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Course: MATH 616 - Real Analysis
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Homework - Week 5 Exercises
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Exercise 5.1
Find the limit of the integral when $n \to \infty$ with justification.

$$\lim_{n\to\infty} \int_0^n \frac{\cos^2(\frac{x}{n})}{x^2 + \cos^2(\frac{x}{n})} dx$$

Solution: Let

$$f_n(x) = \mathbb{1}_{[0,n]} \frac{\cos^2(\frac{x}{n})}{x^2 + \cos^2(\frac{x}{n})}$$

be a sequence of functions defined on $(0, +\infty)$. It is easy to see that $\lim_{n\to\infty} f_n(x) = \frac{1}{x^2+1}$ for any $x > 0$. Consider the function

$$g(x) = \mathbb{1}_{(0,1)} \cdot 1 + \mathbb{1}_{[1,+\infty)} \cdot \frac{1}{x^2}.$$

For any $x \in [0, 1]$, we have

$$0 \le f_n(x) = \frac{\cos^2(\frac{x}{n})}{x^2 + \cos^2(\frac{x}{n})} \le 1 = g(x).$$

and for any $x \ge 1$, we have

$$0 \le f_n(x) \le \frac{\cos^2(\frac{x}{n})}{x^2 + \cos^2(\frac{x}{n})} \le \frac{1}{x^2} = g(x).$$

So $|f_n(x)| \le g(x)$ for any $x > 0$ and note that

$$\int_0^\infty g(x) dx = \int_0^1 1 dx + \int_1^\infty \frac{1}{x^2} dx = 1 + 1 = 2 < \infty.$$

So $g \in L^1$. By Lebesgue dominated convergence theorem

$$\begin{aligned}
\lim_{n\to\infty} \int_0^n \frac{\cos^2(\frac{x}{n})}{x^2 + \cos^2(\frac{x}{n})} dx &= \lim_{n\to\infty} \int_0^\infty f_n(x) dx \\
&= \lim_{n\to\infty} \int_0^\infty \mathbb{1}_{[0,n]} \frac{\cos^2(\frac{x}{n})}{x^2 + \cos^2(\frac{x}{n})} dx \\
&= \int_0^\infty \lim_{n\to\infty} \mathbb{1}_{[0,n]} \frac{\cos^2(\frac{x}{n})}{x^2 + \cos^2(\frac{x}{n})} dx \\
&= \int_0^\infty \frac{1}{x^2+1} dx \\
&= \lim_{x\to\infty} \arctan x - \arctan 0 \\
&= \frac{\pi}{2}.
\end{aligned}$$

Exercise 5.2

Let f_n be a sequence of real-valued measurable functions in $L^1(\mu)$ on a measure space (X, μ). Suppose $\lim_{n\to\infty} f_n = f$ a.e. and

$$\lim_{n\to\infty} \int_X |f_n| d\mu = \int_X |f| d\mu.$$

Prove that for any measurable set A, we have

$$\lim_{n\to\infty} \int_A |f_n| d\mu = \int_A |f| d\mu.$$

Is that true

$$\lim_{n\to\infty} \int_A f_n d\mu = \int_A f d\mu?$$

Solution: For any measurable set $A \subset X$, we know that $|f_n|$ and $|f|$ are positive measurable functions on A. By Fatou's lemma,

$$\int_X |f| d\mu = \int_X \liminf_{n\to\infty} |f_n| d\mu \leq \liminf_{n\to\infty} \int_X |f_n| d\mu.$$

On the other hand, by Fatou's lemma and use that fact that

$$\lim_{n\to\infty} \int_X |f_n| d\mu = \int_X |f| d\mu,$$

we have

$$\begin{aligned}
0 \leq \int_{X\setminus A} |f| d\mu &= \int_{X\setminus A} \liminf_{n\to\infty} |f_n| d\mu \\
&\leq \liminf_{n\to\infty} \int_{X\setminus A} |f_n| d\mu \\
&= \liminf_{n\to\infty} (\int_X |f_n| d\mu - \int_A |f_n| d\mu) \\
&= \int_X |f| d\mu - \limsup_{n\to\infty} \int_A |f_n| d\mu.
\end{aligned}$$

This implies that

$$\limsup_{n\to\infty} \int_A |f_n| d\mu \leq \int_X |f| d\mu - \int_{X\setminus A} |f| d\mu = \int_A |f| d\mu.$$

Combine this with

$$\int_A |f| d\mu \leq \liminf_{n\to\infty} \int_A |f_n| d\mu,$$

We obtain that for any measurable set $A \subset X$,

$$\lim_{n\to\infty} \int_A |f_n| d\mu = \int_A |f| d\mu.$$

Consider $X = A = \mathbb{R}^1$ and the sequence

$$f_n = \mathbb{1}_{[-n,0)} \cdot (-1) + \mathbb{1}_{(0,n]}.$$

Note that for any n,

$$\int_{\mathbb{R}^1} |f_n| d\mu = 2n < +\infty,$$

So $\{f_n\}$ is a sequence of measurable functions in $L^1(\mu)$ and f_n converges to

$$f = \mathbb{1}_{(-\infty,0)} \cdot (-1) + \mathbb{1}_{(0,+\infty)}$$

almost everywhere and

$$\lim_{n\to\infty} \int_{\mathbb{R}^1} |f_n| d\mu = \lim_{n\to\infty} 2n = +\infty = \int_{\mathbb{R}^1} |f| d\mu.$$

But on the other hand,

$$\int_{\mathbb{R}`1} f_n d\mu = -m([-n,0)) + m((0,n]) = -n + n = 0.$$

So

$$\lim_{n\to\infty} \int_{\mathbb{R}^1} f_n d\mu = 0$$

while

$$\int_{\mathbb{R}^1} f d\mu$$

does not exist because both $\int_{\mathbb{R}^1} f^+ d\mu$ and $\int_{\mathbb{R}^1} f^- d\mu$ are infinite.

Exercise 5,3
Let $E \subset \mathbb{R}^1$ be a measurable set with respect to Lebesgue measure and $m(E) > 0$. Prove that for any $0 < \alpha < 1$, there exists an open interval I such that

$$m(E \cap I) > \alpha m(I).$$

Given $\frac{3}{4} < \alpha < 1$, let I be an open interval with length $b = m(I)$ such that

$$m(E \cap I) > \alpha m(I).$$

Prove that the set

$$E - E := \{x - y : x, y \in E\}$$

contains an open interval $(-\frac{b}{2}, \frac{b}{2})$.

Solution: We first assume $m(E) < \infty$. Assume the opposite that for any open interval $I \subset \mathbb{R}$, we have

$$m(E \cap I) \leq \alpha m(I).$$

By the outer regularity of Lebesgue measure, for any $\varepsilon > 0$, there exists an open set $U \supset E$ such that

$$m(E) < m(U) < m(E) + \varepsilon.$$

We know that the open set $U \subset \mathbb{R}$ can be written as

$$U = \bigcup_{n=1}^{\infty} I_n$$

where I_n are open intervals and $I_i \cap I_j = \varnothing$ if $i \neq j$. Then

$$\begin{aligned} m(E) &= m(E \cap V) \\ &= \sum_{n=1}^{\infty} m(E \cap I_n) \\ &\leq \alpha \sum_{n=1}^{\infty} m(I_n) \\ &\leq \alpha m(V) \\ &< \alpha m(E) + \alpha\varepsilon. \end{aligned}$$

Let $\varepsilon \to 0$, and we obtain that $m(E) \leq \alpha m(E)$, but $0 < \alpha < 1$. A contradiction. Thus, there exists an open interval I such that

$$m(E \cap I) > \alpha m(I).$$

Now assume $m(E) = +\infty$. Then for a large number N, consider the measurable set

$$E' := E \cap (-N, N)$$

with finite measure. We have proved that there exists an open interval I such that

$$m(E' \cap I) > \alpha m(I).$$

If we choose N large enough, then

$$m(E \cap I) \geq m(E' \cap I) > \alpha m(I).$$

Now suppose $\frac{3}{4} < \alpha < 1$. For any point $z \in (-\frac{b}{2}, \frac{b}{2})$, to prove that $z \in E - E$, it is the same as proving there exists $x, y \in E$ such that $x = y + z$, which is the same as

$$E \cap (E + z) \neq \varnothing$$

for any $z \in (-\frac{b}{2}, \frac{b}{2})$. Assume the opposite that there exists $|z| < \frac{b}{2}$ such that

$$E \cap (E + z) = \varnothing.$$

Note that $(E + z) \cap (I + z) = (E \cap I) + z \subset I + z$. Since E does not intersect $E + z$, by the translation invariance of Lebesgue measure,

$$\begin{aligned}
m(I \cup (I+z)) &= 2m((E \cap I) \cup ((E \cap I) + z)) \\
&\geq m(E \cap I) + m((E \cap I) + z) \\
&> \alpha m(I) + \alpha m(I) \\
&= 2\alpha m(I) \\
&> \frac{3}{2} m(I)
\end{aligned}$$

Here $|z| < \frac{b}{2} = \frac{1}{2}m(I)$, so

$$m(I \cup (I+z)) < m(I) + m(I) - \frac{1}{2}m(I) = \frac{3}{2}m(I).$$

This is a contradiction, so $E \cap E + z \neq \varnothing$.