

Exercise 5.1

Assume X and Y are two affine varieties and that $\phi : X \rightarrow Y$ is a morphism. Show that ϕ is a closed embedding if and only if the map $\phi^* : A(Y) \rightarrow A(X)$ between the coordinate rings is surjective.

Solution: Suppose $\phi : X \rightarrow Y$ is a closed embedding. By definition, there exists a closed subvariety $i : Z \hookrightarrow Y$ such that the following triangle commutes:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ & \searrow \cong & \uparrow i \\ & & Z \end{array}$$

Because X is isomorphic to Z , so Z is also affine and the induced map $i^* : A(Y) \rightarrow A(Z)$ is a quotient map of rings and thus surjective. By the main theorem of affine varieties, we have a commuting triangle of coordinate rings:

$$\begin{array}{ccc} A(X) & \xleftarrow{\phi^*} & A(Y) \\ & \nwarrow \cong & \downarrow i^* \\ & & A(Z) \end{array}$$

This implies that the map $\phi^* : A(Y) \rightarrow A(X)$ is surjective.

Conversely, suppose the map $\phi^* : A(Y) \rightarrow A(X)$ is surjective. Then we have a commuting triangle of rings:

$$\begin{array}{ccc} A(X) & \xleftarrow{\phi^*} & A(Y) \\ & \searrow \cong & \downarrow \\ & & A(Y)/\ker \phi^* \end{array}$$

Note that $\ker \phi^*$ is the preimage of the zero ideal $(0) \subseteq A(X)$. And since $A(X)$ is reduced, so $\ker \phi^*$ is a prime ideal in $A(Y)$. This implies that $A(Y)/\ker \phi^*$ is a reduced, finitely generated k -algebra. Consider Z to be the affine variety with the coordinate ring $A(Y)/\ker \phi^*$. Then Z is a closed subvariety of Y and the map ϕ must factor through Z . Moreover, ϕ gives an isomorphism between affine varieties X and Z . Thus, we can conclude that ϕ is a closed embedding.

Exercise 5.3

Consider the map $\phi : \mathbb{A}^1 \rightarrow \mathbb{A}^2$ given as $\phi(t) = (t^2 - 1, t(t^2 - 1))$.

- (a) Show that ϕ is a closed map, but not a closed embedding.
- (b) Exhibit an open covering $\{U_i\}$ of \mathbb{A}^1 such that each restriction ϕ_{U_i} is a closed embedding into some open subset V_i of \mathbb{A}^2 .

Solution:

- (a) Define C to be the curve cut out by $y^2 = x^2(x + 1)$ in \mathbb{A}^2 . We claim that $\text{Im } \phi = C$. Indeed, let $x = t^2 - 1$ and $y = t(t^2 - 1)$, it is easy to check that $y^2 = x^2(x + 1)$. This implies that $\text{Im } \phi \subseteq C$. Conversely, take a point $(x, y) \in C$. If $x = 0$, then $y = 0$, and we know that $\phi(1) = (0, 0)$. So the point $(0, 0) \in \text{Im } \phi$. Suppose $x \neq 0$. Consider the point $\frac{y}{x} \in \mathbb{A}^1$, we have

$$\phi\left(\frac{y}{x}\right) = \left(\frac{y^2}{x^2} - 1, \frac{y}{x}\left(\frac{y^2}{x^2} + 1\right)\right).$$

Note that $(x, y) \in C$, so $y^2 = x^2(x + 1)$. This tells us

$$\left(\frac{y^2}{x^2} - 1, \frac{y}{x}\left(\frac{y^2}{x^2} + 1\right)\right) = \left(\frac{x^2(x + 1)}{x^2} - 1, \frac{y}{x}\left(\frac{x^2(x + 1)}{x^2} - 1\right)\right) = (x, y).$$

This implies that $C \subseteq \text{Im } \phi$. Thus, we can see that $C = \text{Im } \phi$. Note that C is not isomorphic to \mathbb{A}^1 as C is not regular at the point $(0, 0)$, so ϕ cannot be a closed embedding. But ϕ is still a closed map. To see this, consider a closed set $Z \subseteq \mathbb{A}^1$. Without loss of generality, we can assume $Z = Z(f)$ for some $f \in k[t]$. Write

$$\tilde{f}(x, y) = x^{\deg f} f\left(\frac{y}{x}\right) \in k[x, y]$$

Then $\phi(Z) = C \cap Z(\tilde{f})$ is still a closed set.

- (b) Consider the following two distinguished open set $U = D(t - 1)$ and $V = D(t + 1)$ in \mathbb{A}^1 . $\{U, V\}$ is an open cover of \mathbb{A}^1 . We claim that the map

$$\phi : D(t - 1) \xrightarrow{\sim} C \subseteq \mathbb{A}^2$$

is a closed embedding. Consider the following map

$$\begin{aligned} \mathbb{A}^1 - \{1, -1\} &\rightarrow C - (0, 0), \\ t &\mapsto (t^2 - 1, t(t^2 - 1)), \\ \frac{y}{x} &\leftrightarrow (x, y). \end{aligned}$$

In addition, if we map $-1 \in \mathbb{A}^1$ to $(0, 0) \in C$, then we have an isomorphism between $D(t - 1)$ and C . Similarly, if we map $1 \in \mathbb{A}^1$ to $(0, 0) \in C$, then we have an isomorphism between $D(t + 1)$ and C . Thus, ϕ restricting to the open sets U and V are both closed embeddings in \mathbb{A}^2 .

Exercise 5.11

Show that the coordinate ring $A(C(S_{n,m}))$ of the cone over the Segre variety $S_{n,m}$ is not a UFD. Give explicit examples of height one primes that are not principal.

Solution: Let $M = (z_{ij})$ to be a $(n+1) \times (m+1)$ -matrix over k where $0 \leq i \leq n$ and $0 \leq j \leq m$. Suppose $R = k[\{z_{ij}\}_{i,j}]$ and I is the ideal generated by all 2×2 -minors of M . We know that $A(C(S_{n,m})) = R/I$. We need to show that R/I is not a UFD. Consider the element $z_{00} \in R/I$ and we claim that z_{00} is irreducible. Indeed, the ring R is a graded ring and I is a homogeneous ideal, so the quotient ring R/I has a natural grading induced from the grading of R . z_{00} is a degree one element, and all degree 0 elements are in k , thus they must be units. On the other hand, z_{00} is not prime. This can be seen that $(R/I)/(z_{00})$ is not an integral domain as $z_{00}z_{11} - z_{10}z_{01}$ is a generator in the ideal I , so $z_{10}z_{01} = 0$ in $(R/I)/(z_{00})$. This proves that R/I is not a UFD as every irreducible element is prime.

Consider the ideal $J = (z_{00}, z_{10}, \dots, z_{n0})$ in R/I . $(R/I)/J$ is isomorphic to the coordinate ring of the cone $C(S_{n,m-1})$ as we send all elements in the first column of the matrix M to 0. So $(R/I)/J$ is a domain and J is a prime ideal in R/I . The dimension of the cone $C(S_{n,m})$ is $n+m+1$ and the dimension of the quotient $C(S_{n,m-1})$ is $n+m+1-1=n+m$. Passing to the coordinate rings, we know that

$$\text{ht } J \leq \dim A(C(S_{n,m})) - \dim A(C(S_{n,m-1})).$$

So the height of the ideal J is smaller or equal to 1. We already have a chain of prime ideals $0 \subseteq J$. So J is a height one prime ideal, which is not principal.

Exercise 5.12

Show that under the Segre map the fibers of the two projections from $\mathbb{P}^1 \times \mathbb{P}^1$ onto \mathbb{P}^1 embed as lines in \mathbb{P}^1 . Show that if Z is an effective divisor in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(0, n)$ or $(n, 0)$, then Z is a union of lines.

Solution: Let $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the projection to the first factor. $(a : b)$ is a point in \mathbb{P}^1 , the fiber over $(a : b)$ is $\{(a : b)\} \times \mathbb{P}^1$. The Segre map restricting to this fiber is given by

$$\begin{aligned} \sigma : \mathbb{P}^1 &\rightarrow \mathbb{P}^3, \\ (x : y) &\mapsto (ax : ay : bx : by). \end{aligned}$$

We need to show that the image is a line in \mathbb{P}^3 . On the affine open patch $\{x = 1\}$ in \mathbb{P}^1 , the image is parametrized by

$$(a : ay : b : by), \quad y \in k.$$

Without loss of generality, we can assume $a \neq 0$, so it is the same point as

$$(1 : y : \frac{b}{a} : \frac{b}{a}y), \quad y \in k.$$

This gives a line in one of the affine open patch of \mathbb{P}^3 . Similarly, we can argue that the image on another affine patch $\{y = 1\}$ is also a line. This implies that the image of the fibers are lines in \mathbb{P}^3 .

Suppose Z is a codimension 1 subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$ cut out by a bihomogeneous polynomial of degree $(0, n)$. There exists a homogeneous polynomial f of degree n such that the projection to the

second factor

$$\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

sends Z to $Z' = Z(f) \subseteq \mathbb{P}^1$. The fiber of this projection over any point in Z' is isomorphic to \mathbb{P}^1 (the first factor), and the image of this fiber under Segre map is a line from our above discussion. Thus, the image of Z under Segre map is a union of lines, each line corresponding to a point in Z' .

Exercise 5.15

Show that the rational normal curve C_d in \mathbb{P}^d is the intersection of the Segre variety $S_{1,d-1}$ in \mathbb{P}^{2d-1} with an appropriate linear subspace of dimension d .

Solution: Let M be the following matrix:

$$M = \begin{pmatrix} z_{00} & z_{01} & \cdots & z_{0,d-1} \\ z_{10} & z_{11} & \cdots & z_{1,d-1} \end{pmatrix}$$

Let \mathbb{P}^{2d-1} be the projective space with coordinates

$$(z_{00} : \cdots : z_{0,d-1} : z_{10} : \cdots : z_{1,d-1}).$$

Consider the linear subspace $L \subseteq \mathbb{P}^{2d-1}$ defined by the following equations

$$z_{1i} = z_{0,i+1}, \quad 0 \leq i \leq d-2.$$

For $0 \leq j \leq d-1$, choose $t_j = z_{0,j}$ and $t_d = z_{1,d-1}$. Then $L \cong \mathbb{P}^d$ as a subspace of \mathbb{P}^{2d-1} , its coordinates $(t_0 : t_1 : \cdots : t_d)$ are given by the matrix

$$\begin{pmatrix} t_0 & t_1 & \cdots & t_{d-1} \\ t_1 & t_2 & \cdots & t_d \end{pmatrix}$$

The intersection $S_{1,d-1} \cap L$ are defined by the equations $t_I t_J - t_K t_L$ for $I + J = K + L$. These are exactly the equations defining the rational normal curve C_d in $L \cong \mathbb{P}^d$.