Zhengdong Zhang
Email: zhengz@uoregon.edu
Course: MATH 681 - Algebraic Geometry I
Instructor: Professor Nick Addington

Homework - Chapter 4 Exercises
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Exercise 4.3
Let $(a_{10} : a_{11} : a_{12})$ and $(a_{20} : a_{21} : a_{22})$ be two different points in $\mathbb{P}^2$. Show that the line through them has equation

$$\det \begin{pmatrix} x_0 & x_1 & x_2 \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} = 0.$$

Dually, if

$$\begin{aligned} \lambda_{10}x_0 + \lambda_{11}x_1 + \lambda_{12}x_2 &= 0, \\ \lambda_{20}x_0 + \lambda_{21}x_1 + \lambda_{22}x_2 &= 0 \end{aligned}$$

are equations for two different lines, show that the coordinates of the intersection point are the 2×2-minors of the matrix

$$\begin{pmatrix} \lambda_{10} & \lambda_{11} & \lambda_{12} \\ \lambda_{20} & \lambda_{21} & \lambda_{22} \end{pmatrix}.$$

Solution: The two points

$$p = (a_{10} : a_{11} : a_{12}), q = (a_{20} : a_{21} : a_{22}) \in \mathbb{P}^2$$

corresponding to the lines generated by the vector (a_{10}, a_{11}, a_{12}) and (a_{20}, a_{21}, a_{22}) respectively in $\mathbb{A}^3$. Denote by ℓ the line passing through the points $p, q \in \mathbb{P}^2$. ℓ corresponds to a plane passing through the origin in $\mathbb{A}^3$, and any point $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2$ lies in this line ℓ if and only if the vector (x_0, x_1, x_2) generates a line through the origin in this plane. Then we know the row vector of the following matrix is linearly dependent,

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}$$

And its determinant must be 0 since the rank of this matrix is smaller than 3.

Dually, the equations for lines in the projective space $\mathbb{P}^2$ are the equations for planes in $\mathbb{A}^3$. Let $M_1, M_2 \subset \mathbb{A}^3$ be the two planes defined by these two equations respectively. The intersection $M_1 \cap M_2$ is a line ℓ through the origin. The vector $(\lambda_{10}, \lambda_{11}, \lambda_{12})$ is a vector normal to the plane M_1 and the vector $(\lambda_{20}, \lambda_{21}, \lambda_{22})$ is a vector normal to the plane M_2. Note that ℓ is normal to these two vectors and ℓ passes through the origin, so the points on the line ℓ are given by the coordinates

$$(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}, -(\lambda_{10}\lambda_{22} - \lambda_{12}\lambda_{20}), \lambda_{10}\lambda_{21} - \lambda_{11}\lambda_{20})$$

up to the rescaling of $t \in \daleth^*$. The coordinates of the intersection point are given by the 2×2-minors of the matrix

$$\begin{pmatrix} \lambda_{10} & \lambda_{11} & \lambda_{12} \\ \lambda_{20} & \lambda_{21} & \lambda_{22} \end{pmatrix}.$$

Exercise 4.4
Show that n hyperplanes in $\mathbb{P}^n$ always have a common point of intersection. Show that n linearly independent hyperplanes meet in exactly one point.

Solution: The hyperplanes in $\mathbb{P}^n$ can be viewed as hyperplanes through the origin in the affine space $\mathbb{A}^{n+1}$, where lines through the origin in the affine space are points in the projective space. To prove n hyperplanes in $\mathbb{P}^n$ always have a common point of intersection, it is enough to prove that n ⅂-subspace of dimension n in ⅂$^{n+1}$ has a common 1-dimentional subspace. Let $V_1, V_2, \ldots, V_n \subset$ ⅂$^{n+1}$ be subspaces of dimension n, the dimension formula tells us that

$$\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cup V_2) \geq \dim V_1 + \dim V_2 - \dim ⅂^{n+1} = n - 1.$$

Now use $V_1 \cap V_2$ to intersect V_3, similarly, we obtain

$$\dim(V_1 \cap V_2 \cap V_3) \geq \dim(V_1 \cap V_2) + \dim V_3 - (n+1) = n - 2.$$

Repeat this process and we get

$$\dim(\bigcap_{i=1}^{n} V_i) \geq 1.$$

This is exactly what we need.

Now assume the n hyperplanes are linearly independent. In $\mathbb{A}^{n+1}$, note that a hyperplane is uniquely determined by its normal vector. n linearly independent hyperplanes give us n linearly independent vectors in $\mathbb{A}^{n+1}$. Since the vectors in the intersection subspace is normal to every normal vector of these hyperplanes, so the intersection subspace can only be of dimension 1 because $\mathbb{A}^{n+1}$ has dimension $n + 1$. This means in the projective space, these hyperplanes meet at exactly one point.

Exercise 4.13
Show that each rational function in one variable defines a rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ which extends to a morphism $\mathbb{P}^1 \to \mathbb{P}^1$.

Solution: Suppose we have a rational function in one variable $\frac{P(t)}{Q(t)}$ defined on $t \in \mathbb{A}^1$ for $Q(t) \neq 0$. Let $d = \max(\deg P, \deg Q)$. Consider the following map

$$\begin{aligned} \phi : \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1, \\ (x_0 : x_1) &\mapsto (x_0^d P(\frac{x_1}{x_0}) : x_0^d Q(\frac{x_1}{x_0})). \end{aligned}$$

This is well-defined as the right-hand side is the homogenization of the polynomials P and Q. We can write P and Q as product of linear terms and assume they do not have common factors.

Otherwise, just cancel the common factors in the rational function $\frac{P(t)}{Q(t)}$, and it gives us the same rational function. Thus, ϕ can be extended to $\mathbb{P}^1$ as P and Q have no common roots, so no point $(x_0 : x_1)$ is mapped to 0 under ϕ. This implies ϕ is a morphism $\mathbb{P}^1 \to \mathbb{P}^1$.

Exercise 4.14

Let the projection $\mathbb{P}^3$ to $\mathbb{P}^2$ be given by the assignment $(x : y : z : w) \mapsto (x : x + z : w + y)$. Determine the center and describe the projection of the twisted cubic parametrized as $(u : v) \mapsto (u^3 : u^2v : uv^2 : v^3)$.

Solution: The center is $(0 : 1 : 0 : -1)$ as this point is mapped to $(0, 0, 0)$ under the projection. The image of the twisted cubic curve can be parametrized as $(u : v) \mapsto (u^3 : u(u^2+v^2) : v(u^2+v^2))$. Write $(X : Y : Z)$ as homogeneous coordinates in $\mathbb{P}^2$. The parametrization $(u : v) \mapsto (u^3 : u(u^2 + v^2) : v(u^2 + v^2))$ gives a curve C which is the zero locus of the ideal $(XY^2 - Y^3 + XZ^2)$. On the affine patch $\{X = 1\}$, this is the rational nodal curve $Z^2 = Y^3 - Y^2$.