Model Predictive Path Integral Control: A Tutorial

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Introduction

Model predictive path integral (**MPPI**) control is a powerful and popular controller method in robotics and control, because

- it can handle **nonconvex** and **non-smooth** optimization problems,
- and it can be computed in **real-time** with GPU due to its **parallel computation** nature.

[1] Williams, et al. Information theoretic MPC for model-based reinforcement learning. ICRA 2017.

[2] Xue, et al. Full-order sampling-based mpc for torque-level locomotion control via diffusion-style annealing. ICRA 2025.

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Many applications:

- Aggressive driving [1]
- Legged robot control[2]
- Sim2real dexterous retargeting [3]
- Hard control problems in my research (MPPI $>$ Policy gradient $>$ SDP)
- Control with signal temporal logical constraints [4]

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The iteration formula of MPPI is

$$\mathbf{u}_{out} = \frac{\sum_{i=1}^{N} \mathbf{u}_i \exp\big(-J(\mathbf{u}_i)/\lambda\big)}{\sum_{i=1}^{N} \exp\big(-J(\mathbf{u}_i)/\lambda\big)}, \tag{2}$$

where λ is a hyperparameter (called temperature from simulated annealing) and $\{\mathbf{u}_i\}_{i=1}^{N}$ are i.i.d. samples from the following Gaussian distribution $\mathcal{N}(\mathbf{u}_{in}, \mathbf{\Sigma})$, with the current controller $\mathbf{u}_{in}$ and $\mathbf{\Sigma} \succ 0$.

Derive Single MPPI Iteration Based on KL Control[5]

Let $q(\mathbf{u})$ be a given probability density (for $\mathbf{u}_{in}$) on $\mathbb{R}^k$. Let $J : \mathbb{R}^k \to \mathbb{R}$ be measurable and $\lambda > 0$. We consider the optimization problem over probability densities $p(\mathbf{u})$:

$$\begin{aligned} \min_p \int p(\mathbf{u}) J(\mathbf{u})\, d\mathbf{u} + \lambda \int p(\mathbf{u}) \log \frac{p(\mathbf{u})}{q(\mathbf{u})}\, d\mathbf{u} \\ \text{s.t. } \int p(\mathbf{u})\, d\mathbf{u} = 1. \end{aligned} \tag{3}$$

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Define

$$p^\star(\mathbf{u}) := \frac{1}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda} J(\mathbf{u})\big)\Big]}\, q(\mathbf{u}) \exp\Big(-\frac{1}{\lambda} J(\mathbf{u})\Big). \tag{4}$$

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Goal: prove that $p^\star$ is the unique minimizer of (3).

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Variation Calculus

Proof. It can be proved that (3) is a strictly convex infinite-dimensional optimization problem. Then, the feasible stationary point of its Lagrange function is a unique minimizer. Define Lagrangian

$$\mathcal{L}(p, \eta) = \int p(\mathbf{u}) J(\mathbf{u}) \, d\mathbf{u} + \lambda \int p(\mathbf{u}) \log \frac{p(\mathbf{u})}{q(\mathbf{u})} \, d\mathbf{u} + \eta \Big(\int p(\mathbf{u}) \, d\mathbf{u} - 1 \Big), \tag{5}$$

where $\eta \in \mathbb{R}$ is a Lagrange multiplier.

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where $\eta \in \mathbb{R}$ is a Lagrange multiplier. We compute stationarity with respect to $p(\mathbf{u})$. Let $p_\epsilon = p + \epsilon h$ with $\int h(\mathbf{u})\,d\mathbf{u} = 0$.
Using $\frac{d}{dx}(x \log x) = 1 + \log x$, we obtain

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{L}(p_\epsilon,\eta) = \int h(\mathbf{u})\Big[J(\mathbf{u}) + \lambda\big(1 + \log p(\mathbf{u}) - \log q(\mathbf{u})\big) + \eta\Big]\,d\mathbf{u}.$$

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For this to vanish for all admissible h, the bracket must be constant:

$$J(\mathbf{u}) + \lambda\big(1 + \log p(\mathbf{u}) - \log q(\mathbf{u})\big) = c. \tag{6}$$

Solve the Stationarity Condition

Rearranging,

$$\log p(\mathbf{u}) = \log q(\mathbf{u}) - \frac{1}{\lambda}J(\mathbf{u}) + \frac{c-\lambda}{\lambda}.$$

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$$p(\mathbf{u}) = \frac{1}{\mathbb{E}_q\Big[\exp\big(-\tfrac{1}{\lambda} J(\mathbf{u}) \big) \Big]}\, q(\mathbf{u}) \exp\Big(-\frac{1}{\lambda} J(\mathbf{u}) \Big) = p^\star(\mathbf{u}).$$

From Optimal Distribution to Optimal Controller

KL-control yields an *optimal distribution* $p^\star(\mathbf{u})$, but controller implementation must output a deterministic value $\bar{\mathbf{u}}$. Adopt the posterior squared-loss criterion

$$\bar{\mathbf{u}}^\star \in \arg\min_{\bar{\mathbf{u}} \in \mathbb{R}^{mH}} \mathbb{E}_{p^\star}\big[\|\mathbf{u} - \bar{\mathbf{u}}\|_2^2\big]. \tag{7}$$

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Expand the risk:

$$\mathbb{E}_{p^\star}\big[\|\mathbf{u} - \bar{\mathbf{u}}\|_2^2\big] = \mathbb{E}_{p^\star}[\|\mathbf{u}\|_2^2] - 2\bar{\mathbf{u}}^\top \mathbb{E}_{p^\star}[\mathbf{u}] + \|\bar{\mathbf{u}}\|_2^2. \tag{8}$$

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Differentiate w.r.t. $\bar{\mathbf{u}}$:

$$\nabla_{\bar{\mathbf{u}}} = -2\mathbb{E}_{p^\star}[\mathbf{u}] + 2\bar{\mathbf{u}}. \tag{9}$$

Setting $\nabla_{\bar{\mathbf{u}}} = 0$ gives

$$\bar{\mathbf{u}}^\star = \mathbb{E}_{p^\star}[\mathbf{u}].$$

Since the Hessian is $2I \succ 0$, this minimizer is unique.

Convergence of MPPI for Quadratic Programming

Consider a QP

$$\min_{\mathbf{u}} \; J(\mathbf{u}) = \mathbf{u}^\top \mathbf{D} \mathbf{u} + \mathbf{u}^\top \mathbf{d}, \tag{10}$$

with $\mathbf{D} \succ 0$. Draw N i.i.d. samples $\mathbf{u}_i \sim q = \mathcal{N}(\mathbf{u}_{in}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} \succ 0$.

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$$\widehat{\mathbf{u}}_{\text{out}} = \frac{\sum_{i=1}^N \mathbf{u}_i \exp\big(-J(\mathbf{u}_i)/\lambda\big)}{\sum_{i=1}^N \exp\big(-J(\mathbf{u}_i)/\lambda\big)} \text{ and } \mathbf{u}_{\text{out}} = \frac{\mathbb{E}_q\Big[\mathbf{u}\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}. \tag{11}$$

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Theorem 1.[6] Consider the minimizer $\mathbf{u}^\star = \frac{-1}{2}\mathbf{d}\mathbf{D}^{-1}$ of (10). We have

$$\hat{\mathbf{u}}_{out} \xrightarrow{p} \mathbf{u}_{out}, \tag{12}$$

and

$$\mathbf{u}_{out} - \mathbf{u}^\star \xrightarrow{p} (\mathbf{I} + 2\lambda\boldsymbol{\Sigma}\mathbf{D})^{-1}(\mathbf{u}_{in} - \mathbf{u}^\star). \tag{13}$$

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Step 1: Write $p^\star$ exactly

Proof. Define

$$Z := \mathbb{E}_q\Big[\exp\big(-\tfrac{1}{\lambda}J(\mathbf{u})\big)\Big] = \int_{\mathbb{R}^k} q(\mathbf{u})\exp\Big(-\tfrac{1}{\lambda}J(\mathbf{u})\Big)\, d\mathbf{u}. \tag{14}$$

Then the optimal distribution satisfies

$$p^\star(\mathbf{u}) = \frac{1}{Z}\, q(\mathbf{u})\exp\Big(-\tfrac{1}{\lambda}J(\mathbf{u})\Big), \qquad \mathbf{u}_{out} = \mathbb{E}_{p^\star}[\mathbf{u}]. \tag{15}$$

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Because $q(\mathbf{u}) = \mathcal{N}(\mathbf{u}_{in}, \mathbf{\Sigma})$, then

$$q(\mathbf{u}) = \frac{1}{(2\pi)^{k/2}|\mathbf{\Sigma}|^{1/2}}\exp\Big(-\tfrac{1}{2}(\mathbf{u}-\mathbf{u}_{in})^\top\mathbf{\Sigma}^{-1}(\mathbf{u}-\mathbf{u}_{in})\Big). \tag{16}$$

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Using $J(\mathbf{u}) = \mathbf{u}^\top\mathbf{D}\mathbf{u} + \mathbf{u}^\top\mathbf{d}$, we obtain the exact product

$$q(\mathbf{u})\exp\Big(-\tfrac{1}{\lambda}J(\mathbf{u})\Big) = \frac{1}{(2\pi)^{k/2}|\boldsymbol{\Sigma}|^{1/2}}\exp\Big(-\tfrac{1}{2}\mathbf{u}^\top\mathbf{A}\mathbf{u} + \mathbf{u}^\top\mathbf{b} - c_0\Big), \tag{17}$$

where

$$\mathbf{A} := \boldsymbol{\Sigma}^{-1} + \tfrac{2}{\lambda}\mathbf{D}, \qquad \mathbf{b} := \boldsymbol{\Sigma}^{-1}\mathbf{u}_{in} - \tfrac{1}{\lambda}\mathbf{d}, \qquad c_0 := \tfrac{1}{2}\mathbf{u}_{in}^\top\boldsymbol{\Sigma}^{-1}\mathbf{u}_{in}. \tag{18}$$

Step 2: Complete the square and normalize

Complete the square with $\boldsymbol{\mu} := \mathbf{A}^{-1}\mathbf{b}$:

$$-\frac{1}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u} + \mathbf{u}^\top \mathbf{b} = -\frac{1}{2}(\mathbf{u} - \boldsymbol{\mu})^\top \mathbf{A}(\mathbf{u} - \boldsymbol{\mu}) + \frac{1}{2}\mathbf{b}^\top \mathbf{A}^{-1}\mathbf{b}. \tag{19}$$

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Hence

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After a long derivation,

$$Z = \int_{\mathbb{R}^k} q(\mathbf{u}) \exp\Big(-\tfrac{1}{\lambda} J(\mathbf{u})\Big)\, d\mathbf{u} = \frac{|\mathbf{A}^{-1}|^{1/2}}{|\boldsymbol{\Sigma}|^{1/2}} \exp\Big(\tfrac{1}{2}\mathbf{b}^\top \mathbf{A}^{-1}\mathbf{b} - \tfrac{1}{2}\mathbf{u}_{in}^\top \boldsymbol{\Sigma}^{-1}\mathbf{u}_{in}\Big). \tag{21}$$

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Substituting this Z back to (20) yields the fully normalized Gaussian form

$$p^\star(\mathbf{u}) = \frac{|\mathbf{A}|^{1/2}}{(2\pi)^{k/2}} \exp\Big(-\tfrac{1}{2}(\mathbf{u}-\boldsymbol{\mu})^\top \mathbf{A}(\mathbf{u}-\boldsymbol{\mu})\Big) = \mathcal{N}(\boldsymbol{\mu}, \mathbf{A}^{-1}). \tag{22}$$

In particular,

$$\mathbf{u}_{out} = \mathbb{E}_{p^\star}[\mathbf{u}] = \boldsymbol{\mu} = \Big(\boldsymbol{\Sigma}^{-1} + \tfrac{2}{\lambda}\mathbf{D}\Big)^{-1}\Big(\boldsymbol{\Sigma}^{-1}\mathbf{u}_{in} - \tfrac{1}{\lambda}\mathbf{d}\Big). \tag{23}$$

Step 3: Express the mean as a contraction toward $\mathbf{u}^\star$

Recall

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$$\begin{aligned}
\mathbf{u}_{out} &= \left(\boldsymbol{\Sigma}^{-1} + \tfrac{2}{\lambda}\mathbf{D}\right)^{-1}\left(\boldsymbol{\Sigma}^{-1}\mathbf{u}_{in} + \tfrac{2}{\lambda}\mathbf{D}\mathbf{u}^\star\right) && (25)\\
&= (\lambda\mathbf{I} + 2\boldsymbol{\Sigma}\mathbf{D})^{-1}\left(\lambda\mathbf{u}_{in} + 2\boldsymbol{\Sigma}\mathbf{D}\mathbf{u}^\star\right) && (26)\\
&= (\lambda\mathbf{I} + 2\boldsymbol{\Sigma}\mathbf{D})^{-1}\left(\lambda\mathbf{u}_{in} + 2\boldsymbol{\Sigma}\mathbf{D}\mathbf{u}_{in} - 2\boldsymbol{\Sigma}\mathbf{D}\mathbf{u}_{in} + 2\boldsymbol{\Sigma}\mathbf{D}\mathbf{u}^\star\right) && (27)\\
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Therefore,

$$\mathbf{u}_{out} - \mathbf{u}^\star = (\mathbf{I} + 2\lambda\mathbf{\Sigma}\mathbf{D})^{-1}(\mathbf{u}_{in} - \mathbf{u}^\star).$$

Step 4: Convergence in probability and Slutsky

Let the MPPI estimator be

$$\mathbf{u}_{out}^{(N)} = \frac{\frac{1}{N}\sum_{i=1}^{N} \mathbf{u}_i w_i}{\frac{1}{N}\sum_{i=1}^{N} w_i}, \qquad w_i := \exp\Big(-\tfrac{1}{\lambda} J(\mathbf{u}_i)\Big), \qquad \mathbf{u}_i \overset{\text{i.i.d.}}{\sim} q. \tag{29}$$

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Then by the weak LLN,

$$\frac{1}{N}\sum_{i=1}^{N} w_i \xrightarrow{p} \mathbb{E}_q[w], \qquad \frac{1}{N}\sum_{i=1}^{N} \mathbf{u}_i w_i \xrightarrow{p} \mathbb{E}_q[\mathbf{u}w]. \tag{30}$$

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Then, Slutsky's theorem yields

$$\hat{\mathbf{u}}_{out} \xrightarrow{p} \frac{\mathbb{E}_q[\mathbf{u}w]}{\mathbb{E}_q[w]} = \mathbb{E}_{p^\star}[\mathbf{u}] = \mathbf{u}_{out}. \tag{31}$$