Noise Sensitivity of Direct Data-Driven Linear Quadratic Regulator by Semidefinite Programming

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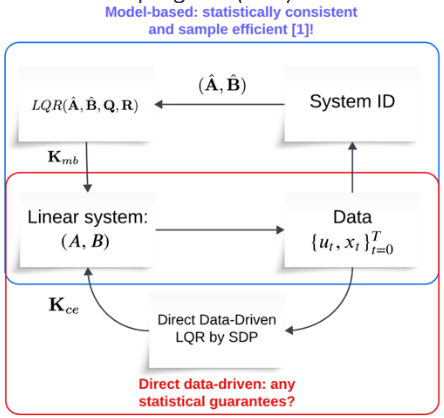
Introduction: We study the noise sensitivity of two kinds of **direct data-driven** (DDD) infinite horizon linear quadratic regulator (LQR) problems: **certainty equivalence** (CE) and **robustness-promoting** (RP) cases, both of which are based on the semidefinite program (SDP).



Problem Setup: Given a discrete-time linear time-invariant (LTI) system:

$$x_{t+1} = Ax_t + Bu_t + w_t,$$

where $x_t \in \mathbb{R}^n, u_t \in \mathbb{R}^m, w_t \sim N(0, \sigma_w^2 \boldsymbol{I}_n),\ \sigma_w^2 \geq 0$. (A, B) is controllable.

LQR problem:

$$\min_{u_0,u_1,\ldots} \lim_{T\to\infty} [\frac{1}{T}\sum_{t=0}^{T}(x_t^T Q x_t + u_t^T R u_t)]$$

s.t. the above LTI system,

where $Q \succ 0, R \succ 0$.

When $(\boldsymbol{A},\boldsymbol{B})$ is known, the above optimal solution is $u_t = -K_{lqr}x_t$ and

$$K_{lqr} = -(R + B^T P B)^{-1} B^T P A,$$

where P is the solution of discrete-time algebraic Riccati equation.

When $(\boldsymbol{A},\boldsymbol{B})$ is unknown, based on the following data matrices:

$$X_0 = [x_0\ x_1\ \ldots x_{T-1}],$$
$$U_0 = [u_0\ u_1\ \ldots u_{T-1}],$$
$$X_1 = [x_1\ x_2\ \ldots x_T],$$

where $u_t \sim N(0, \sigma_u^2 \boldsymbol{I}_m)$, we use DDD LQR to estimate K_{lqr}.

CE DDD LQR ([2]):

$$\min_{X,Y} trace(QX_0Y) + trace(X)$$

$$\text{s.t.}\quad \begin{bmatrix} X_0Y - I_n & X_1Y \\ Y^T X_1^T & X_0Y \end{bmatrix} \succcurlyeq 0$$

$$\begin{bmatrix} X & \sqrt{R}U_0Y \\ (\sqrt{R}U_0Y)^T & X_0Y \end{bmatrix} \succcurlyeq 0.$$

Let Y_{ce}^* denote its optimal solution,

$$K_{ce} := -U_0 Y_{ce}^* (X_0 Y_{ce}^*)^{-1}.$$

Theorem 1 ([2]). Let $[X_0; U_0]$ be full row rank. When $\sigma_w^2 = 0$, $K_{ce} = K_{lqr}$.
(CE DDD LQR is perfect for noiseless case ☺)

Theorem 2 (our work [4]). Assume $\sigma_w^2 > 0$ and $T \geq (m+n)(n+1)+n$,

$$P(K_{ce} = \mathbf{0}_{m\times n}) = 1.$$

(CE DDD LQR is trivial for almost all noise ☹)

RP DDD LQR ([3]):

$$\min_{X,Y,S} trace(QX_0Y) + trace(X) + trace(S)$$

$$\text{s.t.}\quad \begin{bmatrix} X_0Y - I_n & X_1Y \\ Y^T X_1^T & X_0Y \end{bmatrix} \succcurlyeq 0$$

$$\begin{bmatrix} X & \sqrt{R}U_0Y \\ (\sqrt{R}U_0Y)^T & X_0Y \end{bmatrix} \succcurlyeq 0$$

$$\begin{bmatrix} S & Y \\ Y^T & X_0Y \end{bmatrix} \succcurlyeq 0.$$

Let Y_{rp}^* denote its optimal solution,

$$K_{rp} := -U_0 Y_{rp}^* (X_0 Y_{rp}^*)^{-1}.$$

Theorem 3 (our work [4]). Assume $\sigma_w^2 > 0$,

$$\lim_{T\to\infty} P(K_{rp} = \mathbf{0}_{m\times n}) = 1.$$

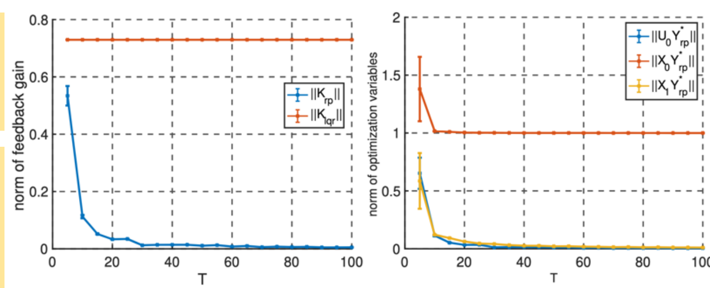
(RP DDD LQR is not statistically consistent ☹)

Key Observation: the following equalities always hold when $\sigma_w^2 > 0$

$$\begin{cases} U_0 Y^* = \mathbf{0}_{m\times n} \\ X_0 Y^* = \boldsymbol{I}_n \\ X_1 Y^* = \mathbf{0}_{n\times n} \end{cases},$$

where Y^* is an optimal solution of CE DDD LQR for any $T \geq (m+n)(n+1)+n$ or an optimal solution of RP DDD LQR for $T \to \infty$.

Experiments for RP DDD LQR: Consider an order-2 single-input unstable system, for which $K_{lqr} = [-0.7112\ -0.2046]$. $\sigma_w^2 = 1$ and $\sigma_u^2 = 1$.



References:
[1] H. Mania, S. Tu, and B. Recht, "Certainty equivalence is efficient for linear quadratic control," NeurIPS, 2019.
[2] C. De Persis and P. Tesi, "Formulas for data-driven control: Stabilization, optimality, and robustness," IEEE TAC, 2019.
[3] C. De Persis and P. Tesi, "Low-complexity learning of linear quadratic regulators from noisy data," Automatica, 2021
[4] X. Zeng, L. Bako, and N. Ozay, "Noise sensitivity of direct data-driven linear quadratic regulator by semidefinite programming," Submitted to ACC 2025 and the extension for TAC.