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Course: MATH 649 - Abstract Algebra
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Homework - Week 5
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Problem 13.3.5
The 5th cyclotomic field $\mathbb{Q}(\zeta_5)$ contains $\sqrt{5}$.

Solution: Let ζ_5 be a 5th primitive root of $x^5 - 1$. The cyclotomic field $\mathbb{Q}(\zeta_5)$ contains all 5 roots: $1, \zeta^5, \zeta_5^2$

Problem 13.3.7
If p is a prime then

$$\Phi_{p^n}(x) = 1 + x^{p^{n-1}} + x^{2p^{n-1}} + \cdots + x^{(p-1)p^{n-1}}.$$

Solution:

Problem 13.3.9
$\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic extension of $\mathbb{Q}$.

Solution:

Problem 13.5.2
Let $\mathbb{K}/\mathbb{k}$ be a field extension. If $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ is algebraically independent over $\mathbb{k}$, and $\alpha \notin \mathbb{k}$ is the element of $k(\alpha_1, \ldots, \alpha_n)$, then α is transcendental over $\mathbb{k}$.

Solution:

Problem 13.5.4
If β is algebraic over $\mathbb{k}(\alpha)$ and β is transcendental over $\mathbb{k}$ then α is algebraic over $\mathbb{k}(\beta)$.

Solution:

Problem 13.5.19
Let $\mathbb{k} \subsetneq \mathbb{F} \subseteq \mathbb{k}(x)$ be field extensions, with x transcendental over $\mathbb{k}$. Then $\mathbb{k}(x)/\mathbb{F}$ is finite.

Solution:

Problem 13.6.5 (Newton's identities)
Let $x_1, \dots, x_n$ be variables, and define power sum symmetric functions

$$p_k = p_k(x_1, \dots, x_n) = x_1^k + \cdots + x_n^k \quad (k \in \mathbb{Z}_{>0}).$$

Prove the *Newton identities*:

$$ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i$$

where e_k are the elementary symmetric functions interpreted 1 if $k = 0$ and as 0 if $k > n$. Deduce that every elementary symmetric function e_k can be written down as a polynomial in $p_1, \dots, p_k$ with rational coefficients. Deduce that every symmetric polynomial can be written down as a polynomial in the power sum symmetric functions.

Solution: Let $x_1, \dots, x_n$ be variables, define the following polynomial

$$f(x) = (x - x_1)(x - x_2) \cdots (x - x_n).$$

Remove the parentheses, and we can rewrite $f(x)$ as

$$f(x) = x^n - e_1 x^{n-1} + \cdots + (-1)^{n-1} e_{n-1} x + (-1)^n e_n.$$

We prove Newton's identities in different cases.

(a) Suppose $k = n$.
We know that for $1 \leq i \leq n$, x_i is the root of f, so it satisfies the following equation.

$$x_i^n - e_1 x_i^{n-1} + \cdots + (-1)^{n-1} e_{n-1} x + (-1)^n e_n = 0. \tag{1}$$

Add all these equations from $i = 1$ to $i = n$, we obtain

$$p_n - e_1 p_{n-1} + \cdots + (-1)^{n-1} e_{n-1} p_1 + (-1)^n n e_n = 0.$$

This is the same as

$$\begin{aligned} (-1)^{n-1} n e_n &= \sum_{i=1}^{n} (-1)^{n-i} e_{n-i} p_i \\ n e_n &= \sum_{i=1}^{n} (-1)^{i-1} e_{n-i} p_i. \end{aligned}$$

(b) Suppose $k > n$.
Let $\alpha_1, \alpha_2, \dots, \alpha_{k-n}$ be a variable. Consider the polynomial

$$g(x) = f(x)(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_{k-n}).$$

In this case, the generic polynomial is e'_j and the power sum function is p'_j for $1 \leq j \leq k$.

From the result in (a), we have an equation

$$ke'_k = \sum_{i=1}^{k} (-1)^{i-1} e'_{k-i} p'_i.$$

Let $\alpha_1 = \alpha_2 = \cdots = \alpha_{k-n} = 0$. Then we have

$$\begin{aligned}
e'_j &= e_j, && \text{if } \ 1 \le j \le n, && (2)\\
e'_j &= 0, && \text{if } \ n+1 \le j \le k, && (3)\\
p'_j &= p_j, && \text{if } \ 1 \le j \le k. && (4)
\end{aligned}$$

Then the equation can be rewritten as

$$0 = ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i.$$

(c) Suppose $k < n$.
Consider the formal derivative $f'(x)$ of $f(x)$, which can be written in two forms:

$$\begin{aligned}
f'(x) &= \sum_{j=1}^{n} \frac{f(x)}{x - x_j},\\
f'(x) &= nx^{n-1} - (n-1)e_1 x^{n-2} + \cdots + (-1)^{n-1} e_{n-1}.
\end{aligned}$$

For $0 \le l \le n-1$, the coefficient in front of x^{n-1-l} is $(-1)^l (n-l) e_l$. For $1 \le j \le n$, $\frac{f(x)}{x-x_j}$ can be written as

$$\frac{f(x)}{x - x_j} = (x - x_1) \cdots (x - x_{j-1})(x - x_{j+1}) \cdots (x - x_n).$$

Remove the parentheses, and we obtain

$$\begin{aligned}
\frac{f(x)}{x - x_j} =& x^{n-1} + (-e_1 + x_j) x^{n-2} + (e_2 - e_1 x_j + x_j^2) x^{n-3}\\
&+ \cdots + ((-1)^l e_l + \sum_{m=1}^{l} (-1)^{l+m} e_{l-m} x_j^m) x^{n-l-1} + \cdots\\
&+ (-1)^{n-1} e_{n-1} + \sum_{m=1}^{n-1} (-1)^{m+n-1} e_{n-m-1} x_j^m.
\end{aligned}$$

Add $j = 1$ to $j = n$ together and we have

$$f'(x) = nx^{n-1} + \sum_{l=1}^{n-1} [(-1)^l n e_l + (\sum_{m=1}^{l} (-1)^{l+m} e_{l-m} p_m)] x^{n-l-1}$$

Comparing coefficients, and we have, for $1 \le l \le n-1$,

$$(-1)^l (n-l) e_l = (-1)^l n e_l + \sum_{m=1}^{l} (-1)^{l+m} e_{l-m} p_m.$$

This is equivalent to

$$le_l = \sum_{m=1}^{l} (-1)^{m-1} e_{l-m} p_m$$

for all $1 \leq l \leq n-1$.

We have proved Newton's identities for $k > 0$. We have

$$\begin{aligned} e_1 &= p_1, \\ 2e_2 &= e_1 p_1 - p_2, \\ 3e_3 &= e_2 p_1 - e_1 p_2 + p_3, \\ &\dots \end{aligned}$$

for all $k > 0$. From this, we can inductively write e_k as a polynomial of $p_1, \dots, p_k$ with rational coefficients. By Theorem 13.6.1, since every symmetric polynomial can be written down as a polynomial in symmetric functions, then it can also be written as a polynomial in power sum functions.