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Course: MATH 682 - Algebraic Geometry II

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Exercise 3.6.6

Show that if $f : X \to Y$ is a morphism, and A_Y us a constant sheaf on Y, then $f^{-1}A_Y$ is a constant sheaf on X.

Let $f : \mathbb{R} \to \mathbb{R}$ be the map $t \mapsto t^2$ (in the Euclidean topology). Compute all stalks of the pushforward of the constant sheaf $f_*\mathbb{Z}$. Deduce that $f_*\mathbb{Z}$ is not a constant sheaf.

Solution: Let $U \subset X$ be an open set. By definition,

$$(f^{-1}A_Y)(U) = \varinjlim_{V \supset f(U)} A_Y(V).$$

If U is connected, then the image $f(U)$ is also connected. This implies that the open set V in the directed system can also be taken as connected, so $A_Y(V) = A$ for any $V \supset f(U)$. Hence,

$$(f^{-1}A_Y)(U) = A$$

for any U connected. A similar argument implies that it is constant on each connected component of U, so $f^{-1}A_Y$ is a constant sheaf on X.

Let $x \in \mathbb{R}$. By definition, the stalk at x can be computed as

$$(f_*\mathbb{Z})_x = \varinjlim_{U \ni x} \mathbb{Z}(f^{-1}(U)).$$

If $x < 0$, then there exists a small enough open interval U containing x such that $f^{-1}(U) = \varnothing$. Thus,

$$(f^*\mathbb{Z})_x = \mathbb{Z}(\varnothing) = 0.$$

If $x = 0$, let U be any small open interval containing 0, then the preimage $f^{-1}(U)$ is also connected. So

$$(f_*\mathbb{Z})_x = \mathbb{Z}.$$

as $\mathbb{Z}$ is a constant sheaf. If $x > 0$, for small enough open neighborhood U of x, the preimage $f^{-1}(U)$ is the disjoint union of two open sets in $\mathbb{R}$, so

$$(f_*\mathbb{Z})_x = \mathbb{Z} \oplus \mathbb{Z}.$$

From this we can see that $f_*\mathbb{Z}$ is not a constant sheaf as $f_*\mathbb{Z}$ has different stalks on a connected open set $(-1, 1)$.

Exercise 3.6.7

Let $X = S^1$ be the unit circle in $\mathbb{C}$. Let $\mathscr{F}$ be the sheaf of continuous $\mathbb{C}$-valued functions on S^1. Let $\mathscr{G}$ be the sheaf of $\mathbb{C}^\times$-valued functions on S^1. Show that the exponential function defines a map of sheaves $exp : \mathscr{F} \to \mathscr{G}$. Show that the image presheaf $U \mapsto \operatorname{Im} exp(U)$ is not a sheaf.

Solution: For any open set $U \subset X$, let $f, g : U \to \mathbb{C}$ be continuous functions on S^1. We have

$$\exp(f(z) + g(z)) = \exp(f(z)) \cdot \exp(g(z))$$

for any $z \in U$. So we have a map

$$\exp : \mathscr{F}(U) \to \mathscr{G}(U)$$

sending any $\mathbb{C}$-valued continuous function to $\mathbb{C}^\times$-valued functions, which is compatible with the group operations. Moreover, exp is compatible with restriction maps on X, so exp defines a morphism a sheaves.

Recall that for any $z \in S^1$, the complex logarithm $\log z$ in a small neighborhood of z if we choose a principal value. But there does not exist a global logarithm function on S^1 as 0 is a branch point for $\log z$. This implies that the presheaf $U \mapsto \operatorname{Im} exp(U)$ does not satisfy the gluing axiom for sheaves, so it is not a sheaf.

Exercise 3.6.15

Let X be a topological space and $x \in X$ a point that is not necessarily closed. Let $\iota : \{x\} \to X$ be the inclusion. Let A be the constant sheaf on $\{x\}$ on the group A. Show that the stalk of $\iota_* A$ at a point $y \in X$ is equal to A if $y \in \bar{x}$ and 0 otherwise.

Solution: If $y \notin \bar{x}$, by definition, there exists an open set $U \ni y$ such that $U \cap \{x\} = \varnothing$. Thus, the stalk at y

$$(\iota_* A)_y = \varinjlim_{U \ni y} A(\iota^{-1}(U)) = \varinjlim_{U \ni y} A(U \cap \{x\}) = 0.$$

On the other hand, if $y \in \bar{x}$, then for any open set $U \ni y$, we have $U \cap \{x\} = \{x\}$. Thus, the stalk at y

$$(\iota_* A)_y = \varinjlim_{U \ni y} A(\iota^{-1}(U)) = \varinjlim_{U \ni y} A(\{x\}) = A.$$

Exercise 3.6.22

Describe the points of the spectrum $X = \operatorname{Spec} \mathbb{C}[x, y]_{(x,y)}$. Compute $\mathscr{O}_X(U)$ where $U = X - \{(x, y)\}$.

Solution: Write $R = \mathbb{C}[x, y]_{(x,y)}$. We know that R is a local ring and the prime ideals in R corresponds to the prime ideals in $\mathbb{C}[x, y]$ contained in the maximal ideal (x, y). It has three types of points in X:

(1) A closed point, the maximal ideal in the local ring R, corresponding to the maximal ideal (x, y) in $\mathbb{C}[x, y]$.

(2) f is an irreducible polynomial in $\mathbb{C}[x,y]$ and $(0,0)$ is a root of f. (f) is a prime ideal but not maximal ideal in $\mathbb{C}[x,y]$, corresponding to a point in R.

(3) The zero ideal (0).

Note that $U = D(x) \cap D(y)$. Indeed, for any prime ideal $\mathfrak{p} \subset R$, either $x \notin \mathfrak{p}$ or $y \notin \mathfrak{p}$, otherwise $(x,y) \in \mathfrak{p}$. Thus, from the left exact sequence of sheaves, we have

$$0 \to \mathcal{O}_X(U) \to \mathcal{O}_X(D(x)) \times \mathcal{O}_X(D(y)) \xrightarrow{\varphi} \mathcal{O}_X(D(xy))$$

Here

$$\mathcal{O}_X(D(x)) \times \mathcal{O}_X(D(y)) \cong R[\frac{1}{x}] \times R[\frac{1}{y}]$$

$$\mathcal{O}_X(D(xy)) \cong R[\frac{1}{xy}].$$

The map φ is given by the difference of the two localization map $R[\frac{1}{x}] \to R[\frac{1}{xy}]$ and $R[\frac{1}{y}] \to R[\frac{1}{xy}]$. Both maps are injective since R is a domain, so

$$\mathcal{O}_X(U) \cong \ker \varphi = R[\frac{1}{x}] \cap R[\frac{1}{y}] \cong R.$$