

Problem 21.2.2

Let I_1, \dots, I_m be ideals in $\mathbb{F}[T_1, \dots, T_n]$. Then $\mathcal{V}(I_1 \cdots I_m) = \mathcal{V}(I_1 \cap \cdots \cap I_m)$.

Solution: We only need to prove the case $m = 2$, the rest can be obtained from induction. To prove $\mathcal{V}(I_1 I_2) = \mathcal{V}(I_1 \cap I_2)$, by Corollary 21.1.10, it is the same as proving

$$\sqrt{I_1 I_2} = \sqrt{I_1 \cap I_2}.$$

Suppose $a \in \sqrt{I_1 I_2}$, then there exists $n \geq 1$ such that $a^n \in I_1 I_2 \subseteq I_1 \cap I_2$. This implies that $a \in \sqrt{I_1 \cap I_2}$. On the other hand, suppose $b \in \sqrt{I_1 \cap I_2} = \sqrt{I_1} \cap \sqrt{I_2}$, then there exists $k, l \geq 1$ such that $b^k \in I_1$ and $b^l \in I_2$. This implies $b^{k+l} = b^k \cdot b^l \in I_1 I_2$, so $b \in \sqrt{I_1 I_2}$. This proves $\sqrt{I_1 I_2} = \sqrt{I_1 \cap I_2}$.

Problem 21.2.3

Let $f \in \mathbb{F}[T_1, \dots, T_n]$. The corresponding *principal open set* is

$$\mathbb{A}^n \setminus \mathcal{V}(f) = \{x \in \mathbb{A}^n \mid f(x) \neq 0\}.$$

Show that each open set in \mathbb{A}^n is finite union of principal open sets, so principal open sets form a base of Zariski topology.

Solution: We know that the Zariski closed sets of \mathbb{A}^n have the form $\mathcal{V}(I)$ for some ideal $I \subseteq \mathbb{F}[T_1, \dots, T_n]$. So for any open set $U \subseteq \mathbb{A}^n$, U can be written as $U = \mathbb{A}^n - \mathcal{V}(I)$ for some radical ideal I . Since $\mathbb{F}[T_1, \dots, T_n]$ is noetherian, I is finitely generated by $f_1, \dots, f_k \in \mathbb{F}[T_1, \dots, T_n]$. This implies

$$\mathcal{V}(I) = \mathcal{V}(f_1, \dots, f_k) = \mathcal{V}(f_1) \cap \cdots \cap \mathcal{V}(f_k).$$

Thus, we can write U as

$$\begin{aligned} U &= \mathbb{A}^n - \mathcal{V}(I) \\ &= \mathbb{A}^n - \mathcal{V}(f_1) \cap \cdots \cap \mathcal{V}(f_k) \\ &= (\mathbb{A}^n - \mathcal{V}(f_1)) \cup \cdots \cup (\mathbb{A}^n - \mathcal{V}(f_k)). \end{aligned}$$

This proves that any Zariski open set can be written as a finite union of principal open sets.

Problem 21.2.13

Let $X = \mathcal{V}(x^2 + y^2 + z^2, xyz) \subseteq \mathbb{A}^3$. Decompose X into irreducible components.

Solution: We need to find all the points $(x, y, z) \in \mathbb{A}^3$ satisfying $x^2 + y^2 + z^2 = 0$ and $xyz = 0$. Since \mathbb{A}^3 has no nilpotents, $xyz = 0$ implies at least one of the coordinates is 0. Suppose $x = 0$. The y and z satisfy the equation $y^2 + z^2 = 0$. Note that \mathbb{F} is algebraically closed, if $\text{char } \mathbb{F} = 2$, then $y + z = 0$. X has three irreducible components

$$X = \mathcal{V}(x + y + z, xyz) = \mathcal{V}(x, y + z) \cup \mathcal{V}(y, x + z) \cup \mathcal{V}(z, x + y).$$

Each of them is isomorphic to \mathbb{A}^1 because the coordinate ring

$$\mathbb{F}[x, y, z]/(x, y + z) \cong \mathbb{F}[y, -y] \cong \mathbb{F}[y].$$

Next, assume $\text{char } \mathbb{F} \neq 2$, then $y^2 + z^2 = (y + iz)(y - iz) = 0$. This is the union of two algebraic sets $\mathcal{V}(y + iz)$ and $\mathcal{V}(y - iz)$. Thus, X has six irreducible components

$$X = \mathcal{V}(x, y + iz) \cup \mathcal{V}(x, y - iz) \cup \mathcal{V}(y, x + iz) \cup \mathcal{V}(y, x - iz) \cup \mathcal{V}(z, x + iy) \cup \mathcal{V}(z, x - iy).$$

Each of them is isomorphic to \mathbb{A}^1 because the coordinate ring

$$\mathbb{F}[x, y, z]/(x, y - iz) \cong \mathbb{F}[z, iz] \cong \mathbb{F}[z].$$

Problem 21.2.14

Let $\text{char } \mathbb{F} \neq 2$. Decompose $\mathcal{V}(x^2 + y^2 + z^2, x^2 - y^2 - z^2 + 1)$ into irreducible components.

Solution: We need to find all the pointd $(x, y, z) \in \mathbb{A}^3$ satisfying $x^2 + y^2 + z^2 = 0$ and $x^2 - y^2 - z^2 + 1 = 0$. From these two equations, we obtain

$$0 = 2x^2 + 1.$$

We know $\text{char } \mathbb{F} \neq 2$. So this equation has two different solutions: $x = \frac{i}{\sqrt{2}}$ and $x = \frac{-i}{\sqrt{2}}$. When $x = \frac{i}{\sqrt{2}}$, y and z satisfy the equation $y^2 + z^2 = \frac{1}{2}$. This is a hyperbola and $(y^2 + z^2 - \frac{1}{2})$ is a prime ideal in $\mathbb{F}[x, y]$ since we proved in Exercise 21.4.14 that

$$\mathbb{F}[y, z]/(y^2 + z^2 - 1) \cong \mathbb{F}[u, v]/(uv - 1) \cong \mathbb{F}[u, u^{-1}].$$

Thus, X has two irreducible components

$$X = \mathcal{V}(x - \frac{i}{\sqrt{2}}, y^2 + z^2 - \frac{1}{2}) \cup \mathcal{V}(x + \frac{i}{\sqrt{2}}, y^2 + z^2 - \frac{1}{2}).$$

Problem 21.3.4

If $f : A \rightarrow B$ is a homomorphism of affine algebras and M is a maximal ideal of B , then $f^{-1}(M)$ is a maximal ideal of A .

Solution: A, B are finitely generated \mathbb{F} -algebras, so B/M is also a finitely generated \mathbb{F} -algebra. We have a map

$$\begin{aligned}\phi : A/f^{-1}(M) &\rightarrow B/M, \\ a + f^{-1}(M) &\mapsto f(a) + M.\end{aligned}$$

This is a well-defined \mathbb{F} -algebra homomorphism. Indeed, suppose $a, b \in A$ and $a - b \in f^{-1}(M)$. This means $f(a - b) = f(a) - f(b) \in M$, so $f(a) + M = f(b) + M$ is the same element in B/M . Moreover, ϕ is injective. Let $a + f^{-1}(M) \in \ker \phi$ and assume $f(a) + M = M$, namely, $f(a) \in M$. Then $a \in f^{-1}(M)$ and $a + f^{-1}(M) = f^{-1}(M)$ is trivial in $A/f^{-1}(M)$.

M is a maximal ideal, so B/M is a field and is a finitely generated \mathbb{F} -algebra. By the first version of Nullstellensatz we proved in class, $\mathbb{F} \subseteq B/M$ is an algebraic and finite extension. We know that $A/f^{-1}(M)$ is a domain as $f^{-1}(M)$ is a prime ideal in A , so we have

$$\mathbb{F} \subseteq A/f^{-1}(M) \subseteq B/M$$

and $A/f^{-1}(M)$ is a subring of B/M . By Exercise 10.1.11, $A/f^{-1}(M)$ is a field, thus $f^{-1}(M)$ is a maximal ideal in A .

Problem 21.4.6

The hyperbola $xy = 1$ and \mathbb{A}^1 are not isomorphic.

Solution: The coordinate ring of the hyperbola $xy = 1$ is

$$\mathbb{F}[x, y]/(xy - 1) \cong \mathbb{F}[x, x^{-1}] \cong \mathbb{F}[x]_x.$$

Here, $\mathbb{F}[x]_x$ is $\mathbb{F}[x]$ localized with respect to the multiplicative set $\{1, x, x^2, \dots\}$. On the other hand, the coordinate ring of \mathbb{A}^1 is $\mathbb{F}[x]$. The two rings $\mathbb{F}[x]_x$ and $\mathbb{F}[x]$ are not isomorphic as $\mathbb{F}[x, x^{-1}] = \mathbb{F}[x]_x$. Suppose

$$\phi : \mathbb{F}[x]_x \rightarrow \mathbb{F}[y]$$

is a map of \mathbb{F} -algebras. We know that the units must be sent to units in $\mathbb{F}[y]$, and x is a unit in $\mathbb{F}[x, x^{-1}]$, so x must be sent to some element in \mathbb{F} . Note that x generates $\mathbb{F}[x]$ as an \mathbb{F} -algebra, so there is no element which can be sent to y . This implies ϕ can never be surjective, so we do not have such isomorphism. This implies that $xy = 1$ and \mathbb{A}^1 are not isomorphic as they have different coordinate rings.

Problem 21.4.14

The circle $x^2 + y^2 = 1$ and \mathbb{A}^1 are isomorphic if and only if $\text{char } \mathbb{F} = 1$.

Solution: Suppose $\text{char } \mathbb{F} = 2$. The radical ideal of $(x^2 + y^2 - 1)$ is $(x + y - 1)$. The coordinate ring

$$\mathbb{F}[x, y]/(x^2 + y^2 - 1) \cong \mathbb{F}[x, y]/(x + y - 1) \cong \mathbb{F}[t]$$

if we consider the isomorphism

$$\begin{aligned}\mathbb{F}[x, y]/(x + y - 1) &\rightarrow \mathbb{F}[t], \\ x &\mapsto t, \\ y &\mapsto t + 1.\end{aligned}$$

This proves the circle $x^2 + y^2 = 1$ is isomorphic to \mathbb{A}^1 if $\text{char } \mathbb{F} = 2$.

Suppose $\text{char } \mathbb{F} \neq 2$. Then consider the following map

$$\begin{aligned}\phi : \mathbb{F}[u, v]/(uv - 1) &\rightarrow \mathbb{F}[x, y]/(x^2 + y^2 - 1), \\ u &\mapsto x + iy, \\ v &\mapsto x - iy.\end{aligned}$$

This map is a regular map since it is given by a polynomial in y and x . It is an isomorphism because it has an inverse

$$\begin{aligned}\phi^{-1} : \mathbb{F}[x, y]/(x^2 + y^2 - 1) &\rightarrow \mathbb{F}[u, v]/(uv - 1), \\ x &\mapsto \frac{1}{2}u + \frac{1}{2}v, \\ y &\mapsto \frac{-i}{2}u + \frac{i}{2}v.\end{aligned}$$

This implies that the circle $x^2 + y^2 = 1$ is isomorphic to the hyperbola $uv = 1$, and we have proved in Exercise 21.4.6 that the hyperbola $uv = 1$ is not isomorphic to \mathbb{A}^1 . So the circle $x^2 + y^2 = 1$ is not isomorphic to \mathbb{A}^1 when $\text{char } \mathbb{F} \neq 2$.