ECE 562: Nonlinear Systems and Control

Chapter 2 Qualitative behavior of linear systems

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statistical learning
+ control system
theory + design

Chapter 2: Big picture

$\dot{x} = f(x) \quad x \in \mathbb{R}^2$

- Focus: qualitative behavior of **second-order** systems via **phase portraits**.
- Linear case: $\dot{x} = Ax$ with $A \in \mathbb{R}^{2\times 2}$. ✓
- Classification by eigen-structure (real/complex, distinct/repeated, zero eigenvalues).

$\mathbb{R}^2$ $\mathbb{R}^3$ ✓ chatgpt

$\mathbb{R}^4$ ✗

2.1 Second order systems: phase portrait viewpoint

Consider the second-order nonlinear system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = f_2(x_1, x_2). \end{cases}$$

Assume the solution $x(t) = (x_1(t), x_2(t))$ with $x(t_0) = x_0$ exists and is unique.

- The **locus** of $x(t)$ on the (x_1, x_2)-plane for $t \geq t_0$ is a curve through x_0.
- The family of all locus curves is the **phase portrait**.

root locus

2.2 Linear planar systems and real Jordan form

$A = M^{-1} J_r M$

$\dot{x} = f(x) \in \mathbb{R}^2$

$x(t) = e^{At} x(0)$

Let

$$\dot{x} = Ax, \qquad A \in \mathbb{R}^{2\times 2}.$$

The solution from initial condition x_0 can be written as

$$x(t) = M \exp(J_r t) M^{-1} x_0,$$

$z(t) = M^{-1} x(t)$

where J_r is the *real Jordan form* of A and M is real, nonsingular with

$$M^{-1}AM = J_r.$$

$z(t) = \exp(J_r t)\, z(0)$

Depending on eigenvalues of A, J_r takes standard forms below.

Real Jordan forms in $\mathbb{R}^{2\times 2}$

- **Two real and distinct eigenvalues**:

$$J_r = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \quad \checkmark$$

- **Complex conjugate eigenvalues** $\underline{\lambda_{1,2} = a \pm jb}$:

??

$$\begin{pmatrix} a+jb & \\ & a-jb \end{pmatrix}$$

$$\underline{J_r = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}}. \quad \checkmark$$

- **Two real and equal eigenvalues**:

$$J_r = \begin{pmatrix} \lambda & k \\ 0 & \lambda \end{pmatrix}, \qquad k \in \{0, 1\}. \quad \checkmark$$

2.2.1 Case I: real eigenvalues λ_1, λ_2

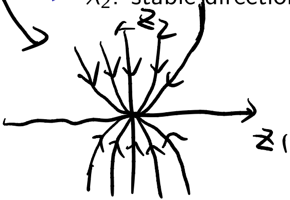
$t = \cdots z_1$

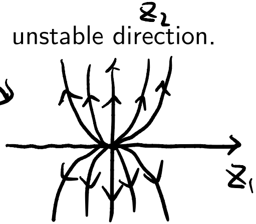
$$z = M^{-1} x \qquad \begin{cases} \dot z_1 = \lambda_1 z_1 \\ \dot z_2 = \lambda_2 z_2 \end{cases} \text{ with } \begin{pmatrix} z_{10} \\ z_{20} \end{pmatrix} \Rightarrow \begin{cases} z_1(t) = z_{10} e^{\lambda_1 t} \\ z_2(t) = z_{20} e^{\lambda_2 t} \end{cases}$$

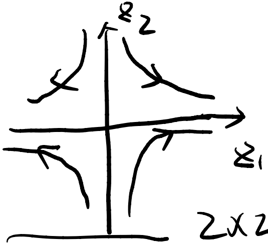
$$\cancel{t} \Rightarrow z_2 = \frac{z_{20}}{z_{10}^{\lambda_2/\lambda_1}} z_1^{\lambda_2/\lambda_1}$$

C

- **Stable node:** $\lambda_2 < \lambda_1 < 0$.
  $\lambda_1 < \lambda_2 < 0$
  - λ_2 is the *fast* eigenvalue, λ_1 the *slow*.
  - Trajectories approach the origin tangent to the *slow* eigenvector.
  - Far from the origin, trajectories are approximately parallel to the *fast* eigenvector.
- **Unstable node:** $0 < \lambda_2 < \lambda_1$ and both positive.
  - Same geometry as stable node but time direction reversed (repelling).
- **Saddle:** $\lambda_2 < 0 < \lambda_1$.
  - λ_2: stable direction, λ_1: unstable direction.







$2 \times 2 \times 2$ R^3

$$\begin{cases} z_1(t) = z_{10} e^{\lambda_1 t} \checkmark \\ z_2(t) = z_{20} \underline{e}^{\lambda_2 t} \end{cases}$$

$$e^{\lambda_1 t} = \frac{z_1}{z_{10}} \qquad t = \frac{1}{\lambda_1} \ln \frac{z_1(t)}{z_{10}}$$

$$z_2(t) = z_{20}\, e^{\lambda_2 \cdot \frac{1}{\lambda_1} \ln \frac{z_1(t)}{z_{10}}}$$

$$= z_{20}\, e^{\frac{\lambda_2}{\lambda_1} \left(\ln z_1(t) - \ln z_{10} \right)}$$

$$= \frac{z_{20}}{e^{\frac{\lambda_2}{\lambda_1} \ln z_{10}}}\, e^{\frac{\lambda_2}{\lambda_1} \ln z_1(t)}$$

$$= \frac{z_{20}}{\left(e^{\ln z_{10}} \right)^{\frac{\lambda_2}{\lambda_1}}}\, e^{\frac{\lambda_2}{\lambda_1} \ln z_1(t)}$$

2.2.2 Case II: complex eigenvalues $\lambda_{1,2} = a \pm jb$

$a+jb \quad a+jb$

$a-jb$

$z = M^{-1} x$

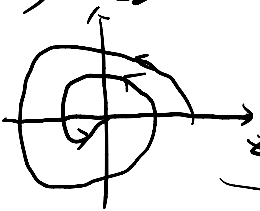
$$\begin{cases} \dot{z}_1 = \alpha z_1 - \beta z_2 \\ \dot{z}_2 = \beta z_1 + \alpha z_2 \end{cases} \xrightarrow{\text{polar coordinates}} \begin{cases} \dot{r} = \alpha r \\ \dot{\theta} = \beta \end{cases}$$

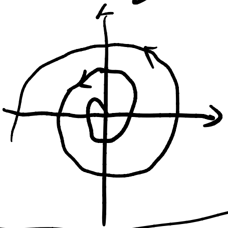
$r = \sqrt{z_1^2 + z_2^2}$

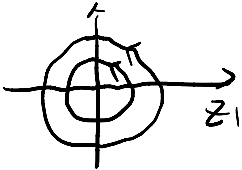
$\theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$

$$\Rightarrow \begin{cases} r(t) = r_0 e^{\alpha t} \\ \theta(t) = \theta_0 + \beta t \end{cases}$$

- **Stable focus:** $a < 0$ (spirals into the origin).
- **Unstable focus:** $a > 0$ (spirals out from the origin).
- **Center:** $a = 0$ (closed orbits in the linear model).







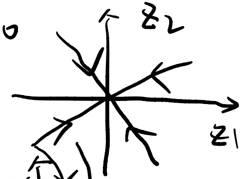
$x_1 \quad x_2$

2.2.3 Case III: repeated nonzero eigenvalue $\lambda_1 = \lambda_2 = \lambda \neq 0$

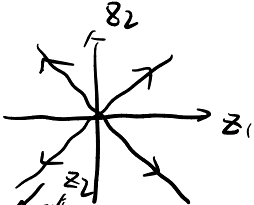
$$\begin{cases} \dot{z}_1 = \lambda z_1 + k z_2 \\ \dot{z}_2 = \lambda z_2 \end{cases} \Longrightarrow \begin{cases} z_1(t) = e^{\lambda t}(z_{10} + k z_{20} t) \\ z_2(t) = e^{\lambda t} z_{20} \end{cases} \xRightarrow{\cancel{t}} z_1 = z_2\left[\frac{z_{10}}{z_{20}} + \frac{k}{\lambda}\ln\frac{z_2}{z_{20}}\right]$$

- Phase portrait is similar to a node, but without the same fast–slow asymptotics.
- Common terminology:
  - $\lambda < 0$: **stable node**
  - $\lambda > 0$: **unstable node**.

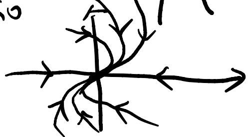
$k=0$ $\lambda<0$



$\lambda>0$



$k\neq0$ $\lambda<0$



$\lambda>0$

2.2.4 Case IV: zero eigenvalue(s)

$Ax=0 \Rightarrow x=0$

$\lambda_1 \quad \lambda_2 \neq 0$

A^{-1} ✓ single equilibri.

when $\lambda_1 = 0 \quad \lambda_2 \neq 0$

$$\begin{cases} \dot z_1 = 0 \\ \dot z_2 = \lambda_2 z_2 \end{cases} \Rightarrow \begin{cases} z_1(t) = z_{10} \\ z_2(t) = z_{20}\, e^{\lambda_2 t} \end{cases}$$

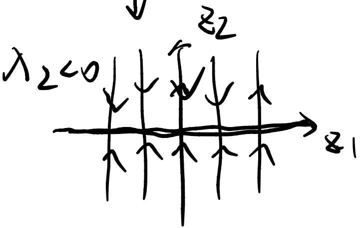
$\lambda_1 = 0$

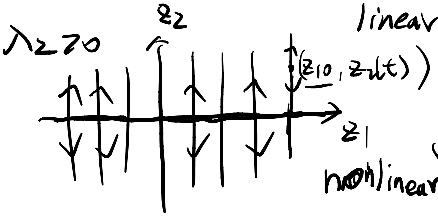
- If A has a nontrivial nullspace, any vector in $\mathcal{N}(A)$ is an equilibrium:

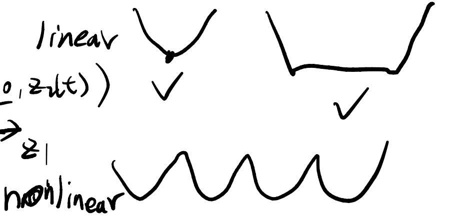
$$Ax = 0 \implies \dot x = 0.$$

A^{-1} ✗ infinite equilibrium

- Hence the system has an **equilibrium space** (not an isolated equilibrium point).







2.3 Linearization near an equilibrium

Consider the nonlinear second-order system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = f_2(x_1, x_2), \end{cases} \quad \mathbb{R}^2$$

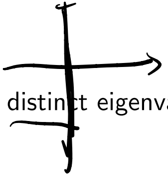
and let $p = (p_1, p_2)$ be an equilibrium point. Assume f_1, f_2 are continuously differentiable.
Let $y_i = x_i - p_i$. Linearization about p yields

$$\dot{y} = Ay, \longrightarrow A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{(x_1, x_2) = (p_1, p_2)}$$

where A is the Jacobian of $f(x)$ evaluated at $x = p$.

2.3 What linearization can (and cannot) conclude

$\lambda_1 \quad \lambda_2$



- If the linearized system has:
  - a stable/unstable node with distinct eigenvalues, or
  - a stable/unstable focus, or
  - a saddle,

  then in a sufficiently small neighborhood of p, the nonlinear trajectories behave qualitatively the same way.

repeated ??? eigenvalues also work

see next slide

- **Exception: center.** If the linearization has a *center*, then linearization is **inconclusive**: the nonlinear behavior near p may differ significantly.

$$\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix} = \alpha \pm \beta i$$

$\alpha \neq 0$

$\alpha = 0$

Hyperbolicity and Hartman–Grobman theorem (not covered in exam)

Definition (Hyperbolic equilibrium point). An equilibrium point p is *hyperbolic* if the real part of all eigenvalues of A is not equal to zero.
Definition (Topological linearization). A *continuous linearization* of $f(x)$ near p is a homeomorphism h such that $h \circ \eta_t \circ h^{-1} = e^{At}$, where η_t is the local flow of $\dot{x} = \mathbf{f}(x)$.
Theorem (Hartman–Grobman theorem). Let f be continuously differentiable and let p be a hyperbolic equilibrium point. Then there exists a neighborhood of p in which f admits a continuous linearization. That is, the nonlinear flow is locally topologically equivalent to its linearization.

Implication: Local phase portraits are determined by the Jacobian for all **hyperbolic equilibrium points**.
It's still an active area![1] [2]

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[1] Kvalheim, Matthew D., and Eduardo D. Sontag. "Global linearization of asymptotically stable systems without hyperbolicity." Systems & Control Letters 203 (2025): 106163.

[2] Liu, Zexiang, Necmiye Ozay, and Eduardo D. Sontag. "Properties of immersions for systems with multiple limit sets with implications to learning Koopman embeddings." Automatica 176 (2025): 112226.