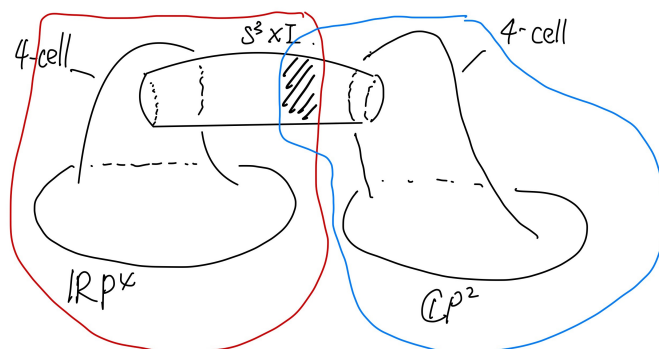


**Problem 1**

Determine all the cohomology groups  $H^*(\mathbb{R}P^4 \# \mathbb{C}P^2)$ .

*Solution:* We first use the Mayer-Vietoris sequences in cohomology as follows:



$$U \cong \mathbb{R}P^4 - * \cong \mathbb{R}P^3 \quad U \cap V = S^3 \times I \cong S^3$$

$$V \cong \mathbb{C}P^2 - * \cong \mathbb{C}P^1$$

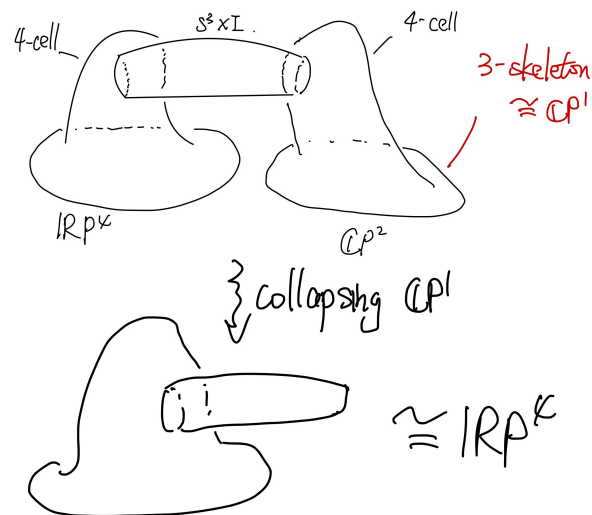
Let  $X = \mathbb{R}P^4 \# \mathbb{C}P^2$ . We have the following long exact sequence in cohomology

	$H^*(X)$	$H^*(\mathbb{R}P^3) \oplus H^*(\mathbb{C}P^1)$	$H^*(S^3)$
0	?	$\longrightarrow \mathbb{Z} \oplus \mathbb{Z}$	$\longrightarrow \mathbb{Z}$
1	?	$\longrightarrow 0 \oplus 0$	$\longrightarrow 0$
2	?	$\longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}$	$\longrightarrow 0$
3	?	$\longrightarrow \mathbb{Z} \oplus 0$	$\longrightarrow \mathbb{Z}$
4	?	$\longrightarrow 0$	$\longrightarrow 0$

Note that  $X$  is connected, so  $H^0(X) = \mathbb{Z}$ . By exactness of the above sequence, we have

$$H^2(X) \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

Now consider collapsing the  $\mathbb{C}P^1 \subseteq \mathbb{C}P^2$  in  $X$  as shown



we have a cofiber sequence

$$\mathbb{C}P^1 \rightarrow X \rightarrow \mathbb{R}P^4$$

This gives us a long exactness sequence in reduced cohomology

	$\tilde{H}^*(\mathbb{R}P^4)$	$\tilde{H}^*(X)$	$\tilde{H}^*(\mathbb{C}P^1)$
0	0	$\longrightarrow$ 0	$\longrightarrow$ 0
1	0	$\longleftarrow$ ? $\longrightarrow$	0
2	$\mathbb{Z}/2$	$\longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2 \longrightarrow$	$\mathbb{Z}$
3	0	$\longleftarrow$ ? $\longrightarrow$	0
4	$\mathbb{Z}/2$	$\longrightarrow$ ? $\longrightarrow$	0

By exactness, we know  $H^1(X) = H^3(X) = 0$  and  $H^4(X) = \mathbb{Z}/2$ . We can summarize the cohomology groups as follows.

$$H^i(\mathbb{R}P^4 \# \mathbb{C}P^2) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}/2, & \text{if } i = 4; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } i = 2; \\ 0, & \text{otherwise.} \end{cases}$$

**Problem 2**

Let  $f : X \rightarrow Y$  be a map, and consider the diagram

$$\begin{array}{ccc} H^k(Y; R) & \xrightarrow{f^*} & H^k(X; R) \\ \downarrow \phi & & \downarrow \phi \\ \text{hom}(H_k(Y), R) & \xrightarrow{\text{hom}(f_*, R)} & \text{hom}(H_k(X), R) \end{array}$$

where the bottom horizontal map is the one obtained by applying the functor  $\text{hom}(-, R)$  to  $f_* : H_k(X) \rightarrow H_k(Y)$ . Here the vertical maps  $\phi$  are the adjoints to the Kronecker pairings, namely the maps that send a cohomology class  $[\alpha]$  to the homomorphism  $[v] \mapsto \alpha(v)$ . Verify that the above diagram commutes.

*Solution:* Let  $\alpha : C_k(Y) \rightarrow R$  be a cocycle in  $Y$  and  $[\alpha]$  be the cohomology class represented by  $\alpha$  in  $H^k(Y; R)$ . By definition, we know that  $f^*([\alpha]) = [\alpha \circ f_\#]$ , where  $f_\# : C_k(X) \rightarrow C_k(Y)$  is the map on the chain complex induced by  $f$ . By definition,  $\phi$  sends  $[\alpha \circ f_\#]$  to the homomorphism  $[v] \mapsto (\alpha \circ f_\#)(v)$  for any  $k$ -cycle  $v \in C_k(X)$ . On the other hand,  $\phi$  sends  $[\alpha]$  to the homomorphism  $[w] \mapsto \alpha(w)$  for any  $k$ -cycle  $w \in C_k(Y)$ . Applying  $\text{hom}(f_*, R)$  to this homomorphism, we obtain a homomorphism sending any  $[v] \in H_k(X)$  to  $\alpha(f_*([v]))$ . Note that by definition, for any  $k$ -cycle  $v \in C_k(X)$ , we have

$$\alpha(f_*([v])) = \alpha([f_\#(v)]) = \alpha(f_\#(v)) = (\alpha \circ f_\#)(v).$$

This proves the commutativity of this diagram.

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**Problem 3**

- (a) Give a  $\Delta$ -complex structure on  $\mathbb{R}P^2$  and use this to write down explicit cocycles  $\alpha \in Z^1(\mathbb{R}P^2; \mathbb{Z}/2)$  and  $\beta \in Z^2(\mathbb{R}P^2; \mathbb{Z}/2)$  that generates the cohomology groups. Compute  $\alpha \cup \alpha$  and decide if it equals  $\beta$  or not in  $H^*(\mathbb{R}P^2; \mathbb{Z}/2)$ .
- (b) Let  $K$  be the Klein bottle. Write down explicit cocycles which represent generators for  $H^*(K; \mathbb{Z}/2)$  and use these to compute all the cup products of these generators.
- (c) If  $R$  is a ring then we can extend the Kronecker pairing to be maps

$$H^k(X; R) \otimes H_k(X; R) \rightarrow R.$$

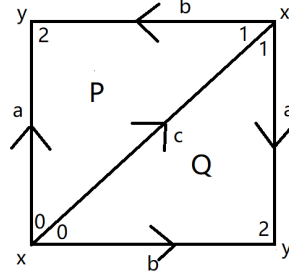
The adjoint is then a map

$$\phi_R : H^k(X; R) \rightarrow \text{hom}(H_k(X), R).$$

When  $X$  is the Klein bottle and  $\mathbb{R} = \mathbb{Z}/2$  determine bases for each  $H^k(X; R)$  and verify by hand that the maps  $\phi$  are isomorphisms for all  $k$ .

*Solution:*

(a) Consider the following  $\Delta$ -complex structure for  $\mathbb{R}P^2$ :



We know from last homework that  $H^1(\mathbb{R}P^2; \mathbb{Z}/2) = H^2(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2$ . So as long as the cocycle  $\alpha, \beta \in Z^*(\mathbb{R}P^2; \mathbb{Z}/2)$  are not zero, they are the generator of the cohomology groups. Consider  $\hat{a} + \hat{c} \in C^1(\mathbb{R}P^2; \mathbb{Z}/2)$ , we check that this is a cocycle.

$$\begin{aligned} (\delta(\hat{a} + \hat{c}))(P) &= (\hat{a} + \hat{c})(\partial P) = (\hat{a} + \hat{c})(-a + b + c) = 1 + 1 = 0; \\ (\delta(\hat{a} + \hat{c}))(Q) &= (\hat{a} + \hat{c})(\partial Q) = (\hat{a} + \hat{c})(a - b + c) = 1 + 1 = 0. \end{aligned}$$

This proves  $\hat{a} + \hat{c} \in Z^1(\mathbb{R}P^2; \mathbb{Z}/2)$  and we need to show that  $\hat{a} + \hat{c}$  is not a coboundary, this is true because

$$\begin{aligned} \delta(\hat{x})(a) &= \delta(\hat{y})(a) = 1, \\ \delta(\hat{x})(b) &= \delta(\hat{y})(b) = 1, \\ \delta(\hat{x})(c) &= \delta(\hat{y})(c) = 0. \end{aligned}$$

So  $\hat{a} + \hat{c}$  cannot be realized as the image of  $C^0(\mathbb{R}P^2; \mathbb{Z}/2)$ . This implies that  $\alpha = [\hat{a} + \hat{c}]$  generates the cohomology group  $H^1(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2$ . Note that  $\mathbb{R}P^2$  does not have 3-simplices, and we know  $H^2(\mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2$ , so  $\hat{Q} \in Z^2(\mathbb{R}P^2; \mathbb{Z}/2)$  and note that

$$\partial P = \partial Q = a + b + c.$$

So  $\hat{Q}$  is not a coboundary because  $\hat{Q}(P) = 0$  and  $\hat{Q}(Q) = 1$ . This implies that  $\beta = [\hat{Q}]$  generates  $H^2(\mathbb{R}P^2; \mathbb{Z}/2)$ . Next, we calculate the cup product.

$$\begin{aligned} (\alpha \cup \alpha)(P) &= \alpha(c) \cdot \alpha(b) = 0; \\ (\alpha \cup \alpha)(Q) &= \alpha(c) \cdot \alpha(a) = 1. \end{aligned}$$

This proves that  $(\hat{a} + \hat{c}) \cup (\hat{a} + \hat{c}) = \hat{Q}$  on the chain level and in the cohomology ring  $H^*(\mathbb{R}P^2; \mathbb{Z}/2)$ , we have  $\alpha \cup \alpha = \beta$ .

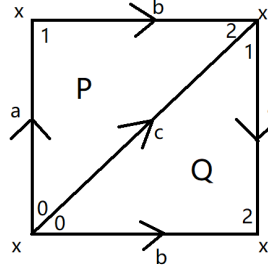
(b) Consider the cellular chain complex of the Klein bottle  $K$

$$0 \rightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix} 2 \\ 0 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \rightarrow 0$$

Apply  $\text{hom}(-, \mathbb{Z}/2)$ , we obtain the cellular cochain complex

$$0 \leftarrow \mathbb{Z}/2 \xleftarrow{0} (\mathbb{Z}/2)^2 \xleftarrow{0} \mathbb{Z}/2 \leftarrow 0.$$

So we know  $H^0(K; \mathbb{Z}/2) = H^2(K; \mathbb{Z}/2) = \mathbb{Z}/2$ , each has one generator and  $H^1(K; \mathbb{Z}/2) = (\mathbb{Z}/2)^2$  having two generators. Consider the following  $\Delta$ -complex structure of  $K$



$K$  only has one 0-simplex  $x$ , so the cocycle  $\hat{x}$  generates  $H^0(K; \mathbb{Z}/2)$ .  $K$  has no 3-simplices, so  $\hat{P}$  is a cocycle and we need to show that  $\hat{P}$  is not a coboundary. Suppose  $\delta(m\hat{a} + n\hat{b} + k\hat{c}) = \hat{P}$ . Then

$$\begin{aligned} 1 &= (m\hat{a} + n\hat{b} + k\hat{c})(P) = (m\hat{a} + n\hat{b} + k\hat{c})(a + b - c) = m + n + k, \\ 0 &= (m\hat{a} + n\hat{b} + k\hat{c})(Q) = (m\hat{a} + n\hat{b} + k\hat{c})(a - b + c) = m + n + k. \end{aligned}$$

This leads to a contradiction and thus  $\hat{P}$  is not a coboundary and generates  $H^2(K; \mathbb{Z}/2)$ . Note that

$$\begin{aligned} \partial P &= a + b - c, \\ \partial Q &= a - b + c. \end{aligned}$$

Consider two cochains  $\alpha = \hat{a} + \hat{b}$  and  $\beta = \hat{a} + \hat{c}$ , they are cocycles because

$$\begin{aligned} 0 &= (\delta\alpha)(P) = (\delta\alpha)(Q), \\ 0 &= (\delta\beta)(P) = (\delta\beta)(Q). \end{aligned}$$

We need to show that  $[\alpha] \neq [\beta]$  in  $H^1(K; \mathbb{Z}/2)$ . Assume the opposite. This means  $\alpha - \beta$  is a coboundary. We only have one 0-simplex, so  $\alpha - \beta = \hat{b} + \hat{c} = \delta\hat{x}$ . But

$$1 = (\hat{b} + \hat{c})(b) = (\delta\hat{x})(b) = \hat{x}(x - x) = 0$$

A contradiction. This tells us  $\alpha, \beta$  generates  $H^1(K; \mathbb{Z}/2)$ .

Next, we calculate the cup product. For dimension reasons, we have  $[\hat{P}] \cup [\hat{P}] = 0$ , and  $[\hat{x}]$  is the unity in the cohomology ring. In degree 1,

$$\begin{aligned} (\alpha \cup \alpha)(P) &= \alpha(a) \cdot \alpha(b) = 1 \cdot 1 = 1; \\ (\alpha \cup \alpha)(Q) &= \alpha(c) \cdot \alpha(a) = 0 \cdot 1 = 0. \end{aligned}$$

This tells us  $\alpha \cup \alpha = \hat{P}$  on the cochain level. So

$$[\alpha] \cup [\alpha] = [\hat{P}].$$

$$\begin{aligned} (\beta \cup \beta)(P) &= \beta(a) \cdot \beta(b) = 1 \cdot 0 = 0; \\ (\beta \cup \beta)(Q) &= \beta(c) \cdot \beta(a) = 1 \cdot 1 = 1. \end{aligned}$$

This tells us  $\beta \cup \beta = \hat{Q}$  on the cochain level, and we know that

$$\begin{aligned} \delta(\hat{a} + \hat{b} + \hat{c})(P) &= (\hat{a} + \hat{b} + \hat{c})(a + b - c) = 1 + 1 + 1 = 1, \\ \delta(\hat{a} + \hat{b} + \hat{c})(Q) &= (\hat{a} + \hat{b} + \hat{c})(a - b + c) = 1 + 1 + 1 = 1. \end{aligned}$$

This implies  $\hat{P} + \hat{Q}$  is a coboundary and  $[\hat{P}] = [\hat{Q}]$  in  $H^2(K; \mathbb{Z}/2)$ . So

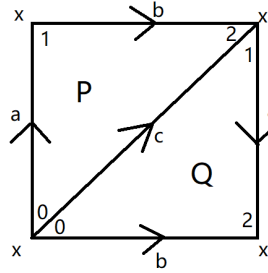
$$[\beta] \cup [\beta] = [\hat{P}].$$

$$\begin{aligned} (\alpha \cup \beta)(P) &= \alpha(a) \cdot \beta(b) = 1 \cdot 0 = 0; \\ (\alpha \cup \beta)(Q) &= \alpha(c) \cdot \beta(a) = 0 \cdot 1 = 0. \end{aligned}$$

This tells us  $\alpha \cup \beta = 0$  on the cochain level, so

$$[\alpha] \cup [\beta] = -([\beta] \cup [\alpha]) = 0.$$

(c) Let  $K$  be the Klein bottle. Consider the  $\Delta$ -complex structure we used in (b)



We have a chain complex with  $\mathbb{Z}/2$ -coefficients

$$0 \rightarrow (\mathbb{Z}/2)^2 \xrightarrow{d} (\mathbb{Z}/2)^3 \xrightarrow{0} \mathbb{Z}/2 \rightarrow 0$$

where the boundary map  $d$  is given by the matrix  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$ . So  $H_2(K; \mathbb{Z}/2) = \ker d$  is generated by the 2-cycle  $P + Q$  and  $H_1(K; \mathbb{Z}/2)$  is generated by  $a, b$  where  $c = a + b$ .  $H_0(K; \mathbb{Z}/2)$  is generated by  $x$ . Note that to check  $\phi$  is an isomorphism, it is the same as checking the Kronecker pairing is a perfect pairing. For  $k = 0$  and  $k = 2$ , it is easy to see because on the

only generator, we have

$$\begin{aligned}([\hat{x}], [x]) &= \hat{x}(x) = 1, \\ ([\hat{P}], P + Q) &= \hat{P}(P + Q) = 1.\end{aligned}$$

For  $k = 1$ , use  $\alpha = \hat{a} + \hat{b}$  and  $\beta = \hat{a} + \hat{c}$  as generators of  $H^1(K; \mathbb{Z}/2)$  as before, we have

$$\begin{aligned}([\alpha], [a]) &= (\hat{a} + \hat{b})(a) = 1, \\ ([\alpha], [b]) &= (\hat{a} + \hat{b})(b) = 1, \\ ([\beta], [a]) &= (\hat{a} + \hat{c})(a) = 1, \\ ([\beta], [b]) &= (\hat{a} + \hat{c})(b) = 0.\end{aligned}$$

This can be written as a matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

which is invertible in  $M_2(\mathbb{Z}/2)$ , so this is also a perfect pairing. Thus, we have proved that  $H^k(K; \mathbb{Z}/2) \rightarrow \text{hom}(H_k(K; \mathbb{Z}/2), \mathbb{Z}/2)$  is an isomorphism for  $k = 0, 1, 2$ .

#### Problem 4

Prove that there does not exist a map  $S^2 \rightarrow T$  that induces an isomorphism on  $H_2$ . In fact, prove this in two different ways: give a proof that uses homotopy groups and give a proof that uses cohomology and the cup product.

*Solution:* In (1) we prove this using homotopy groups and in (2), we prove this using cohomology rings.

(1) We check that the Hurewicz homomorphism is natural.

Claim:  $f : X \rightarrow Y$  is a map of connected, pointed spaces. For  $n \geq 1$ , we have a commutative diagram

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{f_{*,1}} & \pi_n(Y) \\ h_X \downarrow & & \downarrow h_Y \\ H_n(X) & \xrightarrow{f_{*,2}} & H_n(Y) \end{array}$$

Both  $f_{*,1}, f_{*,2}$  are induced by  $f$  and  $h_X, h_Y$  are Hurewicz homomorphism in degree  $n$  for space  $X$  and  $Y$ .

Proof: Let  $u = [\Delta^n / \partial \Delta^n] \in H_n(S^n)$  be the canonical generator. For any  $\sigma : S^n \rightarrow X$ ,  $\sigma$  induces a map  $\sigma_* : H_n(S^n) \rightarrow H_n(X)$  and the Hurewicz homomorphism sends  $[\sigma]$  to  $\sigma_*(u)$ . Note that

$$(f_{*,2} \circ h_X)([\sigma]) = f_{*,2}\sigma_*(u) = (f \circ \sigma)_*(u) = f_{*,1}(\sigma_*(u)).$$

This proves the commutativity. ■

Suppose there exists a homomorphism  $f : S^2 \rightarrow T$  such that  $f_{*,2} : H_2(S^2) \rightarrow H_2(T)$  is an

isomorphism. Apply the claim and we have a commutative diagram

$$\begin{array}{ccc} \pi_2(S^2) & \xrightarrow{f_{*,1}} & \pi_2(T) \\ h_{S^2} \downarrow & & \downarrow h_T \\ H_2(S^2) & \xrightarrow{f_{*,2}} & H_2(T) \end{array}$$

Note that  $S^2$  is 1-connected, so  $h_{S^2}$  is an isomorphism, by our assumption,

$$h_T \circ f_{*,1} = f_{*,2} \circ h_{S^2} : \pi_2(S^2) \rightarrow H_2(T)$$

is an isomorphism between  $\mathbb{Z}$  and  $\mathbb{Z}$ . Consider the fiber sequence

$$\mathbb{Z}^2 \rightarrow \mathbb{R}^2 \rightarrow T$$

This gives us a long exact sequence in homotopy groups and we have  $\pi_2(T) = \pi_2(\mathbb{R}^2) = 0$ . So  $h_T \circ f_{*,1} = 0$ . A contradiction.

- (2) Suppose there exists a homomorphism  $f : S^2 \rightarrow T$  such that  $f_{*,2} : H_2(S^2) \rightarrow H_2(T)$  is an isomorphism. Apply  $\text{hom}(-, \mathbb{Z})$  and we obtain an isomorphism

$$\text{hom}(H_2(T), \mathbb{Z}) \rightarrow \text{hom}(H_2(S^2), \mathbb{Z}).$$

From problem #2, we have a commutative diagram

$$\begin{array}{ccc} H^2(T) & \xrightarrow{f^*} & H^2(S^2) \\ \downarrow \phi & & \downarrow \phi \\ \text{hom}(H_2(T), \mathbb{Z}) & \xrightarrow{\text{hom}(f_*, \mathbb{Z})} & \text{hom}(H_2(S^2), \mathbb{Z}) \end{array}$$

We have seen in class that  $\phi$  is an isomorphism for  $T$  or  $S^2$ , thus,  $f^* : H^2(T) \rightarrow H^2(S^2)$  is isomorphism. By definition,  $f_* : H^*(T) \rightarrow H^*(S^2)$  can be viewed as a map of rings and let  $[\hat{a}], [\hat{b}] \in H^1(T)$  be two generators and  $[\hat{T}] \in H^2(T)$  be the generator of  $H^2(T)$ . We have seen in class that

$$f^*([\hat{T}]) = f^*([\hat{a}] \cup [\hat{b}]) = f^*([\hat{a}]) \cup f^*([\hat{b}]).$$

But  $H^1(S^2) = 0$ , so  $f^*([\hat{a}]) = f^*([\hat{b}]) = 0$ . So  $f^*$  cannot map  $H^2(T)$  isomorphically to  $H^2(S^2)$ . A contradiction.

### Problem 5

Prove that  $\mathbb{R}P^2$  is not a retract of the Klein bottle.

*Solution:* Suppose  $\mathbb{R}P^2$  is a retract of the Klein bottle  $K$ . There exists maps  $i : \mathbb{R}P^2 \rightarrow K$  and  $r : K \rightarrow \mathbb{R}P^2$  such that  $r \circ i$  is the identity map of  $\mathbb{R}P^2$ . This induces a map in cohomology rings



with coefficients  $\mathbb{Z}/2$ .

$$\begin{array}{ccccc} H^*(\mathbb{R}P^2; \mathbb{Z}/2) & \xrightarrow{r^*} & H^*(K; \mathbb{Z}/2) & \xrightarrow{i^*} & H^*(\mathbb{R}P^2; \mathbb{Z}/2) \\ & & \searrow & \nearrow & \\ & & id & & \end{array}$$

Let  $\alpha$  be the generator of  $H^1(\mathbb{R}P^2; \mathbb{Z}/2)$ ,  $T$  be the generator of  $H^2(\mathbb{R}P^2; \mathbb{Z}/2)$ ,  $\beta, \gamma$  be the two generators of  $H^1(K; \mathbb{Z}/2)$  and  $P$  be the generator of  $H^2(K; \mathbb{Z}/2)$ . Since  $i^* \circ r^* = id$ , so  $i^*$  must be surjective and  $r^*$  must be injective. Either  $i^*(\beta) = \alpha$  or  $i^*(\gamma) = \alpha$ . Without loss of generality, we can assume  $i^*(\beta) = \alpha$  and  $i^*(\gamma) = 0$ . This is impossible because from our calculation in problem #3, we have

$$T = \alpha \cup \alpha = i^*(\beta) \cup i^*(\beta) = i^*(\beta \cup \beta) = i^*(P) = i^*(\gamma \cup \gamma) = i^*\gamma \cup i^*\gamma = 0.$$

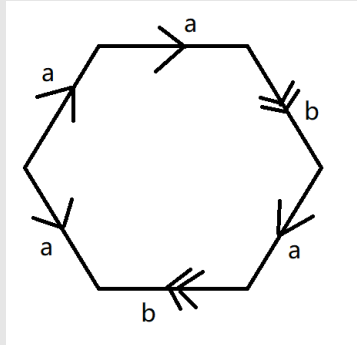
It is also impossible if we maps  $i^*(\gamma) = i^*(\beta) = \alpha$  because

$$T = \alpha \cup \alpha = i^*(\beta) \cup i^*(\gamma) = i^*(\beta \cup \gamma) = 0.$$

So we cannot have such ring homomorphisms.

### Problem 6

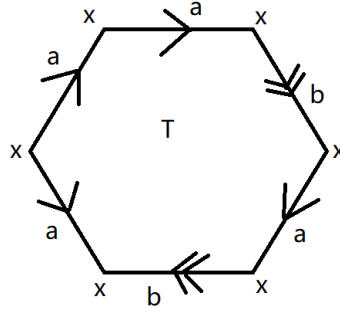
Let  $X$  be obtained by identifying points on the boundary of a solid hexagon, as indicated in the following diagram:



- Calculate the homology and cohomology groups of  $X$  with  $\mathbb{Z}/2$ -coefficients.
- Give a  $\Delta$ -complex structure to  $X$ , for example by placing one point in the center of the hexagon, drawing lines to the outer vertices, and orienting the 2-simplices appropriately.
- Using your  $\Delta$ -complex structure, give a 1-cocycle  $\alpha$  with  $\mathbb{Z}/2$ -coefficients having the property that  $\alpha(a) = 1$  and  $\alpha(b) = 0$ .
- Compute  $\alpha \cup \alpha$  on all the 2-simplices in your picture. Is  $\alpha \cup \alpha$  zero or nonzero in  $H^2(X; \mathbb{Z}/2)$ ? Explain.

*Solution:*

(a) Consider the following cellular structure of  $X$



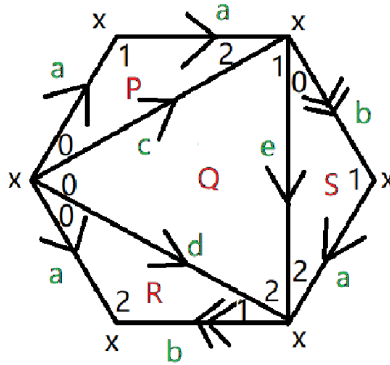
$X$  has one 0-cell  $x$ , two 1-cell  $a, b$  and one 2-cell  $T$ . We have a cellular chain complex with coefficients  $\mathbb{Z}/2$

$$0 \rightarrow \mathbb{Z}/2 \xrightarrow{0} (\mathbb{Z}/2)^2 \xrightarrow{0} \mathbb{Z}/2 \rightarrow 0$$

Note that all boundary maps are 0, so apply  $\text{hom}(-, \mathbb{Z}/2)$  will obtain all 0 coboundary maps. This implies

$$H_i(X; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0, 2; \\ (\mathbb{Z}/2)^2, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases} \quad \left| \quad H^i(X; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0, 2; \\ (\mathbb{Z}/2)^2, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

(b) The following is a  $\Delta$ -complex structure on  $X$



(c) Consider the 1-cochain  $\alpha = \hat{a} + \hat{d} + \hat{e}$  satisfying  $\alpha(a) = 1$  and  $\alpha(b) = 0$ . Let us check this is a cocycle.

$$\begin{aligned} \delta(\hat{a} + \hat{d} + \hat{e})(P) &= (\hat{a} + \hat{d} + \hat{e})(a + a - c) = 1 + 1 = 0, \\ \delta(\hat{a} + \hat{d} + \hat{e})(Q) &= (\hat{a} + \hat{d} + \hat{e})(c + e - d) = 1 + 1 = 0, \\ \delta(\hat{a} + \hat{d} + \hat{e})(S) &= (\hat{a} + \hat{d} + \hat{e})(b + a - e) = 1 + 1 = 0, \\ \delta(\hat{a} + \hat{d} + \hat{e})(R) &= (\hat{a} + \hat{d} + \hat{e})(a - b - d) = 1 + 1 = 0. \end{aligned}$$

This proves  $\alpha = \hat{a} + \hat{d} + \hat{e}$  is a 1-cocycle satisfying  $\alpha(a) = 1$  and  $\alpha(b) = 0$ .

(d) We calculate  $\alpha \cup \alpha$  on each 2-simplices.

$$\begin{aligned}(\alpha \cup \alpha)(P) &= \alpha(a) \cdot \alpha(a) = 1 \cdot 1 = 1, \\(\alpha \cup \alpha)(Q) &= \alpha(c) \cdot \alpha(e) = 0 \cdot 1 = 0, \\(\alpha \cup \alpha)(S) &= \alpha(b) \cdot \alpha(a) = 0 \cdot 1 = 0, \\(\alpha \cup \alpha)(R) &= \alpha(d) \cdot \alpha(b) = 1 \cdot 0 = 0.\end{aligned}$$

This proves that  $\alpha \cup \alpha = \hat{P}$  on the chain level. Suppose  $\sigma \in C^1(X; \mathbb{Z}/2)$  satisfying  $\delta\sigma = \hat{P}$ . Then we have

$$\begin{aligned}(\delta\sigma)(P) &= \sigma(c) = 1, \\(\delta\sigma)(Q) &= \sigma(c) + \sigma(e) + \sigma(d) = 0, \\(\delta\sigma)(S) &= \sigma(b) + \sigma(a) + \sigma(e) = 0, \\(\delta\sigma)(R) &= \sigma(a) + \sigma(b) + \sigma(d) = 0.\end{aligned}$$

We add the last two equations together and obtain

$$\sigma(e) + \sigma(d) = 0$$

But from the second and first equation, we know that  $\sigma(e) + \sigma(d) + 1 = 0$ . This leads to a contradiction, so  $\alpha \cup \alpha = \hat{P}$  is not a coboundary, thus  $[\alpha \cup \alpha] \neq 0$  in  $H^*(X; \mathbb{Z}/2)$ .