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Course: MATH 649 - Abstract Algebra
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**Homework - Week 7**
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**Problem 19.3.18**
If $R$ is an integrally closed domain with quotient field $\mathbb{F}$, and $f, g \in \mathbb{F}[x]$ are monic with $fg \in \mathbb{R}[x]$, then $f, g \in R[x]$.

*Solution:* Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. Suppose $f$ and $g$ can be written as

$$f(x) = \prod_{i=1}^{n}(x - \alpha_i), \quad g(x) = \prod_{j=1}^{m}(x - \beta_j)$$

in $\overline{\mathbb{F}}$. The roots $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ satisfies the monic polynomial $fg \in R[x]$, so they are integral over $R$ in $\overline{F}$. Moreover, the coefficients of $f$ and $g$ can be written as symmetric polynomials of these roots, and the integral elements over $R$ in $\overline{F}$ form a subring, so all the coefficients of $f$ and $g$ are integral over $R$. Both $f, g \in \mathbb{F}[x]$, so $f, g$ have coefficients in $\mathbb{F}$ and are integral over $R$, since $R$ is integrally closed in $\mathbb{F}$, this implies $f, g \in R[x]$.

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**Problem 19.4.8**
Show that the conclusion of the Incomparability theorem fails for the ring extension $\mathbb{F}[x] \subseteq \mathbb{F}[x, y]$.

*Solution:* Consider the ideal $(x)$ and $(x, y)$ in $\mathbb{F}[x, y]$. Note that $\mathbb{F}[x, y]/(x) = \mathbb{F}[y]$ and $\mathbb{F}[x, y]/(x, y) = \mathbb{F}$ are domains, so $(x, y)$ and $(x)$ are prime ideals. We know that

$$(x) = (x) \cap \mathbb{F}[x] = (x, y) \cap \mathbb{F}[x]$$

and $(x) \subseteq (x, y)$. But $y \in (x, y)$ and $y \notin (x, y)$. So $(x)$ and $(x, y)$ are different prime ideals.

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**Problem 19.4.13**
True or false? Let $A \supseteq R$ be an integral ring extension. If every non-zero prime ideal of $R$ is a maximal ideal, then every non-zero prime ideal of $A$ is also maximal.

*Solution:* This is false. Let $R = \mathbb{Z}$ and $A = \mathbb{Z}[x]/(x^2)$. Note that $\mathbb{Z}[x]/(x^2) = \mathbb{Z} \oplus \mathbb{Z}x$ with $x \cdot x = 0$. This implies that $A$ is finitely generated as an $R$-module, so $R \hookrightarrow A$ is an integral extension. We know $R = \mathbb{Z}$ is a PID, so every non-zero prime ideal in $R$ is maximal. On the other hand, consider two ideals $(x)$ and $(x, 2)$ in $A$, we have $A/(x) = \mathbb{Z}$ is a domain and $A/(x, 2) = \mathbb{Z}/2$ is a field. So $(x)$ is a prime ideal and $(x, 2)$ is a maximal ideal with $(x) \subseteq (x, 2)$.

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**Problem 19.4.15**

Consider the ring extension $\mathbb{Z} \subset \mathbb{Z}[\sqrt{5}]$.

(1) Find all prime ideals of $\mathbb{Z}[\sqrt{5}]$ which lie over the prime ideal $(5)$ of $\mathbb{Z}$.

(2) Find all prime ideals of $\mathbb{Z}[\sqrt{5}]$ which lie over the prime ideal $(3)$ of $\mathbb{Z}$.

(3) Find all prime ideals of $\mathbb{Z}[\sqrt{5}]$ which lie over the prime ideal $(2)$ of $\mathbb{Z}$.

*Solution:* Let $R = \mathbb{Z}[\sqrt{5}]$ and $p \subseteq R$ is a prime ideal.

(1) Suppose $p \cap \mathbb{Z} = (5)$. This means $5 \in p$. We know that the radical $\sqrt{(5)} = (\sqrt{5})$ is a prime ideal in $R$ containing 5. Note that $R/(5) = \mathbb{Z}/5$ is a field. This implies that $(\sqrt{5}) \subseteq R$ is the only prime ideal lying over $(5) \subseteq \mathbb{Z}$.

(2) Suppose $p \cap \mathbb{Z} = (3)$. This means $3 \in p$. Note that

$$\begin{aligned} R/(3) &\cong \mathbb{Z}[\sqrt{5}]/(3) \\ &\cong \mathbb{F}_3[\sqrt{5}] \\ &\cong \mathbb{F}_3[x]/(x^2 - 5) \\ &\cong \mathbb{F}_3[x]/(x^2 + 1) \end{aligned}$$

It is easy to check that none of $0, 1, 2$ are not roots of $x^2 + 1$ in $\mathbb{F}_3$. So $x^2 + 1$ is irreducible in $\mathbb{F}_3$ and $R/(3)$ is isomorphic to the degree 2 extension of $\mathbb{F}_3$, which is still a field. This proves that $(3) \subseteq R$ is a maximal ideal and the only prime ideal over $(3) \subseteq \mathbb{Z}$.

(3) Suppose $p \cap \mathbb{Z} = (2)$. This means $2 \in p$ and $(\sqrt{5}+1)(\sqrt{5}-1) = 4 \in p$. Because $p$ is prime, so $\sqrt{5}+1 \in p$. Note that $R/(2, \sqrt{5}+1) = \mathbb{Z}/2$ is a field, so $(2, \sqrt{5}+1) \subseteq R$ is maximal and the only prime ideal lying over $(2) \subseteq \mathbb{Z}$.

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**Problem 20.1.5**

If the ring $R$ is noetherian, then so is the ring $R[[x_1, \ldots, x_n]]$ of formal power series.

*Solution:* Note that $R[[x_1, \ldots, x_n]] = R[[x_1, \ldots, x_{n-1}]][[x_n]]$. We only need to prove the case $n = 1$, the rest can be done by repeating the same proof.

Suppose the ring $R$ is noetherian and $I \subset R[[x]]$ is a proper ideal. Let $f \in R[[x]]$ have non-zero constant term, then $f$ is invertible in $R$, so such $f$ cannot in $I$. For any $i \geq 1$, we define the following subsets in $R$. $a \in J_i$ if and only if there exists an element $f = ax^i + a_{i+1}x^{i+1} + \cdots \in I$.

$$J_i = \left\{ a_i \in R \mid \exists f_{a_i} = a_i x^i + a_{i+1} x^{i+1} + \cdots \in I \right\}$$

Here $i$ is the order of the element $f$. $J_i$ is an ideal in $R$. Indeed, if $a_i, b_i \in J_i$, then the coefficient of the lowest term of $f_{a_i} + f_{b_i}$ is $a_i + b_i$ and has degree $i$, so $a_i + b_i \in J_i$. For any $r \in R$, if $f_{a_i} \in I$, then $rf_{a_i} \in I$ because $I$ is an ideal, so $ra_i \in J_i$. This proves that $J_i \subseteq R$ is an ideal. Moreover, $J_i \subseteq J_{i+1}$ because if $a \in J_i$, then $f_a \in I$ and $xf_a \in I$. This proves that $a \in J_{i+1}$. We obtain an ascending chain of ideals

$$J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots J_n \subseteq \cdots R$$

This chain must stabilize as $R$ is noetherian. Suppose $J_n = J_{n+1} = \cdots$. Let $S_i = \{a_{i,k}\}_{1\le k\le s_i}$ be the generating set of $J_i$. This set is finite for every $i$ because $R$ is noetherian. We need to show that the set

$$K = \left\{ f_{a_{i,k}} \in R[[x]] \mid a_{i,k} \in S_i, 1 \le k \le s_i, 1 \le i \le n \right\}$$

generates $I$. It is easy to see that $K \subseteq I$.

Conversely, let $f \in I$ and the coefficients of degree $i$th term is $a_i$, we need to show that $f$ can be generated from elements in $K$. Without loss of generality, we may assume $\operatorname{ord}(f) = 1$. Define $f_1 = f$ and we have $\operatorname{ord} f_1 = 1$. There exists $\{r_{1,k}\}_{1\le k\le s_1}$ such that $a_1 = \sum_{k=1}^{s_1} r_{1,k} a_{1,k}$, thus we know that

$$\operatorname{ord}(f - \sum_{k=1}^{s_1} r_{1,k} f_{a_{1,k}}) \ge 2.$$

Define $f_2 := f_1 - \sum_{k=1}^{s_1} r_{1,k} f_{a_{1,k}}$ and we have $\operatorname{ord} f_2 = 2$. We can define this continuously

$$f_i = f_{i-1} - \sum_{k=1}^{s_{i-1}} r_{i-1,k} f_{a_{i,k}}$$

for all $i \le n+1$. For $i \ge n+2$, suppose we have already defined $f_{i-1}$ with $\operatorname{ord}(f_{i-1}) \ge i-1$, suppose the coefficient of $(i-1)$th term in $f_{i-1}$ is $a_{i-1} \in J_{i-1}$, we know that $J_{i-1} = J_n$, there exists $\{r_{i-1,k}\}_{1\le k\le s_n}$ such that $a_{i-1} = \sum_{k=1}^{s_n} r_{i-1,k} a_{n,k}$, this implies that

$$f_{i-1} - \sum_{k=1}^{s_n} r_{i-1,k} x^{i-1-n} f_{a_{n,k}}$$

has order $\ge i$. Define an element

$$p_k = \sum_{i=n+2}^{\infty} r_{i,k} x^{i-1-n} \in R[[x]].$$

Then by definition, $f_{n+1} - \sum_{k=1}^{s_n} p_k f_{a_{n,k}}$ has no degree $\ge n+2$ term, thus equal to 0. This proves that $f$ can be written as a finite sum of elements from $K$ with coefficients in $R[[x]]$, since at every step, we only remove finite sum of elements. This proves that $I$ is finitely generated, so $R[[x]]$ is noetherian.