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Course: MATH 616 - Real Analysis
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Homework - Week 1 Exercises
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Exercise 1.2
The Cantor set $\mathcal{C}$ can also be described in terms of ternary expansions.

(a) Every number in $[0, 1]$ has a ternary expansion

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}, \quad \text{where } a_k = 0, 1, 2$$

Note that this decomposition is not unique since, for example,

$$\frac{1}{3} = \sum_{k=2}^{\infty} \frac{2}{3^k}.$$

Prove that $x \in \mathcal{C}$ if and only if x has a representation as above where every a_k is either 0 or 2.

(b) The **Cantor-Lebesgue function** is defined on $\mathcal{C}$ by

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k}, \quad \text{if } x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \quad \text{where } b_k = \frac{a_k}{2}.$$

In this definition, we choose the expansion of x in which $a_k = 0$ or 2. Show that F is well-defined and continuous on $\mathcal{C}$, and moreover, $F(0) = 0$ as well as $F(1) = 1$.

(c) Prove that $F : \mathcal{C} \to [0, 1]$ is surjective, that is, for every $y \in [0, 1]$, there exists $x \in \mathcal{C}$ such that $F(x) = y$.

(d) One can also extend F to be a continuous function on $[0, 1]$ as follows. Note that if (a, b) is an open interval of the complement of $\mathcal{C}$, then $F(a) = F(b)$. Hence, we may define F to have the constant value $F(a)$ in that interval.

Solution:

Exercise 1.5
Suppose E is a given set, and $\mathcal{O}_n$ is the open set:

$$\mathcal{O}_n = \left\{ x : \ d(x, E) < \frac{1}{n} \right\}.$$

Show:

(a) If E is compact, then $m(E) = \lim_{n\to\infty} < \frac{1}{n}$.

(b) However, the conclusion in (a) may be false for E closed and unbounded; or E open and bounded.

Solution:

Exercise 1.6
Using translations and dilations, prove the following: Let B be a ball in $\mathbb{R}^d$ of radius r. Then $m(B) = v_d r^d$, where $v_d = m(B_1)$, and B_1 is the unit ball.

$$B_1 = \left\{ x \in \mathbb{R}^d : |x| < 1 \right\}.$$

Solution:

Exercise 1.9
Give an example of an open set $\mathcal{O}$ with the following property: the boundary of the closure of $\mathcal{O}$ has positive Lebesgue measure.

Solution:

Exercise 1.10
Let $\hat{\mathcal{C}}$ denote a Canton-like set, in particular $m(\hat{\mathcal{C}}) > 0$. Let F_1 denote a piecewise linear and continuous function on $[0,1]$, with $F_1 = 1$ in the complement of the first interval removed in the construction of $\hat{\mathcal{C}}$, $F_1 = 0$ at the center of this interval, and $0 \leq F_1(x) \leq 1$ for all x. Similarly, construct $F_2 = 1$ in the complement of the intervals in stage two of the construction of $\hat{\mathcal{C}}$, with $F_2 = 0$ at the center of these intervals, and $0 \leq F_2 \leq 1$. Continuing this way, let $f_n = F_1 \cdot F_2 \cdots F_n$. Prove the following:

(a) For all $n \geq 1$ and all $x \in [0,1]$, one has$0 \leq f_n(x) \leq 1$ and $f_n(x) \geq f_{n+1}(x)$. Therefore, $f_n(x)$ converges to a limit as $n \to \infty$ which we denote by $f(x)$.

(b) The function is discontinuous at every point of $\hat{\mathcal{C}}$.

Solution:

Exercise 1.13
The following deals with G_δ and F_σ sets.

(a) Show that a closed set is a G_δ and an open set an F_σ.

(b) Give an example of an F_σ which is not a G_δ.

(c) Give an example of a Borel set which is not a G_δ nor an F_σ.

Solution: