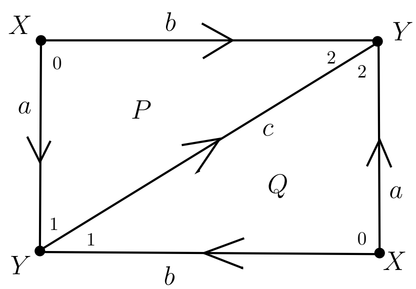
Math 636 Homework #1
Due Friday, April 11

1. Compute the cohomology groups of the following spaces X by writing down explicit cochain complexes (e.g., cellular cochain complexes) and computing the cohomology directly. In each case, compute both $H^*(X)$ and $H^*(X;\mathbb{Z}/2)$.
   (a) S^n
   (b) The genus g torus
   (c) $\mathbb{R}P^n$ and $\mathbb{C}P^n$
   (d) The space $\mathbb{R}P^2\#\mathbb{R}P^2\#\cdots\#\mathbb{R}P^2$ (g copies).

2. The picture below shows a Δ-complex X (note that $X \cong \mathbb{R}P^2$). Note that there are two 0-simplices X and Y, three 1-simplices a, b, and c, and two 2-simplices P and Q.



Write $\hat{X}, \hat{Y}$ for the basis of $C^0(X;\mathbb{Z})$ that is dual to X, Y, and similarly for $\hat{a}, \hat{b}, \hat{c} \in C^1(X;\mathbb{Z})$ and $\hat{P}, \hat{Q} \in C^2(X;\mathbb{Z})$. In this problem we will also work with $C^*(X;\mathbb{Z}/2)$, and $\hat{P}$ could denote either the cochain in $C^2(X;\mathbb{Z})$ or in $C^2(X;\mathbb{Z}/2)$; the precise meaning should always be clear from context.

Answer each of the following questions (and give brief explanations). It will be better for you in the long run to reason out the answers without writing down the cochain complex $C^*_\Delta(X)$ and the matrices for the coboundary operators.

   (a) Is $\hat{a}+\hat{b}+\hat{c}$ a cocycle? What about $\hat{a}+\hat{c}$ and $\hat{a}+\hat{b}$?
   (b) Are the answers to (a) different in $C^*(X;\mathbb{Z}/2)$?
   (c) Cellular cohomology readily shows that $H^1(\mathbb{R}P^2) = 0$, so $H^1(X) = 0$. This means that the cocycle you found in (a) is a coboundary. Find a 0-cochain σ such that $\delta(\sigma)$ is your cocycle.
   (d) Is $\hat{P}$ a coboundary? What about $\hat{P}+\hat{Q}$? What about $\hat{P}+3\hat{Q}$?
   (e) Is it true that $[\hat{P}] = [\hat{Q}]$ in $H^2(X)$?
   (f) Find a 1-cocycle that generates $H^1(X;\mathbb{Z}/2)$.
   (g) We have the subspace $S^1 \subseteq X$ consisting of the 1-cells a and b. We can restrict $\hat{a}$ and $\hat{b}$ to this S^1, and we will call those by the same names. Complete the following:

$$[p\hat{a}+q\hat{b}] = 0 \text{ in } H^1(S^1) \text{ if and only if } \underline{\hspace{3cm}}$$

   (give a condition involving the integers p and q).
   (h) Analyze the map $H^1(X;\mathbb{Z}/2) \to H^1(S^1;\mathbb{Z}/2)$. Is it injective? Surjective?
   (i) Given a cochain $\alpha \in C^1(X;\mathbb{Z}/2) = \mathrm{Hom}(C_1(X),\mathbb{Z}/2)$ and a chain $v \in C_1(X)$ we can evaluate α on v to get an elemenet $\alpha(v) \in \mathbb{Z}/2$. True or False: If $v = [b]+[c]$ then for all cocycles $\alpha, \alpha' \in C^1(X;\mathbb{Z}/2)$ such that $[\alpha] = [\alpha']$ in $H^1(X;\mathbb{Z}/2)$ we have $\alpha(v) = \alpha'(v)$.

(j) Re-do the previous part if $v = [b] - [a]$.

(k) If $[\alpha] = [\alpha']$ in $H^1(X;\mathbb{Z}/2)$ then $\alpha - \alpha' = \delta(\beta)$ for some $\beta \in C^0(X;\mathbb{Z}/2)$. What would you need to know about v in order to show that $\alpha(v) = \alpha'(v)$, no matter what β is?

3. For an abelian group B, write $B[0]$ for the chain complex which has B in degree zero and zeros everywhere else. Write $\Sigma^n B[0]$ for the result of shifting this complex up n spots, so that it has B in degree n.

   Let C_* be a chain complex of abelian groups. Prove that $\alpha \in \mathrm{Hom}(C_n, B)$ is a cocycle in $\mathrm{Hom}(C_*, B)$ if and only if $\alpha\colon C_* \to \Sigma^n B[0]$ is a chain map. Then prove that the set of chain homotopy classes of maps $[C_*, \Sigma^n B[0]]$ is in bijective correspondence with $H^n(\mathrm{Hom}(C_*, B))$

4. Consider a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_3 \arrow[r, "d_A"] \arrow[d, "f_3"'] & A_2 \arrow[r, "d_A"] \arrow[d, "f_2"'] & A_1 \arrow[r] \arrow[d, "f_1"'] & 0 \\
0 \arrow[r] & B_3 \arrow[r, "d_B"] \arrow[d, "g_3"'] & B_2 \arrow[r, "d_B"] \arrow[d, "g_2"'] & B_1 \arrow[r] \arrow[d, "g_1"'] & 0 \\
0 \arrow[r] & C_3 \arrow[r, "d_C"] \arrow[d] & C_2 \arrow[r, "d_C"] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

   in which every row and every column is a chain complex. Assume that all the homology groups in each row are zero except for $H_3(C)$. Also assume that all the homology groups of the columns are zero except for the kernel of the map labelled f_1. Prove that $\ker f_1 \cong H_3(C)$. (Hint: Diagram chase.)

   [Note: I don't want you to write a 10 page proof of this. Try to be efficient in tackling the various things that need to be shown.]