

We do some calculation about Tor and Ext, which will be used in this homework.

- (1) For any abelian group A , $\text{Tor}_1(A, \mathbb{Z}) = \text{Tor}_1(\mathbb{Z}, A) = \text{Ext}^1(\mathbb{Z}, A) = 0$.

Note that the free resolution of \mathbb{Z} is given by

$$0 \rightarrow \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \rightarrow 0.$$

The degree 1 part is already 0, so we have

$$\text{Tor}_1(\mathbb{Z}, A) = \text{Ext}^1(\mathbb{Z}, A) = 0.$$

Suppose

$$0 \rightarrow J_1 \rightarrow J_0 \rightarrow A \rightarrow 0$$

is a free resolution of A and note that for any abelian group B , we have $B \otimes \mathbb{Z} = B$. After tensoring with \mathbb{Z} , we have the following chain complex

$$0 \rightarrow J_1 \rightarrow J_0 \rightarrow 0.$$

This implies $\text{Tor}_1(A, \mathbb{Z}) = 0$.

- (2) Let A be an abelian group and A_t denote the torsion part of A . Then $\text{Ext}^1(A, \mathbb{Z}) = A_t$.

The functor Ext^1 is additive and we have prove that $\text{Ext}^1(\mathbb{Z}, \mathbb{Z}) = 0$. We only need to show that $\text{Ext}^1(\mathbb{Z}/n, \mathbb{Z}) = \mathbb{Z}/n$ for any $n \geq 2$. Consider the free resolution

$$0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow \mathbb{Z}/n \rightarrow 0.$$

Apply $\text{hom}(-, \mathbb{Z})$ and we obtain a cochain complex

$$0 \leftarrow \mathbb{Z} \xleftarrow{n} \mathbb{Z} \leftarrow 0.$$

This implies $\text{Ext}^1(\mathbb{Z}/n, \mathbb{Z}) = \mathbb{Z}/n$.

- (3) For any abelian group A, B , we know that

$$\begin{aligned} \text{Tor}_0(A, B) &= A \otimes B, \\ \text{Ext}^0(A, B) &= \text{hom}(A, B). \end{aligned}$$

We have seen the proof in class.

- (4) For any integers $m, n \geq 2$, we have

$$\text{Tor}_1(\mathbb{Z}/m, \mathbb{Z}/n) = \text{Ext}^1(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/d$$

where d is the greatest common divisor of m and n (If $d = 1$, then $\mathbb{Z}/d = 0$).
Consider the following free resolution of \mathbb{Z}/m

$$0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}/m \rightarrow 0.$$

Apply $-\otimes \mathbb{Z}/n$ and $\text{hom}(-, \mathbb{Z}/n)$, we obtain a chain complex and cochain complex as follows

$$\begin{aligned} 0 \rightarrow \mathbb{Z}/n &\xrightarrow{m} \mathbb{Z}/n \rightarrow 0, \\ 0 \leftarrow \mathbb{Z}/n &\xleftarrow{m} \mathbb{Z}/n \leftarrow 0. \end{aligned}$$

By calculation,

$$\text{Tor}_1(\mathbb{Z}/m, \mathbb{Z}/n) = \text{Ext}^1(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/d.$$

Problem 1

Compute both $\text{Tor}_i(A, B)$ and $\text{Ext}^i(A, B)$ for all i in the following cases:

- (a) $A = \mathbb{Z}/9$ and $B = \mathbb{Z}/6$.
- (b) $A = \mathbb{Z}/9$ and $B = \mathbb{Z}$.
- (c) $A = \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/10$ and $B = \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/6$.

Solution:

- (a) From the discussion at the beginning, we know that

$$\begin{aligned} \text{Tor}_0(\mathbb{Z}/9, \mathbb{Z}/6) &= \mathbb{Z}/9 \otimes \mathbb{Z}/6 = \mathbb{Z}/3, \\ \text{Tor}_1(\mathbb{Z}/9, \mathbb{Z}/6) &= \mathbb{Z}/3, \\ \text{Ext}^0(\mathbb{Z}/9, \mathbb{Z}/6) &= \text{hom}(\mathbb{Z}/9, \mathbb{Z}/6) = \mathbb{Z}/3, \\ \text{Ext}^1(\mathbb{Z}/9, \mathbb{Z}/6) &= \mathbb{Z}/3. \end{aligned}$$

All other Tor and Ext are 0.

- (b) From the discussion at the beginning, we know that

$$\begin{aligned} \text{Tor}_0(\mathbb{Z}/9, \mathbb{Z}) &= \mathbb{Z}/9 \otimes \mathbb{Z} = \mathbb{Z}/9, \\ \text{Tor}_1(\mathbb{Z}/9, \mathbb{Z}) &= 0, \\ \text{Ext}^0(\mathbb{Z}/9, \mathbb{Z}) &= \text{hom}(\mathbb{Z}/9, \mathbb{Z}) = 0, \\ \text{Ext}^1(\mathbb{Z}/9, \mathbb{Z}) &= \mathbb{Z}/9. \end{aligned}$$

All other Tor and Ext are 0.

- (c) Tor and Ext are additive functors, so we can calculate using the results from the discussion.
Let

$$\begin{aligned} A &= \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/10, \\ B &= \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/6. \end{aligned}$$

Use the discussion at the beginning and the fact that Tor and Ext are additive.

$$\begin{aligned}
\mathrm{Tor}_0(A, B) &= A \otimes B \\
&= B \oplus B \oplus ((\mathbb{Z}/4)^2 \oplus \mathbb{Z}/2) \oplus \mathbb{Z}/5 \oplus (\mathbb{Z}/10 \oplus (\mathbb{Z}/2)^2) \\
&= \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^3 \oplus (\mathbb{Z}^3)^2 \oplus (\mathbb{Z}/4)^4 \oplus \mathbb{Z}/5 \oplus (\mathbb{Z}/6)^2 \oplus \mathbb{Z}/10.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tor}_1(A, B) &= \mathrm{Tor}_1(\mathbb{Z}/4 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/10, \mathbb{Z}/3 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/6) \\
&= \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^3.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ext}^0(A, B) &= \mathrm{hom}(A, B) \\
&= B^2 \oplus \mathrm{hom}(\mathbb{Z}/4 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/10, \mathbb{Z}/3 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/6) \\
&= B^2 \oplus \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^3 \\
&= \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/4)^3 \oplus (\mathbb{Z}/6)^2.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ext}^1(A, B) &= \mathrm{Ext}^1(\mathbb{Z}/4 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/10, B) \\
&= \mathbb{Z}/4 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/10 \oplus \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^3 \\
&= (\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/4)^2 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/10.
\end{aligned}$$

All other Ext and Tor are zero.

Problem 2

Let A be an abelian group and let $G_* \rightarrow A \rightarrow 0$ be a free resolution. Let B be another abelian group and let $J_* \rightarrow B \rightarrow 0$ be a free resolution.

- Given a map $f : A \rightarrow B$, prove that there are maps $F_i : G_i \rightarrow J_i$ making all squares commute, we call this chain map $\{F : G_* \rightarrow J_*\}$ a lifting of the map f .
- Prove that if $\{F' : G_* \rightarrow J_*\}$ is another lifting of f then the chain map F and F' are chain homotopic.
- If C is another abelian group one gets an induced map $F \otimes id : G_* \otimes C \rightarrow J_* \otimes C$ and therefore an induced map on homology groups $f_* : \mathrm{Tor}_i(A, C) \rightarrow \mathrm{Tor}_i(B, C)$. Since any two choices of F are homotopic, this f_* is well-defined.

Use the above procedure to calculate the maps

$$\begin{aligned}
j_* : \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) &\rightarrow \mathrm{Tor}_1(\mathbb{Z}/4, \mathbb{Z}/2), \\
k_* : \mathrm{Tor}_1(\mathbb{Z}/4, \mathbb{Z}/2) &\rightarrow \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2).
\end{aligned}$$

induced by the map $j : \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ (sending 1 to 2) and $k : \mathbb{Z}/4 \rightarrow \mathbb{Z}/2$ (sending 1 to 1).

Solution:

(a) We have a diagram with long exact sequences as follows

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{g_2} & G_1 & \xrightarrow{g_1} & G_0 & \xrightarrow{g_0} & A \longrightarrow 0 \\
 & & & & & & \downarrow f \\
 \cdots & \xrightarrow{j_2} & J_1 & \xrightarrow{j_1} & J_0 & \xrightarrow{j_0} & B \longrightarrow 0
 \end{array}$$

we need to construct $f_i : G_i \rightarrow J_i$ from $i = 0$ inductively. Consider the following diagram with solid arrows

$$\begin{array}{ccc}
 & G_0 & \\
 f_0 \swarrow & \downarrow f_{g_0} & \\
 J_0 & \xrightarrow{j_0} & B \longrightarrow 0
 \end{array}$$

$j_0 : J_0 \rightarrow B$ is surjective by exactness and G_0 is projective because it is free, there exists a map $f_0 : G_0 \rightarrow J_0$ such that $j_0 f_0 = f_{g_0}$. We have constructed the first step with a diagram as follows:

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{g_2} & G_1 & \xrightarrow{g_1} & G_0 & \xrightarrow{g_0} & A \longrightarrow 0 \\
 & & & & \downarrow f_0 & & \downarrow f \\
 \cdots & \xrightarrow{j_2} & J_1 & \xrightarrow{j_1} & J_0 & \xrightarrow{j_0} & B \longrightarrow 0
 \end{array}$$

Next, consider the composition $f_0 g_1 : G_1 \rightarrow J_0$, for any $x \in G_1$, by commutativity of the diagram, we have

$$j_0 f_0 g_1(x) = f_{g_0 g_1}(x) = 0$$

because of the exactness of top row. By the exactness of the bottom row, we have

$$f_0 g_1(x) \in \ker j_0 = \operatorname{Im} j_1.$$

This means the map $f_0 g_1$ must factor through $\operatorname{Im} j_1$ and we have the following solid arrow diagram

$$\begin{array}{ccc}
 & G_1 & \\
 f_1 \swarrow & \downarrow f_{0g_1} & \\
 J_1 & \xrightarrow{j_1} & \operatorname{im} j_1 \longrightarrow 0
 \end{array}$$

G_1 is projective so there exists a map $f_1 : G_1 \rightarrow J_1$ such that $j_1 f_1 = f_{0g_1}$. This means we have obtained the next map we need in the following diagram

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{g_2} & G_1 & \xrightarrow{g_1} & G_0 & \xrightarrow{g_0} & A \longrightarrow 0 \\
 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f \\
 \cdots & \xrightarrow{j_2} & J_1 & \xrightarrow{j_1} & J_0 & \xrightarrow{j_0} & B \longrightarrow 0
 \end{array}$$

Repeat the steps inductively and we obtain a chain map $F : G_* \rightarrow J_*$ where in each degree is given by $f_i : G_i \rightarrow J_i$ for $i \geq 0$.

(b) Suppose we have two chain maps

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{g_2} & G_1 & \xrightarrow{g_1} & G_0 & \xrightarrow{g_0} & A \longrightarrow 0 \\
& & \downarrow f'_1 & \downarrow f_1 & \downarrow f'_0 & \downarrow f_0 & \downarrow f \\
\cdots & \xrightarrow{j_2} & J_1 & \xrightarrow{j_1} & J_0 & \xrightarrow{j_0} & B \longrightarrow 0
\end{array}$$

we already have two zero maps $0 : 0 \rightarrow B$ and $0 : A \rightarrow j_0$ satisfying

$$0 + j_0 \circ 0 = f - f$$

We can take $H_{-1} = H_0 = 0$ as the chain homotopy map. For $n \geq 1$, suppose we have already constructed $H_{n-1} : G_{n-1} \rightarrow J_n$ and $H_{n-2} : G_{n-2} \rightarrow J_{n-1}$ satisfying

$$f_{n-1} - f'_{n-1} = H_{n-2}g_{n-1} + j_n H_{n-1}.$$

We want to construct the map $H_n : G_n \rightarrow J_{n+1}$.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & G_{n+1} & \xrightarrow{g_{n+1}} & G_n & \xrightarrow{g_n} & G_{n-1} \xrightarrow{g_{n-1}} G_{n-2} \longrightarrow \cdots \\
& & \downarrow & & \downarrow f'_n & \downarrow f_n & \downarrow f'_{n-1} \\
& & & \swarrow H_n & & \swarrow f'_n & \swarrow H_{n-1} \\
\cdots & \longrightarrow & J_{n+1} & \xrightarrow{j_{n+1}} & J_n & \xrightarrow{j_n} & J_{n-1} \xrightarrow{j_{n-1}} J_{n-2} \longrightarrow \cdots
\end{array}$$

Consider the following map

$$f_n - f'_n - H_{n-1}g_n : G_n \rightarrow J_n.$$

For any $x \in G_n$, use the commutativity of the diagram and the property of H_{n-1} and H_{n-2} , we have

$$\begin{aligned}
(j_n f_n - j_n f'_n - j_n H_{n-1} g_n)(x) &= (j_n f_n)(x) - (j_n f'_n)(x) - (j_n H_{n-1} g_n)(x) \\
&= (f_{n-1} g_n)(x) - (f'_{n-1} g_n)(x) - [(f_{n-1} - f'_{n-1} - H_{n-2} g_{n-1}) g_n](x) \\
&= [(f_{n-1} - f'_{n-1}) g_n](x) - [(f_{n-1} - f'_{n-1}) g_n](x) \\
&= 0.
\end{aligned}$$

This implies that

$$(f_n - f'_n - H_{n-1} g_n)(x) \in \ker j_n = \text{Im } j_{n+1}$$

for all $x \in G_n$ by exactness of the bottom row. Then this map must factor through $\text{Im } j_{n+1}$ and we have a solid arrow diagram

$$\begin{array}{ccc}
& G_n & \\
\swarrow H_n & \downarrow f_n - f'_n - H_{n-1} g_n & \\
J_{n+1} & \xrightarrow{j_{n+1}} \text{Im } j_{n+1} & \longrightarrow 0
\end{array}$$

G_n being projective implies there exists a map $H_n : G_n \rightarrow J_{n+1}$ such that

$$f_n - f'_n = H_{n-1} g_n + j_{n+1} H_n.$$

Repeat this step inductively and we have constructed a chain homotopy between F and F' .

(c) Consider the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} \\ 2 \downarrow & & \downarrow 4 \\ \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} \\ 1 \downarrow & & \downarrow 1 \\ \mathbb{Z}/2 & \xrightarrow{2} & \mathbb{Z}/4 \end{array}$$

The left and right vertical columns are free resolutions of $\mathbb{Z}/2$ and $\mathbb{Z}/4$. Apply $- \otimes \mathbb{Z}/2$ to the resolutions and we get a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/2 & \xrightarrow{1} & \mathbb{Z}/2 \\ 2 \downarrow & & \downarrow 4 \\ \mathbb{Z}/2 & \xrightarrow{2} & \mathbb{Z}/2 \end{array}$$

The map sending 1 to 1 in homology is the identity map, so

$$j_* : \text{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) \rightarrow \text{Tor}_1(\mathbb{Z}/4, \mathbb{Z}/2)$$

is the identity map of $\mathbb{Z}/2$. Similarly, consider the free resolutions of $\mathbb{Z}/4$ and $\mathbb{Z}/4$.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} \\ 4 \downarrow & & \downarrow 2 \\ \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} \\ 1 \downarrow & & \downarrow 1 \\ \mathbb{Z}/4 & \xrightarrow{1} & \mathbb{Z}/2 \end{array}$$

Apply $- \otimes \mathbb{Z}/2$ and we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/2 & \xrightarrow{2} & \mathbb{Z}/2 \\ 4 \downarrow & & \downarrow 2 \\ \mathbb{Z}/2 & \xrightarrow{1} & \mathbb{Z}/2 \end{array}$$

The map sends 1 to 2 is the zero map for $\mathbb{Z}/2$, so

$$k_* : \text{Tor}_1(\mathbb{Z}/4, \mathbb{Z}/2) \rightarrow \text{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2)$$

is the zero map of $\mathbb{Z}/2$.

Problem 3.7

If F is a finitely-generated free abelian group then there is a canonical isomorphism

$$\text{hom}(\text{hom}(F, \mathbb{Z}), \mathbb{Z}) \cong F.$$

So if C is a chain complex consisting of finitely generated, free abelian groups, one gets an induced isomorphism

$$\text{hom}(\text{hom}(C, \mathbb{Z}), \mathbb{Z}) \cong C.$$

Using this, derive a universal coefficient theorem which lets you predict $H_*(C)$ if you know $H^*(\text{hom}(C, \mathbb{Z}))$.

Solution: For all i , we have

$$\begin{aligned} H_i(C) &\cong H_i(\text{hom}(\text{hom}(C, \mathbb{Z}), \mathbb{Z})) \\ &\cong \text{hom}(H^i(\text{hom}(C, \mathbb{Z})), \mathbb{Z}) \oplus \text{Ext}^1(H^{i-1}(\text{hom}(C, \mathbb{Z})), \mathbb{Z}). \end{aligned}$$

Problem 3.8

In this problem we'll use the abbreviations $H^i(\text{hom}(C, \mathcal{A})) = H^i(C; \mathcal{A})$ and $H^i(C) = H^i(C; \mathbb{Z})$.

If F is a finitely -generated free abelian group then there is a canonical isomorphism

$$\text{hom}(F, \mathcal{A}) \cong \text{hom}(F, \mathbb{Z}) \otimes \mathcal{A}.$$

So if C is a chain complex consistin of finitely generated free abelian groups, we have an isomorphism

$$\text{hom}(C, \mathcal{A}) \cong \text{hom}(C, \mathbb{Z}) \otimes \mathcal{A}.$$

Using this, derive a universal coefficient theorem which lets you predict $H^*(C; \mathcal{A})$ if you know $H^*(C)$. The formula should look like

$$H^i(C; \mathcal{A}) \cong [H^?(C) \otimes \mathcal{A}] \oplus [\text{Tor}_1(H^?(C), \mathcal{A})]$$

where you determine the indices marked " ? ".

Solution: For all i , we have

$$H^i(C; \mathcal{A}) \cong [H^i(C) \otimes \mathcal{A}] \oplus [\text{Tor}_1(H^{i-1}(C), \mathcal{A})]$$

Problem 4

Suppose X is a finite CW complex for which

$$H_0(X; \mathbb{Z}/2) = \mathbb{Z}/2, H_1(X; \mathbb{Z}/2) = (\mathbb{Z}/2)^3, H_2(X; \mathbb{Z}/2) = 0, H_3(X; \mathbb{Z}/2) = H_4(X; \mathbb{Z}/2) = \mathbb{Z}/2$$

and $H_i(X; \mathbb{Z}/2) = 0$ for all $i \geq 5$.

- (a) Determine as much as you can about $H_*(X; \mathbb{Z})$.
- (b) Suppose you are also told that $H_2(X; \mathbb{Z}/3) = \mathbb{Z}/3$ and $H_3(X; \mathbb{Z}/3) = 0$. What else can you say about $H_*(X; \mathbb{Z})$ now?
- (c) Suppose Y is a space with finitely-generated homology groups and you are told $H_i(Y; \mathbb{Z}/p) = 0$ for a specific prime p . What can you deduce about $H_i(Y)$ and $H_{i-1}(Y)$?

Solution:

- (a) Use the universal coefficient theorem (UCT) for homology. For $i = 0$, note that $H_{-1}(X) = 0$, we have

$$\mathbb{Z}/2 = H_0(X; \mathbb{Z}/2) \cong H_0(X) \otimes \mathbb{Z}/2.$$

Note that $H_0(X)$ is always free, so $H_0(X) \cong \mathbb{Z}$. For $i = 1$, by UCT, we have

$$(\mathbb{Z}/2)^3 = H_1(X; \mathbb{Z}/2) \cong H_1(X) \otimes \mathbb{Z}/2 \oplus \text{Tor}_1(H_0(X), \mathbb{Z}/2).$$

We know that $H_0(X) = \mathbb{Z}$ and $\text{Tor}_1(\mathbb{Z}, \mathbb{Z}/2) = 0$. So $H_1(X)$ has three generators, each of them is either free or is of even order, plus any generator with finite odd order. Moreover, by UCT in $i = 1$, we have

$$0 = H_2(X; \mathbb{Z}/2) = H_2(X) \otimes \mathbb{Z}/2 \oplus \text{Tor}_1(H_1(X), \mathbb{Z}/2).$$

This implies $\text{Tor}_1(H_1(X), \mathbb{Z}/2) = 0$. If any generator of $H_1(X)$ has finite even order, then $\text{Tor}_1(H_1(X), \mathbb{Z}/2)$ must contain $\mathbb{Z}/2$. So $H_1(X)$ has three free generators and

$$H_1(X) = \mathbb{Z}^3 \oplus \bigoplus_{k \in I_1} \mathbb{Z}/n_{1,k}.$$

From $H_2(X) \otimes \mathbb{Z}/2 = 0$, we know that $H_2(X) = 0$ or $H_2(X) = \bigoplus_{k \in I_2} \mathbb{Z}/n_{2,k}$ for each $n_{2,k} \geq 3$ odd. For $i = 3$, by UCT, we have

$$\mathbb{Z}/2 = H_3(X; \mathbb{Z}/2) = H_3(X) \otimes \mathbb{Z}/2 \oplus \text{Tor}_1(H_2(X), \mathbb{Z}/2).$$

Note that $H_2(X)$ only consists of \mathbb{Z}/n_k for some odd n_k , so $\text{Tor}_1(H_2(X), \mathbb{Z}/2) = 0$. This implies $H_3(X) \otimes \mathbb{Z}/2 = \mathbb{Z}/2$. So either $H_3(X) = \mathbb{Z}$ or $H_3(X) = \mathbb{Z}/m$ for some even number $m \geq 2$, plus some non contributing generators with finite odd order. This will split into two different cases.

Case 1: Assume $H_3(X) = \mathbb{Z}/m$ for some even number $m \geq 2$, plus some odd order generators. For $i = 4$, by UCT, we have

$$\mathbb{Z}/2 = H_4(X; \mathbb{Z}/2) = H_4(X) \otimes \mathbb{Z}/2 \oplus \text{Tor}_1(H_3(X), \mathbb{Z}/2).$$

Note that for any even number $m \geq 2$, we have

$$\mathrm{Tor}_1(H_3(X), \mathbb{Z}/2) = \mathrm{Tor}_1(\mathbb{Z}/m, \mathbb{Z}/2) = \mathbb{Z}/2.$$

This implies that $H_4(X) \otimes \mathbb{Z}/2 = 0$. So $H_4(X) = 0$ or $H_4(X) = \oplus_{k \in I_4} \mathbb{Z}/n_{4,k}$ for some odd numbers $n_{4,k}$. Doing this inductively and we can see that for any $i \geq 5$, we know that

$$H_i(X) = \oplus_{k \in I_k} \mathbb{Z}/n_{i,k}$$

where all $n_{i,k}$ are odd numbers. We summarize as follows

$$H_i(X) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}^3 \oplus \oplus_{k \in I_1} \mathbb{Z}/n_{1,k}, & \text{if } i = 1; \\ 0 \text{ or } \oplus_{k \in I_2} \mathbb{Z}/n_{2,k}, & \text{if } i = 2; \\ \mathbb{Z}/m \oplus \oplus_{k \in I_3} \mathbb{Z}/n_{3,k}, & \text{if } i = 3; \\ 0 \text{ or } \oplus_{k \in I_i} \mathbb{Z}/n_{i,k}, & \text{if } i \geq 4. \end{cases}$$

where $m \geq 2$ is an even number and all $n_{i,k}$ are odd numbers.

Case 2: Assume $H_3(X) = \mathbb{Z}$, plus some odd order generators. Note that in this case $\mathrm{Tor}_1(H_3(X), \mathbb{Z}/2) = 0$. So we have

$$\mathbb{Z}/2 = H_4(X; \mathbb{Z}/2) = H_4(X) \otimes \mathbb{Z}/2.$$

Combine with the fact that

$$0 = H_5(X; \mathbb{Z}/2) = H_5(X) \otimes \mathbb{Z}/2 \oplus \mathrm{Tor}_1(H_4(X), \mathbb{Z}/2).$$

The free part of $H_4(X)$ can only be \mathbb{Z} otherwise the torsion will not vanish. Starting from $H_5(X)$, it follows the same pattern as case 1. We summarize as follows

$$H_i(X) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}^3 \oplus \oplus_{k \in I_1} \mathbb{Z}/n_{1,k}, & \text{if } i = 1; \\ 0 \text{ or } \oplus_{k \in I_2} \mathbb{Z}/n_{2,k}, & \text{if } i = 2; \\ \mathbb{Z} \oplus \oplus_{k \in I_3} \mathbb{Z}/n_{3,k}, & \text{if } i = 3; \\ \mathbb{Z} \oplus \oplus_{k \in I_4} \mathbb{Z}/n_{4,k}, & \text{if } i = 4; \\ 0 \text{ or } \oplus_{k \in I_i} \mathbb{Z}/n_{i,k}, & \text{if } i \geq 5. \end{cases}$$

where all $n_{i,k}$ are odd numbers.

(b) First note that because $0 = H_3(X; \mathbb{Z}/3)$, by UCT, we have

$$0 = H_3(X) \otimes \mathbb{Z}/3 \oplus \mathrm{Tor}_1(H_2(X), \mathbb{Z}/3).$$

So $H_3(X)$ cannot contain \mathbb{Z} , we will only have case 1. $H_2(X)$ does not contain any free part, so

$$\mathrm{Tor}_1(H_2(X), \mathbb{Z}/3) = H_2(X) \otimes \mathbb{Z}/3 = 0$$

By UCT, we have

$$\mathbb{Z}/3 = H_2(X; \mathbb{Z}/3) = \text{Tor}_1(H_1(X), \mathbb{Z}/3).$$

We summarize the additional information as below.

$$\begin{aligned}\text{Tor}_1(H_1(X), \mathbb{Z}/3) &= \mathbb{Z}/3, \\ H_2(X) \otimes \mathbb{Z}/3 &= 0, \\ H_3(X) \otimes \mathbb{Z}/3 &= 0.\end{aligned}$$

So $H_1(X)$ contains and only contains one copy of $\mathbb{Z}/3k$ for some $k \geq 1$. $H_2(X)$ and $H_3(X)$ does not contain any copies of $\mathbb{Z}/3k$ for any $k \geq 1$.

(c) By UCT, we have

$$0 = H_i(Y; \mathbb{Z}/p) = H_i(Y) \otimes \mathbb{Z}/p \oplus \text{Tor}_1(H_{i-1}(Y), \mathbb{Z}/p).$$

So $H_i(Y) = \oplus_{i \in I} \mathbb{Z}/n_i$ for some $n_i \geq 1$ where each n_i is coprime with p . And $H_{i-1}(Y)$ does not contain any \mathbb{Z}/kp for any $k \geq 1$.

Problem 5

(a) For a certain class of n -manifold M , one always has Poincaré Duality:

$$H_i(M; \mathbb{Z}) \cong H^{n-i}(M; \mathbb{Z}).$$

Assuming this, as well as the fact that all the homology groups of M are finitely generated, explain why the Universal Coefficient Theorems then imply that

- (1) the rank of $H_i(M; \mathbb{Z})$ is the same as the rank of $H_{n-i}(M; \mathbb{Z})$, for all i .
- (2) the torsion part of $H_i(M; \mathbb{Z})$ is the same as the torsion part of $H_{n-i-1}(M; \mathbb{Z})$, for all i .

(b) Suppose M is a 5-manifold for which Poincaré Duality holds. Given that $H_0(M) = \mathbb{Z}$, $H_1(M) = \mathbb{Z}^2 \oplus \mathbb{Z}/4$, and $H_2(M) = \mathbb{Z} \oplus \mathbb{Z}/5$, compute $H_i(M)$, $H_i(M; \mathbb{Z}/2)$ and $H_i(M; \mathbb{Z}/5)$ for all i .

Solution:

(a) First we observe that for $i \geq n$, by Poincaré Duality we have

$$H_i(M) = H^{n-i}(M) = 0$$

since $n - i < 0$. For $0 \leq i \leq n$, we have

$$H_i(M) \cong H^{n-i}(M) \cong \text{hom}(H_{n-i}(M), \mathbb{Z}) \oplus \text{Ext}^1(H_{n-i-1}(M), \mathbb{Z}).$$

- (1) We have proved at the beginning that $\text{Ext}^1(H_{n-i-1}(M), \mathbb{Z})$ is either 0 or equal to the direct sum of some finite groups. So the free part can only be detected at $\text{hom}(H_{n-i}(M), \mathbb{Z})$ and note that

$$\text{rank } \text{hom}(H_{n-i}(M), \mathbb{Z}) = \text{rank } H_{n-i}(M).$$

So we know that $H_i(M)$ and $H_{n-i}(M)$ have the same rank.

- (2) Note that $\text{hom}(\mathbb{Z}/n, \mathbb{Z}) = 0$ for any $n \geq 2$. So $\text{hom}(H_{n-i}(M), \mathbb{Z})$ cannot detect any torsion part, and we have proved at the beginning that

$$\text{Ext}^1(H_{n-i-1}(M), \mathbb{Z}) = (H_{n-i-1}(M))_t$$

where $(H_{n-i-1}(M))_t$ means the torsion part of the abelian group $H_{n-i-1}(M)$.

- (b) For an abelian group A , we write $A = A_f \oplus A_t$ where A_f is the free part of A and A_t is the torsion part of A . Therefore, we can calculate

$$\begin{aligned} H_3(M) &= (H_3(M))_f \oplus (H_3(M))_t = (H_2(M))_f \oplus (H_1(M))_t = \mathbb{Z} \oplus \mathbb{Z}/4, \\ H_4(M) &= (H_4(M))_f \oplus (H_4(M))_t = (H_1(M))_f \oplus (H_0(M))_t = \mathbb{Z}^2, \\ H_5(M) &= (H_5(M))_f \oplus (H_5(M))_t = (H_0(M))_f = \mathbb{Z}. \end{aligned}$$

Next, we use UCT for homology to determine $H_*(M; \mathbb{Z}/2)$.

$$\begin{aligned} H_0(M; \mathbb{Z}/2) &= H_0(M) \otimes \mathbb{Z}/2 \oplus \text{Tor}_1(H_{-1}(M), \mathbb{Z}/2) = \mathbb{Z}/2. \\ H_1(M; \mathbb{Z}/2) &= H_1(M) \otimes \mathbb{Z}/2 \oplus \text{Tor}_1(H_0(M), \mathbb{Z}/2) = (\mathbb{Z}/2)^3. \\ H_2(M; \mathbb{Z}/2) &= H_2(M) \otimes \mathbb{Z}/2 \oplus \text{Tor}_1(H_1(M), \mathbb{Z}/2) = (\mathbb{Z}/2)^2. \\ H_3(M; \mathbb{Z}/2) &= H_3(M) \otimes \mathbb{Z}/2 \oplus \text{Tor}_1(H_2(M), \mathbb{Z}/2) = (\mathbb{Z}/2)^2. \\ H_4(M; \mathbb{Z}/2) &= H_4(M) \otimes \mathbb{Z}/2 \oplus \text{Tor}_1(H_3(M), \mathbb{Z}/2) = (\mathbb{Z}/2)^3. \\ H_5(M; \mathbb{Z}/2) &= H_5(M) \otimes \mathbb{Z}/2 \oplus \text{Tor}_1(H_4(M), \mathbb{Z}/2) = \mathbb{Z}/2. \\ H_6(M; \mathbb{Z}/2) &= \text{Tor}_1(H_5(M), \mathbb{Z}/2) = 0. \end{aligned}$$

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Problem 6

Consider $R = \mathbb{Z}/p^2$ and let $M = \mathbb{Z}/p$. Construct a free resolution of M (in the category of R -modules) and use this resolution to compute $\text{Tor}_i^R(M, M)$ for all i .

Solution: Observe that we have the following exact sequence of R -modules

$$\cdots \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{1} \mathbb{Z}/p \rightarrow 0.$$

We check exactness at each spot. The map $\mathbb{Z}/p^2 \xrightarrow{1} \mathbb{Z}/p$ is surjective and the kernel is

$$K = \{0, p, 2p, \dots, (p-1)p\} \subseteq \mathbb{Z}/p^2.$$

The image of the map $\mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2$ is exactly K in \mathbb{Z}/p^2 and the kernel is also the same thing. So we have a \mathbb{Z}/p^2 -free resolution of \mathbb{Z}/p . Tensoring with \mathbb{Z}/p and we obtain a chain complex

$$\cdots \xrightarrow{0} \mathbb{Z}/p \xrightarrow{0} \mathbb{Z}/p \rightarrow 0.$$

So $\text{Tor}_1^{\mathbb{Z}/p^2}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p$ for all $i \geq 0$.