

Exercise 11.5.5

Let \mathbb{K}/\mathbb{k} be a Galois extension, and \mathbb{L}, \mathbb{M} be intermediate fields. Denote by $\mathbb{L} \vee \mathbb{M}$ the minimal subfield of \mathbb{K} containing \mathbb{L} and \mathbb{M} .

- (a) $(\mathbb{L} \cap \mathbb{M})^* = \langle \mathbb{L}^*, \mathbb{M}^* \rangle$.
- (b) $(\mathbb{L} \vee \mathbb{M})^* = \mathbb{L}^* \cap \mathbb{M}^*$.
- (c) Assume that \mathbb{L}/\mathbb{k} is normal. Then $\text{Gal}(\mathbb{L} \vee \mathbb{M}/\mathbb{M}) \cong \text{Gal}(\mathbb{L}/(\mathbb{L} \cap \mathbb{M}))$.

Solution:

- (a) We know that $L \cap M \subseteq L$, by the Galois correspondence, we have $L^* \subseteq (L \cap M)^*$. Similarly, we can see that $M^* \subseteq (L \cap M)^*$. Note that
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Exercise 11.5.6

Let \mathbb{K}/\mathbb{k} be a finite Galois extension and p be a prime number.

- (a) \mathbb{K} has an intermediate subfield \mathbb{L} such that $[\mathbb{K} : \mathbb{L}]$ is a prime power.
- (b) If \mathbb{L}_1 and \mathbb{L}_2 are intermediate subfields with $[\mathbb{K} : \mathbb{L}_1], [\mathbb{K} : \mathbb{L}_2]$ both p -powers, and $[\mathbb{L}_1 : \mathbb{k}], [\mathbb{L}_2 : \mathbb{k}]$ both prime to p , then \mathbb{L}_1 is \mathbb{k} -isomorphic to \mathbb{L}_2 .

Solution:

Exercise 11.5.7

Let $f \in \mathbb{k}[x]$, \mathbb{K}/\mathbb{k} be a splitting field for f over \mathbb{k} , and $G := \text{Gal}(\mathbb{K}/\mathbb{k})$.

1. G acts on the set of the roots of f .
2. G acts transitively if f is irreducible.
3. If f has no multiple roots and G acts transitively then f is irreducible.

Solution:

Exercise 11.6.2

Let \mathbb{k} be a field, $p(x)$ be an irreducible polynomial in $\mathbb{k}[x]$ of degree n , and let \mathbb{K} be a Galois extension of \mathbb{k} containing a root α of $p(x)$. Let $G = \text{Gal}(\mathbb{K}/\mathbb{k})$, and G_α be the set of all $\sigma \in G$ with $\sigma(\alpha) = \alpha$. Then:

- (a) $[G : G_\alpha] = n$;
- (b) $G_\alpha^* = \mathbb{k}(\alpha)$;
- (c) If G_α is normal in G then $p(x)$ splits in the fixed field of G_α .

Solution:

Exercise 11.6.3

Let $\mathbb{k}(\alpha)/\mathbb{k}$ be a field extension obtained by adjoining a root α of an irreducible separable polynomial $f \in \mathbb{k}[x]$. Then there exists an intermediate field $\mathbb{k} \subseteq \mathbb{F} \subseteq \mathbb{k}(\alpha)$ if and only if $\text{Gal}(f; \mathbb{k})$ is imprimitive (as a permutation group on the roots), in which case \mathbb{F} can be chosen so that $[\mathbb{F} : \mathbb{k}]$ is equal to the number of imprimitive blocks.

Solution:

Exercise 11.6.6

Find all subfields of the splitting field of $x^3 - 7$ over \mathbb{Q} . Which of the subfields are normal over \mathbb{Q} ?

Solution:

Exercise 11.6.7

Let \mathbb{K} be a splitting field for $x^4 + 6x^2 + 5$ over \mathbb{Q} . Find subfields of \mathbb{K} .

Solution: