Zhengdong Zhang
Email: zhengz@uoregon.edu
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Instructor: Dr.Daniel Dugger

Homework 7
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Problem 1

If M and N are compact, oriented d-manifold, then the **degree** of a map $f : M \to N$ is defined to be the integer $\deg(f)$ such that $f_*([M]) = \deg f \cdot [N]$.

(a) Suppose that f is not surjective–i.e., there is a point $x \in N$ such that x is not in the image of f. Prove that the degree of f is zero.

(b) Explain how $\deg(f)$ relates to the map $f^* : H^d(N) \to H^d(M)$.

(c) Prove that any map $S^4 \to \mathbb{C}P^2$ must have degree 0.

Solution:

(a) f being not surjective means that there exists a point $y \in N$ such that $f(M) \subseteq N - y$. This implies we have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "f"'] & N - y \arrow[d, hook, "i"] \\ & N \end{tikzcd}$$

where $i : N - y \hookrightarrow N$ is the inclusion map. This induces a commutative diagram in homology groups

$$\begin{tikzcd} H_d(M) \arrow[r, "f_*"] \arrow[dr, "f_*"'] & H_d(N - y) \arrow[d, tail, "i_*"] \\ & H_d(N) \arrow[d, "j_*"] \\ & H_d(N, N - y) \end{tikzcd}$$

The map $j_* : H_d(N) \to H_d(N, N - y)$ is an isomorphism because N is compact and oriented. We know that $j_* \circ i_*$ is the zero map because of the exactness on the vertical map. This implies that f_* is the zero map, so $\deg f = 0$.

(b) By UCT, we have a commutative diagram

$$\begin{tikzcd} H^d(N) \arrow[r, two heads] \arrow[d, "f^*"'] & \hom(H_d(N), \mathbb{Z}) \arrow[d, "\deg f"] \\ H^d(M) \arrow[r, two heads] & \hom(H_d(M), \mathbb{Z}) \end{tikzcd}$$

Both M and N are compact and oriented, so $H_d(M) \cong H_d(N) \cong \mathbb{Z}$. The right vertical map is induced by the map $f_* : H_d(M) \to H_d(N)$, so it is also the multiplication by $\deg f$. By

Poincaré duality, we know that

$$H^d(M) \cong H_0(M) \cong \mathbb{Z}, \quad H^d(N) \cong H_0(N) \cong \mathbb{Z}.$$

This implies the top and bottom horizontal maps in the commutative diagram are isomorphisms. Therefore, if we choose $\widehat{[M]}$ to be the generator of $H^d(M) \cong \mathbb{Z}$ and $\widehat{[N]}$ to be the generator of $H^d(N) \cong \mathbb{Z}$, then the map $f^* : H^d(N) \to H^d(M)$ is sending $\widehat{[N]}$ to $\deg f \cdot \widehat{[M]}$.

(c) Consider a map $f : S^4 \to \mathbb{C}P^2$, which induces a map between cohomology rings

$$f^* : H^*(\mathbb{C}P^2) \to H^*(S^4).$$

We know that $H^*(\mathbb{C}P^2) \cong \mathbb{Z}[x]/(x^3)$ where x is a degree 2 element, then when $* = 4$, we have

$$f^*(x^2) = f^*(x) \cup f^*(x) = 0.$$

because $H^*(S^4)$ has no degree 2 element. This means that

$$f^* : H^4(\mathbb{C}P^2) \to H^4(S^4)$$

is the zero map. We know that both $\mathbb{C}P^2$ and S^4 are compact and orientable. From the discussion in (b), any map $f : S^4 \to \mathbb{C}P^4$ has degree 0.

Problem 2
A topological space is said to be of **finite type** if $H_i(X) = 0$ for all but finitely many values of i, and each nonzero $H_i(X)$ is a finitely-generated abelian group. Recall that the Euler characteristic is then defined to be

$$\chi(X) = \sum_{i=1}^{\infty} (-1)^i \operatorname{rank} H_i(X).$$

Prove that if X and Y are CW-complexes of finite type then so is $X \times Y$, and $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

Solution: By Künneth Theorem, for all i, we have

$$H_i(X \times Y) \cong \sum_{p+q=i} H_p(X) \otimes H_q(Y) \oplus \sum_{p+q=i-1} \operatorname{Tor}_1(H_p(X), H_q(Y)).$$

Since both X and Y are CW-complexes of finite type, only finitely many $H_p(X)$ and $H_q(X)$ are non-zero, this implies $H_i(X \times Y) = 0$ except for finitely many i. Moreover, if Abelian groups A and B are finitely generated, then we know that $A \otimes B$ and $\operatorname{Tor}_1(A, B)$ are also finitely generated. This means for all i, $H_i(X \times Y)$ is finitely generated Abelian group.

We know that for any Abelian group A,

$$\operatorname{rank} A = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q}).$$

Suppose $H_i(X) = 0$ for $i \geq n+1$ and $H_j(Y) = 0$ for $j \geq m+1$. Because $\mathbb{Q}$ is a field, for the space $X \times Y$, by Künneth Theorem, we have

$$\begin{aligned}
\chi(X \times Y) =& \sum_{i=1}^{m+n} (-1)^i \operatorname{rank} H_i(X \times Y) \\
=& \sum_{i=1}^{m+n} (-1)^i \dim_{\mathbb{Q}} H_i(X \times Y; \mathbb{Q}) \\
=& \sum_{i=1}^{m+n} (-1)^i \sum_{p+q=i} \dim_{\mathbb{Q}} H_p(X; \mathbb{Q}) \otimes H_q(Y; \mathbb{Q}) \\
=& \sum_{i=1}^{m+n} (-1)^i \sum_{p+q=i} (\dim_{\mathbb{Q}} H_p(X; \mathbb{Q}) \cdot \dim_{\mathbb{Q}} H_q(Y; \mathbb{Q})) \\
=& \sum_{i=1}^{m+n} \sum_{p+q=i} (-1)^p \dim_{\mathbb{Q}} H_p(X; \mathbb{Q}) \cdot (-1)^q \dim_{\mathbb{Q}} H_q(Y; \mathbb{Q}) \\
=& (\sum_{p=1}^{n} (-1)^p \dim_{\mathbb{Q}} H_p(X; \mathbb{Q})) \cdot (\sum_{q=1}^{m} (-1)^q \dim_{\mathbb{Q}} H_q(Y; \mathbb{Q})) \\
=& \chi(X) \cdot \chi(Y)
\end{aligned}$$

Problem 3
Prove that $\mathbb{C}P^{n-1}$ is not a retract of $\mathbb{C}P^n$.

Solution: Suppose there exists a retract $r : \mathbb{C}P^n \to \mathbb{C}P^{n-1}$ such that the composition

$$\mathbb{C}P^{n-1} \xrightarrow{i} \mathbb{C}P^n \xrightarrow{r} \mathbb{C}P^{n-1}$$

is the identity map, where $i : \mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^{n-1}$ is the inclusion map. This induces maps between cohomology rings

$$H^*(\mathbb{C}P^{n-1}) \xrightarrow{r^*} H^*(\mathbb{C}P^n) \xrightarrow{i^*} H^*(\mathbb{C}P^{n-1})$$

where $i^* \circ r^* = id$. Note that $H^*(\mathbb{C}P^{n-1}) \cong \mathbb{Z}[x]/(x^n)$ and $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[y]/(y^{n+1})$. Suppose $r^*(x) = ky \in H^2(\mathbb{C}P^n)$ for some $k \in \mathbb{Z}$. We have

$$0 = r^*(x^n) = r^*(x)^n = (ky)^n = k^n y^n.$$

We know that $0 \neq y^n$ is the generator of $H^{2n}(\mathbb{C}P^n) \cong \mathbb{Z}$. Thus, $k = 0$. This means r^* is the zero map, which contradicts the assumption that $i^* \circ r^* = id$. Such retract r does not exist.

Problem 4
Prove that there is no self-homeomorphism $\mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$ that reverses the orientation.

Solution: Suppose $f : \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$ is a homeomorphism and f induces a map between cohomology rings

$$f^* : H^*(\mathbb{C}P^{2n}) \to H^*(\mathbb{C}P^{2n})$$

which reverses the orientation. We know that $H^(\mathbb{C}P^{2n}) \cong \mathbb{Z}[x]/(x^{2n+1})$, and x^{2n} generates the group $H^{4n}(\mathbb{C}P^{2n})$. f reversing the orientation means $f^*(x^{2n}) = -x^{2n}$. Assume $f^*(x) = kx \in H^2(\mathbb{C}P^{2n})$ for some $k \in \mathbb{Z}$. Then

$$-x^{2n} = f^*(x^{2n}) = f^*(x)^{2n} = (kx)^{2n} = k^{2n}x^{2n}.$$

This implies $k^{2n} = -1$. No such k exists in $\mathbb{Z}$. Thus, such homeomorphism f does not exist.

There is an algebraic formula

$$(x_1^2 + x_2^2) \cdot (y_1^2 + y_2^2) = (x_1y_1 - x_2y_2)^2 + (x_1y_2 + x_2y_1)^2 \tag{1}$$

which is true for indeterminates x_1, x_2, y_1, y_2 over $\mathbb{R}$. By a **sums of-squares formula** of type $[r, s, n]$ we mean an identity of the form

$$(x_1^2 + x_2^2 + \cdots + x_r^2) \cdot (y_1^2 + y_2^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2.$$

where each z_i is a bilinear expression in the x's and y's. The identity (1) was a formula of type $[2, 2, 2]$. Here is a formula of type $[4, 4, 4]$:

$$\begin{aligned}(x_1^2 + x_2^2 + x_3^2 + x_4^2) \cdot (y_1^2 + y_2^2 + y_3^2 + y_4^2) = &\quad (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4)^2 \\ &= + (x_1y_2 + x_2y_1 - x_3y_4 + x_4y_3)^2 \\ &= + (x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2)^2 \\ &= + (-x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1)^2.\end{aligned}$$

If you try to generalize these examples you will find a formula of type $[8, 8, 8]$, but not one of type $[16, 16, 16]$.

Problem 5

If we have a sums-of-squares formula of type $[r, s, n]$ then we get a bilinear map $\phi : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ such that $\|\phi(x, y)\|^2 = \|x\|^2 \cdot \|y\|^2$ by defining

$$\phi(x_1, \ldots, x_r, y_1, \ldots, y_s) = (z_1, \ldots, z_n)$$

using the bilinear expression z_i.

(a) Explain why ϕ restricts to a map $S^{r-1} \times S^{s-1} \to S^{n-1}$, and then induces a map

$$F : \mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \to \mathbb{R}P^{n-1}.$$

(b) Use singular cohomology to prove that if an $[r, s, n]$ formula exists then $\binom{n}{i}$ must be even for $n - r < i < s$.

(c) With some trouble one can discover a sums-of-squares formula of type $[10, 10, 16]$. Does there exist a better formula of type $[10, 10, 15]$¿

Solution:

(a) Suppose $x \in S^{r-1} \subseteq \mathbb{R}^r$ and $y \in S^{s-1} \subseteq \mathbb{R}^s$. This implies that $\|x\| = \|y\| = 1$. Then

$$\|\phi(x, y)\| = \|x\| \cdot \|y\| = 1 \cdot 1 = 1.$$

So $\phi(x, y) \in S^{n-1}$. This means the map ϕ can be restricted to a map

$$\phi : S^{r-1} \times S^{s-1} \to S^{n-1}.$$

Moreover, for any point $(x, y) \in S^{r-1} \times S^{s-1}$, we have

$$\phi(-x, y) = \phi(x, -y) = -\phi(x, y)$$

because ϕ is bilinear. This means we can identify the antipodal points in each sphere, and obtain a map

$$F : \mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \to \mathbb{R}P^{n-1}.$$

The map F is continuous because the bilinear map ϕ is continuous and F is induced by the quotient map from sphere to the real projective space.

(b) The map F induces a map between cohomology rings with $\mathbb{Z}_2$-coefficients. Using Künneth Theorem, we have a map

$$F^* : H^*(\mathbb{R}P^{n-1}; \mathbb{Z}_2) \to H^*(\mathbb{R}P^{r-1}; \mathbb{Z}_2) \otimes H^*(\mathbb{R}P^{s-1}; \mathbb{Z}_2).$$

This is a map between $\mathbb{Z}/2$-algebras

$$F^* : \mathbb{Z}_2[x]/(x^n) \to \mathbb{Z}_2[y]/(y^r) \otimes \mathbb{Z}_2[z]/(z^s)$$

sending the generator x to $k(y \otimes 1) + l(1 \otimes z)$ for some $k, l \in \mathbb{Z}_2$.

Claim: $k \neq 0$ and $l \neq 0$, namely $k = l = 1$.

Proof: Choose a point $a \in S^{s-1}$ and consider the inclusion map $i : \mathbb{R}^r \hookrightarrow \mathbb{R}^r \times \{a\} \subseteq \mathbb{R}^r \times \mathbb{R}^s$. The composition

$$\mathbb{R}^r \xrightarrow{i} \mathbb{R}^r \times \mathbb{R}^s \xrightarrow{\phi} \mathbb{R}^n$$

is an $\mathbb{R}$-linear map and for all $x \in \mathbb{R}^r$, we have

$$\|(\phi \circ i)(x)\| = \|\phi(x, a)\| = \|x\|^2 \cdot \|a\|^2 = \|x\|^2.$$

meaning that it preserves the norm. This implies $r \leq n$, otherwise the kernel of the map $\phi \circ i$ must be non-zero, and a non-zero element will be mapped to 0, which contradicts the fact that $\phi \circ i$ preserves the norm. Write $g := \phi \circ i$. g can be restrected to a map $S^{r-1} \to S^{n-1}$ and since g is $\mathbb{R}$-linear, it induces a map $G : \mathbb{R}P^{r-1} \to \mathbb{R}P^{n-1}$. To prove $k \neq 0$, it is the same as proving the map induced by g between cohomology rings is not the zero map.

We know $g : \mathbb{R}^r \to \mathbb{R}^n$ with $r \leq n$ is a injective $\mathbb{R}$-linear map since $\ker g = 0$. There exists an invertible matrix $T \in GL_n(\mathbb{R})$ such that the composition

$$\mathbb{R}^r \xrightarrow{g} \mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$$

is an inclusion map, namely $\mathbb{R}^r$ is mapped into the first r coordinates in $\mathbb{R}^n$. Note that every map here is $\mathbb{R}$-linear and injective, this induces a map between real projective spaces

$$\mathbb{R}P^{r-1} \xrightarrow{G} \mathbb{R}P^{n-1} \xrightarrow{t} \mathbb{R}P^{n-1}$$

where t is a homeomorphism as it is induced from an invertible matrix. The composition $t \circ G$ is the inclusion of $(r-1)$-skeleton inside $\mathbb{R}P^{n-1}$ because it is induced from the embedding $T \circ g : \mathbb{R}^r \hookrightarrow \mathbb{R}^n$. Now we have maps between cohomology rings

$$H^*(\mathbb{R}P^{n-1}; \mathbb{Z}_2) \xrightarrow{t^*} H^*(\mathbb{R}P^{n-1}; \mathbb{Z}_2) \xrightarrow{G^*} H^*(\mathbb{R}P^{r-1}; \mathbb{Z}_2).$$

We know here t^* is the identity map between cohomology rings as t is a homeomorphism, and $G^* \circ t^*$ is surjective because $t \circ G : \mathbb{R}P^{r-1} \hookrightarrow \mathbb{R}P^{n-1}$ is the inclusion of $(r-1)$-skeleton. This implies $k \neq 0$ and $k = 1$ since we are working $\mathbb{Z}_2$-coefficients. A similar argument implies $l = 1$. ■

Going back to the map

$$F^* : \mathbb{Z}_2[x]/(x^n) \to \mathbb{Z}_2[y]/(y^r) \otimes \mathbb{Z}_2[z]/(z^s)$$

sending x to $y \otimes 1 + 1 \otimes z$. We have a relation

$$0 = F^*(x^n) = F^*(x)^n = (y \otimes 1 + 1 \otimes z)^n = \sum_{i=0}^{n} \binom{n}{i} y^{n-i} \otimes z^i.$$

in the field $\mathbb{Z}_2$. If $n - r < i < s$, we have $1 \leq i < s$ and $1 \leq n - i < r$, this means $y^{n-i} \otimes z^i \neq 0$ and $\binom{n}{i} = 0$ in $\mathbb{Z}_2$. So if such a formula $[r, s, n]$ exists, then $\binom{n}{i}$ must be even for $n - r < i < s$.

(c) If a formula of type $[10, 10, 15]$ exists, then for $15 - 10 < 6 < 10$, we have that $\binom{15}{6}$ equals to 5005, which is an odd number, so such a formula does not exist.

Problem 6

Suppose $p(x)$ is an irreducible polynomial over $\mathbb{C}$ of degree n, where $n > 1$. Let $E = \mathbb{C}[x]/(p(x))$, which is an algebraic field extension of $\mathbb{C}$ of degree n. Choose a vector space isomorphism $\mathbb{C}^n \cong E$, so that the multiplication on E becomes a bilinear map $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$.

Using singular cohomology rings of appropriate topological spaces, derive a contradiction.

Solution: The multiplication $\mu : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is non-degenerate because it is coming from a multiplication in a field E. Since the map μ is $\mathbb{C}$-bilinear, for any $\lambda \in \mathbb{C}^*$ and $(x, y) \in \mathbb{C}^n \times \mathbb{C}^n$, we have

$$\mu(\lambda x, y) = \mu(x, \lambda y) = \lambda \mu(x, y).$$

This means for any two complex lines $l_1, l_2 \subseteq \mathbb{C}^n$ passing through the origin, they are sent to another line passing through the origin under the map μ. So μ induces a map between complex projective spaces

$$f : \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}.$$

By Künneth Theorem, this further induces a map between cohomology rings

$$f^* : H^*(\mathbb{C}P^{n-1}) \to H^*(\mathbb{C}P^{n-1}) \otimes H^*(\mathbb{C}P^{n-1}).$$

We know that $H^*(\mathbb{C}P^{n-1}) \cong \mathbb{Z}[x]/(x^n)$, so f^* is a map between $\mathbb{Z}$-algebras

$$f^* : \mathbb{Z}[x]/(x^n) \to \mathbb{Z}[y]/(y^n) \otimes \mathbb{Z}[z]/(z^n)$$

sending x to $k(y \otimes 1) + l(1 \otimes z)$ for some $k, l \in \mathbb{Z}$. A similar argument with Problem 5(b) implies that k and l are not zero. So we have

$$0 = f^*(x^n) = f^*(x)^n = (k(y \otimes 1) + l(1 \otimes z))^n = \sum_{i=0}^{n} \binom{n}{i} k^i l^{n-i} y^i \otimes z^{n-i}.$$

Since k, l is non-zero, this implies $\binom{n}{i}$ needs to be 0 for $1 \leq i \leq n-1$, so $n = 1$. This contradicts the assumption that the degree of $p(x)$ is larger than 1.