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Course: MATH 681 - Algebraic Geometry I
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Homework - Chapter 1 Exercises
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Exercise 1.1
Show that $(y-x^2)$ is prime, and hence radical. Let $\alpha \in k$ and let $\mathfrak{a} = (y-x^2, y-\alpha x)$. Show that $\mathfrak{a}$ is a radical ideal when $\alpha \neq 0$, but not when $\alpha = 0$.

Solution: Consider the quotient ring $k[x,y]/(y-x^2) \cong k[x,x^2] \cong k[x]$. It is a domain, so the ideal $(y-x^2)$ is a prime ideal.

For the ideal $\mathfrak{a}$, the quotient ring

$$\begin{aligned} k[x,y]/\mathfrak{a} &\cong k[x,y]/(y-x^2, y-\alpha x) \\ &\cong k[x,x^2]/(x^2-\alpha x) \\ &\cong k[x]/(x(x-\alpha)) \end{aligned}$$

If $\alpha = 0$, then $k[x,y]/\mathfrak{a}$ is not reduced as x is a nilpotent element in $k[x]/(x^2)$, so the ideal $\mathfrak{a} = (y-x^2, y-\alpha x)$ is not radical. If $\alpha \neq 0$, then the ring

$$k[x]/(x(x-\alpha)) \cong k[x]/(x) \oplus k[x]/(x-\alpha) \cong k^2$$

has no nilpotent elements, so the ideal $\mathfrak{a}$ is radical.

Exercise 1.2
Show that a cubic curve in $\mathbb{A}^2(\mathbb{C})$ defined by an equation with real coefficients always has real points. Generalize to curves with real equations of odd degrees.

Solution: Let $f \in \mathbb{C}[x,y]$ be a polynomial with real coefficients. Suppose $\deg f = d$ is odd. Write $X = Z(f) \subseteq \mathbb{A}^2(\mathbb{C})$ to be the curve defined by f. Consider the intersection of X with $Y = Z(x-y)$. Then the points $(x,x) \in X \cap Y$ are the solutions to the following equation $f(x,x) = 0$. Since d is odd, $f(x,x) = 0$ is an odd degree equation in x with real coefficients, and we know that $f(x,x) = 0$ has at least one real solutions. Thus, X must have at least one real point.

Exercise 1.3
Let $f \in k[x_1, \ldots, x_n]$. Show that the ideal (f) is radical if and only if no factor of f is multiple.

Solution: Suppose (f) is a radical ideal. If at least one factor of f is multiple, i.e., there exists $g, h \in k[x_1, \ldots, x_n]$ such that $f = g^N h$ for some integer $N > 1$. We know that $(gh)^N = h^{N-1} f \in (f)$, but $gh \notin (f)$ because every polynomial in (f) must have a factor g to the power at least $N > 1$. A contradiction. So no factor of f is multiple.

Conversely, assume no factor of f is multiple. If (f) is not radical, then there exists $g, h \in k[x_1, \ldots, x_n]$ such that $fh = g^N \in (f)$ but $g \notin (f)$. $k[x_1, \ldots, x_n]$ is a UFD, so we could write $f = f_1 \cdots f_m$ where $f_1, \ldots, f_m$ are irreducible polynomials, at least one f_i for $1 \leq i \leq m$ does not divide g(otherwise $g \in (f)$). Suppose it is f_1. Then $f_1 | f$ but $f_1 \nmid g^N$. A contradiction. This implies that the ideal (f) is radical.

Exercise 1.4
Assume that the characteristic of k is 0. Let $f(x)$ be a polynomial in $k[x]$. Show that the relation $\sqrt{(f)} = (f : f')$ holds. Give a counterexample if k is of positive characteristic.

Solution: Let $d = \gcd(f, f') \in k[x]$. We have the following claim:
Claim: The colon ideal $(f : f')$ is generated by $\frac{f}{d}$.
Proof: Write $f = d \cdot h_1$ and $f' = d \cdot h_2$ for some $h_1, h_2 \in k[x]$ and $(h_1, h_2) = 1$. Then for any $g \in (f : f')$. We know that $gf' \in (f)$. This means there exist $t \in k[x]$ such that

$$gdh_2 = gf' = ft = dh_1 t.$$

That is $gh_2 = h_1 t$. Since h_1 and h_2 are coprime, we know that

$$\frac{f}{d} = h_1 | g.$$

This proves that g is a multiple of h_1. Moreover, $h_1 f' = h_1 h_2 d = f h_2 \in (f)$. So $h_1 \in (f : f')$. This implies that $(h_1) = (f : f')$ because $k[x]$ is a principal ideal domain. ∎

Note that k is algebraically closed and has characteristic 0. Write

$$f(x) = a(x - a_1)^{n_1} \cdots (x - a_k)^{n_k}, \qquad a, a_1, \ldots, a_k \in k, n_1, \ldots, n_k \in \mathbb{Z}_+$$

Then by direct calculation, we have

$$h_1(x) = \frac{f(x)}{d(x)} = (x - a_1) \cdots (x - a_k).$$

Choose $N = \max\{n_1, \ldots, n_k\}$. It is easy to see that $h_1^N \in (f)$. So $(h_1) = (f : f') \subseteq \sqrt{(f)}$.

Conversely, suppose $g \in k[x]$ and for some $N > 0$, $g^N \in (f)$. This means there exists $h \in k[x]$ such that $g^N = fh = dh_1 h$. We know that

$$h_1(x) = (x - a_1) \cdots (x - a_k)$$

has no multiple factors. So $h_1 | g$. This implies $g \in (h_1) = (f : f')$. So we have $\sqrt{(f)} \subseteq (f : f')$. Thus, we can conclude that $\sqrt{(f)} = (f : f')$.

Now assume char $k = p > 0$. Consider $f(x) = x^p \in k[x]$. Note that here we have $f' = 0$. In this case, $\sqrt{(f)} = (x)$ but $(f : f') = k[x]$ because for any $a \in k[x]$, we have

$$af' = 0 \in (x^p).$$

Exercise 1.5
Let $\mathfrak{p}$ be a prime ideal in $k[x_1, \ldots, x_n]$. Show that $\mathfrak{p}$ is the intersection of all the maximal ideals containing it; that is,

$$\mathfrak{p} = \cap_{\mathfrak{p} \subseteq \mathfrak{m}} \mathfrak{m}.$$

Solution: It is easy to see that

$$\mathfrak{p} \subseteq \cap_{\mathfrak{p} \subseteq \mathfrak{m}} \mathfrak{m}.$$

Conversely, we want to show that

$$\cap_{\mathfrak{p} \subseteq \mathfrak{m}} \mathfrak{m} \subseteq \mathfrak{p}.$$

This is the same as proving

$$Z(\mathfrak{p}) \subseteq Z(\cap_{\mathfrak{p} \subseteq \mathfrak{m}} \mathfrak{m}) = \cup_{\mathfrak{p} \subseteq \mathfrak{m}} Z(\mathfrak{m}).$$

For any point $p \in Z(\mathfrak{p})$, by Nullstellensatz, $\{p\} = Z(\mathfrak{m})$ for some maximal ideal $\mathfrak{m} \supseteq \mathfrak{p}$. This implies that $p \in \cup_{\mathfrak{p} \subseteq \mathfrak{m}} Z(\mathfrak{m})$. We are done.

Exercise 1.6
Consider the closed algebraic set in $\mathbb{A}^2$ given by the vanishing of the polynomial

$$P(x) = y^2 - x(x+1)(x-1).$$

Let $\alpha \in \mathbb{C}$ and let $\mathfrak{a} = (x - \alpha, P(x))$. Determine $Z(\mathfrak{a})$ for all α. For which α's is $\mathfrak{a}$ a radical ideal?

Solution: Let $(x, y) \in Z(\mathfrak{a})$. Then we have

$$\begin{aligned} x =& \alpha, \\ y^2 =& \alpha(\alpha+1)(\alpha-1). \end{aligned}$$

If char$k = 2$, then $Z(\mathfrak{a})$ has only one point: $(\alpha, \sqrt{\alpha}(\alpha+1))$. In this case, $\mathfrak{a}$ is not a radical ideal because the quotient ring

$$k[x,y]/\mathfrak{a} \cong k[y]/(y^2 - \alpha(\alpha+1)^2) \cong k[y]/((y - \sqrt{\alpha}(\alpha+1))^2)$$

has a nilpotent polynomial $y - \sqrt{\alpha}(\alpha+1)$.

If $\alpha = 0, 1, -1$, then $Z(\mathfrak{a})$ has only one point: $(\alpha, 0)$. In this case, $\mathfrak{a}$ is not a radical ideal because the quotient ring

$$k[x,y]/\mathfrak{a} \cong k[y]/(y^2)$$

has a nilpotent polynomial y.

If char$k \neq 2$ and $\alpha \neq 0, 1, -1$, let $s = \sqrt{\alpha(\alpha+1)(\alpha-1)}$, then $Z(\mathfrak{a})$ has two points: (α, s) and $(\alpha, -s)$. In this case, $\mathfrak{a}$ is a radical ideal because the quotient ring

$$k[x,y]/\mathfrak{a} \cong k[y]/(y-s)(y+s)) \cong k^2$$

has no nilpotent elements.