Model Predictive Path Integral Control: A Tutorial

Xiong Zeng

January 19, 2026

Introduction

Model predictive path integral (**MPPI**) control is a powerful and popular controller method in robotics and control, because

- it can handle **nonconvex** and **non-smooth** optimization problems,
- and it can be computed in **real-time** with GPU due to its **parallel computation** nature.

[1] Williams, et al. Information theoretic MPC for model-based reinforcement learning. ICRA 2017.

[2] Xue, et al. Full-order sampling-based mpc for torque-level locomotion control via diffusion-style annealing. ICRA 2025.

[3] Pan, et al. pider: Scalable physics-informed dexterous retargeting. ArXiv 2025.

Introduction

Model predictive path integral (**MPPI**) control is a powerful and popular controller method in robotics and control, because

- it can handle **nonconvex** and **non-smooth** optimization problems,
- and it can be computed in **real-time** with GPU due to its **parallel computation** nature.

Many applications:

- Aggressive driving [1]
- Legged robot control[2]
- Sim2real dexterous retargeting [3]
- Hard control problems in my research (MPPI $>$ Policy gradient $>$ SDP)

[1] Williams, et al. Information theoretic MPC for model-based reinforcement learning. ICRA 2017.

[2] Xue, et al. Full-order sampling-based mpc for torque-level locomotion control via diffusion-style annealing. ICRA 2025.

[3] Pan, et al. pider: Scalable physics-informed dexterous retargeting. ArXiv 2025.

Introduction

Consider the optimization problem

$$\min_{\mathbf{u}} \; J(\mathbf{u}) \in \mathbb{R}_{\geq 0}. \tag{1}$$

Introduction

Consider the optimization problem

$$\min_{\mathbf{u}} \; J(\mathbf{u}) \in \mathbb{R}_{\geq 0}. \tag{1}$$

The iteration formula of MPPI is

$$\mathbf{u}_{out} = \frac{\sum_{i=1}^{N} \mathbf{u}_i \exp\big(- J(\mathbf{u}_i)/\lambda\big)}{\sum_{i=1}^{N} \exp\big(- J(\mathbf{u}_i)/\lambda\big)}, \tag{2}$$

where λ is a hyperparameter (called temperature from simulated annealing) and $\{\mathbf{u}_i\}_{i=1}^{N}$ are i.i.d. samples from the following Gaussian distribution $\mathcal{N}(\mathbf{u}_{in}, \mathbf{\Sigma})$, with the current controller $\mathbf{u}_{in}$ and $\mathbf{\Sigma} \succ 0$.

Derive Single MPPI Iteration Based on KL Control[4]

Let $q(\mathbf{u})$ be a given probability density (for $\mathbf{u}_{in}$) on $\mathbb{R}^k$. Let $J : \mathbb{R}^k \to \mathbb{R}$ be measurable and $\lambda > 0$. We consider the optimization problem over probability densities $p(\mathbf{u})$:

$$\begin{aligned} \min_p \int p(\mathbf{u}) J(\mathbf{u})\, d\mathbf{u} + \lambda \int p(\mathbf{u}) \log \frac{p(\mathbf{u})}{q(\mathbf{u})}\, d\mathbf{u} \\ \text{s.t. } \int p(\mathbf{u})\, d\mathbf{u} = 1. \end{aligned} \tag{3}$$

[4] Evangelos & Todorov. Relative entropy and free energy dualities: Connections to path integral and kl control. CDC 2012.

Derive Single MPPI Iteration Based on KL Control[4]

Let $q(\mathbf{u})$ be a given probability density (for $\mathbf{u}_{in}$) on $\mathbb{R}^k$. Let $J : \mathbb{R}^k \to \mathbb{R}$ be measurable and $\lambda > 0$. We consider the optimization problem over probability densities $p(\mathbf{u})$:

$$\begin{aligned} &\min_p \int p(\mathbf{u}) J(\mathbf{u})\, d\mathbf{u} + \lambda \int p(\mathbf{u}) \log \frac{p(\mathbf{u})}{q(\mathbf{u})}\, d\mathbf{u} \\ &\text{s.t. } \int p(\mathbf{u})\, d\mathbf{u} = 1. \end{aligned} \tag{3}$$

Define

$$p^\star(\mathbf{u}) := \frac{1}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}\, q(\mathbf{u}) \exp\Big(-\frac{1}{\lambda}J(\mathbf{u})\Big). \tag{4}$$

[4] Evangelos & Todorov. Relative entropy and free energy dualities: Connections to path integral and kl control. CDC 2012.

Derive Single MPPI Iteration Based on KL Control[4]

Let $q(\mathbf{u})$ be a given probability density (for $\mathbf{u}_{in}$) on $\mathbb{R}^k$. Let $J : \mathbb{R}^k \to \mathbb{R}$ be measurable and $\lambda > 0$. We consider the optimization problem over probability densities $p(\mathbf{u})$:

$$\begin{aligned} \min_p \int p(\mathbf{u})J(\mathbf{u})\, d\mathbf{u} + \lambda \int p(\mathbf{u}) \log \frac{p(\mathbf{u})}{q(\mathbf{u})}\, d\mathbf{u} \\ \text{s.t.} \int p(\mathbf{u})\, d\mathbf{u} = 1. \end{aligned} \tag{3}$$

Define

$$p^\star(\mathbf{u}) := \frac{1}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}\, q(\mathbf{u}) \exp\Big(-\frac{1}{\lambda}J(\mathbf{u})\Big). \tag{4}$$

Goal: prove that $p^\star$ is the unique minimizer of (3).

[4]Evangelos & Todorov. Relative entropy and free energy dualities: Connections to path integral and kl control. CDC 2012.

Variation Calculus

Proof. It can be proved that (3) is a strictly convex infinite-dimensional optimization problem. Then, the stationary point of its Lagrange function is a unique minimizer. Define Lagrangian

$$\mathcal{L}(p,\eta) = \int p(\mathbf{u})J(\mathbf{u})\,d\mathbf{u} + \lambda \int p(\mathbf{u}) \log \frac{p(\mathbf{u})}{q(\mathbf{u})}\,d\mathbf{u} + \eta\Big(\int p(\mathbf{u})\,d\mathbf{u} - 1\Big), \tag{5}$$

where $\eta \in \mathbb{R}$ is the Lagrange multiplier.

Variation Calculus

Proof. It can be proved that (3) is a strictly convex infinite-dimensional optimization problem. Then, the stationary point of its Lagrange function is a unique minimizer. Define Lagrangian

$$\mathcal{L}(p,\eta) = \int p(\mathbf{u})J(\mathbf{u})\,d\mathbf{u} + \lambda \int p(\mathbf{u}) \log \frac{p(\mathbf{u})}{q(\mathbf{u})}\,d\mathbf{u} + \eta\Big(\int p(\mathbf{u})\,d\mathbf{u} - 1\Big), \tag{5}$$

where $\eta \in \mathbb{R}$ is the Lagrange multiplier. We compute stationarity with respect to $p(\mathbf{u})$. Let $p_\epsilon = p + \epsilon h$ with $\int h(\mathbf{u})\,d\mathbf{u} = 0$.
Using $\frac{d}{dx}(x \log x) = 1 + \log x$, we obtain

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{L}(p_\epsilon,\eta) = \int h(\mathbf{u})\Big[J(\mathbf{u}) + \lambda\big(1 + \log p(\mathbf{u}) - \log q(\mathbf{u})\big) + \eta\Big]d\mathbf{u}.$$

Variation Calculus

Proof. It can be proved that (3) is a strictly convex infinite-dimensional optimization problem. Then, the stationary point of its Lagrange function is a unique minimizer. Define Lagrangian

$$\mathcal{L}(p,\eta) = \int p(\mathbf{u})J(\mathbf{u})\,d\mathbf{u} + \lambda \int p(\mathbf{u}) \log \frac{p(\mathbf{u})}{q(\mathbf{u})}\,d\mathbf{u} + \eta\Big(\int p(\mathbf{u})\,d\mathbf{u} - 1\Big), \tag{5}$$

where $\eta \in \mathbb{R}$ is the Lagrange multiplier. We compute stationarity with respect to $p(\mathbf{u})$. Let $p_\epsilon = p + \epsilon h$ with $\int h(\mathbf{u})\,d\mathbf{u} = 0$.
Using $\frac{d}{dx}(x \log x) = 1 + \log x$, we obtain

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{L}(p_\epsilon,\eta) = \int h(\mathbf{u})\Big[J(\mathbf{u}) + \lambda\big(1 + \log p(\mathbf{u}) - \log q(\mathbf{u})\big) + \eta\Big]\,d\mathbf{u}.$$

For this to vanish for all admissible h, the bracket must be constant:

$$J(\mathbf{u}) + \lambda\big(1 + \log p(\mathbf{u}) - \log q(\mathbf{u})\big) = c. \tag{6}$$

Solve the Stationarity Condition

Rearranging,

$$\log p(\mathbf{u}) = \log q(\mathbf{u}) - \frac{1}{\lambda} J(\mathbf{u}) + \frac{c - \lambda}{\lambda}.$$

Solve the Stationarity Condition

Rearranging,

$$\log p(\mathbf{u}) = \log q(\mathbf{u}) - \frac{1}{\lambda} J(\mathbf{u}) + \frac{c - \lambda}{\lambda}.$$

Exponentiating,

$$p(\mathbf{u}) = e^{\frac{c-\lambda}{\lambda}}\, q(\mathbf{u}) \exp\Big(- \tfrac{1}{\lambda} J(\mathbf{u}) \Big).$$

Solve the Stationarity Condition

Rearranging,

$$\log p(\mathbf{u}) = \log q(\mathbf{u}) - \frac{1}{\lambda} J(\mathbf{u}) + \frac{c - \lambda}{\lambda}.$$

Exponentiating,

$$p(\mathbf{u}) = e^{\frac{c-\lambda}{\lambda}}\, q(\mathbf{u}) \exp\Big(- \tfrac{1}{\lambda} J(\mathbf{u}) \Big).$$

Imposing normalization $\int p = 1$ gives

$$e^{\frac{c-\lambda}{\lambda}} = \frac{1}{\mathbb{E}_q\Big[\exp\big(- \tfrac{1}{\lambda} J(\mathbf{u}) \big) \Big]},$$

Solve the Stationarity Condition

Rearranging,

$$\log p(\mathbf{u}) = \log q(\mathbf{u}) - \frac{1}{\lambda} J(\mathbf{u}) + \frac{c - \lambda}{\lambda}.$$

Exponentiating,

$$p(\mathbf{u}) = e^{\frac{c-\lambda}{\lambda}}\, q(\mathbf{u}) \exp\Big(-\tfrac{1}{\lambda} J(\mathbf{u})\Big).$$

Imposing normalization $\int p = 1$ gives

$$e^{\frac{c-\lambda}{\lambda}} = \frac{1}{\mathbb{E}_q\Big[\exp\big(-\tfrac{1}{\lambda} J(\mathbf{u})\big)\Big]},$$

so

$$p(\mathbf{u}) = \frac{1}{\mathbb{E}_q\Big[\exp\big(-\tfrac{1}{\lambda} J(\mathbf{u})\big)\Big]}\, q(\mathbf{u}) \exp\Big(-\frac{1}{\lambda} J(\mathbf{u})\Big) = p^\star(\mathbf{u}).$$

From Optimal Distribution to Optimal Controller

KL-control yields an *optimal distribution* $p^\star(\mathbf{u})$, but controller implementation must output a deterministic value $\bar{\mathbf{u}}$. Adopt the posterior squared-loss criterion

$$\bar{\mathbf{u}}^\star \in \arg\min_{\bar{\mathbf{u}} \in \mathbb{R}^{mH}} \mathbb{E}_{p^\star}\big[\|\mathbf{u} - \bar{\mathbf{u}}\|_2^2\big]. \tag{7}$$

From Optimal Distribution to Optimal Controller

KL-control yields an *optimal distribution* $p^\star(\mathbf{u})$, but controller implementation must output a deterministic value $\bar{\mathbf{u}}$. Adopt the posterior squared-loss criterion

$$\bar{\mathbf{u}}^\star \in \arg\min_{\bar{\mathbf{u}}\in\mathbb{R}^{mH}} \mathbb{E}_{p^\star}\big[\|\mathbf{u}-\bar{\mathbf{u}}\|_2^2\big]. \tag{7}$$

Expand the risk:

$$\mathbb{E}_{p^\star}\big[\|\mathbf{u}-\bar{\mathbf{u}}\|_2^2\big] = \mathbb{E}_{p^\star}[\|\mathbf{u}\|_2^2] - 2\bar{\mathbf{u}}^\top\mathbb{E}_{p^\star}[\mathbf{u}] + \|\bar{\mathbf{u}}\|_2^2. \tag{8}$$

From Optimal Distribution to Optimal Controller

KL-control yields an *optimal distribution* $p^{\star}(\mathbf{u})$, but controller implementation must output a deterministic value $\bar{\mathbf{u}}$. Adopt the posterior squared-loss criterion

$$\bar{\mathbf{u}}^{\star} \in \arg\min_{\bar{\mathbf{u}} \in \mathbb{R}^{mH}} \mathbb{E}_{p^{\star}}\big[\|\mathbf{u} - \bar{\mathbf{u}}\|_2^2\big]. \tag{7}$$

Expand the risk:

$$\mathbb{E}_{p^{\star}}\big[\|\mathbf{u} - \bar{\mathbf{u}}\|_2^2\big] = \mathbb{E}_{p^{\star}}[\|\mathbf{u}\|_2^2] - 2\bar{\mathbf{u}}^{\top}\mathbb{E}_{p^{\star}}[\mathbf{u}] + \|\bar{\mathbf{u}}\|_2^2. \tag{8}$$

Differentiate w.r.t. $\bar{\mathbf{u}}$:

$$\nabla_{\bar{\mathbf{u}}} = -2\mathbb{E}_{p^{\star}}[\mathbf{u}] + 2\bar{\mathbf{u}}. \tag{9}$$

Setting $\nabla_{\bar{\mathbf{u}}} = 0$ gives

$$\bar{\mathbf{u}}^{\star} = \mathbb{E}_{p^{\star}}[\mathbf{u}].$$

Since the Hessian is $2I \succ 0$, this minimizer is unique.

From Optimal Controller to the MPPI Estimator

From the KL-regularized optimization over distributions, the unique minimizer is

$$p^\star(\mathbf{u}) = \frac{1}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}\, q(\mathbf{u}) \exp\Big(-\frac{1}{\lambda}J(\mathbf{u})\Big). \tag{10}$$

From Optimal Controller to the MPPI Estimator

From the KL-regularized optimization over distributions, the unique minimizer is

$$p^\star(\mathbf{u}) = \frac{1}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}\, q(\mathbf{u})\exp\Big(-\frac{1}{\lambda}J(\mathbf{u})\Big). \tag{10}$$

MPPI uses the *posterior mean* under $p^\star$:

$$\mathbf{u}_{\mathrm{out}} = \mathbb{E}_{p^\star}[\mathbf{u}] = \int \mathbf{u}\, p^\star(\mathbf{u})\, d\mathbf{u} = \int \frac{\mathbf{u}\, q(\mathbf{u})\exp\Big(-\frac{1}{\lambda}J(\mathbf{u})\Big)}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}\, d\mathbf{u} = \frac{\mathbb{E}_q\Big[\mathbf{u}\,\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}. \tag{11}$$

From Optimal Controller to the MPPI Estimator

From the KL-regularized optimization over distributions, the unique minimizer is

$$p^\star(\mathbf{u}) = \frac{1}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}\, q(\mathbf{u}) \exp\Big(-\frac{1}{\lambda}J(\mathbf{u})\Big). \tag{10}$$

MPPI uses the *posterior mean* under $p^\star$:

$$\mathbf{u}_{\text{out}} = \mathbb{E}_{p^\star}[\mathbf{u}] = \int \mathbf{u}\, p^\star(\mathbf{u})\, d\mathbf{u} = \int \frac{\mathbf{u}\, q(\mathbf{u}) \exp\Big(-\frac{1}{\lambda}J(\mathbf{u})\Big)}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}\, d\mathbf{u} = \frac{\mathbb{E}_q\Big[\mathbf{u}\, \exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}. \tag{11}$$

Now draw N i.i.d. samples $\mathbf{u}_i \sim q$. By Monte-Carlo approximation of expectations (LLN), the MPPI (self-normalized importance sampling) estimator is

$$\widehat{\mathbf{u}}_{\text{out}} = \frac{\sum_{i=1}^{N} \mathbf{u}_i \exp\big(-J(\mathbf{u}_i)/\lambda\big)}{\sum_{i=1}^{N} \exp\big(-J(\mathbf{u}_i)/\lambda\big)}. \tag{12}$$

Convergence for Quadratic Programming

Consider a QP

$$\min_{\mathbf{u}} \; J(\mathbf{u}) = \mathbf{u}^\top \mathbf{D}\mathbf{u} + \mathbf{u}^\top \mathbf{d}, \tag{13}$$

with $\mathbf{D} \succ 0$. Draw N i.i.d. samples $\mathbf{u}_i \sim q = \mathcal{N}(\mathbf{u}_{in}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} \succ 0$.

[5] Yi et al. CoVO-MPC: Theoretical Analysis of Sampling-based MPC and Optimal Covariance Design. L4DC 2024.

Convergence for Quadratic Programming

Consider a QP

$$\min_{\mathbf{u}} \; J(\mathbf{u}) = \mathbf{u}^\top \mathbf{D}\mathbf{u} + \mathbf{u}^\top \mathbf{d}, \tag{13}$$

with $\mathbf{D} \succ 0$. Draw N i.i.d. samples $\mathbf{u}_i \sim q = \mathcal{N}(\mathbf{u}_{in}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} \succ 0$. Consider the MPPI update

$$\widehat{\mathbf{u}}_{\mathrm{out}} = \frac{\sum_{i=1}^{N} \mathbf{u}_i \exp\big(-J(\mathbf{u}_i)/\lambda\big)}{\sum_{i=1}^{N} \exp\big(-J(\mathbf{u}_i)/\lambda\big)} \text{ and } \mathbf{u}_{\mathrm{out}} = \frac{\mathbb{E}_q\Big[\mathbf{u}\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}. \tag{14}$$

[5] Yi et al. CoVO-MPC: Theoretical Analysis of Sampling-based MPC and Optimal Covariance Design. L4DC 2024.

Convergence for Quadratic Programming

Consider a QP

$$\min_{\mathbf{u}} \; J(\mathbf{u}) = \mathbf{u}^\top \mathbf{D}\mathbf{u} + \mathbf{u}^\top \mathbf{d}, \tag{13}$$

with $\mathbf{D} \succ 0$. Draw N i.i.d. samples $\mathbf{u}_i \sim q = \mathcal{N}(\mathbf{u}_{in}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} \succ 0$. Consider the MPPI update

$$\widehat{\mathbf{u}}_{\mathrm{out}} = \frac{\sum_{i=1}^N \mathbf{u}_i \exp\big(-J(\mathbf{u}_i)/\lambda\big)}{\sum_{i=1}^N \exp\big(-J(\mathbf{u}_i)/\lambda\big)} \text{ and } \mathbf{u}_{\mathrm{out}} = \frac{\mathbb{E}_q\Big[\mathbf{u}\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]}. \tag{14}$$

Theorem 1.[5] Consider the minimize $\mathbf{u}^\star = \frac{-1}{2}\mathbf{d}\mathbf{D}^{-1}$ of (13). We have

$$\hat{\mathbf{u}}_{out} \xrightarrow{p} \mathbf{u}_{out}, \tag{15}$$

and

$$\mathbf{u}_{out} - \mathbf{u}^\star \xrightarrow{p} (\mathbf{I} + 2\lambda\mathbf{\Sigma}\mathbf{D})^{-1}(\mathbf{u}_{in} - \mathbf{u}^\star). \tag{16}$$

[5] Yi et al. CoVO-MPC: Theoretical Analysis of Sampling-based MPC and Optimal Covariance Design. L4DC 2024.

Step 1: Write $p^\star$ explicitly and complete the square

Proof. From the KL-control derivation,

$$\mathbf{u}_{out} = \mathbb{E}_{p^\star}[\mathbf{u}], \qquad p^\star(\mathbf{u}) = \frac{1}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]} q(\mathbf{u}) \exp\Big(-\frac{1}{\lambda}J(\mathbf{u})\Big). \tag{17}$$

Step 1: Write $p^\star$ explicitly and complete the square

Proof. From the KL-control derivation,

$$\mathbf{u}_{out} = \mathbb{E}_{p^\star}[\mathbf{u}], \qquad p^\star(\mathbf{u}) = \frac{1}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]} q(\mathbf{u}) \exp\Big(-\frac{1}{\lambda}J(\mathbf{u})\Big). \tag{17}$$

And $q(\mathbf{u}) = \mathcal{N}(\mathbf{u}_{in}, \boldsymbol{\Sigma})$ yields

$$q(\mathbf{u}) \propto \exp\Big(-\frac{1}{2}(\mathbf{u}-\mathbf{u}_{in})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{u}-\mathbf{u}_{in})\Big). \tag{18}$$

Step 1: Write $p^\star$ explicitly and complete the square

Proof. From the KL-control derivation,

$$\mathbf{u}_{out} = \mathbb{E}_{p^\star}[\mathbf{u}], \qquad p^\star(\mathbf{u}) = \frac{1}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]} q(\mathbf{u}) \exp\Big(-\frac{1}{\lambda}J(\mathbf{u})\Big). \tag{17}$$

And $q(\mathbf{u}) = \mathcal{N}(\mathbf{u}_{in}, \mathbf{\Sigma})$ yields

$$q(\mathbf{u}) \propto \exp\Big(-\frac{1}{2}(\mathbf{u} - \mathbf{u}_{in})^\top \mathbf{\Sigma}^{-1}(\mathbf{u} - \mathbf{u}_{in})\Big). \tag{18}$$

Thus

$$p^\star(\mathbf{u}) \propto \exp\Big(-\frac{1}{\lambda}(\mathbf{u}^\top \mathbf{D}\mathbf{u} + \mathbf{u}^\top \mathbf{d}) - \frac{1}{2}(\mathbf{u} - \mathbf{u}_{in})^\top \mathbf{\Sigma}^{-1}(\mathbf{u} - \mathbf{u}_{in})\Big). \tag{19}$$

Step 1: Write $p^\star$ explicitly and complete the square

Proof. From the KL-control derivation,

$$\mathbf{u}_{out} = \mathbb{E}_{p^\star}[\mathbf{u}], \qquad p^\star(\mathbf{u}) = \frac{1}{\mathbb{E}_q\Big[\exp\big(-\frac{1}{\lambda}J(\mathbf{u})\big)\Big]} q(\mathbf{u}) \exp\Big(-\tfrac{1}{\lambda}J(\mathbf{u})\Big). \tag{17}$$

And $q(\mathbf{u}) = \mathcal{N}(\mathbf{u}_{in}, \mathbf{\Sigma})$ yields

$$q(\mathbf{u}) \propto \exp\Big(-\tfrac{1}{2}(\mathbf{u}-\mathbf{u}_{in})^\top \mathbf{\Sigma}^{-1}(\mathbf{u}-\mathbf{u}_{in})\Big). \tag{18}$$

Thus

$$p^\star(\mathbf{u}) \propto \exp\Big(-\tfrac{1}{\lambda}(\mathbf{u}^\top \mathbf{D}\mathbf{u} + \mathbf{u}^\top \mathbf{d}) - \tfrac{1}{2}(\mathbf{u}-\mathbf{u}_{in})^\top \mathbf{\Sigma}^{-1}(\mathbf{u}-\mathbf{u}_{in})\Big). \tag{19}$$

Expand the Gaussian exponent:

$$(\mathbf{u}-\mathbf{u}_{in})^\top \mathbf{\Sigma}^{-1}(\mathbf{u}-\mathbf{u}_{in}) = \mathbf{u}^\top \mathbf{\Sigma}^{-1}\mathbf{u} - 2\mathbf{u}^\top \mathbf{\Sigma}^{-1}\mathbf{u}_{in} + \mathbf{u}_{in}^\top \mathbf{\Sigma}^{-1}\mathbf{u}_{in}. \tag{20}$$

Then the exponent in $p^\star$ equals

$$-\frac{1}{2}\mathbf{u}^\top\Big(\mathbf{\Sigma}^{-1} + \tfrac{2}{\lambda}\mathbf{D}\Big)\mathbf{u} + \mathbf{u}^\top \mathbf{\Sigma}^{-1}\mathbf{u}_{in} - \frac{1}{\lambda}\mathbf{u}^\top \mathbf{d} \; + \; \text{(constant)}. \tag{21}$$

Step 2: Recognize $p^\star$ as a Gaussian and read off its mean

Let

$$\mathbf{A} := \mathbf{\Sigma}^{-1} + \tfrac{2}{\lambda}\mathbf{D} \succ 0, \qquad \mathbf{b} := \mathbf{\Sigma}^{-1}\mathbf{u}_{in} - \tfrac{1}{\lambda}\mathbf{d}. \tag{22}$$

Drop constants independent of $\mathbf{u}$. Then

$$p^\star(\mathbf{u}) \propto \exp\Big(- \tfrac{1}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u} + \mathbf{u}^\top \mathbf{b}\Big). \tag{23}$$

Step 2: Recognize $p^\star$ as a Gaussian and read off its mean

Let

$$\mathbf{A} := \mathbf{\Sigma}^{-1} + \frac{2}{\lambda}\mathbf{D} \succ 0, \qquad \mathbf{b} := \mathbf{\Sigma}^{-1}\mathbf{u}_{in} - \frac{1}{\lambda}\mathbf{d}. \tag{22}$$

Drop constants independent of $\mathbf{u}$. Then

$$p^\star(\mathbf{u}) \propto \exp\Big(-\frac{1}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u} + \mathbf{u}^\top \mathbf{b}\Big). \tag{23}$$

Complete the square:

$$-\frac{1}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u} + \mathbf{u}^\top \mathbf{b} = -\frac{1}{2}(\mathbf{u} - \mathbf{A}^{-1}\mathbf{b})^\top \mathbf{A}(\mathbf{u} - \mathbf{A}^{-1}\mathbf{b}) + \frac{1}{2}\mathbf{b}^\top \mathbf{A}^{-1}\mathbf{b}. \tag{24}$$

Step 2: Recognize $p^\star$ as a Gaussian and read off its mean

Let

$$\mathbf{A} := \mathbf{\Sigma}^{-1} + \tfrac{2}{\lambda}\mathbf{D} \succ 0, \qquad \mathbf{b} := \mathbf{\Sigma}^{-1}\mathbf{u}_{in} - \tfrac{1}{\lambda}\mathbf{d}. \tag{22}$$

Drop constants independent of $\mathbf{u}$. Then

$$p^\star(\mathbf{u}) \propto \exp\Big(-\tfrac{1}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u} + \mathbf{u}^\top \mathbf{b}\Big). \tag{23}$$

Complete the square:

$$-\tfrac{1}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u} + \mathbf{u}^\top \mathbf{b} = -\tfrac{1}{2}(\mathbf{u} - \mathbf{A}^{-1}\mathbf{b})^\top \mathbf{A}(\mathbf{u} - \mathbf{A}^{-1}\mathbf{b}) + \tfrac{1}{2}\mathbf{b}^\top \mathbf{A}^{-1}\mathbf{b}. \tag{24}$$

By grouping the constant terms, $p^\star(\mathbf{u})$ is a p.d.f. of Gaussian with mean

$$\mathbb{E}_{p^\star}[\mathbf{u}] = \mathbf{A}^{-1}\mathbf{b}. \tag{25}$$

Step 2: Recognize $p^\star$ as a Gaussian and read off its mean

Let

$$\mathbf{A} := \boldsymbol{\Sigma}^{-1} + \tfrac{2}{\lambda}\mathbf{D} \succ 0, \qquad \mathbf{b} := \boldsymbol{\Sigma}^{-1}\mathbf{u}_{in} - \tfrac{1}{\lambda}\mathbf{d}. \tag{22}$$

Drop constants independent of $\mathbf{u}$. Then

$$p^\star(\mathbf{u}) \propto \exp\Big(-\tfrac{1}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u} + \mathbf{u}^\top \mathbf{b} \Big). \tag{23}$$

Complete the square:

$$-\tfrac{1}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u} + \mathbf{u}^\top \mathbf{b} = -\tfrac{1}{2}(\mathbf{u} - \mathbf{A}^{-1}\mathbf{b})^\top \mathbf{A}(\mathbf{u} - \mathbf{A}^{-1}\mathbf{b}) + \tfrac{1}{2}\mathbf{b}^\top \mathbf{A}^{-1}\mathbf{b}. \tag{24}$$

By grouping the constant terms, $p^\star(\mathbf{u})$ is a p.d.f. of Gaussian with mean

$$\mathbb{E}_{p^\star}[\mathbf{u}] = \mathbf{A}^{-1}\mathbf{b}. \tag{25}$$

Substitute:

$$\mathbb{E}_{p^\star}[\mathbf{u}] = \Big(\boldsymbol{\Sigma}^{-1} + \tfrac{2}{\lambda}\mathbf{D} \Big)^{-1} \Big(\boldsymbol{\Sigma}^{-1}\mathbf{u}_{in} - \tfrac{1}{\lambda}\mathbf{d} \Big). \tag{26}$$

Step 3: Express the mean using $\mathbf{u}^\star$ and simplify

$\nabla J(\mathbf{u}^\star) = 0$ yeilds $\mathbf{d} = -2\mathbf{D}\mathbf{u}^\star$. Then $-\frac{1}{\lambda}\mathbf{d} = \frac{2}{\lambda}\mathbf{D}\mathbf{u}^\star$. Hence

$$\mathbb{E}_{p^\star}[\mathbf{u}] = \left(\mathbf{\Sigma}^{-1} + \frac{2}{\lambda}\mathbf{D}\right)^{-1}\left(\mathbf{\Sigma}^{-1}\mathbf{u}_{in} + \frac{2}{\lambda}\mathbf{D}\mathbf{u}^\star\right). \tag{27}$$

Therefore

$$\begin{aligned}
\mathbf{u}_{out} = \mathbb{E}_{p^\star}[\mathbf{u}] &= \lambda(\lambda\mathbf{I} + 2\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\mathbf{u}_{in} + 2(\lambda\mathbf{I} + 2\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{\Sigma}\mathbf{D}\mathbf{u}^\star && (28)\\
&= (\lambda\mathbf{I} + 2\mathbf{\Sigma}\mathbf{D})^{-1}\left(\lambda\mathbf{u}_{in} + 2\mathbf{\Sigma}\mathbf{D}\mathbf{u}^\star\right) && (29)\\
&= (\lambda\mathbf{I} + 2\mathbf{\Sigma}\mathbf{D})^{-1}(\lambda\mathbf{I} + 2\mathbf{\Sigma}\mathbf{D})\mathbf{u}_{in} + (\lambda\mathbf{I} + 2\mathbf{\Sigma}\mathbf{D})^{-1}2\mathbf{\Sigma}\mathbf{D}(\mathbf{u}^\star - \mathbf{u}_{in}) && (30)\\
&= \mathbf{u}_{in} + (\mathbf{I} + 2\lambda\mathbf{\Sigma}\mathbf{D})^{-1}(\mathbf{u}^\star - \mathbf{u}_{in}). && (31)
\end{aligned}$$

Step 6: Convergence in probability and Slutsky

Let the MPPI estimator be

$$\mathbf{u}_{out}^{(N)} = \frac{\frac{1}{N}\sum_{i=1}^{N} \mathbf{u}_i w_i}{\frac{1}{N}\sum_{i=1}^{N} w_i}, \qquad w_i := \exp\Big(-\tfrac{1}{\lambda} J(\mathbf{u}_i)\Big), \qquad \mathbf{u}_i \overset{\text{i.i.d.}}{\sim} q. \tag{32}$$

Step 6: Convergence in probability and Slutsky

Let the MPPI estimator be

$$\mathbf{u}_{out}^{(N)} = \frac{\frac{1}{N}\sum_{i=1}^{N} \mathbf{u}_i w_i}{\frac{1}{N}\sum_{i=1}^{N} w_i}, \qquad w_i := \exp\Big(-\tfrac{1}{\lambda}J(\mathbf{u}_i)\Big), \qquad \mathbf{u}_i \overset{\text{i.i.d.}}{\sim} q. \tag{32}$$

Then by the weak LLN,

$$\frac{1}{N}\sum_{i=1}^{N} w_i \xrightarrow{p} \mathbb{E}_q[w_1], \qquad \frac{1}{N}\sum_{i=1}^{N} \mathbf{u}_i w_i \xrightarrow{p} \mathbb{E}_q[\mathbf{u}w]. \tag{33}$$

Step 6: Convergence in probability and Slutsky

Let the MPPI estimator be

$$\mathbf{u}_{out}^{(N)} = \frac{\frac{1}{N}\sum_{i=1}^{N} \mathbf{u}_i w_i}{\frac{1}{N}\sum_{i=1}^{N} w_i}, \qquad w_i := \exp\Big(-\tfrac{1}{\lambda} J(\mathbf{u}_i)\Big), \qquad \mathbf{u}_i \overset{\text{i.i.d.}}{\sim} q. \tag{32}$$

Then by the weak LLN,

$$\frac{1}{N}\sum_{i=1}^{N} w_i \xrightarrow{p} \mathbb{E}_q[w_1], \qquad \frac{1}{N}\sum_{i=1}^{N} \mathbf{u}_i w_i \xrightarrow{p} \mathbb{E}_q[\mathbf{u}w]. \tag{33}$$

Then, Slutsky's theorem yields

$$\hat{\mathbf{u}}_{out} \xrightarrow{p} \frac{\mathbb{E}_q[\mathbf{u}w]}{\mathbb{E}_q[w]} = \mathbb{E}_{p^\star}[\mathbf{u}]. \tag{34}$$