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Course: MATH 682 - Algebraic Geometry II
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Homework - Chapter 2 Exercises
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Exercise 2.7.2
Describe $\operatorname{Spec} \mathbb{Z}[\frac{1}{18}]$.

Solution: Note that we have a ring isomorphism $\mathbb{Z}[\frac{1}{18}] \cong \mathbb{Z}[x]/(18x-1)$. Let $R = \mathbb{Z}[x]/(18x-1)$. We need to describe $\operatorname{Spec} R$. Consider the ring homomorphism

$$f : \mathbb{Z}[x] \to R.$$

given by the quotient map. We know that prime ideals in R corresponds to prime ideals in $\mathbb{Z}[x]$ containing the ideal $(18x-1)$. It must be of the form $(p, 18x-1)$ where $p \in \mathbb{Z}$ is a prime number. If $p = 2$ or $p = 3$, then $(p, 18x-1) = \mathbb{Z}[x]$, which is not an ideal. When $p \neq 2, 3$, the ideal $(p, 18x-1)$ is a prime ideal in $\mathbb{Z}[x]$, thus corresponds to a prime ideal of R. So $\operatorname{Spec} R$ has closed points corresponds to the maximal ideal $(p, 18x-1)$ in R where $p \neq 2, 3$, and a generic point corresponds to the zero ideal in R (or the ideal $(18x-1)$ in $\mathbb{Z}[x]$).

Exercise 2.7.9
Show that $D(f) = \varnothing$ if and only if f is nilpotent.

Solution: Suppose f is nilpotent. Then there exists $n \geq 1$ such that $f^n = 0 \in \mathfrak{p}$ for any prime ideal $\mathfrak{p} \subset A$. So

$$D(f) = \{\mathfrak{p} \ \text{ prime} \mid f \notin \mathfrak{p}\} = \varnothing.$$

Conversely, suppose $D(f) = \varnothing$. This implies that $f \in \mathfrak{p}$ for all prime ideal $\mathfrak{p}$, so $f \in \cap_{\mathfrak{p} \ \text{ prime}} \mathfrak{p} = \sqrt{(0)}$. There exists $n \geq 1$ such that $f^n = 0$. So f is nilpotent.
