Zhengdong Zhang
Email: zhengz@uoregon.edu
Course: MATH 635 - Algebraic Topology II
Instructor: Dr.Daniel Dugger

Homework 1
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Problem 1
The following turns out to be a complete list of all compact 2-manifold:

$$S^2, T, T\#T, T\#T\#T, \dots, \mathbb{R}P^2, \mathbb{R}P^2\#\mathbb{R}P^2, \mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2, \dots$$

Compute the homology groups and Euler characteristics for each of them.

Solution:

(1) 2-sphere S^2.

For the 2-sphere S^2, it has a CW structure with only one 0-cell and one 2-cell. So the boundary map is all zero and the CW chain complex is as follows:

$$\mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z}.$$

So we have

$$H_i(S^2) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2; \\ 0, & \text{otherwise.} \end{cases}$$

The Euler characteristics is

$$\chi(S^2) = \operatorname{rank} H_0(S^2) + \operatorname{rank} H_2(S^2) = 2.$$

(2) Torus T and real projective space $\mathbb{R}P^2$.

Consider the following CW structure: The torus T has one 0-cell x, two 1-cells a and b, one

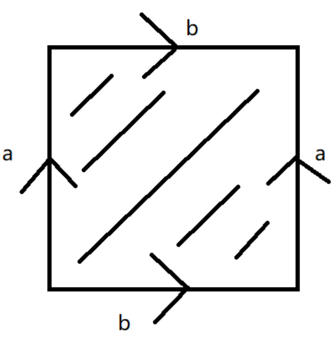


Figure 1: T

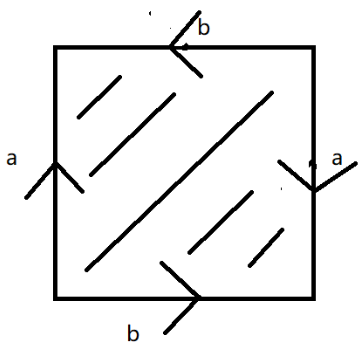


Figure 2: $\mathbb{R}P^2$

2-cell S. We have the following cellular chain complex:

$$\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z}.$$

The boundary map $d_1 = 0$ since we only have one 0-cell. From the figures above, we know that for the torus T, we have

$$d_2(S) = a + b - a - b = 0.$$

So for the torus T, we have

$$H_i(T) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2; \\ \mathbb{Z}^2, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The Euler characteristics is

$$\chi(T) = \operatorname{rank} H_0(T) - \operatorname{rank} H_1(T) + \operatorname{rank} H_2(T) = 0.$$

For the real projective space $\mathbb{RP}^2$, we have two 0-cells x, y, two 1-cells a, b and one 2-cell S. The cellular chain complex is as follows:

$$\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z}^2.$$

The boundary map d_2 sends S to

$$d_2(S) = 2a - 2b$$

and the boundary map

$$\begin{aligned} d_1(a) &= x - y, \\ d_1(b) &= x - y. \end{aligned}$$

So we can calculate the homology groups

$$\begin{aligned} H_0(\mathbb{R}P^2) &= \ker d_0 / \operatorname{Im} d_1 \\ &= \langle x, y \rangle / \langle x - y \rangle \\ &= \langle x - y, y \rangle / \langle x - y \rangle \\ &= \mathbb{Z}. \end{aligned}$$

and

$$\begin{aligned} H_1(\mathbb{R}P^2) &= \ker d_1 / \operatorname{Im} d_2 \\ &= \langle a - b \rangle / \langle 2a - 2b \rangle \\ &= \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

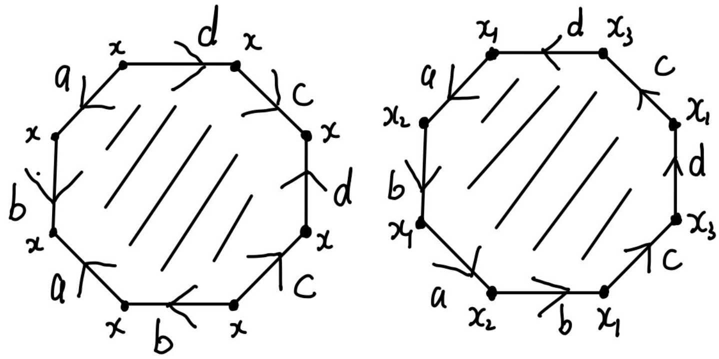
So we have

$$H_i(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The Euler characteristics is

$$\chi(\mathbb{R}P^2) = \operatorname{rank} H_0(\mathbb{RP}^2) - \operatorname{rank} H_1(\mathbb{R}P^2) = 1.$$

(3) The following pictures give CW structures to the connected sum $T\#T$ and $\mathbb{R}P^2\#\mathbb{R}P^2$:



Note that for the connected sum of tori $T\#T$, we have one 0-cell, four 1-cell and one 2-cell, and for the connected sum of real projective space, we have three 0-cells, four 1-cells and one 2-cell. These CW structures can be generalized to n-copies of connected sum $T\#T\#\cdots\#T$ and $\mathbb{R}P^2\#\mathbb{R}P^2\#\cdots\#\mathbb{R}P^2$.

- For $T\#T\#\cdots\#T$, we have one 0-cell x, $2n$ 1-cells $a_1, b_1, a_2, b_2, \dots, a_n, b_n$ and one 2-cell S. The celluar chain complex is given by

$$\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2n} \xrightarrow{d_1} \mathbb{Z}$$

where

$$d_2(S) = (a_1 + b_1 - a_1 - b_1) + (a_2 + b_2 - a_2 - b_2) + \cdots + (a_n + b_n - a_n - b_n) = 0$$

and $d_1 = 0$ since we only have one 0-cell. So the homology group is given by

$$H_i(T\#T\#\cdots\#T) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2; \\ \mathbb{Z}^{2n}, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The Euler characteristics is

$$\chi(T\#T\#T\cdots\#T) = 1 - 2n + 1 = 2 - 2n.$$

- For $Y = \mathbb{R}P^2\#\mathbb{R}P^2\#\cdots\#\mathbb{R}P^2$, we have $n+1$ 0-cells $x_1, x_2, \dots, x_{n+1}$, $2n$ 1-cells $a_1, b_1, a_2, b_2, \dots, a_n$ and one 2-cell S. The cellular complex is given by

$$\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2n} \xrightarrow{d_1} \mathbb{Z}^{n+1}$$

where

$$d_2(S) = 2(a_1 + b_1 + \cdots + a_n + b_n)$$

and

$$d_1(a_i) = x_{i+1} - x_i = -d_1(b_i)$$

for $1 \leq i \leq n+1$ and assume $x_{n+2} = x_1$. We can see that $\ker d_2 = 0$ so $H_2(Y) = 0$. And

$$\begin{aligned} H_0(Y) &= \ker d_0 / \operatorname{Im} d_1 \\ &= \langle x_1, x_2, \ldots, x_{n+1} \rangle / \langle x_2 - x_1, x_3 - x_2, \ldots, x_{n+1} - x_n \rangle \\ &= \langle x_1 \rangle \\ &= \mathbb{Z}. \end{aligned}$$

$$\begin{aligned} H_1(Y) &= \ker d_1 / \operatorname{Im} d_2 \\ &= \langle a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n \rangle / \langle 2(a_1 + b_1 + \cdots a_n + b_n) \rangle \\ &= \langle (a_1 + b_1 + \cdots + a_n + b_n), a_2 + b_2, \ldots, a_n + b_n \rangle / \langle 2(a_1 + b_1 + \cdots a_n + b_n) \rangle \\ &= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{n-1}. \end{aligned}$$

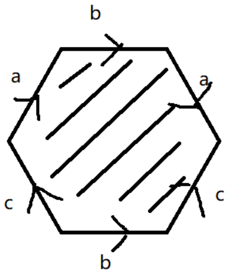
So the homology group of $Y = \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$ with n copies of $\mathbb{R}P^2$ can be summarized as follows:

$$H_i(Y) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{n-1}, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The Euler characteristics is

$$\chi(Y) = 1 - (n-1) = 2 - n.$$

The remaining exercises will hep you get started towards a proof that all compact 2-manifolds are as stated in problem 1.
Suppose one has polygon with labelled edges, representing a quotient space. Assume every edge is labelled, and every label occurs exactly twice. Such a quotient space is a 2-dimensional manifold. If we have a diagram such as



we will represent this by the string "$abac^{-1}b^{-1}c$". Notice that we could also represent it by the string "$b^{-1}cabac^{-1}$", as well as others. When talking about these strings I will use letters x, y, z to stand for one symbol (like a or a^{-1})and letters A, B, C to stand for a block of symbols (like aba or $c^{-1}b^{-1}$).
If B is a block $x_1x_2\cdots x_k$, let B^{-1} denote the block $x_k^{-1}x_{k-1}^{-1}\cdots x_1^{-1}$. Our convention is that $(x^{-1})^{-1} = x$ so that for instance if B is the block abc^{-1} then B^{-1} is $cb^{-1}a^{-1}$.
We will say that two strings are equivalent if the corresponding quotient spaces are homeomorphic. Here are some basic cases of equivalent strings.

(i) The strings $x_1x_2\cdots x_n$ and $x_nx_1x_2\cdots x_{n-1}$ are equivalent.

(ii) If you insert xx^{-1} or $x^{-1}x$ into any string which does not have an x, you get an equivalent string. [Taking this and working backwards, if you remove either $xx^[-1]$ or x^-1x from a given string then you get an equivalent one.]

(iii) The string $ABCBD$ is equivalent to $AxCxD$, assuming the letter x does not appear in A, C or D. (That is, if a block occurs twice then we can rename the whole block to just one letter). Likewise, the string $ABCB^{-1}D$ is equivalent to $AyCy^{-1}D$.

Based on all this, we do the following exercises.

Problem 3
Using (i)-(iii), explain why a string AC is equivalent to both $ABB^{-1}C$ and $AB^{-1}BC$, assuming that none of the letters in B appears in AC.

Solution: Note that $(BB^{-1})^{-1} = B^{-1}B$, we only need to prove that $ABB^{-1}C$ is equivalent to AC. The other situation is just by replacing B with B^{-1}. Let x be a letter that does not appear in A, B, C, by (ii), we have $ABB^{-1}C \cong AB(xx^{-1})B^{-1}C$. Denote the block xx^{-1} by D, by (iii), we have

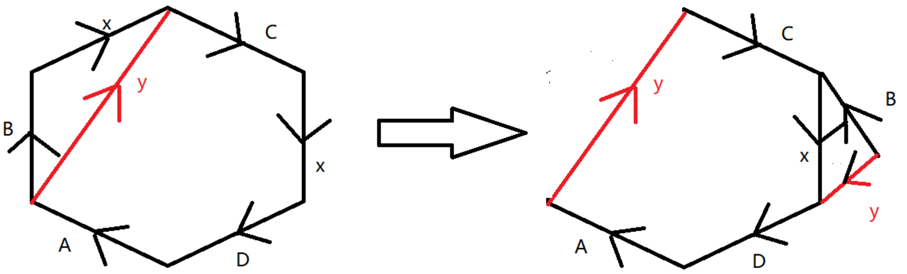
$$ABDB^{-1}C \cong AyDy^{-1}C \cong Ayxx^{-1}y^{-1}C \cong A(yx)(yx)^{-1}C \cong AC$$

since none of the letters in B appears in A and C, so both x and y does not appear in A and C.

Problem 4

Use a topological cut-and-paste arguments, prove that $AxBCxD$ is equivalent to $AyCyB^{-1}D$. Also prove that $AxBCx^{-1}D$ is equivalent to $AyCBy^{-1}D$, and $ABxCx^{-1}D$ is equivalent to $AyCy^{-1}BD$.

Solution: We cut open along the red line y and paste along x as shown in the following picture:

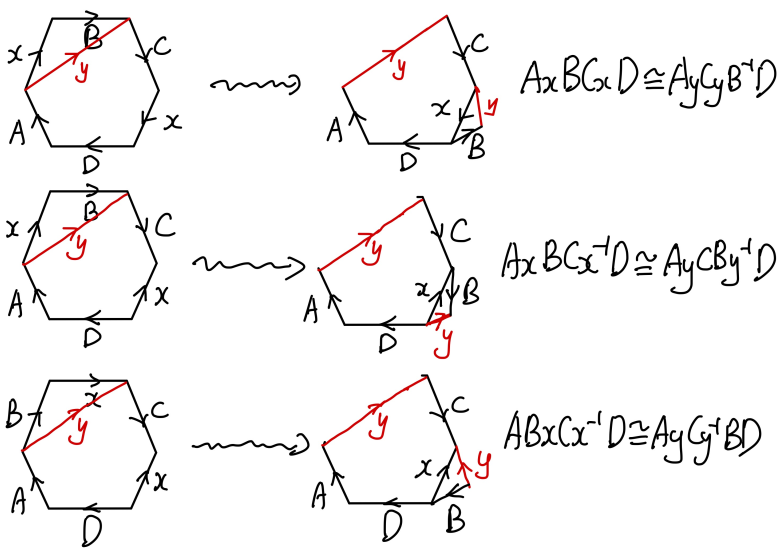


This proves that the string $ABxCxD$ is equivalent to $AyCB^{-1}yD$.

Problem 5

Again using topological cut-and-paste arguments, prove that $AxBCxD$ is equivalent to $AyCyB^{-1}D$. Also prove that $AxBCx^{-1}D$ is equivalent to $AyCBy^{-1}D$, and $ABxCx^{-1}D$ is equivalent to $AyCy^{-1}BD$.

Solution: The cut and paste procedure is shown in the following pictures:



We can summarize the rules you've established in question 2-4 as follows:

(1) $AxBCxD \sim AxCxB^{-1}D$

(2) $ABxCxD \sim AxCB^{-1}xD$

(3) $AxBCx^{-1}D \sim AxCBx^{-1}D$

(4) $ABxCx^{-1}D \sim AxCx^{-1}BD$

The rules can be paraphrased as follows:

- If x occurs twice in a string (without any inverse on it), then a block next to the first x can be moved to a block on the same side of the second x, but the new block has to be inverted.
- If x and x^{-1} occur in a string, then a block next to the first one can be moved to a block on the other side of the second on , and the block does not get inverted.

As an example of using these rules, we can write

$$abacb^{-1}ddc \sim abacdbdc \sim cdbda^{-1}b^{-1}a^{-1}c \sim cdabda^{-1}b^{-1}c.$$

Problem 6

Using (i)-(iii) and (1)-(4), prove the following equivalences:

(a) $abab \sim xx$.

(b) $abab^{-1} \sim xxyy$. (This shows that $K \cong \mathbb{R}P^2 \# \mathbb{R}P^2$)

(c) $aabcb^{-1}c^{-1} \sim xxyyzz$. (This shows that $\mathbb{R}P^2 \# T \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{RP}^2$)

Solution:

(a) Use (iii). Assume A, C, D are empty blocks and $B = ab$. We have $abab \sim xx$.

(b) First we use Rule (1) with $A = D$ is the empty block, $x = a$, $C = b$ and $B^{-1} = b^{-1}$, so we have $abab^{-1} \sim abba$. Then (i) and this tells us $abab^{-1} \sim abba \sim aabb$.

(c) First rewrite $aabcb^{-1}c^{-1}$ into $abcb^{-1}c^{-1}a$ using (i). Then use use Rule (2) with A, D empty, $x = a$, $C = bc$, and $B = cb$, we have

$$abcb^{-1}c^{-1}a \sim cbabca.$$

Then use Rule (2) with $A = c$, $x = b$, C empty, $B = a^{-1}$ and $D = ca$, we have

$$cbabca \sim ca^{-1}bbca \sim caca^{-1}bb.$$

Use Rule (2) with A, C empty, $B = a^{-1}$ and $D = a^{-1}bb$, we have

$$caca^{-1}bb \sim a^{-1}cca^{-1}bb.$$

Finally use Rule (2) with A, C empty, $B = c^{-1}c^{-1}$, $x = a^{-1}$ and $D = bb$, we have

$$c^{-1}c^{-1}a^{-1}a^{-1}bb.$$

Let $c^{-1} = x$, $a^{-1} = y$ and $b = z$ and we are done.

Problem 7

Now we turn to the classification of compact 2-manifolds. Note that S^2 is represented by the string xx^{-1}, which is equivalent to the empty string.

Let S be a string of letters and their 'inverse', in which each letter appears twice (meaning either twice as itself or once as x and once as x^{-1}). Use (i)-(iii) and (1)-(4) to do the following:

(a) Prove that if a letter x appears twice as itself, then S is equivalent to a string in which xx appears as the beginning. [That is, prove that $AxBxC \sim xxD$, for some block D]

(b) Prove that a string of the form $AsCtDs^{-1}Et^{-1}$ is equivalent to one which starts out as $sts^{-1}t^{-1}$.

(c) Let M be the 2-dimensional manifold corresponding to the string

$$abca^{-1}befe^{-1}c^{-1}f.$$

Determine which of the 2-manifold is homeomorphic to M.

(d) Repeat part (c) for the manifold correponding to

$$dbced^{-1}c^{-1}e^{-1}b^{-1}.$$

(e) Explain why any string of the form

$$a_1a_1a_2a_2 \cdots a_ka_ks_1t_1s_1^{-1}t^{-1} \cdots s_rt_rs_r^{-1}t_r^{-1}$$

in which $k \geq 1$ is equivalent to one of the form

$$b_1b_1b_2b_2 \cdots b_nb_n.$$

(f) Think about how you would prove that any string is equivalent to either one of the form

$$a_1a_1a_2a_2 \cdots a_ka_k$$

or

$$s_1t_1s_1^{-1}t_1^{-1} \cdots s_rt_rs_r^{-1}t_r^{-1}.$$

Solution:

(a) Given a string $AxBxC$, use Rule (1), we have

$$AxBxC \sim AxC^{-1}Bx \sim xC^{-1}BxA \sim xA^{-1}C^{-1}Bx \sim xxA^{-1}C^{-1}B \sim xxD$$

where $D = A^{-1}C^{-1}B$.

(b) First we use Rule (4) with $A = A$, $x = s$, $C = CtD$, $B = E$ and $D = t^{-1}$, we have

$$AsCtDs^{-1}Et^{-1} \sim AEsCtDs^{-1}t^{-1}.$$

Then use Rule (4) again with $A = AEs$, $B = C$, $x = t$, $C = Ds^{-1}$,D empty, we have

$$AEsCtDs^{-1}t^{-1} \sim AEstDs^{-1}t^{-1}C.$$

Next use Rule (3) with $A = AEs$, $x = t$,$B = D$,$C = s^{-1}$, $D = C$, we have

$$AEstDs^{-1}t^{-1}C \sim AEsDts^{-1}t^{-1}C.$$

Finally use Rule (4) again with $A = AEs$, $x = t$, $B = D$, $C = s^{-1}$ and $D = C$, we have

$$AEsDts^{-1}t^{-1}C \sim AEsts^{-1}t^{-1}DC \sim sts^{-1}t^{-1}DCAE.$$

(c) First let $A = a$, $x = b$, $B = ca^{-1}$ and $C = efe^{-1}c^{-1}f$. Use (a), then

$$D = A^{-1}C^{-1}B = a^{-1}f^{-1}cef^{-1}e^{-1}ac^{-1}$$

and the original string equals to

$$bba^{-1}f^{-1}cef^{-1}e^{-1}ac^{-1}$$

Use part (a) again with $A = bba^{-1}$, $x = f^{-1}$, $B = ce$ and $C = e^{-1}ac^{-1}$, this time the string becomes

$$f^{-1}f^{-1}A^{-1}C^{-1}B = f^{-1}f^{-1}ab^{-1}b^{-1}ca^{-1}ece.$$

Use part (a) again with $A = f^{-1}f^{-1}ab^{-1}b^{-1}$, $x = c$, $B = a^{-1}e$ and $C = e$, the string becomes

$$ccbba^{-1}ffe^{-1}a^{-1}e$$

Use part (a) again with $A = ccbb$, $x = a^{-1}$, $B = ffe^{-1}$ and $C = e$, the string becomes

$$a^{-1}a^{-1}b^{-1}b^{-1}c^{-1}c^{-1}e^{-1}ffe^{-1}.$$

Use part (a) again with $A = a^{-1}a^{-1}b^{-1}b^{-1}c^{-1}c^{-1}$, $x = e^{-1}$, $B = ff$ and C empty, we have

$$e^{-1}e^{-1}ccbbaaff.$$

This manifold is homeomorphic to the connected sum $\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2$.

(d) Use (i) to rewrite $dbced^{-1}c^{-1}e^{-1}b^{-1}$ into $bced^{-1}c^{-1}e^{-1}b^{-1}d$. Apply part (b) with $s = b$, $t = d^{-1}$,

A, E empty, $C = ce$, $D = c^{-1}e^{-1}$, we have the original string is equivalent to

$$bd^{-1}b^{-1}dc^{-1}e^{-1}ce = (bd^{-1})(bd^{-1})^{-1}(ce)^{-1}(ce),$$

which is just the connected sum $T\#T$.

(e) Since $k \geq 1$, we can always write $a_1a_1a_2a_2 \cdots a_ka_ks_1t_1s_1^{-1}t_1^{-1} \cdots s_rt_rs_r^{-1}t_r^{-1}$ into

$$Aaas_1t_1s_1^{-1}t_1^{-1} \cdots s_rt_rs_r^{-1}t_r^{-1}$$

where $A = a_1a_1 \cdots a_{k-1}a_{k-1}$(When $k = 1$, A is empty). Use (i) to rewrite, we change the string into

$$BAaasts^{-1}t^{-1}$$

where $s = s_1$, $t = t_1$ and $B = s_2t_2s_2^{-1}t_2^{-1} \cdots s_rt_rs_r^{-1}t_r^{-1}$(when $r = 1$, B is empty). Note that the letter a, s, t does not appear in either A or B. So we write BA into a block D. Use (iii), we can rename the block D into one letter z. Use (i) again we have

$$zaasts^{-1}t^{-1} \sim asts^{-1}t^{-1}za.$$

Nest, use Rule (2) with A, D empty, $x = a$, $C = st$ and $B = z^{-1}ts$, we have

$$asts^{-1}t^{-1}za \sim z^{-1}tsasta.$$

Use Rule (2) again with $A = z^{-1}$, $x = t$, C empty, $B = s^{-1}a^{-1}s^{-1}$ and $D = a$, we have

$$z^{-1}tsasta \sim z^{-1}s^{-1}a^{-1}s^{-1}tta.$$

Use Rule (2) and (i) again with $A = z^{-1}$, $x = s^{-1}$, C empty, $B = a$ and $D = tta$, we have

$$z^{-1}s^{-1}a^{-1}s^{-1}tta \sim z^{-1}as^{-1}s^{-1}tta \sim az^{-1}as^{-1}s^{-1}tt.$$

Use Rule (2) and (i) again with A, C empty, $x = a^{-1}$, $B = z$ and $D = s^{-1}s^{-1}tt$, we have

$$az^{-1}as^{-1}s^{-1}tt \sim zaas^{-1}s^{-1}tt \sim BAaas^{-1}s^{-1}tt \sim Aaas^{-1}s^{-1}ttB.$$

Note that this string is similar to our original string but with $k+1$ and $r-1$, since r is finite, repeat this process, we can always turn the original string into

$$b_1b_1 \cdots b_nb_n.$$