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Course: MATH 649 - Abstract Algebra
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Homework - Week 3
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Problem 11.5.5
Let $\mathbb{K}/\Bbbk$ be a Galois extension, and $\mathbb{L}$, $\mathbb{M}$ be intermediate fields. Denote by $\mathbb{L} \vee \mathbb{M}$ the minimal subfield of $\mathbb{K}$ containing $\mathbb{L}$ and $\mathbb{M}$.

(a) $(\mathbb{L} \cap \mathbb{M})^* = \langle \mathbb{L}^*, \mathbb{M}^* \rangle$.

(b) $(\mathbb{L} \vee \mathbb{M})^* = \mathbb{L}^* \cap \mathbb{M}^*$.

(c) Assume that $\mathbb{L}/\Bbbk$ is normal. Then $\mathrm{Gal}(\mathbb{L} \vee \mathbb{M}/\mathbb{M}) \cong \mathrm{Gal}(\mathbb{L}/(\mathbb{L} \cap \mathbb{M}))$.

Solution:

(a) We know that $L \cap M \subseteq L$, by the Galois correspondence, we have $L^* \subseteq (L \cap M)^*$. Similarly, we can see that $M^* \subseteq (L \cap M)^*$. Note that $\langle L^*, M^* \rangle$ is the smallest subgroup containing L^* and M^*. This implies $(L \cap M)^*$ contains $\langle L^*, M^* \rangle$. On the other hand, suppose $a \in \mathbb{K}$ is fixed by every element in the group $\langle L^*, M^* \rangle$, so a is invariant under every element in L^* and M^*. This is the same as $a \in L$ and $a \in M$, so $a \in L \cap M$. This proves $\langle L^*, M^* \rangle^* \subseteq L \cap M$, by Galois correspondence, we have $(L \cap M)^* \subseteq \langle L^*, M^* \rangle$. Thus, we can conclude that $(L \cap M)^* = \langle L^*, M^* \rangle$.

(b) By definition, we know that $L \vee M \supseteq L$ and $L \vee M \supseteq M$, by Galois correspondence, we have $(L \vee M)^* \subseteq L^*$ and $(L \vee M)^* \subseteq M^*$, so $(L \vee M)^* \subseteq L^* \cap M^*$. On the other hand, $L^* \cap M^* \subseteq L^*$ and $L^* \cap M^* \subseteq M^*$, by Galois correspondence, we have $(L^* \cap M^*)^* \supseteq L$ and $(L^* \cap M^*)^* \supseteq M$. Note that $L \vee M$ is the smallest subfield containing L and M, so $(L^* \cap M^*)^* \supseteq L \vee M$, by Galois correspondence, we have $L^* \cap M^* \subseteq (L \vee M)^*$. Thus, we can conclude that $(L \vee M)^* = L^* \cap M^*$.

(c) Consider the field extension $\mathbb{L}/(\mathbb{L} \cap \mathbb{M})/\Bbbk$. We know $\mathbb{L}/\Bbbk$ is normal, so $\mathbb{L}/\mathbb{L} \cap \mathbb{M}$ is also normal. The Galois correspondence and the isomorphisms in (a) and (b) give us two graphs as follows

$$\begin{tikzcd}
& \Bbbk \arrow[d, dash] & \\
& \mathbb{L} \cap \mathbb{M} \arrow[dl, dash, "\text{normal}"'] \arrow[dr, dash] & \\
\mathbb{L} \arrow[dr, dash] & & \mathbb{M} \arrow[dl, dash] \\
& \mathbb{L} \vee \mathbb{M} \arrow[d, dash] & \\
& \mathbb{K} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathrm{Gal}(\mathbb{K}/\Bbbk) \arrow[d, dash] & \\
& \langle \mathbb{L}^*, \mathbb{M}^* \rangle \arrow[dl, dash, "\text{normal}"'] \arrow[dr, dash] & \\
\mathbb{L}^* \arrow[dr, dash] & & \mathbb{M}^* \arrow[dl, dash] \\
& \mathbb{L}^* \cap \mathbb{M}^* \arrow[d, dash] & \\
& \{e\} &
\end{tikzcd}$$

By the second isomorphism theorems in groups, we know that $\mathbb{L}^* \cap \mathbb{M}^*$ is normal in $\mathbb{M}^*$ and we have an isomorphism

$$\langle \mathbb{L}^*, \mathbb{M}^* \rangle / \mathbb{L}^* \cong \mathbb{M}^* / \mathbb{L}^* \cap \mathbb{M}^*.$$

Apply the Galois correspondence again, and we have

$$(\mathbb{L} \cap \mathbb{M})^* / \mathbb{L}^* \cong \mathrm{Gal}(\mathbb{L} / \mathbb{L} \cap \mathbb{M}) \cong (\mathbb{L} \vee \mathbb{M})^* / \mathbb{M}^* \cong \mathrm{Gal}(\mathbb{L} \vee \mathbb{M} / \mathbb{M}).$$

Problem 11.5.6
Let $\mathbb{K}/\mathbb{k}$ be a finite Galois extension and p be a prime number.

(a) $\mathbb{K}$ has an intermediate subfield $\mathbb{L}$ such that $[\mathbb{K} : \mathbb{L}]$ is a prime power.

(b) If $\mathbb{L}_1$ and $\mathbb{L}_{\neq}$ are intermediate subfields with $[\mathbb{K} : \mathbb{L}_1]$, $[\mathbb{K} : \mathbb{L}_2]$ both p-powers, and $[\mathbb{L}_1 : \mathbb{k}]$, $[\mathbb{L}_2 : \mathbb{k}]$ both prime to p, then $\mathbb{L}_1$ is $\mathbb{L}_1$ is $\mathbb{k}$-isomorphic to $\mathbb{L}_2$.

Solution:

Problem 11.5.7
Let $f \in \mathbb{k}[x]$, $\mathbb{K}/\mathbb{k}$ be a splitting field for f over $\mathbb{k}$, and $G := \mathrm{Gal}(\mathbb{K}/\mathbb{k})$.

1. G acts on the set of the roots of f.
2. G acts transitively if f is irreducible.
3. If f has no multiple roots and G acts transitively then f is irreducible.

Solution:

Problem 11.6.2
Let $\mathbb{k}$ be a field, $p(x)$ be an irreducible polynomial in $\mathbb{k}[x]$ of degree n, and let $\mathbb{K}$ be a Galois extension of $\mathbb{k}$ containing a root α of $p(x)$. Let $G = \mathrm{Gal}(\mathbb{K}/\mathbb{k})$, and G_α be the set of all $\sigma \in G$ with $\sigma(\alpha) = \alpha$. Then:

(a) $[G : G_\alpha] = n$;

(b) $G_\alpha^* = \mathbb{k}(\alpha)$;

(c) If G_α is normal in G then $p(x)$ splits in the fixed field of G_α.

Solution:

Problem 11.6.3
Let $\mathbb{k}(\alpha)/\mathbb{k}$ be a field extension obtained by adjoining a root α of an irreducible separable polynomial $f \in \mathbb{k}[x]$. Then there exists an intermediate field $\mathbb{k} \subseteq \mathbb{F} \subseteq \mathbb{k}(\alpha)$ if and only if

Gal(f; $\mathbb{k}$) is imprimitive (as a permutation group on the roots), in which case $\mathbb{F}$ can be chosen so that $[\mathbb{F} : \mathbb{k}]$ is equal to the number of imprimitive blocks.

Solution:

Problem 11.6.6
Find all subfields of the splitting field of $x^3 - 7$ over $\mathbb{Q}$. Which of the subfields are normal over $\mathbb{Q}$?

Solution:

Problem 11.6.7
Let $\mathbb{K}$ be a splitting field for $x^4 + 6x^2 + 5$ over $\mathbb{Q}$. Find subfields of $\mathbb{K}$.

Solution: