Fractals, Vol. 31, No. 7 (2023) 2350066 (14 pages)

DOI: 10.1142/S0218348X23500664

ACCURATE FORMULAS OF HYPER-WIENER INDICES OF SIERPIŃSKI SKELETON NETWORKS

Qin Wang[*,‡] and Qingcheng Zeng[†,§]
[*]*College of Big Data and Software Engineering*
Zhejiang Wanli University
315101 Ningbo, P. R. China
[†]*School of Mathematics and Statistics*
Ningbo University
315211 Ningbo, P. R. China
[‡]*qinwang@126.com*
[§]*2111071003@nbu.edu.cn*

Received February 16, 2023
Accepted April 26, 2023
Published July 27, 2023

Abstract

The hyper-Wiener index on a graph is an important topological invariant that is defined as one half of the sum of the distances and square distances between all pairs of vertices of a graph. In this paper, we develop the discrete version of finite pattern to compute the accurate formulas of the hyper-Wiener indices of the Sierpiński skeleton networks.

Keywords: Fractal Network; Hyper-Wiener Index; Discrete Version of Finite Pattern.

[‡]Corresponding author.

1. INTRODUCTION

Many mathematical works are devoted to the complex network in which the small-world model and scale-free model were proposed by Watts and Strogatz,[1] Barabási and Albert,[2] respectively, refer also to Newman's works.[3,4]

The small-world effect describes the feature on the average shortest path length among vertices of networks. The average shortest path length is related to the Wiener index (by Wiener in 1947)[5] that is the most earliest and studied topological index in *Chemical Graph Theory*. Afterward, the hyper-Wiener index[6] was introduced by Randić in 1993. Suppose $G = (V(G), E(G))$ is a connected graph, and $d(x, y)$ denotes the minimum length of all paths from x to y on G, then the hyper-Wiener index $WW(G)$ of G is defined by

$$WW(G) = \sum_{x,y \in V(G)} \frac{1}{2}(d(x, y) + (d(x, y))^2).$$

Recently, the hyper-Wiener index has gained plenty of attention. For example, Cash, Klavzar and Petkovsek[7] investigated three methods for computing the hyper-Wiener index of molecular graphs; Feng, Liu and Xu[8] discussed the hyper-Wiener index of bicyclic graphs. See Refs. 9 and 10.

The Sierpiński gasket $\mathcal{K}$ is a classical fractal of Hausdorff dimension $\log 3/\log 2$ proposed by Sierpiński in 1915. Suppose

$$\left\{ S_i(x) = \frac{x + a_i}{2} : \mathbb{R}^2 \to \mathbb{R}^2 \right\}_{i=1}^{3} \quad \text{with } a_1 = (0, 0),$$
$$a_2 = (1, 0), \quad a_3 = (1/2, \sqrt{3}/2)$$

is an *Iterated Function System* (IFS) of planar similitudes. Then the Sierpiński gasket $\mathcal{K} = \mathcal{J}(\mathcal{K})$ is the fixed point of the operator $\mathcal{J}(A) = \bigcup_{i=1}^{3} S_i(A)$ for non-empty compact set A. See Fig. 1 for the first two iterations of the Sierpiński gasket.

In 2017, Wang, Yu and Xi[11] provided a new technique named **finite pattern** to compute the average geodesic distance on the Sierpiński gasket $\mathcal{K}$. For self-similar fractals, Xi *et al.*,[12–17] Wang *et al.*,[18,19] Deng *et al.*,[20,21] Xue *et al.*,[22–26] Niu and Li[27] and Dai *et al.*[28,29] discussed the average geodesic distances by using this technique.

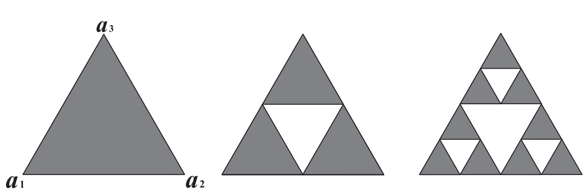


Fig. 1 The initial (solid) triangle $\Delta a_1a_2a_3$ and the first two constructions of the Sierpiński gasket.

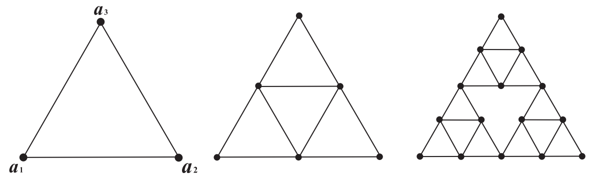


Fig. 2 The Sierpiński skeleton networks G_0, G_1, G_2.

In this paper, we use **discrete version of finite pattern** to obtain the accurate formula of the hyper-Wiener indices of the Sierpiński skeleton networks. For $n \geq 0$, let $G_n = (V_n, E_n)$ be the skeleton of the nth construction of the Sierpiński gasket with $V_n \subset \mathbb{R}^2$ and $d_n(x, y)$ the shortest path length between $x, y \in G_n$. For example, we can see the Sierpiński skeleton networks in Fig. 2. Then we have

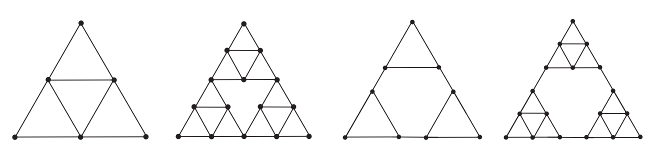
$$\Gamma_n = \sharp V_n = \frac{3}{2}(1 + 3^n). \tag{1.1}$$

It is worth mentioning that the Sierpiński skeleton networks differ significantly from the Hanoi graphs[30] in that the former has three overlapping points. See Fig. 3 for the comparison of Sierpiński skeleton networks and Hanoi graphs.

Using the self-similar structure of Sierpiński skeleton networks and the idea of *discrete version* of finite pattern, we obtain

Theorem 1. *Suppose G_n is the nth Sierpiński skeleton network with $n \geq 0$. Then*

$$\begin{aligned} WW(G_n) &= \frac{1664639}{4353492}2^{2n}3^{2n} + \frac{699}{1180}2^n3^{2n} + \frac{7}{6}2^{2n}3^n \\ &\quad + \frac{1}{3}3^{2n} + \frac{3}{2}2^n3^n + \frac{3}{4}2^{2n} + \frac{3593}{4290}3^n + \frac{3}{4}2^n \\ &\quad - \left(\frac{72717}{527696} + \frac{116067}{8970832}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{2}\right)^n \end{aligned}$$



(a) The Sierpiński skeleton networks **(b)** The Hanoi graphs

Fig. 3 The comparison of networks.

$$+\left(-\frac{72717}{527696}+\frac{116067}{8970832}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{2}\right)^n$$
$$-\left(\frac{225}{12272}+\frac{14409}{208624}\sqrt{17}\right)(5+\sqrt{17})^n$$
$$+\left(-\frac{225}{12272}+\frac{14409}{208624}\sqrt{17}\right)(5-\sqrt{17})^n.$$

We organize the paper as follows. In Sec. 2, we will develop the finite pattern[11] to the *discrete version* and state Lemmas 1–5. In Sec. 3, we shall give the proofs of Lemmas 1–5 listed in Sec. 2. Finally, we will prove Theorem 1 in Sec. 4.

2. PRELIMINARIES

2.1. Notations

Recall an *isomorphism* of graphs G and H is a bijection $f : V(G) \to V(H)$ between the vertex sets of G and H such that $u_1 \overset{G}{\sim} u_2$ if and only if $f(u_1) \overset{H}{\sim} f(u_2)$. We denote

$$G \overset{f}{\simeq} H \quad \text{or} \quad H = f(G).$$

Specifically, suppose $V(G), V(H) \subset \mathbb{R}^2$ and $H = f(G)$ with a translation $f : x \mapsto x + b$ ($b \in \mathbb{R}^2$), we denote

$$H = G + b.$$

Furthermore, if $H = f(G)$ and $f(u^*) = v^*$ for some $u^* \in V(G)$, $v^* \in V(H)$, we denote it by $(G, u^*) \overset{f}{\simeq} (H, v^*)$ or $(G, u^*) \simeq (H, v^*)$. If $(G, a) \simeq (G, b)$, then we say that G is *symmetric* with respect to a and b.

For each $n \geq 1$ and $j \in \{1, 2, 3\}$, let

$$G^{(n,j)} = S_j(G_{n-1}).$$

Fix $b \in \mathbb{R}^2$, let

$$H_n = G_n + b \quad \text{and} \quad H^{(n,j)} = G^{(n,j)} + b \quad \forall\, j \in \{1, 2, 3\}.$$

For simplicity of notation, we continue to write $d_n(x, y)$ for the shortest path length between $x, y \in V(H_n)$. See Fig. 4 for the structures of G_2 and H_2.

We write $\sigma_i^{(0)} = a_i$ for any $i \in \{1, 2, 3\}$. By induction, we let $\sigma_i^{(n)}$ denote $S_i(\sigma_i^{(n-1)})$ and $\sigma_i^{(n,j)}$ denote $S_j(\sigma_i^{(n-1)})$, then we have $\sigma_i^{(n)} = \sigma_i^{(n,i)}$. See Fig. 5 for

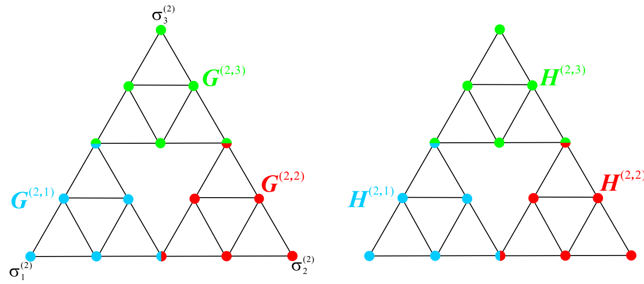


Fig. 4 The structures of G_2 and H_2.

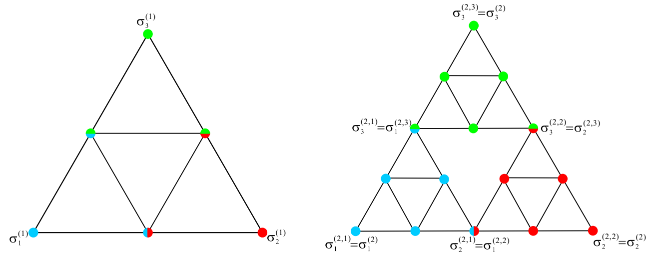


Fig. 5 $\{\sigma_i^{(1)}\}_i$ and $\{\sigma_i^{(2,j)}\}_{i,j}$ in G_1 and G_2, respectively.

$\{\sigma_i^{(1)}\}_i$ and $\{\sigma_i^{(2,j)}\}_{i,j}$ in the Sierpiński skeleton networks. Then we have the following *self-similarity*:

$$(G^{n,j}, \sigma_i^{(n,j)}) \simeq (G_{n-1}, \sigma_i^{(n-1)}) \quad \text{for any } i \text{ and } j. \tag{2.1}$$

Notice for any $n \geq 0$, the graph G_n is symmetric with respect to $\sigma_i^{(n)}$ and $\sigma_j^{(n)}$ for any distinct $i, j \in \{1, 2, 3\}$, i.e.

$$(G_n, \sigma_i^{(n)}) \simeq (G_n, \sigma_j^{(n)}). \tag{2.2}$$

2.2. Discrete Patterns and $r_n^{(k)}, q_n^{(k)}, \alpha_n^{(k)}, \beta_n^{(k)}, \gamma_n^{(k)}$

For abbreviation, we write $x \in G$ to represent $x \in V(G)$. We let

$$r_n^{(k)} = \sum_{x \in G_n} (d_n(x, \sigma_i^{(n)}))^k \quad \forall\, k \geq 1 \text{ and } \forall\, i \in \{1, 2, 3\},$$

independent of the choice of i, and

$$q_n^{(k)} = \sum_{x \in G_n} \min\{d_n(x, \sigma_i^{(n)}), d_n(x, \sigma_{i'}^{(n)})\} \quad \forall\, k \geq 1 \text{ and } \forall\, i \neq i' \in \{1, 2, 3\},$$

independent of the choice of (i, i') with $i \neq i'$. We also write $r_n = r_n^{(1)}, q_n = q_n^{(1)}$.

Inspired by the idea of finite pattern on fractals,[11] we define three *discrete* patterns as follows.

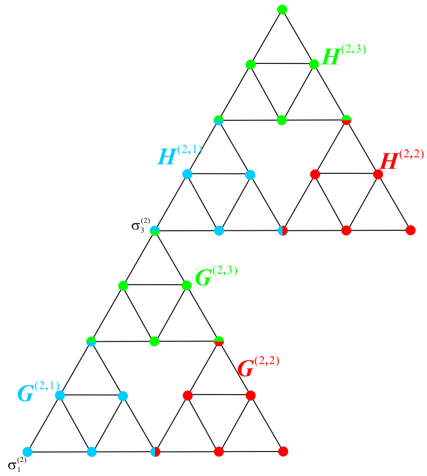


Fig. 6 Discrete Pattern I for $n = 2$.

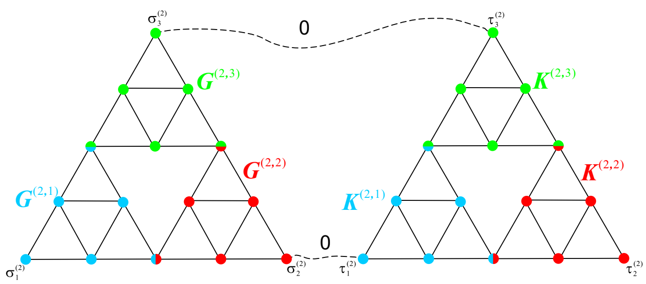


Fig. 7 Discrete Pattern II for $n = 2$.

As in Fig. 6, suppose $H_n = G_n + \sigma_3^{(n)} = G_n + a_3$. For $x \in G_n, y \in H_n$, we let

$$D_{\mathrm{I}}^{(n)}(x,y) = d_n(x, \sigma_3^{(n)}) + d_n(\sigma_3^{(n)}, y)$$

and

$$\alpha_n^{(k)} = \sum_{x\in G_n, y\in H_n} (D_{\mathrm{I}}^{(n)}(x,y))^k, \quad \forall\, k \geq 1.$$

As in Fig. 7, suppose $K_n = G_n + 2\sigma_2^{(n)} = G_n + a_2$ (satisfying $V(G_n) \cap V(K_n) = \varnothing$). Denote $\tau_i^{(n)} = \sigma_i^{(n)} + 2\sigma_2^{(n)}$ for any $i \in \{1,2,3\}$. For $x \in G_n, y \in K_n$, we let

$$\begin{aligned} &D_{\mathrm{II}}^{(n)}(x,y) \\ &\quad = \min\{d_n(x, \sigma_2^{(n)}) + d_n(\tau_1^{(n)}, y), d_n(x, \sigma_3^{(n)}) \\ &\qquad + d_n(\tau_3^{(n)}, y)\} \end{aligned}$$

and

$$\beta_n^{(k)} = \sum_{x\in G_n, y\in K_n} (D_{\mathrm{II}}^{(n)}(x,y))^k, \quad \forall\, k \geq 1.$$

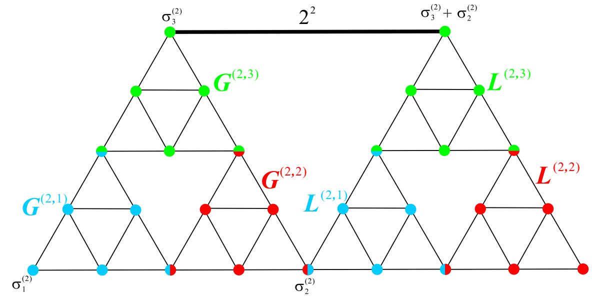


Fig. 8 Discrete Pattern III for $n = 2$.

As in Fig. 8, suppose $L_n = G_n + \sigma_2^{(n)}$. For $x \in G_n, y \in L_n$, we let

$$\begin{aligned} &D_{\mathrm{III}}^{(n)}(x,y) \\ &\quad = \min\{d_n(x, \sigma_2^{(n)}) + d_n(\sigma_2^{(n)}, y), d_n(x, \sigma_3^{(n)}) \\ &\qquad + d_n(\sigma_3^{(n)} + \sigma_2^{(n)}, y) + 2^n\} \end{aligned}$$

and

$$\gamma_n^{(k)} = \sum_{x\in G_n, y\in L_n} (D_{\mathrm{III}}^{(n)}(x,y))^k, \quad \forall\, k \geq 1.$$

To shorten notation, we write $\alpha_n = \alpha_n^{(1)}$, $\beta_n = \beta_n^{(1)}$ and $\gamma_n = \gamma_n^{(1)}$.

2.3. Lemmas 1–5

To prove Theorem 1, we need the following Lemmas 1–5.

Lemma 1. *For $n \geq 1$, we have recursions*

$$r_n = 3r_{n-1} + 3 \cdot 2^{n-1}3^{n-1} - 2^{n-1}$$

and

$$r_n^{(2)} = 3r_{n-1}^{(2)} + 7 \cdot 2^{2(n-1)}3^{n-1} + 2^{2(n-1)}.$$

As consequences, for $n \geq 0$, we have

$$r_n = 2^n 3^n + 2^n \tag{2.3}$$

and

$$r_n^{(2)} = \frac{7}{9}2^{2n}3^n + 2^{2n} + \frac{2}{9}3^n. \tag{2.4}$$

Lemma 2. *For $n \geq 1$, we have recursions*

$$q_n = q_{n-1} + \frac{7}{2}2^{n-1}3^{n-1} + \frac{1}{2}2^{n-1}$$

and

$$\begin{aligned} q_n^{(2)} &= q_{n-1}^{(2)} + \frac{401}{90}2^{2(n-1)}3^{n-1} + \frac{3}{2}2^{2(n-1)} \\ &\quad + \frac{4}{9}3^{n-1} - \frac{2}{5}2^{n-1}. \end{aligned}$$

As consequences, for $n \geq 0$, we have

$$q_n = \frac{7}{10}2^n3^n + \frac{1}{2}2^n - \frac{1}{5} \tag{2.5}$$

and

$$q_n^{(2)} = \frac{401}{990}2^{2n}3^n + \frac{2}{9}3^n + \frac{1}{2}2^{2n} - \frac{2}{5}2^n + \frac{3}{11}.$$

Lemma 3. *For $n \geq 0$, we have*

$$\alpha_n = 3 \cdot 2^n3^{2n} + 6 \cdot 2^n3^n + 3 \cdot 2^n \tag{2.6}$$

and

$$\alpha_n^{(2)} = \frac{2}{3}3^n + \frac{2}{3}3^{2n} + 5 \cdot 2^{2n} + \frac{13}{3}2^{2n}3^n + \frac{13}{3}2^{2n}3^{2n} + 5 \cdot 2^{2n}3^n. \tag{2.7}$$

Lemma 4. *We have recursions*

$$\beta_n = 3\beta_{n-1} + 4\gamma_{n-1} + 24 \cdot 2^{n-1}3^{2(n-1)} - \frac{51}{5}2^{n-1}3^{n-1} - 13 \cdot 2^{n-1} + \frac{6}{5}$$

and

$$\gamma_n = \beta_{n-1} + 2\gamma_{n-1} + 45 \cdot 2^{n-1}3^{2(n-1)} + \frac{93}{5}2^{n-1}3^{n-1} - 2 \cdot 2^{n-1} + \frac{2}{5}.$$

As consequences, for $n \geq 0$, we have

$$\beta_n = \frac{141}{59}2^n3^{2n} + \frac{21}{5}2^n3^n + 2 \times 2^n - \frac{1}{5} - \left(\frac{391}{2006} + \frac{123}{2006}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{2}\right)^n + \left(-\frac{391}{2006} + \frac{123}{2006}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{2}\right)^n$$

and

$$\gamma_n = \frac{699}{236}2^n3^{2n} + \frac{57}{10}2^n3^n + \frac{11}{4} \times 2^n - \frac{1}{5} - \left(\frac{425}{4012} + \frac{67}{4012}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{2}\right)^n + \left(-\frac{425}{4012} + \frac{67}{4012}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{2}\right)^n. \tag{2.8}$$

Lemma 5. *We have recursions*

$$\begin{aligned}\beta_n^{(2)} &= 3\beta_{n-1}^{(2)} + 4\gamma_{n-1}^{(2)} + \frac{26189}{354}2^{2(n-1)}3^{2(n-1)} + \frac{340}{33}2^{2(n-1)}3^{n-1} + \frac{4}{3}3^{2(n-1)} - \frac{29}{2}2^{2(n-1)} \\ &\quad - \frac{8}{3}3^{n-1} + 4 \cdot 2^{n-1} - \frac{18}{11} \\ &\quad - \left(\frac{142}{59} + \frac{626}{1003}\sqrt{17}\right)(5+\sqrt{17})^{n-1} \\ &\quad + \left(-\frac{142}{59} + \frac{626}{1003}\sqrt{17}\right)(5-\sqrt{17})^{n-1}\end{aligned}$$

and

$$\begin{aligned}\gamma_n^{(2)} &= \beta_{n-1}^{(2)} + 2\gamma_{n-1}^{(2)} + \frac{8275}{59}2^{2(n-1)}3^{2(n-1)} + \frac{4331}{55}2^{2(n-1)}3^{n-1} + 4 \cdot 3^{2(n-1)} \\ &\quad + 6 \cdot 2^{2(n-1)} + \frac{4}{5}2^{n-1} - \frac{6}{11} \\ &\quad - \left(\frac{96}{59} + \frac{380}{1003}\sqrt{17}\right)(5+\sqrt{17})^{n-1} \\ &\quad + \left(-\frac{96}{59} + \frac{380}{1003}\sqrt{17}\right)(5-\sqrt{17})^{n-1}.\end{aligned}$$

As consequences, for $n \geq 0$, we have

$$\begin{aligned}\beta_n^{(2)} &= \frac{544513}{197886}2^{2n}3^{2n} + \frac{802}{165}2^{2n}3^n + \frac{2}{3}3^{2n} + \frac{5}{2}2^{2n} + \frac{2}{3}3^n - \frac{4}{5}2^n + \frac{3}{11} \\ &\quad + \left(\frac{25}{2236} + \frac{1675}{38012}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{2}\right)^n \\ &\quad + \left(\frac{25}{2236} - \frac{1675}{38012}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{2}\right)^n \\ &\quad - \left(\frac{111}{236} + \frac{727}{4012}\sqrt{17}\right)(5+\sqrt{17})^n \\ &\quad + \left(-\frac{111}{236} + \frac{727}{4012}\sqrt{17}\right)(5-\sqrt{17})^n\end{aligned}$$

and

$$\begin{aligned}\gamma_n^{(2)} &= \frac{1664639}{395772}2^{2n}3^{2n} + \frac{2759}{330}2^{2n}3^n + \frac{2}{3}3^{2n} + \frac{17}{4}2^{2n} + \frac{2}{3}3^n - \frac{4}{5}2^n + \frac{3}{11} \\ &\quad + \left(\frac{825}{8944} - \frac{625}{152048}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{2}\right)^n\end{aligned}$$

$$+\left(\frac{825}{8944}+\frac{625}{152048}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{2}\right)^n$$
$$-\left(\frac{381}{944}+\frac{837}{16048}\sqrt{17}\right)(5+\sqrt{17})^n$$
$$+\left(-\frac{381}{944}+\frac{837}{16048}\sqrt{17}\right)(5-\sqrt{17})^n. \tag{2.9}$$

3. PROOFS OF LEMMAS 1–5

Denote

$$\partial_n=\{\sigma_1^{(n,2)}(=\sigma_2^{(n,1)}),\sigma_2^{(n,3)}(=\sigma_3^{(n,2)}),$$
$$\sigma_3^{(n,1)}(=\sigma_1^{(n,3)})\}(\subset G_n).$$

Moreover, we denote

$$\partial_n'=\partial_n+2\sigma_2^{(n)}(\subset K_n)$$

and

$$\partial_n''=\partial_n+\sigma_2^{(n)}(\subset L_n).$$

Due to the isomorphisms (2.1) and (2.2) of graphs, we have the following claims.

Claim 1. *For any $i\in\{1,2,3\}$ and $k\geq 1$, we have*

$$\sum_{x,y\in G^{(n,i)}} d_n(x,y)=\xi_{n-1}^{(k)}.$$

Claim 2. *For any $i,j\in\{1,2,3\}$ and $k\geq 1$, we have*

$$\sum_{x\in G^{(n,j)}}(d_n(x,\sigma_i^{(n,j)}))^k=r_{n-1}^{(k)}.$$

Claim 3. *For any $i,i',j\in\{1,2,3\}$ with $i\neq i'$ and $k\geq 1$, we have*

$$\sum_{x\in G^{(n,j)}}(\min\{d_n(x,\sigma_i^{(n,j)}),d_n(x,\sigma_{i'}^{(n,j)})\})^k=q_{n-1}^{(k)}.$$

Claim 4. *For any distinct vertexes $u,v\in\{\sigma_1^{(n)},\sigma_2^{(n)},\sigma_3^{(n)}\}$, we have*

$$d_n(u,v)=\mathrm{diam}(G_n)=2^n.$$

For any $j\in\{1,2,3\}$ and distinct vertexes $x,y\in\{\sigma_1^{(n,j)},\sigma_2^{(n,j)},\sigma_3^{(n,j)}\}$, we have

$$d_n(x,y)=\mathrm{diam}(G^{n,j})=2^{n-1}.$$

As a result of Claim 4, we have the following:

Claim 5. *For each $i\in\{1,2,3\}$, we have*

$$d_n(u,\sigma_i^{(n)})=\begin{cases}d_n(u,\sigma_i^{(n)}) & \text{if } u\in V(G^{n,i}),\\ d_n(u,\sigma_i^{(n,j)})+2^{n-1} & \text{if} u\in V(G^{n,j}),\ \ j\neq i.\end{cases} \tag{3.1}$$

For any distinct $i,i'\in\{1,2,3\}$, we also have

$$\min\{d_n(x,\sigma_i^{(n)}),d_n(x,\sigma_{i'}^{(n)})\}$$
$$=\begin{cases}d_n(x,\sigma_i^{(n)}) & \textit{if } \ u\in V(G^{n,i}),\\ d_n(x,\sigma_{i'}^{(n)}) & \textit{if } \ u\in V(G^{n,i'}),\\ \min\{d_n(x,\sigma_i^{(n,j)}),d_n(x,\sigma_{i'}^{(n,j)})\}+2^{n-1} \\ \quad \textit{if } u\in V(G^{n,j}),\ \ j\notin\{i,i'\}.\end{cases} \tag{3.2}$$

Proof of Lemma 1. By (3.1) of Claim 5, the symmetry (2.2), Claim 2 and (1.1), we have

$$\begin{aligned}r_n&=\sum_{x\in G_n} d_n(x,\sigma_1^{(n)})\\ &=\sum_{x\in G^{n,1}} d_n(x,\sigma_1^{(n)})\\ &\quad+2\sum_{x\in G^{n,2}}(d_n(x,\sigma_1^{(n,2)})+2^{n-1})\\ &\quad-\sum_{x\in\partial_n} d_n(x,\sigma_1^{(n)})\\ &=3r_{n-1}+3\cdot 2^{n-1}3^{n-1}-2^{n-1}.\end{aligned} \tag{3.3}$$

Iterating (3.3) and using $r_0=2$, we can obtain that

$$r_n=2^n3^n+2^n. \tag{3.4}$$

In the same way, we get

$$\begin{aligned}r_n^{(2)}&=\sum_{x\in G_n}(d_n(x,\sigma_1^{(n)}))^2\\ &=\sum_{x\in G^{n,1}}(d_n(x,\sigma_1^{(n)}))^2\\ &\quad+2\sum_{x\in G^{n,2}}(d_n(x,\sigma_1^{(n,2)})+2^{n-1})^2\\ &\quad-\sum_{x\in\partial_n}(d_n(x,\sigma_1^{(n)}))^2\\ &=3r_{n-1}^{(2)}+4\cdot 2^{n-1}r_{n-1}+3\cdot 2^{2(n-1)}3^{n-1}\\ &\quad-3\cdot 2^{2(n-1)}.\end{aligned}$$

By the result (3.4) of r_n, we have

$$r_n^{(2)} = 3r_{n-1}^{(2)} + 7 \cdot 2^{2(n-1)}3^{n-1} + 2^{2(n-1)}. \tag{3.5}$$

Iterating (3.5) and using $r_0^{(2)} = 14$, we can obtain that

$$r_n^{(2)} = \frac{7}{9}2^{2n}3^n + 2^{2n} + \frac{2}{9}3^n. \qquad \square$$

Proof of Lemma 2. We first calculate q_n. By (3.2) of Claim 5, the symmetry (2.2), Claims 2 and 3 and (1.1), we have

$$\begin{aligned}
q_n &= \sum_{x\in G_n} \min\{d_n(x,\sigma_1^{(n)}), d_n(x,\sigma_2^{(n)})\} \\
&= \sum_{x\in G^{(n,3)}} (\min\{d_n(x,\sigma_1^{(n,3)}), d_n(x,\sigma_2^{(n,3)})\} + 2^{n-1}) \\
&\quad + 2\sum_{x\in G^{(n,1)}} d_n(x,\sigma_1^{(n)}) \\
&\quad - \sum_{x\in\partial_n} \min\{d_n(x,\sigma_1^{(n,3)}), d_n(x,\sigma_2^{(n,3)})\} \\
&= q_{n-1} + 2r_{n-1} + \frac{3}{2}2^{n-1}3^{n-1} - \frac{3}{2}2^{n-1}.
\end{aligned}$$

Using the result $r_n = 2^n3^n + 2^n$ (formula (2.3)) we have

$$q_n = q_{n-1} + \frac{7}{2}2^{n-1}3^{n-1} + \frac{1}{2}2^{n-1}. \tag{3.6}$$

Iterating (3.6) and using $q_0 = 1$, we obtain that

$$q_n = \frac{7}{10}2^n3^n + \frac{1}{2}2^n - \frac{1}{5}.$$

In the same way, we have

$$\begin{aligned}
q_n^{(2)} &= \sum_{x\in G_n} (\min\{d_n(x,\sigma_1^{(n)}), d_n(x,\sigma_2^{(n)})\})^2 \\
&= \sum_{x\in G^{(n,3)}} (\min\{d_n(x,\sigma_1^{(n,3)}), d_n(x,\sigma_2^{(n,3)})\} \\
&\quad + 2^{n-1})^2 + 2\sum_{x\in G^{(n,1)}} (d_n(x,\sigma_1^{(n)}))^2 \\
&\quad - \sum_{x\in\partial_n} (\min\{d_n(x,\sigma_1^{(n,3)}), d_n(x,\sigma_2^{(n,3)})\})^2 \\
&= q_{n-1}^{(2)} + 2\cdot 2^{n-1}q_{n-1} + 2r_{n-1}^{(2)} + \frac{3}{2}2^{2(n-1)}3^{n-1} \\
&\quad - \frac{3}{2}2^{2(n-1)}.
\end{aligned}$$

Using the results (2.4) and (2.5) on $r_n^{(2)}$ and q_n, we get

$$\begin{aligned}
q_n^{(2)} &= q_{n-1}^{(2)} + \frac{401}{90}2^{2(n-1)}3^{n-1} + \frac{3}{2}2^{2(n-1)} \\
&\quad + \frac{4}{9}3^{n-1} - \frac{2}{5}2^{n-1}.
\end{aligned} \tag{3.7}$$

Iterating (3.7) and using $q_0^{(2)} = 1$, we obtain that

$$q_n^{(2)} = \frac{401}{990}2^{2n}3^n + \frac{2}{9}3^n + \frac{1}{2}2^{2n} - \frac{2}{5}2^n + \frac{3}{11}. \qquad \square$$

Proof of Lemma 3. Using Claim 2, (1.1) and the result of $r_n = 2^n3^n + 2^n$ (formula (2.3)), we can obtain that

$$\begin{aligned}
\alpha_n &= \sum_{x\in G_n, y\in H_n} (d_n(x,\sigma_3^{(n)}) + d_n(\sigma_1^{(n)}, y-\sigma_3^{(n)})) \\
&= 2\cdot\Gamma_n\cdot r_n \\
&= 3\cdot 2^n3^{2n} + 6\cdot 2^n3^n + 3\cdot 2^n.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
\alpha_n^{(2)} &= \sum_{x\in G_n, y\in H_n} (d_n(x,\sigma_3^{(n)}) + d_n(\sigma_1^{(n)}, y-\sigma_3^{(n)}))^2 \\
&= 2\cdot\Gamma_n\cdot r_n^{(2)} + 2(r_n)^2 \\
&= \frac{2}{3}3^n + \frac{2}{3}3^{2n} + 5\cdot 2^{2n} \\
&\quad + \frac{13}{3}2^{2n}3^n + \frac{13}{3}2^{2n}3^{2n} + 5\cdot 2^{2n}3^n. \qquad \square
\end{aligned}$$

Next we will prove Lemmas 4 and 5.

Note that for any distinct $j, j' \in \{1,2,3\}$, the subgraphs $G^{(n,j)}$ and $G^{(n,j')}$ of G_n share one vertex, i.e.

$$\{\sigma_{j'}^{(n.j)}(=\sigma_j^{(n,j')})\} = G^{(n,j)} \cap G^{(n,j')} \subset G_n.$$

Hence we have

$$\begin{aligned}
\beta_n^{(k)} &= \sum_{x\in G_n, y\in K_n} (D_{\mathrm{II}}^{(n)}(x,y))^k \\
&= \sum_{j,j'=1}^{3} \sum_{\substack{x\in G^{(n,j)} \\ y\in K^{(n,j')}}} (D_{\mathrm{II}}^{(n)}(x,y))^k \\
&\quad - 2\sum_{\substack{x\in\partial_n \\ y\in K_n}} (D_{\mathrm{II}}^{(n)}(x,y))^k + \sum_{\substack{x\in\partial_n \\ y\in\partial_n'}} (D_{\mathrm{II}}^{(n)}(x,y))^k
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\gamma_n^{(k)} &= \sum_{x\in G_n, y\in L_n} (D_{\mathrm{III}}^{(n)}(x,y))^k \\ &= \sum_{j,j'=1}^{3} \sum_{\substack{x\in G^{(n,j)}\\ y\in L^{(n,j')}}} (D_{\mathrm{III}}^{(n)}(x,y))^k \\ &\quad - 2\sum_{\substack{x\in\partial_n\\ y\in L_n}} (D_{\mathrm{III}}^{(n)}(x,y))^k \\ &\quad + \sum_{\substack{x\in\partial_n\\ y\in\partial_n''}} (D_{\mathrm{III}}^{(n)}(x,y))^k. \end{aligned} \tag{3.9}$$

Proof of Lemma 4. To deal with $\sum_{j,j'=1}^{3} \times \sum_{x\in G^{(n,j)}, y\in K^{(n,j')}} D_{\mathrm{II}}^{(n)}(x,y)$, we have Table 1, e.g.

$$\sum_{x\in G^{(n,1)}, y\in K^{(n,1)}} D_{\mathrm{II}}^{(n)}(x,y) = \gamma_{n-1} + 2^{n-1}(\Gamma_{n-1})^2.$$

For $\sum_{x\in\partial_n, y\in K_n} D_{\mathrm{II}}^{(n)}(x,y)$, we also have Table 2, for example

$$\sum_{y\in K^{(n,1)}} D_{\mathrm{II}}^{(n)}(\sigma_1^{(n,2)}, y) = r_{n-1} + 2^{n-1}\Gamma_{n-1}.$$

Therefore, using Tables 1 and 2 and (3.8), we have

$$\begin{aligned}\beta_n &= 2\alpha_{n-1} + 3\beta_{n-1} + 4\gamma_{n-1} + 8\cdot 2^{n-1}\cdot(\Gamma_{n-1})^2 \\ &\quad - 2(6r_{n-1} + 3q_{n-1} + 14\cdot 2^{n-1}\Gamma_{n-1}) \\ &\quad + 20\cdot 2^{n-1}.\end{aligned}$$

By (1.1) and the results (2.3), (2.5) and (2.6) on r_n, q_n and α_n, we obtain that

$$\begin{aligned}\beta_n &= 3\beta_{n-1} + 4\gamma_{n-1} + 24\cdot 2^{n-1}3^{2(n-1)} \\ &\quad - \frac{51}{5}2^{n-1}3^{n-1} - 13\cdot 2^{n-1} + \frac{6}{5}.\end{aligned} \tag{3.10}$$

For $\sum_{j,j'=1}^{3} \sum_{x\in G^{(n,j)}, y\in L^{(n,j')}} D_{\mathrm{III}}^{(n)}(x,y)$, we have Table 3, e.g.

$$\sum_{x\in G^{(n,1)}, y\in L^{(n,1)}} D_{\mathrm{III}}^{(n)}(x,y) = \alpha_{n-1} + 2^{n-1}(\Gamma_{n-1})^2.$$

To deal with $\sum_{x\in\partial_n, y\in L_n} D_{\mathrm{III}}^{(n)}(x,y)$, we also have Table 4, e.g.

$$\sum_{y\in L^{(n,1)}} D_{\mathrm{III}}^{(n)}(\sigma_1^{(n,2)}, y) = r_{n-1} + 2^{n-1}\Gamma_{n-1}.$$

Table 1 Calculation of $\sum_{j,j'=1}^{3} \sum_{x\in G^{(n,j)}, y\in K^{(n,j')}} D_{\mathrm{II}}^{(n)}(x,y)$.

	$G^{(n,1)}$	$G^{(n,2)}$	$G^{(n,3)}$
$K^{(n,1)}$	$\gamma_{n-1} + 2^{n-1}(\Gamma_{n-1})^2$	α_{n-1}	$\beta_{n-1} + 2^{n-1}(\Gamma_{n-1})^2$
$K^{(n,2)}$	$\beta_{n-1} + 2\cdot 2^{n-1}(\Gamma_{n-1})^2$	$\gamma_{n-1} + 2^{n-1}(\Gamma_{n-1})^2$	$\gamma_{n-1} + 2^{n-1}(\Gamma_{n-1})^2$
$K^{(n,3)}$	$\gamma_{n-1} + 2^{n-1}(\Gamma_{n-1})^2$	$\beta_{n-1} + 2^{n-1}(\Gamma_{n-1})^2$	α_{n-1}

Table 2 Calculation of $\sum_{x\in\partial_n, y\in K_n} D_{\mathrm{II}}^{(n)}(x,y)$.

	$\sigma_1^{(n,2)}$	$\sigma_2^{(n,3)}$	$\sigma_3^{(n,1)}$
$K^{(n,1)}$	$r_{n-1} + 2^{n-1}\Gamma_{n-1}$	$r_{n-1} + 2^{n-1}\Gamma_{n-1}$	$q_{n-1} + 2\cdot 2^{n-1}\Gamma_{n-1}$
$K^{(n,2)}$	$r_{n-1} + 2\cdot 2^{n-1}\Gamma_{n-1}$	$q_{n-1} + 2\cdot 2^{n-1}\Gamma_{n-1}$	$r_{n-1} + 2\cdot 2^{n-1}\Gamma_{n-1}$
$K^{(n,3)}$	$q_{n-1} + 2\cdot 2^{n-1}\Gamma_{n-1}$	$r_{n-1} + 2^{n-1}\Gamma_{n-1}$	$r_{n-1} + 2^{n-1}\Gamma_{n-1}$

Table 3 Calculation of $\sum_{j,j'=1}^{3} \sum_{x\in G^{(n,j)}, y\in L^{(n,j')}} D_{\mathrm{III}}^{(n)}(x,y)$.

	$G^{(n,1)}$	$G^{(n,2)}$	$G^{(n,3)}$
$L^{(n,1)}$	$\alpha_{n-1} + 2^{n-1}(\Gamma_{n-1})^2$	α_{n-1}	$\alpha_{n-1} + 2^{n-1}(\Gamma_{n-1})^2$
$L^{(n,2)}$	$\alpha_{n-1} + 2\cdot 2^{n-1}(\Gamma_{n-1})^2$	$\alpha_{n-1} + 2^{n-1}(\Gamma_{n-1})^2$	$\gamma_{n-1} + 2\cdot 2^{n-1}(\Gamma_{n-1})^2$
$L^{(n,3)}$	$\gamma_{n-1} + 2\cdot 2^{n-1}(\Gamma_{n-1})^2$	$\alpha_{n-1} + 2^{n-1}(\Gamma_{n-1})^2$	$\beta_{n-1} + 2\cdot 2^{n-1}(\Gamma_{n-1})^2$

Table 4 Calculation of $\sum_{x\in\partial_n,y\in L_n} D_{\mathrm{III}}^{(n)}(x,y)$.

	$\sigma_1^{(n,2)}$	$\sigma_2^{(n,3)}$	$\sigma_3^{(n,1)}$
$L^{(n,1)}$	$r_{n-1}+2^{n-1}\Gamma_{n-1}$	$r_{n-1}+2^{n-1}\Gamma_{n-1}$	$r_{n-1}+2\cdot 2^{n-1}\Gamma_{n-1}$
$L^{(n,2)}$	$r_{n-1}+2\cdot 2^{n-1}\Gamma_{n-1}$	$r_{n-1}+2\cdot 2^{n-1}\Gamma_{n-1}$	$r_{n-1}+3\cdot 2^{n-1}\Gamma_{n-1}$
$L^{(n,3)}$	$r_{n-1}+2\cdot 2^{n-1}\Gamma_{n-1}$	$r_{n-1}+2\cdot 2^{n-1}\Gamma_{n-1}$	$q_{n-1}+3\cdot 2^{n-1}\Gamma_{n-1}$

Table 5 Calculation of $\sum_{j,j'=1}^{3}\sum_{x\in G^{(n,j)},y\in K^{(n,j')}}(D_{\mathrm{II}}^{(n)}(x,y))^2$.

	$G^{(n,1)}$	$G^{(n,2)}$	$G^{(n,3)}$
$K^{(n,1)}$	$\gamma_{n-1}^{(2)}+2\cdot 2^{n-1}\gamma_{n-1}+2^{2(n-1)}(\Gamma_{n-1})^2$	$\alpha_{n-1}^{(2)}$	$\beta_{n-1}^{(2)}+2\cdot 2^{n-1}\beta_{n-1}+2^{2(n-1)}(\Gamma_{n-1})^2$
$K^{(n,2)}$	$\beta_{n-1}^{(2)}+4\cdot 2^{n}\beta_{n-1}+4\cdot 2^{2(n-1)}(\Gamma_{n-1})^2$	$\gamma_{n-1}^{(2)}+2\cdot 2^{n-1}\gamma_{n-1}+2^{2(n-1)}(\Gamma_{n-1})^2$	$\gamma_{n-1}^{(2)}+2\cdot 2^{n-1}\gamma_{n-1}+2^{2(n-1)}(\Gamma_{n-1})^2$
$K^{(n,3)}$	$\gamma_{n-1}^{(2)}+2\cdot 2^{n-1}\gamma_{n-1}+2^{2(n-1)}(\Gamma_{n-1})^2$	$\beta_{n-1}^{(2)}+2\cdot 2^{n-1}\beta_{n-1}+2^{2(n-1)}(\Gamma_{n-1})^2$	$\alpha_{n-1}^{(2)}$

Therefore, using Tables 3 and 4 and (3.9), we have

$$\begin{aligned}\gamma_n &= 6\alpha_{n-1}+\beta_{n-1}+2\gamma_{n-1}+12\cdot 2^{n-1}\cdot(\Gamma_{n-1})^2\\ &\quad -2(8r_{n-1}+q_{n-1}+18\cdot 2^{n-1}\Gamma_{n-1})\\ &\quad +24\cdot 2^{n-1}.\end{aligned}$$

By (1.1) and the results (2.3), (2.5) and (2.6) on r_n, q_n and α_n, we obtain that

$$\begin{aligned}\gamma_n &= \beta_{n-1}+2\gamma_{n-1}+45\cdot 2^{n-1}3^{2(n-1)}\\ &\quad +\frac{93}{5}2^{n-1}3^{n-1}-2\cdot 2^{n-1}+\frac{2}{5}. \end{aligned}\tag{3.11}$$

Iterating (3.10) and (3.11), and using $\beta_0=8$ and $\gamma_0=11$, we can obtain that

$$\begin{aligned}\beta_n &= \frac{141}{59}2^n3^{2n}+\frac{21}{5}2^n3^n+2\times 2^n-\frac{1}{5}\\ &\quad -\left(\frac{391}{2006}+\frac{123}{2006}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{2}\right)^n\\ &\quad +\left(-\frac{391}{2006}+\frac{123}{2006}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{2}\right)^n\end{aligned}$$

and

$$\begin{aligned}\gamma_n &= \frac{699}{236}2^n3^{2n}+\frac{57}{10}2^n3^n+\frac{11}{4}\times 2^n-\frac{1}{5}\\ &\quad -\left(\frac{425}{4012}+\frac{67}{4012}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{2}\right)^n\\ &\quad +\left(-\frac{425}{4012}+\frac{67}{4012}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{2}\right)^n. \qquad \square\end{aligned}$$

Proof of Lemma 5. For the sum $\sum_{j,j'=1}^{3}\times\sum_{x\in G^{(n,j)},y\in K^{(n,j')}}(D_{\mathrm{II}}^{(n)}(x,y))^2$, we have Table 5, e.g.

$$\begin{aligned}&\sum_{x\in G^{(n,1)},y\in K^{(n,1)}}(D_{\mathrm{II}}^{(n)}(x,y))^2\\ &\quad =\gamma_{n-1}^{(2)}+2\cdot 2^{n-1}\gamma_{n-1}+2^{2(n-1)}(\Gamma_{n-1})^2.\end{aligned}$$

To deal with $\sum_{x\in\partial_n,y\in K_n}(D_{\mathrm{II}}^{(n)}(x,y))^2$, we also have Table 6, for example,

$$\begin{aligned}&\sum_{y\in K^{(n,1)}}(D_{\mathrm{II}}^{(n)}(\sigma_1^{(n,2)},y))^2\\ &\quad =r_{n-1}^{(2)}+2\cdot 2^{n-1}r_{n-1}+2^{2(n-1)}\Gamma_{n-1}.\end{aligned}$$

Therefore, using Tables 5 and 6 and (3.8), we have

$$\begin{aligned}\beta_n^{(2)} &= 2\alpha_{n-1}^{(2)}+3\beta_{n-1}^{(2)}+4\gamma_{n-1}^{(2)}+8\cdot 2^{n-1}\beta_{n-1}\\ &\quad +8\cdot 2^{n-1}\gamma_{n-1}+10\cdot 2^{2(n-1)}(\Gamma_{n-1})^2\\ &\quad -2(6r_{n-1}^{(2)}+16\cdot 2^{n-1}r_{n-1}+3q_{n-1}^{(2)}\\ &\quad +12\cdot 2^{n-1}q_{n-1}+24\cdot 2^{2(n-1)}\Gamma_{n-1})\\ &\quad +46\cdot 2^{2(n-1)}.\end{aligned}$$

Table 6 Calculation of $\sum_{x\in\partial_n, y\in K_n}(D_{\mathrm{II}}^{(n)}(x,y))^2$.

	$\sigma_1^{(n,2)}$	$\sigma_2^{(n,3)}$	$\sigma_3^{(n,1)}$
$K^{(n,1)}$	$r_{n-1}^{(2)} + 2\cdot 2^{n-1}r_{n-1} + 2^{2(n-1)}\Gamma_{n-1}$	$r_{n-1}^{(2)} + 2\cdot 2^{n-1}r_{n-1} + 2^{2(n-1)}\Gamma_{n-1}$	$q_{n-1}^{(2)} + 4\cdot 2^{n-1}q_{n-1} + 4\cdot 2^{2(n-1)}\Gamma_{n-1}$
$K^{(n,2)}$	$r_{n-1}^{(2)} + 4\cdot 2^{n-1}r_{n-1} + 4\cdot 2^{2(n-1)}\Gamma_{n-1}$	$q_{n-1}^{(2)} + 4\cdot 2^{n-1}q_{n-1} + 4\cdot 2^{2(n-1)}\Gamma_{n-1}$	$r_{n-1}^{(2)} + 4\cdot 2^{n-1}r_{n-1} + 4\cdot 2^{2(n-1)}\Gamma_{n-1}$
$K^{(n,3)}$	$q_{n-1}^{(2)} + 4\cdot 2^{n-1}q_{n-1} + 4\cdot 2^{2(n-1)}\Gamma_{n-1}$	$r_{n-1}^{(2)} + 2\cdot 2^{n-1}r_{n-1} + 2^{2(n-1)}\Gamma_{n-1}$	$r_{n-1}^{(2)} + 2\cdot 2^{n-1}r_{n-1} + 2^{2(n-1)}\Gamma_{n-1}$

Table 7 Calculation of $\sum_{j,j'=1}^{3}\sum_{x\in G^{(n,j)}, y\in L^{(n,j')}}(D_{\mathrm{III}}^{(n)}(x,y))^2$.

	$G^{(n,1)}$	$G^{(n,2)}$	$G^{(n,3)}$
$L^{(n,1)}$	$\alpha_{n-1}^{(2)} + 2\cdot 2^{n-1}\alpha_{n-1} + 2^{2(n-1)}(\Gamma_{n-1})^2$	$\alpha_{n-1}^{(2)}$	$\alpha_{n-1}^{(2)} + 2\cdot 2^{n-1}\alpha_{n-1} + 2^{2(n-1)}(\Gamma_{n-1})^2$
$L^{(n,2)}$	$\alpha_{n-1}^{(2)} + 4\cdot 2^{n-1}\alpha_{n-1} + 4\cdot 2^{2(n-1)}(\Gamma_{n-1})^2$	$\alpha_{n-1}^{(2)} + 2\cdot 2^{n-1}\alpha_{n-1} + 2^{2(n-1)}(\Gamma_{n-1})^2$	$\gamma_{n-1}^{(2)} + 4\cdot 2^{n-1}\gamma_{n-1} + 4\cdot 2^{2(n-1)}(\Gamma_{n-1})^2$
$L^{(n,3)}$	$\gamma_{n-1}^{(2)} + 4\cdot 2^{n-1}\gamma_{n-1} + 4\cdot 2^{2(n-1)}(\Gamma_{n-1})^2$	$\alpha_{n-1}^{(2)} + 2\cdot 2^{n-1}\alpha_{n-1} + 2^{2(n-1)}(\Gamma_{n-1})^2$	$\beta_{n-1}^{(2)} + 4\cdot 2^{n-1}\beta_{n-1} + 4\cdot 2^{2(n-1)}(\Gamma_{n-1})^2$

By (1.1), the result of $\alpha_n^{(2)}$ (formula (2.7)), Lemmas 1, 2 and 4, we can obtain

$$\begin{aligned}\beta_n^{(2)} &= 3\beta_{n-1}^{(2)} + 4\gamma_{n-1}^{(2)} + \frac{26189}{354}2^{2(n-1)}3^{2(n-1)}\\ &\quad + \frac{340}{33}2^{2(n-1)}3^{n-1} + \frac{4}{3}3^{2(n-1)} - \frac{29}{2}2^{2(n-1)}\\ &\quad - \frac{8}{3}3^{n-1} + 4\cdot 2^{n-1} - \frac{18}{11}\\ &\quad - \left(\frac{142}{59} + \frac{626}{1003}\sqrt{17}\right)(5+\sqrt{17})^{n-1}\\ &\quad + \left(-\frac{142}{59} + \frac{626}{1003}\sqrt{17}\right)(5-\sqrt{17})^{n-1}.\end{aligned} \tag{3.12}$$

To deal with $\sum_{j,j'=1}^{3}\sum_{x\in G^{(n,j)}, y\in L^{(n,j')}}(D_{\mathrm{III}}^{(n)}(x,y))^2$, we have Table 7, e.g.

$$\sum_{x\in G^{(n,1)}, y\in L^{(n,1)}}(D_{\mathrm{II}}^{(n)}(x,y))^2 = \alpha_{n-1}^{(2)} + 2\cdot 2^{n-1}\alpha_{n-1} + 2^{2(n-1)}(\Gamma_{n-1})^2.$$

For the sum $\sum_{x\in\partial_n, y\in L_n}(D_{\mathrm{III}}^{(n)}(x,y))^2$, we also have Table 8, e.g.

$$\sum_{y\in L^{(n,1)}}(D_{\mathrm{II}}^{(n)}(\sigma_1^{(n,2)},y))^2 = r_{n-1}^{(2)} + 2\cdot 2^{n-1}r_{n-1} + 2^{2(n-1)}\Gamma_{n-1}.$$

Therefore, using Tables 7 and 8 and (3.9), we have

$$\begin{aligned}\gamma_n^{(2)} &= 6\alpha_{n-1}^{(2)} + \beta_{n-1}^{(2)} + 2\gamma_{n-1}^{(2)} + 12\cdot 2^{n-1}\alpha_{n-1}\\ &\quad + 4\cdot 2^{n-1}\beta_{n-1} + 8\cdot 2^{n-1}\gamma_{n-1}\\ &\quad + 20\cdot 2^{2(n-1)}(\Gamma_{n-1})^2 - 2(8r_{n-1}^{(2)} + q_{n-1}^{(2)}\\ &\quad + 30\cdot 2^{n-1}r_{n-1} + 6\cdot 2^{n-1}q_{n-1}\\ &\quad + 40\cdot 2^{2(n-1)}\Gamma_{n-1}) + 68\cdot 2^{2(n-1)}.\end{aligned}$$

By (1.1) and Lemmas 1–4, we have

$$\begin{aligned}\gamma_n^{(2)} &= \beta_{n-1}^{(2)} + 2\gamma_{n-1}^{(2)} + \frac{8275}{59}2^{2(n-1)}3^{2(n-1)}\\ &\quad + \frac{4331}{55}2^{2(n-1)}3^{n-1} + 4\cdot 3^{2(n-1)}\\ &\quad + 6\cdot 2^{2(n-1)} + \frac{4}{5}2^{n-1} - \frac{6}{11}\end{aligned}$$

Table 8 Calculation of $\sum_{x\in\partial_n, y\in L_n}(D_{\mathrm{III}}^{(n)}(x,y))^2$.

	$\sigma_1^{(n,2)}$	$\sigma_2^{(n,3)}$	$\sigma_3^{(n,1)}$
$L^{(n,1)}$	$r_{n-1}^{(2)}+2\cdot 2^{n-1}r_{n-1}$ $+2^{2(n-1)}\Gamma_{n-1}$	$r_{n-1}^{(2)}+2\cdot 2^{n-1}r_{n-1}$ $+2^{2(n-1)}\Gamma_{n-1}$	$r_{n-1}^{(2)}+4\cdot 2^{n-1}r_{n-1}$ $+4\cdot 2^{2(n-1)}\Gamma_{n-1}$
$L^{(n,2)}$	$r_{n-1}^{(2)}+4\cdot 2^{n-1}r_{n-1}$ $+4\cdot 2^{2(n-1)}\Gamma_{n-1}$	$r_{n-1}^{(2)}+4\cdot 2^{n-1}r_{n-1}$ $+4\cdot 2^{2(n-1)}\Gamma_{n-1}$	$r_{n-1}^{(2)}+6\cdot 2^{n-1}r_{n-1}$ $+9\cdot 2^{2(n-1)}\Gamma_{n-1}$
$L^{(n,3)}$	$r_{n-1}^{(2)}+4\cdot 2^{n-1}r_{n-1}$ $+4\cdot 2^{2(n-1)}\Gamma_{n-1}$	$r_{n-1}^{(2)}+4\cdot 2^{n-1}r_{n-1}$ $+4\cdot 2^{2(n-1)}\Gamma_{n-1}$	$q_{n-1}^{(2)}+6\cdot 2^{n-1}q_{n-1}$ $+9\cdot 2^{2(n-1)}\Gamma_{n-1}$

$$
\begin{aligned}
&-\left(\frac{96}{59}+\frac{380}{1003}\sqrt{17}\right)(5+\sqrt{17})^{n-1}\\
&+\left(-\frac{96}{59}+\frac{380}{1003}\sqrt{17}\right)(5-\sqrt{17})^{n-1}.
\end{aligned}
\tag{3.13}
$$

Iterating (3.12) and (3.13), and using $\beta_0^{(2)}=10$ and $\gamma_0^{(2)}=17$, we can obtain that

$$
\begin{aligned}
\beta_n^{(2)} &= \frac{544513}{197886}2^{2n}3^{2n}+\frac{802}{165}2^{2n}3^n+\frac{2}{3}3^{2n}\\
&\quad+\frac{5}{2}2^{2n}+\frac{2}{3}3^n-\frac{4}{5}2^n+\frac{3}{11}\\
&\quad+\left(\frac{25}{2236}+\frac{1675}{38012}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{2}\right)^n\\
&\quad+\left(\frac{25}{2236}-\frac{1675}{38012}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{2}\right)^n\\
&\quad-\left(\frac{111}{236}+\frac{727}{4012}\sqrt{17}\right)(5+\sqrt{17})^n\\
&\quad+\left(-\frac{111}{236}+\frac{727}{4012}\sqrt{17}\right)(5-\sqrt{17})^n
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_n^{(2)} &= \frac{1664639}{395772}2^{2n}3^{2n}+\frac{2759}{330}2^{2n}3^n+\frac{2}{3}3^{2n}\\
&\quad+\frac{17}{4}2^{2n}+\frac{2}{3}3^n-\frac{4}{5}2^n+\frac{3}{11}\\
&\quad+\left(\frac{825}{8944}-\frac{625}{152048}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{2}\right)^n\\
&\quad+\left(\frac{825}{8944}+\frac{625}{152048}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{2}\right)^n\\
&\quad-\left(\frac{381}{944}+\frac{837}{16048}\sqrt{17}\right)(5+\sqrt{17})^n\\
&\quad+\left(-\frac{381}{944}+\frac{837}{16048}\sqrt{17}\right)(5-\sqrt{17})^n. \quad\square
\end{aligned}
$$

4. PROOF OF THEOREM 1

For any $k\ge 1$, we let

$$\xi_n^{(k)}=\sum_{x,y\in G_n}(d_n(x,y))^k.$$

Write $\xi_n=\xi_n^{(1)}$ for simplicity of notation.

To obtain Theorem 1, it is sufficient to calculate ξ_n and $\xi_n^{(2)}$.

4.1. Calculation of ξ_n

Recall $\xi_n=\sum_{x,y\in G_n}d_n(x,y)$. Using the symmetry (2.2) and Claims 1–3, we obtain that

$$
\begin{aligned}
\xi_n &= \sum_{x,y\in G_n}d_n(x,y)\\
&= 3\sum_{x,y\in G^{(n,1)}}d_n(x,y)\\
&\quad+6\left(\sum_{x\in G^{(n,1)},y\in G^{(n,2)}}d_n(x,y)\right.\\
&\quad\left.-\sum_{x\in G^{(n,1)}}d_n(x,\sigma_1^{(n,2)})-\sum_{y\in G^{(n,2)}}d_n(y,\sigma_1^{(n,2)})\right)\\
&\quad-6\sum_{x\in G^{(n,3)}}(d_n(x,\sigma_1^{(n,2)})+2^{n-1})\\
&\quad+\sum_{x,y\in\partial_n}d(x,y)\\
&= 3\xi_{n-1}+6\gamma_{n-1}-12r_{n-1}-6q_{n-1}\\
&\quad-6\cdot 2^{n-1}\Gamma_{n-1}+6\cdot 2^{n-1}.
\end{aligned}
$$

By (1.1) and the results (2.3), (2.5) and (2.8) on r_n, q_n and γ_n, we have

$$\begin{aligned}\xi_n &= 3\xi_{n-1} + \frac{2097}{118}2^{n-1}3^{2(n-1)} + 9\cdot 2^{n-1}3^{n-1}\\ &\quad - \frac{3}{2}2^{n-1} - \left(\frac{75}{118} + \frac{201}{2006}\sqrt{17}\right)2^{-(n-1)}\\ &\quad \times(\sqrt{17}+5)^{n-1} + \left(-\frac{75}{118} + \frac{201}{2006}\sqrt{17}\right)\\ &\quad \times 2^{-(n-1)}(5-\sqrt{17})^{n-1}. \qquad (4.1)\end{aligned}$$

Iterating (4.1) and using $\xi_0 = 6$, we can obtain that

$$\begin{aligned}\xi_n &= \frac{699}{590}2^n3^{2n} + 3\times 2^n3^n + \frac{9}{10}3^n + \frac{3}{2}2^n\\ &\quad - \left(\frac{1173}{4012} + \frac{369}{4012}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{2}\right)^n\\ &\quad + \left(-\frac{1173}{4012} + \frac{369}{4012}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{2}\right)^n. \qquad (4.2)\end{aligned}$$

4.2. Calculation of $\xi_n^{(2)}$

Recall $\xi_n^{(2)} = \sum_{x,y\in G_n}(d_n(x,y))^2$. Using the symmetry (2.2) and Claims 1–3, we obtain that

$$\begin{aligned}\xi_n^{(2)} &= \sum_{x,y\in G_n}(d_n(x,y))^2\\ &= 3\sum_{x,y\in G^{(n,1)}}(d_n(x,y))^2\\ &\quad + 6\left(\sum_{\substack{x\in G^{(n,1)}\\ y\in G^{(n,2)}}}(d_n(x,y))^2 - \sum_{x\in G^{(n,1)}}(d_n(x,\sigma_1^{(n,2)}))^2 - \sum_{y\in G^{(n,2)}}(d_n(y,\sigma_1^{(n,2)}))^2\right)\\ &\quad - 6\sum_{x\in G^{(n,3)}}(d_n(x,\sigma_1^{(n,2)}) + 2^{n-1})^2\\ &\quad + \sum_{x,y\in\partial_n}(d(x,y))^2\\ &= 3\xi_{n-1}^{(2)} + 6\gamma_{n-1}^{(2)} - 12r_{n-1}^{(2)} - 6q_{n-1}^{(2)}\\ &\quad - 12\cdot 2^{n-1}q_{n-1} - 6\cdot 2^{2(n-1)}\Gamma_{n-1}\\ &\quad + 6\cdot 2^{2(n-1)}.\end{aligned}$$

By (1.1), Lemma 2 and the results (2.4) and (2.9) on $r_n^{(2)}$ and $\gamma_n^{(2)}$, we have

$$\begin{aligned}\xi_n^{(2)} &= 3\xi_{n-1}^{(2)} + \frac{1664639}{65962}2^{2(n-1)}3^{2(n-1)}\\ &\quad + 21\cdot 2^{2(n-1)}3^{n-1} + 4\cdot 3^{2(n-1)} + \frac{3}{2}2^{2(n-1)}\\ &\quad + \left(\frac{2475}{4472} - \frac{1875}{76024}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{2}\right)^{n-1}\\ &\quad + \left(\frac{2475}{4472} + \frac{1875}{76024}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{2}\right)^{n-1}\\ &\quad - \left(\frac{1143}{472} + \frac{2511}{8024}\sqrt{17}\right)(5+\sqrt{17})^{n-1}\\ &\quad + \left(-\frac{1143}{472} + \frac{2511}{8024}\sqrt{17}\right)(5-\sqrt{17})^{n-1}. \qquad (4.3)\end{aligned}$$

Iterating (4.3) and using $\xi_0^{(2)} = 6$, we can obtain that

$$\begin{aligned}\xi_n^{(2)} &= \frac{1664639}{2176746}2^{2n}3^{2n} + \frac{7}{3}2^{2n}3^n\\ &\quad + \frac{2}{3}3^{2n} + \frac{3}{2}2^{2n} + \frac{665}{858}3^n\\ &\quad + \left(\frac{75}{4472} + \frac{5025}{76024}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{2}\right)^n\\ &\quad + \left(\frac{75}{4472} - \frac{5025}{76024}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{2}\right)^n\\ &\quad - \left(\frac{225}{6136} + \frac{14409}{104312}\sqrt{17}\right)(5+\sqrt{17})^n\\ &\quad + \left(-\frac{225}{6136} + \frac{14409}{104312}\sqrt{17}\right)(5-\sqrt{17})^n. \qquad (4.4)\end{aligned}$$

4.3. Proof of Theorem 1

Finally, using (4.2) and (4.4), we can obtain

$$\begin{aligned}&\sum_{x,y\in G_n}\frac{1}{2}(d_n(x,y) + (d_n(x,y))^2)\\ &\quad = \frac{1}{2}(\xi_n + \xi_n^{(2)})\end{aligned}$$

$$
\begin{aligned}
&= \frac{1664639}{4353492}2^{2n}3^{2n} + \frac{699}{1180}2^{n}3^{2n} + \frac{7}{6}2^{2n}3^{n} \\
&\quad + \frac{1}{3}3^{2n} + \frac{3}{2}2^{n}3^{n} + \frac{3}{4}2^{2n} + \frac{3593}{4290}3^{n} + \frac{3}{4}2^{n} \\
&\quad - \left(\frac{72717}{527696} + \frac{116067}{8970832}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{2}\right)^{n} \\
&\quad + \left(-\frac{72717}{527696} + \frac{116067}{8970832}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{2}\right)^{n} \\
&\quad - \left(\frac{225}{12272} + \frac{14409}{208624}\sqrt{17}\right)(5+\sqrt{17})^{n} \\
&\quad + \left(-\frac{225}{12272} + \frac{14409}{208624}\sqrt{17}\right)(5-\sqrt{17})^{n}.
\end{aligned}
$$

Theorem 1 is proved.

ACKNOWLEDGMENT

The work is supported by the National Natural Science Foundation of China (No. 12171259).

REFERENCES

1. D. Watts and S. Strogatz, Collective dynamics of 'small-world' networks, *Nature* **393** (1998) 440–442.
2. A. Barabási and R. Albert, Emergence of scaling in random networks, *Science* **286** (1999) 509–512.
3. M. Newman, The structure and function of complex networks, *SIAM Rev.* **45** (2003) 167–256.
4. M. Newman, *Networks*: *An Introduction* (Oxford University Press, New York, 2010).
5. H. Wiener, Structural determination of paraffin boiling points, *J. Am. Chem. Soc.* **69** (1947) 17–20.
6. M. Randić, Novel molecular descriptor for structure — property studies, *Chem. Phys. Lett.* **211** (1993) 478–483.
7. G. Cash, S. Klavzar and M. Petkovsek, Three methods for calculation of the hyper-Wiener index of molecular graphs, *J. Chem. Inf. Comput. Sci.* **42** (2002) 571–576.
8. L. Feng, W. Liu and K. Xu, The hyper-Wiener index of bicyclic graphs, *Util. Math.* **84** (2011) 97–104.
9. N. Tratnik, A method for computing the edge-hyper-Wiener index of partial cubes and an algorithm for benzenoid systems, *Appl. Anal. Discrete Math.* **12** (2018) 126–142.
10. T. Wu and H. Lu, Hyper-Wiener indices of polyphenyl chains and polyphenyl spiders, *Open Math.* **17** (2019) 668–676.
11. S. Wang, Z. Yu and L. Xi, Average geodesic distance of Sierpinski gasket and Sierpinski networks, *Fractals* **25** (2017) 1750044.
12. L. Zhao, S. Wang and L. Xi, Average geodesic distance of Sierpinski carpet, *Fractals* **25** (2017) 1750061.
13. T. Li, K. Jiang and L. Xi, Average distance of self-similar fractal trees, *Fractals* **26** (2018) 1850016.
14. J. Yang, S. Wang, L. Xi and Y. Ye, Average geodesic distance of skeleton networks of Sierpinski tetrahedron, *Physica A* **495** (2018) 269–277.
15. Q. Ye and L. Xi, Average distance of substitution networks, *Fractals* **27** (2019) 1950097.
16. B. Zhao, J. Fan and L. Xi, Average distances of lindstrom snowflake networks, *Fractals* **29** (2021) 2150067.
17. Y. Li, J. Fan and L. Xi, Average geodesic distance on stretched Sierpinski gasket, *Chaos Solitons Fractals* **150** (2021) 111120.
18. J. Zhu, L. Tian and Q. Wang, Average geodesic distance on Sierpinski hexagon and Sierpinski hexagon networks, *Fractals* **27** (2019) 1950077.
19. J. Chen, L. He and Q. Wang, Eccentric distance sum of Sierpinski gasket and Sierpinski network, *Fractals* **27** (2019) 1950016.
20. J. Deng, Q. Ye and Q. Wang, Weighted average geodesic distance of Vicsek network, *Physica A* **527** (2019) 121327.
21. J. Deng and Q. Wang, Asymptotic formula of average distances on fractal networks modeled by Sierpinski tetrahedron, *Fractals* **27** (2019) 1950120.
22. Y. Xue and D. Zhou, Asymptotic formula on average path length of a special network based on Sierpinski carpet, *Fractals* **26** (2018) 1850039.
23. Q. Zhang, Y. Xue, D. Wang and M. Niu, Asymptotic formula on average path length in a hierarchical scale-free network with fractal structure, *Chaos Solitons Fractals* **122** (2019) 196–201.
24. D. Wang and Y. Xue, Average path length and degree distribution of networks generated by random sequence, *Mod. Phys. Lett. B* **35** (2021) 2150347.
25. C. Zeng, Y. Huang and Y. Xue, Fractal networks with hierarchical structure: Mean fermat distance and small-world effect, *Mod. Phys. Lett. B* **36** (2022) 2250109.
26. C. Zeng, Y. Huang and Y. Xue, Average geodesic distance of Sierpinski-type networks, *Fractals* **30** (2022) 2250088.
27. M. Niu and R. Li, The average weighted path length for a class of hierrchical networks, *Fractals* **28** (2020) 2050073.

28. F. Huang, M. Dai, J. Zhu and W. Su, Scaling of average shortest distance of two colored substitution networks, *IEEE Trans. Network Sci. Eng.* **7** (2020) 3067–3073.
29. M. Dai, F. Huang, J. Zhu, Y. Guo, Y. Li, H. Qiu and W. Su, Structure properties and weighted average geodesic distances of the Sierpinski carpet fractal networks, *Phys. Scr.* **95** (2020) 065210.
30. T. Chan, A statistical analysis of the towers of Hanoi problem, *Int. J. Comput. Math.* **28** (1989) 57–65.