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Homework - Chapter 2 Exercises
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Course: MATH 682 - Algebraic Geometry II
Term: Winter 2026
Instructor: Professor Nick Addington
Due Date: Jan 16, 2026

Exercise 2.7.2
Describe $\operatorname{Spec}\mathbb{Z}[\frac{1}{18}]$.

Solution: Note that we have a ring isomorphism $\mathbb{Z}[\frac{1}{18}] \cong \mathbb{Z}[x]/(18x-1)$. Let $R = \mathbb{Z}[x]/(18x-1)$. We need to describe $\operatorname{Spec} R$. Consider the ring homomorphism

$$f : \mathbb{Z}[x] \to R.$$

given by the quotient map. We know that prime ideals in R corresponds to prime ideals in $\mathbb{Z}[x]$ containing the ideal $(18x-1)$. It must be of the form $(p, 18x-1)$ where $p \in \mathbb{Z}$ is a prime number. If $p = 2$ or $p = 3$, then $(p, 18x-1) = \mathbb{Z}[x]$, which is not an ideal. When $p \neq 2, 3$, the ideal $(p, 18x-1)$ is a prime ideal in $\mathbb{Z}[x]$, thus corresponds to a prime ideal of R. So $\operatorname{Spec} R$ has closed points corresponds to the maximal ideal $(p, 18x-1)$ in R where $p \neq 2, 3$, and a generic point corresponds to the zero ideal in R (or the ideal $(18x-1)$ in $\mathbb{Z}[x]$).

Exercise 2.7.8
Let A be a Noetherian ring. Show that $X = \operatorname{Spec} A$ is a finite set, and the topology is the discrete topology if and only if A is an Artinian ring.

Solution: X having the discrete topology means that for any $x \in X$, the set $\{x\}$ is closed, so this is equivalent to that every prime ideal in A is maximal. We first prove a very useful claim.
<u>Claim:</u> Suppose in a ring A, the zero ideal (0) can be written as the product of finitely many maximial ideals, then A is Noetherian if and only id A is Artinian.
<u>Proof:</u> Suppose $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$ where $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ are maximal ideals. Consider the following finite sequence

$$A \supset \mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1 \cdots \mathfrak{m}_n = (0).$$

For any $1 \leq t \leq n$, the factor $\mathfrak{m}_1 \cdots \mathfrak{m}_t / \mathfrak{m}_1 \cdots \mathfrak{m}_{t+1}$ is a vector space over the field $A/\mathfrak{m}_{t+1}$. ■

Exercise 2.7.9
Show that $D(f) = \varnothing$ if and only if f is nilpotent.

Solution: Suppose f is nilpotent. Then there exists $n \geq 1$ such that $f^n = 0 \in \mathfrak{p}$ for any prime ideal $\mathfrak{p} \subset A$. So

$$D(f) = \{\mathfrak{p} \text{ prime} \mid f \notin \mathfrak{p}\} = \varnothing.$$

Conversely, suppose $D(f) = \varnothing$. This implies that $f \in \mathfrak{p}$ for all prime ideal $\mathfrak{p}$, so $f \in \cap_{\mathfrak{p} \text{ prime}} \mathfrak{p} = \sqrt{(0)}$. There exists $n \geq 1$ such that $f^n = 0$. So f is nilpotent.

Exercise 2.7.10
Show that the ideal $\mathfrak{m} = (x, y-1)$ in $A = \mathbb{R}[x,y]/(x^2+y^2-1)$ is not principal.

Solution: We calculate that

$$\mathfrak{m}^2 = (x^2, (y-1)^2, x(y-1)) = (1-y^2, x(y-1), (y-1)^2).$$

It is easy to see that $\mathfrak{m}^2 \subset (y-1)$. Conversely, note that

$$(y^2-1) - (y-1)^2 = (y-1)(y+1-y+1) = 2(y-1).$$

so $y - 1 \in \mathfrak{m}^2$. This implies that $\mathfrak{m}^2 = (y-1)$ is principal. The elements in A can be written as $f(x) + g(x)y$. Suppose $\mathfrak{m} = (f(x) + g(x)y)$ is principal. Then

$$\mathfrak{m}^2 = ((f(x) + g(x)y)^2) = (y-1).$$

Thus, $y - 1$ and $(f(x) + g(x)y)^2$ differ by a unit in $\mathbb{R}$. By choosing $f(x) + g(x)y$ properly, we have

$$(f(x) + g(x)y)^2 = y - 1.$$

By checking the norms $N(f(x) + g(x)y) = f(x)^2 + g(x)^2(x^2-1)$, we have

$$N(y-1) = 1 + x^2 - 1 = x^2$$

Note that norm N is multiplicative, so we need to find an element with norm x. Such element does not exist. So $(x, y-1)$ is not principal.