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Course: MATH 636 - Algebraic Topology III
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Homework 3
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Problem 1
Compute both $\mathrm{Tor}_i(A, B)$ and $\mathrm{Ext}^i(A, B)$ for all i in the following cases:

(a) $A = \mathbb{Z}/9$ and $B = \mathbb{Z}/6$.

(b) $A = \mathbb{Z}/9$ and $B = \mathbb{Z}$.

(c) $A = \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/5 \oplus \mathbb{Z}/10$ and $B = \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/6$.

Solution:

Problem 2
Let A be an abelian group and let $G_* \to A \to 0$ be a free resolution. Let B be another abelian group and let $J_* \to B \to 0$ be a free resolution.

(a) Given a map $f : A \to B$, prove that there are maps $F_i : G_i \to J_i$ making all squares commute, we call this chain map $\{F : G_* \to J_*\}$ a lifting of the map f.

(b) Prove that if $\{F' : G_* \to J_*\}$ is another lifting of f then the chain map F and F' are chain homotopic.

(c) If C is another abelian group one gets an induced map $F \otimes id : G_* \otimes C \to J_* \otimes C$ and therefore an induced map on homology groups $f_* : \mathrm{Tor}_i(A, C) \to \mathrm{Tor}_i(B, C)$. Since any two choices of F are homotopic, this f_* is well-defined.
Use the above procedure to calculate the maps

$$\begin{aligned} j_* &: \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) \to \mathrm{Tor}_1(\mathbb{Z}/4, \mathbb{Z}/2), \\ k_* &: \mathrm{Tor}_1(\mathbb{Z}/4, \mathbb{Z}/2) \to \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2). \end{aligned}$$

induced by the map $j : \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ (sending 1 to 2) and $k : \mathbb{Z}/4 \to \mathbb{Z}/2$ (sending 1 to 1).

Solution:

Problem 3.7
If F is a finitely-generated free abelian group then there is a canonical isomorphism

$$\hom(\hom(F, \mathbb{Z}), \mathbb{Z}) \cong F.$$

So if C is a chain complex consisting of finitely generated, free abelian groups, one gets an induced isomorphism

$$\hom(\hom(C, \mathbb{Z}), \mathbb{Z}) \cong C.$$

Using this, derive a universal coefficient theorem which lets you predict $H_*(C)$ if you know $H^*(\hom(C, \mathbb{X}))$.

Solution:

Problem 3.8

In this problem we'll use the abbreviations $H^i(\hom(C, \mathcal{A})) = H^i(C; \mathcal{A})$ and $H^i(C) = H^i(C; \mathbb{Z})$.

If F is a finitely -generated free abelian group then there is a canonical isomorphism

$$\hom(F, \mathcal{A}) \cong \hom(F, \mathbb{Z}) \otimes \mathbb{Z}.$$

So if C is a chain complex consistin of finitely generated free abelian groups, we have an isomorphism

$$\hom(C, \mathcal{A}) \cong \hom(C, \mathbb{Z}) \otimes \mathcal{A}.$$

Using this, derive a universal coefficient theorem which lets you predict $H^*(C; \mathcal{A})$ if you know $H^*(C)$. The formula should look like

$$H^i(C; \mathcal{A}) \cong [H^?(C) \otimes \mathcal{A}] \oplus [\mathrm{Tor}_1(H^?(C), \mathcal{A})]$$

where you determine the indices marked "?".

Solution:

Problem 4

Suppose X is a finite CW complex for which

$$H_0(X; \mathbb{Z}/2) = \mathbb{Z}/2, H_1(X; \mathbb{Z}/2) = (\mathbb{Z}/2)^3, H_2(X; \mathbb{Z}/2) = 0, H_3(X; \mathbb{Z}/2) = H_4(X; \mathbb{Z}/2) = \mathbb{Z}/2$$

and $H_i(X; \mathbb{Z}/2) = 0$ for all $i \geq 5$.

(a) Determine as much as you can about $H_*(X; \mathbb{Z})$.

(b) Suppose you are also told that $H_2(X; \mathbb{Z}/3) = \mathbb{Z}/3$ and $H_3(X; \mathbb{Z}/3) = 0$. What else can you say about $H_*(X; \mathbb{Z})$ now?

(c) Suppose Y is a space with finitely-generated homology groups and you are told $H_i(Y; \mathbb{Z}/p) = 0$ for a specific prime p. What can you deduce about $H_i(Y)$ and $H_{i-1}(Y)$?

Solution:

Problem 5

(a)