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Course: MATH 616 - Real Analysis
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**Homework - Week 3 Exercises**
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**Exercise 3.1**
Let $f : X \to \mathbb{C}$ be a measurable function, then there exists a sequence of simple functions $\{\phi_n\}$ such that $|\phi_n|$ is nondecreasing with respect to $n$ and $\phi_n \to f$ pointwise.

*Solution:* First we prove this is true for a real-valued function $f$. Define

$$f^+(x) := \max\{0, f(x)\}, \quad f^-(x) := -\min\{f(x), 0\}.$$

Then both $f^+$ and $f^-$ are positive measurable functions on $X$, and $f = f^+ - f^-$ by definition. A theorem we proved in class tells us that there exists a positive nondecreasing sequence $\{s_n\}$ of simple functions such that $0 \leq s_n \leq f^+$ for all $n \geq 1$ and $\lim_{n\to\infty} s_n(x) = f^+(x)$. Similarly, there exists a positive nondecreasing sequence $\{t_n\}$ of simple functions such that $0 \leq t_n \leq f^-$ for all $n$ and $\lim_{n\to\infty} t_n(x) = f^-(x)$. Note that by definition, for any $x \in X$, either $f^+(x) = 0$, or $f^-(x) = 0$. This implies that $s_n(x)t_n(x) = 0$ for all $n$ and all $x \in X$. Consider a sequence $\{s_n - t_n\}$ of simple functions. For any $n$,

$$|s_n - t_n|^2 = s_n^2 + t_n^2 \leq s_{n+1}^2 + t_{n+1}^2 = |s_{n+1} - t_{n+1}|^2.$$

This implies that $|s_n - t_n|$ is nondecreasing and

$$\lim_{n\to\infty} (s_n(x) - t_n(x)) = f^+(x) - f^-(x) = f(x).$$

Now consider $f = u + iv$ to be a complex-valued function, then apply the above construction to $u$ and $v$ separately, obtaining two nondecreasing sequence of simple functions $\{p_n\}$ and $\{q_n\}$, then let $\phi_n = p_n + iq_n$. Here $|\phi_n|$ is decreasing as

$$|\phi_n|^2 = p_n^2 + q_n^2 \leq p_{n+1}^2 + q_{n+1}^2 = |\phi_{n+1}|^2$$

and

$$\lim_{n\to\infty} \phi_n(x) = \lim_{n\to\infty} p_n(x) + i \lim_{n\to\infty} q_n(x) = u(x) + iv(x) = f(x).$$

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**Exercise 3.2**
Compute the limits and justify the computation:

$$\lim_{n\to\infty} \int_0^1 \frac{1 + nx^2}{(1 + x^2)^n} dx, \quad \lim_{n\to\infty} \int_a^\infty \frac{n}{1 + n^2x^2} dx$$

*Solution:*

(1) Define

$$f_n(x) = \frac{1+nx^2}{(1+x^2)^n} = \frac{1+nx^2}{1+nx^2+\sum_{k=2}^{n}\binom{n}{k}x^{2k}}.$$

Write

$$g_n(x) := \frac{\sum_{k=2}^{n}\binom{n}{k}x^{2k}}{1+nx^2}$$

and $f_n(x) = \frac{1}{1+g_n(x)}$. It is easy to see that $g_n(x) \to +\infty$ as $n \to \infty$ for all $x \in (0,1)$, so $f_n(x) \to 1$ as $n \to \infty$ for all $x \in (0,1)$. Moreover, $g_n(x) \geq 0$, so

$$|f(x)| = \frac{1}{1+g_n(x)} \leq 1.$$

And 1 is integrable on the interval $(0,1)$, by Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n}dx = \int_0^1 \lim_{n\to\infty} \frac{1+nx^2}{(1+x^2)^n}dx = 1.$$

(2) Note that for all $n \geq 1$ and all $x \in \mathbb{R}$,

$$\frac{d}{dx}\arctan(nx) = \frac{n}{1+n^2x^2}.$$

Thus, by the fundamental theorem of calculus, write

$$\begin{aligned} I = \lim_{n\to\infty} \int_a^{+\infty} \frac{n}{1+n^2x^2}dx &= \lim_{n\to\infty} \lim_{b\to+\infty} \int_a^b \frac{n}{1+n^2x^2}dx \\ &= \lim_{n\to\infty} \lim_{b\to+\infty} (\arctan(nb) - \arctan(na)) \\ &= \lim_{n\to\infty} (\frac{\pi}{2} - \arctan(na)) \\ &= \frac{\pi}{2} - \lim_{n\to\infty} \arctan(na). \end{aligned}$$

The result can be listed as follows

$$I = \begin{cases} 0, & \text{if } a > 0, \\ \frac{\pi}{2}, & \text{if } a = 0, \\ \pi, & \text{if } a < 0. \end{cases}$$

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**Exercise 1.7**
Suppose $f_n : X \to [0,+\infty]$ is measurable for $n = 1,2,\ldots$. $f_1 \geq f_2 \geq \cdots \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

and show that this conclusion does not follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

*Solution:* Both $f_1$ and $f = \lim_{n\to\infty} f_n$ are positive and measurable, and since for every $n$, $f_n \leq f_1$, we have

$$|f(x)| = f(x) \leq f_1(x)$$

for all $x \in X$. Here $f_1 \in L^1(\mu)$ is an integrable function. By Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

The condition $f_1 \in L^1(\mu)$ is essential. Consider the following function

$$f_n(x) = \chi_{[n,+\infty)}.$$

For all $n$, $[n+1, +\infty) \subsetneq [n, +\infty)$, so $f_n$ is positive and decreasing. It is not hard to see that

$$\lim_{n\to+\infty} f_n(x) = 0.$$

But on the other hand,

$$\lim_{n\to\infty} \int_X f_n d\mu = \lim_{n\to+\infty} m([n, +\infty)) = \lim_{n\to+\infty} +\infty = +\infty.$$

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**Exercise 1.8**
Put $f_n = \chi_E$ if $n$ is odd, and $f_n = 1 - \chi_E$ if $n$ is even. What is the relevance of this example to Fatou's lemma.

*Solution:* This is an example where the inequality in Fatou's lemma can be strict. Indeed, let $E \subsetneq X$ be measurable, and $m(E) < m(X) < +\infty$. On the one hand, for any $x \in X$, either $\chi_E(x) = 0$ if $x \notin E$, or $1 - \chi_E(x) = 0$ if $x \in E$. So

$$\liminf_{n\to\infty} f_n(x) = 0$$

for all $x \in X$. This implies that

$$\int_X \liminf_{n\to\infty} f_n d\mu = 0.$$

On the other hand, we have

$$\int_X f_n d\mu = \begin{cases} m(E), & \text{if } n \text{ is odd} \\ m(E^c), & \text{if } n \text{ is even.} \end{cases}$$

We know both $m(E)$ and $m(E^c)$ is strictly positive by our assumption, so

$$\int_X \liminf_{n\to\infty} f_n d\mu < \liminf_{n\to\infty} \int_X f_n d\mu.$$

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**Exercise 1.10**
Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on $X$, and $f_n \to f$ uniformly on $X$. Prove that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu,$$

and show that the hypothesis '$\mu(X) < \infty$' cannot be omitted.

*Solution:* For any $\varepsilon > 0$, because $f_n$ converges to $f$ uniformly, there exists $N > 0$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{\mu(X)}$$

for any $x \in X$ and $n > N$. Since every $f_n$ is bounded and $\mu(X) < \infty$, the limit $f$ is also bounded on $X$, and the integrals $\int_X f_n d\mu$ and $\int_X f d\mu$ are finite. Moreover, for any $n > N$, we have

$$\begin{aligned}\left|\int_X f_n d\mu - \int_X f d\mu\right| &\leq \int_X |f_n - f| d\mu \\ &< \frac{\varepsilon}{\mu(X)} \int_X d\mu \\ &= \frac{\varepsilon}{\mu(X)} \cdot \mu(X) \\ &= \varepsilon.\end{aligned}$$

This proves that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

Now let $X = [0, +\infty)$ satisfying $\mu(X) = +\infty$. Consider a sequence of functions $\{f_n\}$:

$$f_n := \frac{1}{n}\chi_{[0,n]}.$$

It is easy to see $f_n$ is bounded for every $n \geq 1$, and $f_n \to 0$ uniformly as $n \to \infty$. However, on the one hand,

$$\lim_{n\to\infty} \int_0^\infty f_n d\mu = \lim_{n\to\infty} n \cdot \frac{1}{n} = 1.$$

On the other hand,

$$\int_0^\infty 0 d\mu = 0 \cdot \infty = 0.$$

This shows that the condition $\mu(X) < \infty$ is necessary.

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**Exercise 1.12**
Suppose $f \in L^1(\mu)$. Prove that to each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

*Solution:* Given any $\varepsilon > 0$, consider the sequence of simple functions $\{\phi_n\}$ we constructed in

Exercise 1. They satisfy the condition $\lim_{n\to\infty} \phi_n(x) = f(x)$ for all $x \in E$ and

$$|\phi_n(x)| \leq |f(x)|$$

where $|f|$ is a real-valued integrable function since $f \in L^1(\mu)$. By Lebesgue dominated convergence theorem, there exists some $N > 1$ such that

$$\int_E |f - \phi_N| d\mu < \frac{\varepsilon}{2}.$$

where $\phi_N$ is some simple function on $E$. Suppose we can write

$$\phi_N = \sum_{i=1}^{k} \alpha_i \chi_{A_i}.$$

Then the integral

$$\int_E \phi_N d\mu = \sum_{i=1}^{k} \alpha_i \mu(A_i \cap E) \leq \mu(E)(\sum_{i=1}^{k} \alpha_i).$$

If we choose $\delta < \frac{\varepsilon}{2\sum_{i=1}^{k} \alpha_i}$, then

$$\mu(E)(\sum_{i=1}^{k} \alpha_i) < \delta(\sum_{i=1}^{k} \alpha_i) < \frac{\varepsilon}{2}.$$

Thus, for any $m(E) < \delta$, the integral

$$\begin{aligned}
\int_E |f| d\mu &= \int_E |f - \phi_N + \phi_N| d\mu \\
&\leq \int_E |f - \phi_N| d\mu + \int_E |\phi_N| d\mu \\
&< \frac{\varepsilon}{2} + \mu(E)(\sum_{i=1}^{k} \alpha_i) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$