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Problem 19.2.7
Let S be a multiplicative subset of R and T be a multiplicative subset of $S^{-1}R$. Let

$$S_* = \left\{ r \in R \mid [\frac{r}{s}] \in T \ \text{ for some } \ s \in S \right\}.$$

Then S_* is a multiplicative subset of R and there is a ring isomorphism $S_*^{-1}R \cong T^{-1}(S^{-1}R)$.

Solution: We first prove that S_* is a multiplicative subset of R. Suppose $r_1, r_2 \in S_*$, then there exist $s_1, s_2 \in S$ such that $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in T$. We have $\frac{r_1 r_2}{s_1 s_2} \in T$ since T is a multiplicative subset of $S^{-1}R$. This proves that $r_1 r_2 \in S_*$. So S_* is a multiplicative subset of R.

The elements in $T^{-1}(S^{-1}R)$ can be written as $\frac{\frac{r_1}{s_1}}{\frac{r_2}{s_2}}$ where $\frac{r_2}{s_2} \in T$ and $\frac{r_1}{s_1} \in S^{-1}R$. We define a map

$$f : T^{-1}(S^{-1}R) \to S_*^{-1}R,$$
$$\frac{\frac{r_1}{s_1}}{\frac{r_2}{s_2}} \mapsto \frac{r_1 s_2}{r_2 s_1}.$$

This map is well-defined. Indeed, we know that $\frac{r_2}{s_2} \in T$, so $r_2 \in S_*$. Moreover, since S is a multiplicative subset of R, we know that $s_1 s_2 \in S$, so $\frac{r_2}{s_2} \sim \frac{r_2 s_1}{s_2 s_1} \in T$ in $S^{-1}R$. This proves $r_2 s_1 \in S_*$ and $\frac{r_1 s_2}{r_2 s_1} \in S_*^{-1}R$. Suppose $\frac{\frac{r_1'}{s_1'}}{\frac{r_2'}{s_2'}}$ is another equivalent representative of $\frac{\frac{r_1}{s_1}}{\frac{r_2}{s_2}}$ in $T^{-1}(S^{-1}R)$. Then there exists $\frac{p}{q} \in T$ in $S^{-1}R$ such that

$$\frac{p}{q}(\frac{r_1'}{s_1'} \cdot \frac{r_2}{s_2} - \frac{r_1}{s_1} \cdot \frac{r_2'}{s_2'}) = 0$$

Namely, in $S^{-1}R$, we have

$$\frac{p}{q} \cdot \frac{r_1'}{s_1'} \cdot \frac{r_2}{s_2} \sim \frac{p}{q} \cdot \frac{r_1}{s_1} \cdot \frac{r_2'}{s_2'}$$

There exists $u \in S$ such that $upq(r_1' s_1 r_2 s_2' - r_1 s_1' r_2' s_2) = 0$ in R. Note that $uq^2 \in S$ since S is a multiplicative subset, and $\frac{upq}{uq^2} = \frac{p}{q} \in T$, so $upq \in S_*$. This implies that

$$\frac{r_1 s_2}{r_2 s_1} \sim \frac{r_1' s_2'}{r_2' s_1'}$$

in $S_*^{-1}R$. Thus, the map f is well-defined. It is easy to check that

$$f(\frac{\frac{r_1}{s_1}}{\frac{r_2}{s_2}}) f(\frac{\frac{r_3}{s_3}}{\frac{r_4}{s_4}}) = \frac{r_1 s_2}{r_2 s_1} \cdot \frac{r_3 s_4}{r_4 s_3} = \frac{r_1 s_2 r_3 s_4}{r_2 s_1 r_4 s_3} = f(\frac{\frac{r_1 r_3}{s_1 s_3}}{\frac{r_2 r_4}{s_2 s_4}}).$$

This proves that f is a ring homomorphism.

Next, we want to show that f is injective and surjective. Suppose $f(\frac{\frac{r_1}{s_1}}{\frac{r_2}{s_2}}) = 0$ in $S_*^{-1}R$. This implies there exists $u \in S_*$ such that $ur_1s_2 = 0$. By definition, there exists $s \in S$ such that $\frac{u}{s} \in T$. Since S is multiplicative, we have $\frac{u}{s} \sim \frac{us_2}{ss_2} \in T$ and

$$\frac{us_2}{ss_2} \cdot \frac{r_1}{s_1} = \frac{us_2r_1}{ss_1s_2} = 0.$$

This proves that $\frac{\frac{r_1}{s_1}}{\frac{r_2}{s_2}} = 0$ in $T^{-1}(S^{-1}R)$. So f is injective. For any $p \in S_*$, there exists $s' \in S$ such that $\frac{p}{s'} \in T$. So we have

$$f(\frac{\frac{r}{s'}}{\frac{p}{s'}}) = \frac{rs'}{ps'} = \frac{r}{p}$$

for all $r \in R$ and $p \in S_*$. This proves f is surjective. Therefore, we can conclude that f is a ring isomorphism between $S_*^{-1}R$ and $T^{-1}(S^{-1}R)$.

Problem 19.2.8

Let V be an R-module and S be a multiplicative subset of R. Then the map $j_S : V \to S^{-1}V$, $v \mapsto [\frac{v}{1}]$ is a homomorphism of R-modules and for every R-homomorphism from V to an $S^{-1}R$-module, there exists a unique $S^{-1}R$-module homomorphism $\hat{f} : S^{-1}V \to V'$ such that the following diagram commutes:

$$\begin{tikzcd}
& V \arrow[dl, "j_S"'] \arrow[dr, "f"] & \\
S^{-1}V \arrow[rr, "\hat{f}"'] & & V'
\end{tikzcd}$$

Moreover, this property characterizes $S^{-1}V$ uniquely up to a (unique) isomorphism of $S^{-1}R$-modules.

Solution: We check that j_S is an R-module homomorphism. For any $r \in R$ and $v \in V$, we have

$$rj_S(v) = r \cdot \frac{v}{1} = \frac{rv}{1} = j_S(rv).$$

Now given an R-module homomorphism $f : V \to V'$ where V' is an $S^{-1}R$-module, we define the following map

$$\begin{aligned} \hat{f} : S^{-1}V &\to V', \\ \frac{v}{s} &\mapsto \frac{1}{s} \cdot f(v). \end{aligned}$$

This map $\hat{f}$ is a well-defined $S^{-1}R$-module homomorphism. Indeed, for any $\frac{r'}{s'} \in S^{-1}R$, we have

$$\frac{r'}{s'} \cdot \hat{f}(\frac{v}{s}) = \frac{r'}{s'}\frac{1}{s}f(v) = \frac{r'}{ss'}f(v) = \hat{f}(\frac{r'v}{s's}).$$

Moreover, for any $v \in V$, we have

$$(\hat{f} \circ j_S)(v) = \hat{f}(\frac{v}{1}) = f(v).$$

This implies we have a commutative diagram

$$\begin{tikzcd}
 & V \arrow[dl, "j_S"'] \arrow[dr, "f"] & \\
S^{-1}V \arrow[rr, "\hat{f}"'] & & V'
\end{tikzcd}$$

The uniqueness can be seen from the commutativity of the diagram.

Problem 19.2.13

Let $f : V \to W$ be an R-module homomorphism.

(1) $S^{-1}(\operatorname{Im} f) = \operatorname{Im}(S^{-1}f)$ for any multiplicative subset $S \subset R$.

(2) f is surjective if and only if $f_M : V_M \to W_M$ is surjective for every maximal ideal M of R.

Solution:

(1) Consider a short exact sequence of R-modules

$$0 \to \ker f \to V \xrightarrow{f} \operatorname{Im} f \to 0.$$

The localization is an exact functor, so we have

$$0 \to S^{-1}\ker f \to S^{-1}V \to S^{-1}\operatorname{Im} f \to 0.$$

By Lemma 19.2.12, we know that $S^{-1}\ker f = \ker(S^{-1}f)$ and the cokernel of the map $\ker(S^{-1}f) \to S^{-1}V$ is $\operatorname{Im}(S^{-1}f)$. By exactness, we have

$$S^{-1}\operatorname{Im} f \cong \operatorname{Im}(S^{-1}f).$$

(2) The "only if" part follows from the fact that the localization functor is exact. Conversely, suppose $f_M : V_M \to W_M$ is surjective for every maximal ideal M of R. Consider the short exact sequence

$$0 \to \operatorname{Im} f \to W \to W/\operatorname{Im} f \to 0.$$

Localize at M, and we obtain a short exact sequence

$$0 \to (\operatorname{Im} f)_M \to W_M \to (W/\operatorname{Im} f)_M \to 0$$

This tells us that $(W/\operatorname{Im} f)_M \cong W_M/(\operatorname{Im} f)_M$. By surjectivity of f_M and what we have proved in (1), we have

$$\begin{aligned}0 &= \operatorname{coker} f_M \\ &= W_M/\operatorname{Im} f_M \\ &= W_M/(\operatorname{Im} f)_M \\ &= (W/\operatorname{Im} f)_M \\ &= (\operatorname{coker} f)_M.\end{aligned}$$

This implies that for every maximal ideal M of R, $(\operatorname{coker} f)_M = 0$. By Exercise 19.2.11, we have $\operatorname{coker} f = 0$, so the map $f : V \to W$ is surjective.

Problem 19.2.15
Let S be a proper multiplicative subset of R, and V, W be R-modules. Then

$$S^{-1}V \otimes_{S^{-1}R} S^{-1}W \cong S^{-1}(V \otimes_R W).$$

Solution: We define the following map

$$f : S^{-1}V \otimes_{S^{-1}R} S^{-1}W \to S^{-1}(V \otimes_R W),$$
$$\frac{v}{s_1} \otimes \frac{w}{s_2} \mapsto \frac{v \otimes w}{s_1 s_2}.$$

We check that this is an $S^{-1}R$-module homomorphism. For any $\frac{r}{s} \in S^{-1}R$, we have

$$\frac{r}{s} f(\frac{v}{s_1} \otimes \frac{w}{s_2}) = \frac{r}{s}\frac{v \otimes w}{s_1 s_2} = \frac{rv \otimes w}{s s_1 s_2} = f(\frac{rv}{s s_1} \otimes \frac{w}{s_2}).$$

Next, we show that f is both injective and surjective. Suppose $f(\frac{v}{s_1} \otimes \frac{w}{s_2}) = 0$ for some $\frac{v}{s_1} \in V$ and $\frac{w}{s_2} \in W$. This implies that $\frac{v \otimes w}{s_1 s_2} = 0$ in $S^{-1}(V \otimes_R W)$. There exists $s \in S$ such that $s(v \otimes w) = 0$. This implies that

$$s(\frac{v}{s_1} \otimes \frac{w}{s_2}) = \frac{1}{s_1 s_2}(sv \otimes w) = 0.$$

This proves injectivity. On the other hand, for any $\frac{v \otimes w}{s}$ in $S^{-1}(V \otimes_R W)$, there exists $\frac{v}{s} \in S^{-1}V$ and $\frac{w}{1} \in S^{-1}W$ such that

$$f(\frac{v}{s} \otimes \frac{w}{1}) = \frac{v \otimes w}{s}.$$

This proves that f is surjective. Therefore, we can conclude that f is an $S^{-1}R$-module isomorphism between $S^{-1}V \otimes_{S^{-1}R} S^{-1}W$ and $S^{-1}(V \otimes_R W)$.

Problem 19.2.16
Let V be an R-module. Then V is flat if and only if V_M is flat for every maximal ideal M of R.

Solution: Assume V is flat. Given an injective map $f : A \to B$, by Proposition 19.2.9, for any maximal ideal M of R, we have

$$A \otimes_R V_M = A \otimes_R (R_M \otimes_R V) = (A \otimes_R R_M) \otimes_R V = A_M \otimes V.$$

The isomorphism is functorial, so we have a commutative diagram

$$\begin{tikzcd}
A \otimes_R V_M \arrow[r, "f \otimes id_{V_M}"] \arrow[d, "\cong"'] & B \otimes_R V_M \arrow[d, "\cong"] \\
A_M \otimes_R V \arrow[r, "f_M \otimes id_V"'] & B_M \otimes_R V
\end{tikzcd}$$

Since V is flat, by Lemma 19.2.12, the map $f_M \otimes id_V$ is still injective, and thus the map $f \otimes id_{V_M}$ is injective. This proves that V_M is flat for every maximal ideal M of R.

Conversely, assume V_M is flat for every maximal ideal M. Given an injective map $f : A \to B$, consider the map

$$f_M : A_M \to B_M$$

where M is a maximal ideal M of R. By Lemma 19.2.12, f_M is injective as f is injective. We know that V_M is flat, so the map

$$f_M \otimes id_{V_M} : A_M \otimes_{R_M} V_M \to B_M \otimes_{R_M} V_M$$

is still injective. Note that by Exercise 19.2.15, $A_M \otimes_{R_M} V_M = (A \otimes_R V)_M$. So the map

$$(f \otimes id_V)_M : (A \otimes_R V)_M \to (B \otimes_R V)_M$$

is injective. Use Lemma 19.2.12 again, and we know that the map

$$f \otimes id_V : A \otimes_R V \to B \otimes_R V$$

is injective. This proves that V is flat.

Problem 19.3.3
Let $\alpha \in \mathbb{C}$ be algebraic over $\mathbb{Q}$. Then α is integral over $\mathbb{Z}$ if and only if $\mathrm{irr}(\alpha; \mathbb{Q}) \in \mathbb{Z}[x]$.

Solution: Let f be the irreducible polynomial of α over $\mathbb{Q}$. Suppose $f \in \mathbb{Z}[x]$. Then f can be written as

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

where $a_i \in \mathbb{Z}$ for all i. This proves that α is integral over $\mathbb{Z}$.

Conversely, suppose α is integral over $\mathbb{Z}$. Then there exists a monic polynomial such that α is a root. Let

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_n$$

be the irreducible factor of this polynomial with $f(\alpha) = 0$. Let g be the minimal polynomial of α over $\mathbb{Q}$. We know that $f \in \mathbb{Z}[x] \subseteq \mathbb{Q}[x]$ and $f(\alpha) = 0$, so $g|f$ over $\mathbb{Q}$. This implies there exists $h \in \mathbb{Q}[x]$ such that $gh = f$. Note that f, g, h are monic polynomials. If h is of positive degree, then by Gauss' Lemma, f can be factored into two polynomials in $\mathbb{Z}[x]$, but f is irreducible by assumption, so $h = 1$ and $f = g$. This proves that the minimal polynomial of f over $\mathbb{Q}$ is in $\mathbb{Z}[x]$.