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Course: MATH 648 - Abstract Algebra
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Homework - Week 2
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Problem 14.4.1
Give examples of

(1) a module which is both noetherian and artinian;

(2) a module which is noetherian but not artinian;

(3) a module which is artinian but not noetherian;

(4) a module which is neither artinian nor noetherian.

Solution:

(1) Consider a field $\mathbb{F}$, viewed as an $\mathbb{F}$-module ($\mathbb{F}$-vector space). We know $\mathbb{F}$ only has 0 and $\mathbb{F}$ as its ideal, so $\mathbb{F}$ is both artinian and noetherian.

(2) Consider the ring $\mathbb{Z}$ as a $\mathbb{Z}$-module. It is a PID so any ideal has the form (n) for $n \geq 0$. Note that $(m) \subset (n)$ if and only if $n|m$. Since $\mathbb{Z}$ is a UFD, every positive number n has a unique prime decomposition up to reordering, the ascending chain of submodules must stabilize. So $\mathbb{Z}$ is noetherian. On the other hands, consider the following descending chain of submodules

$$(2) \supset (2^2) \supset (2^3) \supset \cdots$$

This chain never stabilizes so $\mathbb{Z}$ is not artinian.

(3) Let p be a prime number. Consider the $\mathbb{Z}$-module $V = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. V can be written as

$$V = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} = \left\{ \frac{a}{p^n} \mid n \geq 0, 0 \leq a \leq p^n \right\}.$$

Every submodule of V is generated by a single element $\frac{1}{p^k}$, where $k \in \mathbb{N}$ is a positive integer. Note that $\langle \frac{1}{p^k} \rangle \subset \langle \frac{1}{p^e} \rangle$ if and only if $0 \leq k \leq e$. So we have an ascending chain of submodules

$$\langle \frac{1}{p} \rangle \subset \langle \frac{1}{p^2} \rangle \subset \cdots$$

which never stabilizes. This means V is not noetherian. On the other hand, for any submodule $\langle \frac{1}{p^k} \rangle$, the descending chain

$$\langle \frac{1}{p^k} \rangle \supset \langle \frac{1}{p^{k-1}} \rangle \supset \cdots \supset \langle \frac{1}{p} \rangle \supset (0)$$

is the longest possible descending chain, so V is artinian but not noetherian.

(4) Let $\mathbb{F}$ be a field and V be an infinite dimensional $\mathbb{F}$-vector space with a countable basis $B = \{v_1, v_2, \ldots, v_n, \ldots\}$. For any $i \geq 1$, let V_i be the finite dimensional subspace generated by $v_1, v_2, \ldots, v_i$. Then we have an ascending chain of subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$$

which never stabilizes since V is infinite dimensional. So V is not noetherian. Moreover, consider the descending chain

$$V \setminus V_1 \supset V \setminus V_2 \supset \cdots \supset V \setminus V_n \supset \cdots$$

which also never stabilizes because $V \setminus V_n$ is always infinite dimensional for any n. So V is neither noetherian nor artinian.

Problem 14.4.2
If V is a noetherian R-module then any surjective R-module endomorphism of V is an isomorphism.

Solution: The kernel of an R-module homomorphism is a submodule, so we have an ascending chain of submodules

$$\ker f \subset \ker f^2 \subset \ker f^3 \subset \cdots$$

which stabilizes since V is noetherian. This means there exists $N \geq 1$ such that $\ker f^n = \ker f^{n+1}$ for all $n \geq N$. Let $x \in \ker f$. f being surjective tells us that there exists $y \in V$ such that $f^n(y) = x$ for some $n \geq N$. We have $0 = f(x) = f^{n+1}(y)$, so $y \in \ker f^{n+1} = \ker f^n$. This shows that $0 = f^n(y) = x$. We have proved that $\ker f = 0$, namely f is injective. Thus, f is an isomorphism.

Problem 14.4.10
Given an exmaple of

(1) A commutative non-noetherian ring.

(2) A commutative ring which is noetherian but not artinian.

Solution:

(1) Let $R = \mathbb{F}[x_1, x_2, \ldots]$ be a polynomial ring over a field $\mathbb{F}$ with countably many indeterminates. consider the following chain of ideals

$$(x_1) \subset (x_1, x_2) \subset \cdots (x_1, x_2, \ldots, x_n) \subset \cdots$$

which never stabilizes since we have infinitely many variables. So R is not a noetherian ring.

(2) Let $R = \mathbb{F}[x]$ be the polynomial ring over a field $\mathbb{F}$. We know R is a PID, so it is noetherian. On the other hand, consider the descending chain of ideals

$$(x) \supset (x^2) \supset (x^3) \supset \cdots$$

which never stabilizes. So R is not artinian.

Problem 14.4.11
True or false? If R is artinian, then $R[x]$ is artinian.

Solution: This is false. Consider $R = \mathbb{F}$ is a field. We know R is artinian since the only ideals in $\mathbb{F}$ is the zero ideal 0 and $\mathbb{F}$ itself, so R is artinian. But $R[x]$ is not artinian as we have shown in Exercise 14.4.10(2).

Problem 14.4.14
Give an example showing that a submodule of a finitely generated module in general does not have to be finitely generated.

Solution: Let $\mathbb{F}$ be a field of characteristic not equal to 2 and $R = \mathbb{F}[x_1, x_2, \ldots]$ be a polynomial ring over $\mathbb{F}$ with countably many variables, viewed as a regular module. The principal ideal (2) is finitely generated but $(2) \subset (2x_1, 2x_2, \ldots)$. Here $(2x_1, 2x_2, \ldots)$ is not finitely generated since we have a ascending chain

$$(2x_1) \subset (2x_1, 2x_2) \subset \cdots$$

which never stabilizes.

Problem 14.4.17
Let D be a division ring and $R = M_n(D)$. Construct composition series of ${}_RR$ and R_R and conclude that R is left and right artinian.

Solution: First we consider the case ${}_RR$ is a left regular R-module. For any $1 \leq i \leq n$, define $V_i \subset V$ to be the set of matrices with the first i columns being zeros. It is easy to check that they are left submodules of ${}_RR$. Moreover, let $V_0 = V$ and $V_n = 0$, for any $0 \leq i \leq n-1$, the quotient $V_i/V_{i+1} \cong D^n$ as a column vector, by Example 14.1.22, D^n is irreducible as an R-module. So we have a composition series

$$V = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_n = 0$$

This proves that R is left artinian.

Now view R as a right regular module. This time for any $1 \leq i \leq n$, define $V_i \subset V$ to be the set of all matrices with the first i rows being zeros. They are right R-submodules of R, and still we have $V_i/V_{i+1} \cong D^n$ as a row vector. A similar argument as before shows that R is right artinian.

Problem 14.4.19
Let $\ell(V)$ denote the composition length of an R-module V. Suppose that V is an R-module of finite length and let X and Y be submodules of V. Then $\ell(X+Y)+\ell(X\cap Y)=\ell(X)+\ell(Y)$.

Solution: We first prove the following claim:
Claim: Let K, V, Q be submodules of V and suppose we have a short exact sequence

$$0 \to K \xrightarrow{\iota} V \xrightarrow{\pi} Q \to 0.$$

Then we have

$$\ell(V) = \ell(K) + \ell(Q).$$

Proof: V is an R-module of finite length, so V, K, Q as submodules also has finite length. These exists a composition series for K

$$K = K_0 \supset K_1 \supset \cdots \supset K_n = 0.$$

Apply the map ι and we get a sequence

$$\iota(K) = \iota(K_0) \supset \iota(K_1) \supset \cdots \supset \iota(K_n) = 0.$$

Note that because ι is injective, so for any $0 \leq i \leq n-1$, $\iota(K_i)/\iota(K_{i+1}) \cong \iota(K_i/K_{i+1})$ is still simple. So this is a composition series for $\iota(K)$. On the other hand, $V/\iota(K) \cong Q$ is of finite length, so we have a composition series

$$V/\iota(K) = V_0/\iota(K) \supset V_1/\iota(K) \supset \cdots \supset V_m/\iota(K) = 0.$$

By the correspondence theorem for modules, this is equivalent to a sequnce

$$V = V_0 \supset V_1 \supset \cdots \supset V_m = K.$$

and by the third isomorphism theorem, for any $0 \leq i \leq m-1$, $V_i/V_{i+1} \cong \frac{V_i/\iota(K)}{V_{i+1}/\iota(K)}$ is still simple. So we have a composition series for V

$$V = V_0 \supset V_1 \supset \cdots \supset V_m = K = K_0 \supset K_1 \supset \cdots \supset K_n = 0$$

and we can see that $\ell(V) = m + n = \ell(K) + \ell(Q)$. ■

Consider the two short exact sequences of R-modules as follows

$$0 \longrightarrow X \cap Y \longrightarrow X \longrightarrow X/(X \cap Y) \longrightarrow 0$$

$$0 \longrightarrow Y \longrightarrow X + Y \longrightarrow (X+Y)/Y \longrightarrow 0$$

By the previous claim, we have

$$\ell(X) = \ell(X \cap Y) + \ell(X/(X \cap Y)),$$
$$\ell(X+Y) = \ell(Y) + \ell((X+Y)/Y).$$

By the second isomorphism theorem, we have

$$X/(X \cap Y) \cong (X+Y)/Y.$$

So we can write

$$\ell(X) - \ell(X \cap Y) = \ell(X/(X \cap Y)) = \ell((X+Y)/Y) = \ell(X+Y) - \ell(Y).$$

This is equivalent to

$$\ell(X+Y) + \ell(X \cap Y) = \ell(X) + \ell(Y).$$

Problem 14.4.20

If V_1 and V_2 are non-isomorphic irreducible R-modules, then $V_1 \oplus V_2$ has exactly four submodules: (0), $(0) \oplus V_2$, $V_1 \oplus (0)$, and $V_1 \oplus V_2$.

Solution: It is easy to see that $(0), (0) \oplus V_2, V_1 \oplus (0), V$ are all submodules of V, we need to show V does not have any other submodule. Suppose $W = \langle (v_1, v_2) \rangle \subset V$ is a submodule of V and $W \neq 0$, $W \neq (0) \oplus V_2$ and $W \neq V_1 \oplus (0)$. So $v_1 \in V_1$ and $v_2 \in V_2$, v_1 and v_2 are both nonzero. Note that $W \cap (V_1 \oplus 0)$ is a submodule of W and by the second isomorphism theorem, we have

$$W/(W \cap (V_1 \oplus 0)) \cong (W + (V_1 \oplus 0))/(V_1 \oplus 0) \subset V/(V_1 \oplus 0) \cong 0 \oplus V_2.$$

Since V_2 is simple, we know that $W/(W \cap (V_1 \oplus 0)) = 0$ or $W \cap (V_1 \oplus 0) \cong V_2$. If $W/(W \cap (V_1 \oplus 0)) = 0$, namely $W = W \cap (V_1 \oplus 0)$, but we know that $v_2 \neq 0$. So $W/(W \cap (V_1 \oplus 0)) \cong V_2$ is simple and nonzero. We have shown V_2 is a composition factor of W and similarly using $W \cap (0 \oplus V_2)$, we can show that V_1 is also a composition factor of W. By Jordan-Hölder theorem, W must be isomorphic to V. Thus, we can conclude that V does not have any other submodule.

Problem 14.5.14

Let A be the ring of continous functions $\mathbb{R} \to \mathbb{R}$. Prove that A has no non-trivial idempotens, but A is not local.

Solution: Suppose f is an idempotent in A and $f \neq 0$. For any $x \in \mathbb{R}$ satisfying $f(x) \neq 0$, we have $(f(x))^2 = f(x)$. Divide both sides by $f(x)$, we have $f(x) = 1$. Since f is continous everywhere on $\mathbb{R}$, so $f(x) = 1$ for all $x \in \mathbb{R}$. This proves that A does not have any idempotent except for 0 and 1.

Both $f(x) = 1 - x^2$ and $g(x) = x^2$ are non-units in A but $(f+g)(x) = 1$ is a unit. So A is not a local ring.

Problem 14.5.17
Let D be a division ring, and let R be the ring of all 2×2 upper triangular matrices over D. Let $e_1 := E_{1,1}$ and $e_2 := E_{2,2}$.

(1) e_1 and e_2 are orthogonal idempotents with $e_1 + e_2 = 1$, so ${}_RR = Re_1 \oplus Re_2$.

(2) Re_1 is irreducible but Re_2 is not.

(3) $\mathrm{End}_R(Re_1)^{op} \cong \mathrm{End}_R(Re_2)^{op} \cong D$ as rings. Deduce that Re_1 and Re_2 are indecomposable R-modules, and the idempotents e_1 and e_2 are primitive.

(4) Classify irreducible R-modules.

Solution:

(1) We can check that e_1 and e_2 are idempotents by direct computations.
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$
Same for e_2. And we have
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
Note that $e_1e_2 = e_2e_1 = 0$ and $\mathrm{End}_R({}_RR) = R$, by Lemma 14.5.1, we have $R = Re_1 \oplus Re_2$.

(2) Given an upper triangular matrix $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, it is easy to check that $Ae_1 = ae_1$. So $Re_1 \cong D$ and since D is a division ring, the only ideal is the zero ideal and D itself, so Re_1 is simple. For Re_2, we have
$$Ae_2 = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}.$$
Consider the submodule
$$W = \left\{ \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \mid w \in D \right\} \cong D.$$
This is a proper submodule of Re_2 as
$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & aw \\ 0 & 0 \end{pmatrix}.$$
So Re_2 is not irreducible.

(3) We know that $\mathrm{End}_R({}_RR) \cong R$ and e_1, e_2 are idempotents, by Lemma 14.5.4, we only need to show that
$$e_1Re_1 \cong e_2Re_2 \cong D.$$

This can be checked by direct computations, suppose $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$$

So we can see that $e_1Re_1 \cong e_2Re_2 \cong D$. Moreover, we know by Exercise 14.4.16 that R is of finite length, and D is a local ring, so by Proposition 14.5.11 Re_1 and Re_2 are both indecomposable, and by Corollary 14.5.5 e_1 and e_2 are primitive idempotents.

(4) By Exercise 14.1.23, this is the same as classifying the maximal left ideals of R since every irreducible R-module is isomorphic to R/I for some maximal left ideal I in R. Suppose $I \subset R$ is a left ideal.

<u>Claim:</u> For $1 \le i, j \le 2$, let I_{ij} be the set of all (i, j)-entries in I. $I_{ij} \subset D$ is a left ideal in D.

<u>Proof:</u> Let $a, r \in D$, we have

$$\begin{pmatrix} r & * \\ 0 & * \end{pmatrix} \begin{pmatrix} a & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} ra & * \\ 0 & * \end{pmatrix},$$

$$\begin{pmatrix} r & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} * & a \\ 0 & * \end{pmatrix} = \begin{pmatrix} * & ra \\ 0 & * \end{pmatrix},$$

$$\begin{pmatrix} * & * \\ 0 & r \end{pmatrix} \begin{pmatrix} * & * \\ 0 & a \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & ra \end{pmatrix}$$

From this we can see that I_{ij} has to be a left ideal in D. ■

D being a division ring implies the only left ideals are (0) and D. It is easy to check that $\left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}$ is not a left ideal in R. So R only has two maximal left ideal $I = \left\{\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}\right\}$ and $J = \left\{\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}\right\}$. Note that $R/I \cong Re_2$ and $R/J \cong Re_1$. From (b), we know that R/I is not isomorphic to R/J. Thus, we can conclude that we have two nonisomorphic classes of irreducible R-modules, given by Re_1 and Re_2.