Math 636 Homework #7
Due Friday, May 30

1. If M and N are compact, oriented d-manifolds, then the **degree** of a map $f\colon M \to N$ is defined to be the integer $\deg(f)$ such that $f_*([M]) = \deg(f)\cdot[N]$.
   (a) Suppose that f is not surjective—i.e., there is a point $x \in N$ such that x is not in the image of f. Prove that the degree of f is zero.
   (b) Explain how $\deg(f)$ relates to the map $f^*\colon H^d(N) \to H^d(M)$.
   (c) Prove that any map $S^4 \to \mathbb{C}P^2$ must have degree 0.

2. A topological space is said to be of **finite type** if $H_i(X) = 0$ for all but finitely many values of i, and each nonzero $H_i(X)$ is a finitely-generated abelian group. Recall that the Euler characteristic is then defined to be

$$\chi(X) = \sum_{i=0}^{\infty}(-1)^i \operatorname{rank} H_i(X).$$

   Prove that if X and Y are CW-complexes of finite type then so is $X \times Y$, and $\chi(X\times Y) = \chi(X)\cdot\chi(Y)$. [Hint: For the second part, one way is to use $\mathbb{Q}$-coefficients somehow.]

3. Prove that $\mathbb{C}P^{n-1}$ is not a retract of $\mathbb{C}P^n$ (use the cup product somehow).

4. Prove that there is no self-homeomorphism $\mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$ that reverses the orientation.

5. There is an algebraic formula

$$(*) \qquad (x_1^2 + x_2^2)\cdot(y_1^2+y_2^2) = (x_1y_1 - x_2y_2)^2 + (x_1y_2 + x_2y_1)^2$$

   which is true for indeterminates x_1, x_2, y_1, y_2 over $\mathbb{R}$ (that is to say, it is an identity in the polynomial ring $\mathbb{R}[x_1, x_2, y_1, y_2]$). Are there other such identities? By a **sums-of-squares formula** of type $[r, s, n]$ we mean an identity of the form

$$(x_1^2 + x_2^2 + \cdots + x_r^2)\cdot(y_1^2 + y_2^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2$$

   where each z_i is a bilinear expression in the x's and y's. The identity (*) was a formula of type $[2, 2, 2]$. Here is a formula of type $[4, 4, 4]$:

$$\begin{aligned}(x_1^2+x_2^2+x_3^2+x_4^2)\cdot(y_1^2+y_2^2+y_3^2+y_4^2) = {} & (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4)^2 + \\ & (x_1y_2 + x_2y_1 - x_3y_4 + x_4y_3)^2 + \\ & (x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2)^2 + \\ & (-x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1)^2.\end{aligned}$$

   If you try to generalize these examples you will find a formula of type $[8, 8, 8]$, but not one of type $[16, 16, 16]$ (this is a good example where checking the first few cases isn't enough to deduce a general theorem!)

   If we have a sums-of-squares formula of type $[r, s, n]$ then we get a bilinear map $\phi\colon \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ such that $||\phi(x, y)||^2 = ||x||^2 \cdot ||y||^2$ by defining

$$\phi(x_1, \ldots, x_r, y_1, \ldots, y_s) = (z_1, \ldots, z_n)$$

   using the bilinear expressions z_i.

(a) Explain why ϕ restricts to a map $S^{r-1} \times S^{s-1} \to S^{n-1}$, and then induces a map $F\colon \mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \to \mathbb{R}P^{n-1}$.

(b) Use singular cohomology to prove that if an $[r, s, n]$ formula exists then $\binom{n}{i}$ must be even for $n - r < i < s$.

(c) With some trouble one can discover a sums-of-squares formula of type $[10, 10, 16]$. Does there exist a better formula of type $[10, 10, 15]$?

6. In this problem you will give a cohomological proof of the Fundamental Theorem of Algebra, which says that every irreducible polynomial over $\mathbb{C}$ has degree equal to 1.

   Suppose $p(x)$ is an irreducible polynomial over $\mathbb{C}$ of degree n, where $n > 1$. Let $E = \mathbb{C}[x]/(p(x))$, which is an algebraic field extension of $\mathbb{C}$ of degree n. Choose a vector space isomorphism $\mathbb{C}^n \cong E$, so that the multiplication on E becomes a bilinear map $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$.

   Using singular cohomology rings (of appropriate topological spaces of your choosing), derive a contradiction. [Hint: Use projective spaces.]