International Journal of Modern Physics C
(2023) 2350109 (14 pages)

DOI: 10.1142/S0129183123501097

Small-world and scale-free effects of complex networks generated by a self-similar fractal

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Received 4 November 2022
Accepted 24 December 2022
Published 31 January 2023

In this paper, we construct a class of growing networks by the encoding method of the iterated function system based on a planar self-similar fractal, and demonstrate that the networks have small-world and scale-free effects.

Keywords: Fractal networks; scale-free; small-world.

1. Introduction

As we all know, the complex networks have gained plenty of attention owing to their relevance to many realistic systems such as social and biological networks. Two remarkable features of these real-world complex networks are small-world[1] and scale-free effects.[2]

In the research on complex networks, the self-similarity and fractality are studied by many scholars, for example, Song *et al.*[3–5] showed that many realistic networks, such as the WWW and the human brain, have self-similarity and fractal dimension. In addition, we can construct complex networks by using various self-similar fractals, please refer to Refs. 6 and 7 by Zhang *et al.* for Sierpinski networks by Dai *et al.*[8,9] for Koch networks by Zhang *et al.*[10] and Deng *et al.*[11] for Vicsek networks, Dai *et al.*[12] and Xue *et al.*[13] for cube networks.

It is worth mentioning that Xi *et al.*[14,15] provide a new approach based on the encoding method and topological properties of basic geometric objects to construct the Sierpinski networks. Specifically, they encode any basic geometric object $S_\sigma(M)$ by a word σ in symbolic system $\cup_{k=0}^{\infty}\{1,\ldots,n\}^k$ related to the IFS $\{S_1,\ldots,S_n\}$, where M is the initial compact set such that $\cup_{i=1}^{n}(S_iM) \subset M$, then G_t is constructed

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with node set $V_t = \{\sigma : |\sigma| \leq t\}$ and σ and τ are defined to be neighbors depending on the geometric character of intersection of $S_\sigma M$ and $S_\tau M$. From then on, Xue *et al.*,[16–20] Niu and Shao[21] and Huang and Peng[22] demonstrate the scale-free and small-world properties of complex networks generated by self-similar fractals.

In this paper, using the encoding approach we construct our growing networks generated by a two-dimensional self-similar fractal, where the initial geometric object M is a solid square and σ and τ are neighbors if $\partial S_\sigma M \cap \partial S_\tau M \neq \emptyset$. Note that in Ref. 14 M is a solid triangle, whereas in Ref. 15 words σ and τ are neighbors if $\partial S_\sigma M \cap \partial S_\tau M$ contains a line segment.

2. Construction of Evolving Networks

Suppose $\{S_i(x) = x/4 + a_i : \mathbb{R}^2 \to \mathbb{R}^2\}_{i=1}^{8}$ is an IFS on the plane, where $a_1 = (0,0)$, $a_2 = (3/4,0)$, $a_3 = (3/4,3/4)$, $a_4 = (0,3/4)$, $a_5 = (1/4,1/4)$, $a_6 = (1/2,1/4)$, $a_7 = (1/2,1/2)$, $a_8 = (1/4,1/2)$. Then the invariant set of the above IFS denoted by E satisfies $E = \cup_{i=1}^{8} S_i(E)$, see Fig. 1 for the iteration process from $Q = [0,1]^2$.

For a finite word $\sigma = i_1 i_2 \cdots i_k \in \Delta_k = \{1,\ldots,8\}^k$, its length is denoted by $|\sigma|(= k)$. For notational convenience, let $\Delta_0 = \{\emptyset\}$, where $\emptyset$ is the empty word with its length as 0. Fix an integer $t \geq 0$, let

$$V_t = \cup_{k=0}^{t} \Delta_k$$

be a set of all words with length less than or equal to t. Hence its cardinality

$$\sharp V_t = 1 + 8 + 8^2 + \cdots + 8^t = \frac{8^{t+1}-1}{7}. \tag{2.1}$$

For any word $\sigma = i_1 i_2 \cdots i_k \in \Delta_k$, we write

$$S_\sigma = S_{i_1 i_2 \cdots i_k} = S_{i_1} \circ S_{i_2} \cdots \circ S_{i_k} \quad \text{and} \quad Q_\sigma = S_\sigma(Q).$$

We call Q_σ a basic square of rank k. We let $Q_\emptyset = Q$. In this way, we note that the self-similar set

$$E = \bigcap_{k=1}^{\infty} \bigcup_{i_1 i_2 \cdots i_k \in \Delta_k} S_{i_1 i_2 \cdots i_k}(Q) = \bigcap_{k=1}^{\infty} \bigcup_{\sigma \in \Delta_k} Q_\sigma.$$

See Fig. 2.

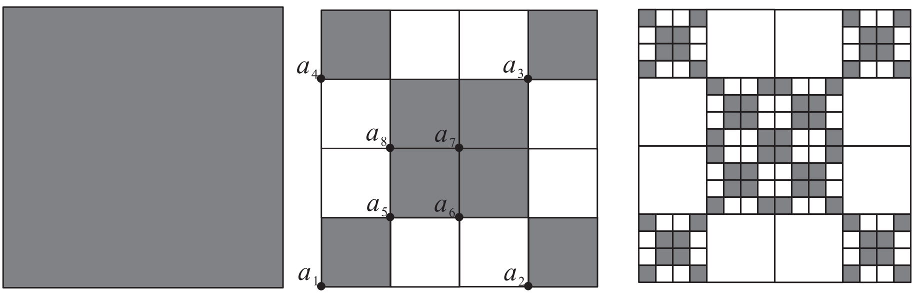

Fig. 1. The first three constructions of the self-similar set E from the initial solid square $Q = [0,1]^2$.

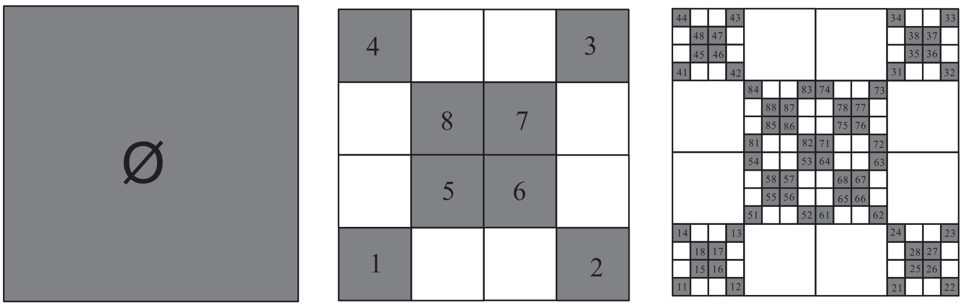

Fig. 2. Encoding of basic squares.

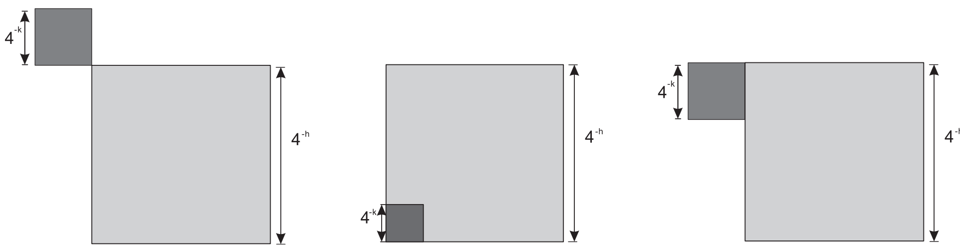

Fig. 3. Three forms of being neighbors.

For any two words $\sigma = i_1 i_2 \cdots i_k$ and $\tau = i_1 i_2 \cdots i_k j_1 j_2 \cdots j_m$ with $\sigma \neq \tau$, we call σ a prefix of τ and write $\sigma \prec \tau$. Moreover, if $\sigma \prec \tau$ and $|\sigma| = |\tau| - 1$, we call σ the father of τ.

For a fixed integer $t \geq 0$, a network G_t with node set V_t can be constructed. For any words $\sigma, \tau \in V_t$ with $\sigma \neq \tau$, we say that between σ and τ, there is an edge if and only if $\partial Q_\sigma \cap \partial Q_\tau \neq \emptyset$ (see Fig. 3), where ∂A stands for the boundary of the set A. In this case, σ and τ are called neighbors and denoted by $\sigma \sim \tau$.

Then we have evolving networks $\{G_t\}_t$, see Fig. 4 for G_5.

For two words $\sigma = i_1 i_2 \cdots i_k$ and $\beta = j_1 j_2 \cdots j_m$, we define the concatenation $\sigma * \beta = i_1 i_2 \cdots i_k j_1 j_2 \cdots j_m$.

Claim 1. *Suppose $\tau = \sigma * \beta$, then $\sigma \sim \tau$ if and only*

$$\beta \in \{1,2\}^{|\beta|}, \{2,3\}^{|\beta|}, \{3,4\}^{|\beta|} \text{ or } \{4,1\}^{|\beta|}.$$

Proof. If $\beta \in \{1,2\}^{|\beta|}$, then Q_τ intersects Q_σ with the bottom line segment, that is $\sigma \sim \tau$. In the same way, if $\beta \in \{2,3\}^{|\beta|}$, $\{3,4\}^{|\beta|}$ or $\{4,1\}^{|\beta|}$, we have $\sigma \sim \tau$. On the other hand, if $\sigma \sim \tau$, then Q_τ intersects Q_σ with some line segment. By the encoding approach, we have $\beta \in \{1,2\}^{|\beta|}$, $\{2,3\}^{|\beta|}$, $\{3,4\}^{|\beta|}$ or $\{4,1\}^{|\beta|}$. □

For example, 12345121 and 12345 are neighbors and 1475641 and 14756 are also neighbors. However, 12546 and 1254 are not neighbors and 45613 and 456 are not neighbors.

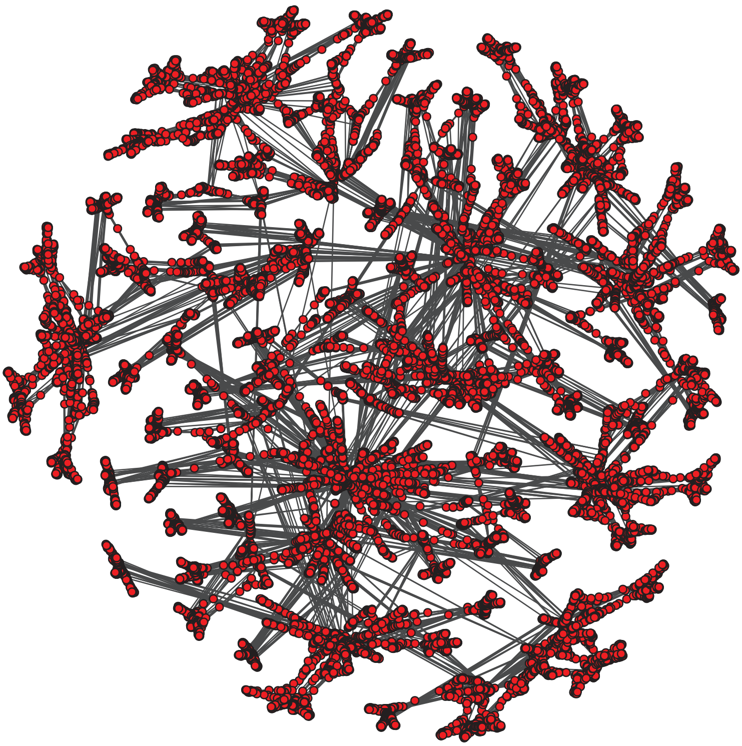

Fig. 4. G_5.

3. Cumulative Degree Distribution

Recall the following definition:

$$P_{\rm cum}(k) = \frac{\sharp\{\sigma : \deg(\sigma) \geq k\}}{\sharp V_t},$$

where $\deg(\sigma)$ denotes the cardinality of neighbors of σ.

Theorem 1. *Suppose* $t \geq 10$, *we have*

$$P_{\rm cum}(u) \propto u^{-3}.$$

Specifically, let $t \geq 10$, *then we have*

$$\left(\frac{7^4}{64}\right)u^{-3} \leq P_{\rm cum}(u) < (8 \times 7^3)u^{-3} \text{ for any } u \in [7 \times 2^{\lceil \frac{t}{2} \rceil}, 7 \times 2^t),$$

where $\lceil x \rceil$ *denotes the integer upwards for* x.

We need the following estimates to obtain Theorem 1.

Claim 2. *Suppose* $\sigma \in \Delta_k$.

(1) *If* $h \le k$, *then we*

$$\sharp\{\tau \in \Delta_h : \tau \sim \sigma\} \le 4.$$

(2) *If* $h > k$, *then we*

$$4 \times 2^{h-k} - 4 \le \sharp\{\tau \in \Delta_h : \tau \sim \sigma\} \le 6 \times 2^{h-k} - 2.$$

Proof. (1) Note that

$$\sharp\{\tau \in \Delta_k : \partial Q_\tau \cap \partial Q_\sigma \text{ is a singleton}\} \le 2, \tag{3.1}$$

$$\sharp\{\tau \in \Delta_k : \partial Q_\tau \cap \partial Q_\sigma \text{ is a line segment}\} \le 2. \tag{3.2}$$

Hence for $h = k$, we can deduce $\sharp\{\tau \in \Delta_h : \tau \sim \sigma\} \le 4$. Since there is only one father for any given child, we obtain $\sharp\{\tau \in \Delta_h : \tau \sim \sigma\} \le 4$ for $h \le k$ by induction.

(2) Note that for $h > k$,

$$\{\tau \in \Delta_h : \sigma \prec \tau, \sigma \sim \tau\} \subset \{\tau \in \Delta_h : \tau \sim \sigma\},$$

we obtain that

$$4 \times 2^{h-k} - 4 = \sharp\{\tau \in \Delta_h : \sigma \prec \tau, \sigma \sim \tau\} \le \sharp\{\tau \in \Delta_h : \tau \sim \sigma\}. \tag{3.3}$$

On the other side, we have

$$\{\tau \in \Delta_h : \sigma \sim \tau\} = \{\tau \in \Delta_h : \sigma \prec \tau, \tau \sim \sigma\} \cup A,$$

where $A = \{\tau \in \Delta_h : \sigma \not\prec \tau, \tau \sim \sigma\}$. Note that

$$A \subset \bigcup_{\alpha \in \Delta_k, \alpha \sim \sigma} \mathcal{A}_\alpha,$$

where

$$\mathcal{A}_\alpha = \{\tau \in \Delta_h : \alpha \prec \tau, \alpha \sim \sigma\}.$$

Notice that $\sharp\mathcal{A}_\alpha = 1$ if $\partial Q_\sigma \cap \partial Q_\alpha$ is a singleton and $\sharp\mathcal{A}_\alpha = 2^{h-k}$ if $\partial Q_\sigma \cap \partial Q_\alpha$ is a line segment. According to (3.1)–(3.3), we have

$$\begin{aligned}\sharp\{\tau \in \Delta_h : \tau \sim \sigma\} &= \sharp\{\tau \in \Delta_h : \sigma \prec \tau, \sigma \sim \tau\} + \sharp A \\ &\le (4 \times 2^{h-k} - 4) + 2 \times 1 + 2 \times 2^{h-k} \\ &= 6 \times 2^{h-k} - 2.\end{aligned}$$

□

Proof of Theorem 1.

We first claim that

$$\{\sigma : \deg(\sigma) > 7 \times 2^{t-k}\} = \{\sigma : |\sigma| \le k\}, \tag{3.4}$$

where

$$k \le \frac{t}{2} \quad \text{and} \quad t \ge 10.$$

Suppose $\sigma \in \Delta_k$ with $t-k \geq t/2 \geq 5$. According to Claim 3, we have

$$\begin{aligned}\deg(\sigma) &\geq \sum_{t\geq h>k} \sharp\{\tau \in \Delta_h : \tau \sim \sigma\} \\ &\geq \sum_{t\geq h>k} (4 \times 2^{h-k} - 4) \\ &= 8 \times 2^{t-k} - 4(t-k) - 8 > 7 \times 2^{t-|\sigma|}\end{aligned}$$

due to $2^x > 4x+8$ for any $x \geq 5$.

On the other hand, notice that $k \leq t/2$ and $t-k \geq t/2 \geq 5$, we obtain that

$$\begin{aligned}\deg(\sigma) &\leq \sum_{t\geq h>k} \sharp\{\tau \in \Delta_h : \tau \sim \sigma\} + \sum_{0\leq h\leq k} \sharp\{\tau \in \Delta_h : \tau \sim \sigma\} \\ &\leq \sum_{t\geq h>k} (6 \times 2^{h-k} - 2) + 4(k+1) \\ &= 12 \times 2^{t-k} - 8 - 2(t-k) + 4k < 14 \times 2^{t-|\sigma|}\end{aligned}$$

due to $2 \times 2^{x/2} > 2x$ for any $x \geq 6$. Therefore, we have

$$7 \times 2^{t-|\sigma|} < \deg(\sigma) < 7 \times 2^{t-|\sigma|+1}. \tag{3.5}$$

Thus (3.4) follows from (3.5).

Let $u \in [7 \times 2^{\lceil \frac{t}{2} \rceil}, 7 \times 2^t)$. We can select k with $1 \leq k \leq \frac{t}{2}$ such that $7 \times 2^{t-k} \leq u < 7 \times 2^{t-k+1}$, then we have

$$\{\sigma : \deg(\sigma) > 7 \times 2^{t-k+1}\} \subset \{\sigma : \deg(\sigma) > u\} \subset \{\sigma : \deg(\sigma) > 7 \times 2^{t-k}\}.$$

Note that $\{\sigma : \deg(\sigma) > 7 \times 2^{t-k+1}\} = \{\sigma : |\sigma| \leq k-1\}$ and $\{\sigma : \deg(\sigma) > 7 \times 2^{t-k}\}$ $= \{\sigma : |\sigma| \leq k\}$ by (3.4). Hence, we have

$$\sharp\{\sigma : |\sigma| \leq k-1\} \leq \sharp\{\sigma : \deg(\sigma) > u\} \leq \sharp\{\sigma : |\sigma| \leq k\}. \tag{3.6}$$

By (3.6), we obtain

$$\frac{8^k - 1}{8^{t+1} - 1} \leq P_{\text{cum}}(u) \leq \frac{8^{k+1} - 1}{8^{t+1} - 1}.$$

Since $-\log_2(u/7) \leq k-t < 1 - \log_2(u/7)$, we have

$$\frac{8^{k+1} - 1}{8^{t+1} - 1} \leq \frac{8^{k+1} - 1 + 1}{8^{t+1} - 1 + 1} = 8^{k-t} < 8 \times 7^3 \times u^{-3},$$

$$\frac{8^k - 1}{8^{t+1} - 1} \geq \frac{7}{8} \times \frac{8^{k-1}}{8^t} = \frac{7}{64} 8^{k-t} \geq \frac{7^4}{64} \times u^{-3}.$$

Then we have

$$\frac{7^4}{64} \times u^{-3} \leq P_{\text{cum}}(u) < 8 \times 7^3 \times u^{-3},$$

for any $7 \times 2^{\lceil \frac{t}{2} \rceil} \leq u < 7 \times 2^t$.

4. Average Clustering Coefficient

Recall that for a node x the notation C_x is the local clustering coefficient defined by

$$C_x = \frac{\sharp\{\{\alpha, \beta\}|\alpha \neq \beta \text{ and } \alpha \sim \beta, \alpha \sim x, \beta \sim x\}}{\deg(x)(\deg(x) - 1)/2}.$$

Fix t and consider the average clustering coefficient $\overline{C}_t = \sum_{\sigma\in V_t} C_\sigma/\sharp V_t$ of G_t. By calculating, we obtain that $\overline{C}_1 = 0.25$.

First, we introduce the following set:

$$l_t = \{\sigma : |\sigma| = t \text{ and } Q_t \cap \partial Q = \emptyset\}.$$

It is easily seen that

$$\sharp l_t = 8^t - 4(2^t - 1).$$

For given $t \geq 2$, we let J_t denote the unions of the sets l_i with $1 \leq i \leq t$, i.e.

$$J_t = \bigcup_{i=1}^{t} l_i = \{\sigma : |\sigma| \leq t \text{ and } Q_t \cap \partial Q = \emptyset\}.$$

By calculating, we obtain that

$$\sharp J_t = \sum_{i=1}^{t}(8^i - 4 \times 2^i + 4) = \frac{8(8^t - 1)}{7} - 8(2^t - 1) + 4t. \tag{4.1}$$

Considering the above sets, we have the following conclusion.

Lemma 1. *For all $t \geq 2$, we have*

$$\sum_{\sigma\in l_t} C_\sigma \geq 8 \sum_{\sigma\in l_{t-1}} C_\sigma. \tag{4.2}$$

Proof. *As shown in Fig. 5, we have*

$$\sum_{\sigma\in l_t} C_\sigma = \sum_{i=1}^{8} \sum_{i\tau\in l_t} C_{i\tau} \geq \sum_{i=1}^{8} \sum_{\tau\in l_{t-1}} C_{i\tau}.$$

In fact, by self-similarity and Jordan curve theorem, we have

$$C_{i\tau} = C_\tau \quad \text{for all } \tau \in l_{t-1}.$$

Hence, (4.2) *holds.* □

Similarly, we also have

$$\sum_{\sigma\in J_t} C_\sigma \geq 8 \times \sum_{\sigma\in J_{t-1}} C_\sigma. \tag{4.3}$$

Then we can obtain a weaker result by Lemma 1.

Lemma 2. *For any $t \geq 2$, we have $\overline{C}_t \geq 0.3691$.*

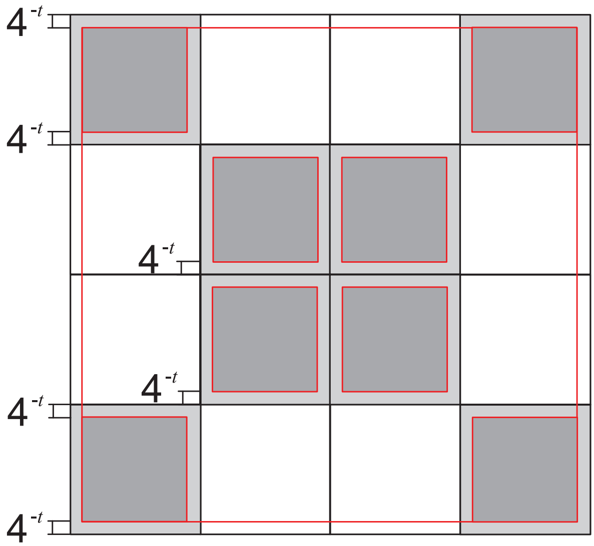

Fig. 5. l_t and il_{t-1} for $i = 1, \ldots, 8$.

Proof. For $t = 2$, we can calculate that

$$\sum_{\sigma \in l_2} C_\sigma = 27.$$

By (4.2), we have

$$\sum_{\sigma \in l_t} C_\sigma \geq 8 \sum_{\sigma \in l_{t-1}} C_\sigma \geq \cdots \geq 8^{t-2} \sum_{\sigma \in l_2} C_\sigma = 8^{t-2} \times 27.$$

Since $\sharp V_t = \frac{8^{t+1}-1}{7} \leq \frac{8^{t+1}}{7}$, we conclude that

$$\frac{\sum_{\sigma \in V_t} C_\sigma}{\sharp V_t} \geq \frac{\sum_{\sigma \in l_t} C_\sigma}{\sharp V_t} \geq \frac{7}{8^{t+1}} \times 27 \times 8^{t-2} = 0.3691 \ldots.$$

$\square$

Let $\alpha_t = \frac{\sharp J_t}{\sharp J_{t-1}}$, and thus

$$e_t = \prod_{i=3}^{t} \frac{\alpha_i}{8} = \frac{\sharp J_t}{8 \sharp J_{t-1}} \cdot \frac{\sharp J_{t-1}}{8 \sharp J_{t-2}} \cdot \frac{\sharp J_{t-2}}{8 \sharp J_{t-3}} \cdots \frac{\sharp J_3}{8 \sharp J_2} = \frac{1}{8^{t-2}} \frac{\sharp J_t}{\sharp J_2}.$$

Since $\sharp J_2 = 56$, then we can find that

$$\lim_{t \to \infty} e_t = \lim_{t \to \infty} \left(\frac{1}{8^{t-2}} \frac{\sharp J_t}{\sharp J_2} \right) = \frac{64}{49}. \tag{4.4}$$

For any t, let $\overline{\beta}_t = \frac{\sum_{\sigma \in J_t} C_\sigma}{\sharp J_t}$, we have

$$0 \leq \overline{\beta}_t \leq 1. \tag{4.5}$$

Then we can obtain the following conclusion.

Lemma 3. $\lim_{t\to\infty} \overline{\beta}_t$ *exists and positive.*

Proof. *Recall* $e_{t-1} = \frac{\sharp J_{t-1}}{8\sharp J_{t-2}} \cdot \frac{\sharp J_{t-2}}{8\sharp J_{t-3}} \cdots \frac{\sharp J_3}{8\sharp J_2}$. *To deduce the* $\lim_{t\to\infty} \overline{\beta}_t$ *exists, we begin by proving that* $\{\overline{\beta}_t \cdot e_t\}_t$ *is nondecreasing. In fact, for any* $t \geq 3$ *and applying* (4.3), *we have*

$$\begin{aligned}
\overline{\beta}_t \cdot e_t &= \left(\frac{\sum_{\sigma\in J_t} C_\sigma}{\sharp J_t}\right) \cdot \left(\prod_{i=3}^{t} \frac{\alpha_i}{8}\right) \\
&= \left(\frac{\sum_{\sigma\in J_t} C_\sigma}{\sharp J_t}\right) \cdot \left(\frac{\sharp J_t}{8\sharp J_{t-1}} \cdot \frac{\sharp J_{t-1}}{8\sharp J_{t-2}} \cdot \frac{\sharp J_{t-2}}{8\sharp J_{t-3}} \cdots \frac{\sharp J_3}{8\sharp J_2}\right) \\
&= \left(\frac{\sum_{\sigma\in J_t} C_\sigma}{8\sharp J_{t-1}}\right) \cdot \left(\frac{\sharp J_{t-1}}{8\sharp J_{t-2}} \cdot \frac{\sharp J_{t-2}}{8\sharp J_{t-3}} \cdots \frac{\sharp J_3}{8\sharp J_2}\right) \\
&= \frac{\sum_{\sigma\in J_t} C_\sigma}{8\sharp J_{t-1}} \cdot e_{t-1} \geq \frac{8\sum_{\sigma\in J_t} C_\sigma}{8\sharp J_{t-1}} \cdot e_{t-1} \\
&= \overline{\beta}_{t-1} \cdot e_{t-1}.
\end{aligned}$$

It follows that $\{\overline{\beta}_t \cdot e_t\}_t$ *is nondecreasing. What is left is to show that* $\{\overline{\beta}_t \cdot e_t\}_t$ *has the upper bound, which follows from* (4.4) *and* (4.5). *We can see that the limit of* $\{\overline{\beta}_t \cdot e_t\}_t$ *exists. By* (4.4), *the limit* $\lim_{t\to\infty}\overline{\beta}_t$ *exists and*

$$\lim_{t\to\infty} \overline{\beta}_t = \frac{\lim\limits_{t\to\infty} (\overline{\beta}_t \cdot e_t)}{\lim\limits_{t\to\infty} e_t} > 0.$$

□

By (2.1) and (4.1), we have

$$\lim_{t\to\infty} \frac{\sharp V_t - \sharp J_t}{\sharp V_t} = 0. \tag{4.6}$$

Theorem 2. *The limit of* $\{\overline{C}_t\}$ *exists and*

$$\overline{C}_t \geq 0.3964 \text{ for any } t \geq 2 \text{ which implies } \lim_{t\to\infty} \overline{C}_t \geq 0.3964. \tag{4.7}$$

Proof. Let us first prove that

$$\lim_{t\to\infty} \frac{\overline{\beta}_t}{\overline{C}_t} = 1. \tag{4.8}$$

Recall the definition of the average clustering coefficient and by Lemma 2, we have

$$\sum_{\sigma\in V_t} C_\sigma = \overline{C}_t \times \sharp V_t \geq (0.3691)\sharp V_t.$$

Furthermore, we obtain that

$$\sum_{\sigma\in V_t\setminus J_t} C_\sigma \leq \sharp\{\sigma : \sigma \in V_t \setminus J_t\} = \sharp V_t - \sharp J_t.$$

By (4.6), we also have

$$0 \le \frac{\sum_{\sigma\in V_t\setminus J_t} C_\sigma}{\sum_{\sigma\in V_t} C_\sigma} \le \frac{\sharp V_t - \sharp J_t}{(0.3691)\sharp V_t} \to 0 \text{ as } t \to \infty.$$

Then we can verify that

$$\begin{aligned}
\lim_{t\to\infty} \frac{\overline{\beta}_t}{\overline{C}_t} &= \lim_{t\to\infty} \frac{(\sum_{\sigma\in J_t} C_\sigma)\sharp V_t}{(\sum_{\sigma\in V_t} C_\sigma)\sharp J_t} \\
&= \lim_{t\to\infty} \frac{(\sum_{\sigma\in V_t} C_\sigma - \sum_{\sigma\in V_t\setminus J_t} C_\sigma)\sharp V_t}{(\sum_{\sigma\in V_t} C_\sigma)\sharp J_t} \\
&= \left(1 - \lim_{t\to\infty} \frac{\sum_{\sigma\in V_t\setminus J_t} C_\sigma}{\sum_{\sigma\in V_t} C_\sigma}\right) \cdot \left(\lim_{t\to\infty} \frac{\sharp V_t}{\sharp J_t}\right) \\
&= 1.
\end{aligned}$$

Using Lemma 3 and (4.8), we see that

$$\lim_{t\to\infty} \overline{C}_t = \lim_{t\to\infty} \left(\frac{\overline{C}_t}{\overline{\beta}_t} \cdot \overline{\beta}_t\right) = \lim_{t\to\infty} \overline{\beta}_t.$$

Therefore, $\lim_{t\to\infty} \overline{C}_t$ exists.

By calculation, we have

$$\sharp J_2 = 56 \text{ and } \overline{\beta}_2 = \frac{29}{56}.$$

For any $t \ge 2$, using $\sharp V_t \le \frac{8^{t+1}}{7}$, we have

$$\overline{C}_t \ge \frac{\sum_{\sigma\in J_t} C_\sigma}{\sharp V_t} \ge \frac{8^{t-2}\sum_{\sigma\in J_2} C_\sigma}{\frac{8^{t+1}}{7}} = \frac{8^{t-2}\overline{\beta}_2 \sharp J_2}{\frac{8^{t+1}}{7}} > 0.3964.$$

Hence (4.7) holds. □

5. Average Path Length

For a word $\sigma \ne \emptyset$, suppose $\sigma = i_1 i_2 \cdots i_{k-1} i_k$ with $|\sigma| = k$. Assume that the last letter $i_k \in \{1,2,3,4\}$, then there is a unique shortest subword $f(\sigma)$ with the properties that

$$f(\sigma) \prec \sigma \text{ and } f(\sigma) \sim \sigma.$$

Otherwise, when the last letter $i_k \in \{5,6,7,8\}$, then we let $f(\sigma) = i_1 i_2 \cdots i_{k-1} i'_k$ with $i'_k = i_k - 4$, in this case $\sigma \sim \sigma' \sim \sigma^-$, where σ^- is the father of σ (see Fig. 2).

Example 1. For $\sigma = 12863 = (12)(8)(6)(3)$, we have

$$\begin{gathered}
f(\sigma) = (12)(8)(6), \quad f^2(\sigma) = (12)(8)(2), \quad f^3(\sigma) = (12)(8), \\
f^4(\sigma) = (12)(4), \quad f^5(\sigma) = (12), \quad f^6(\sigma) = \emptyset.
\end{gathered}$$

For any $\sigma \neq \emptyset$, iterating f repeatedly, we can find a shortest path from σ to $\emptyset$ satisfying

$$\sigma \sim f(\sigma) \sim f^2(\sigma) \sim \cdots f^n(\sigma) = \emptyset,$$

then we write $\omega(\sigma) = n$.

Fix an integer $t \geq 0$. We denote the shortest distance from σ to τ in G_t by $d(\sigma, \tau)$, hence we have

$$\omega(\sigma) = d(\sigma, \emptyset).$$

Since $|f^2(\sigma)| < |\sigma|$, we have

$$d(\sigma, \emptyset) = \omega(\sigma) \leq 2|\sigma|. \tag{5.1}$$

Let $L(\sigma)$ be the minimal step to move from Q_σ to ∂Q, i.e.

$$L(\sigma) = d(\sigma, \emptyset) - 1 = \omega(\sigma) - 1,$$

where $\sigma \neq \emptyset$. Then we have

$$L(\sigma\tau) \geq L(\sigma) + L(\tau). \tag{5.2}$$

Then at time t, the average path length is defined by

$$\bar{d}_t = \frac{\sum_{\sigma \neq \tau \in V_t} d(\sigma, \tau)}{\sharp V_t(\sharp V_t - 1)/2} = \frac{\sum_{\sigma, \tau \in V_t} d(\sigma, \tau)}{\sharp V_t(\sharp V_t - 1)/2}.$$

The following lemma will be needed.

Lemma 4. *Let*

$$\overline{\alpha}_k = \frac{\sum_{|\sigma|=k} L(\sigma)}{\sharp\{\sigma : |\sigma| = k\}},$$

then for all $k \geq 1$, *we have*

$$\overline{\alpha}_k \geq \frac{k}{2}. \tag{5.3}$$

Proof. Note that $\sum_{|\sigma|=1} L(\sigma) = \sharp\{\sigma \nsim \emptyset : |\sigma| = 1\}$, we have

$$\overline{\alpha}_1 = \frac{\sharp\{\sigma \nsim \emptyset : |\sigma| = 1\}}{\sharp\{\sigma : |\sigma| = 1\}} = \frac{4}{8} = \frac{1}{2}.$$

Then (5.3) is proved for $k = 1$.

By induction, we assume that (5.3) is true for $k - 1$. Then any $|\sigma| = k > 1$, suppose $\sigma = \tau\sigma'$ with $|\tau| = 1$ and $|\sigma'| = k - 1$, we have

$$\frac{\sum_{|\sigma|=k} L(\sigma)}{\sharp\{\sigma : |\sigma| = k\}} = \frac{\sum_{|\tau|=1} \sum_{|\sigma'|=k-1} L(\tau\sigma')}{\sum_{|\tau|=1} \sharp\{\sigma' : |\sigma'| = k - 1\}}.$$

Since $L(\tau)$ is either 0 or 1, we discuss the following two cases:
(1) For $\tau \nsim \emptyset$, by (5.2) we have $L(\sigma) \geq L(\sigma') + L(\tau) = L(\sigma') + 1$, then

$$\frac{\sum_{|\sigma'|=k-1} L(\tau\sigma')}{\sharp\{\sigma' : |\sigma'| = k-1\}} \geq \frac{\sum_{|\sigma'|=k-1}(L(\sigma')+1)}{\sharp\{\sigma' : |\sigma'| = k-1\}} = \overline{\alpha}_{k-1} + 1. \tag{5.4}$$

(2) For $\tau \sim \emptyset$, by (5.2) we have $L(\sigma) \geq L(\sigma')$, then

$$\frac{\sum_{|\sigma'|=k-1} L(\tau\sigma')}{\sharp\{\sigma' : |\sigma'| = k-1\}} \geq \frac{\sum_{|\sigma'|=k-1} L(\sigma')}{\sharp\{\sigma' : |\sigma'| = k-1\}} = \overline{\alpha}_{k-1}. \tag{5.5}$$

It follows from (5.4) and (5.5) and the inductive assumption $\overline{\alpha}_k \geq \frac{k-1}{2}$ that

$$\overline{\alpha}_k \geq \frac{4}{8}(\overline{\alpha}_{k-1} + 1) + \frac{4}{8}\overline{\alpha}_{k-1} = \overline{\alpha}_{k-1} + \frac{1}{2} \geq \frac{1}{2}(k-1) + \frac{1}{2} = \frac{k}{2}.$$

□

Suppose $\sigma = i_1\sigma'$ and $\tau = j_1\tau'$ with $i_1 \neq j_1$, using the Jordan curve theorem, we have

$$d(\sigma, \tau) \geq L(\sigma') + L(\tau'). \tag{5.6}$$

Let $W_t = \{\sigma : |\sigma| = t\}$ with $\sharp W_t = 8^t$, then we have

$$\sum_{\{\sigma,\tau\} \subset V_t} d(\sigma, \tau) \geq \sum_{\{\sigma,\tau\} \subset W_t} d(\sigma, \tau). \tag{5.7}$$

It can be easily checked that

$$\frac{\sharp V_t(\sharp V_t - 1)}{2} \leq 4 \cdot \frac{\sharp W_t(\sharp W_t - 1)}{2}. \tag{5.8}$$

The following theorem can be derived by the discussion above.

Theorem 3. *For all $t \geq 1$, we have*

$$\frac{7}{32}(t-1) \leq \overline{d}_t \leq 4t.$$

Proof. For any $\sigma, \tau \in V_t$, by (5.1), we have

$$d(\sigma, \tau) \leq d(\sigma, \emptyset) + d(\emptyset, \tau) \leq 4t.$$

Suppose $\sigma = i_1\sigma'$ and $\tau = j_1\tau'$ with $i_1 \neq j_1$, then by (5.6) and (5.3), we have

$$\begin{aligned}
\sum_{\{\sigma,\tau\} \subset W_t} d(\sigma, \tau) &\geq \sum_{i_1 \neq j_1, \{i_1, j_1\} \subset \{1,\ldots,8\}} \; \sum_{i_1 \prec \sigma, j_1 \prec \tau, \sigma, \tau \in W_t} d(\sigma, \tau) \\
&\geq \binom{8}{2} \times \left(2 \times 8^{2(t-1)} \times \frac{\sum_{|\sigma'|=t-1} L(\sigma')}{\sharp W_{t-1}}\right) \\
&= 56 \times \overline{\alpha}_{t-1} \times 8^{2(t-1)} \\
&= \frac{7}{16} \times 8^{2t}(t-1).
\end{aligned}$$

By (5.7) and (5.8), we obtain

$$\begin{aligned}\frac{\sum_{\{\sigma,\tau\}\subset V_t} d(\sigma,\tau)}{\sharp V_t(\sharp V_t-1)/2} &\geq \frac{1}{4}\,\frac{\sum_{\{\sigma,\tau\}\subset W_t} d(\sigma,\tau)}{\sharp W_t(\sharp W_t-1)/2}\\ &\geq \frac{1}{4}\times\frac{7}{16}\times 8^{2t}\times(t-1)\times\frac{2}{8^t(8^t-1)}\\ &\geq \frac{7}{32}(t-1).\end{aligned}$$

Hence the proof is complete. □

6. Conclusion

In this paper, we construct evolving networks $\{G_t\}_t$ produced from a planar carpet. We take all basic solid squares during the iterative process as node set V_t of network G_t and define the neighbor relationship between any two nodes.

Actually, the cumulative degree distribution $P_{\text{cum}}(u)$ of G_t obeys power law with exponent -3 (Theorem 1), which means our networks have the scale-free effect.

And then, for the small-world effect, we have the following result. The average clustering coefficient $\overline{C}_t$ of G_t is discussed and we can obtain that the limit of $\{\overline{C}_t\}_{t\geq 1}$ exists and

$$\overline{C}_t \geq 0.3964 \text{ for all } t \geq 2 \text{ (Theorem 2)},$$

i.e. 0.3964 is an independent lower bound of $\overline{C}_t$. On the other hand, we obtain that the average path length $\bar{d}_t$ of G_t is proportional to the time t (Theorem 2), however the size of network G_t is $\sharp V_t = \frac{8^{t+1}-1}{7}$, which means

$$\bar{d}_t(\propto t) \ll \sharp V_t(\propto 8^t) \text{ when } t \text{ is large}.$$

In the next work, we will apply our networks to realistic networks.

Acknowledgments

The work is supported by the National Natural Science Foundation of China (Nos. 12171259, 11831007 and 11771226) and K. C. Wong Magna Fund in Ningbo University.

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