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Course: MATH 607 - Homological Algebra
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Homework 2
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Exercise 2.2.1
Show that a chain complex P is a projective object in $\mathcal{C}h$ if and only if it is a split exact complex of projectives.

Solution: Suppose the chain complex $P_\bullet$ is a projective object in $\mathcal{C}h$. We first prove that for any n, P_n is a projective object. Consider a surjection in the abelian category and suppose we have a map $P_n \to B$

$$\begin{tikzcd} & P_n \arrow[d] & \\ A \arrow[r, two heads] & B \arrow[r] & 0 \end{tikzcd}$$

Now view A and B as complexes concentrated at degree n, we obtain a diagram of chain complexes

$$\begin{tikzcd} & P_\bullet \arrow[d] & \\ A \arrow[r, two heads] & B \arrow[r] & 0 \end{tikzcd}$$

By projectivity of $P_\bullet$ in $\mathcal{C}h$, there exists a map $P_\bullet \to A$ such that the diagram commutes. Take the map $P_n \to A$, and we have a commutative diagram in the original abelian category. This implies P_n is projective for any n. To see that $P_\bullet$ is split exact, consider the short exact sequence

$$0 \to P \xrightarrow{j} \mathrm{cone}(P) \to P[-1] \to 0$$

We know that P is projective in $\mathcal{C}h$, so $P[-1]$ is also projective in $\mathcal{C}h$, thus the short exact sequence splits. There exists a map $k : \mathrm{cone}(P) \to P$ such that $k \circ j = id_P$. By exercise 1.5.2, this implies that id_P is nullhomotopic. By exercise 1.4.3, we know that $P_\bullet$ is a split exact complex of projectives.

Conversely, suppose $P_\bullet$ is a split exact complex of projectives. First consider $P_\bullet$ is of the following form

$$0 \to P_n \xrightarrow{\cong} P_{n-1} \to 0$$

Let $f_\bullet : A_\bullet \to B_\bullet$ be levelwise surjection and we have a map $\varphi : P_\bullet \to B_\bullet$, namely a commutative diagram

$$\begin{tikzcd} & P_n \arrow[rr, "p"] \arrow[dd, "\varphi_n"] & & P_{n-1} \arrow[dd, "\varphi_{n-1}"] \\ A_n \arrow[rr, "d_A"] \arrow[dr, two heads, "f_n"'] & & A_{n-1} \arrow[dr, two heads, "f_{n-1}"] & \\ & B_n \arrow[rr, "d_B"'] & & B_{n-1} \end{tikzcd}$$

By projectivity of P_n, there exists a map $g_n : P_n \to A_n$ such that $f_n \circ g_n = \varphi_n$. Note that p is an isomorphism, we define

$$g_{n-1} = d_A \circ g_n \circ p^{-1} : P_{n-1} \to A_{n-1}.$$

We have $g_{n-1} \circ p = d_A \circ g_n$ by definition. Moreover, we have

$$\begin{aligned} f_{n-1} \circ g_{n-1} &= (f_{n-1} \circ d_A) \circ g_n \circ p^{-1} \\ &= d_B \circ (f_n \circ g_n) \circ p^{-1} \\ &= d_B \circ \varphi_n \circ p^{-1} \\ &= \varphi_{n-1} \circ p \circ p^{-1} \\ &= \varphi_{n-1}. \end{aligned}$$

This implies we have a commutative diagram

$$\begin{tikzcd} & P_n \arrow[rr, "p \ \cong"] \arrow[dl, dashed, "g_n"'] \arrow[dd, "\varphi_n"] & & P_{n-1} \arrow[dl, dashed, "g_{n-1}"'] \arrow[dd, "\varphi_{n-1}"] \\ A_n \arrow[rr, "d_A"] \arrow[dr, two heads, "f_n"'] & & A_{n-1} \arrow[dr, two heads, "f_{n-1}"] & \\ & B_n \arrow[rr, "d_B"'] & & B_{n-1} \end{tikzcd}$$

So

$$0 \to P_n \xrightarrow{\cong} P_{n-1} \to 0$$

is a projective object in $\mathcal{Ch}$. In general, consider a split exact complex of projectives

$$\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \to \cdots$$

The split exactness implies that $P_n = \ker d_n \oplus (P_n / \ker d_n)$, and we have an isomorphism

$$P_n / \ker d_n \xrightarrow{\cong} \operatorname{Im} d_n \cong \ker d_{n-1}$$

Note that $\ker d_n$ is projective as it is a summand of a projective module P_n. This implies that the split exact complex $P_\bullet$ of projectives can be written as sums of

$$P(n) : 0 \to P_n / \ker d_n \xrightarrow{\cong} \ker d_{n-1} \to 0$$

Namely, $P_\bullet \cong \bigoplus_{n \in \mathbb{Z}} P(n)$. We have proved that $P(n)$ is a projective object in $\mathcal{Ch}$ for all $n \in \mathbb{Z}$. So P is also a projective object.

Exercise 2.4.2 (Preserving derived functors)
If $U : \mathcal{B} \to \mathcal{C}$ is an exact functor, show that

$$U(L_i F) \cong L_i(UF).$$

Solution: Let A be an object in the abelian category. Consider a projective resolution of A:

$$\cdots \to P_1 \to P_0 \to A \to 0.$$

We know that $(L_iF)(A)$ is the ith homology of the complex

$$\cdots \to F(P_i) \to F(P_{i-1}) \to \cdots$$

and $(L_iUF)(A)$ is the ith homology of the complex

$$\cdots \to UF(P_i) \to UF(P_{i-1}) \to \cdots$$

To show that $U(L_iF)(A) \cong (L_iUF)(A)$, it is the same as showing taking homology of a complex commutes with applying an exact functor U. And it is equivalent as showing that U preserves kernels and cokernels. We show that U preserves kernels, the other part is completely analogous. Let $f : A \to B$ be a map and consider the following exact sequence

$$0 \to \ker f \to A \xrightarrow{f} B$$

Since U is exact, we have an exact sequence

$$0 \to U \ker f \to UA \xrightarrow{Uf} UB$$

This implies that

$$U \ker f \cong \ker Uf.$$

So U preserves kernels.

Exercise 2.4.3 (Dimension shifting)
If $0 \to M \to P \to A \to 0$ is exact with P projective (or F-acyclic), show that $L_iF(A) \cong L_{i-1}F(M)$ for $i \geq 2$ and that $L_1F(A)$ is the kernel of $F(M) \to F(P)$. More generally, show that if

$$0 \to M_m \to P_m \to P_{m-1} \to \cdots \to P_0 \to A \to 0$$

is exact with the P_i projective (or F-acyclic), then $L_iF(A) \cong L_{i-m-1}F(M_m)$ for $i \geq m+2$ and $L_{m+1}F(A)$ is the kernel of $F(M_m) \to F(P_m)$. Conclude that if $P \to A$ is an F-acyclic resolution of A, then $L_iF(A) = H_i(F(P))$.

Solution: From the short exact sequence

$$0 \to M \to P \to A \to 0,$$

we get a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & L_2F(A) \arrow[dll] \\
L_1F(M) \arrow[r] & L_1F(P) \arrow[r] & L_1F(A) \arrow[dll] \\
F(M) \arrow[r] & F(P) \arrow[r] & F(A) \arrow[r] & 0
\end{tikzcd}$$

Here P is projective or F-acyclic, so $L_iF(P) = 0$ for all $i > 0$. Thus, by exactness, $L_iF(A) \cong L_{i-1}F(M)$ for $i \geq 2$ and

$$L_1F(A) \cong \ker(F(M)) \to F(P).$$

More generally, suppose we have an exact complex

$$0 \to M_m \xrightarrow{f} P_m \xrightarrow{d_m} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0.$$

We have a short exact sequence

$$0 \to \ker d_0 \to P_0 \to A \to 0.$$

From what we proved above, we obtain that

$$\begin{aligned} L_iF(A) &\cong L_{i-1}F(\ker d_0), \quad \text{if} \quad i \geq 2, \\ L_1F(A) &\cong \ker(F(\ker d_0) \to F(P_0)). \end{aligned}$$

Now consider the short exact sequence

$$0 \to \ker d_1 \to P_1 \xrightarrow{d_1} \operatorname{Im} d_1 \to 0$$

By exactness, $\operatorname{Im} d_1 \cong \ker d_0$ and we obtain that

$$\begin{aligned} L_iF(A) &\cong L_{i-1}F(\ker d_0) \cong L_{i-2}F(\ker d_1), \quad \text{if} \quad i \geq 3, \\ L_2F(A) &\cong L_1F(\ker d_0) \cong \ker(F(\ker d_1) \to F(P_1)). \end{aligned}$$

Repeat this process. Note that $M_m \cong \ker d_m$, so we get

$$\begin{aligned} L_iF(A) &\cong L_{i-m-1}F(M_m), \quad \text{if} \quad i \geq m+2, \\ L_{m+1}F(A) &\cong \ker(F(M_m) \to F(P_m)). \end{aligned}$$

Suppose

$$\cdots \to P_1 \to P_0 \to A \to 0$$

is a F-acyclic resolution of A. Since F is right exact, we have

$$F(P_1) \to F(P_0) \to F(A) \to 0$$

is still exact, so

$$F(A) \cong L_0F(A) \cong H_0(F(P_\bullet)).$$

For any $i \geq 0$, we have an exact sequence

$$0 \to M_i \to P_i \to P_{i-1} \to \cdots \to P_0 \to A \to 0$$

where $M_i = \ker(P_i \to P_{i-1})$. From what we have proved above, we get

$$L_{i+1}F(A) \cong \ker(F(M_i) \to F(P_i)).$$

On the other hand, note that by exactness

$$M_i = \ker(P_i \to P_{i-1}) \cong \operatorname{Im}(P_{i+1} \to P_i).$$

So we have an exact sequence

$$\cdots \to P_{i+2} \to P_{i+1} \to M_i \to 0.$$

Since F is right exact, we have

$$F(M_i) \cong \operatorname{coker}(F(P_{i+2}) \to F(P_{i+1})) \cong F(P_{i+1})/\operatorname{Im}(F(P_{i+2}) \to F(P_{i+1})).$$

Thus, for $i \geq 0$, we get

$$L_{i+1}F(A) \cong \ker(F(P_{i+1})/\operatorname{Im}(F(P_{i+2}) \to F(P_{i+1})) \to F(P_i)) \cong H_{i+1}(F(P_\bullet)).$$

Exercise 3.1.2
Suppose that R is a commutative domain with field of fractions F. Show that $\operatorname{Tor}_1^R(F/R, B)$ is the torsion submodule

$$\{b \in B : \exists r \neq 0, rb = 0\}$$

of B for every R-module B.

Solution: Note that fraction field F as an R-module is flat since it is the localization of R. So the short exact sequence

$$0 \to R \to F \to F/R \to 0$$

is a flat resolution of F/R. Apply $- \otimes_R B$, we obtain a complex

$$0 \to B \xrightarrow{b \mapsto 1 \otimes b} F \otimes_R B \to 0.$$

So

$$\operatorname{Tor}_1^R(F/R, B) = \ker(B \to F \otimes_R B).$$

Let T be the torsion submodule

$$T = \{b \in B : \exists r \neq 0, rb = 0\}.$$

We claim $T = \ker(B \to F \otimes_R B)$. Indeed, for any $b \in T$, there exists nonzero $r \in R$ such that $rb = 0$. Thus, we have

$$1 \otimes b = r \cdot \frac{1}{r} \otimes b = \frac{1}{r} \otimes rb = 0.$$

So $T \subset \ker(B \to F \otimes_R B)$. On the other hand, suppose $1 \otimes b = 0$ for some $b \in B$, then there exists $0 \neq r \in R$ such that $rb = 0$. Hence, we can conclude that

$$\operatorname{Tor}_1^R(F/R, B) = \{b \in B : \exists r \neq 0, rb = 0\}.$$

Exercise 3.1.3
Show that $\operatorname{Tor}_1^R(R/I, R/J) \cong \frac{I \cap J}{IJ}$ for every right ideal I and left ideal J of R. In particular, $\operatorname{Tor}_1(R/I, R/I) \cong I/I^2$ for every 2-sided ideal I.

Solution: Consider the short exact sequence

$$0 \to I \to R \to R/I \to 0.$$

We have a long exact sequence for Tor:

$$\cdots \to \operatorname{Tor}_1(R, R/J) \to \operatorname{Tor}_1(R/I, R/J) \to I \otimes R/J \to R \otimes R/J \to \cdots$$

Note that $\operatorname{Tor}_1(R, R/J) = 0$ as R is free. So

$$\operatorname{Tor}_1(R/I, R/J) = \ker(I \otimes R/J \to R \otimes R/J)$$

where the map is induced by the inclusion $I \to R$. Now consider the following map between short exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & IJ & \longrightarrow & I & \longrightarrow & I \otimes R/J & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & J & \longrightarrow & R & \longrightarrow & R \otimes R/J & \longrightarrow & 0
\end{array}$$

Let $K = \ker(I \otimes R/J \to R \otimes R/J)$ and note that the map $I \to R$ is injective, by the Snake Lemma, we have an exact sequence

$$0 \to K \to J/IJ \xrightarrow{\varphi} R/I \to \cdots$$

So $K = \ker(\varphi : J/IJ \to R/I)$ where φ is induced by the square

$$\begin{array}{ccccc}
IJ & \longrightarrow & J & \twoheadrightarrow & J/IJ \\
\downarrow & & \downarrow & & \downarrow \varphi \\
I & \longrightarrow & R & \twoheadrightarrow & R/I
\end{array}$$

Consider the map $f : J \to R/I$. IJ is in the $\ker f$, so the map φ is induced by f, namely we have a commutative square

$$\begin{tikzcd} J \arrow[r, "f"] \arrow[d] & R/I \\ J/IJ \arrow[ur, "\varphi"'] & \end{tikzcd}$$

It is not hard to see that $\ker f = I \cap J$, so $\ker \varphi = \ker f/IJ = I \cap J/IJ$. This implies that

$$\mathrm{Tor}_1^R(R/I, R/J) = \frac{I \cap J}{IJ}.$$

In particular, when $I = J$ as two-sided ideals, we obtain

$$\mathrm{Tor}_1^R(R/I, R/I) = I/I^2.$$

Exercise 3.2.2
Show that if $0 \to A \to B \to C \to 0$ is exact and both B and C are flat, then A is also flat.

Solution: For any R-module M, the short exact sequence

$$0 \to A \to B \to C \to 0$$

induces a long exact sequence in Tor

$$\begin{tikzcd} & & \cdots \arrow[dll] & \\ \mathrm{Tor}_2(A, M) \arrow[r] & \mathrm{Tor}_2(B, M) \arrow[r] & \mathrm{Tor}_2(C, M) \arrow[dll] & \\ \mathrm{Tor}_1(A, M) \arrow[r] & \mathrm{Tor}_1(B, M) \arrow[r] & \mathrm{Tor}_1(C, M) \arrow[dll] & \\ A \otimes M \arrow[r] & B \otimes M \arrow[r] & C \otimes M & 0 \end{tikzcd}$$

Since B and C are flat, we have

$$\mathrm{Tor}_i(B, M) = \mathrm{Tor}_i(C, M) = 0$$

for any $i > 0$. This implies that

$$\mathrm{Tor}_i(A, M) = 0$$

for any $i > 0$. So A is a flat R-module.