

Problem 1

Write down a complete description of the homology groups of $\text{Gr}_3(\mathbb{C}^5)$. Determine as many intersection products between the Schubert classes $[\underline{a}]$ as you can. At least do all cases of complementary dimensions, and compute $[1, 2, 2]^2$ (here $\underline{a} = (1, 2, 2)$ is a Schubert symbol, not a jump sequence). Try to do some others.

Solution: Let $0 \leq a_1 \leq a_2 \leq a_3 \leq 2 = 5 - 3$ be the Schubert symbol of $\text{Gr}_3(\mathbb{C}^5)$. We have ten different choices and the homology groups can be summarized as follows

degree	generators of $H_*(\text{Gr}_3(\mathbb{C}^5))$
0	$[0, 0, 0]$
2	$[0, 0, 1]$
4	$[0, 1, 1], [0, 0, 2]$
6	$[0, 1, 2], [1, 1, 1]$
8	$[0, 2, 2], [1, 1, 2]$
10	$[1, 2, 2]$
12	$[2, 2, 2]$

Next, we are going to determine the intersection product in complementary dimension. Note the cohomology ring is Abelian because we only have cohomology in even dimensions. For simplicity, I will only write the representative matrices. We always choose the first Schubert symbol in the standard flag

$$0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \langle e_1, e_2, e_3 \rangle \subseteq \langle e_1, e_2, e_3, e_4 \rangle \subseteq \mathbb{C}^5$$

and the second Schubert symbol in the reverse flag

$$0 \subseteq \langle e_5 \rangle \subseteq \langle e_4, e_5 \rangle \subseteq \langle e_3, e_4, e_5 \rangle \subseteq \langle e_2, e_3, e_4, e_5 \rangle \subseteq \mathbb{C}^5$$

$$(1) [0, 0, 1] \cdot [1, 2, 2]$$

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

So the intersection has only one point and $[0, 0, 1] \cdot [1, 2, 2] = [0, 0, 0]$.

$$(2) [1, 1, 1] \cdot [1, 1, 1]$$

$$\begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & * & * \\ 0 & * & * & * & * \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

So the intersection has only one point and $[1, 1, 1] \cdot [1, 1, 1] = [0, 0, 0]$.

(3) $[0, 1, 2] \cdot [0, 1, 2]$

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & * \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

So the intersection has only one point and $[0, 1, 2] \cdot [0, 1, 2] = [0, 0, 0]$.

(4) $[1, 1, 1] \cdot [0, 1, 2]$

$$\begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

There does not exist a matrix satisfying the given two conditions. So the intersection has only no point and $[1, 1, 1] \cdot [0, 1, 2] = 0$.

(5) $[0, 0, 2] \cdot [0, 2, 2]$

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & * \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

So the intersection has only one point and $[0, 0, 2] \cdot [0, 2, 2] = [0, 0, 0]$.

(6) $[0, 1, 1] \cdot [0, 2, 2]$

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & 0 & * \\ 0 & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = 0$$

So the intersection has no point and $[0, 1, 1] \cdot [0, 2, 2] = 0$.

(7) $[0, 1, 1] \cdot [1, 1, 2]$

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

So the intersection has only one point and $[0, 1, 1] \cdot [1, 1, 2] = [0, 0, 0]$.

(8) $[0, 0, 2] \cdot [1, 1, 2]$

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & * \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & * & * \\ * & * & * & * & * \end{pmatrix} = 0$$

So the intersection has no point and $[0, 1, 1] \cdot [1, 1, 2] = 0$.

(9) In this part we will determine the intersection product $[1, 2, 2]^2$. Note that $[1, 2, 2] \in H_{10}$, so $[1, 2, 2]^2 \in H_{10+10-12} = H_8$. Suppose

$$[1, 2, 2]^2 = A[0, 2, 2] + B[1, 1, 2]$$

for some $A, B \in \mathbb{Z}$. We have

$$\begin{aligned} [1, 2, 2]^2[0, 1, 1] &= A[0, 2, 2][0, 1, 1] + B[1, 1, 2][0, 1, 1] = B, \\ [1, 2, 2]^2[0, 0, 2] &= A[0, 2, 2][0, 0, 2] + B[1, 1, 2][0, 0, 2] = A. \end{aligned}$$

Suppose W is a 3-plane in the intersection $[1, 2, 2]^2[0, 1, 1]$, note that for all the 2's in the Schubert symbol, the condition is automatically satisfied. W needs to satisfy the following condition:

- (i) $\dim W \cap F_2 \geq 1$ for some 2-plane F_2 .
- (ii) $\dim W \cap F'_2 \geq 1$ for some 2-plane F'_2 .
- (iii) $\dim W \cap F'_1 \geq 1$ for some 1-line F'_1 .
- (iv) $\dim W \cap F'_3 \geq 2$ for some 3-plane F'_3 .
- (v) $\dim W \cap F'_4 \geq 3$ for some 4-plane F'_4 .

Here $F''_1 \subseteq F''_3 \subseteq F''_4$. The condition (iii) implies W contains a vector e_1 where $\langle e_1 \rangle = F''_1$. The condition (v) implies that W is contained in a 4-plane F''_4 . For any generic 3-plane $F''_3 \subseteq F''_4$, we have

$$\dim W \cap F''_3 = \dim W + \dim F''_3 - \dim F''_4 = 3 + 3 - 4 = 2.$$

This implies that the condition (iv) is automatically satisfied.

We can see that W is uniquely determined by three lines: $F''_1, W \cap F_2, W \cap F'_2$. Thus, $B = 1$.

On the other hand, suppose W is a 3-plane in the intersection $[1, 2, 2]^2[0, 0, 2]$. W needs to satisfy the following conditions:

- (i) $\dim W \cap F_2 \geq 1$ for some 2-plane F_2 .
- (ii) $\dim W \cap F'_2 \geq 1$ for some 2-plane F'_2 .
- (iii) $\dim W \cap F''_1 \geq 1$ for some 1-line F''_1 .
- (iv) $\dim W \cap F''_2 \geq 2$ for some 2-plane F''_2 .

The condition (iv) implies that W contains a 2-plane F''_2 , and the condition (iii) is automatically true because $F''_1 \subseteq F''_2$. If F_2 intersects with F'_2 , then W is uniquely determined by $F_2 \cap F'_2$ and F''_2 . If F_2 has no intersection with F'_2 , in this case one of them must intersect F''_2 because we are in \mathbb{C}^5 , suppose it is F_2 , then $\dim W \cap F_2 \geq 1$ is automatically satisfied, this means W is uniquely determined by F''_2 and $W \cap F'_2$. In both cases, W is unique. Thus, $A = 1$.

We can conclude that $[1, 2, 2]^2 = [0, 2, 2] + [1, 1, 2]$.

Problem 2

Compute $H_*(\Omega_{\underline{a}})$ where \underline{a} is the Schubert symbol 012, and $\Omega_{\underline{a}} \hookrightarrow \text{Gr}_3(\mathbb{C}^5)$. Observe that $\Omega_{\underline{a}}$ cannot be a manifold, as this would violate Poincaré Duality.

Solution: From the cellular structure of $\text{Gr}_3(\mathbb{C}^5)$, we know that $\Omega_{\bar{a}}$ is of dimension 6, and has 2 4-dimensional cells $[0, 1, 1]$ and $[0, 0, 2]$, but only 1 2-dimensional cell $[0, 0, 1]$. If $\Omega_{\bar{a}}$ is a manifold, then this will violate Poincaré duality as H_2 and H_4 have different ranks.

Problem 3

Fix $n \geq 1$ and $k \leq n$. Let $\eta_k \subseteq \text{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n$ be the subspace of pairs (W, v) where $v \in W$. Let $p : \eta_k \rightarrow \text{Gr}_k(\mathbb{R}^n)$ be the map sending (W, v) to W . Prove that p is a fiber bundle with fiber \mathbb{R}^k .

Solution: Given any k -plane $W \in \text{Gr}_k(\mathbb{R}^n)$, we can view W as a $k \times n$ matrix, where each row in this matrix is a basis of the k -plane W in \mathbb{R}^n . Without loss of generality, we may assume the $k \times k$ -minor from the first k columns is non-degenerate. In this case, we can choose the basis of W in the following way such that W can be viewed as a matrix

$$W = \left[\begin{array}{c|c} I_k & A \end{array} \right]$$

Consider the following subset $U \subseteq \text{Gr}_k(\mathbb{R}^n)$: if we view k -planes as $k \times n$ matrix, then the set U is the set of all k -planes $V \in \text{Gr}_k(\mathbb{R}^n)$ satisfying the first $k \times k$ -minor is non-degenerate. Equivalently, U is the following set

$$U = \{V \in \text{Gr}_k(\mathbb{R}^n) \mid V = \left[\begin{array}{c|c} I_k & A \end{array} \right] \text{ where } A \in M_{k \times (n-k)}(\mathbb{R})\} \cong \mathbb{R}^{k(n-k)}.$$

We need to construct a homeomorphism $p^{-1}(U) \cong U \times \mathbb{R}^k$. For any $(V, v) \in p^{-1}(U)$, the matrix form gives a basis of V

$$\begin{aligned} v_1 &= (1, 0, \dots, 0, *, \dots, *) \\ v_2 &= (0, 1, \dots, 0, *, \dots, *) \\ &\dots \\ v_k &= (0, 0, \dots, 1, *, \dots, *) \end{aligned}$$

v can be written uniquely as

$$v = a_1 v_1 + \dots + a_k v_k$$

for some $a_1, \dots, a_k \in \mathbb{R}$. Define the map

$$\begin{aligned} f : p^{-1}(U) &\rightarrow U \times \mathbb{R}^k, \\ (V, v) &\mapsto (V, (a_1, \dots, a_k)) \end{aligned}$$

Conversely, given any $(a_1, \dots, a_k) \in \mathbb{R}^k$, there exists

$$v = a_1 v_1 + \dots + a_k v_k \in V.$$

This gives an inverse map $f^{-1} : U \times \mathbb{R}^k \rightarrow p^{-1}(U)$. It is easy to see that f and f^{-1} are compatible with the projection map, and it remains to prove both f and f^{-1} are continuous. Consider the

following commutative diagram

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{f} & U \times \mathbb{R}^k \\ \downarrow & \searrow \tilde{f} & \\ \text{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n & & \end{array}$$

The map

$$\tilde{f} : U \times \mathbb{R}^n \rightarrow \text{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^k$$

is sending V to V and (a_1, \dots, a_k) to $v \in V \subseteq \mathbb{R}^n$. Under the basis we choose previously, we can write

$$\begin{aligned} v &= a_1 v_1 + \dots + a_k v_k \\ &= a_1(1, 0, \dots, 0, *, \dots, *) \\ &\quad + a_2(0, 1, \dots, 0, *, \dots, *) \\ &\quad + \dots \\ &\quad + a_k(0, 0, \dots, 1, *, \dots, *) \\ &= (a_1, \dots, a_k, *, \dots, *) \end{aligned}$$

So the map \tilde{f} is just a inclusion map from \mathbb{R}^k to \mathbb{R}^n and thus is an open map and continuous. The composition

$$\tilde{f} \circ f : p^{-1}(U) \rightarrow \text{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n$$

is also an inclusion map and continous, so f is continuous. Similarly, consider the commutative diagram

$$\begin{array}{ccc} U \times \mathbb{R}^k & \xrightarrow{f^{-1}} & p^{-1}(U) \\ & \searrow \widetilde{f^{-1}} & \downarrow \\ & & \text{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n \end{array}$$

The same argument implies that the map $\widetilde{f^{-1}}$ is an inclusion, and thus continuous. The right vertical map is an inclusion of open set, so it is an open map. This implies f^{-1} is also continous.

Problem 4

Let $q : X \rightarrow Q$ be a surjection. Say that a map of spaces $f : X \rightarrow Z$ is "q-compatible" if whenever $q(x) = q(y)$ we have $f(x) = f(y)$ (this says that the identifications made by q are also made by f). The map q is a quotient map if and only if for every space Z and every map $f : X \rightarrow Z$ that is q -compatible, there is a map $\tilde{f} : Q \rightarrow Z$ such that $\tilde{f} \circ q = f$.

Prove that if $q : X \rightarrow Q$ is a quotient map and A is locally compact and Hausdoff, then

$$q \times id : X \times A \rightarrow Q \times A$$

is also a quotient map.

Solution: Let Z be any space and $f : X \times A \rightarrow Z$ be a $(q \times id)$ -compatible map, namely for all

$a \in A$, if $(q \times id)(x, a) = (q \times id)(y, a)$ for some $x, y \in X$, then $f(x, a) = f(y, a) \in Z$. Note that A is locally compact and Hausdoff, we have a bijection

$$\mathcal{Tl}_{\sqrt{}}(X \times A, Z) \cong \mathcal{Tl}_{\sqrt{}}(X, Z^A).$$

The map f is equivalent to a continuous map $g : X \rightarrow Z^A$ sending $x \in X$ to the map $a \mapsto f(x, a)$. We claim that the map g is q -compatible. Indeed, suppose $q(x) = q(y)$ for some $x, y \in X$, then $(q \times id)(x, a) = (q \times id)(y, a)$ for all $a \in A$. Since the map f is $(q \times id)$ -compatible, we have $f(x, a) = f(y, a)$ for all $a \in A$. This implies the two maps $a \mapsto f(x, a)$ and $a \mapsto f(y, a)$ are the same map. So g is q -compatible. We know that $q : X \rightarrow Q$ is a quotient map, so there exists $\tilde{g} : Q \rightarrow Z^A$ such that $\tilde{g} \circ q = g$.

$$\begin{array}{ccc} X & \xrightarrow{g} & Z^A \\ q \downarrow & \nearrow \tilde{g} & \\ Q & & \end{array}$$

The map $\tilde{g} : Q \rightarrow Z^A$ is equivalent to a continuous map $\tilde{f} : Q \times A \rightarrow Z$ sending $(p, a) \in Q \times A$ to $\tilde{g}(p)(a)$. We check that $\tilde{f} \circ (q \times id) = f$, namely the following diagram commutes.

$$\begin{array}{ccc} X \times A & \xrightarrow{f} & Z \\ q \times id \downarrow & \nearrow \tilde{f} & \\ Q \times A & & \end{array}$$

For any $(x, a) \in X \times A$, we have

$$(\tilde{f} \circ (q \times id))(x, a) = \tilde{f}(q(x), a) = \tilde{g}(q(x))(a) = (\tilde{g} \circ q)(x)(a) = g(x)(a).$$

Note that $g(x)$ is an element in Z^A , and $g(x)(a) = f(x, a) \in Z$ because f and g is equivalent under the bijection

$$\mathcal{Tl}_{\sqrt{}}(X \times A, Z) \cong \mathcal{Tl}_{\sqrt{}}(X, Z^A).$$

This proves that the diagram commutes and $q \times id$ is a quotient map.

Problem 5

Let (X, x) be a pointed space. Recall that $PX \subseteq X^I$ is the subspace of paths that end at x . Said differently, PX is defined by the pullback diagram

$$\begin{array}{ccc} PX & \longrightarrow & X^I \\ \downarrow & & \downarrow ev_1 \\ * & \xrightarrow{x} & X \end{array}$$

Convince yourself that maps $W \rightarrow PX$ are in bijective correspondence with maps $CW \rightarrow X$ sending the cone point to x (here CW is the cone on W).

If A is a CW-complex, prove that $ev_0 : PX \rightarrow X$ has the homotopy lifting property with respect to A . In particular, the fact that this holds whenever A is I^n (any $n \geq 0$) implies that

$PX \rightarrow X$ is a Serre fibration.

Solution: Suppose we have a commutative diagram

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{f} & PX \\ \downarrow & & \downarrow ev_0 \\ A \times I & \xrightarrow{g} & X \end{array}$$

For all $a \in A$, the map $f : A \times \{0\} \rightarrow PX$ sends a to a path $f(a) : I \rightarrow X$. This is equivalent to a map $\tilde{f} : CA \times \{0\} \rightarrow X$ sending (a, t) to $f(a)(t)$. This is well-defined because all different paths ends at the same point $x \in X$. We know that A is a CW complex, so $A \times I$ and $C(A \times I)$ are also CW complexes, and $A \times I$ and $C(A \times \{0\})$ are subcomplexes of $C(A \times I)$. The inclusion of the subcomplex

$$C(A \times \{0\}) \cup A \times I \rightarrow C(A \times I)$$

is a homotopy equivalence because $A \times I$ is homotopy equivalent to $A \times \{0\} \subseteq C(A \times \{0\})$, and both cones are contractible. Consider the solid-arrow diagram

$$\begin{array}{ccc} C(A \times \{0\}) \cup A \times I & \xrightarrow{\tilde{f} \cup g} & X \\ \downarrow & \nearrow F & \downarrow \\ C(A \times I) & \xrightarrow{\quad} & * \end{array}$$

\tilde{f} restricting to $A \times \{0\}$ can be viewed as the composition of the map

$$A \times \{0\} \xrightarrow{f} PX \xrightarrow{ev_0} X.$$

By commutativity of the original diagram, this is equal to the map

$$g|_{A \times \{0\}} : A \times \{0\} \rightarrow X.$$

By GLP, we have a lifting $F : C(A \times I) \rightarrow X$ such that $F|_{C(A \times \{0\})} = \tilde{f}$ and $F|_{A \times I} = g$. We know that F is equivalent to a map $\tilde{F} : A \times I \rightarrow PX$ satisfying the following two diagrams

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{f} & PX \\ \downarrow & \nearrow \tilde{F} & \\ A \times I & & \end{array} \quad \begin{array}{ccc} & & PX \\ & \nearrow \tilde{F} & \downarrow ev_0 \\ A \times I & \xrightarrow{g} & X \end{array}$$

This proves that original diagram has a lifting. Thus, we can conclude that $ev_0 : PX \rightarrow X$ has the homotopy lifting property with respect to any CW complex A .