Zhengdong Zhang
Email: zhengz@uoregon.edu

Homework - Week 1
ID: 952091294

Course: MATH 648 - Abstract Algebra
Term: Winter 2025
Instructor: Professor Arkady Berenstein
Due Date: 15^{th} January, 2025

Problem 14.1.13
Let V be a left R-module and X be a subset of V. Then $\operatorname{Ann}(X)$ is a left ideal of R, and if X is a submodule of V then $\operatorname{Ann}(X)$ is a two-sided ideal of R.

Solution: Assume X is a subset of V. Let $a, b \in \operatorname{Ann}(X)$. For any $x \in X$, we have $(a+b)(x) = ax + bx = 0$. Let $r \in R$, we have

$$(ra)x = r(ax) = r0 = 0.$$

This proves that $\operatorname{Ann}(X)$ is a left ideal of R. Now assume X is a submodule of V, we only need to prove that $\operatorname{Ann}(X)$ is a right ideal of R. Note that

$$(ar)x = a(rx) = 0$$

since X being a submodule of V tells us that rx is still an element of X for any $r \in R$. This shows that $\operatorname{Ann}(X)$ is a two-sided ideal of R.

Problem 14.1.15
True or false?

(1) Let R be a commutative ring and $I, J \lhd R$ be two ideals of R. If the modules R/I and R/J are isomorphic then $I = J$.

(2) Let R be a ring and I, J be two left ideals in R. If the modules R/I and R/J are isomorphic then $I = J$.

Solution:

(1) This is true. Let $\phi : R/I \xrightarrow{\sim} R/J$ be an isomorphism of R-modules with $\phi(1+I) = a + J$ for some $a \in R$. For any $j \in J$, we have

$$\phi(j + I) = \phi(j \cdot 1 + I) = j \cdot \phi(1 + I) = ja + J.$$

Note that J is an ideal in R, so $ja + J = J$ as $ja \in J$. Thus, $j + I \in \ker \phi = \{I\}$ because ϕ is an isomorphism. This shows that $j \in I$. We have proved $J \subseteq I$. Use a similar argument and an injective R-module homomorphism $\phi^{-1} : R/J \to R/I$ we can show that $I \subseteq J$. Now we can conclude that $I = J$.

(2) This is false. Consider $R = M_2(\mathbb{R})$ is the 2×2 matrix ring over $\mathbb{R}$. We define

$$I = \left\{ \begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix} \mid p, q \in \mathbb{R} \right\},$$
$$J = \left\{ \begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix} \mid p, q \in \mathbb{R} \right\}.$$

I is a left ideal of R since for any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix} = \begin{pmatrix} ap + bq & 0 \\ cp + dq & 0 \end{pmatrix} \in I.$$

Similar for J. Note that the quotient is a R-module

$$R/I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} + I \right\}$$

and $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ represent the same element in the quotient if $b_1 = b_2$ and $d_1 = d_2$. So the choice of b, d uniquely determines an element in R/I. Similarly, the choice of a, c uniquely determines an element in R/J. Define a map

$$\phi : R/I \to R/J,$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} b & a \\ d & c \end{pmatrix}.$$

It is easy to see that ϕ is both injective and surjective. Moreover, this is an R-module homomorphism since

$$\begin{aligned} \phi(\begin{pmatrix} k & l \\ m & n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}) &= \phi(\begin{pmatrix} ka + lc & kb + ld \\ ma + nc & mb + nd \end{pmatrix}) \\ &= \begin{pmatrix} kb + ld & ka + lc \\ mb + nd & ma + nc \end{pmatrix} \\ &= \begin{pmatrix} k & l \\ m & n \end{pmatrix} \begin{pmatrix} b & a \\ d & c \end{pmatrix} \\ &= \begin{pmatrix} k & l \\ m & n \end{pmatrix} \phi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}). \end{aligned}$$

Thus, R/I and R/J are isomorphic as R-modules but obviously I and J are different left ideals in R.

Problem 14.1.16
Let V be an R-module. A family $(V_i)_{i \in I}$ of submodules of V is a directed system of submodules if for any $i, j \in I$ there exists $k \in I$ such that $V_i \subseteq V_k$ and $V_j \subseteq V_k$. Prove that V is finitely generated if and only if the union $\cup_{i\in I} V_i$ of any directed set of proper submodules is proper. Deduce that a finitely generated module has a maximal proper submodule.

Solution: First we assume V is finitely generated. Suppose the set

$$S = \{v_1, v_2, \ldots, v_n\}$$

is the generating set for the module V. Denote the generating set for V_i is S_i for every $i \in I$. For any $i \in I$, V_i being proper submodules implies that S_i is a proper subset of S.
Claim: There exists $k \in I$ such that for any $i \in I$, $V_i \subseteq V_k$.
Proof: Note that every V_i is finitely generated, so we only need to show that there exists $k \in I$, for any $i \in I$, we have $S_i \subseteq S_k$. We pick S_k as the set with the most elements. This can be done because $(S_i)_{i\in I}$ are all subsets of a finite set S with n elements. Suppose there exists $v \in S_i$ for some i such that $v \notin S_k$. By the definition of the directed system, there must exists some $k' \in I$ such that $V_i \subseteq V_{k'}$ and $V_k \subseteq V_{k'}$. This implies $v \in S_{k'}$ and $S_k \subseteq S_{k'}$. $S_{k'}$ is strictly larger than S_k. This contradicts our choice of S_k. ∎
The above claim tells us that $\cup_{i\in I} V_i = V_k$ is a proper submodule.

Now assume for any directed system of proper submodule $(V_i)_{i\in I}$, the union $\cup_{i\in I} V_i$ is proper and V is not finitely generated. Consider a set of $(V_i)_{i\in I}$ where for each $i \in I$, V_i is a finitely generated submodule of V. This is a directed system because for any $i, j \in I$, the union $V_i \cup V_j$ is also a finitely generated submodule. And since V is not finitely generated, each V_i must be proper. It is easy to see that $\cup_{i\in I} V_i \subseteq V$. We claim that $V = \cup_{i\in I} V_i$ because for every $v \in V$, $v \in \langle v \rangle$ which is a finitely generated submodule. This shows that the union $\cup_{i\in I} V_i = V$ is not proper. A contradiction.

Given a finitely generated nontrivial module V, consider the set $S = \{V_i\}_{i\in I}$ where each V_i is a proper submodule of V.S is non-empty since the zero module 0 is always in S. S has a partial order given by the inclusion of subodules. Consider a totally ordered subset X of S. Note that X is a directed system and we can take the union of all elements in X as an upper bound since previously we have shown that the union is still a proper submodule. By Zorn's lemma, S must has a maximal element, which is a maximal proper submodule of V.

Problem 14.1.17
True or false? As a $\mathbb{Z}$-module, $\mathbb{Q}$ has a maximal proper submodule.

Solution: This is false. Let M be a maximal proper $\mathbb{Z}$-submodule of $\mathbb{Q}$. Then $\mathbb{Q}/M$ is a simple $\mathbb{Z}$-module. By Example 14.1.21, a $\mathbb{Z}$-module $\mathbb{Q}/M$ is simple if and only if $\mathbb{Q}/M \cong C_p$ for some prime p. So we have a surjective homomorphism of $\mathbb{Z}$-modules $\phi : \mathbb{Q} \to C_p$ with $\ker \phi = M$. For any $\frac{m}{n} \in \mathbb{Q}$, we have

$$\phi(\frac{m}{n}) = \phi(\frac{m}{pn} + \frac{m}{pn} + \cdots + \frac{m}{pn}) = p\phi(\frac{m}{pn}) = 0.$$

So ϕ is the zero morphism, which contradicts the surejctivity of ϕ.

Problem 14.1.23
An R-module V is irreducible if and only if $V \cong R/I$ for a maximal left ideal I of R.

Solution: First we assume that V is irreducible. For any $v \in V$, by Exercise 14.1.13, the annihilator $\mathrm{Ann}(v)$ is a left ideal of R. Suppose $\mathrm{Ann}(v) \subseteq J \subseteq R$ is contained in some other left ideal in R. There exists $a \in J$ such that $a \notin \mathrm{Ann}(v)$. Note that $a \notin \mathrm{Ann}(v)$ implies $av \neq 0$, so $v \in \langle av \rangle = V$ since V is simple. This means that there exist $r \in R$ such that $v = rav$. So $1 - ra \in \mathrm{Ann}(v) \subseteq J$. Then $1 = (1 - ra) + ra \in J$ and this shows that $J = R$. We have proved that $\mathrm{Ann}(v)$ is a maximal left ideal of R. We define a map

$$\begin{aligned} \phi : R/\mathrm{Ann}(v) &\to V; \\ r + \mathrm{Ann}(v) &\mapsto rv. \end{aligned}$$

It is obvious that ϕ is a well-defined R-module homomorphism. Suppose $r + \mathrm{Ann}(v) \in \ker \phi$. Then we have

$$\phi(r + \mathrm{Ann}(v)) = rv = 0.$$

This implies $r \in \mathrm{Ann}(v)$, so $r + \mathrm{Ann}(v) = \mathrm{Ann}(v)$ in $R/\mathrm{Ann}(v)$. This proves that ϕ is injective. Moreover, we know that

$$V = \{rv \mid r \in R\}.$$

Suppose $r+\mathrm{Ann}(v)$ and $s+\mathrm{Ann}(v)$ are different representatives for the same element in $R/\mathrm{Ann}(v)$, we know $r - s \in \mathrm{Ann}(v)$ and $(r - s)v = 0$. This means $rv = sv$ is the same element in V. This proves that ϕ is surjective. Thus, we have an R-module isomorphism $V \cong R/\mathrm{Ann}(v)$.

Conversely, we assume $V \cong R/I$ for some maximal left ideal I in R. We need to prove that R/I as a R-module is simple. By Exercise 14.1.14, the correspondence theorem in modules tells us that the set of submodules in R/I corresponds to the set of left ideals containing I, and since I is maximal, R/I only has two submodules: 0 and itself. This proves that $V \cong R/I$ is simple.

Problem 14.1.24
Prove that for every ring R there exists an irreducible R-module.

Solution: By Exercise 14.1.23, we only need to show that every ring R has a maximal left ideal I. Let S be the set of all proper left ideals in R. S is not empty since the zero ideal 0 is always in S. S has a partial order given by the inclusion. Given a totally orderer subset $\{J_i\}_{i \in I}$, we claim that $\cup_{i \in I} J_i$ is an upper bound. Indeed, for any $a \in \cup_{i \in I} J_i$, there must exists $k \in I$ such that $a \in J_k$. Then $ra \in J_k \subseteq \cup_{i \in I} J_i$ for any $r \in R$ since J_k is a left ideal. Moreover, suppose $\cup_{i \in I} J_i = R$. Then $1 \in R = \cup_{i \in I} J_i$. This means there exists $k \in I$ such that $1 \in J_k$, namely $J_k = R$. This contradicts that J_k is a proper left ideal of R. So $\cup_{i \in I} J_i$ is still a proper left ideal of R. By Zorn's lemma, S has a maximal element.

Problem 14.1.25
Let R be a commutative ring. Show that the map $V \mapsto \text{Ann}_R(V)$ induces a bijection between the set of isomorphism classes of irreducible R-modules and the set of maximal ideals of R.

Solution: Let V be a simple R-module and $v \in V$ is a nonzero element.
Claim:

$$\text{Ann}(v) = \text{Ann}(V).$$

Proof: It is obvious that $\text{Ann}(V) \subseteq \text{Ann}(v)$ since $v \in V$. On the other hand, given $r \in \text{Ann}(v)$ satisfying $rv = 0$. We know V is simple so $\langle v \rangle = V$. For any $w \in V$, there exists $r' \in R$ such that $w = r'v$. Now using the fact that R is commutative, we have

$$rw = r(r'v) = (rr')v = (r'r)v = r'(rv) = 0.$$

so $r \in \text{Ann}(V)$. ■

The claim tells us that the map $\phi : V \mapsto \text{Ann}(V) = \text{Ann}(v)$ for some $v \in V$. We have already seen in Exercise 14.1.23 that $\text{Ann}(v)$ is a maximal ideal of R and $V \cong R/\text{Ann}(v)$. Suppose $V \cong W$ as R-modules, then we have $R/\text{Ann}(v) \cong R/\text{Ann}(w)$ for some $v \in V$ and $w \in W$. By Exercise 14.1.15 (1), $\text{Ann}(v) = \text{Ann}(w)$ in R. So the map ϕ is well-defined. Moreover, we can define $\phi^{-1} : I \to R/I$ where I is a maximal ideal of R. Note that $I = \text{Ann}(R/I)$ if we view R/I as an R-module. So this indeed defines an inverse of ϕ. This proves that ϕ is an isomorphism.

Problem 14.2.1
If R is commutative and V is an irreducible R-module, then $V \oplus V$ is not cyclic. Give an example of a non-commutative ring R and an irreducible left R-module V such that $V \oplus V$ is cyclic.

Solution: First assume R is commutative. Let $v, w \in V$ be two different nonzero elements. Since V is simple, we know that $V = \langle v \rangle = \langle w \rangle$. We can write $V \oplus V = \langle v \rangle \oplus \langle w \rangle$. Suppose that $V \oplus V$ is cyclic. Then there exists $a \in \langle v \rangle \oplus \langle w \rangle$ such that for some $r_1, r_2 \in R$, $a = r_1 v + r_2 w$ generates $V \oplus V$. There exists $r_3, r_4 \in R$ such that

$$r_3(r_1 v + r_2 w) = v$$
$$r_4(r_1 v + r_2 w) = w$$

which gives us

$$r_3 r_1 = 1,\ r_3 r_2 = 0,$$
$$r_4 r_1 = 0,\ r_4 r_2 = 1.$$

Note that R is commutative, so we have

$$1 = (r_3 r_1)(r_4 r_2) = (r_3 r_2)(r_4 r_1) = 0.$$

A contradiction. So $V \oplus V$ cannot be cyclic.

Now assume R is not commutative. Let $R = M_2(\mathbb{F})$ be the 2×2 matrix ring over a field $\mathbb{F}$. The column vectors

$$V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in \mathbb{F} \right\}$$

is a simple left R-module. Moreover, we have $V \oplus V \cong M_2(\mathbb{F})$, which can be generated by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as a left $M_2(\mathbb{F})$-module.

Problem 14.2.6
Let $0 \to K \xrightarrow{\iota} V \xrightarrow{\pi} Q \to 0$ be an exact sequence of left R-modules. If Q and K are finitely generated then so is V.

Solution: Since K and Q are finitely generated, we can write

$$K = \langle v_1, v_2, \ldots, v_m \rangle, \;\; Q = \langle w_1, w_2, \ldots, w_n \rangle$$

for some positive integer m, n. There exists $w'_1, w'_2, \ldots, w'_n \in V$ such that $\pi(w'_i) = w_i$ for any $1 \leq i \leq n$ because π is surjective.
Claim: V is generated by the set

$$\{\iota(v_1), \iota(v_2), \ldots, \iota(v_m), w'_1, w'_2, \ldots, w'_n\}.$$

Proof: Note that by exactness, we have $V/\iota(K) \cong Q$. By our choice of generators, $V/\iota(K)$ is generated by

$$w'_1 + \iota(K), w'_2 + \iota(K), \ldots, w'_n + \iota(K).$$

So for every $v \in V$, the coset $v + \iota(K)$ can be written as

$$\begin{aligned} v + \iota(K) &= (a_1 w'_1 + \iota(K)) + (a_2 w'_2 + \iota(K)) + \cdots + (a_n w'_n + \iota(K)) \\ &= (a_1 w'_1 + a_2 w'_2 + \cdots + a_n w'_n) + \iota(K) \end{aligned}$$

where $a_1, a_2, \ldots, a_n \in R$. This means that

$$v - a_1 w'_1 - a_2 w'_2 - \cdots - a_n w'_n \in \iota(K).$$

Note that ι is injective and K is generated by $v_1, \ldots, v_m$, so we can write

$$v - a_1 w'_1 - a_2 w'_2 - \cdots - a_n w'_n = b_1 \iota(v_1) + b_2 \iota(v_2) + \cdots + b_m \iota(v_m).$$

This proves the claim. ■

Problem 14.2.12
The sequence of R-modules and R-module homomorphisms

$$U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

is exact if and only if the corresponding sequence

$$0 \to \hom_R(W, X) \xrightarrow{g^*} \hom_R(V, X) \xrightarrow{f^*} \hom_R(U, X)$$

of abelian groups is exact for every R-module X.

Solution: First we prove the sufficiency. Assume we have an exact sequence of R-modules

$$U \xrightarrow{f} V \xrightarrow{g} W \to 0.$$

Apply the functor $\hom_R(-, X)$ where X is an R-module. We need to prove the following:

(1) $g^* : \hom_R(W, X) \to \hom_R(V, X)$ is injective.
Let $p \in \hom_R(W, x)$ and $p \in \ker g^*$. This is the same as for any $v \in V$, we have $g^*(p)(v) = (p \circ g)(v) = 0$ in X. For any $w \in W$, by exactness we know that $g : V \to W$ is surjective, so there exists $w' \in V$ such that $g(w') = w$. Then we have

$$p(w) = p(g(w')) = (p \circ g)(w') = 0.$$

This proves that $p : W \to X$ is the zero morphism, so $\ker g^* = 0$ and g^* is injective.

(2) We need to show that $\ker f^* = \operatorname{Im} g^*$.
First we show that $\operatorname{Im} g^* \subseteq \ker f^*$. This is equivalent to $f^* \circ g^* = 0$. Note that $\hom_R(-, X)$ is a functor, so

$$f^* \circ g^* = (g \circ f)^* = 0^* = 0$$

by exactness of the original sequence. Next, consider a map $q : V \to X$ satisfying $f^*(q) = q \circ f = 0$. For any $w \in W$, since g is surjective, we can choose $v \in V$ such that $g(v) = w$. Define a map $q' : W \to X$ as follows:

$$q'(w) = q(v).$$

This is well-defined (it does not depend on the choice of v). Suppose there exists another $v' \in V$ such that $g(v') = w$. Then $v - v' \in \ker g = \operatorname{Im} f$ by the exactness of the original sequence. This means there exists $u \in U$ such that $f(u) = v - v'$. So we have

$$q(v) - q(v') = q(v - v') = q(f(u)) = (q \circ f)(u) = 0$$

since $q \in \ker f^*$. We have proved that $\ker f^* \subseteq \operatorname{Im} g^*$. Therefore, $\ker f^* = \operatorname{Im} g^*$.

Now assume for any R-module X, we have an exact sequence

$$0 \to \hom_R(W, X) \xrightarrow{g^*} \hom_R(V, X) \xrightarrow{f^*} \hom_R(U, X).$$

We need to show that the original sequence

$$U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

is exact. We need to prove the following:

(1) g is surjective.
Take $X = \operatorname{coker} g$ and $q : W \to \operatorname{coker} g$ is the canonical projection map. The composition $(q \circ g) = g^*(q) = 0$ by definition of the cokernel and since g^* is injective by the exactness of the sequence, we know that $q = 0$ is the zero map. This implies that $\operatorname{coker} g = 0$, so g is surjective.

(2) We need to prove that $\operatorname{Im} f = \ker g$.
First we need to show that $\operatorname{Im} f \subseteq \ker g$, which is equivalent to $g \circ f = 0$. By exactness, we have
$$f^* \circ g^* = (g \circ f)^* : \hom_R(W, X) \to \hom_R(U, X)$$
is the zero map. Take $X = W$ and we have
$$0 = (g \circ f)^*(id_W) = f \circ g \circ id_W = f \circ g.$$
This proves $\operatorname{Im} f \subseteq \ker g$. On the other hand, consider $p : V \to \operatorname{coker} f$ is the canonical projection. By definition, we know that $f^*(p) = p \circ f = 0$, so $p \in \ker f^* = \operatorname{Im} g^*$ by exactness of the sequence. This means there exists $\phi : W \to \operatorname{coker} f$ such that $g^*(\phi) = \phi \circ g = p$. Now suppose $v \in V$ satisfying $g(v) = 0$. We have
$$p(v) = (\phi \circ g)(v) = \phi(g(v)) = 0.$$
So $v \in \ker p = \ker(V \to \operatorname{coker} f) = \operatorname{Im} f$. We have proved that $\ker g \subseteq \operatorname{Im} f$. Therefore, $\ker g = \operatorname{Im} f$.

Problem 14.2.16
Let G be a finite group, $\mathbb{F}$ a field of characteristic p dividing $|G|$. Then the 1-dimensional submodule of the left regualr module $\mathbb{F}G$ spanned by the element $\sum_{g \in G} g$ is not a direct summand of the regular module.

Solution: Let $x = \sum_{g \in G} g$. Assume $\langle x \rangle$ is a direct summand of $\mathbb{F}G$ as an $\mathbb{F}G$-module, then there exists a $\mathbb{F}G$-submodule C such that $C \oplus \langle x \rangle = \mathbb{F}G$. Suppose an element $\sum_{h \in G} a_h h \in C$ where $a_h \in \mathbb{F}$. We have
$$x(\sum_{h \in G} a_h h) = (\sum_{g \in G} g)(\sum_{h \in G} a_h h) = \sum_{g \in G} \sum_{h \in G} a_h gh = \sum_{k \in G} b_k k.$$
Note that for any $k \in G$, there exists a unique $g = kh^{-1} \in G$ for every $h \in G$ such that $gh = k$. So $b_k = \sum_{h \in G} a_h$ for any $k \in G$. And we can write
$$x(\sum_{h \in G} a_h h) = \sum_{g \in G} (\sum_{h \in G} a_h) g = (\sum_{h \in G} a_h)(\sum_{g \in G} g) = (\sum_{h \in G} a_h) x \in \langle x \rangle.$$

Since $\sum_{h\in G} a_h h \in C$, we can be see that $x(\sum_{h\in G} a_h h) \in C \cap \langle x \rangle = 0$. This implies $0 = \sum_{h\in G} a_h \in \mathbb{F}$. On the other hand, suppose $\sum_{h\in G} b_h h \in \mathbb{F}G$ and similarly we have

$$(\sum_{h\in G} b_h h)x = (\sum_{h\in G} b_h h)(\sum_{g\in G} g) = (\sum_{h\in G} b_h)x \in \langle x \rangle.$$

Now consider $e \in G$ which is the identity element, viewed as an element in $\mathbb{F}G$. $e \notin C$ since the coefficients need to sum to zero and also $e \notin \langle x \rangle$. But $\mathbb{F}G = \langle x \rangle \oplus C$. A contradiction.

Problem 14.3.4
True or false? If K is a free R-module, then every short exact sequence of R-modules of the form $0 \to K \to V \to Q \to 0$ is split.

Solution: This is false. Consider the following short exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/(2) \to 0.$$

Here $K = \mathbb{Z}$ is a free $\mathbb{Z}$-module but this sequence does not split since $\mathbb{Z}$ cannot be written as direct sum of $\mathbb{Z}$ and $\mathbb{Z}/(2)$.

Problem 14.3.9
Suppose R is an integral domain with the property that any maximal R-linearly independent set of vectors in any free R-module is a basis. Prove that R is a field.

Solution: Consider R itself as a free R-module of rank 1. Take $0 =\neq x \in R$. Then the set $\{x\}$ is maximal and R-linearly independent, so it is a basis. This means for any $r \in R$, there exists $y \in R$ such that $yx = r$. Take $r = 1$ and this tells us that every element in R has a multiplicative inverse. Thus, R is a field.

Problem 14.3.10
Give an example of an integral domain R and a submodule of a free R-module that is not free.

Solution: Let k be a field of characteristic 0 and $R = k[x, y]$, which is an integral domain. R itself can be viewed as a free R-module of rank 1. Consider the ideal $I = (x, y) \subseteq R$, which is a submodule of R, and I is not a free R-module. Assume the opposite. Note that R as a free R-module is of rank 1, so I is free, then it must also be of rank 1. This means I can be generated by one element, namely I is a principal ideal in R. Note that x and y are relatively prime in R, so I has be generated by $\gcd(x, y) = 1$. This shows that $I = R$, but I is proper since $2 \in k[x, y]$ is not in I. A contradiction.

Problem 14.3.11
If R is commutative and every submodule of a free R-module is free then R is a PID.

Solution: View R itself as a free R-module of rank 1, then any ideal $I \subseteq R$ is a R-submodule of rank at most 1. This means there exist $a \in I$ such that for any $b \in I$, there exist $r \in R$ such that $ra = b$. This shows that I is generated by a, so I is a principal. Moreover, for any $a \in R$, consider the principal ideal generated by a, it is a free R-module with basis $\{a\}$. If a is a zero divisor, then $\{a\}$ is not R-linearly independent since there exists non zero $r \in R$ such that $ra = 0$. A contradiction. This proves that R is a principal ideal domain.