2^{t-k} < \frac{3}{2} \cdot 3^{t-k}$ and $2t < \frac{3}{2} \cdot 3^{t/2} \leq \frac{3}{2} \cdot 3^{t-k}$ due to

$$t \geq 4, t-k \geq \frac{t}{2} \text{ and } t-k \geq 4. \qquad \square$$

Proof of Theorem 1. Suppose $3^{t-k+1} \leq u < 3^{t-(k-1)+1}$ with some integer $k \leq t/2$, then

$$\sharp\{\sigma : \deg(\sigma) \geq 3^{t-k+2}\} \leq \sharp\{\sigma : \deg(\sigma) \geq u\} \leq \sharp\{\sigma : \deg(\sigma) \geq 3^{t-k+1}\}.$$

According to Lemma 1, we have

$$\frac{\sharp\{\sigma : |\sigma| \leq k-1\}}{\sharp V_t} \leq P_{\text{cum}}(u) \leq \frac{\sharp\{\sigma : |\sigma| \leq k\}}{\sharp V_t},$$

where $\frac{\sharp\{\sigma:|\sigma|\leq k-1\}}{\sharp V_t}, \frac{\sharp\{\sigma:|\sigma|\leq k\}}{\sharp V_t} \propto 5^{k-t}$ and $3^{t-k+1} \leq u < 3^{t-(k-1)+1}$, we obtain that

$$P_{\text{cum}}(u) \propto u^{-\log_3 5}. \qquad \square$$

3. CLUSTERING COEFFICIENT

One feature of the small-world effect is that the average clustering coefficient is not small.

For the network G_t with node set V_t, recall that the average clustering coefficient of G_t is

$$\overline{C}_t = \frac{\sum_{\sigma \in V_t} C_\sigma}{\sharp V_t},$$

where $C_\sigma = \frac{\sharp\{\{x,y\} | \, x\sim y, x\sim\sigma, y\sim\sigma\}}{\deg(\sigma)(\deg(\sigma)-1)/2}$ is the clustering coefficient of the node σ.

Theorem 2. *We have* $\overline{C}_t \geq \frac{88}{375} = 0.2346\ldots$ *for any* $t \geq 2$.

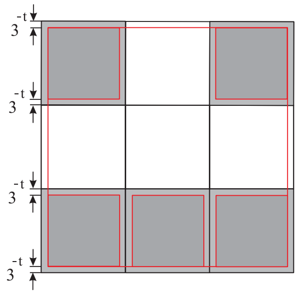

Fig. 5 The self-similarity of l_t and l_{t-1}.

Proof. Given $t > 0$, we consider the following set:

$$l_t = \{\sigma : |\sigma| = t \text{ and } Q_\sigma \cap \partial Q = \emptyset\}. \tag{3.1}$$

When $t = 2$, we can calculate that

$$\sum_{\sigma \in l_2} C_\sigma = \frac{22}{3}.$$

As shown in Fig. 5, we obtain that

$$\sum_{\sigma \in l_t} C_\sigma = \sum_{i=1}^{5} \sum_{i\tau \in l_t} C_{i\tau} \geq \sum_{i=1}^{5} \sum_{\tau \in l_{t-1}} C_{i\tau}.$$

Using Jordan's curve theorem and the self-similarity of E, we see that

$$C_{i\tau} = C_\tau \quad \text{for all } \tau \in l_{t-1}.$$

Then it follows that

$$\sum_{\sigma \in l_t} C_\sigma \geq 5 \sum_{\sigma \in l_{t-1}} C_\sigma,$$

which implies

$$\sum_{\sigma \in l_t} C_\sigma \geq 5 \sum_{\sigma \in l_{t-1}} C_\sigma \geq \cdots \geq 5^{t-2} \sum_{\sigma \in l_2} C_\sigma$$
$$= 5^{t-2} \times \frac{22}{3}.$$

Note $\sharp V_t = \frac{5^{t+1}-1}{4} \leq \frac{5^{t+1}}{4}$, we obtain that

$$\frac{\sum_{\sigma \in V_t} C_\sigma}{\sharp V_t} \geq \frac{\sum_{\sigma \in l_t} C_\sigma}{\sharp V_t} \geq \frac{4}{5^{t+1}} \times \frac{22}{3} \times 5^{t-2}$$
$$= \frac{88}{375} = 0.2346 \ldots. \quad \square$$

4. AVERAGE GEODESIC DISTANCE

One feature of the small-world effect is that the average geodesic distance is much smaller than the size of the network. Recall that for a graph $G = (V, E)$, the geodesic distance $d(x, y)$ between $x, y \in V$ is defined by

$$d(x, y) = \min\{n : x = x_0 \sim x_1 \sim \cdots \sim x_n = y\}.$$

Considering G_t, we let $d_t(\sigma, \tau)$ denote the geodesic distance between σ and τ with $|\sigma|, |\tau| \leq t$. Then by Jordan's curve theorem, we have

$$d_{|\sigma|}(\sigma, \emptyset) = \omega(\sigma).$$

For any $\sigma \neq \emptyset$, we can write $L(\sigma)$ as the least step to move for Q_σ to reach the boundary of $Q = [0, 1]^2$, i.e.

$$L(\sigma) = d_{|\sigma|}(\sigma, \emptyset) - 1 = \omega(\sigma) - 1. \tag{4.1}$$

Note that $\omega(\sigma\tau) \geq \omega(\sigma) + \omega(\tau) - 1$ and (4.1), we have the following remark.

Remark 2. We have $L(\sigma\tau) \geq L(\sigma) + L(\tau)$ for any $\sigma, \tau \in V_t$.

Then the average geodesic distance $\overline{d}_t$ on G_t is defined by

$$\overline{d}_t = \frac{\sum_{\sigma,\tau} d(\sigma, \tau)}{\sharp V_t(\sharp V_t - 1)/2}.$$

First, we consider the words of length k and let

$$\overline{\alpha}_k = \frac{\sum_{|\sigma|=k} L(\sigma)}{\sharp\{\sigma : |\sigma| = k\}}.$$

We have $\overline{\alpha}_1 = 0$ and

$$\overline{\alpha}_2 = \frac{\sharp\{\sigma \nsim \emptyset : |\sigma| = 2\}}{\sharp\{\sigma : |\sigma| = 2\}} = \frac{8}{25}.$$

Lemma 2. *For any $k \geq 1$,*

$$\overline{\alpha}_k \geq \frac{4}{25}(k - 1). \tag{4.2}$$

Proof. For any $|\sigma| = k > 2$, write $\sigma = \tau\sigma'$ with $|\tau| = 2$ and $|\sigma'| = k - 2$, we have

$$\frac{\sum_{|\sigma|=k} L(\sigma)}{\sharp\{\sigma : |\sigma| = k\}} = \frac{\sum_{|\tau|=2} \sum_{|\sigma'|=k-2} L(\tau\sigma')}{\sum_{|\tau|=2} \sharp\{\sigma' : |\sigma'| = k - 2\}}.$$

We will distinguish the following two different cases.

Case I. If $\tau \nsim \emptyset$, then

$$L(\sigma) \geq L(\sigma') + L(\tau) = L(\sigma') + 1,$$

and thus

$$\frac{\sum_{|\sigma'|=k-2} L(\tau\sigma')}{\sharp\{\sigma' : |\sigma'| = k - 2\}} \geq \frac{\sum_{|\sigma'|=k-2} (L(\sigma') + 1)}{\sharp\{\sigma' : |\sigma'| = k - 2\}}$$
$$= \overline{\alpha}_{k-2} + 1. \tag{4.3}$$

Case II. If $\tau \sim \emptyset$, then

$$L(\sigma) \geq L(\sigma') + L(\tau) = L(\sigma'),$$

which implies

$$\frac{\sum_{|\sigma'|=k-2} L(\tau\sigma')}{\sharp\{\sigma' : |\sigma'| = k-2\}} \geq \frac{\sum_{|\sigma'|=k-2} L(\sigma')}{\sharp\{\sigma' : |\sigma'| = k-2\}} = \overline{\alpha}_{k-2}. \tag{4.4}$$

By (4.3), (4.4) and $\overline{\alpha}_1 = 0$, $\overline{\alpha}_2 = \frac{8}{25}$, we have

$$\begin{aligned}\overline{\alpha}_k &\geq \frac{8}{25}(\overline{\alpha}_{k-2} + 1) + \frac{17}{25}\overline{\alpha}_{k-2} = \overline{\alpha}_{k-2} + \frac{8}{25} \\ &\geq \overline{\alpha}_{k-4} + \frac{8}{25} \times 2 \geq \cdots \geq \frac{4}{25}(k-1). \qquad \square\end{aligned}$$

Remark 3. Suppose $\sigma = i_1\sigma'$ and $\tau = j_1\tau'$ with $i_1 \neq j_1$, using Jordan's curve theorem, we have

$$d(\sigma, \tau) \geq L(\sigma') + L(\tau').$$

Theorem 3. *We have*

$$\frac{16}{125}(t-2) \leq \overline{d}_t \leq 2t \quad \textit{for any } t \geq 0.$$

Proof. For any $\sigma, \tau \in V_t$, we notice that

$$d(\sigma, \tau) \leq d(\sigma, \emptyset) + d(\emptyset, \tau) \leq 2t. \tag{4.5}$$

Write $W_t = \{\sigma \in V_t : |\sigma| = t\}$ with $\sharp W_t = 5^t$.

Then by (4.2) and Remark 3, we have

$$\begin{aligned}\sum_{\sigma,\tau \in W_t} d(\sigma, \tau) &\geq \sum_{i_1 \neq j_1 \in \{1,2,\ldots,5\}} \sum_{i_1 \prec \sigma, j_1 \prec \tau, \sigma,\tau \in W_t} d(\sigma, \tau) \\ &\geq 20 \times 5^{2(t-1)} \times \frac{\sum_{|\sigma'|=t-1} L(\sigma')}{\sharp W_{t-1}} \\ &= 20 \times 5^{2(t-1)} \times \overline{\alpha}_{t-1} \\ &\geq \frac{32}{125} \times 5^{2t}(t-2).\end{aligned}$$

Since

$$\sum_{\sigma,\tau \in V_t} d(\sigma, \tau) \geq \sum_{\sigma,\tau \in W_t} d(\sigma, \tau)$$

and $\frac{\sharp V_t(\sharp V_t - 1)}{2} \leq 4(\frac{\sharp W_t(\sharp W_t - 1)}{2})$, we have

$$\begin{aligned}\frac{\sum_{\sigma,\tau \in V_t} d(\sigma, \tau)}{\sharp V_t(\sharp V_t - 1)/2} &\geq \frac{1}{4} \frac{\sum_{\sigma,\tau \in W_t} d(\sigma, \tau)}{\sharp W_t(\sharp W_t - 1)/2} \\ &\geq \frac{1}{4} \times \frac{\frac{32}{125} \times 5^{2t}(t-2)}{5^t(5^t - 1)/2} \\ &\geq \frac{16}{125}(t-2).\end{aligned} \tag{4.6}$$

Therefore, the theorem follows from (4.5) and (4.6). $\square$

5. CONCLUSION

In this paper, using the structure of symbolic system in the IFS which generates the self-similar fractal carpet E, we construct a family of evolving networks $\{G_t\}_t$.

We check the scale-free effect and small-world effect of $\{G_t\}_t$. Fix $t > 0$, for the cumulative degree distribution $P_{\text{cum}}(u)$ of our network G_t, we show that

$$P_{\text{cum}}(u) \propto u^{-\log_3 5},$$

where $\log_3 5$ is the dimension of the fractal carpet E. Moreover, we conclude that the uniform positive lower bound of the average clustering coefficient $\overline{C}_t$ of G_t according to the self-similarity, i.e.

$$\inf_t \overline{C}_t > 0.2346.$$

For the average geodesic distance $\overline{d}_t$ on G_t, we show that

$$\overline{d}_t \propto t,$$

which means that $\overline{d}_t$ is much smaller than the size of $V_t \propto 5^t$.

ACKNOWLEDGMENT

This work is supported by the National Natural Science Foundation of China (No. 11771226).

REFERENCES

1. D. J. Watts and S. H. Strogatz, Collective dynamics of "small-world" networks, *Nature* **393** (1998) 440–442.
2. A.-L. Barabási and R. Albert, Emergence of scaling in random networks, *Science* **286** (1999) 509–512.
3. M. E. J. Newman, The structure and function of complex networks, *SIAM Rev.* **45** (2003) 167–256.
4. B. B. Mandelbrot, How long is the coast of Britain? Statistical self-similarity and fractional dimension, *Science* **156** (1967) 636–638.
5. J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30**(5) (1981) 713–747.
6. C. Song, S. Havlin and H. A. Makse, Self-similarity of complex networks, *Nature* **433** (2005) 392–395.
7. C. Song, S. Havlin and H. A. Makse, Origins of fractality in the growth of complex networks, *Nat. Phys.* **2** (2006) 275–281.
8. C. Song, L. K. Gallos, S. Havlin and H. A. Makse, How to calculate the fractal dimension of a complex network: The box covering algorithm, *J. Stat. Mech., Theory Exp.* **2007** (2007) P03006.

9. Z. Zhang, S. Zhou, L. Fang, J. Guan and Y. Zhang, Maximal planar scale-free Sierpinski networks with small-world effect and power law strength-degree correlation, *Europhys. Lett.* **79** (2007) 38007.
10. Z. Zhang, S. Zhou, Z. Su, T. Zou and J. Guan, Random Sierpinski network with scale-free small-world and modular structure, *Eur. Phys. J. B* **65** (2008) 141–147.
11. J. Guan, Y. Wu, Z. Zhang, S. Zhou and Y. Wu, A unified model for Sierpinski networks with scale-free scaling and small-world effect, *Physica A* **388** (2009) 2571–2578.
12. M. Dai, D. Ye, J. Hou and X. Li, Scaling of average weighted receiving time on double-weighted Koch networks, *Fractals* **23**(2) (2015) 1550011.
13. M. Dai, Q. Xie and L. Xi, Trapping on weighted tetrahedron Koch networks with small-world property, *Fractals* **22**(01n02) (2014) 1450006.
14. Z. Zhang, S. Zhou, L. Chen, M. Yin and J. Guan, The exact solution of the mean geodesic distance for Vicsek fractals, *J. Phys. A Math. Theor.* **41** (2008) 485102.
15. J. Deng, Q. Q. Ye and Q. Wang, Weighted average geodesic distance of Vicsek network, *Physica A* **527** (2019) 121327.
16. A. Le, F. Gao, L. Xi and S. Yin, Complex networks modeled on the Sierpinski gasket, *Physica A* **436** (2015) 646–657.
17. S. Wang, L. Xi, H. Xu and L. Wang, Scale-free and small-world properties of Sierpinski networks, *Physica A* **465** (2017) 690–700.
18. S. Wang, Z. Yu and L. Xi, Average geodesic distance of Sierpinski gasket and Sierpinski networks, *Fractals* **25**(5) (2017) 1750044.
19. L. Zhao, S. Wang and L. Xi, Average geodesic distance of Sierpinski carpet, *Fractals* **25**(6) (2017) 1750061.
20. J. Yang, S. Wang, L. Xi and Y. Ye, Average geodesic distance of skeleton networks of Sierpinski tetrahedron, *Physica A* **495** (2018) 269–277.
21. C. Zeng, Y. Xue and M. Zhou, Fractal networks on Sierpinski-type polygon, *Fractals* **28**(5) (2020) 2050087.
22. K. Cheng, D. R. Chen, Y. M. Xue and Q. Zhang, The scale-free and small-world properties of complex networks on Sierpinski-type hexagon, *Fractals* **28**(3) (2020) 2050054.
23. C. Zeng, M. Zhou and Y. M. Xue, Small-world and scale-free properties of the n-dimensional Sierpinski cube networks, *Fractals* **28**(1) (2020) 2050001.
24. Q. Zhang, Y. M. Xue, D. H. Wang and M. Niu, Asymptotic formula on average path length in a hierarchical scale-free network with fractal structure, *Chaos Solitons Fractals* **122** (2019) 196–201.
25. M. Niu and M. J. Shao, The asymptotic formula on average weighted path length for scale-free modular network, *Modern Phys. Lett. B* **35**(18) (2021) 2150298.
26. L. Huang and L. Peng, Complex networks modeled on a kind of Sierpinski-like carpet, *Fractals* **30**(3) (2022) 2250059.