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Course: MATH 635 - Algebraic Topology II
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Homework 2
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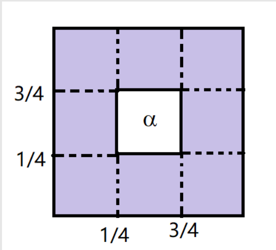
Problem 1

Consider a map $\alpha : (I^2, \partial I^2) \to (X, x)$ and the corresponding element $[\alpha]$ of $\pi_2(X, x)$. The following picture depicts a map $\beta : (I^2, \partial I^2) \to (X, x)$ which is "α scaled down to a center cube of side-length $\frac{1}{2}$, and the basepoint outside of that" (so the shaded region is all mapped to x):

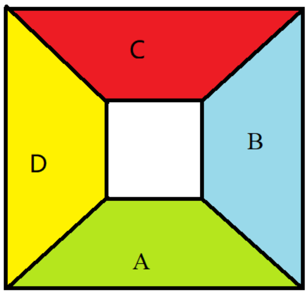


(a) Write down an explicit formula for a map $\lambda : I^2 \to I^2$ such that $\beta = \alpha \circ \lambda$.

(b) Use the Convexity Lemma to give a rigorous proof (no handwaiving!) that $[\alpha] = [\beta]$ as elements of $\pi_2(X, x)$.

Solution:

(a) Write $I^2 = [0, 1] \times [0, 1]$. We first divide the purple part of the square in the problem into four regions:



- We say a point (x, y) is in the region $A \subset I^2$ if it satisfies one of the follwoing conditions:
  (1) $0 \le y \le x \le \frac{1}{4}$;
  (2) $\frac{1}{4} \le x \le \frac{3}{4}, 0 \le y \le \frac{1}{4}$;
  (3) $0 \le y \le -x + 1 \le \frac{1}{4}$.
- We say a point (x, y) is in the region $B \subset I^2$ if it satisfies one of the following conditions:
  (1) $0 \le -x + 1 \le y \le \frac{1}{4}$;
  (2) $\frac{1}{4} \le x \le 1, \frac{1}{4} \le y \le \frac{3}{4}$;
  (3) $0 \le y - \frac{3}{4} \le x - \frac{3}{4} \le \frac{1}{4}$.
- We say a point (x, y) is in the region $C \subset I^2$ if it satisfies one of the following conditions:

(1) $0 \le x - \frac{3}{4} \le y - \frac{3}{4} \le \frac{1}{4}$;
(2) $\frac{1}{4} \le x \le \frac{3}{4}, \frac{3}{4} \le y \le 1$;
(3) $0 \le 1 - y \le x \le \frac{1}{4}$.

- We say a point (x, y) is in the region $D \subset I^2$ if it satisfies one of the following conditions:
  (1) $0 \le x \le 1 - y \le \frac{1}{4}$;
  (2) $0 \le x \le \frac{1}{4}, \frac{1}{4} \le y \le \frac{3}{4}$;
  (3) $0 \le x \le y \le \frac{1}{4}$.

We define a continous map $\lambda : I^2 \to I^2$ as follows

$$\lambda(x, y) = \begin{cases} (\frac{1}{2} - \frac{1}{2}\frac{2x-1}{2y-1}, 0), & \text{if}\,(x, y) \in A; \\ (1, \frac{1}{2} + \frac{1}{2}\frac{2y-1}{2x-1}), & \text{if}\,(x, y) \in B; \\ (\frac{1}{2} + \frac{1}{2}\frac{2x-1}{2y-1}, 1), & \text{if}\,(x, y) \in C; \\ (0, \frac{1}{2} - \frac{1}{2}\frac{2y-1}{2x-1}), & \text{if}\,(x, y) \in D; \\ (2x - \frac{1}{2}, 2y - \frac{1}{2}), & \text{if}\,\frac{1}{4} \le x, y \le \frac{3}{4}. \end{cases}$$

This map λ sends the points in the purple square to the boundary and stretch the middle white square to the whole square, so we have $\beta = \alpha \circ \lambda$.

(b) $(I^2, \partial I^2)$ is a pair of spaces where I^2 is a convex set in $\mathbb{R}^2$ and $id, \lambda : I^2 \to I^2$ are both maps from I^2 to I^2. When restricted to the boundary ∂I^2, both id and λ is the identity $\partial I^2 \to \partial I^2$. By the Convexity Lemma, we have $\lambda \simeq id$ rel. ∂I^2. This implies that

$$\beta = \alpha \circ \lambda \simeq \alpha \circ id = \alpha \text{ rel. } \partial I^2.$$

Thus, α and β are homotopic relative to ∂I^2 and they represent the same element $[\alpha] = [\beta]$ in the homotopy group $\pi_2(X, x)$.

Problem 2
Let $f : (I, \partial I) \to (X, x)$ be a based loop, and let $\bar{f} : I \to X$ be the loop $\bar{f}(t) = f(1 - t)$. Use the Convexity Lemma to give a rigorous and brief proof that $\bar{f} * f$ is homotopic rel ∂I to the constant loop at x.

Solution: Write $I = [0, 1]$ and define a map $\lambda : I \to I$ as follows:

$$\lambda(t) = \begin{cases} 2t, & \text{if}\,0 \le t \le \frac{1}{2}, \\ -2t + 2 & \text{if}\,\frac{1}{2} \le t \le 1. \end{cases}$$

λ is continous and we have

$$(\bar{f} * f)(t) = (f \circ \lambda)(t) = \begin{cases} f(2t), & \text{if}\,0 \le t \le \frac{1}{2}, \\ \bar{f}(2t - 1) = f(-2t + 2), & \text{if}\,\frac{1}{2} \le t \le 1. \end{cases}$$

Consider the constant map $C_0 : I \to I$ which maps everything to $0 \in I$ and we have

$$C_0(0) = 0 = \lambda(0) \ \ , C_0(1) = 0 = \lambda(1).$$

By the Convexity Lemma, we have

$$C_0 \simeq \lambda \text{ rel. } \partial I.$$

This implies that

$$\bar{f} * f = f \circ \lambda \simeq f \circ C_0 = C_x \text{ rel } \partial I$$

where $C_x : I \to X$ is the constant map which sends everything to $x \in X$.

Problem 3

Find the flaw in the following "proof" that $0 = 1$: Fix a natural number n, and let $j : S^n \hookrightarrow \mathbb{R}^{n+1} - 0$ be the inclusion. Of course j is a homotopy equivalence. If $f, g : S^n \to S^n$ are any two maps notice that $jf \simeq jg$ via the straight-line homotopy $H : S^n \times I \to \mathbb{R}^{n+1} - 0$ given by

$$H(x,t) = (1-t) \cdot f(x) + t \cdot g(x)$$

since obviously $H_0 = jf$ and $H_1 = jg$. Now look at the two composites

$$H_n(S^n) \xrightarrow{f_*} H_n(S^n) \xrightarrow{j_*} H_n(\mathbb{R}^{n+1} - 0),$$
$$H_n(S^n) \xrightarrow{g_*} H_n(S^n) \xrightarrow{j_*} H_n(\mathbb{R}^{n+1} - 0).$$

We know $j_* f_* = (jf)_* = (jg)_* = j_* g_*$ since jf is homotopic to jg. But j_* is an isomorphism because j is a homotopy equivalence, therefore $f_* = g_*$. So any two maps $f, g : S^n \to S^n$ has the same degree. In particular, if we take f to be a constant map and g to be the identity, then we obtain $0 = 1$.

Solution: The flaw is the following:
When we prove that $jg \simeq jf$ using the straight-line homotopy, the image of this homotopy H may contain the point $0 \in \mathbb{R}^{n+1}$. This means $(jf)_*$ is not necessarily equal to $(jg)_*$ as both the space S^n and $\mathbb{R}^{n+1} - 0$ do not contain the point 0. The straight-line homotopy we use may not be a homotopy for $jf, jg : S^n \to \mathbb{R}^{n+1} - 0$.

In the case f is the constant map $f(S^n) = p$ and g is the identity. Consider the antipodal point $-p$ and the straight-line homotopy $H(x,t)$, we have

$$H(-p, 1/2) = (1 - 1/2)p + (-p) \cdot 1/2 = 0.$$

So this homotopy is not well-defined.

Problem 4

Let X and Y be two spaces, with basepoint $x \in X$ and $y \in Y$. Prove that when $n \geq 1$ there is an isomorphism of groups

$$\pi_n(X \times Y, (x,y)) \cong \pi_n(X, x) \times \pi_n(Y, y).$$

Solution: Consider the projection maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$. We have $p_X(x, y) = x$ and $p_Y(x, y) = y$. This induces two maps

$$\begin{aligned}(p_X)_* :&\pi_n(X \times Y, (x, y)) \to \pi_n(X, x),\\ (p_Y)_* :&\pi_n(X \times Y, (x, y)) \to \pi_n(Y, y).\end{aligned}$$

Define

$$p = (p_X)_* \times (p_Y)_* : \pi_n(X \times Y, (x, y)) \to \pi_n(X, x) \times \pi_n(Y, y).$$

This is a group homomorphism since both $(p_X)_*$ and $(p_Y)_*$ are group homomorphisms. We need to show that p is both injective and surjective.

- Let $\gamma : (I^n, \partial I^n) \to (X \times Y, (x, y))$ be a continous map and $[\gamma]$ is corresponding element in $\pi_n(X \times Y, (x, y))$. Assume $[\gamma] \in \ker p$. This means $p([\gamma]) = ([C_x], [C_y])$ where $C_x : (I^n, \partial I^n) \to (X, x)$ and $C_y : (I^n, \partial I^n) \to (Y, y)$ are the constant maps. By definition, we have the composition $I^n \xrightarrow{\gamma} X \times Y \xrightarrow{p_X} X$ is homotopic to the constant map C_x and $I^n \xrightarrow{\gamma} X \times Y \xrightarrow{p_Y} Y$ is homotopic to the constant map C_y. Since p_X and p_Y are only projections, so $\gamma : I^n \to X \times Y$ is homotopic to the constant map $C_{x,y}$ sending everything in I^n to the point (x, y) via the product of the previous two homotopies. This shows that $[\gamma]$ is the identity element in $\pi_n(X \times Y, (x, y))$. So p is injective.

- Suppose $([\gamma_1], [\gamma_2]) \in \pi_n(X, x) \times \pi_n(Y, y)$ is an element represented by $\gamma_1 : I^n \to X$ and $\gamma_2 : I^n \to Y$. Define a map

$$\begin{aligned}\gamma : I &\to X \times Y,\\ t &\mapsto (\gamma_1(t), \gamma_2(t)).\end{aligned}$$

  We know that the corresponding homotopy class $[\gamma]$ is an element in $\pi_n(X \times Y, (x, y))$ and we have

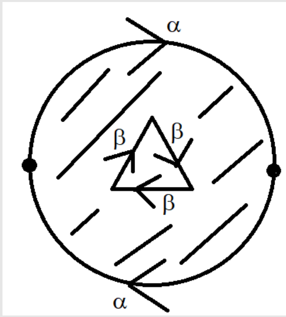
$$p([\gamma]) = ((p_X)_*[\gamma], (p_Y)_*[\gamma]) = ([p_X \circ \gamma], [p_Y \circ \gamma]) = ([\gamma_1], [\gamma_2]).$$

  This proves that p is surjective.

Problem 5

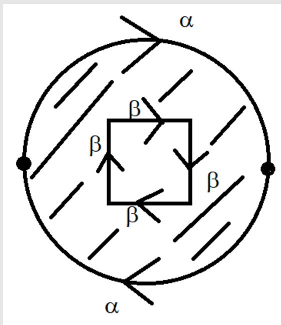
(a) Let T be the torus and M be the Möbius band. Let X be the space obtained by gluing the boundary of M homeomorphically to the "diagonal" circle on the torus (when viewing the torus as a quotient space of I^2 in the standard way, this is the usual diagonal). Compute $H_*(X)$.

(b) Start with the usual model of $\mathbb{R}P^2$ where we have a disk D^2 with antipodal points on the boundary identified. Drill a triangular hole in the middle of the disk and label the side β, β, and β (all oriented clockwise), and let Y be the resulting quotient space:



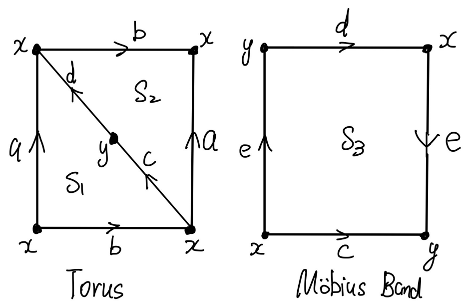
Determine $H_*(Y)$.

(c) Repeat part (b) where the hole is a square rather than a triangle, and where the four sides are again all labelled with β and all oriented clockwise.



Solution:

(a) We use the following cell complex structure:



The space X has two 0-cells x, y, five 1-cells a, b, c, d, e and three 2-cells S_1, S_2, S_3. We have the following cellular chain complex:

$$\mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^5 \xrightarrow{d_1} \mathbb{Z}^2 \xrightarrow{d_0} 0$$

where $d_0(x) = d_0(y) = 0$ and

$$\begin{aligned} d_1(a) &= d_1(b) = x - x = 0, & d_2(S_1) &= -a + b + c + d, \\ d_1(c) &= d_1(e) = y - x, & d_2(S_2) &= c + d + b - a, \\ d_1(d) &= x - y, & d_2(S_3) &= 2e + d - c. \end{aligned}$$

We can see that

$$\begin{aligned} \ker d_1 &= \langle a, b, c + d, e + d \rangle, \\ \operatorname{Im} d_1 &= \langle x - y \rangle, \\ \ker d_2 &= \langle S_1 - S_2 \rangle, \\ \operatorname{Im} d_2 &= \langle a - b - c - d, 2e + d - c. \\ \ker d_0 &= \langle x, y \rangle. \end{aligned}$$

We can calculate the homology group:

$$\begin{aligned} H_0(X) &= \ker d_0 / \operatorname{Im} d_1 \\ &= \langle x, y \rangle / \langle x - y \rangle \\ &= \mathbb{Z}, \\ H_2(X) &= \ker d_2 \\ &= \langle S_1 - S_2 \rangle \\ &= \mathbb{Z}. \end{aligned}$$

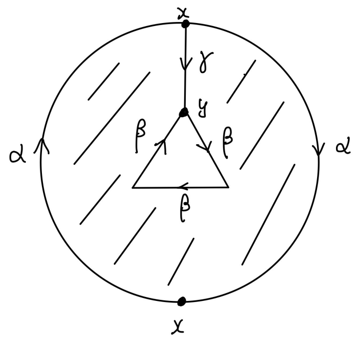
and

$$\begin{aligned} H_1(X) &= \ker d_1 / \operatorname{Im} d_2 \\ &= \langle a, b, c + d, e + d \rangle / \langle a - b - c - d, 2e + d - c \rangle \\ &= \langle b, a - b, e - c, c + d \rangle / \langle a - b - c - d, 2e + d - c \rangle \\ &= \langle b, c + d, e - c, a - b - c - d \rangle / \langle a - b - c - d, 2(e - c) + c + d \rangle \\ &= \langle b, u, v \rangle / \langle u + 2v \rangle \\ &= \langle b, u + v, v \rangle / \langle (u + v) + v \rangle \\ &= \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

So the homology group of X can be summarized as

$$H_i(X) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2; \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

(b) consider the following cell complex structure:



The space Y has two 0-cells x, y, three 1-cells α, β, γ and one 2-cell S. We have the following cellular complex:

$$\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z}^2 \xrightarrow{d_0} 0.$$

The boundary maps are given by

$$\begin{aligned} d_2(S) &= 2\alpha + \gamma - 3\beta - \gamma = 2\alpha - 3\beta, \\ d_1(\alpha) &= d_1(\beta) = 0, \\ d_1(\gamma) &= y - x, \\ d_0 &= 0 \end{aligned}$$

So we can calculate the homology groups

$$H_2(Y) = \ker d_2 = 0.$$

$$\begin{aligned} H_0(Y) &= \ker d_0 / \operatorname{Im} d_1 \\ &= \langle x, y \rangle / \langle y - x \rangle \\ &= \mathbb{Z} \end{aligned}$$

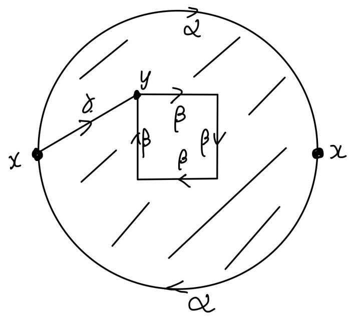
and

$$\begin{aligned} H_1(Y) &= \ker d_1 \operatorname{Im} d_2 \\ &= \langle \alpha, \beta \rangle / \langle 2\alpha - 3\beta \rangle \\ &= \langle \alpha - \beta, \beta \rangle / \langle 2(\alpha - \beta) - \beta \rangle \\ &= \langle \alpha - \beta, \alpha - 2\beta \rangle / \langle \alpha - \beta + \alpha - 2\beta \rangle \\ &= \mathbb{Z}. \end{aligned}$$

So the homology group of Y can be summarized as

$$H_i(Y) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 1; \\ 0, & \text{otherwise.} \end{cases}$$

(c) Now the cell structure is like this:



We denote this space as Z, which has two 0-cells x, y, three 1-cells α, β, γ and one 2-cell S. We have the cellular chain complex:

$$\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z}^2 \xrightarrow{d_0} 0.$$

The boundary maps are given by

$$\begin{aligned} d_2(S) &= 2\alpha + \gamma - 4\beta - \gamma = 2\alpha - 4\beta, \\ d_1(\alpha) &= d_1(\beta) = 0, d_1(\gamma) = y - x, \\ d_0 &= 0. \end{aligned}$$

So we can calculate the homology group of Z:

$$H_2(Z) = \ker d_2 = 0.$$

$$\begin{aligned} H_0(Z) &= \ker d_0 / \operatorname{Im} d_1 \\ &= \langle x, y \rangle / \langle y - x \rangle \\ &= \mathbb{Z}. \end{aligned}$$

and

$$\begin{aligned} H_1(Z) &= \ker d_1 / \operatorname{Im} d_2 \\ &= \langle \alpha, \beta \rangle / \langle 2\alpha - 4\beta \rangle \\ &= \langle \alpha - \beta, \alpha - 2\beta \rangle / \langle 2(\alpha - 2\beta) \rangle \\ &= \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

So the homology group of Z can be summarized as

$$H_i(Z) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 6
Now that you have computed the homology of all compact 2-manifolds, we can start to explore the situation for 3-manifolds a bit.

(a) If X is any space, recall that the suspension ΣX is CX/X, where CX is the cone on X. Prove that $\tilde{H}_i(\Sigma X) = \tilde{H}_{i-1}(X)$ for all i.

(b) If X is a pointed space then the reduced suspension $\tilde{\Sigma} X$ is defined to be $(\Sigma X)/(\Sigma *)$ where $*$ is the base point. Compute $\tilde{H}_i(\tilde{\Sigma} X)$ in terms of $\tilde{H}_i(X)$.

(c) Convince yourself there is a cofiber sequence $S^1 \vee X \hookrightarrow S^1 \times X \to \tilde{\Sigma} X$. Prove that the induced maps $H_*(S^1 \vee X) \to H_*(S^1 \times X)$ are injective for $* \geq 1$ by constructing a splitting.

(d) Calculate $H_*(S^1 \times T)$ and $H_*(S^1 \times \mathbb{R}P^2)$.

Solution:

(a) The pair (CX, X) is a good pair so the quotient map

$$X \to CX \to \Sigma X$$

induces a long exact sequence in the reduced homology groups

$$\cdots \to \tilde{H}_i(X) \to \tilde{H}_i(CX) \to \tilde{H}_i(\Sigma X) \to \tilde{H}_{i-1}(X) \to \tilde{H}_{i-1}(CX) \to \cdots$$

The space CX is contractible, so $\tilde{H}_i(CX) = 0$ for all i. This implies $\tilde{H}_i(\Sigma X) \cong \tilde{H}_{i-1}(X)$ for all i.

(b) Assume for any point $x \in X$, the pair (X, x) is a good pair. Consider the inclusion $* \hookrightarrow X$ and apply the suspension, we get the follwoing:

$$\Sigma * \hookrightarrow \Sigma X \to (\Sigma X)/(\Sigma *) \cong \tilde{\Sigma} X.$$

We have an induced long exact sequence in reduced homology groups

$$\cdots \to \tilde{H}_i(\Sigma *) \to \tilde{H}_i(\Sigma X) \to \tilde{H}_i(\tilde{\Sigma} X) \to \tilde{H}_{i-1}(\Sigma *) \to \cdots$$

Note that $\Sigma *$ is homeomorphic to the interval, which is contractible. So $\tilde{H}_i(\Sigma *) = 0$ for all i. From the long exact sequence and (a) we can see that

$$\tilde{H}_i(\tilde{\Sigma} X) \cong \tilde{H}_i(\Sigma X) \cong \tilde{H}_{i-1}(X)$$

for all i.

(c) Identify $S^1 \times X$ as the quotient space $Y = (I \times X)/\sim$ where $(1, x) \sim (0, x)$ for all $x \in X$. Consider the subspace

$$Y \supset Z = \{(*, t) \mid t \in I\} \cup \{(x, 0) \mid x \in X\}$$

where $* \in X$ is the basepoint we choose. Note that Z can be viewed as S^1 and X glued at the point $(*, 0)$. So $Z \cong S^1 \vee X$. Moreover,

$$\{(*, t) \mid t \in I\} \cong \Sigma *$$

and we have

$$\tilde{\Sigma} X = (\Sigma X)/(\Sigma *) \cong Y/Z \cong (S^1 \times X)/(S^1 \vee X).$$

Thus, we have a cofiber sequence $S^1 \vee X \to S^1 \times X \to \tilde{\Sigma} X$.
Consider two projection maps $p_1 : S^1 \times X \to S^1$ and $p_2 : S^1 \times X \to X$. For all $i \geq 1$, p_1 and p_2 induce maps in homology groups $(p_1)_* : H_i(S^1 \times X) \to H_i(S^1)$ and $(p_2)_* : H_i(S^1 \times X) \to H_i(X)$. Define a map

$$\begin{aligned} \phi : H_i(S^1 \times X) &\to H_i(S^1) \oplus H_i(X), \\ [a] &\mapsto ((p_1)_*[a], (p_2)_*[a]) \end{aligned}$$

We can identify $H_i(S^1 \vee X) \cong H_i(S^1) \oplus H_i(X)$ and ϕ can be viewed as a map $\phi : H_i(S^1 \times X) \to H_i(S^1 \vee X)$. We claim this is the splitting we want. Indeed, write $i : S^1 \vee X \hookrightarrow S^1 \times X$ as the map of inclusion and i_* is the induced map in homology, we know the composition $\phi \circ i_*$ is induced by the following two maps of spaces:

$$\begin{aligned} S^1 \vee X \to S^1 \times X \to S^1, \\ S^1 \vee X \to S^1 \times X \to X. \end{aligned}$$

The coproduct for two pointed spaces is the wedge sum, so we can conclude that $\phi \circ i_*$ is indeed a splitting. And thus, $i_* : H_i(S^1 \vee X) \to H_i(S^1 \times X)$ is injective for $i \geq 1$.

(d) The cofiber sequence $S^1 \vee X \to S^1 \times X \to \tilde{\Sigma} X$ induces a long exact sequence in reduced homology groups and for all $i \geq 1$, we identify $\tilde{H}_i(S^1 \vee X) \cong \tilde{H}_i(S^1) \oplus \tilde{H}_i(X)$ and $\tilde{H}_i(\tilde{\Sigma} X) \cong \tilde{H}_{i-1}(X)$, we have

$$\cdots \to \tilde{H}_i(S^1) \oplus \tilde{H}_i(X) \to \tilde{H}_i(S^1 \times T) \to \tilde{H}_{i-1}(X) \to \tilde{H}_{i-1}(S^1) \oplus \tilde{H}_{i-1}(X) \to \cdots$$

We have shown in (c) that the map for $i \geq 1$, $\tilde{H}_i(S^1) \oplus \tilde{H}_i(T) \to \tilde{H}_i(S^1 \times T)$ is injective, by exactness, the map

$$\tilde{H}_{i+1}(\tilde{\Sigma} T) \cong \tilde{H}_i(T) \to \tilde{H}_i(S^1) \oplus \tilde{H}_i(T)$$

is the zero map. So for $i \geq 2$, we have

$$\tilde{H}_i(S^1 \times T)/(\tilde{H}_i(S^1) \oplus \tilde{H}_i(T)) \cong \tilde{H}_{i-1}(T).$$

When $i = 3$, $\tilde{H}_3(S^1) = \tilde{H}_3(T) = 0$, so

$$H_3(S^1 \times T) \cong \tilde{H}_3(S^1 \times T) \cong \tilde{H}_2(T) \cong \mathbb{Z}.$$

When $i = 2$, $\tilde{H}_2(S^1) = 0$, $\tilde{H}_2(T) = \mathbb{Z}$ and $\tilde{H}_1(T) = \mathbb{Z} \oplus \mathbb{Z}$, so

$$H_2(S^1 \times X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

For $H_0(S^1 \times T)$, we know that S^1 and T are all path-connected, so the product $S^1 \times T$ is also path-connected. This implies that

$$H_0(S^1 \times T) \cong 0.$$

When $i = 1$, note that $\tilde{H}_0(T) = 0$, so the map

$$i_* : \tilde{H}_1(S^1) \oplus \tilde{H}_1(T) \to \tilde{H}_0(T)$$

is an isomorphism, so

$$H_1(S^1 \times T) \cong \tilde{H}_1(S^1 \times T) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

The homology group of $S^1 \times T$ can be summarized as

$$H_i(S^1 \times T) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 3; \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, & \text{if } i = 1, 2; \\ 0, & \text{otherwise.} \end{cases}$$

When $X = \mathbb{R}P^2$, since S^1 and $\mathbb{R}P^2$ are path-connected, so the product $S^1 \times \mathbb{R}P^2$ are also path-connected. This implies

$$H_0(S^1 \times \mathbb{R}P^2) \cong \mathbb{Z}.$$

For $i = 2, 3$, we can still use the isomorphism

$$\tilde{H}_i(S^1 \times \mathbb{R}P^2)/(\tilde{H}_i(S^1) \oplus \tilde{H}_i(\mathbb{R}P^2)) \cong \tilde{H}_{i-1}(\mathbb{R}P^2).$$

Note that for $i = 2, 3$, we have

$$H_3(S^1) = H_2(S^1) = 0 = H_3(\mathbb{R}P^2) = H_2(\mathbb{R}P^2).$$

So

$$\begin{aligned} H_3(S^1 \times \mathbb{R}P^2) &\cong H_2(\mathbb{R}P^2) = 0 \\ H_2(S^1 \times \mathbb{R}P^2) &\cong H_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

For $i = 1$, since $\mathbb{R}P^2$ is path-connected, so the map

$$i_* : H_1(S^1 \vee \mathbb{R}P^2) \to H_1(S^1 \times \mathbb{R}P^2)$$

is an isomorphism and we have $H_1(S^1 \times \mathbb{R}P^2) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The homology groups of $S^1 \times \mathbb{R}P^2$ can be summarized as

$$H_i(S^1 \times \mathbb{R}P^2) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } i = 2; \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$