Zhengdong Zhang
Email: zhengz@uoregon.edu
Course: MATH 635 - Algebraic Topology II
Instructor: Dr.Daniel Dugger

Homework 3
ID: 952091294
Term: Winter 2025
Due Date: 30^{th} January, 2025

Problem 1

Let x be the basepoint of S^n. Identify $S^n \vee S^n$ with the subspace $(S^n \times \{x\}) \cup (\{x\} \times S^n)$ of $S^n \vee S^n$. Prove that the diagram

$$\begin{tikzcd} S^n \arrow[r, "\Delta"] \arrow[dr, "\text{pinch}"'] & S^n \times S^n \\ & S^n \vee S^n \arrow[u, hook, "j"'] \end{tikzcd}$$

commutes up to homotopy, where Δ is the diagonal map and j is the inclusion.

Solution: Embed S^n into $\mathbb{R}^{n+1}$ canonically with $x_1^2 + \cdots + x_n^2 + x_{n+1}^2 = 1$. Choose coordinates properly such that x is on the equator $x_{n+1} = 0$. The pinch map $S^n \to S^n \vee S^n$ collapses the equator $\{x_{n+1} = 0\}$ and sends $\{x_{n+1} \geq 0\}$ to the first S^n in $S^n \vee S^n$ and $\{x_{n+1} \leq 0\}$ to the second $S^n \vee S^n$. For any point $y = (y_1, \ldots, y_{n+1}) \in S^n$, if $y_{n+1} > 0$, then the composition $S^n \xrightarrow{\text{pinch}} S^n \vee S^n \hookrightarrow S^n \times S^n$ sends y to the point (y, x). If $y_{n+1} < 0$, it was sent to (x, y). And if $y_{n+1} = 0$, the point y is on the equator and it was sent to (x, x). Now we need to show that the diagonal map $\Delta : S^n \to S^n \times S^n$ is homotopic to the inclusion map

$$\begin{aligned} i : S^n &\to S^n \times S^n, \\ y &\mapsto (y, x) \end{aligned}$$

Since S^n is path-connected, there exists a continous path $\gamma : I \to S^n$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Define a map

$$\begin{aligned} H : S^n \times I &\to S^n \times S^n, \\ (y, t) &\mapsto (y, \gamma(t)). \end{aligned}$$

H is continous since γ is continous. $H(-, 0) = i$ and $H(-, 1) = \Delta$. This proves that i is homotopic to Δ and similarly, $y \mapsto (x, y)$ is also homotopic to Δ. This proves that the diagram

$$\begin{tikzcd} S^n \arrow[r, "\Delta"] \arrow[dr, "\text{pinch}"'] & S^n \times S^n \\ & S^n \vee S^n \arrow[u, hook, "j"'] \end{tikzcd}$$

is commutative.

Problem 2
Let X be a topological space with a continous map $\mu : X \times X \to X$. Assume there is an element $e \in X$ with the property that $\mu(e, x) = \mu(x, e) = x$ for all $x \in X$. Write $x \cdot y$ for $\mu(x, y)$.

(a) Prove that $\pi_1(X, e)$ is abelian.

(b) If $f, g : (I^n, \partial I^n) \to (X, e)$, let $f \diamond g : I^n \to X$ be given by $(f \diamond g)(a) = f(a) \cdot g(a)$. Prove that the $\diamond$ defines a unital operation on $\pi_n(X, e)$.

(c) Prove this operation on $\pi_n(X, e)$ agrees with the usual one (defined in class for any space X). That is, prove that if $f, g \in \pi_n(X, e)$, then $f * g = f \diamond g$.

Solution:

(a) First we prove a useful claim.

Claim: Let $f, g, h, k : (I^n, \partial I^n) \to (X, e)$ be continous maps. We have

$$(f * g) \cdot (h * k) = (f \cdot h) * (g \cdot k).$$

Proof: Recall by definition

$$(f{*}g)(t) = \begin{cases} g(2t, y), & \text{if } \ 0 \le t \le 1/2, y \in I^{n-1}; \\ f(2t-1, y), & \text{if } \ 1/2 \le t \le 1, y \in I^{n-1}. \end{cases} \quad \text{and} \quad (h{*}k)(t) = \begin{cases} k(2t, y), & \text{if } \ 0 \le t \le 1 \\ h(2t-1, y), & \text{if } \ 1/2 \le t \le \end{cases}$$

So $(f * g) \cdot (h * k)$ can be written as

$$(f * g) \cdot (h * k)(t) = \begin{cases} g(2t, y) \cdot k(2t, y), & \text{if } \ 0 \le t \le 1/2, y \in I^{n-1}; \\ f(2t-1, y) \cdot h(2t-1, y), & \text{if } \ 1/2 \le t \le 1, y \in I^{n-1}. \end{cases}$$

This is exactly the definition of $(f \cdot h) * (g \cdot k)$. ■

We have already know that $\pi_1(X, e)$ has a group structure with the homotopy class of the constant map $[C_e]$ represents the identity element. For any $\beta : (I, \partial I) \to (X, e)$, we have $[C_e * \beta] = [\beta * C_e]$ in $\pi_1(X, e)$. There exists a continous map $H_1 : I \times I \to X$ such that $H_1(x, 0) = \beta(x) * C_e$, $H_1(x, 1) = (C_e * \beta)(x)$ and $H_1(0, t) = H_1(1, t) = e$ for all $t \in I$. Similarly, for any $\gamma : (I, \partial I) \to (X, e)$, there exists a continous map $H_2 : I \times I \to X$ such that $H_2(x, 0) = C_e * \gamma(x)$, $H_2(x, 1) = (\gamma(x) * C_e)$ and $H_2(0, t) = H_2(1, t) = e$ for all $t \in I$. We define a map

$$\begin{aligned} &H : I \times I \to X, \\ &(x, t) \mapsto H_1(x, t) \cdot H_2(x, t). \end{aligned}$$

This map is continous since it is the composition

$$I \times I \xrightarrow{(H_1, H_2)} X \times X \xrightarrow{\mu} X.$$

Moreover, note that by the claim

$$\begin{aligned} H(x,0) &= H_1(x,0)\cdot H_2(x,0) = (\beta(x) * C_e)\cdot(C_e * \gamma(x)) = (\beta * \gamma)(x),\\ H(x,1) &= H_1(x,1)\cdot H_2(x,1) = (C_e * \beta(x))\cdot(\gamma(x) * C_e) = (\gamma * \beta)(x). \end{aligned}$$

And for any $t \in I$, we have $H(0,t) = H(1,t) = H_1(0,t)\cdot H_2(0,t) = e$. Thus, we can conclude that $\beta * \gamma$ and $\gamma * \beta$ represents the same homotopy class in $\pi_1(X,e)$, which means $\pi_1(X,e)$ is abelian.

(b) Let $C_e : (I^n, \partial I) \to (X,e)$ be the constant map at the base point e. For any map $f : (I^n, \partial I^n) \to (X,e)$, we have

$$(C_e \diamond f)(a) = (C_e)(a)\cdot f(a) = e \cdot f(a) = f(a)$$

for any $a \in I$. Same for $f \diamond C_e$. This shows that $\diamond$ defines a unital operation on paths.

(c) Given two paths $f, g : (I^n, \partial I^n) \to (X,e)$, use the claim in part (a) and note that $C_e * f$ is homotopic to f and $g * C_e$ is homotopic to g, we have

$$\begin{aligned} f \diamond g &\simeq (C_e * f) \diamond (g * C_e)\\ &= (C_e \diamond g) * (f \diamond C_e)\\ &= g * f\\ &\simeq f * g. \end{aligned}$$

The last step is because $\pi_n(X,e)$ is abelian for all $n \geq 1$. We have proved $f \diamond g = f * g$ in $\pi_n(X,e)$.

Problem E

ach part below gives a pushout diagram in a specified category $\mathcal{C}$. For each one, identify the pushout with something explicit and (usually) familiar.

(a) $\mathcal{C} = \mathcal{A}\lfloor$ (the category of abelian groups), and the diagram is

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ 0 & \end{tikzcd}$$

(b) $\mathcal{C} = \mathcal{T}\wr_{\surd}$ (the category of topological spaces), A is a subspace of X, and the diagram is

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & X \\ \{*\} & \end{tikzcd}$$

(c) $\mathcal{C} = \mathcal{G}\nabla_{\surd}$ (the category of groups), H is a subgroup of G, and the diagram is

$$\begin{tikzcd} H \arrow[r, tail] \arrow[d] & G \\ \{*\} & \end{tikzcd}$$

(note that the answer is not G/H, this one is a little tricky).

(d) $\mathcal{C} = \mathcal{A}\lfloor$ and the diagram is

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_1"'] & C \\ A \oplus B & \end{tikzcd}$$

where i_1 is the standard inclusion of A into the first summand of $A \oplus B$.

(e) $\mathcal{C} = \mathcal{A}\lfloor$ and the diagram is

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "2"] \arrow[d, "2"'] & \mathbb{Z} \\ \mathbb{Z} & \end{tikzcd}$$

where both maps are multiplication by 2.

(f) $\mathcal{C} = \mathcal{A}\lfloor$ and the diagram is

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "3"] \arrow[d, "\pi"'] & \mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} & \end{tikzcd}$$

where π is the usual quotient map.

(g) $\mathcal{C} = \mathcal{T}\wr_{\surd}$ and the diagram is

$$\begin{tikzcd} S^1 \arrow[r, tail] \arrow[d, "f"'] & D^2 \\ S^1 & \end{tikzcd}$$

where the horizontal map is the inclusion and the vertical map is $f(z) = z^2$ (where S^1 is identified with the unit complex numbers).

(h) Compare and construct the pushouts of

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathbb{Z} \\ \mathbb{Z} & \end{tikzcd}$$

in the category $\mathcal{A}\lfloor$, the category $\mathcal{G}\nabla_{\surd}$, and the category $\mathcal{T}\wr_{\surd}$ (where all three objects in the pushout are given the discrete topology).

Solution: We will use the following construction of pushout in different categories:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \\ B & \end{tikzcd}$$

In $\mathcal{T}\wr_{\surd}$, the pushout can be constructed as the disjoint union $B \sqcup C/\sim$ where $f(a) \sim g(a)$ for all $a \in A$. Similarly, in $\mathcal{A}\lfloor$, the pushout can be constructed as $B \oplus C/\sim$ where $(f(a), 0) \sim (0, g(a))$ for all $a \in A$.

(a) By construction, the pushout is isomorphic to $B \oplus 0/(f(A) = 0)$, which is just the cokernel of the map $A \xrightarrow{f} B$.

(b) By construction, the pushout is isomorphic to $X \sqcup \{*\} / \sim$ where $*$ is identified with the image of A in X. This is the same as the quotient space X/A.

(c) This is a quotient group of G where every elements in the subgroup H is sent to 1. Consider

$$N = \langle ghg^{-1} \mid g \in G, h \in H \rangle.$$

N is the smallest normal subgroup of G containing H. Suppose P is the pushout of $\{1\} \leftarrow H \to G$ in $\mathcal{G}\nabla_{\surd}$:

$$\begin{tikzcd} H \arrow[r] \arrow[d] & G \arrow[d, "f"] \\ 1 \arrow[r] & P \end{tikzcd}$$

$f(H) = \{1\}$ implies $f(N) = \{1\}$. So $P \cong G/N$.

(d) By construction, the pushout is isomorphic to $A \oplus B \oplus C/\sim$ where $(a, 0, 0) \sim (0, f(a), 0)$. This is the same as identifying A as its image in B. So the pushout is given by

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{icl}"'] & C \arrow[d, "\mathrm{icl}"] \\ A \oplus B \arrow[r, "(f,id)"'] & C \oplus B \end{tikzcd}$$

(e) By construction, the pushout is isomorphic to the abelian group $\mathbb{Z} \oplus \mathbb{Z}/(2, -2) \sim (0, 0)$. This is the same as the group $\langle (1, 0), (1, -1) \rangle / \langle 2(1, -1) \rangle \cong \mathbb{Z} \oplus (\mathbb{Z}/2)$.

(f) By construction, the pushout is isomorphic to the abelian group $(\mathbb{Z}/2) \oplus \mathbb{Z}/\sim$ where $(0, 3)$ is identified with $(1, 0)$. The second part is the abelian group $\mathbb{Z}/6\mathbb{Z}$ and the first part $\mathbb{Z}/2\mathbb{Z}$ is identifies with the order 2 subgroup. So the pushout is isomorphic to $\mathbb{Z}/6\mathbb{Z}$.

(g) This is a CW complex X. Its 1-skeleton X_1 is isomorphic to S^1 and a 2-cell D^2 is glued to X_1 via a 2-sheeted covering map. This is exactly the CW complex structure for $\mathbb{R}P^2$, so the pushout is isomorphic to $\mathbb{R}P^2$.

(h) In $\mathbb{A}$, the pushout by construction is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}/\{0\} \cong \mathbb{Z}\oplus\mathbb{Z}$. In $\mathcal{G}\nabla_{\surd}$, the pushout is isomorphic to the free product $\mathbb{Z} * \mathbb{Z}$. In $\mathcal{T}\wr_{\surd}$, the pushout is the wedge sum $\mathbb{Z} \vee \mathbb{Z}$ glued at 0 with discrete topology.

Problem 4

Suppose you are told that the following three squares are pushout diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & P \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ * \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r] \arrow[d] & P \arrow[d] \\ * \arrow[r] & Y \end{tikzcd}$$

Prove that $X \cong Y$. You can give the argument assuming you are in $\mathcal{T}\wr_{\surd}$ if you want, or you can give it in any category with a terminal object $*$; if you do the former, try to only use the categorical properties of pushouts and not anything special about topological spaces.

Solution: We have the following commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & * \arrow[d] \\ C \arrow[r] & P \arrow[r] & Y \end{tikzcd}$$

Note that by uniqueness of the terminal object $*$, the composition of maps $A \to B \to *$ is the same map as $A \to *$, since X is the pushout of the diagram $C \leftarrow A \to *$, there exists a unique map $f : X \to Y$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & * \arrow[d] \arrow[dd, bend left] \\ C \arrow[r] \arrow[drr] & P \arrow[r] & Y \\ & & X \arrow[u, dashed, "\exists! f"] \end{tikzcd}$$

Now consider the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\ C \arrow[r] \arrow[drr, bend right] & P \arrow[dr, dashed, "\exists! g"] & \\ & & X \end{tikzcd}$$

where $B \to X$ is just the composition $B \to * \to X$. This is part of the previous diagram so it commutes. Since P is the pushout of the diagram $C \leftarrow A \to B$, there exists a unique map $g : P \to X$ such that the above diagram commutes. Now let Z be an object in this category together with two maps $P \to Z$ and $* \to Z$ such that the following diagram commutes

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & * \arrow[d] \\
P \arrow[r] & Z
\end{tikzcd}$$

Since Y is the pushout of $P \leftarrow B \to *$, there exists a unique map $h : Y \to Z$ such that the following diagram commutes

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & * \arrow[d] \arrow[ddr, bend left] & \\
P \arrow[r] \arrow[drr, bend right] & Y \arrow[dr, dashed, "\exists! h"'] & \\
& & Z
\end{tikzcd}$$

Now consider the following diagram

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & * \arrow[d] \arrow[drr, bend left] & & \\
P \arrow[r] \arrow[rrr, bend right, "g"'] & Y \arrow[r, "h"] & Z & X \arrow[l, dashed] \arrow[ll, bend left, "f"]
\end{tikzcd}$$

This just combines the previous diagram and we can define a map $h \circ f : X \to Z$ to make the diagram commutes, it is unique because both f and h are unique. This proves X satisfies the universal property of the pushout diagram $* \leftarrow B \to P$, and by uniqueness of the pushout, we have $X \cong Y$.

Problem 5

In this problem we continue our exploration of 3-dimensional manifolds. The ones you know at this point are

$$S^3, \mathbb{R}P^3, S^2 \times S^1, T^g \times S^1, N_r \times S^1$$

where T^g is the genus g torus and $N_r = \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$ (connected sum of r copies of $\mathbb{R}P^2$).

1. Make a table showing the homology groups with $\mathbb{Z}$-coefficients, with $\mathbb{Q}$-coefficients, and with $\mathbb{Z}/2$-coefficients for each of these spaces.

2. Let X be the quotient of the cube $I \times I \times I$ in which one identifies each face with its opposite face via a clockwise 90 degree rotation. Compute the homology groups of X and prove that this 3-manifold is different from all the ones listed above.

Solution:

(a) Consider the standard CW complex structure for a n-sphere ($n \geq 1$): one 0-cell and one n-cell. The boundary map is 0 so it does not change with different coefficients. So for any module

M, we have

$$H_i(S^n; M) = \begin{cases} M, & \text{if } i = 0, n; \\ 0, & \text{otherwise} \end{cases}$$

For $\mathbb{R}P^n$ with $n \geq 2$, consider the CW complex structure with only one k-cell in each dimension $k \leq n$. The attaching map is given by 2-sheeted covering and we can calculate the cellular complex (Hatcher, Chapter 2, Example 2.42):

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \quad \text{if } n \text{ even},$$
$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \quad \text{if } n \text{ odd}.$$

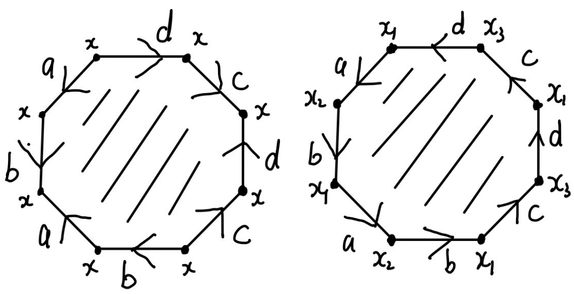
With $\mathbb{Z}/2$-coefficients, every $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ becomes the zero map. With $\mathbb{Q}$-coefficients, every $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ map becomes an isomorphism (2 is invertible in $\mathbb{Q}$). For $n = 2, 3$, we have

$$H_i(\mathbb{R}P^2; \mathbb{Z}) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 1; \\ \mathbb{Z}, & \text{if } i = 0; \\ 0, & \text{otherwise.} \end{cases} \qquad H_i(\mathbb{R}P^3; \mathbb{Z}) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 1; \\ \mathbb{Z}, & \text{if } i = 0, 3; \\ 0, & \text{otherwise.} \end{cases}$$

$$H_i(\mathbb{R}P^2; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0, 1, 2; \\ 0, & \text{otherwise.} \end{cases} \qquad H_i(\mathbb{R}P^3; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0, 1, 2, 3; \\ 0, & \text{otherwise.} \end{cases}$$

$$H_i(\mathbb{R}P^2; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } i = 0; \\ 0, & \text{otherwise.} \end{cases} \qquad H_i(\mathbb{R}P^3; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } i = 0, 3; \\ 0, & \text{otherwise.} \end{cases}$$

Now we consider the connected sum of tori and real projective space, we have already calculate in Homework 1 using the cellular structure and chain complex:



$$T^g : \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z};$$
$$N_r : \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2r} \xrightarrow{d_1} \mathbb{Z}^{r+1}.$$

The cellular does not change with whatever coefficients we take. For N_r, the boundary map is

$$d_2(S) = 2(a_1 + b_1 + \cdots + a_n + b_n)$$

where $a_1, b_1, \ldots, a_n, b_n$ are 1-cells. So $d_2 = 0$ in $\mathbb{Z}/2$-coefficients and $d_2 = (a_1 + b_1 + \cdots + a_n + b_n)$ with $\mathbb{Q}$-coefficients. d_1 does not change. We summarize the homology as below.

$$H_i(T^g;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0,2; \\ \mathbb{Z}^{2g}, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases} \qquad H_i(N_r;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{r-1}, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$H_i(T^g;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } \quad i = 0,2; \\ (\mathbb{Z}/2)^{2g}, & \text{if } \quad i = 1; \\ 0, & \text{otherwise.} \end{cases} \qquad H_i(N_r;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } \quad i = 0,2; \\ (\mathbb{Z}/2)^{r}, & \text{if } \quad i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$H_i(T^g;\mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } \quad i = 0,2; \\ (\mathbb{Q})^{2g}, & \text{if } \quad i = 1; \\ 0, & \text{otherwise.} \end{cases} \qquad H_i(N_r;\mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } \quad i = 0; \\ \mathbb{Q}^{r-1}, & \text{if } \quad i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Now to make the table for these 3-manifolds, use the fact that for any ring R,

$$H_n(X \times S^1; R) \cong H_n(X;R) \oplus H_{n-1}(X;R)$$

where X is a surface.

Table 1: Homology groups with $\mathbb{Z}$-coefficients

	$H_*(S^3)$	$H_*(\mathbb{R}P^3)$	$H_*(S^2 \times S^1)$	$H_*(T^g \times S^1)$	$H_*(N_r \times S^1)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^{2g+1}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^r$
2	0	0	$\mathbb{Z}$	$\mathbb{Z}^{2g+1}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{r-1}$
3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	0

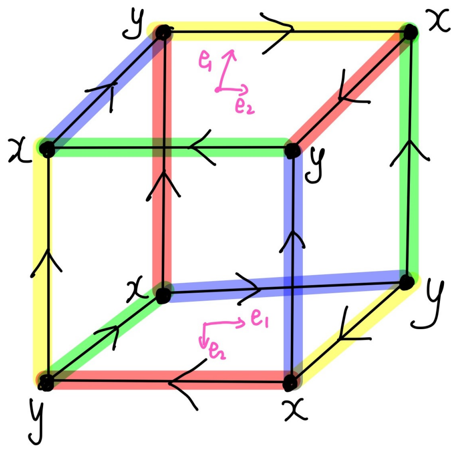
Table 2: Homology groups with $\mathbb{Z}/2$-coefficients

	$H_*(S^3)$	$H_*(\mathbb{R}P^3)$	$H_*(S^2 \times S^1)$	$H_*(T^g \times S^1)$	$H_*(N_r \times S^1)$
0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
1	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^{2g+1}$	$(\mathbb{Z}/2)^{r+1}$
2	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^{2g+1}$	$(\mathbb{Z}/2)^{r+1}$
3	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)$

Table 3: Homology groups with $\mathbb{Q}$-coefficients

	$H_*(S^3)$	$H_*(\mathbb{R}P^3)$	$H_*(S^2 \times S^1)$	$H_*(T^g \times S^1)$	$H_*(N_r \times S^1)$
0	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$
1	0	0	$\mathbb{Q}$	$\mathbb{Q}^{2g+1}$	$\mathbb{Q}^r$
2	0	0	$\mathbb{Q}$	$\mathbb{Q}^{2g+1}$	$\mathbb{Q}^{r-1}$
3	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	0

(b) Let X be the described space. Consider the following cell complex structure



We have two 0-cells x and y, four 1-cells r, e, g, b (red, yellow, green, blue), three 2-cells T, F, S (Top, Front, Sides) and one 3-cell Δ. We have the following cellular complex:

$$\mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^4 \xrightarrow{d_1} \mathbb{Z}^2 \xrightarrow{d_0=0} 0$$

For the boundary maps, we know that

$$d_1(r) = d_1(b) = -d_1(g) = -d_1(e) = y - x$$

and

$$\begin{aligned} d_2(T) &= g + b + e + r, \\ d_2(F) &= e + r - g - b, \\ d_2(S) &= e + b - r - g. \end{aligned}$$

Claim: The boundary map $d_3 = 0$.

Proof: Consider the orientation (pink arrows on the top face) given at the point p by the ordered vector e_1, e_2. This is the same as the ordered vector shown at the bottom face (twisted by how we glue the top and bottom faces). Consider the attaching map of 3-cells. We need to calculate the degree of $S^2 \cong \partial\Delta \to X_2/F, S \cong S^2$. This is a two sheeted covering with the second part twisted by 90 degrees. The orientation on the top face, while moving along the boundary of the 3-cell, will give us the opposite orientation as shown in the picture on the bottom face. So the degree of the gluing map must be 0. ■

To calculate the change of variables, we first do a change of basis. Let $A = e + r$, $B = g + b$ and $C = r - b$. Note that $\ker d_1$ can be generated by A, B, C. And $\operatorname{Im} d_2$ can be written as

$\langle A+B, A-B, A-B-2C\rangle$. So

$$\begin{aligned} H_1(X) &= \ker d_1/\operatorname{Im} d_2 \\ &= \langle A, B, C\rangle/\langle A+B, A-B, A-B-2C\rangle \\ &= \langle A-B, B, C\rangle/\langle 2B, A-B, 2C\rangle \\ &= \langle B, C\rangle/\langle 2B, 2C\rangle \\ &= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

$H_0(X) = H_3(X) = \mathbb{Z}$ since X is path-connected and $d_3 = 0$. And $H_2(X) = \ker d_2 = 0$. To summarize

$$H_i(X) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0, 3; \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } \ i = 1; \\ 0, & \text{if } \ \text{otherwise}. \end{cases}$$

This does not match any homology groups of spaces in the table above, so this space X is different from all the ones listed above.

Problem 6

Conisder the 3-manifold $\mathbb{R}P^3 \# \mathbb{R}P^3$. Construct two cofiber sequences $S^2 \hookrightarrow \mathbb{R}P^3 \# \mathbb{R}P^3 \to \mathbb{R}P^3 \vee \mathbb{R}P^3$ and $X \hookrightarrow \mathbb{R}P^3 \# \mathbb{R}P^3 \to \mathbb{R}P^3$ where $X \simeq \mathbb{R}P^2$. Use these to compute $H_*(\mathbb{R}P^3 \# \mathbb{R}P^3)$.

Solution: The solutions are divided into two parts. Part (1) we show that we have two cofiber sequence

$$\begin{aligned} S^2 &\hookrightarrow \mathbb{R}P^3 \# \mathbb{R}P^3 \to \mathbb{R}P^3 \vee \mathbb{R}P^3 \\ \mathbb{R}P^2 &\hookrightarrow \mathbb{R}P^3 \# \mathbb{R}P^3 \to \mathbb{R}P^3. \end{aligned}$$

In part (2), we use these two cofiber sequence to calculate $H_*(\mathbb{R}P^3 \# \mathbb{R}P^3)$.

(1) The connected sum of two copies of $\mathbb{R}P^3$ is constructed by first deleting one 3-cell from each $\mathbb{R}P^3$, and then glue the boundary of the deleted 3-cells together. Their boundary is homeomorphic to S^2. Collapsing the glued boundary in $\mathbb{R}P^3 \# \mathbb{R}P^3$ is the same as gluing two $\mathbb{R}P^3$ together at one point. This gives us the cofiber sequence

$$S^2 \hookrightarrow \mathbb{R}P^3 \# \mathbb{R}P^3 \to \mathbb{R}P^3 \vee \mathbb{R}P^3$$

Now consider $\mathbb{R}P^3$ with the standard CW complex structure: one cell in each dimension 0,1,2, and 3. The 2-skeleton of $\mathbb{R}P^3$ is isomorphic to $\mathbb{R}P^2$. Identify $\mathbb{R}P^2$ with the 2-skeleton in one copy of $\mathbb{R}P^3$ in the connected sum. Collapsing this 2-skeleton in $\mathbb{R}P^3 \# \mathbb{R}P^3$ gives back one full copy of $\mathbb{R}P^3$. So we have a cofiber sequence

$$\mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3 \# \mathbb{R}P^3 \to \mathbb{R}P^3.$$

(2) The cofiber sequence

$$\mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3 \# \mathbb{R}P^3 \to \mathbb{R}P^3$$

induceds a long exact sequence in reduced homology

$$
\begin{tikzcd}
 & \tilde{H}_*(\mathbb{R}P^2) & \tilde{H}_*(\mathbb{R}P^3 \# \mathbb{R}P^3) & \tilde{H}_*(\mathbb{R}P^3) \\
3 & 0 \arrow[r] & ? \arrow[r] & \mathbb{Z} \arrow[dll] \\
2 & 0 \arrow[r] & ? \arrow[r] & 0 \arrow[dll] \\
1 & \mathbb{Z}/2 \arrow[r] & ? \arrow[r] & \mathbb{Z}/2 \arrow[dll] \\
0 & 0 & &
\end{tikzcd}
$$

We can see that $H_3(\mathbb{R}P^3\#\mathbb{R}P^3) \cong H_3(\mathbb{R}P^3) = \mathbb{Z}$ and $H_2(\mathbb{R}P^3\#\mathbb{R}P^3) = 0$. Another cofiber sequence

$$S^2 \hookrightarrow \mathbb{R}P^3\#\mathbb{R}P^3 \to \mathbb{R}P^3 \vee \mathbb{R}P^3$$

also induces a long exact sequence in reduced homology

$$
\begin{tikzcd}
 & \tilde{H}_*(S^2) & \tilde{H}_*(\mathbb{R}P^3 \# \mathbb{R}P^3) & \tilde{H}_*(\mathbb{R}P^3 \vee \mathbb{R}P^3) \\
3 & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[dll] \\
2 & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
1 & 0 \arrow[r] & ? \arrow[r] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[dll] \\
0 & 0 & &
\end{tikzcd}
$$

By exactness, $H_1(\mathbb{R}P^3\#\mathbb{R}P^3) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Moreover, $\mathbb{R}P^3\#\mathbb{R}P^3$ is path-connected since $\mathbb{R}P^3$ is path connected, so the homology of $\mathbb{R}P^3\#\mathbb{R}P^3$ can be summarized

$$H_i(\mathbb{R}P^3\#\mathbb{R}P^3) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0, 3; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & \text{if } \ i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 7
Identify S^1 with the unit complex numbers, and let $\pi : \mathbb{C} - 0 \to S^1$ be the map $\pi(z) = z/|z|$.
For each $c \in \mathbb{R}_{>0}$ let $j_c : S^1 \hookrightarrow \mathbb{C}$ be the map $j_c(z) = cz$.
Let $f(z)$ be a degree n polynomial with coefficients in $\mathbb{C}$, and for convenience assume f is monic.

Let K be a positive real number that is larger than the norms of all the roots of $f(z)$. Note that if $c > K$ then f maps $j_c(S^1)$ into $\mathbb{C} - 0$, and so we may consider the composite $\pi f j_c$; it is a map $S^1 \to S^1$.

(a) If $K < c < d$ prove that $\deg(\pi f j_c) = \deg(\pi f j_d)$.

(b) Prove that if c is large enough then $f j_c$ is homotopic to the map $z \mapsto z^n$.

(c) Conclude that if $c > K$ then $\deg(\pi f j_c) = n$.

(d) In this last part we will prove the Fundamental Theorem of Algebra. Suppose f does not have any roots at all. Prove that $\pi f j_1$ factors through a contractible space, and use this to deduce a contradiction.

Solution:

(a) Consider the annulus $A = \{z \in \mathbb{C} \mid c \leq |z| \leq d\}$. For any $z \in A$, $|f(z)| > 0$ since $c > K$. Define the following map

$$\begin{aligned} H : S^1 \times I &\to S^1, \\ (z,t) &\mapsto \pi f((1-t)cz + tdz) \end{aligned}$$

Note that for any $0 \leq t \leq 1$, $((1-t)c + td)z \in A$. So $f((1-t)cz + tdz) \in \mathbb{C} - 0$. And $H(z,0) = \pi f j_c$ and $H(z,1) = \pi f j_d$. This proves that $\pi f j_c$ is homotopic to $\pi f j_d$. Thus, we have $\deg \pi f j_c = \deg \pi f j_d$.

(b) Write $g(z) = c^n z^n$. We first prove that $f j_c$ is homotopic to g. Consider following map

$$\begin{aligned} H : S^1 \times I &\to \mathbb{C} - 0, \\ (z,t) &\mapsto g(z) + t(f(cz) - g(z)). \end{aligned}$$

Write $f j_c(z) = f(cz) = c^n z^n + a_{n-1}c^{n-1}z^{n-1} + \cdots + a_0$. Note that for any $0 \leq t \leq 1$, by reverse triangle inequality, we have

$$\begin{aligned} |g(z) + t(f(cz) - g(z))| &= |c^n z^n + t a_{n-1} c^{n-1} z^{n-1} + \cdots + a_0| \\ &\geq c^n |z^n| - c^{n-1} t |a_{n-1} z^{n-1}| - \cdots - |a_0| \\ &= h(c). \end{aligned}$$

View the last thing as a polynomial in c and since $|z| = 1$ is bounded, if c is large enough, $h(c) > 0$. So we can choose a large c such that H is well-defined as the image lies in $\mathbb{C} - 0$.

What remains to show is that for a large enough real constant M, the map $z \mapsto (cz)^n$ is homotopic to $z \mapsto z^n$ on $S^1 = \{|z| = 1\}$. Consider the homotopy

$$\begin{aligned} H : S^1 \times I &\to \mathbb{C} - 0, \\ (z,t) &\mapsto ((1-t)z + ctz)^n. \end{aligned}$$

This homotopy is well-defined since the only $|z| = 1$ implies $|z^n| \neq 0$.

(c) We need to show that $p : S^1 \to S^1$ by sending z to $\frac{z^n}{|z^n|}$ has degree n. Write $z = e^{2\pi i t}$ for

$0 \leq t \leq 1$. We have

$$p(z) = \frac{z^n}{|z^n|} = \frac{e^{2\pi nit}}{|e^{2\pi nit}|} = e^{2\pi nit}.$$

It is easy to see that p is just the n times counterclockwise rotation on S^1. So $\deg(p) = \deg(\pi f j_c) = n$.

(d) Suppose f has no root at all. Then for any $z \in \mathbb{C}$, the map $\frac{f(z)}{|f(z)|}$ is continous and well-defined and $f(z) \neq 0$. So we have a commutative diagram

$$\begin{tikzcd} S^1 \arrow[r, "\pi f(z)=\frac{f(z)}{|f(z)|}"] \arrow[d, hook] & S^1 \\ \mathbb{C} \arrow[ur, "\frac{f(z)}{|f(z)|}"'] & \end{tikzcd}$$

This induces a map in homology

$$\begin{tikzcd} H_1(S^1) \arrow[r, "(\pi f)_*"] \arrow[d, hook] & H_1(S^1) \\ H_1(\mathbb{C}) \arrow[ur] & \end{tikzcd}$$

And since $\mathbb{C}$ is contractible, $H_1(\mathbb{C}) = 0$, so $(\pi f)_*$ is the zero map. We have $\deg(\pi f j_1) = 0$. But $\deg(\pi f j_1) = n$ according to our previous discussion. A contradiction. f must have at least one root in $\mathbb{C}$.