Math 636 Homework #4
Due Friday, May 2

1. Compute all of the homology groups for the spaces in parts (a) and (b) below, and use your calculations to show that
   (a) $\mathbb{R}P^2 \times S^3$ and $\mathbb{R}P^3 \times S^2$ have isomorphic homotopy groups (in all dimensions), but non-isomorphic homology groups.
   (b) $S^4 \times S^2$ and $\mathbb{C}P^3$ have isomorphic homology groups but non-isomorphic homotopy groups. [Hint for the latter: Recall that there is a fiber bundle $S^{2n+1} \to \mathbb{C}P^n$.]

2. Let $I_*$ be the chain complex concentrated in degrees 0 and 1 with $I_1 = \mathbb{Z}\langle e\rangle$, $I_0 = \mathbb{Z}\langle a, b\rangle$, and $d(e) = b - a$ (the elements $e$, $a$, and $b$ are chosen basis elements for the given free abelian groups). Note that this is the simplicial chain complex for $\Delta^1$. Let $C_*$ and $D_*$ be chain complexes.
   (a) Describe the chain complex $I_* \otimes C_*$ by giving the groups in each degree as well as the boundary maps.
   (b) Let $F\colon I_* \otimes C_* \to D_*$ be a chain map. Define $f, g\colon C_* \to D_*$ by $f(x) = F(a \otimes x)$ and $g(x) = F(b \otimes x)$. Likewise, define $s_n\colon C_n \to D_{n+1}$ by $s_n(x) = F(e \otimes x)$. Prove that $f$ and $g$ are chain maps and the collection $\{s_n\}$ is a chain homotopy between $f$ and $g$.

   [Don't write this out, but convince yourself that giving a chain map $I_* \otimes C_* \to D_*$ is *equivalent* to giving the data of two chain maps $C_* \to D_*$ and a chain homotopy between them.]

3. Let $Y$ be the space obtained by starting with $S^3$ and attaching a 4-cell via a map of degree 5: $Y = S^3 \cup_f e^4$ where $f\colon \partial(e^4) \to S^3$ has degree 5. Write down the cellular chain complex for $\mathbb{R}P^3 \times Y$; in particular, specify the rank of each chain group and identify the boundary maps. Compute the homology groups of $\mathbb{R}P^3 \times Y$ (you do not need to compute the homology groups directly from the chain complex, though).

4. Compute both the homology and cohomology groups of the following spaces, both with integral and $\mathbb{Z}/2$ coefficients. Heck, do it with $\mathbb{Z}/3$ coefficients as well.
   (a) $K \times K$, where $K$ is the Klein bottle.
   (b) $K \times T^g$, where $T^g$ is the genus $g$ torus and $K$ is the Klein bottle.
   (c) $K \times \mathbb{R}P^n$.

5. Let $f\colon A_* \to B_*$ be a map of chain complexes. We can regard this as a forming a double complex

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
A_2 \arrow[r] \arrow[d] & B_2 \arrow[d] \\
A_1 \arrow[r] \arrow[d] & B_1 \arrow[d] \\
A_0 \arrow[r] & B_0
\end{tikzcd}$$

by putting zeros in all the "empty" spots. The total complex of this double complex is called the **algebraic mapping cone** of $f$, denoted $Cf$. Specifically, we set $(Cf)_n = A_{n-1} \oplus B_n$ and define $d\colon (Cf)_n \to (Cf)_{n-1}$ by

$$d(a,b) = \big(d_A(a), (-1)^{n-1} f(a) + d_B(b)\big)$$

(one sometimes sees the definition with slightly different signs).

This is an algebraic analog of the mapping cone in topology. If $f\colon A \to B$ is a cellular map of CW-complexes then the topological mapping cone has an evident cell structure where the $i$-cells consist of the $i$-cells of $B$ together with one $i$-cell in the cone on $A$ for every $(i-1)$-cell of $A$; note the parallels with the above algebraic construction.

(a) Explain why there is a short exact sequence of chain complexes $0 \to B_* \hookrightarrow C(f) \to \Sigma A_* \to 0$, where $\Sigma A_*$ is the evident chain complex having $(\Sigma A)_n = A_{n-1}$.

(b) The short exact sequence from (a) gives rise to a long exact sequence in homology groups. This has the form

$$\cdots \to H_i(B) \to H_i(Cf) \to H_i(\Sigma A) \xrightarrow{\partial} H_{i-1}(B) \to \cdots$$

Verify that the connecting homomorphism is really just the map $f_*\colon H_{i-1}(A) \to H_{i-1}(B)$, possibly up to a sign.

The following problem gives some more practice with homological algebra, but in somewhat different settings than you have seen so far. I think it is not nearly as long as it looks.

6. Let $k$ be a field, and let $\mathcal{V}$ denote the category of vector spaces over $k$. Let $I$ be any (small) category, and let $\mathcal{V}^I$ be the category whose objects are functors $I \to \mathcal{V}$ and whose morphisms are natural transformations. We call $\mathcal{V}^I$ the category of "$I$-shaped diagrams in $\mathcal{V}$".

In this problem we will focus on the case where $I$ is the pushout category

$$1 \longleftarrow 0 \longrightarrow 2$$

with three objects and two non-identity maps (as shown above). An object of $\mathcal{V}^I$ is then just a diagram of vector spaces $V_1 \leftarrow V_0 \to V_2$. A map from $[V_1 \leftarrow V_0 \to V_2]$ to $[W_1 \leftarrow W_0 \to W_2]$ is a commutative diagram

$$\begin{tikzcd} V_1 \arrow[d] & V_0 \arrow[l] \arrow[r] \arrow[d] & V_2 \arrow[d] \\ W_1 & W_0 \arrow[l] \arrow[r] & W_2. \end{tikzcd}$$

Let $P\colon \mathcal{V}^I \to \mathcal{V}$ be the pushout functor: $P$ assigns to each diagram its pushout.

Note that if $D_1$ and $D_2$ are diagrams in $\mathcal{V}^I$ and $f\colon D_1 \to D_2$ is a map, then the diagram of kernels and the diagram of cokernels is also an object of $\mathcal{V}^I$ in an evident way. We can talk about chain complexes of objects in $\mathcal{V}^I$, and we can talk about the homology "groups" of such a complex (which are not groups, but rather objects in $\mathcal{V}^I$). We can redo all of homological algebra in this setting, in all cases replacing the category of abelian groups with the category $\mathcal{V}^I$.

(a) Let $F_1$, $F_0$, and $F_2$ be the three diagrams

$$F_1 = [k \leftarrow 0 \to 0] \qquad F_0 = [k \leftarrow k \to k] \qquad F_2 = [0 \leftarrow 0 \to k]$$

where in $F_0$ the maps are the identities. These diagrams are "free" in a certain sense: namely, if $D$ is an object of $\mathcal{V}^I$ then morphisms $F_i \to D$ are in bijective correspondence with elements of $D_i$. Convince yourself that this is true (do not hand anything in).

(b) Let $D = [0 \leftarrow k \to 0]$ and $E = [0 \leftarrow k \to k]$, where in $E$ the nontrivial map is the identity. Determine free resolutions for $D$ and $E$ (in each level your resolution will consist of a sum of $F_i$'s, for various $i$).

(c) Apply the functor $P$ to your resolutions, to produce a chain complex of vector spaces. Compute the homology groups, which are the groups $(L_iP)(D)$ and $(L_iP)(E)$. These are the derived functors of the pushout functor $P$. Confirm in your examples that $L_0P = P$.

(d) Now let $I$ be the category with one object 0 and one non-identity map $t\colon 0 \to 0$ such that $t^2 = id$. Objects of $\mathcal{V}^I$ are then pairs $(W, t)$ consisting of a vector space $W$ and an endomorphism $t\colon W \to W$ such that $t^2 = id$. In $\mathcal{V}^I$ the basic "free" object is $(k^2, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$; this can also be thought of as the vector space $k\langle g, tg\rangle$ where $t(tg) = g$. Let $P\colon \mathcal{V}^I \to \mathcal{V}$ be the colimit functor, sending an object $(W, t)$ to $W/\{x - tx \mid x \in W\}$. Find the free resolution of the object $(k, id)$ and compute $(L_iP)(k, id)$ for all $i \geq 0$ (the answer will depend on the characteristic of $k$).