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Homework - Chapter 2 Exercises
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Exercise 2.3
Show that any Hausdoff space if of dimension zero.

Solution: Let X be a Hausdoff topological space and $x \in X$ is a point. It is easy to see that $\{x\}$ is irreducible. If $X = \{x\}$, then obviously $\dim X = 0$ since we only have one point. For any $y \in X$ that is not x, we have an open set

$$U_y \cap \{x\} = \varnothing$$

because X is Hausdoff. Then we have

$$X \setminus \{x\} = \bigcup_{y \in X, y \neq x} U_y$$

is open in X. This implies $\{x\}$ is closed and irreducible for any $x \in X$. Thus, the only irreducible subsets in X are singletons. Thus, we can conclude that $\dim X = 0$.

Exercise 2.4
Assume that

$$Y = Y_1 \cup \cdots \cup Y_r$$

is the primary decomposition of the Noetherian space Y into irreducible components. Show that

$$\dim Y = \max_{1 \leq i \leq r} \dim Y_i.$$

Solution: For $1 \leq i \leq r$, we know that $Y_i \subseteq Y$, so $\dim Y_i \leq \dim Y$. This implies that

$$\max_{1 \leq i \leq r} \dim Y_i \leq \dim Y.$$

On the other hand, suppose

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_k$$

is a maximal strictly increasing chains of irreducible subsets in Y. Then $Z_k = Y_i$ for some $1 \leq i \leq r$ because $Y_1, \ldots, Y_r$ are irreducible components of Y. Thus, this chain is also a strictly increasing chain of irreducible subsets in Y_i. This proves

$$\dim Y \leq \max_{1 \leq i \leq r} \dim Y_i.$$

Hence, we can conclude that

$$\dim Y = \max_{1 \leq i \leq r} \dim Y_i.$$

Exercise 2.6
Let $X = Z(zx, zy) \subseteq \mathbb{A}^2$. Describe X and determine $\dim X$. Exhibit two maximal chains of irreducible subvarieties of different lengths. Exhibit a hypersurface Z so that $Z \cap X$ is of dimension zero.

Solution: The ideal $I(X) = (zx, zy)$ has the primary decomposition

$$(zx, zy) = (z) \cap (x, y)$$

So X has two irreducible components $X_1 = Z(z)$ and $X_2 = Z(x, y)$. The coordinate ring

$$\begin{aligned} A(X_1) &= k[x, y, z]/(z) \cong k[x, y], \\ A(X_2) &= k[x, y, z]/(x, y) \cong k[z]. \end{aligned}$$

So by proposition 2.46, we have

$$\begin{aligned} \dim X_1 &= \dim A(X_1) = \dim k[x, y] = 2, \\ \dim X_2 &= \dim A(X_2) = \dim k[z] = 1. \end{aligned}$$

This implies that $\dim X = \max_{i=1,2} \dim X_i = 2$. We have the following two maximal chains of irreducible subvarieties

$$\begin{aligned} &Z(x, y, z) \subsetneq Z(y, z) \subsetneq Z(z), \\ &Z(x, y, z) \subsetneq Z(x, y). \end{aligned}$$

Consider the hypersurface $Z = Z(z - 1) \subseteq \mathbb{A}^3$. The intersection

$$Z \cap X = Z(z - 1) \cap Z(zx, zy) = Z(z - 1, zx, zy).$$

Then we have

$$\begin{aligned} \dim Z \cap X &= \dim A(Z \cap X) \\ &= \dim k[x, y, z]/(z - 1, zx, zy) \\ &\cong \dim k[x, y, z]/(z - 1, x, y) \\ &\cong \dim k \\ &= 0. \end{aligned}$$

Exercise 2.11
Let $\psi : \mathbb{A}^3 \to \mathbb{A}^3$ be given as $(x, y, z) \mapsto (yz, xz, xy)$. Find all fibers of ψ and their ideals.

Solution: Let (a, b, c) be a point in $\mathbb{A}^3$.

If $a = b = c = 0$, then the preimage

$$\begin{aligned}\psi^{-1}(0,0,0) &= \{x = y = 0\} \cup \{x = z = 0\} \cup \{y = z = 0\} \\ &= Z(x,y) \cup Z(x,z) \cup Z(y,z).\end{aligned}$$

The fiber over $(0,0,0)$ is the union of three axes. Note that the maximal corresponding to $(0,0,0)$ is $\mathfrak{m}_0 = (x,y,z)$, so we have

$$\psi^*\mathfrak{m}_0 = (x,y) \cap (x,z) \cap (y,z) = (yz, xz, yz).$$

If only one of a, b, c equals 0, the other two are not zero, then the preimage $\psi^{-1}(a,b,c) = \varnothing$, and

$$\psi^*\mathfrak{m}_{(a,b,c)} = Z(\varnothing) = k[x,y,z].$$

If two of a, b, c equal 0, then the preimage

$$\begin{aligned}\psi^{-1}(0,0,c) &= Z(xy - c, z), \\ \psi^{-1}(0,b,0) &= Z(xz - b, y), \\ \psi^{-1}(a,0,0) &= Z(yz - a, x).\end{aligned}$$

The corresponding algebra map

$$\begin{aligned}\psi^*\mathfrak{m}_{(0,0,c)} &= (xy - c, z), \\ \psi^*\mathfrak{m}_{(0,b,0)} &= (xz - b, y), \\ \psi^*\mathfrak{m}_{(a,0,0)} &= (yz - a, x),\end{aligned}$$

If none of a, b, c equal 0, then the preimage of (a,b,c) has two points:

$$\left\{(\sqrt{\frac{bc}{a}}, \sqrt{\frac{ac}{b}}, \sqrt{\frac{ab}{c}}), (-\sqrt{\frac{bc}{a}}, -\sqrt{\frac{ac}{b}}, -\sqrt{\frac{ab}{c}})\right\}.$$

Let $\mathfrak{m}_1, \mathfrak{m}_2$ be the corresponding maximal ideals at these two points. We have

$$\psi^*\mathfrak{m}_{(a,b,c)} = \mathfrak{m}_1 \cap \mathfrak{m}_2.$$

Exercise 2.28
Let $f = y^2 - x(x-1)(x-2)$ and $g = y^2 + (x-1)^2 - 1$. Show that $Z(f,g) = \{(0,0),(2,0)\}$. Determine the primary decomposition of $Z(f,g)$.

Solution: Consider the solutions to $f = g = 0$. We have

$$\begin{aligned} y^2 - x(x-1)(x-2) = y^2 + (x-1)^2 - 1 = 0 \\ (x-1)^2 - 1 + x(x-1)(x-2) = 0 \\ x^3 - 2x^2 = 0 \\ x = 0 \quad \text{or} \quad x = 2 \end{aligned}$$

This implies that $x = 0, y = 0$ and $x = 2, y = 0$ are the only solutions, so

$$Z(f, g) = \{(0,0), (2,0)\}.$$

Note that $f = y^2 - x^3 + 3x^2 - 2x$ and $g = y^2 + x^2 - 2x$. We claim that

$$(f, g) = (y^2, x-2) \cap (y^2 - 2x, x^2).$$

First note that we can write

$$\begin{aligned} (f, g) &= (g - f, g) \\ &= (x^2(x-2), y^2 + x^2 - 2x) \end{aligned}$$

We claim that

$$(x^2(x-2), y^2 + x^2 - 2x) = (x^2, y^2 + x^2 - 2x) \cap (x - 2, y^2 + x^2 - 2x).$$

One direction $\subseteq$ is clear. Conversely, suppose we have an element

$$f_1(y^2 + x^2 - 2x) + g_1 x^2 = f_2(y^2 + x^2 - 2x) + g_2(x-2)$$

for some $f_1, f_2, g_1, g_2 \in k[x, y]$. Then

$$(f_1 - f_2)y^2 + g_1 x^2 = (f_2 - f_1)(x^2 - 2x) + g_2(x-2).$$

The right-hand side is divisible by $x - 2$, so the left-hand side is also divisible by $x - 2$. This implies that $(x-2)|g_1$ and $(x-2)|(f_1 - f_2)$. Similarly, we can prove that $x^2|g_2$. Thus, the claim is proved.

Rewrite

$$\begin{aligned} (y^2 + x^2 - 2x, x - 2) = (y^2, x - 2), \\ (y^2 + x^2 - 2x, x^2) = (y^2 - 2x, x^2). \end{aligned}$$

Note that

$$\begin{aligned} \sqrt{(y^2, x-2)} = (y, x-2), \\ \sqrt{(y^2 - 2x, x^2)} = (y, x). \end{aligned}$$

are maximal, so both two ideals are primary and we obtain a primary decomposition

$$(f, g) = (y^2, x-2) \cap (y^2 - 2x, x^2).$$