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ZAGREB ECCENTRICITY INDICES OF VICSEK NETWORKS

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Abstract

For a connected graph, the first Zagreb eccentricity index is defined as the sum of squares of the eccentricities of the vertices, and the second Zagreb eccentricity index is defined as the sum of the products of the eccentricities of pairs of adjacent vertices. In this paper, by using the self-similarity, we compute the first and second Zagreb eccentricity indices of Vicsek network.

Keywords: Fractal Network; Vicsek Fractal; Zagreb Eccentricity Indices; Self-Similarity.

1. INTRODUCTION

Chemical Graph Theory[1] is the topology branch of mathematical chemistry which applies graph theory to chemical phenomena. In chemical graph theory, topological indices are numerical parameters of a graph which characterize its topology and are usually graph invariant such as the Wiener index,[2] Hosoya index[3] and Zagreb indices.[4,5] Topological indices are used for example in the development of quantitative structure–activity relationships (QSARs) in which the biological activity or other properties of molecules are correlated with their chemical structure. In particular, the Wiener index is related to the average distance of graph, see Refs. 6–14 (by Wang *et al.*).

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Gutman and Trinajstić[4] introduced the Zagreb indices, and Gutman *et al.*[5] elaborated it. Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$, and $\deg(u)$ the degree of $u \in V(G)$. The Zagreb indices of G are defined as

$$M_1(G) = \sum_{u \in V(G)} \deg^2(u),$$

$$M_2(G) = \sum_{uv \in E(G)} \deg(u)\deg(v).$$

Todeschini and Consonni[15,16] revealed that the Zagreb indices and their variants are useful molecular descriptors which found numerous use in QSPR and QSAR studies, also see Ref. 17.

Consider the eccentricity $\varepsilon_G(u) = \varepsilon(u) = \max_{v \in V(G)} d(u, v)$, where $d(u, v)$ is the length of the shortest path connecting u and v. Many mathematical works are devoted to this invariant, see Refs. 18, 19, 20 (by Ye *et al.*), etc. Vukičeivc and Graovac[21] introduced two types of Zagreb eccentricity indices by replacing degrees by eccentricity of the vertices. The first Zagreb eccentricity index of G is defined as

$$\xi_1(G) = \sum_{u \in V(G)} \varepsilon^2(u),$$

while the second Zagreb eccentricity index of G is defined as

$$\xi_2(G) = \sum_{uv \in E(G)} \varepsilon(u)\varepsilon(v).$$

The Vicsek fractal K is a self-similar fractal of Hausdorff dimension $\log 5/\log 3$ arising from a construction similar to that of the Sierpinski carpet, proposed by Vicsek. It has applications including as compact antennas. Suppose $S_i(x, y) = (x, y)/3 + a_i$ with $a_0 = (1, 1)/3$, $a_1 = (0, 1)/3$, $a_2 = (1, 0)/3$, $a_3 = (2, 1)/3$ and $a_4 = (1, 2)/3$ and write $K_i = S_i(K)$. Then $K = \cup_{i=0}^4 K_i$ is generated by unit square $[0, 1]^2$ and the first three steps of construction are shown in Fig. 1.

Vicsek network is a family of growing self-similar graphs. Precisely, given an integer $n \geq 1$, let G_n be a graph of node set $\{i_1 \ldots i_n : i_t = 0, 1, 2, 3 \text{ or } 4\}$ and two nodes $i_1 \ldots i_n$ and $j_1 \ldots j_n$ are neighbors if and only if

$$S_{i_1 \cdots i_n}([0, 1]^2) \cap S_{j_1 \cdots j_n}([0, 1]^2) \text{ is a line segment.}$$

For example, we can see the self-similar growing networks in Fig. 2. Then we have $|V(G_n)| = 5^n$ and $|E(G_n)| = 5^n - 1$.

Using the self-similar structure of Vicsek networks, we obtain the following.

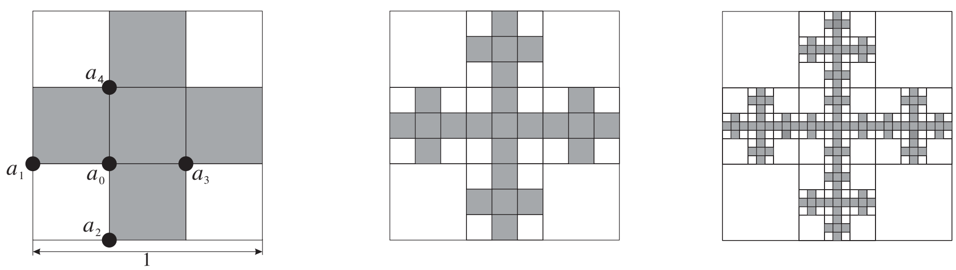

Fig. 1 The first three steps of Vicsek fractal.

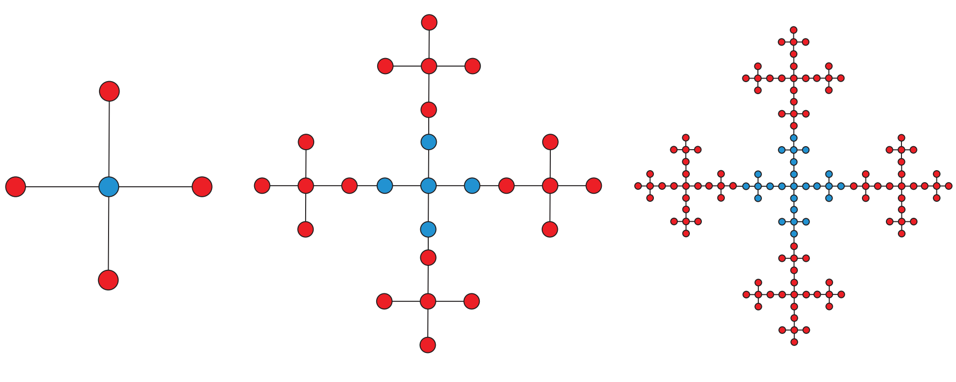

Fig. 2 Vicsek networks G_1, G_2, G_3.

Theorem 1. *For $n \geq 1$, we have*

$$\xi_1(G_n) = \frac{5629}{7700}45^n - \frac{59}{50}15^n + \frac{41}{100}5^n - \frac{1}{7}3^n + \frac{2}{11},$$
$$\xi_2(G_n) = \frac{5629}{7700}45^n - \frac{354}{175}15^n - \frac{1}{4}9^n + \frac{111}{100}5^n + \frac{6}{7}3^n - \frac{131}{308}.$$

2. PRELIMINARIES

Recall an *isomorphism* of graphs G and H is a bijection $f : V(G) \to V(H)$ between the vertex sets of G and H such that $u_1 \overset{G}{\sim} u_2$ if and only if $f(u_1) \overset{H}{\sim} f(u_2)$. Furthermore, if $f(u^*) = v^*$ for some $u^* \in V(G)$, $v^* \in V(H)$, we denote it by $(G, u^*) \overset{f}{\simeq} (H, v^*)$ or $(G, u^*) \simeq (H, v^*)$. If $(G, a) \simeq (G, b)$, then we say that G is *symmetric* with respect to a and b.

As in Fig. 3, for each $n \geq 1$, the graph G_n consists of five subgraphs isomorphic (or similar) to G_{n-1}, denoted by $G^{n,j}$ for $j \in \{0, \ldots, 4\}$, and four edges connecting these five subgraphs. Then, we have $|V(G^{n,j})| = 5^{n-1}$ and $|E(G^{n,j})| = 5^{n-1} - 1$.

As in Fig. 4, for each $n \geq 1$, we denote the leftmost, bottommost, rightmost, and topmost vertices of G_n by $\beta_1^{(n)}$, $\beta_2^{(n)}$, $\beta_3^{(n)}$ and $\beta_4^{(n)}$, respectively. Furthermore, we use $\beta_i^{(n,j)}$ to denote the copy of $\beta_i^{(n)}$ in $G^{n,j}$. Then, we have the following *self-similarity*:

$$(G^{n,j}, \beta_i^{(n,j)}) \simeq (G_{n-1}, \beta_i^{(n-1)}) \quad \text{for any } i \text{ and } j. \tag{2.1}$$

Note for any $n \geq 1$, the graph G_n is symmetric with respect to $\beta_i^{(n)}$ and $\beta_j^{(n)}$ for any distinct $i, j \in \{1, \ldots, 4\}$, i.e.

$$(G_n, \beta_i^{(n)}) \simeq (G_n, \beta_j^{(n)}). \tag{2.2}$$

Recall that $\varepsilon_G(u)$ denotes the eccentricity of u in G, we let

$$\varepsilon_n^+ = \sum_{uv \in E(G_n)} (\varepsilon_{G_n}(u) + \varepsilon_{G_n}(v)),$$
$$\varepsilon_n^{(k)} = \sum_{u \in V(G_n)} \varepsilon_{G_n}^k(u), \quad \forall\, k \geq 1.$$

For any $i \in \{1, \ldots, 4\}$ we let

$$d_n^+ = \sum_{uv \in E(G_n)} (d(u, \beta_i^{(n)}) + d(v, \beta_i^{(n)})),$$
$$d_n^* = \sum_{uv \in E(G_n)} d(u, \beta_i^{(n)}) d(v, \beta_i^{(n)}),$$
$$d_n^{(k)} = \sum_{u \in V(G_n)} d^k(u, \beta_i^{(n)}), \quad \forall\, k \geq 1,$$

which are independent of $i \in \{1, \ldots, 4\}$ due to the above symmetry (2.2).

Because of the above isomorphisms (2.1)–(2.2) of graphs, we have the following Claims 1–4.

Claim 1. *For $k \geq 1$, we have*

$$\sum_{u \in V(G^{n,0})} \varepsilon_{G^{n,0}}^k(u) = \varepsilon_{n-1}^{(k)}.$$

Moreover, we have

$$\sum_{uv \in E(G^{n,0})} (\varepsilon_{G^{n,0}}(u) + \varepsilon_{G^{n,0}}(u)) = \varepsilon_{n-1}^+$$

and

$$\sum_{uv \in E(G^{n,0})} \varepsilon_{G^{n,0}}(u) \varepsilon_{G^{n,0}}(u) = \xi_2(G_{n-1}).$$

Claim 2. *For any $i \in \{1, \ldots, 4\}$, $j \in \{0, \ldots, 4\}$ and $k \geq 1$, we have*

$$\sum_{u \in V(G^{n,j})} d^k(u, \beta_i^{(n,j)}) = d_{n-1}^{(k)}.$$

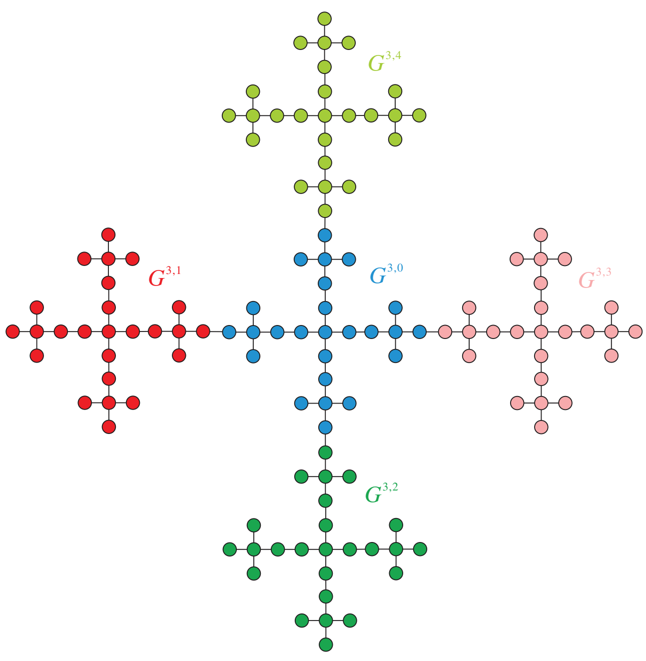

Fig. 3 The structure of G_3.

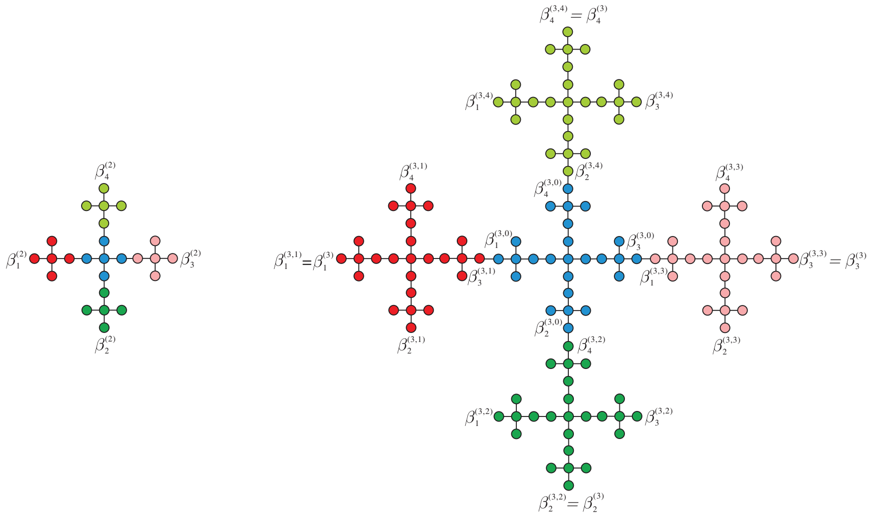

Fig. 4 $\{\beta_i^{(2)}\}_i$ and $\{\beta_i^{(3,j)}\}_{i,j}$ in Vicsek networks.

Claim 3. *For any $i \in \{1,\ldots,4\}$ and $j \in \{0,\ldots,4\}$, we have*

$$\sum_{uv\in E(G^{n,j})}(d(u,\beta_i^{(n,j)})+d(v,\beta_i^{(n,j)})) = d_{n-1}^{+}.$$

Claim 4. *For any $i \in \{1,\ldots,4\}$ and $j \in \{0,\ldots,4\}$, we have*

$$\sum_{uv\in E(G^{n,j})} d(u,\beta_i^{(n,j)})d(v,\beta_i^{(n,j)}) = d_{n-1}^{*}.$$

Claim 5. *For any distinct points $u,v \in \{\beta_1^{(n)},\beta_2^{(n)},\beta_3^{(n)},\beta_4^{(n)}\}$, we have*

$$d(u,v) = \mathrm{diam}(G_n) = 3^n - 1.$$

For any $j \in \{0,\ldots,4\}$ and distinct points $x,y \in \{\beta_1^{(n,j)},\beta_2^{(n,j)},\beta_3^{(n,j)},\beta_4^{(n,j)}\}$, we have

$$d(x,y) = \mathrm{diam}(G^{n,j}) = 3^{n-1} - 1.$$

We let $\sigma = (1,3)(2,4)$ be a permutation on $\{1,\ldots,4\}$. Note that there is an edge $\beta_{\sigma(j)}^{(n,j)}\beta_j^{(n,0)}$ in $E(G_n)$ connecting $G^{n,0}$ and $G^{n,j}$ for any $j \in \{1,\ldots,4\}$. As the result of Claim 5, we have the following.

Claim 6. *We have*

$$\varepsilon_{G_n}(u) = \begin{cases} \varepsilon_{G^{n,0}}(u) + 3^{n-1} & \text{if } u \in V(G^{n,0}), \\ d(u,\beta_{\sigma(j)}^{(n,j)}) + 2\times 3^{n-1} & \text{if } u \in V(G^{n,j}),\ j \in \{1,\ldots,4\}. \end{cases} \tag{2.3}$$

For each $i \in \{1,2,3,4\}$, we also have

$$d(u,\beta_i^{(n)}) = \begin{cases} d(u,\beta_i^{(n)}) & \text{if } u \in V(G^{n,i}), \\ d(u,\beta_i^{(n,0)}) + 3^{n-1} & \text{if } u \in V(G^{n,0}), \\ d(u,\beta_{\sigma(j)}^{(n,j)}) + 2\times 3^{n-1} & \text{if } u \in V(G^{n,j}),\ j \notin \{0,i\}. \end{cases} \tag{2.4}$$

To prove Theorem 1, we need the following Lemmas 1–2.

Lemma 1. *For $n \geq 1$, we have*

$$d_n^{(1)} = \frac{7}{10}(15^n - 5^n) \tag{2.5}$$

and

$$d_n^{(2)} = \frac{57}{100}45^n - \frac{49}{50}15^n + \frac{41}{100}5^n. \tag{2.6}$$

Proof. We first calculate $d_n^{(1)}$. Using (2.4) of Claim 6, the symmetry (2.2) and Claim 2, we have

$$\begin{aligned} d_n^{(1)} &= \sum_{u\in V(G_n)} d(u,\beta_1^{(n)}) \\ &= \sum_{u\in V(G^{n,1})} d(u,\beta_1^{(n)}) \\ &\quad + \sum_{u\in V(G^{n,0})}(d(u,\beta_1^{(n,0)}) + 3^{n-1}) \end{aligned}$$

$$+3\sum_{u\in V(G^{n,2})}(d(u,\beta_{\sigma(2)}^{(n,2)})+2\times 3^{n-1})$$
$$=5d_{n-1}^{(1)}+7\times 15^{n-1}. \tag{2.7}$$

Iterating (2.7) and using $d_1^{(1)}=7$, we have

$$d_n^{(1)}=\frac{7}{10}(15^n-5^n).$$

In the same way, we obtain that

$$\begin{aligned}d_n^{(2)} &= \sum_{u\in V(G_n)} d^2(u,\beta_1^{(n)})\\ &= \sum_{u\in V(G^{n,1})} d^2(u,\beta_1^{(n)})\\ &\quad+\sum_{u\in V(G^{n,0})}(d(u,\beta_1^{(n,0)})+3^{n-1})^2\\ &\quad+3\sum_{u\in V(G^{n,2})}(d(u,\beta_{\sigma(2)}^{(n,2)})+2\times 3^{n-1})^2\\ &= 5d_{n-1}^{(2)}+14\times 3^{n-1}d_{n-1}^{(1)}\\ &\quad+13\times 5^{n-1}\times 9^{n-1}.\end{aligned}$$

By (2.5), we have

$$d_n^{(2)}=5d_{n-1}^{(2)}+\frac{114}{225}45^n-\frac{49}{75}15^n. \tag{2.8}$$

Iterating (2.8) and using $d_1^{(2)}=13$, we obtain

$$d_n^{(2)}=\frac{57}{100}45^n-\frac{49}{50}15^n+\frac{41}{100}5^n. \qquad \square$$

Note that there is an edge $\beta_{\sigma(j)}^{(n,j)}\beta_j^{(n,0)}$ in $E(G_n)$ connecting $G^{n,0}$ and $G^{n,j}$ for any $j\in\{1,\ldots,4\}$.

Lemma 2. *For $n\geq 1$, we have*

$$d_n^{+}=21\times 15^{n-1}-12\times 5^{n-1}+1 \tag{2.9}$$

and

$$d_n^{*}=\frac{513}{20}45^{n-1}+\frac{111}{20}5^{n-1}-\frac{126}{5}15^{n-1}. \tag{2.10}$$

Proof. We first calculate d_n^+. Using (2.4) of Claim 6, the symmetry (2.2) and Claim 3, we have

$$\begin{aligned}d_n^+ &= \sum_{uv\in E(G_n)}(d(u,\beta_1^{(n)})+d(v,\beta_1^{(n)}))\\ &= \sum_{uv\in E(G^{n,1})}(d(u,\beta_1^{(n)})+d(v,\beta_1^{(n)}))\\ &\quad+\sum_{uv\in E(G^{n,0})}(d(u,\beta_1^{(n,0)})+3^{n-1}+d(v,\beta_1^{(n,0)})\\ &\quad+3^{n-1})+3\sum_{uv\in E(G^{n,2})}(d(u,\beta_{\sigma(2)}^{(n,2)})+2\times 3^{n-1}\\ &\quad+d(v,\beta_{\sigma(2)}^{(n,2)})+2\times 3^{n-1})+d(\beta_{\sigma(1)}^{(n,1)},\beta_1^{(n)})\\ &\quad+d(\beta_1^{(n,0)},\beta_1^{(n)})+3d(\beta_{\sigma(2)}^{(n,2)},\beta_1^{(n)})\\ &\quad+3d(\beta_2^{(n,0)},\beta_1^{(n)})\\ &= 5d_{n-1}^{+}+14\times 15^{n-1}-4.\end{aligned} \tag{2.11}$$

Iterating (2.11) and using $d_1^+=10$, we may obtain

$$d_n^+=21\times 15^{n-1}-12\times 5^{n-1}+1.$$

In the same way, using (2.4) of Claim 6, the symmetry (2.2), Claims 3 and 4, we obtain that

$$\begin{aligned}d_n^* &= \sum_{uv\in E(G_n)} d(u,\beta_1^{(n)})d(v,\beta_1^{(n)})\\ &= \sum_{uv\in E(G^{n,1})} d(u,\beta_1^{(n)})d(v,\beta_1^{(n)})\\ &\quad+\sum_{uv\in E(G^{n,0})}(d(u,\beta_1^{(n,0)})+3^{n-1})(d(v,\beta_1^{(n,0)})\\ &\quad+3^{n-1})+3\sum_{uv\in E(G^{n,2})}(d(u,\beta_{\sigma(2)}^{(n,2)})+2\times 3^{n-1})\\ &\quad\times(d(v,\beta_{\sigma(2)}^{(n,2)})+2\times 3^{n-1})+d(\beta_{\sigma(1)}^{(n,1)},\beta_1^{(n)})\\ &\quad\times d(\beta_1^{(n,0)},\beta_1^{(n)})+3d(\beta_{\sigma(2)}^{(n,2)},\beta_1^{(n)})\\ &\quad\times d(\beta_2^{(n,0)},\beta_1^{(n)})\\ &= 5d_{n-1}^{*}+7\times 3^{n-1}d_{n-1}^{+}+13\times 45^{n-1}\\ &\quad-7\times 3^{n-1}.\end{aligned}$$

By (2.9), we obtain

$$d_n^*=5d_{n-1}^*+\frac{38}{75}45^n-\frac{28}{25}15^n. \tag{2.12}$$

Iterating (2.12) and using $d_1^*=6$, we have

$$d_n^*=\frac{513}{20}45^{n-1}+\frac{111}{20}5^{n-1}-\frac{126}{5}15^{n-1}. \qquad \square$$

3. PROOF OF THEOREM 1

3.1. Part I

We will deal with $\xi_1(G_n)$.

Let us first consider $\varepsilon_n^{(1)} = \sum_{u\in V(G_n)} \varepsilon_{G_n}(u)$. By (2.3) of Claim 6, the symmetry (2.2), Claims 1 and 2, we have

$$\begin{aligned}\varepsilon_n^{(1)} &= \sum_{u\in V(G_n)} \varepsilon_{G_n}(u) = \sum_{u\in V(G^{n,0})} (\varepsilon_{G^{n,0}}(u) + 3^{n-1}) \\ &\quad + 4\sum_{u\in V(G^{n,1})} (d(u, \beta_{\sigma(1)}^{(n,1)}) + 2\times 3^{n-1}) \\ &= \varepsilon_{n-1}^{(1)} + 4d_{n-1}^{(1)} + 9\times 5^{n-1}\times 3^{n-1}.\end{aligned}$$

By (2.5), we get

$$\varepsilon_n^{(1)} = \varepsilon_{n-1}^{(1)} + \frac{59}{5}15^{n-1} - \frac{14}{5}5^{n-1}. \tag{3.1}$$

Iterating (3.1) and using $\varepsilon_1^{(1)} = 9$, we obtain that

$$\varepsilon_n^{(1)} = \frac{177}{14}15^{n-1} - \frac{7}{2}5^{n-1} - \frac{1}{7}. \tag{3.2}$$

In the same way, we get

$$\begin{aligned}\xi_1(G_n) &= \sum_{u\in V(G_n)} \varepsilon_{G_n}^2(u) \\ &= \sum_{u\in V(G^{n,0})} (\varepsilon_{G^{n,0}}(u) + 3^{n-1})^2 \\ &\quad + 4\sum_{u\in V(G^{n,1})} (d(u, \beta_{\sigma(1)}^{(n,1)}) + 2\times 3^{n-1})^2 \\ &= \sum_{u\in V(G^{n,0})} \varepsilon_{G^{n,0}}^2(u) + 2\times 3^{n-1} \\ &\quad \times \sum_{u\in V(G^{n,0})} \varepsilon_{G^{n,0}}(u) + 5^{n-1}\times (3^{n-1})^2 \\ &\quad + 4\sum_{u\in V(G^{n,1})} d^2(u, \beta_{\sigma(1)}^{(n,1)}) + 16\times 3^{n-1} \\ &\quad \times \sum_{u\in V(G^{n,1})} d(u, \beta_{\sigma(1)}^{(n,1)}) \\ &\quad + 16\times 5^{n-1}\times (3^{n-1})^2 \\ &= \xi_1(G_{n-1}) + 2\times 3^{n-1}\varepsilon_{n-1}^{(1)} + 4d_{n-1}^{(2)} \\ &\quad + 16\times 3^{n-1} d_{n-1}^{(1)} + 17\times 45^{n-1}.\end{aligned}$$

By (3.2), (2.6) and (2.5), we have

$$\begin{aligned}\xi_1(G_n) &= \xi_1(G_{n-1}) + \frac{5629}{175}45^{n-1} - \frac{413}{25}15^{n-1} \\ &\quad + \frac{41}{25}5^{n-1} - \frac{2}{7}3^{n-1}.\end{aligned} \tag{3.3}$$

Iterating (3.3) and using $\xi_1(G_1) = 17$, we obtain that

$$\xi_1(G_n) = \frac{5629}{7700}45^n - \frac{59}{50}15^n + \frac{41}{100}5^n - \frac{1}{7}3^n + \frac{2}{11}.$$

3.2. Part II

We will deal with $\xi_2(G_n)$.

We first consider $\varepsilon_n^+ = \sum_{uv\in E(G_n)} (\varepsilon_{G_n}(u) + \varepsilon_{G_n}(v))$. Note that there is an edge $\beta_{\sigma(j)}^{(n,j)}\beta_j^{(n,0)}$ in $E(G_n)$ connecting $G^{n,0}$ and $G^{n,j}$ for any $j \in \{1,\ldots,4\}$. Using (2.3) of Claim 6, the symmetry (2.2), Claims 1 and 3, we have

$$\begin{aligned}\varepsilon_n^+ &= \sum_{uv\in E(G_n)} (\varepsilon_{G_n}(u) + \varepsilon_{G_n}(v)) \\ &= \sum_{uv\in E(G^{n,0})} (\varepsilon_{G^{n,0}}(u) + 3^{n-1} + \varepsilon_{G^{n,0}}(v) + 3^{n-1}) \\ &\quad + 4\Bigg(\sum_{uv\in E(G^{n,1})} (d(u, \beta_{\sigma(1)}^{(n,1)}) + 2\times 3^{n-1} \\ &\quad + d(u, \beta_{\sigma(1)}^{(n,1)}) + 2\times 3^{n-1})\Bigg) \\ &\quad + 4(\beta_{\sigma(1)}^{(n,1)} + \varepsilon_{G_n}(\beta_1^{(n,0)})) \\ &= \varepsilon_{n-1}^+ + 4d_{n-1}^+ + 18\times 15^{n-1} \\ &\quad - 18\times 3^{n-1} + 16\times 3^{n-1} - 4.\end{aligned}$$

By (2.9), we get

$$\varepsilon_n^+ = \varepsilon_{n-1}^+ + 354\times 15^{n-2} - 48\times 5^{n-2} - 2\times 3^{n-1}. \tag{3.4}$$

Iterating (3.4) and using $\varepsilon_1^+ = 12$, we obtain that

$$\varepsilon_n^+ = \frac{177}{7}15^{n-1} - 12\times 5^{n-1} - 3^n + \frac{12}{7}. \tag{3.5}$$

In the same way, we get

$$\begin{aligned}\xi_2(G_n) &= \sum_{uv\in E(G_n)} (\varepsilon_{G_n}(u)\varepsilon_{G_n}(v)) \\ &= \sum_{uv\in E(G^{n,0})} ((\varepsilon_{G^{n,0}}(u) + 3^{n-1})(\varepsilon_{G^{n,0}}(v) \\ &\quad + 3^{n-1})) + 4\Bigg(\sum_{uv\in E(G^{n,1})} ((d(u, \beta_{\sigma(1)}^{(n,1)}) \\ &\quad + 2\times 3^{n-1})(d(v, \beta_{\sigma(1)}^{(n,1)}) + 2\times 3^{n-1}))\Bigg)\end{aligned}$$

$$+ 4\beta_{\sigma(1)}^{(n,1)}\varepsilon_{G_n}(\beta_1^{(n,0)})$$
$$= \xi_2(G_{n-1}) + 3^{n-1}\varepsilon_{n-1}^{+} + 4d_{n-1}^{*} + 8 \times 3^{n-1}d_{n-1}^{+} + 17 \times 45^{n-1} - 9^{n-1} - 8 \times 3^{n-1},$$

By (3.5), (2.10) and (2.9), we have

$$\xi_2(G_n) = \xi_2(G_{n-1}) + \frac{5629}{175}45^{n-1} - \frac{708}{25}15^{n-1} - 2 \times 9^{n-1} + \frac{111}{25}5^{n-1} + \frac{12}{7}3^{n-1}. \tag{3.6}$$

Iterating (3.6) and using $\xi_2(G_1) = 8$, we obtain that

$$\xi_2(G_n) = \frac{50661}{1540}45^{n-1} - \frac{1062}{35}15^{n-1} - \frac{9}{4}9^{n-1} + \frac{111}{20}5^{n-1} + \frac{18}{7}3^{n-1} - \frac{131}{308}.$$

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REFERENCES

1. D. Bonchev and D. H. Rouvray (eds.), *Chemical Graph Theory: Introduction and Fundamentals* (Gordon & Breach Science Publishers, New York, 1991).
2. H. Wiener, Structural determination of paraffin boiling points, *J. Am. Chem. Soc.* **69** (1947) 17–20.
3. H. Hosoya, Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons, *Bull. Chem. Soc. Jpn.* **44**(9) (1971) 2332–2339.
4. I. Gutman and N. Trinajstić, Graph theory and molecular orbitals. Total φ-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.
5. I. Gutman, B. Ruščić, N. Trinajstić and C. F. Wilcox, Graph theory and molecular orbitals. XII acyclic polyenes, *J. Chem. Phys.* **62** (1975) 3399–3405.
6. S. Wang, Z. Yu and L. Xi, Average geodesic distance of Sierpinski gasket and Sierpinski networks, *Fractals* **25**(5) (2017) 1750044.
7. J. Yang, S. Wang, L. Xi and Y. Ye, Average geodesic distance of skeleton networks of Sierpinski tetrahedron, *Fractals* **495** (2018) 269–277.
8. J. Deng, Q. Ye and Q. Wang, Weighted average geodesic distance of Vicsek network, *Physica A* **527** (2019) 121327.
9. J. Deng and Q. Wang, Asymptotic formula of average distances on fractal networks modeled by Sierpinski tetrahedron, *Fractal* **27**(7) (2019) 1950120.
10. Q. Zhang, Y. Xue, D. Wang and M. Niu, Asymptotic formula on average path length in a hierarchical scale-free network with fractal structure, *Chaos Solitons Fractals* **122** (2019) 196–201.
11. M. Niu and R. Li, The average weighted path length for a class of hierarchical networks, *Fractals* **28**(4) (2020) 2050073.
12. F. Huang, M. Dai, J. Zhu and W. Su, Scaling of average shortest distance of two colored substitution networks, *IEEE Trans. Netw. Sci. Eng.* **7**(4) (2020) 3067–3073.
13. Y. Liu, M. Dai and Y. Guo, Weighted average geodesic distance of Vicsek network in three-dimensional space, *Int. J. Mod. Phys. B* **35**(5) (2021) 2150077.
14. C. Zeng, Y. Huang and Y. Xue, Fractal networks with hierarchical structure: Mean Fermat distance and small-world effect, *Mod. Phys. Lett. B* **36**(22) (2022) 2250109.
15. R. Todeschini and V. Consonni, *Handbook of Molecular Descriptors* (Wiley-VCH, Weinheim, 2000), p. 124.
16. R. Todeschini and V. Consonni, *Molecular Descriptors for Chemoinformatics*, Vol. 1 (Wiley-VCH, Weinheim, 2009), pp. 237–241.
17. K. C. Das, K. Xu and J. Nam, Zagreb indices of graphs, *Front. Math. China* **10**(3) (2015) 567–582.
18. Q. Ye, L. He, Q. Wang and L. Xi, Asymptotic formula of eccentric distance sum for Vicsek network, *Fractals* **26**(3) (2018) 1850027.
19. J. Chen, L. He and Q. Wang, Eccentric distance sun of Sierpinski gasket and Sierpiński network, *Fractals* **27**(2) (2019) 1950016.
20. W. Ma, Q. Jia, L. Lei and L. Xi, Eccentric Steiner distance sum of Vicsek networks, *Fractals* **30**(7) (2022) 2250129.
21. D. Vukičeivć and A. Graovac, Note on the comparison of the first and second normalized Zagreb eccentricity indices, *Acta Chim. Slov.* **57** (2010) 524–538.