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Course: MATH 649 - Abstract Algebra
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Homework - Week 2
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Exercise 11.1.2
True or false? $\text{Gal}(\Bbbk(x)/\Bbbk) = \{1\}$, where $\Bbbk(x)$ is the field of rational functions.

Solution: This is false. Suppose $\Bbbk$ has characteristic not 2 and consider the field homomorphism $\phi : k(x) \to k(x)$ by sending x to $2x$. ϕ has an inverse ϕ^{-1} sending x to $\frac{1}{2}x$. So ϕ is a field automorphism of $\mathbb{K}$ and fixes the base field $\Bbbk$.

Exercise 11.1.9
Let $\mathbb{K}/\Bbbk$ be a finite field extension. Then $|\text{Gal}(\mathbb{K}/\Bbbk)| \leq [\mathbb{K} : \Bbbk]$, and if $|\text{Gal}(\mathbb{K}/\Bbbk)| < [\mathbb{K} : \Bbbk]$, then the fixed subfield $\text{Gal}(\mathbb{K}/\Bbbk)^*$ properly contains $\Bbbk$.

Solution: Let $G = \text{Gal}(\mathbb{K}/\Bbbk)$ and $[\mathbb{K} : \Bbbk] = n$. We know G is a finite group and assume $|G| = m < \infty$. For ang $g \in G$, we could define a $\mathbb{K}$-linear map

$$\begin{aligned} \mathbb{K} \otimes_{\Bbbk} \mathbb{K} &\to \mathbb{K}, \\ x \otimes y &\mapsto g(x)y \end{aligned}$$

Consider the direct sum of all these distinct $\mathbb{K}$-linear maps $\phi = g_1 \oplus \cdots \oplus g_m : \mathbb{K} \otimes_{\Bbbk} \mathbb{K} \to \mathbb{K}^m$. We need to show that this map ϕ is surjective. Consider the $\mathbb{K}$-linear dual map

$$\phi^* : \text{hom}_{\mathbb{K}}(\mathbb{K}^m, \mathbb{K}) \to \text{hom}_{\mathbb{K}}(\mathbb{K} \otimes_{\Bbbk} \mathbb{K}, \mathbb{K})$$

Identify $\text{hom}_{\mathbb{K}}(\mathbb{K}^m, \mathbb{K}) \cong \mathbb{K}^m$ and by $\text{hom} - \otimes$ adjunction,

$$\begin{aligned} \text{hom}_{\mathbb{K}}(\mathbb{K} \otimes_{\Bbbk} \mathbb{K}, \mathbb{K}) &\cong \text{hom}_{\Bbbk}(\mathbb{K}, \text{hom}_{\mathbb{K}}(\mathbb{K}, \mathbb{K})) \\ &\cong \text{hom}_{\Bbbk}(\mathbb{K}, \mathbb{K}) \end{aligned}$$

Given a m-tuple $(z_1, \ldots, z_m) \in \mathbb{K}^m$, by definition $\phi^*(z_1, \ldots, z_m) \in \text{hom}_{\Bbbk}(\mathbb{K}, \mathbb{K})$ and it sends $x \in \mathbb{K}$ to $z_1 g_1(x) + \cdots + z_m g_m(x) \in \mathbb{K}$. Suppose $(z_1, \ldots, z_m) \in \ker \phi^*$, then $z_1 g_1 + \cdots + z_m g_m$ is the zero map and by Dedekind's Lemma, $z_1 = \cdots = z_m = 0$ since $g_1, \ldots, g_m$ are $\mathbb{K}$-linearly independent. This proves ϕ^* is injective. Thus, ϕ is surjective. so we have

$$\dim_{\mathbb{K}} \mathbb{K} \otimes_{\Bbbk} \mathbb{K} \geq m$$

Since $[\mathbb{K} : \Bbbk] = n$, so $\mathbb{K} \otimes_{\Bbbk} \mathbb{K}$ is a n-dimensional $\mathbb{K}$-vector space, therefore, $|\text{Gal}(\mathbb{K}/\Bbbk)| \leq [\mathbb{K} : \Bbbk]$.

Now suppose $|G| = |\text{Gal}(\mathbb{K}/\Bbbk)| = m < n = [\mathbb{K} : \Bbbk]$. Write $\mathbb{F} = \text{Gal}(\mathbb{K}/\Bbbk)^*$ as the fixed subfield under the automorphism group $\text{Gal}(\mathbb{K}/\Bbbk)$. By Theorem 11.1.6, we have

$$[\mathbb{K} : \mathbb{F}] = |G| = m < n = [\mathbb{K} : \Bbbk]$$

This implies that $\mathbb{F}$ strictly contains $\Bbbk$.

Exercise 11.2.8
Construct subfields of $\mathbb{C}$ which are splitting fields over $\mathbb{Q}$ for the polynomials

(a) $x^3 - 1$

(b) $x^4 - 5x^2 + 6$

(c) $x^6 - 8$

Find the degrees of thoses fields as extensions over $\mathbb{Q}$.

Solution:

(a) Let ξ be the 3rd primitive root of unit. Note that $x^3 - 1$ splits into

$$x^3 - 1 = (x-1)(x-\xi)(x-\xi^2)$$

over $Q(\xi)$. So $Q(\xi)$ is the splitting field and $[\mathbb{Q}(\xi) : \mathbb{Q}] = 2$ as $x^2 + x + 1$ is the irreducible minimal polynomial of ξ over $\mathbb{Q}$.

(b) Note that

$$x^4 - 5x^2 + 6 = (x^2 - 2)(x^2 - 3).$$

Both $x^2 - 2$ and $x^2 - 3$ are irreducible over $\mathbb{Q}$. The splitting field is $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ and we have

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 \cdot 2 = 4.$$

(c) Note that

$$x^6 - 8 = (x^2 - 2)(x^4 + 2x^2 + 4)$$

Let $\xi = e^{(\pi/3)i} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ satisfying $\xi^3 + 1 = 0$. By calculation, over the complex number $\mathbb{C}$, we have

$$x^4 + 2x^2 + 4 = (x - \sqrt{2}\xi)(x + \sqrt{2}\xi)(x - \sqrt{2}\xi^2)(x + \sqrt{2}\xi^2).$$

The minimal polynomial of ξ over $\mathbb{Q}$ is $x^2 - x + 1$. So the splitting field is $\mathbb{Q}(\sqrt{2}, \xi)$, and we have

$$[\mathbb{Q}(\sqrt{2}, \xi) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, \xi) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 \cdot 2 = 4.$$

Exercise 11.3.2
True or false? If $[\mathbb{K} : \Bbbk] = 2$, then $\mathbb{K}/\Bbbk$ is normal.

Solution: This is true. Let $\alpha \in \mathbb{K}$ and $\alpha \notin \Bbbk$. Denote the minial polynomial of α over $\Bbbk$ by f. We have $\deg f = 2$ because $[\mathbb{K} : \Bbbk] = 2$. Suppose f has two roots α and β. We need to show that $\beta \in \Bbbk(\alpha)$. Note that f can be written as

$$f(x) = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta.$$

We know $\alpha+\beta \in \Bbbk$ and $\alpha \in \Bbbk(\alpha)$, so $\beta \in \Bbbk(\alpha)$. This means $\mathbb{K}$ is the splitting field of f over $\Bbbk$, and by Theorem 11.3.3, $\mathbb{K}/\Bbbk$ is normal.

Exercise 11.3.6
Which of the following extensions are normal?

(a) $\mathbb{Q}(x)/\mathbb{Q}$

(b) $\mathbb{Q}(\sqrt{-5})/\mathbb{Q}$

(c) $\mathbb{Q}(\sqrt[7]{5})/\mathbb{Q}$

(d) $\mathbb{Q}(\sqrt{5}, \sqrt[7]{5})/\mathbb{Q}(\sqrt[7]{5})$

(e) $\mathbb{R}(\sqrt{-7})/\mathbb{R}$

Solution:

(a) This is not an algebraic extension so it is not normal.

(b) We know that $\sqrt{-5}$ has minimal polynomial x^2+5 over $\mathbb{Q}$. Note that both of the two roots $\sqrt{-5}$ and $-\sqrt{-5}$ are in the field $\mathbb{Q}(\sqrt{-5})$, so $\mathbb{Q}(\sqrt{-5})$ is the splitting field of the polynomial x^2+5. The field extension $\mathbb{Q}(\sqrt{-2})/\mathbb{Q}$ is normal.

(c) $\sqrt[7]{5}$ has minimal polynomial x^7-5 over $\mathbb{Q}$. Let ξ be the 7th primitive root of unity and $\sqrt[7]{5}\xi$ is a root of x^7+5 but $\xi \notin \mathbb{Q}(\sqrt[7]{5})$. So the field extension $\mathbb{Q}(\sqrt[7]{5})/\mathbb{Q}$ is not normal.

(d) Write $\mathbb{Q}(\sqrt{5}, \sqrt[7]{5}) = \mathbb{Q}(\sqrt[7]{5})(\sqrt{5})$. The minimal polynomial of $\sqrt{5}$ over $\mathbb{Q}(\sqrt[7]{5})$ is x^2-5, so

$$[\mathbb{Q}(\sqrt[7]{5})(\sqrt{5}) : \mathbb{Q}(\sqrt[7]{5})] = 2.$$

According to what we have proved in Exercise 11.3.2, this is a normal extension.

(e) The minimal polynomial of $\sqrt{-7}$ over $\mathbb{R}$ is x^2+7. So the field extension $\mathbb{R}(\sqrt{-7})/\mathbb{R}$ has degree 2 and by Exercise 11.3.2, we know that this is a normal extension.

Exercise (complexification/realification functors)

(a) Construct an isomorphism of rings $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.

(b) Let $\mathrm{Vect}_{\mathbb{R}}$ (resp. $\mathrm{Vect}_{\mathbb{C}}$) denote the category of vector spaces over $\mathbb{R}$ (resp. over $\mathbb{C}$). Consider the *realification* and *complexification* functors

$$R : \mathrm{Vect}_{\mathbb{C}} \to \mathrm{Vect}_{\mathbb{R}}, \quad C : \mathrm{Vect}_{\mathbb{R}} \to \mathrm{Vect}_{\mathbb{C}}$$

where R sends $V \in \mathrm{Vect}_{\mathbb{C}}$ to itself viewed as a real vector space (forgetting part of the structure), while C sends $V \in \mathrm{Vect}_{\mathbb{R}}$ to $V \otimes_{\mathbb{R}} \mathbb{C}$ with the complex structure $z \cdot (v \otimes x) =$

$v \otimes zx$, for $x, z \in \mathbb{C}$, $v \in V$. Construct an isomorphism of functors

$$CR(V) \cong V \oplus \overline{V}$$

for $V \in \mathrm{Vect}_{\mathbb{C}}$, where $\overline{V}$ denote the same space V with the conjugate complex structure, i.e., the multiplication by $z \in \mathbb{C}$ in $\overline{V}$ is given by $z * v := \bar{z} \cdot v$, where $z \mapsto \bar{z}$ is the complex conjugation.

Solution:

(a) We know that $\mathbb{C} \cong \mathbb{R}[x]/(x^2+1)$ as $\mathbb{R}$-algebras. By Chinese Remainder Theorem, we have

$$\begin{aligned}
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{R}[x]/(x^2+1) \otimes_{\mathbb{R}} \mathbb{C} \\
&\cong \mathbb{C}[x]/(x^2+1) \\
&\cong (\mathbb{C}[x]/(x+i) \oplus \mathbb{C}[x]/(x-i)) \\
&\cong \mathbb{C} \oplus \mathbb{C}.
\end{aligned}$$

We still need to prove that

<u>Claim:</u> $\mathbb{R}[x]/(x^2+1) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[x]/(x^2+1)$ is a ring isomorphism.

<u>Proof:</u> We know every element in $\mathbb{R}[x]/(x^2+1)$ can be written as $a+bx$ for some $a, b \in \mathbb{R}$. Consider the map

$$\begin{aligned}
\phi : \mathbb{R}[x]/(x^2+1) \otimes_{\mathbb{R}} \mathbb{C} &\to \mathbb{C}[x]/(x^2+1), \\
(a+bx) \otimes z &\mapsto az + bzx
\end{aligned}$$

This is a well-defined ring map because

$$\begin{aligned}
\phi((a+bx) \otimes z)\phi((c+dx) \otimes w) &= (az+bzx)(cw+dwx) \\
&= aczw + (bzcw + azdw)x + bzdwx^2 \\
&= (aczw - bzdw) + (bzcw + azdw)x \\
&= (ac-bd)zw + (bc+ad)zwx \\
&= \phi(((ac-bd) + (bc+ad)x) \otimes zw) \\
&= \phi(((a+bx) \otimes z)((c+dx) \otimes w))
\end{aligned}$$

It is easy to see ϕ is injective and as an $\mathbb{R}$-vector space, we have

$$\dim_{\mathbb{R}}(\mathbb{R}[x]/(x^2+1) \otimes_{\mathbb{R}} \mathbb{C}) = \dim_{\mathbb{R}} \mathbb{C}[x]/(x^2+1) = 4.$$

So ϕ is an isomorphism. ■

(b) For $\mathbb{C}$-vector spaces V, W, we can check by definition that $\overline{V \oplus W} \cong \overline{V} \oplus \overline{W}$. We know that a $\mathbb{C}$-vector space V can be written as

$$V = \bigoplus_{i \in I} V_i$$

for some index set I where V_i is a one dimensional complex vector space. We only need to prove this result for one dimensional complex vector space. Indeed, we have

$$\begin{aligned} CR(V) &= V \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong (\bigoplus_{i \in I} V_i) \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong \bigoplus_{i \in I} (V_i \otimes_{\mathbb{R}} \mathbb{C}) \\ &\cong \bigoplus_{i \in I} (\overline{V_i} \oplus V_i) \\ &\cong (\bigoplus_{i \in I} \overline{V_i}) \oplus (\bigoplus_{i \in I} V_i) \\ &= \overline{V} \oplus V. \end{aligned}$$

Suppose V is generated by v as a $\mathbb{C}$-vector space. Define the following map

$$\begin{aligned} \phi : R(V) \otimes_{\mathbb{R}} \mathbb{C} &\to \overline{V} \oplus V, \\ u \otimes z &\mapsto (z * u, zu) = (\bar{z}u, zu). \end{aligned}$$

This is a well-defined map because for any $w \in \mathbb{C}$, we have

$$\begin{aligned} \phi(w \cdot (u \otimes z)) &= \phi(u \otimes wz) \\ &= (\overline{wz}u, wzu) \\ &= ((wz) * u, wzu) \\ &= w \cdot (z * u, zu) \\ &= w \cdot \phi(u \otimes z). \end{aligned}$$

We know $R(V)$ is a 2-dimensional $\mathbb{R}$-vector space generated by v and iv. Every $u \in R(V) \otimes_{\mathbb{R}} \mathbb{C}$ can be written as

$$u = v \otimes x + iv \otimes y$$

for some $x, y \in \mathbb{C}$. Suppose $u \in \ker \phi$, then we have

$$0 = \phi(u) = \phi(v \otimes x + iv \otimes y) = \phi(v \otimes x) + \phi(iv \otimes y) = (\bar{x}v + \bar{y}iv, xv + yiv)$$

This implies

$$\begin{cases} \bar{x} + i\bar{y} = 0 \\ x + iy = 0 \end{cases}$$

Note that $x + \bar{x} = 2\mathrm{Re}(x)$ and $x - \bar{x} = 2\mathrm{Im}(x)$, so we have

$$\begin{cases} 0 = 2\mathrm{Re}(x) = 2\mathrm{Re}(y) \\ 0 = 2\mathrm{Im}(x) = 2\mathrm{Im}(y) \end{cases}$$

This implies $x = y = 0$, namely $u = 0$. We have proved $\ker \phi = 0$, thus, ϕ is injective. Moreover,

$$2 = \dim_{\mathbb{C}} CR(V) = \dim_{\mathbb{C}}(\overline{V} \oplus V).$$

This implies ϕ is an isomorphism.