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Homework - Week 9
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Exercise 6.7.2
$Z(Q_{4m}) = \{1, a^m\}$, and $Q_{4m}/Z(Q_{4m}) \cong D_{2m}$.

Solution: When $m = 1$, we have a presentation

$$Q_4 = \langle\langle a, b \,|\, a^2 = 1, b^2 = a, bab^{-1} = a^{-1} = a\rangle\rangle.$$

This is the presentation of the abelian group C_4. In this case, the center $Z(Q_4) = Q_4 \cong C_4$, and $Q_4/Z(Q_4) = Q_4/Q_4 = \{1\}$ is trivial.

When $m \geq 2$, we know that every element of Q_{4m} can be written in the form $a^i b^j$ for $0 \leq i < 2m$ and $j \in \{0, 1\}$. We first show that an element of the form $a^i b$ for $0 \leq i < 2m$ is not in the center $Z(Q_{4m})$. Suppose the opposite is true. Then we have $a \cdot a^i b = a^i b \cdot a$. This implies $ab = ba$, but we know that in Q_{4m}, $ba = a^{-1}b$, so we have $a^{-1}b = ab$. Note that $m \geq 2$, so $a^{-1} \neq a$. A contradiction.

$Z(Q_{4m})$ only has the elements of the form a^i for some $0 \leq i < 2m$. To make a^i commutes with b, we must have

$$a^i = ba^ib^{-1} = a^{-1}ba^{i-1}b^{-1} = \cdots = a^{-i}.$$

This shows that $a^{2i} = 1$. So $2m|2i$ for some $0 \leq i < 2m$ and i can only equal to m or 0. It is easy to see that $a^0 = 1$ is indeed in the center. For a^m, we know that a^m commutes with any elements of the form a^i for $0 \leq i < 2m$, and we have

$$a^m(a^ib) = a^i(a^mb) = a^i(ba^{-m}) = (a^ib)a^m.$$

So we can conclude that $Z(Q_{4m}) = \{1, a^m\}$. The quotient group $Q_{4m}/Z(Q_{4m})$ has the following presentation

$$\langle\langle a, b \,|\, a^m = 1, b^2 = a^m = 1, bab^{-1} = a^{-1}\rangle\rangle.$$

This is the presentation of the dihedral group D_{2m}.

Exercise 6.7.11
Show that the free product $\coprod_{i\in I} G_i$ together with homomorphism $\iota_j : G_j \to \coprod_{i\in I} G_i$, $g \mapsto (g)$ is the coproduct of the family $(G_i)_{i\in I}$ in the category of groups.

Solution: We prove the universal property of $(\coprod_{i\in I} G_i, \iota_j)$. Suppose H is a group and we have a collection of maps $f_j : G_j \to H$ such that for all $j, k \in I$, if we have a map $p_{jk} : G_j \to G_k$, the following diagram commutes:

$$\begin{tikzcd}
G_j \arrow[r, "p_{jk}"] \arrow[d, "f_j"'] & G_k \arrow[dl, "f_k"] \\
H &
\end{tikzcd}$$

Consider a map $f : \coprod_{i\in I} G_i \to H$ defined as follows. We define $f(1) = 1$ and for an alternating word $(g_1, \ldots, g_n)$ where $g_l \in G_{i_l} \setminus \{1\}$ and $i_l \neq i_{l+1}$ for all $1 \leq l < n$, we define

$$f(g_1, \ldots, g_n) = f_{i_1}(g_1)f_{i_2}(g_2)\cdots f_{i_n}(g_n).$$

Note that f defined in this way is the unique f making the following diagram commutes:

$$\begin{tikzcd}
G_j \arrow[rr, "id"] \arrow[dr, "\iota_j"] \arrow[ddr, "f_j"'] & & G_j \arrow[dl, "\iota_j"'] \arrow[ddl, "f_j"] \\
& \coprod_{i\in I} G_i \arrow[d, "f"] & \\
& H &
\end{tikzcd}$$

for any $j \in I$. For any $g \in G_j$, this forces f mapping $(g) \in \coprod_{i\in I} G_i$ to $f(g)$. Given $p_{jk} : G_j \to G_k$, we have a commutative diagram:

$$\begin{tikzcd}
G_j \arrow[rr, "p_{jk}"] \arrow[dr, "\iota_j"] \arrow[ddr, "f_j"'] & & G_k \arrow[dl, "\iota_k"'] \arrow[ddl, "f_k"] \\
& \coprod_{i\in I} G_i \arrow[d, "f"] & \\
& H &
\end{tikzcd}$$

This proves that $\coprod_{i\in I} G_i$ is the coproduct of $(G_i)_{i\in I}$.

Exercise 7.1.8
Prove that no infinite simple group G has a proper subgroup of finite index.

Solution: Suppose we have $H < G$ a proper subgroup of index $2 < k < \infty$. Consider G acts on the set of left coset $X = \{gH \mid g \in G\}$, which is a finite set of k elements:

$$\begin{aligned} G \times X &\to X, \\ g_1 \cdot g_2H &\mapsto g_1g_2H. \end{aligned}$$

This is a well-defined group action and for every $g \in G$,note that if $gg_1H = gg_2H$, then $g_1H = g_2H$ is the same element in X. This implies that g defines a permutation of the elements in X, and we have a map $f : G \to S_k$. This is a group homomorphism since we have

$$(g_1g_2) \cdot gH = g_1 \cdot (g_2g)H.$$

Note that $\ker f$ is normal subgroup of G and since G is simple, $\ker f = \{1\}$ or $\ker f = G$. If $\ker f = \{1\}$, then f is injective but G is infinite and S_k is finite. This is impossible. Now assume $\ker f = G$. This means that f is the trivial map and for any $g \in G$, $g \cdot g'H = gg'H = g'H$ for any $g, g' \in G$. This shows that $H = G$, which contradicts that H is a proper subgroup. This concludes that such H does not exists.

Exercise 7.2.3
Let G be a finite group. We choose an element $g \in G$ randomly. Then replace it and make another random choice of an element $h \in G$. Prove that the probability that g and h commute equals to $k/|G|$, where k is the number of conjugacy classes in G.

Solution: This is the same as asking the probability of randomly choosing two elements $g, h \in G$ with the property that one is in the centralizer of another. For every fixed $g \in G$, the probability of choosing g is $\frac{1}{|G|}$, now we need to choose an element $h \in C_G(g)$. The probability is $\frac{|C_G(g)|}{|G|}$. So the total probability should be summing over all elements $g \in G$, which is

$$\sum_{g \in G} \frac{|C_G(g)|}{|G|^2}.$$

<u>Claim:</u> $\sum_{g \in G} |C_G(g)| = k \cdot |G|$ where k is the number of conjugacy classes in G.
<u>Proof:</u> Let G acts on G by conjugation. By Lemma 7.1.6 (Orbit counting lemma), we have

$$k = \frac{1}{|G|} \sum_{g \in G} |G^g|$$

where $|G^g| = \{h \in G \mid ghg^{-1} = h\} = C_G(g)$. ■

So the probability is

$$\sum_{g \in G} \frac{|C_G(g)|}{|G|^2} = \frac{k|G|}{|G|^2} = \frac{k}{|G|}.$$

Exercise 7.2.4
Suppose that a finite group G has exactly two conjugacy classes. Determine G up to isomorphism.

Solution: If $a \in Z(G)$, then the conjugacy classes of a must be of size 1. We know that $1 \in G$ is in the center $Z(G)$. So this is one of the two conjugacy classes. Suppose the other conjugacy classes also has only 1 element g. This is the same as for any $h \in G$, we have $hgh^{-1} = g$. So $g \in Z(G)$. This means G is an abelian group of order 2. Thus, $G \cong C_2$. Now assume the size of the other conjugacy class is $k \geq 2$. By the class equation

$$|G| = |Z(G)| + [G : C_G(g)]$$

where $g \notin Z(G)$ and by Theorem 7.1.7, $[G : C_G(g)] = \frac{|G|}{|C_G(g)|} = G \cdot g = k \geq 2$. And we have

$$|C_G(g)| = \frac{k+1}{k}.$$

For any $k \geq 2$, $\frac{k+1}{k}$ is not an integer. A contradiction. So the group G can only be isomorphic to C_2.

Exercise 7.5.7
If $H < G$ contains a Sylow p-subgroup of G for each prime p, then $H = G$.

Solution: Suppose the order $|G| = p_1^{a_1} \cdots p_n^{a_n}$. For any $1 \leq i \leq n$, we have a Sylow p-group of order $p_i^{a_i}$. H containing this subrgoup means $p_i^{a_i} || H|$. So we have

$$|H| \geq lcd(p_1^{a_1}, \ldots, p_n^{a_n}) = p_1^{a_1} \cdots p_n^{a_n} = |G|.$$

So we conclude that $H = G$.

Exercise 8.1.10
If G is a finite solvable group, then G contains a non-trivial normal abelian subgroup. If G is not solvable then it contains a normal subgroup H such that $H' = H$.

Solution: Assume G is finite, solvable and simple. Then the only Jordan-Hölder factor G must be cyclic, in which case G itself is a nor-trivial normal abelian group. Now assume G is not simple and let H be a non-trivial proper normal subgroup of G. Consider the derived series of G:

$$G = G^{(0)} > G^{(1)} > \cdots > G^{(n)} = \{1\}.$$

If G is already abelian, then we are done. If G is not abelian, then there exists $1 \leq i \leq n$ such that $H \cap G^{(i)}$ is non-trivial but $H \cap G^{(i+1)} = \{1\}$.
Claim: For $1 \leq i \leq n$, $G^{(i)}$ is a normal subgroup of G.
Proof: For any group G and a group automorphism $\phi : G \to G$. For any $x, y \in G$, we have

$$\phi(xyx^{-1}y^{-1}) = \phi(x)\phi(y)\phi(x)^{-1}\phi(y)^{-1} \in G'.$$

So we can see that $\phi(G') \subset G'$, thus, the commutator subgroup is a characteristic subgroup. In particular, G' is normal in G. So we have $G^{(i)}$ is a characteristic subgroup in $G^{(i-1)}$ for $1 \leq i \leq n$, and by Exercise 6.1.12, $G^{(i)}$ is a characteristic subgroup of G by induction. In particular, $G(i)$ is normal in G. ■

Both $G^{(i)}$ and H are normal subgroups of G, so $H \cap G^{(i)}$ is normal in G. Note that for any $a, b \in H \cap G^{(i)}$, we have $aba^{-1}b^{-1} = 1$ since $H \cap G^{(i+1)}$ is trivial. So $H \cap G^{(i)}$ is a non-trivial normal abelain group in G.

Now assume G is finite and not solvable. Note that we have proved that for every $i \geq 0$, $G^{(i)}$ is a normal subgroup of G. Consider the derived series

$$G = G^{(0)} > G^{(1)} > G^{(2)} > \cdots.$$

Since G is not solvable, this sequence does not terminate and because G is finite, there must exists a $G^{(k)}$ such that $G^{(k+1)} = G^{(k)}$.

Exercise 8.1.12
Let G be a finite group containing elements x and y such that the orders of x, y and xy are pairwise relatively prime (and not all equal to 1). Prove that G is not solvable.

Solution: Let $H = \langle x, y \rangle$ be the finite subgroup generated by x, y. Assume $\mathrm{ord}(x) = a$, $\mathrm{ord}(y) = b$ and $\mathrm{ord}(xy) = c$. a, b, c are pairwise coprime to each other. We know that the quotient group H/H' is abelian since H' is the commutator subgroup. We have $1 = (xH)^a = (yH)^b = (xyH)^c$. So $\mathrm{ord}\,(xH)|a$, $\mathrm{ord}\,(yH)|b$ and $\mathrm{ord}\,(xyH)|c$. Since a, b are coprime, $\mathrm{ord}\,(xH)$ and $\mathrm{ord}\,(yH)$ are also coprime. This means that

$$\mathrm{ord}\,(xyH) = \mathrm{ord}\,(xH \cdot yH) = \mathrm{ord}\,(xH) \cdot \mathrm{ord}\,(yH)$$

is a divisor of c. But a, b, c are pairwise coprime, so we have $a = b = c = 1$. This shows that $H = H'$ and we can conclude that H is not solvable. Therefore, G is also not solvable as H is a subgroup of G.

Exercise 8.2.2
The group U of upper unitriangular $n \times n$ matirces (over any field) is nilpotent.

Solution: Let $A = (a_{ij})_{1 \le i,j \le n}$ and $B = (b_{ij})_{1 \le i,j \le n}$ be two unitriangular matrices. We have

$$\begin{aligned} 0 &= a_{ij} = b_{ij}, \text{ if } 1 \le j < i \le n, \\ 1 &= a_{ii} = b_{ii}, \text{ if } 1 \le i \le n. \end{aligned}$$

Note that for any $1 \le i \le n$, the product $(AB)_{i,i+1}$ can be written as

$$(AB)_{i,i+1} = \sum_{k=1}^{n} a_{i,k} b_{k,i+1} = a_{i,i+1} + b_{i,i+1}.$$

So we have $(A^{-1})_{i,i+1} = -a_{i,i+1}$ and

$$(ABA^{-1}B^{-1})_{i,i+1} = a_{i,i+1} + b_{i,i+1} - a_{i,i+1} - b_{i,i+1} = 0.$$

for all $1 \le i \le n$. The commutator subgroup $\gamma_1(U) = [U, U]$ consists of upper unitriangular matrices A with the property that $a_{i,i+1} = 0$ for all $1 \le i \le n$.

Now use induction and we assume $\gamma_m(U)$ consists of upper unitriangular matrices A with the property $a_{i,i+m} = 0$ for all $1 \le i \le n$. We have proved the case $m = 1$. For $m \ge 2$, assume we have proved the case $m - 1$. Let $A = (a_{ij}) \in \gamma_{m-1}(U)$ and $B = (b_{ij}) \in U$. We have

$$(AB)_{i,i+m} = \sum_{k=1}^{n} a_{i,k} b_{k,i+m}.$$

Note that $a_{i,k} = 0$ if $i + 1 \le k \le i + m - 1$ by assumption and $b_{k,i+m} = 0$ if $k > i + m$. SO

$$(AB)_{i,i+m} = a_{i,i+m} + b_{i,i+m}.$$

This implies that

$$(ABA^{-1}B^{-1})_{i,i+m} = a_{i,i+m} + b_{i,i+m} - a_{i,i+m} - b_{i,i+m} = 0.$$

Now if $m = n$, then $[\gamma_n(G), G] = \{I_n\}$ is the trivial group, and we can conclude that U is nilpotent.

> **Exercise 8.2.13**
> Let G be a finite group. Then G is nilpotent if and only if $N_G(H) \gneq H$ whenever $H \lneq G$.

Solution:

(1) Necessity.

Consider a finite central series for G:

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

such that $G_i/G_{i-1} \leq Z(G/G_{i-1})$ for all $1 \leq i \leq n$. There exists $1 \leq k \leq n$ such that $G_k \leq H$ but G_{k+1} is not a subgroup of H. Note that

$$G_{k+1}/G_k \leq Z(G/G_k) \leq N_{G/G_k}(H/G_k).$$

Claim: $N_{G/G_k}(H/G_k)$ is a subgroup of $N_G(H)/G_k$.

Proof: Let $gG_k \in N_{G/G_k}(H/G_k)$. For any $h \in H$, we have

$$(gG_k)(hG_k)(g^{-1}G_k) = (ghg^{-1})G_k \in H/G_k.$$

There exist $h' \in H$ such that $ghg^{-1}h'^{-1} \in G_k \leq H$. This implies that $ghg^{-1} \in H$, thus $g \in N_G(H)$. ■

Now we have $G_{k+1}/G_k \leq N_G(H)/G_k$. G_{k+1} is a subgroup of $N_G(H)$ and since G_{k+1} is strictly larger than H, we have $N_G(H) \gneq H$.

(2) Sufficiency.

Let $H < G$ be a maximal proper subgroup of G. We know $N_G(H)$ is strictly larger than H. Since H is already maxmial, so $N_G(H) = G$. This means for any $g \in G = N_G(H)$, we have $gHg^{-1} = H$. H is normal in G. By Proposition 8.2.12, we know that G is nilpotent.