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Final Exam Solution – Economics 711

1. (15 points)
   (i) g is a map from type profiles $\theta \in \Theta$ to social alternatives $\chi \in \mathcal{X}$.
   (ii) g is incentive compatible if in the direct mechanism $\mathcal{M}^d = \{\{\Theta_i\}_{i \in \mathcal{A}}, g\}$ with outcome function g, truth telling is a Nash equilibrium. (Remember that outcome functions of direct mechanisms are the same sort of mathematical objects as social choice functions, although we think of the former as being applied to profiles of type announcements, and the latter to profiles of types.)

2. (15 points)
   A VCG mechanism is the direct mechanism whose outcome function $g(\cdot) = (x^*(\cdot), t^V(\cdot))$ consists of an ex post efficient allocation function $x^*(\cdot)$ and the transfer function

$$(\dagger) \qquad t_i^V(\theta) = \sum_{j \neq i} u_j(x^{-i}(\theta_{-i}), \theta_j) - \sum_{j \neq i} u_j(x^*(\theta), \theta_j),$$

where $x^{-i} \colon \Theta_{-i} \to X$ is an "allocation function" that is ex post efficient if agent i is ignored. (Note that in the present problem, $x^*(\theta)$ and $x^{-i}(\theta)$ are uniquely defined when all θ_i^k are distinct.)

In the efficient allocation $x^*(\theta)$, the K units are assigned to the agents with the K highest θ_j^k values. Thus if agent i has $\ell \in \{0, \ldots, K\}$ of these values, he gets ℓ goods. Allocation $x^{-i}(\theta)$ is defined similarly, except that agent i is excluded from consideration.

To determine the VCG transfers, think of the components $\hat{\theta}_j^k$ of the profile $\hat{\theta}$ of type announcements as bids. Then $x^*(\hat{\theta})$ assigns the the goods to the agents with the K highest bids, and $x^{-i}(\hat{\theta})$ assigns the goods to the agents with the K highest bids when i's bids are ignored. Thus under $x^{-i}(\hat{\theta})$, the $\ell \in \{0, \ldots, K\}$ goods that were assigned to i under $x^*(\hat{\theta})$ are assigned to the agents who made the ℓ highest losing bids under $x^*(\hat{\theta})$, ignoring i's own losing bids. By (†), agent i's transfer $t_i^V(\hat{\theta})$ is the sum of these ℓ losing bids of other agents.

3. (20 points)
   The manager's problem is

$$\max_{w_1, \ldots, w_m} \sum_{j=1}^{m} p_j v(x_j - w_j) \ \text{ subject to } \sum_{j=1}^{m} p_j u(w_j) - c \geq 0.$$

Clearly the constraint must hold with equality. The Lagrangian is

$$\mathcal{L}(w,\mu) = \sum_{j=1}^{m} p_j \left(v(x_j - w_j) + \mu(u(w_j) - c) \right),$$

leading to the first order condition

$$(\dagger) \qquad \frac{v'(x_j - w_j)}{u'(w_j)} = \mu \quad \text{for all } j.$$

Now suppose that w_k^* satisfies (†), where $k < m$. Since $x_{k+1} > x_k$, it follows that $v'(x_{k+1} - w_k^*) < v'(x_k - w_k^*)$ (since $v'(\cdot)$ is decreasing), and hence that

$$(\ddagger) \qquad \frac{v'(x_{k+1} - w_k^*)}{u'(w_k^*)} < \mu.$$

Thus if we want to obtain

$$\frac{v'(x_{k+1} - w_{k+1}^*)}{u'(w_{k+1}^*)} = \mu,$$

we must choose $w_{k+1}^* > w_k^*$, since this increases the numerator and decreases the denominator relative to (‡) (since $v'(\cdot)$ and $u'(\cdot)$ are decreasing). (The assumption that $w_1^* > 0$ ensures that all wages in the optimal schedule are positive.)

4. (25 points)

   (i) The principal's problem is to choose a pair of contracts, $((q_\ell, p_\ell), (q_h, p_h))$, that solves

$$\max_{q_\ell, q_h \geq 0;\ p_\ell, p_h} (1 - \pi_h)(p_\ell - c(q_\ell)) + \pi_h(p_h - c(q_h)) \quad \text{subject to}$$

$$\begin{array}{ll} (\mathrm{IC}_\ell) & \theta_\ell q_\ell - p_\ell \geq \theta_\ell q_h - p_h \\ (\mathrm{IC}_h) & \theta_h q_h - p_h \geq \theta_h q_\ell - p_\ell \\ (\mathrm{IR}_\ell) & \theta_\ell q_\ell - p_\ell \geq 0 \\ (\mathrm{IR}_h) & \theta_h q_h - p_h \geq 0 \end{array}$$

   where $\theta_l = 20$, $\theta_h = 50$, and $\pi_h = \frac{1}{3}$. We know that constraints (IC_ℓ) and (IR_ℓ) will bind. This implies that

$$\begin{aligned} &p_\ell = \theta_\ell q_\ell \quad \text{and} \\ &p_h = \theta_h(q_h - q_\ell) + p_\ell = \theta_h(q_h - q_\ell) + \theta_\ell q_\ell, \end{aligned}$$

   reducing the problem to the separable concave problem

$$\max_{q_\ell, q_h \geq 0} (1 - \pi_h)(\theta_\ell q_\ell - c(q_\ell)) + \pi_h(\theta_\ell q_\ell + \theta_h(q_h - q_\ell) - c(q_h))$$

$$\text{subject to } \ q_h \geq q_\ell.$$

The first order condition for q_h is $c'(q_h^*) = \theta_h$, so $q_h = 25$. The first order condition for q_ℓ is

$$c'(q_\ell^*) = \theta_\ell - \frac{\pi_h}{1-\pi_h}(\theta_h - \theta_\ell),$$

so $2q_\ell^* = 20 - \frac{1}{2} \cdot 30 = 5$, and hence $q_\ell^* = 2.5$. Using the formula for the prices above yields $p_\ell^* = \theta_\ell q_\ell^* = 20 \cdot 2.5 = 50$ and $p_h^* = \theta_h(q_h^* - q_\ell^*) + p_\ell^* = 50 \cdot 22.5 + 50 = 1175$.

(ii) Most of the analysis is as in part (i), except that the first order condition for q_ℓ is now $2q_\ell^* = 20 - 1 \cdot 30 = -10$. Thus we obtain the corner solution $q_\ell^* = 0$, and so $p_\ell^* = \theta_\ell q_\ell^* = 0$ and $p_h^* = \theta_h(q_h^* - q_\ell^*) + p_\ell^* = 50 \cdot 25 = 1250$.

(iii) When the principal specifies a nonnull contract for type θ_ℓ, he must pay type θ_h an information rent to prevent her from buying the contract intended for type θ_ℓ. If the probability that the agent is type θ_ℓ is low enough, the principal is better off offering a null contract to type θ_ℓ (and thus earning no profits from this type) in order to avoid paying information rent to type θ_h. This is precisely what happened here when π_h was increased from $\frac{1}{3}$ to $\frac{1}{2}$.

5. (25 points)

(i) The utility of a type θ_P consumer from choosing policy (b, p) is

$$u_P(b,p) = (1-\theta_P)\, v(\underbrace{w-p}_{x_0}) + \theta_P\, v(\underbrace{w-p-\ell+b}_{x_1}).$$

Consider an indifference curve $u_P(p, b) = \bar{u}_P$ of type θ_P in (b, p) space. Expressing p as a function of b and differentiating yields

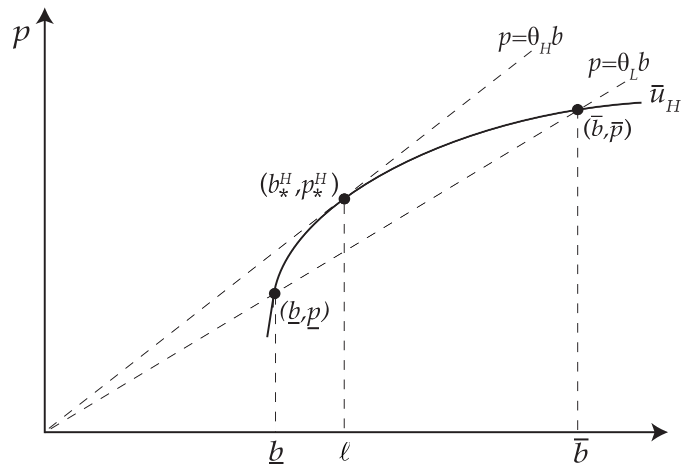
$$\frac{\partial u_P}{\partial b}(b,p) + \frac{\partial u_P}{\partial p}(b,p)\frac{\mathrm{d}p_P}{\mathrm{d}b}(b) = 0.$$

Thus the slope of type P's indifference curve is

$$\frac{\mathrm{d}p_P}{\mathrm{d}b}(b) = -\frac{\frac{\partial u_P}{\partial b}(b,p)}{\frac{\partial u_P}{\partial p}(b,p)} = \frac{\theta_P v'(x_1)}{(1-\theta_P)v'(x_0) + \theta_P v'(x_1)} = \frac{1}{1 + \frac{(1-\theta_P)v'(x_0)}{\theta_P v'(x_1)}} > 0.$$

And thus $\frac{\mathrm{d}p_P}{\mathrm{d}b}(b)$ is increasing in θ_P. In other words, type θ_H's indifference curves cross type θ_L's from below.

(ii) In the figure below, the θ_H's contract is the point on the upper dashed line (since it is actuarially fair) with horizontal component ℓ (since θ_H is fully insured). Type θ_L's contract is somewhere on the lower dashed line (since it is actuarially fair).



(iii) If there is a separating equilibrium, it must be that $(b^L_*, p^L_*) = (\underline{b}, \theta_L \underline{b})$ where $u_H(\underline{b}, \theta_L \underline{b}) = u_H(\ell, \theta_H \ell)$ and $\underline{b} < \ell$, as in the figure above.

We can rule out all other possibilities as follows.

Suppose there is a $\bar{b} > \ell$ such that $u_H(\bar{b}, \theta_L \bar{b}) = u_H(\ell, \theta_H \ell)$. Then applying the single-crossing property at $(\ell, \theta_H \ell)$ shows that θ_L's indifference curve through this contract crosses $p = \theta_L b$ from above at a point southwest of $(\bar{b}, \theta_L \bar{b})$. Thus θ_L strictly prefers $(\ell, \theta_H \ell)$ to any $(b^L_*, \theta_L b^L_*)$ with $b^L_* \geq \bar{b}$. (Alternatively, one can argue that if i offered (b^L_*, p^L_*) with $b^L_* \geq \bar{b}$ and θ_L chose this contract, then by single crossing at (b^L_*, p^L_*), j could cream-skim by slightly reducing b and p.)

If $b^L_* > \underline{b}$ (and $b^L_* < \bar{b}$ if $\bar{b}$ exists), then θ_H strictly prefers (b^L_*, p^L_*) to $(b^H_*, p^H_*) = (\ell, \theta_H \ell)$.

Finally, If $b^L_* < \underline{b}$ and i offers (b^L_*, p^L_*), then since θ_L's indifference curve through (b^L_*, p^L_*) crosses the actuarially fair line from below, j can cream-skim by slightly increasing b and p. (The claim about the indifference curve is true because the function $p_L(\cdot)$ describing θ_L's indifference curve is concave and has slope θ_L at $b = \ell$, and thus slope greater than θ_L at $b = b^L_*$.)