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Exercise 1
Suppose f is a Lebesgue measurable function such that f and $xf(x)$ are both in $L^2(\mathbb{R})$. Prove that $f \in L^1(\mathbb{R})$.

Solution: Write the subset $E = (-\infty, -1) \cup (1, +\infty) \subset \mathbb{R}$. We need to show that

$$\int_{\mathbb{R}} |f(x)|dx = \int_E |f(x)|dx + \int_{-1}^{1} |f(x)|dx < \infty.$$

For the first part on E, use Hölder inequality and note that $xf(x) \in L^2(\mathbb{R})$, we get

$$\begin{aligned}
\int_E |f(x)|dx &= \int_E |\frac{1}{x}| \cdot |xf(x)|dx \\
&\leq (\int_E \frac{1}{x^2}dx)^{\frac{1}{2}}(\int_E |xf(x)|^2 dx)^{\frac{1}{2}} \\
&\leq (\int_{-\infty}^{-1} \frac{1}{x^2}dx + \int_1^{+\infty} \frac{1}{x^2}dx)^{\frac{1}{2}}(\int_{\mathbb{R}} |xf(x)|^2 dx)^{\frac{1}{2}} \\
&= (1+1)^{\frac{1}{2}} \cdot ||xf(x)||_2 \\
&< +\infty.
\end{aligned}$$

For the second part on $[-1, 1]$, use Hölder inequality and note that $f \in L^2(\mathbb{R})$, we get

$$\begin{aligned}
\int_{-1}^{1} |f(x)|dx &= \int_{-1}^{1} 1 \cdot |f(x)|dx \\
&\leq (\int_{-1}^{1} 1^2 dx)^{\frac{1}{2}}(\int_{-1}^{1} |f(x)|^2 dx)^{\frac{1}{2}} \\
&\leq \sqrt{2}(\int_{\mathbb{R}} |f(x)|^2 dx)^{\frac{1}{2}} \\
&= \sqrt{2}||f||_2 \\
&< +\infty.
\end{aligned}$$

Combine these two together, and we get

$$\int_{\mathbb{R}} |f(x)|dx < +\infty.$$

This proves that $f \in L^1(\mathbb{R})$.

Exercise 2

Let $E \subset \mathbb{R}$ be a compact subset. Define, for $r > 0$,

$$E_r = \{x \in \mathbb{R} : d(x, E) < r\}$$

where $d(x, E) = \inf_{y \in E} |x - y|$. Prove that ($m$ is the Lebesgue measure)

$$m(E) = \lim_{r \to 0} m(E_r).$$

Solution: $E \subset \mathbb{R}$ being compact implies that E is closed and bounded. Let $\{r_n\}_{n=1}^{\infty}$ be a positive monotonically decreasing sequence with

$$\lim_{n \to \infty} r_n = 0.$$

Write the function

$$f(r) = m(E_r) = \int_{\mathbb{R}} \chi_{E_r} dm.$$

If we can prove for every such sequence $\{r_n\}$, we have

$$\lim_{n \to \infty} f(r_n) = m(E),$$

then we can conclude that

$$\lim_{r \to 0^+} f(r) = m(E).$$

By definition, $r_{n+1} \leq r_n$ tells us that $E_{r_{n+1}} \subseteq E_{r_n}$, and each E_{r_n} is measurable because $d(x, E)$ is a continuous function, so the sequence of sets $\{E_{r_n}\}$ is a decreasing sequence of measurable sets. Moreover, it is easy to see that E_{r_n} is bounded as E is bounded, and $E \subset \bigcap_{n \geq 1} E_{r_n}$. Conversely, by definition, for every $x \in \bigcap_{n \geq 1} E_{r_n}$ and every n, there exists an element $x_n \in E$ such that $d(x, x_n) < r_n$. Bacause $\lim_{n \to +\infty} r_n = 0$, so we have

$$\lim_{n \to +\infty} d(x, x_n) = 0.$$

This implies that x is a limit point of E, and since E is closed, we have $x \in E$. Hence, we obtain

$$\bigcap_{n \geq 1} E_{r_n} = E, \quad E_{r_n} \searrow E.$$

Therefore, we have

$$m(E) = \lim_{n \to \infty} m(E_{r_n})$$

for any such sequence $\{r_n\}$.

Exercise 3
Let A be a bounded measurable set in $\mathbb{R}$. Prove that

$$\lim_{n\to\infty}\int_A \sin^2(nx)dm = \frac{1}{2}m(A).$$

Solution: Note that because A is bounded, for any $n \geq 1$, we have

$$\int_A \sin^2(nx)dx = \int_A \frac{1-\cos(2nx)}{2}dx = \frac{1}{2}m(A) - \int_A \cos(2nx)dx.$$

To prove that

$$\lim_{n\to\infty}\int_A \sin^2(nx)dm = \frac{1}{2}m(A),$$

we only need to show

$$\lim_{n\to\infty}\int_A \cos(2nx)dx = \lim_{n\to\infty}\int_A \cos nxdx = 0.$$

A is bounded and measurable, so the characteristic function χ_A is a measurable function and

$$||\chi_A||_2 = m(A)^{\frac{1}{2}} < +\infty.$$

Choose a closed interval $[a,b] \supset A$ and let $P = b-a$. We can view χ_A as a p-periodic function on $\mathbb{R}$. Consider the Fourier series of χ_A:

$$a_0 + \sum_{n=1}^{+\infty}(a_n \cos(\frac{2n\pi}{P}x) + b_n \sin(\frac{2n\pi}{P}x))$$

where

$$\begin{aligned} a_n &= \frac{2}{P}\int_a^b \chi_A \cos(\frac{2n\pi}{P}x)dx = \frac{2}{P}\int_A \cos(\frac{2n\pi}{P}x)dx, \\ b_n &= \frac{2}{P}\int_a^b \chi_A \sin(\frac{2n\pi}{P}x)dx = \frac{2}{P}\int_A \sin(\frac{2n\pi}{P}x)dx. \end{aligned}$$

By Paeseval's Theorem, we have

$$||\chi_A||_2^2 = \sum_{n=0}^{+\infty} a_n^2 + \sum_{n=1}^{+\infty} b_n^2 = m(A) < +\infty.$$

This implies that

$$\lim_{n\to\infty}\int_A \cos(\frac{2n\pi}{P}x)dx = \lim_{n\to\infty}\int_A \cos nxdx = 0.$$

Exercise 4
Let l^2 be the Hilbert space

$$\left\{ x = (x_n)_{n=1}^{\infty} : \sum_n |x_n|^2 < \infty, x_n \in \mathbb{C} \right\}$$

where the inner product given by

$$(x, y) = \sum_n x_n \overline{y_n}.$$

(a) Prove that $L^2(T)$ is isomorphic to l^2 as a Hilbert space.

(b) Prove that the unit closed ball in l^2 is not a compact set.

Solution:

(a) Define the 2π-periodic fucntions

$$u_n = e^{int} \in L^2(T), \qquad n \in \mathbb{Z}.$$

The set $\{u_n\}_{n\in\mathbb{Z}}$ is a maximal orthonormal set of $L^2(T)$. Consider the map

$$F : L^2(T) \to l^2,$$
$$u_n \mapsto x_n.$$

The Riesz-Fischer theorem implies that F is an isometry from $L^2(T)$ onto l^2, and the Parsval's identity implies that F gives an isomorphism of Hilbert spaces.

(b) Consider the following sequence $\{(x_n)_k\}_{k=1}^{\infty}$ in l^2: for each fixed k, $(x_n)_k$ is the following sequence:

$$x_n = 1, \quad \text{if} \;\; n = k;$$
$$x_n = 0, \quad \text{otherwise}.$$

It is easy to see that for every $k \geq 1$, $(x_n)_k \in l^2$ and $||(x_n)_k||_2 = 1$, so $\{(x_n)_k\}$ is a sequence in the unit ball of l^2. We claim that it has no convergent subsequences. Indeed, for any $k_1 \neq k_2$, we have

$$||(x_n)_{k_1} - (x_n)_{k_2}||_2 = \sqrt{1+1} = \sqrt{2}.$$

This proves that the unit ball is not compact in l^2.

Exercise 5

Let f_n be a sequence of positive measurable function on a measurable space (X, μ) with a positive Borel measure $\mu(X) < \infty$. Suppose

$$\lim_{n\to\infty} \int_X f_n^2 d\mu = 0.$$

Prove that

$$\lim_{n\to\infty} \int_X f_n \log f_n d\mu = 0.$$

Solution: Write

$$E = \{x \in X : 0 < f(x) < 1\}.$$

Then we can write

$$\int_X f_n \log f_n d\mu = \int_E f_n \log f_n d\mu + \int_{X\setminus E} f_n \log f_n d\mu.$$

Note that on $X \setminus E$, for every n, we have $0 \leq \log f_n \leq f_n$, so

$$0 \leq f_n \log f_n \leq f_n^2.$$

Take the integral and let n goes to ∞, we obtain that

$$0 \leq \lim_{n\to\infty} \int_{X\setminus E} f_n \log f_n d\mu \leq \lim_{n\to\infty} \int_{X\setminus E} f_n^2 d\mu \leq \lim_{n\to\infty} \int_X f_n^2 d\mu = 0.$$

This implies that

$$\lim_{n\to\infty} \int_{X\setminus E} f_n \log f_n d\mu = 0.$$

Therefore, we may assume $0 < f_n < 1$ for every n on X. For every $\varepsilon > 0$, define

$$E(n, \varepsilon) = \{x \in X : 0 < f_n(x) < \varepsilon\}.$$

<u>Claim:</u> For every fixed ε, we have

$$\lim_{n\to\infty} \mu(E(n, \varepsilon)) = \mu(X).$$

<u>Proof:</u> Assume this is not the case. Then there exists a measurable set $F \subset X$ with $0 < \mu(F) < \mu(X) < +\infty$ such that for all $x \in F$ and n large enough, we have $f_n(x) > \frac{\varepsilon}{2}$. Hence

$$\int_X f_n^2 d\mu \geq \int_F f_n^2 d\mu > \frac{\varepsilon^2}{4} m(F) > 0.$$

This contradicts the condition that

$$\lim_{n\to\infty} \int_X f_n^2 d\mu = 0.$$

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Since for all n, we have $0 < f_n < 1$, so the function $f_n \log f_n$ is bounded, namely, there exists a constant $M > 0$ such that

$$|f_n(x) \log f_n(x)| < M$$

for all $x \in X$. Given $\varepsilon > 0$, we know that $x \log x \to 0$ when $x \to 0$, so there exists $\delta > 0$ such that $|x \log x| < \varepsilon$ whenever $0 < x < \delta$. From the above claim, we choose n large enough such that

$$\mu(X - E(n, \delta)) < \varepsilon.$$

Note that in this case, for every $x \in E(n, \delta)$, we have

$$|f_n(x) \log f_n(x)| < \varepsilon.$$

Hence, the integral

$$\begin{aligned}
\int_X f_n \log f_n d\mu &= \int_{E(n,\delta)} f_n \log f_n d\mu + \int_{X - E(n,\delta)} f_n \log f_n d\mu \\
&< \varepsilon m(E(n, \delta)) + M\mu(X - E(n, \delta)) \\
&\leq \varepsilon m(X) + M\varepsilon \\
&= (M + m(X))\varepsilon.
\end{aligned}$$

Let ε goes to 0, and we have proved that

$$\lim_{n \to \infty} \int_X f_n \log f_n d\mu = 0.$$

Exercise 6

(a) Prove that $L^1(\mathbb{R}) \cap L^{2025}(\mathbb{R})$ is a proper subset of $\bigcap_{1<p<2025} L^p(\mathbb{R})$.

(b) Prove that $L^p([0, 1])$ is separable for $p \in [1, +\infty)$ but $L^\infty([0, 1])$ is not separable.

Solution:

(a) Let $f \in L^1(\mathbb{R}) \cap L^{2025}(\mathbb{R})$ and m be the Lebesgue measure. For any $p \in (1, 2025)$, by Hölder inequality, and note that both $||f||_1 < +\infty$ and $||f||_{2025} < +\infty$, we obtain that

$$\begin{aligned}
||f||_p^p &= \int_{\mathbb{R}} |f|^p dm \\
&= \int_{\mathbb{R}} |f|^{\frac{2025-p}{2024}} \cdot |f|^{\frac{2025p-2025}{2024}} dm \\
&\leq (\int_{\mathbb{R}} |f| dm)^{\frac{2025-p}{2024}} \cdot (\int_{\mathbb{R}} |f|^{2025} dm)^{\frac{p-1}{2024}} \\
&= ||f||_1^{\frac{2025-p}{2024}} \cdot ||f||_{2025}^{\frac{2025p-2025}{2024}} \\
&< +\infty.
\end{aligned}$$

This proves that $L^p(\mathbb{R}) \supset L^1(\mathbb{R}) \cap L^{2025}(\mathbb{R})$ for any $p \in (1, 2025)$. Thus,

$$\bigcap_{1<p<2025} L^p(\mathbb{R}) \supset L^1(\mathbb{R}) \cap L^{2025}(\mathbb{R}).$$

Consider the function $f(x) = \frac{1}{x}$ on $[1, +\infty)$ and 0 otherwise. For any $p > 1$, we have

$$\begin{aligned} ||f||_p^p &= \int_{\mathbb{R}} f dm \\ &= \int_1^{+\infty} x^{-p} dx \\ &= \frac{x^{1-p}}{1-p}\Big|_1^{+\infty} \\ &= 0 - \frac{1}{1-p} \\ &= \frac{1}{p-1} \\ &< +\infty. \end{aligned}$$

So $f \in \bigcap_{1<p<2025} L^p(\mathbb{R})$. On the other hand, however, $f \notin L^1(\mathbb{R})$ as

$$||f||_1 = \int_1^{+\infty} \frac{1}{x} dx = +\infty.$$

We can conclude that $L^1(\mathbb{R}) \cap L^{2025}(\mathbb{R})$ is a proper subset of $\bigcap_{1<p<2025} L^p(\mathbb{R})$.

(b) For every function defined on $[0, 1]$, we can view them as periodic function defined on $\mathbb{R}$ with period 1. Consider the collection of functions

$$u_n(t) = e^{i2\pi nt}, \qquad n \in \mathbb{Z}.$$

Let E be the set spanned by $\{u_n\}_{n\in\mathbb{Z}}$ with rational coefficients. We know that E consists of trigonometry polynomials with rational coefficients, so that E is dense in $C([0, 1])$, and because $C([0, 1])$ is dense in $L^p([0, 1])$ for any $1 < p < +\infty$, we can conclude that $L^p([0, 1])$ is separable for any $1 < p < +\infty$.

Next, we want to show that $L^\infty([0, 1])$ is not separable. Consider the first intervals: $I_0 = [0, \frac{1}{2})$ and for any $n \geq 1$, we define

$$I_n = [\sum_{k=1}^{n} \frac{1}{2^k}, \sum_{k=1}^{n+1} \frac{1}{2^k}).$$

Then we have $[0, 1) = \bigcup_{n\geq 0} I_n$. Let $a = \{a_n\}_{n=0}^{\infty}$ be a sequence taking values in $\{0, 1\}$. Define the function

$$f_a(x) = 2a_n, \quad \text{if} \quad x \in I_n.$$

We have $f_a \in L^\infty([0, 1])$ and

$$||f_a - f_b||_\infty \geq 2$$

if $a = a_n$ and $b = \{b_n\}$ are two different sequences taking values in 0 and 1. Let $E = \{f_a\} \subset L^\infty([0, 1])$. Each function $\mathbb{Z}_{\geq 0} \to \{0, 1\}$ gives a unique element in E and any two different element of E satisfies that

$$||f_a - f_b||_\infty > 1.$$

This implies that $L^\infty([0, 1])$ is not separable as it cannot contain a countable and dense subset.