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Course: MATH 636 - Algebraic Topology III
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Homework 5
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Problem 1
The space $S^2 \times S^3$ and $S^2 \vee S^3 \vee S^5$ have isomorphic homology groups. Use the cup product to prove that they are not homotopy equivalent.

Solution:

Problem 2

(a) Let M be an n-dimensional manifold-with-boundary (boundary points have neighborhoods that look like $\{\underline{x} \in \mathbb{R}^n \mid x_n \geq 0\}$). If x is on the boundary of M, prove that $H_n(M, M - x) = 0$.

(b) Let M be a compact, connected, orientable n-manifold. If U is a Euclidean open disk in M, prove that $H_i(M - U) \to H_i(M)$ is an isomorphism when $i < n$ and prove that $H_n(M - U) = 0$. Also, if $A = \partial(M - U)$ prove that the connecting homomorphism $\partial : H_n(M - U, A) \to H_{n-1}(A)$ is an isomorphism.

Solution:

Problem 3
Let M and N be compact, connected n-manifolds, $n \geq 2$. Prove the following:

(a) If M and N are orientable, then so is $M\#N$.

(b) If M and N are non-orientable, then so is $M\#N$.

(c) What happens when M is orientable and N is not? Justify your answer.

Solution:

Problem 4
Suppose $X = Y = S^n$. $H^*(X)$ has no torsion then the Künneth Theorem gives an isomorphism of rings

$$H^*(X) \otimes H^*(Y) \to H^*(X \times Y).$$

For $a \in H^*(X)$ and $b \in H^*(Y)$, the map sends $a \otimes b$ to $\pi_1^*(a) \cup \pi_2^*(b)$. The latter expression is sometimes denoted $a \times b$.

Suppose that S^n has a continuous unital multiplication $\mu : S^n \times S^n \to S^n$. So there is a unit element $e \in S^n$ with property that $\mu(e, x) = x = \mu(x, e)$ for all $x \in S^n$. Said differently, the following diagram is commutative:

$$
\begin{tikzcd}
S^n \times \{*\} \arrow[dr, "id\times j"'] \arrow[drr, bend left, "id"] & & \\
& S^n \times S^n \arrow[r, "\mu"] & S^n \\
\{*\} \times S^n \arrow[ur, "j\times id"] \arrow[urr, bend right, "id"'] & &
\end{tikzcd}
$$

where $j : * \hookrightarrow S^n$ sends the point to e.

(a) Let z be a generator for $H^n(S^n)$. Use the above diagram to prove that $\mu^*(z) = z \otimes 1 + 1 \otimes z$.

(b) Use the fact that μ^* is a ring homomorphism, together with your knowledge of the ring structure on $H^*(S^n \times S^n)$, to conclude that n must be odd.

(c) Consider a multiplication $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ which was unital and had no zero-divisors (that is, $xy = 0 \Rightarrow (x = 0 \text{ or } y = 0)$). Use (b) to prove that no such multiplication exists, assuming that the identity element is nonzero.

Solution:

Problem 5

The relative form of the Künneth Theorem is that there is a natural short exact sequence

$$
0 \to \bigoplus_{p+q=n} H_p(X, A) \otimes H_q(Y, B) \to H_n(X \times Y, X \times B \cup A \times Y)
$$
$$
\to \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(X, A), H_q(Y, B)) \to 0.
$$

(a) Suppose that M is an n-manifold and N is a k-manifold, and that we are given local orientations $\mu_M \in H_n(M, M - x)$ and $\mu_N \in H_k(N, N - y)$, for some $x \in M$ and $y \in N$. Explain how to get an induced local orientation for $M \times N$ at (x, y).

(b) Explain how to get an interesting continuous map $\tilde{M} \times \tilde{N} \to \widetilde{M \times N}$, where $\tilde{M}$ is the space of pairs (m, μ_m) such that $m \in M$ and $\mu_m \in H_n(M, M - m)$ is a generator (and similarly for N, etc.). How many points are in each fiber?

(c) Prove that if M and N are orientable then so is $M \times N$ (we are not assuming compactness here).

Solution:

Problem 6

Consider the space $X = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$, made into a Δ-complex as follows.

Recall that $H^0(X;\mathbb{Z}/2) = H^2(X;\mathbb{Z}/2) = \mathbb{Z}/2$ and $H^1(X;\mathbb{Z}/2) = (\mathbb{Z}/2)^3$. The Universal Coefficients Theorem implies that the standard maps

$$\phi_i : H^i(X;\mathbb{Z}/2) \to \hom(H_i(X;\mathbb{Z}/2), \mathbb{Z}/2)$$

are isomorphisms.

(a) Write down explicit 1-cocycles α, β and γ (with $\mathbb{Z}/2$ coefficients) that map to $\hat{a}$, $\hat{b}$ and $\hat{c}$ under ϕ.

(b) Given a 2-cochain Θ (with $\mathbb{Z}/2$ coefficients), how can one easily determine if Θ is a generator for $H^2(X;\mathbb{Z}/2)$?

(c) Determine a class $u \in H^1(X;\mathbb{Z}/2)$ such that $\alpha \cup u$ is a generator for $H^2(X;\mathbb{Z}/2)$. Then do the same for β and γ.

Solution: