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Problem 18.1.2
Let G be a group and P be a projective $\mathbb{Z}G$-module. If M is a $\mathbb{Z}G$-module, which is projective as a $\mathbb{Z}$-module, then the $\mathbb{Z}G$-module $P \otimes_{\mathbb{Z}} M$ (with diagonal action of G) is projective.

Solution: We first prove a useful fact.
Claim: Let U, V be $\mathbb{Z}G$-module, then there is a G-action on the $\mathbb{Z}$-module $\hom_{\mathbb{Z}}(U, V)$ and we have a natural isomorphism

$$\hom_{\mathbb{Z}}(U, V)^G \cong \hom_{\mathbb{Z}G}(U, V)$$

as abelian groups where $\hom_{\mathbb{Z}}(U, V)$ is the G-invariant set under the previous action.
Proof: We first define a G-action on $\hom_{\mathbb{Z}}(U, V)$. Let $f : U \to V$ be a map of $\mathbb{Z}$-modules, for any $u \in U$, we define

$$(g \cdot f)(u) := g \cdot f(g^{-1}u).$$

This is a well-defined G-action. For $g, h \in G$, we have

$$(g \cdot h \cdot f)(u) = g \cdot (h \cdot f)(g^{-1}u) = g \cdot h \cdot f(h^{-1}g^{-1}u) = (gh) \cdot f((gh)^{-1}u) = ((gh) \cdot f)(u).$$

Consider the G-invariant set $\hom_{\mathbb{Z}}(U, V)^G$ under this action, consider extending the above G-action $\mathbb{Z}$-linearly, and we obtain a $\mathbb{Z}G$-module structure because

$$(gf)(u) = (g(g^{-1} \cdot f))(u) = g \cdot g^{-1} \cdot f(gu) = f(gu)$$

for all $g \in G$ and $u \in U$. Conversely, given a $\mathbb{Z}G$-module homomorphism $h : U \to V$, viewed as a $\mathbb{Z}$-module homomorphism, we need to show that h is G-invariant. Indeed, we have

$$(g \cdot h)(u) = g \cdot h(g^{-1}u) = h(gg^{-1}u) = h(u)$$

for all $g \in G$ and $u \in U$. We have proved there is an isomorphism of abelian groups

$$\hom_{\mathbb{Z}G}(U, V) \cong \hom_{\mathbb{Z}}(U, V)^G.$$

Lastly, we check this isomorphism is natural. Suppose we have $\mathbb{Z}G$-modules U, V_1, V_2 and a $\mathbb{Z}G$-module homomorphism $\phi : V_1 \to V_2$, we have a diagram

$$\begin{tikzcd} \hom_{\mathbb{Z}G}(U, V_1) \arrow[r, "\cong"] \arrow[d] & \hom_{\mathbb{Z}}(U, V_1)^G \arrow[d] \\ \hom_{\mathbb{Z}G}(U, V_2) \arrow[r, "\cong"'] & \hom_{\mathbb{Z}}(U, V_2)^G \end{tikzcd}$$

It is commutative because the isomorphism is taking map f to the same map. ■

Suppose we have two $\mathbb{Z}G$-module U, V and a surjective $\mathbb{Z}G$-homomorphism $f : U \twoheadrightarrow V$. By the adjointness of $\otimes$ and hom and the above claim, we have a commutative diagram

$$\begin{array}{ccccccc}
\hom_{\mathbb{Z}G}(P \otimes_{\mathbb{Z}} M, U) & \xrightarrow{\cong} & \hom_{\mathbb{Z}}(P \otimes_{\mathbb{Z}} M, U)^G & \xrightarrow{\cong} & \hom_{\mathbb{Z}}(P, \hom_{\mathbb{Z}}(M, U))^G & \xrightarrow{\cong} & \hom_{\mathbb{Z}G}(P, \hom_{\mathbb{Z}}(M, U)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\hom_{\mathbb{Z}G}(P \otimes_{\mathbb{Z}} M, V) & \xrightarrow[\cong]{} & \hom_{\mathbb{Z}}(P \otimes_{\mathbb{Z}} M, V)^G & \xrightarrow[\cong]{} & \hom_{\mathbb{Z}}(P, \hom_{\mathbb{Z}}(M, V))^G & \xrightarrow[\cong]{} & \hom_{\mathbb{Z}G}(P, \hom_{\mathbb{Z}}(M, V))
\end{array}$$

M being a projective $\mathbb{Z}$-module implies that $\hom_{\mathbb{Z}}(M, U) \to \hom_{\mathbb{Z}}(M, V)$ is surjective. P being a projective $\mathbb{Z}G$-module implies that

$$\hom_{\mathbb{Z}G}(P, \hom_{\mathbb{Z}}(M, U)) \to \hom_{\mathbb{Z}G}(P, \hom_{\mathbb{Z}}(M, V))$$

is surjective. So the right vertical map is surjective and by commutativity, we know the left vertical map

$$\hom_{\mathbb{Z}G}(P \otimes_{\mathbb{Z}} M, U) \to \hom_{\mathbb{Z}G}(P \otimes_{\mathbb{Z}} M, V)$$

is also surjective. This proves that $P \otimes_{\mathbb{Z}} M$ is a projective $\mathbb{Z}G$-module.

Problem 18.1.4(Restriction is left adjoint to coinduction)
Let S be a subring of a ring R. Define the coinduction functor

$$\begin{aligned}
\mathrm{coind}_S^R : S - \mathbf{Mod} &\to R - \mathbf{Mod}, \\
U &\mapsto \hom_S({}_S R_R, U).
\end{aligned}$$

Prove that coind_S^R is right adjoint to res_S^R.

Solution: By the adjointness of $\otimes$ and hom, we know that the functor $\hom_S({}_S R_R, -)$ is right adjoint to ${}_S R_R \otimes_R -$, so we need to show that res_S^R is isomorphic to ${}_S R_R \otimes_R -$. Let M be a left R-module, viewed as a left S-module, we have an S-module homomorphism

$$\begin{aligned}
\alpha : {}_S R_R \otimes_R M &\to {}_S M, \\
r \otimes m &\mapsto rm.
\end{aligned}$$

By Lemma 17.2.11, this is a functorial isomorphism.

Problem 18.1.5
Let $\phi : S \to R$ be a ring homomorphism. Then R can be regarded as a right S-module, and we have a functor $R \otimes_S - : S - \mathbf{Mod} \to R - \mathbf{Mod}$. Prove that $R \otimes_S -$ is left adjoint to the functor $R - \mathbf{Mod} \to S - \mathbf{Mod}$ obtained by composing the R-action with ϕ.

Solution: By the adjointness of $\otimes$ and hom, if we view R as ${}_R R_S$, a (R, S) bimodule, then ${}_R R_S \otimes_S -$ is left adjoint to the functor

$$\hom_R({}_R R_S, -) : R - \mathbf{Mod} \to S - \mathbf{Mod}.$$

We need to show that this functor is isomorphic to the functor obtained by composing the R-action with ϕ. Let $f : {}_R R_S \to M$ be a R-module homomorphism. For any $s \in S$ and $r \in R$, we have

$$(s \cdot f)(r) = f(r\phi(s)).$$

Recall that we have an R-module isomorphism $\hom_R({}_R R_S, M) \to M$ by sending f to $f(1) = m$. Then the induced left S-module structure on M under this isomorphism is given by

$$s \cdot m = (s \cdot f)(1) = f(\phi(s)) = \phi(s) \cdot m.$$

This is exactly the S-module structure obtained from composing with ϕ.

Problem 18.1.6
In Theorem 18.1.1, Corollary 18.1.3, and Exercise 18.1.4, we have seen three examples of adjoint pairs of functors $(\mathcal{F}, \mathcal{G})$. For each of those pairs explicitly construct the unit and the counit of the adjunction.

Solution: We give the unit and counit of Theorem 18.1.1 in (a), Corollary 18.1.3 in (b), and Exercise 18.1.4 in (c).

(a) Let V be a (R, S)-bimodule and U be a left S-module. By Theorem 18.1.1 and Theorem 5.1.8, We have an isomorphism of abelian groups

$$\alpha : \hom_R(V \otimes_S U, V \otimes_S U) \xrightarrow{\sim} \hom_S(U, \hom_R(V, V \otimes_S U)).$$

The unit

$$\eta : id_{S-\mathbf{Mod}} \Rightarrow \hom_R(V, V \otimes_S -)$$

is a natural transformation given by

$$\eta_U = \alpha(id_{V \otimes_S U}) : U \to \hom_R(V, V \otimes_S U)$$

on each $U \in S - \mathbf{Mod}$. More explicitly, for any $u \in U$ and $v \in V$, we have

$$\eta_U(u)(v) = v \otimes u.$$

Conversely, given a left R-module W, by Theorem 18.1.1, we have an isomorphism of abelian groups

$$\beta : \hom_S(\hom_R(V, W), \hom_R(V, W)) \xrightarrow{\sim} \hom_R(V \otimes_S \hom_R(V, W), W).$$

By theorem 5.1.8, the counit

$$\varepsilon : V \otimes_S \hom_R(V, -) \Rightarrow id_{R-\mathbf{Mod}}$$

is given by

$$\varepsilon_W = \beta(id_{\hom_R(V,W)}) : V \otimes_S \hom_R(V, W) \to W$$

on each $W \in R-\mathbf{Mod}$. More explicitly, for any $v \in V$ and $f \in \hom_R(V, W)$, we have

$$\varepsilon_W(v \otimes f) = f(v).$$

(b) Let S be a subring of R and U be a left S-module. By Corollary 18.1.3, we have an isomorphism of abelian groups

$$\alpha : \hom_R(\mathrm{ind}_S^R U, \mathrm{ind}_S^R U) \xrightarrow{\sim} \hom_S(U, \mathrm{res}_S^R \mathrm{ind}_S^R U).$$

By Theorem 5.1.8, the unit

$$\eta : id_{S-\mathbf{Mod}} \Rightarrow \mathrm{res}_S^R \mathrm{ind}_S^R(-)$$

is given by

$$\eta_U = \alpha(id_{\mathrm{ind}_S^R U}) : U \to \mathrm{res}_S^R \mathrm{ind}_S^R U$$

on each $U \in S-\mathbf{Mod}$. More explicitly, for any $u \in U$, we have

$$\eta_U(u) = 1 \otimes u \in R \otimes_S U$$

where we restrict the action from R to S on $R \otimes_S U$, viewing it as a S-module.

Conversely, given a left R-module V, we have an isomorphism of abelian groups

$$\beta : \hom_S(\mathrm{res}_S^R V, \mathrm{res}_S^R V) \xrightarrow{\sim} \hom_R(\mathrm{ind}_S^R \mathrm{res}_S^R V, V).$$

By Theorem 5.1.8, the counit

$$\varepsilon : \mathrm{ind}_S^R \mathrm{res}_S^R \Rightarrow id_{R-\mathbf{Mod}}$$

is given by

$$\varepsilon_V = \beta(id_{\mathrm{res}_S^R V}) : \mathrm{ind}_S^R \mathrm{res}_S^R V \to V$$

on each $V \in R-\mathbf{Mod}$. More explicitly, for any $v \in V$, we first view v as an element in an S-module V, then note that

$$\mathrm{ind}_S^R \mathrm{res}_S^R V = {}_R R_S \otimes_S {}_S V$$

and we have

$$\varepsilon_V(r \otimes v) = rv.$$

(c) Let $S \subseteq R$ be a subring and U be a left R-module. We have proved in Exercise 18.1.4 that we have an isomorphism of abelian groups

$$\alpha : \hom_S(\mathrm{res}_S^R U, \mathrm{res}_S^R U) \xrightarrow{\sim} \hom_R(U, \mathrm{coind}_S^R \mathrm{res}_S^R U).$$

By Theorem 5.1.8, the unit

$$\eta : id_{R-\mathbf{Mod}} \Rightarrow \mathrm{coind}_S^R \mathrm{res}_S^R$$

is given by

$$\eta_U = \alpha(id_{\mathrm{res}_S^R U}) : U \to \mathrm{coind}_S^R \mathrm{res}_S^R U$$

for each $U \in R-\mathbf{Mod}$. More explicitly, for any $u \in U$, we have

$$\eta_U(u) = f \in \hom_S({}_S R_R, {}_S U)$$

where ${}_SU$ implies that U is viewed as a left S-module and f satisfies $f(1) = u$.

Conversely, given a left S-module V, we have an isomorphism of abelian groups

$$\beta : \hom_R(\mathrm{coind}_S^R V, \mathrm{coind}_S^R V) \xrightarrow{\sim} \hom_S(\mathrm{res}_S^R \mathrm{coind}_S^R V, V).$$

By Theorem 5.1.8, the counit

$$\varepsilon : \mathrm{res}_S^R \mathrm{coind}_S^R \Rightarrow id_{S-\mathbf{Mod}}$$

is given by

$$\varepsilon_V = \beta(id_{\mathrm{coind}_S^R V}) : \mathrm{res}_S^R \mathrm{coind}_S^R V \to V$$

on each $V \in S - \mathbf{Mod}$. More explicitly, we view $\hom_S({}_SR_R, V)$ as a left S-module by restricting the R-action, then for any $f \in \hom_S({}_SR_R, V)$, we have

$$\varepsilon_V(f) = f(1).$$

Problem 18.2.1

Prove that in an additive category, initial and terminal objects are isomorphic, hence an additive category always has a zero object.

Solution: Let I be the initial object and T be the terminal object. By definition, there is a unique morphism $id_I : I \to I$ and since $\hom(I, I)$ is an abelian group, we have $id_I = 0$. Same thing is true for the terminal object T, we have $id_T = 0$. Note that $\hom(I, T)$ and $\hom(T, I)$ are abelian groups, so we have two zero maps $\alpha : I \to T$ and $\beta : T \to I$, note that

$$id_T = 0 = \alpha \circ \beta : T \to T, id_I = 0 = \beta \circ \alpha : I \to I$$

by uniqueness of the map. We have proved that I and T are isomorphic.

Problem 18.2.4

If (X, p_i, q_i) is a biproduct of the X_i, then (X, p_i) is a product of the X_i, and (X, q_i) is a coproduct of the X_i.

Solution: We prove that (X, p_i) is the product of the product of X_i by showing that it satisfies the universal property of the product. Suppose $Y \in \mathrm{Ob}\mathbf{C}$ and we have a family of morphisms $\{f_i : Y \to X_i\}_i$. For any $1 \leq i, j \leq n$, consider the morphism $\sum_{i=1}^n q_i \circ f_i : Y \to X$ and $p_j : X \to X_j$, by definition of biproduct, if $i \neq j$, then $p_j \circ q_i \circ f_i = 0 \circ f_i = 0$. If $i = j$, then $p_i \circ q_i \circ f_i = id_{X_i} \circ f_i = f_i$. This means that

$$p_j \circ (\sum_{i=1}^n q_i \circ f_i) = p_j \circ q_j \circ f_j = f_j.$$

We know that $g = \sum_{i=1}^n q_i \circ f_i$ makes the following diagram commutes:

$$\begin{tikzcd} & Y \arrow[dl, bend right, "f_i"'] \arrow[d, "g"] \arrow[dr, bend left, "f_j"] & \\ & X \arrow[dl, "p_i"] \arrow[dr, "p_j"'] & \\ X_i & & X_j \end{tikzcd}$$

Suppose there exists another map $h : Y \to X$ satisfying $p_i \circ h = f_i$ for all $1 \leq i \leq n$. Then we know

$$h = (\sum_{i=1}^n q_i \circ p_i) \circ h = \sum_{i=1}^n q_i \circ p_i \circ h = \sum_{i=1}^n q_i \circ f_i.$$

So g is unique and this proves the universal property of (X, p_i).

For the coproduct (X, q_i), it is the same proof with arrow reversed. Suppose Y is an object in $\mathbf{C}$ and we have a family of morphisms $\{f_i : X_i \to Y\}_i$. Consider the morphism

$$g = \sum_{i=1}^n f_i \circ p_i.$$

By a similar argument, g is the unique morphism making the following diagram commutes:

$$\begin{tikzcd} & Y & \\ & X \arrow[u, "g"] & \\ X_i \arrow[uur, bend left, "f_i"] \arrow[ur, "q_i"] & & X_j \arrow[uul, bend right, "f_j"'] \arrow[ul, "q_j"'] \end{tikzcd}$$

We have proved the universal property for the coproduct (X_i, q_i).

Problem 18.2.7
The map

$$\hom(X_1, X_1') \oplus \cdots \oplus \hom(X_n, X_n') \to \hom(X_1 \oplus \cdots \oplus X_n, X_1' \oplus \cdots \oplus X_n'),$$
$$(f_1, \ldots, f_n) \mapsto f_1 \oplus \cdots \oplus f_n$$

is an injection of abelian groups.

Solution: Suppose the map in the problem is α. To prove α is injective, we need to find a map

$$\beta : \hom(X_1 \oplus \cdots \oplus X_n, X_1' \oplus \cdots \oplus X_n') \to \hom(X_1, X_1') \oplus \cdots \oplus \hom(X_n, X_n')$$

such that $\beta \circ \alpha = id$. Given a morphism

$$g : X_1 \oplus \cdots X_n \to X_1' \oplus \cdots \oplus X_n',$$

consider the composition $p'_j \circ g \circ q_j : X_j \to X'_j$ for any $1 \leq j \leq n$. In this way, we can define a map

$$\begin{aligned}\beta_j : \hom(X_1 \oplus \cdots \oplus X_n, X'_1 \oplus \cdots \oplus X'_n) &\to \hom(X_j, X'_j), \\ g &\mapsto p'_j \circ g \circ q_j.\end{aligned}$$

And β can be defined as $\beta = (\beta_1, \ldots, \beta_n)$. We need to check that $\beta \circ \alpha = id$. Suppose we have a family of maps $\{f_i : X_i \to X'_i\}_{i=1}^n$, we know that by definition

$$\begin{aligned}(\beta \circ \alpha)(f_1, \ldots, f_n) &= (p'_1 \circ (f_1 \oplus \cdots \oplus f_n) \circ q_1, \ldots, p'_n \circ (f_1 \oplus \cdots \oplus f_n) \circ q_n) \\ &= (f_1, \ldots, f_n).\end{aligned}$$

The last equality is due to Lemma 18.2.6(iii).

Problem 18.2.8
The assignment $(X_1, \ldots, X_n) \mapsto X_1 \oplus \cdots \oplus X_n$ and $(f_1, \ldots, f_n) \mapsto f_1 \oplus \cdots \oplus f_n$ define a fucntor $\mathbf{C}^{\times n} \to \mathbf{C}$.

Solution: We check the assignment

$$\begin{gathered}\mathscr{F} : \mathbf{C}^{\times n} \to \mathbf{C}, \\ (X_1, \ldots, X_n) \mapsto X_1 \oplus \cdots \oplus X_n, \\ (f_1, \ldots, f_n) \mapsto f_1 \oplus \cdots \oplus f_n.\end{gathered}$$

is a functor. Let $(id_1, \ldots, id_n)$ be an identity morphism of $(X_1, \ldots, X_n)$ in $\mathbf{C}^{\times n}$. We need to prove the morphism

$$id_1 \oplus \cdots \oplus id_n : X_1 \oplus \cdots \oplus X_n \to X_1 \oplus \cdots \oplus X_n$$

is the identity morphism for $X_1 \oplus \cdots \oplus X_n$. Note that by Lemma 18.2.6, we have

$$id_1 \oplus \cdots \oplus id_n = \sum_{i=1}^n q_i \circ id_i \circ p_i = \sum_{i=1}^n q_i \circ p_i = id_{X_1 \oplus \cdots \oplus X_n}.$$

Next suppose we have two families of morphisms $\{f_i : X_i \to Y_i\}_{i=1}^n$ and $\{g_i : Y_i \to Z_i\}_{i=1}^n$. Let $(X = X_1 \oplus \cdots \oplus X_n, p_i, q_i)$, $(Y = Y_1 \oplus \cdots \oplus Y_n, p'_i, q'_i)$ and $(Z = Z_1 \oplus \cdots \oplus Z_n, p''_i, q''_i)$ be the corresponding biproduct. For any $1 \le i, j \le n$, we have

$$\begin{aligned}
p''_j \circ \mathcal{F}(g_1, \dots, g_n) \circ \mathcal{F}(f_1, \dots, f_n) \circ q_i &= p''_j \circ (g_1 \oplus \cdots \oplus g_n) \circ (f_1 \circ \cdots \circ f_n) \circ q_j \\
&= p''_j \circ (g_1 \oplus \cdots \oplus g_n) \circ (\sum_{k=1}^{n} q'_k \circ p'_k) \circ (f_1 \circ \cdots \circ f_n) \circ q_j \\
&= \sum_{k=1}^{n} p''_j \circ (g_1 \oplus \cdots \oplus g_n) \circ q'_k \circ p'_k \circ (f_1 \circ \cdots \circ f_n) \circ q_j \\
&= \sum_{k=1}^{n} \delta_{j,k} g_k \circ \delta_{k,i} f_i \\
&= \delta_{j,i}(g_i \circ f_i) \\
&= p''_j \circ ((g_1 \circ f_1) \oplus \cdots \oplus (g_n \circ f_n)) \circ q_i \\
&= p''_j \circ \mathcal{F}((g_1 \circ f_1), \dots, (g_n \circ f_n)) \circ q_i.
\end{aligned}$$

By the uniqueness in Lemma 18.2.6(iii), we know that

$$\mathcal{F}(g_1, \dots, g_n) \circ \mathcal{F}(f_1, \dots, f_n) = \mathcal{F}((g_1 \circ f_1), \dots, (g_n \circ f_n)).$$

This proves that $\mathcal{F}$ is indeed a functor.

Problem 18.2.9
We have $\Delta_X := q_1 + q_2$ and $\nabla_X := p_1 + p_2$.

Solution: Note that

$$\begin{aligned}
p_1 \circ (q_1 + q_2) &= p_1 \circ q_1 + p_1 \circ q_2 \\
&= id_X + 0 \\
&= 0 + id_X \\
&= p_2 \circ q_1 + p_2 \circ q_2 \\
&= p_2 \circ (q_1 + q_2).
\end{aligned}$$

Since Δ_X is the unique morphism satisfying $p_1 \circ \Delta_X = id_X = p_2 \circ \Delta_X$, we can see that $\Delta_X = q_1 + q_2$. Similarly, note that

$$\begin{aligned}
(p_1 + p_2) \circ q_1 &= p_1 \circ q_1 + p_2 \circ q_1 \\
&= id_X + 0 \\
&= 0 + id_X \\
&= p_1 \circ q_2 + p_2 \circ q_2 \\
&= (p_1 + p_2) \circ q_2.
\end{aligned}$$

Since ∇_X is the unique morphism satisfying $\nabla_X \circ q_1 = id_X = \nabla_X \circ q_2$, we can see that $\nabla_X = p_1 + p_2$.

Problem 18.2.18
True or false? If R and R' are rings, $\mathcal{F} : R - \mathbf{Mod} \to R' - \mathbf{Mod}$ is a functor left adjoint to a functor $\mathcal{G} : R' - \mathbf{Mod} \to R - \mathbf{Mod}$, and P is a projective R-module, then $\mathcal{F}P$ is a projective R'-module.

Solution: This is false. Consider the functor

$$\mathbb{Z}/2\mathbb{Z} \otimes - : \mathbb{Z} - \mathbf{Mod} \to \mathbb{Z} - \mathbf{Mod}$$

By adjointness of $\otimes$ and hom, we know that $\mathbb{Z}/2\mathbb{Z} \otimes -$ is left adjoint to the functor $\hom(\mathbb{Z}/2\mathbb{Z}, -)$. We know $\mathbb{Z}$ as a $\mathbb{Z}$-module is projective because $\mathbb{Z}$ is free. But $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}$ is not projective. Consider the surjective quotient map $q : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, by the adjointness, we have a commutative diagram

$$\begin{tikzcd}
\hom(\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}, \mathbb{Z}) \arrow[r, "q_*"] \arrow[d, "\sim"'] & \hom(\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \arrow[d, "\sim"] \\
\hom(\mathbb{Z}, \hom(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})) \arrow[r] \arrow[d, "\sim"'] & \hom(\mathbb{Z}, \hom(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})) \arrow[d, "\sim"] \\
\hom(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \arrow[r] & \hom(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})
\end{tikzcd}$$

We know that the bottom map is not surjective because we only have zero map from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$. This imples q_* is also not surjective, so $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}$ is not a projective $\mathbb{Z}$-module.