Zhengdong Zhang
Email: zhengz@uoregon.edu
Course: MATH 634 - Algebraic Topology
Instructor: Dr.Patricia Hersh

Homework - Week 4
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Problem 2.1.15
For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective. Hence for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.

Solution: Assume $C = 0$. Then $0 \to D \to E$ implies that $\ker(D \to E) = \mathrm{im}(0 \to D) = 0$. So$D \to E$ is injective. Similarly, $\mathrm{im}(A \to B) = \ker(B \to 0) = B$ implies that $A \to B$ is surjective. Conversely, $A \to B$ is surjective implies that $B = \mathrm{im}(A \to B) = \ker(B \to C)$. So $B \to C$ is the zero map. And $\mathrm{im}(B \to C) = \ker(C \to D)$ while $C \to D$ is also the zero map because $\mathrm{im}(C \to D) = \ker(D \to E) = 0$ as $D \to E$ is injective. So $\mathrm{im}(B$
$rightarrowC) = 0$. Thus, we have $C = 0$.

Let (A, X) be a pair of spaces. Consider the short exact sequence of chain complexes:

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X, A) \to 0.$$

We have a long exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A) \arrow[dll] & \\
& H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & H_{n-1}(X, A) \arrow[r] & \cdots
\end{tikzcd}$$

where $H_n(A, X) = 0$ for all n if and only if $H_n(A) \xrightarrow{\sim} H_n(X)$ for all n.

Problem 2.1.16
Show that $H_0(X, A) = 0$ iff A meets each path-component of X.

Solution: As the Exercise above shows that we have a long exact sequence:

$$\cdots \longrightarrow H_0(A) \longrightarrow H_0(X) \longrightarrow H_0(X, A) \longrightarrow 0$$

So $H_0(X, A) = 0$ if and only if $H_0(A) \to H_0(X)$ is surjective. This map is induced by the inclusion $A \hookrightarrow X$. For a topological space X, we have proved in class that $H_0(X) \cong \bigoplus \mathbb{Z}^d$ where each copy of $\mathbb{Z}$ represents a path-component of X. So $H_0(A) \to H_0(X) \cong \bigoplus \mathbb{Z}^d$ is surjective if and only if for each path-component of X, there exist a 0-chain $a \in C_0(A)$ such that a is generated by points in that path-component. This is the same as saying A meets each path-component of X.