Some Statistical Fundamental Limits of Learning for Dynamics and Control

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Some Statistical Fundamental Limits of Learning for Dynamics and Control with Insights for Algorithms Design

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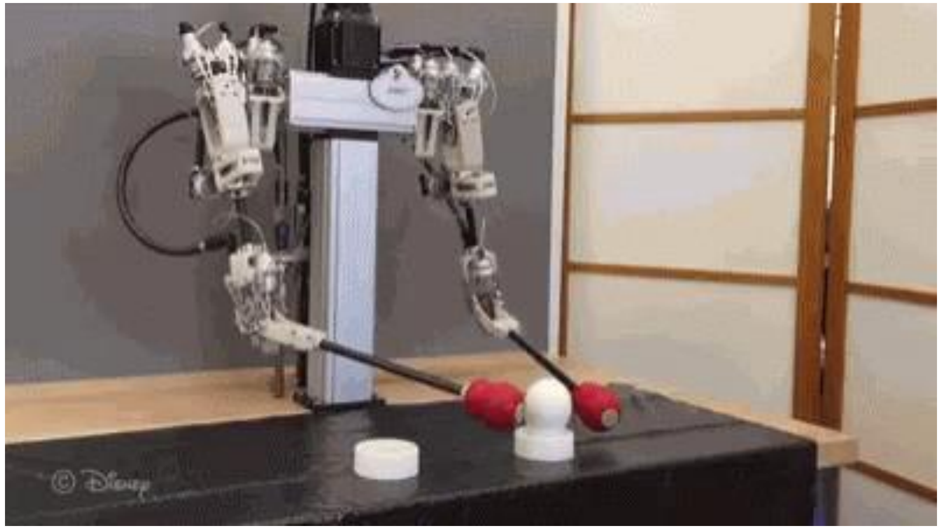
Unlike ChatGPT, the physical agents are sensitive and dangerous



https://tenor.com/search/robot-fail-gifs

Why?

- Unlike cyberspace with discrete states, the physical world is a continuous state space.
- In continuous state space, infinitesimal error can lead to catastrophic failures in stability and safety.
- Learning for infinitesimal error might be arbitrarily hard.

Therefore, we focus on the statistical fundamental limits of agent learning in continuous state space, like system identification and learning-based control, **etc**.

Summary of Statistical Fundamental Limits

- **Statistical Consistency:** Does the algorithm converge to the ground-truth solution with respect to sample size?
- **Statistical Optimality:** If the algorithm is statistically consistent, does the algorithm achieve the minimax sample complexity lower bound?
- **Statistical Hardness:** If the algorithm is statistically optimal, does the optimal sample complexity increase moderately with the system complexity?

Summary of Our Contributions

- Chapter 2 **for Statistical Consistency ---** Noise Sensitivity of the Semidefinite Programs for Direct Data-Driven LQR (*ACC 2025 and TAC 2025*)

- Chapter 3 **for Statistical Optimality ---** System Identification Under Bounded Noise: Optimal Rates Beyond Least Squares *(Submitted to CDC 2025 and L-CSS)*

- Chapter 4 **for Statistical Hardness ---** On the Hardness of Learning to Stabilize Linear Systems *(CDC 2023)*

Chapter 2

Noise Sensitivity of the Semidefinite Programs for Direct Data-Driven LQR

Given a linear time-invariant (LTI) system:

$$x_{t+1} = Ax_t + Bu_t + w_t,$$

where $x_t, w_t \in \mathbb{R}^n, u_t \in \mathbb{R}^m, w_t = 0 \text{ or } w_t \sim \mathcal{N}(0, \sigma_w^2 I_n)$, and (A, B) is controllable.

Linear quadratic regulator (LQR) problem:

$$\min_{u_0, u_1, \ldots} \lim_{T \to \infty} \mathbb{E}\left[\frac{1}{T} \sum_{t=0}^{T} (x_t^T Q x_t + u_t^T R u_t)\right]$$

s.t. the previous LTI system,

where $Q \succ 0, R \succ 0$. When (A, B) is known, the above optimal solution is $u_t = K_{lqr} x_t$, where

$$K_{lqr} = -\left(R + B^T P B\right)^{-1} B^T P A,$$

and P is the PSD solution of discrete-time algebraic Riccati equation.

When (A, B) is unknown

Offline Data matrices:

$$X_0 = [x_0\ x_1\ \ \dots x_{T-1}],$$
$$U_0 = [u_0\ u_1\ \ \dots u_{T-1}],$$
$$X_1 = [x_1\ x_2\ \ \dots x_T],$$

where $u_t \sim N(0, \sigma_u^2 \mathbf{I}_m)$.

Consider direct data-driven (DDD) control.

Certainty Equivalence (CE) DDD LQR

(De Persis & Tesi, TAC 2019):

$$\min_{X,Y}\ \mathrm{trace}(QX_0Y) + \mathrm{trace}(X)$$
$$\text{s.t.}\ \begin{bmatrix} X_0Y - I_n & X_1Y \\ Y^T X_1^T & X_0Y \end{bmatrix} \succcurlyeq 0$$
$$\begin{bmatrix} X & \sqrt{R}U_0Y \\ \left(\sqrt{R}U_0Y\right)^T & X_0Y \end{bmatrix} \succcurlyeq 0.$$

Let Y_{ce}^* be an optimal solution, an estimate of K_{lqr}

$$K_{ce}(T) := -U_0Y_{ce}^*(X_0Y_{ce}^*)^{-1}.$$

Theorem 1 (De Persis & Tesi, 2019). Let $\mathrm{rank}\left(\begin{bmatrix} X_0 \\ U_0 \end{bmatrix}\right) = m + n$. When $w_t = 0$ for all t,

$$K_{ce}(T) = K_{lqr}.$$

(CE DDD LQR is perfect for noiseless case)

Theorem 2 (our work). Assume $\sigma_w^2 > 0$. When $T \geq (m+n)(n+1)+n$,

$$P(K_{ce}(T) = 0_{m\times n}) = 1.$$

(CE DDD LQR is trivial for almost all noise)

Key observation: The following equalities hold for all Y_{ce}^* when $\sigma_w^2 > 0$

$$\begin{cases} U_0 Y_{ce}^* = 0_{m\times n} \\ X_0 Y_{ce}^* = I_n \\ X_1 Y_{ce}^* = 0_{n\times n} \end{cases},$$

for any $T \geq (m+n)(n+1)+n$. Recall $K_{ce} = -U_0 Y_{ce}^* (X_0 Y_{ce}^*)^{-1}$.

$$
\begin{aligned}
&\min_{X,Y}\ \operatorname{trace}(QX_0Y) + \operatorname{trace}(X)\\
&\text{s.t. } \begin{bmatrix} X_0Y - I_n & X_1Y \\ Y^TX_1^T & X_0Y \end{bmatrix} \succcurlyeq 0\\
&\qquad \begin{bmatrix} X & \sqrt{R}U_0Y \\ \left(\sqrt{R}U_0Y\right)^T & X_0Y \end{bmatrix} \succcurlyeq 0.
\end{aligned}
$$

Explanation for key observations:

- $X_0Y - I_n \succcurlyeq 0$ and $X \succcurlyeq 0 \implies \operatorname{trace}(QX_0Y) + \operatorname{trace}(X) \geq \operatorname{trace}(Q)$.

- All Y *with* $\begin{cases} U_0Y = \mathbf{0}_{m\times n} \\ X_0Y = \mathbf{I}_n \\ X_1Y = \mathbf{0}_{n\times n} \end{cases}$ are feasible solutions, for which $\operatorname{trace}(QX_0Y) + \operatorname{trace}(X) = \operatorname{trace}(Q)$.

Robustness-Promoting (RP) DDD LQR (De Persis & Tesi, Automatica 2021):

$$\min_{X,Y,S}\ \mathrm{trace}(QX_0Y) + \mathrm{trace}(X) + \mathrm{trace}(S)$$

$$\text{s.t.}\ \begin{bmatrix} X_0Y - I_n & X_1Y \\ Y^T X_1^T & X_0Y \end{bmatrix} \succcurlyeq 0$$

$$\begin{bmatrix} X & \sqrt{R}U_0Y \\ \left(\sqrt{R}U_0Y\right)^T & X_0Y \end{bmatrix} \succcurlyeq 0$$

$$\begin{bmatrix} S & Y \\ Y^T & X_0Y \end{bmatrix} \succcurlyeq 0.$$

Let Y_{rp}^* denote its optimal solution, $K_{rp}(T) \coloneqq -U_0Y_{rp}^*\left(X_0Y_{rp}^*\right)^{-1}$.

Theorem 3 (our work). Assume $w_t = 0$,

$$\lim_{T\to\infty} P(K_{rp}(T) = K_{lqr}) = 1.$$

(RP DDD LQR is statistically consistent for noiseless case)

Theorem 4 (our work). Assume $\sigma_w^2 > 0$,

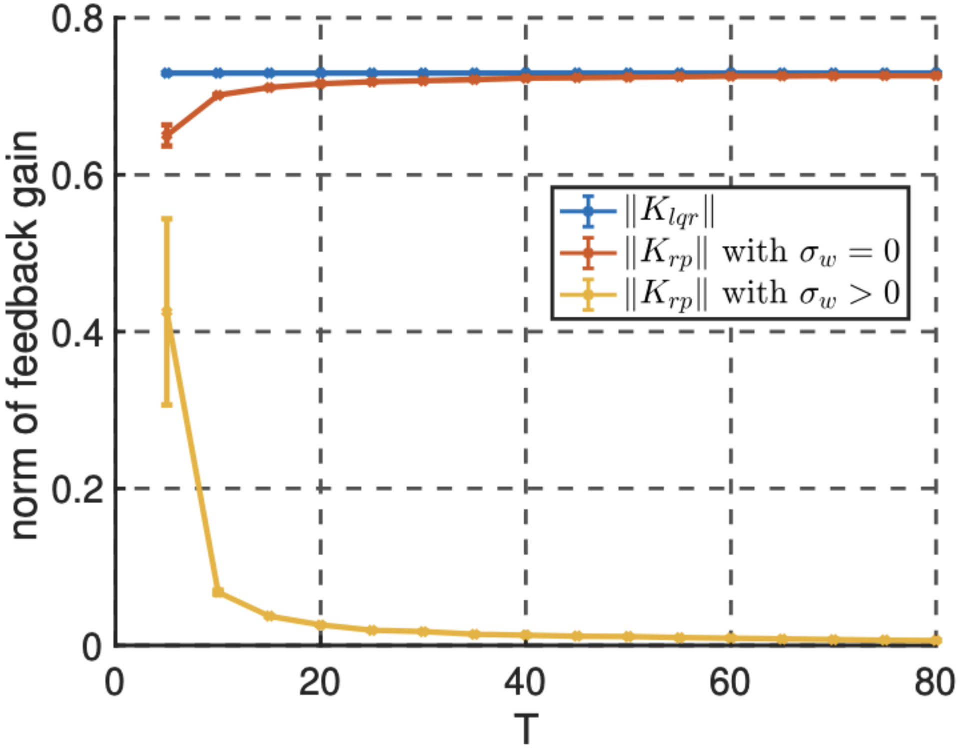
$$\lim_{T\to\infty} \mathrm{P}(K_{rp}(T) = \mathbf{0}_{\boldsymbol{m}\times\boldsymbol{n}}) = 1\,.$$

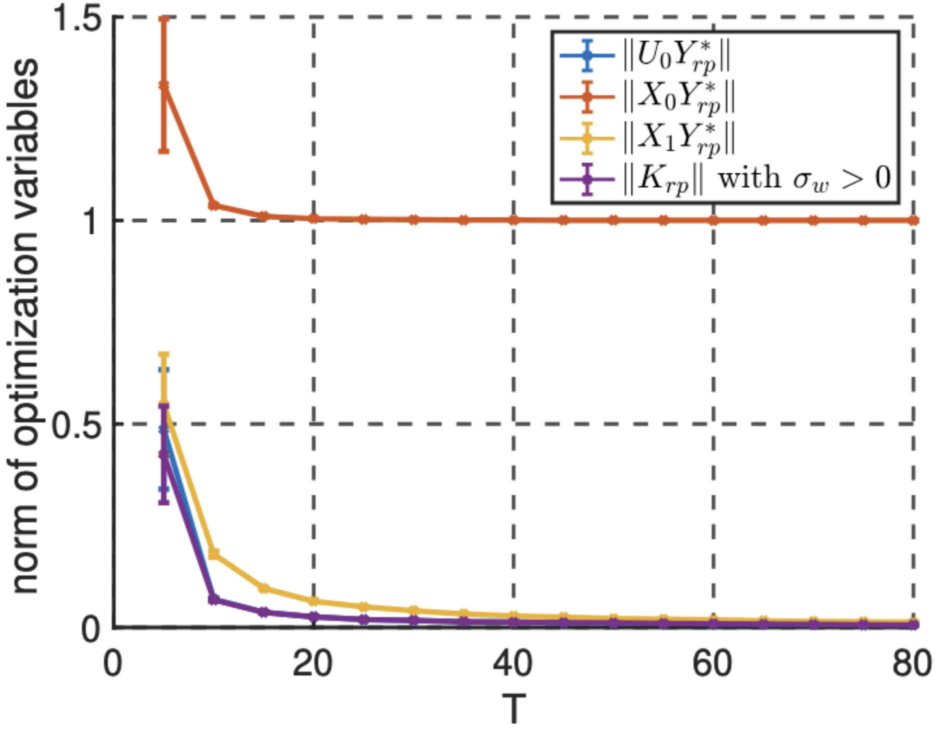
(RP DDD LQR is not statistically consistent for noisy case)

Key Observation: The following equalities always hold when $\sigma_w^2 > 0$:

$$\begin{cases} \lim_{T\to\infty} \mathrm{P}(U_0 Y_{rp}^* = \mathbf{0}_{\boldsymbol{m}\times\boldsymbol{n}}) = 1 \\ \lim_{T\to\infty} \mathrm{P}\left(X_0 Y_{rp}^* = \boldsymbol{I}_{\boldsymbol{n}}\right) = 1 \\ \lim_{T\to\infty} \mathrm{P}(X_1 Y_{rp}^* = \mathbf{0}_{\boldsymbol{n}\times\boldsymbol{n}}) = 1 \end{cases},$$

and recall $\mathrm{K_{rp}}(T) = -\mathrm{U_0 Y_{rp}^*}\left(\mathrm{X_0 Y_{rp}^*}\right)^{-1}$.





Experiments for RP DDD LQR: Consider an order-2 single-input unstable system, for which $K_{lqr} = [-0.7112\ \ -0.2046]$. $\sigma_w^2 = 1$ and $\sigma_u^2 = 1$.

	CE DDD LQR	RP DDD LQR
noiseless ($\mathbf{w}_t = 0$)	$\mathbb{P}(\mathbf{K}_{ce} = \mathbf{K}_{\text{lqr}}) = 1$ (De Persis & Tesi, 2019)	$\mathbf{K}_{rp}(T) \xrightarrow{p} \mathbf{K}_{\text{lqr}}$ (Theorem 3)
noisy ($\sigma_w > 0$)	$\mathbb{P}(\mathbf{K}_{ce} = \mathbf{0}_{m\times n}) = 1$ (Theorem 2)	$\mathbf{K}_{rp}(T) \xrightarrow{p} \mathbf{0}_{m\times n}$ (Theorem 4)

Summary and Future Work

1. Some SDPs for DDD LQR are sensitive to noise.
2. Check the statistical fundamental limits when designing new DDD control algorithms.
3. The fundamental limits for DDD Robust Control by matrix S-lemma (Waarde et al. 2020, Waarde et al. 2023) are unclear now.

Chapter 3

System Identification Under Bounded Noise: Optimal Rates Beyond Least Squares

Consider an unknown LTI system:

$$x_{t+1} = Ax_t + w_t,$$

where $x_t, w_t \in \mathbb{R}^n$. Assume

- $||w_t||_\infty < \bar{w}$ and w_t are i.i.d. for all t,
- $\rho(A) < 1$,
- and for any $\epsilon \in [0, \bar{w}]$, there exists $C > 0$, such that $\forall j \in [n]$,

$$\max\left(P\left(w_t^{(j)} < -\bar{w} + \epsilon\right), P\left(w_t^{(j)} > \bar{w} - \epsilon\right)\right) < C\epsilon.$$

Theorem 5 (our work). Given a single trajectory $\{x_t\}_{t\in[T]}$. $\mathcal{F}_T$ denotes the σ-algebra generated by $\{x_t\}_{t\in[T]}$ and $\hat{A}_T$denotes a $\mathcal{F}_T$ -measurable estimator. Then, $\forall \delta \in (0,1)$ and small $\epsilon > 0$,

$$\sup_{\hat{A}_T} \inf_{A\in R^{n\times n}} P(||\hat{A}_T - A||_2 < \epsilon) \geq 1 - \delta \quad \text{only if} \quad T > \frac{1}{4\bar{w}C\epsilon}(1 - \frac{2\delta}{n}).$$

(The best estimator can achieve $\Omega(\frac{1}{\epsilon})$)

The ordinary least squares (OLS) for scalar case:

$$\hat{a}_T^{OLS} = \mathrm{argmin}_{\mathrm{a}} \sum_{t=1}^{T-1} \left|\left|x_{t+1} - a x_t\right|\right|^2$$

Theorem 6 (our work). Assume $|a| < 1$. Then, $\forall \delta \in (0,1)$ and small $\epsilon > 0$,

$$P(||\hat{a}_T^{OLS} - a||_2 < \epsilon) \geq 1 - \delta \quad \text{only if} \quad T > \Omega(\frac{1}{\epsilon^2}).$$

(OLS only achieves $\Omega(\frac{1}{\epsilon^2})$)

The set membership estimator (SME) based on $\{x_t\}_{t\in[T]}$

$$\mathcal{S}_\mathrm{T} = \left\{A \in \mathbb{R}^{n\times n}: \left|\left|x_{t+1} - A x_t\right|\right|_\infty \leq \bar{w}, \forall t \in [T-1]\right\}.$$

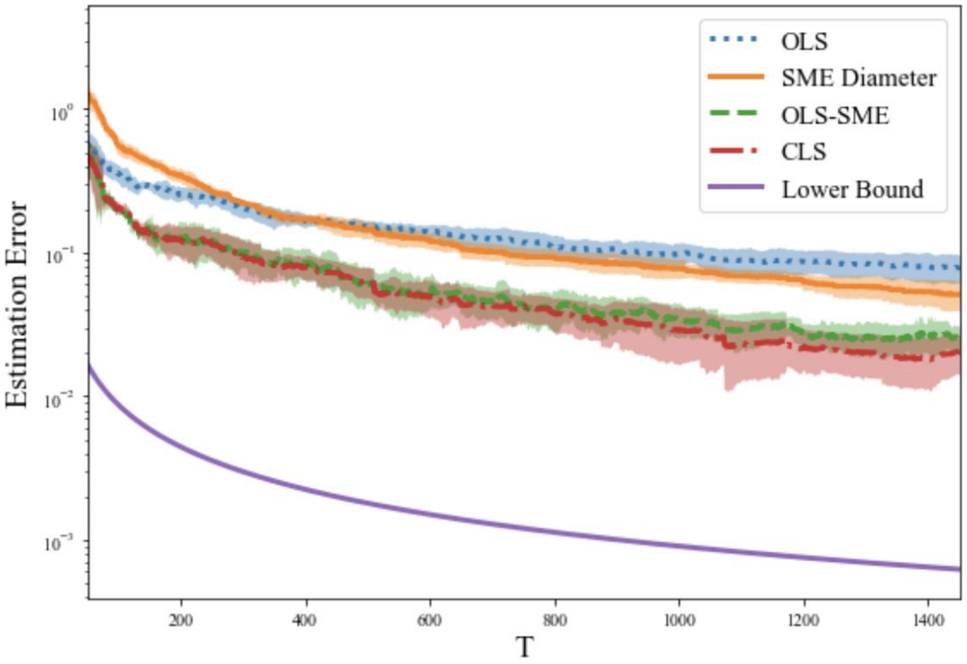
Theorem 7 (Li & Yu et al., ICML 2024). $\forall \delta \in (0,1)$ and small $\epsilon > 0$, with or without knowing $\bar{w}$,

$$\forall \hat{A}_T \in \mathcal{S}_\mathrm{T}, \ \ P(||\hat{A}_T - A||_2 < \epsilon) \geq 1 - \delta \quad \text{if} \quad T > \Omega(\frac{1}{\epsilon}).$$

(SME achieves the optimal $\Omega(\frac{1}{\epsilon})$)

Summary:

		Minimax Lower Bound	Lower Bound for OLS
Regression	Gaussian	$\Omega(1/\sqrt{T})$ (Wainwright, 2019)	$\Omega(1/\sqrt{T})$ (Mourtada, 2022)
	Bounded	$\Omega(1/T)$ (Yi & Neykov, 2024)	$\Omega(1/\sqrt{T})$ (Rudelson & Vershynin, 2008)
LTI Sys Id	Gaussian	$\Omega(1/\sqrt{T})$ (Jedra & Proutiere, 2019)	$\Omega(1/\sqrt{T})$ (Tu et al., 2024)
	Bounded	$\Omega(1/T)$ (Theorem 5)	$\Omega(1/\sqrt{T})$ (Theorem 6)



OLS: $\min_A \sum_{t=1}^{T-1} ||x_{t+1} - Ax_t||^2$

SME: $\mathcal{S}_T = \left\{A \in \mathbb{R}^{n\times n}: ||x_{t+1} - Ax_t||_\infty \le \bar{w}, \forall t \in [T]\right\}$

OLS-SME: $\min_{A\in\mathcal{S}_T} \left|\left|A - \hat{A}_T^{OLS}\right|\right|$

CLS: $\min_{A\in\mathcal{S}_T} \sum_{t=1}^{T-1} ||x_{t+1} - Ax_t||^2$

OLS achieves $O(\frac{1}{\sqrt{T}})$ ☹

SME, OLS-SME, and CLS achieves $O(\frac{1}{T})$ ☺

Chapter 4

On the Hardness of Learning to Stabilize Linear Systems

Given an unknown LTI system

$$x_{t+1} = Ax_t + Bu_t + w_t,$$

where $x_t, w_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, and $w_t \sim \mathcal{N}(0, \sigma_w^2 I_n)$. Assume (A, B) is controllable the energy bounded input $\mathbb{E}[\|u_t\|^2] \leq \sigma_u^2$.

Consider a learning to stabilize algorithm π that

- interacts with the above system for T units of time, and
- outputs a **linear static state feedback** controller $\widehat{K}_T$ at time T.

We want $\rho\left(A + B\widehat{K}_T\right) < 1$.

Theorem 7 (our work). Then $\forall \delta \in [0,0.5]$, we have

$$\sup_{\pi} \inf_{(A,B)} P((A + B\,\widehat{K}_T) \text{ is stable}) \geq 1 - \delta,$$

only if

$$\mathrm{T} \geq \Omega(exp(n)).$$

($exp(n)$ hardness of learning to stabilize with easy SysId)

Comparison with the previous work

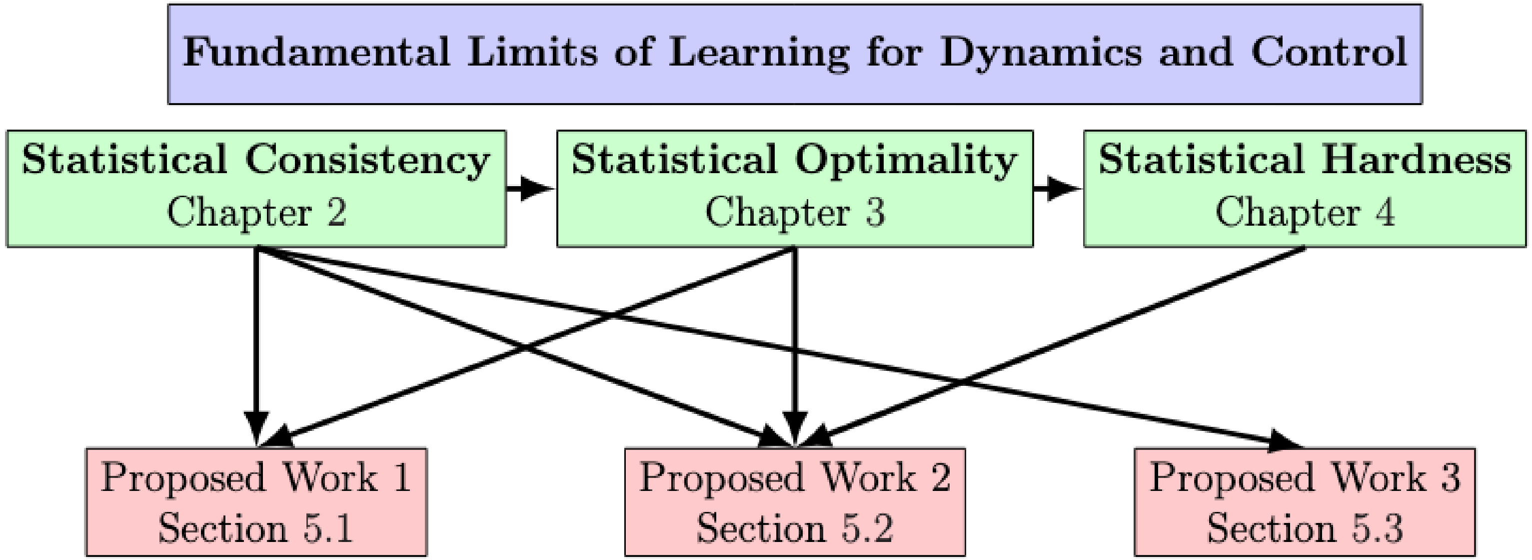
Tsiamis et al., COLT 2022	Hard to identify, then hard to learn to stabilize
Our Work	Hard to distinguish and hard to co-stabilize, then hard to learn to stabilize

Summary

1. System identification is easy, but learning to stabilize by linear static state feedback is still exponentially hard with the state dimension

2. We can try other state feedback, like **online switching** (**Proposed Work 1**) or **historical state feedback** (**Proposed Work 2**)

Summary and Timeline



References

[3] De Persis & Tesi, “Formulas for data-driven control: Stabilization, optimality, and robustness,” TAC, 2019.

[4] De Persis & Tesi, “Low-complexity learning of linear quadratic regulators from noisy data,” Automatica, 2021.

[5] X. Zeng, L. Bako, and N. Ozay, “Noise sensitivity of direct data-driven linear quadratic regulator by semidefinite programming,” ACC 2025 and long version is submitted to TAC.

[6]. Waarde et al., “From noisy data to feedback controllers: Nonconservative design via a matrix s-lemma,” IEEE Transactions on Automatic Control, 2020.

[7]. Waarde et al., “Quadratic matrix inequalities with applications to data-based control,” SIAM Journal on Control and Optimization, 2023.

[8]. A. Tsiamis, I. M. Ziemann, M. Morari, N. Matni, and G. J. Pappas, “Learning to control linear systems can be hard,” COLT 2022.

[9]. Y. Li, J. Yu, L. Conger, T. Kargin, and A. Wierman, “Learning the uncertainty sets of linear control systems via set membership: A non-asymptotic analysis,” ICML 2024.

[10]. Zeng, X., Liu, Z., Du, Z., Ozay, N., & Sznaier, M. On the hardness of learning to Stabilize linear systems. CDC 2023.

[11]. Zeng, X., Yu, J., & Ozay, N. (2025). System Identification Under Bounded Noise: Optimal Rates Beyond Least Squares. ArXiv 2025.