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Homework - Week 7 Exercises
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Exercise 3.3
Assume that ϕ is a continuous real function on (a, b) such that

$$\phi(\frac{x+y}{2}) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y)$$

for all $x, y \in (a, b)$. Prove that ϕ is convex.

Solution: We first prove the following claim.
Claim: For an integer $n \geq 1$ and $m \in \{0, 1, \ldots, 2^n\}$, the function ϕ satisfies

$$\phi(\frac{m}{2^n}x + (1 - \frac{m}{2^n})y) \leq \frac{m}{2^n}\phi(x) + (1 - \frac{m}{2^n})\phi(y).$$

Proof: We prove this by induction on n.

When $n = 1$, m can be 0, 1 or 2. The case $m = 0$ and $m = 2$ is trivial. The case $m = 1$ is given by

$$\phi(\frac{x+y}{2}) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y).$$

Now assume we have proved the case for n, and we want to show that this claim is true for $n+1$. We need to prove the inequality for $0 \leq m \leq 2^{n+1}$. When $0 \leq m \leq 2^n$, note that for any $x, y \in (a, b)$,

$$\frac{m}{2^{n+1}}x + (1 - \frac{m}{2^{n+1}})y = \frac{1}{2}(\frac{m}{2^n}x + (1 - \frac{m}{2^n})y) + \frac{1}{2}y$$

We can use what we have proved for n to show

$$\begin{aligned}
\phi(\frac{m}{2^{n+1}}x + (1 - \frac{m}{2^{n+1}})y) &= \phi(\frac{1}{2}(\frac{m}{2^n}x + (1 - \frac{m}{2^n})y) + \frac{1}{2}y) \\
&\leq \frac{1}{2}\phi(\frac{m}{2^n}x + (1 - \frac{m}{2^n})y) + \frac{1}{2}\phi(y) \\
&\leq \frac{1}{2}[\frac{m}{2^n}\phi(x) + (1 - \frac{m}{2^n})\phi(y)] + \frac{1}{2}\phi(y) \\
&= \frac{m}{2^{n+1}}\phi(x) + (1 - \frac{m}{2^{n+1}})\phi(y).
\end{aligned}$$

When $2^n < m \leq 2^{n+1}$, in this case we write $m = 2^n + k$ where $1 \leq k \leq 2^n$. Note that

$$\begin{aligned}
\frac{m}{2^{n+1}}x + (1 - \frac{m}{2^{n+1}})y &= \frac{1}{2}(1 + \frac{k}{2^n})x + \frac{1}{2}(1 - \frac{k}{2^n})y \\
&= \frac{1}{2}[\frac{k}{2^n}x + (1 - \frac{k}{2^n})y] + \frac{1}{2}x
\end{aligned}$$

Then again using the case n, we can write

$$\begin{aligned}
\phi(\frac{m}{2^{n+1}}x + (1-\frac{m}{2^{n+1}})y) &= \phi(\frac{1}{2}(\frac{k}{2^n}x + (1-\frac{k}{2^n})y) + \frac{1}{2}x) \\
&\leq \frac{1}{2}\phi(\frac{k}{2^n}x + (1-\frac{k}{2^n})y) + \frac{1}{2}\phi(x) \\
&\leq \frac{k}{2^{n+1}}\phi(x) + (\frac{1}{2} - \frac{k}{2^{n+1}})\phi(y) + \frac{1}{2}\phi(x) \\
&= \frac{m}{2^{n+1}}\phi(x) + (1-\frac{m}{2^{n+1}})\phi(y).
\end{aligned}$$

This concludes the proof of the claim. ■

For any real number $\lambda \in [0,1]$, Consider the following sequence in $\mathbb{Q} \cap [0,1]$

$$\lambda_k = \frac{\lfloor 2^k \lambda \rfloor}{2^k}, \quad k = 1, 2, 3, \ldots$$

where $\lfloor \rfloor$ is the floor function. We have $\lambda_k \to \lambda$ as $k \to \infty$. From the claim, we know that for any k,

$$\phi(\lambda_k x + (1-\lambda_k)y) \leq \lambda_k \phi(x) + (1-\lambda_k)\phi(y).$$

Let $k \to \infty$ and use the fact that ϕ is a continuous function, we obatin

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y).$$

This proves that ϕ is a convex function.

Exercise 3.4

Suppose f is a complex measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_X |f|^p d\mu = ||f||_p^p \qquad (0 < p < \infty).$$

Let $E = \{p : \varphi(p) < \infty\}$. Assume $||f||_\infty > 0$.

(a) If $r < p < s$, $r \in E$ and $s \in E$, prove that $p \in E$.

(b) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.

(c) By (a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset $(0, \infty)$?

(d) If $r < p < s$, prove that $||f||_p \leq \max(||f||_r, ||f||_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Assume that $||f||_r < \infty$ for some $r \leq \infty$ and prove that

$$||f||_p \to ||f||_\infty \quad \text{as} \quad p \to \infty.$$

Solution:

(a) We define the following numbers:

$$k = \frac{rs - pr}{s - r} > 0,$$
$$l = \frac{ps - rs}{s - r} > 0,$$
$$m = \frac{s - r}{s - p} > 0,$$
$$n = \frac{s - r}{p - r} > 0,$$

We can check by direct calculations that these numbers satisfy the following:

$$\begin{aligned} k + l &= p, \\ km &= r, \\ ln &= s, \\ \frac{1}{m} + \frac{1}{n} &= 1. \end{aligned}$$

Then by Hölder inequality, we have

$$\begin{aligned} \varphi(p) &= \int_X |f|^p d\mu \\ &= \int_X |f|^k \cdot |f|^l d\mu \\ &\leq (\int_X |f|^{km} d\mu)^{\frac{1}{m}} (\int_X |f|^{ln} d\mu)^{\frac{1}{n}} \\ &= (\int_X |f|^r d\mu)^{\frac{1}{m}} (\int_X |f|^s d\mu)^{\frac{1}{n}} \\ &= \varphi(r)^{\frac{1}{m}} \cdot \varphi(s)^{\frac{1}{n}}. \end{aligned}$$

Here $r, s \in E$, so $\varphi(r), \varphi(s) < \infty$. This implies that $\varphi(p) < \infty$, and we can conclude that $p \in E$.

(b) Let $\lambda \in [0, 1]$ be a real number. Suppose x, y are in the interior of E. Without loss of generality, we can assume $x < y$. Then from (a), we know that $\lambda x + (1 - \lambda)y \in E$ for any $\lambda \in [0, 1]$. Note that log is an increasing function and $\lambda + (1 - \lambda) = 1$, by Hölder inequality, we have

$$\begin{aligned} \log \varphi(\lambda x + (1 - \lambda)y) &= \log \int_X |f|^{\lambda x + (1-\lambda)y} d\mu \\ &= \log \int_X |f|^{\lambda x} \cdot |f|^{(1-\lambda)y} d\mu \\ &\leq \log(\int_X |f|^{\lambda x \cdot \frac{1}{\lambda}} d\mu)^{\lambda} (\int_X |f|^{(1-\lambda)y \cdot (1-\lambda)} d\mu)^{1-\lambda} \\ &= \log(\varphi(x)^{\lambda}) + \log(\varphi(y)^{1-\lambda}) \\ &= \lambda \log \varphi(x) + (1 - \lambda) \log \varphi(y) \end{aligned}$$

This proves that $\log \varphi$ is a convex function.

We first show that φ is continuous on the interior of E. We know that $\log \varphi$ is convex on the interior of E. Let $g(x) = e^x$ be a convex and increasing function. Then $g(\log \varphi) = \phi$ is still a convex function, and we know that convex function is continuous on the interior of E.

The set E is convex by (a). So E must be an interval or a single point. If E is just a single point, then φ is continuous automatically. Assume E is an interval. We have already proved φ is continuous on the interior of this interval, we only need to show that φ is continuous at the side. Assume $E = [a, b]$ where $0 < a < b < \infty$. We need to prove that φ is continuous at a and b.

Consider an increasing sequence

$$b_1 \le b_2 \le \cdots \le b_n \le \cdots$$

with $b_n \to b$ as $n \to \infty$. We need to show that $\varphi(b_n) \to \varphi(b)$, namely

$$\lim_{n\to\infty} \int_X |f|^{b_n} d\mu = \int_X |f|^b d\mu.$$

It is easy to see that

$$\lim_{n\to\infty} |f|^{b_n} = |f|^b.$$

Now consider the following set

$$E := \{x \in X : |f| \le 1\}.$$

Since b_n is increasing, we know that $|f(x)|^{b_n} \le |f(x)|^{b_1}$ for all $x \in E$ and $|f(x)|^{b_n} \le |f|^b$ for all $x \in E^c$. Define a function

$$g(x) := |f|^{b_1} \chi_{x\in E} + |f|^b \chi_{x \in E^c}.$$

Then we have $|f(x)| \le g(x)$ for all $x \in X$ and

$$\int_X g d\mu = \int_E |f|^{b_1} d\mu + \int_{E^c} |f|^b d\mu \le \varphi(b_1) + \varphi(b) < \infty.$$

By Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_X |f|^{b_n} d\mu = \int_X |f|^b d\mu.$$

This implies φ is continuous at b.

The continuity of φ at the point a can be proved similarly using a decreasing sequence a_n.

(c) The set is an interval or a singleton. All intervals are possible. The following cases are all with Lebesgue measure.

   (i) E is an open interval.

Let $X = (0,1)$ and $f(x) = \frac{1}{x}$. For $p \neq 1$, we have

$$\int_0^1 \frac{1}{x^p} dx = \frac{x^{1-p}}{1-p}\Big|_0^1.$$

If $0 < p < 1$, $1-p > 0$, so x^{1-p} evaluated at 0 is 0, so $(0,1) \subseteq E$. If $p > 1$, $1-p < 0$ and

$$\lim_{x \to 0^+} x^{1-p} = +\infty.$$

So $(1,+\infty) \cap E = \varnothing$. For $p = 1$, we know that $\frac{1}{x}$ is not integrable on $(0,1)$. This implies $E = (0,1)$.

(ii) E is an interval which is open on the left and closed on the right.
Let $X = (0, \frac{1}{2})$ and $f(x) = \frac{1}{x(\log x)^2}$. Define

$$g(p) = x^p (\log x)^{2p}, \qquad 0 < p < +\infty.$$

$g(p)$ is a nonnegative function on $(0, \frac{1}{2})$. We have

$$g'(p) = px^{p-1} \cdot \frac{(\log x)^{2p}}{\log x}(\log x + 2).$$

Note that $\log x$ is negative on $(0, \frac{1}{2})$, and $\log x + 2 > 0$. This implies that $g(p)$ is a decreasing function. When $p = 1$, we have

$$\int_0^{\frac{1}{2}} \frac{dx}{x(\log x)^2} = \int_{-\infty}^{\log \frac{1}{2}} \frac{du}{u^2} = (\log 2)^{-1} < +\infty.$$

Moreover, for any $0 < p < 1$, recall that $g(p)$ is decreasing, we have

$$\int_0^{\frac{1}{2}} \frac{dx}{x^p(\log x)^{2p}} = \int_0^{\frac{1}{2}} \frac{dx}{g(p)} \leq \int_0^{\frac{1}{2}} \frac{dx}{g(1)} = \int_0^{\frac{1}{2}} \frac{dx}{x(\log x)^2} < +\infty.$$

This implies that $(0,1] \subseteq E$. When $p > 1$, write $p = 1 + s$ for some $s > 0$, then

$$x^p(\log x)^{2p} = x \cdot x^s (\log x)^{2(1+s)}.$$

Note here when $x \to 0^+$, $x^s(\log x)^{2(1+s)} \to 0$, so there exists a constant M such that $x^s(\log x)^{2(1+s)} < M$ for all $x \in (0, \frac{1}{2})$, and we have

$$\int_0^{\frac{1}{2}} \frac{dx}{x^p(\log x)^{2p}} \geq \frac{1}{M} \int_0^{\frac{1}{2}} \frac{dx}{x} = +\infty.$$

This implies that $E = (0,1]$.

(iii) E is an interval which is closed on the left and open on the right.
Let $X = (e, +\infty)$ and $f(x) = \frac{1}{x(\log x)^2}$. When $p = 1$, we have

$$\int_e^{+\infty} \frac{dx}{x(\log x)^2} = \int_1^{+\infty} \frac{du}{u^2} < +\infty.$$

So $1 \in E$. When $p > 1$, note that on $(e, +\infty)$, $(\log x)^{2p}$ is an increasing function, so

$$\int_e^{+\infty} \frac{dx}{x^p(\log x)^{2p}} \leq \int_e^{+\infty} \frac{dx}{x^p} \frac{e^{1-p}}{1-p} < +\infty.$$

This implies that $[1, +\infty) \subseteq E$. When $0 < p < 1$, note that $x^{p-1}(\log x)^{2p} \to 0$ when $x \to +\infty$, so there exists a constant M such that $x^{p-1}(\log x)^{2p} < M$ for all $x \in (e, +\infty)$. This means

$$x^p(\log x)^{2p} < Mx$$

on $(e, +\infty)$. Thus, we have

$$\int_e^{+\infty} \frac{dx}{x^p(\log x)^{2p}} \geq \int_e^{+\infty} \frac{dx}{Mx} = +\infty.$$

This proves that $E = [1, +\infty)$.

(iv) E can be a singleton.

Let $X = (0, +\infty)$ and

$$f(x) = \frac{1}{x(\log x)^2}(\chi_{(0,\frac{1}{2})}(x) + \chi_{(e,+\infty)}(x)).$$

From what we proved above, we know that

$$E = (0, 1] \cap [1, +\infty] = \{1\}.$$

(v) E is a closed interval.

Let $X = (0, +\infty)$. Write

$$f_1(x) = \frac{1}{x(\log x)^2},$$
$$f_2(x) = \frac{1}{x^{\frac{1}{2}}|\log x|}.$$

Let

$$f(x) = f_2(x)\chi_{(0,\frac{1}{2})}(x) + f_1(x)\chi_{(e,+\infty)}(x).$$

From what we have seen above,

$$E = (0, 2] \cap [1, +\infty) = [1, 2].$$

(d) Assume the opposite. Suppose $||f||_p > ||f||_r$ and $||f||_p > ||f||_s$. From (a), we have proved that

$$||f||_p^p \leq ||f||_r^{\frac{r(s-p)}{s-r}} ||f||_s^{\frac{s(p-r)}{s-r}}.$$

From $||f||_p > ||f||_r$ and $||f||_p > ||f||_s$, we have

$$||f||_r^{p(s-r)} < ||f||_p^{p(s-r)} \leq ||f||_r^{r(s-p)}||f||_s^{s(p-r)}, \tag{1}$$
$$||f||_s^{p(s-r)} < ||f||_p^{p(s-r)} \leq ||f||_r^{r(s-p)}||f||_s^{s(p-r)}. \tag{2}$$

This implies that

$$1 < ||f||_r^{s(r-p)}||f||_s^{s(p-r)} = (\frac{||f||_s}{||f||_r})^{s(p-r)}, \tag{3}$$
$$1 < ||f||_r^{r(s-p)}||f||_s^{r(p-s)} = (\frac{||f||_r}{||f||_s})^{r(s-p)}. \tag{4}$$

Note that here $s(p-r) > 0$ and $r(s-p) > 0$, so we have shown that

$$||f||_r > ||f||_s \quad \text{and} \quad ||f||_s > ||f||_r.$$

This is a contradiction and we can conclude that

$$||f||_p \leq \max(||f||_r, ||f||_s).$$

If $f \in L^r(\mu) \cap L^s(\mu)$, then

$$||f||_p \leq \max(||f||_r, ||f||_s) \leq +\infty.$$

This implies that $f \in L^p(\mu)$, and we can see that $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Fix a small $\varepsilon > 0$ such that $||f||_\infty - \varepsilon > 0$ (This is possible because $||f||_\infty > 0$). Consider the following set

$$E_\varepsilon := \{x \in X : |f(x)| \geq ||f||_\infty - \varepsilon\}.$$

We know that $\mu(E_\varepsilon) > 0$ by definition of essential supreme. Moreover, because $||f||_r < +\infty$ for some $r > 0$, we have

$$+\infty > ||f||_r = (\int_X |f|^r d\mu)^{\frac{1}{r}} \geq (||f||_\infty - \varepsilon) \cdot \mu(E_\varepsilon)^{\frac{1}{r}}$$

This implies that $0 < \mu(E_\varepsilon) < +\infty$ for any small $\varepsilon > 0$. For any $p > 1$, we have

$$||f||_p = (\int_X |f|^p d\mu)^{\frac{1}{p}} \geq (||f||_\infty - \varepsilon) \cdot \mu(E_\varepsilon)^{\frac{1}{p}}.$$

Let $p \to +\infty$, we have

$$\lim_{p\to\infty} ||f||_p \geq ||f||_\infty - \varepsilon$$

for all small ε. Let $\varepsilon \to 0^+$, and we have proved that

$$\lim_{p\to\infty} ||f||_p \geq ||f||_\infty.$$

On the other hand, let

$$A = \{x \in X : |f(x)| \leq ||f||_\infty\}.$$

By definition of essential supreme, we know that $\mu(A) = \mu(X)$, so for all $p > r$ and $p > 1$, we have

$$
\begin{aligned}
||f||_p &= (\int_X |f|^{p-r}|f|^r d\mu)^{\frac{1}{p}} \\
&\leq ||f||_\infty^{\frac{p-r}{p}} \cdot (\int_X |f|^r d\mu)^{\frac{1}{r}\cdot\frac{r}{p}} \\
&= ||f||_\infty^{\frac{p-r}{p}} \cdot ||f||_r^{\frac{r}{p}}
\end{aligned}
$$

Here $||f||_r^{\frac{r}{p}}$ is a positive finite number, let $p \to +\infty$, we have

$$\lim_{p\to\infty} ||f||_p \leq ||f||_\infty.$$

Thus, we can conclude that

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty.$$

Exercise 3.10
Suppose $f_n \in L^p(\mu)$ for $n = 1, 2, \ldots$ and $||f_n - f||_p \to 0$ and $f_n \to g$ a.e. as $n \to \infty$. What relation exists between f and g?

Solution: $f = g$ almost everywhere on X. To prove this, replace f_n with $f_n - f$ and $g - f$ with g, the condition becomes $||f||_p \to 0$ and $f_n \to g$ a.e., we need to show that $g = 0$ a.e. We know that f_n is a Cauchy sequence in $L^p(\mu)$, by Theorem 3.11, there exists a subsequence f_{n_i} such that

$$\lim_{i\to\infty} f_{n_i}(x) = f(x)$$

almost everywhere for some $f \in L^p(\mu)$ and $||f_n - f||_p \to 0$. We know that $||f_n||_p \to 0$, so $f = 0$ almost everywhere. And since we already have a limit $f_n \to g$ almost everywhere. This implies $f = g = 0$ almost everywhere.