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Course: MATH 682 - Algebraic Geometry II
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Homework 1
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Exercise 1

(a) Let $R = \mathbb{Z}[\sqrt{-3}]$. Sketch the lattice in the complex plane.

(b) Prove that R is not integrally closed using

$$\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}.$$

(c) Prove that 2 is irreducible in R by reasoning with norms, where $N(a+b\sqrt{-3}) = a^2+3b^2$. But prove that 2 is not prime by showing that the quotient ring $R/(2)$ is not an integral domain.

(d) Same for $1+\sqrt{-3}$.

(e) Prove that the ideal $\mathfrak{m} = (2, 1+\sqrt{-3})$ is maximal by showing that the quotient ring $R/\mathfrak{m}$ is a field. Prove that $\mathfrak{m}$ is the only prime ideal containing 2 by reasoning about quotient rings. Same for $1+\sqrt{-3}$.

(f) Prove that $\mathfrak{m}^2 = 2\mathfrak{m}$. Find the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $R/\mathfrak{m}$. Prove that the principal ideal (2) and $1+\sqrt{-3}$ are not powers of $\mathfrak{m}$, so they do not factor as products of primes.

(g) Let $S = \mathbb{Z}[\omega]$. Use the fact that S is a principal ideal domain to prove that the Krull dimension of R is 1.

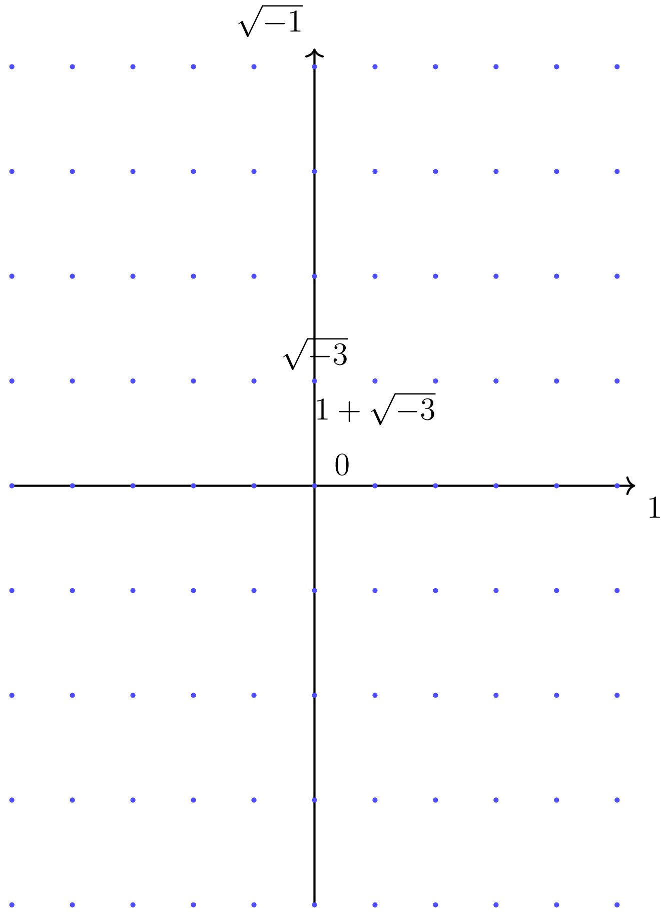
(h) Let $\mathfrak{n} = \mathfrak{m}S$. Prove that $\mathfrak{n}$ is a principal ideal. Is it still prime? Describe the quotient ring $S/\mathfrak{n}$ which should contain $R/\mathfrak{m}$.

(i) Find the dimension of $\mathfrak{n}/\mathfrak{n}^2$ as a vector space over $S/\mathfrak{n}$.

Solution:

(a) The lattice is below:

$$R = \left\{a + b\sqrt{-3} \mid a, b \in \mathbb{Z}\right\}$$



(b) Since $\omega = e^{2\pi i/3}$, it is easy to see that $\omega^3 - 1 = 0$. So the minimal polynomial of ω over R is $x^2 + x + 1$. It is monic so ω is integral over R, but $\omega \notin R$ as 2 is not invertible in R.

(c) Suppose $2 = \alpha\beta$ in R where α, β are not units in R. Then

$$4 = N(2) = N(\alpha\beta) = N(\alpha)N(\beta).$$

Since α, β are not units, so $N(\alpha) \neq 1$ and $N(\beta) \neq 1$. This implies that

$$N(\alpha) = N(\beta) = 2.$$

It is easy to see that there exist no integers $a, b \in \mathbb{Z}$ such that

$$a^2 + 3b^2 = 2.$$

A contradiction. So 2 is irreducible in R.

Now consider the quotient ring $R/(2)$. $1 + \sqrt{-3}$ and $1 - \sqrt{-3}$ are nonzero elements in $R/(2)$ because they are not multiples of 2. But

$$(1 + \sqrt{-3})(1 - \sqrt{-3}) = 4 \equiv 0$$

in $R/(2)$. This implies $R/(2)$ is not an integral domain. So (2) is not a prime ideal in R.

(d) Note that $N(1 + \sqrt{-3}) = N(2) = 4$. The same argument as above implies that $1 + \sqrt{-3}$ is irreducible. For the quotient ring $R/(1 + \sqrt{-3})$. Note that 2 is not a multiple of $1 + \sqrt{-3}$ in

R and

$$2^2 = (1+\sqrt{-3})(1-\sqrt{-3}) \in (1+\sqrt{-3}).$$

So 2 is a nilpotent element in $R/(1+\sqrt{-3})$. This tells us that $1+\sqrt{-3}$ is not prime.

(e) Consider the following map

$$\begin{aligned} f : R &\to \mathbb{F}_2, \\ 1 &\mapsto 1, \\ \sqrt{-3} &\mapsto 1. \end{aligned}$$

It is a well-defined surjective ring map as in R, we have $(\sqrt{-3})^2+3=0$ and in $\mathbb{F}_2$, we have $1^2+3=0$. It is easy to see that $f(2)=f(1)+f(1)=0$ and $f(1+\sqrt{-3})=f(1)+f(\sqrt{-3})=1+1=0$, so

$$\mathfrak{m} = (2, 1+\sqrt{-3}) \subset \ker f.$$

Conversely, suppose $a+b\sqrt{-3} \in R$ satisfies that

$$f(a+b\sqrt{-3}) = af(1)+bf(\sqrt{-3}) = a+b = 0$$

in $\mathbb{F}_2$. So $a+b \in \mathbb{Z}$ is even. Write

$$a+b\sqrt{-3} = a-b+b(1+\sqrt{-3}).$$

Here $a-b$ is also even as $a+b$ is even. So $a+b\sqrt{-3} \in (2, 1+\sqrt{-3})$. This proves that

$$\ker f = (2, 1+\sqrt{-3}) = \mathfrak{m}.$$

Hence, the quotient ring $R/\mathfrak{m} \cong \mathbb{F}_2$ is a field and $\mathfrak{m}$ is a maximal ideal.

Now suppose $\mathfrak{p}$ is a prime ideal and $2 \in \mathfrak{p}$. We have

$$(1+\sqrt{-3})(1-\sqrt{-3}) = 2 \cdot 2 \in \mathfrak{p}.$$

Because $\mathfrak{p}$ is prime, so $1+\sqrt{-3} \in \mathfrak{p}$. So we have

$$\mathfrak{m} = (2, 1+\sqrt{-3}) \subset \mathfrak{p}.$$

We have proved that $\mathfrak{m}$ is maximal, so $\mathfrak{p} = \mathfrak{m}$. Similarly, if $1+\sqrt{-3} \in \mathfrak{p}$, then by the same argument that $2 \in \mathfrak{p}$, so again $\mathfrak{p} = \mathfrak{m}$.

(f) Write

$$\begin{aligned} \mathfrak{m}^2 &= (4, -2+2\sqrt{-3}, 2+2\sqrt{-3}), \\ 2\mathfrak{m} &= (4, 2+2\sqrt{-3}). \end{aligned}$$

It is easy to see that

$$-2+2\sqrt{-3} = (2+2\sqrt{-3}) - 4.$$

So $\mathfrak{m}^2 = 2\mathfrak{m}$. We first prove a claim.

Claim: Any element in the ideal $\mathfrak{m} = (2, 1+\sqrt{-3})$ can be written as $2a + b(1+\sqrt{-3})$ for $a, b \in \mathbb{Z}$.

Proof: By definition, every element in $(2, 1+\sqrt{-3})$ can be written as

$$2a + b(1+\sqrt{-3}) \quad \text{where} \quad a, b \in \mathbb{Z}[\sqrt{-3}].$$

Suppose $a = p + q\sqrt{-3}$ for $p, q \in \mathbb{Z}$. We can rewrite

$$2(p + q\sqrt{-3}) + b(1+\sqrt{-3}) = 2(p-q) + (b+2q)(1+\sqrt{-3}).$$

So we can assume $a \in \mathbb{Z}$. Now suppose $b = m + n\sqrt{-3}$ for $m, n \in \mathbb{Z}$, we can rewrite

$$2a + (m + n\sqrt{-3})(1+\sqrt{-3}) = 2(a-2n) + (m+n)(1+\sqrt{-3}).$$

In this way, we can make both $a, b \in \mathbb{Z}$. ■

Now write an element in $\mathfrak{m}$ as

$$2a + b(1+\sqrt{-3})$$

for $a, b \in \mathbb{Z}$. When view it as an element in $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{m}/2\mathfrak{m}$, it has four possibilities depending on a is even or odd and b is even or odd. This means $\mathfrak{m}/\mathfrak{m}^2$ as an $R/\mathfrak{m}$-vector space has 4 different elements, so its dimension is 2.

By induction, for $n \geq 1$, we have $\mathfrak{m}^n = 2^{n-1}\mathfrak{m}$. Suppose $(2) = \mathfrak{m}^n$, then $2 \in (2^n, 2^{n-1}(1+\sqrt{-3}))$. So there exists $a, b, c, d \in \mathbb{Z}$ such that

$$2 = 2^n(a + b\sqrt{-3}) + 2^{n-1}(c + d\sqrt{-3})(1+\sqrt{-3}).$$

This implies that

$$\begin{aligned} 2^{n-1}(2a + c - 3d) &= 2, \\ 2^{n-1}(2b + c + d) &= 0. \end{aligned}$$

When $n = 1$, we know that $(2) \neq \mathfrak{m}$, so $n = 2$. Then

$$\begin{aligned} 2a + c - 3d &= 1, \\ 2b + c + d &= 0. \end{aligned}$$

Take their difference and we have

$$2a - 2b - 4d = 1.$$

A contradiction. So (2) is not powers of $\mathfrak{m}$. Similarly, if $1+\sqrt{-3} \in \mathfrak{m}^n$, we get two equations:

$$\begin{aligned} 2^{n-1}(2a + c - 3d) &= 1, \\ 2^{n-1}(2b + c + d) &= 1. \end{aligned}$$

This is only possible when $n = 1$, but we know that $(1+\sqrt{-3}) \neq (2, 1+\sqrt{-3})$. So $(1+\sqrt{-3})$ is also not powers of $\mathfrak{m}$.

(g) It is easy to see that R is a subring of S. Note that $S = R[\omega]$, and ω is integral over R as the minimal polynomial of ω over R is $x^2 + x + 1$. So $R \subset S$ is an integral extension. Thus, $\dim R = \dim S = 1$ as S is a principal ideal domain.

(h) Note that
$$\mathfrak{n} = \mathfrak{m}S = (2, 1 + \sqrt{-3}) = (2, -1 + \sqrt{-3}) = (2, 2\omega) = (2).$$
So $\mathfrak{n}$ is a principal ideal in S. Consider the quotient ring $S/\mathfrak{n} = \mathbb{Z}[\omega]/(2) = \mathbb{F}_2[\omega]$. This is the field adjoining a root of the polynomial $x^2 + x + 1$. So $S/(2) \cong \mathbb{F}_4$ is the splitting field of $x^2 + x + 1$, which is the finite field with 4 elements. This implies $\mathfrak{n}$ is a prime ideal.

(i) The field $S/\mathfrak{n}$ has 4 elements $0, 1, \omega, \omega^2 = 1 + \omega$. The vector space $\mathfrak{n}/\mathfrak{n}^2$ has only one generator 2, so its dimension over $S/\mathfrak{n}$ is 1.

Exercise 2

(a) Let $R = \mathbb{R}[x, y]/(y^2 + x^2 - x^3)$. Sketch the curve in $\mathbb{R}^2$.

(b) Prove that R is not integrally closed.

(c) Prove that x is irreducible in R by reasoning about degrees. But prove that x is not prime by showing that the quotient ring $R/(x)$ is not an integral.

(d) Same for y.

(e) Prove that the ideal $\mathfrak{m} = (x, y)$ is maximal by showing that the quotient ring $R/\mathfrak{m}$ is a field. Prove that $\mathfrak{m}$ is the only prime ideal containing x by reasoning about quotient rings.

(f) Find the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $R/\mathfrak{m}$. Prove that the principal ideal (x) is not a power of $\mathfrak{m}$, so it does not factor as a product of primes.

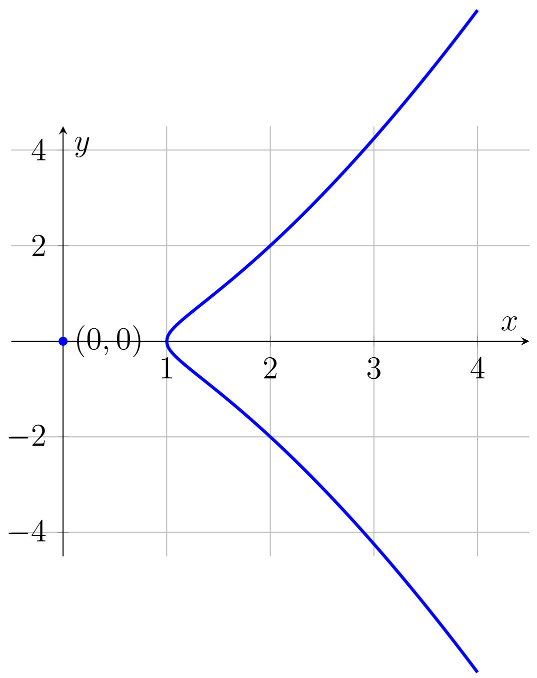
(g) Let $S = \mathbb{R}[z]$. Describe the normalization map $\varphi : R \to S$ that sends y/x to z: where does it send x and y? Prove that the Krull dimension of R is 1.

(h) Let $\mathfrak{n} = \varphi(\mathfrak{m})S$. Prove that $\mathfrak{n}$ is a principal ideal. Is it still prime? Describe the quotient ring $S/\mathfrak{n}$, which should contain $R/\mathfrak{m}$.

(i) Find the dimension of $\mathfrak{n}/\mathfrak{n}^2$ as a vector space over $S/\mathfrak{n}$.

Solution:

(a) The curve $y^2 + x^2 - x^3 = 0$ is sketched below:



(b) Let $z = y/x$ be an element in the fraction field of R. And we have

$$z^2 = \frac{y^2}{x^2} = \frac{x^3 - x^2}{x^2} = x - 1 \in R.$$

Next, we need to show that $z \notin R$. Consider a monomial $x^m y^n \in \mathbb{R}[x, y]$. If $n \geq 2$, we can replace y^2 with $x^3 - x^2$ such that

$$x^m y^n = x^m (x^3 - x^2) y^{n-2}$$

in R. By doing so repeatedly, we can write every element in R in the form of $f(x) + g(x)y$. Suppose there exists $f, g \in \mathbb{R}[x]$ such that

$$\frac{y}{x} = f(x) + g(x)y.$$

Then

$$y = xf(x) + xg(x)y.$$

This means $xg(x) = 1$ for some $g(x) \in \mathbb{R}[x]$. A contradiction. So $z \notin R$. We have found an element z which is not in R but integral over R, so R is not integrally closed.

(c) Suppose x is not irreducible in R. Then there exists $f, g, h, k \in \mathbb{R}[x]$ such that

$$x = (f(x) + g(x)y)(h(x) + k(x)y).$$

Expand and replace y^2 with $x^3 - x^2$, we get

$$x = f(x)h(x) + (x^3 - x^2)g(x)k(x) + [f(x)k(x) + g(x)h(x)]y.$$

Let

$$\deg f = a, \quad \deg g = b, \quad \deg h = c, \quad \deg k = d.$$

From the equation, we get

$$\begin{aligned} a+c &= b+d+3, \\ a+d &= b+c. \end{aligned}$$

Take their difference and we get

$$2c-2d=3$$

A contradiction. So x is irreducible in R. Consider the quotient ring

$$R/(x)=\mathbb{R}[y]/(y^2).$$

It is not an integral domain as y is a nilpotent element. So x is not prime.

(d) We use a similar argument to show that y is irreducible. In this case, we get

$$y=f(x)h(x)+(x^3-x^2)g(x)k(x)+[f(x)k(x)+g(x)h(x)]y.$$

So

$$\begin{aligned} a+c &= b+d+3, \\ a+d &= b+c. \end{aligned}$$

Same argument implies that y is irreducible. Consider the quotient ring

$$R/(y)=\mathbb{R}[x]/(x^2-x^3).$$

This is not an integral domain as x is a zero divisor. So y is not prime.

(e) It is not hard to see that

$$R/\mathfrak{m}=R/(x,y)\cong\mathbb{R}.$$

So $R/\mathfrak{m}$ is a field and $\mathfrak{m}$ is maximal. Let $\mathfrak{p}$ be a prime ideal and $x\in\mathfrak{p}$. We know that $R/\mathfrak{p}$ is an integral domain and $y\in R/(x)$ is nilpotent. Note that

$$R/\mathfrak{p}\subset R/(x)=R[y]/(y^2).$$

So $R/\mathfrak{p}$ is a subring of $R[y]/(y^2)$ and $R/\mathfrak{p}$ does not contain y. So $R/\mathfrak{p}\cong\mathbb{R}$ and $\mathfrak{p}\supset(x,y)=\mathfrak{m}$. We have proved $\mathfrak{m}$ is maximal, so $\mathfrak{p}=\mathfrak{m}$.

(f) Here $\mathfrak{m}=(x,y)$ and $\mathfrak{m}^2=(x^2,xy,y^2)$, so $\mathfrak{m}/\mathfrak{m}^2$ only contains degree 1 polynomials and we have

$$\mathfrak{m}/\mathfrak{m}^2\cong\mathbb{R}x\oplus\mathbb{R}y.$$

This implies that $\mathfrak{m}/\mathfrak{m}^2$ is a 2-dimensional vector space over $R/\mathfrak{m}\cong\mathbb{R}$.

For $n\geq 1$, the ideal $\mathfrak{m}^n$ is generated by all monomials x^py^{n-p} for $0\leq p\leq n$. For $n-p\geq 2$, we can replace y^2 with x^3-x^2. Note that this does not decrease the degree of the monomial, so if $x\in\mathfrak{m}^n$, n has to be 1. But we know $(x)\neq(x,y)=\mathfrak{m}$, so (x) is not a power of $\mathfrak{m}$, thus it does not factor as a product of primes.

(g) Consider the following map:

$$\begin{aligned} f : \mathbb{R}[x, y] &\to S = \mathbb{R}[z], \\ x &\mapsto z^2 + 1, \\ y &\mapsto z(z^2 + 1). \end{aligned}$$

It is easy to see that

$$\begin{aligned} &f(y)^2 + f(x)^2 - f(x)^3 \\ =&z^2(z^2 + 1)^2 + (z^2 + 1)^2 - (z^2 + 1)^3 \\ =&(z^2 + 1)^2(z^2 + 1 - z^2 - 1) \\ =&0. \end{aligned}$$

So $y^2 + x^2 - x^3 \in \ker f$ and the ideal $(y^2 + x^2 - x^3) \subset \ker f$. We know that the Krull dimenison of S is 1, and the image of f is a subring of S. This implies that $\ker f$ is a prime ideal of $\mathbb{R}[x, y]$ and the Krull dimension of the quotient ring $\mathbb{R}[x, y]/\ker f$ is also 1 as the image is not the field $\mathbb{R}$. By

$$\operatorname{ht} \ker f + \dim \mathbb{R}[x, y]/\ker f = \operatorname{ht} \ker f + 1 \leq \dim \mathbb{R}[x, y] = 2,$$

We know that the height of the prime ideal $\ker f$ is smaller or equal to 1. The ideal $(y^2+x^2-x^3)$ is a prime ideal contained in $\ker f$ since $y^2 + x^2 - x^3$ is irreducible, so $\ker f = (y^2 + x^2 - x^3)$. The map f induces a well-defined map

$$\begin{aligned} \varphi :&\mathbb{R}[x, y]/(y^2 + x^2 - x^3) = R \to S, \\ &x \mapsto z^2 + 1, \\ &y \mapsto z(z^2 + 1). \end{aligned}$$

By construction, the map φ sends y/x to z and is injective. So S is an integral extension of R and $\dim R = \dim S = 1$.

(h) $\mathfrak{n} = (z^2 + 1, z(z^2 + 1)) = (z^2 + 1)$. This implies that $\mathfrak{n}$ is a principal ideal generated by $z^2 + 1$. Consider the quotient ring $S/\mathfrak{n} = \mathbb{R}[z]/(z^2 + 1) \cong \mathbb{C}$. It is a field, so the ideal $\mathfrak{n}$ is a maximal ideal, thus $\mathfrak{n}$ is prime.

(i) For any $f \in \mathbb{R}[z]$, use Euclidean algorithm, we can write

$$f(z) = g(z)(z^2 + 1) + r(z)$$

where $r(z) = az + b$ for some $a, b \in \mathbb{R}$. We know every element in $\mathfrak{n}$ can be written as

$$f(z)(z^2 + 1) = g(z)(z^2 + 1)^2 + (a + bz)(z^2 + 1).$$

Here $a + bz$ can be viewed as elements in $S/\mathfrak{n} = \mathbb{R}[z]/(z^2 + 1)$. The vector space $\mathfrak{n}/\mathfrak{n}^2$ is generated by one base $z^2 + 1$, so the dimension is 1.

Exercise 3
Find a square-free integer D with $D \equiv 1 \pmod 4$ such that the maximal ideal $\mathfrak{m} = (2, 1+\sqrt{D})$ in $R = \mathbb{Z}[\sqrt{D}]$ splits when you extend to $S = \mathbb{Z}[\frac{1+\sqrt{D}}{2}]$?

Solution: Let $D = -7$. We have $-7 \equiv 1 \pmod 4$. Let $f : R \to S$ be the extension map and

$$\mathfrak{n} = f(\mathfrak{m})S = (2, 1+\sqrt{-7}) = (2, 2 \cdot \frac{1+\sqrt{-7}}{2}) = (2).$$

Note that 2 is not irreducible in S as

$$\frac{1+\sqrt{-7}}{2} \cdot \frac{1-\sqrt{-7}}{2} = 2.$$

So the ideal $\mathfrak{n}$ splits into $(\frac{1+\sqrt{-7}}{2})$ and $(\frac{1-\sqrt{-7}}{2})$.