Notes on Readings about Commutative Algebra

February 5, 2026

Contents

Chapter 1

Regular Sequence and the Koszul Complex

All rings are assumed to be Noetherian in this chapter.

Definition 1.0.1 (Regular Sequence)**.** Let M be an R-module. An ordered sequence of elements $x_1, \dots, x_n \in R$ is called a **regular sequence** on M (or an M-sequence) if the following 2 conditions are satosfied:

(1) $(x_1, \dots, x_n)M \neq M$,

(2) For $i = 1, \dots, n$, x_i is a nonzerodivisor on $M/(x_1, \dots, x_{i-1})M$ (Assume $x_0 = 0$).

Note that here x_i cannot be a zerodivisor in M for $1 \leq i \leq n$.

We will use a homological tool called the Koszul complex to study the regular sequence.

1.1 Koszul complexes of length 1 and 2

For any $x \in R$, we can define a chain complex $K(x)$ as follows:

$$0 \to R \xrightarrow{x} R$$

The homology in the middle is denoted by $H^0(K(x))$ (Note that here we view the complex as cochain complex and the index follows as cohomology) and given by the ideal quotient $(0 : x)$. Given another element $y \in R$, y gives a map between chain complexes:

$$\begin{tikzcd}
K(x): & 0 \arrow[r] & R \arrow[r, "x"] \arrow[d, "y"'] & R \arrow[d, "y"] \\
K(x): & 0 \arrow[r] & R \arrow[r, "x"] & R
\end{tikzcd}$$

We can form the mapping cone of this chain map and obtain a complex

$$K(x,y) : 0 \to R \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} R \oplus R \xrightarrow{(-x,y)} R \to 0.$$

This is different from the usual way Koszul complex is written

$$0 \to R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R \oplus R \xrightarrow{(x,y)} R \to 0.$$

This version will be denoted as $K'(\varphi)$ where $\varphi : R^2 \to R$ is given by (x, y). The two versions are isomorphic as complexes.

The mapping cone $K(x, y)$ can be fitted into a short exact sequence

$$0 \to K(x)[-1] \to K(x,y) \to K(x) \to 0$$

and induces a long exact sequence in homology:

$$\cdots \to H^0(K(x)) \xrightarrow{\delta} H^0(K(x)) \to H^1(K(x,y)) \to H^1(K(x)) \to \cdots$$

where the connecting homomorphism δ is induced by the chain map $y : K(x) \to K(x)$.

Let us discuss the homology groups in more details. The group $H^0(K(x)) = (0 : x)$ gives the annihilators of x, and it is not hard to see that $H^0(K(x, y)) = (0 : (x, y))$. If x is not a zerodivisor, then $H^0(K(x, y)) = 0$.

What about the homology group $H^1(K(x, y))$? An element $(a, b) \in \mathbb{R}^2$ is in the kernel if and only if $-ax + by = 0$ by definition. This implies that $b \in (x : y)$. Conversely, if $b \in (x : y)$, then there exists $a \in R$ such that $by = ax$, so (a, b) is in the kernel. Assume that x is not a zero divisor in R. In this case, the element a is uniquely determined by b, and the association $b \mapsto a$ is a module homomorphism. So the kernel is $(x : y)$.

On the other hand, the image is of the form $(cy, cx) \in R^2$, which is contained in the kernel. We know $b = cx$ completely determines this element, so the element in the image, if viewed as subsets of $(x : y)$, must be from the ideal (x). So the image is isomorphic to the ideal (x), and we can write

$$H^1(K(x,y)) = (x : y)/(x).$$

In particular, under the assumption that x is not a zerodivisor, the condition $H^1(K(x, y)) = 0$ is equivalent to the sequence x, y satisfies the 2nd condition in the definition of regular sequence when $M = R$. Note that this does not necessarily mean x, y is a regular sequence, but in some cases, it is.

Theorem 1.1.1. *If R is a Noetherian local ring and x, y are in the maximal ideal, then x, y is a regular sequence if and only if $H^1(K(x, y)) = 0$.*

Proof. Assume $H^1(K(x, y)) = 0$. From the long exact sequence, we know

$$y : H^0(K(x)) \to H^0(K(x))$$

is an isomorphism. This implies that

$$yH^0(K(x)) = H^0(K(x)).$$

Note that R is a Noetherian local ring and y is in the maximal ideal. By Nakayama's lemma, we have $H^0(K(x)) = 0$. So x is not a zerodivisor. Both x, y are in the maximal ideal, so $(x, y)R \neq R$. This implies that x, y is a regular sequence. The converse has already been shown from the above discussion. □

Remark. From the presentation of $K(x, y)$, it is not hard to see that $K(x, y)$ is isomorphic to $K(y, x)$. Under the hypothesis of the above theorem, x, y is a regular sequence if and only if y, x is a regular sequence. Note that in general, we cannot permute the order of elements in a regular sequence if the ring is not local.

Corollary 1.1.2. *If R is a Noetherian local ring and $x_1, \ldots, x_r$ is a regular sequence in the maximal ideal of R, then any permutation of $x_1, \ldots, x_r$ is again a regular sequence.*

Example 1.1.3. An example.

1.2 Koszul Complex in General

Now we build the Koszul complex in general. Let N be an R-module, and $\wedge N$ is the exterior algebra.

Definition 1.2.1 (Koszul complex)**.** Given an R-module N and $x \in N$, we define the Koszul complex to be the complex

$$K(x) : 0 \to R \to N \to \wedge^2 N \to \cdots \to \wedge^i N \xrightarrow{d_x} \wedge^{i+1} N \to \cdots$$

where the differential d_x sends an element a to $x \wedge a$. In particular, $d_x(1) = x \in N$. If $N \cong R^n$ is a free module of rank n, we can write

$$x = (x_1, \ldots, x_n) \in R^n \cong N.$$

In this case, we write $K(x_1, \ldots, x_n)$ instead of $K(x)$.

The functoriality of the Koszul complex is obvious from the functoriality of the exterior algebra. Suppose $f : N \to M$ is a map of R-modules satisfying $f(x) = y$, then the map

$$\wedge f : \wedge N \to \wedge M$$

preserves the differential and thus can be viewed as a map of complexes.

Proposition 1.2.2. *Let N be a free module of rank n. We have*

$$H^n(K(x_1, \ldots, x_n)) = R/(x_1, \ldots, x_n).$$

Proof. Consider the right hand side of the Koszul complex

$$\cdots \to \wedge^{n-1} N \to \wedge^n N \to 0.$$

Let $e_1, \ldots, e_n$ be a basis of $N \cong R^n$. Then $\wedge^n \cong R$ has rank 1 and is generated by $e_1 \wedge \cdots \wedge e_n$. And $\wedge^{n-1} N$ has a basis

$$e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n, \quad 1 \leq i \leq n.$$

So $\wedge^{n-1} N$ is isomorphic to R^n. We know that

$$x = (x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i e_i.$$

By definition, the differential d_x sends $e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n$ to

$$(\sum_{i=1}^{n} x_i e_i) \wedge e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n = \pm x_i e_1 \wedge \cdots \wedge e_n.$$

So the image of d_x is the ideal $(x_1, \ldots, x_n)$ and the homology group

$$H^n(K(x_1, \ldots, x_n)) = R/(x_1, \ldots, x_n).$$

□

Think about our discussion on the Koszul complex of length 2, it is reasonable to expect the homology of the Koszul complex has something to do with the regular sequences. In general, it does not detect whether $x_1, \ldots, x_n$ is a regular sequence, but somehow it detects the lengths of the maximal regular sequence in the ideal $(x_1, \ldots, x_n)$.

Theorem 1.2.3. *Let M be a finitely generated module over R. If*

$$H^i(M \otimes K(x_1, \ldots, x_n)) = 0$$

for all $j < r$ and

$$H^r(M \otimes K(x_1, \ldots, x_n)) \neq 0,$$

then every maximal M-sequence in $I = (x_1, \ldots, x_n) \subset R$ has length r.

Proof. Later. $\square$

Corollary 1.2.4. *If $x_1, \ldots, x_n$ is an M-sequence, then $M \otimes K(x_1, \ldots, x_n)$ is exact except at the extreme right,i.e.*

$$H^j(M \otimes K(x_1, \ldots, x_n)) = 0, \quad j < n,$$
$$H^n(M \otimes K(x_1, \ldots, x_n)) = M/(x_1, \ldots, x_n)M.$$

Proof. $K(x_1, \ldots, x_n)$ is a chain complex of free modules, tensoring with it is exact. $\square$

The converse of the theorem 1.2.3 is not true in general, but it does hold if R is a local ring. A more general version will be given later. Recall that $I = (x_1, \ldots, x_n) \subset R$ is an ideal in M. Assume

$$H^n(M \otimes K(x_1, \ldots, x_n)) = M/IM \neq 0.$$

In this case, the length r from the above theorem is positive, and the length if all maximal M-sequence is the same.

Definition 1.2.5 (Depth)**.** Let R be a Noetherian ring, $I \subset R$ be an ideal and M be an R-module. If $IM \neq M$, we define the depth of I on M, written $\operatorname{depth}(I, M)$, to be the length of any maximal M-sequence in I. If $M = R$, we shall simply speak of the depth of I. If $IM = M$, we adopt the convention that $\operatorname{depth}(I, M) = \infty$.

In the local case, the theorem 1.2.3 can be strengthened.

Theorem 1.2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be an R-module. Suppose $x_1, \ldots, x_n \in \mathfrak{m}$. If for some k,*

$$H^k(M \otimes K(x_1, \ldots, x_n)) = 0,$$

then

$$H^j(M \otimes K(x_1, \dots, x_n)) = 0$$

for all $j \leq k$. In particular, if $H^{n-1}(M \otimes K(x_1, \dots, x_n)) = 0$, then $x_1, \dots, x_n$ is an M-sequence.

Proof. Later. □

The depth of an ideal I is a kind of arithmetic measure of the size of I, while the codimension of I is a geometric measure. Like the codimension, the depth depends only on the radical of I. The theorem 1.2.3 implies that an ideal with r generators can have depth at most r. We shall see in general that

$$\operatorname{depth} I \leq \operatorname{codim} I.$$

Corollary 1.2.7 (Geometric nature of depth)**.** *(a) If $x_1, \dots, x_r$ is an M-sequence, then the sequence $x_1^t, \dots, x_r^t$ is an M-sequence for any positive integer t.*

(b) If I is an ideal of R and $J = \sqrt{I}$ is its radical, we have

$$\operatorname{depth}(I, M) = \operatorname{depth}(J, M).$$

Proof. (a) We do an induction on r to reduce to the local case. For $r = 1$, we know that a power of nonzero divisor is still a nonzero divisor. For $r \geq 2$, assume that $x_1^t, \dots, x_{r-1}^t$ is an M-sequence for any positive integer t. We need to show that x_r is a nonzero divisor on $M/(x_1^t, \dots, x_{r-1}^t)M$. We know that $x_1, \dots, x_r$ is an M-sequence, so there exists a maximal ideal $\mathfrak{m}$ such that $(x_1, \dots, x_r) \subset \mathfrak{m}$. Localizing at $\mathfrak{m}$ does not change if x_r is a nonzero divisor on $M/(x_1^t, \dots, x_{r-1}^t)M$, so we may assume R is a local ring and $x_1, \dots, x_r$ are contained in the maximal ideal.

If $x_1, \dots, x_r$ is an M-sequence, then $x_1, \dots, x_{r-1}, {x_r}^t$ is also an M-sequence. In a local ring R, we can permute the order of elements for any regular sequence. So $x_r^t, x_1, \dots, x_{r-1}$ is also an M-sequence. Repeating the argument, and we obtain that ${x_1}^t, {x_2}^t, \dots, x_r^t$ is an M-sequence.

(b) Since $I \subset J$, we have $\operatorname{depth}(I, M) \leq \operatorname{depth}(J, M)$. Conversely, if $x_1, \dots, x_r$ is an M-sequence in J, then for large enough t, we know that $x_1^t, \dots, x_r^t$ is also an M-sequence and $x_1^t, \dots, {x_r}^t$ are in I, so $\operatorname{depth}(J, M) \leq \operatorname{depth}(I, M)$.

□

1.3 Building Koszul complex from parts

We briefly describe the tensor product of two complexes and use it to prove Theorem 1.2.3.

Proposition 1.3.1. *If $N = N' \oplus N''$, then $\wedge N = \wedge N' \oplus \wedge N''$ as skew-commutative algebra. If $x' \in N'$ and $x'' \in N''$, then let $x = (x', x'') \in N$, we have*

$$K(x) = K(x') \otimes K(x'')$$

as complexes.

Proof. Algebra. Omitted. □

Corollary 1.3.2. *Let $y_1, \dots, y_r$ be elements of the ideal generated by $x_1, \dots, x_n \in R$, and M be any R-module. Then*

$$H^*(M \otimes K(x_1, \dots, x_n, y_1, \dots, y_r)) \cong H^*(M \otimes K(x_1, \dots, x_n)) \otimes \wedge R^r.$$

as graded modules. In particular, for each i, we have

$$H^i(M \otimes K(x_1, \dots, x_n, y_1, \dots, y_r)) \cong \sum_{i=j+k} H^k(M \otimes K(x_1, \dots, x_n)) \otimes \wedge^j R^r.$$

Thus,

$$H^i(M \otimes K(x_1, \dots, x_n, y_1, \dots, y_r)) = 0$$

iff

$$H^k(M \otimes K(x_1, \dots, x_n)) = 0$$

for all k with $i - r \leq k \leq i$.

1.4 Duality and Homotopy

Recall that the Koszul complex is defined via an element $x \in R$, which can be viewed as a map $x : R \to N$ for an R-module N. We can define a dual version

Chapter 2

Depth, Codimension and Cohen-Macaulay Rings

Same as the previous chapter, all rings are assumed to be Noetherian.

2.1 Depth

Let R be a ring, and $I \subset R$ is an ideal. M is a finitely generated R-module such that $IM \neq M$. Recall from the last chapter that the depth $\operatorname{depth}(I, M)$ is the length of the maximal M-sequence. It can be charaterized in terms of the vanishing of the homology of the Koszul complex. We want to explore the behavior of depth under localization.

Lemma 2.1.1. *R is a ring and P is a prime ideal in the support of a finitely generated R-module M, then any M-sequence in P localizes to a M_P-sequence. Thus for any ideal $I \subset P$, we have* $\operatorname{depth}(I, M) \leq \operatorname{depth}(I_P, M_P)$*, the latter taken in the ring R_P. In general, the inequality may be strict, but for any ideal I there exists a maximal ideal P in the support of M such that* $\operatorname{depth}(I, M) = \operatorname{depth}(I_P, M_P)$*. In particular, if P is a maximal ideal, then* $\operatorname{depth}(P, M) = \operatorname{depth}(P_P, M_P)$.

Proof. □

There is a lemma for the depth similar to the princiapl ideal theorem of codimension.

Lemma 2.1.2. *Let $(R, \mathfrak{m})$ be a local ring. M is a finitely generated R-module, $I \subset R$ is an ideal and $y \in \mathfrak{m}$. Then*

$$\operatorname{depth}(I + (y), M) \leq \operatorname{depth}(I, M) + 1.$$

Proof. □

2.2 Depth and the Vanishing of Ext

Proposition 2.2.1. *Let R be a ring and M, N be finitely generated R-modules. If* $\operatorname{ann} M + \operatorname{ann} N = R$*, then* $\operatorname{Ext}^r_R(M,N) = 0$ *for every r. Otherwise,* $\operatorname{depth}(\operatorname{ann} M, N)$ *is the smallest number r such that* $\operatorname{Ext}^r_R(M,N) \neq 0$.

Proof. Note that if $s \in \operatorname{ann} M$ or $s \in \operatorname{ann} N$, then s also annihilates $\operatorname{Hom}_R(M,N)$. This tells us that s annihilates $\operatorname{Ext}^r_R(M,N)$ for any r.
Claim: $\operatorname{ann} M + \operatorname{ann} N = R$ if and only if $\operatorname{ann}(M)N = N$.
Proof: Suppose $\operatorname{ann}(M)N = N$. By Nakayama's lemma, there exists $r \in R$ such that $(1-r)N = 0$. Thus $1 - r \in \operatorname{ann} N$, and $1 \in \operatorname{ann} M + \operatorname{ann} N$. This proves $\operatorname{ann} M + \operatorname{ann} N = R$. Conversely, suppose $\operatorname{ann} M + \operatorname{ann} N = R$. We can write $1 = r + s$ where $r \in \operatorname{ann} M$ and $s \in \operatorname{ann} N$, hence

$$rN = (r+s)N = N.$$

This implies that $\operatorname{ann}(M)N = N$. ■

Suppose $\operatorname{ann} M + \operatorname{ann} N \neq R$. This means that $\operatorname{ann}(M)N \neq N$, so the depth

$$d = \operatorname{depth}(\operatorname{ann} M, N) < \infty.$$

We do induction on d. If $d = 0$, We need to show that

$$\operatorname{Ext}^0_R(M,N) = \operatorname{Hom}_R(M,N) \neq 0.$$

Since $\operatorname{depth}(\operatorname{ann} M, N) = 0$, this means that $\operatorname{ann} M$ are all zero divisors for N, so $\operatorname{ann} M$ is contained in one of the associated primes $\mathfrak{p}$ of N. Note that for finitely generated modules M, N, localization commutes with taking hom, so it is enough to prove this after localizing at $\mathfrak{p}$. Note that $\mathfrak{p} = \operatorname{ann}_R(m)$ for some $m \in N$, so the localized $N_\mathfrak{p}$ must contain a copy of the field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ generated by m in $R_\mathfrak{p}$. On the other hand, we know that $M_\mathfrak{p} \neq 0$, by Nakayama's lemma, $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \neq 0$ as a vector space over $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. This implies we have a nonzero homomorphism from $M_\mathfrak{p}$ to $N_\mathfrak{p}$.

Now assume $d \geq 1$ and let $x \in \operatorname{ann} M$ be a nonzero divisor on N. We have $\operatorname{ann}(M)(N/xN) \neq N/xN$ and $\operatorname{depth}(\operatorname{ann} M, N/xN) = d - 1$. By induction $\operatorname{Ext}^{d-1}_R(M, N/xN) \neq 0$, and

$$\operatorname{Ext}^{<d-1}_R(M, N/xN) = 0.$$

Consider the short exact sequence

$$0 \to N \xrightarrow{x} N \to N/xN \to 0.$$

Apply $\mathrm{Ext}_R(M, -)$ and we obtain a long exact sequence. Note that $x \in \operatorname{ann} M$, so it annihilates $\mathrm{Ext}^j_R(M, N)$. The induced map will be the zero map. Thus, we get a collection of short exact sequence

$$0 \to \mathrm{Ext}^{j-1}_R(M, N) \to \mathrm{Ext}^{j-1}_R(M, N/xN) \to \mathrm{Ext}^j_R(M, N) \to 0$$

for all $j \geq 1$. By induction on d, we can see that

$$\mathrm{Ext}^i_R(M, N) = 0$$

for all $i < d$ and

$$\mathrm{Ext}^d_R(M, N) \cong \mathrm{Ext}^{d-1}_R(M, N/xN) \neq 0.$$

□

Example 2.2.2. Let k be a field and $R = k[x, y, z]$. Consider the ideal $I = (xy, yz) = (x, z) \cap (y)$. We have a free resolution for R/I:

$$0 \leftarrow R/I \leftarrow R \xleftarrow{(xy \quad yz)} R^2 \xleftarrow{\begin{pmatrix} -z \\ x \end{pmatrix}} R \leftarrow 0.$$

The $\mathrm{Ext}_R(R/I, R)$ can be computed from the following complex

$$R \xrightarrow{\begin{pmatrix} xy \\ yz \end{pmatrix}} R^2 \xrightarrow{(-z \quad x)} R$$

So $\mathrm{Hom}(R/I, R) = 0$, $\mathrm{Ext}^1_R(R/I, R) = R/(y)$ and $\mathrm{Ext}^2_R(R/I, R) = R/(x, z)$.

Example 2.2.3. Let $x_1, \ldots, x_n$ be a regular sequence in R. The Koszul complex $K(x_1, \ldots, x_n)$ is a free resolution of $R/(x_1, \ldots, x_n)$. Write $N = R/(x_1, \ldots, x_n)$. We know that $\mathrm{Ext}_R(N, M)$ can be computed via the homology of the complex $\mathrm{Hom}_R(K(x_1, \ldots, x_n), M)$. Recall that the Koszul complex is isomorphic to its dual, we have

$$\mathrm{Hom}_R(K(x_1, \ldots, x_n), M) \cong M \otimes_R K(x_1, \ldots, x_n).$$

And we recover a previous theorem. On the other hand, $\mathrm{depth} I = 1$ as $xy \in I$ is a nonzero divisor in R, for any $x \in I$, x is a zero divisor in $R/(xy)$.

Corollary 2.2.4. *For any nonzero module M, $\mathrm{pd}_R M \geq \mathrm{depth} \operatorname{ann} M$ where $\mathrm{pd}_R M$ is the projective dimension of M.*

Proof. Take $N = R$. □

2.3 Cohen-Macaulay Rings

Suppose $(R, \mathfrak{m})$ is a regular local ring (that is, the number of generators of $\mathfrak{m}$ equals $\dim R$), then any minimal set of generators for $\mathfrak{m}$ is a regular sequence. And we have $\operatorname{depth}\mathfrak{m} = \operatorname{codim}\mathfrak{m}$. This equality also holds in some other rings.

Theorem 2.3.1. *Let R be a ring such that $\operatorname{depth}\mathfrak{m} = \operatorname{codim}\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$. If $I \subset R$ is a proper ideal, then $\operatorname{depth}I = \operatorname{codim}I$.*

Proof. We have already proved that $\operatorname{depth}I \leq \operatorname{codim}I$. We need to prove $\operatorname{depth}I \geq \operatorname{codim}I$.

We know $I \subset \mathfrak{m}$, so localizing at $\mathfrak{m}$ does not change $\operatorname{codim}I$. Similarly, by lemma 2.1.1, localizing also does not change $\operatorname{depth}I$. We may assume $(R, \mathfrak{m})$ is a local ring with $I \subset \mathfrak{m}$. If $\sqrt{I} = \mathfrak{m}$, then $\operatorname{codim}I = \operatorname{codim}\mathfrak{m}$, and by Corollary 1.2.7, we have

$$\operatorname{depth}I = \operatorname{depth}\sqrt{I} = \operatorname{depth}\mathfrak{m} = \operatorname{codim}\mathfrak{m} = \operatorname{codim}I.$$

Now suppose $\sqrt{I} \subsetneq \mathfrak{m}$. By Noetherian induction, we may assume the theorem holds for all ideals strictly larger than I. Since $\mathfrak{m}$ is not a minimal prime over I, by prime avoidance, there exists $x \in \mathfrak{m}$ such that x is not in any minimal prime over I. Thus, by induction and Lemma 2.1.2, we have

$$\operatorname{depth}I + 1 \geq \operatorname{depth}(I + (x)) = \operatorname{codim}(I + (x)) = \operatorname{codim}I + 1.$$

This implies that

$$\operatorname{depth}I \geq \operatorname{codim}I.$$

□

Rings satisfying the hypothesis in the above theorem have a special name: Cohen-Macaulay rings.

Definition 2.3.2 (Cohen-Macaulay rings)**.** A ring such that $\operatorname{depth}\mathfrak{m} = \operatorname{codim}\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ is called a **Cohen-Macaulay** ring.

The property of Cohen-Macaulay is local in some sense and can be passed to polynomial rings. Below all Cohen-Macaulay will be abbreviated as CM.

Proposition 2.3.3. *R is CM iff $R_{\mathfrak{m}}$ is CM for every maximal ideal $\mathfrak{m}$ of R, and then $R_{\mathfrak{p}}$ is CM for every prime ideal $\mathfrak{p}$ of R. A local ring is CM iff its completion if CM.*

Proof. Suppose R is CM and $\mathfrak{p} \subset R$ is a prime idea. By lemma 2.1.1, we have

$$\operatorname{codim}\mathfrak{p}_{\mathfrak{p}} = \operatorname{codim}\mathfrak{p} = \operatorname{depth}\mathfrak{p} \leq \operatorname{depth}\mathfrak{p}_{\mathfrak{p}} \leq \operatorname{codim}\mathfrak{p}_{\mathfrak{p}}.$$

They are all equalities so $R_\mathfrak{p}$ is CM. Conversely, if $R_\mathfrak{m}$ is CM for every maximal ideal $\mathfrak{m}$, then by lemma 2.1.1, we have

$$\operatorname{depth}\mathfrak{m} = \operatorname{depth}\mathfrak{m}_\mathfrak{m} = \operatorname{codim}\mathfrak{m}_\mathfrak{m} = \operatorname{codim}\mathfrak{m}.$$

This proves that R is also CM.

Let $(R, \mathfrak{m})$ be a local ring and $(\hat{R}, \hat{\mathfrak{m}})$ be its completion. We know that $\operatorname{codim}\mathfrak{m} = \operatorname{codim}\hat{\mathfrak{m}}$ since $\dim R = \dim \hat{R}$. Next, we want to show that $\operatorname{depth}(\mathfrak{m}, R) = \operatorname{depth}(\hat{\mathfrak{m}}, \hat{R})$. Let $x_1, \ldots, x_n$ be generators of $\mathfrak{m}$ and $K(x_1, \ldots, x_n)$ be the Koszul complex of the ideal $\mathfrak{m}$ in R. We know that $\hat{K} = \hat{R} \otimes_R K(x_1, \ldots, x_n)$ is the Koszul complex of $\hat{\mathfrak{m}}$ in $\hat{R}$, and

$$H^*(\hat{K}) = \hat{R} \otimes_R K(x_1, \ldots, x_n).$$

Note that $R \to \hat{R}$ is faithfully flat for a local ring R. This implies

$$\operatorname{depth}(\mathfrak{m}, R) = \operatorname{depth}(\hat{\mathfrak{m}}, \hat{R}).$$

□

Proposition 2.3.4 (CM can pass to polynomial rings). *A ring R is CM iff the polynomial ring $R[x]$ is CM.*

Proof. If $R[x]$ is CM, and we know that x is a nonzerodivisor, so $R = R[x]/(x)$ is also CM.

Suppose R is CM. It is suffice to prove $R[x]_\mathfrak{m}$ is CM for every maximal ideal $\mathfrak{m} \subset R[x]$. Let $\mathfrak{m}$ be a maximal ideal and $\mathfrak{n} = \mathfrak{m} \cap R$. The complement of $\mathfrak{n}$ in R is contained in the complement of $\mathfrak{m}$ in $R[x]$, we have

$$R[x]_\mathfrak{m} = (R_\mathfrak{n}[x])_\mathfrak{m}.$$

So we may assume R is a local ring with the maximal ideal $\mathfrak{n}$. Note that

$$R[x]/\mathfrak{n}R[x] = (R/\mathfrak{n})[x]$$

is a PID. So modulo $\mathfrak{n}$ the ideal $\mathfrak{m}$ is generated by a monic polynomial $f(x)$. Let $x_1, \ldots, x_n$ be an R-sequence in $\mathfrak{n}$, then it is also an $R[x]$-sequence in $\mathfrak{m}$ since $R[x]$ is a free R-module. Moreover, note that $f(x)$ is a nonzerodivisor in $(R/I)[x]$ for any ideal $I \subset R$, so

$$x_1, \ldots, x_n, f(x)$$

is an $R[x]$-sequence in $(\mathfrak{m})$, and this implies that

$$\operatorname{depth}(\mathfrak{m}, R[x]) \geq \operatorname{depth}(\mathfrak{n}, R) + 1.$$

On the other hand, by the principal ideal theorem and R is CM, we have

$$\operatorname{codim}\mathfrak{m} \leq \operatorname{codim}\mathfrak{n} + 1 = \operatorname{depth}(\mathfrak{n}, R) + 1 \leq \operatorname{depth}(\mathfrak{m}, R[x]).$$

Thus, $R[x]$ is CM. □

In the next part, we discuss some nice properties and applications for CM rings.

Definition 2.3.5 (Catenary)**.** A ring R is **catenary**, or has the saturated chain condition, if given any prime ideals $\mathfrak{p} \subset \mathfrak{q}$ of R, the maximal chains of prime ideals between $\mathfrak{p}$ and $\mathfrak{q}$ all have the same length. R is **universally catenary** if every finitely generated R-algebras is catenary.

Corollary 2.3.6. *CM rings are universally catenary. Moreover, in a local CM ring, any two maximal chains of prime ideals have equal length, and every associated prime of R is minimal.*

Proof. By 2.3.4, we know that the polynomial ring over a CM ring is still CM. Let R be a CM ring. To prove any finitely generated R-algebra is CM, it is thus suffice to prove that any holomorphic image S of R is catenary. Any two maximal chains of prime ideals in S can be pulled back to two maximal chains of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ in R, and by 2.3.3, we may localize R at the prime ideal $\mathfrak{q}$ without changing the chains. So we may assume R is a local CM ring, and the first statement follows from the second.

second □

Definition 2.3.7 (Equidimensional)**.** A ring R is **equidimensional** if all its maximal ideals have the same codimension and all its minimal prime ideals have the same codimension.

For a CM ring R, the second condition in the definition follows from the first by Corollary 2.3.6. If R_1 and R_2 are CM, the direct product $R_1 \times R_2$ is also CM. So a CM ring need not be equidimensional, but we have the following:

Corollary 2.3.8. *Any local CM ring is equidimensional.*

Geometrically speaking, if a variety X is locally CM at a point p (in the sense that the local ring $\mathcal{O}_{X,p}$ is CM), then p cannot lie on two components of different dimensions. For example, the ring $\mathbb{C}[x, y, z]/(xy, yz)$ is not CM at the origin. We can get a lot of information about an ideal I if we know the ring R/I is CM.

Proposition 2.3.9. *Let R be a CM ring. If $I = (x_1, \ldots, x_n)$ is an ideal generated by n elements in a CM ring R such that* $\operatorname{codim} I = n$*, the largest possible value, then R/I is a CM ring.*

Proof. □

Corollary 2.3.10 (Unmixedness Theorem)**.** *Let R be a ring. If $I = (x_1, \ldots, x_n)$ is an ideal generated by n elements such that* $\operatorname{codim} I = n$*, then all minimal primes of I have codimension n. If R is CM, then every associated prime of I is minimal over I.*