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Course: MATH 634 - Algebraic Topology
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Homework - Week 9
ID: 952091294
Term: Fall 2024
Due Date: 5^{th} December, 2024

Problem 2.2.16
Let $\Delta^n = [v_0, \dots, v_n]$ have its natural Δ-complex structure with k-simplices $[v_{i_0}, \dots, v_{i_k}]$ for $i_0 < \cdots < i_k$. Compute the ranks of the simplicial (or cellular) chain groups $\Delta_i(\Delta^n)$ and the subgroups of cycles and boundaries. [Hint:Pascal's triangle.] Apply this to show that the k-skeleton of Δ^n has homology groups $\tilde{H}_i((\Delta^n)^k)$ equal to 0 for $i < k$ and free of rank $\binom{n}{k+1}$ for $i = k$.

Solution: Let C_k denote the kth simplicial chain group and we have a chain complex of abelian groups

$$0 \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \cdots \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

For $0 \le k \le n$, C_k is generated by k-simplices in a standard n-simplex Δ^n, choosing a k-simplex is the same as choosing $(k+1)$ vertices, so $\operatorname{rank} C_k = \binom{n+1}{k+1}$. Now write $Z_k = \ker d_k \subset C_k$ as the subgroup of k-cycles and $B_k = \operatorname{Im} d_{k+1} \subset C_k$ as the subgroup of k-boundaries. The $H_k = Z_k/B_k$ is the k-th simplicial homology group of Δ_n. We have two short exact sequences

$$0 \longrightarrow Z_k \longrightarrow C_k \longrightarrow B_{k-1} \longrightarrow 0$$

$$0 \longrightarrow B_k \longrightarrow Z_k \longrightarrow H_k \longrightarrow 0$$

This gives us

$$\begin{aligned} \operatorname{rank} Z_k &= \operatorname{rank} B_k + \operatorname{rank} H_k, \\ \operatorname{rank} C_k &= \operatorname{rank} Z_k + \operatorname{rank} B_{k-1}. \end{aligned}$$

Note that Δ^n is contractible so $\operatorname{rank} H_0 = 1$ and $\operatorname{rank} H_k = 0$ for $k \neq 0$. Moreover, $B_{-1} = \operatorname{Im} d_0 = 0$, so

$$\begin{aligned} \operatorname{rank} Z_0 &= \operatorname{rank} C_0 = \binom{n+1}{1} = n+1, \\ \operatorname{rank} B_0 &= \operatorname{rank} Z_0 - \operatorname{rank} H_0 = n+1-1 = n. \end{aligned}$$

For $k > 0$, we can inductively calculate

$$\begin{aligned} \operatorname{rank} Z_k &= \operatorname{rank} C_k - \operatorname{rank} B_{k-1}, \\ \operatorname{rank} B_k &= \operatorname{rank} Z_k - \operatorname{rank} H_k = \operatorname{rank} Z_k \end{aligned}$$

Using the law of Pascal's triangle, we can see that for $1 \leq k \leq n$,

$$\begin{aligned}
\operatorname{rank} Z_k &= \binom{n+1}{k+1} - \binom{n}{k} = \binom{n}{k+1}, \\
\operatorname{rank} B_k &= \operatorname{rank} Z_k = \binom{n}{k+1}.
\end{aligned}$$

The simplicial chain complex of the k-skeleton $(\Delta^n)^k$ is the tuncated version

$$0 \xrightarrow{0} C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

So for $0 \leq i \leq k-1$, we have

$$\tilde{H}_i((\Delta^n)^k) = \tilde{H}_i(\Delta^n) = 0.$$

and

$$\tilde{H}_k((\Delta^n)^k) = \ker d_k$$

is free abelian and has rank $\binom{n}{k+1}$.

Problem 2.2.17

Show the isomorphism between cellular and singular homology is natural in the following sense: A map $f : X \to Y$ that is cellular, satisfying $f(X^n) \subset Y^n$ for all n, induces a chain map $f_\sharp$ between the cellular chain complexes of X and Y, and the map $f_* : H_n^{CW}(X) \to H_n^{CW}(Y)$ induced by this chain map corresponds to $f_* : H_n(X) \to H_n(Y)$ under the isomorphism $H_n^{CW} \approx H_n$.

Solution: For every n, we have a map of pairs:

$$f : (X^{n+1}, X^n) \to (Y^{n+1}, Y^n)$$

which induces the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & C_n(X^n) \arrow[r] \arrow[d] & C_n(X^{n+1}) \arrow[r] \arrow[d] & C_n(X^n, X^{n+1}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & C_n(Y^n) \arrow[r] & C_n(Y^{n+1}) \arrow[r] & C_n(Y^n, Y^{n+1}) \arrow[r] & 0
\end{tikzcd}$$

The naturality of the induced long exact sequence gives us

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H_n(X^n) \arrow[r, "j_n"] \arrow[d, "f'_n"] & H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[d, "f_n"] & H_{n-1}(X^{n-1}) \arrow[r] \arrow[d, "f'_{n-1}"] & \cdots \\
0 \arrow[r] & H_n(Y^n) \arrow[r, "j'_n"] & H_n(Y^n, Y^{n-1}) \arrow[r, "\partial'_n"] & H_{n-1}(Y^{n-1}) \arrow[r] & \cdots
\end{tikzcd}$$

Using $f_n \circ j_n = j'_n \circ f'_n$ and $f'_{n-1} \circ \partial_n = \partial'_n \circ f_n$, we claim that we have a chain map $f_\sharp$ between cellular chain complex:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[r, "d_n"] \arrow[d, "f_n"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(Y^{n+1}, Y^n) \arrow[r, "d'_{n+1}"] & H_n(Y^n, Y^{n-1}) \arrow[r, "d'_n"] & H_{n-1}(Y^{n-1}, Y^{n-2}) \arrow[r] & \cdots
\end{tikzcd}$$

To check this diagram indeed commutes, we can see that by the definition of the boundary map

$$\begin{aligned}
f_n \circ d_{n+1} &= f_n \circ (j_n \circ \partial_{n+1}) \\
&= (f_n \circ j_n) \circ \partial_{n+1} \\
&= (j'_n \circ f'_n) \circ \partial_{n+1} \\
&= j'_n \circ (f'_n \circ \partial_{n+1}) \\
&= j'_n \circ (\partial'_{n+1} \circ f_{n+1}) \\
&= (j'_n \circ \partial'_{n+1}) \circ f_{n+1} \\
&= d'_{n+1} \circ f_{n+1}
\end{aligned}$$

This chain map induces a map $f_* : H_n^{CW}(X) \to H_n^{CW}(Y)$ for every n.

To see that this map corresponds to the map $f_* : H_n(X) \to H_n(Y)$ under the isomorphism $H_n^{CW} \cong H_n$, recall from Theorem 2.35, $H_n(X)$ is identified with $H_n(X)/\operatorname{Im} \partial_{n+1}$, using the naturality of induced long exact sequence, we have

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & H_n(X^n) \arrow[r] \arrow[d] & H_n(X^{n+1}) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & H_{n+1}(Y^{n+1}, Y^n) \arrow[r, "\partial'_{n+1}"] & H_n(Y^n) \arrow[r] & H_n(Y^{n+1}) \arrow[r] & 0
\end{tikzcd}$$

This shows that the map $f_* : H_n(X) \to H_n(Y)$ is equivalent to the map

$$f_* : H_n(X^n)/\operatorname{Im} \partial_{n+1} \to H_n(Y^n)/\operatorname{Im} \partial'_{n+1}.$$

We know that $j_n : H_n(X^n) \xrightarrow{\sim} H_n(X^n, X^{n-1})$ induces an isomorphism $j_{n,*} : H_n(X) \xrightarrow{\sim} H_n^{CW}(X)$, the following diagram commutes:

$$\begin{tikzcd}
H_n(X^n)/\operatorname{Im} \partial_{n+1} \arrow[r, "j_{n,*}"] \arrow[d] & \ker d_n/\operatorname{Im} d_{n+1} \arrow[d] \\
H_n(Y^n)/\operatorname{Im} \partial'_{n+1} \arrow[r, "j'_{n,*}"] & \ker d'_n/\operatorname{Im} d'_{n+1}
\end{tikzcd}$$

And the commutativity comes from the fact that j_n commutes with f_* in the induced long exact sequence.

Problem 2.2.18

For a CW pair (X, A) show there is a relative cellular chain complex formed by the groups

$H_i(X^i, X^{i-1} \cup A^i)$, having homology groups isomorphic to $H_n(X, A)$.

Solution: We first establish some preliminary facts similar to lemma 2.34 in the book.
Claim:

(1) $H_k(X^n, X^{n-1} \cup A^n) = 0$ for $k > n$ and is free abelian for $k = n$, with a basis in one-to-one correspondence with n-cells in X excluding the n-cells in A.

(2) $H_k(X^n \cup A^{n+1}, A^{n+1}) \cong H_k(X^n, A^n) = 0$ for $k > n$. If X is finite dimensional, then $H_k(X, A) = 0$ for $k > \dim X$.

(3) The map $H_k(X^n, A^n) \to H_k(X, A)$ induced by the inclusion of pairs $(X^n, A^n) \hookrightarrow (X, A)$ is an isomorphism for $k < n$ and surjective for $k = n$.

Proof:

(1) Since (X, A) is a CW pair, $(X^n, X^{n-1} \cup A^n)$ is also a good pair. We have a isomorphism

$$\tilde{H}_k(X^n, X^{n-1} \cup A^n) \xrightarrow{\sim} \tilde{H}_k(X^n/(X^{n-1} \cup A^n)).$$

Note that the $(n-1)$-skeleton along with any n-cells in A collasped into a point in the quotient space $X^n/(X^{n-1} \cup A^n)$. This space is a wedge sum of n-spheres corresponding each n-cells in X excluding the n-cells in A. So $H_k(X^n, X^{n-1} \cup A^n) = 0$ if $k > n$ and is free abelain if $k = n$.

(2) Note that A is a subcomplex of X, so we have $A^{n+1} \cap X^n = A^n$. The isomorphism $H_k(X^n \cup A^{n+1}, A^{n+1}) \cong H_k(X^n, A^n)$ is given by the excision. Using the fact that (X^n, A^n) is a good pair, we have $H_k(X^n, A^n) \cong \tilde{H}_k(X^n/A^n)$. The quotient space X^n/A^n inhabits a natural CW complex structure of dimensional n, so

$$H_k(X^n, A^n) \cong \tilde{H}_k(X^n/A^n) = 0$$

if $k > n$.

(3) The natural inclusions $A^n \to A$ and $X^n \to X$ give a map between long exact sequences.

$$\begin{tikzcd}
H_k(A^n) \arrow[r] \arrow[d] & H_k(X^n) \arrow[r] \arrow[d] & H_k(X^n, A^n) \arrow[r] \arrow[d] & H_{k-1}(A^n) \arrow[r] \arrow[d] & H_{k-1}(X^n) \arrow[d] \\
H_k(A) \arrow[r] & H_k(X) \arrow[r] & H_k(X, A) \arrow[r] & H_{k-1}(A) \arrow[r] & H_{k-1}(X)
\end{tikzcd}$$

By Lemma 2.34, the outer four maps are isomorphism when $k < n$, by 5 lemma, the middle map is also an isomorphism.

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Consider the triple $A^n \subset X^{n-1} \cup A^n \subset X^n$, which induces a long exact sequence of relative homology groups

$$\cdots \to H_k(X^{n-1} \cup A^n, A^n) \to H_k(X^n, A^n) \to H_k(X^n, X^{n-1} \cup A^n) \to H_{k-1}(X^{n-1} \cup A^n, A^n) \to \cdots.$$

Consider the following diagram

$$\begin{tikzcd}
H_n(X^{n+1}, X^n \cup A^{n+1}) = 0 & & \\
H_n(X^{n+1}, A^{n+1}) \cong H_n(X, A) \arrow[u] & H_n(X^{n-1} \cup A^n, A^n) = 0 \arrow[d] & \\
H_n(X^n \cup A^{n+1}, A^{n+1}) \arrow[u, two heads] \arrow[r, "\text{excision}", "e_n"'] & H_n(X^n, A^n) \arrow[d, hook, "j_n"] & \\
H_{n+1}(X^{n+1}, X^n \cup A^{n+1}) \arrow[u, "\partial_{n+1}"] \arrow[r, "d_{n+1}"] & H_n(X^n, X^{n-1} \cup A^n) \arrow[r, "d_n"] \arrow[d, "\partial_n"] & H_{n-1}(X^{n-1}, X^{n-2} \cup A^{n-1}) \\
& H_{n-1}(X^{n-1} \cup A^n, A^n) \arrow[r, "\text{excision}", "e_{n-1}"'] & H_{n-1}(X^{n-1}, A^{n-1}) \arrow[u, hook, "j_{n-1}"'] \\
& & H_{n-1}(X^{n-2} \cup A^{n-1}, A^{n-1}) = 0 \arrow[u]
\end{tikzcd}$$

Note that the vertical column in the above diagram is exact. We define the n-th chain group as $H_n(X^n, X^{n-1} \cup A^n)$, which is free abelian with generators corresponding to n-cells in X but not in A. The boundary map d_n is defined to be the cocomposition $d_n = j_{n-1} \circ e_{n-1} \circ \partial_n$. It is easy to see that we have $d_n \circ d_{n+1} = 0$ since

$$d_n \circ d_{n+1} = j_{n-1} \circ e_{n-1} \circ (\partial_n \circ j_n) \circ e_n \circ \partial_{n+1} = 0.$$

Finally we are going to show that the homology of the above chain complex is isomorphic to the relative homology group $H_\bullet(X, A)$. Because j_{n-1} is injective and e_{n-1} is an isomorphism, we have

$$\ker d_n = \ker \partial_n = \operatorname{Im} j_n \cong H_n(X^n, A^n) \cong H_n(X^n \cup A^{n+1}, A^{n+1}).$$

Again j_n is injective implies that $\operatorname{Im} d_{n+1} = \operatorname{Im} \partial_{n+1}$, so

$$\ker d_n / \operatorname{Im} d_{n+1} \cong H_n(X^n \cup A^{n+1}, A^{n+1}) / \operatorname{Im} \partial_{n+1} \cong H_n(X^{n+1}, A^{n+1}) \cong H_n(X, A).$$

Problem 2.2.20
For finite CW complexes X and Y, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

Solution: Suppose X has dimension m and in the dimension $0 \le i \le m$, the number of i-cells is denoted by a_i. Similarly, Y has dimension n and the number of j-cells is denoted by b_j. So by definition of Euler characteristics, we have

$$\begin{aligned}
\chi(X)\chi(Y) &= (\sum_{i=0}^{m} (-1)^i a_i)(\sum_{j=0}^{n} (-1)^j b_j) \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{i+j} a_i b_j
\end{aligned}$$

On the other hand, we know that the product $X \times Y$ is a CW complex of dimension mn, and it has $\sum_{i+j=k} a_i b_j$ k-cells for each $0 \le k \le mn$. So by definition

$$\begin{aligned}\chi(X \times Y) &= \sum_{k=0}^{mn} (-1)^k (\sum_{i+j=k} a_i b_j) \\ &= \sum_{k=0}^{mn} \sum_{i+j=k} (-1)^{i+j} a_i b_j\end{aligned}$$

reordering the summation and we can see that they are equal.

Problem 2.2.27
The short exact sequence

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$$

always split, but why does this not always yield splittings

$$H_n(X) \approx H_n(A) \oplus H_n(X, A).$$

Solution: Recall that $C_n(X, A) = C_n(X)/C_n(A)$ is generated by the maps $\Delta^n \to X$ whose image is not completely in A. So it is a free abelian group and the short exact sequence

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$$

splits. However, there is no reason that the for a general pair (X, A), we have

$$H_n(X) \approx H_n(A) \oplus H_n(X, A).$$

Consider the following case X is a closed 2-disk D^2 and $A \subset X$ is its boundary $\partial D^2 = S^1$. This is a good pair and by Proposition 2.22, $H_2(X, A)$ is isomorphic to $\tilde{H}_2(X/A) \cong \tilde{H}_2(S^2) = \mathbb{Z}$. But X is contractible and $H_2(X) = 0$.

Problem 2.3.2
Define a candidate for a reduced homology theory on CW complexes by $\tilde{h}_n(X) = \prod_i \tilde{H}_i(X)/\oplus_i \tilde{H}_i(X)$. Thus $\tilde{h}_n(X)$ is independent of n and is zero if X is finite dimensional, but is not identically zero, for example for $X = \vee_i S^i$. Show that the axiom for a homology theory are satisfied except that the wedge axiom fails.

Solution: We check the first two axioms and give a counter example for the third axiom.

(1) Let X and Y be two homotopy equivalent spaces. For every i, we have $\tilde{H}_i(X) \cong \tilde{H}_i(Y)$ by the homotopy invariance of reduced singular homology. Then

$$\tilde{h}_n(X) = \prod_i \tilde{H}_i(X)/\oplus_i \tilde{H}_i(X) \cong \prod_i \tilde{H}_i(Y)/\oplus_i \tilde{H}_i(Y) = \tilde{h}_n(Y)$$

for all n.

(2) Let (X, A) be a CW pair. For each i, we have a connecting homomorphsim $\partial_i : \tilde{H}_i(X/A) \to \tilde{H}_{i-1}(A)$. Consider a homomorphsim

$$\partial : \prod_i \tilde{H}_i(X/A) \to \prod_i \tilde{H}_{i-1}(A)$$

where on each component $\tilde{H}_i(X/A)$, it is just ∂_i. The exactness is preseved in the long exact sequence. Same as the naturality.

(3) Consider $X = \vee_i S^i$. For every n, by Corollary 2.25, we have

$$\tilde{H}_k(\vee_i S^i) \cong \oplus_i \tilde{H}_k(S^i).$$

Note that $\tilde{H}_k(S^i) = \mathbb{Z}$ for $i = k$ and equal to 0 otherwise. So we have $\tilde{H}_k(X) = \mathbb{Z}$ for every $k \geq 0$. This implies that

$$\tilde{h}_n(X) = (\prod_k \mathbb{Z})/(\oplus_k \mathbb{Z})$$

is non trivial. On the other hand, $\tilde{h}_n(S^i) = \mathbb{Z}/\mathbb{Z} = 0$ is trivial. So we have

$$\oplus_i \tilde{h}_n(S^i) \neq \tilde{h}_n(\vee_i S^i).$$

Problem 2.3.3
Show that if $\tilde{h}$ is a reduced homology theory, then $\tilde{h}_n(point) = 0$ for all n. Deduce that there are suspension isomorphism $\tilde{h}_n(X) \approx \tilde{h}_{n+1}(SX)$ for all n.

Solution: Let X be a CW complex and consider the identity map $id : X \to X$. This gives a CW pair (X, X) and note that $X/X = \{point\}$. By the second axiom of homology theory we have a long exact sequence

$$\cdots \xrightarrow{\partial} \tilde{h}_n(X) \xrightarrow{id_*} \tilde{h}_n(X) \xrightarrow{q_*} \tilde{h}_n(point) \xrightarrow{\partial} \tilde{h}_{n-1}(X) \to \cdots$$

We know that id_* is an isomorphism and using exactness, we can see that $\tilde{h}_n(point) = 0$ for all n.

Let CX denote the cone of X and we know that it is homotopy equivalent to a point. We have $\tilde{h}_n(CX) \cong \tilde{h}_n(point) = 0$. Consider the quotient space CX/X and by the second axiom, we have a long exact sequence

$$\cdots \to \tilde{h}_{n+1}(CX) \to \tilde{h}_{n+1}(CX/X) \to \tilde{h}_n(X) \to \tilde{h}_n(CX) \to \cdots$$

By exactness we have $\tilde{h}_{n+1}(CX/X) \cong \tilde{h}_n(X)$. Now Consider the suspension SX and two cones $A \cong B \cong CX$ whose intersection is homeomorphic to $A \cap B = X$. We have a homeomorphism $SX/CX \cong CX/X$. Moreover, we apply the second axiom to the quotient space SX/CX

$$\cdots \to \tilde{h}_{n+1}(CX) \to \tilde{h}_{n+1}(SX) \to \tilde{h}_{n+1}(SX/CX) \to \tilde{h}_n(CS) \to \cdots$$

By exactness we have $\tilde{h}_{n+1}(SX) \cong \tilde{h}_{n+1}(SX/CX) \cong \tilde{h}_n(X)$ for all n.