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Homework - Chapter 3 Exercises
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Course: MATH 681 - Algebraic Geometry I
Term: Fall 2025
Instructor: Professor Nick Addington
Due Date: Oct 22nd, 2025

Exercise 3.2
Let R be a UFD. Show that any prime ideal of height one is principal.

Solution: Let $\mathfrak{p}$ be a prime ideal in R. The zero ideal (0) is properly contained in $\mathfrak{p}$, so there exists a nonzero element $x \in \mathfrak{p}$. Since R is a UFD, x can be written as

$$x = ux_1 \cdots x_n$$

where $u \in R$ is a unit and $x_1, \dots, x_n$ are irreducible elements in R. We know that $\mathfrak{p}$ is a prime ideal, so at least one $x_i \in \mathfrak{p}$ for $1 \leq i \leq n$. Without loss of generality, we can assume $x_1 \in \mathfrak{p}$. The principal ideal $(x_1) \subset \mathfrak{p}$ is also prime because x_1 is irreducible, and since $\mathfrak{p}$ has height 1, $(x_1) = \mathfrak{p}$, namely the prime ideal $\mathfrak{p}$ is principal.

Exercise 3.4
Show that the composition of two composable morphism is a morphism. Show that morphisms having $\mathbb{A}^1$ as target are just the regular functions.

Solution: Let $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms between prevarieties. Let $U \subset Z$ be an open set and $f \in \mathcal{O}_Z(U)$ is a regular function. Then $f \circ \psi$ is a regular function on $\psi^{-1}(U)$ since ψ is a morphism. Similarly,

$$f \circ (\psi \circ \phi) = (f \circ \psi) \circ \phi$$

is a regular function on $(\psi \circ \phi)^{-1}(U)$. So $\psi \circ \phi$ is a morphism.

Let $\phi : X \to \mathbb{A}^1$ be a morphism. By lemma 3.37, we can choose an affine open cover $\{U_i\}_{i \in I}$ of X and prove ϕ on each $U_i \to \mathbb{A}_1$ is just regular functions in $\mathcal{O}_X(U_i)$. This is true because on each affine U_i, the regular functions are coming from the coordinate ring $A(U_i)$, which can be viewed as maps from U_i to $\mathbb{A}^1$, and it can be viewed as a morphism because composition of polynomials is still a polynomial.

Exercise 3.5
Let f be a regular function without zeros on the prevariety X. Show that $\frac{1}{f}$ is a regular function.

Solution: Choose an affine open cover $\{U_i\}_{i \in I}$. On each U_i, f can be written as $\frac{p_i}{q_i} \in k(U_i)$. Because f has no zeros, so $\frac{q_i}{p_i}$ is also in $k(U_i)$. For any $i, j \in I$, $\frac{q_i}{p_i} = \frac{q_j}{p_j}$ on $U_i \cap U_j$ because $\frac{p_i}{q_i} = \frac{p_j}{q_j}$ as f is a regular function on X. This implies that we can patch it together and obtain a regular function $\frac{1}{f}$ on X.

Exercise 3.19
Consider the curve C in $\mathbb{A}^2$ whose equation is $y^2 - x^3$. Show that C can be parametrized by the map

$$\begin{aligned}\phi : \mathbb{A}^1 &\to \mathbb{A}^2,\\ t &\mapsto (t^2, t^3).\end{aligned}$$

Describe the map $\phi^* : A(C) \to A(\mathbb{A}^1)$. Show that ϕ is bijective but not an isomorphism. Show that the function field of C equals $k(t)$.

Solution: For all $t \in \mathbb{A}^1$, it is easy to see that the point $\phi(t) = (t^2, t^3)$ is a point on C, so $\operatorname{Im}\phi \in C$. Moreover, ϕ is injective because

$$\begin{cases} t_1^2 = t_2^2 \\ t_1^3 = t_2^3 \end{cases}$$

implies that $t_1 = t_2$. Conversely, suppose (a, b) is a point on C. If $a = b = 0$, choose $t = 0$ and $\phi(0) = (a, b)$. If $a \neq 0$ and $b \neq 0$, the equation $x^2 = a$ has two different solutions in $\mathbb{C}$. Suppose t is such a solution. Note that

$$t^6 = (t^2)^3 = a^3 = b^2.$$

This implies that either $t^3 = b$ or $t^3 = -b$. Choose the solution t satisfying $t^3 = b$. Thus, we find a preimage $t \in \mathbb{A}^1$. This proves that C can be parametrized by the map ϕ which is a bijective map.

The map $\phi^* : A(C) \to A(\mathbb{A}^1)$ is given by

$$\phi^* : k[x, y]/(y^2 - x^3) \to k[t], \tag{1}$$
$$x \mapsto t^2, \tag{2}$$
$$y \mapsto t^3. \tag{3}$$

This map ϕ^* is not surjective as $t \in k[t]$ is not in the image. Hence, ϕ is not an isomorphism of affine varieties.

The image of $\phi^*(A(C))$ is isomorphic to the subring of $k[t]$ with only degree ≥ 2 part. It is an integral domain. Let F be the field of fractions for this subring, which is isomorphic to the function field of C. We claim that it is isomorphic to $k(t)$. Indeed, note that

$$t = t^3 \cdot (t^2)^{-1}, \quad t^{-1} = (t^3)^{-1} \cdot t^2.$$

This implies F is subfield of $k(t)$ containing t and t^{-1}, so $F = k(t)$.