Math 636 Homework #2
Due Friday, April 18

1. Determine all of the cohomology groups $H^*(\mathbb{R}P^4 \# \mathbb{C}P^2)$.

   [Suggestion: It is easy to calculate the groups $H^*(\mathbb{R}P^4)$ and $H^*(\mathbb{C}P^2)$. Then find a cofiber sequence $A \hookrightarrow X \to X/A$ where you can use the associated long exact sequence on cohomology to get your answer. You can probably find at least three different cofiber sequences to try, and one of them works better than the others.]

2. Let $f\colon X \to Y$ be a map, and consider the diagram

$$\begin{tikzcd}
H^k(Y;R) \arrow[r, "f^*"] \arrow[d, "\phi"'] & H^k(X;R) \arrow[d, "\phi"] \\
\mathrm{Hom}(H_k(Y),R) \arrow[r, "\mathrm{Hom}(f_*,R)"] & \mathrm{Hom}(H_k(X),R)
\end{tikzcd}$$

   where the bottom horizontal map is the one obtained by applying the functor $\mathrm{Hom}(-,R)$ to $f_*\colon H_k(X) \to H_k(Y)$. Here the vertical maps ϕ are the adjoints to the Kronecker pairings, namely the maps that send a cohomology class $[\alpha]$ to the homomorphism $[v] \mapsto \alpha(v)$ (these were discussed in class). Verify that the above diagram commutes.

3. (a) Give a Δ-complex structure on $\mathbb{R}P^2$ and use this to write down explicit cocycles $\alpha \in Z^1(\mathbb{R}P^2;\mathbb{Z}/2)$ and $\beta \in Z^2(\mathbb{R}P^2;\mathbb{Z}/2)$ that generate the cohomology groups. Compute $\alpha \cup \alpha$ and decide if it equals β or not in $H^*(\mathbb{R}P^2;\mathbb{Z}/2)$.

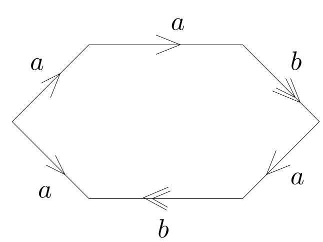
   (b) Let K be the Klein bottle. Write down explicit cocycles which represent generators for $H^*(K;\mathbb{Z}/2)$ and use these to compute all the cup products of these generators.

   (c) If R is a ring then we can extend the Kronecker pairing to be maps $H^k(X;R) \otimes H_k(X;R) \to R$. The adjoint is then a map $\phi_R\colon H^k(X;R) \to \mathrm{Hom}(H_k(X;R),R)$. When X is the Klein bottle and $R = \mathbb{Z}/2$ determine bases for each $H_k(X;R)$ and verify by hand that the maps ϕ are isomorphisms for all k. [We will eventually prove that when $R = \mathbb{Z}/p$ (p a prime) or $R = \mathbb{Q}$, these ϕ_R maps are isomorphisms for every space X.]

4. Prove that there does not exist a map $S^2 \to T$ that induces an isomorphism on H_2. In fact, prove this in two different ways: give a proof that uses homotopy groups and give a proof that uses cohomology and the cup product. (Note that you can use problem #1 to pass from homology information to cohomology information).

5. Prove that $\mathbb{R}P^2$ is not a retract of the Klein bottle (Hint: Examine what happens on cohomology rings with $\mathbb{Z}/2$-coefficients.)

6. Let X be obtained by identifying points on the boundary of a solid hexagon, as indicated in the following diagram:



(a) Calculate the homology and cohomology groups of X with $\mathbb{Z}/2$-coefficients.

(b) Give a Δ-complex structure to X, for example by placing one point in the center of the hexagon, drawing lines to the outer vertices, and orienting the 2-simplices appropriately.

(c) Using your Δ-complex structure, give a 1-cocycle α with $\mathbb{Z}/2$-coefficients having the property that $\alpha(a) = 1$ and $\alpha(b) = 0$. Make sure you give a cocycle, not just a cochain!

(d) Compute $\alpha \cup \alpha$ on all the 2-simplices in your picture. Is $\alpha \cup \alpha$ zero or nonzero in $H^2(X; \mathbb{Z}/2)$? Explain. [For the last part, one approach is to determine by brute force whether your cocycle is a coboundary for the given Δ-complex. This is slightly painful, but it builds character. An alternative approach is to use the Kronecker adjoints $\phi_{\mathbb{Z}/2}$ from #3c, and in particular use that they are always isomorphisms (even though we haven't yet proven that).]