Zhengdong Zhang
Email: zhengz@uoregon.edu
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Instructor: Professor Arkady Berenstein

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Problem 17.1.3
Let $(V_i)_{i\in I}$ be a family of R-modules. Then:

(1) $\oplus_{i\in I} V_i$ is projective if and only if each V_i is projective.

(2) $\prod_{i\in I} V_i$ is injective if and only if each V_i is injective.

Solution:

(1) Note that $\oplus_{i\in I} V_i$ is the coproduct of a family of R-modules $(V_i)_{i\in I}$ and thus has the universal property. Assume $\oplus_{i\in I} V_i$ is projective. Fix $k \in I$. Let $\pi : V \twoheadrightarrow W$ be a surjective map and $\phi : V_k \to W$ is an R-module homomorphism. Define a family of R-module homomorphisms $p_i : V_i \to W$ for any $i \in I$ as follows: if $i = k$, then $p_i = \phi$. Otherwise p_i is the zero map. By universal property of $\oplus_{i\in I} V_i$, we have a unique map $p : \oplus_{i\in I} V_i \to W$ such that $p \circ j_k = \phi$ where $j_k : V_k \to \oplus_{i\in I} V_i$ is the canonical inclusion.

$$\begin{tikzcd}
& V_k \arrow[d, "j_k"] \arrow[dd, bend left, "\phi"] & \\
& \oplus_{i\in I} V_i \arrow[d, "p"] \arrow[dl, dashed, "\psi"'] & \\
V \arrow[r, two heads, "\pi"'] & W \arrow[r] & 0
\end{tikzcd}$$

By projectivity of $\oplus_{i\in I} V_i$, there exists a map $\psi : \oplus_{i\in I} V_i \to V$ such that $\pi \circ \psi = p$. So for any $v \in V_k$, we have

$$\phi(v) = (p \circ j_k)(v) = (\pi \circ \psi \circ j_k)(v).$$

This implies $\psi \circ j_k$ is a map making the following diagram commutes:

$$\begin{tikzcd}
& V_k \arrow[d, "\phi"] \arrow[dl, dashed, "\psi\circ j_k"'] & \\
V \arrow[r, two heads, "\pi"'] & W \arrow[r] & 0
\end{tikzcd}$$

This proves that V_k is projective for any $k \in I$.

On the other hand, assume for each $i \in I$, V_i is a projective R-module. Suppose $\pi : V \to W$ is surjective and we have a homomorphism $\phi : \oplus_{i\in I} V_i \to W$. For each $i \in I$, consider the composition with canonical inclusion $\phi \circ j_i : V_i \to W$. By the projectivity of V_i, there exists a map $\psi_i : V_i \to V$ such that $\pi \circ \psi_i = \phi \circ j_i$.

$$\begin{tikzcd}
& V_i \arrow[d, "j_i"] \arrow[ddl, dashed, "\psi_i"'] & \\
& \oplus_{i\in I} V_i \arrow[d, "\phi"] & \\
V \arrow[r, two heads, "\pi"'] & W \arrow[r] & 0
\end{tikzcd}$$

The universal property tells us there exists a map $\psi : \oplus_{i\in I} V_i \to V$ such that $\psi \circ j_i = \psi_i$. We claim that $p \circ \psi = \phi$. For any $v \in \oplus_{i\in I} V_i$, v can be written as $v = \sum_{i\in I} v_i$ for each $v_i \in V_i$. Then

$$\begin{aligned}
(\pi \circ \psi)(v) &= (\pi \circ \psi)(\sum_{i\in I} v_i) \\
&= \sum_{i\in I} (\pi \circ \psi \circ j_i)(v_i) \\
&= \sum_{i\in I} (\pi \circ \psi_i)(v_i) \\
&= \sum_{i\in I} (\phi \circ j_i)(v_i) \\
&= \phi(\sum_{i\in I} j_i(v_i)) \\
&= \phi(v)
\end{aligned}$$

We have the following commutative diagram

$$\begin{tikzcd}
& V_i \arrow[d, "j_i"] \arrow[ddl, dashed, "\psi_i"'] & \\
& \oplus_{i\in I} V_i \arrow[d, "\phi"] \arrow[dl, dashed, "\psi"] & \\
V \arrow[r, two heads, "\pi"'] & W \arrow[r] & 0
\end{tikzcd}$$

This proves that $\oplus_{i\in I} V_i$ is projective.

(2) Assume $\prod_{i\in I} V_i$ is injective. Fix $k \in I$. Given an injective homomorphism $\iota : W \to V$ and a homomorphism $\phi : W \to V_k$, for any $k \neq i \in I$, consider a family of zero maps $W \to V_i$. By the universal property of the product $\prod_{i\in I} V_i$, we have a unique map $q : W \to \prod_{i\in I} V_i$ such that $p_k \circ q = \phi$ where $p_k : \prod_{i\in I} V_i \to V_k$ is the canonical projection.

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r, hook, "\iota"] \arrow[d, "q"] \arrow[dd, bend right=60, "\phi"'] & V \\
& \prod_{i\in I} V_i \arrow[d, "p_k"] & \\
& V_k &
\end{tikzcd}$$

$\prod_{i\in I} V_i$ being injective implies that there exists $\psi : V \to \prod_{i\in I} V_i$ such that $\psi \circ \iota = q$. We claim we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r, hook, "\iota"] \arrow[d, "\phi"'] & V \arrow[dl, dashed, "p_k \circ \psi"] \\
& V_k &
\end{tikzcd}$$

Indeed, given any $w \in W$, we have

$$\begin{aligned}
(p_k \circ \psi \circ \iota)(w) &= (p_k)(\psi \circ \iota)(w) \\
&= (p_k \circ q)(w) \\
&= \phi(w).
\end{aligned}$$

This proves that for each $i \in I$, V_i is injective.

Conversely, assume for any $i \in I$, each V_i is injective. Given an injective homomorphism $\iota : W \to V$ and a homomorphism $\phi : W \to \prod_{i\in I} V_i$, consider the composition $p_i \circ \phi : W \to V_i$ where $p_i : \prod_{i\in I} V_i \to V_i$ is the canonical projection. Since V_i is injective, there exists a homomorphism $\psi_i : v \to V_i$ such that $p_i \circ \phi = \psi_i \circ \iota$.

$$\begin{tikzcd}
& V_i & \\
& \prod_{i\in I} V_i \arrow[u, "p_i"'] & \\
0 \arrow[r] & W \arrow[r, hook, "\iota"'] \arrow[u, "\phi"'] & V \arrow[uul, dashed, "\psi_i"']
\end{tikzcd}$$

We have a family of homomorphisms $(\psi_i)_{i\in I}$. By the universal property of product, there exists a unique homomorphism $\psi : V \to \prod_{i\in I} V_i$ such that for any $i \in I$, we have $p_i \circ \psi = \psi_i$. We claim that $\psi \circ \iota = \phi$, namely the following diagram commutes

$$\begin{tikzcd}
& \prod_{i\in I} V_i & \\
0 \arrow[r] & W \arrow[r, hook, "\iota"'] \arrow[u, "\phi"] & V \arrow[ul, "\psi"']
\end{tikzcd}$$

For any $w \in W$ and $i \in I$, we have

$$\begin{aligned}
(p_i \circ \psi \circ \iota)(w) &= (\psi_i \circ \iota)(w) \\
&= (p_i \circ \phi)(w).
\end{aligned}$$

Since $p_i : \prod_{i\in I} V_i \to V_i$ is the projection. We have $\psi \circ \iota = \phi$. This proves that $\prod_{i\in I} V_i$ is injective.

Problem 17.1.5
Let P, I and V be R-modules.

(1) If I is injective and $I \subseteq V$ is a submodule then I is a summand of V.

(2) If P is projective and $P \cong V/W$ for some submodule $W \subseteq V$ then W is a summand of V.

Solution:

(1) We have an exact sequence

$$0 \longrightarrow I \xhookrightarrow{\iota} V$$

where $\iota : I \to V$ is the inclusion. By Theorem 17.1.4, I being injective implies that there exists $\psi : V \to I$ such that $\psi \circ \iota = id_I$. We know that $\ker \psi$ is a submodule of V and by the first isomorphism theorem for modules we have $V/\ker \psi \cong I$. We have a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker \psi \arrow[r] & V \arrow[r, shift left, "\psi"] & I \arrow[l, shift left, "\iota"] \arrow[r] & 0 \end{tikzcd}$$

Note that $\psi \circ \iota = id_i$ implies that this sequence splits by Lemma 14.2.8. So we have $\ker \psi \oplus I = V$. This proves that I is a summand of V.

(2) $P \cong V/W$ implies that we have a short exact sequence

$$0 \longrightarrow W \xhookrightarrow{i} V \xrightarrow{\pi} P \longrightarrow 0$$

where i is the inclusion and π is the projection. By Theorem 17.1.4, there exists a map $\psi : P \to V$ such that $\pi \circ \psi = id_P$. By Lemma 14.2.8, this sequence must split and we have $V \cong P \oplus W$ and thus, W is a summand of V.

Problem 17.1.8 (Schanuel's Lemma)
Let

$$0 \to B \xrightarrow{j} P \xrightarrow{\pi} A \to 0,$$
$$0 \to B' \xrightarrow{j'} P' \xrightarrow{\pi'} A \to 0.$$

be two short exact sequences of R-modules with P and P' projective. Then $B \oplus P' \cong B' \oplus P$.

Solution: Consider the suset $W \subseteq P \times P'$ satisfying the following property: for any $(a, b) \in P \times P'$, we have $\pi(a) = \pi'(b)$. We claim that W is a submodule of $P \times P'$. Indeed, for any $r \in R$ and $(a, b) \in W$, we have

$$\pi(ra) = r\pi(a) = r\pi'(b) = \pi'(rb).$$

This implies $r(a, b) = (ra, rb) \in W$. So W is a submodule of $P \times P'$. Consider the composition

$$f : W \to P \times P' \to P,$$
$$g : W \to P \times P' \to P'.$$

where the first map is the inclusion of submodules and the second map is the projection. We are going to show that f and g are surjective. For any $p \in P$, $\pi(p)$ is an element in A. Since π' is surjective, there exists $p' \in P'$ such that $\pi(p) = \pi'(p')$. Note that by definition, (p, p') is in W, and $f(p, p') = p$. This proves that f is surjective. Similarly, we can show that g is surjective.

Next, we are going to show that the kernel of the map $f : W \to P$ is isomorphic to B' and the kernel of $g : W \to P'$ is isomorphic to B. Suppose $(a, b) \in \ker f$, namely $f(a, b) = a = 0 \in P$. Then by definition of W, we have

$$0 = \pi(0) = \pi(a) = \pi'(b).$$

This tells us $b \in \ker \pi' = B'$ by exactness. Conversely, suppose $b \in B' = \ker \pi'$ and consider $(0, b) \in W$, we have $f(0, b) = 0 \in P$. So $(0, b) \in \ker f$. This implies that the map $B' \to W$ sending $b \in B'$ to $(0, b) \in W$ is exactly the kernel of $f : W \to P$. Similarly, we can prove that B is isomorphic to the kernel of $g : W \to P'$. We have the following two short exact sequence:

$$0 \to B' \to W \xrightarrow{f} P \to 0,$$
$$0 \to B \to W \xrightarrow{g} P' \to 0.$$

Note that P and P' is projective, so by Theorem 17.1.7, these two short exact sequences splits and we have

$$B' \oplus P \cong W \cong B \oplus P'.$$

Problem 17.1.10
If every irreducible R-module is projective then R is semisimple artinian.

Solution: We want to prove that R is semisimple artinian, by Lemma 16.2.1, it is the same as proving the left regualr module ${}_RR$ is semisimple. By Theorem 14.2.19, it suffice to show that R is the sum of all of its simple submodules. Consider S is the sum of all simple submodules in R. Note that for every element $a \in S$, a can be written as $a = \sum v_i$ for some $v_i \in V_i$ where each V_i is a simple ideal of R. So for any $r \in R$, we have $rv_i \in V_i$ for any i. So $ra = \sum rv_i \in S$. This proves that S is an ideal in R. Assume $S \neq R$, then S must be contained in some maximal ideal M in R. Consider the following short exact sequence

$$0 \to M \to R \to R/M \to 0.$$

Since M is maximal, so R/M is isomorphic to a simple R-module by Exercise 14.1.23. We have known that every simple R-module is projective, so R/M is projective and the above short exact sequence splits. We have

$$R \cong M \oplus R/M.$$

This shows that R/M is a simple R-submodule of R which is not in S. A contradiction. So $S = R$ and we have proved R is the sum of all its simple submodules, thus R is semisimple artinian.

Problem 17.1.11
True or false? Every short exact sequence of $\mathbb{C}[x]/(x^2-1)$-modules is split.

Solution: This is true. The ideal $(x^2-1) = (x-1)\cap(x+1)$. The polynomials $x-1$ and $x+1$ are coprime in $\mathbb{C}[x]$. By the Chinese remainder theorem, we have

$$\mathbb{C}[x]/(x^2-1) \cong \mathbb{C}[x]/(x-1) \times \mathbb{C}[x]/(x+1) \cong \mathbb{C}\times\mathbb{C}.$$

By the Wedderburn-Artin theorem for algebras and Corollary 16.2.15, $\mathbb{C}[x]/(x^2-1)$ is semisimple artinian and by Proposition 17.1.9, every $\mathbb{C}[x]/(x^2-1)$-module is projective. By Theorem 17.1.7, every short exact sequence of $\mathbb{C}[x]/(x^2-1)$-modules must split.

Problem 17.1.16
Let P be a projective R-module, and

$$\cdots \to V_{n-1} \xrightarrow{f_n} V_n \xrightarrow{f_{n+1}} V_{n+1} \to \cdots$$

be an exact complex of R-modules and R-module homomorphisms. Prove that the corresponding complex

$$\cdots \to \hom_R(P, V_{n-1}) \xrightarrow{(f_n)_*} \hom_R(P, V_n) \xrightarrow{(f_{n+1})_*} \hom(P, V_{n+1}) \to \cdots$$

is exact. Formulate and prove the dual statement involving injective modules.

Solution: We first prove two useful claims.
Claim: $\hom_R(P, \ker f_{n+1}) = \ker(f_{n+1})_*$. Here we view $\hom(P, \ker f_{n+1})$ as a subset of $\hom(P, V_n)$.
Proof: Given $g : P \to \ker f_{n+1} \hookrightarrow V_n$, the composition

$$f_{n+1}\circ g : P \to V_{n+1}$$

must be the zero map since it factors through $\ker f_{n+1}$. So we have $\hom_R(P, \ker f_{n+1}) \subseteq \ker(f_{n+1})_*$.
Conversely, given $g : P \to V_n$ such that

$$f_{n+1}\circ g : P \to V_{n+1}$$

is the zero map, by the universal property of the kernel, g must factor through $\ker f_{n+1}$, so g can be viewed as an element in $\hom_R(P, \ker f_{n+1})$. This proves that

$$\hom(P, \ker f_{n+1}) = \ker(f_{n+1})_*.$$

■

Claim: $\hom_R(P, \operatorname{Im} f_n) = \operatorname{Im}(f_n)_*$. Here $\hom_R(P, \operatorname{Im} f_n)$ is viewed as a subset of $\hom_R(P, V_n)$.

Proof: Given $g : P \to \operatorname{Im} f_n$, consider the following diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "g'"'] \arrow[d, "g"] & \\ V_{n-1} \arrow[r, two heads, "f_n"'] & \mathrm{im} f_n \arrow[r] & 0 \end{tikzcd}$$

Because P is projective, there exists $g' : P \to V_{n-1}$ such that the above diagram commutes. This proves that $\hom_R(P, \operatorname{Im} f_n) \subseteq \operatorname{Im}(f_n)_*$. Conversely, consider a composition $P \xrightarrow{h} V_{n-1} \xrightarrow{f_n} V_n$. We need to show that $f_n \circ h \in \hom_R(P, \operatorname{Im} f_n)$. This is true since $\operatorname{Im}(f_n \circ h)$ must be contained in $\operatorname{Im} f_n$. Thus, $\hom_R(P, \operatorname{Im} f_n) = \operatorname{Im}(f_n)_*$. ■

The exactness of the original sequence tells us that $\ker f_{n+1} = \operatorname{Im} f_n$ and by the claims, we have

$$\ker(f_{n+1})_* = \hom_R(P, \ker f_{n+1}) = \hom_R(P, \operatorname{Im} f_n) = \operatorname{Im}(f_n)_*.$$

Therefore, we have an exact sequence

$$\cdots \to \hom_R(P, V_{n-1}) \xrightarrow{(f_n)_*} \hom_R(P, V_n) \xrightarrow{(f_{n+1})_*} \hom(P, V_{n+1}) \to \cdots$$

Problem 17.1.19
Consider $\mathbb{Z}/2\mathbb{Z}$ as a $\mathbb{Z}/6\mathbb{Z}$-module via the natural projection $\mathbb{Z}/6\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$. Then $\mathbb{Z}/2\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module.

Solution: By the Chinese remainder Theorem, we have

$$\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

By Theorem 17.1.17, $\mathbb{Z}/2\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module.

Problem 17.1.21
If $e \in R$ is an idempotent then Re is a projective R-module.

Solution: By Lemma 14.5.1, e being an idempotent implies that

$$R = Re \oplus R(1-e).$$

By Theorem 17.1.17, Re is a projective R-module.

Problem 17.1.26
Let R be a domain and $\mathbb{F}$ be its field of fractions. Prove that $\mathbb{F}$ is an injective R-module.

Solution: Let $I \subset R$ be an ideal and $\phi : I \to \mathbb{F}$ is a R-module homomorphism. If I is the zero ideal and ϕ is the zero map, then ϕ can be extended to the zero map $R \to \mathbb{F}$. Assume I contains nonzero elements. Pick $a \in I$ and $a \neq 0$, note that R is a domain so $a \in I \subset R$ is invertible in $\mathbb{F}$ since $\mathbb{F}$ is the fraction field of R. Define

$$\begin{aligned} \psi : R &\to \mathbb{F}, \\ b &\mapsto \frac{b\phi(a)}{a}. \end{aligned}$$

For any $r \in R$, we have

$$\psi(rb) = \frac{rb\phi(a)}{a} = r\psi(b).$$

This is a R-module homomorphism. Moreover, for any $b \in I$, note that R is commutative and we have

$$\psi(b) = \frac{b\phi(a)}{a} = \frac{\phi(ba)}{a} = \frac{a\phi(b)}{a} = \phi(b).$$

We have a commutative diagram

$$\begin{tikzcd} I \arrow[r, "\phi"] \arrow[d, hook] & \mathbb{F} \\ R \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

By Baer's Criterion, $\mathbb{F}$ is an injective R-module.

Problem 17.1.27
Let $n \geq 1$. Then $\mathbb{Z}/n\mathbb{Z}$ is an injective $\mathbb{Z}/n\mathbb{Z}$-module.

Solution: The ideals in $\mathbb{Z}/n\mathbb{Z}$ are of the form $\mathbb{Z}/d\mathbb{Z}$ where $d|n$. Given a homomorphism $\phi : \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, we have $d\phi(1) = 0 \in \mathbb{Z}/n\mathbb{Z}$. This implies $n|(d\phi(1))$. There exists $r \in \mathbb{Z}$ such that $rn = d\phi(1)$. Note that $r < n$ since $d\phi(1) < dn$. We define a homomorphism $\psi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ with $\psi(1) = r$. We have the following diagram

$$\begin{tikzcd} \mathbb{Z}/d\mathbb{Z} \arrow[r, "\phi"] \arrow[d, "i"'] & \mathbb{Z}/n\mathbb{Z} \\ \mathbb{Z}/n\mathbb{Z} \arrow[ur, "\psi"'] & \end{tikzcd}$$

where i sends 1 to $\frac{n}{d}$, thus identify $\mathbb{Z}/d\mathbb{Z}$ as an ideal $\frac{n}{d}\mathbb{Z}/n\mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$. It is easy to check that for any $a \in \mathbb{Z}/d\mathbb{Z}$, we have

$$(\psi \circ i)(a) = \psi(\frac{n}{d}a) = \frac{n}{d}ra = \phi(1)a = \phi(a).$$

By Baer's Criterion, $\mathbb{Z}/n\mathbb{Z}$ is injective.