**Zhengdong Zhang**
Email: zhengz@uoregon.edu
Course: MATH 616 - Real Analysis
Instructor: Professor Weiyong He

**Homework - Week 2 Exercises**
ID: 952091294
Term: Fall 2025
Due Date: Oct 15th, 2025

---

**Exercise 1.0**
Let $f_n(x) = (nx)^{-2}(1 - \cos(nx))$. Find the value of

$$\lim_{n\to\infty} \int_0^{\infty} f_n(x)\, dx$$

*Solution:* Note that for any $n \geq 1$ and $x \in (0, +\infty)$, we have

$$|f_n(x)| \leq \frac{2}{n^2x^2}$$

Thus, for any $x \in (0, +\infty)$,

$$\lim_{n\to\infty} f_n(x) = 0.$$

When $x \in (0, 1)$, note that for all $n \geq 1$

$$|f_n(x)| = \frac{|1 - \cos nx|}{n^2x^2} \leq \frac{\frac{1}{2}n^2x^2 + \frac{1}{4}n^4x^4 + o(x^4)}{n^2x^2} \leq \frac{1}{2}.$$

And for $x \in (1, +\infty)$, note that for all $n \geq 1$

$$|f_n(x)| = \frac{|1 - \cos nx|}{n^2x^2} \leq \frac{1}{x^2}$$

Define

$$g(x) = \mathbb{1}_{(0,1)}\frac{1}{2} + \mathbb{1}_{(1,+\infty)}\frac{1}{x^2}$$

where $\mathbb{1}$ is the characteristic function. We observe that

$$|f_n(x)| \leq g(x)$$

for all $n \geq 1$. Here $g$ is measurable and positive and

$$\int_0^{+\infty} g(x)dx = \int_0^1 \frac{1}{2}dx + \int_1^{+\infty} \frac{1}{x^2}dx = \frac{1}{2} + 1 < +\infty.$$

So $g$ is integrable, and by Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_0^{+\infty} f_n(x)dx = \int_0^{+\infty} \lim_{n\to\infty} f_n(x)dx = 0.$$

---

**Exercise 1.1**
Does there exist an infinite $\sigma$-algebra which has only countably many members?

*Solution:* No, such $\sigma$-algebra does not exist. Assume a $\sigma$-algebra $\mathfrak{M}$ has only countably many members on a set $X$. Write

$$\mathfrak{M} = \{X_1, X_2, \ldots, X_n, \ldots\}$$

For any $x \in X$, let $A_x := \bigcap_{x \in X_i} X_i$. $A_x$ is not empty and $A_x \in \mathfrak{M}$ because it is the countable intersection of members in $\mathfrak{M}$. By definition, if $x \in X_i$ for any $X_i$, then we must have $A_x \subset X_i$. Suppose $y \in X$ and $y \neq x$. If $y \in A_x$, then $A_y = A_x$. Indeed, $A_x$ is a member of $\mathfrak{M}$, so $A_y \subseteq A_x$. If $x \notin A_y$, then $x \in A_x \setminus A_y$ which is not contained in $A_x$. This contradicts that $A_x$ is the intersection of all sets containing $x$. So $x \in A_y$, and this implies $A_x \subseteq A_y$, thus $A_x = A_y$.

Write $X = \bigcup_{x \in X} A_x$. From what we discuss above, for $x \neq y$, either $A_x = A_y$ or $A_x \cap A_y = \varnothing$. Thus, we can write $X = \bigcup_{i \in I} A_{x_i}$ where $I$ is the index set and $A_{x_i} \cap A_{x_j} = \varnothing$ for $i \neq j$ in $I$.

Assume $I$ is finite. For any $Y \in \mathfrak{M}$, we have $Y = \bigcup_{x \in Y} A_x$. So $Y$ can be written in the form $\bigcup_{i \in J} A_{x_i}$ for some $J \subseteq I$. Since $I$ is finite, this implies that $\mathfrak{M}$ only has finitely many members.

Assume $I$ is countably infinite. Note that for $I_1, I_2 \subset I$,

$$\bigcup_{i \in I_1} A_{x_i} = \bigcup_{j \in I_2} A_{x_j}$$

if and only if $I_1 = I_2$. The cardinality of the power sets of $I$ must be uncountably many, so $\mathfrak{M}$ has at least uncountably many memebrs.

Assume $I$ is uncountably infinite. Note that every $A_{x_i}$ is a different member of $\mathfrak{M}$ by our choice, so again $\mathfrak{M}$ has uncountably many members.

This is a contradiction. Hence, we conclude that such $\sigma$-algebra $\mathfrak{M}$ does not exist.

---

**Exercise 1.3**
Prove that if $f$ is a real function on a measurable space $X$ such that $\{x : f(x) \geq r\}$ is measurable for every rational $r$, then $f$ is measurable.

*Solution:* Let $O \subset \mathbb{R}$ be an open set. We know that $O$ can be written as the countable union of disjoint open intervals. To prove $f$ is measurable, we only need to prove that $f^{-1}(a, b)$ is measurable for any open interval $(a, b)$. Note that

$$(a, b) = (a, +\infty) \cap [b, +\infty)^c.$$

So we need to show that $f^{-1}(a, +\infty)$ and $f^{-1}[b, +\infty)$ is measurable. If $a$ is a rational number, then $f^{-1}(a, +\infty)$ is already measurable. If $a$ is an irrational number, consider a decreasing sequence of rational numbers $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{Q}$ such that

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq a$$

and $\lim_{n \to \infty} a_n = a$. Such sequence exists because the rational numbers are dense in $\mathbb{R}$. Therefore, we have

$$f^{-1}(a, +\infty) = f^{-1}\left(\bigcup_{n=1}^{\infty} (a_n, +\infty)\right) = \bigcup_{n=1}^{\infty} f^{-1}(a_n, +\infty).$$

Here $f^{-1}(a_n, +\infty)$ is measurable for all $n \geq 1$, so $f^{-1}(a, +\infty)$ is also measurable.

Similarly, choose an increasing sequence $\{b_n\}_{n=1}^{\infty} \subseteq \mathbb{Q}$ such that

$$b_1 \leq b_2 \leq \cdots \leq b_n \leq \cdots \leq b$$

and $\lim_{n\to\infty} b_n = b$. We have

$$f^{-1}[b, +\infty) = f^{-1}(\bigcap_{n=1}^{\infty}(b_n, +\infty)) = \bigcap_{n=1}^{\infty} f^{-1}(b_n, +\infty).$$

Here $f^{-1}(b_n, +\infty)$ is measurable for all $n \geq 1$, so $f^{-1}[b, +\infty)$ is also measurable.

---

**Exercise 1.5**

(a) Suppose $f : X \to [-\infty, \infty]$ and $g : X \to [-\infty, \infty]$ are measurable. Prove that the sets

$$\{x : f(x) < g(x)\}\,, \quad \{x : f(x) = g(x)\}$$

are measurable.

(b) Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

*Solution:*

(a) If we can prove the function

$$\begin{aligned} f - g : X &\to \mathbb{R}, \\ x &\mapsto f(x) - g(x) \end{aligned}$$

is measurable, then the two sets

$$\{x : f(x) < g(x)\}\,, \quad \{x : f(x) = g(x)\}$$

are measurable because $(-\infty, 0)$ is an open set, and

$$\{x : f(x) - g(x) = 0\} = \{x : f(x) - g(x) < 0\}^c \cap \{x : f(x) - g(x) > 0\}^c$$

where both are measurable.

To see that $f - g$ is measurable, consider the following two functions

$$\begin{aligned} F : X &\to \mathbb{R}^2, \\ x &\mapsto (f(x), g(x)). \end{aligned} \qquad\qquad \begin{aligned} h : \mathbb{R}^2 &\to \mathbb{R}, \\ (u, v) &\mapsto u - v. \end{aligned}$$

We can see that $f - g = h \circ F$ where $F$ is measurable (we proved it in class) and $h$ is continuous, so $f - g$ is also measurable.

(b) Given a sequence of measurable functions $\{f_n\}_{n=1}^{\infty}$, we know that the functions

$$\limsup_{n\to\infty} f_n(x), \quad \liminf_{n\to\infty} f_n(x)$$

are measurable. Consider the set

$$E = \left\{ x \in X : \liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) \right\}.$$

This is exactly the set of points on which $\{f_n\}$ is converging pointwise. Thus, $E$ is measurable from what we proved in (a).