Math 636 Homework #5
Due Friday, May 16

1. The spaces $S^2 \times S^3$ and $S^2 \vee S^3 \vee S^5$ have isomorphic homology groups. Use the cup product to prove that they are not homotopy equivalent.

2. Here is some practice with the relative groups we've been using in orientation theory. Try to be completely rigorous in this problem.

   (a) Let M be an n-dimensional manifold-with-boundary (boundary points have neighborhoods that look like $\{\underline{x} \in \mathbb{R}^n \,|\, x_n \geq 0\}$). If x is on the boundary of M, prove that $H_n(M, M-x) = 0$.

   (b) Let M be a compact, connected, orientable n-manifold. By an "Euclidean open disk" in M we will mean an open set $U \subseteq M$ that is contained in some Euclidean chart V and is homeomorphic to an open disk under some homeomorphism $V \cong \mathbb{R}^n$.

   If U is a Euclidean open disk in M, prove that $H_i(M-U) \to H_i(M)$ is an isomorphism when $i < n$ and prove that $H_n(M-U) = 0$. Also, if $A = \partial(M-U)$ prove that the connecting homomorphism $\partial\colon H_n(M-U, A) \to H_{n-1}(A)$ is an isomorphism. [Hint for some parts of this: Get $M - x$ into the picture somehow.]

3. Let M and N be compact, connected n-manifolds, $n \geq 2$. Prove the following:

   (a) If M and N are orientable, then so is $M\#N$.

   (b) If M and N are non-orientable, then so is $M\#N$.

   (c) What happens when M is orientable and N is not? Justify your answer.

   [Hint: For (a) and (b) use a conveniently chosen cofiber sequence involving $M\#N$. For (c), let U be a Euclidean open disk where M and N are being sewn together, and consider the long exact sequence for the pair $(M\#N, M-U)$.]

Recall that if $H^*(X)$ has no torsion then the Künneth Theorem gives an isomorphism of rings $H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$. For $a \in H^*(X)$ and $b \in H^*(Y)$, the map sends $a \otimes b$ to $\pi_1^*(a) \cup \pi_2^*(b)$. The latter expression is sometimes denoted $a \times b$ and called the "cross product" or "external cup product". Because of the isomorphism, it is common to mix the notations $a \times b$ and $a \otimes b$ in this case.

The following problem will use this isomorphism when $X = Y = S^n$.

4. Suppose that S^n has a continuous unital multiplication $\mu\colon S^n \times S^n \to S^n$. So there is a unit element $e \in S^n$ with the property that $\mu(e, x) = x = \mu(x, e)$ for all $x \in S^n$. Said differently, the following diagram is commutative:

$$\begin{tikzcd}
S^n \times \{*\} \arrow[dr, "id\times j"'] \arrow[drr, bend left, "id"] & & \\
& S^n \times S^n \arrow[r, "\mu"] & S^n \\
\{*\} \times S^n \arrow[ur, "j\times id"] \arrow[urr, bend right, "id"'] & &
\end{tikzcd}$$

   where $j\colon * \hookrightarrow S^n$ sends the point to e.

   (a) Let z be a generator for $H^n(S^n)$. Use the above diagram to prove that $\mu^*(z) = z \otimes 1 + 1 \otimes z$. [Hint: $\mu^*(z) = k_1(z \otimes 1) + k_2(1 \otimes z)$ for some $k_1, k_2 \in \mathbb{Z}$. Compute k_1 and k_2.]

(b) Use the fact that μ^* is a ring homomorphism, together with your knowledge of the ring structure on $H^*(S^n \times S^n)$, to conclude that n must be odd. (This involves being careful with some signs).

(c) Before Hamilton discovered the quaternions, he spent a long time searching for a multiplication $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ which was unital and had no zero-divisors (that is, $xy = 0 \Rightarrow (x = 0 \text{ or } y = 0)$). Use (b) to prove that no such multiplication exists, assuming that the identity element is nonzero. [Hamilton was actually looking for a bilinear multiplication, and that condition implies that $0 \cdot x = 0$ for all x—so 0 could never be the identity for such a multiplication].

5. The relative form of the Künneth Theorem is that there is a natural short exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(X,A) \otimes H_q(Y,B) \to H_n(X \times Y, X \times B \cup A \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(X,A), H_q(Y,B)) \to 0.$$

(a) Suppose that M is an n-manifold and N is a k-manifold, and that we are given local orientations $u_M \in H_n(M, M-x)$ and $u_N \in H_k(N, N-y)$, for some $x \in M$ and $y \in N$. Explain how to get an induced local orientation for $M \times N$ at (x,y).

(b) Explain how to get an interesting continuous map $\tilde{M} \times \tilde{N} \to \widetilde{M \times N}$, where $\tilde{M}$ is the space of pairs (m, μ_m) such that $m \in M$ and $\mu_m \in H_n(M, M-m)$ is a generator (and similarly for $\tilde{N}$, etc). How many points are in each fiber?

(c) Prove that if M and N are orientable then so is $M \times N$ (note that we are not assuming compactness here).

6. Consider the space $X = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$, made into a Δ-complex by taking a hexagon with outer boundary labelled clockwise *aabbcc*, putting a vertex at the center and drawing all lines (directed outward) to the vertices of the hexagon. 1-simplices on the hexagon are directed clockwise.

Recall that $H^0(X;\mathbb{Z}/2) = H^2(X;\mathbb{Z}/2) = \mathbb{Z}/2$ and $H^1(X;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^3$. The Universal Coefficient Theorem implies that the standard maps $\phi_i \colon H^i(X;\mathbb{Z}/2) \to \mathrm{Hom}(H_i(X;\mathbb{Z}/2), \mathbb{Z}/2)$ are isomorphisms.

When answering the following questions, it is best NOT to write down the entire simplicial chain complex for X and grind out cohomology groups that way.

(a) Write down explicit 1-cocycles α, β, and γ (with $\mathbb{Z}/2$ coefficients) that map to $\hat{a}$, $\hat{b}$, and $\hat{c}$ under ϕ.

(b) Given a 2-cochain Θ (with $\mathbb{Z}/2$ coefficients), how can one easily determine if Θ is a generator for $H^2(X;\mathbb{Z}/2)$? Of course you should explain your answer.

(c) Determine a class $u \in H^1(X;\mathbb{Z}/2)$ such that $\alpha \cup u$ is a generator for $H^2(X;\mathbb{Z}/2)$. Then do the same for β and γ.