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A .699-approximation algorithm for Max-Bisection

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Abstract. We present a .699-approximation algorithm for Max-Bisection, i.e., partitioning the nodes of a weighted graph into two blocks of equal cardinality so as to maximize the weights of crossing edges. This is an improved result from the .651-approximation algorithm of Frieze and Jerrum and the semidefinite programming relaxation of Goemans and Williamson.

Key words. graph bisection – polynomial-time approximation algorithm – semidefinite programming

1. Introduction

Consider the Max-Bisection problem on an undirected graph $G = (V, E)$ with non-negative weights w_{ij} for each edge in E (and $w_{ij} = 0$ if $(i, j) \notin E$), which is the problem of partitioning the even number of nodes in V into two sets S and $V \setminus S$ of equal cardinality so that

$$w(S) := \sum_{i\in S,\ j\in V\setminus S} w_{ij}$$

is maximized. This problem can be formulated by assigning each node a binary variable x_j:

$$\text{(MB)}\qquad \begin{aligned} w^* := \text{Maximize } & \frac{1}{4}\sum_{i,j} w_{ij}(1 - x_i x_j) \\ \text{subject to } & \sum_{j=1}^{n} x_j = 0 \quad \text{or} \quad e^T x = 0 \\ & x_j^2 = 1,\ \ j = 1, \ldots, n, \end{aligned}$$

where $e \in \Re^n$ (n even) is the column vector of all ones, superscript ^{T} is the transpose operator. Note that x_j takes either 1 or -1, so that we can choose either $S = \{j : x_j = 1\}$ or $S = \{j : x_j = -1\}$. The constraint $e^T x = 0$ ensures that $|S| = |V \setminus S|$.

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A problem of this type arises from many network planning, circuit design, and scheduling applications. In particular, the popular and widely used Min-Bisection problem sometimes can be solved by finding the Max-Bisection over the complementary graph of G.

Recently, Frieze and Jerrum [2] generalized Goemans and Williamson's Max-Cut algorithm [3] to provide a randomized .651-approximation algorithm for Max-Bisection, that is, they designed a polynomial-time algorithm to provide a random bisection S such that the expectation $\mathrm{Ex}[w(S)] \geq .651 \cdot w^*$. Their algorithm is based on using the semidefinite programming relaxation of the above binary quadratic program:

$$\text{(SDP)} \qquad \begin{aligned} w^{SD} := \text{Maximize } & \frac{1}{4}\sum_{i,j} w_{ij}(1 - X_{ij}) \\ \text{subject to } & ee^T \bullet X = 0, \\ & X_{jj} = 1, \quad j = 1, \ldots, n, \; X \succeq 0. \end{aligned} \tag{1}$$

Here, the unknown $X \in \Re^{n\times n}$ is a symmetric matrix. Furthermore, $\bullet$ is the matrix inner product $Q \bullet X = \mathrm{trace}(QX)$, and $X \succeq Z$ means that $X - Z$ is positive semidefinite. Obviously, (SDP) is a relaxation of (MB), since for any feasible solution x of (MB), $X = xx^T$ is feasible for (SDP); so that $w^{SD} \geq w^*$.

What complicates matters in Max-Bisection, comparing to Max-Cut, is that two objectives are actually sought – the objective value of $w(S)$ and the size of S. Therefore, in any (randomized) rounding method at the beginning, we need to balance the (expected) quality of $w(S)$ and the (expected) size of S. We want high $w(S)$; but, at the same time, zero or small difference between $|S|$ and $n/2$, since otherwise we have to either add or subtract nodes from S, resulting in a deterioration of $w(S)$ at the end. Our method is built upon a careful balance of the two, plus an improved proof technique.

2. The .651-method of Frieze and Jerrum

We first review the Frieze and Jerrum method, and then proceed with our improved method.

Let $\bar{X}$ be an optimal solution of Problem (SDP). Since $\bar{X}$ is positive semidefinite, the randomization method of Goemans and Williamson [3] essentially generates a random vector u from a multivariate normal distribution with 0 mean and covariance matrix $\bar{X}$, that is,

$$u \in N(0, \bar{X}),$$

then assign

$$\hat{x} = \mathrm{sign}(u), \tag{2}$$

where

$$\mathrm{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Finally, they select the block $S = \{i : \hat{x}_i = 1\}$ or $S = \{i : \hat{x}_i = -1\}$.

Then, one can prove (see Goemans and Williamson [3] and Frieze and Jerrum [2]; also see Bertsimas and Ye [1]):

$$\mathrm{Ex}[\hat{x}_i \hat{x}_j] = \frac{2}{\pi} \arcsin(\bar{X}_{ij}), \quad i, j = 1, 2, \ldots, n, \tag{3}$$

and

$$1 - \frac{2}{\pi} \arcsin(\bar{X}_{ij}) \geq \alpha(1)(1 - \bar{X}_{ij}),$$

where

$$\alpha(1) := \min_{-1 \leq y < 1} \frac{1 - \frac{2}{\pi} \arcsin(y)}{1 - y} \geq .878567$$

which is a special case of Definition (15) in Sect. 5.

Thus,

$$\begin{aligned}
\mathrm{Ex}[w(S)] &= \frac{1}{4} \sum_{i,j} w_{ij} \left(1 - \frac{2}{\pi} \arcsin(\bar{X}_{ij})\right) \\
&\geq \frac{1}{4} \sum_{i,j} w_{ij} \cdot \alpha(1)(1 - \bar{X}_{ij}) \\
&= \alpha(1) \cdot w^{SD} \\
&\geq \alpha(1) \cdot w^*;
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
\mathrm{Ex}\left[\frac{n^2}{4} - \frac{(e^T \hat{x})^2}{4}\right] &= \frac{1}{4} \sum_{i,j} \left(1 - \frac{2}{\pi} \arcsin(\bar{X}_{ij})\right) \\
&\geq \frac{1}{4} \sum_{i,j} \alpha(1)(1 - \bar{X}_{ij}) \\
&= \alpha(1) \cdot \frac{n^2}{4},
\end{aligned} \tag{5}$$

since

$$\sum_{i,j} \bar{X}_{ij} = ee^T \bullet \bar{X} = 0.$$

However, $\hat{x}$ may not satisfy $e^T \hat{x} = 0$, i.e., S may not be a bisection. Then, using a greedy method, Frieze and Jerrum have adjusted S by swapping nodes from the majority block into the minority block until they are equally sized. Note that inequality (5) assures that not too many nodes need to be swapped. Also, in selecting swapping nodes, Frieze and Jerrum make sure that the least weighted node gets swapped first. More precisely, (w.l.o.g) let $|S| \geq n/2$; and for each $i \in S$, let $\zeta(i) = \sum_{j \notin S} w_{ij}$

and $S = \{i_1, i_2, \ldots, i_{|S|}\}$, where $\zeta(i_1) \geq \zeta(i_2) \geq \ldots \geq \zeta(i_{|S|})$. Then, assign $\tilde{S} = \{i_1, i_2, \ldots, i_{n/2}\}$. Clearly, the construction of bisection $\tilde{S}$ guarantees that

$$w(\tilde{S}) \geq \frac{n \cdot w(S)}{2|S|}, \quad n/2 \leq |S| \leq n. \tag{6}$$

In order to analyze the quality of bisection $\tilde{S}$, they define two random variables:

$$w := w(S) = \frac{1}{4}\sum_{i,j} w_{ij}(1 - \hat{x}_i\hat{x}_j)$$

and

$$m := |S|(n - |S|) = \frac{n^2}{4} - \frac{(e^T\hat{x})^2}{4} = \frac{1}{4}\sum_{i,j}(1 - \hat{x}_i\hat{x}_j)$$

(since $(e^T\hat{x})^2 = (2|S| - n)^2$). Then, from (4) and (5),

$$\mathrm{Ex}[w] \geq \alpha(1) \cdot w^* \quad \text{and} \quad \mathrm{Ex}[m] \geq \alpha(1) \cdot m^*, \quad \text{where} \quad m^* = \frac{n^2}{4}.$$

Thus, if a new random variable

$$z = \frac{w}{w^*} + \frac{m}{m^*}, \tag{7}$$

then

$$\mathrm{Ex}[z] \geq 2\alpha(1), \quad \text{and} \quad z \leq 3$$

since $w/w^* \leq 2|S|/n \leq 2$ and $m/m^* \leq 1$.

For simplicity, Frieze and Jerrum's analysis can be described as follows. They repeatedly generate samples $u \in N(0, \bar{X})$, create $\hat{x}$ or S using the randomization method (2), construct $\tilde{S}$ using the swapping procedure, and record the largest value of $w(\tilde{S})$ among these samples. Since $\mathrm{Ex}[z] \geq 2\alpha(1)$ and $z \leq 3$, they are almost sure to have one z to meet its expectation before too long, i.e., to have

$$z \geq 2\alpha(1). \tag{8}$$

Moreover, when (8) holds and suppose

$$w(S) = \lambda w^*,$$

which from (7) and (8) implies that

$$\frac{m}{m^*} \geq 2\alpha(1) - \lambda.$$

Suppose that $|S| = \delta n$; then the above inequality implies

$$\lambda \geq 2\alpha(1) - 4\delta(1 - \delta). \tag{9}$$

Applying (6) and (9), one can see that

$$\begin{aligned} w(\tilde{S}) &\geq \frac{w(S)}{2\delta} \\ &= \frac{\lambda w^*}{2\delta} \\ &\geq \frac{(2\alpha(1) - 4\delta(1-\delta))w^*}{2\delta} \\ &\geq 2(\sqrt{2\alpha(1)} - 1)w^*. \end{aligned}$$

The last inequality follows from simple calculus that $\delta = \sqrt{2\alpha(1)}/2$ yields the minimal value for $(2\alpha(1) - 4\delta(1-\delta))/(2\delta)$ when $0 \leq \delta \leq 1$. Note that for $\alpha(1) \geq .878567$, $2(\sqrt{2\alpha(1)} - 1) > .651$. Since the largest $w(\tilde{S})$ in the process is at least as good as the one who meets $z \geq 2\alpha(1)$, they proceed to prove a .651-approximation algorithm for Max-Bisection.

3. A modified rounding and improved analyses

Our improved rounding method is to use a convex combination of $\bar{X}$ and a positive semidefinite matrix P as the covariance matrix to generate u and $\hat{x}$, i.e,

$$\begin{aligned} u &\in N(0, \theta\bar{X} + (1-\theta)P), \\ \hat{x} &= \text{sign}(u), \\ S = \{i : \hat{x}_i = 1\} &\quad \text{or} \quad S = \{i : \hat{x}_i = -1\} \end{aligned}$$

such that $|S| \geq n/2$, and then $\tilde{S}$ from the Frieze and Jerrum swapping procedure.

One choice of P is

$$P = \frac{n}{n-1}\Big(I - \frac{1}{n}ee^T\Big),$$

where I is the n-dimensional identity matrix. Note that $I - \frac{1}{n}ee^T$ is the projection matrix onto the null space of $e^T x = 0$, and P is a feasible solution for (SDP). The other choice is $P = I$, which was also proposed in Nesterov [4], and by Zwick [5] for approximating the Max-Cut problem when the graph is sparse.

Again, the overall performance of the rounding method is determined by two factors: the expected quality of $w(S)$ and how much S need to be downsized. The convex combination parameter θ used in $\theta\bar{X} + (1-\theta)P$ provides a balance between these two factors. Typically, the more use of $\bar{X}$ in the combination results in higher expected $w(S)$ but larger expected difference between $|S|$ and $n/2$; and the more use of P results in less expected $w(S)$ and more accurate $|S|$.

More precisely, our hope is that we could provide two new inequalities in replacing (4) and (5):

$$\text{Ex}[w(S)] = \frac{1}{4}\sum_{i,j} w_{ij}(1 - \text{Ex}[\hat{x}_i\hat{x}_j]) \geq \alpha \cdot w^* \tag{10}$$

and

$$\mathrm{Ex}[|S|(n-|S|)] = \mathrm{Ex}\left[\frac{n^2}{4} - \frac{(e^T\hat{x})^2}{4}\right] = \frac{1}{4}\sum_{i,j}(1-\mathrm{Ex}[\hat{x}_i\hat{x}_j]) \geq \beta\cdot\frac{n^2}{4}, \tag{11}$$

such that α would be slightly less than $\alpha(1)$ but β would be significantly greater than $\alpha(1)$. Thus, we could give a better overall bound than .651 for Max-Bisection.

Before proceed, we prove a technical result. Again, let two random variables

$$w := w(S) = \frac{1}{4}\sum_{i,j} w_{ij}(1-\hat{x}_i\hat{x}_j) \quad \text{and} \quad m := |S|(n-|S|) = \frac{1}{4}\sum_{i,j}(1-\hat{x}_i\hat{x}_j).$$

Then, from (10) and (11),

$$\mathrm{Ex}[w] \geq \alpha\cdot w^* \quad \text{and} \quad \mathrm{Ex}[m] \geq \beta\cdot m^*, \quad \text{where} \quad m^* = \frac{n^2}{4}.$$

Furthermore, for a given parameter $\gamma \geq 0$, let new random variable

$$z(\gamma) = \frac{w}{w^*} + \gamma\frac{m}{m^*}. \tag{12}$$

Then, we have

$$\mathrm{Ex}[z(\gamma)] \geq \alpha + \gamma\beta \quad \text{and} \quad z \leq 2+\gamma.$$

Note that Frieze and Jerrum used $\gamma = 1$ in their analyses.

Now we prove the following lemma:

Lemma 1. *Assume (10) and (11) hold. Then, for any given $\gamma \geq \alpha/(4-\beta)$, if random variable $z(\gamma)$ meets its expectation, i.e., $z(\gamma) \geq \alpha+\gamma\beta$, then*

$$w(\tilde{S}) \geq 2\left(\sqrt{\gamma(\alpha+\gamma\beta)} - \gamma\right)\cdot w^*.$$

In particular, if

$$\gamma = \frac{\alpha}{2\beta}\left(\frac{1}{\sqrt{1-\beta}} - 1\right)$$

(which is greater than $\alpha/(4-\beta)$ since $\beta > 0$), then

$$w(\tilde{S}) \geq \frac{\alpha}{1+\sqrt{1-\beta}}\cdot w^*.$$

Proof. Suppose

$$w(S) = \lambda w^* \quad \text{and} \quad |S| = \delta n,$$

which from (12) and $z(\gamma) \geq \alpha+\gamma\beta$ implies that

$$\lambda \geq \alpha + \gamma\beta - 4\gamma\delta(1-\delta).$$

Applying (6) we see that

$$\begin{aligned} w(\tilde{S}) &\geq \frac{w(S)}{2\delta} \\ &= \frac{\lambda w^*}{2\delta} \\ &\geq \frac{\alpha + \gamma\beta - 4\gamma\delta(1-\delta)}{2\delta} \cdot w^* \\ &\geq 2(\sqrt{\gamma(\alpha+\gamma\beta)} - \gamma) \cdot w^*. \end{aligned}$$

The last inequality follows from simple calculus that

$$\delta = \frac{\sqrt{\alpha + \gamma\beta}}{2\sqrt{\gamma}}$$

yields the minimal value for $(\alpha + \gamma\beta - 4\gamma\delta(1-\delta))/(2\delta)$ in the interval $[0, 1]$, if $\gamma \geq \alpha/(4-\beta)$.

In particular, substitute

$$\gamma = \frac{\alpha}{2\beta}\left(\frac{1}{\sqrt{1-\beta}} - 1\right)$$

into the first inequality, we have the second desired result in the lemma.

□

The motivation to select $\gamma = \frac{\alpha}{2\beta}(\frac{1}{\sqrt{1-\beta}} - 1)$ is that it yields the maximal value for $2(\sqrt{\gamma(\alpha+\gamma\beta)} - \gamma)$. In fact, when both $\alpha = \beta = \alpha(1) \geq .878567$ as in the case of Frieze and Jerrum,

$$\frac{\alpha}{1 + \sqrt{1-\beta}} > 0.6515,$$

which is just slightly better than $2(\sqrt{2\alpha(1)} - 1) \geq 0.6511$ proved by Frieze and Jerrum. So their choice $\gamma = 1$ is almost "optimal". We emphasize that γ is only used in the analysis of the quality bound, and is not used in the rounding method.

4. A simple .5-approximation

To see the impact of θ in the new rounding method, we analyze the other extreme case where $\theta = 0$ and

$$P = \frac{n}{n-1}\left(I - \frac{1}{n}ee^T\right).$$

That is, we generate $u \in N(0, P)$, then $\hat{x}$ and S. Now, we have

$$\mathrm{Ex}[w(S)] = \mathrm{Ex}\left[\frac{1}{4}\sum_{i,j} w_{ij}(1 - \hat{x}_i\hat{x}_j)\right] = \frac{1}{4}\left(1 + \frac{2}{\pi}\arcsin\left(\frac{1}{n-1}\right)\right)\sum_{i\neq j} w_{ij} \geq .5 \cdot w^* \tag{13}$$

and

$$\mathrm{Ex}\left[\frac{n^2}{4} - \frac{(e^T\hat{x})^2}{4}\right] \geq \frac{n^2}{4} - \frac{n}{4} + \frac{n(n-1)}{4}\frac{2}{\pi}\arcsin\left(\frac{1}{n-1}\right) \geq \left(1-\frac{1}{n}\right)\cdot\frac{n^2}{4}, \quad (14)$$

where from (3) we have used the facts that

$$\mathrm{Ex}[\hat{x}_i\hat{x}_j] = \frac{2}{\pi}\arcsin\left(\frac{-1}{n-1}\right), \quad i \neq j$$

and

$$\frac{1}{2}\sum_{i\neq j} w_{ij} = \sum_{i<j} w_{ij} \geq w^*.$$

In other words, we have in Lemma 1

$$\alpha = .5 \quad \text{and} \quad \beta = 1 - \frac{1}{n}.$$

Comparing (13) and (14) to (4) and (5), here the first inequality on $w(S)$ is worse, .5 vs .878567; but the second inequality is substantially better, $1-\frac{1}{n}$ vs .878567; i.e., $\hat{x}$ here is a bisection with probability almost 1 when n is large. Using Lemma 1, we see that the method is a

$$\frac{\alpha}{1+\sqrt{1-\beta}} > \frac{.5}{1+\sqrt{1/n}}$$

approximation method. The same ratio can be established for $P = I$.

5. A .699-approximation

We now prove our main result. For simplicity, we will use $P = I$ in our rounding method. Therefore, we discuss using the convex combination of $\theta\bar{X} + (1-\theta)I$ as the covariance matrix to generate u, $\hat{x}$, S and $\tilde{S}$ for a given $0 \leq \theta \leq 1$, i.e.,

$$u \in N(0, \theta\bar{X} + (1-\theta)P),$$
$$\hat{x} = \mathrm{sign}(u),$$
$$S = \{i : \hat{x}_i = 1\} \quad \text{or} \quad S = \{i : \hat{x}_i = -1\}$$

such that $|S| \geq n/2$, and then $\tilde{S}$ from the Frieze and Jerrum swapping procedure.

Define

$$\alpha(\theta) := \min_{-1\leq y<1} \frac{1-\frac{2}{\pi}\arcsin(\theta y)}{1-y}; \quad (15)$$

and

$$\beta(\theta) := \left(1-\frac{1}{n}\right)b(\theta) + c(\theta), \quad (16)$$

where

$$b(\theta) = 1 - \frac{2}{\pi}\arcsin(\theta) \quad \text{and} \quad c(\theta) = \min_{-1 \le y < 1} \frac{2}{\pi}\frac{\arcsin(\theta) - \arcsin(\theta y)}{1 - y}.$$

Note that $\alpha(1) = \beta(1) \ge .878567$ as shown in Goemans and Williamson [3]; and $\alpha(0) = .5$ and $\beta(0) = 1 - \frac{1}{n}$. Similarly, one can also verify that

$$\alpha(.89) \ge .835578, \quad b(.89) \ge .301408 \quad \text{and} \quad c(.89) \ge .660695.$$

We now prove another technical lemma:

Lemma 2. *For any given* $0 \le \theta \le 1$ *in our rounding method, inequalities (10) and (11) hold for*

$$\alpha = \alpha(\theta) \quad \textit{and} \quad \beta = \beta(\theta).$$

Proof. Since $1 - \frac{2}{\pi}\arcsin(\theta) \ge 0$ for any $0 \le \theta \le 1$, from definition (15) we have

$$\begin{aligned}
\mathrm{Ex}[w(S)] &= \frac{1}{4}\sum_{i,j} w_{ij}\left(1 - \frac{2}{\pi}\arcsin(\bar{X}_{ij})\right)\\
&\ge \frac{1}{4}\sum_{i,j} w_{ij}\cdot\alpha(\theta)(1 - \bar{X}_{ij})\\
&= \alpha(\theta)\cdot w^{SD}\\
&\ge \alpha(\theta)\cdot w^{*}.
\end{aligned}$$

Noting that

$$\sum_{i \ne j} \bar{X}_{ij} = -n,$$

from the definition (16) we have

$$\begin{aligned}
\mathrm{Ex}[|S|(n - |S|)] &= \mathrm{Ex}\left[\frac{n^2}{4} - \frac{(e^T\hat{x})^2}{4}\right]\\
&= \frac{1}{4}\sum_{i \ne j}\left(1 - \frac{2}{\pi}\arcsin(\theta\bar{X}_{ij})\right)\\
&= \frac{1}{4}\sum_{i \ne j}\left(1 - \frac{2}{\pi}\arcsin(\theta) + \frac{2}{\pi}\arcsin(\theta) - \frac{2}{\pi}\arcsin(\theta\bar{X}_{ij})\right)\\
&= \frac{1}{4}\sum_{i \ne j}\left(b(\theta) + \left(\frac{2}{\pi}\arcsin(\theta) - \frac{2}{\pi}\arcsin(\theta\bar{X}_{ij})\right)\right)\\
&\ge \frac{1}{4}\sum_{i \ne j}\left(b(\theta) + c(\theta)(1 - \bar{X}_{ij})\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\big((n^2-n)b(\theta) + (n^2-n)c(\theta) + nc(\theta)\big) \\
&= \left(\left(1-\frac{1}{n}\right)b(\theta) + c(\theta)\right)\cdot\frac{n^2}{4} \\
&= \beta(\theta)\cdot\frac{n^2}{4}.
\end{aligned}
$$

□

Lemmas 1 and 2 together imply that for any given θ between 0 and 1, our rounding method will generate a

$$w(\tilde{S}) \geq \frac{\alpha(\theta)}{1+\sqrt{1-\beta(\theta)}}\cdot w^*$$

as soon as (bounded) $z(\gamma)$ of (12) meets its expectation. Thus, we can set θ to a value θ^* in [0, 1] such that $\frac{\alpha(\theta)}{1+\sqrt{1-\beta(\theta)}}$ is maximized, that is, let

$$\theta^* = \arg\max_{\theta\in[0,1]} \frac{\alpha(\theta)}{1+\sqrt{1-\beta(\theta)}}$$

and

$$r^{MB} = \frac{\alpha(\theta^*)}{1+\sqrt{1-\beta(\theta^*)}}.$$

In particular, if $\theta = .89$ is selected in the new rounding method,

$$\alpha(.89) > .8355,$$

and for n sufficiently large ($\geq 10^4$)

$$\beta(.89) = \left(1-\frac{1}{n}\right)b(.89) + c(.89) > .9620,$$

which imply

$$r^{MB} = \frac{\alpha(\theta^*)}{1+\sqrt{1-\beta(\theta^*)}} \geq \frac{\alpha(.89)}{1+\sqrt{1-\beta(.89)}} > .69920.$$

This bound, together with Frieze and Jerrum's analysis [2], yield our final result:

Theorem 1. *There is a polynomial-time approximation algorithm for Max-Bisection whose expected cut is at least r^{MB} times the maximal bisection cut, if the number of nodes in the graph is sufficiently large. In particular, if parameter $\theta = .89$ is used, our rounding method is a .699-approximation for Max-Bisection.*

6. Final remarks

The reader may ask why we have used two different formulations in defining $\alpha(\theta)$ of (15) and $\beta(\theta)$ of (16). The reason is that we have no control on the ratio, ρ, of the maximal bisection cut w^* over the total weight $\sum_{i<j} w_{ij}$, i.e.,

$$\rho := \frac{w^*}{\sum_{i<j} w_{ij}}.$$

Note that ρ ranges from $1/2$ to 1. Indeed, using the second derivation in Lemma 2, we can also prove that in Lemma 1

$$\alpha \geq \frac{1}{2\rho} b(\theta) + c(\theta).$$

Thus, in the worse case $\rho = 1$, we can only establish

$$\alpha \geq \frac{1}{2} b(\theta) + c(\theta).$$

Then, for $\theta = .89$, we have $\alpha \geq .8113$, which is less than .8355 established by using the first derivation.

However, if $\rho \leq .8$, then we have $\alpha \geq .8490$. For Max-Bisection on these graphs, our method is a .710 approximation for setting $\theta = .89$. This bound can be further improved by setting a smaller θ. In general, the quality bound improves as ρ decreases. When ρ near $1/2$, we have a close to 1 approximation if $\theta = 0$ is chosen, since $b(0) = 1$, $c(0) = 0$, $\alpha \geq \frac{1}{2\rho}$ and $\beta \geq 1 - \frac{1}{n}$.

In any case, we can run our rounding method 100 times for parameter $\theta = .00, .01, \ldots, .98, .99$ and report the best rounding solution among the 100 tries. This will ensure us to produce a solution with a near best guarantee, but without the need to know ρ.

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