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Course: MATH 635 - Algebraic Topology II
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Homework 6
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Problem 1
Compute π_1 of each of the following spaces.

(a) Take S^2 and identify the north and south poles.

(b) Take two copies of S^2, identify the two north poles together, and then also identify the two south poles together.

(c) Take $\mathbb{R}^3$ and remove three lines through the origin.

(d) $\mathbb{R}^n$ with k points removed (do $n = 2$ and $n > 2$ separately).

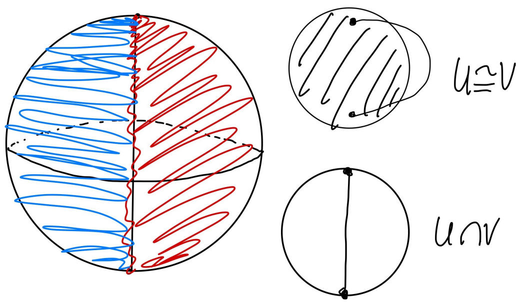
(e) A torus with two points removed.

(f) Take $S^4 \times S^1$ and remove one point.

(g) $\mathbb{R}P^4$ with two points removed.

Solution:

(a) Let X be the quotient space of S^2 where the north pole and the south pole are identified. An interval is contractible, so X is homotopic equivalent to the space S^2 in which the north pole and the south pole are connected by an interval. Consider the left half sphere U and the right half sphere V in S^2 whose intersection is an annulus. $X = U \cup V$ where U $congV$ is homeomorphic to a disk on which two points are connected by a line, and the intersection is homotopic equivalent to S^1 where two points are connected by a line, as show in the picture.

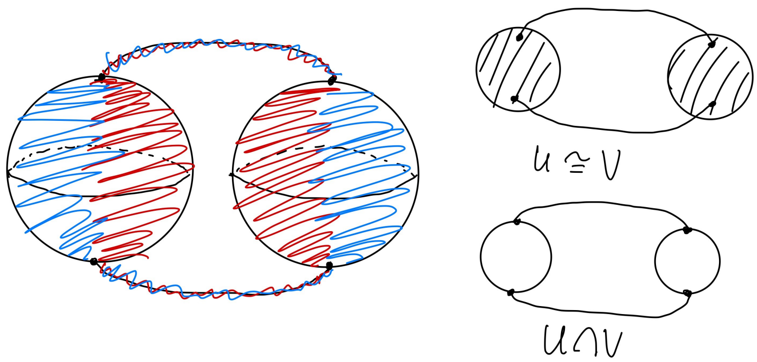


We know both $U, V, U \cap V$ are path-connected, so we assume the basepoint is chosen in $U \cap V$ and omit the notation in the calculation. Note that $U \cong V \simeq S^1$ and $U \cap V \simeq S^1 \vee S^1$. By Van Kampen Theorem, we have a pushout square in groups

$$\begin{tikzcd}
\pi_1(S^1 \vee S^1) \arrow[r, "i"] \arrow[d, "j"] & \pi_1(S^1) \arrow[d] \\
\pi_1(S^1) \arrow[r] & \pi_1(X)
\end{tikzcd}$$

We have $\pi_1(X) = \mathbb{Z} * \mathbb{Z}/(\operatorname{Im} i \sim \operatorname{Im} j)$. Note that the two maps i, j induced by inclusion $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$. i and j have the same image by symmetry and the induced map is surjective. So $\operatorname{Im} i = \operatorname{Im} j = \mathbb{Z}$. This implies $\pi_1(X) = \mathbb{Z}$.

(b) Let X be the quotient space of $S^2 \sqcup S^2$ where the two north poles and the two south poles are identified respectively. Same as previous. X is homotopic equivalent to the space $S^2 \sqcup S^2$ where there are two lines connecting north poles and south poles respectively. Consider U and V indicated in the pictures by red and blue. We can see that $U \cong V$ is homeomorphic to two disks connected by two lines, thus homotopic equivalent to S^1 since a disk is contractible. The intersection $U \cap V$ is homotopic equivalent to two S^1 connected by two lines, thus further homotopic equivalent to the wedge sum $S^1 \vee S^1 \vee S^1$.



We know that $U \cap V, U, V$ are path-connected, so we assume the base point is chosen in $V \cap U$ and omit the notation in the calculation. By Van Kampen Theorem, we have a pushout square in groups

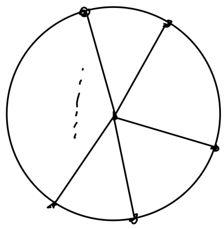
$$\begin{tikzcd}
\pi_1(S^1 \vee S^1 \vee S^1) \arrow[r, "i"] \arrow[d, "j"'] & \pi_1(S^1) \arrow[d] \\
\pi_1(S^1) \arrow[r] & \pi_1(X)
\end{tikzcd}$$

We have $\pi_1(S^1 \vee S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. So $\pi_1(X) = \mathbb{Z}/(\operatorname{Im} i \sim \operatorname{Im} j)$. i and j have the same image by symmetry and the map i must be sujective since it is induced by inclusion of spaces. So we have $\pi_1(X) = \mathbb{Z}$.

(c) We know that $\mathbb{R}^3$ is homotopic to D^3. Let X be the space of D^3 removing three lines passing through this solid ball. These three lines pass through the one common point in D^3. We first remove the commom point in three lines from inside D^3. Since D^3 removing one point inside is homotopic equivalent to S^2, we know that X is homotopic equivalent to the S^2 removing 6 points. Note that $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$. Thus, S^2 removing 6 points is homotopic equivalent to $\mathbb{R}^2$ removing 5 points. And from what we have discussed in part (d), we know that $\pi_1(X) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{5 \text{ times}}$.

(d) Let X be $\mathbb{R}^n$ removing k points. X is always path-connected, so we omit the basepoint notation in the calculations. When $n = 2$, if $k = 0$, then $\mathbb{R}^2$ is contractible, so $\pi_1(\mathbb{R}^2) = *$ is trivial. If $k \geq 1$, without loss of generality, we may assume all k points $x_1, \ldots, x_k$ are on the circle centered at the origin with radius 1. We know that $\mathbb{R}^3$ is homotopic equivalent to D^2, which can be viewed as a disk centered at the origin with radius 2. Divide the disk

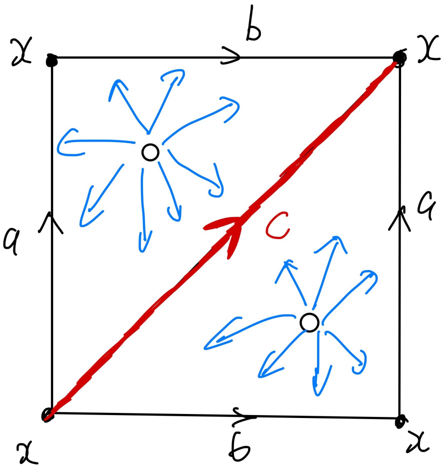
D^2 into fan-shaped sectors with having one and only one x_i inside each sector. This can be done since the points removed are discrete. Then removing each point inside the fan-shaped sector can be viewed as homotopic to removing the surfaces occupying that secton. Thus, X is homotopic equivalent to S^1 with k lines connecting to the center. Choose the center as the basepoint and it is easy to see that X is homotopic equivalent to k copies of S^1 wedged at one point. By Van Kampen Theorem and what we proved in class, we know $\pi_1(X) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k \text{ times}}$.



Now assume $n \geq 2$. When $k = 0$, we know that $\mathbb{R}^n$ is contractible. So $\pi_1(\mathbb{R}^n) = *$ is trivial. For $k \geq 1$, using the same method as above, we can see that X is homotopic equivalent to k copies of S^{n-1} wedged at one point. For any $n \geq 2$, S^{n-1} is simply-connected, so by Van Kampen Theorem, the wedge sum is also simply-connected. We have $\pi_1(X) = *$ is trivial. To summarize, we have

$$\pi_1(X) = \begin{cases} \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k \text{ times}}, & \text{if } \ n = 2, k \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

(e) Let X be the space of torus T removing two points. Note that X is still path-connected, so we omit the choice of basepoint. Consider the standard cell complex structure for the torus T as shown. Additionally, the 0-cell x are connected by a red line, which is homeomorphic to S^1 in the 2-cell, thus dividing the 2-cell into two sectors. Next, remove one point from each of the two sectors. X is homotopic equivalent to the 1-skeleton of this cell complex, which has one 0-cell x, and three 1-cells a, b, c. From this we can see that X is homotopic equivalent to $S^1 \vee S^1 \vee S^1$. By Van Kampen Theorem, $\pi_1(X) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.



(f) Let $M = S^4 \times S^1$ and $x \in M$ be a point. We know both S^4 and S^1 are path-connected, so M is also path-connected. We omit the choice of basepoint from now on. Note that M is a 5-manifold. There exists an open neighborhood U of x in M such that $U \cong \mathbb{R}^5$. Let $V = M - \{x\}$ which is also open in M. We have

$$U \cap V \cong U - \{x\} \cong \mathbb{R}^5 - \{*\}.$$

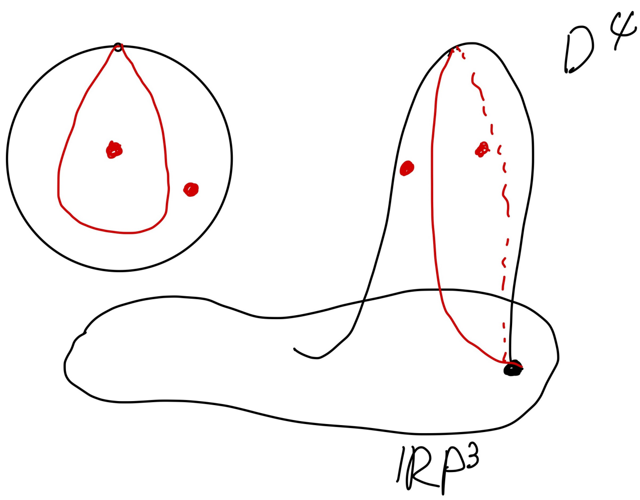
So $U \cap V$ is homotopic equivalent to S^4. By Van Kampen Theorem, we have a pushout square of groups

$$\begin{tikzcd} \pi_1(S^4) \arrow[r] \arrow[d] & \pi_1(U) \arrow[d] \\ \pi_1(M - \{x\}) \arrow[r] & \pi_1(M) \end{tikzcd}$$

Note that $\pi_1(S^4) = *$ is trivial and $\pi_1(U) = \pi_1(\mathbb{R}^5) = *$ is also trivial. So we have

$$\pi_1(M - \{x\}) = \pi_1(M) = \pi_1(S^4 \times S^1) = \pi_1(S^4) \times \pi_1(S^1) = \mathbb{Z}.$$

(g) Let X be the space of $\mathbb{R}P^4$ removing two points. Consider the standard CW complex structure on $\mathbb{R}P^4$. We know that $\mathbb{R}P^4$ can be viewed as $\mathbb{R}P^3$ attaching a 4-cell D^4 via a degree 0 boundary map $S^3 \to S^3$. We choose a base point $x \in \mathbb{R}P^3$ and one S^3 inside the interior of D^4 (red in picture), denoted by Y. We know that $Y \cap \partial D^4 = \{x\}$. Choose one point inside the space bounded by Y and another point in the interior of D^4 but not in Y. Remove these two points and we obtain X. We know that D^4 removing these two points is homotopic equivalent to the union $Y \cup \partial D^4$. Note that $\partial D^4 = S^3$ is glued to the 3-skeleton $\mathbb{R}P^3$, so X is homotopic equivalent to $\mathbb{R}P^3 \vee Y = \mathbb{R}P^3 \vee S^3$.



By Van Kampen Theorem, we have

$$\pi_1(X) = \pi_1(\mathbb{R}P^3) * \pi_1(\mathbb{S}^3) = \pi_1(\mathbb{R}P^3).$$

We know $\mathbb{R}P^2$ is the 2-skeleton of $\mathbb{R}P^3$, so $\pi_1(X) = \pi_1(\mathbb{R}P^3) \cong \pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$.

Problem 2
Let p and q be relatively prime, positive integers. Consider the 3-disk $D^3 \subseteq \mathbb{R}^3$, and regard it as the space

$$D^3 = \left\{(z,r) \mid z \in \mathbb{C}, r \in \mathbb{R}, |z|^2 + r^2 \leq 1\right\}.$$

Let $\zeta = e^{2\pi i/p}$, and let $L(p,q)$ be the quotient space of X where one identifies

$$(z,r) \sim (\zeta^q z, -r)$$

if $r \geq 0$ and $|z|^2 + r^2 = 1$. Convince yourself that $L(p,q)$ is a 3-manifold. The space $L(p,q)$ is called a lens space. Note that $L(2,1) = \mathbb{R}P^3$.
Let $\rho : D^3 \to L(p,q)$ be the quotient map. Let

$$\begin{aligned} X_0 &= \{\rho(1,0)\}, \\ X_1 &= \left\{\rho(z,0) \mid z \in \mathbb{C}, |z|^2 = 1\right\}, \\ X_2 &= \left\{\rho(z,r) \mid |z|^2 + r^2 = 1\right\}. \end{aligned}$$

Convince yourself that

$$\varnothing \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X_3 = L(p,q)$$

is a CW-structure on $L(p,q)$, with exactly one cell in each dimension. Use this to compute $H_*(L(p,q))$ as well as $\pi_1(L(p,q))$.

Solution: The solutions are divided into three parts. In part (1), we prove that $L(p.q)$ is indeed a 3-manifold. In part (2), we prove that the given structure in the problem is a CW structure on $L(p,q)$. Lastly, in part (3), we calculate the homology and fundamental groups of $L(p,q)$.

(1) We have already know that D^3 is a 3-manifold with boundary, to show that $L(p,q)$ is a 3-manifold, we need to show that for every point $(z,r) \in \partial D^3$ with $|z| = 1$, after the identification, it has an open neighborhood which is homeomorphic to $\mathbb{R}^3$. For points (z,r) satisfying $r < 0$, it is identifies with exactly one point in the upper half sphere, so it has a neighborhood homeomorphic to $\mathbb{R}^3$. For each point $(z,0)$ with $|z|^2 = 1$ on the equator, note that if we choose an open neighborhood small enough, no points in this neighborhood are identified with each other, so we can piece together $p-1$ such neighborhood to get $\mathbb{R}^3$ as each of them can be chosen as being homeomorphic to a half ball.

(2) It is easy to see that X_0 is just a point and X_1 just S^1 with $|z| = 1$ and $r = 0$ in $L(p,q)$. Note that for point $z \in L(p,q)$ satisfying $|z| = 1$, we identify the points z with $e^{2\pi i \frac{q}{p}} z$. Since p and q are coprime, we divide this circle into p copies of S^1 and view them as just one S^1. When attaching a 2-cell to X_1, we use a degree p map $S^1 \to S^1$. Note that we only have one 2-cell because if $L(p,q)$ is viewed as a quotient space $D^3/\sim$, for $r \geq 0$, all points (z,r) in lower half sphere in ∂D^3 is identified with a point $(e^{2\pi i \frac{q}{p}} z, -r)$ in the upper half sphere. Finally, we attach a 3-cell to the 2-skeleton X_2, which corresponding to the interior of the quotient space $D^3/\sim$. This gives us a CW structure for $L(p,q)$:

$$\varnothing \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X_3 = L(p,q).$$

(3) The cellular chain complex from the above CW structure is given by

$$0 \to \mathbb{Z} \xrightarrow{d_3} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

Note that $d_3 = 0$ because this is a chain complex. So the homology can be calculated as

$$H_i(L(p,q)) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 3; \\ \mathbb{Z}/p\mathbb{Z}, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

For the fundamental group $\pi_1(L(p,q))$, from the CW structure, we know that $\pi_1(L(p,q)) = \pi_1(X_2) = \pi_1(X_1)/\sim$ where $\sim$ is given by the attaching map $S^1 \to S^1$. We have seen that it is a degree p map, so the fundamental group $\pi_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}$.

Problem 3

Suppose W is a space and $h : W \to W$ is a homeomorphism. Let X be the quotient space of $W \times I$ where we identify $(w, 0) \sim (h(w), 1)$ for all $w \in W$. Note that if W is a d-manifold then X is a $(d+1)$-manifold. If $W = S^2$ then $\deg(h)$ is either 1 or -1. Compute $H_*(X)$ in both cases.

Solution: Let X be the quotient space of $S^2 \times I$ obtained in this way. Consider the open sets $U = S^2 \times (\frac{1}{4}, \frac{3}{4})$ and $V = S^2 \times ((0, \frac{1}{3}) \cup (\frac{2}{3}, 1))$. We have $X = U \cup V$. Note that $S^2 \times \{0\}$ is identified with $S^2 \times \{1\}$ via a homeomorphism h, so we know that both U and V are homotopic equivalent to S^2. Moreover, we have

$$U \cap V = S^2 \times ((\frac{1}{4}, \frac{1}{3}) \cup (\frac{2}{3}, \frac{3}{4})) \simeq S^2 \sqcup S^2.$$

By Mayer-Vietoris, we have a long exact sequence in homology

$$\begin{tikzcd}
 & \tilde{H}_*(U \cap V) & \tilde{H}_*(U) \oplus \tilde{H}_*(V) & \tilde{H}_*(X) \\
3 & 0 \arrow[r] & 0 \arrow[r] & ? \arrow[dll] \\
2 & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "{i_*, j_*}"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & ? \arrow[dll] \\
1 & 0 \arrow[r] & 0 \arrow[r] & ? \arrow[dll] \\
0 & \mathbb{Z} \arrow[r] & 0 &
\end{tikzcd}$$

It can be observed that

$$H_1(X) \cong \tilde{H}_0(U \cap V) = \tilde{H}_0(S^2 \sqcup S^2) = \mathbb{Z}.$$

X is path-connected since $S^2 \times I$ is path-connected, so $H_0(X) = \mathbb{Z}$. For the rest of the homology groups, we need to determine the map in homology

$$H_2(S^2 \sqcup S^2) \to H_2(S^2) \oplus H_2(S^2)$$

which is induced by the inclusion $i : U \cap V \to U$ and $j : U \cap V \to V$. Choose the spheres $S^2 \times \left\{\frac{2}{7}\right\}$ and $S^2 \times \left\{\frac{5}{7}\right\}$ in $U \cap V$ as the generators of the homology group $H_2(U \cap V) = \mathbb{Z} \oplus \mathbb{Z}$. We know that $H_2(U) = H_2(S^2) = \mathbb{Z}$ has only one generator, so the spheres at $S^2 \times \left\{\frac{2}{7}\right\}$ and the sphere at $S^2 \times \left\{\frac{5}{7}\right\}$ is the same generator in $H_2(U)$, so i_* can be described as

$$\begin{aligned} i_* : \mathbb{Z} \oplus \mathbb{Z} &\to \mathbb{Z}; \\ (1,0) &\mapsto 1, \\ (0,1) &\mapsto 1. \end{aligned}$$

For V, we know $H_2(V) = \mathbb{Z}$ have one generator and we choose the sphere at $S^2 \times \left\{\frac{5}{7}\right\}$. So one of the generator $S^2 \times \left\{\frac{5}{7}\right\}$ of $H_*(U \cap V)$ is sent to itself, and for the other generator $S^2 \times \left\{\frac{2}{7}\right\}$, firstly it is sent to the sphere at $S^2 \times \{0\}$, then the induced map in homology $h_* : H_2(S^2) \to H_2(S^2)$ sends it to the homology group of the sphere at $S^2 \times \{1\}$, which is the same as the generator at $S^2 \times \left\{\frac{5}{7}\right\}$. Note that $\deg h$ is just how we sent $1 \in H_2(S^2)$, so j_* can be described as

$$\begin{aligned} j_* : \mathbb{Z} \oplus \mathbb{Z} &\to \mathbb{Z}; \\ (1,0) &\mapsto \deg h, \\ (0,1) &\mapsto 1. \end{aligned}$$

So the map in homology $H_2(S^2 \sqcup S^2) \xrightarrow{i_*, j_*} H_2(S^2) \oplus H_2(S^2)$ can be summarized as a matrix $\begin{pmatrix} 1 & \deg h \\ 1 & 1 \end{pmatrix}$. Note that $H_3(X) = \ker(i_*, j_*)$ and $H_2(X) = \operatorname{coker}(i_*, j_*)$.

When $\deg h = 1$, the matrix is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and we have

$$H_i(X) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0,1,2,3; \\ 0, & \text{otherwise.} \end{cases}$$

And when $\deg h = -1$, the matrix is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and we have

$$H_i(X) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0,1; \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \ i = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 4

Regard $\mathbb{R}^7$ as contained in $\mathbb{R}^8$ in the usual way. For every line ℓ through the origin in $\mathbb{R}^8$, I want to associate a line $F(\ell)$ through the origin in $\mathbb{R}^7$. I want this assignment to be continous, and when $\ell \subseteq \mathbb{R}^7$, I want $F(\ell) = \ell$. Prove that no such assignment F can exist.

Solution: Recall that for $n \geq 2$, each point in $\mathbb{R}P^n$ can be viewed as a line through the origin in $\mathbb{R}^{n+1}$. To get the map F described in the problem, it is the same as defining a continous map $f : \mathbb{R}P^7 \to \mathbb{R}P^6$. F restricted to lines in $\mathbb{R}^7$ being the identity means that f restricted to $\mathbb{R}P^6$ is the identity map. Namely we have a commutative diagram

$$\begin{tikzcd} \mathbb{R}P^7 \arrow[r, "f"] & \mathbb{R}P^6 \\ \mathbb{R}P^6 \arrow[u, hook, "i"] \arrow[ur, "id"'] & \end{tikzcd}$$

where $i : \mathbb{R}P^6 \to \mathbb{R}P^7$ is the inclusion map. Apply $\pi_6(-, *)$, and we have a commutative diagram in homotopy groups

$$\begin{tikzcd} \pi_6(\mathbb{R}P^7, *) \arrow[r, "f_*"] & \pi_6(\mathbb{R}P^6, *) \\ \pi_6(\mathbb{R}P^6, *) \arrow[u, hook, "i_*"] \arrow[ur, "id"'] & \end{tikzcd}$$

Note that for $\pi_6(\mathbb{R}P^6) = \pi_6(S^6) = \mathbb{Z}$ and $\pi_6(\mathbb{R}P^7) = \pi_6(S^7) = 0$. So this diagram cannot commute since the identity map cannot be obtained from the zero map composed with anything.

Problem 5
Let V be a finite dimensional real vector space and assume it's equipped with a continous product $V \times V \to V$ satisfying the following three conditions:

(i) $(rv) \cdot w = r(v \cdot w) = v \cdot (rw)$ for all $v, w \in V$, $r \in \mathbb{R}$;

(ii) There exists an element $1 \in V$ such that $1 \cdot v = v \cdot 1 = v$ for all $v \in V$;

(iii) $vw = 0$ if and only if either $v = 0$ or $w = 0$.

Pick a basis $e_1, \ldots, e_n$ for V such that $e_1 = 1$, and define a norm on V by

$$||r_1 e_1 + \cdots + r_n e_n|| = \sqrt{r_1^2 + \cdots + r_n^2}.$$

Let $S(V) = \{v \in V : ||v|| = 1\}$.

(a) Show that under our assumptions on V there is a continous map $\theta : S(V) \times S(V) \to S(V)$ with the property that $\theta(-1, v) = -v$ and $\theta(1, v) = v$ for all $v \in S(V)$.

(b) Use part (a) to show that if $\dim_{\mathbb{R}} V \geq 2$, then the identity map $S(V) \to S(V)$ is homotopic to the antipodal map. Conclude that $\dim_{\mathbb{R}} V$ is either even or equal to 1.

Solution:

(a) For any $v, w \in S(V)$, we define $\theta(v, w) = \dfrac{v \cdot w}{||v \cdot w||}$. This map is a well-defined map from

$S(V) \times S(V)$ to $S(V)$ because

$$||\theta(v,w)|| = \frac{||v \cdot w||}{||v \cdot w||} = 1.$$

For any $v \in S(V)$, suppose $v = r_1e_1 + \cdots + r_ne_n$ satisfying $r_1^2 + \cdots + r_n^2 = 1$. We have

$$(-1) \cdot v = -(1 \cdot v) = -v = -r_1e_1 - r_2e_2 - \cdots - r_ne_n.$$

So we know that

$$|| - v|| = \sqrt{(-r_1)^2 + \cdots + (-r_n)^2} = \sqrt{r_1^2 + \cdots + r_n^2} = 1.$$

We can calculate

$$\theta(1,v) = \frac{1 \cdot v}{||1 \cdot v||} = \frac{v}{||v||} = v,$$
$$\theta(-1,v) = \frac{(-1) \cdot v}{||(-1) \cdot v||} = \frac{-v}{|| - v||} = -v.$$

The last thing we need to show is that θ define in this way is continous for all $(v,w) \in S(V) \times S(V)$. Since θ is symmetric, we only need to show that for any $w \in S(V)$, $\theta(-,w) : S(V) \to S(V)$ is a continous function.

(b) Suppose $\dim_{\mathbb{R}} V \geq 2$. For any $-1 \leq t \leq 1$, define $w(t) = te_1 + \sqrt{1-t^2}e_2 \in S(V)$, which can be viewed as a continous function of t. We have a map

$$H : S(V) \times [-1,1] \to S(V),$$
$$(v,t) \mapsto \theta(v,w(t)).$$

This map is continous because w and θ are continous. For any $v \in S(V)$, we can see that

$$H(v,-1) = \theta(v,w(-1)) = \frac{(-1) \cdot v}{||(-1) \cdot v||} = -v$$

is the antipodal map and

$$H(v,1) = \theta(v,w(1)) = \frac{1 \cdot v}{||1 \cdot v||} = v$$

is the identity map. This proves that if $\dim_{\mathbb{R}} V \geq 2$, the identity map $S(V) \to S(V)$ is homotopic to the antipodal map. Suppose $\dim_{\mathbb{R}} V = n \geq 2$. We know that $S(V) \cong S^{n-1}$. So the degree of the identity map $S(V) \xrightarrow{id} S(V)$ is 1 and the degree of the antipodal map is $(-1)^{n-1+1} = (-1)^n$. Thus, this implies n must be even if $n \geq 2$. We know that we have a product for 1-dimensional real vector space because in this case, V is isomorphic to $\mathbb{R}$, and $\mathbb{R}$ is a field.