

Problem 1

Compute all of the homology groups for the spaces in parts (a) and (b) below, and use your calculations to show that

- (a) $\mathbb{R}P^2 \times S^3$ and $\mathbb{R}P^3 \times S^2$ have isomorphic homotopy groups (in all dimensions), but non-isomorphic homology groups.
- (b) $S^4 \times S^2$ and $\mathbb{C}P^3$ have isomorphic homology groups but non-isomorphic homotopy groups.

Solution:

- (a) Let $X = \mathbb{R}P^2 \times S^3$ and $Y = \mathbb{R}P^3 \times S^2$. It is easy to see that both X and Y are path-connected, so $\pi_0(X) = \pi_0(Y) = *$. By direct calculation, we have

$$\begin{aligned}\pi_1(X) &= \pi_1(\mathbb{R}P^2) \times \pi_1(S^3) = 0, \\ \pi_1(Y) &= \pi_1(\mathbb{R}P^3) \times \pi_1(S^2) = 0.\end{aligned}$$

This implies $\pi_1(X) \cong \pi_1(Y)$. Recall that for all $n \geq 2$, the universal covering space of $\mathbb{R}P^n$ is S^n . So the universal covering space of X and Y are both isomorphic to $S^2 \times S^3 \cong S^3 \times S^2$. The long exact sequence in homotopy groups tells us that

$$\pi_n(X) \cong \pi_n(Y) \cong \pi_n(S^3 \times S^2)$$

for all $n \geq 2$. Thus, we can conclude that X and Y have the same homotopy groups.

For the homology groups, note that the homology groups of S^3 and S^2 are all free. By Künneth theorem, we have

$$\begin{aligned}H_n(X) &= \bigoplus_{p+q=n} H_p(\mathbb{R}P^2) \otimes H_q(S^3), \\ H_n(Y) &= \bigoplus_{p+q=n} H_p(\mathbb{R}P^3) \otimes H_q(S^2).\end{aligned}$$

The homology groups of each space is listed below: We can obtain the of X and Y by tensoring

	$H_*(\mathbb{R}P^2)$	$H_*(S^3)$		$H_*(\mathbb{R}P^3)$	$H_*(S^2)$
3	0	\mathbb{Z}	3	\mathbb{Z}	0
2	0	0	2	0	\mathbb{Z}
1	$\mathbb{Z}/2$	0	1	$\mathbb{Z}/2$	0
0	\mathbb{Z}	\mathbb{Z}	0	\mathbb{Z}	\mathbb{Z}

at each degree, this gives us

$$H_i(X) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 3; \\ \mathbb{Z}/2, & \text{if } i = 1, 4; \\ 0, & \text{otherwise.} \end{cases} \quad H_i(Y) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2, 5; \\ \mathbb{Z}/2, & \text{if } i = 1; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

From this we can see that X and Y have non-isomorphic homology groups.

- (b) We know that $\mathbb{C}P^3$ has a cellular structure with one 0-cell, one 2-cell, one 4-cell, and one 6-cell. The boundary maps in the cellular chain complex are all zero, so $H_i(\mathbb{C}P^3) = \mathbb{Z}$ for $i = 0, 2, 4, 6$ and 0 otherwise. For the space $S^4 \times S^2$, use Künneth theorem and note that S^2 does not have torsion in homology, so $H_i(S^4 \times S^2) = \mathbb{Z}$ for $i = 0, 2, 4, 6$ and 0 otherwise. This shows that $S^4 \times S^2$ and $\mathbb{C}P^3$ have isomorphic homology groups.

Recall that we have a fibration $S^1 \rightarrow S^7 \rightarrow \mathbb{C}P^3$. This induces a long exact sequence in homotopy groups

$$\cdots \rightarrow \pi_2(S^7) \rightarrow \pi_2(\mathbb{C}P^3) \rightarrow \pi_1(S^1) \rightarrow \pi_1(S^7) \rightarrow \cdots$$

Note that $\pi_2(S^7) = \pi_1(S^7) = 0$ and $\pi_1(S^1) = \mathbb{Z}$. This implies that $\pi_2(\mathbb{C}P^3) = \mathbb{Z}$. On the other hand, we know that

$$\pi_2(S^4 \times S^2) \cong \pi_2(S^4) \times \pi_2(S^2) = 0.$$

This implies that $S^4 \times S^2$ and $\mathbb{C}P^3$ have non-isomorphic homotopy groups.

Problem 2

Let I_* be the chain complex concentrated in degree 0 and 1 with $I_1 = \mathbb{Z}\langle e \rangle$, $I_0 = \mathbb{Z}\langle a, b \rangle$, and $d(e) = b - a$. Note that this is the simplicial chain complex for Δ_1 . Let C_* and D_* be chain complexes.

- Describe the chain complex $I_* \otimes C_*$ by giving the groups in each degree as well as the boundary maps.
- Let $F : I_* \otimes C_* \rightarrow D_*$ be a chain map. Define $f, g : C_* \rightarrow D_*$ by $f(x) = F(a \otimes x)$ and $g(x) = F(b \otimes x)$. Likewise, define $s_n : C_n \rightarrow D_{n+1}$ by $s_n : C_n \rightarrow D_{n+1}$ by $s_n(x) = F(e \otimes x)$. Prove that f and g are chain maps and the collection $\{s_n\}$ is a chain homotopy between f and g .

Solution:

- (a) We denote both the boundary map in C_* by d_C . Consider the double complex $I_* \otimes C_*$ first.

$$\begin{array}{ccccc}
& \vdots & & \vdots & & \vdots \\
& \downarrow & & \downarrow & & \downarrow \\
I_0 \otimes C_2 & \longleftarrow & I_1 \otimes C_2 & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
I_0 \otimes C_1 & \longleftarrow & I_1 \otimes C_1 & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
I_0 \otimes C_0 & \longleftarrow & I_1 \otimes C_0 & \longleftarrow & 0
\end{array}$$

The vertical boundary map d_v is $id \otimes d_C$ and the horizontal boundary map d_h is $d \otimes id$. Let T_* be the total complex of this double complex, then in each degree we have

$$T_n = I_0 \otimes C_n \oplus I_1 \otimes C_{n-1}.$$

We know that $I_0 = \mathbb{Z}\langle a, b \rangle$, so $I_0 \otimes C_n$ is isomorphic to $(C_n)^2$ where the isomorphism is given by sending $a \otimes x$ to x and $b \otimes y$ to y for all $x, y \in C_n$. Similarly, $I_1 = \mathbb{Z}\langle e \rangle$, so $I_1 \otimes C_{n-1}$ is isomorphic to C_{n-1} where the isomorphism is given by sending $e \otimes z$ to z for all $z \in C_{n-1}$. The boundary map in the total complex is given by $d_t(x) = d_h(x) + (-1)^p d_v(x)$ for $x \in I_p \otimes C_q$. For $a \otimes x, b \otimes y$ in $I_0 \otimes C_n$ and $e \otimes z \in I_1 \otimes C_{n-1}$, we have

$$\begin{aligned}
d_t(a \otimes x) &= d_C(x), \\
d_t(b \otimes y) &= d_C(y), \\
d_t(e \otimes z) &= (b - a) \otimes x - e \otimes d_C(z).
\end{aligned}$$

- (b) Write the boundary maps in C_* as d_C and boundary maps in D_* as d_D . For any $n \in \mathbb{Z}$, we know F is a chain map, so we have a commutative diagram

$$\begin{array}{ccc}
I_0 \otimes C_n & \xrightarrow{id \otimes d_C} & I_0 \otimes C_{n-1} \\
F \downarrow & & \downarrow F \\
D_n & \xrightarrow{d_D} & D_{n-1}
\end{array}$$

For any $x \in C_n$, we have

$$(d_D \circ F)(a \otimes x) = [F \circ (id \otimes d_C)(a \otimes x)] = F(a \otimes d_C(x)).$$

By definition this is equivalent to

$$(d_D \circ f)(x) = (f \circ d_C)(x).$$

Namely, we have a commutative diagram

$$\begin{array}{ccc} C_n & \xrightarrow{d_C} & C_{n-1} \\ f \downarrow & & \downarrow f \\ D_n & \xrightarrow{d_D} & D_{n-1} \end{array}$$

This proves f is a chain map. By a similar argument, g is also a chain map.

Next, to show that s_n defines a chain homotopy between f and g , we need to show for any n , there exists a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_C} & C_n & \xrightarrow{d_C} & C_{n-1} \longrightarrow \cdots \\ & & f \downarrow \quad \downarrow g & \swarrow s_n & f \downarrow \quad \downarrow g & \swarrow s_{n-1} & f \downarrow \quad \downarrow g \\ \cdots & \longrightarrow & D_{n+1} & \xrightarrow{d_D} & D_n & \xrightarrow{d_D} & D_{n-1} \longrightarrow \cdots \end{array}$$

For any $x \in C_n$, we have

$$g(x) - f(x) = F(b \otimes x) - F(a \otimes x) = F((b - a) \otimes x).$$

On the other hand, use the fact that F is a chain map, we have

$$\begin{aligned} (d_D \circ s_n)(x) + (s_{n-1} \circ d_C)(x) &= (d_D \circ F)(e \otimes x) + F(e \otimes d_C(x)) \\ &= (F \circ d_C)(e \otimes x) + F(e \otimes d_C(x)) \\ &= F((b - a) \otimes x) - F(e \otimes d_C(x)) + F(e \otimes d_C(x)) \\ &= F((b - a) \otimes x). \end{aligned}$$

This proves that

$$g - f = d_D \circ s_n + s_{n-1} \circ d_C.$$

The collection of s_n is a chain homotopy between f and g .

Problem 3

Let Y be the space obtained by starting with S^3 and attaching a 4-cell via a map of degree 5: $Y = S^3 \cup_f e^4$ where $f : \partial(e^4) \rightarrow S^3$ has degree 5. Write down the cellular chain complex for $\mathbb{R}P^3 \otimes Y$; in particular, specify the rank of each chain group and identify the boundary maps. Compute the homotopy groups of $\mathbb{R}P^3 \otimes Y$ and specify the rank of each chain group and identify the boundary maps. Compute the homology groups of $\mathbb{R}P^3 \otimes Y$.

Solution: The space Y has a cellular chain complex

$$0 \rightarrow \mathbb{Z}\langle e^4 \rangle \xrightarrow{5} \mathbb{Z}\langle e^3 \rangle \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{Z}\langle e^0 \rangle \rightarrow 0.$$

where e^i are cells in Y for $i = 0, 3, 4$. The real projective space $\mathbb{R}P^3$ has the following cellular chain complex

$$0 \rightarrow \mathbb{Z}\langle f^3 \rangle \xrightarrow{0} \mathbb{Z}\langle f^2 \rangle \xrightarrow{2} \mathbb{Z}\langle f^1 \rangle \xrightarrow{0} \mathbb{Z}\langle f^0 \rangle \rightarrow 0.$$

The tensor product of these two chain complex is the double complex

$$\begin{array}{ccccccc}
\mathbb{Z}\langle f^0 \otimes e^4 \rangle & \xleftarrow{0} & \mathbb{Z}\langle f^1 \otimes e^4 \rangle & \xleftarrow{2 \otimes id} & \mathbb{Z}\langle f^2 \otimes e^4 \rangle & \xleftarrow{0} & \mathbb{Z}\langle f^3 \otimes e^4 \rangle \\
id \otimes 5 \downarrow & & id \otimes 5 \downarrow & & id \otimes 5 \downarrow & & id \otimes 5 \downarrow \\
\mathbb{Z}\langle f^0 \otimes e^3 \rangle & \xleftarrow{0} & \mathbb{Z}\langle f^1 \otimes e^3 \rangle & \xleftarrow{2 \otimes id} & \mathbb{Z}\langle f^2 \otimes e^3 \rangle & \xleftarrow{0} & \mathbb{Z}\langle f^3 \otimes e^3 \rangle \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}\langle f^0 \otimes e^0 \rangle & \xleftarrow{0} & \mathbb{Z}\langle f^1 \otimes e^0 \rangle & \xleftarrow{2 \otimes id} & \mathbb{Z}\langle f^2 \otimes e^0 \rangle & \xleftarrow{0} & \mathbb{Z}\langle f^3 \otimes e^0 \rangle
\end{array}$$

Denote the total chain complex by (T_n, d_n) . we have

$$T_n = \begin{cases} \mathbb{Z}\langle f^0 \otimes e^0 \rangle \cong \mathbb{Z}, & \text{if } n = 0; \\ \mathbb{Z}\langle f^1 \otimes e^0 \rangle \cong \mathbb{Z}, & \text{if } n = 1; \\ \mathbb{Z}\langle f^2 \otimes e^0 \rangle \cong \mathbb{Z}, & \text{if } n = 2; \\ \mathbb{Z}\langle f^0 \otimes e^3, f^3 \otimes e^0 \rangle \cong \mathbb{Z}^2, & \text{if } n = 3; \\ \mathbb{Z}\langle f^0 \otimes e^4, f^1 \otimes e^3 \rangle \cong \mathbb{Z}^2, & \text{if } n = 4; \\ \mathbb{Z}\langle f^1 \otimes e^4, f^2 \otimes e^3 \rangle \cong \mathbb{Z}^2, & \text{if } n = 5; \\ \mathbb{Z}\langle f^2 \otimes e^4, f^3 \otimes e^3 \rangle \cong \mathbb{Z}^2, & \text{if } n = 6; \\ \mathbb{Z}\langle f^3 \otimes e^4 \rangle \cong \mathbb{Z}, & \text{if } n = 7. \end{cases}$$

For $1 \leq n \leq 7$, the boundary map d_n is given by the formula

$$d_n(f^i \otimes e^j) = d(f^i) \otimes e^j + (-1)^i f^i \otimes d(e^j).$$

The boundary map d_n is given by the following table: To calculate the homology groups of $\mathbb{R}P^3 \times Y$,

i	d_i
1	0
2	2
3	$\begin{pmatrix} 0 & 0 \end{pmatrix}$
4	$\begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$
5	$\begin{pmatrix} 0 & 2 \\ -5 & 0 \end{pmatrix}$
6	$\begin{pmatrix} 2 & 0 \\ 5 & 0 \end{pmatrix}$
7	$\begin{pmatrix} 0 \\ -5 \end{pmatrix}$

we first write down the homology groups of $\mathbb{R}P^3$ and Y .

	$H_*(\mathbb{R}P^3)$	$H_*(Y)$
0	\mathbb{Z}	\mathbb{Z}
1	$\mathbb{Z}/2$	0
2	0	0
3	\mathbb{Z}	$\mathbb{Z}/5$
4	0	0

We calculate their tensor products and Tor_1 respectively. Note that $\text{Tor}_1(\mathbb{Z}/5, \mathbb{Z}/2) = 0$, so we do not have any terms coming from Tor_1 . The homology groups of $\mathbb{R}P^3 \times Y$ can be summarized as follows

$$H_i(\mathbb{R}P^3 \times Y) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}/2, & \text{if } i = 1; \\ \mathbb{Z} \oplus \mathbb{Z}/5, & \text{if } i = 3; \\ \mathbb{Z}/5, & \text{if } i = 6; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 4

Compute both the homology and cohomology groups of the following spaces, both with integral and $\mathbb{Z}/2$ coefficients. Heck, do it with $\mathbb{Z}/3$ coefficients as well.

- (a) $K \times K$, where K is the Klein bottle.
- (b) $K \times T^g$, where T^g is the genus g torus and K is the Klein bottle.
- (c) $K \times \mathbb{R}P^n$.

Solution:

- (a) We can use UCT for homology to calculate the homology groups of K in different coefficients and we summarized them as follows.

	$H_*(K)$	$H_*(K; \mathbb{Z}/2)$	$H_*(K; \mathbb{Z}/3)$
0	\mathbb{Z}	$\mathbb{Z}/2$	$\mathbb{Z}/3$
1	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/3$
2	0	$\mathbb{Z}/2$	0

From this we know that the tensor product is

$$H_p(K) \otimes H_q(K) = \begin{cases} \mathbb{Z}, & \text{if } p + q = 0; \\ \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2, & \text{if } p + q = 1; \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^3, & \text{if } p + q = 2; \\ 0, & \text{otherwise.} \end{cases}$$

The only non trivial Tor_1 is given by $\text{Tor}_1(H_1(K), H_1(K)) = \mathbb{Z}/2$. Thus, the homology groups of K is

$$H_i(K \times K; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2, & \text{if } i = 1; \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^3, & \text{if } i = 2; \\ \mathbb{Z}/2, & \text{if } i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we use UCT for homology to calculate the homology groups with $\mathbb{Z}/2$ and $\mathbb{Z}/3$ coefficients.

$$H_i(K \times K; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0; \\ (\mathbb{Z}/2)^4, & \text{if } i = 1; \\ (\mathbb{Z}/2)^6, & \text{if } i = 2; \\ (\mathbb{Z}/2)^4, & \text{if } i = 3; \\ (\mathbb{Z}/2), & \text{if } i = 4; \\ 0, & \text{otherwise.} \end{cases} \quad H_i(K \times K; \mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & \text{if } i = 0; \\ (\mathbb{Z}/3)^2, & \text{if } i = 1; \\ \mathbb{Z}/3, & \text{if } i = 2; \\ 0, & \text{otherwise.} \end{cases}$$

(b) The homology of K and T^g are as follows:

	$H_*(K)$	$H_*(T^g)$
0	\mathbb{Z}	\mathbb{Z}
1	$\mathbb{Z} \oplus \mathbb{Z}/2$	\mathbb{Z}^{2g}
2	0	\mathbb{Z}

Note that $H_*(T^g)$ are all free, so by Künneth theorem, we have

$$H_*(K \times T^g) \cong \bigoplus_{i+j=n} H_i(K) \otimes H_j(T^g).$$

Thus, we conclude $H_*(K \times T^g)$ as follows:

$$H_i(K \times T^g) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}^{2g+1} \oplus \mathbb{Z}/2, & \text{if } i = 1; \\ \mathbb{Z}^{2g+1} \oplus (\mathbb{Z}/2)^{2g}, & \text{if } i = 2; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we use UCT for homology to calculate different coefficients.

$$H_i(K \times T^g; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0; \\ (\mathbb{Z}/2)^{2g+2}, & \text{if } i = 1; \\ (\mathbb{Z}/2)^{4g+2}, & \text{if } i = 2; \\ (\mathbb{Z}/2)^{2g+2}, & \text{if } i = 3; \\ \mathbb{Z}/2, & \text{if } i = 4; \\ 0, & \text{otherwise.} \end{cases} \quad H_i(K \times T^g; \mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & \text{if } i = 0; \\ (\mathbb{Z}/3)^{2g+1}, & \text{if } i = 1; \\ (\mathbb{Z}/3)^{2g+1}, & \text{if } i = 2; \\ \mathbb{Z}/3, & \text{if } i = 3; \\ 0, & \text{otherwise.} \end{cases}$$

(c) $\mathbb{R}P^n$ has different homology groups when n is odd or even.

(1) Suppose n is even.

The homology groups of $\mathbb{R}P^n$ can be summarized as follows:

- $H_0(\mathbb{R}P^n) = \mathbb{Z}$.
- $H_i(\mathbb{R}P^n) = \mathbb{Z}/2$ if $i \geq 0$ and i is odd.
- 0 otherwise.

Use Künneth theorem, we can calculate the homology groups of $K \times \mathbb{R}P^n$:

$$H_i(K \times \mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^2, & \text{if } i = 1; \\ (\mathbb{Z}/2)^2, & \text{if } 2 \leq i \leq n; \\ \mathbb{Z}/2, & \text{if } i = n + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we use UCT for homology to calculate the homology groups with other coefficients.

$$H_i(K \times \mathbb{R}P^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0; \\ (\mathbb{Z}/2)^3, & \text{if } i = 1; \\ (\mathbb{Z}/2)^4, & \text{if } 2 \leq i \leq n; \\ (\mathbb{Z}/2)^3, & \text{if } i = n + 1; \\ \mathbb{Z}/2, & \text{if } i = n + 2; \\ 0, & \text{otherwise.} \end{cases}$$

$$H_i(K \times \mathbb{R}P^n; \mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & \text{if } i = 0, 1; \\ 0, & \text{otherwise.} \end{cases}$$

(2) Suppose n is odd.

The homology groups of $\mathbb{R}P^n$ can be summarized as follows:

- $H_0(\mathbb{R}P^n) = H_n(\mathbb{R}P^n) = \mathbb{Z}$.
- $H_i(\mathbb{R}P^n) = \mathbb{Z}/2$ for $1 \leq i \leq n - 1$ if i is odd.
- 0 otherwise.

Use Künneth theorem, we can calculate the homology groups of $K \times \mathbb{R}P^n$:

$$H_i(K \times \mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z} \oplus (\mathbb{Z}/2)^2, & \text{if } i = 1; \\ (\mathbb{Z}/2)^2, & \text{if } 2 \leq i \leq n-1; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } i = n; \\ \mathbb{Z} \oplus \mathbb{Z}/2, & \text{if } i = n+1; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we use UCT for homology to calculate the homology groups with coefficients:

$$H_i(K \times \mathbb{R}P^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & \text{if } i = 0; \\ (\mathbb{Z}/2)^3, & \text{if } i = 1; \\ (\mathbb{Z}/2)^4, & \text{if } 2 \leq i \leq n; \\ (\mathbb{Z}/2)^3, & \text{if } i = n+1; \\ \mathbb{Z}/2, & \text{if } i = n+2; \\ 0, & \text{otherwise.} \end{cases}$$

$$H_i(K \times \mathbb{R}P^n; \mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & \text{if } i = 0, 1, n, n+1; \\ 0, & \text{otherwise.} \end{cases}$$

Problem 5

Let $f : A_* \rightarrow B_*$ be a map of chain complexes. We can regard this as forming a double complex

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & B_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & B_0 \end{array}$$

by putting zeros in all the "empty" spots. The total complex of this double complex is called the **algebraic mapping cone** of f , denoted Cf . Specifically, we set $(Cf)_n = A_{n-1} \oplus B_n$ and define $d : (Cf)_n \rightarrow (Cf)_{n-1}$ by

$$d(a, b) = (d_A(a), (-1)^{n-1}f(a) + d_B(b))$$

(a) Explain why there is a short exact sequence of chain complexes

$$0 \rightarrow B_* \hookrightarrow C(f) \rightarrow \Sigma A_* \rightarrow 0,$$

where ΣA_* is the evident chain complex having $(\Sigma A)_n = A_{n-1}$.

(b) The short exact sequence from (a) gives rise to a long exact sequence in homology groups. This has the form

$$\cdots \rightarrow H_i(B) \rightarrow H_i(Cf) \rightarrow H_i(\Sigma A) \xrightarrow{\partial} H_{i-1}(B) \rightarrow \cdots$$

Verify that the connecting homomorphism is really just the map $f_* : H_{i-1}(A) \rightarrow H_{i-1}(B)$, possibly up to a sign.

Solution:

(a) We need to prove that for any $n \geq 0$, we have the following commutative diagrams where the top row and bottom row is exact.

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_n & \xrightarrow{i_n} & A_{n-1} \oplus B_n & \xrightarrow{p_n} & A_{n-1} \longrightarrow 0 \\ & & \downarrow d_B & & \downarrow d & & \downarrow d_A \\ 0 & \longrightarrow & B_{n-1} & \xrightarrow{i_{n-1}} & A_{n-2} \oplus B_{n-1} & \xrightarrow{p_{n-1}} & A_{n-2} \longrightarrow 0 \end{array}$$

We choose $i_n : B_n \rightarrow A_{n-1} \oplus B_n$ as the inclusion $b \mapsto (0, b)$ and $p_n : A_{n-1} \oplus B_n \rightarrow A_{n-1}$ as the projection $(a, b) \mapsto a$. It is easy to see the top row and the bottom row is exact. For any $(a, b) \in A_{n-1} \oplus B_n$, we have

$$\begin{aligned} (p_{n-1} \circ d)(a, b) &= p_{n-1}(d_A(a), (-1)^{n-1}f(a) + d_B(b)) \\ &= d_A(a) \\ &= (d_A \circ p_n)(a, b). \end{aligned}$$

This proves the right square commutes. Moreover, for any $b \in B_n$, we have

$$\begin{aligned} (d \circ i_n)(b) &= d(0, b) \\ &= (0, 0 + d_B(b)) \\ &= (i_{n-1} \circ d_B)(b). \end{aligned}$$

This proves the left square commutes. Thus, we have a short exact sequence of chain complex

$$0 \rightarrow B_* \rightarrow (Cf)_* \rightarrow \Sigma A_* \rightarrow 0$$

where $(Cf)_n = A_{n-1} \oplus B_n$ and $(\Sigma A)_n = A_{n-1}$ for all n .

(b) By the snake lemma, we have a long exact sequence of homology groups deduced from the short exact sequence of chain complexes

$$0 \rightarrow B_* \rightarrow (Cf)_* \rightarrow \Sigma A_* \rightarrow 0.$$

Take $a \in \ker d_A \subseteq A_{n-1}$, we specify how to define $\partial a \in B_{n-1}$ from the snake lemma. We take the preimage $(a, 0) \in (Cf)_n$, send it to $d(a, 0) = (0, (-1)^{n-1}f(a)) \in (Cf)_{n-1}$, lastly we take the preimage $(-1)^{n-1}f(a) \in B_{n-1}$. Thus, we can conclude that the map

$$\begin{aligned}\partial : H_{n-1}(A) &\rightarrow H_{n-1}(B), \\ [a] &\mapsto [(-1)^{n-1}f(a)]\end{aligned}$$

This implies the connecting homomorphism is just the map induced by f

$$f_* : H_{n-1}(A) \rightarrow H_{n-1}(B)$$

up to a sign.

Problem 6

Let k be a field, and let \mathcal{V} denote the category of vector spaces over k . Let I be any (small) category, and let \mathcal{V}^I be the category whose objects are functors $I \rightarrow \mathcal{V}$ and whose morphisms are natural transformations. We call \mathcal{V}^I the category of " I -shaped diagram in \mathcal{V} ".

In this problem we will focus on the case where I is the pushout category

$$1 \leftarrow 0 \rightarrow 2$$

with three objects and two non-identity maps (as shown above). An object of \mathcal{V}^I is then just a diagram of vector spaces $V_1 \leftarrow V_0 \rightarrow V_2$. A map from $[V_1 \leftarrow V_0 \rightarrow V_2]$ to $[W_1 \leftarrow W_0 \rightarrow W_2]$ is a commutative diagram

$$\begin{array}{ccccc} V_1 & \longleftarrow & V_0 & \longrightarrow & V_2 \\ \downarrow & & \downarrow & & \downarrow \\ W_1 & \longleftarrow & W_0 & \longrightarrow & W_2 \end{array}$$

Let $P : \mathcal{V}^I \rightarrow \mathcal{V}$ be the pushout functor. P assigns each diagram its pushout.

- (a) Let F_1 , F_0 and F_2 be the three diagrams

$$F_1 : [k \leftarrow 0 \rightarrow 0] \quad F_0 = [k \leftarrow k \rightarrow k] \quad F_2 = [0 \leftarrow 0 \rightarrow k]$$

where in F_0 the maps are the identities. These diagrams are "free" in a certain sense: namely, if D is an object of \mathcal{V}^I then morphisms $F_i \rightarrow D$ are in bijective correspondence with elements of D_i . Convince yourself that this is true.

- (b) Let $D = [0 \leftarrow k \rightarrow 0]$ and $E = [0 \leftarrow k \rightarrow k]$, where in E the nontrivial map is the identity. Determine free resolutions for D and E .
- (c) Apply the functor P to your resolution, to produce a chain complex of vector spaces. Compute the homology groups, which are the groups $(L_i P)(D)$ and $(L_i P)(E)$. These are the derived functor of the pushout functor P . Confirm in your example that $L_0 P = P$.

- (d) Now let I be the category with one object 0 and one non-identity map $t : 0 \rightarrow 0$ such that $t^2 = id$. Objects of \mathcal{V}^I are then pairs (W, t) consisting of a vector space W and an endomorphism $t : W \rightarrow W$ such that $t^2 = id$. In \mathcal{V}^I the basic "free" object is $(k^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$; this can also be thought of as the vector space $k\langle g, tg \rangle$ where $t(tg) = g$. Let $P : \mathcal{V}^I \rightarrow \mathcal{V}$ be the colimit functor, sending an object (W, t) to $W / \{x - tx \mid x \in W\}$. Find the free resolution of the object (k, id) and compute $(L_i P)(k, id)$ for all $i \geq 0$.

Solution:

(a)

(b) Consider the following sequence

$$0 \rightarrow F_1 \oplus F_2 \rightarrow F_0 \rightarrow D \rightarrow 0.$$

Note that $F_1 \oplus F_2$ is the following diagram $[k \leftarrow 0 \rightarrow k]$ Namely the following diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & \longleftarrow & 0 & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 F_1 \oplus F_2 & & k & \longleftarrow & 0 & \longrightarrow & k & \\
 & \downarrow & \downarrow id & & \downarrow & & \downarrow id & \\
 & F_0 & k & \xleftarrow{id} & k & \xrightarrow{id} & k & \\
 & \downarrow & \downarrow & & \downarrow id & & \downarrow & \\
 & D & 0 & \longleftarrow & k & \longrightarrow & 0 & \\
 & \downarrow & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & 0 & \longleftarrow & 0 & \longrightarrow & 0 &
 \end{array}$$

The vertical columns are exact because we only have isomorphisms. For E , consider the following sequence

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow E \rightarrow 0.$$

This can be written as the diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & \longleftarrow & 0 & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & F_1 & & k & \longleftarrow & 0 & \longrightarrow & 0 \\
 & \downarrow & \downarrow id & & \downarrow & & \downarrow & \\
 & F_0 & k & \xleftarrow{id} & k & \xrightarrow{id} & k & \\
 & \downarrow & \downarrow & & \downarrow id & & \downarrow id & \\
 & E & 0 & \longleftarrow & k & \xrightarrow{id} & k & \\
 & \downarrow & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & 0 & \longleftarrow & 0 & \longrightarrow & 0 &
 \end{array}$$

This is a free resolution for E .

(c) Apply the pushout functor to the free resolution

$$0 \rightarrow F_1 \oplus F_2 \oplus F_0 \rightarrow 0.$$

The pushout of $F_1 \oplus F_2$ is k^2 and the pushout of F_0 is k . The map $F_1 \oplus F_2 \rightarrow F_0$ induces a map p between pushouts

$$\begin{array}{ccccc}
 k & \xleftarrow{\quad} & 0 & \xrightarrow{\quad} & k \\
 \downarrow id & \searrow & & \swarrow & \downarrow id \\
 & & k^2 & & \\
 & & \downarrow p & & \\
 k & \xleftarrow{\quad} & k & \xrightarrow{\quad} & k \\
 \downarrow id & \searrow & & \swarrow & \downarrow id \\
 & & k & &
 \end{array}$$

We can see from the diagram that $p = (id, id)$, so p is surjective, so we have $(L_0P)(D) = P(D) = 0$ and $(L_1P)(D) = k$.

Apply the pushout functor to the free resolution

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow E \rightarrow 0.$$

The map $F_1 \rightarrow F_0$ induces a map between pushouts

$$\begin{array}{ccccc}
 k & \xleftarrow{\quad} & 0 & \xrightarrow{\quad} & 0 \\
 \downarrow id & \searrow & & \swarrow & \downarrow id \\
 & & k & & \\
 & & \downarrow id & & \\
 k & \xleftarrow{\quad} & k & \xrightarrow{\quad} & k \\
 \downarrow id & \searrow & & \swarrow & \downarrow id \\
 & & k & &
 \end{array}$$

This map must be identity, so we have

$$(L_1P)(E) = (L_0P)(E) = P(E) = 0.$$