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HAUSDORFF DIMENSION OF A FAMILY OF NETWORKS

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Abstract

For a family of networks $\{G_n\}_{n\geq 1}$, we define the Hausdorff dimension of $\{G_n\}_{n\geq 1}$ inspired by the Frostman's characteristics of potential for Hausdorff dimension of fractals on Euclidean spaces. We prove that our Hausdorff dimension of the touching networks is $\log m/\log N$. Our definition is quite different from the fractal dimension defined for real-world networks.

Keywords: Fractal Network; Dimension; Touching Networks.

1. INTRODUCTION

1.1. Fractal Dimensions

The essential idea of fractal dimensions has a long history in mathematics, but the term itself was proposed by Mandelbrot.[1–3] The Hausdorff dimension is a fractal dimension, that was first introduced in 1919 by Hausdorff.[4] Bouligand[5] adapted the Minkowski content to non-integral dimension in 1928, and the more usual definition of box dimension was given in 1932 by Pontrjagin and Schnirelman.[6] The packing dimension was introduced by Tricot[7] in 1982. The Assouad dimension was presented by Assouad in his Ph.D. thesis,[8] and then Lü and Xi[9] defined the quasi-Assouad dimension. It is worth pointing out that Frostman[10]

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provided characteristics of potential for the Hausdorff dimension

$$\dim_{\mathrm{H}}(E) = \sup\left\{\alpha : \exists \text{ Borel probability measure } \nu \text{ supported on } E \text{ s.t. } \int_{(x,y)\in E\times E} \frac{d\nu(x)d\nu(y)}{|x-y|^\alpha} < \infty\right\}, \tag{1.1}$$

where $\dim_{\mathrm{H}}(E)$ is the Hausdorff dimension of E.

Many works have been devoted to the fractal dimensions of *realistic networks*. Song *et al.*[11] revealed that many real-world networks have self-similarity and fractality, and Gallos *et al.*[12] gave a review of fractality of complex networks. The algorithms to numerically calculate the fractal dimension of complex networks have been proposed, for example, the compact box burning algorithm (CBB)[13,14] was applied to calculate the fractal dimension of complex networks through the minimum box-covering; Kim *et al.*[15] improved the CBB algorithm to investigate the fractal scaling property in scale-free networks; Gao *et al.*[16] gave the minimum ball-covering approach to calculate the fractal dimensions of complex networks. Considering a network as a graph $G = (V, E)$ equipped with the shortest distance d, we let an l-box A denote a subset of V such that the shortest distance of any two points in the subset is less than l, an l-ball centered at x_0 the subset $\{y : d(y, x_0) < l\}$. Let N_l be the smallest number of l-boxes needed to cover V, and B_l the smallest number of l-balls needed to cover V. Suppose that

$$\sharp V/N_l \sim l^{d_{\mathrm{B}}} \quad \text{and} \quad \sharp V/B_l \sim l^{d_{\mathrm{ball}}},$$

where d_{B} is the fractal dimension defined by Song *et al.*,[11] and d_{ball} is defined by Gao *et al.*[16] Following the above relations, we can investigate whether a network is fractal and what is the fractal dimension of the network. For example, the fractal dimensions of the WWW, the human brain, metabolic network are $d_{\mathrm{B}} = 4.1$, 3.7 and 3.4, respectively.

1.2. Hausdorff Dimension of a Family of Networks

Suppose $\widetilde{G} = \{G_n = (V_n, E_n)\}_{n\geq 0}$ is a family of networks. Let $d_n(x, y)$ be the shortest distance between x and y on G_n, which is the shortest length of path from x to y on G_n. We let $\mathrm{diam}(G_n)$ denote the diameter of G_n, i.e. $\mathrm{diam}(G_n) = \max_{x,y\in G_n} d_n(x, y)$.

We introduce the Hausdorff dimension of a family of networks $\widetilde{G}$ inspired by (1.1).

Definition 1. For a family of networks $\widetilde{G} = \{G_n = (V_n, E_n)\}_{n\geq 0}$ with $\sharp V_n \to \infty$, we define the Hausdorff dimension $\dim(\widetilde{G})$ of $\widetilde{G}$ to be

$$\dim(\widetilde{G}) = \sup\left\{\alpha : \underline{\lim}_{n\to\infty} \frac{1}{(\sharp V_n)^2} \sum_{u,v\in V_n, u\neq v} \frac{1}{(d_n(u,v)/\mathrm{diam}(G_n))^\alpha} < \infty\right\}.$$

Our main result is stated as follows (for the definition of touching networks and conditions (A1) and (A2), see Sec. 1.3).

Theorem 1. *Suppose $\{S_i : \mathbb{R}^l \to \mathbb{R}^l\}_{i=1}^m$ is an Iterated Function System (IFS) of similitudes, satisfying the open set condition, and $K = \bigcup_{i=1}^m S_i(K)$ is the self-similar set with respect to the IFS. For every $1 \leq i \leq m$, we suppose*

$$|S_i(x) - S_i(y)| = |x - y|/N, \quad \text{for all } x, y \in \mathbb{R}^l \tag{1.2}$$

for some $N \in \mathbb{N}$ with $N \geq 2$. If the touching networks $\widetilde{G} = \{G_n\}_{n\geq 0}$ satisfy conditions (A1) *and* (A2), *then*

$$\dim(\widetilde{G}) = \dim_{\mathrm{H}}(K) = \frac{\log m}{\log N}.$$

Example 1. The Hausdorff dimensions of skeleton networks of Sierpinski Gasket, touching networks generated from Vicsek fractal, skeleton networks of Sierpinski Hexagon are $\log 3/\log 2$, $\log 5/\log 3$ and $\log 6/\log 3$, respectively. See Figs. 1–3.

1.3. Touching Networks

Recall the notion of touching networks.[17,18] Let $\{S_i\}_{i=1}^m$ be the IFS in the assumption of Theorem 1

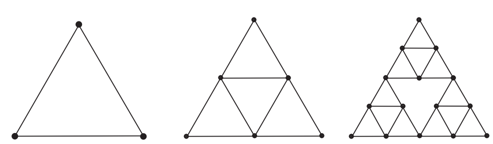

Fig. 1 Skeleton networks of Sierpinski Gasket ($N = 2$, $k_0 = 0$).

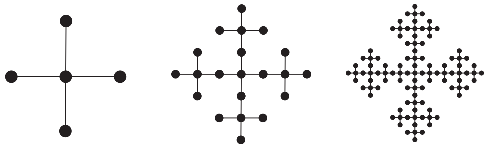

Fig. 2 Touching networks generated from Vicsek fractal ($N = 3, k_0 = 0$).

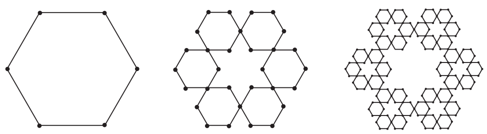

Fig. 3 Skeleton networks of Sierpinski Hexagon ($N = 3, k_0 = 1$).

and $\mu = \sum_{i=1}^{m} \frac{1}{m}\mu \circ S_i^{-1}$ the self-similar probability measure on K. Assume that $G_0 = (V_0, E_0)$ is the initial graph with a finite vertex set $V_0 \subset \mathbb{R}^l$ and an edge set E_0. By induction, given $G_n = (V_n, E_n)$, let

$$V_{n+1} = \bigcup_{i=1}^{m} S_i(V_n)$$

and $u \overset{n+1}{\sim} u'$ for $u, u' \in V_{n+1}$ if and only if there exists i such that $u = S_i(v)$, $u' = S_i(v')$ with $v \overset{n}{\sim} v'$ for $v, v' \in V_n$. Let $d_{n+1}(x, y)$ denote the shortest distance on G_{n+1} and write $\widetilde{d}_{n+1}(x, y) = d_{n+1}(x, y)/N^{n+1}$.

Definition 2. We call that $\{G_n\}_{n\geq 0}$ defined above touching networks, if

(1) $V_0 \subset V_1 = \bigcup_{i=1}^{m} S_i(V_0)$;
(2) G_0, G_1 are connected graphs;
(3) $\widetilde{d}_{k_0}(u, v) = \widetilde{d}_{k_0+1}(u, v)$, $\forall\, u, v \in V_0$ for some integer $k_0 \geq 0$;
(4) $S_i(V_n) \cap S_j(V_n) = S_i(V_0) \cap S_j(V_0)$ for $\forall\, n \geq 1$ and $\forall\, i \neq j$;
(5) $\forall\, u, v \in S_i(V_{k_0})$ there exists a shortest path on G_{k_0+1} within $S_i(G_{k_0})$ from u to v.

We let

$$\partial = \bigcup_{i<j} (S_i(V_0) \cap S_j(V_0)) \quad \text{with } \sharp\partial < \infty, \partial_i = \partial \cap S_i(V_0)$$

and $\partial_{i_1\cdots i_n} = \bigcup_{j_1\cdots j_n \neq i_1\cdots i_n} (S_{i_1\cdots i_n}(V_0) \cap S_{j_1\cdots j_n}(V_0))$ with $\sharp\partial_{i_1\cdots i_n} < \infty$.

Condition (A1). If $u \in \partial_i$, then $S_i^{-1}(u) \in V_0 \backslash \partial$ and $S_i^{-1}(u)$ is the fixed point of S_j with $j \in \mathbb{N} \cap [1, m]$.

Condition (A2). If $K_i \cap K_j \neq \varnothing$ with $i \neq j$, then

$$K_i \cap K_j = \partial_i \cap \partial_j$$

and there is an angle $\theta_0 \in (0, \frac{\pi}{2})$ such that for any $u \in \partial_i \cap \partial_j$, and for any $v_i \in K_i$ and any $v_j \in K_j$ with $v_i \neq u$ and $v_j \neq u$, the angle between $\overrightarrow{uv_i}$ and $\overrightarrow{uv_j}$ is greater than or equal to θ_0. As a result, $\sharp(K_i \cap K_j) \leq 1$, for any $i \neq j$.

Remark 1. For example, above skeleton networks of Sierpinski Gasket, touching networks generated from Vicsek fractal, skeleton networks of Sierpinski Hexagon satisfy conditions (A1) and (A2). See Fig. 4.

Note that $V_0 \subset V_1 \subset V_2 \subset \cdots$, let $V^* = \bigcup_{j=0}^{\infty} V_j$. Let $\widetilde{d}(x, y) = \lim_{n\to\infty} \widetilde{d}_n(x, y)$ for $x, y \in V^*$. Let $(\widetilde{K}, \widetilde{d})$ be the completion of $(V^*, \widetilde{d})$, and $\widetilde{S}_i : \widetilde{K} \to \widetilde{K}$ the continuous extension of $S_i|_{V^*} : V^* \to V^*$. We call $\widetilde{d}$ the ultimate distance on $\widetilde{K}$. Assume that μ_n is the uniform discrete probability measure on V_n, i.e. $\mu_n(u) = \frac{1}{\sharp V_n}$, $\forall\, u \in V_n$.

Then we have the following results[17]:

(R1) Let $\mathrm{diam}(\widetilde{K}) = \sup_{x,y\in\widetilde{K}} \widetilde{d}(x, y)$, we have

$$\mathrm{diam}(\widetilde{K}) < \infty. \tag{1.3}$$

(R2) $(\widetilde{K}, \widetilde{d})$ is a compact metric space with $\widetilde{K} = \bigcup_{i=1}^{m} \widetilde{S}_i(\widetilde{K})$, and for any i,

$$\widetilde{d}(\widetilde{S}_i x, \widetilde{S}_i y) = \widetilde{d}(x, y)/N, \quad \forall\, x, y \in \widetilde{K}. \tag{1.4}$$

(R3) For any $u, v \in V_n$ and any $k \geq k_0$, we have

$$\widetilde{d}(u, v) = \widetilde{d}_{n+k}(u, v). \tag{1.5}$$

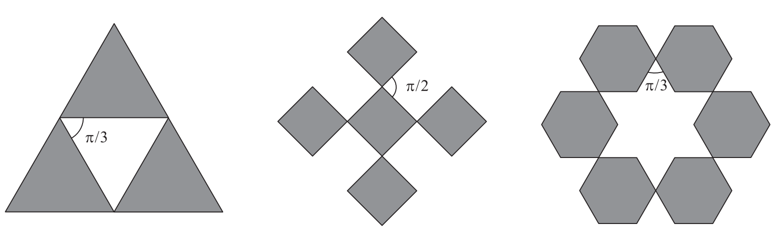

Fig. 4 $\theta_0 = \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{6}$ for Sierpinski Gasket, Vicsek fractal and Sierpinski Hexagon, respectively.

(R4) The sequence of measures $\{\mu_n\}_n$ are weakly convergent to a probability measure

$$\widetilde{\mu} = \sum_{i=1}^{m} \frac{1}{m} \widetilde{\mu} \circ \widetilde{S}_i^{-1} \quad \text{on } \widetilde{K}$$

satisfying $\widetilde{\mu}(\widetilde{S}_i \widetilde{K} \cap \widetilde{S}_j \widetilde{K}) = 0, \forall\, i \neq j$ and

$$\widetilde{\mu}(\widetilde{S}_i E) = \frac{1}{m} \widetilde{\mu}(E) \quad \text{for every Borel set } E \subset \widetilde{K}.$$

Moreover, we have

$$\widetilde{\mu}(\widetilde{S}_{i_1 \cdots i_n} \widetilde{K} \cap \widetilde{S}_{j_1 \cdots j_n} \widetilde{K}) = 0, \quad \forall\, i_1 \cdots i_n \neq j_1 \cdots j_n.$$

We organize the paper as follows. Section 2 is the preliminaries including the result that there exists a positive constant c such that

$$c^{-1}|x-y| \leq \widetilde{d}(x,y) \leq c|x-y| \quad \forall\, x, y \in V^*.$$

To prove Theorem 1, we give the proofs of $\dim(\widetilde{G}) \leq \log m / \log N$ and $\dim(\widetilde{G}) \geq \log m / \log N$ in Secs. 3 and 4, respectively.

2. PRELIMINARIES

Notation 1. Let $x_\lambda \asymp y_\lambda$ denote that $\{x_\lambda\}_{\lambda \in \Lambda}$ is comparable to $\{y_\lambda\}_{\lambda \in \Lambda}$, i.e. there exist two independent constants c_1 and c_2 such that

$$c_1 y_\lambda \leq x_\lambda \leq c_2 y_\lambda, \ \forall\, \lambda \in \Lambda.$$

If only the left or right side of the above inequality holds, then they are denoted as $x_\lambda \gtrsim y_\lambda$ and $x_\lambda \lesssim y_\lambda$, respectively.

Denote

$$\begin{aligned} S_{i_1 i_2 \cdots i_k} &= S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_k}, \\ \widetilde{S}_{i_1 i_2 \cdots i_k} &= \widetilde{S}_{i_1} \circ \widetilde{S}_{i_2} \circ \cdots \circ \widetilde{S}_{i_k}, \\ V^*_{i_1 i_2 \cdots i_k} &= S_{i_1 i_2 \cdots i_k}(V^*), \\ V^{(n)}_{i_1 i_2 \cdots i_k} &= V^*_{i_1 i_2 \cdots i_k} \cap V_n \end{aligned}$$

and

$$K_{i_1 i_2 \cdots i_k} = S_{i_1 i_2 \cdots i_k}(K), \quad \widetilde{K}_{i_1 i_2 \cdots i_k} = S_{i_1 i_2 \cdots i_k}(\widetilde{K}),$$

where $i_1 i_2 \cdots i_k$ is composed of the letters in $\{1, 2, \ldots, m\}$. We write

$$s = \dim_{\mathrm{H}}(K) = \frac{\log m}{\log N}.$$

Recall the following standard result.

Lemma 1 (Theorem 4.13(a), Ref. 19). *Let F be a subset of $\mathbb{R}^l$. If μ is a mass distribution on F with $\int_{(x,y) \in F \times F} \frac{d\mu(x) d\mu(y)}{|x-y|^\alpha} < \infty$, then $\mathcal{H}^\alpha(F) = \infty$ and $\dim_{\mathrm{H}}(F) \geq \alpha$.*

To prove Theorem 1, we need to establish several propositions. For $\alpha \geq 0$, let

$$\mathcal{I}_\alpha(\mu) = \int_{(x,y) \in K \times K} \frac{d\mu(x) d\mu(y)}{|x-y|^\alpha} \quad \text{and}$$

$$\mathcal{J}_\alpha(\widetilde{\mu}) = \int_{(x,y) \in \widetilde{K} \times \widetilde{K}} \frac{d\widetilde{\mu}(x) d\widetilde{\mu}(y)}{\widetilde{d}(x,y)^\alpha}.$$

Proposition 1. *We have*

$$\sup\{\alpha : \mathcal{I}_\alpha(\mu) < \infty\} = s. \tag{2.1}$$

Proof. Note that the IFS satisfies the open set condition (which yields $0 < \mathcal{H}^s(K) < \infty$) and μ is the natural self-similar probability measure, hence μ is Ahlfors–David regular, i.e. there exist two constants c_1 and c_2 such that

$$c_1 r^s \leq \mu(B_r(x)) \leq c_2 r^s \tag{2.2}$$

for all $x \in K$ and $0 < r <$diam(K), where $B_r(x)$ is the closed ball with center x and radius r, and diam(K) denotes the diameter of K.

We first show

$$\sup\{\alpha : \mathcal{I}_\alpha(\mu) < \infty\} \geq s. \tag{2.3}$$

Let $\phi_\alpha(x) = \int_{y \in K} \frac{d\mu(y)}{|x-y|^\alpha}$ for $x \in K$. For any $\alpha < s$, after integrating by parts and using the Ahlfors–David regularity (2.2), we have

$$\begin{aligned} \phi_\alpha(x) &= \int_{|x-y| \leq 1} \frac{d\mu(y)}{|x-y|^\alpha} + \int_{|x-y|>1} \frac{d\mu(y)}{|x-y|^\alpha} \\ &\leq \int_0^1 r^{-\alpha} d\mu(B_r(x)) + \mu(K) \\ &= [r^{-\alpha} \mu(B_r(x))]_0^1 \\ &\quad + \alpha \int_0^1 r^{-(\alpha+1)} \mu(B_r(x)) dr + 1 \\ &\leq 2c_2 + c_2 \alpha \int_0^1 r^{s-\alpha-1} dr + 1 \\ &= c_2 \left(2 + \frac{\alpha}{s-\alpha}\right) + 1 = c. \end{aligned}$$

Thus, $\phi_\alpha(x) \leq c$ for all $x \in K$, so that

$$\begin{aligned} \mathcal{I}_\alpha(\mu) &= \int_{x \in K} \phi_\alpha(x) d\mu(x) \\ &\leq c\mu(K) \leq c < \infty, \quad \forall\, \alpha < s. \end{aligned}$$

Hence, we obtain that (2.3).

On the other hand, using $0 < \mathcal{H}^s(K) < \infty$ and Lemma 1, we have

$$\mathcal{I}_s(\mu) = \infty, \tag{2.4}$$

otherwise $\mathcal{H}^s(K) = \infty$. Since

$$\mathcal{I}_\alpha(\mu) = \int_{|x-y|\le 1} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} + \int_{|x-y|>1} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}$$

and $\int_{|x-y|>1} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} \le (\mu(K))^2 = 1$, we obtain that

$$\mathcal{I}_\alpha(\mu) = \infty \quad \text{if and only if} \quad \int_{|x-y|\le 1} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} = \infty. \tag{2.5}$$

Note that for any $\alpha \ge s$ and $|x-y| \le 1$,

$$\frac{1}{|x-y|^s} \le \frac{1}{|x-y|^\alpha},$$

hence by (2.4) and (2.5) we have

$$\mathcal{I}_\alpha(\mu) = \infty, \quad \forall\, \alpha \ge s. \tag{2.6}$$

Thus, (2.1) follows from (2.3) and (2.6). □

Suppose $A \in \partial_i$, according to condition (A1), $u = S_i^{-1}(A) \in V_0 \backslash \partial$ is the fixed point for some S_p.

Lemma 2. *For any $x \in V_i^*$ with $x \ne A$, we have*

$$\widetilde{d}(x, A) \asymp |x - A|.$$

To prove Lemma 2, it is sufficient to show the following.

Claim 1. For any $x \in V_i^*$ with $x \ne A$, if $x \in V^*_{i[p]^{k-1}} \backslash V^*_{i[p]^k}$ for some $k \ge 1$, then

$$|x - A| \asymp \frac{1}{N^k} \quad \text{and} \quad \widetilde{d}(x, A) \asymp \frac{1}{N^k}. \tag{2.7}$$

Here $[p]^k$ denotes the word composed of k letters p for $k \ge 0$.

Proof. By the self-similarities (1.2) and (1.4), we have

$$|x - A| = \frac{1}{N^k}|y - u| \quad \text{and} \quad \widetilde{d}(x, A) = \frac{1}{N^k}\widetilde{d}(y, u),$$

where $y = S^{-1}_{i[p]^{k-1}}(x) \notin V_p^*$ as $x \in V^*_{i[p]^{k-1}} \backslash V^*_{i[p]^k}$ and $u = S_i^{-1}(A) = S^{-1}_{i[p]^{k-1}}(A) \in V_p^*$ as $S_p(u) = u$. To prove (2.7), it is sufficient to show

$$|y - u| \asymp 1 \quad \text{and} \quad \widetilde{d}(y, u) \asymp 1.$$

(1) We first prove $\widetilde{d}(y, u) \asymp 1$.

For $y, u \in V^*$, take n large enough such that $y, u \in V_n$. Since $u \in V_p^* \backslash \partial$ and $y \notin V_p^*$, we conclude that the shortest path from y to u on G_{n+k_0} shall pass through one vertex in

$$\partial_p = S_p(V_0) \cap \partial = S_p(V_{n+k_0-1}) \cap \partial (\subset \partial)$$

independent of n due to condition (4) of Definition 2. Notice ∂_p is finite and $u \notin \partial_p$, using $\mathrm{diam}(\widetilde{K}) < \infty$ (formula (1.3)) and $\widetilde{d}(u, v) = \widetilde{d}_{n+k}(u, v)$ (formula (1.5)), we have

$$\widetilde{d}(y, u) \le \mathrm{diam}(\widetilde{K}) < \infty$$

and

$$\widetilde{d}(y, u) = \widetilde{d}_{n+k_0}(y, u) \ge \min_{z \in \partial_p} \widetilde{d}_{n+k_0}(z, u) = \min_{z \in \partial_p} \widetilde{d}(z, u) \ge \min_{\substack{z_1, z_2 \in \partial \\ z_1 \ne z_2}} \widetilde{d}(z_1, z_2) > 0.$$

Hence, we obtain that $\widetilde{d}(y, u) \asymp 1$.

(2) Next, we prove $|y - u| \asymp 1$.

According to condition (A2), we have

$$u \notin \bigcup_{t \ne p} K_t. \tag{2.8}$$

Otherwise, if $u \in K_{t_0}$ for some $t_0 \ne p$, then $u \in K_{t_0} \cap K_p = \partial_{t_0} \cap \partial_p \subset \partial$, this contradicts the fact $u \notin \partial$. Thus, by the compactness of K and (2.8), we have

$$0 < \inf_{z \in \bigcup_{t \ne p} K_t} |u - z| \le |y - u| \le \mathrm{diam}(K) < \infty.$$

Hence, we obtain that $|y - u| \asymp 1$. □

Proposition 2. *The ultimate distance is comparable to Euclidean distance in V^*, i.e.*

$$\widetilde{d}(x, y) \asymp |x - y| \quad \textit{for all } x, y \in V^*.$$

Proof. Let $x \ne y \in V^*$, there is a word $i_1 \cdots i_n$ such that $x, y \in V^*_{i_1 \cdots i_n}$ and $x \in V^*_{i_1 \cdots i_n j_1}$, $y \in V^*_{i_1 \cdots i_n j_2}$ $(j_1 \ne j_2)$. Let $x' = S^{-1}_{i_1 \cdots i_n} x \in V^*_{j_1}$, $y' = S^{-1}_{i_1 \cdots i_n} y \in V^*_{j_2}$. By the self-similarities (1.2) and (1.4), we can obtain that

$$|x - y| = N^{-n}|x' - y'| \quad \text{and}$$
$$\widetilde{d}(x, y) = N^{-n}\widetilde{d}(x', y').$$

To prove $\widetilde{d}(x, y) \asymp |x - y|$, it is sufficient to show

$$\widetilde{d}(x', y') \asymp |x' - y'|.$$

Suppose $x', y' \in V_{n'}$ with n' large enough.

We will distinguish two different cases.

Case 1. If $\{A\} = K_{j_1} \cap K_{j_2} = \partial_{j_1} \cap \partial_{j_2}$. By Lemma 2, we have

$$\widetilde{d}(x', A) \asymp |x' - A|, \quad \widetilde{d}(y', A) \asymp |y' - A|. \tag{2.9}$$

We first consider the Euclidean distance $|x' - y'|$. Suppose $\theta(\geq \theta_0)$ is the angle between $\overrightarrow{Ax'}$ and $\overrightarrow{Ay'}$, by the cosine theorem and condition (A2), we have

$$\begin{aligned}
&(|x' - A| + |y' - A|)^2 \\
&\quad \geq |x' - y'|^2 \\
&\quad = |x' - A|^2 + |y' - A|^2 \\
&\qquad - 2|x' - A||y' - A| \cos\theta \\
&\quad \geq |x' - A|^2 + |y' - A|^2 \\
&\qquad - 2|x' - A||y' - A| \cos\theta_0 \\
&\quad = (1 - \cos\theta_0)(|x' - A|^2 + |y' - A|^2) \\
&\qquad + \cos\theta_0(|x' - A| - |y' - A|)^2 \\
&\quad \geq \frac{1 - \cos\theta_0}{2}(|x' - A| + |y' - A|)^2.
\end{aligned}$$

Hence, we obtain that

$$|x' - y'| \asymp |x' - A| + |y' - A|. \tag{2.10}$$

Next, we consider the ultimate distance $\widetilde{d}(x', y')$.

Subcase 1.1. If the shortest path on $G_{n'+k_0}$ from x' to y' passes through A, then $\widetilde{d}(x', y') = \widetilde{d}_{n'+k_0}(x', y') = \widetilde{d}_{n'+k_0}(x', A) + \widetilde{d}_{n'+k_0}(A, y') = \widetilde{d}(x', A) + \widetilde{d}(A, y')$, and by (2.9) and (2.10), we have

$$\widetilde{d}(x', y') \asymp |x' - y'|.$$

Subcase 1.2. If the shortest path on $G_{n'+k_0}$ from x' to y' passes outside of $V^*_{j_1}$ and $V^*_{j_2}$, then we can conclude that the path shall pass through distinct $B_1, B_2, \ldots, B_t \in \partial$ $(t \geq 2)$ by condition (4) of Definition 2. Note that ∂ is a finite set, and by $\mathrm{diam}(\widetilde{K}) < \infty$ (formula (1.3)) and $\widetilde{d}(u, v) = \widetilde{d}_{n+k}(u, v)$ (formula (1.5)), we have

$$\widetilde{d}(x', y') \leq \mathrm{diam}(\widetilde{K}) < \infty$$

and

$$\begin{aligned}
\widetilde{d}(x', y') &= \widetilde{d}_{n'+k_0}(x', y') \geq \widetilde{d}_{n'+k_0}(B_1, B_2) \\
&= \widetilde{d}(B_1, B_2) \geq \min_{\substack{z_1, z_2 \in \partial \\ z_1 \neq z_2}} \widetilde{d}(z_1, z_2) > 0.
\end{aligned}$$

Hence, we have

$$\widetilde{d}(x', y') \asymp 1. \tag{2.11}$$

Note that

$$\widetilde{d}(x', y') \leq \widetilde{d}(x', A) + \widetilde{d}(A, y') \leq 2\,\mathrm{diam}(\widetilde{K}) < \infty$$

and using (2.11), we have

$$\widetilde{d}(x', A) + \widetilde{d}(A, y') \asymp 1. \tag{2.12}$$

By (2.9), (2.10) and (2.12), we can obtain that

$$|x' - y'| \asymp 1. \tag{2.13}$$

Therefore, it follows from (2.11) and (2.13)

$$\widetilde{d}(x', y') \asymp |x' - y'|.$$

Case 2. If $K_{j_1} \cap K_{j_2} = \varnothing$, then by the compactness of K we have

$$\begin{aligned}
\infty > \mathrm{diam}(K) &\geq |x' - y'| \\
&\geq \min_{K_i \cap K_j = \varnothing} \mathrm{dist}_{||}(K_i, K_j) > 0,
\end{aligned}$$

where $\mathrm{dist}_{||}(D_1, D_2) = \inf_{x \in D_1, y \in D_2} |x - y|$. Hence, we obtain that

$$|x' - y'| \asymp 1. \tag{2.14}$$

Since $K_{j_1} \cap K_{j_2} = \varnothing$ implies $V^*_{j_1} \cap V^*_{j_2} = \varnothing$, we can conclude that the shortest path on $G_{n'+k_0}$ from x' to y shall pass through distinct $B_1, B_2, \ldots, B_q \in \partial$ $(q \geq 2)$ according to condition (4) of Definition 2. Note that ∂ is finite, using $\mathrm{diam}(\widetilde{K}) < \infty$ (formula (1.3)) and $\widetilde{d}(u, v) = \widetilde{d}_{n+k}(u, v)$ (formula (1.5)), we have

$$\widetilde{d}(x', y') \leq \mathrm{diam}(\widetilde{K}) < \infty$$

and

$$\begin{aligned}
\widetilde{d}(x', y') &= \widetilde{d}_{n'+k_0}(x', y') \\
&\geq \widetilde{d}_{n'+k_0}(B_1, B_2) \\
&= \widetilde{d}(B_1, B_2) \geq \min_{\substack{z_1, z_2 \in \partial \\ z_1 \neq z_2}} \widetilde{d}(z_1, z_2) > 0.
\end{aligned}$$

Hence, we obtain that

$$\widetilde{d}(x', y') \asymp 1. \tag{2.15}$$

Therefore, it follows from (2.14) and (2.15)

$$\widetilde{d}(x', y') \asymp |x' - y'|. \qquad \square$$

Corollary 1. *We have the following result*:

(1) $K = \widetilde{K}$ *and* $S = \widetilde{S}$. *Specifically,*

$$\widetilde{d}(x, y) \asymp |x - y| \quad \text{for all } x, y \in K = \widetilde{K}. \tag{2.16}$$

(2) $\mu = \widetilde{\mu}$.
(3) *For any* $\alpha > 0$,

$$\mathcal{I}_\alpha(\mu) < \infty \quad \text{if and only if } \mathcal{J}_\alpha(\widetilde{\mu}) < \infty. \tag{2.17}$$

As a consequence, we have

$$\sup\{\alpha : \mathcal{J}_\alpha(\widetilde{\mu}) < \infty\} = s. \tag{2.18}$$

Proof. Note that $(K, ||)$ and $(\widetilde{K}, \widetilde{d})$ are the completions of $(V^*, ||)$ and $(V^*, \widetilde{d})$, respectively, and by Proposition 2, we may obtain that $K = \widetilde{K}$. Since $\widetilde{S}_i : \widetilde{K} \to \widetilde{K}$ is the continuous extension of $S_i|_{V^*} : V^* \to V^*$, we get

$$S_i = \widetilde{S}_i.$$

Furthermore, we obtain (2.16). Since the IFS satisfies the open set condition, we have

$$\mu = \frac{1}{m}\sum_{i=1}^m \mu \circ S_i^{-1}.$$

Using $K = \widetilde{K}$, $S_i = \widetilde{S}_i$, $\mu = \frac{1}{m}\sum_{i=1}^m \mu \circ S_i^{-1}$, $\widetilde{\mu} = \sum_{i=1}^m \frac{1}{m}\widetilde{\mu} \circ \widetilde{S}_i^{-1}$ and the fact that the operator $\mathcal{F}(\nu) = \frac{1}{m}\sum_{i=1}^m \nu \circ S_i^{-1}$ is compressive with respect to the Hutchinson metric $d_{\mathrm{H}}(\nu_1, \nu_2) = \sup_{f \in \mathrm{Lip}1}(\int f d\nu_1 - \int f d\nu_2)$,[20] then $\mathcal{F}$ has a unique fixed point, i.e.

$$\mu = \widetilde{\mu}.$$

Therefore, (2.17) follows from $K = \widetilde{K}$, $\mu = \widetilde{\mu}$ and (2.16). Moreover, by Proposition 1 and (2.17), we can obtain that

$$\sup\{\alpha : \mathcal{J}_\alpha(\widetilde{\mu}) < \infty\} = s. \qquad \square$$

Proposition 3. *We have*

$$\widetilde{d}_n(x,y) \gtrsim \widetilde{d}(x,y)(= \widetilde{d}_{n+k_0}(x,y)) \quad \text{for any } x, y \in V_n.$$

Proof. By induction and $\widetilde{d}(u,v) = \widetilde{d}_{n+k}(u,v)$ (formula (1.5)), it is sufficient to prove there is a constant $c_1 > 0$ such that

$$\widetilde{d}_n(x,y) \geq c_1 \widetilde{d}_{n+1}(x,y) \quad \text{for any } x, y \in V_n (\subset V_{n+1}).$$

Given $x, y \in V_n$, there is a shortest path on G_n from x to y as follows:

$$x = x_0 \overset{n}{\sim} x_1 \overset{n}{\sim} x_2 \overset{n}{\sim} \cdots \overset{n}{\sim} x_t = y \quad \text{with } t = d_n(x,y).$$

Fix i, we notice $x_i \overset{n}{\sim} x_{i+1}$, then there exists a word $j_1 j_2 \cdots j_n$ such that

$$x_i = S_{j_1 j_2 \cdots j_n}(u) \quad \text{and} \quad x_{i+1} = S_{j_1 j_2 \cdots j_n}(v),$$

where $u, v \in V_0 (\subset V_1)$ with $u \overset{0}{\sim} v$. Since $u, v \in V_1$ and G_1 is connected, there is a shortest path on G_1 from u to v, that is

$$u = w_0 \overset{1}{\sim} w_1 \overset{1}{\sim} w_2 \overset{1}{\sim} \cdots \overset{1}{\sim} w_q = v \quad \text{with} \quad q = d_1(u,v) \leq \mathrm{diam}(G_1).$$

Then we obtain the following path on G_{n+1} from x_i to x_{i+1},

$$x_i = S_{j_1 j_2 \cdots j_n} w_0 \overset{n+1}{\sim} S_{j_1 j_2 \cdots j_n} w_1 \overset{n+1}{\sim} S_{j_1 j_2 \cdots j_n} w_2 \overset{n+1}{\sim} \cdots \overset{n+1}{\sim} S_{j_1 j_2 \cdots j_n} w_q = x_{i+1}.$$

Hence, we obtain that

$$d_{n+1}(x_i, x_{i+1}) \leq \mathrm{diam}(G_1).$$

Therefore, we have

$$d_{n+1}(x,y) \leq \sum_{i=0}^{t-1} d_{n+1}(x_i, x_{i+1}) \leq \sum_{i=0}^{t-1} \mathrm{diam}(G_1) \leq \mathrm{diam}(G_1) d_n(x,y),$$

which implies $\widetilde{d}_{n+1}(x,y) \leq \frac{\mathrm{diam}(G_1)}{N}\widetilde{d}_n(x,y)$. Let $c_1 = \frac{N}{\mathrm{diam}(G_1)}$, we have

$$\widetilde{d}_n(x,y) \geq c_1 \widetilde{d}_{n+1}(x,y), \forall\, x, y \in V_n. \qquad \square$$

Recall the following.

Proposition 4 (Proposition 2, Ref. 17). *Given the graph G_n with $n \geq k_0 + 1$, then for any i and any $x, y \in S_i(V_{n-1})$, we can take a shortest path within $S_i(G_{n-1})$ from x to y.*

Inductively, we obtain the following.

Corollary 2. *Given the graph G_{n+k_0} with $n \geq 1$, then for any $i_1 i_2 \cdots i_k$ $(0 \leq k \leq n-1)$ and any $x, y \in S_{i_1 i_2 \cdots i_k}(V_{n+k_0-k})$, we can take a shortest path within $S_{i_1 i_2 \cdots i_k}(G_{n+k_0-k}) \subset G_{n+k_0}$ from x to y.*

3. PROOF OF $\dim(\widetilde{G}) \leq \log m / \log N$

Proposition 5. *For any $\alpha > 0$, we have*

$$\mathcal{J}_\alpha(\widetilde{\mu}) \lesssim \underline{\lim}_{n\to\infty} \frac{1}{(\sharp V_n)^2} \sum_{x,y \in V_n, x \neq y} \frac{1}{\widetilde{d}_n(x,y)^\alpha}. \tag{3.1}$$

Proof. Let $\varepsilon_n = \frac{200\,\mathrm{diam}(\widetilde{K})}{N^n}$. Then we have

$$\chi_{\{(x,y)\in\widetilde{K}\times\widetilde{K}:\widetilde{d}(x,y)\geq\varepsilon_n\}}\frac{1}{\widetilde{d}(x,y)^\alpha} \to \chi_{\{(x,y)\in\widetilde{K}\times\widetilde{K}:x\neq y\}}\frac{1}{\widetilde{d}(x,y)^\alpha}, \quad \text{as } n\to\infty.$$

Using the monotone convergence theorem, we have

$$\mathcal{J}_\alpha(\widetilde{\mu}) = \lim_{n\to\infty}\int_{\widetilde{d}(x,y)\geq\varepsilon_n}\frac{d\widetilde{\mu}(x)d\widetilde{\mu}(y)}{\widetilde{d}(x,y)^\alpha}. \tag{3.2}$$

Take $x_{i_1\cdots i_n}\in V^{(n)}_{i_1i_2\cdots i_n}$. For $x\in\widetilde{K}_{i_1\cdots i_n}$, let

$$f_n(x) = \begin{cases} x_{i_1\cdots i_n} & \text{if } x\in\widetilde{K}_{i_1\cdots i_n}\backslash\partial_{i_1\cdots i_n},\\ x & \text{if } x\in\partial_{i_1\cdots i_n}.\end{cases}$$

Notice that the measure of boundary $\widetilde{\mu}(\partial_{i_1\cdots i_n}) = 0$, since the boundary is a finite set. Hence,

$$f_n(x)|_{\widetilde{K}_{i_1\cdots i_n}} \overset{\widetilde{\mu}-\text{a.e.}}{\equiv} x_{i_1\cdots i_n}. \tag{3.3}$$

We also have

$$f_n(x)\to x \quad \text{as } n\to\infty. \tag{3.4}$$

In fact, (3.4) follows from that

$$\widetilde{d}(f_n(x),x)\leq \mathrm{diam}(\widetilde{K}_{i_1\cdots i_n}) = \frac{\mathrm{diam}(\widetilde{K})}{N^n}.$$

Denote $\mathrm{dist}_{\widetilde{d}}(D_1,D_2) = \inf_{x\in D_1,y\in D}\widetilde{d}(x,y)$. Fix n_0, using the Fatou's lemma, (3.3), (3.4), $\widetilde{\mu}(\widetilde{K}_{i_1\cdots i_n}) = m^{-n}$ and $\sharp V_n \asymp m^n$, we have

$$\begin{aligned}
&\int_{|x-y|\geq\varepsilon_{n_0}}\frac{d\widetilde{\mu}(x)d\widetilde{\mu}(y)}{\widetilde{d}(x,y)^\alpha}\\
&\quad\leq \underline{\lim}_{n\to\infty}\int_{|x-y|\geq\varepsilon_{n_0}}\frac{d\widetilde{\mu}(x)d\widetilde{\mu}(y)}{\widetilde{d}(f_n(x),f_n(y))^\alpha}\\
&\quad\leq \underline{\lim}_{n\to\infty}\sum_{(i_1\cdots i_n,j_1\cdots j_n)}\int_{\substack{|x-y|\geq\varepsilon_{n_0}\\ x\in\widetilde{K}_{i_1\cdots i_n},y\in\widetilde{K}_{j_1\cdots j_n}}}\\
&\qquad\times\frac{d\widetilde{\mu}(x)d\widetilde{\mu}(y)}{\widetilde{d}(f_n(x),f_n(y))^\alpha}\\
&\quad\leq \underline{\lim}_{n\to\infty}\sum_{\substack{(i_1\cdots i_n,j_1\cdots j_n)\\ \mathrm{dist}_{\widetilde{d}}(\widetilde{K}_{i_1\cdots i_n},\widetilde{K}_{j_1\cdots j_n})>\frac{50\,\mathrm{diam}(\widetilde{K})}{N^{n_0}}}}\\
&\qquad\times\int_{(x,y)\in\widetilde{K}_{i_1\cdots i_n}\times\widetilde{K}_{j_1\cdots j_n}}\frac{d\widetilde{\mu}(x)d\widetilde{\mu}(y)}{\widetilde{d}(f_n(x),f_n(y))^\alpha}\\
&\quad\leq \underline{\lim}_{n\to\infty}\frac{1}{m^{2n}}\\
&\qquad\times\sum_{\substack{(i_1\cdots i_n,j_1\cdots j_n)\\ \mathrm{dist}_{\widetilde{d}}(\widetilde{K}_{i_1\cdots i_n},\widetilde{K}_{j_1\cdots j_n})>\frac{50\,\mathrm{diam}(\widetilde{K})}{N^{n_0}}}}\\
&\qquad\times\frac{1}{\widetilde{d}(f_n(x),f_n(y))^\alpha}\\
&\quad\lesssim \underline{\lim}_{n\to\infty}\frac{1}{(\sharp V_n)^2}\sum_{u,v\in V_n,u\neq v}\frac{1}{\widetilde{d}(u,v)^\alpha}.
\end{aligned}$$

Note that $\sharp V_{n+k_0}\asymp m^{-(n+k_0)}\asymp m^{-n}\asymp\sharp V_n$, then we have

$$\begin{aligned}
&\underline{\lim}_{n\to\infty}\frac{1}{(\sharp V_n)^2}\sum_{u,v\in V_n,u\neq v}\frac{1}{\widetilde{d}(u,v)^\alpha}\\
&\quad\lesssim \underline{\lim}_{n\to\infty}\frac{1}{(\sharp V_{n+k_0})^2}\sum_{u,v\in V_n,u\neq v}\frac{1}{\widetilde{d}_{n+k_0}(u,v)^\alpha}\\
&\quad\leq \underline{\lim}_{n\to\infty}\frac{1}{(\sharp V_{n+k_0})^2}\sum_{u,v\in V_{n+k_0},u\neq v}\frac{1}{\widetilde{d}_{n+k_0}(u,v)^\alpha}\\
&\quad\leq \underline{\lim}_{n\to\infty}\frac{1}{(\sharp V_n)^2}\sum_{u,v\in V_n,u\neq v}\frac{1}{\widetilde{d}_n(u,v)^\alpha}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int_{|x-y|\geq\varepsilon_{n_0}}\frac{d\widetilde{\mu}(x)d\widetilde{\mu}(y)}{\widetilde{d}(x,y)^\alpha}\\
&\quad\lesssim \underline{\lim}_{n\to\infty}\frac{1}{(\sharp V_n)^2}\\
&\qquad\times\sum_{u,v\in V_n,u\neq v}\frac{1}{\widetilde{d}_n(u,v)^\alpha}, \quad \forall\, n_0\in\mathbb{N}.
\end{aligned}\tag{3.5}$$

Therefore, (3.1) follows form (3.2) and (3.5). □

Given $\alpha > s = \log m/\log N$, using $\sup\{\alpha : \mathcal{J}_\alpha(\widetilde{\mu}) < \infty\} = s$ (formula (2.18)) we can immediately obtain that

$$\mathcal{J}_\alpha(\widetilde{\mu}) = \infty.$$

Then by Proposition 5 we have

$$\underline{\lim}_{n\to\infty}\frac{1}{(\sharp V_n)^2}\sum_{u,v\in V_n,u\neq v}\frac{1}{\widetilde{d}_n(u,v)^\alpha}\geq\mathcal{J}_\alpha(\widetilde{\mu}) = \infty.$$

Therefore, we have

$$\dim(\widetilde{G})\leq s.$$

4. PROOF OF $\dim(\widetilde{G}) \geq \log m/\log N$

To prove $\dim(\widetilde{G}) \geq s = \frac{\log m}{\log N}$, by Proposition 3, it is sufficient to show

$$\underline{\lim}_{n\to\infty}\frac{1}{(\sharp V_n)^2}\sum_{u,v\in V_n,u\neq v}\frac{1}{\widetilde{d}(u,v)^\alpha}<\infty,\quad \forall\,\alpha<s.$$

For any $u \neq v \in V_n$, there is a word $\mathbf{i}$ of length $|\mathbf{i}| = k$ $(0 \leq k \leq n-1)$ such that $u \in V^{(n)}_{\mathbf{i}*j_1}$ and $v \in V^{(n)}_{\mathbf{i}*j_2}$ with $j_1 \neq j_2$. According to condition (4) of Definition 2 and Corollary 2, there are finitely many vertexes $B_1, B_2, \ldots, B_t \in S_{\mathbf{i}}(\partial)$ $(t \geq 1)$ with $B_1 \in S_{\mathbf{i}}(\partial_{j_1})$, such that the shortest path within $S_{\mathbf{i}}(G_{n+k_0-k}) (\subset G_{n+k_0})$ from u to v passes $B_1, B_2, \ldots, B_t$. It follows from $\widetilde{d}(u,v) = \widetilde{d}_{n+k}(u,v)$ (formula (1.5)) that

$$\begin{aligned}\widetilde{d}(u,v) &= \widetilde{d}(u,B_1)+\widetilde{d}(B_1,B_2)\\ &\quad+\cdots+\widetilde{d}(B_{t-1},B_t)+\widetilde{d}(B_t,v)\geq\widetilde{d}(u,B_1).\end{aligned}$$

Suppose $A = S^{-1}_{\mathbf{i}}(B_1) \in \partial_{j_1}$, according to condition (A1), $S^{-1}_{j_1}(A)$ is the fixed point of S_p for some $p = p_{\mathbf{i},j_1,A} \in \mathbb{N}\cap[1,m]$. If $u \in V^*_{\mathbf{i}*j_1*[p]^q}\backslash V^*_{\mathbf{i}*j_1*[p]^{q+1}}$ $(0 \leq q \leq n-k-2)$, applying $x = S^{-1}_{\mathbf{i}}(u) \in V^*_{j_1[p]^q}\backslash V^*_{j_1[p]^{q+1}}$ and $A = S^{-1}_{\mathbf{i}}(B_1) \in \partial_{j_1}$ to Claim 1, we obtain that

$$\widetilde{d}(S^{-1}_{\mathbf{i}}(u),S^{-1}_{\mathbf{i}}(B_1))\asymp\frac{1}{N^q},$$

which implies $\widetilde{d}(u,B_1) = \widetilde{d}(S_{\mathbf{i}}(S^{-1}_{\mathbf{i}}(u)),S_{\mathbf{i}}(S^{-1}_{\mathbf{i}}(B_1))) = \frac{1}{N^k}\widetilde{d}(S^{-1}_{\mathbf{i}}(u),S^{-1}_{\mathbf{i}}(B_1)) \asymp \frac{1}{N^{k+q}}$. Hence,

$$\widetilde{d}(u,v)\geq\widetilde{d}(u,B_1)\asymp\frac{1}{N^{k+q}}.\tag{4.1}$$

We also have $\sharp V_n \asymp m^n$ and

$$\begin{aligned}\sharp V^{(n)}_{\mathbf{i}*j_2}\asymp m^{n-k},\quad &\sharp(V^{(n)}_{\mathbf{i}*j_1*[p]^q}\backslash V^*_{\mathbf{i}*j_1*[p]^{q+1}})\\ &\leq\sharp V^{(n)}_{\mathbf{i}*j_1*[p]^q}\asymp m^{n-(k+q)}.\end{aligned}\tag{4.2}$$

Then using (4.1) and (4.2) we have

$$\begin{aligned}&\frac{1}{(\sharp V_n)^2}\sum_{u,v\in V_n,u\neq v}\frac{1}{\widetilde{d}(u,v)^\alpha}\\
&=\frac{1}{(\sharp V_n)^2}\sum_{k=0}^{n-1}\sum_{|\mathbf{i}|=k}\sum_{j_1\neq j_2}\sum_{u\in V^{(n)}_{\mathbf{i}*\mathbf{j}_1},v\in V^{(n)}_{\mathbf{i}*\mathbf{j}_2}}\frac{1}{\widetilde{d}(u,v)^\alpha}\\
&=\frac{1}{(\sharp V_n)^2}\sum_{k=0}^{n-1}\sum_{|\mathbf{i}|=k}\sum_{j_1\neq j_2}\sum_{v\in V^{(n)}_{\mathbf{i}*\mathbf{j}_2}}\sum_{A\in\partial_{j_1}}\\
&\quad\times\sum_{q=0}^{n-k-2}\sum_{u\in V^{(n)}_{\mathbf{i}*\mathbf{j}_1*[p]^q}\backslash V^*_{\mathbf{i}*\mathbf{j}_1*[p]^{q+1}}}\frac{1}{\widetilde{d}(u,v)^\alpha}\end{aligned}$$

(here $A\in\partial_{j_1}$ and $S_p(S^{-1}_{j_1}A) = S^{-1}_{j_1}A)$)

$$\begin{aligned}&\lesssim\frac{1}{m^{2n}}\sum_{k=0}^{n-1}\sum_{|\mathbf{i}|=k}\sum_{j_1\neq j_2}\sum_{A\in\partial_{j_1}}\sum_{q=0}^{n-k-2}\left(\frac{1}{N^{k+q}}\right)^{-\alpha}\\
&\quad\cdot\sharp(V^{(n)}_{\mathbf{i}*j_1*[p]^q}\backslash V^*_{\mathbf{i}*j_1*[p]^{q+1}})\cdot\sharp V^{(n)}_{\mathbf{i}*j_2}\\
&\lesssim\frac{1}{m^{2n}}\sum_{k=0}^{n-1}\sum_{|\mathbf{i}|=k}\sum_{q=0}^{n-k-2}N^{(k+q)\alpha}\\
&\quad\cdot m^{n-(k+q)}\cdot m^{n-k}\\
&\lesssim\sum_{k=0}^{n-1}m^k\sum_{q=0}^{n-k-2}N^{(k+q)\alpha}\cdot m^{-(k+q)}\cdot m^{-k}\\
&\lesssim\sum_{k=0}^{n-1}N^{k\alpha}m^{-k}\sum_{q=0}^{n-k-2}\left(\frac{N^\alpha}{m}\right)^q\\
&\lesssim\left(\sum_{k=0}^{n}\left(\frac{N^\alpha}{m}\right)^k\right)^2<\infty.\end{aligned}$$

Notice $\alpha < s$, which implies $N^\alpha < m$. Letting $n \to \infty$, we have

$$\begin{aligned}&\underline{\lim}_{n\to\infty}\frac{1}{(\sharp V_n)^2}\sum_{u,v\in V_n,u\neq v}\frac{1}{\widetilde{d}(u,v)^\alpha}\\
&\quad\lesssim\left(\sum_{k=0}^{\infty}\left(\frac{N^\alpha}{m}\right)^k\right)^2<\infty,\quad\forall\,\alpha<s.\end{aligned}$$

Hence, we have

$$\dim(\widetilde{G})\geq s.$$

5. REMARK

We say that $u \in V_0$ is of unique first code if there is a unique $i \in \{1,2,\ldots,m\}$ such that $v = S^{-1}_i(u) \in V_0$. Let

$$\begin{aligned}\Xi=\{u\in V_0: u=S_i(v)\text{ is of unique first code}\\ \text{with } v\in V_0\backslash\partial\}.\end{aligned}$$

We say that u is an isolate vertex, if $u \in \Xi$ and there is an infinite sequence $i_1 i_2 \cdots i_k \cdots$ such that $u_k = S^{-1}_{i_1 i_2 \cdots i_k}(u) \in \Xi$ for each $k \in \mathbb{N}$. In fact, the sequence $i_1 i_2 \cdots i_k \cdots$ is eventually periodic.

Condition (A1$'$). If $u \in \partial_i$, then u is an isolate vertex. We give the comment that the results of this paper are also valid when condition (A1) is replaced with condition (A1$'$).

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