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Course: MATH 634 - Algebraic Topology
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Homework - Week 6
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Problem 2.1.24
Show that each n-simplex in the barycentric subdivision of Δ^n is defined by n inequalities $t_{i_0} \le t_{i_1} \le \cdots \le t_{i_n}$ in its barycentric coordinates, where $(i_0, \ldots, i_n)$ is a permutation of $(0, \ldots, n)$.

Solution: Let $[v_0 \ldots, v_n]$ be a standard n-simplex. We prove this using the induction on n.

When $n = 1$, under barycentric coordinates, the 1-simplex is an interval $[v_0, v_1]$ with two vertices $v_0 = (1, 0)$ and $v_1 = (0, 1)$. The barycenter is $(\frac{1}{2}, \frac{1}{2})$. After barycentric subdivision, the 2 1-simplices are just $(t, 1-t)$ given by $0 \le t \le \frac{1}{2}$ and $\frac{1}{2} \le t \le 1$ respectively. So it satifies the assumption.

Now assume $n \ge 2$ and we have prove the case for $n-1$. The barycenter for $[v_0, \ldots, v_n]$ has coordinates $b = (\frac{1}{n+1}, \ldots, \frac{1}{n+1})$. Consider one of its faces $[v_0, \ldots, \hat{v}_k, \ldots, v_n]$, by our assumption we know that each $(n-1)$-simplex after the barycentric subdivision in this face is given by an equality $0 \le t_{i_0} \le \cdots \le t_{i_{k-1}} \le t_{i_{k+1}} \le \cdots \le t_{i_n}$ where $(i_0, i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n)$ is a permutation of $(0, 1, \ldots, k-1, k+1, \ldots, n)$. Fix such a $(n-1)$-simplex Δ^{n-1} (namely, an inequality as above), we will try to describe any point $x = (t_0, t_1, \ldots, t_n)$ in the n-simplex formed using vertices from Δ^{n-1} and b. Consider the line passing through x and b and it intersects with Δ^{n-1} at the point $y \in \Delta^{n-1}$. By colinearity we can write the coordinate

$$y = (\frac{t_0 - t_k}{n+1}, \frac{t_1 - t_k}{n+1}, \ldots, \frac{t_{k-1} - t_k}{n+1}, 0, \frac{t_{k+1} - t_k}{n+1}, \ldots, \frac{t_n - t_k}{n+1}).$$

The inequality implies that

$$0 \le \frac{t_{i_0} - t_k}{n+1} \le \frac{t_{i_1} - t_k}{n+1} \le \cdots \le \frac{t_{i_{k-1}} - t_k}{n+1} \le \frac{t_{i_{k+1}} - t_k}{n+1} \le \cdots \le \frac{t_{i_n} - t_k}{n+1} \le 1.$$

Combine this with the requirements that the coordinate $\frac{t_{i_j} - t_k}{n+1} \ge 0$ for all $j = 0, 1, \ldots, k-1, k+1, \ldots, n$ gives us a total order

$$0 \le t_k \le t_{i_0} \le t_{i_1} \le \cdots \le t_{i_{k-1}} \le t_{i_{k+1}} \le \cdots \le t_{i_n} \le 1.$$

Varying $k = 0, 1, \ldots, n$ and repeat the same process for each face we run through all the n-simplex in the barycentric subdivision, giving us an inequality as above each time. We are done.

Problem 2.1.26
Show that $H_1(X, A)$ is not isomorphic to $\tilde{H}_1(X/A)$ if $X = [0, 1]$ and A is the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ together with its limit 0.

Solution: We first use the long exact sequence for relative homology to calculate $H_1(X, A)$

$$\cdots \longrightarrow H_1(X) \longrightarrow H_1(X, A) \longrightarrow H_0(A) \longrightarrow H_0(X) \longrightarrow \cdots$$

We know that $X = [0, 1]$ is contractible, so $H_1(X) = 0$. Note that A is countable many points $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, so by Proposition 2.7, $H_0(A) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}$. We have an injective map $\partial : H_1(X, A) \to$

$H_0(A) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}$. On the other hand, the quotient space X/A is homeomorphic to the shrinking wedge of circles, each interval $[\frac{1}{n+1}, \frac{1}{n}]$ in X/A is a small circle, the radius of which is shrinking as n gets larger. Denote this circle as C_n. Consider the retraction $r_n : X/A \to C_n$ collapsing all circles except C_n to the point represented by A. The induced map in homology $r_{n,*} : H_1(X/A) \to H_1(C_n) \cong \mathbb{Z}$ is surjective. Moreover, by the universal property of products, we have a surjective map $r_* : H_1(X/A) \to \prod_{i=1}^{\infty} \mathbb{Z}$. Note that for any topological space, $H_1(X/A) \cong \tilde{H}_1(X/A)$. And $\prod_{i=1}^{\infty} \mathbb{Z}$ is not isomorphic to $\bigoplus_{i=1}^{\infty} \mathbb{Z}$, so $H_1(X, A)$ cannot be isomorphic to $\tilde{H}_1(X/A)$.

Problem 2.1.30

In each of the following commutative diagrams assume that all maps but one are isomorphisms. Show that the remaining map must be isomorphism as well.

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \\ & C \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C \arrow[r] & D \arrow[u] \end{tikzcd}$$

Solution:

(1)

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\ & C \arrow[ur, "h"'] & \end{tikzcd}$$

(a) Assume g, h are isomorphisms, then $f = h \circ g$ is also an isomorphism since it is the composition of two isomorphisms.

(b) Assume f, g are isomorphisms. g is an isomorphism implies that there exist a map $g^{-1} : C \to A$ such that $g \circ g^{-1} = id_C$, then

$$h = h \circ id_C = h \circ (g \circ g^{-1}) = (h \circ g) \circ g^{-1} = f \circ g^{-1}$$

where both f and g^{-1} are isomorphisms, so is h.

(c) Assume f, h are isomorphisms. h is an isomorphism implies that there exists a map $h^{-1} : B \to C$ such that $h^{-1} \circ h = id_C$., then

$$g = id_C \circ g = (h^{-1} \circ h) \circ g = h^{-1} \circ (h \circ g) = h^{-1} \circ f$$

where both f and h^{-1} are isomorphisms, so is g.

(2)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "g"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

Assume i, g, h are isomorphisms. Then view the composition $h \circ i$ as one isomorphism, and we are back to the situation (c) in (1).
Assume i, f, g are isomorphisms. Then view the composition $g \circ f$ as one isomorphism, and we are back to the situation (b) in (1).

Assume f, i, h are isomorphisms. Then view the composition $h \circ i$ as one isomorphism, and we are back to the situation (b) in (1).
Assume f, g, h are isomorphisms. Then view the composition $g \circ f$ as one isomorphism, and we are back to the situation (c) in (1).

(3)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \\ C \arrow[r, "h"'] & D \arrow[u, "g"'] \end{tikzcd}$$

Assume i, g, h are isomorphisms, then $f = g \circ h \circ i$ is also an isomorphism since it is the composition of isomorphisms.
Assume i, f, h are isomorphisms, then view the composition $h \circ i$ as one isomorphism, and we are back to the situation (b) in (1).
Assume i, f, g are isomorphisms. i, g are isomorphisms implies that there exist $g^{-1} : B \to D$ and $i^{-1} : C \to A$ such that $g^{-1} \circ g = id_D$ and $i \circ i^{-1} = id_C$. Now we have

$$h = id_D \circ h \circ id_C = g^{-1} \circ g \circ h \circ i \circ i^{-1} = g^{-1} \circ (g \circ h \circ i) \circ i^{-1} = g^{-1} \circ f \circ i^{-1}$$

where i^{-1}, f, g^{-1} are isomorphisms, so is h.
Assume f, g, h are isomorphisms, then view the composition $g \circ h$ as one isomorphism, and we are back to the situation (c) in (1).

Problem 2.1.31
Using the notation of the five lemma, give an example where the maps α, β, δ and ε are zero but γ is nonzero. This can be done with short exact sequencs in which all the groups are either $\mathbb{Z}$ or 0.

Solution: Consider the following diagrams:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\sim"] \arrow[d, "0"'] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "0"'] & \mathbb{Z} \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathbb{Z} \arrow[r] \arrow[d, "0"] & 0 \arrow[d, "0"] \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\sim"'] & \mathbb{Z} \arrow[r, "0"'] & \mathbb{Z} \arrow[r, "\sim"'] & \mathbb{Z} \end{tikzcd}$$

The top row and the bottom row are exact. And we have $\alpha = \beta = \delta = \varepsilon = 0$, and $\gamma : \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$ is an isomorphism and nonzero.

Problem 2.2.1
Prove the Brouwer fixed point theorem for maps $f : D^n \to D^n$ by applying degree theory to the map $S^n \to S^n$ that sends both the northern and southern hemispheres of S^n to the southern hemisphere via f.

Solution: Denote the descirbed map by $\bar{f} : S^n \to S^n$. Since $\bar{f}$ is not surjective, so we know $\deg \bar{f} = 0$. Moreover, $\bar{f}$ has no fix point unless $\deg \bar{f} = (-1)^{n+1}$. There exist $x \in S^n$ such that $\bar{f}(x) = x$. Note that x cannot be in the northern hemisphere because the northern hemisphere is not in the image of $\bar{f}$. And we know that when $\bar{f}$ restricts to the southern hemisphere, it is just the map $f : D^n \to D^n$, so we can conclude that f has a fixed point.