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Course: MATH 636 - Algebraic Topology III
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Homework 4
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Problem 1
Compute all of the homology groups for the spaces in parts (a) and (b) below, and use your calculations to show that

(a) $\mathbb{R}P^2 \times S^3$ and $\mathbb{R}P^3 \times S^2$ have isomorphic homotopy groups (in all dimensions), but non-isomorphic homology groups.

(b) $S^4 \times S^2$ and $\mathbb{C}P^3$ have isomorphic homology groups but non-isomorphic homotopy groups.

Solution:

(a) Let $X = \mathbb{R}P^2 \times S^3$ and $Y = \mathbb{R}P^3 \times S^2$. It is easy to see that both X and Y are path-connected, so $\pi_0(X) = \pi_0(Y) = *$. By direct calcualtion, we have

$$\pi_1(X) = \pi_1(\mathbb{R}P^2) \times \pi_1(S^3) = \mathbb{Z}/2 \oplus \mathbb{Z},$$
$$\pi_1(Y) = \pi_1(\mathbb{R}P^3) \times \pi_1(S^2) = \mathbb{Z}/2 \oplus \mathbb{Z}.$$

This implies $\pi_1(X) \cong \pi_1(Y)$. Recall that for all $n \geq 2$, the universal covering space of $\mathbb{R}P^n$ is S^n. So the universal covering space of X and Y are both isomorphic to $S^2 \times S^3 \cong S^3 \times S^2$. The long exact sequence in homotopy groups tells us that

$$\pi_n(X) \cong \pi_n(Y) \cong \pi_n(S^3 \times S^2)$$

for all $n \geq 2$. Thus, we can conclude that X and Y have the same homotopy groups.

For the homology groups, note that the homology groups of S^3 and S^2 are all free. By Künneth theorem, we have

$$H_n(X) = \bigoplus_{p+q=n} H_p(\mathbb{R}P^2) \otimes H_q(S^3),$$
$$H_n(Y) = \bigoplus_{p+q=n} H_p(\mathbb{R}P^3) \otimes H_q(S^2).$$

The homology groups of each space is listed below:

	$H_*(\mathbb{R}P^2)$	$H_*(S^3)$
3	0	$\mathbb{Z}$
2	0	0
1	$\mathbb{Z}/2$	0
0	$\mathbb{Z}$	$\mathbb{Z}$

	$H_*(\mathbb{R}P^3)$	$H_*(S^2)$
3	$\mathbb{Z}$	0
2	0	$\mathbb{Z}$
1	$\mathbb{Z}/2$	0
0	$\mathbb{Z}$	$\mathbb{Z}$

Problem 2

Let I_* be the chain complex concentrated in degree 0 and 1 with $I_1 = \mathbb{Z}\langle e\rangle$, $I_0 = \mathbb{Z}\langle a, b\rangle$, and $d(e) = b - a$. Note that this is the simplicial chain complex for Δ_1. Let C_* and D_* be chain complexes.

(a) Describe the chain complex $I_* \otimes C_*$ by giving the groups in each degree as well as the boundary maps.

(b) Let $F : I_* \otimes C_* \to D_*$ be a chain map. Define $f, g : C_* \to D_*$ by $f(x) = F(a \otimes x)$ and $g(x) = F(b\otimes x)$. Likewise, define $s_n : C_n \to D_{n+1}$ by $s_n : C_n \to D_{n+1}$ by $s_n(x) = F(e\otimes x)$. Prove that f and g are chain maps and the collection $\{s_n\}$ is a chain homotopy between f and g.

Solution:

(a) We denote bothe the boundary map in C_* and I_* by d. Consider the double complex $I_* \otimes C_*$ first.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
I_0 \otimes C_2 \arrow[d] & I_1 \otimes C_2 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
I_0 \otimes C_1 \arrow[d] & I_1 \otimes C_1 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
I_0 \otimes C_0 & I_1 \otimes C_0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

The vertical boundary map d_v is $id \otimes d$ and the horizontal boundary map d_h is $d \otimes id$. Let T_* be the total complex of this double complex, then in each degree we have

$$T_n = I_0 \otimes C_n \oplus I_1 \otimes C_{n-1}.$$

Problem 3

Let Y be the space obtained by starting with S^3 and attaching a 4-cell via a map of degree $5 : Y = S^3 \cup_f e^4$ where $f : \partial(e^4) \to S^3$ has degree 5. Write down the cellular chain complex for $\mathbb{R}p^3 \otimes Y$; in particular, specify the rank of each chain group and identify the boundary maps. Compute the homotopy groups of specify the rank of each chain group and identify the boundary maps. Compute the homology groups of $\mathbb{R}P^3 \otimes Y$.

Solution:

Problem 4

Compute both the homology and cohomology groups of the following spaces, both with integral and $\mathbb{Z}/2$ coefficients. Heck, do it with $\mathbb{Z}/3$ coefficients as well.

(a) $K \times K$, where K is the Klein bottle.

(b) $K \times T^g$, where T^g is the genus g torus and K is the Klein bottle.

(c) $K \times \mathbb{R}P^n$.

Solution:

Problem 5

Let $f : A_* \to B_*$ be a map of chain complexes. We can regard this as forming a double complex

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & B_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & B_0 \end{array}$$

by putting zeros in all the "empty" spots. The total complex of this double complex is called the **algebraic mapping cone** of f, denoted Cf. Specifically, we set $(Cf)_n = A_{n-1} \oplus B_n$ and define $d : (Cf)_n \to (Cf)_{n-1}$ by

$$d(a, b) = (d_A(a), (-1)^{n-1} f(a) + d_B(b))$$

(a) Explain why there is a short exact sequence of chain complexes

$$0 \to B_* \hookrightarrow C(f) \to \Sigma A_* \to 0,$$

where ΣA_* is the evident chain complex having $(\Sigma A)_n = A_{n-1}$.

(b) The short exact sequence from (a) gives rise to a long exact sequence in homology groups. This has the form

$$\cdots \to H_i(B) \to H_i(Cf) \to H_i(\Sigma A) \xrightarrow{\partial} H_{i-1}(B) \to \cdots$$

Verify that the connecting homomorphism is really just the map $f_* : H_{i-1}(A) \to H_{i-1}(B)$, possibly up to a sign.

Solution:

Problem 6

Let k be a field, and let $\mathcal{V}$ denote the category of vector spaces over k. Let I be any (small) category, and let $\mathcal{V}^I$ be the category whose objects are functors $I \to \mathcal{V}$ and whose morphisms

are natural transformations. We call $\mathcal{V}^I$ the category of "I-shaped diagram in $\mathcal{V}$".
In this problem we will focus on the case where I is the pushout category

$$1 \leftarrow 0 \rightarrow 2$$

with three objects and two non-identity maps (as shown above). An object of $\mathcal{V}^I$ is then just a diagram of vector spaces $V_1 \leftarrow V_0 \rightarrow V_2$. A map from $[V_1 \leftarrow V_0 \rightarrow V_2]$ to $[W_1 \leftarrow W_0 \rightarrow W_2]$ is a commutative diagram

$$\begin{tikzcd} V_1 \arrow[d] & V_0 \arrow[l] \arrow[r] \arrow[d] & V_2 \arrow[d] \\ W_1 & W_0 \arrow[l] \arrow[r] & W_2 \end{tikzcd}$$

Let $P : \mathcal{V}^I \rightarrow \mathcal{V}$ be the pushout functor. P assigns each diagram its pushout.

(a) Let F_1, F_0 and F_2 be the three diagrams

$$F_1 : [k \leftarrow 0 \rightarrow 0] \quad F_0 = [k \leftarrow k \rightarrow k] \quad F_2 = [0 \leftarrow 0 \rightarrow 0]$$

where in F_0 the maps are the identities. These diagrams are "free" in a certain sense: namely, if D is an object of $\mathcal{V}^I$ then morphisms $F_i \rightarrow D$ are in bijective correspondence with elements of D_i. Convince yourself that this is true.

(b) Let $D = [0 \leftarrow k \rightarrow 0]$ and $E = [0 \leftarrow k \rightarrow k]$, where in E the nontrivial map is the identity. Determine free resolutions for D and E.

(c) Apply the functor P to your resolution, to produce a chain complex of vector spaces. Compute the homology groups, which are the groups $(L_iP)(D)$ and $(L_iP)(E)$. These are the derived functor of the psuhout functor P. Confirm in your example that $L_0P = P$.

(d) Now let I be the category with one object 0 and one non-identity map $t : 0 \rightarrow 0$ such that $t^2 = id$. Objects of $\mathcal{V}^I$ are then pairs (W, t) consisting of a vector space W and an endomorphism $t : W \rightarrow W$ such that $t^2 = id$. In $\mathcal{V}^I$ the basic "free" object is $(k^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$; this can also be thought of as the vector space $k\langle g, tg\rangle$ where $t(tg) = g$. Let $P : \mathcal{V}^I \rightarrow \mathcal{V}$ be the colimit functor, sending an object (W, t) to $W/\{x - tx \mid x \in W\}$. Find the free resolution of the object (k, id) and compute $(L_iP)(k, id)$ for all $i \geq 0$.

Solution: