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Course: MATH 649 - Abstract Algebra
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Homework - Week 9
ID: 952091294
Term: Spring 2025
Due Date: 4^{th} June, 2025

Problem 21.2.2
Let $I_1, \dots, I_m$ be ideals in $\mathbb{F}[T_1, \dots, T_n]$. Then $\mathcal{V}(I_1 \cdots I_m) = \mathcal{V}(I_1 \cap \cdots \cap I_m)$.

Solution: We only need to prove the case $m = 2$, the rest can be obtained from induction. To prove $\mathcal{V}(I_1 I_2) = \mathcal{V}(I_1 \cap I_2)$, by Corollary 21.1.10, it is the same as proving

$$\sqrt{I_1 I_2} = \sqrt{I_1 \cap I_2}.$$

Suppose $a \in \sqrt{I_1 I_2}$, then there exists $n \geq 1$ such that $a^n \in I_1 I_2 \subseteq I_1 \cap I_2$. This implies that $a \in \sqrt{I_1 \cap I_2}$. On the other hand, suppose $b \in \sqrt{I_1 \cap I_2} = \sqrt{I_1} \cap \sqrt{I_2}$, then there exists $k, l \geq 1$ such that $b^k \in I_1$ and $b^l \in I_2$. This implies $b^{k+l} = b^k \cdot b^l \in I_1 I_2$, so $b \in \sqrt{I_1 I_2}$. This proves $\sqrt{I_1 I_2} = \sqrt{I_1 \cap I_2}$.

Problem 21.2.3
Let $f \in \mathbb{F}[T_1, \dots, T_n]$. The corresponding *principal open set* is

$$\mathbb{A}^n \setminus \mathcal{V}(f) = \{x \in \mathbb{A}^n \mid f(x) \neq 0\}.$$

Show that each open set in $\mathbb{A}^n$ is finite union of principal open sets, so principal open sets form a base of Zariski topology.

Solution: We know that the Zariski closed sets of $\mathbb{A}^n$ have the form $\mathcal{V}(I)$ for some ideal $I \subseteq \mathbb{F}[T_1, \dots, T_n]$. So for any open set $U \subseteq \mathbb{A}^n$, U can be written as $U = \mathbb{A}^n - \mathcal{V}(I)$ for some radical ideal I. Since $\mathbb{F}[T_1, \dots, T_n]$ is noetherian, I is finitely generated by $f_1, \dots, f_k \in \mathbb{F}[T_1, \dots, T_n]$. This implies

$$\mathcal{V}(I) = \mathcal{V}(f_1, \dots, f_k) = \mathcal{V}(f_1) \cap \cdots \cap \mathcal{V}(f_k).$$

Thus, we can write U as

$$\begin{aligned} U &= \mathbb{A}^n - \mathcal{V}(I) \\ &= \mathbb{A}^n - \mathcal{V}(f_1) \cap \cdots \cap \mathcal{V}(f_k) \\ &= (\mathbb{A}^n - \mathcal{V}(f_1)) \cup \cdots \cup (\mathbb{A}^n - \mathcal{V}(f_k)). \end{aligned}$$

This proves that any Zariski open set can be written as a finite union of principal open sets.

Problem 21.2.13

Let $X = \mathcal{V}(x^2+y^2+z^2, xyz) \subseteq \mathbb{A}^3$. Decompose X into irreducible components.

Solution: We need to find all the points $(x, y, z) \in \mathbb{A}^3$ satisfying $x^2+y^2+z^2 = 0$ and $xyz = 0$. Since $\mathbb{A}^3$ has no nilpotents, $xyz = 0$ implies at least one of the coordinates is 0. Suppose $x = 0$. The y and z satisfy the equation $y^2+z^2 = 0$. Note that $\mathbb{F}$ is algebraically closed, if char $\mathbb{F} = 2$, then $y+z = 0$. X has three irreducible components

$$X = \mathcal{V}(x+y+z, xyz) = \mathcal{V}(x, y+z) \cup \mathcal{V}(y, x+z) \cup \mathcal{V}(z, x+y).$$

Each of them is isomorphic to $\mathbb{A}^1$ because the coordinate ring

$$\mathbb{F}[x, y, z]/(x, y+z) \cong \mathbb{F}[y, -y] \cong \mathbb{F}[y].$$

Next, assume char $\mathbb{F} \neq 2$, then $y^2+z^2 = (y+iz)(y-iz) = 0$. This is the union of two algebraic sets $\mathcal{V}(y+iz)$ and $\mathcal{V}(y-iz)$. Thus, X has six irreducible components

$$X = \mathcal{V}(x, y+iz) \cup \mathcal{V}(x, y-iz) \cup \mathcal{V}(y, x+iz) \cup \mathcal{V}(y, x-iz) \cup \mathcal{V}(z, x+iy) \cup \mathcal{V}(z, x-iy).$$

Each of them is isomorphic to $\mathbb{A}^1$ because the coordinate ring

$$\mathbb{F}[x, y, z]/(x, y-iz) \cong \mathbb{F}[z, iz] \cong \mathbb{F}[z].$$

Problem 21.2.14
Let $\operatorname{char}\mathbb{F} \neq 2$. Decompose $\mathcal{V}(x^2+y^2+z^2, x^2-y^2-z^2+1)$ into irreducible components.

Solution: We need to find all the pointd $(x, y, z) \in \mathbb{A}^3$ satisfying $x^2+y^2+z^2 = 0$ and $x^2-y^2-z^2+1 = 0$. From these two equations, we obtain

$$0 = 2x^2 + 1.$$

We know $\operatorname{char}\mathbb{F} \neq 2$. So this equation has two different solutions: $x = \frac{i}{\sqrt{2}}$ and $x = \frac{-i}{\sqrt{2}}$. When $x = \frac{i}{\sqrt{2}}$, y and z satisfy the equation $y^2+z^2 = \frac{1}{2}$. This is a hyperbola and $(y^2+z^2-\frac{1}{2})$ is a prime ideal in $\mathbb{F}[x, y]$ since we proved in Exercise 21.4.14 that

$$\mathbb{F}[y, z]/(y^2+z^2-1) \cong \mathbb{F}[u, v]/(uv-1) \cong \mathbb{F}[u, u^{-1}].$$

Thus, X has two irreducible components

$$X = \mathcal{V}(x - \frac{i}{\sqrt{2}}, y^2+z^2-\frac{1}{2}) \cup \mathcal{V}(x + \frac{i}{\sqrt{2}}, y^2+z^2-\frac{1}{2}).$$

Problem 21.3.4

If $f : A \to B$ is a homomorphism of affine algebras and M is a maximal ideal of B, then $f^{-1}(M)$ is a maximal ideal of A.

Solution: A, B are finitely generated $\mathbb{F}$-algebras, so B/M is also a finitely generated $\mathbb{F}$-algebra. We have a map

$$\begin{aligned} \phi : A/f^{-1}(M) &\to B/M, \\ a + f^{-1}(M) &\mapsto f(a) + M. \end{aligned}$$

This is a well-defined $\mathbb{F}$-algebra homomorphism. Indeed, suppose $a, b \in A$ and $a - b \in f^{-1}(M)$. This means $f(a-b) = f(a) - f(b) \in M$, so $f(a) + M = f(b) + M$ is the same element in B/M. Moreover, ϕ is injective. Let $a + f^{-1}(M) \in \ker \phi$ and assume $f(a) + M = M$, namely, $f(a) \in M$. Then $a \in f^{-1}(M)$ and $a + f^{-1}(M) = f^{-1}(M)$ is trivial in $A/f^{-1}(M)$.

M is a maximal ideal, so B/M is a field and is a finitely generator $\mathbb{F}$-algebra. By the first version of Nullstellensatz we proved in class, $\mathbb{F} \subseteq B/M$ is an algebraic and finite extension. We know that $A/f^{-1}(M)$ is a domain as $f^{-1}(M)$ is a prime ideal in A, so we have

$$\mathbb{F} \subseteq A/f^{-1}(M) \subseteq B/M$$

and $A/f^{-1}(M)$ is a subring of B/M. By Exercise 10.1.11, $A/f^{-1}(M)$ is a field, thus $f^{-1}(M)$ is a maximal ideal in A.

Problem 21.4.6
The hyperbola $xy = 1$ and $\mathbb{A}^1$ are not isomorphic.

Solution: The coordinate ring of the hyperbola $xy = 1$ is

$$\mathbb{F}[x, y]/(xy - 1) \cong \mathbb{F}[x, x^{-1}] \cong \mathbb{F}[x]_x.$$

Here, $\mathbb{F}[x]_x$ is $\mathbb{F}[x]$ localized with respect to the multiplicative set $\{1, x, x^2, \ldots\}$. On the other hand, the coordinate ring of $\mathbb{A}^1$ is $\mathbb{F}[x]$. The two rings $\mathbb{F}[x]_x$ and $\mathbb{F}[x]$ are not isomorphic as $\mathbb{F}[x, x^{-1}] = \mathbb{F}[x]_x$. Suppose

$$\phi : \mathbb{F}[x]_x \to \mathbb{F}[y]$$

is a map of $\mathbb{F}$-algebras. We know that the units must be sent to units in $\mathbb{F}[y]$, and x is a unit in $\mathbb{F}[x, x^{-1}]$, so x must be sent to some element in $\mathbb{F}$. Note that x generateds $\mathbb{F}[x]$ as an $\mathbb{F}$-algebra, so there is no element which can be sent to y. This implies ϕ can never be surjective, so we do not have such isomorphism. This implies that $xy = 1$ and $\mathbb{A}^1$ are not isomorphic as they have different coordinate rings.

Problem 21.4.14
The circle $x^2 + y^2 = 1$ and $\mathbb{A}^1$ are isomorphic if and only if $\operatorname{char} \mathbb{F} = 2$.

Solution: Suppose $\operatorname{char} \mathbb{F} = 2$. The radical ideal of $(x^2 + y^2 - 1)$ is $(x + y - 1)$. The coordinate ring

$$\mathbb{F}[x, y]/(x^2 + y^2 - 1) \cong \mathbb{F}[x, y]/(x + y - 1) \cong \mathbb{F}[t]$$

if we consider the isomorphism

$$\begin{aligned} \mathbb{F}[x,y]/(x+y-1) &\to \mathbb{F}[t], \\ x &\mapsto t, \\ y &\mapsto t+1. \end{aligned}$$

This proves the circle $x^2+y^2=1$ is isomorphic to $\mathbb{A}^1$ if $\operatorname{char}\mathbb{F}=2$.

Suppose $\operatorname{char}\mathbb{F}\neq 2$. Then consider the following map

$$\begin{aligned} \phi : \mathbb{F}[u,v]/(uv-1) &\to \mathbb{F}[x,y]/(x^2+y^2-1), \\ u &\mapsto x+iy, \\ v &\mapsto x-iy. \end{aligned}$$

This map is a regular map since it is given by a polynomial in y and x. It is an isomorphism because it has an inverse

$$\begin{aligned} \phi^{-1} : \mathbb{F}[x,y]/(x^2+y^2-1) &\to \mathbb{F}[u,v]/(uv-1), \\ x &\mapsto \frac{1}{2}u+\frac{1}{2}v, \\ y &\mapsto \frac{-i}{2}u+\frac{i}{2}v. \end{aligned}$$

This implies that the circle $x^2+y^2=1$ is isomorphic to the hyperbola $uv=1$, and we have proved in Exercise 21.4.6 that the hyperbola $uv=1$ is not isomorphic to $\mathbb{A}^1$. So the circle $x^2+y^2=1$ is not isomorphic to $\mathbb{A}^1$ when $\operatorname{char}\mathbb{F}\neq 2$.