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HAUSDORFF DIMENSIONS OF FLOWER NETWORKS AND HANOI GRAPHS

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Abstract

Zeng and Xi introduced the Hausdorff dimension of a family of networks and investigated the dimensions of touching networks. In this paper, using the self-similarity and induction we obtain the Hausdorff dimension of flower networks and Hanoi graphs, which are not touching networks.

Keywords: Fractal Network; Hausdorff Dimension; Self-Similarity.

1. INTRODUCTION

Recently complex networks have gained plenty of attention from scholars. There are two extraordinary properties revealed in real-world complex networks, the small-world effect[1] and the scale-free effect,[2] also see Ref. 3.

Mandelbrot[4] introduced notions of the fractal, the fractal dimension and the self-similarity, and then Hutchinson[5] characterized deterministic self-similar sets in terms of the iterated function system (IFS) and calculated the Hausdorff and box dimensions of self-similar sets under the open set

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condition. Song, Havlin and Makse[6] discovered the fractal dimension and self-similarity on many real-world networks, also see Ref. 7 for the recent development. From then on, a lot of self-similar fractals have been used to model evolving complex networks, such as Sierpinski networks,[8–10] Koch networks[11,12] and Vicsek networks.[13,14]

Bao, Xi and Zhao[15] introduced the notion of touching networks including skeleton networks of Sierpinski gasket, Sierpinski tetrahedron, Sierpinski hexagon and Lindström Snowflake. On touching networks, Fan, Xi and Zhao[16] established the theory of finite geodesic patterns on the limit space to calculate the average distances. Refer to Refs. 17–26 for the average distances of fractal networks.

According to the dimension theory of fractal geometry, we always have $\dim_{\mathrm{H}} G = 0$ for any countable network (or graph) G. To avoid this problem, Zeng and Xi[27] defined the Hausdorff dimension of a family of networks based on the potential theory and discussed the dimension of the touching networks. Let $\widetilde{G} = \{G_n = (V_n, E_n)\}_{n\geq 0}$ be a family of networks, and $d_n(x, y)$ the shortest length of all paths from x to y on G_n. Write the diameter $\mathrm{diam}(G_n) = \max_{x,y\in G_n} d_n(x, y)$.

Definition 1. For a family of networks $\widetilde{G} = \{G_n = (V_n, E_n)\}_{n\geq 0}$ with $\sharp V_n \to \infty$, we define its Hausdorff dimension $\dim(\widetilde{G})$ to be

$$\dim(\widetilde{G}) = \sup\left\{\alpha : \varliminf_{n\to\infty} \frac{1}{(\sharp V_n)^2} \times \sum_{u,v\in V_n, u\neq v} \frac{1}{(d_n(u, v)/\mathrm{diam}(G_n))^{\alpha}} < \infty\right\}.$$

In this paper, we will investigate the Hausdorff dimension of the flower networks and Hanoi graphs, which are not the touching networks based on the self-similarity of these fractal networks. Roughly speaking, the self-similarity in a family of graphs $\{G_n\}_n$ means the *isomorphism* between G_n and the subgraph of G_{n+1}. Recall that an *isomorphism* of graphs G and H is a bijection $f : V(G) \to V(H)$ between the vertex sets of G and H such that $u_1 \overset{G}{\sim} u_2$ if and only if $f(u_1) \overset{H}{\sim} f(u_2)$. We denote $G \overset{f}{\simeq} H$ or $H = f(G)$. Furthermore, if $H = f(G)$ and $f(u^*) = v^*$ for some $u^* \in V(G)$, $v^* \in V(H)$, we denote it by $(G, u^*) \overset{f}{\simeq} (H, v^*)$ or $(G, u^*) \simeq (H, v^*)$. If $(G, a) \simeq (G, b)$, then we say that G is *symmetric* with respect to a and b.

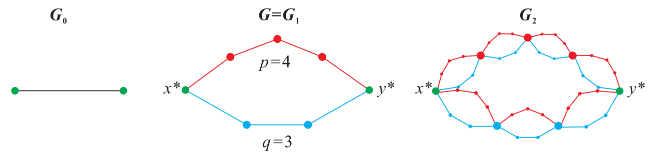

Fig. 1 Iterated structure of $(4, 3)$-flower networks.

Recall the (p, q)-flower networks[28] as follows. Given a *cycle* graph $G = (V, E)$ with $\sharp E = \sharp V = p + q$ $(p, q \geq 2)$ as the initial pattern, we fix x^* and y^* as Fig. 1, there are two parallel paths connecting x^* and y^* with lengths p and q, respectively. Let $G_1 = G$. By induction, assume that G_n has been defined, we replace every edge of G_n by G, identifying x^*, y^* with two nodes of edge. Then we obtain $\widetilde{G} = \{G_n = (V(G_n), E(G_n))\}_n$. See the $(4, 3)$-flower networks in Fig. 1.

The Hanoi graphs[29] are undirected graphs whose vertices represent the possible states of the Tower of Hanoi puzzle, and whose edges represent admissible moves between pairs of states. For simplicity of notation, we denote the Hanoi graph for a puzzle with 3 disks on n towers by H_n. See Fig. 2 for H_0, H_1, H_2. Let $\widetilde{H} = \{H_n = (V(H_n), E(H_n))\}_n$. Suppose $0 < \lambda < 1/2$ and $A_1 = (0, 0)$, $A_2 = (1, 0)$, $A_3 = (1/2, \sqrt{3}/2)$. Assume $\{S_i(x) = \lambda x + (1 - \lambda)A_i\}_{i=1}^3$ is an IFS of planar similitudes, and $K = \bigcup_{i=1}^3 S_i(K)$ is corresponding the self-similar set iterated from the equilateral triangle $\triangle A_1A_2A_3$. In fact, the Hanoi graphs are also skeleton networks of the fractal K.

Our main results are stated as follows.

Theorem 1. *Suppose $p \geq q$, then the Hausdorff dimension of the (p, q)-flower networks $\widetilde{G}$ is* $\log(p + q)/\log q$.

Theorem 2. *The Hausdorff dimension of the Hanoi graphs $\widetilde{H}$ is* $\log 3/\log 2$.

Remark 1. The Hanoi graphs and skeleton networks of Sierpinski gasket have the same Hausdorff dimension. In fact, they look like the Sierpinski gasket when observed from infinity.

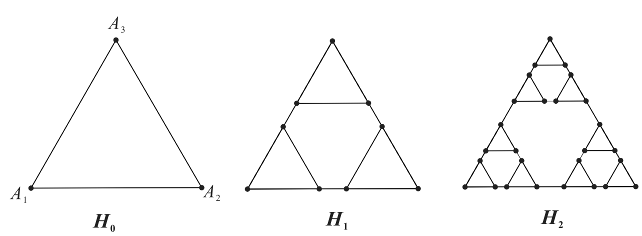

Fig. 2 The Hanoi graphs.

2. PROOF OF THEOREM 1

Notation 1. Let $x_\lambda \asymp y_\lambda$ denote that $\{x_\lambda\}_{\lambda\in\Lambda}$ is comparable to $\{y_\lambda\}_{\lambda\in\Lambda}$, i.e. there exist two independent constants $c_1, c_2 > 0$ such that

$$c_1 y_\lambda \le x_\lambda \le c_2 y_\lambda, \quad \forall\, \lambda \in \Lambda.$$

If only the left or right side of the above inequality holds, then they are denoted as $x_\lambda \gtrsim y_\lambda$ and $x_\lambda \lesssim y_\lambda$, respectively.

Using Notation 1, for (p, q)-flower networks $\{G_n = (V(G_n), E(G_n))\}_{n\ge 0}$ we have

$$\sharp V(G_n) \asymp (p+q)^n. \tag{2.1}$$

Note that $p \ge q$, we can obtain that

$$\mathrm{diam}(G_n) \asymp q^n = d_n(x^*, y^*). \tag{2.2}$$

As in Fig. 3, for each n, we denote the $p+q$ copies of G_{n-1} on G_n by $G^{n,j}$ for $j \in \{1, 2, \dots, p+q\}$. Then we have the following self-similarity:

$$G^{n,j} \overset{f_{n,j}}{\simeq} G_{n-1} \quad \text{for any } j. \tag{2.3}$$

Moreover, for any $j \in \{1, 2, \dots, p+q\}$, we have

$$\sharp V(G^{n,j}) = \sharp V(G_{n-1}) \asymp (p+q)^{n-1} \tag{2.4}$$

and

$$\begin{aligned} \mathrm{diam}(G^{n,j}) &= \mathrm{diam}(G_{n-1}) \asymp q^{n-1} \\ &= d_n(f_{n,j}(x^*), f_{n,j}(y^*)). \end{aligned} \tag{2.5}$$

Let $\delta_n = \sum_{x,y\in G_n} \frac{q^{n\alpha}}{(d_n(x,y))^\alpha}$.

As in Fig. 4, suppose G_n and its copy L_n share one vertex. We denote

$$\beta_n = \sum_{x\in G_n, y\in L_n} \frac{q^{n\alpha}}{(d_n(x,y))^\alpha}.$$

Because of the above isomorphism (2.3) of graphs, we have the following claims:

Claim 1. *For any $j \in \{1, 2, \dots, p+q\}$ and $\alpha > 0$, we have*

$$\sum_{x,y\in G^{n,j}} \frac{q^{(n-1)\alpha}}{(d_n(x,y))^\alpha} = \delta_{n-1}.$$

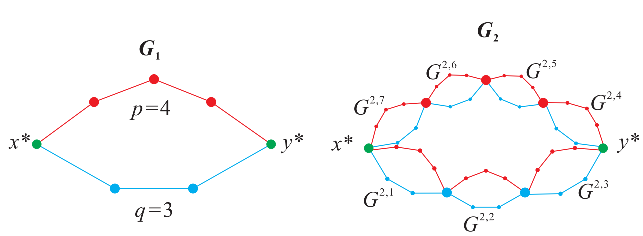

Fig. 3 The structure of G_2.

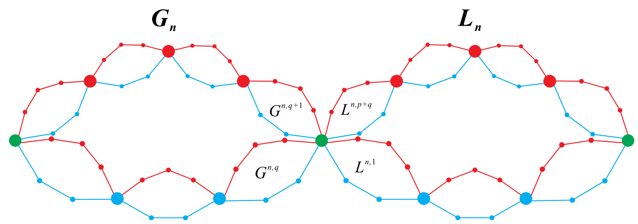

Fig. 4 The structure of G_n and L_n.

Claim 2. *For any $j \in \{1, 2, \dots, p+q\}$ and $\alpha > 0$, we have*

$$\sum_{x\in G^{n,j}, y\in G^{n,j+1}} \frac{q^{(n-1)\alpha}}{(d_n(x,y))^\alpha} = \beta_{n-1},$$

where we let $j+1 = 1$ if $j = p+q$. Furthermore, we have

$$\sum_{x\in E, y\in F} \frac{q^{(n-1)\alpha}}{(d_n(x,y))^\alpha} = \beta_{n-1},$$

where $E \in \{G^{n,q}, G^{n,q+1}\}$ and $F \in \{L^{n,1}, L^{n,p+q}\}$.

Claim 3. *For any $j_1, j_2 \in \{1, 2, \dots, p+q\}$ with $j_2 - j_1 \ge 2$ and $\alpha > 0$, we have*

$$\sum_{x\in G^{n,j_1}, y\in G^{n,j_2}} \frac{q^{(n-1)\alpha}}{(d_n(x,y))^\alpha} \le 2\beta_{n-1}.$$

Proof. As in Fig. 5, suppose y' is the copy of y in the piece near x, then $d_n(A, y) = d_n(A', y')$ and thus $d_n(x, y) \ge d_n(x, A') + d_n(A, y) = d_n(x, A') + d_n(A', y') = d_n(x, y')$. In the same way, we have $d_n(x, z) \ge d_n(x, z')$ if $p = q$ and x, z lie in a pair of mirror pieces, respectively. □

Part I: $\dim(\widetilde{G}) \ge \log(p+q)/\log q$.

Suppose $0 < \alpha < \log(p+q)/\log q$. By (2.1) and (2.2), there exists constant $b_1 > 0$ such that

$$\sharp V(G_n) \le b_1 (p+q)^n. \tag{2.6}$$

Let $\mathcal{A}_n = (G^{n,q} \cup G^{n,q+1}) \times (L^{n,1} \cup L^{n,p+q})$ and $\mathcal{B}_n = (G_n \times L_n) \backslash \mathcal{A}_n$. Using Claim 2 and (2.4)–(2.6),

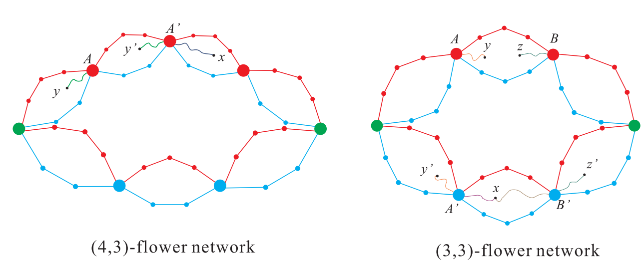

Fig. 5 The structure of G^{n,j_1} and G^{n,j_2}.

we can obtain that

$$\begin{aligned}\beta_n &= \sum_{x\in G_n, y\in L_n} \frac{q^{n\alpha}}{(d_n(x,y))^\alpha} \\ &\le \sum_{(x,y)\in\mathcal{A}_n} \frac{q^{n\alpha}}{(d_n(x,y))^\alpha} + \sum_{(x,y)\in\mathcal{B}_n} \frac{q^{n\alpha}}{(d_n(x,y))^\alpha} \\ &\le 4q\beta_{n-1} + ((p+q)^2-4)(\sharp V(G_{n-1}))^2 \\ &\quad \times \frac{q^{n\alpha}}{d_n(f_{n,j}(x^*), f_{n,j}(y^*))} \\ &\le 4q\beta_{n-1} + b_1 \frac{((p+q)^2-4)q^\alpha}{(p+q)^2}(p+q)^{2n}. \end{aligned} \tag{2.7}$$

Iterating (2.7), we can obtain that

$$\begin{aligned}\beta_n &\le \beta_0 4^n q^{n\alpha} + b_1 \frac{((p+q)^2-4)q^\alpha}{(p+q)^2}(p+q)^{2n} \\ &\quad \times \sum_{k=0}^{n-1}\left(\frac{4q}{(p+q)^2}\right)^k \\ &\le \beta_0 4^n q^{n\alpha} + b_1 \frac{((p+q)^2-4)q^\alpha}{(p+q)^2} \\ &\quad \times \frac{1}{1-\frac{4q}{(p+q)^2}}(p+q)^{2n} \\ &= \beta_0 4^n q^{n\alpha} + b_2(p+q)^{2n}, \end{aligned} \tag{2.8}$$

where $b_2 = b_1 \frac{((p+q)^2-4)q^\alpha}{(p+q)^2}\frac{1}{1-\frac{4q}{(p+q)^2}}$.

Using Claims 1–3 and (2.8), we have

$$\begin{aligned}\delta_n &= \sum_{x,y\in G_n} \frac{q^{n\alpha}}{(d_n(x,y))^\alpha} \\ &\le \sum_{j=1}^{p+q}\sum_{x,y\in G^{n,j}} \frac{q^{n\alpha}}{(d_n(x,y))^\alpha} \\ &\quad + \sum_{1\le j_1<j_2\le p+q}\ \sum_{x\in G^{n,j_1}, y\in G^{n,j_2}} \frac{q^{n\alpha}}{(d_n(x,y))^\alpha} \\ &\le (p+q)q^\alpha\delta_{n-1} + (p+q)(p+q-1)\cdot 2\beta_{n-1} \\ &\le (p+q)q^\alpha\delta_{n-1} + \frac{\beta_0(p+q)(p+q-1)}{2q^\alpha}4^n q^{n\alpha} \\ &\quad + \frac{2b_2(p+q-1)}{(p+q)}(p+q)^{2n}. \end{aligned} \tag{2.9}$$

We will distinguish two different cases $p+q>4$ or $p+q=4$.

(1) If $p+q>4$, iterating (2.9), we can obtain that

$$\begin{aligned}\delta_n &\le (p+q)^n q^{n\alpha}\delta_0 \\ &\quad + \frac{\beta_0(p+q)(p+q-1)}{2q^\alpha}4^n q^{n\alpha}\sum_{k=0}^{n-1}\left(\frac{p+q}{4}\right)^k \\ &\quad + \frac{2b_2(p+q-1)}{(p+q)}(p+q)^{2n}\sum_{k=0}^{n-1}\left(\frac{q}{p+q}\right)^k \\ &\le c_1(p+q)^n q^{n\alpha} + c_2 4^n q^{n\alpha}\frac{(p+q)^n}{4^n} + c_3(p+q)^{2n} \\ &= (c_1+c_2)(p+q)^n q^{n\alpha} + c_3(p+q)^{2n}, \end{aligned} \tag{2.10}$$

where $c_1 = \delta_0$, $c_2 = \frac{\beta_0(p+q)(p+q-1)}{2q^\alpha}\frac{1}{\frac{p+q}{4}-1}$ and $c_3 = \frac{2b_2(p+q-1)}{(p+q)}\frac{1}{1-\frac{q}{p+q}}$.

Note that $\alpha < \log(p+q)/\log q$, using (2.2) and (2.10), we have

$$\begin{aligned}&\frac{1}{(\sharp V(G_n))^2}\sum_{x,y\in G_n}\frac{(\mathrm{diam}(G_n))^\alpha}{(d_n(x,y))^\alpha} \\ &\quad \lesssim \frac{\delta_n}{(p+q)^{2n}} \\ &\quad \le \frac{(c_1+c_2)(p+q)^n q^{n\alpha} + c_3(p+q)^{2n}}{(p+q)^{2n}} \\ &\quad = (c_1+c_2)\frac{q^{n\alpha}}{(p+q)^n} + c_3 \to c_3, \quad \text{as } n\to\infty. \end{aligned}$$

(2) If $p+q=4$, in the same way, we have

$$\delta_n \le c_4 n(p+q)^n q^{n\alpha} + c_3(p+q)^{2n},$$

where $c_4 = \delta_0 + \frac{\beta_0(p+q)(p+q-1)}{2q^\alpha}$. And we also obtain that

$$\begin{aligned}&\frac{1}{(\sharp V(G_n))^2}\sum_{x,y\in G_n}\frac{(\mathrm{diam}(G_n))^\alpha}{(d_n(x,y))^\alpha} \\ &\quad \lesssim \frac{\delta_n}{(p+q)^{2n}} \\ &\quad \le \frac{c_4 n(p+q)^n q^{n\alpha} + c_3(p+q)^{2n}}{(p+q)^{2n}} \\ &\quad = c_4 n\frac{q^{n\alpha}}{(p+q)^n} + c_3 \to c_3, \quad \text{as } n\to\infty. \end{aligned}$$

Hence we obtain

$$\dim(\widetilde{G}) \ge \log(p+q)/\log q. \tag{2.11}$$

Part II: $\dim(\widetilde{G}) \le \log(p+q)/\log q$.

If $\alpha > \log(p+q)/\log q$, using Claim 1, we have

$$\delta_n = \sum_{x,y\in G_n} \frac{q^{n\alpha}}{(d_n(x,y))^\alpha} \ge \sum_{j=1}^{p+q} \sum_{x,y\in G^{n,j}} \frac{q^{n\alpha}}{(d_n(x,y))^\alpha}$$
$$= (p+q)q^\alpha \delta_{n-1}. \tag{2.12}$$

Iterating (2.12), we can obtain that

$$\delta_n \ge (p+q)^n q^{n\alpha} \delta_0.$$

Hence by (2.2) and (2.10) we have

$$\frac{1}{(\sharp V(G_n))^2} \sum_{x,y\in G_n} \frac{(\mathrm{diam}(G_n))^\alpha}{(d_n(x,y))^\alpha}$$
$$\gtrsim \frac{\delta_n}{(p+q)^{2n}}$$
$$\ge \frac{(p+q)^n q^{n\alpha}\delta_0}{(p+q)^{2n}}$$
$$\gtrsim \frac{q^{n\alpha}}{(p+q)^n} \to \infty, \quad \text{as } n \to \infty,$$

which implies

$$\dim(\widetilde{G}) \le \log(p+q)/\log q. \tag{2.13}$$

Therefore, Theorem 1 follows from (2.11) and (2.13).

3. PROOF OF THEOREM 2

Using Notation 1, we have

$$\sharp V(H_n) = 3^{n+1} \asymp 3^n \tag{3.1}$$

and

$$\mathrm{diam}(H_n) = 2^{n+1} - 1 \asymp 2^n. \tag{3.2}$$

As in Fig. 6, for each n, we denote the 3 copies of H_{n-1} on H_n by $H^{n,j}$ for $j \in \{1,2,3\}$. Then we have

$$\sharp V(H^{n,j}) = \sharp V(H_{n-1}) = 3^n \tag{3.3}$$

and

$$\mathrm{diam}(H^{n,j}) = \mathrm{diam}(H_{n-1}) = 2^n - 1. \tag{3.4}$$

We write $\sigma_i^{(0)} = A_i$ for any $i \in \{1,2,3\}$. By induction, we let $\sigma_i^{(n)}$ denote $S_i(\sigma_i^{(n-1)})$ and $\sigma_i^{(n,j)}$ denote $S_j(\sigma_i^{(n-1)})$, then we have $\sigma_i^{(n)} = \sigma_i^{(n,i)}$. See Fig. 7 for

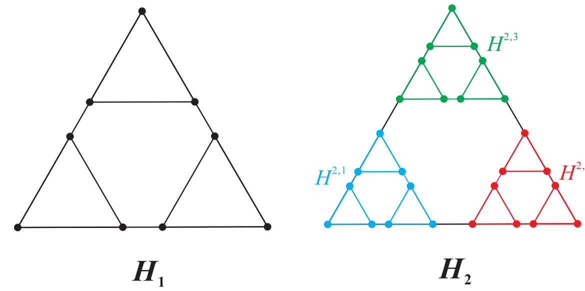

Fig. 6 The structure of H_2.

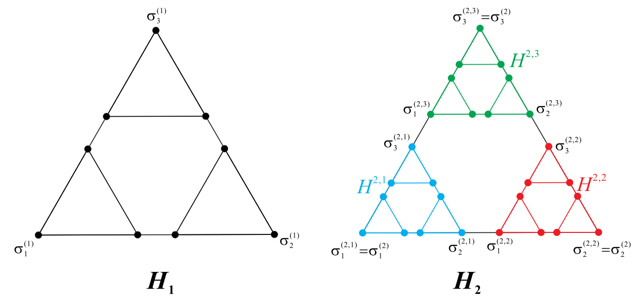

Fig. 7 $\{\sigma_i^{(1)}\}_i$ and $\{\sigma_i^{(2,j)}\}_{i,j}$ in H_1 and H_2, respectively.

$\{\sigma_i^{(1)}\}_i$ and $\{\sigma_i^{(2,j)}\}_{i,j}$ in the Hanoi graphs. Then we have the following self-similarity:

$$(H^{n,j}, \sigma_i^{(n,j)}) \simeq (H_{n-1}, \sigma_i^{(n-1)}) \quad \text{for any } i \text{ and } j. \tag{3.5}$$

Notice for any $n \ge 0$, the graph H_n is symmetric with respect to $\sigma_i^{(n)}$ and $\sigma_j^{(n)}$ for any distinct $i,j \in \{1,2,3\}$, i.e.

$$(H_n, \sigma_i^{(n)}) \simeq (H_n, \sigma_j^{(n)}). \tag{3.6}$$

Let

$$\xi_n = \sum_{x,y\in H_n} \frac{2^{n\alpha}}{(d_n(x,y))^\alpha}.$$

For any $i \ne j \in \{1,2,3\}$, we let

$$\gamma_n = \sum_{y\in H_n} \frac{2^{n\alpha}}{(d_n(y,A_i))^\alpha}$$

and

$$\tau_n = \sum_{x\in H^{n,i}, y\in H^{n,j}} \frac{2^{n\alpha}}{(d_n(x,y))^\alpha}.$$

Because of the above isomorphism (3.5) and (3.6) of graphs, we have the following claims.

Claim 4. *For any $j \in \{1,2,3\}$ and $\alpha > 0$, we have*

$$\sum_{x,y\in H^{n,j}} \frac{2^{(n-1)\alpha}}{(d_n(x,y))^\alpha} = \xi_{n-1}.$$

Claim 5. *For any $i, j \in \{1,2,3\}$ and $\alpha > 0$, we have*

$$\sum_{y\in H^{n,j}} \frac{2^{(n-1)\alpha}}{(d_n(y,\sigma_i^{n,j}))^\alpha} = \gamma_{n-1}.$$

Claim 6. *For any $j \in \{1,2,3\}$ and $\alpha > 0$, we have*

$$\sum_{x\in H^{n,j}} \frac{1}{(\min_{1\le i\le 3}\{d_n(x,\sigma_i^{(n,j)})\})^\alpha} = \sum_{x\in H_{n-1}} \frac{1}{(\min_{1\le i\le 3}\{d_{n-1}(x,\sigma_i^{(n-1)})\})^\alpha}.$$

Part I: $\dim(\widetilde{H}) \ge \log 3/\log 2$.

Using Claim 5, the symmetry of the Hanoi graphs (3.6), (3.3) and (3.4), we have

$$\begin{aligned}\gamma_n &= \sum_{y\in H_n} \frac{2^{n\alpha}}{(d_n(y,\sigma_1^{(n)}))^\alpha}\\ &= \sum_{y\in H^{n,1}} \frac{2^{n\alpha}}{(d_n(y,\sigma_1^{(n)}))^\alpha} + 2\sum_{y\in H^{n,2}} \frac{2^{n\alpha}}{(d_n(y,\sigma_1^{(n)}))^\alpha}\\ &\le 2^\alpha \sum_{y\in H^{n,1}} \frac{2^{(n-1)\alpha}}{(d_n(y,\sigma_1^{(n,1)}))^\alpha}\\ &\quad + 2\sum_{y\in H^{n,2}} \frac{2^{n\alpha}}{(d_n(\sigma_1^{(n,1)},\sigma_1^{(n,2)}))^\alpha}\\ &\le 2^\alpha\gamma_{n-1} + 2\sharp V(H_{n-1})\frac{2^{n\alpha}}{(\mathrm{diam}(H^{n,1})+1)^\alpha}\\ &= 2^\alpha\gamma_{n-1} + 2\cdot 3^n. \end{aligned} \tag{3.7}$$

Iterating (3.7), by $\alpha < \log 3/\log 2$ we can obtain that

$$\begin{aligned}\gamma_n &\le 2^{n\alpha}\gamma_0 + 2\cdot 3^n\left(1 + \frac{2^\alpha}{3} + \cdots + \left(\frac{2^\alpha}{3}\right)^{n-1}\right)\\ &\le 2^{n\alpha}\gamma_0 + 2\cdot 3^n\cdot\frac{1}{1-\frac{2^\alpha}{3}}\\ &\le C_1(2^{n\alpha}+3^n),\end{aligned} \tag{3.8}$$

where $C_1 = \gamma_0 + \frac{2}{1-\frac{2^\alpha}{3}}$.

Let

$$\mathcal{D}_i^n = \left\{x\in H_{n-1} : d_{n-1}(x,\sigma_i^{(n-1,1)}) = \min_{1\le i\le 3} d_{n-1}(x,\sigma_i^{(n-1,1)})\right\}.$$

Note that

$$d_n(x,y) \ge \min_{1\le i\le 3}\{d_n(x,\sigma_i^{(n,1)})\} \quad \forall x\in H^{n,1},\ y\in H^{n,2}. \tag{3.9}$$

Then using Claims 5 and 6, (3.1), (3.8) and (3.9), we have

$$\begin{aligned}\tau_n &= \sum_{x\in H^{n,1},y\in H^{n,2}} \frac{2^{n\alpha}}{(d_n(x,y))^\alpha}\\ &\le \sharp V(H_{n-1})\sum_{x\in H^{n,1}} \frac{2^{n\alpha}}{(\min_{1\le i\le 3}\{d_n(x,\sigma_i^{(n,1)})\})^\alpha}\\ &= 3^n\sum_{x\in H_{n-1}} \frac{2^{n\alpha}}{(\min_{1\le i\le 3}\{d_{n-1}(x,\sigma_i^{(n-1,1)})\})^\alpha}\\ &\le 3^n\sum_{i=1}^3\sum_{x\in\mathcal{D}_i^n} \frac{2^{n\alpha}}{(\min_{1\le i\le 3}\{d_{n-1}(x,\sigma_i^{(n-1,1)})\})^\alpha}\\ &\le 3^n\cdot 3\cdot\sum_{x\in\mathcal{D}_1^n} \frac{2^{n\alpha}}{(\min_{1\le i\le 3}\{d_{n-1}(x,\sigma_i^{(n-1,1)})\})^\alpha}\\ &\le 3^n\cdot 3\cdot 2^{2\alpha}\sum_{x\in H^{n-1,1}} \frac{2^{(n-2)\alpha}}{(d_{n-1}(x,\sigma_1^{(n-1,1)}))^\alpha}\\ &= 3\cdot 2^{2\alpha}\cdot 3^n\gamma_{n-2} \le C_2(2^{n\alpha}3^n + 3^{2n}),\end{aligned} \tag{3.10}$$

where $C_2 = 3\cdot 2^{2\alpha}C_1$.

Using Claim 4 and (3.10), we have

$$\begin{aligned}\xi_n &= \sum_{x,y\in H_n} \frac{2^{n\alpha}}{(d_n(x,y))^\alpha}\\ &\le 2^\alpha\sum_{i=1}^3\sum_{x,y\in H^{n,1}} \frac{2^{(n-1)\alpha}}{(d_n(x,y))^\alpha}\\ &\quad + \sum_{1\le i<j\le 3}\sum_{x\in H^{n,i},y\in H^{n,j}} \frac{2^{n\alpha}}{(d_n(x,y))^\alpha}\\ &= 2^\alpha\cdot 3\xi_{n-1} + 6\tau_{n-1}\\ &\le 2^\alpha\cdot 3\xi_{n-1} + 6C_2(2^{(n-1)\alpha}3^{n-1} + 3^{2(n-1)})\\ &= 2^\alpha\cdot 3\xi_{n-1} + \frac{2C_2}{2^\alpha}2^{n\alpha}3^n + \frac{2C_2}{3}3^{2n}.\end{aligned} \tag{3.11}$$

Iterating (3.11) and using $\alpha < \log 3/\log 2$, we can obtain that

$$\xi_n \le 2^{n\alpha}3^n\xi_0 + \frac{2C_2}{2^\alpha}n\cdot 2^{n\alpha}3^n$$

$$+\frac{2C_2}{3}3^{2n}\left(1+\frac{2^\alpha}{3}+\cdots+\left(\frac{2^\alpha}{3}\right)^{n-1}\right)$$
$$\leq 2^{n\alpha}3^n\xi_0+\frac{2C_2}{2^\alpha}n\cdot 2^{n\alpha}3^n+\frac{2C_2}{3}3^{2n}\frac{1}{1-\frac{2^\alpha}{3}}$$
$$\leq C_3 n\cdot 2^{n\alpha}3^n+C_4 3^{2n},$$

where $C_3=\xi_0+\frac{2C_2}{2^\alpha}$ and $C_4=\frac{2C_2}{3}\frac{1}{1-\frac{2^\alpha}{3}}$.

Note that $\alpha<\log 3/\log 2$, using (3.1), (3.2) and (3.11), we have

$$\frac{1}{(\sharp V(H_n))^2}\sum_{x,y\in H_n}\frac{(\mathrm{diam}(H_n))^\alpha}{(d_n(x,y))^\alpha}$$
$$\lesssim \frac{\xi_n}{3^{2n}}$$
$$\leq \frac{C_3 n\cdot 2^{n\alpha}3^n+C_4 3^{2n}}{3^{2n}}$$
$$\leq C_3\cdot n\frac{2^{n\alpha}}{3^n}+C_4\to C_4,\quad \text{as } n\to\infty.$$

Hence we obtain

$$\dim(\widetilde{H})\geq \log 3/\log 2. \tag{3.12}$$

Part II: $\dim(\widetilde{H})\leq \log 3/\log 2$.

Suppose $\alpha>\log 3/\log 2$, using Claim 4 we have

$$\xi_n=\sum_{x,y\in H_n}\frac{2^{n\alpha}}{(d_n(x,y))^\alpha}\geq 2^\alpha\sum_{j=1}^{3}\frac{2^{(n-1)\alpha}}{(d_n(x,y))^\alpha}$$
$$=2^\alpha 3\xi_{n-1}. \tag{3.13}$$

Iterating (3.13), we can obtain that

$$\xi_n\geq 2^{n\alpha}3^n\xi_0. \tag{3.14}$$

Note that $\alpha<\log 3/\log 2$, using (3.1), (3.2) and (3.14), we have

$$\frac{1}{(\sharp V(H_n))^2}\sum_{x,y\in H_n}\frac{(\mathrm{diam}(H_n))^\alpha}{(d_n(x,y))^\alpha}$$
$$\gtrsim\frac{\xi_n}{3^{2n}}\geq\frac{2^{n\alpha}3^n\xi_0}{3^{2n}}\gtrsim\frac{2^{n\alpha}}{3^n}\to\infty,\quad \text{as } n\to\infty.$$

Hence we can obtain that

$$\dim(\widetilde{H})\leq \log 3/\log 2. \tag{3.15}$$

Therefore, Theorem 2 follows from (3.12) and (3.15).

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