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Problem 2.2.2
Given a map $f : S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. Construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.

Solution: Assume the opposite. If f does not have a fix point, then f is homotopic to the antipodal map a. For every point $x \in S^{2n}$, we know that $f(x) \neq a(x)$, so f is also homotopic to $a \circ a = id$. This implies the antipodal map a is homotopic to id, but we know

$$\deg a = (-1)^{2n+1} = -1 \neq \deg id.$$

This gives us a contradiction.

Now given a map $g : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$, Consider the quotient map $q : S^{2n} \to \mathbb{RP}^{2n}$ and the composition:

$$S^{2n} \xrightarrow{q} \mathbb{RP}^{2n} \xrightarrow{g} \mathbb{RP}^{2n}.$$

Note that q is also the universal covering. By Proposition 1.33 in the book, $g \circ q : S^{2n} \to \mathbb{RP}^{2n}$ can be lifted to S^{2n} if and only if

$$(g \circ q)_*(\pi_1(S^{2n})) \subseteq q_*(\pi_1(S^{2n})).$$

This can be done because S^{2n} is simply connected for $n > 0$. So we have a commutative diagram:

$$\begin{tikzcd} & & S^{2n} \arrow[d, "q"] \\ S^{2n} \arrow[r, "q"'] \arrow[urr, dashed, "\bar{g}"] & \mathbb{RP}^{2n} \arrow[r, "g"'] & \mathbb{RP}^{2n} \end{tikzcd}$$

There exists a point $x \in S^{2n}$ such that $\bar{g}(x) = x$ or $\bar{g}(x) = -x$, and in $\mathbb{RP}^{2n}$ we know x and $-x$ is identified, so g has a fixed point.

Consider the linear map $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by the following matrix A:

$$\begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}$$

A has no eigenvalue in $\mathbb{R}$, so it does not have eigenvectors. View every point in $\mathbb{RP}^{2n-1}$ as a line through origin in $\mathbb{R}^{2n}$. The linear map f defines a map $\bar{f} : \mathbb{RP}^{2n-1} \to \mathbb{RP}^{2n-1}$ has no fixed point because every line in $\mathbb{R}^{2n}$ is mapped to another line under f (f has no eigenvector).

Problem 2.2.4
Construct a surjective map $S^n \to S^n$ of degree zero, for each $n \geq 1$.

Solution: When $n = 1$, parametrize the unit circle S^1 in complex coordinates e^{it} for $0 \leq t \leq 2\pi$ and consider the following map from S^1 to S^1:

$$f_1(e^{it}) = \begin{cases} e^{2it}, & 0 \leq t \leq \pi, \\ e^{-2i(t-\pi)}, & \pi \leq t \leq 2\pi. \end{cases}$$

It is obvious that f_1 is continous and surjective. We claim that $\deg f_1 = 0$. Indeed, consider the point $z = -1 = e^{i\pi} = e^{-i\pi}$, it has two preimage $t_1 = \frac{\pi}{2}$ and $t_2 = \frac{3\pi}{2}$. Choose a neighbourhood U_1 and U_2 around t_1 and t_2 repesctively and let $U_1 \cap U_2 = \varnothing$. f_1 on U_1 is a homeomorphism with the same direction but on U_2 is a homeomorphism with the reverse direction, so by local degree theorem, $\deg f_1 = \deg f|_{t_1} + \deg f|_{t_2} = 1 + (-1) = 0$. $f_1 : S^1 \to S^1$ is a surjective map of degree zero.

Define $f_n : S^n \to S^n$ inductively as $f_n = Sf_{n-1}$ where S is the suspension. By Proposition 2.33, we have

$$\deg f_n = \deg f_{n-1} = \cdots = \deg f_1 = 0.$$

And for every n, every f_n is surjective since f_1 is surjective.

Problem 2.2.6
Show that every map $S^n \to S^n$ can be homotoped to have a fixed point if $n > 0$.

Solution: We embed S^n into R^{n+1} via the set

$$S^n = \left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,|\, x_0^2 + \cdots + x_n^2 = 1\right\}.$$

For $0 \leq i < j \leq n$, let R_{ij} denote the rotation by degree π of S^n with respect to the axis normal to the $x_i x_j$-plane.

Claim: Every R_{ij} is homotopic to the identity map $id : S^n \to S^n$.

Proof: Without loss of generality, assume $i = 0$ and $j = 1$. Write $r = \sqrt{1 - \sum_{k=2}^n x_k^2}$. We have a change of variable

$$x_0 = r\cos(\alpha),$$
$$x_1 = r\sin(\alpha)$$

for $0 \leq \alpha \leq 2\pi$. Consider the following homotopy F:

$$F : I \times S^n \to S^n,$$
$$(t, r, \alpha) \mapsto (r, \alpha + t\pi).$$

F is continous and $F(0,-) = id$, $F(1,-)$ changes only the coordinates x_0, x_1 and is just a rotation by π. ■

Note that for $(x_0, \ldots, x_n) \in S^n$, we have

$$R_{ij}(x_0, \ldots, x_n) = (x_0, \ldots, -x_i, \ldots, -x_j, \ldots, x_n).$$

Given a map $f : S^n \to S^n$, suppose n is an even number. If f has a fixed point, then we are done. If f does not have a fixed point, then by Exercise 2.2.2, we know that there exists some point $x \in S^n$ such that $f(x) = -x$. In this case $(R_{n-1,n} \circ \cdots \circ R_{23} \circ R_{01} \circ f)(x) = x$ since R_{ij} flip the signs of x_i, x_j and n is an even number. From the claim, we know that for any $0 \leq i < j \leq n$, $R_{ij} \circ f$ is homotopic to $id \circ f = f$.

Now assume n is an odd number. If f does not have a fixed point, then f is homotopic to the antipodal map $a : S^n \to S^n,\ a(x) = -x$. Consider

$$S^{n-1} = \left\{(0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \,|\, x_1^2 + \cdots + x_n^2 = 1\right\} \subseteq S^n.$$

We know that $a(0, x_1, \ldots, x_n) = (0, -x_1, \ldots, -x_n)$ is still the antipodal map when restricted S^{n-1}, by the previous discussion on even spheres, we know we can f can be homotoped to have a fixed point.

Problem 2.2.8
A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \to \mathbb{C}$, can always be extended to a continous map of one-point compactification $\hat{f} : S^2 \to S^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.

Solution: Identify S^2 with the complex projective space $\mathbb{CP}^1$. $\mathbb{C} \subset \mathbb{CP}^1$ has projective coordinates $[z : 1]$, and the point $\infty \in \mathbb{CP}^1$ has coordinate $[0 : 1]$. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial $f(z) = a_n z^n + \cdots + a_1 z + a_0$. We can extend f to a map $\mathbb{CP}^1 \to \mathbb{CP}^1$ by sending $[z : w]$ to $[\sum_{i=0}^n a_i z^i w^{n-i} : w^n]$. This map is well-defined and continous and if restricted to $\mathbb{C} \cong \{[z : w] \in \mathbb{CP}^1 \,|\, w = 1\}$, this map is just $f : \mathbb{C} \to \mathbb{C}$. We call it $\hat{f}$. Let S be the preimage of $[0 : 1]$ under $\hat{f}$. We have

$$S = \left\{[z : 1] \in \mathbb{CP}^1 \,|\, z \in \mathbb{C} \text{ and } \hat{f}(z) = 0\right\}.$$

Since $\mathbb{C}$ is algebraically closed, we can write

$$f(z) = c(z - z_1)^{i_1}(z - z_2)^{i_2} \cdots (z - z_k)^{i_k}$$

where $z_1, z_2, \ldots, z_k \in \mathbb{C}$, $c \in C^*$ and $i_1, i_2, \ldots, i_k$ are positive integers satisfying $i_1 + \cdots 1_k = n$.
<u>Claim:</u> Given a polynomial $f : \mathbb{C} \to \mathbb{C}$ where $f(z) = c(z - a)^n$ for some $n \geq 2$. The extension $\hat{f}$ is homotopic to an extension of a polynomial g which has n different roots in $\mathbb{C}$.
<u>Proof:</u> Without loss of generality, assume $a = 0$. Choose a very small real positive $\varepsilon > 0$ and n different complex numbers $a_1, \ldots, a_n$ such that for $i = 1, 2, \ldots, n$, we have $|a_i| < \varepsilon$. Define $g(z) = \prod_{i=1}^n (z - a_i)$ and a map

$$F : I \times \mathbb{C} \to \mathbb{C}\,,\ (t, z) \mapsto tg(z) + (1 - t)f(z).$$

This map is obviously continous on $\mathbb{C}$ and we need to show that it can be extended continouly to $\mathbb{CP}^1$. This is the same as showing that the zeros of F is bounded for $t \in [0,1]$. This is true because f and g have the same degree, so $\frac{f(z)}{g(z)}$ and $\frac{g(z)}{f(z)}$ is always bounded when $|z| \to \infty$. ■

From the above claim, we can assume $f(z) = \prod_{i=1}^n (z - z_i)$ for different $z_1, z_2, \ldots, z_n \in \mathbb{C}$. Choose a neighborhood U of z_1 whcih does not contain $z_2, \ldots, z_n$. Locally on U, f can be written as $z \mapsto z - z_1$ is an orientation-preserving homeomorphism (it is just a translation), so $\deg f|_{z_1} = 1$. By the local degree theorem, we have $\deg f = \sum_{i=1}^n \deg f|_{z_i} = n$, which is equal to the degree of f as a polynomial.

Now consider a polynomial

$$f(z) = (z - z_0)^m (z - z_1) \cdots (z - z_n)$$

where $z_0, z_1, \ldots, z_n \in \mathbb{C}$ are different complex numbers and $m \geq 2$ is an integer. From the previous discussion we know that $\deg \hat{f} = m + n$ and the local degree at $z_1, \ldots, z_n$ is 1. So by local degree theorem

$$\deg \hat{f}|_{z_0} = \deg \hat{f} - \sum_{k=1}^n \deg \hat{f}|_{z_k} = m + n - n = m.$$

Problem 2.2.9
Compute the homology groups of the following 2-complexes.

1. The quotient of S^2 obtained by identifying north and south poles to a point.
2. $S^1 \times (S^1 \vee S^1)$.
3. The space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interiors of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.
4. The quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by $\frac{2\pi}{m}$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $\frac{2\pi}{n}$ rotation.

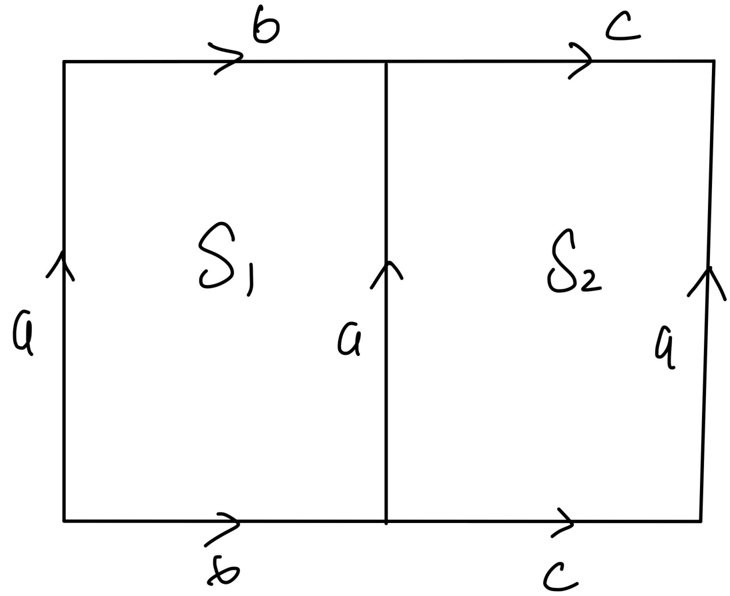
Solution: Write each of the following 2-complex as X.

1. Connect the north and south pole with one meridian and consider the following CW complex structure on X. X has one 0-cell, one 1-cell and one 2-cell. The 0-cell is the point obtained from identifying the north and south pole. The 1-cell is the meridian we choose, and the 2-cell is the S^2 that we are given at first. The boundary map $d_1 = 0$ since we only have one 0-cell. To compute the boundary map d_2, we compute the degree of the attaching map $\partial S^2 = S^1 \to S^1 = X^1$. Note that this maps the half of S^1 in one direction to the whole S^1 and another half in the opposite direction, so the degree is 0, which means $d_2 = 0$. We have the following cellular chain complex:

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

So we have $H_n(X) = \mathbb{Z}$ for $n = 0, 1, 2$ and 0 for all else.

2.



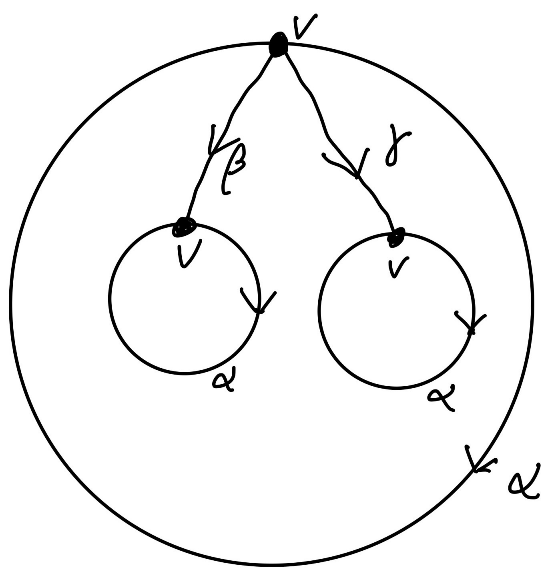
Consider the CW complex structure of X above with one 0-cell v, three 1-cell a, b, c and two 2-cells S_1, S_2. The cellular boundary map $d_1 = 0$ since we have one 0-cell. And we can see from the diagram that $d_2(S_1) = a + b - a - b = 0$ and $d_2(S_2) = a + c - a - c$. So we have the following cellular chain complex:

$$0 \to \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0.$$

So we have

$$H_n(X) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = 2, \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, & n = 1 \\ \mathbb{Z}, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

3.



Consider the CW complex struture of X given as above. It has one 0-cell v, three 1-cell α, β, γ and one 2-cell S. The boundary map $d_1 = 0$ since we only have one 0-cell. And

$$d_2(S) = \alpha + \beta - \alpha - \beta + \gamma - \alpha - \gamma = -\alpha.$$

So We have the following cellular chain complex:

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \to 0.$$

We can see that $\ker d_2 = 0$ and $\operatorname{im} d_2 = \langle -\alpha \rangle$. So $H_0(X) = \mathbb{Z}$ and

$$H_1(X) = \langle \alpha, \beta, \gamma \rangle / \langle -\alpha \rangle = \mathbb{Z} \oplus \mathbb{Z}.$$

All other homology groups are zero.

4. Consider the CW complex structure on $S^1 \times S^1$ with one 0-cell, two 1-cell and one 2-cell. The boundary map $d_1 = 0$ and d_2 is given by the word $aba^{-1}b^{-1}$, where a and b are 1-cells corresponding to two S^1. Identifying points in a that differ by $\frac{2\pi}{m}$ rotation is the same as making the attaching map into a^m. So for X, d_2 is given by the word $a^m b^n a^{-m} b^{-n}$. This map is still 0 since a, a^{-1} and b, b^{-1} appear the same times. So we have a cellular chain complex:

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \to 0.$$

So we have $H_n(X) = \mathbb{Z}$ for $n = 0, 2$ and $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$. All other homology groups are zero.

Problem 2.2.14
A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all $x \in S^n$ is called an even map. Show that an even map $S^n \to S^n$ must have even degree, and that the degree must in fact be zero when n is even. When n is odd, show there exist even maps of any given even degree.

Solution: Consider the quotient map $q : S^n \to \mathbb{RP}^n$ by identifying $x \sim -x$. Since $f(x) = f(-x)$ for all $x \in S^n$, f must factor through q:

$$S^n \xrightarrow{q} \mathbb{RP}^n \xrightarrow{f'} S^n.$$

This induces a map in top homology:

$$H_n(S^n) \xrightarrow{q_*} H_n(\mathbb{RP}^n) \xrightarrow{f'_*} H_n(S^n).$$

Note that $q : S^n \to \mathbb{RP}^n$ is the two-sheeted covering map. Write $X = \mathbb{RP}^n$ and consider the CW complex structure of X with only one cell in each dimension up to n. Note that every k-skeleton $X^k \cong \mathbb{RP}^k$ for $0 \leq k \leq n$. From Example 2.42, we could identify $H_n(X) = H_n(X^n)$ as the kernel

$d_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$. Consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
H_n(S^n) \arrow[r, "q_*"] & H_n(X) \arrow[d, "j_n"'] & \\
 & H_n(X^n, X^{n-1}) \arrow[r, "d_n"] \arrow[dr, "\partial_n"'] & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & H_{n-1}(X^{n-1}) \arrow[u, "j_{n-1}"'] \\
 & & 0 \arrow[u]
\end{tikzcd}$$

We know that $j_n \circ q_*$ is induced by

$$S^n \xrightarrow{q} \mathbb{RP}^n \to \mathbb{RP}^n/\mathbb{RP}^{n-1} \cong S^n.$$

From the Example 2.42, this map has degree $1+(-1)^n$ and note that j_n is injective, so $q_* : H_n(S^n) \cong \mathbb{Z} \to H_n(\mathbb{RP}^n)$ is either the zero map if n is even or a map sending $1 \in H_n(S^n)$ to $2 \in H_n(\mathbb{RP}^n) \cong \mathbb{Z}$ if n is odd. Therefore, we could conclude that an even map always has even degree, and if n is even, $\deg f = 0$ since $H_n(\mathbb{RP}^n) = 0$.

Now assume n is odd. The Example 2.32 tells us that for every integer $k \in \mathbb{Z}$, we have a map $S^1 \to S^1$ of degree k. Since Suspension preserves degree, we have obtained a map $f_k : S^n \to S^n$ of degree k. Consider the following composition $f = f_k \circ p \circ q$:

$$S^n \xrightarrow{q} \mathbb{RP}^n \xrightarrow{p} \mathbb{RP}^n/\mathbb{RP}^{n-1} \cong S^n \xrightarrow{f_k} S^n.$$

From the previous discussion, we know that

$$\deg f = \deg f_k \cdot \deg(p \circ p) = 2k.$$

We have constructed a map of degree $2k$ for every integer $k \in \mathbb{Z}$.

Problem 2.2.15
Show that if X is a CW compelx then $H_n(X^n)$ is free by identifying it with the kernel of the cellular boundary map $H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$.

Solution: Consider the following commutative diagram:

$$
\begin{tikzcd}
H_n(X^{n-1}) = 0 \arrow[dr] & & & & \\
& H_n(X^n) \arrow[dr, "j_n"'] & & & \\
& & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
& & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & \\
& & H_{n-1}(X^{n-2}) = 0 \arrow[ur] & &
\end{tikzcd}
$$

We know by definition $d_n = j_{n-1} \circ \partial_n$, and from the diagram we can see that j_{n-1} is injective. So $\ker d_n = \ker \partial_n$. By exactness, we know that $\ker \partial_n = \mathrm{im} j_n$ and since $H_n(X^{n-1}) = 0$, j_n is injective and we can identify $H_n(X^n)$ as the kernel of d_n. By Lemma 2.34, $H_n(X^n, x^{n-1})$ is free abelian, so $H_n(X^n) \cong \ker d_n$ as a subgroup of $H_n(X^n, X^{n-1})$ is also free.