Zhengdong Zhang
Email: zhengz@uoregon.edu
Course: MATH 636 - Algebraic Topology III
Instructor: Dr.Daniel Dugger

Homework 6
ID: 952091294
Term: Spring 2025
Due Date: 23^{th} May, 2025

Problem 1
Write down a complete description of the homology groups of $\mathrm{Gr}_3(\mathbb{C}^5)$. Determine as many intersection products between the Schubert classes $[\underline{a}]$ as you can. At least do all cases of complementary dimensions, and compute $[1,2,2]^2$ (here $\underline{a} = (1,2,2)$ is a Schubert symbol, not a jump sequence). Try to do some others.

Solution: Let $0 \leq a_1 \leq a_2 \leq a_3 \leq 2 = 5 - 3$ be the Schubert symbol of $\mathrm{Gr}_3(\mathbb{C}^5)$. We have ten different choices and the homology groups can be summarized as follows

degree	generators of $H_*(\mathrm{Gr}_3(\mathbb{C}^5)$
0	$[0,0,0]$
2	$[0,0,1]$
4	$[0,1,1], [0,0,2]$
6	$[0,1,2], [1,1,1]$
8	$[0,2,2], [1,1,2]$
10	$[1,2,2]$
12	$[2,2,2]$

Next, we are going to determine the intersection product in complementary dimension. Note the cohomology ring is Abelian because we only have cohomology in even dimensions. For simplicity, I will only write the representative matrices. We always choose the first Schubert symbol in the standard flag

$$0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \langle e_1, e_2, e_3 \rangle \subseteq \langle e_1, e_2, e_3, e_4 \rangle \subseteq \mathbb{C}^5$$

and the second Schubert symbol in the reverse flag

$$0 \subseteq \langle e_5 \rangle \subseteq \langle e_4, e_5 \rangle \subseteq \langle e_3, e_4, e_5 \rangle \subseteq \langle e_2, e_3, e_4, e_5 \rangle \subseteq \mathbb{C}^5$$

(1) $[0,0,1] \cdot [1,2,2]$

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

So the intersection has only one point and $[0,0,1] \cdot [1,2,2] = [0,0,0]$.

(2) $[1,1,1] \cdot [1,1,1]$

$$\begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & * & * \\ 0 & * & * & * & * \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

So the intersection has only one point and $[1,1,1]\cdot[1,1,1]=[0,0,0]$.

(3) $[0,1,2]\cdot[0,1,2]$

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & * \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

So the intersection has only one point and $[0,1,2]\cdot[0,1,2]=[0,0,0]$.

(4) $[1,1,1]\cdot[0,1,2]$

$$\begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

There does not exist a matrix satisfying the given two conditions. So the intersection has only no point and $[1,1,1]\cdot[0,1,2]=0$.

(5) $[0,0,2]\cdot[0,2,2]$

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & * \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

So the intersection has only one point and $[0,0,2]\cdot[0,2,2]=[0,0,0]$.

(6) $[0,1,1]\cdot[0,2,2]$

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = 0$$

So the intersection has no point and $[0,1,1]\cdot[0,2,2]=0$.

(7) $[0,1,1]\cdot[1,1,2]$

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

So the intersection has only one point and $[0,1,1]\cdot[1,1,2]=[0,0,0]$.

(8) $[0,0,2]\cdot[1,1,2]$

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & * \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & * & * \\ * & * & * & * & * \end{pmatrix} = 0$$

So the intersection has no point and $[0,1,1]\cdot[1,1,2]=0$.

(9) In this part we will determine the intersection product $[1,2,2]^2$. Note that $[1,2,2]\in H_{10}$, so $[1,2,2]^2\in H_{10+10-12}=H_8$. Suppose

$$[1,2,2]^2 = A[0,2,2]+B[1,1,2]$$

for some $A, B \in \mathbb{Z}$. We have

$$[1,2,2]^2[0,1,1] = A[0,2,2][0,1,1] + B[1,1,2][0,1,1] = B,$$
$$[1,2,2]^2[0,0,2] = A[0,2,2][0,0,2] + B[1,1,2][0,0,2] = A.$$

Suppose W is a 3-plane in the intersection $[1,2,2]^2[0,1,1]$, note that for all the 2's in the Schubert symbol, the condition is automatically satisfied. W needs to satisfy the following condition:

(i) $\dim W \cap F_2 \geq 1$ for some 2-plane F_2.

(ii) $\dim W \cap F_2' \geq 1$ for some 2-plane F_2'.

(iii) $\dim W \cap F_1' \geq 1$ for some 1-line F_1''.

(iv) $\dim W \cap F_3' \geq 2$ for some 3-plane F_3''.

(v) $\dim W \cap F_4' \geq 3$ for some 4-plane F_4''.

Here $F_1'' \subseteq F_3'' \subseteq F_4''$. The condition (iii) implies W contains a vector e_1 where $\langle e_1 \rangle = F_1''$. The condition (v) implies that W is contained in a 4-plane F_4''. For any generic 3-plane $F_3'' \subseteq F_4''$, we have

$$\dim W \cap F_3'' = \dim W + \dim F_3'' - \dim F_4'' = 3 + 3 - 4 = 2.$$

This implies that the condition (iv) is automatically satisfied.

We can see that W is uniquely determined by three lines: F_1'', $W \cap F_2$, $W \cap F_2'$. Thus, $B = 1$.

On the other hand, suppose W is a 3-plane in the intersection $[1,2,2]^2[0,0,2]$. W needs to satisfy the following conditions:

(i) $\dim W \cap F_2 \geq 1$ for some 2-plane F_2.

(ii) $\dim W \cap F_2' \geq 1$ for some 2-plane F_2'.

(iii) $\dim W \cap F_1'' \geq 1$ for some 1-line F_1''.

(iv) $\dim W \cap F_2'' \geq 2$ for some 2-plane F_2''.

The condition (iv) implies that W contains a 2-plane F_2'', and the condition (iii) is automatically true because $F_1'' \subseteq F_2''$. If F_2 intersects with F_2', then W is uniquely determined by $F_2 \cap F_2'$ and F_2''. If F_2 has no intersection with F_2', in this case one of them must intersect F_2'' because we are in $\mathbb{C}^5$, suppose it is F_2, then $\dim W \cap F_2 \geq 1$ is automatically satisfied, this means W is uniquely determined by F_2'' and $W \cap F_2'$. In both cases, W is unique. Thus, $A = 1$.

We can conclude that $[1,2,2]^2 = [0,2,2] + [1,1,2]$.

Problem 2
Compute $H_*(\Omega_{\underline{a}})$ where $\underline{a}$ is the Schubert symbol 012, and $\Omega_{\underline{a}} \hookrightarrow \mathrm{Gr}_3(\mathbb{C}^5)$. Observe that $\Omega_{\underline{a}}$ cannot be a manifold, as this would violate Poincaré Duality.

Solution: From the cellular structure of $\mathrm{Gr}_3(\mathbb{C}^5)$, we know that $\Omega_{\underline{a}}$is of dimension 6, and has 2 4-dimensional cells $[0,1,1]$ and $[0,0,2]$, but only 1 2-dimensional cell $[0,0,1]$. If $\Omega_{\overline{a}}$ is a manifold, then this will violate Poincaré duality as H_2 and H_4 have different ranks.

Problem 3

Fix $n \geq 1$ and $k \leq n$. Let $\eta_k \subseteq \mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n$ be the subspace of pairs (W, v) where $v \in W$. Let $p : \eta_K \to \mathrm{Gr}_k(\mathbb{R}^n)$ be the map sending (W, v) to W. Prove that p is a fiber bundle with fiber $\mathbb{R}^k$.

Solution: Given any k-plane $W \in \mathrm{Gr}_k(\mathbb{R}^n)$, we can view W as a $k \times n$ matrix, where each row in this matrix is a basis of the k-plane W in $\mathbb{R}^n$. Without loss of generality, we may assume the $k \times k$-minor from the first k columns is non-degenerate. In this case, we can choose the basis of W in the following way such that W can be viewed as a matrix

$$W = \left[\begin{array}{c|c} I_k & A \end{array} \right]$$

Consider the following subset $U \subseteq \mathrm{Gr}_k(\mathbb{R}^n)$: if we view k-planes as $k \times n$ matrix, then the set U is the set of all k-planes $V \in \mathrm{Gr}_k(\mathbb{R}^n)$ satisfying the first $k \times k$-minor is non-degenerate. Equivalently, U is the following set

$$U = \left\{ V \in \mathrm{Gr}_k(\mathbb{R}^n) \mid V = \left[\begin{array}{c|c} I_k & A \end{array} \right] \text{ where } A \in M_{k \times (n-k)}(\mathbb{R}) \right\} \cong \mathbb{R}^{k(n-k)}.$$

We need to construct a homeomorphism $p^{-1}(U) \cong U \times \mathbb{R}^k$. For any $(V, v) \in p^{-1}(U)$, the matrix form gives a basis of V

$$\begin{aligned} v_1 &= (1, 0, \ldots, 0, *, \ldots, *) \\ v_2 &= (0, 1, \ldots, 0, *, \ldots, *) \\ \cdots & \\ v_k &= (0, 0, \ldots, 1, *, \ldots, *) \end{aligned}$$

v can be written uniquely as

$$v = a_1 v_1 + \cdots + a_k v_k$$

for some $a_1, \ldots, a_k \in \mathbb{R}$. Define the map

$$\begin{aligned} f : p^{-1}(U) &\to U \times \mathbb{R}^k, \\ (V, v) &\mapsto (V, (a_1, \ldots, a_k)) \end{aligned}$$

Conversely, given any $(a_1, \ldots, a_k) \in \mathbb{R}^k$, there exists

$$v = a_1 v_1 + \cdots + a_k v_k \in V.$$

This gives an inverse map $f^{-1} : U \times \mathbb{R}^k \to p^{-1}(U)$. It is easy to see that f and f^{-1} are compatible with the projection map, and it remains to prove both f and f^{-1} are continuous. Consider the following commutative diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "f"] \arrow[d, tail] & U \times \mathbb{R}^k \\ U \times \mathbb{R}^n \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

The map

$$\tilde{f} : U \times \mathbb{R}^k \to U \times \mathbb{R}^n$$

is sending V to V and $v \in V \subseteq \mathbb{R}^n$ to $(a_1, \dots, a_k)$. Under the basis we choose previously, we can write

$$\begin{aligned} v = \quad & a_1 v_1 + \cdots + a_k v_k \\ = \quad & a_1(1, 0, \dots, 0, *, \dots, *) \\ & + a_2(0, 1, \dots, 0, *, \dots, *) \\ & + \cdots \\ & + a_k(0, 0, \dots, 1, *, \dots, *) \\ =& (a_1, \dots, a_k, *, \dots, *) \end{aligned}$$

So the map $\tilde{f}$ is just a projection map from $\mathbb{R}^n$ to $\mathbb{R}^k$ and thus continuous. The composition with the inclusion map is also continous, so f is continuous. Similarly, consider the commutative diagram

$$\begin{tikzcd} U \times \mathbb{R}^k \arrow[r, "f^{-1}"] \arrow[dr, "\widetilde{f^{-1}}"'] & p^{-1}(U) \arrow[d, tail] \\ & \mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n \end{tikzcd}$$

The same argument implies that the map $\widetilde{f^{-1}}$ is an inclusion, and thus continuous. The right vertical map is an inclusion of open set, so it is an open map. This implies f^{-1} is also continous.

Problem 4

Let $q : X \to Q$ be a surjection. Say that a map of spaces $f : X \to Z$ is "q-compatible" if whenever $q(x) = q(y)$ we have $f(x) = f(y)$ (this says that the identifications made by q are also made by f). The map q is a quotient map if and only if for every space Z and every map $f : X \to Z$ that is q-compatible, there is a map $\tilde{f} : Q \to Z$ such that $\tilde{f} \circ q = f$.

Prove that if $q : X \to Q$ is a quotient map and A is locally compact and Hausdoff, then

$$q \times id : X \times A \to Q \times A$$

is also a quotient map.

Solution: Let Z be any space and $f : X \times A \to Z$ be a $(q \times id)$-compatible map, namely for all $a \in A$, if $(q \times id)(x, a) = (q \times id)(y, a)$ for some $x, y \in X$, then $f(x, a) = f(y, a) \in Z$. Note that A is locally compact and Hausdoff, we have a bijection

$$\mathcal{T}op(X \times A, Z) \cong \mathcal{T}op(X, Z^A).$$

The map f is equivalent to a continuous map $g : X \to Z^A$ sending $x \in X$ to the map $a \mapsto f(x, a)$. We claim that the map g is q-compatible. Indeed, suppose $q(x) = q(y)$ for some $x, y \in X$, then $(q \times id)(x, a) = (q \times id)(y, a)$ for all $a \in A$. Since the map f is $(q \times id)$-compatible, we have $f(x, a) = f(y, a)$ for all $a \in A$. This implies the two maps $a \mapsto f(x, a)$ and $a \mapsto f(y, a)$ are the same map. So g is q-compatible. We know that $q : X \to Q$ is a quotient map, so there exists $\tilde{g} : Q \to Z^A$ such that $\tilde{g} \circ q = g$.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "q"'] & Z^A \\ Q \arrow[ur, dashed, "\tilde{g}"'] & \end{tikzcd}$$

The map $\tilde{g} : Q \to Z^A$ is equivalent to a continuous map $\tilde{f} : Q \times A \to Z$ sending $(p, a) \in Q \times A$ to $\tilde{g}(p)(a)$. We check that $\tilde{f} \circ (q \times id) = f$, namely the following diagram commutes.

$$\begin{tikzcd} X \times A \arrow[r, "f"] \arrow[d, "q \times id"'] & Z \\ Q \times A \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

For any $(x, a) \in X \times A$, we have

$$(\tilde{f} \circ (q \times id))(x, a) = \tilde{f}(q(x), a) = \tilde{g}(q(x))(a) = (\tilde{g} \circ q)(x)(a) = g(x)(a).$$

Note that $g(x)$ is an element in Z^A, and $g(x)(a) = f(x, a) \in Z$ because f and g is equivalent under the bijection

$$\mathcal{T}op(X \times A, Z) \cong \mathcal{T}op(X, Z^A).$$

This proves that the diagram commutes and $q \times id$ is a quotient map.

Problem 5

Let (X, x) be a pointed space. Recall that $PX \subseteq X'$ is the subspace of paths that end at x. Said differently, PX is defined by the pullback diagram

$$\begin{tikzcd} PX \arrow[r] \arrow[d] & X^I \arrow[d, "ev_1"] \\ * \arrow[r, "x"'] & X \end{tikzcd}$$

Convince yourself that maps $W \to PX$ are in bijective correspondence with maps $CW \to X$ sending the cone point to x (here CW is the cone on W).

If A is a CW-complex, prove that $ev_0 : PX \to X$ has the homotopy lifting property with respect to A. In particular, the fact that this holds whenever A is I^n (any $n \geq 0$) implies that $PX \to X$ is a Serre fibration.

Solution: Suppose we have a commutative diagram

$$\begin{tikzcd} A \times \{0\} \arrow[r, "f"] \arrow[d, tail] & PX \arrow[d, "ev_0"] \\ A \times I \arrow[r, "g"'] & X \end{tikzcd}$$

For all $a \in A$, the map $f : A \times \{0\} \to PX$ sends a to a path $f(a) : I \to X$. This is equivalent to a map $\tilde{f} : CA \times \{0\} \to X$ sending (a, t) to $f(a)(t)$. This is well-defined because all different paths ends at the same point $x \in X$. We know that A is a CW complex, so $A \times I$ and $C(A \times I)$ are also CW complexes, and $A \times I$ and $C(A \times \{0\})$ are subcomplexes of $C(A \times I)$. The inclusion of the subcomplex

$$C(A \times \{0\}) \cup A \times I \to C(A \times I)$$

is a homotopy equivalence because $A \times I$ is homotopy equivalent to $A \times \{0\} \subseteq C(A \times \{0\})$, and both cones are contractible. Consider the solid-arrow diagram

$$\begin{tikzcd} C(A \times \{0\}) \cup A \times I \arrow[r, "\tilde{f} \cup g"] \arrow[d] & X \arrow[d] \\ C(A \times I) \arrow[r] \arrow[ur, dashed, "F"'] & * \end{tikzcd}$$

$\tilde{f}$ restricting to $A \times \{0\}$ can be viewed as the composition of the map

$$A \times \{0\} \xrightarrow{f} PX \xrightarrow{ev_0} X.$$

By commutativity of the original diagram, this is equal to the map

$$g|_{A \times \{0\}} : A \times \{0\} \to X.$$

By GLP, we have a lifting $F : C(A \times I) \to X$ such that $F|_{C(A \times \{0\})} = \tilde{f}$ and $F|_{A \times I} = g$. We know that F is equivalent to a map $\tilde{F} : A \times I \to PX$ satisfying the following two diagrams

$$\begin{tikzcd} A \times \{0\} \arrow[r, "f"] \arrow[d] & PX \\ A \times I \arrow[ur, "\tilde{F}"'] & \end{tikzcd} \qquad \begin{tikzcd} & PX \arrow[d, "ev_0"] \\ A \times I \arrow[ur, "\tilde{F}"] \arrow[r, "g"'] & X \end{tikzcd}$$

This proves that original diagram has a lifting. Thus, we can conclude that $ev_0 : PX \to X$ has the homotopy lifting property with respect to any CW complex A.