Math 636 Homework #3
Due Friday, April 25

1. Compute both $\mathrm{Tor}_i(A,B)$ and $\mathrm{Ext}^i(A,B)$ (for all i) in the following cases:
   (a) $A=\mathbb{Z}/9$ and $B=\mathbb{Z}/6$.
   (b) $A=\mathbb{Z}/9$ and $B=\mathbb{Z}$.
   (c) $A=\mathbb{Z}^2\oplus\mathbb{Z}/4\oplus\mathbb{Z}/5\oplus\mathbb{Z}/10$ and $B=\mathbb{Z}\oplus\mathbb{Z}/3\oplus\mathbb{Z}/4\oplus\mathbb{Z}/6$.

2. Let A be an abelian group and let $G_*\to A\to 0$ be a free resolution. Let B be another abelian group and let $J_*\to B\to 0$ be a free resolution.
   (a) Given a map $f\colon A\to B$, prove that there are maps $F_i\colon G_i\to J_i$ making all squares commute—we call the chain map $\{F\colon G_*\to J_*\}$ a "lifting" of the map f.
   (b) Prove that if $\{F'\colon G_*\to J_*\}$ is another lifting of f then the chain maps F and F' are chain homotopic.
   (c) If C is another abelian group one gets an induced map $F\otimes id\colon G_*\otimes C\to J_*\otimes C$ and therefore an induced map on homology groups $f_*\colon \mathrm{Tor}_i(A,C)\to\mathrm{Tor}_i(B,C)$. Since any two choices for F are chain homotopic, this f_* is well-defined.
   Use the above procedure to calculate the maps
   $$j_*\colon \mathrm{Tor}_1(\mathbb{Z}/2,\mathbb{Z}/2)\to\mathrm{Tor}_1(\mathbb{Z}/4,\mathbb{Z}/2)\quad\text{and}\quad k_*\colon \mathrm{Tor}_1(\mathbb{Z}/4,\mathbb{Z}/2)\to\mathrm{Tor}_1(\mathbb{Z}/2,\mathbb{Z}/2)$$
   induced by the maps $j\colon\mathbb{Z}/2\hookrightarrow\mathbb{Z}/4$ (sending 1 to 2) and $k\colon\mathbb{Z}/4\to\mathbb{Z}/2$ (sending 1 to 1).

3. Do Exercises 3.7 and 3.8 from the "Notes on Homological Algebra" on the course webpage. Just state the answers, don't bother to give proofs (but make sure you understand why the answers are correct).

4. Suppose X is a finite CW-complex for which
   $$H_0(X;\mathbb{Z}/2)=\mathbb{Z}/2,\quad H_1(X;\mathbb{Z}/2)=(\mathbb{Z}/2)^3,\quad H_2(X;\mathbb{Z}/2)=0,\quad H_3(X;\mathbb{Z}/2)=H_4(X;\mathbb{Z}/2)=\mathbb{Z}/2$$
   and $H_i(X;\mathbb{Z}/2)=0$ for all $i\geq 5$.
   (a) Determine as much as you can about $H_*(X;\mathbb{Z})$.
   (b) Suppose you are also told that $H_2(X;\mathbb{Z}/3)=\mathbb{Z}/3$ and $H_3(X;\mathbb{Z}/3)=0$. What else can you say about $H_*(X;\mathbb{Z})$ now?
   (c) Suppose Y is a space with finitely-generated homology groups and you are told $H_i(Y;\mathbb{Z}/p)=0$ for a specific prime p. What can you deduce about $H_i(Y)$ and $H_{i-1}(Y)$?

5. (a) In a few weeks we will prove that for a certain class of n-manifolds M one always has Poincaré Duality: $H_i(M;\mathbb{Z})\cong H^{n-i}(M;\mathbb{Z})$. Assuming this, as well as the fact that all the homology groups of M are finitely-generated, explain why the Universal Coefficient Theorems then imply that
      (1) the rank of $H_i(M;\mathbb{Z})$ is the same as the rank of $H_{n-i}(M;\mathbb{Z})$, for all i.
      (2) the torsion part of $H_i(M;\mathbb{Z})$ is the same as the torsion part of $H_{n-i-1}(M;\mathbb{Z})$, for all i.
   (b) Suppose M is a 5-manifold for which Poincaré Duality holds. Given that $H_0(M)=\mathbb{Z}$, $H_1(M)=\mathbb{Z}^2\oplus\mathbb{Z}/4$, and $H_2(M)=\mathbb{Z}\oplus\mathbb{Z}/5$, compute $H_i(M)$, $H_i(M;\mathbb{Z}/2)$, and $H_i(M;\mathbb{Z}/5)$ for all i.

6. Instead of doing homological algebra in the context of abelian groups, we can do it in the context of R-modules for any ring R. In this problem consider $R=\mathbb{Z}/p^2$, so that an R-module is just an abelian group where all elements are killed by p^2. The free R-modules are the ones of the form $\bigoplus_i\mathbb{Z}/p^2$.
   Let $M=\mathbb{Z}/p$. Construct a free resolution of M (in the category of R-modules) and use this to compute $\mathrm{Tor}_i^R(M,M)$ for all i (the superscript R just is a reminder that all computations are to be done in the category of R-modules). You may use the fact that the tensor in R-modules is the same as the tensor in abelian groups.