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Course: MATH 636 - Algebraic Topology III
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Homework 7
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Problem 1
If M and N are compact, oriented d-manifold, then the **degree** of a map $f : M \to N$ is defined to be the integer $\deg(f)$ such that $f_*([M]) = \deg f \cdot [N]$.

(a) Suppose that f is not surjective–i.e., there is a point $x \in N$ such that x is not in the image of f. Prove that the degree of f is zero.

(b) Explain how $\deg(f)$ relates to the map $f^* : H^d(N) \to H^d(M)$.

(c) Prove that any map $S^4 \to \mathbb{C}P^2$ must have degree 0.

Solution:

Problem 2
A topological space is said to be of **finite type** if $H_i(X) = 0$ for all but finitely many values of i, and each nonzero $H_i(X)$ is a finitely-generated abelian group. Recall that the Euler characteristic is then defined to be

$$\chi(X) = \sum_{i=1}^{\infty} (-1)^i \operatorname{rank} H_i(X).$$

Prove that if X and Y are CW-complexes of finite type then so is $X \times Y$, and $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

Solution:

Problem 3
Prove that $\mathbb{C}P^{n-1}$ is not a retract of $\mathbb{C}P^n$.

Solution:

Problem 4
Prove that there is no self-homeomorphism $\mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$ that reverses the orientation.

Solution:

There is an algebraic formula

$$(x_1^2 + x_2^2) \cdot (y_1^2 + y_2^2) = (x_1y_1 - x_2y_2)^2 + (x_1y_2 + x_2y_1)^2 \tag{1}$$

which is true for indeterminates x_1, x_2, y_1, y_2 over $\mathbb{R}$. By a **sumsof-squares formula** of type $[r, s, n]$ we mean an identity of the form

$$(x_1^2 + x_2^2 + \cdots + x_r^2) \cdot (y_1^2 + y_2^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2.$$

where each z_i is a bilinear expression in the x's and y's. The identity (1) was a formula of type $[2, 2, 2]$. Here is a formula of type $[4, 4, 4]$:

$$\begin{aligned}(x_1^2 + x_2^2 + x_3^2 + x_4^2) \cdot (y_1^2 + y_2^2 + y_3^2 + y_4^2) = &\quad (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4)^2 \\ &= + (x_1y_2 + x_2y_1 - x_3y_4 + x_4y_3)^2 \\ &= + (x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2)^2 \\ &= + (-x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1)^2.\end{aligned}$$

If you try to generalize these examples you will find a formula of type $[8, 8, 8]$, but not one of type $[16, 16, 16]$.

Problem 5

If we have a sums-of-squares formula of type $[r, s, n]$ then we get a bilinear map $\phi : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ such that $\|\phi(x, y)\|^2 = \|x\|^2 \cdot \|y\|^2$ by defining

$$\phi(x_1, \ldots, x_r, y_1, \ldots, y_s) = (z_1, \ldots, z_n)$$

using the bilinear expression z_i.

(a) Explain why ϕ restricts to a map $S^{r-1} \times S^{s-1} \to S^{n-1}$, and then induces a map

$$F : \mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \to \mathbb{R}P^{n-1}.$$

(b) Use singular cohomology to prove that if an $[r, s, n]$ formula exists then $\binom{n}{i}$ must be even for $n - r < i < s$.

(c) With some trouble one can discover a sums-of-squares formula of type $[10, 10, 16]$. Does there exist a better formula of type $[10, 10, 15]$¿

Solution:

Problem 6

Suppose $p(x)$ is an irreducible polynomial over $\mathbb{C}$ of degree n, where $n > 1$. Let $E = \mathbb{C}[x]/(p(x))$, which is an algebraic field extension of $\mathbb{C}$ of degree n. Choose a vector space isomorphism $\mathbb{C}^n \cong E$, so that the multiplication on E becomes a bilinear map $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$.

Using singular cohomology rings of appropriate topological spaces, derive a contradiction.

Solution: