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Homework - Week 8 Exercises
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Exercise 3.14
Suppose $1 < p < \infty$. $f \in L^p = L^p((0, +\infty))$ relative to Lebesgue measure, and

$$F(x) = \frac{1}{x}\int_0^x f(t)dt \qquad (0 < x < +\infty).$$

(a) Prove Hardy's inequality

$$||F||_p \leq \frac{p}{p-1}||f||_p$$

which shows that the mapping $f \to F$ carries L^p into L^p.

(b) Prove that equality holds only if $f = 0$ a.e.

(c) Prove that the constant $\frac{p}{p-1}$ cannot be replaced by a smaller one.

(d) If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

Solution:

(a) Assume first $f \geq 0$ and $f \in C_c((0, +\infty))$. Let $p, q \in (1, +\infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Using integration by parts, we have

$$\begin{aligned}
||F||_p^p &= \int_0^\infty F^p(x)dx \\
&= xF^p(x)\Big|_0^\infty - p\int_0^\infty xF^{p-1}(x)F'(x)dx \\
&= \frac{1}{x^{p-1}}\Big(\int_0^x f(t)dt\Big)\Big|_0^\infty - p\int_0^\infty xF^{p-1}(x)F'(x)dx.
\end{aligned}$$

Note that $f \geq 0$ and f has compact support, so $\int_0^\infty f(t)dt$ is bounded and

$$\lim_{x\to\infty} \frac{1}{x^{p-1}}\int_0^x f(t)dt = 0.$$

On the other hand, since $f(x)$ has compact support, there exists some small $\varepsilon > 0$ such that $f(x) = 0$ for all $x \in (0, \varepsilon)$. This implies that

$$\lim_{x\to 0^+} \frac{1}{x^{p-1}}\int_0^x f(t)dt = 0.$$

By definition of $F(x)$, we have

$$xF(x) = \int_0^x f(t)dt.$$

Take derivative with respect to x at both sides, and we get

$$F(x) + xF'(x) = f(x).$$

Hence, we have

$$\begin{aligned}||F||_p^p &= -p\int_0^\infty F^{p-1}(x)(f(x) - F(x))dx \\ &= -p\int_0^\infty F^{p-1}f(x) - F^p(x)dx \\ &= p||F||_p^p - p\int_0^\infty F^{p-1}(x)f(x)dx.\end{aligned}$$

Hence, use Hölder inequality and we obtain that

$$\begin{aligned}||F||_p^p &= \frac{p}{p-1}\int_0^\infty F^{p-1}(x)f(x)dx \\ &\leq \frac{p}{p-1}(\int_0^\infty F^{q(p-1)}(x)dx)^{\frac{1}{q}}(\int_0^\infty f^p(x)dx)^{\frac{1}{p}} \\ &= \frac{p}{p-1}||F||_p^{\frac{p}{q}}||f||_p\end{aligned}$$

Therefore, we obtain that

$$||F||_p^{p-\frac{p}{q}} = ||F||_p \leq \frac{p}{p-1}||f||_p.$$

Now assume $f \geq 0$ and $f \in L^p$. We know that by 3.14 Theorem, $C_c((0, +\infty))$ is dense in L^p, so there exists a sequence $\{f_n\}_{n=1}^\infty$ of continuous functions with compact support such that

$$\lim_{n\to\infty} ||f_n - f||_p^p = \lim_{n\to\infty}\int_0^\infty |f_n(x) - f(x)|^p dx = 0.$$

This implies that $||f_n - f||_\infty \to 0$ when $n \to \infty$. So we have

$$\begin{aligned}\lim_{n\to\infty} ||F_n - F||_p^p &= \lim_{n\to\infty}\int_0^\infty \frac{1}{x^p}\left|\int_0^x |f_n(t) - f(t)|dt\right|^p dx \\ &\leq \lim_{n\to\infty}\int_0^\infty \frac{1}{x^p}\cdot ||f_n - f||_\infty^p \cdot x^p dx \\ &= \lim_{n\to\infty}\int_0^\infty ||f_n - f||_\infty dx \\ &= 0\cdot\infty \\ &= 0.\end{aligned}$$

Hence, by Minkovski inequality and what we have proved for continuous functions with compact support, for all $n \geq 1$, we have

$$\begin{aligned}
||F||_p &= ||F - F_n + F_n||_p \\
&\leq ||F - F_n||_p + ||F_n||_p \\
&\leq ||F - F_n||_p + \frac{p}{p-1}||f_n||_p \\
&= ||F - F_n||_p + \frac{p}{p-1}||f_n - f + f||_p \\
&\leq ||F - F_n||_p + \frac{p}{p-1}||f_n - f||_p + \frac{p}{p-1}||f||_p
\end{aligned}$$

Let n goes to ∞, and we obtain that

$$||F||_p \leq \frac{p}{p-1}||f||_p.$$

Finally, for $f \in L^p$ is not necessarily positive, we have

$$\begin{aligned}
||F||_p &= \left(\int_0^\infty \frac{1}{x^p}\left|\int_0^x f(t)dt\right|^p dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_0^\infty \frac{1}{x^p}\left|\int_0^x |f(t)|dt\right|^p dx\right)^{\frac{1}{p}} \\
&\leq \frac{p}{p-1}\left(\int_0^\infty |f(x)|^p dx\right)^{\frac{1}{p}} \\
&= \frac{p}{p-1}||f||_p.
\end{aligned}$$

(b) We use the Hölder inequality in part (a), and the equality holds when the equality holds in the Hölder inequality. This happens when

$$\alpha F^p(x) = f^p(x)$$

for some $\alpha > 0$ and for $x \in (0, +\infty)$ almost everywhere. This is the same as saying

$$F(x) = \frac{1}{x}\int_0^x \alpha F(t)dt$$

for $0 < x < +\infty$ almost everywhere. Note that $F(x)$ is always continuous by definition, and in this case $F(x)$ is differentiable, so

$$xF'(x) = (\alpha - 1)F(x).$$

Note that in part (a), we prove that

$$||F||_p^p = \int_0^\infty F^p(x)dx = \frac{p}{p-1}\int_0^\infty F^{p-1}(x)f(x)dx.$$

This implies that

$$\alpha = \frac{p-1}{p}.$$

So we have a differential equation

$$xF'(x) = -\frac{1}{p}F(x).$$

Suppose $F(x) = \frac{p-1}{p}f(x)$ is a positive function and f is equal 0 almost everywhere. Then the differential equation can be written as

$$\frac{F'(x)}{F(x)} = -\frac{1}{p}x^{-1}.$$

It has a solution $F(x) = Cx^{-\frac{1}{p}}$ for some constant C, but in this case, $f(x) = \frac{pC}{p-1}x^{-\frac{1}{p}}$ is not in L^p. A contradiction. So $f = 0$ almost everywhere.

Next, assume f is not necessarily positive. Then apply the same above argument to $|f|$. And we find the same result.

(c) Consider the following function: for a large number $A > 1$, $f(x) = x^{-\frac{1}{p}}$ on the closed interval $[1, A]$ and 0 elsewhere. Then

$$F(x) = \begin{cases} 0, & \text{if } x \in (0,1]; \\ \frac{p}{p-1}(x^{-\frac{1}{p}} - x^{-1}), & \text{if } x \in (1, A]; \\ \frac{p}{p-1}(A^{1-\frac{1}{p}} - 1)x^{-1}, & \text{if } x \in (A, +\infty). \end{cases}$$

Then

$$\begin{aligned} ||F||_p^p &= (\frac{p}{p-1})^p \int_1^A (x^{-\frac{1}{p}} - x^{-1})^p dx + (\frac{p}{p-1})^p (A^{1-\frac{1}{p}} - 1)^p \int_A^{+\infty} x^{-p} dx \\ &= (\frac{p}{p-1})^p \int_1^A (x^{-\frac{1}{p}} - x^{-1})^p dx + (\frac{p}{p-1})^p (A^{1-\frac{1}{p}} - 1)^p (\frac{A^{1-p}}{p-1}) \\ &\geq (\frac{p}{p-1})^p \int_1^A (x^{-\frac{1}{p}} - x^{-1})^p dx. \end{aligned}$$

Hence

$$\frac{||F||_p^p}{||f||_p^p} \geq (\frac{p}{p-1})^p \frac{\int_1^A (x^{-\frac{1}{p}} - x^{-1})^p dx}{\log A}.$$

The last thing we need to show is that

$$\lim_{A\to+\infty} \frac{\int_1^A (x^{-\frac{1}{p}} - x^{-1})^p dx}{\log A} = 1.$$

For any $B < A$ large enough, note that

$$\begin{aligned}
\log A &= \int_1^A (x^{-\frac{1}{p}})^p dx \\
&> \int_1^A (x^{-\frac{1}{p}} - x^{-1})^p dx \\
&> \int_B^A (x^{-\frac{1}{p}} - x^{-1})^p dx \\
&> \int_B^A (x^{-\frac{1}{p}} - B^{\frac{1}{p}-1} x^{-\frac{1}{p}})^p dx \\
&= (1 - B^{\frac{1}{p}-1})^p \int_B^A x^{-1} dx \\
&= (1 - B^{\frac{1}{p}-1})^p (\log A - \log B).
\end{aligned}$$

Divide both sides by $\log A$, and we have

$$1 > \frac{\int_1^A (x^{-\frac{1}{p}} - x^{-1})^p dx}{\log A} > (1 - B^{\frac{1}{p}-1})^p (1 - \frac{\log B}{\log A}).$$

Choose $B = A - \varepsilon$ and let $\varepsilon \to 0$ and $A \to +\infty$. We know that

$$\lim_{B \to +\infty} (1 - B^{\frac{1}{p}-1})^p = 1.$$

Thus, we can conclude that

$$\lim_{A \to +\infty} \frac{\int_1^A (x^{-\frac{1}{p}} - x^{-1})^p dx}{\log A} = 1.$$

(d) Since $f > 0$, there exists some $\delta > 0$ and some measurable set $E \subseteq X$ with $\mu(E) > 0$ such that $f(x) > \delta$ for all $x \in E$. Then

$$F(x) = \frac{1}{x} \int_0^x f(t) dt \geq \frac{1}{x} \cdot \delta m(E).$$

Here $\frac{\delta m(E)}{x}$ is not in L^1 as $\delta m(E) > 0$. So $F \notin L^1$.

Exercise 3.16

Prove Egoroff's theorem: If $\mu(X) < \infty$, if $\{f_n\}$ is a sequence of complex measurable functions which converges pointwise at every point of X, and if $\varepsilon > 0$, there is a measurable set $E \subset X$, with $\mu(X - E) < \varepsilon$ such that $\{f_n\}$ converges uniformly on E.

Solution: For any $n, k \geq 1$, consider the following set:

$$S(n,k) = \bigcap_{i,j>n} \left\{ x \in X : |f_i(x) - f_j(x)| < \frac{1}{k} \right\}.$$

For every fixed k, we have

$$\mu(S(n,k)) \to \mu(X), \qquad n \to \infty$$

because $\{f_n\}$ converges pointwise. Given any $\varepsilon > 0$, for every fixed k, there exists $n_k \geq 1$ such that

$$\mu(X) - \mu(S(n_k,k)) = \mu(X - S(n_k,k)) < \frac{\varepsilon}{2^k}$$

since $\mu(X)$ is finite. Let $E = \cap_{k\geq 1} S(n_k,k)$. We have

$$\mu(X-E) = \mu(X - \cap_{k\geq 1} S(n_k,k)) \leq \sum_{k\geq 1} \mu(X - S(n_k,k)) < \sum_{k\geq 1} \frac{\varepsilon}{2^k} \leq \varepsilon.$$

We claim that $\{f_n\}$ is uniformly convergent on E. Indeed, for any $\delta > 0$, there exists p such that $\frac{1}{p} < \delta$. Hence, for any $i, j > n_p$, we have

$$|f_i(x) - f_j(x)| < \frac{1}{p} < \delta$$

for any $x \in E \subseteq S(n_p, p)$. This proves that $\{f_n\}$ converges uniformly on E.

This theorem is not true if $\mu(X) = +\infty$. Consider $X = [0, +\infty)$ and $f_n = \chi_{[n-1,n]}$. f_n converges to 0 pointwise. Suppose there exists a set $E \subseteq (0, +\infty)$ such that $\mu(X - E) < 1$ and f_n converges uniformly on E. Choose $\delta = \frac{1}{2}$, there exists N such that for any $n > N$, we have $|f_n(x)| < \frac{1}{2}$ for any $x \in E$. This implies that $E \subseteq [0, N]$. This is a contradiction because $\mu(X - E) < 1$ and $\mu(X) = +\infty$. So such a set E cannot exist.

Now assume we have a family of functions $\{f_t\}$ where $t \in \mathbb{R}^+$ with pointwise convergence

$$\lim_{t\to\infty} f_t(x) = f(x)$$

for all $x \in X$, and $t \to f_t(x)$ is continuous for every $x \in X$. Consider the set

$$S(r,k) = \bigcap_{t>r} \left\{ x \in X : |f_r(x) - f(x)| < \frac{1}{k} \right\}$$

where $k \in \mathbb{Z}_+$ and $r \in \mathbb{R}^+$. For every fixed k, we have

$$\mu(S(r,k)) \to \mu(X)$$

as $r \to +\infty$ since $t \to f_t(x)$ is continuous and f_t converges to f pointwise. The rest of the proof is similar.

Exercise 4.1
If M is a closed subspace of H, prove that $M = (M^\perp)^\perp$. Is there a similar statement for subspaces M which are not necessarily closed?

Solution: Fix any $v \in M$, we know that by definition $(v, w) = 0$ for any $w \in M^\perp$. This proves that $v \in (M^\perp)^\perp$. Thus, $M \subseteq (M^\perp)^\perp$. Conversely, suppose $v \in (M^\perp)^\perp$. We know that M is a closed subspace of H, so H has a decomposition

$$H = M \oplus M^\perp.$$

By 4.11 Theorem, v has a unique decomposition $v = v_1 + v_2$ where $v_1 \in M$ and $v_2 \in M^\perp$. So here $v - v_1 = v_2 \in M^\perp$. On the other hand, $v \in (M^\perp)^\perp$ and we have proved that $v_1 \in M \subseteq (M^\perp)^\perp$. Since $(M^\perp)^\perp$ is a subspace of H, we can say that $v - v_1 \in (M^\perp)^\perp$. We have proved that

$$v - v_1 \in (M^\perp)^\perp \cap M^\perp.$$

Note that $M^\perp$ is also a closed subspace of H, so H has a decomposition

$$H = M^\perp \oplus (M^\perp)^\perp.$$

This implies that $v - v_1 = 0$, so $v = v_1 \in M$. Hence, $(M^\perp)^\perp \subseteq M$, and we can conclude that $M = (M^\perp)^\perp$.

The statement is not true if M is not closed. Let $H = \ell^2(\mathbb{N})$ be the Hilbert space of sequences

$$\left\{ x = (a_n)_{n=1}^\infty : a_n \in \mathbb{R}, \sum_{n=1}^\infty |a_n|^2 < +\infty \right\}.$$

Consider the subspace M consisting of sequences with only finitely many nonzero entries. It is easy to see that this is a linear subspace of H. Let $\{x_N\}_{N=1}^\infty \subseteq H$ be the following:

$$x_N = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N}, 0, \ldots).$$

Suppose $x = \lim_{N\to\infty} x_N$, and it is not hard to see that

$$||x||^2 = \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6} < +\infty.$$

So $x \in H$ but $x \notin M$. This shows that M is not closed. Suppose $y \in H$ satisfying that $(x, y) = 0$ for all $x \in M$. Then we claim that all entries of y must be 0. Indeed, if for some $i \geq 1$, $y = (b_n)_{n=1}^\infty$ has some $b_i \neq 0$. Consider the following element $x \in M$ where only the ith entry equal $b_i \neq 0$. Then $(x, y) = b_i^2 \neq 0$. A contradiction. So $y = 0 \in H$. This proves that $M^\perp = 0$, and thus

$$M \subsetneq H = (0)^\perp = (M^\perp)^\perp.$$

Exercise 4.3
Show that $L^p(T)$ is separable if $1 \leq p < \infty$, but that $L^\infty(T)$ is not separable.

Solution: Let $I = (a, b)$ be an open interval satisfying $-\pi < a < b < \pi$ where $a, b \in \mathbb{Q}$ and χ_I is the characteristic function. We know that the set of all such functions χ_I is a countable set. Let A be the set of all finite linear combinations of such functions with rational coefficients. A is also countable and $A \subset L^p(T)$ for $1 \leq p < +\infty$. We want to show that A is dense in $L^p(T)$.

Given a function $f \in L^p(T)$ and any $\varepsilon > 0$, by 3.14 Theorem, the compactly supported continuous functions $C_c([-\pi, \pi])$ is dense in $L^p(T)$, so there exists a continuous function with compact support

$$g : [-\pi, \pi] \to \mathbb{C}$$

such that $||f - g||_p < \frac{\varepsilon}{2}$. Note that the support supp$g$ is a compact subset of $[-\pi, \pi]$. Now use finitely many small open intervals with rational endpoints to cover supp$g$, satisfying the variation of g is smaller than $\frac{\varepsilon}{2}$ on every small open interval (this can be done because g is continuous). Then there exists a function $h \in A$ such that $||h - g||_\infty < \frac{\varepsilon}{2}$ by construction and

$$\begin{aligned} ||h-g||_p &= \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(t) - g(t)|^p dt \right\}^{\frac{1}{p}} \\ &\leq \left\{ \frac{1}{2\pi} ||h-g||_\infty^p \cdot 2\pi \right\}^{\frac{1}{p}} \\ &< \frac{\varepsilon}{2} \end{aligned}$$

Hence, By Minkovski inequality, we have

$$||f - h||_p \leq ||f - g||_p + ||g - h||_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that A is countable and dense in $L^p(T)$, so $L^p(T)$ is separable.

To show that $L^\infty(T)$ is not separable, we use the following claim:

Claim: Let X be a complete metric space. If there exists a family $\{E_i\}_{i \in I} \subseteq X$ of open sets where I is an uncountable set satisfying $E_i \cap E_j = \varnothing$ for any $i \neq j$. Then X is not separable.

Proof: Assume the opposite that X is separable, then there exists a countable dense subset $U = \{x_n : 1 \leq n\} \subseteq X$. For every $i \in I$, the intersection $E_i \cap U$ is non-empty as U is dense in X. Namely, there exists a positive integer m such that $x_m \in E_i \cap U$. This defines function

$$\begin{aligned} I &\to \mathbb{N}, \\ i &\mapsto m. \end{aligned}$$

Moreover, this function is injective. Indeed, if i, j in I have the same image m, then $x_m \in E_i \cap E_j$. We know by assumption that$E_i \cap E_j = \varnothing$ for $i \neq j$. So $i = j$. This is contradiction because we cannot have an injective function from uncountable set I to the countable set $\mathbb{N}$. ■

We need to construct an uncountable disjoint family of open subsets in $L^\infty(T)$. We first divide the interval $[-\pi, \pi)$ into a disjoint family $\bigcup_{n=0}^\infty$ where $I_0 = [-\pi, 0)$ and

$$I_n = [-\pi + (\sum_{k=1}^{n} \frac{1}{2^k}) \cdot 2\pi, -\pi + (\sum_{k=1}^{n+1} \frac{1}{2^k}) \cdot 2\pi), \quad n \geq 1.$$

It is easy to see that $[-\pi, \pi) = \bigcup_{n=0}^\infty I_n$ and $I_i \cap I_j = \varnothing$ if $i \neq j$. Let $\{a_n\}_{n=0}^\infty$ be a sequence taking values in only 0 or 1. For any such a sequence $a = (a_n)$, define a function

$$f_a(x) = 2a_n, \quad \text{if } \ x \in I_n.$$

Obviously $f_a \in L^\infty(T)$ and if $a \neq b$ are two different sequences, then f_a and f_b must differ on some interval I_n, so

$$||f_a - f_b||_\infty \geq 2.$$

For any sequence Let E_a be the set

$$E_a = \{f_b : ||f_b - f_a||_\infty < 1\} \subset L^\infty(T).$$

E_a is an open ball centered at f_a with radius 1 in $L^\infty(T)$. We claim that the family $\{E_a\}_a$ is what we are looking for. From our discussion above we can see that this is a disjoint family. The choice of a can be made from all functions $\mathbb{Z}_{\geq 0} \to \{0, 1\}$ with cardinality $2^{\aleph_0}$. This is uncountable.

Exercise 4.5
If $M = \{x : Lx = 0\}$ where L is a continuous linear functional on H. Prove that $M^\perp$ is a vector space of dimension 1 (unless $M = H$).

Solution: By 4.12 Theorem, there exists a unique $y \in H$ such that

$$Lx = (x, y)$$

for all $x \in H$. Hence, $M = \ker L$ can be written as

$$M = \{x \in H : (x, y) = 0\} = y^\perp.$$

Note that the one dimensional subspace $\langle y \rangle$ is closed, from exercise 4.1, we know that

$$(M^\perp)^\perp = (y^\perp)^\perp = \langle y \rangle.$$

This proves that $M^\perp$ is a vector space of dimension 1.

Exercise 4.9
If $A \subset [0, 2\pi]$ and A is measurable, prove that

$$\lim_{n\to\infty} \int_A \cos nx dx = \lim_{n\to\infty} \int_A \sin nx dx = 0.$$

Solution: Consider the Fourier series for the characteristic function χ_A:

$$\sum_{-\infty}^{+\infty} c_n e^{int} = a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} \chi_A e^{-int}dt = \frac{1}{2\pi}\int_A e^{-int}dt,$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} \chi_A \cos ntdt = \frac{1}{\pi}\int_A \cos ntdt,$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} \chi_A \sin ntdt = \frac{1}{\pi}\int_A \sin ntdt.$$

The Parseval theorem implies that

$$\mu(A) = ||\chi_A||_2^2 = \sum_{-\infty}^{+\infty} |c_n|^2 = \sum_{n=0}^{\infty} |a_n|^2 + \sum_{n=1}^{\infty} |b_n|^2 < +\infty.$$

So both $\sum_{n=0}^{\infty} |a_n|^2$ and $\sum_{n=1}^{\infty} |b_n|^2$ is finite. Hence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\int_A \cos nxdx = 0,$$
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\int_A \sin nxdx = 0.$$

Exercise 4.11
Find a nonempty closed set E in $L^2(T)$ that contains no element of smallest norm.

Solution: Let $\{u_k\}_{k=1}^{\infty}$ be a maximal orthonormal set in $L^2(T)$ with $||u_k||_2 = 1$ for all k. Consider the set

$$f_k = \left\{(1+\frac{1}{k})u_k\right\}_{k=1}^{\infty}.$$

This is a closed set as the maximal orthonormal set is closed. And we have

$$||f_k||_2 = (1+\frac{1}{k})||u_k||_2 = 1+\frac{1}{k}.$$

So the norm is decreasing, but no element has norm 1.
