Zhengdong Zhang
Email: zhengz@uoregon.edu
Course: MATH 649 - Abstract Algebra
Instructor: Professor Sasha Polishchuk

Homework - Week 8
ID: 952091294
Term: Spring 2025
Due Date: 28^{th} May, 2025

Problem 20.2.7
If R is a PID then an ideal Q in R is primary if and only if $\sqrt{Q}$ is prime.

Solution: From Lemma 20.2.3, we know that if Q in R is primary, then $\sqrt{Q}$ is prime. Conversely, assume $\sqrt{Q}$ is a prime ideal, then $\sqrt{Q} = (p)$ for some prime element $p \in R$ since R is a PID. $p \in \sqrt{Q}$ implies that there exists some $n > 0$ such that $p^n \in Q$. Let $k \in \mathbb{Z}_+$ be the smallest positive integer such that $p^k \in Q$. We claim that $Q = (p^k)$. Suppose $Q = (a)$ for some $a \in R$. We know that $p^k \in Q$, so $a|p^k$. Since p is prime in R, $a = p^{k'}$ for $1 \leq k' \leq k$. The way we choose k implies that $k' = k$. So $Q = (p^k)$. Suppose $rs \in Q = (p^k)$, then there exists $b \in R$ such that $p^k b = rs$. If $r \notin (p) = \sqrt{Q}$, this means $p^k \nmid r$. Since p is prime, $p^k|s$ and this implies $s \in Q$. We have proved that Q is primary.

Problem 20.2.9
If $\sqrt{I}$ is a maximal ideal, then I is $\sqrt{I}$-primary.

Solution: Proving I is $\sqrt{I}$-primary is the same as proving that every zero divisor in R/I is nilpotent. $\sqrt{I}$ is a maximal ideal containing I in R, so $\sqrt{I}/I$ is a maximal ideal in R/I. Moreover, it is the unique maximal ideal in R/I. Suppose m is another maximal ideal in R/I, then m corresponds to a prime ideal $p \subseteq R$ containing I. We know that $\sqrt{I}$ is the intersection of all prime ideals containing I, so $p \supseteq \sqrt{I}$. This contradicts that $\sqrt{I}$ is maximal. So such m does not exists. Thus, R/I is a local ring with the unique maximal ideal $\sqrt{I}/I$.

Suppose $a, b \in R - I$ and $ab \in I$, in this case $a + I$ and $b + I$ are zero divisors in R/I. Assume $b + I$ is not nilpotent in R/I, this implies that $b \notin \sqrt{I}$, so $b + I \notin \sqrt{I}/I$. In this case, $b + I$ must be a unit in R/I because $(R/I)/(\sqrt{I}/I)$ is a field. This contradicts that $b + I$ is a zero divisor in R/I. So $b + I$ must be nilpotent and thus I is $\sqrt{I}$-primary.

Problem 20.2.16
The ideal $(4, 2x, x^2)$ in the ring $\mathbb{Z}[x]$ is primary but not irreducible.

Solution: We first prove the following claim.
Claim:

$$(4, 2x, x^2) = (4, x) \cap (2, x^2).$$

Proof: Note that $(2x, x^2) \subseteq (x)$, so $(4, 2x, x^2) \subseteq (4, x)$. Similarly, $(4, 2x) \subseteq (2)$, so $(4, 2x, x^2) \subseteq (2, x^2)$. This proves that $(4, 2x, x^2) \subseteq (4, x) \cap (2, x^2)$. Conversely, suppose $r \in (4, x) \cap (2, x^2)$. $r \in (2, x^2)$ implies there exists $f, g \in \mathbb{Z}[x]$ such that $r = 2f + x^2 g \in (4, x)$. This means that

$2f \in (4,x)$. Either $2|f$ or $x|f$. In both cases, $2f \in (4,2x) \subseteq (4,2x,x^2)$. This implies that $r = 2f + x^2 g \in (4,2x,x^2)$. So $(4,2x,x^2) \supseteq (4,x) \cap (2,x^2)$. ■

Next, note that $\sqrt{(4,x)} = \sqrt{(2,x^2)} = (x,2)$. And $\mathbb{Z}[x]/(2,x) \cong \mathbb{Z}/2$ is a field, so $(x,2)$ is maximal. By Exercise 20.2.9, $(4,x)$ and $(2,x^2)$ are both $(2,x)$-primary ideals. By Lemma 20.2.10, the intersection

$$(4,2x,x^2) = (4,x) \cap (2,x^2)$$

is also a $(2,x)$-primary ideal, but it is not irreducible as $(4,2x,x^2)$ is properly contained in two ideals $(4,x)$ and $(2,x^2)$.

Problem 20.2.18
Represent the ideal $(9,3x+3)$ in $\mathbb{Z}[x]$ as the intersection of primary ideals.

Solution: We claim that $(9,3x+3) = (3) \cap (9,x+1)$. 3 is a prime element in $\mathbb{Z}[x]$ so (3) is a prime ideal thus primary. Similarly, Note that

$$\mathbb{Z}[x]/(9,x+1) \cong \mathbb{Z}/9.$$

The zero divisors in $\mathbb{Z}/9$ are 3 and 6, both of them are nilpotent since $3^2 = 9$ and $6^2 = 9 \cdot 4$. This proves that $(9,x+1) \subseteq \mathbb{Z}[x]$ is a primary ideal. Since 3 divides 9 and $3|(x+1)$, so $(9,3x+3) \subseteq (3)$. On the other hand, $(x+1)|(3x+3)$, so $(3x+3) \subseteq (x+1)$, thus $(9,3x+3) \subseteq (9,x+1)$. This proves that $(9,3x+3) \subseteq (3) \cap (9,x+1)$. Conversely, for any $r \in (3) \cap (9,x+1)$, there exists $f,g \in \mathbb{Z}[x]$ such that $r = 9f + (x+1)g$. We know that $3|r$, this means $3|(x+1)g$, so $3|g$ and $3x+3|(x+1)g$. This proves that $r \in (9,3x+3)$. Therefore, we have found a primary decomposition

$$(9,3x+3) = (3) \cap (9,x+1).$$

Problem 20.3.6
Let P be a prime ideal. Then $P^{(n)}$ is the smallest P-primary ideal containing P^n.

Solution: By definition, $P^{(n)} \supseteq P^n$, and by Lemma 20.3.5, $P^{(n)}$ is a P-primary ideal. Suppose Q is a P-primary ideal containing P^n. We need to prove that $Q \supseteq P^{(n)}$. For any $r \in P^{(n)}$, there exists $s \in R - P$ such that $rs \in P^n \subseteq Q$. We know $s \notin P = \sqrt{Q}$, and since Q is P-primary, this implies $r \in Q$. Thus, we have proved every P-primary ideal containing P^n will contain $P^{(n)}$, namely, $P^{(n)}$ is the smallest P-primary ideal containing P^n.

Problem 20.3.18
If $R \subseteq A$ is an integral extension of noetherian rings then $\dim R = \dim A$.

Solution: Given a chain of strict inclusions of prime ideals

$$p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n \subsetneq R.$$

We claim that there exists a chain of strict inclusion of prime ideals of the same length in A. We start with $p_0 \subsetneq R$, by Lying Over Theorem, there exists a prime ideal $q_0 \subsetneq A$ such that $q_0 \cap R = p_0$. Next, consider the inclusion $p_0 \subsetneq p_1$, by Going Up Theorem, there exists a prime ideal $q_1 \supseteq q_0$ in A such that $q_1 \cap R = p_1$. Note that here $p_0 \neq p_1$, so $q_0 \subsetneq q_1$ is a strict inclusion of prime ideals as they are pulled back to different ideals in R. Repeat this step, and we can construct a chain of strict inclusions of prime ideals in A:

$$q_0 \subsetneq q_1 \subsetneq \cdots \subsetneq q_n \subsetneq A.$$

This proves that $\dim A \geq \dim R$. On the other hand, consider a chain of strict inclusions of prime ideals in A:

$$q_0 \subsetneq q_1 \subsetneq \cdots \subsetneq q_n \subsetneq A.$$

We know that the pullback of prime ideals are still prime ideals, so we have a chain of prime ideals in R:

$$q_0 \cap R \subseteq q_1 \cap R \subseteq \cdots \subseteq q_n \cap R \subseteq R.$$

Write $p_i := q_i \cap R$ for $1 \leq i \leq n$. We are going to show that $p_i \subseteq p_{i+1}$ are strict inclusions for all i. Suppose $p_i = p_{i+1}$ for some i. This means $q_i \cap R = q_{i+1} \cap R$, by Incomparability Theorem, $q_i = q_{i+1}$. This contradicts the assumption that $q_i \subsetneq q_{i+1}$ is a strict inclusion. So $p_i \subsetneq p_{i+1}$ for all i. We have a chain of strict inclusions of prime ideals in R:

$$p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n \subsetneq R.$$

This implies $\dim R \geq \dim A$. Thus, we can conclude that $\dim R = \dim A$.

Problem 21.1.14
Let I and J be ideals of $A = \mathbb{C}[x, y]$ and $\mathcal{V}(I) \cap \mathcal{V}(J) = \varnothing$. Show that $A/(I \cap J) = A/I \times A/J$.

Solution: By Proposition 21.2.1, we know that

$$\varnothing = \mathcal{V}(1) = \mathcal{V}(I) \cap \mathcal{V}(J) = \mathcal{V}(I + J).$$

By Corollary 21.1.10, this means $\sqrt{I+J} = \sqrt{(1)} = \sqrt{A} = A$. Note that $1 \in A = \sqrt{I+J}$, so $1 = 1^n \in I + J$ for some $n > 0$. This implies that $I + J = A$. By the Chinese Remainder Theorem, we have

$$A/(I \cap J) \cong A/I \times A/J.$$