Noise Sensitivity of the Semidefinite Programs for Direct Data-Driven LQR

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Given a discrete-time linear time-invariant (LTI) system:

$$x_{t+1} = Ax_t + Bu_t + w_t,$$

where $x_t \in \mathbb{R}^n, u_t \in \mathbb{R}^m, w_t \in \mathbb{R}^n$.

Assumptions:

1. $w_t \sim \mathcal{N}(0, \sigma_w^2 I_n)$, $\sigma_w \geq 0$

2. (A, B) is controllable

Linear quadratic regulator (LQR) problem:

$$\min_{u_0,u_1,\dots} \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}(x_t^T Q x_t + u_t^T R u_t)\right]$$

s.t. the previous LTI system,

where $Q \succ 0, R \succ 0$.

When (A, B) is **known**, the above optimal solution is $u_t = K_{lqr} x_t$, where

$$K_{lqr} = -\left(R + B^T P B\right)^{-1} B^T P A,$$

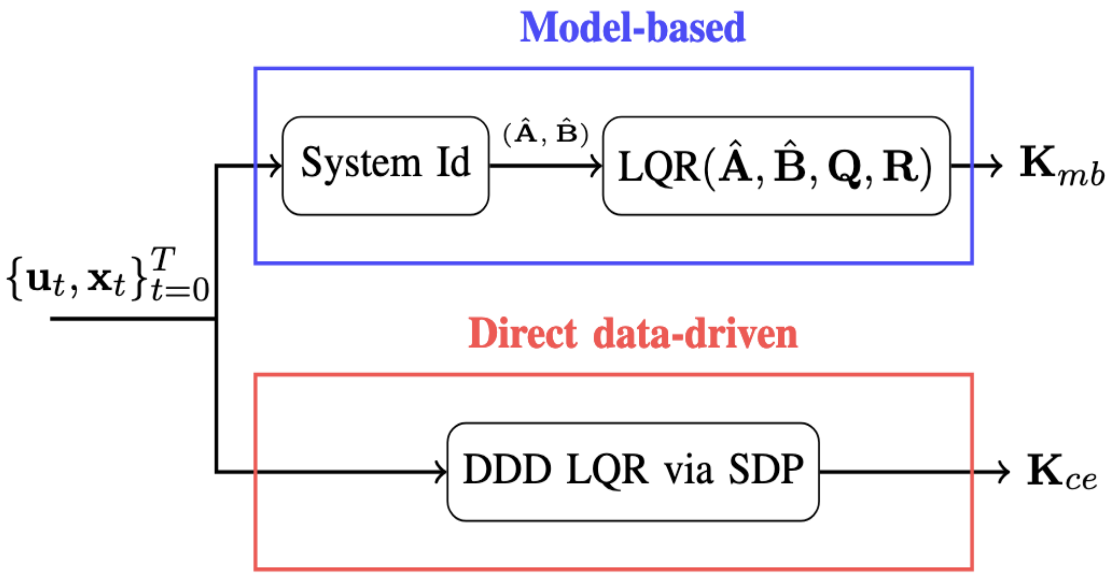
where P is the solution of discrete-time algebraic Riccati equation.

When (A, B) is **unknown**, we need **learning-based control**!

Two **paradigms** for learning-based control:

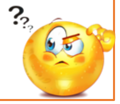
1. Certainty equivalence (CE): ***pretend*** that data is noise-free, utilize the estimated model or the estimated control policy directly

2. Robust control: bound the effect of the noise, find a controller that achieves the desired properties for ***all possible*** estimates



- **Model-based:** Certainty equivalence is statistically consistent and more sample-efficient than robust control (Mania et al., Dean et al.)

- **Model-free:** Statistical properties of CE and robust direct data-driven (DDD) LQR are still unclear



CE DDD LQR (De Persis & Tesi 2019):

$$\min_{X,Y} \ \mathrm{trace}(QX_0Y) + \mathrm{trace}(X)$$

$$\text{s.t.} \begin{bmatrix} X_0Y - I_n & X_1Y \\ Y^TX_1^T & X_0Y \end{bmatrix} \succcurlyeq 0$$

$$\begin{bmatrix} X & \sqrt{R}U_0Y \\ \left(\sqrt{R}U_0Y\right)^T & X_0Y \end{bmatrix} \succcurlyeq 0.$$

Let Y_{ce}^* denote its optimal solution,

$$K_{ce} := -U_0Y_{ce}^*(X_0Y_{ce}^*)^{-1}.$$

Data matrices:

$$X_0 = [x_0 \ x_1 \ \ldots x_{T-1}],$$
$$U_0 = [u_0 \ u_1 \ \ldots u_{T-1}],$$
$$X_1 = [x_1 \ x_2 \ \ldots x_T],$$

where $u_t \sim N(0, \sigma_u^2 \mathbf{I}_m)$.

Theorem 1 (De Persis & Tesi 2019). Let $\mathrm{rank}\left(\begin{bmatrix} X_0 \\ U_0 \end{bmatrix}\right) = m + n$. When $w_t = 0$ for all t,

$$K_{ce} = K_{lqr}.$$

(CE DDD LQR is perfect for noiseless case)

Theorem 2 (our work). Assume $\sigma_w^2 > 0$. When $T \geq (m+n)(n+1)+n$,

$$P(K_{ce} = \mathbf{0}_{\mathbf{m}\times\mathbf{n}}) = 1.$$

(CE DDD LQR is trivial for almost all noise)

Key Observation: the following equalities always hold when $\sigma_w > 0$

$$\begin{cases} U_0 Y_{ce}^* = \mathbf{0}_{\mathbf{m}\times\mathbf{n}} \\ X_0 Y_{ce}^* = \mathbf{I}_{\mathbf{n}} \\ X_1 Y_{ce}^* = \mathbf{0}_{\mathbf{n}\times\mathbf{n}} \end{cases},$$

for any $T \geq (m+n)(n+1)+n$.

Robustness-Promoting (RP) DDD LQR (De Persis & Tesi 2021):

$$\min_{X,Y,S} \ \mathrm{trace}(QX_0Y) + \mathrm{trace}(X) + \mathrm{trace}(S)$$

$$\text{s.t.} \begin{bmatrix} X_0Y - I_n & X_1Y \\ Y^T X_1^T & X_0Y \end{bmatrix} \succcurlyeq 0$$

$$\begin{bmatrix} X & \sqrt{R}U_0Y \\ \left(\sqrt{R}U_0Y\right)^T & X_0Y \end{bmatrix} \succcurlyeq 0$$

$$\begin{bmatrix} S & Y \\ Y^T & X_0Y \end{bmatrix} \succcurlyeq 0.$$

Let Y_{rp}^* denote its optimal solution,

$$K_{rp} \coloneqq -U_0Y_{rp}^*\left(X_0Y_{rp}^*\right)^{-1}.$$

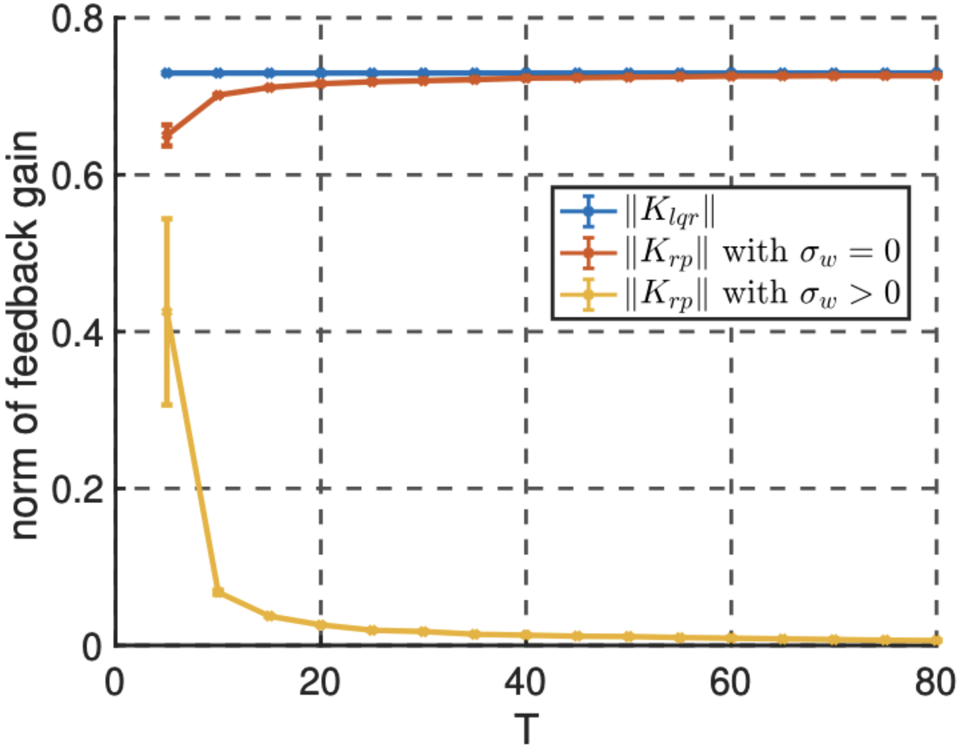
Theorem 3 (our work). Assume $\sigma_w^2 > 0$,

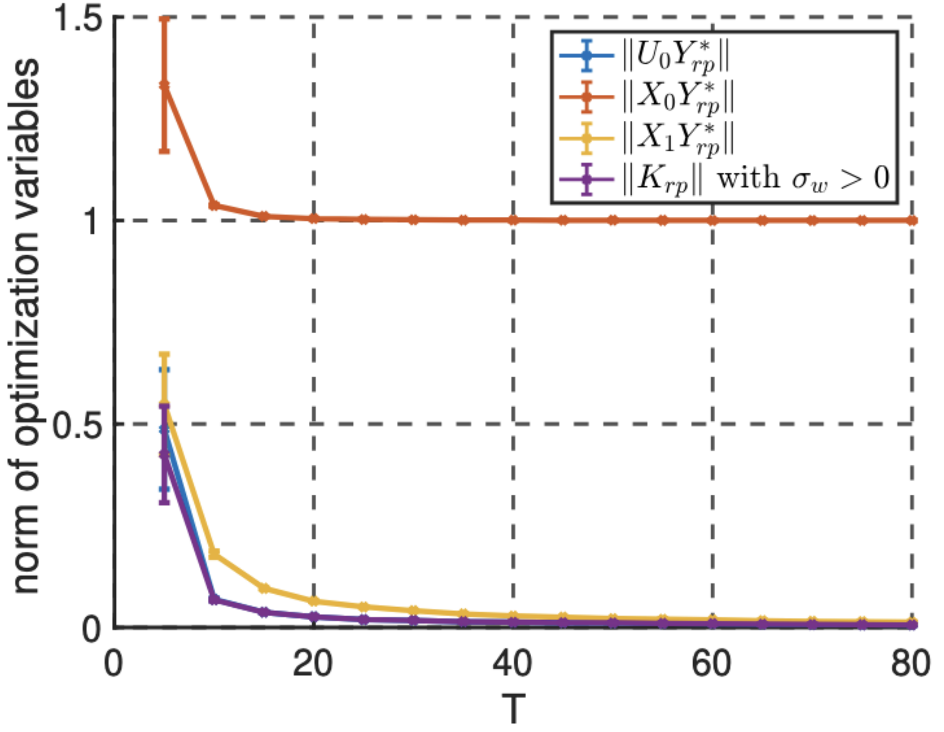
$$K_{rp} \xrightarrow{p} \boldsymbol{0}_{\boldsymbol{m \times n}} \,.$$

(RP DDD LQR is not statistically consistent)

Key Observation: the following equalities always hold when $\sigma_w^2 > 0$

$$\begin{cases} U_0 Y_{rp}^* \xrightarrow{p} \boldsymbol{0}_{\boldsymbol{m \times n}} \\ X_0 Y_{rp}^* \xrightarrow{p} \boldsymbol{I}_{\boldsymbol{n}} \\ X_1 Y_{rp}^* \xrightarrow{p} \boldsymbol{0}_{\boldsymbol{n \times n}} \end{cases} .$$





Experiments for RP DDD LQR: Consider an order-2 single-input unstable system, for which $K_{lqr} = [-0.7112\ -0.2046]$. $\sigma_w^2 = 1$ and $\sigma_u^2 = 1$.

On-Going and Future Work:

Statistical analysis of **robust direct data-driven control** by matrix S-lemma (Waarde et al. 2020, Waarde et al. 2023)

Remark: There are some recent works (Dorfler et al. 2023, Zhao et al. 2024) for direct data-driven control designs, whose optimization problems are **equivalent** to model-based CE control. Thus, their statistical analyses are the same with model-based CE control.

References:

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Thank you!

Two **performance measures** for learning-based control:

1. **Statistical consistency**: When the number of samples approaches ∞, does the estimation error approach 0?

2. **Sample complexity**: How many samples do we need to get a "good" estimate?