Zhengdong Zhang
Email: zhengz@uoregon.edu
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Instructor: Professor Arkady Berenstein

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Problem 16.3.2
Compute $J(\mathbb{C}[x])$ and $J(\mathbb{C}[[x]])$.

Solution: We know that $\mathbb{C}[x]$ is a PID. For $f, g \in \mathbb{C}[x]$, we know that $(f) \supset (g)$ if and only if $f|g$. Since $\mathbb{C}$ is algebraically closed, every polynomial in $\mathbb{C}[x]$ can be written as a product of linear terms. So the maximal ideals in $\mathbb{C}[x]$ are of the form $(x-a)$ where $a \in \mathbb{C}$ is a complex number. So the Jacobson radical

$$J(\mathbb{C}[x]) = \bigcap_{a \in \mathbb{C}} (x-a).$$

Note that for $a \neq b \in \mathbb{C}$, the intersection

$$(x-a) \cap (x-b) = ((x-a)(x-b))$$

So $J(\mathbb{C}[x])$ is generated by the product of all linear terms $x-c$ where $c \in \mathbb{C}$. Because $\mathbb{C}$ has infinitely many elements, so this is impossible as every polynomial only has finitely many terms. So $J(\mathbb{C}[x]) = (0)$.

For $\mathbb{C}[[x]]$, we first prove the following:
Claim: Any proper ideal in $\mathbb{C}[[x]]$ must be of the form (x^p) for $p \geq 1$.
Proof: Write every element in $\mathbb{C}[[x]]$ as

$$t = \sum_{k=0}^{\infty} a_k x^k.$$

Suppose $a_0 \neq 0$. We are going to show that t is invertible in $\mathbb{C}[[x]]$. We define an element $s = \sum_{j=0}^{\infty} b_j x^j$ inductively as follows:

$$\begin{aligned} b_0 a_0 &= 1, \\ b_1 a_0 + b_0 a_1 &= 0, \\ b_2 a_0 + b_1 a_1 + b_0 a_2 &= 0, \\ &\cdots \end{aligned}$$

For each $j \geq 1$, we can obtain b_j by solving a linear equation

$$b_j a_0 + b_{j-1} a_1 + \cdots + b_0 a_j = 0.$$

This defines an element $s = \sum_{j=0}^{\infty} b_j x^j \in \mathbb{C}[[x]]$, and we have

$$\begin{aligned}
st &= (\sum_{k=0}^{\infty} a_k x^k)(\sum_{j=0}^{\infty} b_j x^j) \\
&= a_0 b_0 + (b_1 a_0 + b_0 a_1)x + \cdots + (b_j a_0 + b_{j-1} a_1 + \cdots + b_0 a_j)x^j + \cdots \\
&= 1.
\end{aligned}$$

This proves that for $a_0 \neq 0$, $m = \sum_{k=0}^{\infty} a_k x^k$ is a unit in $\mathbb{C}[[x]]$. Let $I \subset \mathbb{C}[[x]]$ be a proper ideal. I does not contain any unit, so for any $\sum_{k=0}^{\infty} a_k x^k \in I$, $a_0 = 0$. Define

$$p := \min \left\{ p \in \mathbb{Z}_{>0} \mid \sum_{k=0} a_k x^k \in I \quad \text{and} \quad a_p \neq 0 \right\}.$$

We know $p \geq 1$ since I cannot contain units. Note that x^p divides all the elements in I and by definition, there exists an element $\sum_{k=0}^{\infty} a_k x^k \in I$ such that $a_0 = \cdots = a_{p-1} = 0$ and $a_p \neq 0$, this element can be written as $x^p(a_p + a_{p+1}x + \cdots)$ where $a_p + a_{p+1}x + \cdots$ is invertible, so we have proved $I = (x^p)$. ■

By the claim, we know that any proper ideal in $\mathbb{C}[[x]]$ must be of the form (x^p) for $p \geq 1$. And $(x^p) \supset (x^q)$ if and only if $p \leq q$. So $\mathbb{C}[[x]]$ has only one maximal ideal (x), and the Jacobson radical $J(\mathbb{C}[[x]]) = (x)$.

Problem 16.3.3
True or false? $J(R_1 \times \cdots \times R_n) = J(R_1) \times \cdots \times J(R_n)$.

Solution: This is true. We only need to show that $J(R_1 \times R_2) = J(R_1) \times J(R_2)$ and obtain the rest by induction. Let $I \subset R_1 \times R_2$ be an ideal. Let $(a, b) \in I$ and $(r_1, r_2) \in R_1 \times R_2$, we know that $(r_1 a, r_2 b) \in I$. Consider two projections $\pi_1 : R_1 \times R_2 \to R_1$ and $\pi_2 : R_1 \times R_2 \to R_2$. The previous discussion tells us that $\pi_1(I)$ is an ideal in R_1 and $\pi_2(I)$ is an ideal in R_2. So I must be of the form

$$\{(a, b) \in R_1 \times R_2 \mid a \in I_1, b \in I_2\}$$

where I_1 is an ideal in R_1 and I_2 is an ideal in R_2. So the maximal ideals in $R_1 \times R_2$ can only be of the following two forms: $m_1 \times R_2$ or $R_1 \times m_2$, where m_1 is a maximal ideal in R_1 and m_2 is a maximal ideal in R_2. So the Jacobson radical

$$\begin{aligned}
J(R_1 \times R_2) &= (\bigcap_{m_1 \subset R_1} m_1 \times R_2) \bigcap (\bigcap_{m_2 \subset R_2} R_1 \times m_2) \\
&= \bigcap_{m_1 \subset R_1} \bigcap_{m_2 \subset R_2} m_1 \times m_2 \\
&= (\bigcap_{m_1 \subset R_1} m_1) \times (\bigcap_{m_2 \subset R_2} m_2) \\
&= J(R_1) \times J(R_2).
\end{aligned}$$

Problem 16.3.9
Prove that $J(R)$ contains no non-zero idempotent.

Solution: Suppose $e \in J(R)$ is a non-zero idempotent. By Proposition 16.3.7, we know there exists a left inverse $v \in R$ such that $v(1-e) = 1$. By Proposition 16.3.8, $1+e$ is a unit in R, so there exists $u \in R$ such that $u(1+e) = (1+e)u = 1$. Note that since e is an idempotent, $(1-e)(1+e) = 1 - e^2 = 1 - e$. So we have

$$1 = u(1+e) = uv(1-e)(1+e) = uv(1-e) = u.$$

This implies $1 + e = 1$, so $e = 1 \in J(R)$, which means $J(R) = R$. A contradiction.

Problem 16.3.14
True or false? If $r \in R$ is a nilpotent element then the left ideal of R generated by r is nilpotent.

Solution: This is false. Consider the matrix ring $M_2(\mathbb{R})$ over $\mathbb{R}$. $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ is a nilpotent element in $M_2(\mathbb{R})$ because

$$A^2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Consider the ideal I generated by A. Note that

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \in I.$$

<u>Claim:</u> For any $n \geq 1$, we have

$$B^n = \begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}$$

<u>Proof:</u> We prove this by indcution on n. $n = 1$ is obvious. For $n \geq 2$, suppose we have already know

$$B^{n-1} = \begin{pmatrix} 2^{n-2} & -2^{n-2} \\ -2^{n-2} & 2^{n-2} \end{pmatrix}.$$

Then

$$B^n = B \cdot B^{n-1} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2^{n-2} & -2^{n-2} \\ -2^{n-2} & 2^{n-2} \end{pmatrix} = \begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}.$$

We are done. ■

The claim implies that for any $n \geq 1$, $B^n \neq 0$. So the ideal I is not nilpotent.

Problem 16.3.18
Calculate the Jacobson radical of the ring $\mathbb{Z}/m\mathbb{Z}$.

Solution: Suppose $m = p_1^{n_1} \cdots p_k^{n_k}$ where $2 \leq p_1 \leq p_2 \leq \cdots \leq p_k$ are primes in $\mathbb{Z}$ and $n_1, n_2, \ldots, n_k \geq 1$ are positive integers. This decomposition is unique since $\mathbb{Z}$ is a UFD. By Chinese Remainder

Theorem, we have

$$\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/p_1^{n_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{n_k}\mathbb{Z}.$$

We have proved in Exercise 16.3.3 that

$$J(\mathbb{Z}/m\mathbb{Z}) = J(\mathbb{Z}/p_1^{n_1}\mathbb{Z}) \times \cdots \times J(\mathbb{Z}/p_k^{n_k}\mathbb{Z}).$$

Next, we are going to determine the Jacobson radical for the ring $\mathbb{Z}/p^n\mathbb{Z}$ where p is prime number and $n \geq 1$ is a positive integer.
<u>Claim:</u> $J(\mathbb{Z}/p^n\mathbb{Z}) = (p) = p\mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}/p^{n-1}\mathbb{Z}$.
<u>Proof:</u> $\mathbb{Z}/p^n\mathbb{Z}$ is commutative, so for any $pa \in \mathbb{Z}/p^n\mathbb{Z}$, we have

$$(pa)^n = p^n a^n = 0.$$

This means pa is nilpotent. By Corollary 16.3.16, $J(\mathbb{Z}/p^n\mathbb{Z})$ must contain all elements of the form pa, namely $(p) \subset J(\mathbb{Z}/p^n\mathbb{Z})$. Moreover, p being a prime number tells us that (p) is a maximal ideal in $\mathbb{Z}/p^n\mathbb{Z}$, so $J(\mathbb{Z}/p^n\mathbb{Z})$ as the intersection of all maximal ideals must be contained in (p). This proves that

$$J(\mathbb{Z}/p^n\mathbb{Z}) = (p) = p\mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}/p^{n-1}\mathbb{Z}.$$

■

From the claim, the Jacobson radical

$$J(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/p_1^{n_1-1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{n_k-1}\mathbb{Z}.$$

Problem 16.3.19
let A be the algebra of lower triangular $n \times n$ matrices over a field $\mathbb{F}$. Then $J(A)$ is the subset of matrices in A with all diagonal entries zero.

Solution: Let $I \subset A$ be the subset of all matrices in A with diagonal entries zero. For $M = (M_{ij})_{1 \leq i,j \leq n} \in A$ and $N = (N_{ij})_{1 \leq i,j \leq n} \in I$, for any $1 \leq i \leq n$, we have

$$(MN)_{ii} = \sum_{k=1}^{n} M_{ik} N_{ki}.$$

M being a lower triangular matrix implies that for $k > i$, $M_{ik} = 0$. Diagonal entries in N being zero implies that for $k \leq i$, $N_{ki} = 0$. Therefore, $(MN)_{ii} = 0$ for all $1 \leq i \leq n$. This proves $MN \in I$. So I is an ideal in A. Moreover, consider A/I. For $M_1, M_2 \in A$, $M_1 - M_2 \in I$ if and only if the diagonal entries of M_1 and M_2 are the same, namely

$$(M_1)_{ii} = (M_2)_{ii}$$

for all $1 \leq i \leq n$. This means A/I can be viewed as the following set of diagonal matrices

$$\left\{\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \mid a_1, a_2, \ldots, a_n \in \mathbb{F}\right\}.$$

The multiplication is given by

$$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}\begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{pmatrix} = \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{pmatrix}\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & & & \\ & a_2b_2 & & \\ & & \ddots & \\ & & & a_nb_n \end{pmatrix}$$

So we know that $A/I \cong \mathbb{F}^n$ is a field. This proves that I is a maximal ideal in A. If we can prove that every matrix in I is nilpotent, then by Proposition 16.3.16, we are done because $J(A)$ contains a maximal ideal I, then $J(A) = I$.

Claim: Every matrix $N \in I$ is nilpotent.

Proof: We prove this by induction on n. For $n = 2$, let $a, b \in \mathbb{F}$, we have

$$\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $n \geq 3$, suppose we have proved the claim for $(n-1) \times (n-1)$ matrices. Let $N, L \in I$ be two matrices. For $1 \leq i \leq n-1$, we know that the $(i+1, i)$-entry of NL can be calculated as

$$(NL)_{i+1,i} = \sum_{k=1}^{n} N_{i+1,k} L_{ki}.$$

$N, L \in I$ implies that for $i+1 \leq k$, $N_{i+1,k} = 0$ and for $k \leq i$, $L_{ki} = 0$. This proves that $(NL)_{i+1,i} = 0$ for $1 \leq i \leq n-1$. This means NL can be written as the following form

$$\begin{pmatrix} 0 & \cdots & 0 \\ & N' & \vdots \\ & & 0 \end{pmatrix}$$

where N' is a $(n-1) \times (n-1)$ lower triangular matrix with diagonal entries zero. Now choose n even and large enough, $N_1 N_2 \cdots N_n \in I$ can be written as

$$(N_1 N_2) \cdots (N_{n-1} N_n)$$

where each pair can be viewed as a $(n-1) \times (n-1)$ matrix. By our assumption, this must equal to 0. ■

Problem 16.3.22
Let A be a non-commutative finite dimensional algebra over $\mathbb{C}$ such that the left regular module ${}_AA$ has length two. What can you say about A?

Solution: A is finite dimensional over $\mathbb{C}$, so A must be Artinian. Suppose $J(A) = 0$, by Theorem 16.3.21, A is left semisimple. By Wedderburn-Artin Theorem, A is isomorphic to

$$M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_k}(\mathbb{C}).$$

Since A is non-commutative, so at least one of $n_1, n_2, \ldots, n_k$ is bigger or equal to 2. Without loss of generality, we can assume $n_1 \geq 2$. By Proposition 16.2.6, we know that $\mathbb{C}^2$ of the column vectors is the only simple $M_2(\mathbb{C})$-module up to isomorphism, so we have a composition series of length 2 for $M_2(\mathbb{C})$

$$M_2(\mathbb{C}) \supset \left\{ \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \right\} \supset 0.$$

Since A as a left regular module has length 2, so $A \cong M_2(\mathbb{C})$.

Now assume $J(A) \neq 0$. Consider the following composition series of A as a left regular module

$$A \supset B \supset 0.$$

A/B being a simple A-module tells us that B must be a maximal ideal in A. Suppose $m \subset A$ is a maximal ideal, then $m \cap B$ is an ideal in A and we have

$$B \supset (m \cap B) \supset 0.$$

Because B is simple as a A-module, so $m \cap B = 0$ or $m \cap B = B$. We have already assumed $J(A)$ as an intersection of all maximal ideals is not zero, so $m \cap B = B$. This means $m = B$. So B is the unique maximal ideal in A and we have $J(A) = B$. Note that $J(A)$ is a simple A-module, by Exercise 14.1.23, $J(A)$ is isomorphic to A/m for some maximal ideal in A, and there is only one unique maximal ideal $J(A)$, so we have $A/J(A) \cong J(A)$ as simple A-modules. We know that $J(A/J(A)) = 0$ by Proposition 16.3.6. $A/J(A)$ is artinian and thus semisimple. Note that $A/J(A)$ has length 1 and by Wedderburn-Artin theorem, we have

$$J(A) \cong A/J(A) \cong \mathbb{C}.$$

This means $A \cong \mathbb{C}^2$ and commutative. A contradiction. Therefore, the only possible case is that $A \cong M_2(\mathbb{C})$ with $J(A) = 0$.

Problem 16.3.23
True or false? An artinian ring has a finite number of irreducible modules up to isomorphism.

Solution: This is true. Suppose R is an Artinian ring. We have proved in Exercise 14.1.23 that any simple R-module is isomorphic to R/I where I is a maximal left ideal in R. So we only need to that R has finitely many maximal left ideals up to isomorphism. Hopkins-Levitzki Theorem tells us that R is also Noetherian. So R as a left regular module has finite length. For every maximal ideal $m \subset R$, we know that R/m is a field, so there exists a Jordan-Hölder series

$$R = J_0 \supset m \supset m_1 \supset \cdots m_n = 0.$$

By Jordan-Hölder Theorem, R/m must be one of the Jordan Hölder factors, and finite length implies there only exists finitely many Jordan-Hölder factors up to isomorphism. So we only have finitely many

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Problem 16.3.26
True or false? If R is an artinian ring having no non-zero nilpotent elements then R is a direct sum of division rings.

Solution: This is true. R is artinian, by Lemma 16.3.17, $J(R)$ must be nilpotent. But R does not contain any non-zero nilpotent element, so $J(R) = 0$. By Theorem 16.3.21, R is left semisimple, By Wedderburn-Artin Theorem, R is a direct sum of matrix rings

$$M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k).$$

Suppose one of $n_1, \dots, n_k \geq 2$, without loss of generality $n_1 \geq 2$. We can pick an element $A \in M_{n_1}(D_1)$ with only one nonzero entry at the left bottom corner, namely $(n, 1)$-entry. By calculation, we have $A^2 = 0$. This contradicts that R does not have non-zero nilpotents, so $n_1 = \cdots = n_k = 1$. Thus, R is a direct sum of division rings.

Problem 16.3.29
Let A be a finite dimensional algebra over $\mathbb{C}$. For each $a \in A$, consider the linear operator $L_a : A \to A$, $b \mapsto ab$. For all $a, b \in A$, define $(a|b) := \text{tr } (L_a L_b)$, the trace of the linear operator $L_a L_b$.

(1) $(-|-)$ is a symmetric bilinear form on A.

(2) The radical $\text{Rad}(-|-) := \{a \in A \mid (a, b) = 0 \text{ for all } b \in A\}$ is an ideal in A.

(3) $\text{Rad}(-|-) = J(A)$.

Solution:

(1) Let $c_1, c_2 \in \mathbb{C}$ and $a_1, a_2 \in A$. For any $b \in A$, we have $(c_1a_1 + c_2a_2)b = c_1a_1b + c_2a_2b$. This proves that for any $a \in A$,

$$\begin{aligned} L_a : A &\to A, \\ b &\mapsto ab. \end{aligned}$$

is a $\mathbb{C}$-linear operator. Suppose A is a n-dimensional algebra over $\mathbb{C}$. The operator L_a can be represented as a $n \times n$ matrix with entries in $\mathbb{C}$.

<u>Claim:</u> For any$a, b \in A$, we have $\text{tr } (L_a L_b) = \text{tr } (L_b L_a)$ where $\text{tr } (L_a L_b)$ is the trace of the matrix product $L_a L_b$.

Proof: By the previous discussion, we know that L_a can be written as a $n\times n$ matrix $(a_{ij})_{1\leq i,j\leq n}$. Similarly, $L_b = (b_{ij})_{1\leq i,j\leq n}$. Then the trace can be calculated as

$$\begin{aligned}\text{tr }(L_aL_b) &= \text{tr }(\sum_{k=1}^{n} a_{ik}b_{kj})\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki}\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ki}a_{ik}\\ &= \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik}\\ &= \text{tr }(L_bL_a)\end{aligned}$$

■

For any $a_1, a_2, b \in A$ and $c_1, c_2 \in \mathbb{C}$, by linearity of L_a, we have

$$L_{c_1a_1+c_2a_2} = c_1L_{a_1} + c_2L_{a_2}.$$

Note that the trace of a matrix is also a $\mathbb{C}$-linear operator, so we have

$$(c_1a_1 + c_2a_2|b) = \text{tr }(L_{c_1a_1+c_2a_2}L_b) = c_1(a_1|b) + c_2(a_2|b).$$

From the above claim, we know that $(-|-)$ is also symmetric, so $(-|-)$ is a symmetric bilinear form.

(2) For any $a \in \text{Rad}(-|-)$ and $b, c \in A$, let L_a, L_b, L_c be the corresponding $n\times n$ matrices over $\mathbb{C}$. Note that $L_cL_b = L_{cb}$ is also a linear operator related to the element $cb \in A$, so we have

$$0 = (a|cb) = \text{tr }(L_aL_{cb}) = \text{tr }(L_aL_cL_b) = \text{tr }(L_{ac}L_b) = (ac|b)$$

by definition of $\text{Rad}(-|-)$ and associativity of matrix multiplication. This proves that $ac \in \text{Rad}(-|-)$ and $\text{Rad}(-|-)$ is a right ideal of A. On the other hand, from the claim we know that

$$0 = (a|bc) = \text{tr }(L_a(L_{bc})) = \text{tr }((L_aL_b)L_c) = \text{tr }(L_c(L_aL_b)) = \text{tr }(L_{ca}L_b) = (ca|b).$$

This proves that $ca \in \text{Rad}(-|-)$ and $\text{Rad}(-|-)$ is a left ideal of A. Thus, $\text{Rad}(-|-)$ is a two sided ideal of A.

(3) Let $a \in J(A)$. We need to show that $(a|b) = 0$ for all $b \in A$. A being a finite dimensional algebra over $\mathbb{C}$ implies that A is artinian. By Lemma 16.3.17, $J(A)$ is a nilpotent ideal of A. So $ab \in J(A)$ is nilpotent and the matrix $L_{ab} = L_aL_b$ is nilpotent. There exists $m \in \mathbb{Z}_{>0}$ such that $(L_{ab})^m = 0$. The monomial x^m divides the minimal polymial of L_{ab} and since the minimal polynomial divides the characteristic polynomial, we can conclude that the characteristic polynomial of L_{ab} is x^n. All the eigenvalues of L_{ab} is zero and since the trace can be calculated as the sum of all eigenvalues, we know that $\text{tr }(L_{ab}) = \text{tr }(L_aL_b) = 0$. This

proves that $J(A) \subset \mathrm{Rad}(-|-)$.

On the other hand, let $a \in \mathrm{Rad}(-|-)$. For any $b \in A$ and $k \in \mathbb{Z}_{>0}$, we have

$$\mathrm{tr}\ ((L_{ab})^k) = \mathrm{tr}\ (L_a L_b \cdots L_a L_b) = (a|bab\cdots) = 0.$$

<u>Claim:</u> Let M be an $n \times n$ matrix over $\mathbb{C}$. If $\mathrm{tr}\ (M^k) = 0$ for $k = 1, 2, \ldots, n$. Then M is a nilpotent matrix.

<u>Proof:</u> Assume the opposite, M is not nilpotent. Note that over $\mathbb{C}$ M must have n eigenvalues. If M is not nilpotent, then M must have at least one nonzero eigenvalue (otherwise the characteristic polynomial of M will be x^n and M is nilpotent). Suppose $\lambda_1, \lambda_2, \ldots, \lambda_r$ are different nonzero eigenvalues of M with multiplicity $n_1, n_2, \ldots, n_r$ for $r \geq 1$. For any $1 \leq i \leq r$ and $1 \leq k \leq n$, note that

$$M^k v_i = M^{k-1}\lambda_i v_i = \lambda_i M^{k-1} v_i = \cdots = \lambda_i^k v_i.$$

where v_i is the corresponding eigenvector to the eigenvalue λ_i. This tells us that M^k has nonzero eigenvalue $\lambda_1^k, \ldots, \lambda_r^k$ with multiplicity $n_1, \ldots, n_r$. From the assumeption, we know that

$$0 = \mathrm{tr}\ (M) = \mathrm{tr}\ (M^2) = \cdots = \mathrm{tr}\ (M^n).$$

This gives us n equations

$$\begin{aligned} n_1\lambda_1 + n_2\lambda_2 + \cdots + n_r\lambda_r &= 0, \\ n_1\lambda_1^2 + n_2\lambda_2^2 + \cdots + n_r\lambda_r^2 &= 0, \\ &\cdots \\ n_1\lambda_1^n + n_2\lambda_2^n + \cdots + n_r\lambda_r^n &= 0. \end{aligned}$$

This can be rewritten in the matrix form

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^n & \lambda_2^n & \cdots & \lambda_r^n \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This can be viewed as a system of n equations and the coefficient matrix is denoted by N. Note that by Exercise 4.3.13 (Vandermonde determinant), we have

$$\det N = \lambda_1\lambda_2\cdots\lambda_r \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_r^{n-1} \end{pmatrix} = \lambda_1\lambda_2\cdots\lambda_r \prod_{1\leq i<j\leq n} (\lambda_i - \lambda_j).$$

Recall that we choose $\lambda_1, \ldots, \lambda_r$ to be nonzero and different from each other, so $\det N \neq 0$ and N is invertible. This implies the system of equations has a unqiue solution

$$n_1 = n_2 = \cdots = n_r = 0.$$

This contradicts our assumption M has nonzero eigenvalues. So M must be nilpotent. ■

From the claim we know that $(L_{ab})^n = 0$ as a matrix. Similarly for L_{ba}, we have

$$\begin{aligned}
\text{tr }(L_{ba}) &= \text{tr }(L_b L_a) = \text{tr }(L_a L_b) = 0, \\
\text{tr }((L_{ba})^2) &= \text{tr }((L_b L_a L_b) L_a) = \text{tr }(L_a (L_b L_a L_b)) = 0, \\
&\dots \\
\text{tr }((L_{ba})^n) &= \text{tr }((L_{ba})^{n-1} L_b L_a) = \text{tr }(L_a (L_{ba})^{n-1} L_b) = 0.
\end{aligned}$$

By the claim, $(L_{ba})^n = 0$. So $\text{Rad}(-|-)$ is a two-sided ideal where every element is nilpotent. For $I_n + L_{ab} \in M_n(\mathbb{C})$, we have

$$(I_n - L_{ab} + (L_a b)^2 - \cdots + (-1)^{n-1}(L_{ab})^{n-1})(I_n + L_{ab}) = I_n + (L_{ab})^n = I_n.$$

So $I_n + L_{ab}$ is a unit. Same for $I_n + L_{ba}$. By Propostion 16.3.8, since $J(A)$ is the largest two-sided ideal containing such elements, we have $ab \in J(A)$ and $ba \in J(A)$. We have proved $\text{Rad}(-|-) \subset J(A)$ and therefore, $J(A) = \text{Rad}(-|-)$.