Notes on ∞-categories

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Contents

Chapter 1

Introduction

These notes are for a reading course on ∞-categories, based on Chapter 14: a short course on ∞-categories by Moritz Groth [1].

Chapter 2

2 models for $(\infty, 1)$-categories

In this part, we introduce two models for $(\infty, 1)$-categories. One is based on simplicial sets satisfying certain horn extension properties, the other is based on simplicial categories. These two models come with two model structure respectively, and the coherent nerve construction of Cordier can be used to show that they are Quillen equivalent.

2.1 Basics on ∞-categories

Let us first introduce some basic notions related to simplicial sets.

Definition 2.1.1 (Category of finite ordinals)**.** We define a category Δ as follows:

1. The objects are linearly order sets of the form $[n]$ for every integer $n \geq 0$

$$[n] := (0 < 1 < \cdots < n).$$

2. Given $m, n \geq 0$, a morphism $f : [m] \to [n]$ in Δ is an order-preserving map. Said differently, for any $0 \leq i \leq j \leq m$, we have $0 \leq f(i) \leq f(j) \leq n$.

There are two important classes of maps: one is the unique coface map

$$d^k : [n-1] \to [n], \quad 0 \leq k \leq n$$

which does not have k in its image and is injective. The other is the unique codegeneracy map

$$s^k : [n+1] \to [n], \quad 0 \leq k \leq n$$

which hits k twice and is surjective. They satisfy a list of cosimplicial identities, see [2]. Now we define the simplicial sets as contravariant functors.

Definition 2.1.2 (Simplicial sets)**.** A **simplicial set** X is a contravariant functor $X : \Delta^{op} \to \mathcal{S}ets$. They form a category where the morphisms are natural transformations. For a simplicial set X, we write $X_n := X([n])$ as its values. The face maps are $d_k = X(d^k)$ and the degeneracy maps are $s_k = X(s^k)$. They satisfy the corresponding simplicial identities.

We can associate a simplicial set to a category using the nerve construction. For each fixed $n \geq 0$, we can view $[n]$ as a category as follows:

- We have $n + 1$ objects: the number $0, 1, \dots, n$.
- For $0 \leq i \leq j \leq n$, we have a unique morphism $i \to j$ (if $i = j$, this is the unique identity morphism of i). The composition of morphisms is given by the transitivity of $\leq$.

Example 2.1.3. Let $\mathcal{C}$ be a category. The nerve of $\mathcal{C}$ is a simplicial set $N(\mathcal{C})$ given by

$$N(\mathcal{C})_n := \mathrm{Fun}([n], \mathcal{C})$$

for each $n \geq 0$. Namely, all functors from the category $[n]$ to the category $\mathcal{C}$. Here $N(\mathcal{C})_n$ can be viewed as a string of n composable morphisms in $\mathcal{C}$:

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n$$

As a special case, consider $\mathcal{C} = \Delta$. Fix $n \geq 0$, the nerve $N([n])$ is a simplicial set. For every $m \geq 0$, we have

$$N([n])_m = \mathrm{Fun}([m], [n]) = \mathrm{Hom}_\Delta([m], [n]).$$

Because the functors from the category $[m]$ to $[n]$ are exactly the morphism from $[m]$ to $[n]$ if we view them as objects in Δ. This simplicial set $N([n]) = \mathrm{Hom}_\Delta(-, [n])$ is usually denoted as Δ^n, which is the simplicial set represented by $[n] \in \Delta$.

Given a simplicial set X, by the Yoneda lemma, the simplicial maps $\Delta^n \to X$ classify n-simplices of X in the sense that we have a natural bijection

$$\mathrm{Hom}_{s\mathcal{S}et}(\Delta^n, X) \cong X_n.$$

Let $\mathcal{C}$ be a category. The vertices $N(\mathcal{C})_0$ in the simplicial set $N(\mathcal{C})$ can be viewed as objects in $\mathcal{C}$ and the edges $N(\mathcal{C})_1$ can be viewed as morphisms in $\mathcal{C}$. Consider the following triangle in the $[2]$:

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

$N(\mathcal{C})_2$ consists of functors defined on the above triangle. The face map $d_1 : N(\mathcal{C})_2 \to N(C)_1$ is essentially given by composition: how we can compose two morphism into one morphism in $\mathcal{C}$. Strictly speaking, a 2-simplex in $N(\mathcal{C})$ is a pair of composable morphisms together with their composition.

Now let $\mathcal{C}at$ be the category of categories and $s\mathcal{S}et$ be the category of simplicial sets.

Lemma 2.1.4. *The nerve functor $N : \mathcal{C}at \to s\mathcal{S}et$ is fully faithful and hence induces an equivalence onto its essential image.*

Proof. The full proof can be found at [5, Tag 002Z]. We only give a sketch here. □

Our next goal is to understand the essential image. We next introduce two construction from Δ^n. Δ^n contains two subcomplexes: the boundary $\partial\Delta^n$ and the k-th horn Λ^n_k. It is not hard to see that the boundary $\partial\Delta^n$ can be identified with the coequalizer

$$\coprod_{0 \le i < j \le n} \Delta^{n-2} \rightrightarrows \coprod_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n.$$

The kth n-horn $\Lambda^n_k \subseteq \partial\Delta^n$ for $0 \le k \le n$ is obtained from $\partial\Delta^n$ by removing the kth face: the face opposite to the vertex k. The kth horn can also be identified with the coequalizer:

$$\coprod_{0 \le i < j \le n} \Delta^{n-2} \rightrightarrows \coprod_{i \ne k} \Delta^{n-1} \to \Lambda^n_k.$$

Example 2.1.5. Assume the dimension $n = 2$ and $\mathcal{C}$ is a category. We have 3 different horns $\Lambda^2_k \to N(\mathcal{C})$ for $0 \le k \le 2$. From our previous discussion on nerves, we know that they look like the following:

$$\begin{tikzcd} & c_1 & \\ c_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & c_2 \end{tikzcd} \qquad \begin{tikzcd} & c_1 \arrow[dr, "g"] & \\ c_0 \arrow[ur, "f"] & & c_2 \end{tikzcd} \qquad \begin{tikzcd} & c_1 \arrow[dr, "g"] & \\ c_0 \arrow[rr, "h"'] & & c_2 \end{tikzcd}$$

Here c_0, c_1, c_2 are objects in $\mathcal{C}$, and f, g, h are morphisms in $\mathcal{C}$. The composition $h = g \circ f$ in $\mathcal{C}$ uniquely extends the any horn $\Lambda^2_1 \to N(\mathcal{C})$ to an entire 2-simplex $\sigma : \Delta^2 \to N(\mathcal{C})$, i.e., there is a unique dashing arrow making the diagram

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & N(\mathcal{C}) \\ \Delta^2 \arrow[ur, dashed, "\exists!\sigma"'] & \end{tikzcd}$$

commute. The composition $h = g \circ f$ is given by the new face $d_1(\sigma) : \Delta^1 \to N(\mathcal{C})$. Next, we consider the horn $\Lambda_0^2 \to N(\mathcal{C})$, the existence of the extension $c_1 \to c_2$ is equivalent to f has a left inverse in $\mathcal{C}$. Similar observations can be made to $\Lambda_2^2 \to N(\mathcal{C})$.

The above example leads us to the following definition:

Definition 2.1.6 (Inner horns and outer horns)**.** Given $n \geq 0$, the horns $\Lambda_k^n \subseteq \Delta^n$ for $0 < k < n$ are called **inner horns** and Λ_0^n and Λ_n^n are called **outer horns**.

It turns out that the extension properties in the above example describe the essential image of the nerve functor. Let $\mathcal{G}rpd$ be the category of groupoids, where every morphism is invertible.

Proposition 2.1.7. *Let X be a simplicial set.*

(i) There is an isomorphism $X \cong N(\mathcal{C})$ for some $\mathcal{C} \in \mathcal{C}at$ if and only if every inner horn $\Lambda_k^n \to X$, $0 < k < n$, can be uniquely extended to an n-simplex $\Delta^n \to X$.

(ii) There is an isomorphism $X \cong N(\mathcal{G})$ for some $\mathcal{G} \in \mathcal{G}prd$ if and only if every horn $\Lambda_k^n \to X$, $0 \leq k \leq n$ can be uniquely extended to an n-simplex $\Delta^n \to X$.

This is very similar to the definition of Kan complex.

Definition 2.1.8 (Kan complex)**.** A simplicial set X is a **Kan complex** if every horn $\Lambda_k^n \to X$ for $0 \leq k \leq n$ can be extended to an n-simplex $\Delta^n \to X$.

Remark. Note that for Kan complexes, the extension is not unique, unlike for the nerves of categories. We do not want uniqueness in the definition of ∞-categories. The composition of morphisms, in some sense, is not strictly unique, but only up to 'homotopy'.

Let $\mathcal{K}an \subseteq s\mathcal{S}et$ be the full subcategory spanned by the Kan complexes. We have the following commutative diagram of fully faithful functors

$$\begin{tikzcd} \mathcal{G}rpd \arrow[r] \arrow[d, "N"'] & \mathcal{C}at \arrow[d, "N"] \\ \mathcal{K}an \arrow[r] & s\mathcal{S}et \end{tikzcd}$$

Now we can define the ∞-categories.

Definition 2.1.9 (∞-categories)**.** A simplicial set $\mathcal{C}$ is an **∞-category** if every inner horn $\Lambda_k^n \to \mathcal{C}$ for $0 < k < n$ can be extended be an n-simplex $\Delta^n \to \mathcal{C}$.

Any ordinary category $\mathcal{C}$ gives rise to an ∞-category $N(\mathcal{C})$. There exists interesting ∞-categories not of this form. In particular, every simplicial model category has an underlying ∞-category. Let us first introduce some basic terminology for ∞-categories. They are similar to the nerves we discussed before.

Let $\mathcal{C}$ be an ∞-category.

- The objects are the vertices $x \in \mathcal{C}_0$, and the morphisms are 1-simplices $f \in \mathcal{C}_1$.

- We have the face maps

$$\begin{aligned} s = d_1 :& \mathcal{C}_1 \to \mathcal{C}_0, \\ t = d_0 :& \mathcal{C}_1 \to \mathcal{C}_0 \end{aligned}$$

  where s sends a morphism to its source, and t to its target. More formally, we define the set of morphisms for objects x, y as the pullback square

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d, "(s,t)"] \\ * \arrow[r, "(x,y)"'] & \mathcal{C}_0 \times \mathcal{C}_0 \end{tikzcd}$$

- The degeneracy map

$$id = s_0 : \mathcal{C}_0 \to \mathcal{C}_1$$

  is the identity map.

Let us take a closer look at the composition for $\mathcal{C}$. Consider two morphisms $f : x \to y$ and $g : y \to z$. These morphisms together define an inner horn:

$$\lambda = (g, \cdot, f) : \Lambda^2_1 \to \mathcal{C}$$

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] & & z \end{tikzcd}$$

Here $d_0\lambda = g$ and $d_1\lambda = f$. By definition of the ∞-category $\mathcal{C}$, such horn can be *non-uniquely* extended to a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$. The new face obtained $d_1(\sigma)$ is then a candidate composition of g and f. Note that this is not uniquely determined, unlike in ordinary categories. We only demand that any choice of such a composition is equally good: the space of all such choices is to be contractible. Later we will give a more precise statement.

We now want to describe the homotopy category $\mathrm{Ho}(\mathcal{C})$ of an ∞-category $\mathcal{C}$. From what we discuss in A, the nerve functor has a left adjoint $\tau_1 : s\mathcal{S}et \to \mathcal{C}at$, called the **fundamental category functor** or the **categorical realization functor** (Some place may call it the first truncation). It is given by a formula

$$\tau_1(X) = \mathrm{colim}_{(\delta/X)}[-] \circ p.$$

Surprisingly, the image of a simplicial set X under this functor τ_1 is completely determined by its 0-, $1-$, $2-$simplices and the maps between them. We can do this as follows:

Given a simplicial set X, define ob $\tau_1 X$ to be X_0. Morphisms in $\tau_1 X$ are freely generated by the set X_1 subject to the relations given by the elements of X_2. For example, the degeneracy map $s_0 : X_0 \to X_1$ picks out the identity morphisms for every object. The face maps $d_1, d_0 : X_1 \to X_0$ assign every morphism a domain and a codomain. Next, we take the free graph on X_0 generated by the arrows X_1 and impose relations $h = gf$ if there exists a 2-simplex $x \in X_2$ such that $xd^2 = f$, xd^0 and $xd^1 = h$.

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "h"'] & & 2 \end{tikzcd}$$

It is not hard to verify that $\tau_1 X$ is indeed a category. In particular, a morphism in $\tau_1 X$ can be represented by a finite chain of 1-simplices of X. If the simplicial set X happens to be an ∞-category, there is a simple description of the fundamental category $\tau_1(X)$. The morphisms in $\tau_1(X)$ can be represented by actual 1-simplices. We need the definition of homotopies in ∞-categories.

Definition 2.1.10 (homotopy)**.** Two morphisms $f, g : x \to y$ in an ∞-category $\mathcal{C}$ are homotopic (write $f \simeq g$) if there is a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$ with boundary $\partial\sigma = (g, f, id_x)$, i.e. the boundary looks like

$$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"] & & y \end{tikzcd}$$

Any such 2-simplex σ is a **homotopy** from f to g, denoted by $\sigma : f \to g$.

This indeed defines an equivalence relationship for morphisms in an ∞-category.

Proposition 2.1.11. *Let $\mathcal{C}$ be an ∞-category and $x, y \in \mathcal{C}$. The homotopy relation is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(x, y)$. The **homotopy class** of a morphism $f : x \to y$ is denoted by $[f]$.*

Proof. The proof relies on the horn extension property of $\mathcal{C}$ and is completely technical. We first establish the **constant homotopy** for any morphism $f : x \to x$. Apply the degeneracy map s_0, we obtain a 2-simplex $s_0 f : \Delta^2 \to \mathcal{C}$. The simplicial identities implies that

$$d_0 s_0 f = d_1 s_0 f = f$$

and

$$d_2 s_0 f = s_0 d_1 f = s_0 x = id_x$$

by definition. This implies that the following diagram

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

is the constant homotopy of f. So $f \simeq f$ for every morphism.

To prove the reflexivity of $\simeq$, suppose $f, g : x \to y$ and $f \simeq g$, consider the following horn $\Lambda_2^3 \to \mathcal{C}$ in the figure 2.1. The horn Λ_2^3 does not have the bottom face, the other three faces are given by $g \simeq g$, $f \simeq g$ and $id_x \simeq id_x$. The horn extension property gives us the bottom face, and it exactly says that $g \simeq f$. A similar horn $\Lambda_2^3 \to \mathcal{C}$ in the figure 2.2 can prove transitivity of $\simeq$. $\square$

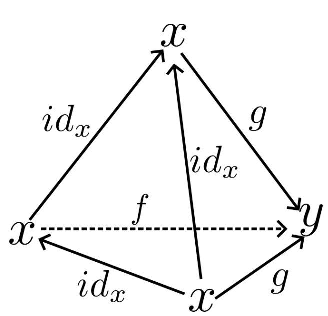

Figure 2.1: Symmetry of $\simeq$

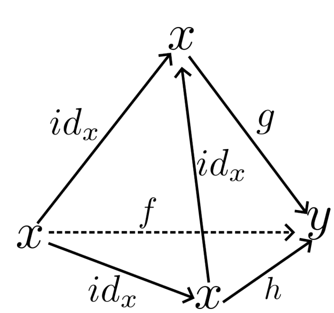

Figure 2.2: Transitivity of $\simeq$

Remark. A quick observation is that for morphisms $f, g : x \to y$, $f \simeq g$ is equivalent to the following condition: there exists a 2-simplex such that

$$\begin{tikzcd} & y \arrow[dr, "id_y"] & \\ x \arrow[ur, "g"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

This can be proved using the horn $\Lambda_2^3 \to \mathcal{C}$ in the figure 2.3.

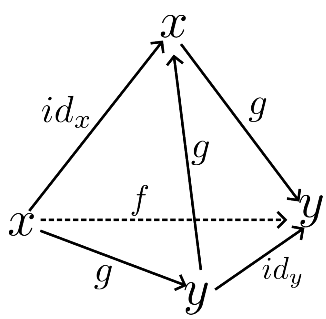

Figure 2.3: Another definition of $\simeq$

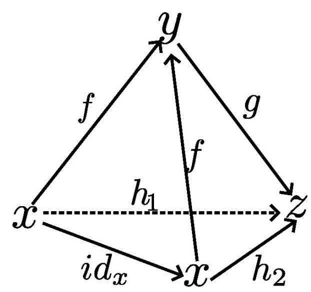

Figure 2.4: Candidate for composition

Next, we want to define the homotopy category $\mathrm{Ho}(\mathcal{C})$ by passing to the homotopy classes of morphisms. We have a lot of things to check, but first, we can show that the choice of different candidate composition of the representatives, is homotopic by using the horn extension property, as shown in figure 2.4.

We just showed that although the choice of composition in an ∞-category is not unique, two such choices are always homotopic. Assume that we can assemble all such choices into a space. What we just proved is the connectedness of the space, i.e., π_0 is trivial. The extension property with respect to higher dimensional inner horns guarantees the higher connectivity of this space, finally making it weakly contractible. Later we will see a more precise statement. Now we can understand what it means to be a homotopy category of an ∞-category.

Proposition 2.1.12 (homotopy category)**.** *Let $\mathcal{C}$ be an ∞-category. There is an ordinary category* $\mathrm{Ho}(\mathcal{C})$*, the* ***homotopy category*** *of $\mathcal{C}$, with the same objects as $\mathcal{C}$ and morphisms the homotopy classes of morphisms in $\mathcal{C}$. Composition and identities are given by*

$$[g] \circ [f] = [g \circ f] \quad \text{and} \quad id_x := [id_x] = [s_0 x]$$

where $g \circ f$ is an arbitrary candidate composition of g and f. Furthermore, there is a natural isomorphism of categories

$$\mathrm{Ho}(\mathcal{C}) \cong \tau_1(\mathcal{C}).$$

Remark. Let $\mathcal{C}$ be an ∞-category and x, y are objects in $\mathcal{C}$. A morphism $f : x \to y$ can be viewed as an element $f \in \mathcal{C}_1$ satisfying $d_1 f = x$ and $d_0 f = y$. Recall a homotopy between $f, g : x \to y$ is defined as

$$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"] & & y \end{tikzcd}$$

This can be viewed as an element $\sigma \in \mathcal{C}_2$ satisfying two vertices are x and one vertex is y. This can be interpreted as a 2-morphism from x to y. Similarly, for Figure 2.1, this is a 3-morphism, or a 2-homotopy (homotopy between the homotopy we defined). If we denote

$$\sigma : f \simeq g, \quad \tilde{\sigma} : g \simeq f.$$

Then this 3-morphism gives a 2-homotopy between $\sigma \circ \tilde{\sigma}$ and the constant homotopy of g. A similar argument can show that all higher morphism is invertible.

The above remark shows that there exists some space of morphisms, assembled from the sets of all n-morphisms from $n \geq 2$. Moreover, the higher morphisms are all invertible, giving a model of $(\infty, 1)$-category. We now try to be more precise in this situation. Recall the adjunction we have in Appendix A.

Fix a simplicial set Y and consider the covariant functor $\mathcal{F} : \Delta \to s\mathcal{S}et$ given by

$$\begin{aligned} [n] &\mapsto \Delta^n \times Y, \\ f &\mapsto f \times id_Y. \end{aligned}$$

The left Kan extension L of $\mathcal{F}$ along the Yoneda embedding y gives a composition of the functor $- \times Y$ with the Yoneda embedding y. So in this case, L is isomorphic to $- \times Y$. L has a right adjoint and this is exactly how we define the simplicial mapping space functor:

Definition 2.1.13 (Mapping space)**.** Let us denote by

$$\mathrm{Map}(-,-) : s\mathcal{S}et^{op} \times s\mathcal{S}et \to s\mathcal{S}et$$

the simplicial mapping space functor (it is an internal hom) by

$$\mathrm{Map}(X, Y)_n := \mathrm{Hom}_{s\mathcal{S}et}(\Delta^n \times X, Y)$$

for any $X, Y \in s\mathcal{S}et$ and any $n \geq 0$. So the vertices are maps, edges are homotopies, and higher dimensional simplices are 'higher homotopies'.

Theorem 2.1.14. *Let $i : \Lambda^2_1 \to \Delta^2$ be the inclusion of horns. A simplicial set X is an ∞-category if and only if the restriction map*

$$i^* : \mathrm{Map}(\Delta^2, X) \to \mathrm{Map}(\Lambda^2_1, X)$$

is an acyclic Kan fibration.

This theorem can be understood in the following way. The simplicial set $\mathrm{Map}(\Lambda^2_1, X)$ can be viewed as the space of composition problems and $\mathrm{Map}(\Delta^2, X)$ is the space of solutions to composition problems. The theorem implies that these two spaces are homotopic equivalent in some sense. More specifically, let $f : x \to y$ and $g : y \to z$ be a pair of composable arrows in $\mathcal{C}$. The vertex $\lambda : \Delta^0 \to \mathrm{Map}(\Lambda^2_1, X)$ is the associated horn. Consider the following pullback square

$$\begin{tikzcd} F_\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\Delta^2, X) \arrow[d, "i^*"] \\ \Delta^0 \arrow[r, "\lambda"'] & \mathrm{Map}(\Lambda^2_1, X) \end{tikzcd}$$

The pullback F_λ can be viewed as the space of all possible composition of g and f. The above theorem tells us that i^* is a weak equivalence and a fibration, so the pullback $F_\lambda \to \Delta^0$ is also a weak equivalence. This implies this space F_λ is contractible. To summarize:

(i) A composition exists (We have a pullback and the pullback space is not empty).

(ii) Any two choices of compositions are homotopic (F_λ has trivial π_0).

(iii) The higher homotopies are unique up to homotopy (F_λ is contractible).

This implies that any candidate in the candidate composition we choose is as good as any other. This motivates us to define the equivalence in an ∞-category.

Definition 2.1.15 (Equivalence)**.** A morphism $f : x \to y$ in an ∞-category $\mathcal{C}$ is an **equivalence** if $[f] : x \to y$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$.

It is obvious that identities are equivalences. Given two homotopic maps $f_1 \simeq f_2$, f_1 is an equivalence if and only if f_2 is an equivalence. Surprisingly, the requirement of being an equivalence is very similar to what we require to be an isomorphism in an ordinary category. More specifically, $f : x \to y$ is an equivalence in an ∞-category if and only if there exists a morphism $g : y \to x$ such that there are 2-simplices with boundaries as such:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "id_x"'] & & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "g"] \arrow[rr, "id_y"'] & & y \end{tikzcd}$$

There are no requirements for any higher morphisms. The proof is complicated. Now we return to the concepts of ∞-groupoids.

Definition 2.1.16 (∞-groupoids)**.** An ∞-category is an ∞-groupoid if the homotopy category is a groupoid.

By definition, an ∞-category is an ∞-groupoid if and only if all the morphisms are equivalences. An important result from Joyal states that the outer horns can be extended if certain edges in an n-simplex become equivalence.

Proposition 2.1.17. *Let $\mathcal{C}$ be an ∞-category. Suppose $\lambda : \Lambda_0^n \to \mathcal{C}$, $n \geq 2$ is an outer horn. If the edge given by the vertices $\{0,1\}$ is an equivalence for this horn λ, then λ can be extended to a simplex $\Delta^n \to \mathcal{C}$.*

A similar statement can be made using another outer horn Λ_n^n. The idea at the beginning that all spaces are ∞-groupoids can be made into the following precise statement.

Proposition 2.1.18. *An ∞-category is an ∞-groupoids if and only if it is a Kan complex.*

The following is an interesting example of ∞-groupoids.

Example 2.1.19 (Singular complex of a topological space)**.** Let $\mathcal{C} = \mathcal{T}op$ be the category of topological spaces, which is cocomplete. Consider the functor $\mathcal{F} : \Delta \to \mathcal{T}op$ mapping $[n]$ to the standard n-simplex Δ^n_{Top} in $\mathbb{R}^{n+1}$:

$$\Delta^n_{Top} = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \right\}.$$

The functor of evaluation at $\mathcal{F}$ is exactly

$$\mathcal{F}^*([n]) = \mathrm{Hom}_{\mathcal{T}op}(\Delta^n_{Top}, -)$$

which is the singular functor $\mathrm{Sing}_n(-)$. It is right adjoint to the geometric realization functor $|-|$:

$$|X| = \operatorname*{colim}_{\Delta^n \to X \text{ in } \Delta/X} \Delta^n_{\mathrm{Top}}$$

and we recover the familiar adjunction formula

$$\mathrm{Hom}_{\mathcal{T}op}(|X|, Y) \cong \mathrm{Hom}_{s\mathcal{S}et}(X, \mathrm{Sing}(Y)).$$

Moreover, in the category of $\mathcal{T}op$, we know that we have the horn extension properties for all horns

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r] \arrow[d] & X \\ {|\Delta^n|} \arrow[ur, dashed] & \end{tikzcd}$$

Because $|\Lambda_k^n| \to |\Delta^n|$ is a cofibration and a weak equivalence. Use the adjunction formula, we obtain the horn extension properties for all horns to $\mathrm{Sing}_\bullet(X)$. This proves $\mathrm{Sing}_\bullet(X)$ is a Kan complex, and thus an ∞-groupoid.

2.2 Simplicial categories and the relations to ∞-categories

In this section, we provide an alternate approach to a theory of $(\infty, 1)$-categories, using **simplicial categories**, or simplicially enriched categories.

Chapter 3

Categorical constructions with ∞-categories

This chapter focuses on extending some key constructions and notions from the ordinary category theory to the ∞-categories (or more generally, the simplicial sets). The following principals need to be satisfied:

(a) The extensions are compatible with the fully faithful nerve functor $N : \mathcal{C}at \to s\mathcal{S}et$.

(b) A homotopy coherent version of ordinary category theory.

(c) They are compatible with the inclusion from ∞-categories to the simplicial sets.

(d) They are invariant under equivalence of ∞-categories.

We will first introduce the functors between ∞-categories, and try to convince the reader why this is a homotopy coherent concept. Next, we are going to define the limits and colimits for the ∞-categories, using the overcategories and undercategories. Lastly, we discuss the concepts of Kan extension in the setting of ∞-categories.

3.1 Functors

Recall that $infty$-categories are simplicial sets with right lifting properties with respect to inner horn inclusions. We can use the definition of maps between simplicial sets to define functors between ∞-categories.

Definition 3.1.1 (Functors)**.** Let K be a simplicial set and $\mathcal{C}$ be an ∞-category. A **functor**

$$F : K \to \mathcal{C}$$

is a map of simplicial set. Similarly, a **natural transformation** is a map of simplicial sets

$$\eta : \Delta^1 \times K \to \mathcal{C}.$$

More generally, the **space of functors** $\mathrm{Fun}(K, \mathcal{C})$ is a simplicial set whose n-simplices are given by

$$\mathrm{Fun}(K, \mathcal{C})_n := \mathrm{Map}_{s\mathcal{S}et}(K, \mathcal{C})_n = \mathrm{Hom}_{s\mathcal{S}et}(\Delta^n \times K, \mathcal{C}).$$

This definition agrees with the classical definition of functors if the simplicial set is the nerve of an ordinary category.

Lemma 3.1.2. *For categories A, B, there is a natural isomorphism of simplicial sets*

$$N(\mathrm{Fun}(A, B)) \cong \mathrm{Fun}(NA, NB).$$

Proof. For any $[n] \in \Delta$, we have the following natural bijections:

$$\begin{aligned} N(\mathrm{Fun}(A, B))_n &= \mathrm{Hom}_{Cat}([n], \mathrm{Fun}(A, B)) \\ &\cong \mathrm{Hom}_{Cat}([n] \times A, B) \\ &\cong \mathrm{Hom}_{sSet}(N([n] \times A), NB) \\ &\cong \mathrm{Hom}_{sSet}(N[n] \times NA, NB) \\ &\cong \mathrm{Hom}_{sSet}(\Delta^n \times NA, NB) \\ &= \mathrm{Fun}(NA, NB). \end{aligned}$$

Here we use the adjointness of $\mathrm{Fun}(A, -)$ and $- \times A$, the full faithfulness of N, the fact that N preserves products, and the isomorphism $N[n] \cong \Delta^n$. $\square$

The following example illustrates that the definition of functor encodes the homotopy coherent information we want.

Example 3.1.3. Let $A \in \mathcal{C}at$ be an ordinary category and $\mathcal{M}$ be a locally fibrant simplicial category. The coherent nerve $N_\Delta(\mathcal{M})$ is an ∞-category. Consider the functor $F : NA \to N_\Delta(\mathcal{M})$. We are going to see how the arrows in A are mapped to diagrams in $N_\Delta(\mathcal{M})$.

(a) For any arrow $x \to y$ in A, it can be viewed as a 1-simplex $s : \Delta^1 \to NA$, and thus $F(s) : \Delta^1 \to N_\Delta(\mathcal{M})$ is a 1-simplex.

(b) Similarly, any composable arrows $x \xrightarrow{f} y \xrightarrow{g} z$ gives rise to a 2-simplex $\sigma : \Delta^2 \to NA$, and we can obtain a 2-simplex $F(\sigma) : \Delta^2 \to N_\Delta(\mathcal{M})$. By the adjointness of $(C[-], N_\Delta)$, this

2-simplex can be viewed as a map $C[\Delta^2] \to \mathcal{M}$, namely a diagram

$$\begin{tikzcd} & Fy \arrow[dr, "Fg"] \arrow[d, phantom, "\Uparrow"] & \\ Fx \arrow[ur, "Ff"] \arrow[rr, "F(g\circ f)"'] & {} & Fz \end{tikzcd}$$

Note that here it also encodes the information of a specific homotopy from $F(g \circ f)$ to $Fg \circ Ff$.

(c) Now consider 3 composable arrows

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$$

in A. The composition can be viewed as a 3-simplex $\tau : \Delta^3 \to NA$: Now apply F, and we

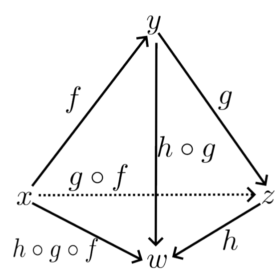

Figure 3.1: Composition of 3 arrows

obtain a 3-simplex $F(\tau) : \Delta^3 \to N_\Delta(\mathcal{M})$, each face has a homotopy, and this 3-simplex can be viewed as the commutative diagram

$$\begin{tikzcd} F(h\circ g\circ f) \arrow[r] \arrow[d] & F(h)\circ F(g\circ f) \arrow[d] \\ F(h\circ g)\circ F(f) \arrow[r] & F(h)\circ F(g)\circ F(f) \end{tikzcd}$$

where arrows are homotopy in each face. The commutativity comes from the fact that $\mathrm{Map}_{C[\Delta^3]}(0,3)$ is isomorphic to the product $\Delta^1 \times \Delta^1$.

The following proposition tells us that the equivalence of ∞-categories behave well under the definition of functors.

Proposition 3.1.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and K and L be simplicial sets.*

(i) The simplicial set $\mathrm{Fun}(K, \mathcal{C})$ *is an ∞-category.*

(ii) If $\mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories, then the induced map

$$\mathrm{Fun}(K, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{D})$$

is also an equivalence of ∞-categories.

(iii) If $K \to L$ is a categorical equivalence of simplicial sets, then the induced map

$$\mathrm{Fun}(L, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C})$$

is an equivalence of ∞-categories.

The proof relies on a careful analysis of the Joyal model structure for simplicial sets.

One part that is worth pointing out is that

3.2 Slice categories

The purpose of this section is to establish some useful constructions to help define limits and colimits in the ∞-categories in the next part. The way to do that is to realize the colimits as initial objects in the undercategories. Most things we discussed below can be dualized. Recall the classical situation in the ordinary category theory. Let A be an ordinary category and fix an object $X \in A$. We can form an overcategory $A_{/X}$ as follows:

- Objects are morphisms $Y \to X$ in A.
- Morphisms from an object $Y \to X$ to $Z \to X$ is a triangle:

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\ & X & \end{tikzcd}$$

A similar but dual construction gives the undercategory $A_{X/}$ (consider maps from X). The overcategory $A_{/X}$ can be characterized using an adjointness relation. For that, we need to first define the join construction.

Definition 3.2.1 (Join in ordinary categories)**.** Let A and B be categories. We can form a new category $A \star B$, the **join** of A and B, as follows. The objects of $A \star B$ are the disjoint union of objects in A and objects in B. For the morphisms, we have the following:

$$\mathrm{Hom}_{A\star B}(x,y)=\begin{cases}\mathrm{Hom}_A(x,y), & \text{if } \ x,y\in A,\\ \mathrm{Hom}_B(x,y), & \text{if } \ x,y\in B,\\ *, & \text{if } \ x\in A, y\in B,\\ \varnothing, & \text{if } \ x\in B, y\in A.\end{cases}$$

The composition is completely determined by requiring that A and B are full subcategories of $A\star B$ in the obvious way.

Note that this construction is not symmetric in A and B. Here is a key example.

Example 3.2.2. Let A be a category and $B=\{\bullet\}$ be the terminal category in $\mathcal{C}at$ (one object with one identity morphism). We write $A^{\triangleright}=A\star\{\bullet\}$ as the **right cone** or **cocone** on A. It can be viewed as adding a new terminal object ∞ to A. The dual concept is the **left cone** or **cone** on A, denoted by $A^{\triangleleft}:=\{\bullet\}\star A$. They play an essential role in the definition of colimits and limits.

Let B be a category. Note that to specify an object X in the category A is the same as giving a functor

$$x:\{\bullet\}\to A$$

sending the point to X. We have the following bijection

$$\mathrm{Hom}_{\mathcal{C}at}(B,A_{/X})\cong\mathrm{Hom}_{\mathcal{C}at_x}(B\star\{\bullet\},A)$$

where on the right-hand side, we only take those functors satisfying that the composition

$$\{\bullet\}\to B\star\{\bullet\}\to A$$

sends the point to X. To give a sketch why this works, take a morphism $s:a\to b$ in B and a functor $F:B\to A_{/X}$, we can write the morphism $F(s)$ in $A_{/X}$ as a triangle

$$\begin{tikzcd} F(a) \arrow[rr] \arrow[dr] & & F(b) \arrow[dl] \\ & X & \end{tikzcd}$$

On the right-hand side, the inclusion functor $B\to B\star\{\bullet\}$ sends the morphism $s:a\to b$ to a triangle

$$\begin{tikzcd} a \arrow[rr, "s"] \arrow[dr] & & b \arrow[dl] \\ & \bullet & \end{tikzcd}$$

Now if we apply a functor sending $\bullet$ to $X \in A$, then we get the same triangle. The inverse direction is similar. We want to use this adjointness to define a similar thing for simplicial sets. First, we extend the join construction to simplicial sets.

Definition 3.2.3 (Join of simplicial sets)**.** Let K and L be simplicial sets. The simplicial set $K \star L$, called the join of K and L, is defined as follows: for each nonempty finite linearly ordered set J, we set

$$(K \star L)(J) = \coprod_{J = I \cup I'} K(I) \times L(I')$$

where the union is taken over all decomposition of J into disjoint subset I and I', satisfying $i < i'$ for all $i \in I$ and $i' \in I'$. Here we allow the possibility that I or I' is empty, in which case

$$K(\varnothing) = L(\varnothing) = *.$$

More concretely, the n-simplices in the simplicial set $K \star L$ are given by

$$(K \star L)_n = K_n \cup L_n \cup \bigcup_{i+j=n-1} K_i \times L_j.$$

The join construction for simplicial sets can be characterized by the following two properties:

Proposition 3.2.4. *(i) The partial join functors*

$$K \star (-) : s\mathcal{S}et \to s\mathcal{S}et_{K/},$$
$$(-) \star L : s\mathcal{S}et \to s\mathcal{S}et_{L/}$$

preserves colimits. Here $s\mathcal{S}et_{K/}$ and $s\mathcal{S}et_{L/}$ are undercategories for simplicial sets K and L.

(ii) For the standard simplices, we have

$$\Delta^i \star \Delta^j \cong \Delta^{i+j+1}.$$

The isomorphism is compatible with the inclusion of Δ^i and Δ^j.

Proof. Both of these proofs can be done by carefully examining the definition. Note that in (i), the product

$$\times : s\mathcal{S}et \times s\mathcal{S}et \to s\mathcal{S}et$$

is left adjoint, so it commutes with colimits. □

This definition of join can be viewed as the correct generalization of joins of categories in the following sense.

Lemma 3.2.5. *The nerve is compatible with the join construction in that there is a natural isomorphism*

$$N(A \star B) \to N(A) \star N(B).$$

Proof. We check the n-simplices in the simplicial sets. Recall that

$$N(A \star B)_n = \mathrm{Fun}([n], A \star B).$$

The way $[n]$ maps to $A \star B$ preserving linear order is exactly given by the partition of

$$\{0 < 1 < 2 < \cdots < n\}$$

while allowing empty set. □

Remark. Recall the functor $C[-]$ sending a simplicial set to a simplicial category is colimit-preserving. For simplicial sets K and L, the simplicial category $C[K \star L]$ contains $C[K]$ and $C[L]$ as full subcategories, but only a unique map

$$C[K \star L] \to C[K] \star C[L]$$

(not isomorphism). This can be shown as an equivalence of simplicial categories [3, Corollary 4.2.1.4].

The join construction has nice properties with respect to ∞-categories.

Proposition 3.2.6. *(i) If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then the join $\mathcal{C} \star \mathcal{D}$ is again an ∞-category.*

(ii) If $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{D} \to \mathcal{D}'$ are equivalences of ∞-categories, then the induced map

$$F \star G : \mathcal{C} \star \mathcal{D} \to \mathcal{C}' \star \mathcal{D}'$$

is also an equivalence of ∞-categories.

Proof. □

Example 3.2.7. (i) Similar to the ordinary category theory, we call $K^{\triangleright} := K \star \Delta^0$ as the **right cone** or **cocone** on K for a simplicial set K. Dually, $L^{\triangleleft} := \Delta^0 \star L$ is the **left cone** or **cone** on a simplicial set L.

(ii) Now let $K = \Lambda_0^2$ be a horn in Δ^2. It can be viewed as the following:

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 & & 2 \end{tikzcd}$$

The cocone $(\Lambda_0^2)^{\triangleright}$ is isomorphic to a square $\square \cong \Delta^1 \times \Delta^1$:

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] \arrow[dd] & \\ 1 \arrow[dr] & \Leftarrow \quad \Rightarrow & 2 \arrow[dl] \\ & \bullet & \end{tikzcd}$$

If $\mathcal{M}$ is a simplicial category then a diagram in $N_{\Delta}(\mathcal{M})$ consists of 5 morphisms and 2 homotopies, commuting up to some coherent homotopy. We also have some dualizing square.

Recall that previously we mentioned fixing an object X in an ordinary category A gives us the overcategory $A_{/X}$ with a bijection

$$\mathrm{Hom}_{\mathcal{C}at}(B, A_{/X}) \cong \mathrm{Hom}_{\mathcal{C}at_x}(B \star \{\bullet\}, A)$$

Here the object X can be identified with a functor $\{\bullet\} \to A$. We want to generalize this concept to simplicial sets and to any maps between simplicial sets, not just the inclusion of Δ^0.

Definition 3.2.8 (Overcategory)**.** Let K and L be simplicial sets and $p : K \to L$ be a map of simplicial sets. Define a simplicial set

$$(L_{/p})_n := \mathrm{Hom}_{s\mathcal{S}et_p}(\Delta^n \star K, L).$$

Note that here we only take maps that satisfy the composition

$$K \to \Delta^n \star K \to L$$

is equal to p.

The universal property holds

$$\mathrm{Hom}_{s\mathcal{S}et}(S, L_{/p}) \cong \mathrm{Hom}_{s\mathcal{S}et_p}(S \star K, L)$$

for any simplicial set S. We can prove this by first note it holds for any standard simplices Δ^n by definition, and any simplicial set can be viewed as the colimit of standard simplices. The following proposition shows that this construction is well-behaved:

Proposition 3.2.9. *Let K be a simplicial set and $\mathcal{C}$ is an ∞-category. Suppose $p : K \to \mathcal{C}$ is a map of simplicial set. Then $\mathcal{C}_{/p}$ is an ∞-category. Moreover, if $q : \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories, then the induced map*

$$\mathcal{C}_{/p} \to \mathcal{D}_{/qp}$$

is also an equivalence of ∞-categories.

Example 3.2.10. We can quickly check the definition of overcategories is compatible with the classical definition. Let $\mathcal{C}$ be an ∞-category and x is an object in $\mathcal{C}$. The objects, or 0-simplices in $\mathcal{C}_{/x}$, are given by

$$(\mathcal{C}_{/x})_0 = \mathrm{Hom}_{s\mathcal{S}et_x}(\Delta^0 \star \{\bullet\}, \mathcal{C}).$$

In the right-hand side, we require $\bullet$ is mapped to $x \in \mathcal{C}$, so it is given by a morphism $y \to x$ where y is the image of Δ^0. The morphisms in $(\mathcal{C}_{/x})$ are 1-simplices:

$$(\mathcal{C}_{/x})_1 = \mathrm{Hom}_{s\mathcal{S}et_x}(\Delta^1 \star \{\bullet\}, \mathcal{C}).$$

So it is a triangle

$$\begin{tikzcd} y_1 \arrow[rr] \arrow[dr] & & y_2 \arrow[dl] \\ & x & \end{tikzcd}$$

This is exactly what we want.

Dually, we can form the undercategories $\mathcal{C}_{p/}$, and together they are called the slice ∞-categories. These constructions are compatible with the nerve functor.

Lemma 3.2.11. *If $p : A \to B$ is a functor or ordinary categories, then there is a natural isomorphism of simplicial sets*

$$N(B_{/p}) \cong N(B)_{/N(p)}$$

Proof. □

3.3 Colimits and limits

Now we can discuss the final and initial objects in ∞-categories, and use them to define colimits and limits. Recall that in an ordinary category, the final object, if exists, admits a unique morphism from all objects, and it is unique up to unique isomorphism. In ∞-categories, we need a less strict version.

Definition 3.3.1 (Final object)**.** An object $x \in \mathcal{C}$ of an ∞-category $\mathcal{C}$ is a **final object** if the canonical map $\mathcal{C}_{/x} \to \mathcal{C}$ is an acyclic fibration of simplicial sets.

This definition has some equivalent reinterpretations.

Proposition 3.3.2. *The following is equivalent for an object $x \in \mathcal{C}$ of an ∞-category $\mathcal{C}$.*

(i) The object x is final.

(ii) The mapping spaces $\mathrm{Map}_{\mathcal{C}}(x', x)$ are acyclic Kan complexes for all $x' \in \mathcal{C}$.

(iii) Every simplicial sphere $\alpha : \partial\Delta^n \to \mathcal{C}$ such that $\alpha(n) = x$ can be filled to an entire n-simplex $\Delta^n \to \mathcal{C}$.

Now before discuss what 'uniqueness' means in the setting of ∞-categories, we first need to mention full subcategories of an ∞-category.

need to fill

Corollary 3.3.3. *Let $\mathcal{C}$ be an ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ be the full subcategory spanned by the final objects of $\mathcal{C}$. Then $\mathcal{D}$ is empty or a contractible Kan complex.*

Lemma 3.3.4. *Let A be a category. An object $a \in A$ is final if and only if $N(a) \in N(A)$ is final.*

Now we can define the colimits in ∞-categories. Recall that in an ordinary category, the colimit of an orodinary functor $p : A \to B$ consists of an object $\mathrm{colim}_A p$ in B together with a universal cocone. Said differently, this can be viewed as an initial object in the category of $B_{p/}$ of cocones on p. We can extend this definition to ∞-categories.

Definition 3.3.5 (colimits)**.** Let K be a simplicial set and $\mathcal{C}$ be an ∞-category. A **colimit** of a diagram $p : K \to \mathcal{C}$ is an initial object in $\mathcal{C}_{p/}$. An ∞-category is **cocomplete** if it admits colimits of all small diagrams.

Remark. A colimit of a diagram $p : K \to \mathcal{C}$ is an object of $\mathcal{C}_{p/}$. We may identify this object with a map $\bar{p} : K^{\triangleright} \to \mathcal{C}$ extending p. In general, we say that a map $\bar{p} : K^{\triangleright} \to \mathcal{C}$ is a colimit diagram if it is a colimit of $p = \bar{p}|_K$. Sometimes we abuse notations and call $\bar{p}(\infty) \in \mathcal{C}$ as a colimit of p.

It is immediately from Corollary 3.3.3 that for every $p : K \to \mathcal{C}$ the full subcategory $\mathcal{D} \subseteq \mathcal{C}_{p/}$ spanned by the colimits of p is empty or a contractible Kan complex. Also, by definition, the nerve functor is compatible with the notion of colimits.

Definition 3.3.6 (pushout and pullback)**.** Let $\mathcal{C}$ be an ∞-category and $q : \square \to \mathcal{C}$ is a square.

(i) The square q is a **pushout** if $q : (\Lambda^2_0)^{\triangleright} \to \mathcal{C}$ is a colimiting cocone.

(ii) The square q is a **pullback** if $q : (\Lambda^2_2)^{\triangleleft} \to \mathcal{C}$ is a limiting cone.

Appendix A

Some category theory

Let $\mathscr{A}$ be a small category. We call the category of contravariant functors

$$\mathscr{P}(\mathscr{A}) := \mathrm{Fun}(\mathscr{A}^{op}, \mathscr{S}et)$$

the presheaf category over $\mathscr{A}$. Suppose $\mathscr{C}$ is a cocomplete category, i.e., $\mathscr{C}$ admits all small colimits. We have the Yoneda embedding: $y : \mathscr{A} \to \mathscr{P}(\mathscr{A})$. Given a covariant functor $\mathscr{F} : \mathscr{A} \to \mathscr{C}$, we want to extend $\mathscr{F}$ to $\mathscr{P}(\mathscr{A})$, namely construct the following dotted arrow, making the diagram

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathscr{F}"] \arrow[d, "y"'] & \mathscr{C} \\ \mathscr{P}(\mathscr{A}) \arrow[ur, dashed] & \end{tikzcd}$$

commute. Even better, this extension admits a right adjoint. The statement will be made precise, following the book [4]. We first show that for a presheaf $X \in \mathscr{P}(A)$, X can be written as the colimit of representable functors. For any object $a \in \mathscr{A}$, we denote by h_a the representable functor $\mathrm{Hom}_{\mathscr{A}}(-, a)$ in $\mathscr{P}(\mathscr{A})$.

Definition A.0.1 (Category of elements)**.** Let $\mathscr{A}$ be a category and $X \in \mathscr{P}(\mathscr{A})$ is a presheaf over $\mathscr{A}$. We define the category of elements of X (denoted by $\mathscr{A}/X$) as follows:

- The objects are (a, s) where a is an object of $\mathscr{A}$ and s is a morphism in $\mathrm{Hom}_{\mathscr{P}(\mathscr{A})}(h_a, X)$, which can be viewed as a section $s \in X(a)$ by Yoneda lemma.

- The morphisms in $\mathscr{A}/X$ are given by a morphism $u : a \to b$ in $\mathscr{A}$. By Yoneda lemma, is a commutative diagram

$$\begin{tikzcd} h_a \arrow[rr, "u_*"] \arrow[dr, "s"'] & & h_b \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

Naturally we have a faithful functor

$$\mathcal{A}/X \to \mathcal{P}(\mathcal{A})$$

sending (a, s) to h_a. The collection of maps

$$s : h_a \to X$$

for $(a, s) \in \mathcal{A}/X$ gives a cocone over X. And we have the following.

Proposition A.0.2. *Any $(a, s) \in \mathrm{Ob}(\mathcal{A}/X)$ gives rise to a morphism $s : h_a \to X$. If $(a, s), (b, t)$ are two objects in $\mathcal{A}/X$ and $u : a \to b$ is a morphism in $\mathcal{A}$, we have a commutative triangle*

$$\begin{tikzcd} h_a \arrow[rr, "u"] \arrow[dr, "s"'] & & h_b \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

Moreover, X can be written as colimit of such functors.

$$X \cong \mathrm{colim}_{(a,s)\in\mathcal{A}/X} h_a$$

If $\mathcal{A}$ is a small category, then $\mathcal{P}(\mathcal{A})$ is a locally small category. The covariant functor $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ gives us a functor of evaluation at $\mathcal{F}$:

$$\begin{aligned} \mathcal{F}^* : \mathcal{C} &\to \mathcal{P}(\mathcal{A}), \\ Y &\mapsto \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(-), Y) \end{aligned}$$

Proposition A.0.3 (Kan)**.** *The functor $\mathcal{F}^*$ has a left adjoint*

$$\mathcal{F}_! : \mathcal{P}(\mathcal{A}) \to \mathcal{C}.$$

Moreover, for any object a in $\mathcal{A}$, there is a unique natural isomorphism

$$\mathcal{F}(a) \cong \mathcal{F}_!(h_a)$$

such that, for any object $Y \in \mathcal{C}$, the induced bijection gives the adjunction formula

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}_!(h_a), Y) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(a), Y) \cong \mathrm{Hom}_{\mathcal{P}(\mathcal{A})}(h_a, \mathcal{F}^*(Y)).$$

Proof. Let $X \in \mathcal{P}(\mathcal{A})$ be a presheaf. By the previous proposition, we know that

$$X \cong \mathrm{colim}_{(a,s)} h_a.$$

Let $p : \mathcal{A}/X \to \mathcal{A}$ be the projection functor sending (a,s) to $a \in \mathcal{A}$. Define

$$\begin{aligned} \mathcal{F}_!(X) : &= \mathrm{colim}_{(a,s)}(\mathcal{F} \circ p)(a,s) \\ &= \mathrm{colim}_{(a,s)}\mathcal{F}(a). \end{aligned}$$

When $X = h_a$ is representable, we get the canonical isomorphism

$$\mathcal{F}_!(h_a) \cong \mathcal{F}(a)$$

because (a, id) is the final object in $\mathcal{A}/h_a$. Recall that the Hom functor preserves limits, so we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}_!(X), Y) &\cong \mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{(a,s)}\mathcal{F}(a), Y) \\ &\cong \lim_{(a,s)} \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(a), Y) \\ &\cong \lim_{(a,s)} \mathrm{Hom}_{\mathcal{P}(\mathcal{A})}(h_a, \mathcal{F}^*(Y)) \\ &\cong \mathrm{Hom}_{\mathcal{P}(\mathcal{A})}(\mathrm{colim}_{(a,s)} h_a, \mathcal{F}^*(Y)) \\ &\cong \mathrm{Hom}_{\mathcal{P}(\mathcal{A})}(X, \mathcal{F}^*(Y)). \end{aligned}$$

This proves that $\mathcal{F}_!$ is left adjoint to $\mathcal{F}^*$. $\square$

Now we consider $\mathcal{P}(\mathcal{A})$ is the category of simplicial sets, i.e., when $\mathcal{A}$ is the category of finite ordinals. Let $\mathcal{F} : \Delta \to \mathcal{C}$ be a covariant functor from Δ to a cocomplete, locally small category. Suppose $Y \in \mathcal{C}$ is an object, we can define

$$(\mathcal{F}^*Y)_n := \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}[n], Y).$$

We have $d_i : (\mathcal{F}^*Y)_n \to (\mathcal{F}^*Y)_{n-1}$ is given by the precomposition with $\mathcal{F}d^i : \mathcal{F}[n-1] \to \mathcal{F}[n]$ and $s_i : (\mathcal{F}^*Y)_n \to (\mathcal{F}^*Y)_{n+1}$ is given by the precomposition with $\mathcal{F}s^i : \mathcal{F}[n+1] \to \mathcal{F}[n]$. Checking the simplicial identities implies that $\mathcal{F}^*Y$ is a simplicial set. The left adjoint $\mathcal{F}_!$ is the left Kan extension of $\mathcal{F}$ along the Yoneda embedding y:

$$\begin{tikzcd} \Delta \arrow[r, "\mathcal{F}"] \arrow[d, "y"'] & \mathcal{C} \\ s\mathcal{S}et \arrow[ur, dashed, "L"] & \end{tikzcd}$$

From our discussion above, the left Kan extension L is just $\mathcal{F}_!$, and it is left adjoint to $\mathcal{F}^*$. There is a more explicit construction of L using the copower, see [6].

Definition A.0.4 (copower)**.** For any set S and object $c \in \mathcal{C}$, the **copower** or **tensor** of c by S, denoted $S \cdot c$, is simply the coproduct $\bigsqcup_S c$: copies of c indexed by S.

If X is a simplicial set, we may form copowers

$$X_m \cdot \mathcal{F}[n]$$

for any $n, m \geq 0$. A morphism $f : [n] \to [m]$ induces a map

$$f_* : X[m] \cdot \mathcal{F}[n] \to X[m] \cdot \mathcal{F}[m]$$

and a map

$$f^* : X[m] \cdot \mathcal{F}[n] \to X[n] \cdot \mathcal{F}[n].$$

The left Kan extension for a simplicial set X is

$$LX := \operatorname{coeq}\Big(\bigsqcup_{f:[n]\to[m]} X_m \cdot \mathcal{F}[n] \underset{f^*}{\overset{f_*}{\rightrightarrows}} \bigsqcup_{[n]} X_n \cdot \mathcal{F}[n] \Big).$$

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