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Problem 2.1.11.
Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Solution: Let $i : A \hookrightarrow X$ be the inclusion map. Since $A \subset X$ is a retract, i has a left inverse $r : X \to A$ such that $r \circ i = \mathrm{id}_A$. We have an induced map on nth singular homology group:

$$\begin{tikzcd} H_n(A) \arrow[r, "i_*"] \arrow[rr, bend right, "id"'] & H_n(X) \arrow[r, "r_*"] & H_n(A) \end{tikzcd}$$

Suppose $a \in \ker i_*$, we have $id(a) = (r_* \circ i_*)(a) = r_*(i_* a) = r_*(0) = 0$. This means $a = 0$. Thus $\ker i_* = 0$ and i_* is injective.

Problem 2.1.12
Show that chain homotopy of chain maps is an equivalence relation.

Solution: We need to show that chain homotopy is reflexive, symmetric and transitive. Suppose $f, g, h : (C_\bullet, \partial^C) \to (D_\bullet, \partial^D)$ are chain maps between chain complexes $C_\bullet$ and $D_\bullet$.
1.(reflexivity) Consider a collection of zero maps

$$\{0 : C_n \to D_{n+1} | n \in \mathbb{N}\}.$$

We have that for all $n \in \mathbb{N}$,

$$0 = f_n - f_n = \partial^D \circ 0 + 0 \circ \partial^C.$$

This proves that chain homotopy is reflexive.
2. (symmetry) Suppose f is chain homotopic to g. We have a collection of group homomorphisms

$$\{\psi_n : C_n \to D_{n+1} | n \in \mathbb{N}\}$$

such that

$$f_n - g_n = \partial^D \circ \psi_n + \psi_{n-1} \circ \partial^C.$$

Consider the collection of group homomorphisms

$$\{-\psi_n : C_n \to D_{n+1} | n \in \mathbb{N}\}$$

and they satisfy

$$g_n - f_n = \partial^D \circ (-\psi_n) + (-\psi_{n-1}) \circ \partial^C.$$

This proves that chain homotopy is symmetric.
3. (transitivity) Suppose f is chain homotopic to g with the collection of group homomorphisms

$$\{\psi_n : C_n \to D_{n+1} | n \in \mathbb{N}\}$$

and g is chain homotopic to h with the collection of group homomorphisms

$$\{\phi_n : C_n \to D_{n+1} | n \in \mathbb{N}\}.$$

Consider the collection of group homomorphisms

$$\{\psi_n + \phi_n : C_n \to D_{n+1} | n \in \mathbb{N}\}$$

and we have

$$\begin{aligned} f_n - h_n &= f_n - g_n + g_n - h_n \\ &= \partial^D \circ \psi_n + \psi_{n-1} \circ \partial^C \\ &\quad + \partial^D \circ \phi_n + \phi_{n-1} \circ \partial^C \\ &= \partial^D \circ (\psi_n + \phi_n) + (\psi_{n-1} + \phi_{n-1}) \circ \partial^C. \end{aligned}$$

Thus f is chain homotopic to h. This proves that chain homotopy is transitive.

> **Problem 2.1.13**
> Verify that $f \simeq g$ implies $f_* = g_*$ for induced homomorphisms of reduced homology groups.

Solution: Let $f, g : X \to Y$ be homotopic maps between topological spaces X and Y. For $n \geq 1$, by Theorem 2.10 and by definition $\tilde{H}_n(X) = H_n(X)$ and $\tilde{H}_n(Y) = H_n(Y)$, we have:

$$f_* = g_* : \tilde{H}_n(X) \to \tilde{H}_n(Y).$$

Now we need to show that f, g induce the same map between $\tilde{H}_0(X)$ and $H_0\tilde{(}Y)$. According to the proof of Theorem 2.10 in the book, we only need to show that there exists a chain homotopic map ψ_{-1} : in degree 0 such that

$$f_\sharp - g_\sharp = \partial_1^D \circ \psi_0 + \psi_{-1} \circ \varepsilon.$$

as shown in the following diagram of augmented chain complex:

$$\begin{tikzcd} C_1(X) \arrow[r] & C_0(X) \arrow[r, "\varepsilon"] \arrow[dl, "\psi_0"] \arrow[d, shift right, "f_\sharp"'] \arrow[d, shift left, "g_\sharp"] & \mathbb{Z} \arrow[r] \arrow[dl, "\psi_{-1}"] & 0 \\ D_1(Y) \arrow[r, "\partial_1^D"'] & D_0(Y) \arrow[r, "\varepsilon"'] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

Recall that $f_\sharp$ is chain homotopic to $g_\sharp$ in the usual singular chain comlex, we have

$$f_\sharp - g_\sharp = \partial_1^D \circ \psi_0 + 0 \circ \partial_0^C.$$

Take ψ_{-1} to be the zero map and we have a chain homotopy. This shows that $f_* = g_*$ is also true for 0th reduced homology groups.

> **Problem 2.1.14**
> Determine whether there exists a short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}_8 \oplus \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$. More generally, determine which abelian groups A fit into a short exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ with p prime. What about the case of short exact sequencs $0 \to \mathbb{Z} \to A \to \mathbb{Z}_n \to 0$?

Solution: Suppose we have a short exact sequence:

$$0 \longrightarrow \mathbb{Z}_4 \xrightarrow{f} \mathbb{Z}_8 \oplus \mathbb{Z}_2 \xrightarrow{g} \mathbb{Z}_4 \longrightarrow 0$$

where $f(1) = (2, 1)$ and $g(1, 0) = 1, g(0, 1) = 2$. It is obvious that f is injective and g is surjective. Now to prove this sequence is exact, we need to show that $\mathrm{im} f = \ker g$. Let $0 \leq a \leq 7$ and $0 \leq b \leq 1$ be integers.

Any element $(a, b) \in \mathbb{Z}_8 \oplus \mathbb{Z}_2$ can be written as $a(1, 0) + b(0, 1)$. So $f(a, b) = af(1, 0) + bf(0, 1) = a + 2b$. The condition $(a, b) \in \text{ker}g$ is equivalent to $4|(a + 2b)$. Test all the possibilities and we could find out that

$$\text{ker}g = \{(0, 0), (2, 1), (4, 0), (6, 1)\} = \text{im}f.$$

This shows that the sequence is exact.

Now consider a short exact sequence of abelian groups:

$$0 \longrightarrow \mathbb{Z}_{p^m} \xrightarrow{f} A \xrightarrow{g} \mathbb{Z}_{p^n} \longrightarrow 0$$

f is an injective map and view it as a normal subgroup of A. We can see that $A/\mathbb{Z}_{p^m} \cong \mathbb{Z}_{p^n}$. The group order of A must be $|A| = p^{m+n}$.

1. Assume A is generated by one element, i.e. A is a cyclic group of order p^{m+n}. In this case, A is isomorphic to $\mathbb{Z}_{p^{m+n}}$. And we define $f(1) = p^n \in \mathbb{Z}_{p^{m+n}}$. Identify $\mathbb{Z}_{p^n}$ with $\mathbb{Z}_{p^{m+n}}/\mathbb{Z}_{p^m}$ and we have the short exact sequence.
2. Assume A is generated by 2 elements. Since $|A| = p^{m+n}$, we could write $A = \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^{m+n-k}}$. Without loss of generality, assume $k \geq m + n - k$. Since f is injective, $\mathbb{Z}_{p^k}$ must have a subgroup isomorphic to $\mathbb{Z}_{p^m}$. This implies $k \geq m$. Thus, $m + n - k$ must be smaller than n. So to have a surjective map $A \to \mathbb{Z}_{p^n}$, k must also be larger than n. So we have $k \geq \max\{m, n\}$. Now we define $f(1) = (p^{k-m}, 1)$ and g is the canonical quotient map. We need to show that the quotient group $A/\mathbb{Z}_{p^m}$ is isomorphic to $\mathbb{Z}_{p^n}$. Note that $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^{m+n-k}}$ is generated by $(1, 0)$ and $(0, 1)$, now identify the image of $\mathbb{Z}_{p^m}$ in A as the cyclic group generated by $(p^{k-m}, 1)$. So the quotient group is generated by $(1, 0) + \langle(p^{k-m}, 1)\rangle$ and $(0, 1) + \langle(p^{k-m}, 1)\rangle$. But $(0, 1) + p^{m+n-k}(p^{k-m}, 1) = (p^n, 0) \in \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^{m+n-k}}$, this shows that the group generated by $(0, 1) + \langle(p^{k-m}, 1)\rangle$ is contained in the group generated by $(1, 0) + \langle(p^{k-m}, 1)\rangle$. So the quotient group is a cyclic group generated by $(1, 0) + \langle(p^{k-m}, 1)\rangle$, which has order p^n, so it is isomorphic to $\mathbb{Z}_{p^n}$.
3. Assume A has 3 generators. Let $f(1) = (a, b, c)$. We know that the folloing three elements must generated the quotient group:

   $$\begin{aligned} A &= (1, 0, 0) + \mathbb{Z}_{p^m}, \\ B &= (0, 1, 0) + \mathbb{Z}_{p^m}, \\ C &= (0, 0, 1) + \mathbb{Z}_{p^m}, \end{aligned}$$

   Since the quotient group is cyclic, B and C must be contained in $\langle A\rangle$. Note that $p \geq 2$, so either b or c must be 0. But in this case, $\ker g$ will be larger than $\text{im}f$. A contradiction. Similar arguments show that A cannot be generated by more than 2 elements.

Now consider we have a short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} A \xrightarrow{g} \mathbb{Z}_n \longrightarrow 0$$

f being injective means that A must contain $\mathbb{Z}$ as a subgroup. A similar argument as 3. above shows that A has at most two generators. If A is cyclic and it has $\mathbb{Z}$ as a subgroup, then $A = \mathbb{Z}$. In this case $f(1) = n$ and g is the quotient map. If A is generated by 2 elements. Note that the torsion part cannot be kill by quotient, so A must both contain $\mathbb{Z}$ and $\mathbb{Z}_n$ as a subgroup. So $A = \mathbb{Z} \oplus \mathbb{Z}_n$.

Problem 5

Calculate the homology with integer coefficients for the chain complex

$$0 \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

with boundary maps $\partial_3 = \partial_0 = 0$ and $\partial_2 = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$ and $\partial_1 = \begin{pmatrix} 4 & 8 \end{pmatrix}$.

Solution: Suppose for free abelian groups,

$$C_2 = \mathbb{Z} = \langle F \rangle, C_1 = \mathbb{Z}^2 = \langle L_1, L_2 \rangle, C_0 = \mathbb{Z} = \langle v \rangle.$$

We have $\partial_2 F = 8L_1 - 4L_2$ and $\partial_1 L_1 = 4v, \partial_1 L_2 = 8v$. Then $\ker\partial_1 = \langle L_2 - 2L_1 \rangle$ and $\mathrm{im}\partial_1 = \langle 4v \rangle$. Similarly, $\ker\partial_2 = 0$ and $\mathrm{im}\partial_2 = \langle 8L_1 - 4L_2 \rangle$. Now we can calculate the homology group.

$$\begin{aligned} H_0(C_\bullet) &= \ker\partial_0/\mathrm{im}\partial_1 \\ &= \langle v \rangle / \langle 4v \rangle \\ &= \mathbb{Z}/4\mathbb{Z}. \end{aligned}$$

$$\begin{aligned} H_1(C_\bullet) &= \ker\partial_1/\mathrm{im}\partial_2 \\ &= \langle L_2 - 2L_1 \rangle / \langle 8L_1 - 4L_2 \rangle \\ &= \langle 2L_1 - L_2 \rangle / \langle 4(2L_1 - L_2) \rangle \\ &= \mathbb{Z}/4\mathbb{Z}. \end{aligned}$$

$$\begin{aligned} H_2(C_\bullet) &= \ker\partial_2/\mathrm{im}\partial_3 \\ &= \ker\partial_2 \\ &= 0. \end{aligned}$$

In conclusion, we have

$$H_n(C_\bullet) = \begin{cases} \mathbb{Z}/4\mathbb{Z} & n = 0 \text{ or } n = 1, \\ 0. & \text{otherwise} \end{cases}$$

Problem 6

Calculate the homology with integer coefficients for the chain complex

$$0 \xrightarrow{\partial_4} \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z}^3 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

with boundary maps $\partial_0 = \partial_4 = 0$ and $\partial_3 = \begin{pmatrix} 4 \\ -6 \\ 10 \end{pmatrix}$ and $\partial_2 = \begin{pmatrix} -2 & -23 & -13 \\ 2 & 23 & 13 \\ 4 & 16 & 8 \end{pmatrix}$ and $\partial_1 = \begin{pmatrix} 2 & 2 & 0 \end{pmatrix}$.

Solution: First write down the generators:

$$\begin{aligned} C_0 &= \mathbb{Z} = \langle v \rangle \\ C_1 &= \mathbb{Z}^3 = \langle L_1, L_2, L_3 \rangle \\ C_2 &= \mathbb{Z}^3 = \langle F_1, F_2, F_3 \rangle \\ C_4 &= \mathbb{Z} = \langle T \rangle \end{aligned}$$

We have already know $\partial_0 = \partial_4 = 0$. And $\partial_3 T = 4F_1 - 6F_2 + 10F_3$, so

$$\ker\partial_3 = 0, \ \mathrm{im}\partial_3 = \langle 4F_1 - 6F_2 + 10F_3 \rangle.$$

And we have $\partial_1 L_1 = 2v$, $\partial_1 L_2 = 2v$ and $\partial_1 L_3 = 0$, so

$$\ker\partial_1 = \langle L_2 - L_1, L_3 \rangle, \ \mathrm{im}\partial_1 = \langle 2v \rangle.$$

Finally, for ∂_2, we have

$$\begin{aligned}\partial_2 F_1 &= -2L_1 + 2L_2 + 4L_3\\ \partial_2 F_2 &= -23L_1 + 23L_2 + 16L_3\\ \partial_2 F_3 &= -13L_1 + 13L_2 + 8L_3\end{aligned}$$

Now we do a base change. The base L_1, L_2, L_3 is changed into $L_2 - L_1, L1, L_3$ and the base F_1, F_2, F_3 is change into $F_1, F_2 - 4F_1, F_3 - 2F_1$. Now we have

$$\begin{aligned}\partial_2 F_1 &= 2(L_2 - L_1) + 4L_3\\ \partial_2 F_2 - 4F_1 &= 15(L_2 - L_1)\\ \partial_2 F_3 - 2F_1 &= 9(L_2 - L_1)\end{aligned}$$

So we can calculate:

$$\begin{aligned}\ker\partial_2 &= \langle 3(F_2 - 4F_1) - 5(F_3 - 2F_1)\rangle\\ &= \langle -2F_1 + 3F_2 - 5F_3\rangle\end{aligned}$$

and

$$\begin{aligned}\mathrm{im}\partial_2 &= \langle 3(L_2 - L_1), 2(L_2 - L_1) + 4L_3\rangle\\ &= \langle 3(L_2 - L_1), (L_2 - L_1) - 4L_3\rangle\end{aligned}$$

Now we calculate the homology of the chain complex:

$$\begin{aligned}H_0(C_\bullet) &= \ker\partial_0/\mathrm{im}\partial_1\\ &= \langle v\rangle/\langle 2v\rangle\\ &= \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

$$\begin{aligned}H_1(C_\bullet) &= \ker\partial_1/\mathrm{im}\partial_2\\ &= \frac{\langle L_2 - L_1, L_3\rangle}{\langle 3(L_2 - L_1), (L_2 - L_1) - 4L_3\rangle}\\ &= \frac{\langle (L_2 - L_1) - 4L_3, L_3\rangle}{\langle 3[(L_2 - L_1) - 4L_3] + 12L_3, (L_2 - L_1) - 4L_3\rangle}\\ &= \langle L_3\rangle/\langle 12L_3\rangle\\ &= \mathbb{Z}/12\mathbb{Z}.\end{aligned}$$

$$\begin{aligned}H_2(C_\bullet) &= \ker\partial_2/\mathrm{im}\partial_3\\ &= \frac{\langle -2F_1 + 3F_2 - 5F_3\rangle}{\langle 4F_1 - 6F_2 + 10F_3\rangle}\\ &= \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

$$\begin{aligned}H_3(C_\bullet) &= \ker\partial_3/\mathrm{im}\partial_4\\ &= 0\end{aligned}$$

In conclusion, we have

$$H_n(C_\bullet) = \begin{cases}\mathbb{Z}/2\mathbb{Z} & n = 0 \text{ or } n = 2,\\ \mathbb{Z}/12\mathbb{Z} & n = 1,\\ 0. & \text{otherwise}\end{cases}$$