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Course: MATH 636 - Algebraic Topology III
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**Homework 1**
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**Problem 1**
Compute the cohomology groups of the following spaces $X$ by writing down the explicit cochain complexes (e.g. cellular cochain complexes) and computing the cohomology directly. In each case, compute both $H^*(X)$ and $H^*(X;\mathbb{Z}/2)$.

(a) $S^n$

(b) The genus $g$ torus

(c) $\mathbb{R}P^n$ and $\mathbb{C}P^n$

(d) The space $\mathbb{R}P^n\#\cdots\mathbb{R}P^n$ ($g$ copies)

*Solution:*

(a) When $n=0$, $S^0$ is the disjoint union of two points. So the cochain complex $C^*(S^0;\mathbb{Z})$ is given by

$$0\to \hom(\mathbb{Z}^2,\mathbb{Z})\to 0$$

Note that $\hom(\mathbb{Z}^2,\mathbb{Z})\cong\mathbb{Z}^2$, so we have $H^0(S^0)=\mathbb{Z}^2$ and all other cohomology groups vanishes. Similarly, we have $H^0(S^0;\mathbb{Z}/2)=(\mathbb{Z}/2)^2$ and all other cohomology groups of coefficient $\mathbb{Z}/2$ are zero. For $n\geq 1$, consider the cellular chain complex of $S^n$ with only one 0-cell and one $n$-cell.

$$0\to\mathbb{Z}\to\cdots\to\mathbb{Z}\to 0$$

All boundary maps are zero. Apply the functor $\hom(-,\mathbb{Z})$ and we get the cochain complex

$$0\to\mathbb{Z}\to\cdots\to\mathbb{Z}\to 0$$

Similarly, apply $\hom(-,\mathbb{Z}/2)$ and we get the cochain complex with coefficient $\mathbb{Z}/2$

$$0\to\mathbb{Z}/2\to\cdots\to\mathbb{Z}/2\to 0$$

So the cohomology groups for $S^n$ ($n\geq 1$) can be summarized as follows

$$H^i(S^n)=\begin{cases}\mathbb{Z}, & \text{if } i=0,n;\\ 0, & \text{otherwise.}\end{cases} \qquad\Bigg|\qquad H^i(S^n;\mathbb{Z}/2)=\begin{cases}\mathbb{Z}/2, & \text{if } i=0,n;\\ 0, & \text{otherwise.}\end{cases}$$

(b) Let $X$ be the genus $g$ torus. Consider the standard cellular structure on $X$ with one 0-cell, $2g$ 1-cells and one 2-cells. The chain complex is given by

$$0\to\mathbb{Z}\xrightarrow{0}\mathbb{Z}^{2g}\xrightarrow{0}\mathbb{Z}\to 0$$

Apply $\hom(-,\mathbb{Z})$ and $\hom(-,\mathbb{Z}/2)$ respectively and we have the following cochain complexes

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \to 0$$

$$0 \to \mathbb{Z}/2 \xrightarrow{0} (\mathbb{Z}/2)^{2g} \xrightarrow{0} \mathbb{Z}/2 \to 0$$

So the cohomology groups of $X$ can be summarized as follows

$$H^i(X) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2; \\ \mathbb{Z}^{2g}, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases} \qquad \Bigg| \qquad H^i(X;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0, 2; \\ (\mathbb{Z}/2)^{2g} & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

(c) Consider the standard cellular structure on $\mathbb{C}P^n$ with one cell in each even dimension from 0 to $2n$. The chain complex is given by

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to \cdots \to \mathbb{Z} \to 0$$

All boundary maps are zero. Apply $\hom(-,\mathbb{Z})$ and $\hom(-,\mathbb{Z}/2)$ respectively, we have cochain complexes

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to \cdots \to \mathbb{Z} \to 0$$

$$0 \to \mathbb{Z}/2 \to 0 \to \mathbb{Z}/2 \to \cdots \to \mathbb{Z}/2 \to 0$$

So the cohomology groups of $\mathbb{C}P^n$ can be summarized as follows

$$H^i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & \text{if } 0 \le i \le 2n \text{ and } i \text{ even;} \\ 0, & \text{otherwise.} \end{cases}$$

$$H^i(\mathbb{C}P^n;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } 0 \le i \le 2n \text{ and } i \text{ even;} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the standard cellular structure on $\mathbb{R}P^n$ with one cell in each dimension from 0 to $n$. The chain complex is given by

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0, \quad n \text{ even}$$

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0, \quad n \text{ odd}$$

Apply $\hom(-,\mathbb{Z}/2)$, and all boundary maps become 0. Both odd and even cases give us the same cochain complex

$$0 \to \mathbb{Z}/2 \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}/2 \to 0$$

So the cohomology groups with coefficient $\mathbb{Z}/2$ are given by

$$H^i(\mathbb{R}P^n;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0, 1, \ldots, n; \\ 0, & \text{otherwise.} \end{cases}$$

Next, apply $\hom(-,\mathbb{Z})$ to the chain complexes and we get

$$0 \leftarrow \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \cdots \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0, \quad n \ \text{ even}$$

$$0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \cdots \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0, \quad n \ \text{ odd}$$

So the cohomology groups of $\mathbb{R}P^n$ can be summarized as follows

$$H^i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0; \\ \mathbb{Z}/2, & \text{if } \ 1 \le i \le n-1 \ \text{ and } \ i \ \text{ even}; \\ 0, & \text{if } \ 1 \le i \le n-1 \ \text{ and } \ i \ \text{ odd}; \\ \mathbb{Z}/2, & \text{if } \ i = n \ \text{ even}; \\ \mathbb{Z}, & \text{if } \ i = n \ \text{ odd}; \\ 0, & \text{otherwise.} \end{cases}$$

(d) Let $X$ be the connected sum of $g$ copies of $\mathbb{R}P^2$. Consider the standard cellular structure on $X$ with one 0-cell, $g$ 1-cells and one 2-cell. The chain complex is given by

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^g \xrightarrow{0} \mathbb{Z} \to 0$$

The boundary map $d_2$ is given by a matrix $\begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix}$. Apply $\hom(-,\mathbb{Z}/2)$ and the coboundary map becomes 0, so the cochain complex with coefficient $\mathbb{Z}/2$ is given by

$$0 \leftarrow \mathbb{Z}/2 \xleftarrow{0} (\mathbb{Z}/2)^g \xleftarrow{0} \mathbb{Z}/2 \leftarrow 0$$

The cohomology groups of $X$ with coefficient $\mathbb{Z}/2$ are

$$H^i(X;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } \ i = 0, 2; \\ (\mathbb{Z}/2)^g, & \text{if } \ i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Next, apply the functor $\hom(-,\mathbb{Z})$ and we get the cochain complex

$$0 \leftarrow \mathbb{Z} \xleftarrow{\delta_1} \mathbb{Z}^g \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

where $\delta_1$ is given by the row matrix $\begin{pmatrix} 2 & \cdots & 2 \end{pmatrix}$. So the image of $\delta_1$ in $\mathbb{Z}$ is $2\mathbb{Z}$ and $\ker \delta_1 = \mathbb{Z}^{g-1}$. The cohomology groups

$$H^i(X) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0; \\ \mathbb{Z}^{g-1}, & \text{if } \ i = 1; \\ \mathbb{Z}/2, & \text{if } \ i = 2; \\ 0, & \text{otherwise.} \end{cases}$$

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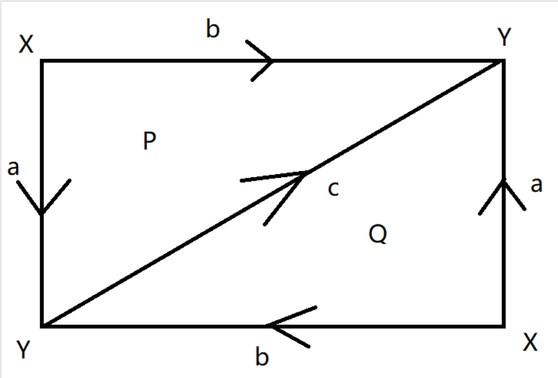
**Problem 2**

The picture below shows a $\Delta$-complex $X$ (note that $X \cong \mathbb{R}P^2$). Note that there are two 0-simplices $X$ and $Y$, three 1-simplices $a, b$ and $c$ and two 2-simplices $P$ and $Q$.




Write $\hat{X}, \hat{Y}$ for the basis of $C^0(X;\mathbb{Z})$ that is dual to $X, Y$, and similarly for $\hat{a}, \hat{b}, \hat{c} \in C^1(X;\mathbb{Z})$ and $\hat{P}, \hat{Q} \in C^2(X;\mathbb{Z})$. In this problem we will also work with $C^*(X;\mathbb{Z}/2)$, and $\hat{P}$ could denote either the cochain in $C^2(X;\mathbb{Z})$ or in $C^2(X;\mathbb{Z}/2)$.

(a) Is $\hat{a} + \hat{b} + \hat{c}$ a cocyle? What about $\hat{a} + \hat{c}$ and $\hat{a} + \hat{b}$?

(b) Are the answers to (a) different in $C^*(X;\mathbb{Z}/2)$?

(c) Cellular cohomology readily shows that $H^1(\mathbb{R}P^2) = 0$, so $H^1(X) = 0$. This means that the cocyle you found in (a) is a coboundary. Find a 0-cochain $\sigma$ such that $\delta(\sigma)$ is your cocycle.

(d) Is $\hat{P}$ a coboundary? What about $\hat{P} + \hat{Q}$ and $\hat{P} + 3\hat{Q}$?

(e) Is it true that $[\hat{P}] = [\hat{Q}]$ in $H^2(X)$?

(f) Find a 1-cocycle that generates $H^1(X;\mathbb{Z}/2)$.

(g) We have the subspace $S^1 \subseteq X$ consisting of the 1-cells $a$ and $b$. We can restrict $\hat{a}$ and $\hat{b}$ to this $S^1$, and we will call those by the same name. Complete the following

$$[p\hat{a} + q\hat{b}] = 0 \text{ in } H^1(S^1) \text{ if and only if} \underline{\hspace{4cm}}$$

(h) Analyze the map $H^1(X;\mathbb{Z}/2) \to H^1(S^1;\mathbb{Z}/2)$. Is it injective? Surjective?

(i) Given a cochain $\alpha \in C^1(X;\mathbb{Z}/2) = \hom(C_1(X), \mathbb{Z}/2)$ and a chain $v \in C_1(X)$ we can evaluate $\alpha$ on $v$ to get an element $\alpha(v) \in \mathbb{Z}/2$. True or false: If $v = [b] + [c]$ then for all cocycles $\alpha, \alpha' \in C^1(X;\mathbb{Z}/2)$ such that $[\alpha] = [\alpha']$ in $H^1(X;\mathbb{Z}/2)$ we have $\alpha(v) = \alpha'(v)$.

(j) Re-do the previous part if $v = [b] - [a]$.

(k) If $[\alpha] = [\alpha']$ in $H^1(X;\mathbb{Z}/2)$ then $\alpha - \alpha' = \delta(\beta)$ for some $\beta \in C^0(X;\mathbb{Z}/2)$. What would you need to know about $v$ in order to show that $\alpha(v) = \alpha'(v)$, no matter what $\beta$ is?

*Solution:*

(a) Let $\partial$ be the differential in the chain complex and $\delta$ be the codifferential in the cochain complex. We can calculate by definition

$$\delta(\hat{a}+\hat{b}+\hat{c})(P) = (\hat{a}+\hat{b}+\hat{c})(\partial P) = (\hat{a}+\hat{b}+\hat{c})(a+c-b) = 1$$

This implies $\delta(\hat{a}+\hat{b}+\hat{c})$ is not the zero map, so it is not a cocycle. Similarly, we have

$$\delta(\hat{a}+\hat{c})(P) = (\hat{a}+\hat{c})(a+c-b) = 2$$

So $\hat{a}+\hat{c}$ is not a cocycle. For $\hat{a}+\hat{b}$, we have

$$\delta(\hat{a}+\hat{b})(P) = (\hat{a}+\hat{b})(a+c-b) = 0$$

$$\delta(\hat{a}+\hat{b})(Q) = (\hat{a}+\hat{b})(b+c-a) = 0$$

We know all the 2-chains are generated by $P$ and $Q$, so this implies $\delta(\hat{a}+\hat{b}) = 0$ for all 2-chains. Thus, $\hat{a}+\hat{b}$ is a cocycle.

(b) Now we take the coefficient to be $\mathbb{Z}/2$. $\hat{a}+\hat{b}$ is still a cocycle and $\hat{a}+\hat{b}+\hat{c}$ is still not a cocycle by the same calculation. For $\hat{a}+\hat{c}$, we have

$$\delta(\hat{a}+\hat{c})(P) = (\hat{a}+\hat{c})(a+c-b) = 0$$

$$\delta(\hat{a}+\hat{c})(Q) = (\hat{a}+\hat{c})(b+c-a) = 0$$

So in this case, $\hat{a}+\hat{c}$ becomes a cocycle.

(c) Suppose we have a 0-cochain $m\hat{X}+n\hat{Y}$ satisfying $\delta(m\hat{X}+n\hat{Y}) = \hat{a}+\hat{b}$. Then we have

$$\begin{aligned}
\delta(m\hat{X}+n\hat{Y})(a) &= (m\hat{X}+n\hat{Y})(Y-X) = n-m = 1,\\
\delta(m\hat{X}+n\hat{Y})(b) &= (m\hat{X}+n\hat{Y})(Y-X) = n-m = 1,\\
\delta(m\hat{X}+n\hat{Y})(c) &= (m\hat{X}+n\hat{Y})(Y-Y) = 0.
\end{aligned}$$

We can choose $n=1$ and $m=0$ as a solution. So $\delta(\hat{Y}) = \hat{a}+\hat{b}$.

(d) Suppose there is a 1-cochain $k\hat{a}+m\hat{b}+n\hat{c}$ satisfying $\delta(k\hat{a}+m\hat{b}+n\hat{c}) = \hat{P}$. Then we have

$$\begin{aligned}
\delta(k\hat{a}+m\hat{b}+n\hat{c})(P) &= (k\hat{a}+m\hat{b}+n\hat{c})(a+c-b) = k+n-m = 1,\\
\delta(k\hat{a}+m\hat{b}+n\hat{c})(Q) &= (k\hat{a}+m\hat{b}+n\hat{c})(b+c-a) = n+m-k = 0.
\end{aligned}$$

Combine two equations together and we have $2n = 1$. There is no solution in $\mathbb{Z}$, so such a 1-cochain does not exist. This implies $\hat{P}$ is not a coboundary. Similarly, for $\hat{P}+\hat{Q}$, we get two equations

$$\begin{aligned}
k+n-m &= 1,\\
m+n-k &= 1.
\end{aligned}$$

Combine two equations and we get $n=1$. So $\hat{P}+\hat{Q}$ is a coboundary and we have $\delta(\hat{c}) = \hat{P}+\hat{Q}$.

For $\hat{P}+3\hat{Q}$, we have two equations

$$k+n-m=1,$$
$$m+n-k=3.$$

Combine two equations and we get $n=2$. So $\hat{P}+3\hat{Q}$ is a coboundary and we have $\delta(\hat{b}+2\hat{c})=\hat{P}+3\hat{Q}$.

(e) This is the same as asking if $[\hat{P}-\hat{Q}]=0$ in $H^2(X)$. Or equivalently, is $\hat{P}-\hat{Q}$ a coboundary? Use the same notation for the previous question and write down the equations

$$k+n-m=1,$$
$$m+n-k=-1.$$

Combine two equations and we get $n=0$. So $\hat{P}-\hat{Q}$ is a coboundary and we have $\delta(\hat{a})=\hat{P}-\hat{Q}$. This proves that $[\hat{P}]=[\hat{Q}]$ in $H^2(X)$.

(f) We know $X\cong\mathbb{R}P^2$, so $H^1(X;\mathbb{Z}/2)\cong\mathbb{Z}/2$ is generated by a non zero element. We have calculated in (b) that $\hat{a}+\hat{c}$ is a cocycle. We need to show that $\hat{a}+\hat{c}$ is not a coboundary. Suppose $m\hat{X}+n\hat{Y}$ is a 0-cochain such that $\delta(m\hat{X}+n\hat{Y})=\hat{a}+\hat{c}$. Then we have

$$\delta(m\hat{X}+n\hat{Y})(a)=(m\hat{X}+n\hat{Y})(Y-X)=1,$$
$$\delta(m\hat{X}+n\hat{Y})(b)=(m\hat{X}+n\hat{Y})(Y-X)=0.$$

We have no solutions for these two equations so $\hat{a}+\hat{c}$ is not a coboundary. This implies that $[\hat{a}+\hat{c}]$ is a nonzero element in $H^1(X;\mathbb{Z}/2)$, so it is a generator.

(g) $[p\hat{a}+q\hat{b}]=0$ in $H^1(S^1)$ is the same as $p\hat{a}+q\hat{b}$ is a 1-coboundary in $C^1(S^1)$. Suppose we have a 0-cochain $m\hat{X}+n\hat{Y}$ such that

$$\delta(m\hat{X}+n\hat{Y})=p\hat{a}+q\hat{b}$$

This is equivalent to

$$\delta(m\hat{X}+n\hat{Y})(a)=(m\hat{X}+n\hat{Y})(Y-X)=m-n=p,$$
$$\delta(m\hat{X}+n\hat{Y})(b)=(m\hat{X}+n\hat{Y})(Y-X)=m-n=q.$$

The two equations having solutions in $\mathbb{Z}$ is equivalent to $p=q$. So we can conclude that

$$[p\hat{a}+q\hat{b}]=0 \text{ in } H^1(S^1) \text{ if and only if } p=q$$

(h) We know $X\cong\mathbb{R}P^2$, so $H^1(X;\mathbb{Z}/2)\cong H^1(S^1;\mathbb{Z}/2)\cong\mathbb{Z}/2$. So the map $i^*:H^1(X;\mathbb{Z}/2)\to H^1(S^1;\mathbb{Z}/2)$ induced by the inclusion $i:S^1\to X$ is either the zero map or an isomorphism. We have already shown in (f) that $\hat{a}+\hat{c}$ is a generator of $H^1(X;\mathbb{Z}/2)$. Note that $S^1$ does not have $c$ as a 1-cell, so $i^*(\hat{a}+\hat{c})=\hat{a}$ in $H^1(S^1;\mathbb{Z}/2)$. By (g), we know that $\hat{a}$ is not a coboundary. This proves that $i^*$ is not a zero map, so $i^*$ is an isomorphism, which is both injective and surjective.

(i) This is false. From (a), (b), (c) and (f), we know that $\hat{a}+\hat{c}$ and $\hat{a}+\hat{b}$ are both cocycles in $C^1(X;\mathbb{Z}/2)$, and $\hat{a}+\hat{b}$ is also a coboundary. So $\hat{a}+\hat{c}+\hat{a}+\hat{b}=\hat{b}+\hat{c}$ represents the same cohomology class as $\hat{a}+\hat{c}$ in $H^1(X;\mathbb{Z}/2)$, namely $[\hat{a}+\hat{c}]=[\hat{b}+\hat{c}]$ in $H^1(X;\mathbb{Z}/2)$. And by calculation, we have

$$(\hat{a}+\hat{c})(v)=(\hat{a}+\hat{c})([b]+[c])=1$$

$$(\hat{b}+\hat{c})(v)=(\hat{b}+\hat{c})([b]+[c])=0$$

This shows the claim is false.

(j) This is true. Suppose $\alpha, \alpha' \in C^1(X;\mathbb{Z}/2)$ are cocycles representing the same cohomology class. Then $\alpha-\alpha'$ is a coboundary. There exists a 0-chain $m\hat{X}+n\hat{Y}$ such that $\delta(m\hat{x}+n\hat{Y})=\alpha-\alpha'$. Then for $v=[b]-[a]$, we have

$$\alpha(v)-\alpha'(v)=(\alpha-\alpha')([b]-[a])=\delta(m\hat{X}+n\hat{Y})([b]-[a])=(m\hat{X}+n\hat{Y})(Y-X-Y+X)=0.$$

(k) We have

$$0=\alpha(v)-\alpha'(v)=\delta(\beta)(v)=\beta(\partial v)$$

for all $\beta\in C^0(X;\mathbb{Z}/2)$. This implies $\partial v=0$, namely $v$ is a coboundary in $C^1(X)$.

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**Problem 3**

For an abelian group $B$, write $B[0]$ for the chain complex which has $B$ in degree zero and zeros everywhere else. Write $\Sigma^n B[0]$ for the result of shifting this complex up $n$ spots, so that it has $B$ in degree $n$.

Let $C_*$ be a chain complex of abelian groups. Prove that $\alpha \in \hom(C_n, B)$ is a cocyle in $\hom(C_*, B)$ if and only if $\alpha : C_* \to \Sigma^n B[0]$ is a chain map. Then prove that the set of chain homotopy classes of maps $[C_*, \Sigma^n B[0]]$ is in bijective correspondence with $H^n(\hom(C_*, B))$.

*Solution:* We divided the proof into two parts. Each part prove one claim.

(a) Let $\alpha\in\hom(C_n,B)$ be a cocycle. We define a chain map

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "0"'] & C_n \arrow[r, "\partial"] \arrow[d, "\alpha"'] & C_{n-1} \arrow[r] \arrow[d, "0"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r, "0"'] & B \arrow[r, "0"'] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

To show that this is a well-defined chain map, we only need to check the commutativity of the two squares above. Recall that $\alpha$ is a cocycle in $\hom(C_*,B)$, so we have

$$0=\delta\alpha=\alpha\circ\partial$$

by definition of the codifferential. This tells us the above two squares commutes. Conversely, suppose we have a well-defined chain map $\alpha : C_*\to\Sigma^n B[0]$, then the commutativity of the square tells us that $\alpha\circ\partial=0$. This is equivalent to $\delta\alpha=0$, so $\alpha$ viewed as a map $\alpha: C_n\to B$ is a cocycle.

(b) From (a), we know that we can define a map

$$\begin{aligned}\phi : H^n(\hom(C_*, B)) &\to [C_*, \Sigma^n B[0]],\\ [\alpha] &\mapsto [\alpha]_h\end{aligned}$$

where $[\alpha]$ is the cohomology class of the cocycle $\alpha$ and $[\alpha]_h$ is the chain homotopy class of the chain map $\alpha$. To show this map is well-defined, consider $\alpha, \alpha'$ are two cocycles representing the same cohomology class, then there exists $\beta \in \hom(C_{n-1}, B)$ such that $\delta(\beta) = \alpha - \alpha'$. Consider the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "0"'] & C_n \arrow[r, "\partial"] \arrow[dl, "0"'] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\alpha'"] & C_{n-1} \arrow[r] \arrow[dl, "\beta"'] \arrow[d, "0"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r, "0"'] & B \arrow[r, "0"'] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Note that

$$\alpha - \alpha' = \delta(\beta) = \beta \circ \partial + 0 \circ 0$$

So $\beta$ gives a chain homotopy between the chain map $\alpha$ and $\alpha'$. This implies $[\alpha]_h = [\alpha']_h$. Conversely, from (a), we can define a map

$$\begin{aligned}\psi : [C_*, \Sigma^n B[0]] &\to H^n(\hom(C_*, B)),\\ [\alpha]_h &\mapsto [\alpha]\end{aligned}$$

By the previous discussion we know that if two chain maps $\alpha, \alpha'$ are chain homotopic, then there exists a map $\beta : C_{n-1} \to B$ such that

$$\beta \circ \partial + 0 \circ 0 = \alpha - \alpha'$$

So $\psi$ is also well-defined. It is easy to check that $\psi \circ \phi = id$ and $\phi \circ \psi = id$. Thus, we can conclude that $[C_*, \Sigma^n b[0]]$ is bijective to $H^n(\hom(C_*, B))$.

**Problem 4**
Consider a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_3 \arrow[r, "d_A"] \arrow[d, "f_3"'] & A_2 \arrow[r, "d_A"] \arrow[d, "f_2"'] & A_1 \arrow[r] \arrow[d, "f_1"'] & 0 \\
0 \arrow[r] & B_3 \arrow[r, "d_B"] \arrow[d, "g_3"'] & B_2 \arrow[r, "d_B"] \arrow[d, "g_2"'] & B_1 \arrow[r] \arrow[d, "g_1"'] & 0 \\
0 \arrow[r] & C_3 \arrow[r, "d_C"] \arrow[d] & C_2 \arrow[r, "d_C"] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which every row and every column is a chain complex. Assume that all the homology groups in each row are zero except for $H_3(C)$. Also assume that all the homology groups of the columns are zero except for the kernel of the map labelled $f_1$. Prove that $\ker f_1 \cong H_3(C)$.

*Solution:* We first define a map $\phi : \ker f_1 \to H_3(C)$. Let $a \in \ker f_1$, the exactness implies that $d_A : A_2 \to A_1$ is surjective, so there exists $a' \in A_2$ such that $d_A(a') = a$. We know that $f_2(a') \in B_2$ and by commutativity of the square we have

$$d_B(f_2(a')) = f_1(d_A(a')) = f_1(a) = 0$$

since $a \in \ker f_1$. This implies $f_2(a') \in \ker d_B$ and by exactness there exists a unique $b \in B_3$ such that $d_B(b) = f_2(a')$. We define

$$\begin{aligned}\ker f_1 &\to C_3,\\ a &\mapsto g_3(b).\end{aligned}$$

Note that $d_C(g_3(b)) = g_2(d_B(b)) = g_2(f_2(a')) = 0$ by exactness of the middle column. So we obtain a map $\phi : \ker f_1 \to H_3(C)$ because $C_4 = 0$. We need to check the map is well defined.

We check that the map $\phi$ does not depend on the choice of $a' \in A_2$. Suppose $a', a'' \in A_2$ with $a = d_A(a') = d_A(a'')$. Then $a' - a'' \in \ker d_A$ and by exactness, there exists $c \in A_3$ such that $d_A(c) = a' - a''$. This implies

$$f_2(a' - a'') = f_2(d_A(c)) = d_B(f_3(c))$$

Suppose $b, b' \in B_3$ such that $d_B(b) = f_2(a')$ and $d_B(b') = f_2(a'')$. Then we have

$$d_B(b - b') = f_2(a' - a'') = d_B(f_3(c))$$

Because the middle row is exact, so $d_B$ is injective. This implies $b - b' = f_3(c)$. Then

$$g_3(b) - g_3(b') = g_3(f_3(c)) = 0$$

To see this is a group homomorphism, consider $x, y \in C_3$, we need to show that $\phi(x + y) =$

$\phi(x) + \phi(y)$. Note that we can choose $x', y', x' + y' \in A_2$ as the preimage of $x, y, x + y$ since the choice does not matter in the final result. All other processes are completely determined by a group homomorphism, so $\phi$ is indeed a group homomorphism.

We define another map $\psi : H_3(C) \to \ker f_1$. Rewrite the whole diagram in the following way

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_3 \arrow[r, "f_3"] \arrow[d, "d_A"] & B_3 \arrow[r, "g_3"] \arrow[d, "d_B"] & C_3 \arrow[r] \arrow[d, "d_C"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "f_2"] \arrow[d, "d_A"] & B_2 \arrow[r, "g_2"] \arrow[d, "d_B"] & C_2 \arrow[r] \arrow[d, "d_C"] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d] & B_1 \arrow[r, "g_1"] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Note that we still have exactness everywhere except for $C_3$ vertically and $A_1$ horizontally. Following the same argument as above we can define a map $\psi : H_3(C) \to \ker f_1$. By comparison with defintiion of these maps, we can see that $\psi \circ \phi = id$ and $\phi \circ \psi = id$. This proves that $\phi$ is an isomorphism and $\ker f_1 \cong H_3(C)$.