

Model Predictive Path Integral Control: A Tutorial

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1 Introduction

2 Motivation and Optimality

3 Convergence for Quadratic Programming

Introduction

Model predictive path integral (**MPPI**) control is a powerful and popular controller method in robotics and control, because

- it can handle **nonconvex** and **non-smooth** optimization problems,
- and it can be computed in **real-time** with GPU due to its **parallel computation** nature.

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Many applications:

- Aggressive driving ¹
- Legged robot control²
- Sim2real dexterous retargeting ³
- Hard control problems in my research (MPPI > Policy gradient > SDP)

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The iteration formula of MPPI is

$$\mathbf{u}_{out} = \frac{\sum_{i=1}^N \mathbf{u}_i \exp(-J(\mathbf{u}_i)/\lambda)}{\sum_{i=1}^N \exp(-J(\mathbf{u}_i)/\lambda)}, \quad (2)$$

where λ is a hyperparameter (called temperature from simulated annealing) and $\{\mathbf{u}_i\}_{i=1}^N$ are i.i.d. samples from the following Gaussian distribution $\mathcal{N}(\mathbf{u}_{in}, \Sigma)$, with the current controller \mathbf{u}_{in} and $\Sigma \succ 0$.

Derive MPPI Iteration Based on KL Control⁴

Let $q(\mathbf{u})$ be a given probability density (for \mathbf{u}_{in}) on \mathbb{R}^k . Let $J : \mathbb{R}^k \rightarrow \mathbb{R}$ be measurable and $\lambda > 0$. We consider the optimization problem over probability densities $p(\mathbf{u})$:

$$\begin{aligned} \min_p & \int p(\mathbf{u}) J(\mathbf{u}) d\mathbf{u} + \lambda \int p(\mathbf{u}) \log \frac{p(\mathbf{u})}{q(\mathbf{u})} d\mathbf{u} \\ \text{s.t. } & p(\mathbf{u}) \geq 0, \quad \int p(\mathbf{u}) d\mathbf{u} = 1. \end{aligned} \tag{3}$$

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Define

$$p^*(\mathbf{u}) := \frac{1}{\mathbb{E}_q \left[\exp \left(-\frac{1}{\lambda} J(\mathbf{u}) \right) \right]} q(\mathbf{u}) \exp \left(-\frac{1}{\lambda} J(\mathbf{u}) \right). \tag{4}$$

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Goal: prove that p^* is the unique minimizer of (3).

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Variation Calculus

Proof. Introduce a Lagrange multiplier $\eta \in \mathbb{R}$. Define Lagrangian

$$\mathcal{L}(p, \eta) = \int p(\mathbf{u}) J(\mathbf{u}) d\mathbf{u} + \lambda \int p(\mathbf{u}) \log \frac{p(\mathbf{u})}{q(\mathbf{u})} d\mathbf{u} + \eta \left(\int p(\mathbf{u}) d\mathbf{u} - 1 \right). \quad (5)$$

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We compute stationarity with respect to $p(\mathbf{u})$. Let $p_\epsilon = p + \epsilon h$ with $\int h(\mathbf{u}) d\mathbf{u} = 0$. Using $\frac{d}{dx}(x \log x) = 1 + \log x$, we obtain

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} \mathcal{L}(p_\epsilon, \eta) = \int h(\mathbf{u}) \left[J(\mathbf{u}) + \lambda(1 + \log p(\mathbf{u}) - \log q(\mathbf{u})) + \eta \right] d\mathbf{u}.$$

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For this to vanish for all admissible h , the bracket must be constant:

$$J(\mathbf{u}) + \lambda(1 + \log p(\mathbf{u}) - \log q(\mathbf{u})) = c. \quad (6)$$

Solve the Stationarity Condition

Rearranging,

$$\log p(\mathbf{u}) = \log q(\mathbf{u}) - \frac{1}{\lambda} J(\mathbf{u}) + \frac{1}{c}.$$

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so

$$p(\mathbf{u}) = \frac{1}{\mathbb{E}_q\left[\exp\left(-\frac{1}{\lambda} J(\mathbf{u})\right)\right]} q(\mathbf{u}) \exp\left(-\frac{1}{\lambda} J(\mathbf{u})\right) = p^*(\mathbf{u}).$$

From Optimal Distribution to Optimal Controller

KL-control yields an *optimal distribution* $p^*(\mathbf{u})$, but controller implementation must output a deterministic value $\bar{\mathbf{u}}$. Adopt the posterior squared-loss criterion

$$\bar{\mathbf{u}}^* \in \arg \min_{\bar{\mathbf{u}} \in \mathbb{R}^{mH}} \mathbb{E}_{p^*} [\|\mathbf{u} - \bar{\mathbf{u}}\|_2^2]. \quad (7)$$

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Expand the risk:

$$\mathbb{E}_{p^*} [\|\mathbf{u} - \bar{\mathbf{u}}\|_2^2] = \mathbb{E}_{p^*} [\|\mathbf{u}\|_2^2] - 2\bar{\mathbf{u}}^\top \mathbb{E}_{p^*} [\mathbf{u}] + \|\bar{\mathbf{u}}\|_2^2. \quad (8)$$

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Differentiate w.r.t. $\bar{\mathbf{u}}$:

$$\nabla_{\bar{\mathbf{u}}} = -2\mathbb{E}_{p^*} [\mathbf{u}] + 2\bar{\mathbf{u}}. \quad (9)$$

Setting $\nabla_{\bar{\mathbf{u}}} = 0$ gives

$$\bar{\mathbf{u}}^* = \mathbb{E}_{p^*} [\mathbf{u}].$$

Since the Hessian is $2I \succ 0$, this minimizer is unique.

From Optimal Controller to the MPPI Estimator

From the KL-regularized optimization over distributions, the unique minimizer is

$$p^*(\mathbf{u}) = \frac{1}{\mathbb{E}_q \left[\exp \left(-\frac{1}{\lambda} J(\mathbf{u}) \right) \right]} q(\mathbf{u}) \exp \left(-\frac{1}{\lambda} J(\mathbf{u}) \right). \quad (10)$$

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MPPI uses the *posterior mean* under p^* :

$$\mathbf{u}_{\text{out}} = \mathbb{E}_{p^*} [\mathbf{u}] = \int \mathbf{u} p^*(\mathbf{u}) d\mathbf{u} = \int \frac{\mathbf{u} q(\mathbf{u}) \exp \left(-\frac{1}{\lambda} J(\mathbf{u}) \right)}{\mathbb{E}_q \left[\exp \left(-\frac{1}{\lambda} J(\mathbf{u}) \right) \right]} d\mathbf{u} = \frac{\mathbb{E}_q [\mathbf{u} \exp \left(-\frac{1}{\lambda} J(\mathbf{u}) \right)]}{\mathbb{E}_q \left[\exp \left(-\frac{1}{\lambda} J(\mathbf{u}) \right) \right]}. \quad (11)$$

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Now draw N i.i.d. samples $\mathbf{u}_i \sim q$. By Monte-Carlo approximation of expectations (LLN), the MPPI (self-normalized importance sampling) estimator is

$$\hat{\mathbf{u}}_{\text{out}} = \frac{\sum_{i=1}^N \mathbf{u}_i \exp(-J(\mathbf{u}_i)/\lambda)}{\sum_{i=1}^N \exp(-J(\mathbf{u}_i)/\lambda)}. \quad (12)$$

Convergence for Quadratic Programming

Consider a QP

$$\min_{\mathbf{u}} J(\mathbf{u}) = \mathbf{u}^\top \mathbf{D} \mathbf{u} + \mathbf{u}^\top \mathbf{d}, \quad (13)$$

with $\mathbf{D} \succ 0$. Draw N i.i.d. samples $\mathbf{u}_i \sim q = \mathcal{N}(\mathbf{u}_{in}, \Sigma)$ with $\Sigma \succ 0$.

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Theorem 1.⁵ Consider the minimize $\mathbf{u}^* = \frac{-1}{2} \mathbf{d} \mathbf{D}^{-1}$ of (13). We have

$$\hat{\mathbf{u}}_{out} \xrightarrow{P} \mathbf{u}_{out}, \quad (15)$$

and

$$\mathbf{u}_{out} - \mathbf{u}^* \xrightarrow{P} (\mathbf{I} + 2\lambda \Sigma \mathbf{D})^{-1} (\mathbf{u}_{in} - \mathbf{u}^*). \quad (16)$$

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Step 1: Write p^* explicitly and complete the square

Proof. From the KL-control derivation,

$$\mathbf{u}_{out} = \mathbb{E}_{p^*}[\mathbf{u}], \quad p^*(\mathbf{u}) = \frac{1}{\mathbb{E}_q\left[\exp\left(-\frac{1}{\lambda}J(\mathbf{u})\right)\right]} q(\mathbf{u}) \exp\left(-\frac{1}{\lambda}J(\mathbf{u})\right). \quad (17)$$

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And $q(\mathbf{u}) = \mathcal{N}(\mathbf{u}_{in}, \boldsymbol{\Sigma})$ yields

$$q(\mathbf{u}) \propto \exp\left(-\frac{1}{2}(\mathbf{u} - \mathbf{u}_{in})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{u} - \mathbf{u}_{in})\right). \quad (18)$$

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Thus

$$p^*(\mathbf{u}) \propto \exp\left(-\frac{1}{\lambda}(\mathbf{u}^\top \mathbf{D}\mathbf{u} + \mathbf{u}^\top \mathbf{d}) - \frac{1}{2}(\mathbf{u} - \mathbf{u}_{in})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{u} - \mathbf{u}_{in})\right). \quad (19)$$

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Expand the Gaussian exponent:

$$(\mathbf{u} - \mathbf{u}_{in})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{u} - \mathbf{u}_{in}) = \mathbf{u}^\top \boldsymbol{\Sigma}^{-1}\mathbf{u} - 2\mathbf{u}^\top \boldsymbol{\Sigma}^{-1}\mathbf{u}_{in} + \mathbf{u}_{in}^\top \boldsymbol{\Sigma}^{-1}\mathbf{u}_{in}. \quad (20)$$

Then the exponent in p^* equals

$$-\frac{1}{2}\mathbf{u}^\top \left(\boldsymbol{\Sigma}^{-1} + \frac{2}{\lambda}\mathbf{D}\right)\mathbf{u} + \mathbf{u}^\top \boldsymbol{\Sigma}^{-1}\mathbf{u}_{in} - \frac{1}{\lambda}\mathbf{u}^\top \mathbf{d} + (\text{constant}). \quad (21)$$

Step 2: Recognize p^* as a Gaussian and read off its mean

Let

$$\mathbf{A} := \boldsymbol{\Sigma}^{-1} + \frac{2}{\lambda} \mathbf{D} \succ 0, \quad \mathbf{b} := \boldsymbol{\Sigma}^{-1} \mathbf{u}_{in} - \frac{1}{\lambda} \mathbf{d}. \quad (22)$$

Drop constants independent of \mathbf{u} . Then

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Complete the square:

$$-\frac{1}{2}\mathbf{u}^\top \mathbf{A} \mathbf{u} + \mathbf{u}^\top \mathbf{b} = -\frac{1}{2}(\mathbf{u} - \mathbf{A}^{-1} \mathbf{b})^\top \mathbf{A} (\mathbf{u} - \mathbf{A}^{-1} \mathbf{b}) + \frac{1}{2} \mathbf{b}^\top \mathbf{A}^{-1} \mathbf{b}. \quad (24)$$

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By grouping the constant terms, $p^*(\mathbf{u})$ is a p.d.f. of Gaussian with mean

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By grouping the constant terms, $p^*(\mathbf{u})$ is a p.d.f. of Gaussian with mean

$$\mathbb{E}_{p^*}[\mathbf{u}] = \mathbf{A}^{-1} \mathbf{b}. \quad (25)$$

Substitute:

$$\mathbb{E}_{p^*}[\mathbf{u}] = \left(\boldsymbol{\Sigma}^{-1} + \frac{2}{\lambda} \mathbf{D}\right)^{-1} \left(\boldsymbol{\Sigma}^{-1} \mathbf{u}_{in} - \frac{1}{\lambda} \mathbf{d}\right). \quad (26)$$

Step 3: Express the mean using \mathbf{u}^* and simplify

$\nabla J(\mathbf{u}^*) = 0$ yeilds $\mathbf{d} = -2\mathbf{D}\mathbf{u}^*$. Then $-\frac{1}{\lambda}\mathbf{d} = \frac{2}{\lambda}\mathbf{D}\mathbf{u}^*$. Hence

$$\mathbb{E}_{p^*}[\mathbf{u}] = \left(\boldsymbol{\Sigma}^{-1} + \frac{2}{\lambda} \mathbf{D} \right)^{-1} \left(\boldsymbol{\Sigma}^{-1} \mathbf{u}_{in} + \frac{2}{\lambda} \mathbf{D}\mathbf{u}^* \right). \quad (27)$$

Therefore

$$\mathbf{u}_{out} = \mathbb{E}_{p^*}[\mathbf{u}] = \lambda(\lambda\mathbf{I} + 2\boldsymbol{\Sigma}\mathbf{D})^{-1} \boldsymbol{\Sigma} \boldsymbol{\Sigma}^{-1} \mathbf{u}_{in} + 2(\lambda\mathbf{I} + 2\boldsymbol{\Sigma}\mathbf{D})^{-1} \boldsymbol{\Sigma} \mathbf{D} \mathbf{u}^* \quad (28)$$

$$= (\lambda\mathbf{I} + 2\boldsymbol{\Sigma}\mathbf{D})^{-1} \left(\lambda \mathbf{u}_{in} + 2\boldsymbol{\Sigma} \mathbf{D} \mathbf{u}^* \right) \quad (29)$$

$$= (\lambda\mathbf{I} + 2\boldsymbol{\Sigma}\mathbf{D})^{-1} (\lambda\mathbf{I} + 2\boldsymbol{\Sigma}\mathbf{D}) \mathbf{u}_{in} + (\lambda\mathbf{I} + 2\boldsymbol{\Sigma}\mathbf{D})^{-1} 2\boldsymbol{\Sigma} \mathbf{D} (\mathbf{u}^* - \mathbf{u}_{in}) \quad (30)$$

$$= \mathbf{u}_{in} + (\mathbf{I} + 2\lambda\boldsymbol{\Sigma}\mathbf{D})^{-1} (\mathbf{u}^* - \mathbf{u}_{in}). \quad (31)$$

Step 6: Convergence in probability and Slutsky

Let the MPPI estimator be

$$\mathbf{u}_{out}^{(N)} = \frac{\frac{1}{N} \sum_{i=1}^N \mathbf{u}_i w_i}{\frac{1}{N} \sum_{i=1}^N w_i}, \quad w_i := \exp\left(-\frac{1}{\lambda} J(\mathbf{u}_i)\right), \quad \mathbf{u}_i \stackrel{\text{i.i.d.}}{\sim} q. \quad (32)$$

Step 6: Convergence in probability and Slutsky

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Then by the weak LLN,

$$\frac{1}{N} \sum_{i=1}^N w_i \xrightarrow{P} \mathbb{E}_q[w_1], \quad \frac{1}{N} \sum_{i=1}^N \mathbf{u}_i w_i \xrightarrow{P} \mathbb{E}_q[\mathbf{u}w]. \quad (33)$$

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Then, Slutsky's theorem yields

$$\hat{\mathbf{u}}_{out} \xrightarrow{P} \frac{\mathbb{E}_q[\mathbf{u}w]}{\mathbb{E}_q[w]} = \mathbb{E}_{p^\star}[\mathbf{u}]. \quad (34)$$