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Homework - Week 1 Exercises
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Exercise 1.2
The Cantor set $\mathcal{C}$ can also be described in terms of ternary expansions.

(a) Every number in $[0,1]$ has a ternary expansion

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}, \qquad \text{where } a_k = 0, 1, 2$$

Note that this decomposition is not unique since, for example,

$$\frac{1}{3} = \sum_{k=2}^{\infty} \frac{2}{3^k}.$$

Prove that $x \in \mathcal{C}$ if and only if x has a representation as above where every a_k is either 0 or 2.

(b) The **Cantor-Lebesgue function** is defined on $\mathcal{C}$ by

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k}, \quad \text{if } x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \quad \text{where } b_k = \frac{a_k}{2}.$$

In this definition, we choose the expansion of x in which $a_k = 0$ or 2. Show that F is well-defined and continuous on $\mathcal{C}$, and moreover, $F(0) = 0$ as well as $F(1) = 1$.

(c) Prove that $F : \mathcal{C} \to [0,1]$ is surjective, that is, for every $y \in [0,1]$, there exists $x \in \mathcal{C}$ such that $F(x) = y$.

(d) One can also extend F to be a continuous function on $[0,1]$ as follows. Note that if (a,b) is an open interval of the complement of $\mathcal{C}$, then $F(a) = F(b)$. Hence, we may define F to have the constant value $F(a)$ in that interval.

Solution:

(a) Let $x \in \mathcal{C} = \cap_{k=0}^{\infty} C_k$. $x \in \mathcal{C}$ is the same as $x \in C_k$ for all $k \geq 0$. Let us start with $k = 1$. We choose $a_1 = 0$ or $a_1 = 2$ based on x is in which interval of C_1. More specifically, if $0 \leq x \leq \frac{1}{3}$, choose $a_1 = 0$. If $\frac{2}{3} \leq x \leq 1$, choose $a_1 = 2$. For $k = 2$, the interval containing x is again become two parts. If x is in the smaller interval (the one closer to 0), choose $a_2 = 0$. If x is in the larger interval (the one closer to 1), choose $a_2 = 2$. Repeat this process, and in each step n, we write

$$x_n = \sum_{k=1}^{n} \frac{a_k}{3^k}.$$

We need to show that $\lim_{n\to\infty} x_n = x$. Note that by the way we choose each a_k, we have

$$x_n \leq x \leq x_n + \frac{3}{3^n}.$$

Let n approaches ∞, and we have proved that

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

where everywhere a_k is either 0 or 2.

Conversely, assume x has such a representation. We need to show that $x \in C_n$ for all $n \geq 1$. For every $n \geq 1$, write

$$x_n = \sum_{k=1}^{n} \frac{a_k}{3^k}.$$

We know x_n is an end point in C_n, and x satisfies

$$x_n \leq x \leq x_n + \frac{3}{3^n}.$$

Thus, $x \in C_n$ for every n. This proves that $x \in \mathcal{C}$.

(b) From (1), we know every point $x \in \mathcal{C}$ has a representation

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

where a_k is 0 or 2. By the way we define the representation, the choice is a_k is unique, so this function F is well-defined. Moreover, if $x = 0$, then all a_k in its representation is 0, so

$$F(0) = \sum_{k=1}^{\infty} \frac{0}{2^k} = 0.$$

Similarly, if $x = 1$, then all a_k in its representation is 2, so

$$F(1) = F(\sum_{k=1}^{\infty} \frac{2}{3^k}) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

Lastly, we need to show that F is continuous. Fix a point $x \in \mathcal{C}$, for any $\varepsilon > 0$, there exists large enough $n \in \mathbb{Z}_+$ such that

$$\frac{1}{3^{n+1}} \leq \frac{1}{2^n} < \varepsilon.$$

Consider the open set

$$U = (x - \frac{1}{3^{n+1}}, x + \frac{1}{3^{n+1}}) \cap \mathcal{C} \subseteq \mathcal{C}.$$

For any $y \in U$, we know that $|x - y| < \frac{1}{3^n}$, so x, y must belong to the same interval for $C_1, C_2, \ldots, C_n$. This implies that in the representations of x and y, the choices of a_k for

$1 \le k \le n$ are the same. Therefore, we have

$$\begin{aligned}|F(x) - F(y)| &\le \sum_{k=n+1}^{\infty} \frac{1}{2^k} \\ &\le 1 - (\sum_{k=1}^{n} \frac{1}{2^k}) \\ &\le 1 - (1 - \frac{1}{2^n}) \\ &\le \frac{1}{2^n} \\ &< \varepsilon.\end{aligned}$$

This proves that F is continuous.

(c) For any $y \in [0,1]$, if y has a representation

$$y = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$$

where b_k is 0 or 1, then we can choose $a_k = 2b_k$ and obtain a number $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k} \in \mathcal{C}$ satisfying $F(x) = y$. So we only need to show that every $y \in [0,1]$ has such a representation. 0 has a representation by setting all $b_k = 0$. Now assume $y \in (0,1]$. Let's start with b_1. If $0 < y \le \frac{1}{2}$, choose $b_1 = 0$. If $\frac{1}{2} < y \le 1$, choose $b_1 = 1$. For b_2, divide the interval y was in into 2 parts again, and if y is in the smaller interval, choose $b_2 = 0$, if y is in the larger interval, choose $b_2 = 1$. Repeat this process, and we obtain a sequence $\sum_{k=1}^{\infty} \frac{b_k}{2^k}$. Note that by our choice, for all $n \ge 1$, we have

$$\sum_{k=1}^{n} \frac{b_k}{2^k} < y \le \sum_{k=1}^{n} \frac{b_k}{2^k} + \frac{2}{2^n}.$$

Thus, we obtain a representation for $y \in [0,1]$.

(d) We need to show that if we extend F in this way, it is still continuous. We use a very similar proof as (2). In this case, we just choose the open neighborhood of x as

$$U = (x - \frac{1}{3^{n+1}}, x + \frac{1}{3^{n+1}}).$$

For any $y \in U$, if $y \in \mathcal{C}$, then the same proof works. If $y \notin \mathcal{C}$, note that the complement of Cantor set $\mathcal{C}$ is a countable union of disjoint open intervals, so $y \in (a,b)$ for some $a, b \in \mathcal{C}$. We have

$$|F(x) - F(y)| = |F(x) - F(a)| = |F(x) - F(b)|.$$

Thus, we could use the same steps as in (2).

Exercise 1.5
Suppose E is a given set, and $\mathcal{O}_n$ is the open set:

$$\mathcal{O}_n = \left\{x : \ d(x, E) < \frac{1}{n}\right\}.$$

Show:

(a) If E is compact, then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$.

(b) However, the conclusion in (a) may be false for E closed and unbounded; or E open and bounded.

Solution:

(a) E is closed and bounded, so E is measurable and $m(E) < +\infty$. For every n, $d(\mathcal{O}_n, E) < \frac{1}{n}$, so $\mathcal{O}_n$ is also bounded, and we have

$$\mathcal{O}_1 \supseteq \mathcal{O}_2 \supseteq \cdots \supseteq \mathcal{O}_n \supseteq \cdots \supseteq E.$$

It is obvious that $E \subseteq \cap_{n=1}^{\infty} \mathcal{O}_n$. Conversely, for any $x \in \cap_{n=1}^{\infty} \mathcal{O}_n$, $d(x, E) < \frac{1}{n}$ for all $n \geq 1$. For every $n \geq 1$, we can find $x_n \in E$ such that $|x_n - x| < \frac{1}{n}$. Let $n \to \infty$, and we have $\lim_{n\to\infty} x_n = x$. This proves that x is a limit point of E, and because E is closed, so $x \in E$. Therefore, $\cap_{n=1}^{\infty} \mathcal{O}_n = E$.

Now, we show that E is open. For any $x \in \mathcal{O}_n$, choose an open ball $B_r(x)$ centered at x with radius

$$r = \frac{1}{3}(\frac{1}{n} - d(x, E)).$$

Then for any $y \in B_r(x)$, we have

$$\begin{aligned}
d(y, E) &= \inf_{z\in E} |y - z| \\
&\leq \inf_{z\in E} (|y - x| + |x - z|) \\
&\leq r + \inf_{z\in E} |x - z| \\
&= r + d(x, E) \\
&\leq \frac{1}{3n} + \frac{2}{3} d(x, E) \\
&< \frac{1}{n}.
\end{aligned}$$

This proves $y \in \mathcal{O}_n$. So $\mathcal{O}_n$ is an open set, thus measurable and $m(\mathcal{O}_n) < +\infty$. By Corollary 3.3 (ii), we have

$$m(E) = \lim_{n\to\infty} m(\mathcal{O}_n).$$

(b) Consider

$$E = \{1, 2, 3, \ldots\}.$$

E is closed and unbounded. We know that $m(E) = 0$ because E is countable. Then

$$\mathcal{O}_n = \cup_{k=1}^{\infty}(k - \frac{1}{n}, k + \frac{1}{n}).$$

We know that for each k, $m((k - \frac{1}{n}, k + \frac{1}{n})) = \frac{2}{n}$, so

$$m(\mathcal{O}_n) - m(E) > m(\cup_{k=1}^{n}(k - \frac{1}{n}, k + \frac{1}{n})) = 2.$$

This implies

$$\lim_{n\to\infty} m(\mathcal{O}_n) \neq m(E).$$

Next, note that $\mathbb{Q} \cap (0, 1)$ is countable, so it can be listed as a sequence $\{x_n\}_{n=1}^{\infty}$. Let

$$E = \bigcup_{n=1}^{\infty}(x_n - \frac{1}{2^{n+2}}, x_n + \frac{1}{2^{n+2}})$$

E is open, thus measurable, and

$$m(E) \leq \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}.$$

For every $n \geq 1$, $\mathcal{O}_n$ must contain $[0, 1]$ because the rational numbers are dense in $(0, 1)$. So $m(\mathcal{O}_n) \geq 1 > \frac{1}{2} \geq m(E)$ for all $n \geq 1$. This implies that

$$\lim_{n\to\infty} m(\mathcal{O}_n) \neq m(E).$$

Exercise 1.6
Using translations and dilations, prove the following: Let B be a ball in $\mathbb{R}^d$ of radius r. Then $m(B) = v_d r^d$, where $v_d = m(B_1)$, and B_1 is the unit ball.

$$B_1 = \left\{x \in \mathbb{R}^d : |x| < 1\right\}.$$

Solution: For any $\varepsilon > 0$, there exists a countable union of almost disjoint closed cubes $\{Q_j\}_{j=1}^{\infty}$ such that

$$B_1 \subseteq \bigcup_{j=1}^{\infty} Q_j$$

and

$$|m(B_1) - \sum_{j=1}^{\infty} |Q_j|| < \frac{\varepsilon}{r^d}.$$

For each $j \geq 1$, suppose Q_j is a closed cube centered at $(x_1, \ldots, x_d)$ with side length l. Define $\widetilde{Q_j}$ as the closed cube centered at $(rx_1, \ldots, rx_n)$ with side length rl. We obtain a new sequence $\left\{\widetilde{Q_j}\right\}_{j=1}^{\infty}$. We claim they are still almost disjoint. Indeed, for every point $T = (t_1, \ldots, t_d)$ in a cube Q_j, the new corresponding point has the coordinate $\tilde{T} = (rt_1, \ldots, rt_n)$. If $\tilde{T}$ is in the interior of two different new cubes, then T must be also in the interior of two different old cubes because both the coordinates and the side length are multiplied by r in this process. For every point $(y_1, \ldots, y_n) \in B$, $(\frac{y_1}{r}, \ldots, \frac{y_n}{r})$ is a point in B_1. So

$$\bigcup_{j=1}^{\infty} \widetilde{Q_j} \supset B$$

is a cover for B as

$$\bigcup_{j=1}^{\infty} Q_j \supset B_1$$

is a cover for B_1. Moreover, for every j, we have

$$|\widetilde{Q_j}| = r^d |Q_j|$$

by definition. So

$$|m(B) - \sum_{j=1}^{\infty} |\widetilde{Q_j}|| < \frac{\varepsilon}{r^d} \cdot r^d = \varepsilon.$$

This implies that

$$m(B) = r^d m(B_1) = v_d r^d.$$

Exercise 1.9
Give an example of an open set $\mathcal{O}$ with the following property: the boundary of the closure of $\mathcal{O}$ has positive Lebesgue measure.

Solution: Consider the example we constructed in Exercise 1.5 (b). The open set

$$E = \bigcup_{n=1}^{\infty} (x_n - \frac{1}{2^{n+2}}, x_n + \frac{1}{2^{n+2}})$$

where $\{x_n\}_{n=1}^{\infty}$ is a sequence of all rational numbers in $(0, 1)$. We have

$$m(E) \leq \frac{1}{2}.$$

On the other hand, the closure $\bar{E}$ must contain $[0, 1]$ because the rational numbers are dense in $(0, 1)$. Thus, the boundary $\bar{E} - E$ must have positive Lebesgue measure.

Exercise 1.10
Let $\hat{C}$ denote a Canton-like set, in particular $m(\hat{C}) > 0$. Let F_1 denote a piecewise linear and continuous function on $[0, 1]$, with $F_1 = 1$ in the complement of the first interval removed in the construction of $\hat{C}$, $F_1 = 0$ at the center of this interval, and $0 \le F_1(x) \le 1$ for all x. Similarly, construct $F_2 = 1$ in the complement of the intervals in stage two of the construction of $\hat{C}$, with $F_2 = 0$ at the center of these intervals, and $0 \le F_2 \le 1$. Continuing this way, let $f_n = F_1 \cdot F_2 \cdots F_n$. Prove the following:

(a) For all $n \ge 1$ and all $x \in [0, 1]$, one has $0 \le f_n(x) \le 1$ and $f_n(x) \ge f_{n+1}(x)$. Therefore, $f_n(x)$ converges to a limit as $n \to \infty$ which we denote by $f(x)$.

(b) The function is discontinuous at every point of $\hat{C}$.

Solution:

(a) For all $n \ge 1$ and all $x \in [0, 1]$, we know that by definition $0 \le F_n(x) \le 1$. So we have

$$0 \le f_n(x) = F_1(x) \cdot F_2(x) \cdots F_n(x) \le 1.$$

Moreover,

$$f_{n+1}(x) = f_n(x) \cdot F_{n+1}(x) \le f_n(x).$$

This implies that $f_n(x)$ is a bounded decreasing sequence, so that $\lim_{n\to\infty} f_n(x) = f(x)$ exists.

(b) Let $x \in \hat{C}$. By definition, $f_n(x) = 1$ for all $n \ge 1$, so $f(x) = 1$. For every $n \ge 1$, denote by $\hat{C}_n$ the set obtained in nth stage of the construction of $\hat{C}$. Consider the open set $(x - \frac{2}{3^n}, x + \frac{2}{3^n})$, it must contain one of the center of a removed interval in the nth stage, because the farthest possible distance between the center and a point in $\hat{C}$ is $\frac{1}{2^n} < \frac{2}{3^n}$. Choose x_n equal to this center, we have $f(x_n) = 0$ for all $n \ge 1$ by definition. Note that $|x_n - x| < 2 \cdot \frac{2}{3^n}$. Let $n \to \infty$. We get a sequence $\{x_n\}_{n=1}^{\infty}$ converging to x but $0 = f(x_n)$ does not converge to $f(x) = 1$. This implies that f is not continuous at $x \in \hat{C}$.

Exercise 1.13
The following deals with G_δ and F_σ sets.

(a) Show that a closed set is a G_δ and an open set an F_σ.

(b) Give an example of an F_σ which is not a G_δ.

(c) Give an example of a Borel set which is not a G_δ nor an F_σ.

Solution:

(a) Let O be an open set. By Theorem 1.4, O can be written as the union of closed cubes, so O is an F_σ. Let E be a closed set. Then E^c is an open set. By Theorem 1.4, we can write

$$E^c = \bigcup_{j=1}^{\infty} Q_j$$

where Q_j is a closed cube for all $j \geq 1$. Take the complement, and we have

$$E = (E^c)^c = (\bigcup_{j=1}^{\infty} Q_j)^c = \bigcap_{j=1}^{\infty} Q_j^c$$

where Q_j^c is open for all $j \geq 1$. This proves that any closed set is a G_δ.

(b) Consider all the rational numbers

$$E = \mathbb{Q} \cap [0, 1]$$

in the open interval $[0, 1]$. E is countable, so it can be written as the countable union of closed sets, where each closed set is the singleton. This implies that E is a F_σ. Suppose E is a G_δ, then there exists a sequence of open sets $\{O_n\}_{n=1}^{\infty}$ such that

$$E = \bigcap_{n=1}^{\infty} O_n.$$

Take the closure at both sides, and since the rational numbers are dense in $[0, 1]$, we have

$$[0, 1] = \bigcap_{n=1}^{\infty} \overline{O_n}.$$

This means for $n \geq 1$, every $\overline{O_n}$ must contain $[0, 1]$. Choose

$$O'_n = O_1 \cap O_2 \cap \cdots \cap O_n.$$

Then $\{O'_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets and $\bigcap_{n=1}^{\infty} O'_n = E$. Each O'_n is measurable, and has finite Lebesgue measure. By Corollary 3.3 (ii), we have

$$0 = m(E) = \lim_{n \to \infty} m(O'_n) \geq m([0, 1]) = 1.$$

A contradiction. So E is not a G_δ.

(c) Let $E' = \mathbb{Q} \cap [-1, 0]$. A similar argument as above shows that E' is an F_σ but not a G_δ. Take $F = [-1, 0] - E'$. Then F is a G_δ but not an F_σ. Consider the set $E \cup F$. E is the set is all rational numbers in $[0, 1]$, so it is countable and thus a Borel set. Similarly, F is the complement of a Borel set, so F is also a Borel set. This implies that $E \cup F$ is also a Borel set. Suppose $E \cup F$ is a G_δ. We can write

$$E \cup F = \bigcup_{j=1}^{\infty} O_j$$

where O_j is open for all $j \geq 1$. Note that $[0, 1]$ is also a G_δ, so we can write

$$[0, 1] = \bigcup_{k=1}^{\infty} P_k$$

where P_k is open for all $k \geq 1$. Then

$$\begin{aligned} E &= (E \cup F) \cap [0,1] \\ &= (\bigcup_{j=1}^{\infty} O_j) \cap (\bigcup_{k=1}^{\infty} P_k) \\ &= \bigcup_{j,k=1}^{\infty} O_j \cap P_k. \end{aligned}$$

Here $O_j \cap P_k$ is open for any $j, k \geq 1$. This is a contradiction because E is not a G_δ. A similar argument can show that $E \cup F$ is also not an F_σ.