

**Problem 1**

Compute all the homology groups for the spaces in parts (a) and (b) below, and use your calculations to show that

- (a)  $\mathbb{R}P^2 \times S^3$  and  $\mathbb{R}P^3 \times S^2$  have isomorphic homotopy groups (in all dimensions), but non-isomorphic homology groups.
- (b)  $S^4 \times S^2$  and  $\mathbb{C}P^3$  have isomorphic homology groups but non-isomorphic homotopy groups.

*Solution:*

- (a) Let  $X = \mathbb{R}P^2 \times S^3$  and  $Y = \mathbb{R}P^3 \times S^2$ . It is easy to see that both  $X$  and  $Y$  are path-connected, so  $\pi_0(X) = \pi_0(Y) = *$ . By direct calculation, we have

$$\begin{aligned}\pi_1(X) &= \pi_1(\mathbb{R}P^2) \times \pi_1(S^3) = \mathbb{Z}/2, \\ \pi_1(Y) &= \pi_1(\mathbb{R}P^3) \times \pi_1(S^2) = \mathbb{Z}/2.\end{aligned}$$

This implies  $\pi_1(X) \cong \pi_1(Y)$ . Recall that for all  $n \geq 2$ , the universal covering space of  $\mathbb{R}P^n$  is  $S^n$ . So the universal covering space of  $X$  and  $Y$  are both isomorphic to  $S^2 \times S^3 \cong S^3 \times S^2$ . The long exact sequence in homotopy groups tells us that

$$\pi_n(X) \cong \pi_n(Y) \cong \pi_n(S^3 \times S^2)$$

for all  $n \geq 2$ . Thus, we can conclude that  $X$  and  $Y$  have the same homotopy groups.

For the homology groups, note that the homology groups of  $S^3$  and  $S^2$  are all free. By Künneth theorem, we have

$$\begin{aligned}H_n(X) &= \bigoplus_{p+q=n} H_p(\mathbb{R}P^2) \otimes H_q(S^3), \\ H_n(Y) &= \bigoplus_{p+q=n} H_p(\mathbb{R}P^3) \otimes H_q(S^2).\end{aligned}$$

The homology groups of each space is listed below: We can obtain the of  $X$  and  $Y$  by tensoring

	$H_*(\mathbb{R}P^2)$	$H_*(S^3)$		$H_*(\mathbb{R}P^3)$	$H_*(S^2)$
3	0	$\mathbb{Z}$	3	$\mathbb{Z}$	0
2	0	0	2	0	$\mathbb{Z}$
1	$\mathbb{Z}/2$	0	1	$\mathbb{Z}/2$	0
0	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$

at each degree, this gives us

$$H_i(X) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 3; \\ \mathbb{Z}/2, & \text{if } i = 1, 4; \\ 0, & \text{otherwise.} \end{cases} \quad \left| \quad H_i(Y) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2, 5; \\ \mathbb{Z}/2, & \text{if } i = 1; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

From this we can see that  $X$  and  $Y$  have non-isomorphic homology groups.

- (b) We know that  $\mathbb{C}P^3$  has a cellular structure with one 0-cell, one 2-cell, one 4-cell, and one 6-cell. The boundary maps in the cellular chain complex are all zero, so  $H_i(\mathbb{C}P^3) = \mathbb{Z}$  for  $i = 0, 2, 4, 6$  and 0 otherwise. For the space  $S^4 \times S^2$ , use Künneth theorem and note that  $S^2$  does not have torsion in homology, so  $H_i(S^4 \times S^2) = \mathbb{Z}$  for  $i = 0, 2, 4, 6$  and 0 otherwise. This shows that  $S^4 \times S^2$  and  $\mathbb{C}P^3$  have isomorphic homology groups.

Recall that we have a fibration  $S^1 \rightarrow S^7 \rightarrow \mathbb{C}P^3$ . This induces a long exact sequence in homotopy groups

$$\cdots \rightarrow \pi_3(S^1) \rightarrow \pi_3(S^7) \rightarrow \pi_3(\mathbb{C}P^3) \rightarrow \pi_2(S^1) \rightarrow \cdots$$

Note that  $\pi_3(S^1) = \pi_2(S^1)$  is trivial. This implies that  $\pi_3(\mathbb{C}P^3) \cong \pi_3(S^7) = \{1\}$  is also trivial. On the other hand, we know that

$$\pi_3(S^4 \times S^2) \cong \pi_3(S^4) \times \pi_3(S^2) = \mathbb{Z}.$$

This implies that  $S^4 \times S^2$  and  $\mathbb{C}P^3$  have non-isomorphic homotopy groups.

### Problem 2

Let  $I_*$  be the chain complex concentrated in degree 0 and 1 with  $I_1 = \mathbb{Z}\langle e \rangle$ ,  $I_0 = \mathbb{Z}\langle a, b \rangle$ , and  $d(e) = b - a$ . Note that this is the simplicial chain complex for  $\Delta_1$ . Let  $C_*$  and  $D_*$  be chain complexes.

- Describe the chain complex  $I_* \otimes C_*$  by giving the groups in each degree as well as the boundary maps.
- Let  $F : I_* \otimes C_* \rightarrow D_*$  be a chain map. Define  $f, g : C_* \rightarrow D_*$  by  $f(x) = F(a \otimes x)$  and  $g(x) = F(b \otimes x)$ . Likewise, define  $s_n : C_n \rightarrow D_{n+1}$  by  $s_n : C_n \rightarrow D_{n+1}$  by  $s_n(x) = F(e \otimes x)$ . Prove that  $f$  and  $g$  are chain maps and the collection  $\{s_n\}$  is a chain homotopy between  $f$  and  $g$ .

*Solution:*

- (a) We denote both the boundary map in  $C_*$  by  $d_C$ . Consider the double complex  $I_* \otimes C_*$  first.

$$\begin{array}{ccccc}
& \vdots & & \vdots & & \vdots \\
& \downarrow & & \downarrow & & \downarrow \\
I_0 \otimes C_2 & \longleftarrow & I_1 \otimes C_2 & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
I_0 \otimes C_1 & \longleftarrow & I_1 \otimes C_1 & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
I_0 \otimes C_0 & \longleftarrow & I_1 \otimes C_0 & \longleftarrow & 0
\end{array}$$

The vertical boundary map  $d_v$  is  $id \otimes d_C$  and the horizontal boundary map  $d_h$  is  $d \otimes id$ . Let  $T_*$  be the total complex of this double complex, then in each degree we have

$$T_n = I_0 \otimes C_n \oplus I_1 \otimes C_{n-1}.$$

We know that  $I_0 = \mathbb{Z}\langle a, b \rangle$ , so  $I_0 \otimes C_n$  is isomorphic to  $(C_n)^2$  where the isomorphism is given by sending  $a \otimes x$  to  $x$  and  $b \otimes y$  to  $y$  for all  $x, y \in C_n$ . Similarly,  $I_1 = \mathbb{Z}\langle e \rangle$ , so  $I_1 \otimes C_{n-1}$  is isomorphic to  $C_{n-1}$  where the isomorphism is given by sending  $e \otimes z$  to  $z$  for all  $z \in C_{n-1}$ . The boundary map in the total complex is given by  $d_t(x) = d_h(x) + (-1)^p d_v(x)$  for  $x \in I_p \otimes C_q$ . For  $a \otimes x, b \otimes y$  in  $I_0 \otimes C_n$  and  $e \otimes z \in I_1 \otimes C_{n-1}$ , we have

$$\begin{aligned}
d_t(a \otimes x) &= d_C(x), \\
d_t(b \otimes y) &= d_C(y), \\
d_t(e \otimes z) &= (b - a) \otimes x - e \otimes d_C(z).
\end{aligned}$$

- (b) Write the boundary maps in  $C_*$  as  $d_C$  and boundary maps in  $D_*$  as  $d_D$ . For any  $n \in \mathbb{Z}$ , we know  $F$  is a chain map, so we have a commutative diagram

$$\begin{array}{ccc}
I_0 \otimes C_n & \xrightarrow{id \otimes d_C} & I_0 \otimes C_{n-1} \\
F \downarrow & & \downarrow F \\
D_n & \xrightarrow{d_D} & D_{n-1}
\end{array}$$

For any  $x \in C_n$ , we have

$$(d_D \circ F)(a \otimes x) = [F \circ (id \otimes d_C)(a \otimes x)] = F(a \otimes d_C(x)).$$

By definition this is equivalent to

$$(d_D \circ f)(x) = (f \circ d_C)(x).$$

Namely, we have a commutative diagram

$$\begin{array}{ccc} C_n & \xrightarrow{d_C} & C_{n-1} \\ f \downarrow & & \downarrow f \\ D_n & \xrightarrow{d_D} & D_{n-1} \end{array}$$

This proves  $f$  is a chain map. By a similar argument,  $g$  is also a chain map.

Next, to show that  $s_n$  defines a chain homotopy between  $f$  and  $g$ , we need to show for any  $n$ , there exists a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_C} & C_n & \xrightarrow{d_C} & C_{n-1} \longrightarrow \cdots \\ & & f \downarrow \quad \downarrow g & \swarrow s_n & f \downarrow \quad \downarrow g & \swarrow s_{n-1} & f \downarrow \quad \downarrow g \\ \cdots & \longrightarrow & D_{n+1} & \xrightarrow{d_D} & D_n & \xrightarrow{d_D} & D_{n-1} \longrightarrow \cdots \end{array}$$

For any  $x \in C_n$ , we have

$$g(x) - f(x) = F(b \otimes x) - F(a \otimes x) = F((b - a) \otimes x).$$

On the other hand, use the fact that  $F$  is a chain map, we have

$$\begin{aligned} (d_D \circ s_n)(x) + (s_{n-1} \circ d_C)(x) &= (d_D \circ F)(e \otimes x) + F(e \otimes d_C(x)) \\ &= (F \circ d_C)(e \otimes x) + F(e \otimes d_C(x)) \\ &= F((b - a) \otimes x) - F(e \otimes d_C(x)) + F(e \otimes d_C(x)) \\ &= F((b - a) \otimes x). \end{aligned}$$

This proves that

$$g - f = d_D \circ s_n + s_{n-1} \circ d_C.$$

The collection of  $s_n$  is a chain homotopy between  $f$  and  $g$ .

### Problem 3

Let  $Y$  be the space obtained by starting with  $S^3$  and attaching a 4-cell via a map of degree 5 :  $Y = S^3 \cup_f e^4$  where  $f : \partial(e^4) \rightarrow S^3$  has degree 5. Write down the cellular chain complex for  $\mathbb{R}P^3 \times Y$ ; in particular, specify the rank of each chain group and identify the boundary maps. Compute the homotopy groups of  $\mathbb{R}P^3 \times Y$  and specify the rank of each chain group and identify the boundary maps. Compute the homology groups of  $\mathbb{R}P^3 \times Y$ .

*Solution:* The space  $Y$  has a cellular chain complex

$$0 \rightarrow \mathbb{Z}\langle e^4 \rangle \xrightarrow{5} \mathbb{Z}\langle e^3 \rangle \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{Z}\langle e^0 \rangle \rightarrow 0.$$

where  $e^i$  are cells in  $Y$  for  $i = 0, 3, 4$ . The real projective space  $\mathbb{R}P^3$  has the following cellular chain complex

$$0 \rightarrow \mathbb{Z}\langle f^3 \rangle \xrightarrow{0} \mathbb{Z}\langle f^2 \rangle \xrightarrow{2} \mathbb{Z}\langle f^1 \rangle \xrightarrow{0} \mathbb{Z}\langle f^0 \rangle \rightarrow 0.$$

The tensor product of these two chain complex is the double complex

$$\begin{array}{ccccccc}
\mathbb{Z}\langle f^0 \otimes e^4 \rangle & \xleftarrow{0} & \mathbb{Z}\langle f^1 \otimes e^4 \rangle & \xleftarrow{2 \otimes id} & \mathbb{Z}\langle f^2 \otimes e^4 \rangle & \xleftarrow{0} & \mathbb{Z}\langle f^3 \otimes e^4 \rangle \\
id \otimes 5 \downarrow & & id \otimes 5 \downarrow & & id \otimes 5 \downarrow & & id \otimes 5 \downarrow \\
\mathbb{Z}\langle f^0 \otimes e^3 \rangle & \xleftarrow{0} & \mathbb{Z}\langle f^1 \otimes e^3 \rangle & \xleftarrow{2 \otimes id} & \mathbb{Z}\langle f^2 \otimes e^3 \rangle & \xleftarrow{0} & \mathbb{Z}\langle f^3 \otimes e^3 \rangle \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}\langle f^0 \otimes e^0 \rangle & \xleftarrow{0} & \mathbb{Z}\langle f^1 \otimes e^0 \rangle & \xleftarrow{2 \otimes id} & \mathbb{Z}\langle f^2 \otimes e^0 \rangle & \xleftarrow{0} & \mathbb{Z}\langle f^3 \otimes e^0 \rangle
\end{array}$$

Denote the total chain complex by  $(T_n, d_n)$ . we have

$$T_n = \begin{cases} \mathbb{Z}\langle f^0 \otimes e^0 \rangle \cong \mathbb{Z}, & \text{if } n = 0; \\ \mathbb{Z}\langle f^1 \otimes e^0 \rangle \cong \mathbb{Z}, & \text{if } n = 1; \\ \mathbb{Z}\langle f^2 \otimes e^0 \rangle \cong \mathbb{Z}, & \text{if } n = 2; \\ \mathbb{Z}\langle f^0 \otimes e^3, f^3 \otimes e^0 \rangle \cong \mathbb{Z}^2, & \text{if } n = 3; \\ \mathbb{Z}\langle f^0 \otimes e^4, f^1 \otimes e^3 \rangle \cong \mathbb{Z}^2, & \text{if } n = 4; \\ \mathbb{Z}\langle f^1 \otimes e^4, f^2 \otimes e^3 \rangle \cong \mathbb{Z}^2, & \text{if } n = 5; \\ \mathbb{Z}\langle f^2 \otimes e^4, f^3 \otimes e^3 \rangle \cong \mathbb{Z}^2, & \text{if } n = 6; \\ \mathbb{Z}\langle f^3 \otimes e^4 \rangle \cong \mathbb{Z}, & \text{if } n = 7. \end{cases}$$

For  $1 \leq n \leq 7$ , the boundary map  $d_n$  is given by the formula

$$d_n(f^i \otimes e^j) = d(f^i) \otimes e^j + (-1)^i f^i \otimes d(e^j).$$

The boundary map  $d_n$  is given by the following table: To calculate the homology groups of  $\mathbb{R}P^3 \times Y$ ,

$i$	$d_i$
1	0
2	2
3	$\begin{pmatrix} 0 & 0 \end{pmatrix}$
4	$\begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$
5	$\begin{pmatrix} 0 & 2 \\ -5 & 0 \end{pmatrix}$
6	$\begin{pmatrix} 2 & 0 \\ 5 & 0 \end{pmatrix}$
7	$\begin{pmatrix} 0 \\ -5 \end{pmatrix}$

we first write down the homology groups of  $\mathbb{R}P^3$  and  $Y$ .

	$H_*(\mathbb{R}P^3)$	$H_*(Y)$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}/2$	0
2	0	0
3	$\mathbb{Z}$	$\mathbb{Z}/5$
4	0	0

We calculate their tensor products and  $\text{Tor}_1$  respectively. Note that  $\text{Tor}_1(\mathbb{Z}/5, \mathbb{Z}/2) = 0$ , so we do not have any terms coming from  $\text{Tor}_1$ . The homology groups of  $\mathbb{R}P^3 \times Y$  can be summarized as follows

$$H_i(\mathbb{R}P^3 \times Y) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}/2, & \text{if } i = 1; \\ \mathbb{Z} \oplus \mathbb{Z}/5, & \text{if } i = 3; \\ \mathbb{Z}/5, & \text{if } i = 6; \\ 0, & \text{otherwise.} \end{cases}$$


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#### Problem 4

Compute both the homology and cohomology groups of the following spaces, both with integral and  $\mathbb{Z}/2$  coefficients. Heck, do it with  $\mathbb{Z}/3$  coefficients as well.

- (a)  $K \times K$ , where  $K$  is the Klein bottle.
- (b)  $K \times T^g$ , where  $T^g$  is the genus  $g$  torus and  $K$  is the Klein bottle.
- (c)  $K \times \mathbb{R}P^n$ .

*Solution:*

- (a) We can use UCT for homology to calculate the homology groups of  $K$  in different coefficients, and we summarized them as follows.

	$H_*(K)$	$H_*(K; \mathbb{Z}/2)$	$H_*(K; \mathbb{Z}/3)$
0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$
1	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/3$
2	0	$\mathbb{Z}/2$	0

From this we know that the tensor product is

$$H_p(K) \otimes H_q(K) = \begin{cases} \mathbb{Z}, & \text{if } p + q = 0; \\ \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2, & \text{if } p + q = 1; \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^3, & \text{if } p + q = 2; \\ 0, & \text{otherwise.} \end{cases}$$

The only non-trivial  $\text{Tor}_1$  is given by  $\text{Tor}_1(H_1(K), H_1(K)) = \mathbb{Z}/2$ . Thus, the homology groups of  $K$  is

$$H_i(K \times K; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2, & \text{if } i = 1; \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^3, & \text{if } i = 2; \\ \mathbb{Z}/2, & \text{if } i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we use UCT for homology to calculate the homology groups with  $\mathbb{Z}/2$  and  $\mathbb{Z}/3$  coefficients.

$$H_i(K \times K; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0; \\ (\mathbb{Z}/2)^4, & \text{if } i = 1; \\ (\mathbb{Z}/2)^6, & \text{if } i = 2; \\ (\mathbb{Z}/2)^4, & \text{if } i = 3; \\ (\mathbb{Z}/2), & \text{if } i = 4; \\ 0, & \text{otherwise.} \end{cases} \quad H_i(K \times K; \mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & \text{if } i = 0; \\ (\mathbb{Z}/3)^2, & \text{if } i = 1; \\ \mathbb{Z}/3, & \text{if } i = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Lastly, we use UCT for cohomology to calculate the cohomology groups.

	$H^*(K \times K)$	$H^*(K \times K; \mathbb{Z}/2)$	$H^*(K \times K; \mathbb{Z}/3)$
0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$
1	$\mathbb{Z}^2$	$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/3)^2$
2	$\mathbb{Z} \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^6$	$\mathbb{Z}/3$
3	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^4$	0
4	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0

(b) The homology of  $K$  and  $T^g$  are as follows:

	$H_*(K)$	$H_*(T^g)$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}^{2g}$
2	0	$\mathbb{Z}$

Note that  $H_*(T^g)$  are all free, so by Künneth theorem, we have

$$H_*(K \times T^g) \cong \bigoplus_{i+j=n} H_i(K) \otimes H_j(T^g).$$

Thus, we conclude  $H_*(K \times T^g)$  as follows:

$$H_i(K \times T^g) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}^{2g+1} \oplus \mathbb{Z}/2, & \text{if } i = 1; \\ \mathbb{Z}^{2g+1} \oplus (\mathbb{Z}/2)^{2g}, & \text{if } i = 2; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we use UCT for homology to calculate different coefficients.

$$H_i(K \times T^g; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0; \\ (\mathbb{Z}/2)^{2g+2}, & \text{if } i = 1; \\ (\mathbb{Z}/2)^{4g+2}, & \text{if } i = 2; \\ (\mathbb{Z}/2)^{2g+2}, & \text{if } i = 3; \\ \mathbb{Z}/2, & \text{if } i = 4; \\ 0, & \text{otherwise.} \end{cases} \quad H_i(K \times T^g; \mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & \text{if } i = 0; \\ (\mathbb{Z}/3)^{2g+1}, & \text{if } i = 1; \\ (\mathbb{Z}/3)^{2g+1}, & \text{if } i = 2; \\ \mathbb{Z}/3, & \text{if } i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

Lastly, we use UCT to calculate the cohomology groups.

	$H^*(K \times T^g)$	$H^*(K \times T^g; \mathbb{Z}/2)$	$H^*(K \times T^g; \mathbb{Z}/3)$
0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$
1	$\mathbb{Z}^{2g+1}$	$(\mathbb{Z}/2)^{2g+2}$	$(\mathbb{Z}/3)^{2g+1}$
2	$\mathbb{Z}^{2g+1} \oplus \mathbb{Z}/2$	$(\mathbb{Z}/2)^{4g+2}$	$(\mathbb{Z}/3)^{2g+1}$
3	$\mathbb{Z} \oplus (\mathbb{Z}/2)^{2g}$	$(\mathbb{Z}/2)^{2g+2}$	$\mathbb{Z}/3$
4	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0

(c)  $\mathbb{R}P^n$  has different homology groups when  $n$  is odd or even.

(1) Suppose  $n \geq 2$  is even.

The homology groups of  $\mathbb{R}P^n$  can be summarized as follows:

- $H_0(\mathbb{R}P^n) = \mathbb{Z}$ .
- $H_i(\mathbb{R}P^n) = \mathbb{Z}/2$  if  $i \geq 0$  and  $i$  is odd.
- 0 otherwise.

Use Künneth theorem, we can calculate the homology groups of  $K \times \mathbb{R}P^n$ :

$$H_i(K \times \mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^2, & \text{if } i = 1; \\ (\mathbb{Z}/2)^2, & \text{if } 2 \leq i \leq n; \\ \mathbb{Z}/2, & \text{if } i = n + 1; \\ 0, & \text{otherwise.} \end{cases}$$



Next, we use UCT for homology to calculate the homology groups with other coefficients.

$$H_i(K \times \mathbb{R}P^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0; \\ (\mathbb{Z}/2)^3, & \text{if } i = 1; \\ (\mathbb{Z}/2)^4, & \text{if } 2 \leq i \leq n; \\ (\mathbb{Z}/2)^3, & \text{if } i = n + 1; \\ \mathbb{Z}/2, & \text{if } i = n + 2; \\ 0, & \text{otherwise.} \end{cases}$$

$$H_i(K \times \mathbb{R}P^n; \mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & \text{if } i = 0, 1; \\ 0, & \text{otherwise.} \end{cases}$$

Lastly, we use UCT for cohomology to calculate the cohomology groups.

	$H^*(K \times \mathbb{R}P^n)$	$H^*(K \times \mathbb{R}P^n; \mathbb{Z}/2)$	$H^*(K \times \mathbb{R}P^n; \mathbb{Z}/3)$
0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$
1	$\mathbb{Z}$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/3$
2	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^4$	0
$\vdots$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^4$	0
$n$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^4$	0
$n + 1$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	0
$n + 2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0

(2) Suppose  $n \geq 3$  is odd.

The homology groups of  $\mathbb{R}P^n$  can be summarized as follows:

- $H_0(\mathbb{R}P^n) = H_n(\mathbb{R}P^n) = \mathbb{Z}$ .
- $H_i(\mathbb{R}P^n) = \mathbb{Z}/2$  for  $1 \leq i \leq n - 1$  if  $i$  is odd.
- 0 otherwise.

Use Künneth theorem, we can calculate the homology groups of  $K \times \mathbb{R}P^n$ :

$$H_i(K \times \mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^2, & \text{if } i = 1; \\ (\mathbb{Z}/2)^2, & \text{if } 2 \leq i \leq n - 1; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } i = n; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } i = n + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we use UCT for homology to calculate the homology groups with coefficients:

$$H_i(K \times \mathbb{R}P^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0; \\ (\mathbb{Z}/2)^3, & \text{if } i = 1; \\ (\mathbb{Z}/2)^4, & \text{if } 2 \leq i \leq n; \\ (\mathbb{Z}/2)^3, & \text{if } i = n + 1; \\ \mathbb{Z}/2, & \text{if } i = n + 2; \\ 0, & \text{otherwise.} \end{cases}$$

$$H_i(K \times \mathbb{R}P^n; \mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & \text{if } i = 0, 1, n, n + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Lastly, we use UCT for cohomology to calculate the cohomology groups.

	$H^*(K \times \mathbb{R}P^n)$	$H^*(K \times \mathbb{R}P^n; \mathbb{Z}/2)$	$H^*(K \times \mathbb{R}P^n; \mathbb{Z}/3)$
0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$
1	$\mathbb{Z}$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/3$
2	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^4$	0
$\vdots$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^4$	0
$n - 1$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^4$	0
$n$	$\mathbb{Z} \oplus (\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^4$	$\mathbb{Z}/3$
$n + 1$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/3$
$n + 2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0

(3) The last case is  $n = 1$ . We have  $\mathbb{R}P^1 \cong S^1$ . The homology of  $S^1$  is free, so we have

$$H_i(K \times \mathbb{R}P^1) = \bigoplus_{p+q=i} H_p(K) \otimes H_q(S^1) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}^2 \oplus \mathbb{Z}/2, & \text{if } i = 1; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } i = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Use UCT for homology we can calculate the homology groups with different coefficients:

$$H_i(K \times \mathbb{R}P^1; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0; \\ (\mathbb{Z}/2)^3, & \text{if } i = 1; \\ (\mathbb{Z}/2)^3, & \text{if } i = 2; \\ \mathbb{Z}/2, & \text{if } i = 3; \\ 0, & \text{otherwise.} \end{cases} \quad H_i(K \times \mathbb{R}P^1; \mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & \text{if } i = 0; \\ (\mathbb{Z}/3)^2, & \text{if } i = 1; \\ \mathbb{Z}/3, & \text{if } i = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Lastly, we use UCT for cohomology to calculate the cohomology groups.

	$H^*(K \times \mathbb{R}P^1)$	$H^*(K \times \mathbb{R}P^1; \mathbb{Z}/2)$	$H^*(K \times \mathbb{R}P^1; \mathbb{Z}/3)$
0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$
1	$\mathbb{Z}^2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/3)^2$
2	$\mathbb{Z} \oplus \mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/3$
3	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0

### Problem 5

Let  $f : A_* \rightarrow B_*$  be a map of chain complexes. We can regard this as forming a double complex

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 A_2 & \longrightarrow & B_2 \\
 \downarrow & & \downarrow \\
 A_1 & \longrightarrow & B_1 \\
 \downarrow & & \downarrow \\
 A_0 & \longrightarrow & B_0
 \end{array}$$

by putting zeros in all the "empty" spots. The total complex of this double complex is called the **algebraic mapping cone** of  $f$ , denoted  $Cf$ . Specifically, we set  $(Cf)_n = A_{n-1} \oplus B_n$  and define  $d : (Cf)_n \rightarrow (Cf)_{n-1}$  by

$$d(a, b) = (d_A(a), (-1)^{n-1}f(a) + d_B(b))$$

- (a) Explain why there is a short exact sequence of chain complexes

$$0 \rightarrow B_* \hookrightarrow Cf \rightarrow \Sigma A_* \rightarrow 0,$$

where  $\Sigma A_*$  is the evident chain complex having  $(\Sigma A)_n = A_{n-1}$ .

- (b) The short exact sequence from (a) gives rise to a long exact sequence in homology groups. This has the form

$$\cdots \rightarrow H_i(B) \rightarrow H_i(Cf) \rightarrow H_i(\Sigma A) \xrightarrow{\partial} H_{i-1}(B) \rightarrow \cdots$$

Verify that the connecting homomorphism is really just the map  $f_* : H_{i-1}(A) \rightarrow H_{i-1}(B)$ , possibly up to a sign.

*Solution:*

- (a) We need to prove that for any  $n \geq 0$ , we have the following commutative diagrams where the

top row and bottom row is exact.

$$\begin{array}{ccccccc}
0 & \longrightarrow & B_n & \xrightarrow{i_n} & A_{n-1} \oplus B_n & \xrightarrow{p_n} & A_{n-1} \longrightarrow 0 \\
& & \downarrow d_B & & \downarrow d & & \downarrow d_A \\
0 & \longrightarrow & B_{n-1} & \xrightarrow{i_{n-1}} & A_{n-2} \oplus B_{n-1} & \xrightarrow{p_{n-1}} & A_{n-2} \longrightarrow 0
\end{array}$$

We choose  $i_n : B_n \rightarrow A_{n-1} \oplus B_n$  as the inclusion  $b \mapsto (0, b)$  and  $p_n : A_{n-1} \oplus B_n \rightarrow A_{n-1}$  as the projection  $(a, b) \mapsto a$ . It is easy to see the top row and the bottom row is exact. For any  $(a, b) \in A_{n-1} \oplus B_n$ , we have

$$\begin{aligned}
(p_{n-1} \circ d)(a, b) &= p_{n-1}(d_A(a), (-1)^{n-1}f(a) + d_B(b)) \\
&= d_A(a) \\
&= (d_A \circ p_n)(a, b).
\end{aligned}$$

This proves the right square commutes. Moreover, for any  $b \in B_n$ , we have

$$\begin{aligned}
(d \circ i_n)(b) &= d(0, b) \\
&= (0, 0 + d_B(b)) \\
&= (i_{n-1} \circ d_B)(b).
\end{aligned}$$

This proves the left square commutes. Thus, we have a short exact sequence of chain complex

$$0 \rightarrow B_* \rightarrow (Cf)_* \rightarrow \Sigma A_* \rightarrow 0$$

where  $(Cf)_n = A_{n-1} \oplus B_n$  and  $(\Sigma A)_n = A_{n-1}$  for all  $n$ .

- (b) By the snake lemma, we have a long exact sequence of homology groups deduced from the short exact sequence of chain complexes

$$0 \rightarrow B_* \rightarrow (Cf)_* \rightarrow \Sigma A_* \rightarrow 0.$$

Take  $a \in \ker d_A \subseteq A_{n-1}$ , we specify how to define  $\partial a \in B_{n-1}$  from the snake lemma. We take the preimage  $(a, 0) \in (Cf)_n$ , send it to  $d(a, 0) = (0, (-1)^{n-1}f(a)) \in (Cf)_{n-1}$ , lastly we take the preimage  $(-1)^{n-1}f(a) \in B_{n-1}$ . Thus, we can conclude that the map

$$\begin{aligned}
\partial : H_{n-1}(A) &\rightarrow H_{n-1}(B), \\
[a] &\mapsto [(-1)^{n-1}f(a)]
\end{aligned}$$

This implies the connecting homomorphism is just the map induced by  $f$

$$f_* : H_{n-1}(A) \rightarrow H_{n-1}(B)$$

up to a sign.

### Problem 6

Let  $k$  be a field, and let  $\mathcal{V}$  denote the category of vector spaces over  $k$ . Let  $I$  be any (small) category, and let  $\mathcal{V}^I$  be the category whose objects are functors  $I \rightarrow \mathcal{V}$  and whose morphisms are natural transformations. We call  $\mathcal{V}^I$  the category of " $I$ -shaped diagram in  $\mathcal{V}$ ".

In this problem we will focus on the case where  $I$  is the pushout category

$$1 \leftarrow 0 \rightarrow 2$$

with three objects and two non-identity maps (as shown above). An object of  $\mathcal{V}^I$  is then just a diagram of vector spaces  $V_1 \leftarrow V_0 \rightarrow V_2$ . A map from  $[V_1 \leftarrow V_0 \rightarrow V_2]$  to  $[W_1 \leftarrow W_0 \rightarrow W_2]$  is a commutative diagram

$$\begin{array}{ccccc} V_1 & \longleftarrow & V_0 & \longrightarrow & V_2 \\ \downarrow & & \downarrow & & \downarrow \\ W_1 & \longleftarrow & W_0 & \longrightarrow & W_2 \end{array}$$

Let  $P : \mathcal{V}^I \rightarrow \mathcal{V}$  be the pushout functor.  $P$  assigns each diagram its pushout.

- (a) Let  $F_1, F_0$  and  $F_2$  be the three diagrams

$$F_1 : [k \leftarrow 0 \rightarrow 0] \quad F_0 : [k \leftarrow k \rightarrow k] \quad F_2 : [0 \leftarrow 0 \rightarrow k]$$

where in  $F_0$  the maps are the identities. These diagrams are "free" in a certain sense: namely, if  $D$  is an object of  $\mathcal{V}^I$  then morphisms  $F_i \rightarrow D$  are in bijective correspondence with elements of  $D_i$ . Convince yourself that this is true.

- (b) Let  $D = [0 \leftarrow k \rightarrow 0]$  and  $E = [0 \leftarrow k \rightarrow k]$ , where in  $E$  the nontrivial map is the identity. Determine free resolutions for  $D$  and  $E$ .
- (c) Apply the functor  $P$  to your resolution, to produce a chain complex of vector spaces. Compute the homology groups, which are the groups  $(L_i P)(D)$  and  $(L_i P)(E)$ . These are the derived functor of the pushout functor  $P$ . Confirm in your example that  $L_0 P = P$ .
- (d) Now let  $I$  be the category with one object  $0$  and one non-identity map  $t : 0 \rightarrow 0$  such that  $t^2 = id$ . Objects of  $\mathcal{V}^I$  are then pairs  $(W, t)$  consisting of a vector space  $W$  and an endomorphism  $t : W \rightarrow W$  such that  $t^2 = id$ . In  $\mathcal{V}^I$  the basic "free" object is  $(k^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ ; this can also be thought of as the vector space  $k\langle g, tg \rangle$  where  $t(tg) = g$ . Let  $P : \mathcal{V}^I \rightarrow \mathcal{V}$  be the colimit functor, sending an object  $(W, t)$  to  $W / \{x - tx \mid x \in W\}$ . Find the free resolution of the object  $(k, id)$  and compute  $(L_i P)(k, id)$  for all  $i \geq 0$ .

*Solution:*

- (a)

- (b) Consider the following sequence

$$0 \rightarrow F_1 \oplus F_2 \rightarrow F_0 \rightarrow D \rightarrow 0.$$

Note that  $F_1 \oplus F_2$  is the following diagram  $[k \leftarrow 0 \rightarrow k]$  Namely the following diagram

$$\begin{array}{ccccccc}
0 & & 0 & \longleftarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_1 \oplus F_2 & & k & \longleftarrow & 0 & \longrightarrow & k \\
\downarrow & & \downarrow \textit{id} & & \downarrow & & \downarrow \textit{id} \\
F_0 & & k & \xleftarrow{\textit{id}} & k & \xrightarrow{\textit{id}} & k \\
\downarrow & & \downarrow & & \downarrow \textit{id} & & \downarrow \\
D & & 0 & \longleftarrow & k & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & \longleftarrow & 0 & \longrightarrow & 0
\end{array}$$

The vertical columns are exact because we only have isomorphisms. For  $E$ , consider the following sequence

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow E \rightarrow 0.$$

This can be written as the diagram

$$\begin{array}{ccccccc}
0 & & 0 & \longleftarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_1 & & k & \longleftarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \textit{id} & & \downarrow & & \downarrow \\
F_0 & & k & \xleftarrow{\textit{id}} & k & \xrightarrow{\textit{id}} & k \\
\downarrow & & \downarrow & & \downarrow \textit{id} & & \downarrow \textit{id} \\
E & & 0 & \longleftarrow & k & \xrightarrow{\textit{id}} & k \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & \longleftarrow & 0 & \longrightarrow & 0
\end{array}$$

This is a free resolution for  $E$ .

(c) Apply the pushout functor to the free resolution

$$0 \rightarrow F_1 \oplus F_2 \rightarrow F_0 \rightarrow 0.$$

The pushout of  $F_1 \oplus F_2$  is  $k^2$  and the pushout of  $F_0$  is  $k$ . The map  $F_1 \oplus F_2 \rightarrow F_0$  induces a map  $p$  between pushouts

$$\begin{array}{ccccc}
k & \longleftarrow & 0 & \longrightarrow & k \\
\downarrow \textit{id} & \searrow & & \swarrow & \downarrow \textit{id} \\
& & k^2 & & \\
& \swarrow p & & \searrow & \\
k & \longleftarrow & k & \longrightarrow & k \\
& \swarrow \textit{id} & & \searrow \textit{id} & \\
& & k & & 
\end{array}$$

We can see from the diagram that  $p = (id, id)$ , so  $p$  is surjective, so we have  $(L_0P)(D) = P(D) = 0$  and  $(L_1P)(D) = k$ .

Apply the pushout functor to the free resolution

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow E \rightarrow 0.$$

The map  $F_1 \rightarrow F_0$  induces a map between pushouts

$$\begin{array}{ccccc} k & \xleftarrow{\quad} & 0 & \xrightarrow{\quad} & 0 \\ & \searrow & & \swarrow & \\ & & k & & \\ & \swarrow & id & \searrow & \\ k & \xleftarrow{\quad} & k & \xrightarrow{\quad} & k \\ & \searrow & & \swarrow & \\ & & k & & \end{array}$$

*(Note: The diagram shows a commutative square with additional arrows. The top row is  $k \xleftarrow{\quad} 0 \xrightarrow{\quad} 0$ . The middle row is  $k \xleftarrow{\quad} k \xrightarrow{\quad} k$ . The bottom row is  $k \xleftarrow{\quad} k \xrightarrow{\quad} k$ . Vertical arrows are  $id$  from  $k$  to  $k$  on the left and right. Diagonal arrows are  $id$  from  $0$  to  $k$  in the middle. There are also arrows from  $k$  to  $k$  in the middle and bottom rows.)*

This map must be identity, so we have

$$(L_1P)(E) = (L_0P)(E) = P(E) = 0.$$

(d) Consider the following free resolution of  $k \xrightarrow{id} k$ :

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\quad} & k^2 & \xrightarrow{\quad} & k^2 & \xrightarrow{\quad} & k^2 & \xrightarrow{\quad} & k^2 & \xrightarrow{\quad} & k & \xrightarrow{\quad} & 0 \\ & & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & & \begin{pmatrix} 1 & 1 \end{pmatrix} & & \\ & & \text{t} & & \text{t} & & \text{t} & & \text{t} & & id & & \end{array}$$

*(Note: The diagram shows a sequence of maps between  $k^2$  and  $k$ . The maps are labeled with matrices. The first four maps are  $k^2 \rightarrow k^2$  with matrices  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ , and  $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ . The fifth map is  $k^2 \rightarrow k$  with matrix  $\begin{pmatrix} 1 & 1 \end{pmatrix}$ . The final map is  $k \rightarrow 0$ . The maps are labeled with  $t$  and  $id$  above them.)*

Let  $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ . Both  $A$  and  $B$  are compatible with the map  $t$  because

$$At = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = tA,$$

$$Bt = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = Bt.$$

Moreover, the sequence is exact at every spot. Apply the colimit functor  $P$ , the map  $A =$

$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} : k^2 \rightarrow k^2$  will give you the following diagram

$$\begin{array}{ccccc}
 k^2 & \xrightarrow{t} & k^2 & & \\
 \downarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & \searrow (1 \ 1) & \swarrow (1 \ 1) & & \downarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\
 & & k & \xrightarrow{0} & \\
 & & \uparrow t & & \\
 k^2 & \xrightarrow{t} & k^2 & & \\
 \searrow (1 \ 1) & & \swarrow (1 \ 1) & & \\
 & & k & & 
 \end{array}$$

The map  $P(A)$  is the zero map. Similarly, we apply the colimit functor  $P$  to the map  $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : k^2 \rightarrow k^2$ :

$$\begin{array}{ccccc}
 k^2 & \xrightarrow{t} & k^2 & & \\
 \downarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \searrow (1 \ 1) & \swarrow (1 \ 1) & & \downarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\
 & & k & \xrightarrow{2} & \\
 & & \uparrow t & & \\
 k^2 & \xrightarrow{t} & k^2 & & \\
 \searrow (1 \ 1) & & \swarrow (1 \ 1) & & \\
 & & k & & 
 \end{array}$$

Thus, apply the colimit functor  $P$  to the free resolution, and we obtain a chain complex

$$\cdots \xrightarrow{2} k \xrightarrow{0} k \xrightarrow{2} k \xrightarrow{0} k \rightarrow 0$$

- When the characteristic of  $k$  is 2, all the boundary maps are zero, we have  $(L_i P)(k, id) = k$  for all  $i \geq 0$ .
- When the characteristic of  $k$  is not equal to 2, this means 2 is invertible in  $k$ , so all 2 maps are isomorphisms. In this case we have  $(L_i P)(k, id) = 0$  for all  $i \geq 1$ , and  $(L_0 P)(k, id) = P(k, id) = k$ .