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Homework 9
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Course: MATH 635 - Algebraic Topology II
Term: Winter 2025
Instructor: Dr.Daniel Dugger
Due Date: 13^{th} March, 2025

Problem 1

(a) Let $p_1 : S^1 \to S^1$ and $p_2 : S^1 \to S^1$ be given by $p_1(z) = z^{15}$ and $p_2(z) = z^6$. Is there a continous map $f : S^1 \to S^1$ making the diagram

$$\begin{tikzcd} S^1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & S^1 \arrow[dl, "p_2"] \\ & S^1 & \end{tikzcd}$$

commute? Explain why or why not.

(b) If T is the torus, use covering space theory to prove that every map $\mathbb{R}P^5 \to T$ is homotopic to a constant map.

Solution:

(a) This is impossible. We know that $\deg p_1 = 15$ and $\deg p_2 = 6$. If such a map $f : S^1 \to S^1$ exists, then we have $p_2 \circ f = p_1$. This implies that

$$(\deg p_2)(\deg f) = \deg p_1.$$

Thus, $\deg f = 15/6 \notin \mathbb{Z}$. This contradicts the definition of degree.

(b) Given a map $f : \mathbb{R}P^5 \to T$, note that $\mathbb{R}P^5$ and T are path-connected, so we can viewed f as a pointed map. $\mathbb{R}P^5$ is pointed at x, T is pointed at b, and we have $f(x) = b$. Let $p : (\mathbb{R}^2, e) \to (T, b)$ be the universal covering space where $e \in p^{-1}(b)$ is a point in the fiber over b. The map f induces a map between fundamental groups

$$f_* : \pi_1(\mathbb{R}P^5, x) \to \pi_1(T, b)$$

where $\pi_1(\mathbb{R}P^5, x) \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(T, b) \cong \mathbb{Z}$. We know the group $\mathbb{Z}$ has no torsion, so f_* must be the zero map. This implies

$$f_*(\pi_1(\mathbb{R}P^5, x)) = 0 \subseteq 0 = p_*(\pi_1(\mathbb{R}^2, e))$$

since $\mathbb{R}^2$ is simply connected. By the map lifting lemma, there exists a lifting $\tilde{f} : \mathbb{R}P^5 \to \mathbb{R}^2$ such that $p \circ \tilde{f} = f$, namely the following diagram commutes:

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ \mathbb{R}P^5 \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & T \end{tikzcd}$$

We know that $\mathbb{R}^2$ is contractible, by the convexity lemma, $\tilde{f}$ is nullhomotopic. There exists $H : \mathbb{R}P^5 \times I \to \mathbb{R}^2$ such that $H(-,0) = \tilde{f}$ and $H(-,1) = C_e$ the constant map. The composition $p \circ H : \mathbb{R}P^5 \times I \to T$ gives a homotopy between f and the constant map C_b. This proves that f is nullhomotopic.

Problem 2

Let B be the figure-eight space, with b the wedge point and basic loops α and β. We know that $\pi_1(B,b)$ is the free group on the two generators α and β. Draw a picture showing the pointed covering space $p : E \to B$ having $p_*(\pi_1(E,e)) = H$ for each of the following subgroups (in each case make clear what the basepoint e is in your picture).

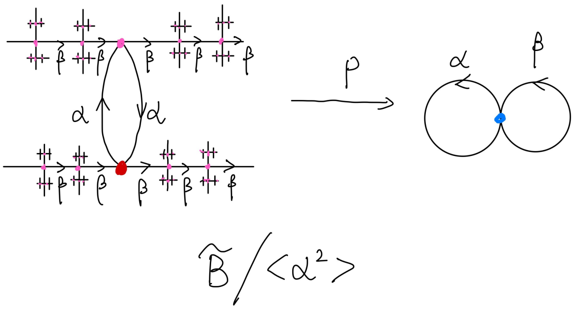
(a) $H = \langle \alpha^2 \rangle$

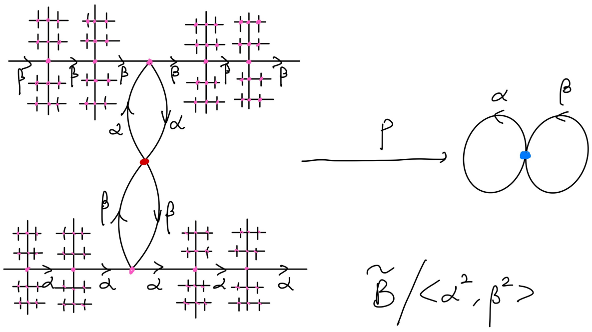
(b) $H = \langle \alpha^2, \beta^2 \rangle$

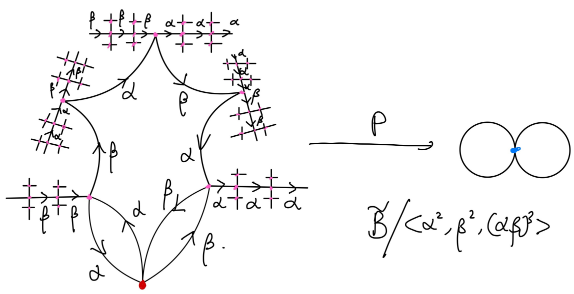
(c) $H = \langle \alpha^2, \beta^2, (\alpha\beta)^3$

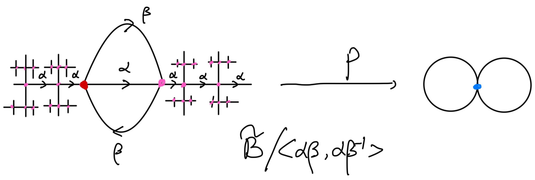
(d) $H = \langle \alpha\beta, \alpha\beta^{-1}$

Solution: The pictures are as follows. The base point $b \in B$ is the blue point, and the basepoint $e \in E$ is the red point. All the preimage of b in E is the pink points.









Problem 3
Recall the universal covering space for $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

(a) $\mathbb{R}P^2 \vee \mathbb{R}P^2$ has a regular 8-fold covering space whose automorphism group is isomorphic to the dihedral group
$$D_4 = \langle p, q \mid p^2 = q^2 = 1, (pq)^4 = 1\rangle.$$
Find the covering space and compute the homology groups.

(b) Given an example of a non-regular 4-fold cover of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

Solution:

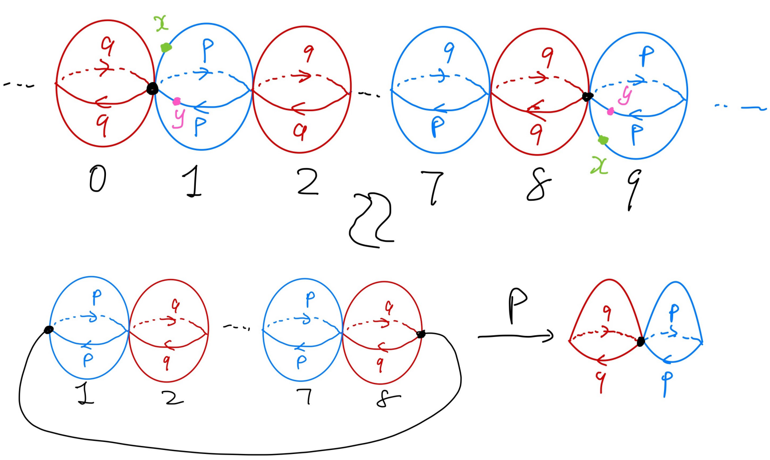
(a) Let $B = \mathbb{R}P^2 \vee \mathbb{R}P^2$ and $G = \pi_1(B) \cong \mathbb{Z}/2 * \mathbb{Z}/2$ be the fundamental group. This is a regular covering, so we know that $\mathrm{Aut}_B(E) = G/H \cong D_4$ for some normal subgroup $H \trianglelefteq G$. Let $f : G \to D_4$ be the quotient map, we have a short exact sequence of groups

$$1 \longrightarrow H \longrightarrow G \xrightarrow{f} D_4 \longrightarrow 1$$

G is generated by 2 elements of order 2. Assume G and D_4 have the following presentation

$$G = \langle p, q \mid p^2 = q^2 = 1\rangle,$$
$$D_4 = \langle p, q \mid p^2 = q^2 = 1, (pq)^4 = 1\rangle.$$

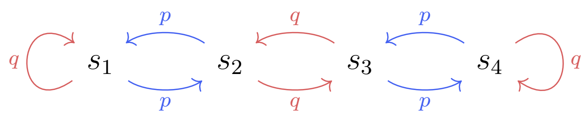
then f sends the generators p, q to p, q. We can see that the kernel $H = \langle (pq)^4\rangle$. So this regular 8-fold covering space is isomorphic to $\tilde{B}/\langle (pq)^4\rangle$ where $\tilde{B}$ is the universal covering space of B. From the Homework#8 we know that $\tilde{B}$ is an infinite wedge of 2-spheres.



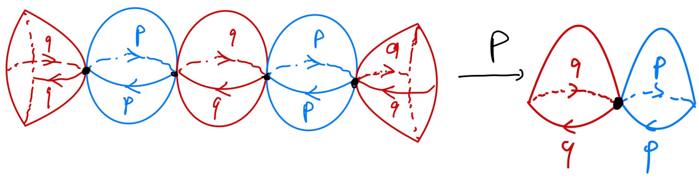
From the picture we can see that $\tilde{B}/\langle (pq)^4 \rangle$ is homotopic equivalent to eight S^2 wedged together with a line connecting the starting and ending point. This is homotopic equivalent to $(\vee_8 S^2) \vee S^1$. So the homology groups are

$$H_i(\tilde{B}/\langle (pq)^4 \rangle) = \begin{cases} \mathbb{Z}^8, & \text{if } \; i = 2, \\ \mathbb{Z}, & \text{if } \; i = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Consider the group G acts on the following Cayley graph of size 4:



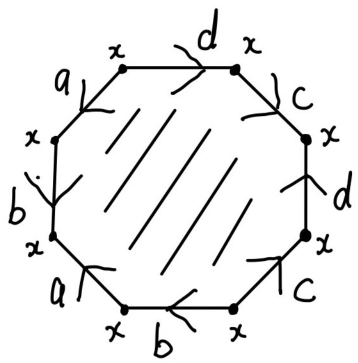
This corresponds to the following path-connected covering space.



Note that $\text{Stab}(s_1) = \langle q, p^2 \rangle$ and $\text{Stab}(s_2) = \langle p^2, q^2 \rangle$. They have different stabilizers, so this is not a regular covering space.

Problem 4
Prove that the genus 2 torus does not admit a path-connected, regular covering space whose automorphism group is $(\mathbb{Z}/3)^5$.

Solution: Let B be the genus 2 torus and B has a CW structure as follows



The fundamental group G can be calculated

$$G = \pi_1(B) = \langle a, b, c, d \mid bab^{-1}a^{-1}dcd^{-1}c^{-1} = 1 \rangle$$

Assume we have a path-connected, regular covering space $p : E \to B$. It is regular so the automorphism group $(\mathbb{Z}/3)^5 = \mathrm{Aut}_B(E) \cong G/H$ for some normal subgroup H. We have a surjective group homomorphism $f : G \twoheadrightarrow (\mathbb{Z}/3)^5$. Note that $(\mathbb{Z}/3)^5$ is an abelian group, so the map f must factor through the abelianization $\pi_1(B)_{ab} = \langle a, b, c, d \rangle = \mathbb{Z}^4$. Moreover, every element in $(\mathbb{Z}/3)^5$ has order 3 except the identity element, so the map must factor through $(\mathbb{Z}/3)^4$, we have an commutative diagram

$$\begin{tikzcd} G \arrow[r, two heads, "f"] \arrow[d, two heads] & (\mathbb{Z}/3)^5 \\ (\mathbb{Z}/3)^4 \arrow[ur, two heads, "\tilde{f}"'] & \end{tikzcd}$$

This means we have a surjective map $(\mathbb{Z}/3)^4 \to (\mathbb{Z}/3)^5$, by the structure theorem of abelian groups, this is impossible, so we do not have such a covering.

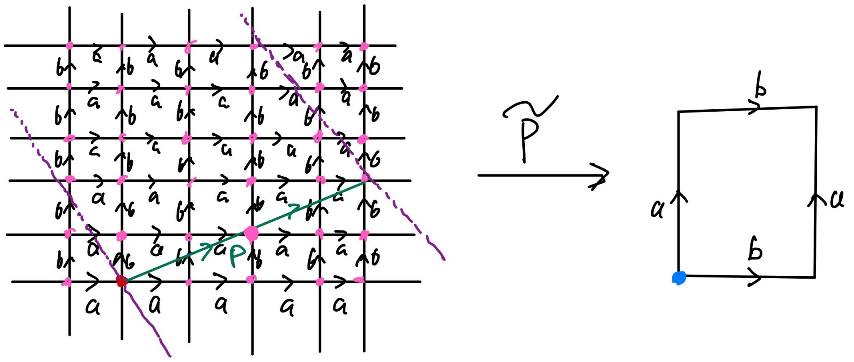
Problem 5

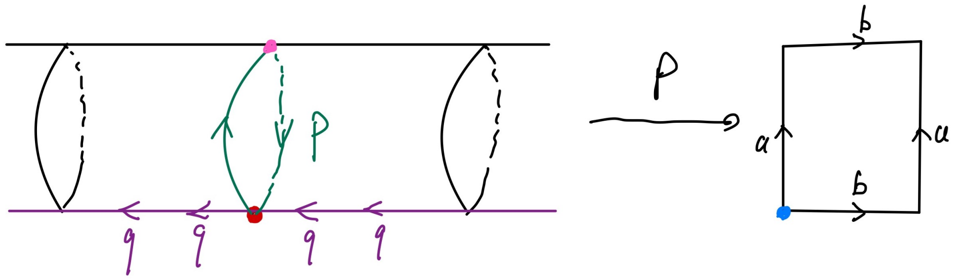
Recall that one has an isomorphism $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ in which the generators $(1, 0)$ and $(0, 1)$ correspond to the usual fundamental loops in the torus. Describe (preferably by drawing a picture) the covering space $p : E \to S^1 \times S^1$ for which $p_*(\pi_1(E, e)) = \langle (2, 4) \rangle$. In your picture of E, indicate a generator for $\pi_1(E)$. Identify the group $\mathrm{Aut}(E)$, and give a geometric description of some generators for this group in terms of your picture.

Solution: Let $T = S^1 \times S^1$ be the torus and $G = \pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ be its fundamental group with generators $(1, 0)$ and $(0, 1)$. Conisder the universal covering space $\tilde{p} : \mathbb{R}^2 \to T$. Write $b \in T$ as the base point in T. Establish an coordinate system in $\mathbb{R}^2$, the point $(0, 0)$ is the base point in $\mathbb{R}^2$ and the integer points are the fiber $\tilde{p}^{-1}(b)$ over $b \in T$. By the classification theorem for covering spaces over T, the subgroup $\langle (2, 4) \rangle$ corresponds to the covering space $E = \mathbb{R}^2/\langle (2, 4) \rangle$. This is an abelian group, so the orbit space under this group action is the same as the quotient space

$$\mathbb{R}^2/\sim \cong S^1 \times \mathbb{R}$$

where $(x, y) \sim (x + 2, y + 4)$ for all $(x, y) \in \mathbb{R}^2$. As shown in the following picture, this gives us an infinite cylinder





The generator for $\pi_1(E)$ can be viewed as a straight line in the $\mathbb{R}^2$ grid from the point $(0,0)$ to $(2,4)$ (they get identified in the quotient space E). Or the green circles in the infinite cylinders. Moreover, since G is abelian, every subgroup is normal, so the covering $p : E \to T$ is normal. We have $\mathrm{Aut}_T(E) \cong G/\langle(2,4)\rangle$, which is

$$\mathrm{Aut}_T(E) = \langle(1,0),(0,1)\rangle/\langle(2,4)\rangle = \langle(1,2),(0,1)\rangle/\langle(2,4)\rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}.$$

Suppose $\mathrm{Aut}_T(E)$ has two generators p and q with $p^2 = 1$ and $\langle q\rangle = \mathbb{Z}$. For the $\mathbb{R}^2/\sim$ model, p corresponds to the translation of $\mathbb{R}^2$ in the direction from $(0,0)$ to $(1,2)$ (green line), q corresponds to the translation in the direction perpendicular to the green line (purple line). For the infinite cylinder model $S^1 \times \mathbb{R}$, p corresponds to the rotation by 180 degrees (red point to pink point), and q corresponds to the translation along with the $\mathbb{R}$ direction (purple).

Problem 6

Let T be the torus, and $p : \mathbb{R}^2 \to T$ the map $p(x,y) = (e^{2\pi ix}, e^{2\pi iy})$.

(a) Let $\sigma : T \to T$ be an automorphism that fixes $p(0,0) \in T$. Using covering space theory (or otherwise), prove that there is an automorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(0,0) = (0,0)$ and the diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\phi} & \mathbb{R}^2 \\ \downarrow & & \downarrow \\ T & \xrightarrow{\sigma} & T \end{array}$$

commutes. If $\sigma^n = id$, explain why $\phi^n = id$.

(b) Let X be the quotient space $(T \times I)/\sim$, where the quotient relation has $(t, 1) \sim (\sigma(t), 0)$. Describe as best you can, the universal covering space of X.

(c) Prove that $\pi_1(X)$ contains $\mathbb{Z}^2$ as a subgroup. If $\phi(x) = Ax$ for some non-identity matrix A in $GL_2(\mathbb{Z})$, prove that $\pi_1(X)$ is non-abelian.

(d) What is $\pi_3(X)$?

Solution:

(a) Let $p : \mathbb{R}^2 \to T$ be the universal covering space of the torus T. Consider the following diagram of pointed spaces:

$$\begin{tikzcd} & & \mathbb{R}^2 \arrow[d, "p"] \\ \mathbb{R}^2 \arrow[urr, dashed, "\exists!\phi"] \arrow[r, "p"'] & T \arrow[r, "\sigma"'] & T \end{tikzcd}$$

Write $b = p(0,0) \in T$ as the base point in the torus. It is easy to check that

$$\sigma(p(0,0)) = \sigma(b) = b = p(0,0).$$

By the map lifting lemma, there exists a unique $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $p\phi = \sigma p$ and $\phi(0,0) = (0,0)$ (the base point is mapped to the base point). Now assume $\sigma^n = id$, consider the following diagram

$$\begin{tikzcd} & & \mathbb{R}^2 \arrow[d, "p"] \\ \mathbb{R}^2 \arrow[urr, dashed, "\phi"] \arrow[r, "p"'] & T \arrow[r, "\sigma^n = id"'] & T \end{tikzcd}$$

By the map lifting lemma, we know that $id : \mathbb{R}^2 \to \mathbb{R}^2$ is the unique map making the diagram commutes. On the other hand, consider the following diagram

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\phi^n"] \arrow[d, "p"'] & \mathbb{R}^2 \arrow[d, "p"] \\ T \arrow[r, "\sigma^n"'] & T \end{tikzcd}$$

It commutes because

$$\sigma^n p = \sigma^{n-1}(\sigma p) = \sigma^{n-1} p\phi = \cdots = p\phi^n.$$

By the uniqueness of the lifted map, we know that $\phi^n = id$.

(b) The universal space is given by $p_2 : \mathbb{R}^3 \to X$. Write $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$. For $\mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$, p_2 can be described as

$$p_2 = p \times id : \mathbb{R}^2 \times \{0\} \to T \times \{0\}$$

where $p : \mathbb{R}^2 \to T$ is the universal covering space of T. For $\mathbb{R}^2 \times \{1\} \subseteq \mathbb{R}^3$, p_2 can be described as

$$p_2 = p \circ \phi : \mathbb{R}^2 \times \{1\} \to \mathbb{R}^2 \times \{1\} \to T \times \{0\}$$

where ϕ is the lifting from part (a). This is well-defind because we know from (a) that $p\phi = \sigma p$, so p_2 in this case is the same as

$$\sigma p = \mathbb{R}^2 \times \{1\} \to T \times \{1\} \to T \times \{0\}\,.$$

For any $0 < z < 1$, we define $p_2(x, y) = p(x, y)$ just as the universal covering space $p : \mathbb{R}^2 \to T$. For $0 \leq z \leq 1$, we define

$$\begin{aligned} p_2 : \mathbb{R}^2 \times [0, 1] &\to T \times I/\sim, \\ ((x, y), z) &\mapsto (p_2(x, y), e^{2\pi i z}). \end{aligned}$$

Let $n \in \mathbb{Z}$. Similarly as above, for any $\mathbb{R}^2 \times \{n\} \subseteq \mathbb{R}$, we can define

$$p_2 = p \circ \phi^n : \mathbb{R}^2 \times \{n\} \to T \times \{0\}\,.$$

This is well-defined from our previous discussion and part (a). Now we obtained the whole covering space

$$\begin{aligned} p_2 : \mathbb{R}^2 \times \mathbb{R} &\to T \times I/\sim, \\ ((x, y), z) &\mapsto (p_2(x, y), e^{2\pi i z}). \end{aligned}$$

Here when $n - 1 \leq z < n$, we have $p_2 = p \circ \phi^{n-1} : \mathbb{R}^2 \times \{z\} \to T \times \{0\}$.

(c) Note that by the classification theorem for covering space, $\mathrm{Aut}_X(\mathbb{R}^3) \cong \pi_1(X)/p_{2*}(\pi_1(\mathbb{R}^3)) = \pi_1(X)$. And consider the translation of $\mathbb{R}^3$ by 1 in the x direction, this defines an automorphism of the covering space $p_2 : \mathbb{R}^3 \to X$, and it generates a subgroup isomorphic to $\mathbb{Z}$ in $\mathrm{Aut}_X(\mathbb{R}^3)$. Same for the translation in the y direction by 1. We can see that $\pi_1(X)$ contains $\mathbb{Z}^2$ as a subgroup.

Now assume $A = \phi : \mathbb{R}^2 \to \mathbb{R}^2$ is in $GL_2(\mathbb{Z})$ and is not the identity matrix. A defines an automorphism of covering space by sending (x, y, z) to $(A(x, y), z)$. For $m, n \in \mathbb{Z}$, note that $(A(x + m, y + n), z) \neq (A(x, y) + (m, n), z)$ in general, so $\pi_1(X)$ is not an abelian group.

(d) The covering space $\mathbb{R}^3 \xrightarrow{p_2} X$ is a fiber bundle and we have a long exact sequence in homotopy groups. The fiber is discrete so for $i \geq 2$, we have

$$\pi_i(\mathbb{R}^3) \cong \pi_i(X).$$

When $i = 3$, we have $\pi_3(X) = 0$ is trivial since $\mathbb{R}^3$ is contractible.