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## **Problem 17.1.3**

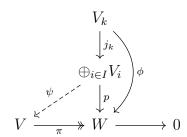
Let  $(V_i)_{i \in I}$  be a family of R-modules. Then:

(1)  $\bigoplus_{i \in I} V_i$  is projective if and only if each  $V_i$  is projective.

(2)  $\prod_{i \in I} V_i$  is injective if and only if each  $V_i$  is injective.

#### Solution:

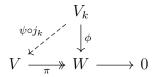
(1) Note that  $\bigoplus_{i \in I} V_i$  is the coproduct of a family of R-modules  $(V_i)_{i \in I}$  and thus has the universal property. Assume  $\bigoplus_{i \in I} V_i$  is projective. Fix  $k \in I$ . Let  $\pi : V \twoheadrightarrow W$  be a surjective map and  $\phi : V_k \to W$  is an R-module homomorphism. Define a family of R-module homomorphisms  $p_i : V_i \to W$  for any  $i \in I$  as follows: if i = k, then  $p_i = \phi$ . Otherwise  $p_i$  is the zero map. By universal property of  $\bigoplus_{i \in I} V_i$ , we have a unique map  $p : \bigoplus_{i \in I} V_i \to W$  such that  $p \circ j_k = \phi$  where  $j_k : V_k \to \bigoplus_{i \in I} V_i$  is the canonical inclusion.



By projectivity of  $\bigoplus_{i\in I} V_i$ , there exists a map  $\psi: \bigoplus_{i\in I} V_i \to V$  such that  $\pi \circ \psi = p$ . So for any  $v \in V_k$ , we have

$$\phi(v) = (p \circ j_k)(v) = (\pi \circ \psi \circ j_k)(v).$$

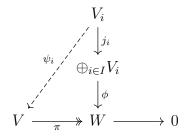
This implies  $\psi \circ j_k$  is a map making the following diagram commutes:



This proves that  $V_k$  is projective for any  $k \in I$ .

On the other hand, assume for each  $i \in I$ ,  $V_i$  is a projective R-module. Suppose  $\pi: V \to W$  is surjective and we have a homomorphism  $\phi: \bigoplus_{i \in I} V_i \to W$ . For each  $i \in I$ , consider the composition with canonical inclusion  $\phi \circ j_i: V_i \to W$ . By the projectivity of  $V_i$ , there exists

a map  $\psi_i: V_i \to V$  such that  $\pi \circ \psi_i = \phi \circ j_i$ .



The universal property tells us there exists a map  $\psi: \bigoplus_{i \in I} V_i \to V$  such that  $\psi \circ j_i = \psi_i$ . We claim that  $p \circ \psi = \phi$ . For any  $v \in \bigoplus_{i \in I} V_i$ , v can be written as  $v = \sum_{i \in I} v_i$  for each  $v_i \in V_i$ . Then

$$(\pi \circ \psi)(v) = (\pi \circ \psi)(\sum_{i \in I} v_i)$$

$$= \sum_{i \in I} (\pi \circ \psi \circ j_i)(v_i)$$

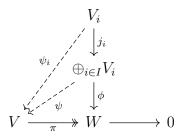
$$= \sum_{i \in I} (\pi \circ \psi_i)(v_i)$$

$$= \sum_{i \in I} (\phi \circ j_i)(v_i)$$

$$= \phi(\sum_{i \in I} j_i(v_i))$$

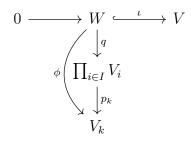
$$= \phi(v)$$

We have the following commutative diagram



This proves that  $\bigoplus_{i \in I} V_i$  is projective.

(2) Assume  $\prod_{i \in I} V_i$  is injective. Fix  $k \in I$ . Given an injective homomorphism  $\iota : W \to V$  and a homomorphism  $\phi : W \to V_k$ , for any  $k \neq i \in I$ , consider a family of zero maps  $W \to V_i$ . By the universal property of the product  $\prod_{i \in I} V_i$ , we have a unique map  $q : W \to \prod_{i \in I} V_i$  such that  $p_k \circ q = \phi$  where  $p_k : \prod_{i \in I} V_i \to V_k$  is the canonical projection.



 $\prod_{i \in I} V_i$  being injective implies that there exists  $\psi : V \to \prod_{i \in I} V_i$  such that  $\psi \circ \iota = q$ . We claim we have the following commutative diagram

$$0 \longrightarrow W \xrightarrow{\iota} V$$

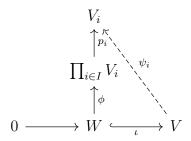
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Indeed, given any  $w \in W$ , we have

$$(p_k \circ \psi \circ \iota)(w) = (p_k)(\psi \circ \iota)(w)$$
$$= (p_k \circ q)(w)$$
$$= \phi(w).$$

This proves that for each  $i \in I$ ,  $V_i$  is injective.

Conversely, assume for any  $i \in I$ , each  $V_i$  is injective. Given an injective homomorphism  $\iota: W \to V$  and a homomorphism  $\phi: W \to \prod_{i \in I} V_i$ , consider the composition  $p_i \circ \phi: W \to V_i$  where  $p_i: \prod_{i \in I} V_i \to V_i$  is the canonical projection. Since  $V_i$  is injective, there exists a homomorphism  $\psi_i: v \to V_i$  such that  $p_i \circ \phi = \psi_i \circ \iota$ .



We have a family of homomorphisms  $(\psi_i)_{i\in I}$ . By the universal property of product, there exists a unique homomorphism  $\psi: V \to \prod_{i\in I} V_i$  such that for any  $i\in I$ , we have  $p_i \circ \psi = \psi_i$ . We claim that  $\psi \circ \iota = \phi$ , namely the following diagram commutes

$$\prod_{i \in I} V_i$$

$$\downarrow \phi \qquad \qquad \downarrow \psi$$

$$0 \longrightarrow W \stackrel{\psi}{\longleftrightarrow} V$$

For any  $w \in W$  and  $i \in I$ , we have

$$(p_i \circ \psi \circ \iota)(w) = (\psi_i \circ \iota)(w)$$
$$= (p_i \circ \phi)(w).$$

Since  $p_i: \prod_{i\in I} V_i \to V_i$  is the projection. We have  $\psi \circ \iota = \phi$ . This proves that  $\prod_{i\in I} V_i$  is injective.

## **Problem 17.1.5**

Let P, I and V be R-modules.

- (1) If I is injective and  $I \subseteq V$  is a submodule then I is a summand of V.
- (2) If P is projective and  $P \cong V/W$  for some submodule  $W \subseteq V$  then W is a summand of V.

Solution:

(1) We have an exact sequence

$$0 \longrightarrow I \stackrel{\iota}{\longleftrightarrow} V$$

where  $\iota: I \to V$  is the inclusion. By Theorem 17.1.4, I being injective implies that there exists  $\psi: V \to I$  such that  $\psi \circ \iota = id_I$ . We know that  $\ker \psi$  is a submodule of V and by the first isomorphism theorem for modules we have  $V/\ker \psi \cong I$ . We have a short exact sequence

$$0 \longrightarrow \ker \psi \longrightarrow V \xrightarrow{\psi} I \longrightarrow 0$$

Note that  $\psi \circ \iota = id_i$  implies that this sequence splits by Lemma 14.2.8. So we have  $\ker \psi \oplus I = V$ . This proves that I is a summand of V.

(2)  $P \cong V/W$  implies that we have a short exact sequence

$$0 \longrightarrow W \stackrel{i}{\longleftrightarrow} V \stackrel{\pi}{\longrightarrow} P \longrightarrow 0$$

where i is the inclusion and  $\pi$  is the projection. By Theorem 17.1.4, there exists a map  $\psi: P \to V$  such that  $\pi \circ \psi = id_P$ . By Lemma 14.2.8, this sequence must split and we have  $V \cong P \oplus W$  and thus, W is a summand of V.

## Problem 17.1.8 (Schanuel's Lemma)

Let

$$0 \to B \xrightarrow{j} P \xrightarrow{\pi} A \to 0,$$
$$0 \to B' \xrightarrow{j'} P' \xrightarrow{\pi'} A \to 0$$

be two short exact sequences of R-modules with P and P' projective. Then  $B \oplus P' \cong B' \oplus P$ .

Solution: Consider the suset  $W \subseteq P \times P'$  satisfying the following property: for any  $(a,b) \in P \times P'$ , we have  $\pi(a) = \pi'(b)$ . We claim that W is a submodule of  $P \times P'$ . Indeed, for any  $r \in R$  and  $(a,b) \in W$ , we have

$$\pi(ra) = r\pi(a) = r\pi'(b) = \pi'(rb).$$

This implies  $r(a,b) = (ra,rb) \in W$ . So W is a submodule of  $P \times P'$ . Consider the composition

$$f: W \to P \times P' \to P,$$
  
 $g: W \to P \times P' \to P'.$ 

where the first map is the inclusion of submodules and the second map is the projection. We are going to show that f and g are surjective. For any  $p \in P$ ,  $\pi(p)$  is an element in A. Since  $\pi'$  is surjective, there exists  $p' \in P'$  such that  $\pi(p) = \pi'(p')$ . Note that by definition, (p, p') is in W, and f(p, p') = p. This proves that f is surjective. Similarly, we can show that g is surjective.

Next, we are going to show that the kernel of the map  $f: W \to P$  is isomorphic to B' and the kernel of  $g: W \to P'$  is isomorphic to B. Suppose  $(a, b) \in \ker f$ , namely  $f(a, b) = a = 0 \in P$ . Then by definition of W, we have

$$0 = \pi(0) = \pi(a) = \pi'(b).$$

This tells us  $b \in \ker \pi' = B'$  by exactness. Conversely, suppose  $b \in B' = \ker \pi'$  and consider  $(0,b) \in W$ , we have  $f(0,b) = 0 \in P$ . So  $(0,b) \in \ker f$ . This implies that the map  $B' \to W$  sending  $b \in B'$  to  $(0,b) \in W$  is exactly the kernel of  $f: W \to P$ . Similarly, we can prove that B is isomorphic to the kernel of  $g: W \to P'$ . We have the following two short exact sequence:

$$0 \to B' \to W \xrightarrow{f} P \to 0,$$
  
$$0 \to B \to W \xrightarrow{g} P' \to 0.$$

Note that P and P' is projective, so by Theorem 17.1.7, these two short exact sequences splits and we have

$$B' \oplus P \cong W \cong B \oplus P'$$
.

#### Problem 17.1.10

If every irreducible R-module is projective then R is semisimple artinian.

Solution: We want to prove that R is semisimple artinian, by Lemma 16.2.1, it is the same as proving the left regualr module R is semisimple. By Theorem 14.2.19, it suffice to show that R is the sum of all of its simple submodules. Consider S is the sum of all simple submodules in R. Note that for every element  $a \in S$ , a can be written as  $a = \sum v_i$  for some  $v_i \in V_i$  where each  $V_i$  is a simple ideal of R. So for any  $r \in R$ , we have  $rv_i \in V_i$  for any i. So  $ra = \sum rv_i \in S$ . This proves that S is an ideal in R. Assume  $S \neq R$ , then S must be contained in some maximal ideal M in R. Consider the following short exact sequence

$$0 \to M \to R \to R/M \to 0$$
.

Since M is maximal, so R/M is isomorphic to a simple R-module by Exercise 14.1.23. We have known that every simple R-module is projective, so R/M is projective and the above short exact sequence splits. We have

$$R \cong M \oplus R/M$$
.

This shows that R/M is a simple R-submodule of R which is not in S. A contradiction. So S = R and we have proved R is the sum of all its simple submodules, thus R is semisimple artinian.

## Problem 17.1.11

True or false? Every short exact sequence of  $\mathbb{C}[x]/(x^2-1)$ -modules is split.

Solution: This is true. The ideal  $(x^2 - 1) = (x - 1) \cap (x + 1)$ . The polynomials x - 1 and x + 1 are coprime in  $\mathbb{C}[x]$ . By the Chinese remainder theorem, we have

$$\mathbb{C}[x]/(x^2-1) \cong \mathbb{C}[x]/(x-1) \times \mathbb{C}[x]/(x+1) \cong \mathbb{C} \times \mathbb{C}.$$

By the Wedderburn-Artin theorem for algebras and Corollary 16.2.15,  $\mathbb{C}[x]/(x^2-1)$  is semisimple artinian and by Proposition 17.1.9, every  $\mathbb{C}[x]/(x^2-1)$ -module is projective. By Theorem 17.1.7, every short exact sequence of  $\mathbb{C}[x]/(x^2-1)$ -modules must split.

## Problem 17.1.16

Let P be a projective R-module, and

$$\cdots \to V_{n-1} \xrightarrow{f_n} V_n \xrightarrow{f_{n+1}} V_{n+1} \to \cdots$$

be an exact complex of R-modules and R-module homomorphisms. Prove that the corresponding complex

$$\cdots \to \operatorname{hom}_R(P, V_{n-1}) \xrightarrow{(f_n)_*} \operatorname{hom}_R(P, V_n) \xrightarrow{(f_{n+1})_*} \operatorname{hom}(P, V_{n+1}) \to \cdots$$

is exact. Formulate and prove the dual statement involving injective modules.

Solution: We first prove two useful claims.

<u>Claim:</u> hom<sub>R</sub> $(P, \ker f_{n+1}) = \ker(f_{n+1})_*$ . Here we view hom $(P, \ker f_{n+1})$  as a subset of hom $(P, V_n)$ . <u>Proof:</u> Given  $g: P \to \ker f_{n+1} \hookrightarrow V_n$ , the composition

$$f_{n+1} \circ g : P \to V_{n+1}$$

must be the zero map since it factors through  $\ker f_{n+1}$ . So we have  $\hom_R(P, \ker f_{n+1}) \subseteq \ker(f_{n+1})_*$ . Conversely, given  $g: P \to V_n$  such that

$$f_{n+1} \circ q : P \to V_{n+1}$$

is the zero map, by the universal property of the kernel, g must factor through ker  $f_{n+1}$ , so g can be viewed as an element in  $hom_R(P, \ker f_{n+1})$ . This proves that

$$hom(P, \ker f_{n+1}) = \ker(f_{n+1})_*.$$

<u>Claim:</u>  $hom_R(P, Im f_n) = Im (f_n)_*$ . Here  $hom_R(P, Im f_n)$  is viewed as a subset of  $hom_R(P, V_n)$ .

<u>Proof:</u> Given  $g: P \to \text{Im } f_n$ , consider the following diagram

$$V_{n-1} \xrightarrow{g' \qquad \qquad \downarrow g} P$$

$$V_{n-1} \xrightarrow{f_n} \inf_{f_n} \longrightarrow 0$$

Because P is projective, there exists  $g': P \to V_{n-1}$  such that the above diagram commutes. This proves that  $\hom_R(P, \operatorname{Im} f_n) \subseteq \operatorname{Im}(f_n)_*$ . Conversely, consider a composition  $P \xrightarrow{h} V_{n-1} \xrightarrow{f_n} V_n$ . We need to show that  $f_n \circ h \in \hom_R(P, \operatorname{Im} f_n)$ . This is true since  $\operatorname{Im}(f_n \circ h)$  must be contained in  $\operatorname{Im} f_n$ . Thus,  $\hom_R(P, \operatorname{Im} f_n) = \operatorname{Im}(f_n)_*$ .

The exactness of the original sequence tells us that ker  $f_{n+1} = \text{Im } f_n$  and by the claims, we have

$$\ker(f_{n+1})_* = \hom_R(P, \ker f_{n+1}) = \hom_R(P, \operatorname{Im} f_n) = \operatorname{Im} (f_n)_*.$$

Therefore, we have an exact sequence

$$\cdots \to \operatorname{hom}_R(P, V_{n-1}) \xrightarrow{(f_n)_*} \operatorname{hom}_R(P, V_n) \xrightarrow{(f_{n+1})_*} \operatorname{hom}(P, V_{n+1}) \to \cdots$$

#### Problem 17.1.19

Consider  $\mathbb{Z}/2\mathbb{Z}$  as a  $\mathbb{Z}/6\mathbb{Z}$ -module via the natural projection  $\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ . Then  $\mathbb{Z}/2\mathbb{Z}$  is a projective  $\mathbb{Z}/6\mathbb{Z}$ -module.

Solution: By the Chinese remainder Theorem, we have

$$\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

By Theorem 17.1.17,  $\mathbb{Z}/2\mathbb{Z}$  is a projective  $\mathbb{Z}/6\mathbb{Z}$ -module.

## Problem 17.1.21

If  $e \in R$  is an idempotent then Re is a projective R-module.

Solution: By Lemma 14.5.1, e being an idempotent implies that

$$R = Re \oplus R(1 - e)$$
.

By Theorem 17.1.17, Re is a projective R-module.

### Problem 17.1.26

Let R be a domain and  $\mathbb{F}$  be its field of fractions. Prove that  $\mathbb{F}$  is an injective R-module.

Solution: Let  $I \subset R$  be an ideal and  $\phi: I \to \mathbb{F}$  is a R-module homomorphism. If I is the zero ideal and  $\phi$  is the zero map, then  $\phi$  can be extended to the zero map  $R \to \mathbb{F}$ . Assume I contains nonzero

elements. Pick  $a \in I$  and  $a \neq 0$ , note that R is a domain so  $a \in I \subset R$  is invertible in  $\mathbb{F}$  since  $\mathbb{F}$  is the fraction field of R. Define

$$\psi: R \to \mathbb{F},$$

$$b \mapsto \frac{b\phi(a)}{a}.$$

For any  $r \in R$ , we have

$$\psi(rb) = \frac{rb\phi(a)}{a} = r\psi(b).$$

This is a R-module homomorphism. Moreover, for any  $b \in I$ , note that R is commutative and we have

$$\psi(b) = \frac{b\phi(a)}{a} = \frac{\phi(ba)}{a} = \frac{a\phi(b)}{a} = \phi(b).$$

We have a commutative diagram



By Baer's Criterion,  $\mathbb{F}$  is an injective R-module.

### Problem 17.1.27

Let  $n \geq 1$ . Then  $\mathbb{Z}/n\mathbb{Z}$  is an injective  $\mathbb{Z}/n\mathbb{Z}$ -module.

Solution: The ideals in  $\mathbb{Z}/n\mathbb{Z}$  are of the form  $\mathbb{Z}/d\mathbb{Z}$  where d|n. Given a homomorphism  $\phi: \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ , we have  $d\phi(1) = 0 \in \mathbb{Z}/n\mathbb{Z}$ . This implies  $n|(d\phi(1))$ . There exists  $r \in \mathbb{Z}$  such that  $rn = d\phi(1)$ . Note that r < n since  $d\phi(1) < dn$ . We define a homomorphism  $\psi: \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$  with  $\psi(1) = r$ . We have the following diagram

$$\mathbb{Z}/d\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/n\mathbb{Z}$$

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where i sends 1 to  $\frac{n}{d}$ , thus identify  $\mathbb{Z}/d\mathbb{Z}$  as an ideal  $\frac{n}{d}\mathbb{Z}/n\mathbb{Z}$  in  $\mathbb{Z}/n\mathbb{Z}$ . It is easy to check that for any  $a \in \mathbb{Z}/d\mathbb{Z}$ , we have

$$(\psi \circ i)(a) = \psi(\frac{n}{d}a) = \frac{n}{d}ra = \phi(1)a = \phi(a).$$

By Baer's Criterion,  $\mathbb{Z}/n\mathbb{Z}$  is injective.