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Course: MATH 636 - Algebraic Topology III
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Homework 2
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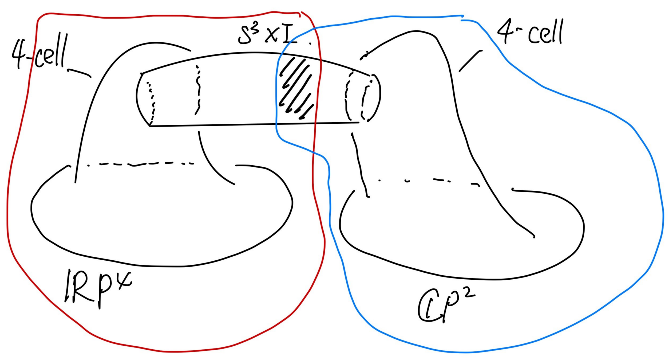
Problem 1
Determine all the cohomology groups $H^*(\mathbb{R}P^4\#\mathbb{C}P^2)$.

Solution: We first use the Mayer-Vietoris sequences in cohomology as follows:



$U \simeq \mathbb{R}P^4 - * \simeq \mathbb{R}P^3$ $\quad U \cap V = S^3 \times I \simeq S^3$

$V \simeq \mathbb{C}P^2 - * \simeq \mathbb{C}P^1$

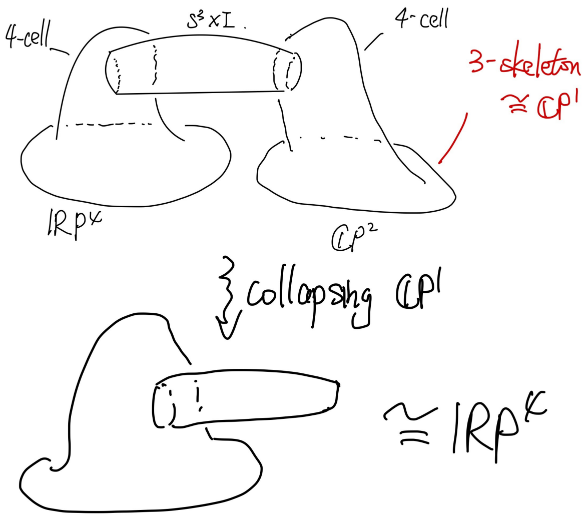
Let $X = \mathbb{R}P^4\#\mathbb{C}P^2$. We have the following long exact sequence in cohomology

$$\begin{tikzcd}
 & H^*(X) & H^*(\mathbb{R}P^3)\oplus H^*(\mathbb{C}P^1) & H^*(S^3) \\
0 & ? \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[dll] \\
1 & ? \arrow[r] & 0\oplus 0 \arrow[r] & 0 \arrow[dll] \\
2 & ? \arrow[r] & \mathbb{Z}/2\oplus\mathbb{Z} \arrow[r] & 0 \arrow[dll] \\
3 & ? \arrow[r] & \mathbb{Z}\oplus 0 \arrow[r] & \mathbb{Z} \arrow[dll] \\
4 & ? \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Note that X is connected, so $H^0(X) = \mathbb{Z}$. By exactness of the above sequence, we have

$$H^2(X) \cong \mathbb{Z}\oplus\mathbb{Z}/2.$$

Now consider collapsing the $\mathbb{C}P^1 \subseteq \mathbb{C}P^2$ in X as shown



we have a cofiber sequence

$$\mathbb{C}P^1 \to X \to \mathbb{R}P^4$$

This gives us a long exactness sequence in reduced cohomology

$$\begin{tikzcd}
 & \tilde{H}^*(\mathbb{R}P^4) & \tilde{H}^*(X) & \tilde{H}^*(\mathbb{C}P^1) \\
0 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
1 & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[dll] \\
2 & \mathbb{Z}/2 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z}/2 \arrow[r] & \mathbb{Z} \arrow[dll] \\
3 & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[dll] \\
4 & \mathbb{Z}/2 \arrow[r] & ? \arrow[r] & 0
\end{tikzcd}$$

By exactness, we know $H^1(X) = H^3(X) = 0$ and $H^4(X) = \mathbb{Z}/2$. We can summarize the cohomology groups as follows.

$$H^i(\mathbb{R}P^4 \# \mathbb{C}P^2) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0; \\ \mathbb{Z}/2, & \text{if } \ i = 4; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } \ i = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 2
Let $f : X \to Y$ be a map, and consider the diagram

$$\begin{tikzcd} H^k(Y;R) \arrow[r, "f^*"] \arrow[d, "\phi"] & H^k(X;R) \arrow[d, "\phi"'] \\ \hom(H_k(Y), R) \arrow[r, "\hom(f_*,R)"] & \hom(H_k(X), R) \end{tikzcd}$$

where the bottom horizontal map is the one obtained by applying the functor $\hom(-, R)$ to $f_* : H_k(X) \to H_k(Y)$. Here the vertical maps ϕ are the adjoints to the Kronecker pairings, namely the maps that send a cohomology class $[\alpha]$ to the homomorphism $[v] \mapsto \alpha(v)$. Verify that the above diagram commutes.

Solution: Let $\alpha : C_k(Y) \to R$ be a cocycle in Y and $[\alpha]$ be the cohomology class represented by α in $H^k(Y;R)$. By defintiion, we know that $f^*([\alpha]) = [\alpha \circ f_\#]$, where $f_\# : C_k(X) \to C_k(Y)$ is the map on the chain complex induced by f. By definition, ϕ sends $[\alpha \circ f_\#]$ to the homomorphism $[v] \mapsto (\alpha \circ f_\#)(v)$ for any k-cycle $v \in C_k(X)$. On the other hand, ϕ sends $[\alpha]$ to the homomorphism $[w] \mapsto \alpha(w)$ for any k-cycle $w \in C_k(Y)$. Applying $\hom(f_*, R)$ to this homomorphism, we obtain a homomorphism sending any $[v] \in H_k(X)$ to $\alpha(f_*([v]))$. Note that by definition, for any k-cycle $v \in C_k(X)$, we have

$$\alpha(f_*([v])) = \alpha([f_\#(v)]) = \alpha(f_\#(v)) = (\alpha \circ f_\#)(v).$$

This proves the commutativity of this diagram.

Problem 3

(a) Give a Δ-complex structure on $\mathbb{R}P^2$ and use this to write down explicit cocycles $\alpha \in Z^1(\mathbb{R}P^2; \mathbb{Z}/2)$ and $\beta \in Z^2(\mathbb{R}P^2; \mathbb{Z}/2)$ that generates the cohomology groups. Compute $\alpha \cup \alpha$ and decide if it equals β or not in $H^*(\mathbb{R}P^2; \mathbb{Z}/2)$.

(b) Let K be the Klein bottle. Write down explicit cocycles whcih represent generators for $H^*(K; \mathbb{Z}/2)$ and use these to compute all the cup products of these generators.

(c) If R is a ring then we can extend the Kronecker pairing to be maps

$$H^k(X;R) \otimes H_k(X;R) \to R.$$

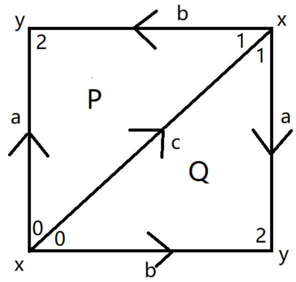
The adjoint is then a map

$$\phi_R : H^k(X;R) \to \hom(H_k(X), R).$$

When X is the Klein bottle and $\mathbb{R} = \mathbb{Z}/2$ determine bases for each $H^k(X;R)$ and verify by hand that the maps ϕ are isomorphisms for all k.

Solution:

(a) Consider the following Δ-complex structure for $\mathbb{R}P^2$:



We know from last homework that $H^1(\mathbb{R}P^2;\mathbb{Z}/2) = H^2(\mathbb{R}P^2;\mathbb{Z}/2) = \mathbb{Z}/2$. So as long as the cocycle $\alpha, \beta \in Z^*(\mathbb{R}P^2;\mathbb{Z}/2)$ are not zero, they are the generator of the cohomology groups. Consider $\hat{a}+\hat{c} \in C^1(\mathbb{R}P^2;\mathbb{Z}/2)$, we check that this is a cocycle.

$$\begin{aligned}(\delta(\hat{a}+\hat{c}))(P) &= (\hat{a}+\hat{c})(\partial P) = (\hat{a}+\hat{c})(-a+b+c) = 1+1 = 0;\\ (\delta(\hat{a}+\hat{c}))(Q) &= (\hat{a}+\hat{c})(\partial Q) = (\hat{a}+\hat{c})(a-b+c) = 1+1 = 0.\end{aligned}$$

This proves $\hat{a}+\hat{c} \in Z^1(\mathbb{R}P^2;\mathbb{Z}/2)$ and we need to show that $\hat{a}+\hat{c}$ is not a coboundary, this is true because

$$\begin{aligned}\delta(\hat{x})(a) &= \delta(\hat{y})(a) = 1,\\ \delta(\hat{x})(b) &= \delta(\hat{y})(b) = 1,\\ \delta(\hat{x})(c) &= \delta(\hat{y})(c) = 0.\end{aligned}$$

So $\hat{a}+\hat{c}$ cannot be realized as the image of $C^0(\mathbb{R}P^2;\mathbb{Z}/2)$. This implies that $\alpha = [\hat{a}+\hat{c}]$ generates the cohomology group $H^1(\mathbb{R}P^2;\mathbb{Z}/2) = \mathbb{Z}/2$. Note that $\mathbb{R}P^2$ does not have 3-simplices, and we know $H^2(\mathbb{R}P^2;\mathbb{Z}/2) = \mathbb{Z}/2$, so $\hat{Q} \in Z^2(\mathbb{R}P^2;\mathbb{Z}/2)$ and note that

$$\partial P = \partial Q = a+b+c.$$

So $\hat{Q}$ is not a coboundary because $\hat{Q}(P) = 0$ and $\hat{Q}(Q) = 1$. This implies that $\beta = [\hat{Q}]$ generates $H^2(\mathbb{R}P^2;\mathbb{Z}/2)$. Next, we calculate the cup product.

$$\begin{aligned}(\alpha \cup \alpha)(P) &= \alpha(c) \cdot \alpha(b) = 0;\\ (\alpha \cup \alpha)(Q) &= \alpha(c) \cdot \alpha(a) = 1.\end{aligned}$$

This proves that $(\hat{a}+\hat{c}) \cup (\hat{a}+\hat{c}) = \hat{Q}$ on the chain level and in the cohomology ring $H^*(\mathbb{R}P^2;\mathbb{Z}/2)$, we have $\alpha \cup \alpha = \beta$.

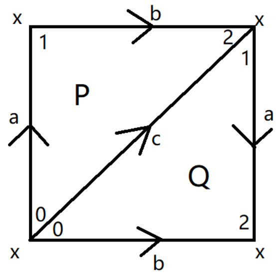
(b) Consider the cellular chain complex of the Klein bottle K

$$0 \to \mathbb{Z} \xrightarrow{\begin{pmatrix}2\\0\end{pmatrix}} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \to 0$$

Apply $\hom(-,\mathbb{Z}/2)$, we obtain the cellular cochain complex

$$0 \leftarrow \mathbb{Z}/2 \xleftarrow{0} (\mathbb{Z}/2)^2 \xleftarrow{0} \mathbb{Z}/2 \leftarrow 0.$$

So we know $H^0(K;\mathbb{Z}/2) = H^2(K;\mathbb{Z}/2) = \mathbb{Z}/2$, each has one generator and $H^1(K;\mathbb{Z}/2) = (\mathbb{Z}/2)^2$ having two generators. Consider the following Δ-complex structure of K



K only has one 0-simplex x, so the cocycle $\hat{x}$ generates $H^0(K;\mathbb{Z}/2)$. K has no 3-simplices, so $\hat{P}$ is a cocycle and we need to show that $\hat{P}$ is not a coboundary. Suppose $\delta(m\hat{a}+n\hat{b}+k\hat{c}) = \hat{P}$. Then

$$\begin{aligned} 1 &= (m\hat{a}+n\hat{b}+k\hat{c})(P) = (m\hat{a}+n\hat{b}+k\hat{c})(a+b-c) = m+n+k, \\ 0 &= (m\hat{a}+n\hat{b}+k\hat{c})(Q) = (m\hat{a}+n\hat{b}+k\hat{c})(a-b+c) = m+n+k. \end{aligned}$$

This leads to a contradiction and thus $\hat{P}$ is not a coboundary and generates $H^2(K;\mathbb{Z}/2)$. Note that

$$\begin{aligned} \partial P &= a+b-c, \\ \partial Q &= a-b+c. \end{aligned}$$

Consider two cochains $\alpha = \hat{a}+\hat{b}$ and $\beta = \hat{a}+\hat{c}$, they are cocycles because

$$\begin{aligned} 0 =&(\delta\alpha)(P) = (\delta\alpha)(Q), \\ 0 =&(\delta\beta)(P) = (\delta\beta)(Q). \end{aligned}$$

We need to show that $[\alpha] \neq [\beta]$ in $H^1(K;\mathbb{Z}/2)$. Assume the opposite. This means $\alpha-\beta$ is a coboundary. We only have one 0-simplex, so $\alpha-\beta = \hat{b}+\hat{c} = \delta\hat{x}$. But

$$1 = (\hat{b}+\hat{c})(b) = (\delta\hat{x})(b) = \hat{x}(x-x) = 0$$

A contradiction. This tells us α, β generates $H^1(K;\mathbb{Z}/2)$.

Next, we calculate the cup product. For dimension reasons, we have $[\hat{P}] \cup [\hat{P}] = 0$, and $[\hat{x}]$ is the unity in the cohomology ring. In degree 1,

$$\begin{aligned} (\alpha\cup\alpha)(P) &= \alpha(a)\cdot\alpha(b) = 1\cdot 1 = 1; \\ (\alpha\cup\alpha)(Q) &= \alpha(c)\cdot\alpha(a) = 0\cdot 1 = 0. \end{aligned}$$

This tells us $\alpha \cup \alpha = \hat{P}$ on the cochain level. So

$$[\alpha] \cup [\alpha] = [\hat{P}].$$

$$(\beta \cup \beta)(P) = \beta(a) \cdot \beta(b) = 1 \cdot 0 = 0;$$
$$(\beta \cup \beta)(Q) = \beta(c) \cdot \beta(a) = 1 \cdot 1 = 1.$$

This tells us $\beta \cup \beta = \hat{Q}$ on the cochain level, and we know that

$$\delta(\hat{a} + \hat{b} + \hat{c})(P) = (\hat{a} + \hat{b} + \hat{c})(a + b - c) = 1 + 1 + 1 = 1,$$
$$\delta(\hat{a} + \hat{b} + \hat{c})(Q) = (\hat{a} + \hat{b} + \hat{c})(a - b + c) = 1 + 1 + 1 = 1.$$

This implies $\hat{P} + \hat{Q}$ is a coboundary and $[\hat{P}] = [\hat{Q}]$ in $H^2(K; \mathbb{Z}/2)$. So

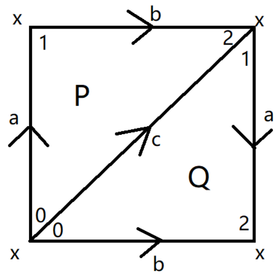
$$[\beta] \cup [\beta] = [\hat{P}].$$

$$(\alpha \cup \beta)(P) = \alpha(a) \cdot \beta(b) = 1 \cdot 0 = 0;$$
$$(\alpha \cup \beta)(Q) = \alpha(c) \cdot \beta(a) = 0 \cdot 1 = 0.$$

This tells us $\alpha \cup \beta = 0$ on the cochain level, so

$$[\alpha] \cup [\beta] = -([\beta] \cup [\alpha]) = 0.$$

(c) Let K be the Klein bottle. Consider the Δ-complex structure we used in (b)



We have a chain complex with $\mathbb{Z}/2$-coefficients

$$0 \to (\mathbb{Z}/2)^2 \xrightarrow{d} (\mathbb{Z}/2)^3 \xrightarrow{0} \mathbb{Z}/2 \to 0$$

where the boundary map d is given by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$. So $H_2(K; \mathbb{Z}/2) = \ker d$ is generated by the 2-cycle $P + Q$ and $H_1(K; \mathbb{Z}/2)$ is generated by a, b where $c = a + b$. $H_0(K; \mathbb{Z}/2)$ is generated by x. Note that to check ϕ is an isomorphism, it is the same as checking the Kronecker pairing is a perfect pairing. For $k = 0$ and $k = 2$, it is easy to see because on the only generator, we have

$$([\hat{x}], [x]) = \hat{x}(x) = 1,$$
$$([\hat{P}], P+Q) = \hat{P}(P+Q) = 1.$$

For $k = 1$, use $\alpha = \hat{a} + \hat{b}$ and $\beta = \hat{a} + \hat{c}$ as generators of $H^1(K; \mathbb{Z}/2)$ as before, we have

$$([\alpha], [a]) = (\hat{a} + \hat{b})(a) = 1,$$
$$([\alpha], [b]) = (\hat{a} + \hat{b})(b) = 1,$$
$$([\beta], [a]) = (\hat{a} + \hat{c})(a) = 1,$$
$$([\beta], [b]) = (\hat{a} + \hat{c})(b) = 0.$$

This can be written as a matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

which is invertible in $M_2(\mathbb{Z}/2)$, so this is also a perfect pairing. Thus, we have proved that $H^k(K; \mathbb{Z}/2) \to \hom(H_k(K; \mathbb{Z}/2), \mathbb{Z}/2)$ is an isomorphism for $k = 0, 1, 2$.

Problem 4
Prove that there does not exist a map $S^2 \to T$ that induces an isomorphism on H_2. In fact, prove this in two different ways: give a proof that uses homotopy groups and give a proof that uses cohomology and the cup product.

Solution: In (1) we prove this using homotopy groups and in (2), we prove this using cohomology rings.

(1) We check that the Hurewicz homomorphism is natural.

Claim: $f : X \to Y$ is a map of connected, pointed spaces. For $n \geq 1$, we have a commutative diagram

$$\begin{tikzcd} \pi_n(X) \arrow[r, "f_{*,1}"] \arrow[d, "h_X"'] & \pi_n(Y) \arrow[d, "h_Y"] \\ H_n(X) \arrow[r, "f_{*,2}"'] & H_n(Y) \end{tikzcd}$$

Both $f_{*,1}, f_{*,2}$ are induced by f and h_X, h_Y are Hurewicz homomorphism in degree n for space X and Y.

Proof: Let $u = [\Delta^n / \partial \Delta^n] \in H_n(S^n)$ be the canonical generator. For any $\sigma : S^n \to X$, σ induces a map $\sigma_* : H_n(S^n) \to H_n(X)$ and the Hurewicz homomorphism sends $[\sigma]$ to $\sigma_*(u)$. Note that

$$(f_{*,2} \circ h_X)([\sigma]) = f_{*,2}\sigma_*(u) = (f \circ \sigma)_*(u) = f_{*,1}(\sigma_*(u)).$$

This proves the commutativity. ∎

Suppose there exists a homomorphism $f : S^2 \to T$ such that $f_{*,2} : H_2(S^2) \to H_2(T)$ is an isomorphism. Apply the claim and we have a commutative diagram

$$\begin{array}{ccc} \pi_2(S^2) & \xrightarrow{f_{*,1}} & \pi_2(T) \\ {\scriptstyle h_{S^2}}\downarrow & & \downarrow{\scriptstyle h_T} \\ H_2(S^2) & \xrightarrow[f_{*,2}]{} & H_2(T) \end{array}$$

Note that S^2 is 1-connected, so h_{S^2} is an isomophism, by our assumption,

$$h_T \circ f_{*,1} = f_{*,2} \circ h_{S^2} : \pi_2(S^2) \to H_2(T)$$

is an isomorphism between $\mathbb{Z}$ and $\mathbb{Z}$. Consider the fiber sequence

$$\mathbb{Z}^2 \to \mathbb{R}^2 \to T$$

This gives us a long exact sequence in homotopy groups and we have $\pi_2(T) = \pi_2(\mathbb{R}^2) = 0$. So $h_T \circ f_{*,1} = 0$. A contradiction.

(2) Suppose there exists a homomorphism $f : S^2 \to T$ such that $f_{*,2} : H_2(S^2) \to H_2(T)$ is an isomorphism. Apply $\hom(-, \mathbb{Z})$ and we obtain an isomorphism

$$\hom(H_2(T), \mathbb{Z}) \to \hom(H_2(S^2), \mathbb{Z}).$$

From problem #2, we have a commutative diagram

$$\begin{array}{ccc} H^2(T) & \xrightarrow{f^*} & H^2(S^2) \\ \downarrow{\scriptstyle \phi} & & {\scriptstyle \phi}\downarrow \\ \hom(H_2(T), \mathbb{Z}) & \xrightarrow{\hom(f_*, \mathbb{Z})} & \hom(H_2(S^2), \mathbb{Z}) \end{array}$$

We have seen in class that ϕ is an isomorphism for T or S^2, thus, $f^* : H^2(T) \to H^2(S^2)$ is isomorphism. By definition, $f_* : H^*(T) \to H^*(S^2)$ can be viewed as a map of rings and let $[\hat{a}], [\hat{b}] \in H^1(T)$ be two generators and $[\hat{T}] \in H^2(T)$ be the generator of $H^2(T)$. We have seen in class that

$$f^*([\hat{T}]) = f^*([\hat{a}] \cup [\hat{b}]) = f^*([\hat{a}]) \cup f^*([\hat{b}]).$$

But $H^1(S^2) = 0$, so $f^*([\hat{a}]) = f^*([\hat{b}]) = 0$. So f^* cannot map $H^2(T)$ isomorphically to $H^2(S^2)$. A contradiction.

Problem 5
Prove that $\mathbb{R}P^2$ is not a retract of the Klein bottle.

Solution: Suppose $\mathbb{R}P^2$ is a retract of the Klein bottle K. There exists maps $i : \mathbb{R}P^2 \to K$ and $r : K \to \mathbb{R}P^2$ such that $r \circ i$ is the identity map of $\mathbb{R}P^2$. This induces a map in cohomology rings with coefficients $\mathbb{Z}/2$.

$$\begin{tikzcd} H^*(\mathbb{R}P^2;\mathbb{Z}/2) \arrow[r, "r^*"] \arrow[rr, bend right, "id"'] & H^*(K;\mathbb{Z}/2) \arrow[r, "i^*"] & H^*(\mathbb{R}P^2;\mathbb{Z}/2) \end{tikzcd}$$

Let α be the generator of $H^1(\mathbb{R}P^2;\mathbb{Z}/2)$, T be the generator of $H^2(\mathbb{R}P^2;\mathbb{Z}/2)$, β,γ be the two generators of $H^1(K;\mathbb{Z}/2)$ and P be the generator of $H^2(K;\mathbb{Z}/2)$. Since $i^*\circ r^* = id$, so i^* must be surjective and r^* must be injective. Either $i^*(\beta)=\alpha$ or $i^*(\gamma)=\alpha$. Without loss of generality, we can assume $i^*(\beta)=\alpha$ and $i^*(\gamma)=0$. This is impossible because from our calculation in problem #3, we have

$$T=\alpha\cup\alpha=i^*(\beta)\cup i^*(\beta)=i^*(\beta\cup\beta)=i^*(P)=i^*(\gamma\cup\gamma)=i^*\gamma\cup i^*\gamma=0.$$

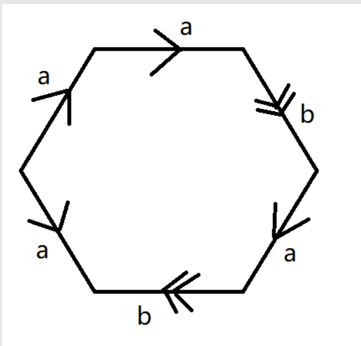
It is also impossible if we maps $i^*(\gamma)=i^*(\beta)=\alpha$ because

$$T=\alpha\cup\alpha=i^*(\beta)\cup i^*(\gamma)=i^*(\beta\cup\gamma)=0.$$

So we cannot have such ring homomorphisms.

Problem 6

Let X be obtained by identifying points on the bounadaryof a solid hexagon, as indicted in the following diagram:



(a) Calculate the homology and cohomology groups of X with $\mathbb{Z}/2$-coefficients.

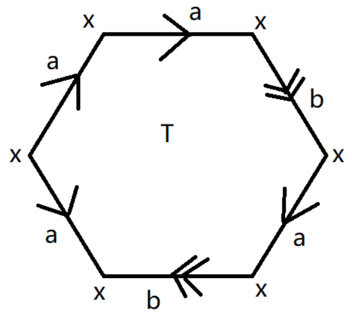
(b) Give a Δ-complex structure to X, for example by placing one point in the center of the hexagon, drawing lines to the outer vertices, and orienting the 2-simplices appropriately.

(c) Using your Δ-complex structure, give a 1-cocycle α with $\mathbb{Z}/2$-coefficients having the property that $\alpha(a)=1$ and $\alpha(b)=0$.

(d) Compute$\alpha\cup\alpha$ on all the 2-simplices in your picture. Is $\alpha\cup\alpha$ zero or nonzero in $H^2(X;\mathbb{Z}/2)$? Explain.

Solution:

(a) Consider the following celluar structure of X



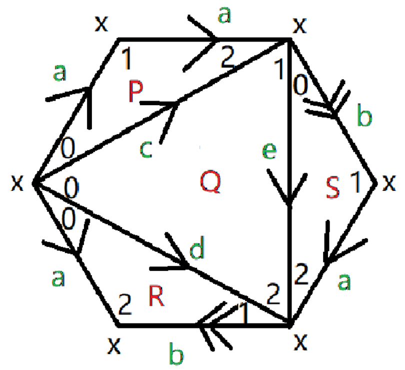
X has one 0-cell x, two 1-cell a, b and one 2-cell T. We have a cellular chain complex with coefficients $\mathbb{Z}/2$

$$0 \to \mathbb{Z}/2 \xrightarrow{0} (\mathbb{Z}/2)^2 \xrightarrow{0} \mathbb{Z}/2 \to 0$$

Note that all boundary maps are 0, so apply $\hom(-, \mathbb{Z}/2)$ will obtain all 0 coboundary maps. This implies

$$H_i(X; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0, 2; \\ (\mathbb{Z}/2)^2, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases} \qquad H^i(X; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0, 2; \\ (\mathbb{Z}/2)^2, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

(b) The following is a Δ-complex structure on X



(c) Consider the 1-cochain $\alpha = \hat{a} + \hat{d} + \hat{e}$ satisfying $\alpha(a) = 1$ and $\alpha(b) = 0$. Let us check this is a cocycle.

$$\begin{aligned}
\delta(\hat{a} + \hat{d} + \hat{e})(P) &= (\hat{a} + \hat{d} + \hat{e})(a + a - c) = 1 + 1 = 0, \\
\delta(\hat{a} + \hat{d} + \hat{e})(Q) &= (\hat{a} + \hat{d} + \hat{e})(c + e - d) = 1 + 1 = 0, \\
\delta(\hat{a} + \hat{d} + \hat{e})(S) &= (\hat{a} + \hat{d} + \hat{e})(b + a - e) = 1 + 1 = 0, \\
\delta(\hat{a} + \hat{d} + \hat{e})(R) &= (\hat{a} + \hat{d} + \hat{e})(a - b - d) = 1 + 1 = 0.
\end{aligned}$$

This proves $\alpha = \hat{a} + \hat{d} + \hat{e}$ is a 1-cocycle satisfying $\alpha(a) = 1$ and $\alpha(b) = 0$.

(d) We calculate $\alpha \cup \alpha$ on each 2-simplices.

$$\begin{aligned}
(\alpha \cup \alpha)(P) &= \alpha(a) \cdot \alpha(a) = 1 \cdot 1 = 1, \\
(\alpha \cup \alpha)(Q) &= \alpha(c) \cdot \alpha(e) = 0 \cdot 1 = 0, \\
(\alpha \cup \alpha)(S) &= \alpha(b) \cdot \alpha(a) = 0 \cdot 1 = 0, \\
(\alpha \cup \alpha)(R) &= \alpha(d) \cdot \alpha(b) = 1 \cdot 0 = 0.
\end{aligned}$$

This proves that $\alpha \cup \alpha = \hat{P}$ on the chain level. Suppose $\sigma \in C^1(X; \mathbb{Z}/2)$ satisfying $\delta\sigma = \hat{P}$. Then we have

$$\begin{aligned}
(\delta\sigma)(P) &= \sigma(c) = 1, \\
(\delta\sigma)(Q) &= \sigma(c) + \sigma(e) + \sigma(d) = 0, \\
(\delta\sigma)(S) &= \sigma(b) + \sigma(a) + \sigma(e) = 0, \\
(\delta\sigma)(R) &= \sigma(a) + \sigma(b) + \sigma(d) = 0.
\end{aligned}$$

We add the last two equations together and obtain

$$\sigma(e) + \sigma(d) = 0$$

But from the second and first equation, we know that $\sigma(e) + \sigma(d) + 1 = 0$. This leads to a contradiction, so $\alpha \cup \alpha = \hat{P}$ is not a coboundary, thus $[\alpha \cup \alpha] \neq 0$ in $H^*(X; \mathbb{Z}/2)$.