Notes on ∞-categories

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Contents

Chapter 1

Introduction

These notes are for a reading course on ∞-categories, based on Chapter 14: a short course on ∞-categories by Moritz Groth [1].

Chapter 2

2 models for $(\infty, 1)$-categories

In this part, we introduce two models for $(\infty, 1)$-categories. One is based on simplicial sets satisfying certain horn extension properties, the other is based on simplicial categories. These two models come with two model structure respectively, and the coherent nerve construction of Cordier can be used to show that they are Quillen equivalent.

2.1 Basics on ∞-categories

Let us first introduce some basic notions related to simplicial sets.

Definition 2.1.1 (Category of finite ordinals)**.** We define a category Δ as follows:

1. The objects are linearly order sets of the form $[n]$ for every integer $n \geq 0$

$$[n] := (0 < 1 < \cdots < n).$$

2. Given $m, n \geq 0$, a morphism $f : [m] \to [n]$ in Δ is an order-preserving map. Said differently, for any $0 \leq i \leq j \leq m$, we have $0 \leq f(i) \leq f(j) \leq n$.

There are two important classes of maps: one is the unique coface map

$$d^k : [n-1] \to [n], \quad 0 \leq k \leq n$$

which does not have k in its image and is injective. The other is the unique codegeneracy map

$$s^k : [n+1] \to [n], \quad 0 \leq k \leq n$$

which hits k twice and is surjective. They satisfy a list of cosimplicial identities, see [2]. Now we define the simplicial sets as contravariant functors.

Definition 2.1.2 (Simplicial sets)**.** A **simplicial set** X is a contravariant functor $X : \Delta^{op} \to \mathcal{S}ets$. They form a category where the morphisms are natural transformations. For a simplicial set X, we write $X_n := X([n])$ as its values. The face maps are $d_k = X(d^k)$ and the degeneracy maps are $s_k = X(s^k)$. They satisfy the corresponding simplicial identities.

We can associate a simplicial set to a category using the nerve construction. For each fixed $n \geq 0$, we can view $[n]$ as a category as follows:

- We have $n + 1$ objects: the number $0, 1, \ldots, n$.
- For $0 \leq i \leq j \leq n$, we have a unique morphism $i \to j$ (if $i = j$, this is the unique identity morphism of i). The composition of morphisms is given by the transitivity of $\leq$.

Example 2.1.3. Let $\mathcal{C}$ be a category. The nerve of $\mathcal{C}$ is a simplicial set $N(\mathcal{C})$ given by

$$N(\mathcal{C})_n := \mathrm{Fun}([n], \mathcal{C})$$

for each $n \geq 0$. Namely, all functors from the category $[n]$ to the category $\mathcal{C}$. Here $N(\mathcal{C})_n$ can be viewed as a string of n composable morphisms in $\mathcal{C}$:

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n$$

As a special case, consider $\mathcal{C} = \Delta$. Fix $n \geq 0$, the nerve $N([n])$ is a simplicial set. For every $m \geq 0$, we have

$$N([n])_m = \mathrm{Fun}([m], [n]) = \mathrm{Hom}_\Delta([m], [n]).$$

Because the functors from the category $[m]$ to $[n]$ are exactly the morphism from $[m]$ to $[n]$ if we view them as objects in Δ. This simplicial set $N([n]) = \mathrm{Hom}_\Delta(-, [n])$ is usually denoted as Δ^n, which is the simplicial set represented by $[n] \in \Delta$.

Given a simplicial set X, by the Yoneda lemma, the simplicial maps $\Delta^n \to X$ classify n-simplices of X in the sense that we have a natural bijection

$$\mathrm{Hom}_{s\mathcal{S}et}(\Delta^n, X) \cong X_n.$$

Let $\mathcal{C}$ be a category. The vertices $N(\mathcal{C})_0$ in the simplicial set $N(\mathcal{C})$ can be viewed as objects in $\mathcal{C}$ and the edges $N(\mathcal{C})_1$ can be viewed as morphisms in $\mathcal{C}$. Consider the following triangle in the $[2]$:

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

$N(\mathcal{C})_2$ consists of functors defined on the above triangle. The face map $d_1 : N(\mathcal{C})_2 \to N(C)_1$ is essentially given by composition: how we can compose two morphism into one morphism in $\mathcal{C}$. Strictly speaking, a 2-simplex in $N(\mathcal{C})$ is a pair of composable morphisms together with their composition.

Now let $\mathcal{C}at$ be the category of categories and $s\mathcal{S}et$ be the category of simplicial sets.

Lemma 2.1.4. *The nerve functor $N : \mathcal{C}at \to s\mathcal{S}et$ is fully faithful and hence induces an equivalence onto its essential image.*

Proof. The full proof can be found at [4, Tag 002Z]. We only give a sketch here. □

Our next goal is to understand the essential image. We next introduce two construction from Δ^n. Δ^n contains two subcomplexes: the boundary $\partial\Delta^n$ and the k-th horn Λ^n_k. It is not hard to see that the boundary $\partial\Delta^n$ can be identified with the coequalizer

$$\coprod_{0 \le i < j \le n} \Delta^{n-2} \rightrightarrows \coprod_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n.$$

The kth n-horn $\Lambda^n_k \subseteq \partial\Delta^n$ for $0 \le k \le n$ is obtained from $\partial\Delta^n$ by removing the kth face: the face opposite to the vertex k. The kth horn can also be identified with the coequalizer:

$$\coprod_{0 \le i < j \le n} \Delta^{n-2} \rightrightarrows \coprod_{i \ne k} \Delta^{n-1} \to \Lambda^n_k.$$

Example 2.1.5. Assume the dimension $n = 2$ and $\mathcal{C}$ is a category. We have 3 different horns $\Lambda^2_k \to N(\mathcal{C})$ for $0 \le k \le 2$. From our previous discussion on nerves, we know that they look like the following:

$$\begin{tikzcd} & c_1 & \\ c_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & c_2 \end{tikzcd} \qquad \begin{tikzcd} & c_1 \arrow[dr, "g"] & \\ c_0 \arrow[ur, "f"] & & c_2 \end{tikzcd} \qquad \begin{tikzcd} & c_1 \arrow[dr, "g"] & \\ c_0 \arrow[rr, "h"'] & & c_2 \end{tikzcd}$$

Here c_0, c_1, c_2 are objects in $\mathcal{C}$, and f, g, h are morphisms in $\mathcal{C}$. The composition $h = g \circ f$ in $\mathcal{C}$ uniquely extends the any horn $\Lambda^2_1 \to N(\mathcal{C})$ to an entire 2-simplex $\sigma : \Delta^2 \to N(\mathcal{C})$, i.e., there is a unique dashing arrow making the diagram

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & N(\mathcal{C}) \\ \Delta^2 \arrow[ur, dashed, "\exists!\sigma"'] & \end{tikzcd}$$

commute. The composition $h = g \circ f$ is given by the new face $d_1(\sigma) : \Delta^1 \to N(\mathcal{C})$. Next, we consider the horn $\Lambda^2_0 \to N(\mathcal{C})$, the existence of the extension $c_1 \to c_2$ is equivalent to f has a left inverse in $\mathcal{C}$. Similar observations can be made to $\Lambda^2_2 \to N(\mathcal{C})$.

The above example leads us to the following definition:

Definition 2.1.6 (Inner horns and outer horns)**.** Given $n \geq 0$, the horns $\Lambda^n_k \subseteq \Delta^n$ for $0 < k < n$ are called **inner horns** and Λ^n_0 and Λ^n_n are called **outer horns**.

It turns out that the extension properties in the above example describe the essential image of the nerve functor. Let $\mathcal{Grpd}$ be the category of groupoids, where every morphism is invertible.

Proposition 2.1.7. *Let X be a simplicial set.*

(i) There is an isomorphism $X \cong N(\mathcal{C})$ for some $\mathcal{C} \in \mathcal{Cat}$ if and only if every inner horn $\Lambda^n_k \to X$, $0 < k < n$, can be uniquely extended to an n-simplex $\Delta^n \to X$.

(ii) There is an isomorphism $X \cong N(\mathcal{G})$ for some $\mathcal{G} \in \mathcal{Grpd}$ if and only if every horn $\Lambda^n_k \to X$, $0 \leq k \leq n$ can be uniquely extended to an n-simplex $\Delta^n \to X$.

This is very similar to the definition of Kan complex.

Definition 2.1.8 (Kan complex)**.** A simplicial set X is a **Kan complex** if every horn $\Lambda^n_k \to X$ for $0 \leq k \leq n$ can be extended to an n-simplex $\Delta^n \to X$.

Remark. Note that for Kan complexes, the extension is not unique, unlike for the nerves of categories. We do not want uniqueness in the definition of ∞-categories. The composition of morphisms, in some sense, is not strictly unique, but only up to 'homotopy'.

Let $\mathcal{Kan} \subseteq s\mathcal{Set}$ be the full subcategory spanned by the Kan complexes. We have the following commutative diagram of fully faithful functors

$$\begin{tikzcd} \mathcal{Grpd} \arrow[r] \arrow[d, "N"'] & \mathcal{Cat} \arrow[d, "N"] \\ \mathcal{Kan} \arrow[r] & s\mathcal{Set} \end{tikzcd}$$

Now we can define the ∞-categories.

Definition 2.1.9 (∞-categories)**.** A simplicial set $\mathcal{C}$ is an **∞-category** if every inner horn $\Lambda^n_k \to \mathcal{C}$ for $0 < k < n$ can be extended be an n-simplex $\Delta^n \to \mathcal{C}$.

Any ordinary category $\mathcal{C}$ gives rise to an ∞-category $N(\mathcal{C})$. There exists interesting ∞-categories not of this form. In particular, every simplicial model category has an underlying ∞-category. Let us first introduce some basic terminology for ∞-categories. They are similar to the nerves we discussed before.

Let $\mathcal{C}$ be an ∞-category.

- The objects are the vertices $x \in \mathcal{C}_0$, and the morphisms are 1-simplices $f \in \mathcal{C}_1$.

- We have the face maps

$$s = d_1 : \mathcal{C}_1 \to \mathcal{C}_0,$$
$$t = d_0 : \mathcal{C}_1 \to \mathcal{C}_0$$

  where s sends a morphism to its source, and t to its target. More formally, we define the set of morphisms for objects x, y as the pullback square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d, "{(s,t)}"] \\
* \arrow[r, "{(x,y)}"'] & \mathcal{C}_0 \times \mathcal{C}_0
\end{tikzcd}$$

- The degeneracy map

$$id = s_0 : \mathcal{C}_0 \to \mathcal{C}_1$$

  is the identity map.

Let us take a closer look at the composition for $\mathcal{C}$. Consider two morphisms $f : x \to y$ and $g : y \to z$. These morphisms together define an inner horn:

$$\lambda = (g, \cdot, f) : \Lambda^2_1 \to \mathcal{C}$$

$$\begin{tikzcd}
 & y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] & & z
\end{tikzcd}$$

Here $d_0\lambda = g$ and $d_1\lambda = f$. By definition of the ∞-category $\mathcal{C}$, such horn can be *non-uniquely* extended to a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$. The new face obtained $d_1(\sigma)$ is then a candidate composition of g and f. Note that this is not uniquely determined, unlike in ordinary categories. We only demand that any choice of such a composition is equally good: the space of all such choices is to be contractible. Later we will give a more precise statement.

We now want to describe the homotopy category $\mathrm{Ho}(\mathcal{C})$ of an ∞-category $\mathcal{C}$. From what we discuss in A, the nerve functor has a left adjoint $\tau_1 : s\mathcal{S}et \to \mathcal{C}at$, called the **fundamental category functor** or the **categorical realization functor** (Some place may call it the first truncation). It is given by a formula

$$\tau_1(X) = \mathrm{colim}_{(\delta/X)}[-] \circ p.$$

Surprisingly, the image of a simplicial set X under this functor τ_1 is completely determined by its 0-, 1−, 2−simplices and the maps between them. We can do this as follows:

Given a simplicial set X, define ob $\tau_1 X$ to be X_0. Morphisms in $\tau_1 X$ are freely generated by the set X_1 subject to the relations given by the elements of X_2. For example, the degeneracy map $s_0 : X_0 \to X_1$ picks out the identity morphisms for every object. The face maps $d_1, d_0 : X_1 \to X_0$ assign every morphism a domain and a codomain. Next, we take the free graph on X_0 generated by the arrows X_1 and impose relations $h = gf$ if there exists a 2-simplex $x \in X_2$ such that $xd^2 = f$, xd^0 and $xd^1 = h$.

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "h"'] & & 2 \end{tikzcd}$$

It is not hard to verify that $\tau_1 X$ is indeed a category. In particular, a morphism in $\tau_1 X$ can be represented by a finite chain of 1-simplices of X. If the simplicial set X happens to be an ∞-category, there is a simple description of the fundamental category $\tau_1(X)$. The morphisms in $\tau_1(X)$ can be represented by actual 1-simplices. We need the definition of homotopies in ∞-categories.

Definition 2.1.10 (homotopy)**.** Two morphisms $f, g : x \to y$ in an ∞-category $\mathcal{C}$ are homotopic (write $f \simeq g$) if there is a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$ with boundary $\partial\sigma = (g, f, id_x)$, i.e. the boundary looks like

$$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"] & & y \end{tikzcd}$$

Any such 2-simplex σ is a **homotopy** from f to g, denoted by $\sigma : f \to g$.

This indeed defines an equivalence relationship for morphisms in an ∞-category.

Proposition 2.1.11. *Let $\mathcal{C}$ be an ∞-category and $x, y \in \mathcal{C}$. The homotopy relation is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(x, y)$. The* ***homotopy class*** *of a morphism $f : x \to y$ is denoted by $[f]$.*

Proof. The proof relies on the horn extension property of $\mathcal{C}$ and is completely technical. We first establish the **constant homotopy** for any morphism $f : x \to x$. Apply the degeneracy map s_0, we obtain a 2-simplex $s_0 f : \Delta^2 \to \mathcal{C}$. The simplicial identities implies that

$$d_0 s_0 f = d_1 s_0 f = f$$

and

$$d_2 s_0 f = s_0 d_1 f = s_0 x = id_x$$

by definition. This implies that the following diagram

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

is the constant homotopy of f. So $f \simeq f$ for every morphism.

To prove the reflexivity of $\simeq$, suppose $f, g : x \to y$ and $f \simeq g$, consider the following horn $\Lambda^3_2 \to \mathcal{C}$ in the figure 2.1. The horn Λ^3_2 does not have the bottom face, the other three faces are given by $g \simeq g$, $f \simeq g$ and $id_x \simeq id_x$. The horn extension property gives us the bottom face, and it exactly says that $g \simeq f$. A similar horn $\Lambda^3_2 \to \mathcal{C}$ in the figure 2.2 can prove transitivity of $\simeq$. □

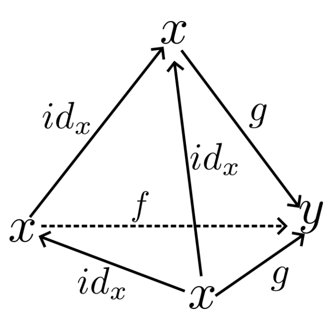

Figure 2.1: Symmetry of $\simeq$

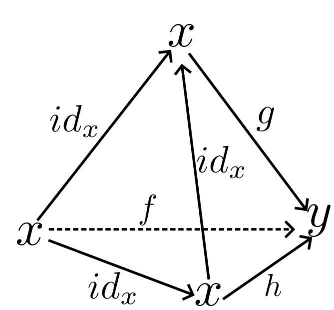

Figure 2.2: Transitivity of $\simeq$

Remark. A quick observation is that for morphisms $f, g : x \to y$, $f \simeq g$ is equivalent to the following condition: there exists a 2-simplex such that

$$\begin{tikzcd} & y \arrow[dr, "id_y"] & \\ x \arrow[ur, "g"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

This can be proved using the horn $\Lambda^3_2 \to \mathcal{C}$ in the figure 2.3.

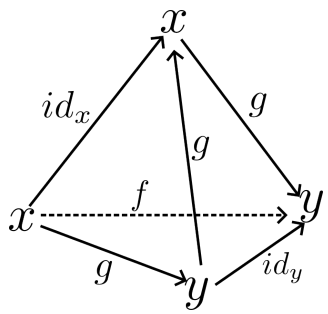

Figure 2.3: Another definition of $\simeq$

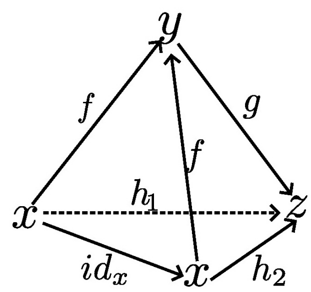

Figure 2.4: Candidate for composition

Next, we want to define the homotopy category $\mathrm{Ho}(\mathcal{C})$ by passing to the homotopy classes of morphisms. We have a lot of things to check, but first, we can show that the choice of different candidate composition of the representatives, is homotopic by using the horn extension property, as shown in figure 2.4.

We just showed that although the choice of composition in an ∞-category is not unique, two such choices are always homotopic. Assume that we can assemble all such choices into a space. What we just proved is the connectedness of the space, i.e., π_0 is trivial. The extension property with respect to higher dimensional inner horns guarantees the higher connectivity of this space, finally making it weakly contractible. Later we will see a more precise statement. Now we can understand what it means to be a homotopy category of an ∞-category.

Proposition 2.1.12 (homotopy category)**.** *Let $\mathcal{C}$ be an ∞-category. There is an ordinary category* $\mathrm{Ho}(\mathcal{C})$*, the* ***homotopy category*** *of $\mathcal{C}$, with the same objects as $\mathcal{C}$ and morphisms the homotopy classes of morphisms in $\mathcal{C}$. Composition and identities are given by*

$$[g] \circ [f] = [g \circ f] \quad \text{and} \quad id_x := [id_x] = [s_0 x]$$

where $g \circ f$ is an arbitrary candidate composition of g and f. Furthermore, there is a natural isomorphism of categories

$$\mathrm{Ho}(\mathcal{C}) \cong \tau_1(\mathcal{C}).$$

Remark. Let $\mathcal{C}$ be an ∞-category and x, y are objects in $\mathcal{C}$. A morphism $f : x \to y$ can be viewed as an element $f \in \mathcal{C}_1$ satisfying $d_1 f = x$ and $d_0 f = y$. Recall a homotopy between $f, g : x \to y$ is defined as

$$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

This can be viewed as an element $\sigma \in \mathcal{C}_2$ satisfying two vertices are x and one vertex is y. This can be interpreted as a 2-morphism from x to y. Similarly, for Figure 2.1, this is a 3-morphism, or a 2-homotopy (homotopy between the homotopy we defined). If we denote

$$\sigma : f \simeq g, \quad \tilde{\sigma} : g \simeq f.$$

Then this 3-morphism gives a 2-homotopy between $\sigma \circ \tilde{\sigma}$ and the constant homotopy of g. A similar argument can show that all higher morphism is invertible.

The above remark shows that there exists some space of morphisms, assembled from the sets of all n-morphisms from $n \geq 2$. Moreover, the higher morphisms are all invertible, giving a model of $(\infty, 1)$-category. We now try to be more precise in this situation. Recall the adjunction we have in Appendix A.

Fix a simplicial set Y and consider the covariant functor $\mathscr{F} : \Delta \to s\mathcal{S}et$ given by

$$\begin{aligned} [n] &\mapsto \Delta^n \times Y, \\ f &\mapsto f \times id_Y. \end{aligned}$$

The left Kan extension L of $\mathscr{F}$ along the Yoneda embedding y gives a composition of the functor $- \times Y$ with the Yoneda embedding y. So in this case, L is isomorphic to $- \times Y$. L has a right adjoint and this is exactly how we define the simplicial mapping space functor:

Definition 2.1.13 (Mapping space)**.** Let us denote by

$$\mathrm{Map}(-, -) : s\mathcal{S}et^{op} \times s\mathcal{S}et \to s\mathcal{S}et$$

the simplicial mapping space functor (it is an internal hom) by

$$\mathrm{Map}(X, Y)_n := \mathrm{Hom}_{s\mathcal{S}et}(\Delta^n \times X, Y)$$

for any $X, Y \in s\mathcal{S}et$ and any $n \geq 0$. So the vertices are maps, edges are homotopies, and higher dimensional simplices are 'higher homotopies'.

Theorem 2.1.14. *Let $i : \Lambda^2_1 \to \Delta^2$ be the inclusion of horns. A simplicial set X is an ∞-category if and only if the restriction map*

$$i^* : \mathrm{Map}(\Delta^2, X) \to \mathrm{Map}(\Lambda^2_1, X)$$

is an acyclic Kan fibration.

This theorem can be understood in the following way. The simplicial set $\mathrm{Map}(\Lambda^2_1, X)$ can be viewed as the space of composition problems and $\mathrm{Map}(\Delta^2, X)$ is the space of solutions to composition problems. The theorem implies that these two spaces are homotopic equivalent in some sense. More specifically, let $f : x \to y$ and $g : y \to z$ be a pair of composable arrows in $\mathcal{C}$. The vertex $\lambda : \Delta^0 \to \mathrm{Map}(\Lambda^2_1, X)$ is the associated horn. Consider the following pullback square

$$\begin{tikzcd} F_\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\Delta^2, X) \arrow[d, "i^*"] \\ \Delta^0 \arrow[r, "\lambda"'] & \mathrm{Map}(\Lambda^2_1, X) \end{tikzcd}$$

The pullback F_λ can be viewed as the space of all possible composition of g and f. The above theorem tells us that i^* is a weak equivalence and a fibration, so the pullback $F_\lambda \to \Delta^0$ is also a weak equivalence. This implies this space F_λ is contractible. To summarize:

(i) A composition exists (We have a pullback and the pullback space is not empty).

(ii) Any two choices of compositions are homotopic (F_λ has trivial π_0).

(iii) The higher homotopies are unique up to homotopy (F_λ is contractible).

This implies that any candidate in the candidate composition we choose is as good as any other. This motivates us to define the equivalence in an ∞-category.

Definition 2.1.15 (Equivalence)**.** A morphism $f : x \to y$ in an ∞-category $\mathcal{C}$ is an **equivalence** if $[f] : x \to y$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$.

It is obvious that identities are equivalences. Given two homotopic maps $f_1 \simeq f_2$, f_1 is an equivalence if and only if f_2 is an equivalence. Surprisingly, the requirement of being an equivalence is very similar to what we require to be an isomorphism in an ordinary category. More specifically, $f : x \to y$ is an equivalence in an ∞-category if and only if there exists a morphism $g : y \to x$ such that there are 2-simplices with boundaries as such:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "id_x"'] & & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "g"] \arrow[rr, "id_y"'] & & y \end{tikzcd}$$

There are no requirements for any higher morphisms. The proof is complicated. Now we return to the concepts of ∞-groupoids.

Definition 2.1.16 (∞-groupoids)**.** An ∞-category is an ∞-groupoid if the homotopy category is a groupoid.

By definition, an ∞-category is an ∞-groupoid if and only if all the morphisms are equivalences. An important result from Joyal states that the outer horns can be extended if certain edges in an n-simplex become equivalence.

Proposition 2.1.17. *Let $\mathcal{C}$ be an ∞-category. Suppose $\lambda : \Lambda_0^n \to \mathcal{C}$, $n \geq 2$ is an outer horn. If the edge given by the vertices $\{0, 1\}$ is an equivalence for this horn λ, then λ can be extended to a simplex $\Delta^n \to \mathcal{C}$.*

A similar statement can be made using another outer horn Λ_n^n. The idea at the beginning that all spaces are ∞-groupoids can be made into the following precise statement.

Proposition 2.1.18. *An ∞-category is an ∞-groupoids if and only if it is a Kan complex.*

The following is an interesting example of ∞-groupoids.

Example 2.1.19 (Singular complex of a topological space)**.** Let $\mathcal{C} = \mathcal{T}op$ be the category of topological spaces, which is cocomplete. Consider the functor $\mathcal{F} : \Delta \to \mathcal{T}op$ mapping $[n]$ to the standard n-simplex Δ^n_{Top} in $\mathbb{R}^{n+1}$:

$$\Delta^n_{Top} = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \right\}.$$

The functor of evaluation at $\mathcal{F}$ is exactly

$$\mathcal{F}^*([n]) = \mathrm{Hom}_{\mathcal{T}op}(\Delta^n_{Top}, -)$$

which is the singular functor $\mathrm{Sing}_n(-)$. It is right adjoint to the geometric realization functor $|-|$:

$$|X| = \operatorname*{colim}_{\Delta^n \to X \text{ in } \Delta/X} \Delta^n_{\mathrm{Top}}$$

and we recover the familiar adjunction formula

$$\mathrm{Hom}_{\mathcal{T}op}(|X|, Y) \cong \mathrm{Hom}_{s\mathcal{S}et}(X, \mathrm{Sing}(Y)).$$

Moreover, in the category of $\mathcal{T}op$, we know that we have the horn extension properties for all horns

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r] \arrow[d] & X \\ {|\Delta^n|} \arrow[ur, dashed] & \end{tikzcd}$$

Because $|\Lambda_k^n| \to |\Delta^n|$ is a cofibration and a weak equivalence. Use the adjunction formula, we obtain the horn extension properties for all horns to $\mathrm{Sing}_\bullet(X)$. This proves $\mathrm{Sing}_\bullet(X)$ is a Kan complex, and thus an ∞-groupoid.

2.2 Simplicial categories and the relations to ∞-categories

In this section, we provide an alternate approach to a theory of $(\infty, 1)$-categories, using **simplicial categories**, or simplicially enriched categories.

Appendix A

Some category theory

Let $\mathcal{A}$ be a small category. We call the category of contravariant functors

$$\mathcal{P}(\mathcal{A}) := \mathrm{Fun}(\mathcal{A}^{op}, \mathcal{S}et)$$

the presheaf category over $\mathcal{A}$. Suppose $\mathcal{C}$ is a cocomplete category, i.e., $\mathcal{C}$ admits all small colimits. We have the Yoneda embedding: $y : \mathcal{A} \to \mathcal{P}(\mathcal{A})$. Given a covariant functor $\mathcal{F} : \mathcal{A} \to \mathcal{C}$, we want to extend $\mathcal{F}$ to $\mathcal{P}(\mathcal{A})$, namely construct the following dotted arrow, making the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathcal{F}"] \arrow[d, "y"'] & \mathcal{C} \\
\mathcal{P}(\mathcal{A}) \arrow[ur, dashed] &
\end{tikzcd}$$

commute. Even better, this extension admits a right adjoint. The statement will be made precise, following the book [3]. We first show that for a presheaf $X \in \mathcal{P}(A)$, X can be written as the colimit of representable functors. For any object $a \in \mathcal{A}$, we denote by h_a the representable functor $\mathrm{Hom}_{\mathcal{A}}(-, a)$ in $\mathcal{P}(\mathcal{A})$.

Definition A.0.1 (Category of elements)**.** Let $\mathcal{A}$ be a category and $X \in \mathcal{P}(\mathcal{A})$ is a presheaf over $\mathcal{A}$. We define the category of elements of X (denoted by $\mathcal{A}/X$) as follows:

- The objects are (a, s) where a is an object of $\mathcal{A}$ and s is a morphism in $\mathrm{Hom}_{\mathcal{P}(\mathcal{A})}(h_a, X)$, which can be viewed as a section $s \in X(a)$ by Yoneda lemma.

- The morphisms in $\mathcal{A}/X$ are given by a morphism $u : a \to b$ in $\mathcal{A}$. By Yoneda lemma, is a commutative diagram

$$\begin{tikzcd}
h_a \arrow[rr, "u_*"] \arrow[dr, "s"'] & & h_b \arrow[dl, "t"] \\
& X &
\end{tikzcd}$$

Naturally we have a faithful functor

$$\mathcal{A}/X \to \mathcal{P}(\mathcal{A})$$

sending (a, s) to h_a. The collection of maps

$$s : h_a \to X$$

for $(a, s) \in \mathcal{A}/X$ gives a cocone over X. And we have the following.

Proposition A.0.2. *Any $(a, s) \in \mathrm{Ob}(\mathcal{A}/X)$ gives rise to a morphism $s : h_a \to X$. If $(a, s), (b, t)$ are two objects in $\mathcal{A}/X$ and $u : a \to b$ is a morphism in $\mathcal{A}$, we have a commutative triangle*

$$\begin{tikzcd} h_a \arrow[rr, "u"] \arrow[dr, "s"'] & & h_b \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

Moreover, X can be written as colimit of such functors.

$$X \cong \mathrm{colim}_{(a,s)\in\mathcal{A}/X} h_a$$

If $\mathcal{A}$ is a small category, then $\mathcal{P}(\mathcal{A})$ is a locally small category. The covariant functor $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ gives us a functor of evaluation at $\mathcal{F}$:

$$\mathcal{F}^* : \mathcal{C} \to \mathcal{P}(\mathcal{A}),$$
$$Y \mapsto \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(-), Y)$$

Proposition A.0.3 (Kan)**.** *The functor $\mathcal{F}^*$ has a left adjoint*

$$\mathcal{F}_! : \mathcal{P}(\mathcal{A}) \to \mathcal{C}.$$

Moreover, for any object a in $\mathcal{A}$, there is a unique natural isomorphism

$$\mathcal{F}(a) \cong \mathcal{F}_!(h_a)$$

such that, for any object $Y \in \mathcal{C}$, the induced bijection gives the adjunction formula

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}_!(h_a), Y) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(a), Y) \cong \mathrm{Hom}_{\mathcal{P}(\mathcal{A})}(h_a, \mathcal{F}^*(Y)).$$

Proof. Let $X \in \mathscr{P}(\mathscr{A})$ be a presheaf. By the previous proposition, we know that

$$X \cong \operatorname{colim}_{(a,s)} h_a.$$

Let $p : \mathscr{A}/X \to \mathscr{A}$ be the projection functor sending (a, s) to $a \in \mathscr{A}$. Define

$$\begin{aligned} \mathscr{F}_!(X) &:= \operatorname{colim}_{(a,s)} (\mathscr{F} \circ p)(a, s) \\ &= \operatorname{colim}_{(a,s)} \mathscr{F}(a). \end{aligned}$$

When $X = h_a$ is representable, we get the canonical isomorphism

$$\mathscr{F}_!(h_a) \cong \mathscr{F}(a)$$

because (a, id) is the final object in $\mathscr{A}/h_a$. Recall that the Hom functor preserves limits, so we have

$$\begin{aligned} \operatorname{Hom}_{\mathscr{C}}(\mathscr{F}_!(X), Y) &\cong \operatorname{Hom}_{\mathscr{C}}(\operatorname{colim}_{(a,s)} \mathscr{F}(a), Y) \\ &\cong \lim_{(a,s)} \operatorname{Hom}_{\mathscr{C}}(\mathscr{F}(a), Y) \\ &\cong \lim_{(a,s)} \operatorname{Hom}_{\mathscr{P}(\mathscr{A})}(h_a, \mathscr{F}^*(Y)) \\ &\cong \operatorname{Hom}_{\mathscr{P}(\mathscr{A})}(\operatorname{colim}_{(a,s)} h_a, \mathscr{F}^*(Y)) \\ &\cong \operatorname{Hom}_{\mathscr{P}(\mathscr{A})}(X, \mathscr{F}^*(Y)). \end{aligned}$$

This proves that $\mathscr{F}_!$ is left adjoint to $\mathscr{F}^*$. □

Now we consider $\mathscr{P}(\mathscr{A})$ is the category of simplicial sets, i.e., when $\mathscr{A}$ is the category of finite ordinals. Let $\mathscr{F} : \Delta \to \mathscr{C}$ be a covariant functor from Δ to a cocomplete, locally small category. Suppose $Y \in \mathscr{C}$ is an object, we can define

$$(\mathscr{F}^* Y)_n := \operatorname{Hom}_{\mathscr{C}}(\mathscr{F}[n], Y).$$

We have $d_i : (\mathscr{F}^* Y)_n \to (\mathscr{F}^* Y)_{n-1}$ is given by the precomposition with $\mathscr{F} d^i : \mathscr{F}[n-1] \to \mathscr{F}[n]$ and $s_i : (\mathscr{F}^* Y)_n \to (\mathscr{F}^* Y)_{n+1}$ is given by the precomposition with $\mathscr{F} s^i : \mathscr{F}[n+1] \to \mathscr{F}[n]$. Checking the simplicial identities implies that $\mathscr{F}^* Y$ is a simplicial set. The left adjoint $\mathscr{F}_!$ is the left Kan extension of $\mathscr{F}$ along the Yoneda embedding y:

$$\begin{tikzcd} {} \arrow[r, "\mathscr{F}"] \arrow[d, "y"'] & \mathscr{C} \\ s\mathscr{S}et \arrow[ur, dashed, "L"] & \end{tikzcd}$$

From our discussion above, the left Kan extension L is just $\mathcal{F}_!$, and it is left adjoint to $\mathcal{F}^*$. There is a more explicit construction of L using the copower, see [5].

Definition A.0.4 (copower)**.** For any set S and object $c \in \mathcal{C}$, the **copower** or **tensor** of c by S, denoted $S \cdot c$, is simply the coproduct $\bigsqcup_S c$: copies of c indexed by S.

If X is a simplicial set, we may form copowers

$$X_m \cdot \mathcal{F}[n]$$

for any $n, m \geq 0$. A morphism $f : [n] \to [m]$ induces a map

$$f_* : X[m] \cdot \mathcal{F}[n] \to X[m] \cdot \mathcal{F}[m]$$

and a map

$$f^* : X[m] \cdot \mathcal{F}[n] \to X[n] \cdot \mathcal{F}[n].$$

The left Kan extension for a simplicial set X is

$$LX := \operatorname{coeq}\Big(\bigsqcup_{f:[n]\to[m]} X_m \cdot \mathcal{F}[n] \underset{f^*}{\overset{f_*}{\rightrightarrows}} \bigsqcup_{[n]} X_n \cdot \mathcal{F}[n]\Big).$$

References

[1] E. M. Friedlander and Daniel R. Grayson, eds. *Handbook of K-theory*. Berlin ; New York: Springer, 2005. 2 pp. ISBN: 978-3-540-23019-9.

[2] Paul G. Goerss and John F. Jardine. *Simplicial Homotopy Theory*. Basel: Birkhäuser Basel, 2009. ISBN: 978-3-0346-0188-7 978-3-0346-0189-4. DOI: 10.1007/978-3-0346-0189-4. URL: http://link.springer.com/10.1007/978-3-0346-0189-4 (visited on 09/28/2025).

[3] Denis-Charles Cisinski. *Higher Categories and Homotopical Algebra*. 1st ed. Cambridge University Press, Apr. 30, 2019. ISBN: 978-1-108-58873-7 978-1-108-47320-0. DOI: 10.1017/9781108588737. URL: https://www.cambridge.org/core/product/identifier/9781108588737/type/book (visited on 09/25/2025).

[4] Jacob Lurie. "Kerodon". 2025. URL: https://kerodon.net.

[5] Emily Riehl. "A Leisurely Introduction to Simplicial Sets". Lecture Notes. URL: https://emilyriehl.github.io/files/ssets.pdf.