Math 636 Homework #6
Due Friday, May 23

1. Write down a complete description of the homology groups of $\mathrm{Gr}_3(\mathbb{C}^5)$. Determine as many intersection products between the Schubert classes $[\underline{a}]$ as you can. At least do all cases of complementary dimensions, and compute $[1,2,2]^2$ (here $\underline{a} = (1,2,2)$ is a Schubert symbol, not a jump sequence). Try to do some of the others. It is okay if you don't want to justify everything, just pick a few of the calculations to explain.

2. Compute $H_*(\Omega_{\underline{a}})$ where $\underline{a}$ is the Schubert symbol 012, and $\Omega_{\underline{a}} \hookrightarrow \mathrm{Gr}_3(\mathbb{C}^5)$. Observe that $\Omega_{\underline{a}}$ cannot be a manifold, as this would violate Poincaré Duality.

3. Fix $n \geq 1$ and $k \leq n$. Let $\eta_k \subseteq \mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n$ be the subspace of pairs (W, v) where $v \in W$. Let $p\colon \eta_k \to \mathrm{Gr}_k(\mathbb{R}^n)$ be the map sending (W, v) to W. Prove that p is a fiber bundle with fiber $\mathbb{R}^k$.

   [Hint: Let $W \in \mathrm{Gr}_k(\mathbb{R}^n)$ and represent W as the row space of a matrix as we have been doing. Some $k \times k$ minor of the matrix is nonzero, and without loss of generality we can assume it is the minor determined by the first k columns. Your job is to produce a neighborhood $W \in U \subseteq \mathrm{Gr}_k(\mathbb{R}^n)$ and a homeomorphism $p^{-1}(U) \cong U \times \mathbb{R}^k$ that is compatible with the bundle projections. Take U to be the Euclidean neighborhood that we produced in class when showing that $\mathrm{Gr}_k(\mathbb{R}^n)$ is a manifold. (Or see the discussion from Milnor-Stasheff, linked on the course webpage).]

4. Let $q\colon X \to Q$ be a surjection. Say that a map of spaces $f\colon X \to Z$ is "q-compatible" if whenever $q(x) = q(y)$ we have $f(x) = f(y)$ (this says that the identifications made by q are also made by f). The map q is a quotient map if and only if for every space Z and every map $f\colon X \to Z$ that is q-compatible, there is a map $\tilde{f}\colon Q \to Z$ such that $\tilde{f} \circ q = f$.

   Prove that if $q\colon X \to Q$ is a quotient map and A is locally compact and Hausdorff, then $q \times id\colon X \times A \to Q \times A$ is also a quotient map.

   [Hint: This is very hard to do directly, but if you use function spaces then there is a proof that takes only a few lines. Recall the bijection $\mathcal{T}op(X \times A, Z) \cong \mathcal{T}op(X, Z^A)$.]

5. Let (X, x) be a pointed space. Recall that $PX \subseteq X^I$ is the subspace of paths that end at x. Said differently, PX is defined by the pullback diagram

$$\begin{tikzcd} PX \arrow[r] \arrow[d] & X^I \arrow[d, "ev_1"] \\ * \arrow[r, "x"] & X. \end{tikzcd}$$

   Convince yourself (don't hand anything in) that maps $W \to PX$ are in bijective correspondence with maps $CW \to X$ sending the cone point to x (here CW is the cone on W, as usual).

   If A is a CW-complex, prove that $ev_0\colon PX \to X$ has the homotopy lifting property with respect to A; recall this means that any diagram

$$\begin{tikzcd} A \times \{0\} \arrow[r] \arrow[d, tail] & PX \arrow[d, "ev_0"] \\ A \times I \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

   has a lifting as shown. In particular, the fact that this holds whenever A is I^n (any $n \geq 0$) implies that $PX \to X$ is a Serre fibration.

[Hint: The general strategy for such problems is "change all maps to PX into maps to X (at the penalty of changing the domains), do an argument entirely in the context of X, and then change back to maps into PX". Said differently, you are trying to produce a map $A \times I \to PX$ and this is equivalent to producing a map $C(A \times I) \to X$ sending the cone point to the basepoint. Build this map up from the base data, using the GLP for the map $X \to *$.

If you get stuck, try the case $A = *$ first. Then try $A = I$.]