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Course: MATH 607 - Homological Algebra
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Homework 1
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Exercise 1.2.5
Give an elementary proof that $\mathrm{Tot}(C)$ is acyclic whenever C is a bounded double complex with eexact rows (or exact columns).

Solution: Without loss of generality, we may assume the bounded double complex C has exact rows. Consider the total complex $\mathrm{Tot}(C)$, for any n, we want to show that $H_n(\mathrm{Tot}(C)) = 0$. Write

$$\mathrm{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}.$$

For any element $a \in \mathrm{Tot}(C)_n$, a can be written as $a = \oplus_{p+q=n} a_{p,q}$. Since C is bounded, there exists $p_0, p_1 \in \mathbb{Z}$ such that for any $p < p_0$ and $p > p_1$, we have $C_{p,n-p} = 0$. Suppose $a \in \mathrm{Tot}(C)_n$ is in the kernel of the map

$$d = d^h + d^v : \mathrm{Tot}(C)_n \to \mathrm{Tot}(C)_{n-1}.$$

So we have

$$(d^h + d^v)a_{p,q} = d^h a_{p,q} + d^v a_{p,q} = 0$$

for all $p + q = n$ and $p_0 \leq p \leq p_1$. To prove that $H_n(\mathrm{Tot}(C)) = 0$, we need to show that every $a_{p,q}$ is the image of some element in $\mathrm{Tot}(C)_{n+1}$.

For $p = p_0$, consider the following diagram:

$$\begin{tikzcd}
0 \arrow[d] & & & \\
C_{p_0-1,n-p_0} & C_{p_0,n-p_0} \arrow[l, "d^h"'] \arrow[d, "d^v"] & C_{p_0+1,n-p_0} \arrow[l, "d^h"'] \arrow[d, "d^v"] & \\
& C_{p_0,n-p_0-1} & C_{p_0+1,n-p_0-1} \arrow[l, "d^h"'] & C_{p_0+2,n-p_0-1} \arrow[l, "d^h"']
\end{tikzcd}$$

Note that $d^h a_{p_0,n-p_0} = 0$ in $C_{p_0-1,n-p_0}$ because $C_{p_0-1,n-p_0+1} = 0$. By exactness of rows, there exists an element $b \in C_{p_0+1,n-p_0}$ such that $d^h b = a_{p_0,n-p_0}$.

For $p = p_0 + 1$, we know that

$$d^h a_{p_0+1,n-p_0-1} + d^v a_{p_0,n-p_0} = 0.$$

Replace $a_{p_0,n-p_0}$ with $d_h b$ and use the fact that $d^v d^h + d^h d^v = 0$, we obtain

$$\begin{aligned}
d^h a_{p_0+1,n-p_0-1} + d^v d^h b &= 0, \\
d^h a_{p_0+1,n-p_0-1} - d^h d^v b &= 0, \\
d^h (a_{p_0+1,n-p_0-1} - d^v b) &= 0.
\end{aligned}$$

By exactness of rows, there exists $c \in C_{p_0+2,n-p_0-1}$ such that

$$d^h c = a_{p_0+1,n-p_0-1} - d^v b.$$

This implies

$$a_{p_0+1,n-p_0-1} = d^h c + d^v b.$$

So $a_{p_0+1,n-p_0-1}$ is also in the image of some element in $\mathrm{Tot}(C)_{n+1}$. Using a similar argument we can prove step by step that for any $p_0 \leq p \leq p_1$, $a_{p,n-p}$ is the image of some element in $\mathrm{Tot}(C)_{n+1}$. So $H_n(\mathrm{Tot}(C)) = 0$, and thus the total complex $\mathrm{Tot}(C)$ is acyclic.

Exercise 1.2.6

Give examples of

(1) a second quadrant double complex C with exact columns such that $\mathrm{Tot}^{\prod}(C)$ is acyclic but $\mathrm{Tot}^{\oplus}(C)$ is not;

(2) a second quadrant double complex C with exact rows such that $\mathrm{Tot}^{\oplus}(C)$ is acyclic but $\mathrm{Tot}^{\prod}(C)$ is not;

(3) a double complex (in the entire plane) for which every row and column is exact, yet neither $\mathrm{Tot}^{\prod}(C)$ nor $\mathrm{Tot}^{\oplus}(C)$ is acyclic.

Solution:

(1) Consider the following double complex C

$$\begin{tikzcd}
\cdots \arrow[d] & & \\
\mathbb{Z} & \mathbb{Z} \arrow[l] \arrow[d] & \\
& \mathbb{Z} & \mathbb{Z} \arrow[l] \arrow[d] \\
& & \mathbb{Z}
\end{tikzcd}$$

where all maps are isomorphisms of $\mathbb{Z}$. Taking different total complexes, we get two maps

$$\alpha : \prod_{n\geq 0} \mathbb{Z} \to \prod_{n\geq 0} \mathbb{Z},$$
$$\beta : \bigoplus_{n\geq 0} \mathbb{Z} \to \bigoplus_{n\geq 0} \mathbb{Z}.$$

In both cases, the map is given by

$$(x_0, x_1, x_2, \ldots) \mapsto (x_0, x_0 + x_1, x_1 + x_2, \ldots).$$

In $\mathrm{Tot}^{\prod}(C)$, we allow any such sequence, while in $\mathrm{Tot}^{\oplus}(C)$, we only allow sequence with finitely many nonzero terms. It is easy to see both α and β are injective. The element

$(0, 0, 0, \ldots)$ has a unique preimage as the equations

$$\begin{aligned} x_0 &= 0, \\ x_0 + x_1 &= 0, \\ x_1 + x_2 &= 0, \\ &\cdots \end{aligned}$$

has a unique solution $x_0 = x_1 = x_2 = \cdots = 0$. A similar argument can show that α is also surjective. On the other hand, consider the preimage of $(1, 0, 0, \ldots)$, the preimage is $(1, -1, 1, -1, \ldots)$ which is an element in $\prod_{n\geq 0} \mathbb{Z}$ but not $\bigoplus_{n\geq 0} \mathbb{Z}$, so β is not surjective. This implies $\mathrm{Tot}^{\Pi}(C)$ is acyclic but $\mathrm{Tot}^{\oplus}(C)$ is not.

(2) Consider the double compelx C

$$\begin{tikzcd} \cdots \arrow[d] & & \\ \mathbb{Z} & \mathbb{Z} \arrow[l] \arrow[d] & \\ & \mathbb{Z} & \mathbb{Z} \arrow[l] \end{tikzcd}$$

where all maps are isomorphisms. Taking different total complexes, we get two maps

$$\begin{aligned} \alpha &: \prod_{n\geq 0} \mathbb{Z} \to \prod_{n\geq 0} \mathbb{Z}, \\ \beta &: \bigoplus_{n\geq 0} \mathbb{Z} \to \bigoplus_{n\geq 0} \mathbb{Z}. \end{aligned}$$

In both cases, the map is given by

$$(x_0, x_1, x_2, \ldots) \mapsto (x_0 + x_1, x_1 + x_2, x_2 + x_3, \ldots).$$

It is easy to see both α and β are surjective if we take $x_0 = 0$ and solve equations. However, consider the following equations

$$\begin{aligned} x_0 + x_1 &= 0, \\ x_1 + x_2 &= 0, \\ x_2 + x_3 &= 0, \\ &\cdots \end{aligned}$$

If we know that at most only finitely many terms are not zero, then these equations have a unique solution $(0, 0, \ldots)$. Otherwise, we have solutions like $(1, -1, 1, -1, \ldots)$. Thus α is not an isomorphism but β is an isomorphism, so $\mathrm{Tot}^{\oplus}(C)$ is acyclic but $\mathrm{Tot}^{\Pi}(C)$ is not.

(3) Consider a double complex C combined from above two

$$\begin{tikzcd}
\cdots \arrow[d] & & \\
\mathbb{Z} & \mathbb{Z} \arrow[l] \arrow[d] & \\
& \mathbb{Z} & \mathbb{Z} \arrow[l] \arrow[d] \\
& & \cdots
\end{tikzcd}$$

Taking different total complexes, we get two maps

$$\alpha : \prod_{n\geq 0} \mathbb{Z} \to \prod_{n\geq 0} \mathbb{Z},$$
$$\beta : \bigoplus_{n\geq 0} \mathbb{Z} \to \bigoplus_{n\geq 0} \mathbb{Z}.$$

In both cases, the map is given by

$$(\ldots, x_{-1}, x_0, x_1, x_2, \ldots) \mapsto (\ldots, x_{-1}+x_0, x_0+x_1, x_1+x_2, x_2+x_3, \ldots).$$

A similar argument as above two shows that α is not injective and β is not surjective, so while the double complex C has exact rows and columns, neither $\mathrm{Tot}^{\Pi}(C)$ nor $\mathrm{Tot}^{\oplus}(C)$ is acyclic.

Exercise 1.3.5
Let f be a morphism of chain complexes. Show that if $\ker f$ and $\operatorname{coker} f$ are acyclic, then f is a quasi-isomorphism. Is the converse true?

Solution: Let $f_\bullet : A_\bullet \to B_\bullet$ be a chain map between complexes such that the complexes $\ker f_\bullet$ and $\operatorname{coker} f_\bullet$ are acyclic. Note that the category of chain complexes is an abelian category, so the map $f_\bullet$ has a factorization:

$$\begin{tikzcd}
0 \arrow[r] & \ker f_\bullet \arrow[r, hook] & A_\bullet \arrow[r, "f_\bullet"] \arrow[d, two heads, "p"'] & B_\bullet \arrow[r, two heads] & \operatorname{coker} f_\bullet \arrow[r] & 0 \\
& & \mathrm{coim} f_\bullet \arrow[r, "\cong"'] & \operatorname{Im} f_\bullet \arrow[u, hook, "q"'] & &
\end{tikzcd}$$

The short exact sequence

$$0 \to \ker f_\bullet \to A_\bullet \xrightarrow{p} \mathrm{coim} f_\bullet \to 0$$

induces a long exact sequence in homology

$$\cdots \to H_n(\ker f_\bullet) \to H_n(A_\bullet) \xrightarrow{p_*} H_n(\mathrm{coim} f_\bullet) \to H_{n-1}(\ker f_\bullet) \to \cdots$$

We know that the complex $\ker f_\bullet$ is acyclic, so the map

$$p_* : H_n(A_\bullet) \to H_n(\text{coim} f_\bullet)$$

is an isomorphism for all n. Similarly, the map

$$q_* : H_n(\operatorname{Im} f_\bullet) \to H_n(B_\bullet)$$

is an isomorphism for all n. The factorization of the chain map induces a factorization of the induced map $f_* : H_n(A_\bullet) \to H_n(B_\bullet)$, namely

$$\begin{tikzcd}
H_n(A_\bullet) \arrow[r, "f_*"] \arrow[d, "p_*"'] & H_n(B_\bullet) \\
H_n(\text{coim} f_\bullet) \arrow[r, "\cong"'] & H_n(\operatorname{Im} f_\bullet) \arrow[u, "q_*"']
\end{tikzcd}$$

The bottom row map is induced by the isomorphism $\text{coim} f_\bullet \to \operatorname{Im} f_\bullet$. Both p_* and q_* are isomorphisms for all n, so f_* is also an isomorphism. This proves that f is a quasi-isomorphism.

Exercise 1.3.6
Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of double complexes of modules. Show that there is a short exact sequence of total complexes, and conclude that if $\text{Tot}(C)$ is acyclic, then $\text{Tot}(A) \to \text{Tot}(B)$ is a quasi-isomorphism.

Solution: We need to prove that the functor $\text{Tot}(-)$ from double complexes to chain complexes preserves kernel and cokernel. Let $f : A \to B$ be a map of double complexes. Consider the double complex $\ker(f : A \to B)$. On each (p, q)-place, it is a submodule of $A_{p,q}$. Note that taking total complexes preserves objects and maps at each level, so the kernel of the map $\text{Tot}(A) \to \text{Tot}(B)$ is the total complex of the kernel $\ker(f : A \to B)$. Similar argument for the cokernel. So we get a short exact sequence of total complexes:

$$0 \to \text{Tot}(A) \to \text{Tot}(B) \to \text{Tot}(C) \to 0.$$

If $\text{Tot}(C)$ is acyclic, from the long exact sequence in homology. For all n, we have an isomorphism

$$H_n(\text{Tot}(A)) \to H_n(\text{Tot}(B)).$$

So the map $\text{Tot}(A) \to \text{Tot}(B)$ is a quasi-isomorphism.

Exercise 1.5.2

Let $f : C \to D$ be a map of complexes. Show that f is null homotopic if and only if f extends to a map $(-s, f) : \text{cone}(C) \to D$.

Solution: We first prove the necessity. Suppose f extends to a map $(-s, f) : \text{cone}(C) \to D$. Then we have a commutative of diagrams, together with the short exact sequence for the cone(C):

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d, "f"'] & \text{cone}(C) \arrow[r] \arrow[dl, "{(-s,f)}"] & C[-1] \arrow[r] & 0 \\
& D & & &
\end{tikzcd}$$

For any n, this induces a commutative diagram in homology where the row is exact:

$$\begin{tikzcd}
\cdots & H_n(C) \arrow[r, "\partial"] & H_n(C) \arrow[r] \arrow[d, "f_*"'] & H_n(\text{cone}(C)) \arrow[r] \arrow[dl, "{(-s_*,f_*)}"] & H_{n-1}(C) \arrow[r, "\partial"] & H_{n-1}(C) \arrow[r] & \cdots \\
& & H_n(D) & & & &
\end{tikzcd}$$

Note that the connecting homomorphism ∂ is induced by the identity map $id_C : C \to C$, so by exactness, $H_n(\text{cone}(C)) = 0$. So $f_* : H_n(C) \to H_n(D)$ factors through 0. This implies $f_* = 0$ for all n and f is null homotopic.

Next we prove the sufficiency. Assume f is null homotopic. This means there exists $s : C_{n-1} \to D_n$ for all n such that $f = sd_C + d_D s$. We want to show that $(-s, f) : \text{cone}(C) = C_{n-1} \oplus C_n \to D_n$ defines a chain map, namely the following diagram commutes:

$$\begin{tikzcd}
C_{n-1} \oplus C_n \arrow[r, "{(-s,f)}"] \arrow[d] & D_n \arrow[d, "d_D"] \\
C_{n-2} \oplus C_{n-1} \arrow[r, "{(-s,f)}"'] & D_{n-1}
\end{tikzcd}$$

Let $(a, b) \in C_{n-1} \oplus C_n$. Then the image in $C_{n-2} \oplus C_{n-1}$ is $(-d_C(a), d_C(b) - a)$, so its image in D_{n-1} is $sd_C(a) + fd_C(b) - f(a)$. On the other hand, the image of (a, b) in D_n is $-s(a) + f(b)$, and thus it maps to $-d_D s(a) + d_D f(b)$. Using $f = sd_C + d_D s$ and f is a chain map, we obtain that

$$\begin{aligned}
(sd_C(a) + fd_C(b) - f(a)) - (-d_D s(a) + d_D f(b)) &= (sd_C + d_D s - f)(a) + (fd_C - d_D f)(b) \\
&= 0.
\end{aligned}$$

This implies $(-s, f)$ is a chain map between cone(C) and D. Moreover, it is easy to check that for all n, the diagram

$$\begin{tikzcd}
C_n \arrow[r, "f"] \arrow[d] & D_n \\
C_{n-1} \oplus C_n \arrow[ur] &
\end{tikzcd}$$

commutes. So f can be extends to a chain map $(-s, f)$.

Exercise Additional
Let $C \to B \to A$ be morphisms in an abelian category. Prove using axioms of abelian categories (and facts proved in class) that if the induced morphism

$$\operatorname{coker}(C \to B) \to \operatorname{coker}(C \to A)$$

is surjective, then $B \to A$ is surjective.

Solution: Consider the composition of morphisms $C \xrightarrow{f} B \xrightarrow{g} A$. We have the following commutative square:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "id"'] & B \arrow[d, "g"] \\
C \arrow[r, "g\circ f"'] & A
\end{tikzcd}$$

Taking cokernels of each row. From what we have proved in class, there exists a unique map

$$\varphi : \operatorname{coker} f \to \operatorname{coker}(g \circ f)$$

such that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "id"'] & B \arrow[r] \arrow[d, "g"] & \operatorname{coker}(C \xrightarrow{f} B) \arrow[d, two heads, "\varphi"] \\
C \arrow[r, "g\circ f"'] & A \arrow[r] & \operatorname{coker}(C \xrightarrow{g\circ f} A)
\end{tikzcd}$$

Now taking the cokernel of the vertical maps

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "id"'] & B \arrow[r, two heads] \arrow[d, "g"] & \operatorname{coker}(C \xrightarrow{f} B) \arrow[d, two heads, "\varphi"] \\
C \arrow[r, "g\circ f"'] & A \arrow[r, two heads] \arrow[d, two heads] & \operatorname{coker}(C \xrightarrow{g\circ f} A) \arrow[d] \\
 & \operatorname{coker}(B \xrightarrow{g} A) & \operatorname{coker}\varphi
\end{tikzcd}$$

Consider the composition

$$B \xrightarrow{g} A \to \operatorname{coker}(C \xrightarrow{g\circ f} A) \to \operatorname{coker}\varphi.$$

From the commutativity of the diagram, this is the same as

$$B \to \operatorname{coker}(C \xrightarrow{f} B) \to \operatorname{coker}(C \xrightarrow{g\circ f} A) \to \operatorname{coker}\varphi,$$

which is 0 by definition of $\operatorname{coker}\varphi$. By universal property of cokernels, this map must factor through

$\operatorname{coker}(B \xrightarrow{g} A)$, namely we have the following commutative diagram:

$$\begin{tikzcd} B \arrow[r, "g"] & A \arrow[r, two heads] \arrow[d] & \operatorname{coker}(B \xrightarrow{g} A) \\ & \operatorname{coker}\varphi \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

The map $\operatorname{coker}\varphi \to \operatorname{coker}(B \xrightarrow{g} A)$ is also an epimorphism, following from the following claim.
Claim: Suppose we have a composition of morphisms: $X \xrightarrow{a} Y \xrightarrow{b} Z$. If $b \circ a$ is an epimorphism, then b is an epimorphism.
Proof:

■ Suppose given an object W and two morphisms $f, g : Z \to W$ such that $f \circ b = g \circ b$. We need to show that $f = g$. Compose with a, we get

$$f \circ b \circ a = g \circ b \circ a.$$

We know $b \circ a$ is an epimorphism, so $f = g$.

Note that $\operatorname{coker}\varphi = 0$ as the map φ is an epimorphism, so the map

$$\operatorname{coker}\varphi = 0 \to \operatorname{coker}(B \xrightarrow{g} A)$$

is an epimorphism. This implies that $\operatorname{coker}(B \xrightarrow{g} A) = 0$, so the map $g : B \to A$ is an epimorphism.