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Course: MATH 635 - Algebraic Topology II
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Homework 4
ID: 952091294
Term: Winter 2025
Due Date: 6^{th} February, 2025

Problem 1
Prove that every map $\mathbb{R}P^6 \to S^6$ is homotopic to a map that is constant on the subspace $\mathbb{R}P^5 \subseteq \mathbb{R}P^6$.

Solution: Given a map $f : \mathbb{R}P^6 \to S^6$, by CAT, f is homotopic to a cellular map $f' : \mathbb{R}P^6 \to S^6$. Note that S^6 has only one 0-cell and one 6-cell, so the 5-skeleton of S^6 is a 0-cell, while the 5-skeleton of $\mathbb{R}P^6$ is homeomorphic to $\mathbb{R}P^5$. the cellularity implies that f' is constant on $\mathbb{R}P^5 \subseteq \mathbb{R}P^6$.

Problem 2
If S_1 and S_2 and S_3 are pointed set, then a sequence $S_1 \xrightarrow{f} S_2 \xrightarrow{g} S_3$ is said to be exact (in the middle spot) if $\operatorname{Im} f = g^{-1}(*)$.

Let (X, A) be a relative CW complex, and choose a basepoint of A (also regard as a basepoint of X). Use HEP to prove that for any pointed space Z, the evident sequence

$$[X/A, Z]_* \xrightarrow{f} [X, Z]_* \xrightarrow{g} [A, Z]_*$$

is exact in the middle spot. Here $[-,-]_*$ denotes homotopy classes of maps relative to the basepoint.

Solution: Let $\pi : X \to X/A$ be the quotient map. Take $\alpha : X/A \to Z$ be a pointed map. We know by definition that $f([\alpha]) = [\alpha \circ \pi]$. And

$$(g \circ f)([\alpha]) = g([\alpha \circ \pi]) = [\alpha \circ \pi|_A].$$

We know that the map $\alpha \circ \pi$ factors through X/A, this means that it sends every point in A to the base point in Z. So we have $\operatorname{Im} f \subset g^{-1}(*)$.

On the other hand, consider $\beta : X \to Z$ with the property that $\beta|_A$ is homotopic to the constant map. We need to show that there exists $\gamma : X/A \to Z$ such that $f([\gamma]) = [\beta]$. β being homotopic to the constant map C_* implies there exists $h : A \times I \to Z$ such that $h(-,0) = \beta|_A$ and $h(-,1) = C_*$. We have the following diagram:

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, "\beta \cup h"] \arrow[d] & Z \\ X \times I \arrow[ur, dashed, "\exists H"'] & \end{tikzcd}$$

By HEP, we have a homotopy $H : X \times I \to Z$ such that $H(-,0) = \beta$. Take $\delta = H(-,1) : X \to Z$, since H is extended from h, we know for any $x \in A$, $\delta(x) = H(x,1) = h(x,1) = *$ is the constant map. So δ factors through X/A, namely there exists $\gamma : X/A \to Z$ such that $\gamma \circ \pi = \delta$. This is the same as saying $f([\gamma]) = [\beta]$. We have proved that $g^{-1}(*) \subset \operatorname{Im} f$. Thus, we can conclude that $\operatorname{Im} f = g^{-1}(*)$.

Problem 3

Let $X_1 \hookrightarrow X_2 \hookrightarrow$ be a sequence of CW inclusions (each (X_i, X_{i-1}) is a relative CW complex). Let $X = \operatorname{colim}_n X_n$. Each inclusion $X_i \hookrightarrow X$ induces maps $[X, Z] \to [X_i, Z]$ for any space Z, and together these yield a map $\phi : [X, Z] \to \lim_n [X_n, Z]$. Use HEP to prove that ϕ is surjective.

Solution: For the limit, we have te following diagram:

$$\begin{tikzcd}
 & & \lim_n[X_n, Z] \arrow[dll, "p_1"'] \arrow[dl, "p_2"] \arrow[dr, "p_n"] & & \\
{[X_1, Z]} & {[X_2, Z]} \arrow[l, "j_2"] & \cdots \arrow[l, "j_3"] & {[X_n, Z]} \arrow[l, "j_n"] & \cdots \arrow[l, "j_{n+1}"]
\end{tikzcd}$$

Take an element $s \in \lim_n [X_n, Z]$, we denote $s_i := p_i s \in [X_i, Z]$. Let $k_i : X_i \hookrightarrow X_{i+1}$ be the inclusion of ith skeleton into $(i+1)$th skeleton of X. By the commutativity of the above diagram, we know that $j_2([s_2]) = [s_2 \circ k_1] = [s_1]$. There exists a homotopy $h_1 : X_1 \times I \to Z$ such that $h_1(-, 1) = s_1(-)$ and $h_2(-, 0) = (s_2 \circ k_1)(-)$. Consider the following diagram:

$$\begin{tikzcd}
X_2 \times \{0\} \cup X_1 \times I \arrow[r, "s_2 \cup h_1"] \arrow[d] & Z \\
X_2 \times I \arrow[ur, dashed, "\exists h_2"'] &
\end{tikzcd}$$

Note that for any $x \in X_1$, $h_1(x, 0) = (s_2 \circ k_1)(x)$. By HEP, we have a homotopy $h_2 : X_2 \times I \to Z$ such that $h_2(-, 0) = s_2(-)$ and for any $x \in X_1$, $h_2(x, 1) = h_1(x, 1) = s_1(x)$, this means we have a commutative diagram:

$$\begin{tikzcd}
X_1 \arrow[r, hook, "k_1"] \arrow[d, "s_1"'] & X_2 \arrow[dl, "h_2(-,1) \simeq s_2"] \\
Z &
\end{tikzcd}$$

We can construct $h_3, h_4, \ldots$ consecutively in this way and obtain a diagram as follows

$$\begin{tikzcd}
X_1 \arrow[r, "k_1"] \arrow[d, "s_1"'] & X_2 \arrow[r, "k_2"] \arrow[dl, "h_2(-,1)"] & \cdots \arrow[r, "k_{n-1}"] & X_n \arrow[r, "k_n"] \arrow[dlll, "h_n(-,1)"] & \cdots \\
Z & & & &
\end{tikzcd}$$

where for any $1 \leq i$, $h_i(-, 1)$ is homotopic to s_i. By the universal property of $X = \operatorname{colim}_n X_n$, we have a unique map $f : X \to Z$ such that the following diagram commutes:

$$\begin{tikzcd}
X_1 \arrow[r, hook, "k_1"] \arrow[drr, bend right, "q_1"] \arrow[ddrr, bend right, "h_1(-,1)"'] & X_2 \arrow[r, hook, "k_2"] \arrow[dr, "q_2"] \arrow[ddr, bend right, "h_2(-,1)"] & \cdots \arrow[r, hook, "k_{n-1}"] & X_n \arrow[r, hook, "k_n"] \arrow[dl, "q_n"'] \arrow[ddl, "h_n(-,1)"] & \cdots \\
 & & X \arrow[d, dashed, "f"'] & & \\
 & & Z & &
\end{tikzcd}$$

Note that f precompose with the canonical map $f \circ q_i : X_i \to X \to Z$ is equal to $h_i(-,1) \simeq s_i$. By the uniqueness of limit, this implies that $\phi(f) = s$ since $p_i(\phi(f)) = [s_i] \in [X_i, Z]$.

Problem 4
Regard $\mathbb{R}P^3$ as a subspace of $\mathbb{R}P6$ in the usual way. Take two copies of $\mathbb{R}P^6$ and glue their 3-skeletons together (via the identity map), to make a new space X. Compute the groups $H_*(X)$.

Solution: The space X has the following CW complex structure: for $i \leq 3$, it has one i-cell in each dimension and the attatching map is the same as the cell structure for $\mathbb{R}P^3$. For $3 \leq i \leq 6$, X has two i-cells in each dimension and each i-cell glued to the $(i-1)$-skeleton X_{i-1} in the same way as what happens in $\mathbb{R}P^6$. So the cellular chain complex can be written as:

$$\mathbb{Z}^2 \xrightarrow{(2,2)} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{(2,2)} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

So the homology groups can be calculated as

$$H_i(X) = \begin{cases} \mathbb{Z}, & \text{if} \quad i = 0, 4; \\ \mathbb{Z}/2, & \text{if} \quad i = 1, 3; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & \text{if} \quad i = 5; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 5
Suppose (X, A) is a pair for which HEP holds. Let $j : A \hookrightarrow X$ be the inclusion, and let C_j denote the mapping cone of j. Let $p : C_j \to X/A$ be the projection that collapse CA down to the basepoint. Use HEP to produce a map $q : X/A \to C_j$ such that p and q are part of a homotopy equivalence.

Solution: Consider the canonical inclusion $i : X \to C_j = X \cup_j CA$. Identify $CA = (A \times I)/(A \times \{1\})$ and the quotient map $h : A \times I \to CA \supseteq C_j$ can be viewed as a homotopy on the subspace $A \subseteq X$.

We know that $h(-,0) = i|_A$ is just the inclusion j.

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, "i \cup h"] \arrow[d] & C_j \\ X \times I \arrow[ur, dashed, "\exists H"'] & \end{tikzcd}$$

By HEP, there exists a map $H : X \times I \to C_j$ such that the above diagram commutes. Note that $H(-,1) : X \to C_j$ maps the subspace $A \subseteq X$ to the peak in $CA \subseteq C_j$ since $h(-,1)$ is the constant map on A. So $H(-,1)$ must factor through the quotient space X/A, and we take $q = H(-,1) : X/A \to C_j$. Next, we need to show that p and q give us a homotopy equivalence between X/A and C_j.

View $pq = p \circ H(-,1) : X \to C_j \to X/A$. We know by construction that $H(-,1)$ is homotopic to $H(-,0) : X \to C_j$ which is just the inclusion. So $pq \simeq p \circ H(-,0) : X \to X/A$ is just the quotient map. When factoring through X/A, pq is homotopic to the identity $id : X/A \to X/A$. On the other hand, we need to show that qp is homotopic to the identity $C_j \to C_j$. Consider the following map $K : C_j \times I \to C_j$ constructed in this way: for any $x \in X$ and $t \in I$, $K(x,t) = H(x,t)$. For $(y,s) \in CA = A \times I/\sim$, write $(y,s,t) \in CA \times I = (A \times I/\sim) \times I$ and $v_0 := A \times \{1\} \in CA$ is the point at the top of the cone. We send (y,s,t) to $(1-s)H(y,t) + sv_0$ where $y \in A \subseteq X$. We check that this indeed defines a map $K : C_j \times I \to C_j$. Note that $C_j = X \cup_j CA$. We need to check that for any $t \in I$, the image $A \times \{0\} \subseteq CA$ must be sent to the same points as $A \subseteq X$. This is true because $K(y,0,t) = H(y,t) = K(y,t)$ if we identify $y \sim (y,0) \in A \times \{0\} \cong A \xrightarrow{j} X$. K is continous by construction. Note that when $t = 0$, on X, $K(-,0) = H(-,0)$ is just the inclusion $X \hookrightarrow C_j$ and on CA, $K(y,s,0) = (1-s)H(y,0) + sv_0 = (1-s)y + sv_0$ just maps the same line in CA to the same line, so $K(-,0) : C_j \to C_j$ is the identity. When $t = 1$, note that $K(y,1) = H(y,1) = v_0$ for any $y \in A$ by construction of H, and $K(-,1)$ is the composition $qp : C_j \to C_j$. Thus, we have constructed a homotopy K between qp and the identity. This proves that p and q give a homotopy equivalence between X/A and C_j.

Problem 6

Suppose M and N are n-dimensional manifold with boundary, and $h : \partial M \to \partial N$ is a homeomorphism. Then one gets a new manifold (without boundary) by gluing M and N together along h. That is, one takes the space $M \cup_h N = [M \sqcup N]/\sim$ where the quotient relation is $x \sim h(x)$ for $x \in \partial M$.

Let $M = N = D^2 \times S^1$. Then $\partial M = \partial N = S^1 \times S^1$. Let p, q, a, b be integers such that $qa - pb = 1$, and let $h : S^1 \times S^1 \to S^1 \times S^1$ be given by

$$(e^{ix}, e^{iy}) \mapsto (e^{i(ax+by)}, e^{i(px+qy)}).$$

The condition that $qa - pb = 1$ implies that h is a homeomorphism. Compute $H_*(M \cup_h N)$.

Solution:
<u>Claim:</u> h is a homeomorphism.

Proof: Consider the following map h':

$$\begin{aligned} h' : S^1 \times S^1 &\to S^1 \times S^1, \\ (e^{ix}, e^{iy}) &\mapsto (e^{i(qx-by)}, e^{i(-px+ay)}). \end{aligned}$$

This map is continous by definition and we can check that

$$\begin{aligned} (e^{ix}, e^{iy}) \xrightarrow{h'\circ h} & h'(e^{i(ax+by)}, e^{i(px+qy)}) \\ &= ((e^{i(ax+by)})^q \cdot (e^{i(px+qy)})^{-b}, (e^{i(ax+by)})^{-p} \cdot (e^{i(px+qy)})^a) \\ &= (e^{i(aq-bp)x} \cdot e^{i(bq-bq)y}, e^{i(-ap+pa)x} \cdot e^{i(-bp+qa)y}) \\ &= (e^{ix}, e^{iy}). \end{aligned}$$

Similarly, we can check that $h \circ h'$ is also the identity. So h is a homeomorphism. ■

Write $M = D^2_M \times S^1$ and $N = D^2_N \times S^1$. Let 0 denote the center of the disks. Note that $M - 0 := (D^2_M - 0) \times S^1$ and $N - 0 := (D^2_N - 0) \times S^1$ are open and deforms retract into the boundary ∂M and ∂N resprectively. Take $U = M \cup_h (N - 0)$ and $V = (M - 0) \cup_h N$. We have $U \cup V = M \cup_h N$ and $U \cap V$ is homotopic equivalent to the glued boundary $\partial M = \partial N = S^1 \times S^1 \cong T$, which is homeomorphic to a torus T. Since $D^2_M - 0 \cong D^2_N - 0$ is contractible, U and V are homotopic equivalent to $\{*\} \times S^1$. The Mayer-Vietoris sequence gives us the following long exact sequence in reduced homology:

$$\begin{tikzcd}
 & \tilde{H}_*(S^1 \times S^1) \cong \tilde{H}_*(T) & \tilde{H}_*(S^1) \oplus \tilde{H}_*(S^1) & \tilde{H}_*(M \cup_h N) \\
3 & 0 \arrow[r] & 0 \arrow[r] & ? \arrow[dll] \\
2 & \mathbb{Z} \arrow[r] & 0 \arrow[r] & ? \arrow[dll] \\
1 & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "i"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & ? \arrow[dll] \\
0 & 0 & &
\end{tikzcd}$$

Note that both M and N are path-connected, from the long exact sequence we know that

$$H_0(M \cup_h N) = H_3(M \cup_h N) = \mathbb{Z}.$$

For the rest of the homology groups, we need to determine the homeomorphism $i : H_1(S^1 \times S^1) \to H_1(S^1) \oplus H_1(S^1)$. i is induced by the compostion of maps

$$\begin{tikzcd}
S^1 \times S^1 \arrow[r, hook] \arrow[d, "h"'] & D^2 \times S^1 \cong M \\
S^1 \times S^1 \arrow[r, hook] & D^2 \times S^1 \cong N
\end{tikzcd}$$

Passing to the first homology groups, we can see that

$$\begin{tikzcd} H_1(S^1 \times S^1) \arrow[r, "p_1"] \arrow[d, "h_*"'] & H_1(D^2 \times S^1) \cong H_1(S^1) \\ H_1(S^1 \times S^1) \arrow[r, "p_2"'] & H_1(D^2 \times S^1) \cong H_1(S^1) \end{tikzcd}$$

We know that $H_1(S^1 \times S^1)$ has two generators corresponding to each S^1. We can see from the diagram that p_1 and p_2 just project the generators to the second factor. Suppose α, β generates $H_1(S^1 \times S^1)$. From the diagram we can see that $p_1(\alpha, \beta) = \beta$ and $p_2(h_*(\alpha, \beta)) = p_2(a\alpha + b\beta, p\alpha + q\beta) = p\alpha + q\beta$. So the map

$$i : H_1(S^1 \times S^1) \to H_1(S^1) \oplus H_1(S^1)$$

in the Mayer-Vietoris sequence is given by $(\alpha, \beta) \mapsto (\beta, p\alpha + q\beta)$. Alternatively, this map can be viewed as a matrix $A = \begin{pmatrix} 0 & 1 \\ p & q \end{pmatrix}$ from $\mathbb{Z}^2$ to $\mathbb{Z}^2$. If $p = 0$, then $\det A = 0$ and $\ker i = \mathbb{Z}$ and $\operatorname{coker} i = \langle \alpha, \beta \rangle / \langle \beta, q\beta \rangle = \mathbb{Z}$. If $p = 1$ or $p = -1$, then A is invertible, and this implies i is an isomorphism, so $\ker i = \operatorname{coker} i = 0$. If $p \neq 0, 1, -1$, then $\ker i = 0$ and $\operatorname{coker} i = \mathbb{Z}/p\mathbb{Z}$. We can summarize the homology groups as follows.
If $p = 0$, then

$$H_i(M \cup_h N) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0, 1, 2, 3; \\ 0, & \text{otherwise.} \end{cases}$$

If $p = 1$ or $p = -1$, then

$$H_i(M \cup_h N) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0, 3; \\ 0, & \text{otherwise.} \end{cases}$$

If $p \neq 0, 1, -1$, then

$$H_i(M \cup_h N) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0, 3; \\ \mathbb{Z}/p\mathbb{Z}, & \text{if } \ i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 7
Let (X, A) be a CW pair, and Y be any space. Suppose given $f : X \to Y$ and $h : A \times I \to Y$ such that $h|_{A \times 0} = f|_A$. The HEP says that there exists an $H : X \times I \to Y$ such that $H|_{A \times I} = h$ and $H_0 = f$. Now suppose that $h' : A \times I \to Y$ is another map such that $h'|_{A \times 0} = f|_A$, and let $H' : X \times I \to Y$ be an extension of h' just as H was an extension of h. Prove that if h is homotopic to h' relative to $A \times \{0\}$, via a homotopy called λ, then H' can be chosen so that it is homotopic to H relative to $X \times \{0\}$ through a homotopy Λ that extends λ.

Solution: Consider the pair of spaces $(X \times I, (X \times \{0\}) \cup (A \times I))$. This is a CW pair since (X, A) is a CW pair and we can choose a CW complex structure for $X \times I$ and $A \times I$. Note that the map $f : X \to Y$ can be extended to a map $F : X \times 0 \times I \to Y$ with $F(x, 0, t) = f(x)$ for any $x \in X$ and any $t \in I$. Consider the map $H : X \times I \to Y$ and the homotopy $F \cup \lambda : (X \times \{0\} \times I) \cup (A \times I \times I) \to Y$.

We know that

$$(F \cup \lambda)|_0 = f \cup h = H|_{(X\times\{0\})\cup(A\times I)}.$$

The following diagram:

$$\begin{tikzcd}
(X \times 0 \times I) \cup (A \times I \times I) \cup (X \times I \times 0) \arrow[r, "F\cup\lambda\cup H"] \arrow[d, hook] & Y \\
X \times I \times I \arrow[ur, dashed, "\exists\Lambda"'] &
\end{tikzcd}$$

implies that there exists a homotopy $\Lambda : X \times I \times I \to Y$ such that $\Lambda(x,t,0) = H(x,t)$ and $\Lambda|_{X\times\{0\}\cup(A\times I)} = F \cup \lambda$. We choose $H'(x,t) = \Lambda(x,t,1)$. This proves that H' is homotopic to H relative to $X \times \{0\}$ through a a homotopy Λ that extends λ.