Zhengdong Zhang
Email: zhengz@uoregon.edu
Course: MATH 648 - Abstract Algebra
Instructor: Professor Arkady Berenstein

Homework - Week 7
ID: 952091294
Term: Winter 2025
Due Date: 26^{th} February, 2025

Problem 17.2.5
True or false? $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for any $m \in \mathbb{Z}_{>0}$.

Solution: This is true. From Example 17.2.4 in the book, we know that

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}/m\mathbb{Q}.$$

Note that $\mathbb{Q}$ is a field and m is invertible in $\mathbb{Q}$, so $m\mathbb{Q} \cong \mathbb{Q}$ and we have

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

Problem 17.2.6
$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as abelian groups.

Solution: We define a map

$$\begin{aligned} f : \mathbb{Q} \times \mathbb{Q} &\to \mathbb{Q}, \\ (\frac{p}{q}, \frac{r}{s}) &\mapsto \frac{pr}{qs}. \end{aligned}$$

where $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$. For any $m \in \mathbb{Z}$, we have

$$mf(\frac{p}{q}, \frac{r}{s}) = \frac{mpr}{qs} = f(\frac{mp}{q}, \frac{r}{s}) = f(\frac{p}{q}, \frac{mr}{s}).$$

And for $\frac{p_1}{q_1}, \frac{p_2}{q_2} \in \mathbb{Q}$, we have

$$\begin{aligned} f(\frac{p_1}{q_1} + \frac{p_2}{q_2}, \frac{r}{s}) &= (\frac{p_1}{q_1} + \frac{p_2}{q_2}) \cdot \frac{r}{s} \\ &= \frac{p_1 r}{q_1 s} + \frac{p_2 r}{q_2 s} \\ &= f(\frac{p_1}{q_1}, \frac{r}{s}) + f(\frac{p_2}{q_2}, \frac{r}{s}) \end{aligned}$$

By symmmetry, this is also true for the second component. Thus, we can conclude that f is a $\mathbb{Z}$-balanced map between $\mathbb{Z}$-modules. By the universal property of tensor product, there exists a map $\tilde{f} : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ between abelian groups sending $\frac{p}{q} \otimes \frac{r}{s}$ to $\frac{pq}{rs}$. Consider a map

$$\begin{aligned} g : \mathbb{Q} &\to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}, \\ \frac{p}{q} &\mapsto \frac{p}{q} \otimes 1 \end{aligned}$$

It is easy to check this is also a map between abelian groups. For $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$, we have

$$\begin{aligned}(g \circ \tilde{f})(\frac{p}{q} \otimes \frac{r}{s}) &= g(\frac{pr}{qs}) \\ &= \frac{pr}{qs} \otimes 1 \\ &= (r \cdot \frac{p}{qs}) \otimes 1 \\ &= \frac{p}{qs} \otimes (s \cdot \frac{r}{s}) \\ &= (s \cdot \frac{p}{qs}) \otimes \frac{r}{s} \\ &= \frac{p}{q} \otimes \frac{r}{s}.\end{aligned}$$

This proves that $g \circ \tilde{f} = id$. Conversely, we know that

$$(\tilde{f} \circ g)(\frac{p}{q}) = \tilde{f}(\frac{p}{q} \otimes 1) = \frac{p}{q}.$$

So $\tilde{f} \circ g = id$. This proves that $\tilde{f}$ is an isomorphism between abelian groups. Therefore we can conclude that

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$$

as abelian groups.

Problem 17.2.7

If I is a right ideal of a ring R and V is a left R-module, then there is an isomorphism of abelian groups

$$R/I \otimes_R V \cong V/IV,$$

where IV is the subgroup of V generated by all elements xv with $x \in I$ and $v \in V$.

Solution: We define a map

$$\begin{aligned}f : R/I \times V &\to V/IV, \\ (r + I, v) &\mapsto rv + IV.\end{aligned}$$

We first check f is well-defined. Suppose $r_1 + I$ and $r_2 + I$ is the same element in R/I, this means $r_1 - r_2 \in I$. Then for any $v \in V$, we have $r_1v - r_2v = (r_1 - r_2)v \in IV$. This means $r_1v + IV$ and $r_2v + IV$ is the same element in V/IV. Given $s \in R$, we have

$$f(rs + I, v) = rsv + IV = f(r + I, sv).$$

So f is a R-balanced map. By the universal property of tensor product, there exists a unique abelian group homomorphism $\tilde{f} : R/I \otimes_R V \to V/IV$ sending $(r + I) \otimes v$ to $rv + IV$. Next, we are going to show that $\tilde{f}$ is an isomorphism.

Given $(r + I) \otimes v \in R/I \otimes_R V$, if $\tilde{f}((r + I) \otimes v) = 0 \in V/IV$, then $rv \in IV$. By definition this means $r \in I$, so $r+I$ is the zero element in R/I and we have $(r+I)\otimes v = (0+I)\otimes v = 0 \in R/I\otimes_R V$.

This proves that $\tilde{f}$ is injective. Conversely, given $w + IV \in V/IV$, consider $(1+I)\otimes w \in R/I \otimes_R V$, we have

$$\tilde{f}((1+I)\otimes w) = w + IV.$$

This proves $\tilde{f}$ is surjective. Thus, we can conclude that $\tilde{f}$ is an isomorphism between abelian groups and

$$R/I \otimes_R V \cong V/IV.$$

Problem 17.2.13
Prove:

(1) If M is any $\mathbb{Z}$-module, then $M \otimes_{\mathbb{Z}} \mathbb{Q}$ is an injective $\mathbb{Z}$-module.

(2) Deduce that given an injective $\mathbb{Z}$-module homomorphism $f : M \to N$, there exists a $\mathbb{Z}$-module homomorphism $\alpha : N \to M \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\alpha(f(m)) = m \otimes 1$.

(3) Let $\mu : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ be the product map, and

$$\beta := (id \otimes \mu) \circ (\alpha \otimes id) : N \otimes_{\mathbb{Z}} \mathbb{Q} \to M \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Then $\beta \circ (f \otimes id)$ is identity on $M \otimes_{\mathbb{Z}} \mathbb{Q}$.

(4) Deduce that $f \otimes id$ is injective and $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

Solution:

(1) Let $n \geq 0$ be an integer. (n) is an ideal in $\mathbb{Z}$, viewed as a $\mathbb{Z}$-module. Suppose we have a $\mathbb{Z}$-module homomorphism $p : (n) \to M \otimes_{\mathbb{Z}} \mathbb{Q}$. We know that p is completely determined by the image of $n \in (n)$. Assume $p(n) = \sum_{i=1}^{k} m_i \otimes \frac{p_i}{q_i}$. Consider a map $\tilde{p} : \mathbb{Z} \to M \otimes_{\mathbb{Z}} \mathbb{Q}$ by sending $1 \in \mathbb{Z}$ to $\sum_{i=1}^{k} m_i \otimes \frac{p_i}{nq_i}$. This is a $\mathbb{Z}$-module homomorphism and

$$\begin{aligned}
\tilde{p}(n) &= n(\sum_{i=1}^{k} m_i \otimes \frac{p_i}{nq_i}) \\
&= \sum_{i=1}^{k} m_i \otimes n \cdot \frac{p_i}{nq_i} \\
&= \sum_{i=1}^{k} m_i \otimes \frac{p_i}{q_i} \\
&= p(n).
\end{aligned}$$

Namely we have a commutative diagram

$$\begin{tikzcd}
(n) \arrow[r, hook] \arrow[dr, "p"'] & \mathbb{Z} \arrow[d, "\tilde{p}"] \\
& M \otimes_{\mathbb{Z}} \mathbb{Q}
\end{tikzcd}$$

Note that $\mathbb{Z}$ is a PID and every ideal in $\mathbb{Z}$ has the form (n) for some $n \in \mathbb{Z}$. We have proved every $\mathbb{Z}$-module homomorphism p can be extended to a $\mathbb{Z}$-module homomorphism $\tilde{p}$. By Baer's Criterion, $M \otimes_{\mathbb{Z}} \mathbb{Q}$ is an injective $\mathbb{Z}$-mdoule.

(2) We use the definition of injective modules. Consider the following diagram of solid arrows

$$\begin{tikzcd}
 & M \otimes_{\mathbb{Z}} \mathbb{Q} & \\
0 \arrow[r] & M \arrow[u, "i"] \arrow[r, "f"'] & N \arrow[ul, dashed, "\alpha"']
\end{tikzcd}$$

where $i : M \to M \otimes_{\mathbb{Z}} \mathbb{Q}$ is the inclusion map sending any $m \in M$ to $m \otimes 1 \in M \otimes_{\mathbb{Z}} \mathbb{Q}$. There exists a $\mathbb{Z}$-module homomorphism $\alpha : N \to M \otimes_{\mathbb{Z}} \mathbb{Q}$ such that the above diagram commutes. For any $m \in M$, we have

$$\alpha(f(m)) = i(m) = m \otimes 1.$$

(3) $- \otimes_{\mathbb{Z}} \mathbb{Q}$ is a functor and consider the following diagram

$$\begin{tikzcd}
M \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[d, "f \otimes id_{\mathbb{Q}}"] \arrow[ddd, bend right=60, "\gamma"'] \\
N \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[d, "\alpha \otimes id_{\mathbb{Q}}"] \arrow[dd, bend left=60, "\beta"] \\
M \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[d, "id_M \otimes \mu"] \\
M \otimes_{\mathbb{Z}} \mathbb{Q}
\end{tikzcd}$$

where $\mu : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ is the product map. For any $m \in M$ and $\frac{p}{q} \in \mathbb{Q}$, we have

$$\begin{aligned}
\gamma(m \otimes \frac{p}{q}) &= (\beta \circ (f \otimes id))(m \otimes \frac{p}{q}) \\
&= \beta(f(m) \otimes \frac{p}{q}) \\
&= (id \otimes \mu) \circ (\alpha \otimes id)(f(m) \otimes \frac{p}{q}) \\
&= (id \otimes \mu)(\alpha(f(m)) \otimes \frac{p}{q}) \\
&= (id \otimes \mu)(m \otimes 1 \otimes \frac{p}{q}) \\
&= m \otimes \frac{p}{q}.
\end{aligned}$$

So $\gamma = id : M \otimes_{\mathbb{Z}} \mathbb{Q} \to M \otimes_{\mathbb{Z}} \mathbb{Q}$ is the identity.

(4) We first prove the following claim.

<u>Claim:</u> Suppose $p : X \to Y$ and $q : Y \to Z$ are two maps of $\mathbb{Z}$-modules. If $q \circ p$ is injective, then p is injective.

<u>Proof:</u> Let $x \in \ker p$. We have $p(x) = 0 \in Y$. This implies that $(q \circ p)(x) = q(0) = 0$. So

$x \in \ker(q \circ p)$. Since $q \circ p$ is injective, so $x = 0$. This means p is also injective. ■

Use the claim above, because $id = \gamma = \beta \circ (f \otimes id)$ is injective, we can conclude that $f \otimes id$ is also injective. And since $- \otimes_{\mathbb{Z}} \mathbb{Q}$ sends injective maps to injective maps, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

Problem 17.2.20
Prove that a free module is flat. Then prove that a projective module is flat.

Solution: We will use Exercise 17.2.18. We prove it in the claim.
Claim: Let $(V_i)_{i \in I}$ be a family of R-modules. Then $\oplus_{i \in I} V_i$ is flat if and only if all V_i are flat.
Proof: Let $f : M \to N$ be an injective R-module homomorphism. Given a family of R-modules $(V_i)_{i \in I}$, by Theorem 17.2.16, we have a commutative diagram

$$\begin{tikzcd} M \otimes_R (\oplus_{i \in I} V_i) \arrow[r, "f \otimes id"] \arrow[d, "\alpha_M"'] & N \otimes_R (\oplus_{i \in I} V_i) \arrow[d, "\alpha_N"] \\ \oplus_{i \in I}(M \otimes_R V_i) \arrow[r, "\oplus(f \otimes id)"] & \oplus_{i \in I}(N \otimes_R V_i) \end{tikzcd}$$

where α_M, α_N are isomorphism of abelian groups. Assume $\oplus_{i \in I} V_i$ is flat, this means $f \otimes id$ in the top row is injective. Then

$$\alpha_N \circ (f \otimes id) = \oplus(f \otimes id) \circ \alpha_M$$

is also injective because α_N is an isomorphism. And since α_M is also an isomorphism, we know that $\oplus(f \otimes id)$ is injective. Conversely, if $\oplus(f \otimes id)$ is injective, by the same argument, we can see that $f \otimes id$ in the top row is injective. This proves that $\oplus_{i \in I} V_i$ is flat if and only if all V_i are flat. ■

Let $f : M \to N$ be an injective R-module homomorphism. We know that $M \otimes_R R \cong M$ and $N \otimes_R R \cong R$, this isomorphism is functorial so we have a commutative diagram

$$\begin{tikzcd} M \otimes_R R \arrow[r, "f \otimes id"] \arrow[d, "\sim"'] & N \otimes_R R \arrow[d, "\sim"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

We can see that $f \otimes id$ is also injective. So R is a flat R-module. By the claim we know that $\oplus_{i \in I} R$ is also flat. Suppose P is a projective R-module, this is equivalent to that there exists an R-module P' such that $P \oplus P' = \oplus_{i \in I} R$. We have already known that the free module $\oplus_{i \in I} R$ is flat, by the claim we know P is also flat.

Problem 17.3.2
If R is commutative and I, J are ideals in R, then there is an isomorphism of R-modules

$$R/I \otimes_R R/J \cong R/(I + J).$$

Solution: From what we know in Exercise 17.2.7, we can see that have an isomorphism of abelian groups

$$R/I \otimes_R R/J \cong (R/J)/I(R/J).$$

Since R is commutative, both sides can be viewed as an R-module and we the isomorphism we defined before is a R-module isomorphism.
Claim: If R is commmutative and $I, J \subset R$ are ideals, then we have the following R-module isomorphisms.

$$I(R/J) \cong I/(I \cap J) \cong (I+J)/J.$$

Proof: We define a map of R-modules. For any $a(b+I) \in I(R/J)$ where $a \in I$ and $b+J \in R/J$,

$$\begin{aligned} f : I(R/J) &\to I/(I \cap J), \\ a(b+J) &\mapsto ab + I \cap J. \end{aligned}$$

We check this is well-defined. Suppose $b_1 + J$ and $b_2 + J$ are two representatives for the same element in R/J. This means $b_1 - b_2 \in J$. Then we know

$$f(a(b_1+J)) - f(a(b_2+J)) = a(b_1 - b_2) + I \cap J.$$

Since $a \in I$ and $b_1 - b_2 \in J$, $a(b_1 - b_2) \in I \cap J$, so $f(a(b_1+J)) = f(a(b_2+J))$. Next, we are going to show this is an isomorphism. For any $a \in I$, consider the map

$$\begin{aligned} g : I/(I \cap J) &\to I(R/J), \\ (a + I \cap J) &\mapsto a(1+J). \end{aligned}$$

This is well-defined. Indeed, suppose for $a_1, a_2 \in I$, $a_1 + I \cap J$ and $a_2 + I \cap J$ represents the same element in $I/(I \cap J)$. This means $a_1 - a_2 \in I \cap J$. Then the image

$$g(a_1 + I \cap J) - g(a_2 + I \cap J) = (a_1 - a_2)(1 + J).$$

Note that $a_1 - a_2 \in I \cap J \subset J$, so $(a_1 - a_2)(1+J) = (a_1 - a_2) + J = J$ is the zero element in R/J. So g is a well-define R-module homomorphism. Moreover, we have $f \circ g = id$ and for any $a \in I$ and $b + J \in R/J$,

$$g(ab + I \cap J) = ab(1+J) = a(b+J).$$

This proves that f is an isomorphism of R-modules and

$$I(R/J) \cong I/(I \cap J) \cong (I+J)/J.$$

The next isomorphism is by the second isomorphism theorem in commutative rings. ■

Note that $J \subset I + J \subset R$, use the third isomorphism theorem and we have

$$R/I \otimes_R R/J \cong (R/J)/I(R/J) \cong (R/J)/(I + J/J) \cong R/(I+J).$$

Problem 17.3.10
$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$ as $\mathbb{C}$-algebras.

Solution: $\mathbb{H}$ is a 4-dimensional $\mathbb{R}$-vector space with standard $\mathbb{R}$-basis $\{1, i, j, k\}$ and $\mathbb{C}$ is a $\mathbb{R}$-vector space with $\mathbb{R}$-basis $\{1, \sqrt{-1}\}$. By Theorem 17.3.4, $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ has a $\mathbb{R}$-basis

$$\begin{gathered} 1 \otimes 1, i \otimes 1, j \otimes 1, k \otimes 1, \\ 1 \otimes i, i \otimes i, j \otimes i, k \otimes i. \end{gathered}$$

If we view $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ as a $\mathbb{C}$-algebra, then for any $a \in \{1, i, j, k\}$, we have

$$i(a \otimes 1) = a \otimes i.$$

Thus,

$$1 \otimes 1, i \otimes 1, j \otimes 1, k \otimes 1$$

is a basis for $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ as a $\mathbb{C}$-algebra. Consider the following map

$$\begin{aligned} f : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} &\to M_2(\mathbb{C}), \\ 1 \otimes 1 &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ i \otimes 1 &\mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\ j \otimes 1 &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ k \otimes 1 &\mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \end{aligned}$$

It is easy to check that

$$f(i \otimes 1)^2 = f(j \otimes 1)^2 = f(k \otimes 1)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\begin{aligned} f(i \otimes 1)f(j \otimes 1) = -f(j \otimes 1)f(i \otimes 1) &= \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} = f(k \otimes 1), \\ f(j \otimes 1)f(k \otimes 1) = -f(k \otimes 1)f(j \otimes 1) &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = f(i \otimes 1), \\ f(k \otimes 1)f(i \otimes 1) = -f(i \otimes 1)f(k \otimes 1) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = f(j \otimes 1). \end{aligned}$$

So f defines a $\mathbb{C}$-algebra homomorphism. Next, we are going to show f is surjective. Let $M \subset M_2(\mathbb{C})$ be the subspace generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We can see that

$$\frac{1}{2}(f(1\otimes 1)-if(i\otimes 1))=\frac{1}{2}(\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}-1&0\\0&1\end{pmatrix})=\begin{pmatrix}1&0\\0&0\end{pmatrix},$$

$$\frac{1}{2}(f(1\otimes 1)+if(i\otimes 1))=\frac{1}{2}(\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}-1&0\\0&1\end{pmatrix})=\begin{pmatrix}0&0\\0&1\end{pmatrix},$$

$$\frac{1}{2}(f(j\otimes 1)-if(k\otimes 1))=\frac{1}{2}(\begin{pmatrix}0&1\\-1&0\end{pmatrix}-\begin{pmatrix}0&-1\\-1&0\end{pmatrix})=\begin{pmatrix}0&1\\0&0\end{pmatrix},$$

$$-\frac{1}{2}(f(j\otimes 1)+if(k\otimes 1))=-\frac{1}{2}(\begin{pmatrix}0&1\\-1&0\end{pmatrix}+\begin{pmatrix}0&-1\\-1&0\end{pmatrix})=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

We know that $M_2(\mathbb{C})$ can be generated by

$$\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

as a $\mathbb{C}$-algebra. This proves that $M=M_2(\mathbb{C})$ and f is surjective. Note that

$$\dim_{\mathbb{C}}(\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C})=4=\dim_{\mathbb{C}}(M_2(\mathbb{C})).$$

So we have

$$\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}\cong M_2(\mathbb{C})$$

as $\mathbb{C}$-algebras.

Problem 17.3.11

$M_n(R)\otimes_R M_m(R)\cong M_{mn}(R)$ for a commutative ring R.

Solution: Let $A=(a_{ij})_{1\le i,j\le n}\in M_n(R)$ and $B=(b_{kl})_{1\le k,l\le m}\in M_m(R)$. We define a map

$$f: M_n(R)\times M_m(R)\to M_{mn}(R),$$
$$(A,B)\mapsto\begin{pmatrix}a_{11}B&a_{12}B&\cdots&a_{1n}B\\a_{21}B&a_{22}B&\cdots&a_{2n}B\\\vdots&\vdots&\ddots&\vdots\\a_{n1}B&a_{n2}B&\cdots&a_{nn}B\end{pmatrix}.$$

where the image is a block matrix and for $1\le i,j\le n$, each block $a_{ij}B$ is a $m\times m$ matrix with (k,l)-entry $a_{ij}b_{kl}$ for $1\le k,l\le m$. For any $r\in R$, we have

$$f(Ar,B)=\begin{pmatrix}a_{11}rB&a_{12}rB&\cdots&a_{1n}rB\\a_{21}rB&a_{22}rB&\cdots&a_{2n}rB\\\vdots&\vdots&\ddots&\vdots\\a_{n1}rB&a_{n2}rB&\cdots&a_{nn}rB\end{pmatrix}=f(A,rB).$$

So f is a R-balanced map and by universal property of tensor product, we have an R-module homomorphism $\tilde{f} : M_n(R) \otimes_R M_m(R) \to M_{mn}(R)$. Suppose $A \otimes B \in M_n(R) \otimes_R M_m(R)$ satisfying $\tilde{f}(A \otimes B) = 0$, namely $a_{ij}B$ is the zero matrix for any $1 \le i, j \le n$. Let E_{ij} be the $n \times n$ matrix with 1 at the (i, j)-entry and all other entries are zero. Then we can write

$$\begin{aligned} A \otimes B &= (\sum_{1 \le i,j \le n} a_{ij} E_{ij}) \otimes B \\ &= \sum_{1 \le i,j \le n} (a_{ij} E_{ij} \otimes B) \\ &= \sum_{1 \le i,j \le n} E_{ij} \otimes a_{ij} B \\ &= \sum_(1 \le i, j \le n) E_{ij} \otimes 0 \\ &= 0 \end{aligned}$$

This proves $\ker \tilde{f} = 0$, so $\tilde{f}$ is injective. Note that both the matrix algebra $M_n(R)$ is a free R-module of rank n^2, so we have

$$\operatorname{rank}(M_n(R) \otimes_R M_m(R)) = \operatorname{rank} M_n(R) \cdot \operatorname{rank} M_m(R) = n^2 \cdot m^2 = (mn)^2 = \operatorname{rank} M_{mn}(R).$$

This proves that $\tilde{f}$ is an R-module isomorphism. The last thing we need to show is that $\tilde{f}$ is compatible with matrix multiplication. Let $A \otimes B, C \otimes D \in M_n(R) \otimes_R M_m(R)$, suppose $A = (a_{ij})_{1 \le i,j \le n}$ and $C = (c_{ij})_{1 \le i,j \le n}$. Then we have

$$\tilde{f}(A \otimes B) \cdot \tilde{f}(C \otimes D) = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix} \begin{pmatrix} c_{11}D & c_{12}D & \cdots & c_{1n}D \\ c_{21}D & c_{22}D & \cdots & c_{2n}D \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1}D & c_{n2}D & \cdots & c_{nn}D \end{pmatrix}$$

Viewed as a block matrix, for $1 \le i, j \le n$, the (i, j)-block entry is

$$\sum_{k=1}^{n} (a_{ik}B)(c_{kj}D) = \sum_{k=1}^{n} (a_{ik}c_{kj})BD = (AC)_{ij}BD$$

where $(AC)_{ij}$ means the (i, j)-entry for the matrix AC. On the other hand, we know that

$$\tilde{f}(AC \otimes BD) = \begin{pmatrix} (AC)_{11}BD & (AC)_{12}BD & \cdots & (AC)_{1n}BD \\ (AC)_{21}BD & (AC)_{22}BD & \cdots & (AC)_{2n}BD \\ \vdots & \vdots & \ddots & \vdots \\ (AC)_{n1}BD & (AC)_{n2}BD & \cdots & (AC)_{nn}BD \end{pmatrix}$$

This proves that

$$\tilde{f}(A \otimes B) \cdot \tilde{f}(C \otimes D) = \tilde{f}(AC \otimes BD).$$

So $\tilde{f}$ is indeed an matrix algebra homomorphism.

Problem 17.3.13
True or false? $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as $\mathbb{Z}$-algebras.

Solution: This is true. The isomorphism we defined in Exercise 17.2.6 is the map

$$f : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q},$$
$$\frac{p}{q} \otimes \frac{r}{s} \mapsto \frac{pq}{rs}$$

We need to show f is compatible with multiplication. Suppose $\frac{p_1}{q_1} \otimes \frac{r_1}{s_1}, \frac{p_2}{q_2} \otimes \frac{r_2}{s_2} \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, we have

$$\begin{aligned} f(\frac{p_1}{q_1} \otimes \frac{r_1}{s_1}) \cdot f(\frac{p_2}{q_2} \otimes \frac{r_2}{s_2}) &= \frac{p_1 r_1}{q_1 s_1} \cdot \frac{p_2 r_2}{q_2 s_2} \\ &= \frac{p_1 p_2 r_1 r_2}{q_1 q_2 s_1 s_2} \\ &= f(\frac{p_1 p_2}{q_1 q_2} \otimes \frac{r_1 r_2}{s_1 s_2}) \\ &= f((\frac{p_1}{q_1} \otimes \frac{r_1}{s_1}) \cdot (\frac{p_2}{q_2} \otimes \frac{r_2}{s_2})) \end{aligned}$$

Now we know that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as $\mathbb{Z}$-algebras.

Problem 17.3.21
True or false? Let A and B be a finite dimensional semisimple algerbras over an alegbraically closed field $\mathbb{F}$. Then every finite dimenisonal $A \otimes B$-module is of the form $V \boxtimes W$ for some A-module V and some B-module W.

Solution: This is false. Consider $A = M_2(\mathbb{C}) \times M_3(\mathbb{C})$ and $B = M_3(\mathbb{C}) \times M_4(\mathbb{C})$. By Wedderburn-Artin Theorem for algebras, A and B are semisimple. By Proposition 16.2.8, simple A-modules up to isomorphism have the form $\mathbb{C}^2, \mathbb{C}^3$ and simple B-modules up to isomorphism have the form $\mathbb{C}^3, \mathbb{C}^4$. Consider the $A \otimes_{\mathbb{C}} B$-module $M = (\mathbb{C}^2 \boxtimes \mathbb{C}^4) \oplus (\mathbb{C}^3 \boxtimes \mathbb{C}^3)$. By Theorem 17.3.20, we know that both $\mathbb{C}^2 \boxtimes \mathbb{C}^4$ and $\mathbb{C}^3 \boxtimes \mathbb{C}^3$ are simple $A \otimes_{\mathbb{C}} B$-modules, so M is a finite dimensional $A \otimes_{\mathbb{F}} B$-module. Now we show that M cannot be written as $V \boxtimes W$ where V is a A-module and W is a B-module. Note that A and B are semisimple, every A-module and B-module are semisimple, namely they can be written as direct sum of simple A-modules and simple B-modules. For dimension reasons, we have

$$\dim_{\mathbb{C}} M = 17 = \dim_{\mathbb{C}} V \cdot \dim_{\mathbb{C}} W.$$

So one of $\dim_{\mathbb{C}} V$ or $\dim_{\mathbb{C}} W$ has to be 1, but $\mathbb{C}$ is not a simple A-module or simple B-module. Thus, M cannot be written in this form.