INTERSECTION THEORY

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1 INTRODUCTION

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To motivate the study, we first give some examples or applications in intersection theory.

1.1 EXAMPLE. Bezout theorem.

1.2 EXAMPLE. (27 lines on cubic surface) To prove this theorem, we need to consider Grassmannian $\mathrm{G}r(1,3)$ parametrizing lines in $\mathbb{C}\mathrm{P}^1$. Then given any point $z \in \mathrm{G}r(1,3)$ we can write four linear equations on $\mathbb{A}^{20}$ parametrizing

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For more visit `https://zenith-john.github.io/post/algebraic_geometry_seminar_2022_fall`.

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cubic surfaces, such that four equations are spontaneously zero if and only if this line is contained in the corresponding cubic surface. The equations can be glued to give a rank 4 vector bundles over $Gr(1,3)$. Every element in $\mathbb{A}^{20}$ gives a global section of $Gr(1,3)$. To get number 27, we need to compute the intersection number of a general section and zero section of the vector bundle.

1.3 EXAMPLE. (Gromov-Witten theory) It is a natural question to determine the number of genus g curves in a scheme X probably with some constraints. To do so, we first construct the moduli space $M_g(X)$ parametrizing all the genus g curves in X. Every constraint determines an algebraic cycles in the moduli space $M_g(X)$. To get final number, we just take the intersection of all algebraic cycles. If the dimension of intersections is zero, then we conclude the number of points is the number of the genus g curves with constraints.

In this introduction, we also want to discuss the relation or difference between algebraic geometry and algebraic topology. We may consider an embedded non-contractible $M \cong S^1$ in 2-dimensional real torus T^2. How do we compute self-interesetion of M? There are basicly two ways to do so: by homological methods and by perturbation. However, we do not have homology in algebraic geometry. On the other hand, it is also only possible to perturbate the submanifold to tranversal intersection only in differential manifolds. For example, we can consider

$$i : \mathbb{C}P^1 \to \mathscr{O}_{\mathbb{C}P^1}(-1) \cong Bl_0(\mathbb{A}^1)$$

Here $E = i(\mathbb{C}P^1)$ is the zero section and exceptional divisor. A standard result of complex geometry says the self-intersection $E \cdot E = -1$. However, E can not be deformed in the realm of algebraic geometry, because $\Gamma(\mathbb{C}P^1, \mathscr{O}_{\mathbb{C}P^1}(-1)) = 0$. Therefore, we can not explain "intersection number" by perturbation.

Instead, much of the information is not contained in the $U \cap V$ part bu in the higher homological part. We have Serre intersection formula.

1.4 THEOREM. *Given a regular scheme* X *and subscheme* $Y, Z \subset X$ *with defining ideal* $\mathscr{I}, \mathscr{J}$. *Then intersection multiplicity at a generic point* x *of* $Y \cap Z$ *is*

$$m(x, Y, Z) = \sum_{i \geq 0} (-1)^i l_{\mathscr{O}_{X,x}} \operatorname{Tor}_i^{\mathscr{O}_{X,x}}(\mathscr{O}_{X,x}/\mathscr{I}_x, \mathscr{O}_{X,x}/\mathscr{J}_x)$$

Here l *denotes length.*

2 ALGEBRAIC CYCLES

2.1 *Cycles*

Let X be an algebraic scheme. A *k-cycle* on X is a finite formal sum

$$\sum n_i[\mathrm{V}_i]$$

where V_i are k-dimensional subvarieties (i.e. irreducible reduced scheme) of X and n_i are integers. The group of k-cycles, denoted by $\mathrm{Z}_k\mathrm{X}$, is a free abelian group. For any $(k+1)$-dimensional subvariety W of X, and any $r \in \mathrm{R}(\mathrm{W})^*$ a rational function, we can associate it with a k-cycle

$$[\operatorname{div}(r)] = \sum \operatorname{ord}_{\mathrm{V}}(r)[\mathrm{V}]$$

The sum is over all codimensional 1 subvarieties V of W. The sum is always finite. $\operatorname{ord}_{\mathrm{V}}(r)$ denote the vanishing order of r at V. Let A be the local ring $\mathrm{A} := \mathscr{O}_{\mathrm{V},\mathrm{W}}$. If $r \in \mathrm{A}$, then $\operatorname{ord}_{\mathrm{V}}(r) = l_{\mathrm{A}}(\mathrm{A}/(r))$, and extend to R(X) by

$$\operatorname{ord}_{\mathrm{V}}(a/b) = \operatorname{ord}_{\mathrm{V}}(a) - \operatorname{ord}_{\mathrm{V}}(b)$$

A k-cycle α is *rationally equivalent to zero*, written as $\alpha \sim 0$, if there are a finite number of $(k+1)$-dimensional subvarieties W_i of X, and $r_i \in \mathrm{R}(\mathrm{W}_i)^*$, such that

$$\alpha = \sum [\operatorname{div}(r_i)]$$

The cycles rationally equivalent to zero form a subgroup $\operatorname{Rat}_k \mathrm{X}$ of $\mathrm{Z}_k\mathrm{X}$. The *group of k-cycles modulo rational equivalence* on X is the quotient group

$$\mathrm{A}_k\mathrm{X} = \mathrm{Z}_k\mathrm{X}/\operatorname{Rat}_k \mathrm{X}$$

and its element is called *cycle class*.

We denote

$$\mathrm{Z}_*\mathrm{X} = \oplus \mathrm{Z}_k\mathrm{X}, \quad \mathrm{A}_*\mathrm{X} = \oplus \mathrm{A}_k\mathrm{X}$$

A cycle is called *positive* if it is nonzero and all coefficients n_i are positive. A cycle class is called *positive* if it can be represented by a positive cycle.

2.1 EXAMPLE.

$$\mathrm{A}_k\mathrm{X} = \mathrm{A}_k\mathrm{X}_{red}$$

2.2 EXAMPLE. If X is nonsingular along V, then $\mathscr{O}_{\mathrm{V},\mathrm{X}}$ is a DVR. For $r \in \mathrm{R}(\mathrm{X})^*$, $r = ut^m$, $u \in \mathrm{A}^*$, $m \in \mathbb{Z}$ and t uniformizer. Then $\operatorname{ord}_{\mathrm{V}}(r) = m$. In general, we only have

$$\operatorname{ord}_{\mathrm{V}}(r) \geq \max\{n \mid r \in \mathscr{M}^n_{\mathrm{V},\mathrm{X}}\}$$

2.3 EXAMPLE. Let $\tilde{X} \to X$ be normalization of X. Then

$$\mathrm{ord}_V(r) = \sum \mathrm{ord}_{\tilde{V}}(r)[R(\tilde{V}) : R(V)]$$

where sum is taken over all subvarieties $\tilde{V}$ of $\tilde{X}$ which map onto V.

2.4 EXAMPLE. Let X_1 and X_2 be closed subschemes of X, then there are exact sequence

$$A_k(X_1 \cap X_2) \to A_k X \oplus A_k X_2 \to A_k(X_1 \cup X_2) \to 0$$

2.2 *Operations of algebraic cycles*

Let $f : X \to Y$ be a proper morphism. For subvariety $V \subset X$, the image $f(V)$ is closed subvariety W of Y. Then we define

$$\deg(V/W) = \begin{cases} [R(V) : R(W)] & \text{if } \dim W = \dim V \\ 0 & \text{if } \dim W < \dim V \end{cases}$$

Define $f_*[V] = \deg(V/W)[W]$ which extends to

$$f_* : Z_k X \to Z_k Y$$

and induces

$$f_* : A_k X \to A_k Y$$

(For proof, see [Ful98, Theorem 1.3, Proposition 1.4])

If X is proper over $S = \mathrm{Spec}\, K$ and $\alpha = \sum_P n_P[P]$ is a zero-cycle on X, then we define *degree* of α by

$$\deg(\alpha) = \int_X \alpha = \sum_P n_P[R(P) : K]$$

We can define

$$\int_X : A_* X \to \mathbb{Z}$$

by defining $\int_X \alpha = 0$ for $\alpha \in A_k X$, $k > 0$.

Now let $f : X \to Y$ be a flat morphism (of relative dimension n).

2.5 NOTATION. In later part, when we say $f : X \to Y$ is flat, we assumes f is flat of relative dimension n for some $n \in \mathbb{Z}$.

We then define $f^*[V] = [f^{-1}(V)]$ of dimension $\dim V + n$. Then it extends to a homomorphism

$$f^* : Z_k Y \to Z_{k+n} X$$

and induces

$$f^* : A_k Y \to A_{k+n} X$$

(For proof, see [Ful98, Theorem 1.7])

We can also define the exterior product

$$Z_k X \otimes Z_k Y \to Z_{k+l}(X \times Y)$$

given by

$$[V] \times [W] \to [V \times X]$$

and induces

$$A_k X \otimes A_l Y \to A_{k+l}(X \times Y)$$

2.6 EXAMPLE. (Properness is essential) We may consider $X = \mathbb{P}^1_K \cup_{\mathbb{A}^1_K} \mathbb{P}^1_K \to \operatorname{Spec} K$, and rational function $r = x/y$ on X. Then we have $f_*[\operatorname{div}(r)] \neq 0$.

2.7 EXAMPLE. Let X be a nonsingular curve of genus g. Then $A_0 X = \operatorname{Pic}(X)$. Notice that for $g > 0$, $A_0 X$ is not finite generated, in contrast with the case of homology.

2.8 EXAMPLE. Let $f : X' \to X$ be a finite and flat morphism; each point of X has an neighborhood U such that coordinate ring of $f^{-1}(U)$ is a finite generated free module over coordinate ring of U. We say f has degree d if the rank of this module is d for all such U. Then

$$f_* f^*[V] = d[V]$$

2.9 EXAMPLE. (Proposition 1.8 of [Ful98]) Let Y be a closed subscheme of X and $U = X - Y$. Let $i : Y \to X$, $j : U \to X$ be the inclusion. Then the sequence

$$A_k Y \xrightarrow{i_*} A_k X \xrightarrow{j^*} A_k U \to 0$$

is exact.

Proof. Since any subvariety V of U extends to a subvariety $\bar{V}$ of X, so the sequence

$$Z_k Y \to Z_k X \to Z_k U \to 0$$

is exact. To see the middle exactness passes to cycle classes, we show that if $\alpha \in Z_k X$ such that $j^*\alpha \sim 0$, then it is in the image of i_*. By assumption, we have

$$j^*\alpha = \sum[\operatorname{div}(r_i)]$$

for $r_i \in R(W_i)^*$. Since $R(\bar{W}_i) = R(W_i)$, we have $\bar{r_i} \in R(\bar{W}_i)$ and

$$j^*(\alpha - \sum[\operatorname{div}(\bar{r_i})]) = 0$$

in $Z_k U$. Therefore

$$\alpha - \sum[\operatorname{div}(\bar{r_i})] = i_*\beta$$

for some $\beta \in Z_k Y$, which complete the proof. □

2.10 EXAMPLE. (Propositon 1.7 of [Ful98]) Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

be a fibre square, with g flat and f proper. Then

$$f'_* g'^* \alpha = g^* f_* \alpha$$

2.11 EXAMPLE. (Proposition 1.9 of [Ful98]) Let $p : E \to X$ be an affine bundle of rank n. Then the flat pull-back

$$p^* : A_k X \to A_{k+n} E$$

is surjective for all k.

Sketch of proof. Continuously applying exact sequence

$$A_k Y \to A_k X \to A_k U \to 0$$

and Noetherian reduction, we may assume that X is affine. By consider composition, $X \times \mathbb{A}^n \to X \times \mathbb{A}^{n-1} \to X$, we may assume $E = X \times \mathbb{A}^1$. Then just do the commutative algebra stuff to figure it out. □

2.12 EXAMPLE. $A_k(\mathbb{A}^n) = \begin{cases} 0 & \text{for } k < n \\ \mathbb{Z} & \text{for } k = n \end{cases}$

2.13 EXAMPLE. $A_k(\mathbb{P}^n) = \mathbb{Z}$ for $k \le n$.

2.14 EXAMPLE. Let $H \subset \mathbb{P}^n$ be a hypersurface of degree d. Then

$$A_{n-1}(\mathbb{P}^n - H) = \mathbb{Z}/d\mathbb{Z}$$

2.15 EXAMPLE.

$$(f \times g)_*(\alpha \times \beta) = f_*\alpha \times f_*\beta$$

$$(f \times g)^*(\alpha \times \beta) = f^*\alpha \times f^*\beta$$

2.3 *Alternative description of rational equivalence*

Let X be any scheme and $X_1, \cdots, X_t$ irreducible components of X. The local rings $\mathcal{O}_{X_i,X}$ are zero dimensional. The *geometric multiplicity* m_i is defined to be

$$m_i = l_{\mathcal{O}_{X_i,X}}(\mathcal{O}_{X_i,X})$$

The *fundamental cycle* [X] of X is the cycle

$$[X] = \sum_{i=1}^{t} m_i [X_{i,red}]$$

Now we give another description of rational equivalence.

2.16 PROPOSITION. *(Proposition 1.6 of [Ful98]) A cycle α in $Z_k X$ is rationally equivalent to zero if and only if there are $(k+1)$-dimensional subvarieties $V_1, \cdots, V_t \subset X \times \mathbb{P}^1$ such that the projection $V_i \to \mathbb{P}^1$ is dominant, with*

$$\alpha = \sum_{i=1}^{t} [V_i(0)] - [V_i(\infty)]$$

in $Z_k X$.

REFERENCES

[Ful98] William Fulton, *Intersection theory*, second ed., Springer, New York Heidelberg, 1998, Literaturverz. S. 442 - 461. - Ursprünglich als: Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge.