Zhengdong Zhang
Email: zhengz@uoregon.edu
Course: MATH 636 - Algebraic Topology III
Instructor: Dr.Daniel Dugger

Homework 8
ID: 952091294
Term: Spring 2025
Due Date: 6^{th} June, 2025

Problem 1
Suppose that M is a compact 3-manifold with $\pi_1(M) \cong \mathbb{Z}/5$.

(a) Prove that M is orientable, and then calculate all of the homology and cohomology groups of M.

(b) Prove that every map $M \to \mathbb{R}P^3$ has even degree.

Solution:

Problem 2

(a) Explain why the Euler characteristic of an odd-dimensional compact manifold must be zero.

(b) Suppose that M is a $(2d+1)$-dimensional compact manifold, and let $W = \partial M$. Let X be the manifold obtained by gluing two copies of M together along their boundary. Using Mayer-Vietoris (or otherwise) prove that $\chi(W) \equiv \chi(X)$ mod 2, and so deduce that $\chi(W)$ must be even.

Solution:

Problem 3
Suppose that there is a fiber bundle $p : X \to S^8$ with fiber S^3.

(a) Prove that X is an orientable manifold.

(b) Prove that $H_*(X)$ is isomorphic to $H_*(S^3 \times S^8)$.

Solution:

Problem 4
Compute the cohomology ring of $\mathbb{R}P^4 \vee S^5$ with $\mathbb{Z}/2$-coefficients. Then use this to prove that $\mathbb{R}P^4 \vee S^5$ is not homotopy equivalent to a compact manifold.

Solution:

Problem 5
Suppose that X is a compact, orientable n-manifold and that $S^n \to X$ is a map of positive degree. Prove that $H_*(X;\mathbb{Q}) \cong H_*(S^n;\mathbb{Q})$.

Solution:

Problem 6
Find the mistake in the following "proof" that $0 = 1$:

Let $A : S^2 \to S^2$ be the antipodal map, and $p : S^2 \to \mathbb{R}P^2$ the projection. Consider the diagram

$$\begin{array}{ccc} \pi_2(S^2) & \xrightarrow{A_*} & \pi_2(S^2) \\ \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_2} \\ H_2(S^2) & \xrightarrow{A_*} & H_2(S^2) \end{array}$$

where h_2 is the Hurewicz map. We know that h_2 is an isomorphism, and we know that the lower map A_* is multiplication by $(-1)^3$. So it follows that the upper A_* is also multiplication by (-1).

Next consider the diagram

$$\begin{tikzcd} \pi_2(S^2) \arrow[rr, "A_*"] \arrow[dr, "p_*"'] & & \pi_2(S^2) \arrow[dl, "p_*"] \\ & \pi_2(\mathbb{R}P^2) & \end{tikzcd}$$

This commutes because of functoriality, since $p \circ A = p$. We know from the long exact sequence for the fibration $p : S^2 \to \mathbb{R}P^2$ that p_* is an isomorphism. Let $g \in \pi_2(S^2)$ be a generator. Then we have

$$p_*(g) = p_*(A_*(g)) = p_*(-g) = -p_*(g).$$

But $\pi_2(\mathbb{R}P^2) \cong \pi_2(S^2) \cong \mathbb{Z}$, and so the above equation implies $p_*(g) = 0$. Therefore p_* is the zero map. But we have already said that p_* is an isomorphism, therefore $\pi_2(\mathbb{R}P^2) = 0$. Since we have also said that $\pi_2(\mathbb{R}P^2) \cong \mathbb{Z}$, it must be that $\mathbb{Z} \cong 0$. So $\mathbb{Z}$ has only one element and, in particular, $0 = 1$.

Solution: