

**Problem 19.2.7**

Let  $S$  be a multiplicative subset of  $R$  and  $T$  be a multiplicative subset of  $S^{-1}R$ . Let

$$S_* = \left\{ r \in R \mid \begin{bmatrix} r \\ s \end{bmatrix} \in T \text{ for some } s \in S \right\}.$$

Then  $S_*$  is a multiplicative subset of  $R$  and there is a ring isomorphism  $S_*^{-1}R \cong T^{-1}(S^{-1}R)$ .

*Solution:* We first prove that  $S_*$  is a multiplicative subset of  $R$ . Suppose  $r_1, r_2 \in S_*$ , then there exist  $s_1, s_2 \in S$  such that  $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in T$ . We have  $\frac{r_1 r_2}{s_1 s_2} \in T$  since  $T$  is a multiplicative subset of  $S^{-1}R$ . This proves that  $r_1 r_2 \in S_*$ . So  $S_*$  is a multiplicative subset of  $R$ .

The elements in  $T^{-1}(S^{-1}R)$  can be written as  $\frac{\frac{r_1}{s_1}}{\frac{r_2}{s_2}}$  where  $\frac{r_2}{s_2} \in T$  and  $\frac{r_1}{s_1} \in S^{-1}R$ . We define a map

$$f : T^{-1}(S^{-1}R) \rightarrow S_*^{-1}R,$$

$$\frac{\frac{r_1}{s_1}}{\frac{r_2}{s_2}} \mapsto \frac{r_1 s_2}{r_2 s_1}.$$

This map is well-defined. Indeed, we know that  $\frac{r_2}{s_2} \in T$ , so  $r_2 \in S_*$ . Moreover, since  $S$  is a multiplicative subset of  $R$ , we know that  $s_1 s_2 \in S$ , so  $\frac{r_2}{s_2} \sim \frac{r_2 s_1}{s_2 s_1} \in T$  in  $S^{-1}R$ . This proves  $r_2 s_1 \in S_*$

and  $\frac{r_1 s_2}{r_2 s_1} \in S_*^{-1}R$ . Suppose  $\frac{\frac{r'_1}{s'_1}}{\frac{r'_2}{s'_2}}$  is another equivalent representative of  $\frac{\frac{r_1}{s_1}}{\frac{r_2}{s_2}}$  in  $T^{-1}(S^{-1}R)$ . Then there exists  $\frac{p}{q} \in T$  in  $S^{-1}R$  such that

$$\frac{p}{q} \left( \frac{r'_1}{s'_1} \cdot \frac{r_2}{s_2} - \frac{r_1}{s_1} \cdot \frac{r'_2}{s'_2} \right) = 0$$

Namely, in  $S^{-1}R$ , we have

$$\frac{p}{q} \cdot \frac{r'_1}{s'_1} \cdot \frac{r_2}{s_2} \sim \frac{p}{q} \cdot \frac{r_1}{s_1} \cdot \frac{r'_2}{s'_2}$$

There exists  $u \in S$  such that  $upq(r'_1 s_1 r_2 s'_2 - r_1 s'_1 r'_2 s_2) = 0$  in  $R$ . Note that  $uq^2 \in S$  since  $S$  is a multiplicative subset, and  $\frac{upq}{uq^2} = \frac{p}{q} \in T$ , so  $upq \in S_*$ . This implies that

$$\frac{r_1 s_2}{r_2 s_1} \sim \frac{r'_1 s'_2}{r'_2 s'_1}$$

in  $S_*^{-1}R$ . Thus, the map  $f$  is well-defined. It is easy to check that

$$f\left(\frac{\frac{r_1}{s_1}}{\frac{r_2}{s_2}}\right)f\left(\frac{\frac{r_3}{s_3}}{\frac{r_4}{s_4}}\right) = \frac{r_1 s_2}{r_2 s_1} \cdot \frac{r_3 s_4}{r_4 s_3} = \frac{r_1 s_2 r_3 s_4}{r_2 s_1 r_4 s_3} = f\left(\frac{\frac{r_1 r_3}{s_1 s_3}}{\frac{\frac{r_2 r_4}{s_2 s_4}}}\right).$$

This proves that  $f$  is a ring homomorphism.

Next, we want to show that  $f$  is injective and surjective. Suppose  $f(\frac{r_1}{\frac{s_1}{r_2}}) = 0$  in  $S_*^{-1}R$ . This implies there exists  $u \in S_*$  such that  $ur_1s_2 = 0$ . By definition, there exists  $s \in S$  such that  $\frac{u}{s} \in T$ . Since  $S$  is multiplicative, we have  $\frac{u}{s} \sim \frac{us_2}{ss_2} \in T$  and

$$\frac{us_2}{ss_2} \cdot \frac{r_1}{s_1} = \frac{us_2r_1}{ss_1s_2} = 0.$$

This proves that  $\frac{r_1}{\frac{s_1}{r_2}} = 0$  in  $T^{-1}(S^{-1}R)$ . So  $f$  is injective. For any  $p \in S_*$ , there exists  $s' \in S$  such that  $\frac{p}{s'} \in T$ . So we have

$$f\left(\frac{r}{\frac{p}{s'}}\right) = \frac{rs'}{ps'} = \frac{r}{p}$$

for all  $r \in R$  and  $p \in S_*$ . This proves  $f$  is surjective. Therefore, we can conclude that  $f$  is a ring isomorphism between  $S_*^{-1}R$  and  $T^{-1}(S^{-1}R)$ .

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### Problem 19.2.8

Let  $V$  be an  $R$ -module and  $S$  be a multiplicative subset of  $R$ . Then the map  $j_S : V \rightarrow S^{-1}V$ ,  $v \mapsto [\frac{v}{1}]$  is a homomorphism of  $R$ -modules and for every  $R$ -homomorphism from  $V$  to an  $S^{-1}R$ -module, there exists a unique  $S^{-1}R$ -module homomorphism  $\hat{f} : S^{-1}V \rightarrow V'$  such that the following diagram commutes:

$$\begin{array}{ccc} & V & \\ j_S \swarrow & & \searrow f \\ S^{-1}V & \xrightarrow{\hat{f}} & V' \end{array}$$

Moreover, this property characterizes  $S^{-1}V$  uniquely up to a (unique) isomorphism of  $S^{-1}R$ -modules.

*Solution:* We check that  $j_S$  is an  $R$ -module homomorphism. For any  $r \in R$  and  $v \in V$ , we have

$$rj_S(v) = r \cdot \frac{v}{1} = \frac{rv}{1} = j_S(rv).$$

Now given an  $R$ -module homomorphism  $f : V \rightarrow V'$  where  $V'$  is an  $S^{-1}R$ -module, we define the following map

$$\begin{aligned} \hat{f} : S^{-1}V &\rightarrow V', \\ \frac{v}{s} &\mapsto \frac{1}{s} \cdot f(v). \end{aligned}$$

This map  $\hat{f}$  is a well-defined  $S^{-1}R$ -module homomorphism. Indeed, for any  $\frac{r'}{s'} \in S^{-1}R$ , we have

$$\frac{r'}{s'} \cdot \hat{f}\left(\frac{v}{s}\right) = \frac{r'}{s'} \frac{1}{s} f(v) = \frac{r'}{ss'} f(v) = \hat{f}\left(\frac{r'v}{s's}\right).$$

Moreover, for any  $v \in V$ , we have

$$(\hat{f} \circ j_S)(v) = \hat{f}\left(\frac{v}{1}\right) = f(v).$$

This implies we have a commutative diagram

$$\begin{array}{ccc} & V & \\ \swarrow js & & \searrow f \\ S^{-1}V & \xrightarrow{\quad \bar{f} \quad} & V' \end{array}$$

The uniqueness can be seen from the commutativity of the diagram.

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**Problem 19.2.13**

Let  $f : V \rightarrow W$  be an  $R$ -module homomorphism.

- (1)  $S^{-1}(\text{Im } f) = \text{Im } (S^{-1}f)$  for any multiplicative subset  $S \subset R$ .
- (2)  $f$  is surjective if and only if  $f_M : V_M \rightarrow W_M$  is surjective for every maximal ideal  $M$  of  $R$ .

*Solution:*

- (1) Consider a short exact sequence of  $R$ -modules

$$0 \rightarrow \ker f \rightarrow V \xrightarrow{f} \text{Im } f \rightarrow 0.$$

The localization is an exact functor, so we have

$$0 \rightarrow S^{-1}\ker f \rightarrow S^{-1}V \rightarrow S^{-1}\text{Im } f \rightarrow 0.$$

By Lemma 19.2.12, we know that  $S^{-1}\ker f = \ker(S^{-1}f)$  and the cokernel of the map  $\ker(S^{-1}f) \rightarrow S^{-1}V$  is  $\text{Im } (S^{-1}f)$ . By exactness, we have

$$S^{-1}\text{Im } f \cong \text{Im } (S^{-1}f).$$

- (2) The "only if" part follows from the fact that the localization functor is exact. Conversely, suppose  $f_M : V_M \rightarrow W_M$  is surjective for every maximal ideal  $M$  of  $R$ . Consider the short exact sequence

$$0 \rightarrow \text{Im } f \rightarrow W \rightarrow W/\text{Im } f \rightarrow 0.$$

Localize at  $M$ , and we obtain a short exact sequence

$$0 \rightarrow (\text{Im } f)_M \rightarrow W_M \rightarrow (W/\text{Im } f)_M \rightarrow 0$$

This tells us that  $(W/\text{Im } f)_M \cong W_M/(\text{Im } f)_M$ . By surjectivity of  $f_M$  and what we have proved

in (1), we have

$$\begin{aligned}
0 &= \operatorname{coker} f_M \\
&= W_M / \operatorname{Im} f_M \\
&= W_M / (\operatorname{Im} f)_M \\
&= (W / \operatorname{Im} f)_M \\
&= (\operatorname{coker} f)_M.
\end{aligned}$$

This implies that for every maximal ideal  $M$  of  $R$ ,  $(\operatorname{coker} f)_M = 0$ . By Exercise 19.2.11, we have  $\operatorname{coker} f = 0$ , so the map  $f : V \rightarrow W$  is surjective.

**Problem 19.2.15**

Let  $S$  be a proper multiplicative subset of  $R$ , and  $V, W$  be  $R$ -modules. Then

$$S^{-1}V \otimes_{S^{-1}R} S^{-1}W \cong S^{-1}(V \otimes_R W).$$

*Solution:* We define the following map

$$\begin{aligned}
f : S^{-1}V \otimes_{S^{-1}R} S^{-1}W &\rightarrow S^{-1}(V \otimes_R W), \\
\frac{v}{s_1} \otimes \frac{w}{s_2} &\mapsto \frac{v \otimes w}{s_1 s_2}.
\end{aligned}$$

We check that this is an  $S^{-1}R$ -module homomorphism. For any  $\frac{r}{s} \in S^{-1}R$ , we have

$$\frac{r}{s} f\left(\frac{v}{s_1} \otimes \frac{w}{s_2}\right) = \frac{r}{s} \frac{v \otimes w}{s_1 s_2} = \frac{rv \otimes w}{ss_1 s_2} = f\left(\frac{rv}{ss_1} \otimes \frac{w}{s_2}\right).$$

Next, we show that  $f$  is both injective and surjective. Suppose  $f\left(\frac{v}{s_1} \otimes \frac{w}{s_2}\right) = 0$  for some  $\frac{v}{s_1} \in V$  and  $\frac{w}{s_2} \in W$ . This implies that  $\frac{v \otimes w}{s_1 s_2} = 0$  in  $S^{-1}(V \otimes_R W)$ . There exists  $s \in S$  such that  $s(v \otimes w) = 0$ . This implies that

$$s\left(\frac{v}{s_1} \otimes \frac{w}{s_2}\right) = \frac{1}{s_1 s_2}(sv \otimes w) = 0.$$

This proves injectivity. On the other hand, for any  $\frac{v \otimes w}{s}$  in  $S^{-1}(V \otimes_R W)$ , there exists  $\frac{v}{s} \in S^{-1}V$  and  $\frac{w}{1} \in S^{-1}W$  such that

$$f\left(\frac{v}{s} \otimes \frac{w}{1}\right) = \frac{v \otimes w}{s}.$$

This proves that  $f$  is surjective. Therefore, we can conclude that  $f$  is an  $S^{-1}R$ -module isomorphism between  $S^{-1}V \otimes_{S^{-1}R} S^{-1}W$  and  $S^{-1}(V \otimes_R W)$ .

**Problem 19.2.16**

Let  $V$  be an  $R$ -module. Then  $V$  is flat if and only if  $V_M$  is flat for every maximal ideal  $M$  of  $R$ .

*Solution:* Assume  $V$  is flat. Given an injective map  $f : A \rightarrow B$ , by Proposition 19.2.9, for any maximal ideal  $M$  of  $R$ , we have

$$A \otimes_R V_M = A \otimes_R (R_M \otimes_R V) = (A \otimes_R R_M) \otimes_R V = A_M \otimes_R V.$$

The isomorphism is functorial, so we have a commutative diagram

$$\begin{array}{ccc} A \otimes_R V_M & \xrightarrow{f \otimes id_{V_M}} & B \otimes_R V_M \\ \cong \downarrow & & \downarrow \cong \\ A_M \otimes_R V & \xrightarrow{f_M \otimes id_V} & B_M \otimes_R V \end{array}$$

Since  $V$  is flat, by Lemma 19.2.12, the map  $f_M \otimes id_V$  is still injective, and thus the map  $f \otimes id_{V_M}$  is injective. This proves that  $V_M$  is flat for every maximal ideal  $M$  of  $R$ .

Conversely, assume  $V_M$  is flat for every maximal ideal  $M$ . Given an injective map  $f : A \rightarrow B$ , consider the map

$$f_M : A_M \rightarrow B_M$$

where  $M$  is a maximal ideal  $M$  of  $R$ . By Lemma 19.2.12,  $f_M$  is injective as  $f$  is injective. We know that  $V_M$  is flat, so the map

$$f_M \otimes id_{V_M} : A_M \otimes_{R_M} V_M \rightarrow B_M \otimes_{R_M} V_M$$

is still injective. Note that by Exercise 19.2.15,  $A_M \otimes_{R_M} V_M = (A \otimes_R V)_M$ . So the map

$$(f \otimes id_V)_M : (A \otimes_R V)_M \rightarrow (B \otimes_R V)_M$$

is injective. Use Lemma 19.2.12 again, and we know that the map

$$f \otimes id_V : A \otimes_R V \rightarrow B \otimes_R V$$

is injective. This proves that  $V$  is flat.

### Problem 19.3.3

Let  $\alpha \in \mathbb{C}$  be algebraic over  $\mathbb{Q}$ . Then  $\alpha$  is integral over  $\mathbb{Z}$  if and only if  $\text{irr}(\alpha; \mathbb{Q}) \in \mathbb{Z}[x]$ .

*Solution:* Let  $f$  be the irreducible polynomial of  $\alpha$  over  $\mathbb{Q}$ . Suppose  $f \in \mathbb{Z}[x]$ . Then  $f$  can be written as

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

where  $a_i \in \mathbb{Z}$  for all  $i$ . This proves that  $\alpha$  is integral over  $\mathbb{Z}$ .

Conversely, suppose  $\alpha$  is integral over  $\mathbb{Z}$ . Then there exists a monic polynomial such that  $\alpha$  is a root. Let

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_n$$

be the irreducible factor of this polynomial with  $f(\alpha) = 0$ . Let  $g$  be the minimal polynomial of  $\alpha$  over  $\mathbb{Q}$ . We know that  $f \in \mathbb{Z}[x] \subseteq \mathbb{Q}[x]$  and  $f(\alpha) = 0$ , so  $g|f$  over  $\mathbb{Q}$ . This implies  $\deg g \leq \deg f$ . On the other hand, there exists  $m \in \mathbb{Z}$  such that  $mg \in \mathbb{Z}[x]$  and since  $f$  is irreducible over  $\mathbb{Z}$ , we

have  $f|mg$  over  $\mathbb{Z}$ . This implies  $\deg f \leq \deg mg = \deg g$ . So we have  $\deg f = \deg g$  and this tells us that  $f = g$ .