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Course: MATH 636 - Algebraic Topology III
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Homework 5
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Problem 1

The space $S^2 \times S^3$ and $S^2 \vee S^3 \vee S^5$ have isomorphic homology groups. Use the cup product to prove that they are not homotopy equivalent.

Solution: By Künneth Theorem, $H^i(S^2 \times S^3) = \mathbb{Z}$ for $i = 0, 2, 3, 5$. The space $Y = S^2 \vee S^3 \vee S^5$ has a cellular structure with one cell in dimension $0, 2, 3, 5$. So its cohomology groups are the same. Suppose e^i $(i = 0, 2, 3, 5)$ are corresponding cells in ith dimension. Then $[e^2] \cup [e^3] = 0$ because they come from different parts of the wedge product. On the other hand, because the cohomology groups of spheres are free Abelian groups, so we have an isomorphism of rings

$$H^*(S^2 \times S^3) \cong \bigoplus_{i+j=*} H^i(S^2) \otimes H^j(S^3).$$

Let $x \in H^0(S^2) = \mathbb{Z}$ and $y \in H^0(S^3) = \mathbb{Z}$ be the generators. They are multiplicative unit in the cohomology rings. Let $a \in H^2(S^2) = \mathbb{Z}$ and $b \in H^3(S^3) = \mathbb{Z}$ be generators. In the tensor product, we have

$$(a \otimes y) \cup (x \otimes b) = (a \cup x) \otimes (y \cup b) = a \otimes b$$

where $a \otimes b$ is the generator of $\mathbb{Z} = H^2(S^2) \otimes H_3(S^3) \cong H^5(S^2 \times S^3)$. This shows that $S^2 \times S^3$ and Y have different cohomology rings, so they are not homotopy equivalent.

Problem 2

(a) Let M be an n-dimensional manifold-with-boundary (boundary points have neighborhoods that look like $\{\underline{x} \in \mathbb{R}^n \mid x_n \geq 0\}$). If x is on the boundary of M, prove that $H_n(M, M - x) = 0$.

(b) Let M be a compact, connected, orientable n-manifold. If U is a Euclidean open disk in M, prove that $H_i(M - U) \to H_i(M)$ is an isomorphism when $i < n$ and prove that $H_n(M - U) = 0$. Also, if $A = \partial(M - U)$ prove that the connecting homomorphism $\partial : H_n(M - U, A) \to H_{n-1}(A)$ is an isomorphism.

Solution:

(a) Suppose $x \in M$ is a boundary point and $U \subseteq M$ is a neighborhood isomorphic to

$$\{\underline{x} \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Consider $M - \bar{U} \subset M - x \subset M$ and use excision, we have

$$H_n(M, M - x) \cong H_n(\bar{U}, \bar{U} - x).$$

Without loss of generality, we can assume $n = 2$ and

$$\bar{U} = \left\{ z = re^{i\alpha} \in \mathbb{C} = \mathbb{R}^2 \mid 0 \le r \le 1, 0 \le \alpha \le \pi \right\}$$

and $x = (0, 0) \in \bar{U}$. Consider the homotopy defined via

$$\begin{aligned} H : (\bar{U} - x) \times I &\to \bar{U} - x, \\ z &\mapsto \frac{z}{|z|} \end{aligned}$$

This is a deformation retract to the half circle

$$\left\{ z = e^{i\alpha} \mid 0 \le \alpha \le \pi \right\}.$$

The half circle is contractible, so $\bar{U} - x$ is also contractible. For higher n, we can prove $\bar{U} - x$ is still contractible in a similar way. Now consider the long exact sequence for the pair $(\bar{U} - x, \bar{U})$

$$\begin{tikzcd} \cdots \arrow[r] & H_n(\bar{U} - x) \arrow[r] & H_n(\bar{U}) \arrow[r] & H_n(\bar{U}, \bar{U} - x) \arrow[dll] \\ & H_{n-1}(\bar{U} - x) \arrow[r] & H_{n-1}(\bar{U}) \arrow[r] & \cdots \end{tikzcd}$$

If $n \ge 2$, note that both $\bar{U}$ and $\bar{U} - x$ are contractible, so $H_n(\bar{U}) = H_{n-1}(\bar{U} - x) = 0$, this implies that $H_n(\bar{U}, \bar{U} - x) = 0$. If $n = 1$, then $H_0(\bar{U} - x) \to H_0(\bar{U})$ is an isomorphism and $H_1(\bar{U}) = 0$, so $H_1(\bar{U}, \bar{U} - x) = 0$. This proves that

$$H_n(M, M - x) \cong H_n(\bar{U}, \bar{U} - x) = 0$$

for all $n \ge 1$.

(b) Consider the quotient space

$$M/M - U \cong U/\partial U \cong S^n$$

because U is the Euclidean open disk. We have an isomorphism for all $*$

$$H_*(M, M - U) \cong \tilde{H}_*(M/M - U) \cong \tilde{H}_*(S^n).$$

Consider the long exact sequence of the pair

$$\begin{tikzcd} & \cdots \arrow[r] & \tilde{H}_{n+1}(S^n) \arrow[dll] \\ H_n(M - U) \arrow[r] & H_n(M) \arrow[r] & \tilde{H}_n(S^n) \arrow[dll] \\ H_{n-1}(M - x) \arrow[r] & H_{n-1}(M) \arrow[r] & \tilde{H}_{n-1}(S^n) \arrow[dll] \\ \cdots & & \end{tikzcd}$$

$\tilde{H}_i(S^n) = 0$ for all $i \leq n-1$ and $i \geq n+1$, to prove that $H_n(M-U) = 0$ and $H_i(M-U) \to H_i(M)$ is an isomorphism for all $i \leq n-1$, we only need to show that

$$H_n(M) \to H_n(M, M-U) \cong \tilde{H}_n(S^n)$$

is an isomorphism. Let $x \in U$ be a point. Consider the following square

$$\begin{tikzcd} M-U \arrow[r] \arrow[d] & M \arrow[d, "id"] \\ M-x \arrow[r] & M \end{tikzcd}$$

All three maps are inclusions. This induces a commutative diagram in homology groups

$$\begin{tikzcd} H_n(M) \arrow[r] \arrow[d, "id"'] & H_n(M, M-U) \arrow[d] \\ H_n(M) \arrow[r] & H_n(M, M-x) \end{tikzcd}$$

Note that M is orientable, so the map $H_n(M) \to H_n(M, M-x)$ is an isomorphism for all $x \in M$. This implies that the composition

$$H_n(M) \to H_n(M, M-U) \to H_n(M, M-x)$$

is an isomorphism. Moreover, U is a Euclidean open disk, so U is homotopy equivalent to a point. This implies that $M-U$ is homotopy equivalent to $M-x$, so $H_n(M, M-U) \to H_n(M, M-x)$ is also an isomorphism. Thus, we have proved that $H_n(M) \to H_n(M, M-U)$ is an isomorphism.

Now consider $A = \partial(M-U)$. Consider the long exact sequence in homology groups coming from the pair $(A, M-U)$

$$H_n(M-U) \to H_n(M-U, A) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(M-U) \to \cdots$$

We have already proved $H_n(M-U) = 0$, to show that ∂ is an isomorphism, we only need to show that the map

$$i_* : H_{n-1}(A) \to H_{n-1}(M-U)$$

is the zero map where $i : A \to M-U$ is the inclusion. Consider the following commutative diagram of spaces

$$\begin{tikzcd} A \arrow[r] \arrow[d] & M-U \arrow[d] \\ M \arrow[r, "id"'] & M \end{tikzcd}$$

All three maps are inclusions. This induces a commutative diagram in homology groups

$$\begin{tikzcd}
H_{n-1}(A) \arrow[r] \arrow[d] & H_{n-1}(M-U) \arrow[d, "\cong"] \\
H_{n-1}(M) \arrow[r, "id"'] & H_{n-1}(M)
\end{tikzcd}$$

We have proved that the right vertical map is an isomorphism. Note that in M, the boundary of $M-U$ is the same as the boundary of the closed disk $\bar{U}$, and $\bar{U}$ is contractible, so the map $H_{n-1}(A) \to H_{n-1}(M)$ is the zero map. This implies that $H_{n-1}(A) \to H_{n-1}(M-U)$ is also the zero map.

Problem 3
Let M and N be compact, connected n-manifolds, $n \geq 2$. Prove the following:

(a) If M and N are orientable, then so is $M\#N$.

(b) If M and N are non-orientable, then so is $M\#N$.

(c) What happens when M is orientable and N is not? Justify your answer.

Solution:

(a) If M, N are orientable, let U be the disk they glued together, whose boundary is isomorphic to S^{n-1}. We have a cofiber sequence

$$S^{n-1} \to M\#N \to M \vee N.$$

This induces a long exact sequence in homology groups

$$H_n(S^{n-1}) \to H_n(M\#N) \to H_n(M \vee N) \to H_{n-1}(S^{n-1}) \to \cdots$$

We know that $H_n(S^{n-1}) = 0$ and $H_n(M \vee N) = H_n(M) \oplus H_n(N)$ for $n \geq 2$. Because both M and N are orientable, $H_n(M \vee N) = \mathbb{Z}^2$. If $H_n(M\#N) = 0$, then the map $H_n(M \vee N) \to H_{n-1}(S^{n-1})$ is injective, namely we have an injective map $\mathbb{Z}^2 \to \mathbb{Z}$. This is impossible. So $H_n(M\#N) = \mathbb{Z}$. This implies $M\#N$ is orientable.

(b) Use the same long exact sequence as (a), and now because M, N are both non-orientable, we have

$$H_n(M \vee N) = H_n(M) \oplus H_n(N) = 0.$$

This implies that $H_n(M\#N) = 0$ and thus $M\#N$ is also non-orientable.

(c) Let U be the Euclidean disk M and N glued together, then we have a cofiber sequence

$$M - U \to M\#N \to N.$$

This induces a long exact sequence in homology groups

$$\cdots \to H_n(M-U) \to H_n(M\#N) \to H_n(N) \to \cdots$$

We have proved in the previous problem that if M is orientable, then $H_n(M-U)=0$ for an open Euclidean disk $U \subseteq M$. And $H_n(N)=0$ because N is non-orientable. This implies that $H_n(M\#N)=0$, and thus we can conclude that $M\#N$ is non-orientable.

Problem 4

Suppose $X = Y = S^n$. $H^*(X)$ has no torsion then the Künneth Theorem gives an isomorphism of rings

$$H^*(X) \otimes H^*(Y) \to H^*(X \times Y).$$

For $a \in H^*(X)$ and $b \in H^*(Y)$, the map sends $a \otimes b$ to $\pi_1^*(a) \cup \pi_2^*(b)$. The latter expression is sometimes denoted $a \times b$.

Suppose that S^n has a continuous unital multiplication $\mu : S^n \times S^n \to S^n$. So there is a unit element $e \in S^n$ with property that $\mu(e,x) = x = \mu(x,e)$ for all $x \in S^n$. Said differently, the following diagram is commutative:

$$\begin{tikzcd}
S^n \times \{*\} \arrow[dr, "id\times j"] \arrow[drr, bend left, "id"] & & \\
& S^n \times S^n \arrow[r, "\mu"] & S^n \\
\{*\} \times S^n \arrow[ur, "j\times id"] \arrow[urr, bend right, "id"'] & &
\end{tikzcd}$$

where $j : * \hookrightarrow S^n$ sends the point to e.

(a) Let z be a generator for $H^n(S^n)$. Use the above diagram to prove that $\mu^*(z) = z\otimes 1 + 1 \otimes z$.

(b) Use the fact that μ^* is a ring homomorphism, together with your knowledge of the ring structure on $H^*(S^n \times S^n)$, to conclude that n must be odd.

(c) Consider a multiplication $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ which was unital and had no zero-divisors (that is, $xy = 0 \Rightarrow (x = 0 \text{ or } y = 0)$). Use (b) to prove that no such multiplication exists, assuming that the identity element is nonzero.

Solution:

(a) Let $1 \in H^0(S^n)$ be the generator and the identity element in the cohomology ring $H^*(S^n)$. Then by Künneth Theorem, the cohomology group

$$H^n(S^n \times S^n) = H^0 \otimes H^n(S^n) \oplus H^n(S^n) \otimes H^0(S^n)$$

has two generators $z \otimes 1$ and $1 \otimes z$. So $\mu^*(z) = k_1(z \otimes 1) + k_2(1 \otimes z)$ for some integer k_1, k_2. Consider the map $id : S^n \times \{*\} \to S^n$, the induced map in cohomology $id : H^n(S^n) \to$

$H^n(S^n \times \{*\})$ maps z to $z \otimes 1$. On the other hand, the map

$$id \times j : S^n \times \{*\} \to S^n \times S^n$$

induceds a map

$$id \otimes j^* : H^n(S^n \times S^n) \to H^n(S^n \times \{*\})$$

sending $z \otimes 1$ to $z \otimes 1$ and $1 \otimes z$ to 0 because the cohomology group of $H^n(S^n \times \{*\})$ is isomorphic to $H^n(S^n) \otimes H^0(\{*\})$, having no $(0, n)$ part. Since we have a commutative diagram

$$\begin{tikzcd}
S^n \times \{*\} \arrow[r, "id"] \arrow[d, "id\times j"'] & S^n \\
S^n \times S^n \arrow[ur, "\mu"'] &
\end{tikzcd}$$

which induces a commutative diagram in cohomology groups

$$\begin{tikzcd}
H^n(S^n \times \{*\}) & H^n(S^n) \arrow[l, "id"'] \arrow[dl, "\mu^*"] \\
H^n(S^n \times S^n) \arrow[u, "id\times j^*"] &
\end{tikzcd}$$

This proves that $k_1 = 1$. Similarly, we can prove $k_2 = 1$. Thus, $\mu^*(z) = z \otimes 1 + 1 \otimes z$.

(b) We know that in the cohomology ring $H^*(S^n)$, $z \cup z = 0$ for the generator $z \in h^n(S^n)$. The map μ^* is a homomorphism of rings, so

$$\begin{aligned}
0 &= \mu^*(z \cup z) \\
&= \mu^*(z) \cup \mu^*(z) \\
&= (z \otimes 1 + 1 \otimes z) \cup (z \otimes 1 + 1 \otimes z) \\
&= (z \cup z) \otimes 1 + 1 \otimes (z \cup z) + (z \otimes 1) \cup (1 \otimes z) + (1 \otimes z) \cup (z \otimes 1) \\
&= (z \otimes 1) \cup (1 \otimes z) + (1 \otimes z) \cup (z \otimes 1).
\end{aligned}$$

Because the cohomology ring $H^*(S^n \times S^n)$ is graded commutative, so

$$(1 \otimes z) \cup (z \otimes 1) = (-1)^{n^2} (z \otimes 1) \cup (1 \otimes z).$$

This implies that $(-1)^{n^2} = -1$, so n must be odd.

(c) The multiplication $m : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ has no zero divisors, so m can be restricted to a multiplication

$$m : \mathbb{R}^3 - 0 \times \mathbb{R}^3 - 0 \to \mathbb{R}^3 - 0.$$

m defines a map

$$m' : \mathbb{R}^3 - 0 \times \mathbb{R}^3 - 0 \to S^2 \subseteq \mathbb{R}^3 - 0$$

by sending (x, y) to $\frac{m(x,y)}{|m(x,y)|}$. Note that if we restrict m' to $S^2 \times S^2$, then m' gives an unital multiplication $m' : S^2 \times S^2 \to S^2$, but we have proved in (b) that such a multiplication cannot exist in even dimensions.

Problem 5

The relative form of the Künneth Theorem is that there is a natural short exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(X,A) \otimes H_q(Y,B) \to H_n(X \times Y, X \times B \cup A \times Y)$$
$$\to \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(X,A), H_q(Y,B)) \to 0.$$

(a) Suppose that M is an n-manifold and N is a k-manifold, and that we are given local orientations $\mu_M \in H_n(M, M-x)$ and $\mu_N \in H_k(N, N-y)$, for some $x \in M$ and $y \in N$. Explain how to get an induced local orientation for $M \times N$ at (x,y).

(b) Explain how to get an interesting continuous map $\tilde{M} \times \tilde{N} \to \widetilde{M \times N}$, where $\tilde{M}$ is the space of pairs (m, μ_m) such that $m \in M$ and $\mu_m \in H_n(M, M-m)$ is a generator (and similarly for N, etc.). How many points are in each fiber?

(c) Prove that if M and N are orientable then so is $M \times N$ (we are not assuming compactness here).

Solution:

(a) Consider the tensor product $\mu_M \otimes \mu_N \in H_n(M, M-x) \otimes H_k(N, N-y)$. Let

$$f : H_n(M, M-x) \otimes H_k(N, N-y) \to H_{n+k}(M \times N, M \times (N-y) \cup (M-x) \times N)$$

be the map coming from the natural short exact sequence. We need to show that

$$M \times N - (x,y) = M \times (N-y) \cup (M-x) \times N.$$

Indeed, it is easy to see that $M \times (N-y) \subseteq M \times N$, and for any point $(m,n) \in M \times (N-y)$, we know that $n \neq y$, so $(m,n) \neq (x,y)$, this means $(m,n) \in M \times N - (x,y)$. This proves that

$$M \times (N-y) \subseteq M \times N - (x,y).$$

Similarly, we can prove that

$$(M-x) \times N \subseteq M \times N - (x,y).$$

So we have

$$M \times (N-y) \cup (M-x) \times N \subseteq M \times N - (x,y).$$

On the other hand, given a point $(m,n) \in M \times N - (x,y)$, if $m \neq x$, then $(m,n) \in (M-x) \times N$. If $m = x$, then we know that $n \neq y$, otherwise $(m,n) = (x,y)$, so in this case $(m,n) \in M \times (N-y)$. This proves that

$$M \times N - (x,y) \subseteq M \times (N-y) \cup (M-x) \times N.$$

Therefore, we can conclude that

$$M \times N - (x,y) = M \times (N-y) \cup (M-x) \times N$$

and $f(\mu_M \otimes \mu_N)$ is an element of $H_{n+k}(M \times N, M \times N - (x,y))$. It is also a generator because the map

$$f : H_n(M, M-x) \otimes H_k(N, N-y) \to H_{n+k}(M \times N, M \times (N-y) \cup (M-x) \times N)$$

is an isomorphism because the top dimensional homology groups do not have torsion for manifolds. Therefore, $f(\mu_M \otimes \mu_N)$ is a generator, which is the same as a local orientation of $M \times N$ at the point (x,y).

(b) We define a map

$$\begin{gathered} g : \tilde{M} \times \tilde{N} \to \widetilde{M \times N}, \\ ((m, \mu_m), (n, \mu_n)) \mapsto ((m,n), f(\mu_m \otimes \mu_n)). \end{gathered}$$

We have proved in (a) that $f(\mu_m \otimes \mu_n)$ is a local orientation of $M \times N$ at the point (m,n), so this map is well-defined.

Next, we show the map g is continuous. Fix a point $(m,n) \in M \times N$ and choose an open neighborhood $U \times V \subseteq M \times N$ where $U \subseteq M$ and $V \subseteq N$ are open. Let $\tilde{f}$ be the map

$$\tilde{f} : H_n(M, M-U) \otimes H_k(N, N-V) \to H_{n+k}(M \times N, M \times N - U \times V).$$

We have a commutative diagram

$$\begin{array}{ccc} H_n(M, M-U) \otimes H_k(N, N-V) & \xrightarrow{\tilde{f}} & H_{n+k}(M \times N, M \times N - U \times V) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ H_n(M, M-x) \otimes H_k(N, N-y) & \xrightarrow[f]{} & H_{n+k}(M \times N, M \times N - (x,y)) \end{array}$$

All the maps are isomorphisms and the vertical map i, j are induced from the inclusion $M - U \to M - x$ and $N - V \to N - y$. The point $((m,n), f(\mu_m \otimes \mu_n))$ in $\widetilde{M \times N}$ has an open neighborhood

$$W = \{((a,b), \mu) \mid (a,b) \in U \times V, \mu \in H_{n+k}(M \times N, M \times N - U \times V)\}$$

satisfying μ is a generator and $j(\mu) = f(\mu_m \otimes \mu_n)$. Since $\tilde{f}$ is an isomorphism, there exists unique $\nu_U \in H_n(M, M-U)$ and $\nu_V \in H_k(N, N-V)$, both generators, such that

$$\tilde{f}(\nu_U \otimes \nu_V) = \mu.$$

The preimage of W can be written as the disjoint union of $W_1, W_2 \subseteq \tilde{M} \times \tilde{N}$:

$$\begin{aligned} W_1 &= \{((s, \nu_U), (t, \nu_V)) \mid s \in U, t \in V\}, \\ W_2 &= \{((s, -\nu_U), (t, -\nu_V)) \mid s \in U, t \in V\}. \end{aligned}$$

We know that ν_U maps to μ_s for all $s \in U$ by the commutative diagram above. Similar for ν_V. And

$$\tilde{f}(\nu_U \otimes \nu_V) = \tilde{f}(-\nu_U \otimes -\nu_V) = \mu.$$

This proves that $W_1, W_2 \subseteq \tilde{M} \times \tilde{N}$ are open sets and $g^{-1}(W) = W_1 \sqcup W_2$. So the map g is continuous, and we have two points in each fiber.

(c) Consider the covering map $p_M : \tilde{M} \to M$, if M is orientable, then there exists a continuous map $s_M : M \to \tilde{M}$ such that $p_M \circ s_M = id_M$. Similar for N. Now consider the map

$$s_M \times s_N : M \times N \to \tilde{M} \times \tilde{N}.$$

The composition $g \circ (s_M \times s_N) : M \times N \to M \tilde{\times} N$ is a section of the covering map

$$p_M \times p_N : \widetilde{M \times N} \to M \times N.$$

Indeed, we can check that

$$\begin{aligned}
(p_M \times p_N) \circ (g \circ (s_M \times s_N)) &= (p_M \times p_N) \circ g \circ (s_M \times s_N) \\
&= (p_M \circ s_M) \times (p_N \circ s_N) \\
&= id_M \times id_N \\
&= id_{M \times N}
\end{aligned}$$

So the covering map

$$p_{M \times N} : \widetilde{M \times N} \to M \times N$$

has a section, thus the manifold $M \times N$ is orientable.

Consider the space $X = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$, made into a Δ-complex as follows.

Recall that $H^0(X; \mathbb{Z}/2) = H^2(X; \mathbb{Z}/2) = \mathbb{Z}/2$ and $H^1(X; \mathbb{Z}/2) = (\mathbb{Z}/2)^3$. The Universal Coefficients Theorem implies that the standard maps

$$\phi_i : H^i(X; \mathbb{Z}/2) \to \hom(H_i(X; \mathbb{Z}/2), \mathbb{Z}/2)$$

are isomorphisms.

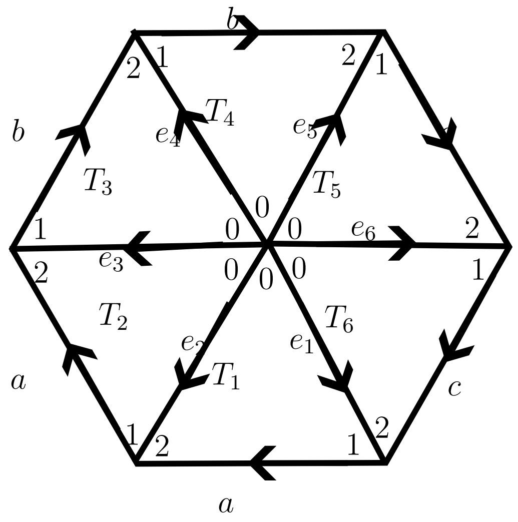
Problem 6

(a) Write down explicit 1-cocycles α, β and γ (with $\mathbb{Z}/2$ coefficients) that map to $\hat{a}$, $\hat{b}$ and $\hat{c}$ under ϕ.

(b) Given a 2-cochain Θ (with $\mathbb{Z}/2$ coefficients), how can one easily determine if Θ is a generator for $H^2(X; \mathbb{Z}/2)$?

(c) Determine a class $u \in H^1(X; \mathbb{Z}/2)$ such that $\alpha \cup u$ is a generator for $H^2(X; \mathbb{Z}/2)$. Then do the same for β and γ.

Solution:



(a) Let $\alpha = \hat{a} + \hat{e_2}$. We check that α is a cocycle and not a coboundary. For $3 \leq i \leq 6$, we know that $(\delta\alpha)(T_i) = 0$ because their boundary does not contain a or e_2. And we have

$$(\delta\alpha)(T_1) = (\hat{a} + \hat{e_2})(e_1 + a - e_2) = 1 - 1 = 0$$
$$(\delta\alpha)(T_2) = (\hat{a} + \hat{e_2})(e_2 + a - e_3) = 1 + 1 = 2 = 0.$$

So α is a cocycle with $\mathbb{Z}/2$-coefficients. On the other hand, suppose there exists $m\hat{x} + n\hat{y}$ for some $m, n \in \mathbb{Z}$ such that $\delta(m\hat{x} + n\hat{y}) = \hat{a} + \hat{e_2}$, then we have

$$1 = (\hat{a} + \hat{e_2})(a) = (\delta(m\hat{x} + n\hat{y}))(a) = (m\hat{x} + n\hat{y})(x - x) = 0$$

A contradiction. So $\hat{a} + \hat{x}$ is a 1-cocycle but not a coboundary. And we know

$$(\hat{a} + \hat{e_2})(a) = 1,$$
$$(\hat{a} + \hat{e_2})(b) = 0,$$
$$(\hat{a} + \hat{e_2})(c) = 0.$$

This implies that $\phi(\alpha) = \hat{a}$ in $\hom(H_1(X; \mathbb{Z}/2), \mathbb{Z}/2)$. By symmetry, we can see that $\beta = \hat{b} + \hat{e_4}$ maps to $\hat{b}$ and $\gamma = \hat{c} + \hat{e_6}$ maps to $\hat{c}$ under ϕ.

(b) We know that ϕ is an isomorphism, so we can check whether Θ is a generator of $H^2(X; \mathbb{Z}/2)$ by checking whether the image $\phi(\Theta)$ is the generator of $\hom(H_2(X; \mathbb{Z}/2), \mathbb{Z}/2)$. We know that the sum $\sum_{i=1}^{6} T_i$ is the generator of the homology group $H_2(X; \mathbb{Z}/2)$, so we can check that whether $\Theta(\sum_{i=1}^{6} T_i) = 1$. If it equals to 1, then Θ is a generator of $H^2(X; \mathbb{Z}/2)$, and if $\Theta(\sum_{i=1}^{6} T_i) = 0$, then Θ is not a generator.

(c) We check that $\alpha \cup \alpha$ is a generator in $H^2(X;\mathbb{Z}/2)$. By calculation, we have

$$
\begin{aligned}
&(\alpha \cup \alpha)(T_1 + \cdots + T_6) \\
=&\alpha(e_1)\alpha(a) + \alpha(e_2)\alpha(a) + \alpha(e_3)\alpha(b) + \alpha(e_4)\alpha(b) + \alpha(e_5)\alpha(c) + \alpha(e_6)\alpha(c) \\
=&\alpha(e_2)\alpha(a) \\
=&1.
\end{aligned}
$$

This implies that $\alpha \cup \alpha$ is a generator of $H^2(X;\mathbb{Z}/2)$. Similarly, $\beta \cup \beta$ and $\gamma \cup \gamma$ are also generators of $H^2(X;\mathbb{Z}/2)$. So we can choose $u = \alpha, \beta, \gamma \in H^1(X;\mathbb{Z}/2)$.