

# Model Predictive Path Integral Control: A Tutorial

Xiong Zeng

January 13, 2026

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- 3 Convergence for Quadratic Programming

# Introduction

Model predictive path integral (**MPPI**) control is a powerful and popular controller method in robotics and control, because

- it can handle **nonconvex** and **non-smooth** optimization problems,
- and it can be computed in **real-time** with GPU due to its **parallel computation** nature.

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Many applications:

- Aggressive driving <sup>1</sup>
- Legged robot control<sup>2</sup>
- Sim2real dexterous retargeting <sup>3</sup>
- Hard control problems in my research (MPPI > Policy gradient > SDP)

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The iteration formula of MPPI is

$$\mathbf{u}_{out} = \frac{\sum_{i=1}^N \mathbf{u}_i \exp(-J(\mathbf{u}_i)/\lambda)}{\sum_{i=1}^N \exp(-J(\mathbf{u}_i)/\lambda)}, \quad (2)$$

where  $\lambda$  is a hyperparameter (called temperature from simulated annealing) and  $\{\mathbf{u}_i\}_{i=1}^N$  are i.i.d. samples from the following Gaussian distribution  $\mathcal{N}(\mathbf{u}_{in}, \mathbf{\Sigma})$ , with the current controller  $\mathbf{u}_{in}$  and  $\mathbf{\Sigma} \succ 0$ .

## Derive MPPI Iteration Based on KL Control<sup>4</sup>

Let  $q(\mathbf{u})$  be a given probability density (for  $\mathbf{u}_{in}$ ) on  $\mathbb{R}^k$ . Let  $J : \mathbb{R}^k \rightarrow \mathbb{R}$  be measurable and  $\lambda > 0$ . We consider the optimization problem over probability densities  $p(\mathbf{u})$ :

$$\begin{aligned} \min_p \quad & \int p(\mathbf{u}) J(\mathbf{u}) d\mathbf{u} + \lambda \int p(\mathbf{u}) \log \frac{p(\mathbf{u})}{q(\mathbf{u})} d\mathbf{u} \\ \text{s.t. } \quad & p(\mathbf{u}) \geq 0, \quad \int p(\mathbf{u}) d\mathbf{u} = 1. \end{aligned} \tag{3}$$

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Define

$$p^*(\mathbf{u}) := \frac{1}{\mathbb{E}_q \left[ \exp \left( -\frac{1}{\lambda} J(\mathbf{u}) \right) \right]} q(\mathbf{u}) \exp \left( -\frac{1}{\lambda} J(\mathbf{u}) \right). \tag{4}$$

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**Goal:** prove that  $p^*$  is the unique minimizer of (3).

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## Variation Calculus

**Proof.** Introduce a Lagrange multiplier  $\eta \in \mathbb{R}$ . Define Lagrangian

$$\mathcal{L}(p, \eta) = \int p(\mathbf{u}) J(\mathbf{u}) d\mathbf{u} + \lambda \int p(\mathbf{u}) \log \frac{p(\mathbf{u})}{q(\mathbf{u})} d\mathbf{u} + \eta \left( \int p(\mathbf{u}) d\mathbf{u} - 1 \right). \quad (5)$$

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We compute stationarity with respect to  $p(\mathbf{u})$ . Let  $p_\epsilon = p + \epsilon h$  with  $\int h(\mathbf{u}) d\mathbf{u} = 0$ . Using  $\frac{d}{dx}(x \log x) = 1 + \log x$ , we obtain

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathcal{L}(p_\epsilon, \eta) = \int h(\mathbf{u}) \left[ J(\mathbf{u}) + \lambda (1 + \log p(\mathbf{u}) - \log q(\mathbf{u})) + \eta \right] d\mathbf{u}.$$

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For this to vanish for all admissible  $h$ , the bracket must be constant:

$$J(\mathbf{u}) + \lambda(1 + \log p(\mathbf{u}) - \log q(\mathbf{u})) = c. \quad (6)$$

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Rearranging,

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so

$$p(\mathbf{u}) = \frac{1}{\mathbb{E}_q\left[\exp\left(-\frac{1}{\lambda} J(\mathbf{u})\right)\right]} q(\mathbf{u}) \exp\left(-\frac{1}{\lambda} J(\mathbf{u})\right) = p^*(\mathbf{u}).$$

## From Optimal Distribution to Optimal Controller

KL-control yields an *optimal distribution*  $p^*(\mathbf{u})$ , but controller implementation must output a deterministic value  $\bar{\mathbf{u}}$ . Adopt the posterior squared-loss criterion

$$\bar{\mathbf{u}}^* \in \arg \min_{\bar{\mathbf{u}} \in \mathbb{R}^{mH}} \mathbb{E}_{p^*}[\|\mathbf{u} - \bar{\mathbf{u}}\|_2^2]. \quad (7)$$

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Expand the risk:

$$\mathbb{E}_{p^*}[\|\mathbf{u} - \bar{\mathbf{u}}\|_2^2] = \mathbb{E}_{p^*}[\|\mathbf{u}\|_2^2] - 2\bar{\mathbf{u}}^\top \mathbb{E}_{p^*}[\mathbf{u}] + \|\bar{\mathbf{u}}\|_2^2. \quad (8)$$

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Differentiate w.r.t.  $\bar{\mathbf{u}}$ :

$$\nabla_{\bar{\mathbf{u}}} = -2\mathbb{E}_{p^*}[\mathbf{u}] + 2\bar{\mathbf{u}}. \quad (9)$$

Setting  $\nabla_{\bar{\mathbf{u}}} = 0$  gives

$$\bar{\mathbf{u}}^* = \mathbb{E}_{p^*}[\mathbf{u}].$$

Since the Hessian is  $2I \succ 0$ , this minimizer is unique.

## From Optimal Controller to the MPPI Estimator

From the KL-regularized optimization over distributions, the unique minimizer is

$$p^*(\mathbf{u}) = \frac{1}{\mathbb{E}_q\left[\exp\left(-\frac{1}{\lambda}J(\mathbf{u})\right)\right]} q(\mathbf{u}) \exp\left(-\frac{1}{\lambda}J(\mathbf{u})\right). \quad (10)$$

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MPPI uses the *posterior mean* under  $p^*$ :

$$\mathbf{u}_{\text{out}} = \mathbb{E}_{p^*}[\mathbf{u}] = \int \mathbf{u} p^*(\mathbf{u}) d\mathbf{u} = \int \frac{\mathbf{u} q(\mathbf{u}) \exp\left(-\frac{1}{\lambda}J(\mathbf{u})\right)}{\mathbb{E}_q\left[\exp\left(-\frac{1}{\lambda}J(\mathbf{u})\right)\right]} d\mathbf{u} = \frac{\mathbb{E}_q\left[\mathbf{u} \exp\left(-\frac{1}{\lambda}J(\mathbf{u})\right)\right]}{\mathbb{E}_q\left[\exp\left(-\frac{1}{\lambda}J(\mathbf{u})\right)\right]}. \quad (11)$$

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Now draw  $N$  i.i.d. samples  $\mathbf{u}_i \sim q$ . By Monte-Carlo approximation of expectations (LLN), the MPPI (self-normalized importance sampling) estimator is

$$\hat{\mathbf{u}}_{\text{out}} = \frac{\sum_{i=1}^N \mathbf{u}_i \exp\left(-J(\mathbf{u}_i)/\lambda\right)}{\sum_{i=1}^N \exp\left(-J(\mathbf{u}_i)/\lambda\right)}. \quad (12)$$

# Convergence for Quadratic Programming

Consider a QP

$$\min_{\mathbf{u}} J(\mathbf{u}) = \mathbf{u}^\top \mathbf{D} \mathbf{u} + \mathbf{u}^\top \mathbf{d}, \quad (13)$$

with  $\mathbf{D} \succ 0$ . Draw  $N$  i.i.d. samples  $\mathbf{u}_i \sim q = \mathcal{N}(\mathbf{u}_{in}, \mathbf{\Sigma})$  with  $\mathbf{\Sigma} \succ 0$ .

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<sup>5</sup>Yi et al. CoVO-MPC: Theoretical Analysis of Sampling-based MPC and Optimal Covariance Design. L4DC 2024.

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**Theorem 1.**<sup>5</sup> Consider the minimize  $\mathbf{u}^* = \frac{-1}{2} \mathbf{d} \mathbf{D}^{-1}$  of (13). We have

$$\hat{\mathbf{u}}_{out} \xrightarrow{P} \mathbf{u}_{out}, \quad (15)$$

and

$$\mathbf{u}_{out} - \mathbf{u}^* \xrightarrow{P} (\mathbf{I} + 2\lambda \mathbf{\Sigma} \mathbf{D})^{-1} (\mathbf{u}_{in} - \mathbf{u}^*). \quad (16)$$

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## Step 1: Write $p^*$ explicitly and complete the square

**Proof.** From the KL-control derivation,

$$\mathbf{u}_{out} = \mathbb{E}_{p^*}[\mathbf{u}], \quad p^*(\mathbf{u}) = \frac{1}{\mathbb{E}_q\left[\exp\left(-\frac{1}{\lambda}J(\mathbf{u})\right)\right]} q(\mathbf{u}) \exp\left(-\frac{1}{\lambda}J(\mathbf{u})\right). \quad (17)$$

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And  $q(\mathbf{u}) = \mathcal{N}(\mathbf{u}_{in}, \mathbf{\Sigma})$  yields

$$q(\mathbf{u}) \propto \exp\left(-\frac{1}{2}(\mathbf{u} - \mathbf{u}_{in})^\top \mathbf{\Sigma}^{-1}(\mathbf{u} - \mathbf{u}_{in})\right). \quad (18)$$

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Thus

$$p^*(\mathbf{u}) \propto \exp\left(-\frac{1}{\lambda}(\mathbf{u}^\top \mathbf{D}\mathbf{u} + \mathbf{u}^\top \mathbf{d}) - \frac{1}{2}(\mathbf{u} - \mathbf{u}_{in})^\top \mathbf{\Sigma}^{-1}(\mathbf{u} - \mathbf{u}_{in})\right). \quad (19)$$

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Expand the Gaussian exponent:

$$(\mathbf{u} - \mathbf{u}_{in})^\top \mathbf{\Sigma}^{-1}(\mathbf{u} - \mathbf{u}_{in}) = \mathbf{u}^\top \mathbf{\Sigma}^{-1}\mathbf{u} - 2\mathbf{u}^\top \mathbf{\Sigma}^{-1}\mathbf{u}_{in} + \mathbf{u}_{in}^\top \mathbf{\Sigma}^{-1}\mathbf{u}_{in}. \quad (20)$$

Then the exponent in  $p^*$  equals

$$-\frac{1}{2}\mathbf{u}^\top \left(\mathbf{\Sigma}^{-1} + \frac{2}{\lambda}\mathbf{D}\right)\mathbf{u} + \mathbf{u}^\top \mathbf{\Sigma}^{-1}\mathbf{u}_{in} - \frac{1}{\lambda}\mathbf{u}^\top \mathbf{d} + (\text{constant}). \quad (21)$$

## Step 2: Recognize $p^\star$ as a Gaussian and read off its mean

Let

$$\mathbf{A} := \boldsymbol{\Sigma}^{-1} + \frac{2}{\lambda} \mathbf{D} \succ 0, \quad \mathbf{b} := \boldsymbol{\Sigma}^{-1} \mathbf{u}_{in} - \frac{1}{\lambda} \mathbf{d}. \quad (22)$$

Drop constants independent of  $\mathbf{u}$ . Then

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By grouping the constant terms,  $p^*(\mathbf{u})$  is a p.d.f. of Gaussian with mean

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Substitute:

$$\mathbb{E}_{p^*}[\mathbf{u}] = \left(\boldsymbol{\Sigma}^{-1} + \frac{2}{\lambda} \mathbf{D}\right)^{-1} \left(\boldsymbol{\Sigma}^{-1} \mathbf{u}_{in} - \frac{1}{\lambda} \mathbf{d}\right). \quad (26)$$

### Step 3: Express the mean using $\mathbf{u}^*$ and simplify

$\nabla J(\mathbf{u}^*) = 0$  yields  $\mathbf{d} = -2\mathbf{D}\mathbf{u}^*$ . Then  $-\frac{1}{\lambda}\mathbf{d} = \frac{2}{\lambda}\mathbf{D}\mathbf{u}^*$ . Hence

$$\mathbb{E}_{p^*}[\mathbf{u}] = \left(\mathbf{\Sigma}^{-1} + \frac{2}{\lambda}\mathbf{D}\right)^{-1} \left(\mathbf{\Sigma}^{-1}\mathbf{u}_{in} + \frac{2}{\lambda}\mathbf{D}\mathbf{u}^*\right). \quad (27)$$

Therefore

$$\mathbf{u}_{out} = \mathbb{E}_{p^*}[\mathbf{u}] = \lambda(\lambda\mathbf{I} + 2\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\mathbf{u}_{in} + 2(\lambda\mathbf{I} + 2\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{\Sigma}\mathbf{D}\mathbf{u}^* \quad (28)$$

$$= (\lambda\mathbf{I} + 2\mathbf{\Sigma}\mathbf{D})^{-1} \left( \lambda\mathbf{u}_{in} + 2\mathbf{\Sigma}\mathbf{D}\mathbf{u}^* \right) \quad (29)$$

$$= (\lambda\mathbf{I} + 2\mathbf{\Sigma}\mathbf{D})^{-1}(\lambda\mathbf{I} + 2\mathbf{\Sigma}\mathbf{D})\mathbf{u}_{in} + (\lambda\mathbf{I} + 2\mathbf{\Sigma}\mathbf{D})^{-1}2\mathbf{\Sigma}\mathbf{D}(\mathbf{u}^* - \mathbf{u}_{in}) \quad (30)$$

$$= \mathbf{u}_{in} + (\mathbf{I} + 2\lambda\mathbf{\Sigma}\mathbf{D})^{-1}(\mathbf{u}^* - \mathbf{u}_{in}). \quad (31)$$

## Step 6: Convergence in probability and Slutsky

Let the MPPI estimator be

$$\mathbf{u}_{out}^{(N)} = \frac{\frac{1}{N} \sum_{i=1}^N \mathbf{u}_i w_i}{\frac{1}{N} \sum_{i=1}^N w_i}, \quad w_i := \exp\left(-\frac{1}{\lambda} J(\mathbf{u}_i)\right), \quad \mathbf{u}_i \stackrel{\text{i.i.d.}}{\sim} q. \quad (32)$$

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Then by the weak LLN,

$$\frac{1}{N} \sum_{i=1}^N w_i \xrightarrow{p} \mathbb{E}_q[w_1], \quad \frac{1}{N} \sum_{i=1}^N \mathbf{u}_i w_i \xrightarrow{p} \mathbb{E}_q[\mathbf{u}w]. \quad (33)$$

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Then, Slutsky's theorem yields

$$\hat{\mathbf{u}}_{out} \xrightarrow{p} \frac{\mathbb{E}_q[\mathbf{u}w]}{\mathbb{E}_q[w]} = \mathbb{E}_{p^*}[\mathbf{u}]. \quad (34)$$