Zhengdong Zhang
Email: zhengz@uoregon.edu
Course: MATH 648 - Abstract Algebra
Instructor: Professor Arkady Berenstein

Homework - Week 9
ID: 952091294
Term: Winter 2025
Due Date: 12^{th} March, 2025

Problem 24.1.4
Let V and W be finite dimensional $\mathbb{F}G$-modules. Define a $\mathbb{F}G$-module structure on $\hom_{\mathbb{F}}(V, W)$, so that $\hom_{\mathbb{F}}(V, W) \cong V^* \otimes W$ as $\mathbb{F}G$-module and $\hom_{\mathbb{F}G}(V, W) = \hom_{\mathbb{F}}(V, W)^G$.

Solution: We define a $\mathbb{F}G$-module structure on $\hom_{\mathbb{F}}(V, W)$. For any $g \in G$, $f \in \hom_{\mathbb{F}}(V, W)$ and $v \in V$, we define

$$(g \cdot f)(v) = g \cdot f(g^{-1}v).$$

This is a well-defined $\mathbb{F}G$-module structure. Indeed, for any $h, g \in G$, we have

$$\begin{aligned}(h \cdot (g \cdot f))(v) &= h \cdot (g \cdot f)(h^{-1}v) \\ &= h \cdot g \cdot f(g^{-1}h^{-1}v) \\ &= (hg) \cdot f((hg)^{-1}v) \\ &= ((hg \cdot f))(v).\end{aligned}$$

This $\mathbb{F}G$-module structure is compatible with the isomorphism $\hom_{\mathbb{F}}(V, W) \cong V^* \otimes W$. Indeed, consider the isomorphism

$$\begin{aligned}\phi : V^* \otimes W &\to \hom_{\mathbb{F}}(V, W), \\ (l \otimes w) &\mapsto (v \mapsto l(v)w).\end{aligned}$$

We check that the G-action we defined above is compatible. For any $g \in G$, $v \in V$, $l \in V^*$ and $w \in W$, we have

$$\begin{aligned}\phi(g \cdot (l \otimes w))(v) &= \phi((g \cdot l) \otimes gw)(v) \\ &= (g \cdot l)(v)gw \\ &= l(g^{-1}v)gw \\ &= g \cdot l(g^{-1}v)w \\ &= (g \cdot (\phi(l \otimes w)))(v).\end{aligned}$$

So we know that the isomorphism ϕ is also a $\mathbb{F}G$-module isomorphism.

Consider a map between $\mathbb{F}G$-modules

$$\begin{aligned}\hom_{\mathbb{F}G}(V, W) &\to \hom_{\mathbb{F}}(V, W), \\ \alpha &\mapsto \alpha.\end{aligned}$$

We first prove that for any $\alpha \in \hom_{\mathbb{F}G}(V, W)$, if we view α as an element in $\hom_{\mathbb{F}}(V, W)$, then α must be invariant under the G-action we defined above. Indeed, for any $g \in G$ and $v \in V$, we have

$$(g \cdot \alpha)(v) = g\alpha(g^{-1}v) = \alpha(gg^{-1}v) = \alpha(v).$$

We have a well-defined $\mathbb{F}G$-module homomorphism $\psi : \hom_{\mathbb{F}G}(V, W) \to \hom_{\mathbb{F}}(V, W)^G$. We check ψ is injective. Suppose $\alpha \in \ker \psi$, note that the zero map is invariant under the G-action, by definition of ψ, $\alpha = 0$. Next, we check that ψ is surjective. Given a G-invariant $\mathbb{F}$-linear map β, for any $g \in G$ and $v \in V$, we have

$$g\beta(v) = g(g^{-1} \cdot \beta)(v) = gg^{-1}\beta(gv) = \beta(gv).$$

This proves that β is also a homomorphism if we view V and W as $\mathbb{F}G$-modules. Thus, ψ is an isomorphism.

Problem 24.1.7
Let G be a finite group and $H \leq G$. Then each irreducible $\mathbb{F}G$-module is a quotient of a module induced from an irreducible $\mathbb{F}H$-module.

Solution: Let V be an irreducible $\mathbb{F}G$-module. We know that $\mathrm{res}_H^G V$ is a finite dimensional $\mathbb{F}H$-module, so it is artinian. Every descending chain must stabilizes. Thus, there exists an irreducible $\mathbb{F}H$ submodule $W \subseteq \mathrm{res}_H^G V$. Consider the $\mathbb{F}H$-module homomorphism

$$W \to \mathrm{res}_H^G V$$

defined by inclusion. By adjointness of res_H^G and ind_H^G, we know there exists an $\mathbb{F}G$-module homomorphism

$$f : \mathrm{ind}_H^G W \to V$$

which is not the zero homomorphism. So there exists $v \in V$ such that $f(a) = v$ for some $a \in \mathrm{ind}_H^G W$. Note that V is irreducible, so for any $v' \in V$, there exists $\sum k_g g$ such that $\sum k_g gv = v'$. Therefore, we have

$$f(\sum k_g ga) = \sum k_g gf(a) = \sum k_g gv = v'.$$

This proves that f is surjective. Thus, we can write

$$V \cong \mathrm{ind}_H^G W / \ker f$$

where W is an irreducible $\mathbb{F}H$-module.

Problem 24.1.10
If G acts transitively on a set X with a point stabilizer H, then the permutation module $\mathbb{F}X$ is isomorphic to the induced module $\mathrm{ind}_H^G 1_H$.

Solution: Let $H = \mathrm{Stab}(x)$ be the stabilizer for the point $x \in X$. For $\mathrm{ind}_H^G 1_H = \mathbb{F}G \otimes_{\mathbb{F}H} 1_H$, we define a map

$$\begin{aligned} \phi : \mathrm{ind}_H^G 1_H &\to \mathbb{F}X, \\ g \otimes 1 &\mapsto g \cdot x. \end{aligned}$$

For any $h \in G$, we have

$$h\phi(g \otimes 1) = h(g \cdot x) = (hg) \cdot x = \phi(hg \otimes 1).$$

This is a well-defined $\mathbb{F}G$-module homomorphism. Moreover, since the G-action on X is transive, we know that ϕ is surjective. By theorem 7.1.7, we know that $X \cong G/H$ is an isomorphism of G-sets. So we have $\dim_{\mathbb{F}} \mathbb{F}X = |G/H|$. Let $g_1, \ldots, g_k$ be coset representatives for G/H. By Lemma 24.1.9, $\{g_1 \otimes 1, \ldots, g_k \otimes 1\}$ is a basis for $\operatorname{ind}_H^G 1_H$. This tells us

$$\dim_{\mathbb{F}} \operatorname{ind}_H^G 1_H = \dim_{\mathbb{F}} \mathbb{F}X = |G/H|.$$

So ϕ is an isomorphism by dimension reasons. We have proved $\operatorname{ind}_H^G 1_H$ is isomorphic to $\mathbb{F}X$ as $\mathbb{F}G$-modules.

Problem 24.2.4
Show that the map $g \mapsto \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ defines a representation of the cyclic group $C_3 = \langle g \rangle$. Prove that this representation is irreducible over the field of real numbers.

Solution: Let $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. To check that $\rho : g \to A$ is a representation of C_3, we need to check that

$$I_3 = \rho(1) = \rho(g^3) = \rho(g)^3 = A^3.$$

And indeed we have

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

To show that ρ is an irreducible $\mathbb{R}C_3$ representation, suppose the corresponding $\mathbb{C}C_3$-module V is generated by v_1, v_2 as $\mathbb{C}$-vector space. If V has a nontrivial submodule, then it must be one dimensional and generated by $av_1 + bv_2$ where $a, b \in \mathbb{C}$. Since it is a submodule of V, we have

$$A \begin{pmatrix} a \\ b \end{pmatrix} \in \langle av_1 + bv_2 \rangle.$$

So

$$A \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}$$

for some $k \in \mathbb{R}$. k must be an eigenvalue of A but A only have two complex eigenvalues. This is a contradiction and thus, V is irreducible.

Problem 24.2.6
True or false? A non-abelian group of order 55 has exactly five one-dimensional complex representations up to isomorphism.

Solution: This is true. Let G be a non-abelian group of order 55. We know that $|G| = 55 = 5 \times 11$. By Sylow's theory, we know that the Sylow 11-subgroup must be unique and is isomorphic to the cyclic group C_{11}. There are two cases for the Sylow 5-subgroup. Either we have a unique Sylow 5-subgroup, or we have 11 Sylow 5-subgroup, each of them isomorphic to C_5 and conjugate to each other. Suppose G has a unique Sylow 5-subgroup C_5 and a unique Sylow 11-subgroup

C_{11}. By Proposition 7.5.16(2), Both Sylow subgroups are normal and G is the direct product $G \cong C_5 \times C_{11} \cong C_{55}$, which is abelian. So G must have 11 Sylow 5-subgroup. In this case, C_{11} is the smallest normal subgroup of G such that $G/C_{11} \cong C_5$ is abelian. So the commutator subgroup $G' = C_{11}$. From Exercise 24.2.5, we know that G has exactly five 1-dimensional complex representations up to isomorphism.

Problem 24.2.8(Irreducible representation of dihedral groups)
Let

$$D_{2n} = \langle\langle a, b \mid a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle\rangle$$

be the dihedral group, $\varepsilon := e^{2\pi i/n}$, and set

$$B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_j := \begin{pmatrix} \varepsilon^j & 0 \\ 0 & \varepsilon^{-j} \end{pmatrix} \quad (1 \leq j < n).$$

(1) Show that for $j = 1, \ldots, n-1$ there is a matrix representation $\rho_j : D_{2n} \to GL_2(\mathbb{C})$ such that $\rho_j(a) = A_j$ and $\rho_j(b) = B$.

(2) Use Schur's Lemma to prove that $\rho_1, \ldots, \rho_{n-1}$ are irreducible unless n is even and $j = n/2$.

(3) Use Schur's Lemma to prove that the representation $\rho_1, \ldots, \rho_{\lfloor (n-1)/2 \rfloor}$ are pairwise non-isomorphic.

(4) If $n = 2k$ is even, then D_{2n} has four non-isomorphic one-dimensional representations, which together with $\rho_1, \ldots, \rho_{k-1}$ give a complete and irrdundant list of irreducible $\mathbb{C}D_{2n}$-modules up to isomorphism.

(5) If $n = 2k+1$ is odd, then D_{2n} has two non-isomorphic representations, which together with $\rho_1, \ldots, \rho_k$ give a complete and irredundant list of irreducible $\mathbb{C}D_{2n}$-modules up to isomorphism.

Solution:

(1) We need to check that B and A_j satisfy the relations for the dihedral group D_{2n}. We have

$$\begin{aligned}
B^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = I_2, \\
A^n &= \begin{pmatrix} \varepsilon^j & 0 \\ 0 & \varepsilon^{-j} \end{pmatrix}^n = \begin{pmatrix} \varepsilon^{nj} & 0 \\ 0 & \varepsilon^{-nj} \end{pmatrix} = I_2. \\
BAB^{-1} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon^j & 0 \\ 0 & \varepsilon^{-j} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon^{-j} & 0 \\ 0 & \varepsilon^j \end{pmatrix} = A^{-1}.
\end{aligned}$$

This proves ρ_j is a matrix representation for all $1 \leq j \leq n-1$.

(2) Let V_j be the corresponding $\mathbb{C}D_{2n}$-module with repesct to the representation ρ_j. Suppose it has an $\mathbb{F}$-basis v_1, v_2. If V_j has a non-trivial 1-dimensional submodule W, then W must be generated by $av_1 + bv_2$ for some $a, b \in \mathbb{C}$. We know that W is irreducible because it is

1-dimensional, by Schur's lemma, A_j acts on W by a scalar $\lambda \in \mathbb{C}$. We have

$$A_j \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \varepsilon^j & 0 \\ 0 & \varepsilon^{-j} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \varepsilon^j a \\ \varepsilon^{-j} b \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}.$$

This means $\varepsilon^{2j} = 1$, namely, $e^{2\pi i \cdot \frac{2j}{n}} = 1$. We have $2j|n$. Since $1 \leq j \leq n-1$, we have $2j = n$. Thus, V_j is not irreducible if and only if n is even and $j = n/2$.

(3) Let $n = 2k$ and $1 \leq i < j \leq k-1$. Let V_i, V_j be the $\mathbb{C}D_{2n}$-modules corresponding to the irreducible representation ρ_i and ρ_j. Let $\phi : V_i \to V_j$ be an isomorphism of $\mathbb{C}D_{2n}$-modules, represented by an invertible matrix $M \in GL_2(\mathbb{C})$. For any $v \in V_1$, M being compatible with D_{2n}-action tells us that

$$A_j(Mv) = M(A_i v).$$

This implies $A_j M - M A_i \in \operatorname{Ann} V_i$. Because V_i is a two dimensional $\mathbb{C}$-vector space, so $\operatorname{Ann} V_i = 0$. We have

$$A_i = M^{-1} A_j M.$$

Thus, $\operatorname{tr}(A_i) = \operatorname{tr}(A_j)$. This is a contradiction because $1 \leq i < j \leq k-1$ and $\varepsilon^i + \varepsilon^{-i} \neq \varepsilon^j + \varepsilon^{-j}$.

(4) We calculate the commutator subgroup D'_{2n}. Note that $aba^{-1}b = a^2 \in D'_{2n}$. So the cyclic subgroup generated by a^2 is contained in D'_{2n}. From a claim below by calculating conjugacy classes we know that $\langle a^2 \rangle$ is normal in D_{2n}. Consider the quotient group $D_{2n}/\langle a^2 \rangle$, we can check that $D_{2n}/\langle a^2 \rangle$ is an abelian group by direct computation. Since the commutator subgroup is the smallest subgroup such that D_{2n}/D'_{2n} is abelian. This proves that $D'_{2n} = \langle a^2 \rangle$. When $n = 2k$, we have

$$|D_{4k}/D'_{4k}| = 4k/k = 4.$$

By Exercise 24.2.5(3), there exists four isomorphism classes of 1-dimensional complex representation of D_{2n}. To show that this is a complete list of irreducible representations for $\mathbb{C}D_{2n}$, by Theorem 24.2.2, we need to calculate the conjugacy classes for D_{2n}.

<u>Claim:</u> If $n = 2k$ is even, then the conjuagcy classes of D_{2n} is given by

$$\{1\}, \{a, a^{2k-1}\}, \ldots, \{a^{k-1}, a^{k+1}\}, \{a^k\}, \{b, a^2b, \ldots, a^{2k-2}b\}, \{ab, a^3b, \ldots, a^{2k-1}b\}.$$

If $n = 2k+1$ is odd, then the conjugacy classes of D_{2n} is given by

$$\{1\}, \{a, a^{2k}\}, \{a^2, a^{2k-1}\} \ldots, \{a^k, a^{k+1}\}, \{b, ab, \ldots, a^{2k}b\}.$$

<u>Proof:</u> Assume $n = 2k$ is even, we first show $\{a^i, a^{2k-i}\}$ is a conjugate class for $1 \leq i \leq k$. Note that all elements in D_{2n} can be written as $a^p b^q$, it is easy to see the conjugate action of a^p sends a^i to a^i, so we only need to consider the conjugate action by $a^p b$ for $0 \leq p \leq 2k-1$. We have

$$(a^p b) a^i (a^p b)^{-1} = a^p b a^i b a^{2k-p} = a^{p-i+2k-p} = a^{2k-i}.$$

Similarly, we have

$$(a^p b) a^{2k-i} (a^p b)^{-1} = a^{2k-2k+i} = a^i.$$

This proves $\{a^i, a^{2k-i}\}$ is a conjugate class for $i = 0, 1, \ldots, k$. We need to show the rest two is also conjugate classes. Consider the conjugate class of b, we have

$$\begin{aligned} aba^{-1} &= aab = a^2b, \\ a(a^2b)a^{-1} &= aa^2ab = a^4b, \\ &\cdots \\ a(a^{2k-2}b)a^{-1} &= a^{2k}b = b. \end{aligned}$$

And for any $0 \leq i \leq k-1$, we have

$$b(a^{2i}b)b = ba^{2i} = a^{2k-2i}b.$$

This calculates all elements insider the conjuagcy class of b. Lastly, for the conjugacy class of ab, we have

$$\begin{aligned} a(ab)a^{-1} &= a^3b, \\ a(a^3b)a^{-1} &= a^5b, \\ &\cdots \\ a(a^{2k-1}b)a^{-1} &= ab. \end{aligned}$$

Also, for $0 \leq i \leq k-1$, we have

$$b(a^{2i+1}b)b = a^{2k-2i-1}b = a^{2(k-1-i)+1}.$$

This calculates the last conjugate class.

Now assume $n = 2k+1$ is odd. A similar calculation shows that we have $k+1$ conjugacy classes

$$\{a^i, a^{2k+1-i}\}$$

for $i = 0, 1, \ldots, k$. We need to show that rest elements are in the same conjugacy class. Note that

$$\begin{aligned} aba^{-1} &= a^2b, \\ &\cdots \\ a(a^{2k}b)a^{-1} &= a^{2k+2}b = ab, \\ &\cdots \\ a(a^{2k-1}b)a^{-1} &= a^{2k+1}b = b. \end{aligned}$$

This proves the rest elements are in the same conjugacy class. ■

From the claim we know D_{4k} has $k+3$ conjuagcy classes, so this is a complete and irredundant list of all the $\mathbb{C}D_{2n}$-modules up to isomorphism.

(5) Similar to what we have discussed above, the commutator group is generated by $\langle a^2 \rangle$. When $n = 2k+1$ is odd, note that a has odd order, so in this case a^2 can generate the element

$a = a^{2k+2}$, so the commutator subgroup

$$|D'_{2n}| = 2k + 1.$$

So D_{4k+2} has exactly two 1-dimensional complex representations up to isomorphism. From the claim above, we know that it has $k+2$ conjugacy classes, so this is a complete and irredundant list of irreducible complex representations.

Problem 24.3.3
Describe the character of the two-dimensional irreducible $\mathbb{C}S_3$-module.

Solution: We can calculate directly from Example 24.2.7, suppose V is the irreducible 2-dimensional $\mathbb{C}S_3$-module, then we know that

$$\chi_V((1)) = \text{tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2, \quad \chi_V((12)) = \text{tr} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = 0,$$
$$\chi_V((23)) = \text{tr} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = 0, \quad \chi_V((13)) = \text{tr} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = 0,$$
$$\chi_V((123)) = \text{tr} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = -1, \quad \chi_V((1)) = \text{tr} \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = -1.$$

Problem 24.3.7
If $g \in G$ is an involution then $\chi(g) \in \mathbb{Z}$ and $\chi(g) \equiv \chi(1) \pmod 2$ for any character χ.

Solution: Let d be the dimension of the representation and $A = \rho(g)$ be the corresponding matrix representation for $g \in G$. g being an involution tells us that $A^2 - I = 0$. So the minimal of polynomial of A is a factor of $x^2 - 1$ with roots 1 and -1. We know that the characteistic polynomial and the minimal polynomial have the same roots and the trace of A is just the sum of all the roots of the characteistic polynomial, counting multiplicity. This means tr (A) is the sum of 1 and -1, so $\chi(g) = \text{tr } (A)$ is an integer. Suppose the minimal polynomial of A is $x - 1$ or $x + 1$. In this case $A = I$ or $A = -I$, so they have the same parity. Now assume the minimal polynomial is $x^2 - 1$. When $d = 2$, the characteistic polynomial of A is also $x^2 - 1$, so 2 and tr $(A) = 0$ has the same parity. Now suppose $d \geq 3$. We can write the characteistic polynomial as $(x^2 - 1)p(x)$ where $p(x)$ is a product of $x - 1$ and $x + 1$ and $\deg p(x) = d - 2$. Note that when the dimension d increase 1, we add 1 or -1 to the trace, so the parity of the trace and the parity of the dimension is always the same. This proves

$$\chi(g) \equiv \chi(1) = d \pmod 2.$$

Problem 24.3.12
Let C_n be the cyclic group of order n. Write dow explicit formulas for the central idempotents $e_1, \dots, e_n \in \mathbb{C}C_n$.

Solution: $C_n = \langle g \rangle$ is an abelian group, so every irreducible $\mathbb{C}C_n$ module has dimensional 1. From Example 24.2.3, let ξ be the nth primitive root of 1, then the irreducible C_n representation ρ_i is given by $\rho_i(g) = \xi^i$. Since every ρ_i is 1-dimensional, for any $0 \le k \le n-1$, we have $\chi_i(g^k) = \rho_i(g^k) = \xi^{ik}$. And

$$\chi_i(g^{-k}) = \chi_i(g^{n-k}) = \xi^{i(n-k)}.$$

By Lemma 24.3.11, we know that the central idempotent

$$e_i = \frac{1}{n}\sum_{k=0}^{n-1} \chi_i(g^{n-k})g^k = \frac{1}{n}\sum_{k=0}^{n-1} \xi^{i(n-k)}g^k.$$

Problem 24.3.18
For any finite dimensional $\mathbb{C}G$-modules V and W, we have

$$\dim \hom_{\mathbb{C}G}(V, W) = (\chi_V, \chi_W).$$

Solution: By the Wedderburn-Artin theorem, every finite dimenisonal $\mathbb{C}G$-module is semisimple and

$$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C}).$$

Let $\{L_1, \cdots, L_r\}$ be a complete and irrdundant set of irreducible $\mathbb{C}G$-modules. We know both V and W are semisimple, assume $V = \oplus_i V_i$ and $W = \oplus_j W_j$ where $V_i, W_j \in \{L_1, \dots, L_r\}$ are irreducible $\mathbb{C}G$-modules for any i, j. On the left hand side, we have

$$\hom_{\mathbb{C}G}(V, W) = \hom_{\mathbb{C}G}(\oplus_i V_i, \oplus_j W_j) = \oplus_{i,j} \hom_{\mathbb{C}G}(V_i, W_j).$$

On the left hand side, by Exercise 23.3.4, we have

$$(\chi_V, \chi_W) = (\chi_{\oplus_i V_i}, \chi_{\oplus_j W_j}) = (\sum_i \chi_{V_i}, \sum_j \chi_{W_j}) = \sum_{i,j} (\chi_{V_i}, \chi_{W_j})$$

since $(-,-)$ is an inner product. We only need to show that for two irreducible $\mathbb{C}G$-module L_i, L_j and the corresponding character χ_i, χ_j, we have

$$\dim \hom_{\mathbb{C}G}(L_i, L_j) = (\chi_i, \chi_j).$$

This is true. By Schur's lemma, we know that

$$\hom_{\mathbb{C}G}(L_i, L_j) = \begin{cases} \mathbb{C}, & \text{if } \ i = j, \\ 0, & \text{if } \ i \neq j. \end{cases}$$

By Theorem 24.3.16, we know that $\chi_1, \dots, \chi_r$ are orthonormal with respect to the inner product

$(-,-)$, so we also have

$$(\chi_i, \chi_j) = \begin{cases} 1, & \text{if } \ i = j, \\ 0, & \text{if } \ i \neq j. \end{cases}$$

This proves that

$$\dim \hom_{\mathbb{C}G}(L_i, L_j) = (\chi_i, \chi_j).$$