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Course: MATH 616 - Real Analysis
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Homework - Week 2 Exercises
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Exercise 1.0
Let $f_n(x) = (nx)^{-2}(1 - \cos(nx))$. Find the value of

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx$$

Solution:

Exercise 1.1
Does there exist an infinite σ-algebra which has only countably many members?

Solution: No, such σ-algebra does not exist. Assume a σ-algebra $\mathfrak{M}$ has only countably many members on a set X. Write

$$\mathfrak{M} = \{X_1, X_2, \ldots, X_n, \ldots\}$$

For any $x \in X$, let $A_x := \bigcap_{x\in X_i} X_i$. A_x is not empty and $A_x \in \mathfrak{M}$ because it is the countable intersection of members in $\mathfrak{M}$. By definition, if $x \in X_i$ for any X_i, then we must have $A_x \subset X_i$. Suppose $y \in X$ and $y \neq x$. If $y \in A_x$, then $A_y = A_x$. Indeed, A_x is a member of $\mathfrak{M}$, so $A_y \subseteq A_x$. If $x \notin A_y$, then $x \in A_x \setminus A_y$ which is not contained in A_x. This contradicts that A_x is the intersection of all sets containing x. So $x \in A_y$, and this implies $A_x \subseteq A_y$, thus $A_x = A_y$.

Write $X = \bigcup_{x\in X} A_x$. From what we discuss above, for $x \neq y$, either $A_x = A_y$ or $A_x \cap A_y = \varnothing$. Thus, we can write $X = \bigcup_{i\in I} A_{x_i}$ where I is the index set and $A_{x_i} \cap A_{x_j} = \varnothing$ for $i \neq j$ in I.

Assume I is finite. For any $Y \in \mathfrak{M}$, we have $Y = \bigcup_{x\in Y} A_x$. So Y can be written in the form $\bigcup_{i\in J} A_{x_i}$ for some $J \subseteq I$. Since I is finite, this implies that $\mathfrak{M}$ only has finitely many members.

Assume I is countably infinite. Note that for $I_1, I_2 \subset I$,

$$\bigcup_{i\in I_1} A_{x_i} = \bigcup_{j\in I_2} A_{x_j}$$

if and only if $I_1 = I_2$. The cardinality of the power sets of I must be uncountably many, so $\mathfrak{M}$ has at least uncountably many memebrs.

Assume I is uncountably infinite. Note that every A_{x_i} is a different member of $\mathfrak{M}$ by our choice, so again $\mathfrak{M}$ has uncountably many members.

This is a contradiction. Hence, we conclude that such σ-algebra $\mathfrak{M}$ does not exist.

Exercise 1.3

Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

Solution:

Exercise 1.4

Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, and prove the following assertions:

(a) $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$.

(b) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$ provided none of the sums is of the form $\infty - \infty$.

(c) If $a_n \leq b_n$ for all n, then

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$$

Show by an example that strict inequality can hold in (b).

Solution:

Exercise 1.5

(a) Suppose $f : X \to [-\infty, \infty]$ and $g : X \to [-\infty, \infty]$ are measurable. Prove that the sets

$$\{x : f(x) < g(x)\}, \quad \{x : f(x) = g(x)\}$$

are measurable.

(b) Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

Solution:

Exercise 1.6

Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that $\mathfrak{M}$ is a σ-algebra in X and that μ is a measure on $\mathfrak{M}$. Describe the corresponding measurable functions and their integrals.

Solution: