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Course: MATH 616 - Real Analysis
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Homework - Week 6 Exercises
ID: 952091294
Term: Fall 2025
Due Date: Nov 12th, 2025

Exercise 6.1
Let f be a nonnegative measurable function on a locally compact, Hausdoff space X with a positive Borel measure μ, such that $f > 0$ almost everywhere with respect to μ. Prove that for any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, f) > 0$ such that if E is a measurable subset of X with $\mu(E) > \varepsilon$, then $\int_E f d\mu > \delta$.

Solution: Define for every $n \geq 1$,

$$A_n := \left\{ x \in X : f(x) \geq \frac{1}{n} \right\}.$$

Each A_n is measurable as f is a measurable function. It is easy to see that $A_n \subset A_{n+1}$ for every n and

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n.$$

Denote this limit $A := \bigcup_{n=1}^{\infty} A_n$. Since $f > 0$ almost everywhere on X, we have $\mu(X \setminus A) = 0$, namely

$$\lim_{n\to\infty} \mu(A_n) = \mu(X).$$

For any $\varepsilon > 0$, let E be any measurable set with $\mu(E) > \varepsilon$. Note that

$$0 \leq \mu(E \cap (X \setminus A)) \leq \mu(X \setminus A) = 0.$$

This implies that

$$\begin{aligned}
\mu(E) &= \mu(E \cap A) + \mu(E \cap (X \setminus A)) \\
&= \mu(E \cap A) \\
&= \mu(E \cap \bigcup_{n=1}^{\infty} A_n) \\
&= \mu(\bigcup_{n=1}^{\infty} (E \cap A_n)) \\
&> \varepsilon.
\end{aligned}$$

Write $E_n := E \cap A_n$ and we have $E_n \subset E_{n+1}$ for every n. Then

$$\mu(\bigcup_{n=1}^{\infty} E_n) > \varepsilon.$$

There exists an integer $N > 0$ such that

$$\mu(E_N) = m(E \cap A_N) > \frac{\varepsilon}{2}.$$

And note that

$$\int_A f d\mu \geq 0$$

for any measurable subset $A \subset X$ because $f > 0$ almost everywhere. This tells us that

$$\begin{aligned}\int_E f d\mu &= \int_{E \cap A_N} f d\mu + \int_{E \cap (X \setminus A_N)} f d\mu \\ &\geq \frac{1}{N} \cdot \mu(E \cap A_N) \\ &> \frac{\varepsilon}{2N}.\end{aligned}$$

We have proved that there exists $\delta = \frac{\varepsilon}{2N}$, for any measurable set $E \subset X$ with $\mu(E) > \varepsilon$, we have

$$\int_E f d\mu > \delta.$$

Exercise 6.2
In this problem, m stands for the Lebesgue measure.

(a) For any $\alpha \in (0, 1)$, construct an open dense set $V \subset [0, 1]$ such that $m(V) = \alpha$.

(b) Let E be a measurable set of $\mathbb{R}$ with $m(E) > 0$. For any $\alpha \in (0, 1)$, there exists an open interval I such that $m(E \cap I) > \alpha m(I)$.

Solution:

(a) For any $\alpha \in (0, 1)$, we do a Cantor-like construction on the closed interval $[0, 1]$.

Let V_1 be an open interval centered at $\frac{1}{2}$ with length $\frac{\alpha}{2}$. The set $E_1 = [0, 1] \setminus V_1$ is the union of 2 disjoint closed interval.

In each of the closed interval in E_1, take the open interval centered at the center of the closed interval with length $\frac{\alpha}{8}$. Let V_2 be the union of these 2 open intervals, and we can see that $m(V_2) = \frac{\alpha}{4}$. The set $E_2 = [0, 1] \setminus (V_1 \cup V_2)$ is the union of 4 disjoint closed intervals.

Repeat the above steps. For a general n, E_{n-1} is the disjoint union of 2^{n-1} closed intervals. In each closed interval, we take the centered open interval with length $\frac{\alpha}{2^{2n-1}}$, and let V_n be the union of all 2^{n-1} open intervals, and it is not hard to see that

$$m(V_n) = \frac{\alpha}{2^{2n-1}} \cdot 2^{n-1} = \frac{\alpha}{2^n}.$$

Note that $V_i \cap V_j = \varnothing$ for $i \neq j$. Thus, take $V := \bigcup_{n=1}^{\infty} V_n$ and we have

$$m(V) = m(\bigcup_{n=1}^{\infty}) = \sum_{n=1}^{\infty} m(V_n) = \sum_{n=1}^{\infty} \frac{\alpha}{2^n} = \alpha.$$

It is obvious V is open in $[0,1]$ because it is the countable union of open interval. For every point $x \in [0,1] \setminus V$, x must be the endpoint of an open interval in V, so it is a limit point of V. This implies that V is dense in $[0,1]$. Hence, we can conclude that V is an open dense set in $[0,1]$ with $m(V) = \alpha$.

(b) We first assume $0 < m(E) < +\infty$. Assume the opposite that for any open interval $I \subset \mathbb{R}$, we have

$$m(E \cap I) \leq \alpha m(I).$$

By the outer regularity of Lebesgue measure, for any $\varepsilon > 0$, there exists an open set $V \supset E$ such that

$$0 < m(E) < m(V) < m(E) + \varepsilon.$$

Note that the open set V can be written as

$$E = \bigcup_{n=1}^{\infty} I_n$$

where every I_n is an open interval and $I_i \cap I_j = \varnothing$ for $i \neq j$. Then

$$\begin{aligned} m(E) &= m(V \cap E) \\ &= m\Big(\bigcup_{n=1}^{\infty}(I_n \cap E)\Big) \\ &= \sum_{n=1}^{\infty} m(I_n \cap E) \\ &\leq \sum_{n=1}^{\infty} \alpha m(I_n) \\ &= \alpha m(V) \\ &< \alpha m(E) + \alpha\varepsilon. \end{aligned}$$

Let $\varepsilon \to 0$, and we obtain $0 < m(E) < \alpha m(E)$. A contradiction.

Now assume $m(E) = +\infty$. Take $E' := E \cap [-N, N]$ for sufficiently large N such that $0 < m(E') < +\infty$. By the above proof that there exists an open interval I such that

$$m(E \cap I) \geq m(E' \cap I) > \alpha m(I).$$

Exercise 6.3
Let $f(x) = \frac{\sin x}{x}$ (clearly $f(0) = 1$) for $x \in \mathbb{R}$.

(a) Prove that $f(x)$ is not an L^1 function on $\mathbb{R}$. That is

$$\int_{\mathbb{R}} |f| dm = \infty.$$

(b) Justify that, on the other hand, the improper integral $\int_0^\infty f dx$ is well defined.

Solution:

(a) We have

$$\begin{aligned}\int_{\mathbb{R}} |f| dm &= \sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+2)\pi} \frac{|\sin x|}{x} dx \\ &\geq \sum_{k=0}^{\infty} \frac{1}{(2k+2)\pi} \int_0^{2\pi} |\sin x| dx \\ &= C \sum_{k=0}^{\infty} \frac{1}{k+1}\end{aligned}$$

where

$$C = \frac{1}{2\pi} \int_0^{2\pi} |\sin x| dx$$

is a finite positive constant. We know the series $\sum_{k=0}^{\infty} \frac{1}{k+1}$ diverges. So

$$\int_{\mathbb{R}} |f| dm = +\infty.$$

(b) We need to show that

$$\lim_{A \to +\infty} \int_0^A \frac{\sin x}{x} dx$$

exists and is finite. Suppose $A > 2\pi$. We can write

$$\int_0^A \frac{\sin x}{x} dx = \int_0^{2\pi} \frac{\sin x}{x} dx + \int_{2\pi}^A \frac{\sin x}{x} dx.$$

Note that the function $\frac{\sin x}{x}$ is continuous and bounded on the closed interval $[0, 2\pi]$, so

$$\int_0^{2\pi} \frac{\sin x}{x} dx$$

is a finite value. On the other hand, we have

$$\begin{aligned}\int_{2\pi}^{A} \frac{\sin x}{x} dx &= \int_{2\pi}^{A} \frac{d(-\cos x)}{x} \\ &= \frac{-\cos x}{x}\Big|_{2\pi}^{A} - \int_{2\pi}^{A} \frac{\cos x}{x^2} dx \\ &= -\frac{\cos A}{A} - \frac{1}{2\pi} - \int_{2\pi}^{A} \frac{\cos x}{x^2} dx.\end{aligned}$$

Write

$$C = \int_{0}^{2\pi} \frac{\sin x}{x} dx - \frac{1}{2\pi}$$

to be a constant, and thus,

$$\begin{aligned}\lim_{A\to\infty} \int_{0}^{A} \frac{\sin x}{x} dx &= C - \lim_{A\to\infty} (\frac{\cos A}{A} + \int_{2\pi}^{A} \frac{\cos x}{x^2} dx) \\ &= C - \lim_{A\to\infty} \int_{2\pi}^{A} \frac{\cos x}{x^2} dx\end{aligned}$$

Note that

$$\int_{2\pi}^{\infty} \frac{|\cos x|}{x^2} dx \leq \int_{2\pi}^{\infty} \frac{1}{x^2} dx = \frac{1}{2\pi}.$$

So $\frac{\cos x}{x} \in L^1$ and

$$\lim_{A\to\infty} \int_{2\pi}^{A} \frac{\cos x}{x^2} dx$$

exists. This proves that $\int_0^\infty \frac{\sin x}{x} dx$ is finite.