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Homework - Chapter 2 Exercises
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Instructor: Professor Nick Addington
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Exercise 2.3
Show that any Hausdoff space if of dimension zero.

Solution: Let X be a Hausdoff topological space and $x \in X$ is a point. It is easy to see that $\{x\}$ is irreducible. If $X = \{x\}$, then obviously $\dim X = 0$ since we only have one point. For any $y \in X$ that is not x, we have an open set

$$U_y \cap \{x\} = \varnothing$$

because X is Hausdoff. Then we have

$$X \setminus \{x\} = \bigcup_{y \in X, y \neq x} U_y$$

is open in X. This implies $\{x\}$ is closed and irreducible for any $x \in X$. Thus, the only irreducible subsets in X are singletons. Thus, we can conclude that $\dim X = 0$.

Exercise 2.4
Assume that

$$Y = Y_1 \cup \cdots \cup Y_r$$

is the primary decomposition of the Noetherian space Y into irreducible components. Show that

$$\dim Y = \max_{1 \leq i \leq r} \dim Y_i.$$

Solution: For $1 \leq i \leq r$, we know that $Y_i \subseteq Y$, so $\dim Y_i \leq \dim Y$. This implies that

$$\max_{1 \leq i \leq r} \dim Y_i \leq \dim Y.$$

On the other hand, suppose

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_k$$

is a maximal strictly increasing chains of irreducible subsets in Y. Then $Z_k = Y_i$ for some $1 \leq i \leq r$ because $Y_1, \ldots, Y_r$ are irreducible components of Y. Thus, this chain is also a strictly increasing chain of irreducible subsets in Y_i. This proves

$$\dim Y \leq \max_{1 \leq i \leq r} \dim Y_i.$$

Hence, we can conclude that

$$\dim Y = \max_{1 \leq i \leq r} \dim Y_i.$$

Exercise 2.6
Let $X = Z(zx, zy) \subseteq \mathbb{A}^2$. Describe X and determine $\dim X$. Exhibit two maximal chains of irreducible subvarieties of different lengths. Exhibit a hypersurface Z so that $Z \cap X$ is of dimension zero.

Solution: The ideal $I(X) = (zx, zy)$ has the primary decomposition

$$(zx, zy) = (z) \cap (x, y)$$

So X has two irreducible components $X_1 = Z(z)$ and $X_2 = Z(x, y)$. The coordinate ring

$$\begin{aligned} A(X_1) &= k[x, y, z]/(z) \cong k[x, y], \\ A(X_2) &= k[x, y, z]/(x, y) \cong k[z]. \end{aligned}$$

So by proposition 2.46, we have

$$\begin{aligned} \dim X_1 &= \dim A(X_1) = \dim k[x, y] = 2, \\ \dim X_2 &= \dim A(X_2) = \dim k[z] = 1. \end{aligned}$$

This implies that $\dim X = \max_{i=1,2} \dim X_i = 2$. We have the following two maximal chains of irreducible subvarieties

$$\begin{aligned} &Z(x, y, z) \subsetneq Z(y, z) \subsetneq Z(z), \\ &Z(x, y, z) \subsetneq Z(x, y). \end{aligned}$$

Consider the hypersurface $Z = Z(z-1) \subseteq \mathbb{A}^3$. The intersection

$$Z \cap X = Z(z-1) \cap Z(zx, zy) = Z(z-1, zx, zy).$$

Then we have

$$\begin{aligned} \dim Z \cap X &= \dim A(Z \cap X) \\ &= \dim k[x, y, z]/(z-1, zx, zy) \\ &\cong \dim k[x, y, z]/(z-1, x, y) \\ &\cong \dim k \\ &= 0. \end{aligned}$$

Exercise 2.11
Let $\psi : \mathbb{A}^3 \to \mathbb{A}^3$ be given as $(x, y, z) \mapsto (yz, xz, xy)$. Find all fibers of ψ and their ideals.

Solution: Let (a, b, c) be a point in $\mathbb{A}^3$.

If $a = b = c = 0$, then the preimage

$$\begin{aligned}\psi^{-1}(0,0,0) &= \{x = y = 0\} \cup \{x = z = 0\} \cup \{y = z = 0\} \\ &= Z(x,y) \cup Z(x,z) \cup Z(y,z).\end{aligned}$$

The fiber over $(0,0,0)$ is the union of three axes. Note that the maximal corresponding to $(0,0,0)$ is $\mathfrak{m}_0 = (x,y,z)$, so we have

$$\psi^* \mathfrak{m}_0 = (x,y) \cap (x,z) \cap (y,z) = (yz, xz, yz).$$

If only one of a,b,c equals 0, the other two are not zero, then the preimage $\psi^{-1}(a,b,c) = \varnothing$, and

$$\psi^* \mathfrak{m}_{(a,b,c)} = Z(\varnothing) = k[x,y,z].$$

If two of a,b,c equal 0, then the preimage

$$\begin{aligned}\psi^{-1}(0,0,c) &= Z(xy - c, z), \\ \psi^{-1}(0,b,0) &= Z(xz - b, y), \\ \psi^{-1}(a,0,0) &= Z(yz - a, x).\end{aligned}$$

The corresponding algebra map

$$\begin{aligned}\psi^* \mathfrak{m}_{(0,0,c)} &= (xy - c, z), \\ \psi^* \mathfrak{m}_{(0,b,0)} &= (xz - b, y), \\ \psi^* \mathfrak{m}_{(a,0,0)} &= (yz - a, x),\end{aligned}$$

If none of a,b,c equal 0, then the preimage of (a,b,c) has two points:

$$\left\{ (\sqrt{\frac{bc}{a}}, \sqrt{\frac{ac}{b}}, \sqrt{\frac{ab}{c}}), (-\sqrt{\frac{bc}{a}}, -\sqrt{\frac{ac}{b}}, -\sqrt{\frac{ab}{c}}) \right\}.$$

Let $\mathfrak{m}_1, \mathfrak{m}_2$ be the corresponding maximal ideals at these two points. We have

$$\psi^* \mathfrak{m}_{(a,b,c)} = \mathfrak{m}_1 \cap \mathfrak{m}_2.$$