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Course: MATH 636 - Algebraic Topology III
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Homework 4
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Problem 1
Compute all the homology groups for the spaces in parts (a) and (b) below, and use your calculations to show that

(a) $\mathbb{R}P^2 \times S^3$ and $\mathbb{R}P^3 \times S^2$ have isomorphic homotopy groups (in all dimensions), but non-isomorphic homology groups.

(b) $S^4 \times S^2$ and $\mathbb{C}P^3$ have isomorphic homology groups but non-isomorphic homotopy groups.

Solution:

(a) Let $X = \mathbb{R}P^2 \times S^3$ and $Y = \mathbb{R}P^3 \times S^2$. It is easy to see that both X and Y are path-connected, so $\pi_0(X) = \pi_0(Y) = *$. By direct calculation, we have

$$\pi_1(X) = \pi_1(\mathbb{R}P^2) \times \pi_1(S^3) = 0,$$
$$\pi_1(Y) = \pi_1(\mathbb{R}P^3) \times \pi_1(S^2) = 0.$$

This implies $\pi_1(X) \cong \pi_1(Y)$. Recall that for all $n \geq 2$, the universal covering space of $\mathbb{R}P^n$ is S^n. So the universal covering space of X and Y are both isomorphic to $S^2 \times S^3 \cong S^3 \times S^2$. The long exact sequence in homotopy groups tells us that

$$\pi_n(X) \cong \pi_n(Y) \cong \pi_n(S^3 \times S^2)$$

for all $n \geq 2$. Thus, we can conclude that X and Y have the same homotopy groups.

For the homology groups, note that the homology groups of S^3 and S^2 are all free. By Künneth theorem, we have

$$H_n(X) = \bigoplus_{p+q=n} H_p(\mathbb{R}P^2) \otimes H_q(S^3),$$
$$H_n(Y) = \bigoplus_{p+q=n} H_p(\mathbb{R}P^3) \otimes H_q(S^2).$$

The homology groups of each space is listed below: We can obtain the of X and Y by tensoring

	$H_*(\mathbb{R}P^2)$	$H_*(S^3)$
3	0	$\mathbb{Z}$
2	0	0
1	$\mathbb{Z}/2$	0
0	$\mathbb{Z}$	$\mathbb{Z}$

	$H_*(\mathbb{R}P^3)$	$H_*(S^2)$
3	$\mathbb{Z}$	0
2	0	$\mathbb{Z}$
1	$\mathbb{Z}/2$	0
0	$\mathbb{Z}$	$\mathbb{Z}$

at each degree, this gives us

$$H_i(X) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 3; \\ \mathbb{Z}/2, & \text{if } i = 1, 4; \\ 0, & \text{otherwise.} \end{cases} \qquad H_i(Y) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2, 5; \\ \mathbb{Z}/2, & \text{if } i = 1; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

From this we can see that X and Y have non-isomorphic homology groups.

(b) We know that $\mathbb{C}P^3$ has a cellular structure with one 0-cell, one 2-cell, one 4-cell, and one 6-cell. The boundary maps in the cellular chain complex are all zero, so $H_i(\mathbb{C}P^3) = \mathbb{Z}$ for $i = 0, 2, 4, 6$ and 0 otherwise. For the space $S^4 \times S^2$, use Künneth theorem and note that S^2 does not have torsion in homology, so $H_i(S^4 \times S^2) = \mathbb{Z}$ for $i = 0, 2, 4, 6$ and 0 otherwise. This shows that $S^4 \times S^2$ and $\mathbb{C}P^3$ have isomorphic homology groups.

Recall that we have a fibration $S^1 \to S^7 \to \mathbb{C}P^3$. This induces a long exact sequence in homotopy groups

$$\cdots \to \pi_3(S^1) \to \pi_3(S^7) \to \pi_3(\mathbb{C}P^3) \to \pi_2(S^1) \to \cdots$$

Note that $\pi_3(S^1) = \pi_2(S^1)$ is trivial. This implies that $\pi_3(\mathbb{C}P^3) \cong \pi_3(S^7) = \{1\}$ is also trivial. On the other hand, we know that

$$\pi_3(S^4 \times S^2) \cong \pi_3(S^4) \times \pi_3(S^2) = \mathbb{Z}.$$

This implies that $S^4 \times S^2$ and $\mathbb{C}P^3$ have non-isomorphic homotopy groups.

Problem 2

Let I_* be the chain complex concentrated in degree 0 and 1 with $I_1 = \mathbb{Z}\langle e\rangle$, $I_0 = \mathbb{Z}\langle a, b\rangle$, and $d(e) = b - a$. Note that this is the simplicial chain complex for Δ_1. Let C_* and D_* be chain complexes.

(a) Describe the chain complex $I_* \otimes C_*$ by giving the groups in each degree as well as the boundary maps.

(b) Let $F : I_* \otimes C_* \to D_*$ be a chain map. Define $f, g : C_* \to D_*$ by $f(x) = F(a \otimes x)$ and $g(x) = F(b \otimes x)$. Likewise, define $s_n : C_n \to D_{n+1}$ by $s_n : C_n \to D_{n+1}$ by $s_n(x) = F(e \otimes x)$. Prove that f and g are chain maps and the collection $\{s_n\}$ is a chain homotopy between f and g.

Solution:

(a) We denote both the boundary map in C_* by d_C. Consider the double complex $I_* \otimes C_*$ first.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
I_0 \otimes C_2 \arrow[d] & I_1 \otimes C_2 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
I_0 \otimes C_1 \arrow[d] & I_1 \otimes C_1 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
I_0 \otimes C_0 & I_1 \otimes C_0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

The vertical boundary map d_v is $id \otimes d_C$ and the horizontal boundary map d_h is $d \otimes id$. Let T_* be the total complex of this double complex, then in each degree we have

$$T_n = I_0 \otimes C_n \oplus I_1 \otimes C_{n-1}.$$

We know that $I_0 = \mathbb{Z}\langle a, b \rangle$, so $I_0 \times C_n$ is isomorphic to $(C_n)^2$ where the isomorphism is given by sending $a \otimes x$ to x and $b \otimes y$ to y for all $x, y \in C_n$. Similarly, $I_1 = \mathbb{Z}\langle e \rangle$, so $I_1 \otimes C_{n-1}$ is isomorphic to C_{n-1} where the isomorphism is given by sending $e \otimes z$ to z for all $z \in C_{n-1}$. The boundary map in the total complex is given by $d_t(x) = d_h(x) + (-1)^p d_v(x)$ for $x \in I_p \otimes C_q$. For $a \otimes x$, $b \otimes y$ in $I_0 \otimes C_n$ and $e \otimes z \in I_1 \otimes C_{n-1}$, we have

$$\begin{aligned}
d_t(a \otimes x) &= d_C(x), \\
d_t(b \otimes y) &= d_C(y), \\
d_t(e \otimes z) &= (b - a) \otimes x - e \otimes d_C(z).
\end{aligned}$$

(b) Write the boundary maps in C_* as d_C and boundary maps in D_* as d_D. For any $n \in \mathbb{Z}$, we know F is a chain map, so we have a commutative diagram

$$\begin{tikzcd}
I_0 \otimes C_n \arrow[r, "id\otimes d_C"] \arrow[d, "F"'] & I_0 \otimes C_{n-1} \arrow[d, "F"] \\
D_n \arrow[r, "d_D"'] & D_{n-1}
\end{tikzcd}$$

For any $x \in C_n$, we have

$$(d_D \circ F)(a \otimes x) = [F \circ (id \otimes d_C)(a \otimes x)] = F(a \otimes d_C(x)).$$

By definition this is equivalent to

$$(d_D \circ f)(x) = (f \circ d_C)(x).$$

Namely, we have a commutative diagram

$$\begin{tikzcd}
C_n \arrow[r, "d_C"] \arrow[d, "f"'] & C_{n-1} \arrow[d, "f"] \\
D_n \arrow[r, "d_D"'] & D_{n-1}
\end{tikzcd}$$

This proves f is a chain map. By a similar argument, g is also a chain map.

Next, to show that s_n defines a chain homotopy between f and g, we need to show for any n, there exists a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_C"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & C_n \arrow[r, "d_C"] \arrow[dl, "s_n"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & C_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d_D"'] & D_n \arrow[r, "d_D"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

For any $x \in C_n$, we have

$$g(x) - f(x) = F(b \otimes x) - F(a \otimes x) = F((b-a) \otimes x).$$

On the other hand, use the fact that F is a chain map, we have

$$\begin{aligned}
(d_D \circ s_n)(x) + (s_{n-1} \circ d_C)(x) &= (d_D \circ F)(e \otimes x) + F(e \otimes d_C(x)) \\
&= (F \circ d_C)(e \otimes x) + F(e \otimes d_C(x)) \\
&= F((b-a) \otimes x) - F(e \otimes d_C(x)) + F(e \otimes d_C(x)) \\
&= F((b-a) \otimes x).
\end{aligned}$$

This proves that

$$g - f = d_D \circ s_n + s_{n-1} \circ d_C.$$

The collection of s_n is a chain homotopy between f and g.

Problem 3

Let Y be the space obtained by starting with S^3 and attaching a 4-cell via a map of degree $5: Y = S^3 \cup_f e^4$ where $f : \partial(e^4) \to S^3$ has degree 5. Write down the cellular chain complex for $\mathbb{R}p^3 \otimes Y$; in particular, specify the rank of each chain group and identify the boundary maps. Compute the homotopy groups of specify the rank of each chain group and identify the boundary maps. Compute the homology groups of $\mathbb{R}P^3 \otimes Y$.

Solution: The space Y has a cellular chain complex

$$0 \to \mathbb{Z}\langle e^4 \rangle \xrightarrow{5} \mathbb{Z}\langle e^3 \rangle \to 0 \to 0 \to \mathbb{Z}\langle e^0 \rangle \to 0.$$

where e^i are cells in Y for $i = 0, 3, 4$. The real projective space $\mathbb{R}P^3$ has the following cellular chain complex

$$0 \to \mathbb{Z}\langle f^3 \rangle \xrightarrow{0} \mathbb{Z}\langle f^2 \rangle \xrightarrow{2} \mathbb{Z}\langle f^1 \rangle \xrightarrow{0} \mathbb{Z}\langle f^0 \rangle \to 0.$$

The tensor product of these two chain complex is the double complex

$$\begin{tikzcd}
\mathbb{Z}\langle f^0 \otimes e^4\rangle \arrow[d, "id\otimes 5"'] & \mathbb{Z}\langle f^1 \otimes e^4\rangle \arrow[l, "0"'] \arrow[d, "id\otimes 5"'] & \mathbb{Z}\langle f^2 \otimes e^4\rangle \arrow[l, "2\otimes id"'] \arrow[d, "id\otimes 5"'] & \mathbb{Z}\langle f^3 \otimes e^4\rangle \arrow[l, "0"'] \arrow[d, "id\otimes 5"'] \\
\mathbb{Z}\langle f^0 \otimes e^3\rangle \arrow[d] & \mathbb{Z}\langle f^1 \otimes e^3\rangle \arrow[l, "0"'] \arrow[d] & \mathbb{Z}\langle f^2 \otimes e^3\rangle \arrow[l, "2\otimes id"'] \arrow[d] & \mathbb{Z}\langle f^3 \otimes e^3\rangle \arrow[l, "0"'] \arrow[d] \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
\mathbb{Z}\langle f^0 \otimes e^0\rangle & \mathbb{Z}\langle f^1 \otimes e^0\rangle \arrow[l, "0"'] & \mathbb{Z}\langle f^2 \otimes e^0\rangle \arrow[l, "2\otimes id"'] & \mathbb{Z}\langle f^3 \otimes e^0\rangle \arrow[l, "0"']
\end{tikzcd}$$

Denote the total chain complex by (T_n, d_n). we have

$$T_n = \begin{cases} \mathbb{Z}\langle f^0 \otimes e^0\rangle \cong \mathbb{Z}, & \text{if } n = 0; \\ \mathbb{Z}\langle f^1 \otimes e^0\rangle \cong \mathbb{Z}, & \text{if } n = 1; \\ \mathbb{Z}\langle f^2 \otimes e^0\rangle \cong \mathbb{Z}, & \text{if } n = 2; \\ \mathbb{Z}\langle f^0 \otimes e^3, f^3 \otimes e^0\rangle \cong \mathbb{Z}^2, & \text{if } n = 3; \\ \mathbb{Z}\langle f^0 \otimes e^4, f^1 \otimes e^3\rangle \cong \mathbb{Z}^2, & \text{if } n = 4; \\ \mathbb{Z}\langle f^1 \otimes e^4, f^2 \otimes e^3\rangle \cong \mathbb{Z}^2, & \text{if } n = 5; \\ \mathbb{Z}\langle f^2 \otimes e^4, f^3 \otimes e^3\rangle \cong \mathbb{Z}^2, & \text{if } n = 6; \\ \mathbb{Z}\langle f^3 \otimes e^4\rangle \cong \mathbb{Z}, & \text{if } n = 7. \end{cases}$$

For $1 \le n \le 7$, the boundary map d_n is given by the formula

$$d_n(f^i \otimes e^j) = d(f^i) \otimes e^j + (-1)^i f^i \otimes d(e^j).$$

The boundary map d_n is given by the following table: To calculate the homology groups of $\mathbb{R}P^3 \times Y$,

i	d_i
1	0
2	2
3	$\begin{pmatrix} 0 & 0 \end{pmatrix}$
4	$\begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$
5	$\begin{pmatrix} 0 & 2 \\ -5 & 0 \end{pmatrix}$
6	$\begin{pmatrix} 2 & 0 \\ 5 & 0 \end{pmatrix}$
7	$\begin{pmatrix} 0 \\ -5 \end{pmatrix}$

we first write down the homology groups of $\mathbb{R}P^3$ and Y.

	$H_*(\mathbb{R}P^3)$	$H_*(Y)$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}/2$	0
2	0	0
3	$\mathbb{Z}$	$\mathbb{Z}/5$
4	0	0

We calculate their tensor products and Tor_1 respectively. Note that $\mathrm{Tor}_1(\mathbb{Z}/5, \mathbb{Z}/2) = 0$, so we do not have any terms coming from Tor_1. The homology groups of $\mathbb{R}P^3 \times Y$ can be summarized as follows

$$H_i(\mathbb{R}P^3 \times Y) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}/2, & \text{if } i = 1; \\ \mathbb{Z} \oplus \mathbb{Z}/5, & \text{if } i = 3; \\ \mathbb{Z}/5, & \text{if } i = 6; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 4
Compute both the homology and cohomology groups of the following spaces, both with integral and $\mathbb{Z}/2$ coefficients. Heck, do it with $\mathbb{Z}/3$ coefficients as well.

(a) $K \times K$, where K is the Klein bottle.

(b) $K \times T^g$, where T^g is the genus g torus and K is the Klein bottle.

(c) $K \times \mathbb{R}P^n$.

Solution:

(a) We can use UCT for homology to calculate the homology groups of K in different coefficients, and we summarized them as follows.

	$H_*(K)$	$H_*(K;\mathbb{Z}/2)$	$H_*(K:\mathbb{Z}/3)$
0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$
1	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/3$
2	0	$\mathbb{Z}/2$	0

From this we know that the tensor product is

$$H_p(K) \otimes H_q(K) = \begin{cases} \mathbb{Z}, & \text{if } p+q = 0; \\ \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2, & \text{if } p+q = 1; \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^3, & \text{if } p+q = 2; \\ 0, & \text{otherwise.} \end{cases}$$

The only non-trivial Tor_1 is given by $\mathrm{Tor}_1(H_1(K), H_1(K)) = \mathbb{Z}/2$. Thus, the homology groups of K is

$$H_i(K \times K; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0; \\ \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2, & \text{if } \ i = 1; \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^3, & \text{if } \ i = 2; \\ \mathbb{Z}/2, & \text{if } \ i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we use UCT for homology to calculate the homology groups with $\mathbb{Z}/2$ and $\mathbb{Z}/3$ coefficients.

$$H_i(K \times K; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } \ i = 0; \\ (\mathbb{Z}/2)^4, & \text{if } \ i = 1; \\ (\mathbb{Z}/2)^6, & \text{if } \ i = 2; \\ (\mathbb{Z}/2)^4, & \text{if } \ i = 3; \\ (\mathbb{Z}/2), & \text{if } \ i = 4; \\ 0, & \text{otherwise.} \end{cases} \qquad H_i(K \times K; \mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & \text{if } \ i = 0; \\ (\mathbb{Z}/3)^2, & \text{if } \ i = 1; \\ \mathbb{Z}/3, & \text{if } \ i = 2; \\ 0, & \text{otherwise.} \end{cases}$$

(b) The homology of K and T^g are as follows:

	$H_*(K)$	$H_*(T^g)$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}^{2g}$
2	0	$\mathbb{Z}$

Note that $H_*(T^g)$ are all free, so by Künneth theorem, we have

$$H_*(K \times T^g) \cong \bigoplus_{i+j=n} H_i(K) \otimes H_j(T^g).$$

Thus, we conclude $H_*(K \times T^g)$ as follows:

$$H_i(K \times T^g) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0; \\ \mathbb{Z}^{2g+1} \oplus \mathbb{Z}/2, & \text{if } \ i = 1; \\ \mathbb{Z}^{2g+1} \oplus (\mathbb{Z}/2)^{2g}, & \text{if } \ i = 2; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } \ i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we use UCT for homology to calculate different coefficients.

$$H_i(K\times T^g;\mathbb{Z}/2)=\begin{cases}\mathbb{Z}/2, & \text{if } i=0;\\ (\mathbb{Z}/2)^{2g+2}, & \text{if } i=1;\\ (\mathbb{Z}/2)^{4g+2}, & \text{if } i=2;\\ (\mathbb{Z}/2)^{2g+2}, & \text{if } i=3;\\ \mathbb{Z}/2, & \text{if } i=4;\\ 0, & \text{otherwise.}\end{cases} \quad H_i(K\times T^g;\mathbb{Z}/3)=\begin{cases}\mathbb{Z}/3, & \text{if } i=0;\\ (\mathbb{Z}/3)^{2g+1}, & \text{if } i=1;\\ (\mathbb{Z}/3)^{2g+1}, & \text{if } i=2;\\ \mathbb{Z}/3, & \text{if } i=3;\\ 0, & \text{otherwise.}\end{cases}$$

(c) $\mathbb{R}P^n$ has different homology groups when n is odd or even.

(1) Suppose n is even.
The homology groups of $\mathbb{R}P^n$ can be summarized as follows:

- $H_0(\mathbb{R}P^n)=\mathbb{Z}$.
- $H_i(\mathbb{R}P^n)=\mathbb{Z}/2$ if $i\geq 0$ and i is odd.
- 0 otherwise.

Use Künneth theorem, we can calculate the homology groups of $K\times\mathbb{R}P^n$:

$$H_i(K\times\mathbb{R}P^n)=\begin{cases}\mathbb{Z}, & \text{if } i=0;\\ \mathbb{Z}\oplus(\mathbb{Z}/2)^2, & \text{if } i=1;\\ (\mathbb{Z}/2)^2, & \text{if } 2\leq i\leq n;\\ \mathbb{Z}/2, & \text{if } i=n+1;\\ 0, & \text{otherwise.}\end{cases}$$

Next, we use UCT for homology to calculate the homology groups with other coefficients.

$$H_i(K\times\mathbb{R}P^n;\mathbb{Z}/2)=\begin{cases}\mathbb{Z}/2, & \text{if } i=0;\\ (\mathbb{Z}/2)^3, & \text{if } i=1;\\ (\mathbb{Z}/2)^4, & \text{if } 2\leq i\leq n;\\ (\mathbb{Z}/2)^3, & \text{if } i=n+1;\\ \mathbb{Z}/2, & \text{if } i=n+2;\\ 0, & \text{otherwise.}\end{cases}$$

$$H_i(K\times\mathbb{R}P^n;\mathbb{Z}/3)=\begin{cases}\mathbb{Z}/3, & \text{if } i=0,1;\\ 0, & \text{otherwise.}\end{cases}$$

(2) Suppose n is odd.
The homology groups of $\mathbb{R}P^n$ can be summarized as follows:

- $H_0(\mathbb{R}P^n)=H_n(\mathbb{R}P^n)=\mathbb{Z}$.
- $H_i(\mathbb{R}P^n)=\mathbb{Z}/2$ for $1\leq i\leq n-1$ if i is odd.
- 0 otherwise.

Use Künneth theorem, we can calculate the homology groups of $K \times \mathbb{R}P^n$:

$$H_i(K \times \mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & \text{if } \ i = 0; \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^2, & \text{if } \ i = 1; \\ (\mathbb{Z}/2)^2, & \text{if } \ 2 \leq i \leq n-1; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } \ i = n; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } \ i = n+1; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we use UCT for homology to calculate the homology groups with coefficients:

$$H_i(K \times \mathbb{R}P^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } \ i = 0; \\ (\mathbb{Z}/2)^3, & \text{if } \ i = 1; \\ (\mathbb{Z}/2)^4, & \text{if } \ 2 \leq i \leq n; \\ (\mathbb{Z}/2)^3, & \text{if } \ i = n+1; \\ \mathbb{Z}/2, & \text{if } \ i = n+2; \\ 0, & \text{otherwise.} \end{cases}$$

$$H_i(K \times \mathbb{R}P^n; \mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & \text{if } \ i = 0, 1, n, n+1; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 5

Let $f : A_* \to B_*$ be a map of chain complexes. We can regard this as forming a double complex

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & B_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & B_0 \end{array}$$

by putting zeros in all the "empty" spots. The total complex of this double complex is called the **algebraic mapping cone** of f, denoted Cf. Specifically, we set $(Cf)_n = A_{n-1} \oplus B_n$ and define $d : (Cf)_n \to (Cf)_{n-1}$ by

$$d(a, b) = (d_A(a), (-1)^{n-1} f(a) + d_B(b))$$

(a) Explain why there is a short exact sequence of chain complexes

$$0 \to B_* \hookrightarrow C(f) \to \Sigma A_* \to 0,$$

where ΣA_* is the evident chain complex having $(\Sigma A)_n = A_{n-1}$.

(b) The short exact sequence from (a) gives rise to a long exact sequence in homology groups. This has the form

$$\cdots \to H_i(B) \to H_i(Cf) \to H_i(\Sigma A) \xrightarrow{\partial} H_{i-1}(B) \to \cdots$$

Verify that the connecting homomorphism is really just the map $f_* : H_{i-1}(A) \to H_{i-1}(B)$, possibly up to a sign.

Solution:

(a) We need to prove that for any $n \geq 0$, we have the following commutative diagrams where the top row and bottom row is exact.

$$\begin{tikzcd}
0 \arrow[r] & B_n \arrow[r, "i_n"] \arrow[d, "d_B"'] & A_{n-1} \oplus B_n \arrow[r, "p_n"] \arrow[d, "d"] & A_{n-1} \arrow[r] \arrow[d, "d_A"] & 0 \\
0 \arrow[r] & B_{n-1} \arrow[r, "i_{n-1}"'] & A_{n-2} \oplus B_{n-1} \arrow[r, "p_{n-1}"'] & A_{n-2} \arrow[r] & 0
\end{tikzcd}$$

We choose $i_n : B_n \to A_{n-1} \oplus B_n$ as the inclusion $b \mapsto (0, b)$ and $p_n : A_{n-1} \oplus B_n \to A_{n-1}$ as the projection $(a, b) \mapsto a$. It is easy to see the top row and the bottom row is exact. For any $(a, b) \in A_{n-1} \oplus B_n$, we have

$$\begin{aligned}
(p_{n-1} \circ d)(a, b) &= p_{n-1}(d_A(a), (-1)^{n-1} f(a) + d_B(b)) \\
&= d_A(a) \\
&= (d_A \circ p_n)(a, b).
\end{aligned}$$

This proves the right square commutes. Moreover, for any $b \in B_n$, we have

$$\begin{aligned}
(d \circ i_n)(b) &= d(0, b) \\
&= (0, 0 + d_B(b)) \\
&= (i_{n-1} \circ d_B)(b).
\end{aligned}$$

This proves the left square commutes. Thus, we have a short exact sequence of chain complex

$$0 \to B_* \to (Cf)_* \to \Sigma A_* \to 0$$

where $(Cf)_n = A_{n-1} \oplus B_n$ and $(\Sigma A)_n = A_{n-1}$ for all n.

(b) By the snake lemma, we have a long exact sequence of homology groups deduced from the short exact sequence of chain complexes

$$0 \to B_* \to (Cf)_* \to \Sigma A_* \to 0.$$

Take $a \in \ker d_A \subseteq A_{n-1}$, we specify how to define $\partial a \in B_{n-1}$ from the snake lemma. We take the preimage $(a, 0) \in (Cf)_n$, send it to $d(a, 0) = (0, (-1)^{n-1} f(a)) \in (Cf)_{n-1}$, lastly we take the preimage $(-1)^{n-1} f(a) \in B_{n-1}$. Thus, we can conclude that the map

$$\begin{aligned} \partial : H_{n-1}(A) \to H_{n-1}(B), \\ [a] \mapsto [(-1)^{n-1} f(a)] \end{aligned}$$

This implies the connecting homomorphism is just the map induced by f

$$f_* : H_{n-1}(A) \to H_{n-1}(B)$$

up to a sign.

Problem 6

Let k be a field, and let $\mathcal{V}$ denote the category of vector spaces over k. Let I be any (small) category, and let $\mathcal{V}^I$ be the category whose objects are functors $I \to \mathcal{V}$ and whose morphisms are natural transformations. We call $\mathcal{V}^I$ the category of "I-shaped diagram in $\mathcal{V}$".

In this problem we will focus on the case where I is the pushout category

$$1 \leftarrow 0 \rightarrow 2$$

with three objects and two non-identity maps (as shown above). An object of $\mathcal{V}^I$ is then just a diagram of vector spaces $V_1 \leftarrow V_0 \rightarrow V_2$. A map from $[V_1 \leftarrow V_0 \rightarrow V_2]$ to $[W_1 \leftarrow W_0 \rightarrow W_2]$ is a commutative diagram

$$\begin{tikzcd} V_1 \arrow[d] & V_0 \arrow[l] \arrow[r] \arrow[d] & V_2 \arrow[d] \\ W_1 & W_0 \arrow[l] \arrow[r] & W_2 \end{tikzcd}$$

Let $P : \mathcal{V}^I \to \mathcal{V}$ be the pushout functor. P assigns each diagram its pushout.

(a) Let F_1, F_0 and F_2 be the three diagrams

$$F_1 : [k \leftarrow 0 \rightarrow 0] \quad F_0 = [k \leftarrow k \rightarrow k] \quad F_2 = [0 \leftarrow 0 \rightarrow k]$$

where in F_0 the maps are the identities. These diagrams are "free" in a certain sense: namely, if D is an object of $\mathcal{V}^I$ then morphisms $F_i \to D$ are in bijective correspondence with elements of D_i. Convince yourself that this is true.

(b) Let $D = [0 \leftarrow k \rightarrow 0]$ and $E = [0 \leftarrow k \rightarrow k]$, where in E the nontrivial map is the identity. Determine free resolutions for D and E.

(c) Apply the functor P to your resolution, to produce a chain complex of vector spaces. Compute the homology groups, which are the groups $(L_i P)(D)$ and $(L_i P)(E)$. These are the derived functor of the pushout functor P. Confirm in your example that $L_0 P = P$.

(d) Now let I be the category with one object 0 and one non-identity map $t : 0 \to 0$ such that $t^2 = id$. Objects of $\mathcal{V}^I$ are then pairs (W, t) consisting of a vector space W and an endomorphism $t : W \to W$ such that $t^2 = id$. In $\mathcal{V}^I$ the basic "free" object is $(k^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$; this can also be thought of as the vector space $k\langle g, tg \rangle$ where $t(tg) = g$. Let $P : \mathcal{V}^I \to \mathcal{V}$ be the colimit functor, sending an object (W, t) to $W/\{x - tx \mid x \in W\}$. Find the free resolution of the object (k, id) and compute $(L_iP)(k, id)$ for all $i \geq 0$.

Solution:

(a)

(b) Consider the following sequence

$$0 \to F_1 \oplus F_2 \to F_0 \to D \to 0.$$

Note that $F_1 \oplus F_2$ is the following diagram $[k \leftarrow 0 \to k]$ Namely the following diagram

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[l] \arrow[r] \arrow[d] & 0 \arrow[d] \\
F_1 \oplus F_2 \arrow[d] & & k \arrow[d, "id"] & 0 \arrow[l] \arrow[r] \arrow[d] & k \arrow[d, "id"'] \\
F_0 \arrow[d] & & k \arrow[d] & k \arrow[l, "id"'] \arrow[r, "id"] \arrow[d, "id"] & k \arrow[d] \\
D \arrow[d] & & 0 \arrow[d] & k \arrow[l] \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 & & 0 & 0 \arrow[l] \arrow[r] & 0
\end{tikzcd}$$

The vertical columns are exact because we only have isomorphisms. For E, consider the following sequence

$$0 \to F_1 \to F_0 \to E \to 0.$$

This can be written as the diagram

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[l] \arrow[r] \arrow[d] & 0 \arrow[d] \\
F_1 \arrow[d] & & k \arrow[d, "id"] & 0 \arrow[l] \arrow[r] \arrow[d] & 0 \arrow[d] \\
F_0 \arrow[d] & & k \arrow[d] & k \arrow[l, "id"'] \arrow[r, "id"] \arrow[d, "id"] & k \arrow[d, "id"] \\
E \arrow[d] & & 0 \arrow[d] & k \arrow[l] \arrow[r, "id"] \arrow[d] & k \arrow[d] \\
0 & & 0 & 0 \arrow[l] \arrow[r] & 0
\end{tikzcd}$$

This is a free resolution for E.

(c) Apply the pushout functor to the free resolution

$$0 \to F_1 \oplus F_2 \oplus F_0 \to 0.$$

The pushout of $F_1 \oplus F_2$ is k^2 and the pushout of F_0 is k. The map $F_1 \oplus F_2 \to F_0$ induces a map p between pushouts

$$\begin{tikzcd}
k \arrow[dd, "id"'] \arrow[drr, hook] & & 0 \arrow[ll] \arrow[rr] & & k \arrow[dd, "id"] \arrow[dll, hook'] \\
& & k^2 \arrow[dd, bend right, "p"'] & & \\
k \arrow[drr, "id"] & & k \arrow[ll] \arrow[rr] & & k \arrow[dll, "id"'] \\
& & k & &
\end{tikzcd}$$

We can see from the diagram that $p = (id, id)$, so p is surjective, so we have $(L_0P)(D) = P(D) = 0$ and $(L_1P)(D) = k$.

Apply the pushout functor to the free resolution

$$0 \to F_1 \to F_0 \to E \to 0.$$

The map $F_1 \to F_0$ induces a map between pushouts

$$\begin{tikzcd}
k \arrow[dd, "id"'] \arrow[drr, hook] & & 0 \arrow[ll] \arrow[rr] & & 0 \arrow[dd] \arrow[dll] \\
& & k \arrow[dd, bend right, "id"'] & & \\
k \arrow[drr, "id"] & & k \arrow[ll] \arrow[rr] & & k \arrow[dll, "id"'] \\
& & k & &
\end{tikzcd}$$

This map must be identity, so we have

$$(L_1P)(E) = (L_0P)(E) = P(E) = 0.$$

(d) Consider the following free resolution of $k \xrightarrow{id} k$:

$$\begin{tikzcd}[column sep=large]
\cdots \arrow[r, "\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}"'] & k^2 \arrow[loop above, "t"] \arrow[r, "\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}"'] & k^2 \arrow[loop above, "t"] \arrow[r, "\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}"'] & k^2 \arrow[loop above, "t"] \arrow[r, "\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}"'] & k^2 \arrow[loop above, "t"] \arrow[r, two heads, "\begin{pmatrix} 1 & 1 \end{pmatrix}"'] & k \arrow[loop above, "id"] \arrow[r] & 0
\end{tikzcd}$$

Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Both A and B are compatible with the map t because

$$At = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = tA,$$

$$Bt = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = Bt.$$

Moreover, the sequence is exact at every spot. Apply the colimit functor P, the map $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} : k^2 \to k^2$ will give you the following diagram

$$\begin{tikzcd}
k^2 \arrow[rr, "t"] \arrow[dr, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] \arrow[dd, "\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}"'] & & k^2 \arrow[dl, "\begin{pmatrix} 1 & 1 \end{pmatrix}"'] \arrow[dd, "\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}"] \\
& k \arrow[dd, bend right, "0"'] & \\
k^2 \arrow[rr, "t"'] \arrow[dr, "\begin{pmatrix} 1 & 1 \end{pmatrix}"'] & & k^2 \arrow[dl, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] \\
& k &
\end{tikzcd}$$

The map $P(A)$ is the zero map. Similarly, we apply the colimit functor P to the map $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : k^2 \to k^2$:

$$\begin{tikzcd}
k^2 \arrow[rr, "t"] \arrow[dr, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] \arrow[dd, "\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}"'] & & k^2 \arrow[dl, "\begin{pmatrix} 1 & 1 \end{pmatrix}"'] \arrow[dd, "\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}"] \\
& k \arrow[dd, bend right, "2"'] & \\
k^2 \arrow[rr, "t"'] \arrow[dr, "\begin{pmatrix} 1 & 1 \end{pmatrix}"'] & & k^2 \arrow[dl, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] \\
& k &
\end{tikzcd}$$

Thus, apply the colimit functor P to the free resolution, and we obtain a chain complex

$$k \to 0$$