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Course: MATH 616 - Real Analysis
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Homework - Week 4 Exercises
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Exercise 4.1
Find the limit of the integral and justify your answer

$$\lim_{n\to\infty}\int_0^\infty \frac{1-\cos(nx)}{nx}dx$$

Solution: We do a change of variable by setting $u = nx$. This gives

$$\int_0^\infty \frac{1-cos(nx)}{nx}dx = \int_0^\infty \frac{1-\cos u}{nu}du = \frac{1}{n}\int_0^\infty \frac{1-\cos u}{u}du.$$

Note that $\frac{1-\cos u}{u} \geq 0$ for any $u > 0$, so we have

$$\begin{aligned}
\int_0^\infty \frac{1-\cos u}{u}du &\geq \sum_{k=0}^{\infty}\int_{\frac{\pi}{2}+2k\pi}^{\frac{3\pi}{2}+2k\pi} \frac{1-\cos u}{u}du \\
&\geq \sum_{k=0}^{\infty}\int_{\frac{\pi}{2}+2k\pi}^{\frac{3\pi}{2}+2k\pi} \frac{1}{u}du \\
&\geq \sum_{k=0}^{\infty} \frac{1}{\frac{3\pi}{2}+2k\pi}\cdot\left(\frac{3\pi}{2}-\frac{\pi}{2}\right) \\
&= \sum_{k=0}^{\infty}\frac{2}{3+4k}
\end{aligned}$$

We know that the sequence $\sum_0^\infty \frac{2}{4k+3}$ diverges, so

$$\int_0^\infty \frac{1-\cos u}{u}du = +\infty.$$

And

$$\begin{aligned}
\lim_{n\to\infty}\int_0^\infty \frac{1-\cos(nx)}{nx}dx &= \lim_{n\to\infty}\frac{1}{n}\int_0^\infty \frac{1-\cos u}{u}du \\
&= \lim_{n\to\infty}\left(\frac{1}{n}\cdot +\infty\right) \\
&= \lim_{n\to\infty} +\infty \\
&= +\infty.
\end{aligned}$$

Exercise 4.2
If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., suppose $|f_n| \le g_n$ and $\int g_n \to \int g$, prove

$$\int f_n \to \int f.$$

Solution: $|f_n| \le g_n$ implies that $g_n - f_n \ge 0$ and $g_n + f_n \ge 0$, so both $g_n + f_n$ and $g_n - f_n$ are positive measurable functions. By Fatou's lemma and note that $\int g_n \to \int g$, we have

$$\begin{aligned}
\int g + \int f &= \int \liminf_{n\to\infty}(g_n + f_n) \le \liminf_{n\to\infty} \int g_n + f_n = \int g + \liminf_{n\to\infty} \int f_n, \\
\int g - \int f &= \int \liminf_{n\to\infty}(g_n - f_n) \le \liminf_{n\to\infty} \int g_n - f_n = \int f - \limsup_{n\to\infty} \int f_n
\end{aligned}$$

So

$$\int f \le \liminf_{n\to\infty} \int f_n \le \limsup_{n\to\infty} \int f_n \le \int f.$$

This implies that $\int f_n \to \int f$ as $n \to \infty$.

Exercise 4.3
Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.

Solution: By the inverse triangular inequality, for any n, we have

$$||f_n| - |f|| \le |f_n - f|.$$

Thus, for any n,

$$\left| \int |f_n| - \int |f| \right| \le \int ||f_n| - |f|| \le \int |f_n - f|.$$

Assume $\int |f_n - f| \to 0$, then $\int |f_n| \to \int |f|$ by the above inequality.

On the other hand, assume $\int |f_n| \to \int |f|$. Note that for any n

$$|f_n - f| \le |f_n| + |f|.$$

Define $g_n = |f_n| + |f| - |f_n - f|$ which are positive measurable functions, and because $f_n \to f$ almost everywhere, $g_n \to 2|f|$ almost everywhere. By Fatou's lemma and $\int |f_n| \to \int |f|$, we have

$$\int 2|f| = \int \liminf_{n\to\infty} g_n \le \liminf_{n\to\infty} \int |f_n| + |f| - |f_n - f| = \int 2|f| - \limsup_{n\to\infty} \int |f_n - f|.$$

So $\limsup_{n\to\infty} |f_n - f| \le 0$ and this implies that $\int |f_n - f| \to 0$.

Exercise 2.3
Let X be a metric space with metric ρ. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf \{\rho(x, y) : y \in E\}.$$

Show that ρ_E is a uniformly continuous function on X. If A and B are disjoint nonempty closed subsets of X, examine the relevance of the fucntion

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

Solution: For any $\varepsilon > 0$, suppose $x, z \in X$ and $\rho(x, z) = \rho(z, x) < \delta = \varepsilon$, then by definition of ρ_E and the triangular inequality for metric ρ, the following inequality works for any y

$$\begin{aligned}\rho_E(x) &= \inf \{\rho(x, y) : y \in E\} \\ &\leq \rho(x, y) \\ &\leq \rho(x, z) + \rho(z, y)\end{aligned}$$

This implies that $\rho(z, y) \geq \rho_E(x) - \rho(x, z)$ for any $y \in E$, and by definition of inf, we have

$$\rho_E(z) = \inf \{\rho(z, y) : y \in E\} \geq \rho_E(x) - \rho(x, z).$$

This means

$$\rho(x, z) \geq \rho_E(x) - \rho_E(z).$$

Similarly, by swapping the place of x and z, we obtain

$$\rho(x, z) = \rho(z, x) \geq \rho_E(z) - \rho_E(x).$$

Thus, we can see that

$$|\rho_E(x) - \rho_E(z)| \leq \rho(x, z) < \varepsilon$$

This proves that the function ρ_E is uniformly continuous on X.

Let A be a closed set in X. $\rho_A(x) = 0$ implies that x is a limit point of A, and since A is closed, $x \in A$. It is not hard to see that $\rho_A(x) = 0$ if and only if $x \in A$. For the function f, $f(x) = 1$ if $x \in B$ and $f(x) = 0$ if $x \in A$, and $0 \leq f(x) \leq 1$ for any $x \in X$. f can be viewed as a continuous function supported on the open set $X \setminus A$ and be constant 1 on the closed set B. This is a generalization of Urysohn's lemma for metric space.

Exercise 2.8
Construct a Borel set $E \subset \mathbb{R}^1$ such that

$$0 < m(E \cap I) < m(I)$$

for every nonempty segment I. Is it possible to have $m(E) < \infty$ for such a set?

Solution: Consider the set of rational intervals

$$E = \{(a, b) : a, b \text{ are rational numbers and } a < b\}$$

The set $\mathbb{Q} \times \mathbb{Q}$ is countable, so E only contains countably many intervals. Suppose the elements of E can be listed as

$$\{I_n\}_{n=1}^{\infty}$$

For $n = 1$, choose two disjoint closed intervals $J_1, K_1 \subset I_1$ such that $J_1 \cap K_1 = \varnothing$ and $m(J_1) = m(K_1) < \frac{1}{3}m(I_1)$. Do a Cantor-like construction on J_1 and K_1 to obtain a Cantor-like set A_1 and B_1 with positive measure $0 < m(A_1) = m(B_1) < \frac{1}{3}m(I_1)$ and $A_1 \cap B_1 = \varnothing$.

For the second rational interval I_2, A_1 and B_1 are not dense in I_2 because A_1 and B_1 are closed, there exists some open interval $(a_2, b_2) \subset I_2 \setminus (A_1 \cup B_1)$. Choose two closed intervals with measure smaller than $\frac{1}{3^2}m(I_1)$, and do a Cantor-like construction similar as before, and obtain two Cantor-like sets A_2 B_2 with positive measure $0 < m(A_2) = m(B_2) < \frac{1}{3^2}m(I_1)$ and $A_2 \cap B_2 = \varnothing$. Next consider the set $I_3 \setminus (A_1 \cup B_1 \cup A_2 \cup B_2)$ and repeat this construction. We obtain two sequences $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ satisfying the following property:

- $A_i \cap A_j = \varnothing$ and $B_i \cap B_j = \varnothing$ for $i \neq j$.
- $A_i \cap B_i = \varnothing$ for any i.
- $0 < m(A_i) = m(B_i) < \frac{1}{3^i}m(I_1)$ for any i.

Take

$$A = \bigcup_{n=1}^{\infty} A_n.$$

Take any segment $I \subset \mathbb{R}$, I must contain a rational interval I_n. Recall that $A_n \subset A$ is constructed from a subinterval of I_n, and we have

$$0 < m(A_n) \leq m(A \cap I) < m(A \cap I) + m(B_n) \leq m(I_n) < m(I).$$

And $m(A)$ is finite as

$$m(A) = m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n) \leq (\sum_{n=1}^{\infty} \frac{1}{3^n})m(I_1) = \frac{1}{2}m(I_1).$$