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Homework 3
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We do some calculation about Tor and Ext, which will be used in this homework.

(1) For any abelian group A, $\text{Tor}_1(A, \mathbb{Z}) = \text{Tor}_1(\mathbb{Z}, A) = \text{Ext}^1(\mathbb{Z}, A) = 0$.
Note that the free resolution of $\mathbb{Z}$ is given by

$$0 \to \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \to 0.$$

The degree 1 part is already 0, so we have

$$\text{Tor}_1(\mathbb{Z}, A) = \text{Ext}^1(\mathbb{Z}, A) = 0.$$

Suppose

$$0 \to J_1 \to J_0 \to A \to 0$$

is a free resolution of A and note that for any abelian group B, we have $B \otimes \mathbb{Z} = B$. After tensoring with $\mathbb{Z}$, we have the following chain complex

$$0 \to J_1 \to J_0 \to 0.$$

This implies $\text{Tor}_1(A, \mathbb{Z}) = 0$.

(2) Let A be an abelian group and A_t denote the torsion part of A. Then $\text{Ext}^1(A, \mathbb{Z}) = A_t$.
The functor Ext^1 is additive and we have prove that $\text{Ext}^1(\mathbb{Z}, \mathbb{Z}) = 0$. We only need to show that $\text{Ext}^1(\mathbb{Z}/n, \mathbb{Z}) = \mathbb{Z}/n$ for any $n \geq 2$. Consider the free resolution

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0.$$

Apply $\hom(-, \mathbb{Z})$ and we obtain a cochain complex

$$0 \leftarrow \mathbb{Z} \xleftarrow{n} \mathbb{Z} \leftarrow 0.$$

This implies $\text{Ext}^1(\mathbb{Z}/n, \mathbb{Z}) = \mathbb{Z}/n$.

(3) For any abelian group A, B, we know that

$$\text{Tor}_0(A, B) = A \otimes B,$$
$$\text{Ext}^0(A, B) = \hom(A, B).$$

We have seen the proof in class.

(4) For any integers $m, n \geq 2$, we have

$$\text{Tor}_1(\mathbb{Z}/m, \mathbb{Z}/n) = \text{Ext}^1(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/d$$

where d is the greatest common divisor of m and n (If $d = 1$, then $\mathbb{Z}/d = 0$).
Consider the following free resolution of $\mathbb{Z}/m$

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/n \to 0.$$

Apply $- \otimes \mathbb{Z}/n$ and $\hom(-, \mathbb{Z}/n)$, we obtain a chain complex and cochain complex as follows

$$0 \to \mathbb{Z}/n \xrightarrow{m} \mathbb{Z}/n \to 0,$$
$$0 \leftarrow \mathbb{Z}/n \xleftarrow{m} \mathbb{Z}/n \leftarrow 0.$$

By calculation,

$$\mathrm{Tor}_1(\mathbb{Z}/m, \mathbb{Z}/n) = \mathrm{Ext}^1(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/d.$$

Problem 1
Compute both $\mathrm{Tor}_i(A, B)$ and $\mathrm{Ext}^i(A, B)$ for all i in the following cases:

(a) $A = \mathbb{Z}/9$ and $B = \mathbb{Z}/6$.

(b) $A = \mathbb{Z}/9$ and $B = \mathbb{Z}$.

(c) $A = \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/10$ and $B = \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/6$.

Solution:

(a) From the discussion at the beginning, we know that

$$\begin{aligned}
\mathrm{Tor}_0(\mathbb{Z}/9, \mathbb{Z}/6) &= \mathbb{Z}/9 \otimes \mathbb{Z}/6 = \mathbb{Z}/3,\\
\mathrm{Tor}_1(\mathbb{Z}/9, \mathbb{Z}/6) &= \mathbb{Z}/3,\\
\mathrm{Ext}^0(\mathbb{Z}/9, \mathbb{Z}/6) &= \hom(\mathbb{Z}/9, \mathbb{Z}/6) = \mathbb{Z}/3,\\
\mathrm{Ext}^1(\mathbb{Z}/9, \mathbb{Z}/6) &= \mathbb{Z}/3.
\end{aligned}$$

All other Tor and Ext are 0.

(b) From the discussion at the beginning, we know that

$$\begin{aligned}
\mathrm{Tor}_0(\mathbb{Z}/9, \mathbb{Z}) &= \mathbb{Z}/9 \otimes \mathbb{Z} = \mathbb{Z}/9,\\
\mathrm{Tor}_1(\mathbb{Z}/9, \mathbb{Z}) &= 0,\\
\mathrm{Ext}^0(\mathbb{Z}/9, \mathbb{Z}) &= \hom(\mathbb{Z}/9, \mathbb{Z}) = 0,\\
\mathrm{Ext}^1(\mathbb{Z}/9, \mathbb{Z}) &= \mathbb{Z}/9.
\end{aligned}$$

All other Tor and Ext are 0.

(c) Tor and Ext are additive functors, so we can calculte using the results from the discussion. Let

$$A = \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/10,$$
$$B = \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/6.$$

Use the discussion at the beginning and the fact that Tor and Ext are additive.

$$\begin{aligned}\mathrm{Tor}_0(A,B) &= A\otimes B\\ &= B\oplus B\oplus((\mathbb{Z}/4)^2\oplus\mathbb{Z}/2)\oplus\mathbb{Z}/5\oplus(\mathbb{Z}/10\oplus(\mathbb{Z}/2)^2)\\ &= \mathbb{Z}^2\oplus(\mathbb{Z}/2)^3\oplus(\mathbb{Z}^3)^2\oplus(\mathbb{Z}/4)^4\oplus\mathbb{Z}/5\oplus(\mathbb{Z}/6)^2\oplus\mathbb{Z}/10.\end{aligned}$$

$$\begin{aligned}\mathrm{Tor}_1(A,B) &= \mathrm{Tor}_1(\mathbb{Z}/4\oplus\mathbb{Z}/5\oplus\mathbb{Z}/10,\mathbb{Z}/3\oplus\mathbb{Z}/4\oplus\mathbb{Z}/6)\\ &= \mathbb{Z}/4\oplus(\mathbb{Z}/2)^3.\end{aligned}$$

$$\begin{aligned}\mathrm{Ext}^0(A,B) &= \hom(A,B)\\ &= B^2\oplus\hom(\mathbb{Z}/4\oplus\mathbb{Z}/5\oplus\mathbb{Z}/10,\mathbb{Z}/3\oplus\mathbb{Z}/4\oplus\mathbb{Z}/6)\\ &= B^2\oplus\mathbb{Z}/4\oplus(\mathbb{Z}/2)^3\\ &= \mathbb{Z}^2\oplus(\mathbb{Z}/2)^3\oplus(\mathbb{Z}/3)^2\oplus(\mathbb{Z}/4)^3\oplus(\mathbb{Z}/6)^2.\end{aligned}$$

$$\begin{aligned}\mathrm{Ext}^1(A,B) &= \mathrm{Ext}^1(\mathbb{Z}/4\oplus\mathbb{Z}/5\oplus\mathbb{Z}/10,B)\\ &= \mathbb{Z}/4\oplus\mathbb{Z}/5\oplus\mathbb{Z}/10\oplus\mathbb{Z}/4\oplus(\mathbb{Z}/2)^3\\ &= (\mathbb{Z}/2)^3\oplus(\mathbb{Z}/4)^2\oplus\mathbb{Z}/5\oplus\mathbb{Z}/10.\end{aligned}$$

All other Ext and Tor are zero.

Problem 2

Let A be an abelian group and let $G_*\to A\to 0$ be a free resolution. Let B be another abelian group and let $J_*\to B\to 0$ be a free resolution.

(a) Given a map $f:A\to B$, prove that there are maps $F_i:G_i\to J_i$ making all squares commute, we call this chain map $\{F:G_*\to J_*\}$ a lifting of the map f.

(b) Prove that if $\{F':G_*\to J_*\}$ is another lifting of f then the chain map F and F' are chain homotopic.

(c) If C is another abelian group one gets an induced map $F\otimes id:G_*\otimes C\to J_*\otimes C$ and therefore an induced map on homology groups $f_*:\mathrm{Tor}_i(A,C)\to\mathrm{Tor}_i(B,C)$. Since any two choices of F are homotopic, this f_* is well-defined.
Use the above procedure to calculate the maps

$$\begin{aligned}j_* &:\mathrm{Tor}_1(\mathbb{Z}/2,\mathbb{Z}/2)\to\mathrm{Tor}_1(\mathbb{Z}/4,\mathbb{Z}/2),\\ k_* &:\mathrm{Tor}_1(\mathbb{Z}/4,\mathbb{Z}/2)\to\mathrm{Tor}_1(\mathbb{Z}/2,\mathbb{Z}/2).\end{aligned}$$

induced by the map $j:\mathbb{Z}/2\hookrightarrow\mathbb{Z}/4$ (sending 1 to 2) and $k:\mathbb{Z}/4\to\mathbb{Z}/2$ (sending 1 to 1).

Solution:

(a) We have a diagram with long exact sequences as follows

$$\begin{tikzcd}
\cdots \arrow[r, "g_2"] & G_1 \arrow[r, "g_1"] & G_0 \arrow[r, "g_0"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[d, "0"] \\
\cdots \arrow[r, "j_2"] & J_1 \arrow[r, "j_1"] & J_0 \arrow[r, "j_0"] & B \arrow[r] & 0
\end{tikzcd}$$

we need to construct $f_i : G_i \to J_i$ from $i = 0$ inductively. Consider the following diagram with solid arrows

$$\begin{tikzcd}
& G_0 \arrow[dl, dashed, "f_0"'] \arrow[d, "fg_0"] & \\
J_0 \arrow[r, two heads, "j_0"'] & B \arrow[r] & 0
\end{tikzcd}$$

$j_0 : J_0 \to B$ is surjective by exactness and G_0 is projective because it is free, there exists a map $f_0 : G_0 \to J_0$ such that $j_0 f_0 = f g_0$. We have constructed the first step with a diagram as follows:

$$\begin{tikzcd}
\cdots \arrow[r, "g_2"] & G_1 \arrow[r, "g_1"] & G_0 \arrow[r, "g_0"] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[d, "0"] \\
\cdots \arrow[r, "j_2"] & J_1 \arrow[r, "j_1"] & J_0 \arrow[r, "j_0"] & B \arrow[r] & 0
\end{tikzcd}$$

Next, consider the composition $f_0 g_1 : G_1 \to J_0$, for any $x \in G_1$, by commutativity of the diagram, we have

$$j_0 f_0 g_1(x) = f g_0 g_1(x) = 0$$

because of the exactness of top row. By the exactness of the bottom row, we have

$$f_0 g_1(x) \in \ker j_0 = \operatorname{Im} j_1.$$

This means the map $f_0 g_1$ must factor through $\operatorname{Im} j_1$ and we have the following solid arrow diagram

$$\begin{tikzcd}
& G_1 \arrow[dl, dashed, "f_1"'] \arrow[d, "f_0 g_1"] & \\
J_1 \arrow[r, two heads, "j_1"'] & \operatorname{im} j_1 \arrow[r] & 0
\end{tikzcd}$$

G_1 is projective so there exists a map $f_1 : G_1 \to J_1$ such that $j_1 f_1 = f_0 g_1$. This means we have obtained the next map we need in the following diagram

$$\begin{tikzcd}
\cdots \arrow[r, "g_2"] & G_1 \arrow[r, "g_1"] \arrow[d, "f_1"] & G_0 \arrow[r, "g_0"] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[d, "0"] \\
\cdots \arrow[r, "j_2"] & J_1 \arrow[r, "j_1"] & J_0 \arrow[r, "j_0"] & B \arrow[r] & 0
\end{tikzcd}$$

Repeat the steps inductively and we obtain a chain map $F : G_* \to J_*$ where in each degree is given by $f_i : G_i \to J_i$ for $i \geq 0$.

(b) Suppose we have two chain maps

$$\begin{tikzcd}
\cdots \arrow[r, "g_2"] & G_1 \arrow[r, "g_1"] \arrow[d, shift right, "f_1'"'] \arrow[d, shift left, "f_1"] & G_0 \arrow[r, "g_0"] \arrow[d, shift right, "f_0'"'] \arrow[d, shift left, "f_0"] & A \arrow[r] \arrow[d, "f"] \arrow[dl, "0"'] & 0 \arrow[d, "0"] \arrow[dl, "0"'] \\
\cdots \arrow[r, "j_2"] & J_1 \arrow[r, "j_1"] & J_0 \arrow[r, "j_0"] & B \arrow[r] & 0
\end{tikzcd}$$

we already have two zero maps $0 : 0 \to B$ and $0 : A \to j_0$ satisfying

$$0 + j_0 \circ 0 = f - f$$

We can take $H_{-1} = H_0 = 0$ as the chain homotopy map. For $n \geq 1$, suppose we have already constructed $H_{n-1} : G_{n-1} \to J_n$ and $H_{n-2} : G_{n-2} \to J_{n-1}$ satisfying

$$f_{n-1} - f'_{n-1} = H_{n-2}g_{n-1} + j_n H_{n-1}.$$

We want to construct the map $H_n : G_n \to J_{n+1}$.

$$\begin{tikzcd}
\cdots \arrow[r] & G_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, shift right] \arrow[d, shift left] & G_n \arrow[r, "g_n"] \arrow[d, shift right, "f_n'"'] \arrow[d, shift left, "f_n"] \arrow[dl, dashed, "H_n"'] & G_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, shift right, "f_{n-1}'"'] \arrow[d, shift left, "f_{n-1}"] \arrow[dl, "H_{n-1}"'] & G_{n-2} \arrow[r] \arrow[d, shift right, "f_{n-2}'"'] \arrow[d, shift left, "f_{n-2}"] \arrow[dl, "H_{n-2}"'] & \cdots \\
\cdots \arrow[r] & J_{n+1} \arrow[r, "j_{n+1}"'] & J_n \arrow[r, "j_n"'] & J_{n-1} \arrow[r, "j_{n-1}"'] & J_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

Consider the following map

$$f_n - f'_n - H_{n-1}g_n : G_n \to J_n.$$

For any $x \in G_n$, use the commutativity of the diagram and the property of H_{n-1} and H_{n-2}, we have

$$\begin{aligned}
(j_n f_n - j_n f'_n - j_n H_{n-1} g_n)(x) &= (j_n f_n)(x) - (j_n f'_n)(x) - (j_n H_{n-1} g_n)(x) \\
&= (f_{n-1}g_n)(x) - (f'_{n-1}g_n)(x) - [(f_{n-1} - f'_{n-1} - H_{n-2}g_{n-1})g_n](x) \\
&= [(f_{n-1} - f'_{n-1})g_n](x) - [(f_{n-1} - f'_{n-1})g_n](x) \\
&= 0.
\end{aligned}$$

This implies that

$$(f_n - f'_n - H_{n-1}g_n)(x) \in \ker j_n = \operatorname{Im} j_{n+1}$$

for all $x \in G_n$ by exactness of the bottom row. Then this map must factor though $\operatorname{Im} j_{n+1}$ and we have a solid arrow diagram

$$\begin{tikzcd}
 & G_n \arrow[d, "f_n - f'_n - H_{n-1}g_n"] \arrow[dl, dashed, "H_n"'] & \\
J_{n+1} \arrow[r, two heads, "j_{n+1}"'] & \operatorname{im} j_{n+1} \arrow[r] & 0
\end{tikzcd}$$

G_n being projective implies there exists a map $H_n : G_n \to J_{n+1}$ such that

$$f_n - f'_n = H_{n-1}g_n + j_{n+1}H_n.$$

Repeat this step inductively and we have constructed a chain homotopy between F and F'.

(c) Consider the following commutative diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "1"] \arrow[d, "2"'] & \mathbb{Z} \arrow[d, "4"] \\ \mathbb{Z} \arrow[r, "2"] \arrow[d, "1"'] & \mathbb{Z} \arrow[d, "1"] \\ \mathbb{Z}/2 \arrow[r, "2"'] & \mathbb{Z}/4 \end{tikzcd}$$

The left and right vertical columns are free resolutions of $\mathbb{Z}/2$ and $\mathbb{Z}/4$. Apply $- \otimes \mathbb{Z}/2$ to the resolutions and we get a commutative diagram

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r, "1"] \arrow[d, "2"'] & \mathbb{Z}/2 \arrow[d, "4"] \\ \mathbb{Z}/2 \arrow[r, "2"] & \mathbb{Z}/2 \end{tikzcd}$$

The map sending 1 to 1 in homology is the identity map, so

$$j_* : \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) \to \mathrm{Tor}_1(\mathbb{Z}/4, \mathbb{Z}/2)$$

is the identity map of $\mathbb{Z}/2$. Similarly, consider the free resolutions of $\mathbb{Z}/4$ and $\mathbb{Z}/4$.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "2"] \arrow[d, "4"'] & \mathbb{Z} \arrow[d, "2"] \\ \mathbb{Z} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{Z} \arrow[d, "1"] \\ \mathbb{Z}/4 \arrow[r, "1"'] & \mathbb{Z}/2 \end{tikzcd}$$

Apply $- \otimes \mathbb{Z}/2$ and we obtain a commutative diagram

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r, "2"] \arrow[d, "4"'] & \mathbb{Z}/2 \arrow[d, "2"] \\ \mathbb{Z}/2 \arrow[r, "1"] & \mathbb{Z}/2 \end{tikzcd}$$

The map sends 1 to 2 is the zero map for $\mathbb{Z}/2$, so

$$k_* : \mathrm{Tor}_1(\mathbb{Z}/4, \mathbb{Z}/2) \to \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2)$$

is the zero map of $\mathbb{Z}/2$.

Problem 3.7
If F is a finitely-generated free abelian group then there is a canonical isomorphism

$$\hom(\hom(F,\mathbb{Z}),\mathbb{Z}) \cong F.$$

So if C is a chain complex consisting of finitely generated, free abelian groups, one gets an induced isomorphism

$$\hom(\hom(C,\mathbb{Z}),\mathbb{Z}) \cong C.$$

Using this, derive a universal coefficient theorem which lets you predict $H_*(C)$ if you know $H^*(\hom(C,\mathbb{Z}))$.

Solution: For all i, we have

$$\begin{aligned} H_i(C) &\cong H_i(\hom(\hom(C,\mathbb{Z})),\mathbb{Z}) \\ &\cong \hom(H^i(\hom(C,\mathbb{Z})),\mathbb{Z}) \oplus \mathrm{Ext}^1(H^{i+1}(\hom(C,\mathbb{Z})),\mathbb{Z}). \end{aligned}$$

Problem 3.8
In this problem we'll use the abbreviations $H^i(\hom(C,\mathcal{A})) = H^i(C;\mathcal{A})$ and $H^i(C) = H^i(C;\mathbb{Z})$.

If F is a finitely -generated free abelian group then there is a canonical isomorphism

$$\hom(F,\mathcal{A}) \cong \hom(F,\mathbb{Z}) \otimes \mathbb{Z}.$$

So if C is a chain complex consistin of finitely generated free abelian groups, we have an isomorphism

$$\hom(C,\mathcal{A}) \cong \hom(C,\mathbb{Z}) \otimes \mathcal{A}.$$

Using this, derive a universal coefficient theorem which lets you predict $H^*(C;\mathcal{A})$ if you know $H^*(C)$. The formula should look like

$$H^i(C;\mathcal{A}) \cong [H^?(C) \otimes \mathcal{A}] \oplus [\mathrm{Tor}_1(H^?(C),\mathcal{A})]$$

where you determine the indices marked "?".

Solution: For all i, we have

$$H^i(C;\mathcal{A}) \cong [H^i(C) \otimes \mathcal{A}] \oplus [\mathrm{Tor}_1(H^{i+1}(C),\mathcal{A})]$$

Problem 4
Suppose X is a finite CW complex for which

$$H_0(X;\mathbb{Z}/2)=\mathbb{Z}/2, H_1(X;\mathbb{Z}/2)=(\mathbb{Z}/2)^3, H_2(X;\mathbb{Z}/2)=0, H_3(X;\mathbb{Z}/2)=H_4(X;\mathbb{Z}/2)=\mathbb{Z}/2$$

and $H_i(X;\mathbb{Z}/2)=0$ for all $i\geq 5$.

(a) Determine as much as you can about $H_*(X;\mathbb{Z})$.

(b) Suppose you are also told that $H_2(X;\mathbb{Z}/3)=\mathbb{Z}/3$ and $H_3(X;\mathbb{Z}/3)=0$. What else can you say about $H_*(X;\mathbb{Z})$ now?

(c) Suppose Y is a space with finitely-generated homology groups and you are told $H_i(Y;\mathbb{Z}/p)=0$ for a specific prime p. What can you deduce about $H_i(Y)$ and $H_{i-1}(Y)$?

Solution:

(a) Use the universal coefficient theorem (UCT) for homology. For $i=0$, note that $H_{-1}(X)=0$, we have

$$\mathbb{Z}/2=H_0(X;\mathbb{Z}/2)\cong H_0(X)\otimes\mathbb{Z}/2.$$

Note that $H_0(X)$ is always free, so $H_0(X)\cong\mathbb{Z}$. For $i=1$, by UCT, we have

$$(\mathbb{Z}/2)^3=H_1(X;\mathbb{Z}/2)\cong H_1(X)\otimes\mathbb{Z}/2\oplus\mathrm{Tor}_1(H_0(X),\mathbb{Z}/2).$$

We know that $H_0(X)=\mathbb{Z}$ and $\mathrm{Tor}_1(\mathbb{Z},\mathbb{Z}/2)=0$. So $H_1(X)$ has three generators, each of them is either free or is of even order, plus any generator with finite odd order. Moreover, by UCT in $i=1$, we have

$$0=H_2(X;\mathbb{Z}/2)=H_2(X)\otimes\mathbb{Z}/2\oplus\mathrm{Tor}_1(H_1(X),\mathbb{Z}/2).$$

This implies $\mathrm{Tor}_1(H_1(X),\mathbb{Z}/2)=0$. If any generator of $H_1(X)$ has finite even order, then $\mathrm{Tor}_1(H_1(X),\mathbb{Z}/2)$ must contain $\mathbb{Z}/2$. So $H_1(X)$ has three free generators and

$$H_1(X)=\mathbb{Z}^3\oplus\oplus_{k\in I_1}\mathbb{Z}/n_{1,k}.$$

From $H_2(X)\otimes\mathbb{Z}/2=0$, we know that $H_2(X)=0$ or $H_2(X)=\oplus_{k\in I_2}\mathbb{Z}/n_{2,k}$ for each $n_{2,k}\geq 3$ odd. For $i=3$, by UCT, we have

$$\mathbb{Z}/2=H_3(X;\mathbb{Z}/2)=H_3(X)\otimes\mathbb{Z}/2\oplus\mathrm{Tor}_1(H_2(X),\mathbb{Z}/2).$$

Note that $H_2(X)$ only consists of $\mathbb{Z}/n_k$ for some odd n_k, so $\mathrm{Tor}_1(H_2(X),\mathbb{Z}/2)=0$. This implies $H_3(X)\otimes\mathbb{Z}/2=\mathbb{Z}/2$. So either $H_3(X)=\mathbb{Z}$ or $H_3(X)=\mathbb{Z}/m$ for some even number $m\geq 2$, plus some non contributing generators with finite odd order. This will split into two different cases.

Case 1: Assume $H_3(X)=\mathbb{Z}/m$ for some even number $m\geq 2$, plus some odd order generators. For $i=4$, by UCT, we have

$$\mathbb{Z}/2=H_4(X;\mathbb{Z}/2)=H_4(X)\otimes\mathbb{Z}/2\oplus\mathrm{Tor}_1(H_3(X),\mathbb{Z}/2).$$

Note that for any even number $m \geq 2$, we have

$$\mathrm{Tor}_1(H_3(X), \mathbb{Z}/2) = \mathrm{Tor}_1(\mathbb{Z}/m, \mathbb{Z}/2) = \mathbb{Z}/2.$$

This implies that $H_4(X) \otimes \mathbb{Z}/2 = 0$. So $H_4(X) = 0$ or $H_4(X) = \oplus_{k\in I_4}\mathbb{Z}/n_{4,k}$ for some odd numbers $n_{4,k}$. Doing this inductively and we can see that for any $i \geq 5$, we know that

$$H_i(X) = \oplus_{k\in I_k}\mathbb{Z}/n_{i,k}$$

where all $n_{i,k}$ are odd numbers. We summarize as follows

$$H_i(X) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0; \\ \mathbb{Z}^3 \oplus \oplus_{k\in I_1}\mathbb{Z}/n_{1,k}, & \text{if } \ i = 1; \\ 0 \ \text{ or } \ \oplus_{k\in I_2}\mathbb{Z}/n_{2,k}, & \text{if } \ i = 2; \\ \mathbb{Z}/m \oplus \oplus_{k\in I_3}\mathbb{Z}/n_{3,k}, & \text{if } \ i = 3; \\ 0 \ \text{ or } \ \oplus_{k\in I_i}\mathbb{Z}/n_{i,k}, & \text{if } \ i \geq 4. \end{cases}$$

where $m \geq 2$ is an even number, and for all i, I_i is a finite set and all $n_{i,k}$ are odd numbers since X is a finite CW complex.

Case 2: Assume $H_3(X) = \mathbb{Z}$, plus some odd order generators. Note that in this case $\mathrm{Tor}_1(H_3(X), \mathbb{Z}/2) = 0$. So we have

$$\mathbb{Z}/2 = H_4(X; \mathbb{Z}/2) = H_4(X) \otimes \mathbb{Z}/2.$$

Combine with the fact that

$$0 = H_5(X; \mathbb{Z}/2) = H_5(X) \otimes \mathbb{Z}/2 \oplus \mathrm{Tor}_1(H_4(X), \mathbb{Z}/2).$$

The free part of $H_4(X)$ can only be $\mathbb{Z}$ otherwise the torsion will not vanish. Starting from $H_5(X)$, it follows the same pattern as case 1. We summarize as follows

$$H_i(X) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0; \\ \mathbb{Z}^3 \oplus \oplus_{k\in I_1}\mathbb{Z}/n_{1,k}, & \text{if } \ i = 1; \\ 0 \ \text{ or } \ \oplus_{k\in I_2}\mathbb{Z}/n_{2,k}, & \text{if } \ i = 2; \\ \mathbb{Z} \oplus \oplus_{k\in I_3}\mathbb{Z}/n_{3,k}, & \text{if } \ i = 3; \\ \mathbb{Z} \oplus \oplus_{k\in I_4}\mathbb{Z}/n_{4,k}, & \text{if } \ i = 4; \\ 0 \ \text{ or } \ \oplus_{k\in I_i}\mathbb{Z}/n_{i,k}, & \text{if } \ i \geq 5. \end{cases}$$

where for all i, I_i is a finite set and $n_{i,k}$ are odd numbers since X is a finite CW complex.

(b) First note that because $0 = H_3(X; \mathbb{Z}/3)$, by UCT, we have

$$0 = H_3(X) \otimes \mathbb{Z}/3 \oplus \mathrm{Tor}_1(H_2(X), \mathbb{Z}/3).$$

So $H_3(X)$ cannot contain $\mathbb{Z}$, we will only have case 1. $H_2(X)$ does not contain any free part, so

$$\mathrm{Tor}_1(H_2(X), \mathbb{Z}/3) = H_2(X) \otimes \mathbb{Z}/3 = 0$$

By UCT, we have

$$\mathbb{Z}/3 = H_2(X; \mathbb{Z}/3) = \mathrm{Tor}_1(H_1(X), \mathbb{Z}/3).$$

We summarize the additional information as below.

$$\begin{aligned} \mathrm{Tor}_1(H_1(X), \mathbb{Z}/3) &= \mathbb{Z}/3, \\ H_2(X) \otimes \mathbb{Z}/3 &= 0, \\ H_3(X) \otimes \mathbb{Z}/3 &= 0. \end{aligned}$$

So $H_1(X)$ contains and only contains one copy of $\mathbb{Z}/3k$ for some $k \geq 1$. $H_2(X)$ and $H_3(X)$ does not contain any copies of $\mathbb{Z}/3k$ for any $k \geq 1$.

(c) By UCT, we have

$$0 = H_i(Y; \mathbb{Z}/p) = H_i(Y) \otimes \mathbb{Z}/p \oplus \mathrm{Tor}_1(H_{i-1}(Y), \mathbb{Z}/p).$$

So $H_i(Y) = \oplus_{i \in I} \mathbb{Z}/n_i$ for some $n_i \geq 1$ where each n_i is coprime with p. And $H_{i-1}(Y)$ does not contain any $\mathbb{Z}/kp$ for any $k \geq 1$.

Problem 5

(a) For a certain class of n-manifold M, one always has Poincaré Duality:

$$H_i(M; \mathbb{Z}) \cong H^{n-i}(M; \mathbb{Z}).$$

Assuming this, as well as the fact that all the homology groups of M are finitely generated, explain why the Universal Coefficient Theorems then imply that

(1) the rank of $H_i(M; \mathbb{Z})$ is the same as the rank of $H_{n-i}(M; \mathbb{Z})$, for all i.

(2) the torsion part of $H_i(M; \mathbb{Z})$ is the same as the torsion part of $H_{n-i-1}(M; \mathbb{Z})$, for all i.

(b) Suppose M is a 5-manifold for which Poincaré Duality holds. Given that $H_0(M) = \mathbb{Z}$, $H_1(M) = \mathbb{Z}^2 \oplus \mathbb{Z}/4$, and $H_2(M) = \mathbb{Z} \oplus \mathbb{Z}/5$, compute $H_i(M)$, $H_i(M; \mathbb{Z}/2)$ and $H_i(M; \mathbb{Z}/5)$ for all i.

Solution:

(a) First we obeserve that for $i \geq n$, by Poincaré Duality we have

$$H_i(M) = H^{n-i}(M) = 0$$

since $n - i < 0$. For $0 \leq i \leq n$, we have

$$H_i(M) \cong H^{n-i}(M) \cong \hom(H_{n-i}(M), \mathbb{Z}) \oplus \mathrm{Ext}^1(H_{n-i-1}(M), \mathbb{Z}).$$

(1) We have proved at the beginning that $\operatorname{Ext}^1(H_{n-i-1}(M), \mathbb{Z})$ is either 0 or equal to the direct sum of some finite groups. So the free part can only be detected at $\hom(H_{n-i}(M), \mathbb{Z})$ and note that

$$\operatorname{rank} \hom(H_{n-i}(M), \mathbb{Z}) = \operatorname{rank} H_{n-i}(M).$$

So we know that $H_i(M)$ and $H_{n-i}(M)$ have the same rank.

(2) Note that $\hom(\mathbb{Z}/n, \mathbb{Z}) = 0$ for any $n \geq 2$. So $\hom(H_{n-i}(M), \mathbb{Z})$ cannnot detect any torsion part, and we have proved at the beginning that

$$\operatorname{Ext}^1(H_{n-i-1}(M), \mathbb{Z}) = (H_{n-i-1}(M))_t$$

where $(H_{n-i-1}(M))_t$ means the torsion part of the abelian group $H_{n-i-1}(M)$.

(b) For an abelian group A, we write $A = A_f \oplus A_t$ where A_f is the free part of A and A_t is the torsion part of A. Therefore, we can calculate

$$\begin{aligned}
H_3(M) &= (H_3(M))_f \oplus (H_3(M))_t = (H_2(M))_f \oplus (H_1(M))_t = \mathbb{Z} \oplus \mathbb{Z}/4, \\
H_4(M) &= (H_4(M))_f \oplus (H_4(M))_t = (H_1(M))_f \oplus (H_0(N))_t = \mathbb{Z}^2, \\
H_5(M) &= (H_5(M))_f \oplus (H_5(M))_t = (H_0(M))_f = \mathbb{Z}.
\end{aligned}$$

Next, we use UCT for homology to determine $H_*(M; \mathbb{Z}/2)$.

$$\begin{aligned}
H_0(M; \mathbb{Z}/2) &= H_0(M) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}_1(H_{-1}(M), \mathbb{Z}/2) = \mathbb{Z}/2. \\
H_1(M; \mathbb{Z}/2) &= H_1(M) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}_1(H_0(M), \mathbb{Z}/2) = (\mathbb{Z}/2)^3. \\
H_2(M; \mathbb{Z}/2) &= H_2(M) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}_1(H_1(M), \mathbb{Z}/2) = (\mathbb{Z}/2)^2. \\
H_3(M; \mathbb{Z}/2) &= H_3(M) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}_1(H_2(M), \mathbb{Z}/2) = (\mathbb{Z}/2)^2. \\
H_4(M; \mathbb{Z}/2) &= H_4(M) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}_1(H_3(M), \mathbb{Z}/2) = (\mathbb{Z}/2)^3. \\
H_5(M; \mathbb{Z}/2) &= H_5(M) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}_1(H_4(M), \mathbb{Z}/2) = \mathbb{Z}/2. \\
H_6(M; \mathbb{Z}/2) &= \operatorname{Tor}_1(H_5(M), \mathbb{Z}/2) = 0.
\end{aligned}$$

$$\begin{aligned}
H_0(M; \mathbb{Z}/3) &= H_0(M) \otimes \mathbb{Z}/3 \oplus \operatorname{Tor}_1(H_{-1}(M), \mathbb{Z}/3) = \mathbb{Z}/3. \\
H_1(M; \mathbb{Z}/3) &= H_1(M) \otimes \mathbb{Z}/3 \oplus \operatorname{Tor}_1(H_0(M), \mathbb{Z}/3) = (\mathbb{Z}/3)^2. \\
H_2(M; \mathbb{Z}/3) &= H_2(M) \otimes \mathbb{Z}/3 \oplus \operatorname{Tor}_1(H_1(M), \mathbb{Z}/3) = \mathbb{Z}/3. \\
H_3(M; \mathbb{Z}/3) &= H_3(M) \otimes \mathbb{Z}/3 \oplus \operatorname{Tor}_1(H_2(M), \mathbb{Z}/3) = \mathbb{Z}/3. \\
H_4(M; \mathbb{Z}/3) &= H_4(M) \otimes \mathbb{Z}/3 \oplus \operatorname{Tor}_1(H_3(M), \mathbb{Z}/3) = (\mathbb{Z}/3)^2. \\
H_5(M; \mathbb{Z}/3) &= H_5(M) \otimes \mathbb{Z}/3 \oplus \operatorname{Tor}_1(H_4(M), \mathbb{Z}/3) = \mathbb{Z}/3. \\
H_6(M; \mathbb{Z}/3) &= \operatorname{Tor}_1(H_5(M), \mathbb{Z}/3) = 0.
\end{aligned}$$

Problem 6
Consider $R = \mathbb{Z}/p^2$ and let $M = \mathbb{Z}/p$. Construct a free resolution of M (in the category of R-modules) and use this resolution to compute $\mathrm{Tor}_i^R(M, M)$ for all i.

Solution: Observe that we have the following exact sequence of R-modules

$$\cdots \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{1} \mathbb{Z}/p \to 0.$$

We check exactness at each spot. The map $\mathbb{Z}/p^2 \xrightarrow{1} \mathbb{Z}/p$ is surjective and the kernel is

$$K = \{0, p, 2p, \ldots, (p-1)p\} \subseteq \mathbb{Z}/p^2.$$

The image of the map $\mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2$ is exactly K in $\mathbb{Z}/p^2$ and the kernel is also the same thing. So we have a $\mathbb{Z}/p^2$-free resolution of $\mathbb{Z}/p$. Tensoring with $\mathbb{Z}/p$ and we obtain a chain complex

$$\cdots \xrightarrow{0} \mathbb{Z}/p \xrightarrow{0} \mathbb{Z}/p \to 0.$$

So $\mathrm{Tor}_1^{\mathbb{Z}/p^2}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p$ for all $i \geq 0$.