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Course: MATH 636 - Algebraic Topology III
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Homework 8
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Problem 1
Suppose that M is a compact 3-manifold with $\pi_1(M) \cong \mathbb{Z}/5$.

(a) Prove that M is orientable, and then calculate all of the homology and cohomology groups of M.

(b) Prove that every map $M \to \mathbb{R}P^3$ has even degree.

Solution:

(a) $\pi_1(M) \cong \mathbb{Z}/5$, so there does not exist $\pi_1(M)$-set of index 2 because 2 does not divide 5. This implies every degree 2 covering space of M must be disconnected, so $\tilde{M} \to M$ is the trivial covering map with 2 connected components. This implies M is orientable.

By Hurewicz theorem, we know that $H_1(M)$ is the abelianization of $\pi_1(M)$. So

$$H_1(M) \cong \pi_1(M) \cong \mathbb{Z}/5.$$

M being orientable implies that $H_3(M) \cong \mathbb{Z}$. By UCT, we have

$$H^1(M) \cong \hom(H_1(M), \mathbb{Z}) \oplus \operatorname{Ext}^1(H_0(M), \mathbb{Z}) \cong 0.$$

By Poincaré duality, $H_2(M) \cong H^1(M) \cong 0$. We have all the homology groups of M and use Poincaré duality again, we can obtain all the cohomology groups.

	$H_*(M)$	$H^*(M)$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}/5$	0
2	0	$\mathbb{Z}/5$
3	$\mathbb{Z}$	$\mathbb{Z}$

(b) By UCT, we have

$$H^1(M; \mathbb{Z}/2) \cong H^1(M) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \oplus \operatorname{Tor}_1(H^2(M), \mathbb{Z}/2) \cong 0.$$

Given a map $f : M \to \mathbb{R}P^3$, the induced map

$$f^* : H^*(\mathbb{R}P^3; \mathbb{Z}/2) \to H^*(M; \mathbb{Z}/2)$$

must be the zero map because we know $H^*(\mathbb{R}P^3; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x]/(x^4)$, and it is generated by x in degree 1, but $H^1(M) = 0$. By naturality of UCT, we have a commutative diagram

$$\begin{tikzcd}
H^3(\mathbb{R}P^3) \otimes \mathbb{Z}/2 \arrow[r, hook] \arrow[d] & H^3(\mathbb{R}P^3; \mathbb{Z}/2) \arrow[d, "0"] \\
H^3(M) \otimes \mathbb{Z}/2 \arrow[r, hook] & H^3(M; \mathbb{Z}/2)
\end{tikzcd}$$

We know that $H^3(M) \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$ and the map $H^3(M) \otimes \mathbb{Z}/2 \to H^3(M; \mathbb{Z}/2)$ is injective, so it cannot be the zero map. Thus,

$$H^3(\mathbb{R}P^3) \otimes \mathbb{Z}/2 \to H^3(M) \otimes \mathbb{Z}/2$$

must be the zero map. This implies $H^3(\mathbb{R}P^3) \to H^3(M)$ is given by multiplication of an even number, namely the map $f : M \to \mathbb{R}P^3$ has even degree.

Problem 2

(a) Explain why the Euler characteristic of an odd-dimensional compact manifold must be zero.

(b) Suppose that M is a $(2d+1)$-dimensional compact manifold, and let $W = \partial M$. Let X be the manifold obtained by gluing two copies of M together along their boundary. Using Mayer-Vietoris (or otherwise) prove that $\chi(W) \equiv \chi(X)$ mod 2, and so deduce that $\chi(W)$ must be even.

Solution:

(a) Suppose M is a compact mainifold of dimension $2n-1$ where $n \geq 1$. We have proved in class that

$$\begin{aligned}
\chi(M) &= \sum_{i=0}^{2n-1} (-1)^i \text{rank}\, H_i(M) \\
&= \sum_{i=0}^{2n-1} (-1)^i \dim_{\mathbb{Z}_2} H_i(M; \mathbb{Z}/2).
\end{aligned}$$

M is $\mathbb{Z}_2$-orientable, by Poincaré duality, we know that

$$H_{2n-1-i}(M; \mathbb{Z}/2) \cong H^i(M; \mathbb{Z}/2).$$

Moreover, by UCT and note that $\mathbb{Z}/2$ is a field, we have an isomorphism

$$H^i(M; \mathbb{Z}/2) \xrightarrow{\cong} \hom_{\mathbb{Z}/2}(H_i(M; \mathbb{Z}/2), \mathbb{Z}/2).$$

So we have

$$\dim_{\mathbb{Z}/2} H_i(M; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^i(M; \mathbb{Z}/2).$$

Combine these two together, and we have

$$\dim_{\mathbb{Z}/2} H_{2n-1-i}(M;\mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2).$$

Note that $(-1)^i + (-1)^{2n-1-i} = 0$, therefore, we have

$$\begin{aligned}
\chi(M) &= \sum_{i=0}^{2n-1} (-1)^i \dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2) \\
&= \sum_{i=0}^{n-1} [(-1)^i \dim_{\mathbb{Z}} /2H_i(M;\mathbb{Z}/2) + (-1)^{2n-1-i} \dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2)] \\
&= 0
\end{aligned}$$

(b) X has an open cover $U \cup V$ where $U \cong V$ is homotopy equivalent to M and $U \cap V$ is homotopy equivalent to ∂M. We have the Mayer-Vietoris sequence

$$\cdots \to H_k(\partial M) \to H_k(M) \oplus H_k(M) \to H_k(X) \to H_{k-1}(\partial M) \to \cdots$$

<u>Claim:</u> Suppose $n > 0$ and we have a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & 0 \arrow[dll] \\
A_n \arrow[r] & B_n \arrow[r] & C_n \arrow[dll] \\
A_{n-1} \arrow[r] & \cdots \arrow[r] & C_1 \arrow[dll] \\
A_0 \arrow[r] & B_0 \arrow[r] & C_0 \arrow[dll] \\
0 & &
\end{tikzcd}$$

Here A_i, B_i and C_i are finitely generated abelian groups. Then

$$\chi(B) = \chi(A) + \chi(C).$$

<u>Proof:</u> We need to prove that

$$\sum_{i=0}^{n}(-1)^i \text{rank}\, A_i - \sum_{j=0}^{n}(-1)^j \text{rank}\, B_j + \sum_{k=0}^{n}(-1)^k \text{rank}\, C_k = \sum_{i=0}^{n}(-1)^i(\text{rank}\, A_i - \text{rank}\, B_i + \text{rank}\, C_i) = 0.$$

This is equivalent as proving the following fact: Given an exact sequence of finitely generated abelian groups

$$0 \to X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} X_0 \xrightarrow{f_0} 0$$

We have

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \text{rank}\, X_i = 0.$$

By the first isomorphism theorem, for any $0 \le i \le n$, we have

$$X_i / \ker f_i \cong \operatorname{Im} f_i.$$

So by exactness,

$$\begin{aligned} \text{rank}\, X_i &= \text{rank ker}\, f_i + \text{rank}\, \text{Im}\, f_i \\ &= \text{rank ker}\, f_i + \text{rank}\, \text{Im}\, f_{i-1}. \end{aligned}$$

Thus,

$$\begin{aligned} \chi(X) = &\sum_{i=0}^{n} (-1)^i \text{rank}\, X_i \\ = &\quad \text{rank ker}\, f_0 \\ &- \text{rank ker}\, f_0 - \text{rank ker}\, f_1 \\ &+ \text{rank ker}\, f_1 + \text{rank ker}\, f_2 \\ &\cdots \\ &+ (-1)^n \text{rank ker}\, f_{n-1} + (-1)^n \text{rank ker}\, f_n \\ =&(-1)^n \text{rank ker}\, f_n. \end{aligned}$$

Note that $f_n : X_n \to X_{n-1}$ is injective because $X_{n+1} = 0$. This proves that $\chi(X) = 0$. ■

By the claim, we know that

$$\chi(X) + \chi(\partial M) = 2\chi(M)$$

This implies $\chi(X) \equiv \chi(\partial M) \pmod 2$. And note that here X is a closed $(2d+1)$-dimensional manifold, so $\chi(X) = 0$ from what we have proved in (a). So $\chi(\partial M)$ must be even.

Problem 3

Suppose that there is a fiber bundle $p : X \xrightarrow{p} S^8$ with fiber S^3.

(a) Prove that X is an orientable manifold.

(b) Prove that $H_*(X)$ is isomorphic to $H_*(S^3 \times S^8)$.

Solution:

(a) The fiber bundle $S^3 \to X \xrightarrow{p} S^8$ implies that X is a 11-dimensional manifold and induces a long exact sequence in homotopy groups

$$\cdots \to \pi_1(S^3) \to \pi_1(X) \to \pi_1(S^8) \to \cdots$$

We know that $\pi_1(S^3) \cong \pi_1(S^*) \cong \{*\}$ is trivial. This implies $\pi_1(X) = \{*\}$ is trivial, so X does not have degree 2 connected coverings, namely the orientation covering $\tilde{X} \to X$ is a trivial covering, so X is orientable.

(b) Let D_+ and D_- be the open upper and lower half hemispheres of S^8. Let $U := p^{-1}(D_+)$ and $V := p^{-1}(D_-)$. $U \cup V$ is an open cover of X. Note that $p^{-1}(D_+) \xrightarrow{p} D_+$ is a subbundle of $X \xrightarrow{p} S^8$, since $D_+ \cong \mathbb{R}^8$ is contracitble, $U \xrightarrow{p} D_+$ is isomorphic to the trivial bundle $S^3 \times D_+ \to D_+$. The space U is homotopy equivalent to S^3. Same for V. Moreover, $p^{-1}(D_+ \cap D_-) \to D_+ \cap D_-$ is a subbundle of $p^{-1}(D_+) \to D_+$, which is isomorphic to a trivial bundle, so it is also isomorphic to a trivial bundle. Thus, we have $p^{-1}(D_+ \cap D_-)$ is homotopy equivalent to $S^7 \times S^3$ as $D_+ \cap D_-$ is homotopy equivalent to $S^7 \subseteq S^8$. Consider the Mayer-Vietoris sequence given by the cover $p^{-1}(D_+) \cup p^{-1}(D_-)$ on X:

$$\cdots \to H_k(S^7 \times S^3) \to H_k(S^3) \oplus H_k(S^3) \to H_k(X) \to H_{k-1}(S^7 \times S^3) \to \cdots$$

For $k \geq 4$, $H_k(S^3) = 0$. So $H_k(X) \cong H_{k-1}(S^3 \times S^7)$ for $k \geq 5$. Namely, $H_8(X) \cong H_7(S^7 \times S^3) \cong \mathbb{Z}$ and $H_{11}(X) \cong H_{10}(S^7 \times S^3) \cong \mathbb{Z}$, else $H_k(X) = 0$ for $k \geq 5$. In addition, by the same argument, $H_1(X) = H_2(X) = 0$ since $H_1(S^3) = H_2(S^3) = 0$, and $H_0(X) \cong \mathbb{Z}$ because X is connected. We need to determine $H_3(X)$ and $H_4(X)$ from the following exact sequence:

$$0 \to H_4(X) \to H_3(S^3 \times S^7) \to H_3(S^3) \oplus H_3(S^3) \to H_3(X) \to 0.$$

That is

$$0 \to H_4(X) \to \mathbb{Z} \to \mathbb{Z}^2 \to H_3(X) \to 0.$$

By exactness, $H_4(X) \to \mathbb{Z}$ is injective, so $H_4(X) = 0$ or $H_4(X) \cong \mathbb{Z}$. By Poincaré duality, $H_4(X)$ and $H_7(X)$ has the same rank, and since $H_7(X) = 0$, $H_4(X) = 0$. $H_3(X)$ is the cokernel of an injective map $\mathbb{Z} \to \mathbb{Z}^2$. By Poincaré duality, $\operatorname{rank} H_3(X) = \operatorname{rank} H_8(X) = 1$. By UCT,

$$H^8(X) \cong \hom(H_8(X), \mathbb{Z}) \oplus \operatorname{Ext}^1(H_7(X), \mathbb{Z}) \cong \mathbb{Z}$$

does not have any torsion. So by Poincaré duality, $H_3(X) \cong H^8(X)$ also does not have torsion. This implies $H_3(X) \cong \mathbb{Z}$. We can summarize that $H_*(X) = 0$ except

$$H_0(X) \cong H_3(X) \cong H_8(X) \cong H_{11}(X) \cong \mathbb{Z}.$$

This means $H_*(X)$ is isomorphic to $H_*(S^3 \times S^8)$ for all $*$.

Problem 4
Compute the cohomology ring of $\mathbb{R}P^4 \vee S^5$ with $\mathbb{Z}/2$-coefficients. Then use this to prove that $\mathbb{R}P^4 \vee S^5$ is not homotopy equivalent to a compact manifold.

Solution: We know that $H^*(\mathbb{R}P^4; \mathbb{Z}/2) = (\mathbb{Z}/2)[x]/(x^5)$ and $H^*(S^5; \mathbb{Z}/2) = (\mathbb{Z}/2)[y]/(y^2)$ (y is in degree 5). Since

$$\tilde{H}^*(\mathbb{R}P^4 \vee S^5; \mathbb{Z}/2) \cong \tilde{H}^*(\mathbb{R}P^4; \mathbb{Z}/2) \oplus \tilde{H}^*(S^5; \mathbb{Z}/2)$$

and $\mathbb{R}P^4 \vee S^5$ is still connected, we know that

$$H^*(\mathbb{R}P^4 \vee S^5; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x, y]/(x^5, y^2, xy)$$

where x is in degree 1 and y is in degree 5. Suppose $\mathbb{R}P^4 \vee S^5$ is homotopy equivalent to a compact manifold, then it is $\mathbb{Z}/2$-orientable. By Poincaré duality, the pairing

$$H^1(\mathbb{R}P^4 \vee S^5; \mathbb{Z}/2) \otimes H^4(\mathbb{R}P^4 \vee S^5; \mathbb{Z}/2) \xrightarrow{\cup} H^5(\mathbb{R}P^4 \vee S^5; \mathbb{Z}/2) \cong \mathbb{Z}/2$$

is a perfect pairing. Here $H^1(\mathbb{R}P^4 \vee S^5; \mathbb{Z}/2)$ is generated by x and $H^4(\mathbb{R}P^4 \vee S^5; \mathbb{Z}/2)$ is generated by x^4, and $x \cup x^4 = x^5 = 0$. A contradiction.

Problem 5
Suppose that X is a compact, orientable n-manifold and that $S^n \to X$ is a map of positive degree. Prove that $H_*(X; \mathbb{Q}) \cong H_*(S^n; \mathbb{Q})$.

Solution: Given a map $f : S^n \to X$ of positive degree, we prove that the induced map

$$f^* : H^k(X; \mathbb{Q}) \to H^k(S^n; \mathbb{Q})$$

is an injective map between $\mathbb{Q}$-vector space for $0 \leq k \leq n$. Suppose $a \in \ker f^*$. If $a \neq 0$ in $H^k(X; \mathbb{Q})$, then by Poincaré duality, there exists $a' \in H^{n-k}(X; \mathbb{Q})$ such that $a \cup a' = \widehat{[X]}$ where $\widehat{[X]}$ is the cohomological fundamantal class of X. Then we have

$$(\deg f)\widehat{[S^n]} = f^*(\widehat{[X]}) = f^*(a \cup a') = f^*(a) \cup f^*(a') = 0.$$

Here $\deg f > 0$. We have a contradiction. So $a = 0$. This proves that f^* is injective for all k. Since $H^k(S^n; \mathbb{Q}) = 0$ for $1 \leq k \leq n-1$, we know that $H^k(X; \mathbb{Q}) = 0$ for $1 \leq k \leq n-1$. Moreover, X being connected and orientable implies that $H^0(X; \mathbb{Q}) \cong H^n(X; \mathbb{Q}) \cong \mathbb{Q}$. By Poincaré duality, we have

$$H_*(S^n; \mathbb{Q}) \cong H_*(X; \mathbb{Q}).$$

Problem 6
Find the mistake in the following "proof" that $0 = 1$:

Let $A : S^2 \to S^2$ be the antipodal map, and $p : S^2 \to \mathbb{R}P^2$ the projection. Consider the diagram

$$\begin{array}{ccc} \pi_2(S^2) & \xrightarrow{A_*} & \pi_2(S^2) \\ {\scriptstyle h_2}\downarrow & & {\scriptstyle h_2}\downarrow \\ H_2(S^2) & \xrightarrow{A_*} & H_2(S^2) \end{array}$$

where h_2 is the Hurewicz map. We know that h_2 is an isomorphism, and we know that the lower map A_* is multiplication by $(-1)^3$. So it follows that the upper A_* is also multiplication by (-1).

Next consider the diagram

$$\begin{tikzcd} \pi_2(S^2) \arrow[rr, "A_*"] \arrow[dr, "p_*"'] & & \pi_2(S^2) \arrow[dl, "p_*"] \\ & \pi_2(\mathbb{R}P^2) & \end{tikzcd}$$

This commutes because of functoriality, since $p \circ A = p$. We know from the long exact sequence for the fibration $p : S^2 \to \mathbb{R}P^2$ that p_* is an isomorphism. Let $g \in \pi_2(S^2)$ be a generator. Then we have

$$p_*(g) = p_*(A_*(g)) = p_*(-g) = -p_*(g).$$

But $\pi_2(\mathbb{R}P^2) \cong \pi_2(S^2) \cong \mathbb{Z}$, and so the above equation implies $p_*(g) = 0$. Therefore p_* is the zero map. But we have already said that p_* is an isomorphism, therefore $\pi_2(\mathbb{R}P^2) = 0$. Since we have also said that $\pi_2(\mathbb{R}P^2) \cong \mathbb{Z}$, it must be that $\mathbb{Z} \cong 0$. So $\mathbb{Z}$ has only one element and, in particular, $0 = 1$.

Solution: Choose a point $x_0 \in S^2$ as the base point. The commutative triangle between pointed space is

$$\begin{tikzcd} (S^2, x_0) \arrow[rr, "A"] \arrow[dr, "p_1"'] & & (S^2, -x_0) \arrow[dl, "p_2"] \\ & (\mathbb{R}P^2, \tilde{x_0}) & \end{tikzcd}$$

Note here p_1 and p_2 are different as pointed maps. This induces a commutative triangle in homotopy groups

$$\begin{tikzcd} \pi_2(S^2, x_0) \arrow[rr, "A_*"] \arrow[dr, "p_{1,*}"'] & & \pi_2(S^2, -x_0) \arrow[dl, "p_{2,*}"] \\ & \pi_2(\mathbb{R}P^2, \tilde{x_0}) & \end{tikzcd}$$

Let $g \in \pi_2(S^2, x_0)$ be the generator. We have

$$p_{1,*}(g) = p_{2,*}(-g).$$

Here $p_{1,*}$ and $p_{2,*}$ are different isomorphisms as we choose different base point for S^2.