Notes for Homological Mirror Symmetry

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1 Introduction to homological mirror symmetry (2022-02-27 Su Weilin)

1.1 GW invariants

Let (M, ω, J) be a symplectic manifold with symplectic form ω and compatible almost complex structure J. Gromov-Witten invariants are roughly the number

$$\sharp\{(\Sigma, u) \mid u : \Sigma \to M \text{ (pseudo-)holomorphic + constraints } \}$$

These invariants are introduced by Gromov around 1985, who proves that the zero dimensional part of above moduli space is finite. In mirror symmetry, Gromov-Witten invariants belong to A-model.

Example 1.1.1

$$\sharp\{\deg 1 \text{ curves } u : \mathbb{CP}^1 \to \mathbb{CP}^n \text{ passing through 2 generic points}\} = 1$$

However, in general the Gromov-Witten invariants are very difficult to compute because the moduli space of (pseudo-)holomorphic curves is far from smooth and intersections are not transversal. So people want to find some indirect ways to compute these invariants.

1.2 Mirror symmetry

Now consider (M, Ω) which is a complex manifold where Ω is the complex structure. We can consider the sheaf cohomology and peroid integration of differential forms. Period integral belongs to B-models.

Suggested by physicists, there exists a diagram

$$\begin{tikzcd}
\text{GW invariants} \arrow[d, leftrightarrow] & \text{symplectic manifold } M \arrow[l] \arrow[d, leftrightarrow, "\text{Mirror}"] \\
\text{period integral} & \text{complex manifold } M^\vee \arrow[l]
\end{tikzcd}$$

where M and $M^\vee$ are called *mirror dual*. The problem of computing GW invariants can be transformed to the calculation of period integral. We now go to consider Kähler manifold (M, ω, Ω), where ω is symplectic form and Ω is complex structure where symplectic form and complex structure are compatible. Physically, above correspondence is from the duality of 2 dimensional supersymmetric field theory and is checked for quintic 3-fold. The question is why they coincide mathematically?

1.3 Homological mirror symmetry

The idea is to replace the space by some kind of categories. The correct "category" is the A_∞-category, introduced by Stasheff in 1963 to study group like topological spaces.

Definition 1.3.1

An A_∞**-category** is following collection of data.

1. A set of objects.
2. Morphisms between objects are $\mathbb{Z}$-graded linear space $\hom(X, Y)$.
3. m_k the "composition" of morphisms

$$m_k : \hom(X_0, X_1) \otimes \cdots \otimes \hom(X_{k-1}, X_k) \to \hom(X_0, X_k)$$

satisfying the A_∞-relation

$$\sum_{i,j} (-1)^{\sum_1^i |x_l|+1} m_{k+1-j}(x_1, \cdots, x_i, m_j(x_{i+1}, \ldots, x_{i+j}), x_{i+j+1}, \ldots, x_k) = 0$$

here $|x_i|$ denote the grading of x_i. ♣

for exmple, when $k = 1$, we have $m_1(m_1(x)) = 0$ that is $m_1^2 = 0$ is a differential. For $k = 2$, the A_∞-relation gives Leibniz rule. It is more convenient to consider on homology level by differential m_1. That is $\mathrm{Hom}^*(X, Y) = H^*(\hom(X, Y), m_1)$ and define the composition to be

$$\mathrm{Hom}^*(X, Y) \otimes \mathrm{Hom}^*(Y, Z) \to \mathrm{Hom}^*(X, Z)$$
$$[x] \otimes [y] \to [x] \cdot [y] := (-1)^{|x|}[m_2(x, y)]$$

In above notation, the $k = 3$ relation gives associativity $([x] \cdot [y]) \cdot [z] = [x] \cdot ([y] \cdot [z])$.

Let $(M^\vee, \Omega)$ be a complex manifold, we can view it as an algebraic variety and consider the derived category of coherent sheaves of the variety which can be enhanced to be a dg-category. A rough definition of derived category is that the objects are bounded complexes of coherent sheaves and morphisms are $\mathrm{Ext}^*(E, F)$ and composition conditions. It is obtained from category of complexes by formally inverting the quasi-isomorphisms with some additional universal properties. This is the story on the complex side.

The story on the symplectic side is Fukaya category. The objects of Fukaya category is $\{L \subset M \mid L \text{ (compact) Lagrangian } \}$. The morphism is generated by intersection points, that is

$$\hom(L_1, L_2) = R\langle L_1 \cap L_2 \rangle$$

for transveral intersections. A Fukaya category is a A_∞ category with coefficients of the composition map couting holomorphic discs satisfying some relations. Kontsevich suggest the two categories are related.

Conjecture 1.3.2. Homological Mirror Symmetry, [Kon95]

For any Calabi-Yau M there exists a mirror dual $M^\vee$ such that

$$\mathrm{Fuk(M^\vee) \cong D^b(CohM) \quad Fuk(M) \cong D^b(CohM^\vee)}$$

♠

The above diagram is completed to be the following

$$\begin{tikzcd}
\text{GW invariants} \arrow[d, leftrightarrow] & \text{symplectic manifold } M \arrow[l] \arrow[r] \arrow[d, leftrightarrow, "\text{Mirror}"] & \mathrm{Fuk(M)} \arrow[d, leftrightarrow, "\text{HMS}"] \\
\text{period integral} & \text{complex manifold } M^\vee \arrow[l] \arrow[r] & D^b(\mathrm{CohM}^\vee)
\end{tikzcd}$$

To go from period integral to Gromov-Witten invariants, we want to get some information of symplectic manifold M from its Fukaya category. We consider Hochschild cohomology. For associative algebras, we define

Definition 1.3.3

We define **Hochschild complex** (HC_*, b) for a k-algebra to be

$$\mathrm{HC}_p(A) = A^{\otimes(p+1)}$$

$$d_i : a_0 \otimes \cdots \otimes a_p \to a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p, \quad (i = 0, \cdots, p-1)$$

$$d_p : a_0 \otimes \cdots \otimes a_p \to a_p a_0 \otimes \cdots \otimes a_{p-1}$$

$$b = \sum_i (-1)^i d_i$$

Then we define the homology of above complex to be **Hochschild homology** and cohomology of dual complex to be **Hochshild cohomology**, denoted by HH_* and HH^* respectively. ♣

We can extend above definition to A_∞-categories. Following conjecture relates Fukaya category to the geometry of original symplectic manifold.

Conjecture 1.3.4. Kontsevich?

$$H^*(M) \cong \mathrm{HH}^*(\mathrm{Fuk(M)})$$

♠

2 Derived category and triangulated category (2022-03-06 Zhang Nantao)

2.1 Derived category

First, recall the complex of an abelian category $\mathcal{A}$ is of the form

$$A^* : \cdots \to A^{n-2} \xrightarrow{d^{i-1}} A^{n-2} \xrightarrow{d^i} A^n \to \cdots$$

satisfying $d^i \circ d^{i-1} = 0$. Morphism between the complex A^* and B^* are a series of morphisms $f^i : A^i \to B^i$ making the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-2} \arrow[r] \arrow[d, "f^{n-2}"] & A^{n-1} \arrow[r] \arrow[d, "f^{n-1}"] & A^n \arrow[r] \arrow[d, "f^n"] & \cdots \\
\cdots \arrow[r] & B^{n-2} \arrow[r] & B^{n-1} \arrow[r] & B^n \arrow[r] & \cdots
\end{tikzcd}$$

where the differentials are omitted.

We then have a category of complex denoted by $Kom(\mathcal{A})$ where objects are complexes of $\mathcal{A}$ and morphisms are given as above.

There exists a natural functor called **shift functor**, $T : Kom(\mathcal{A}) \to Kom(\mathcal{A})$, such that

$$(T(A^*))^i := A^{i+1}$$

$$\mathrm{d}^*_{T(A^*)} := -\,\mathrm{d}^{i+1}_A$$

For $f^* : A^* \to B^*$, we have

$$T(f^*) = f^{i+1}$$

Obviously, T is an equivalence of category. Usually, we denoted $T(A)$ by $A[1]$ and $T(f)$ by $f[1]$, and we use $A[n]$ and $A[-1]$ in an obvious way.

Definition 2.1.1

Recall, the ith **cohomology** of A^* denoted by $H^i(A^*) := \frac{\ker(\mathrm{d}^i)}{\mathrm{im}(\mathrm{d}^{i-1})} \in \mathcal{A}$.

A^* is called **acyclic** if $H^i(A^*) = 0$ for all $i \in \mathbb{Z}$.

$f^* : A^* \to B^*$ induces morphisms $H^i(f) : H^i(A) \to H^i(B)$ if all induced morphism are isomorphisms then we call f a **quasi-isomorphism** (qis for short).

Remark 2.1.2

There exists complexes with same cohomology group but not quasi-isomorphic. For example

$$\mathbb{C}[x,y]^{\oplus 2} \xrightarrow{(x,y)} \mathbb{C}[x,y]$$

$$\mathbb{C}[x,y] \xrightarrow{0} \mathbb{C}$$

We first give a definition of derived category by universal properties.

Definition 2.1.3

The **derived category** of $\mathcal{A}$ is a category $D(\mathcal{A})$ with a functor $Q : Kom(\mathcal{A}) \to D(\mathcal{A})$, such that

1. If $A^* \to B^*$ qis in $Kom(\mathcal{A})$ then $Q(f)$ is an isomorphism in $D(\mathcal{A})$.
2. Any functor $F : Kom(\mathcal{A}) \to D$ satisfying condition (1) uniquely factor through Q. That is there exists unique G making the following diagram commutes

$$\begin{tikzcd} Kom(\mathcal{A}) \arrow[rr, "Q"] \arrow[dr, "F"'] & & D(\mathcal{A}) \arrow[dl, dashed, "\exists! G"] \\ & D & \end{tikzcd}$$

Before giving a construction of derived categories, we notice that the cohomology is well defined in derived category and $\mathcal{A} \to Kom(\mathcal{A}) \to D(\mathcal{A})$ is a full subcategory.

Definition 2.1.4

Given an abelian category $\mathcal{A}$, we define **homotopy category** $K(\mathcal{A})$ to be following data

$$\mathrm{Ob}(K(\mathcal{A})) := \mathrm{Ob}(Kom(\mathcal{A}))$$

$$\mathrm{Hom}_{K(\mathcal{A})}(A^*, B^*) := \mathrm{Hom}_{Kom(\mathcal{A})}(A^*, B^*)/\sim$$

where $\sim$ denote the homotopy equivalence.

Recall two morphism of complexes $f, g : A^* \to B^*$ are called **homotopy equivalent** if there exists a collection of homomorphisms $h^i : A^i \to B^{i-1}$ such that $f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$.

Notice that if $f \circ g \sim \mathrm{id}$ and $g \circ f \sim \mathrm{id}$, then f and g are all quasi-isomorphisms.

Now we give another definition of derived category.

Definition 2.1.5

A **derived category** is the following collection of data

$$\mathrm{Ob}(D(\mathcal{A})) := \mathrm{Ob}(Kom(\mathcal{A}))$$

$$\mathrm{Hom}_{D(\mathcal{A})}(A^*, B^*) = \left\{ \begin{tikzcd} & C^* \arrow[dl, "qis"'] \arrow[dr] & \\ A^* & & B^* \end{tikzcd} \right\} / \sim$$

two moprhisms are equivalent if there exists following commutative diagram in $K(\mathcal{A})$.

$$\begin{tikzcd}
 & & C^* \arrow[dl, "qis"'] \arrow[dr] & & \\
 & C_1^* \arrow[dl, "qis"'] \arrow[drrr] & & C_2^* \arrow[dlll, "qis"] \arrow[dr] & \\
A^* & & & & B^*
\end{tikzcd}$$

The composition of morphisms are given by

$$\begin{tikzcd}
 & & C_0^* \arrow[dl, "qis"'] \arrow[dr] & & \\
 & C_1^* \arrow[dl, "qis"'] \arrow[dr] & & C_2^* \arrow[dl, "qis"'] \arrow[dr] & \\
A^* & & B^* & & C^*
\end{tikzcd}$$

The associativity of the composition is obvious. To check that $D(\mathcal{A})$ is indeed a category, we only need to check that

1. C_0^* exists.
2. The composition is unique.

To address the above two questions, we introduce the notion of mapping cone.

Definition 2.1.6

Let $f^* : A^* \to B^*$ we define the **mapping cone** $C(f)^i := A^{i+1} \oplus B^i$ and

$$\mathrm{d}^*_{C(f)} := \begin{pmatrix} -\mathrm{d}_A^{i+1} & 0 \\ f^{i+1} & \mathrm{d}_B^i \end{pmatrix}$$

Given a morphism $f : A^* \to B^*$, we have long exact sequence

$$\to H^i(A^*) \to H^i(B^*) \to H^i(C(f)) \to H^{i+1}(A^*) \to \dots$$

Proposition 2.1.7

Let $f^* : A^* \to B^*$ a morphism, $C(f)$ its mapping cone, and diagram of solid arrows

$$\begin{tikzcd}
B^* \arrow[r, "\tau"] \arrow[d, "="] & C(f) \arrow[r, "\pi"] \arrow[d, "="] & A^*[1] \arrow[r, "-f"] \arrow[d, dashed, "g"] & B^*[1] \arrow[d, "="] \\
B^* \arrow[r, "\tau"'] & C(f) \arrow[r, "\tau_\tau"'] & C(\tau) \arrow[r, "\pi'"'] & B^*[1]
\end{tikzcd}$$

Then there eixsts an isomorphism in $K(\mathcal{A})$, $g : A^*[1] \to C(\tau)$ making the diagram commutes in $K(\mathcal{A})$.

Proof

Let $g = (-f^{i+1}, \mathrm{id}, 0)$ and check the commutativity.

Remark 2.1.8

The above isomorphism exists in $K(\mathcal{A})$ but not in $Kom(\mathcal{A})$ so we need to start from homotopy category instead of category of complexes.

Proposition 2.1.9

Given a diagram

$$\begin{tikzcd} & C^* \arrow[d, "g"] \\ A^* \arrow[r, "f \text{ qis}"] & B^* \end{tikzcd}$$

there exists C_0^* fill the following diagram

$$\begin{tikzcd} C_0^* \arrow[r, "\text{qis}"] \arrow[d] & C^* \arrow[d, "g"] \\ A^* \arrow[r, "f \text{ qis}"] & B^* \end{tikzcd}$$

Proof

We fill the diagram gradually, first we have

$$\begin{tikzcd} & C^* \arrow[d, "g"] & & \\ A^* \arrow[r, "f"] & B^* \arrow[r, "\tau"] & C(f) \arrow[r] & A^*[1] \end{tikzcd}$$

By Proposition 2.1.7, we can fill to the following diagram by isomorphism $A^*[1] \cong C(\tau \circ g)$.

$$\begin{tikzcd} & C^* \arrow[r, "\tau\circ g"'] \arrow[d, "g"] & C(f) \arrow[r] \arrow[d] & C(\tau\circ g) \arrow[d] \\ A^* \arrow[r, "f"] & B^* \arrow[r, "\tau"] & C(f) \arrow[r] & A^*[1] \end{tikzcd}$$

And finally,

$$\begin{tikzcd} C(\tau\circ g)[-1] \arrow[r] \arrow[d] & C^* \arrow[r, "\tau\circ g"'] \arrow[d, "g"] & C(f) \arrow[r] \arrow[d] & C(\tau\circ g) \arrow[d] \\ A^* \arrow[r, "f"] & B^* \arrow[r, "\tau"] & C(f) \arrow[r] & A^*[1] \end{tikzcd}$$

where $C(\tau \circ g)[-1]$ is the required C_0^* in the proposition. We only need to show that $C(\tau \circ g)[-1] \to C^*$ is a quasi-isomoprhism, but it follows from long exact sequence and the five lemma.

We now solve the existence of the composition, but the uniqueness problem is still left.

Definition 2.1.10

A class of morphism $S \subset \mathrm{Mor}(\mathcal{A})$ is said to be **localizing** if

1. S is closed under compositions and $\mathrm{id}_X \in S$ for every $S \in \mathrm{Ob}(\mathcal{A})$.
2. Excision condition, that is for any $f \in \mathrm{Mor}(\mathcal{A})$ and $s \in S$, there exists $g \in \mathrm{Mor}(\mathcal{A})$ and $t \in S$ such that

$$\begin{tikzcd} C_0^* \arrow[r, "g"] \arrow[d, "t"] & C^* \arrow[d, "s"] \\ A^* \arrow[r, "f"] & B^* \end{tikzcd}$$

   is commutative.
3. Let $f, g \in \mathrm{Hom}(X, Y)$, the existence of $s \in S$ such that $sf = sg$ is equivalent to the existence of $t \in S$ with $ft = gt$.

Remark 2.1.11

Quasi-isomorphisms don't form a localizing class in $Kom(\mathcal{A})$ but in $K(\mathcal{A})$.

Only condition 3 need to be checked for quasi-isomorphism class. Given $f^*: A^* \to B^*$ in $K(\mathcal{A})$ and a quasi-isomorphism $s: B^* \to \bar{B}^*$ with $sf = 0$, we want to show that there exists $t: \bar{A}^* \to A^*$ with $ft = 0$. To see that we only need to see the following diagram

$$\begin{tikzcd}
C(s)[-1] \arrow[r, "\tau[-1]"] \arrow[d, "\cong"] & B^* \arrow[r, "s"] & \bar{B}^* \\
C(s)[-1] & A^* \arrow[l, "g"] \arrow[u, "f"] & C(g)[-1] \arrow[l, "t"]
\end{tikzcd}$$

where $g^i : A^* \to B^i \oplus \bar{B}^{i-1}$ is the map

$$g^i : (a^i) \to (f^i(a^i), -h^i(a^i))$$

where $h^i : A^i \to \bar{B}^{i-1}$ is the homotopy bewteen sf and 0. Then $ft = \tau[-1]gt = 0$ and t is quasi-isomorphism by long exact sequence.

Remark 2.1.12

The derived category is additive but not ableian, as it does not always have kernels and cokernels. But it is a triangulated category, which we will introduce later.

2.2 Triangulated category

Definition 2.2.1

A **triangulated category** is an additve category D with an additive functor $T: D \to D$ called the **shift functor** and a set of **distinguished triangles** satisfying 4 axioms. In convention, we use notation $A[1] := T(A)$ and $f[1] := T(f)$.

1. (a) Any triangle of the form

   $$A \xrightarrow{\text{id}} A \to 0 \to A[1]$$

   is distinguished.

   (b) Any triangle isomorphic to a distinguished triangle is distinguished.

   (c) Any morphism $f: A \to B$ can be completed to a distinguised triangle

   $$A \xrightarrow{f} B \to C \to A[1]$$

2. $$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

   is a distinguised triangle if and only if

   $$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{f[1]} B[1]$$

   is.

3. If there exists a commutative diagram of solid arrows of the following diagram,

   $$\begin{tikzcd}
   A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & A[1] \arrow[d] \\
   A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
   \end{tikzcd}$$

   then there exists a dahsed arrows $h: C \to C''$ (not necessarily unique) to complete the diagram.

4. Suppose given distinguished triangles

$$X \xrightarrow{f} Y \to Z' \to X[1]$$

$$Y \xrightarrow{g} Z \to X' \to Y[1]$$

$$X \xrightarrow{g \circ f} Z \to Y' \to X[1]$$

then there exists a distinguished triangle

$$Z' \to Y' \to X' \to Z'[1]$$

such that the following diagram is commutative

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"] & Y \arrow[r] \arrow[d, "g"] & Z' \arrow[r] \arrow[d] & X[1] \arrow[d, "\mathrm{id}_X[1]"] \\
X \arrow[r, "g \circ f"] \arrow[d, "f"] & Z \arrow[r] \arrow[d, "\mathrm{id}_Z"] & Y' \arrow[r] \arrow[d] & X[1] \arrow[d, "f[1]"] \\
Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & Y[1] \arrow[d] \\
Z' \arrow[r] & Y' \arrow[r] & X' \arrow[r] & Z'[1]
\end{tikzcd}$$

Remark 2.2.2

Basically, the 4th axiom means that $(A/C)/(B/C) = A/B$ in abstract algebra. That axiom is sometimes called octahedral axiom because it can be arranged into an octahedral.

Proposition 2.2.3

A derived category is a triangulated category where the distinguished triangles are those triangles isomorphic to

$$A \xrightarrow{f} B \to C(f) \to A[1]$$

Remark 2.2.4

It may be interesting to know other examples of triangulated categories. One example is the derived category of A_∞-category and another may be given by stable homotopy category. The basic idea is from the cofiber sequence in topology.

$$X \to Y \to C(f) \to \Sigma X \to \Sigma Y \to \cdots$$

and making $T := \Sigma$. However, Σ is not invertible, so we have to do some additional work to make category of topological space a triangulated category.

Lemma 2.2.5

For distinguished triangle

$$A \to B \to C \to A[1]$$

the composition $A \to C = 0$.

Proof

Consider the following diagram

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, dashed] & A[1] \arrow[d] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1]
\end{tikzcd}
$$

♣

Proposition 2.2.6

Let $A \to B \to C \to A[1]$ be a distinguished triangle, and $A_0 \in D$, then

$$\mathrm{Hom}(A_0, A) \to \mathrm{Hom}(A_0, B) \to \mathrm{Hom}(A_0, C)$$

$$\mathrm{Hom}(C, A_0) \to \mathrm{Hom}(B, A_0) \to \mathrm{Hom}(A, A_0)$$

are exact.

♠

Proof

By Lemma 2.2.5, we have the composition of maps are equal to 0. To show that it is exact, we only need to consider the following diagram

$$
\begin{tikzcd}
A_0 \arrow[r] \arrow[d, dashed] & A_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A_0[1] \arrow[d, dashed] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1]
\end{tikzcd}
$$

♣

Definition 2.2.7

An additive functor $F : D \to D'$ between triangulated categories D and D' is called **exact** if the following two conditionsare satisfied:

1. There exists a functor isomorphism

$$F \circ T_D \cong T_{D'} \circ F$$

2. Any distinguished triangle in D is mapped to distinguished triangle in D'.

♣

Proposition 2.2.8

Let $F : D \to D'$ be an exact functor. If $F \dashv H$, then $H : D' \to D$ is exact. Similar result holds for $G \dashv F$.

♠

Proof

We first check the commutativity with shift functor.

$$
\begin{aligned}
\mathrm{Hom}(A, H(T'(B))) &\cong \mathrm{Hom}(F(A), T'(B)) \\
&\cong \mathrm{Hom}(T'^{-1}(F(A)), B) \\
&\cong \mathrm{Hom}(F(T^{-1}(A)), B) \\
&\cong \mathrm{Hom}(T^{-1}(A), H(B)) \\
&\mathrm{Hom}(A, T(H(B)))
\end{aligned}
$$

Then we check that H maps distinguished triangles to distinguished triangles. Let $A \to B \to C \to A[1]$

distinguished in D'. We can completed to a distinguished triangle

$$H(A) \to H(B) \to C_0 \to H(A)[1]$$

in D. By applying F, and use adjoint property and exactness of F, we have

$$\begin{tikzcd}
F(H(A)) \arrow[r] \arrow[d] & F(H(B)) \arrow[r] \arrow[d] & F(C_0) \arrow[r] \arrow[d, dashed] & F(H(A))[1] \arrow[d] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1]
\end{tikzcd}$$

Applying H, combine two diagram and using adjointness $h : \mathrm{id} \to H \circ F$, we have

$$\begin{tikzcd}
HA \arrow[r] \arrow[d] \arrow[dd, bend right] & HB \arrow[r] \arrow[d] \arrow[dd, bend right] & C_0 \arrow[r] \arrow[d] \arrow[dd, bend right, "\xi"'] & HA[1] \arrow[d] \\
HFHA \arrow[r] \arrow[d] & HFHB \arrow[r] \arrow[d] & HFC_0 \arrow[r] \arrow[d] & HFHA[1] \arrow[d] \\
HA \arrow[r] & HB \arrow[r] & HC \arrow[r] & HA[1]
\end{tikzcd}$$

The curved morphisms are isomorphism. And by exact sequence and five lemma, we have

$$\mathrm{Hom}(A_0, C_0) \cong \mathrm{Hom}(A_0, H(C))$$

for all A_0 and hence

$$\xi : C_0 \cong H(C)$$

is an isomorphism. And therefore $H(A) \to H(B) \to H(C) \to H(A)[1]$ is isomorphic to $H(A) \to H(B) \to C_0 \to H(A)[1]$ and is therefore distinguished. ♣

Definition 2.2.9

Two triangulated categories D and D' are **equivalent** if there exists an exact equivalence $F : D \to D'$. If D is a triangulated category the set $\mathrm{Aut}(D)$ of isomorphism classes of equivalence $F : D \to D$ forms the **group of autoequivalence**.
A subcategory $D' \subset D$ of a triangulated category is a **triangulated subcategory** if D' admits the structure of triangulated category such that the inclusion $D' \hookrightarrow D$ is exact. ♣

We say a set of objects S **generates** triangulated category D if any triangulated subcategory of D contains S is D itself.

2.3 References

Chapter 1 of [KSH94], Chapter 1 and 2 of [Huy06], Chapter 3 of [GM03].

3 Derived functors and some examples of derived category of coherent sheaves (2022-03-13 Nantao Zhang)

3.1 Derived functors

To define the derived functor, we first give a definition of boundedness.

Definition 3.1.1

Let $Kom^*(\mathcal{A})$ with $* = +, -$ or b be the category of complexes A^* with $A^i = 0$ for $i \ll 0$, $i \gg 0$, $|i| \gg 0$, and called **bounded from below**, **bounded from above** and **bounded**. We can define subcategory $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ similarly.

Proposition 3.1.2

The natural functor $D^*(\mathcal{A}) \to D(\mathcal{A})$ defines equivalence of $D^*(A)$ with the full triangulated subcategories of all complexes $A^* \in D(\mathcal{A})$ with $H^i(A^*) = 0$ for $i \ll 0$, $i \gg 0$, $|i| \gg 0$.

Remark 3.1.3

The above proposition is not true, if we replace $D(\mathcal{A})$ by $K(\mathcal{A})$.

We now give a formal definition of derived functors.

Lemma 3.1.4

Let $F : K^*(\mathcal{A}) \to K^*(B)$ be an exact functor of triangulated categories. Then F naturally induces a commutative diagra

$$\begin{tikzcd} K^+(\mathcal{A}) \arrow[r] \arrow[d] & K^+(\mathcal{B}) \arrow[d] \\ D^+(\mathcal{A}) \arrow[r] & D^+(\mathcal{B}) \end{tikzcd}$$

if one the following two conditions holds (in fact two conditions are equivalent)

1. F maps quasi-isomorphism to quasi-isomorphism.

2. F maps acyclic complex to acyclic complex.

For $F : \mathcal{A} \to \mathcal{B}$ left exact, we have $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ satisfy above lemma and therefore induces a derived functor.

$$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

Dually, for right exact functor we have

$$LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$$

The construction above is quite formal, we reuiqre some more explicit process for computation. Recall an abelian category contains enough injective if for any object $A \in \mathcal{A}$ there exists an injective morphism $A \to I$ with $I \in \mathcal{A}$ injective.

Proposition 3.1.5

Supose A is an abelian category with enough injectives. For any $A^* \in K^+(\mathcal{A})$, there exists a complex $I^* \in K^+(\mathcal{A})$ with $I^i \in \mathcal{A}$ injective objects and quasi-isomorphism $A^* \to I^*$.

Proof

We construct the injective resolution directly. We may assume that first nonzero element in A^* is A^0 or we shifted A to make it so. at position 0, we consider

$$\begin{tikzcd} 0 \arrow[r] & A^0 \arrow[rr] \arrow[dd, hook] & & A^1 \arrow[dl] \arrow[dd, dashed] \\ & & I^0 \coprod_{A^0} A^1 \arrow[dr, hook] & \\ 0 \arrow[r] & I^0 \arrow[ur] \arrow[rr, dashed] & & I^1 \end{tikzcd}$$

And if we already have I^i, we have step $i+1$ by following construction.

$$\begin{tikzcd}
{} \arrow[r] & A^i \arrow[rrr] \arrow[dd] \arrow[dr] & & & A^{i+1} \arrow[dl] \arrow[dd, dashed] \\
& & \operatorname{coker} d_I^{i-1} \arrow[r] & \operatorname{coker} d_I^{i-1} \coprod_{A^i} A^{i+1} \arrow[dr, hook] & \\
{} \arrow[r] & I^i \arrow[ur] \arrow[rrr, dashed] & & & I^{i+1}
\end{tikzcd}$$

Then you may check it is indeed a quasi-isomorphism. For details, you may consult [GM03]. ♣

Lemma 3.1.6

Suppose $A^* \to B^*$ quuasi-isomorphism between two complexes in $K^+(\mathcal{A})$ then for any complex I^* of injective objects I^i with $I^i = 0$ for $i \ll 0$ the induced map

$$\mathrm{Hom}_{K(\mathcal{A})}(B^*, I^*) \cong \mathrm{Hom}_{K(\mathcal{A})}(A^*, I^*)$$

is bijective. ♠

Proof

By distinguished triangles and long exact sequence, we only need to prove that for acyclic C^*, we have $\mathrm{Hom}(C^*, I^*) = 0$. Let $g \in \mathrm{Hom}(C^*, I^*)$, we show that it is homotopic to 0 map. We argue by induction, first, for small enough i, we have $C^i = I^i = 0$, which may serve as the start of the induction. If we have h^j for $j \leq i$. Then we have $g^i - d_I^{i-1} \circ h^i : C^i \to I^i$ factor through C^i/C^{i-1} by acyclic property of C^i. Then by injectivity of I^i, we may lift it to $h^{i+1} : C^{i+1} \to I^i$ such that $g^i - d_I^{i-1} \circ h^i = h^{i+1} \circ d_C^i$. ♣

Lemma 3.1.7

Let $A^*, I^* \in Kom^+(\mathcal{A})$ such that all I^i are injective. Then

$$\mathrm{Hom}_{K(\mathcal{A})}(A^*, I^*) \cong \mathrm{Hom}_{D(\mathcal{A})}(A^*, I^*)$$

♠

Proof

$$\begin{tikzcd}
& B^* \arrow[dl, "qis"'] \arrow[dr] & \\
A^* \arrow[rr, dashed] & & I^*
\end{tikzcd}$$

For any roof consisting of solid lines, it is equivalent to a dashed line representing a morphism in $\mathrm{Hom}(A^*, I^*)$. ♣

Proposition 3.1.8

If $\mathcal{A}$ is an abelian category with enough injectives. Then the functor

$$\iota : K^+(\mathcal{I}) \to D^+(\mathcal{A})$$

is an equivalence. $\mathcal{I}$ is the full subcategory of all injectives of $\mathcal{A}$. ♠

Now we come back to derived functor. We have another definition for derived functor. Consider the diagram

$$
\begin{tikzcd}
K^+(\mathcal{I}_\mathcal{A}) \arrow[r, hook] & K^+(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "Q_\mathcal{A}"] & K^+(\mathcal{B}) \arrow[d, "Q_\mathcal{B}"] \\
& K^+(\mathcal{A}) \arrow[ul, "\iota^{-1}"] \arrow[r, dashed] & K^+(\mathcal{B})
\end{tikzcd}
$$

And we define $RF := Q_B \circ K(F) \circ \iota^{-1}$ is a well defined **derived functor**.

And we define

$$R^i F(A^*) := H^i(RF(A^*))$$

And object $A \in \mathcal{A}$ is called **F-acyclic** if $R^i F(A) = 0$ for $i \neq 0$.

Remark 3.1.9

We can develop the dual theory for left exact functor and $D^-(\mathcal{A})$ with $\mathcal{A}$ having enough projectives. However, it is not so useful in algebraic geometry because category of coherent sheaves may not have enough projectives! [Har08, Chapter III]

By above remark, we will need a more general framework to do derived functors.

Definition 3.1.10

Let $F : \mathcal{A} \to \mathcal{B}$. A class of objects $\mathcal{I}_F \subset \mathcal{A}$ stable under finite sum is **F-adpated** if the following conditions hold:

1. If $A^* \in K^+(\mathcal{A})$ acyclic with $A^* \in \mathcal{I}_F$ for all i, then $F(A^*)$ is acyclic.
2. Any object in A can be embedded into an object of ci_F.

Let $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$. We define triangulated subcategory $K_F \subset K^+(\mathcal{A})$ **F-adpated** if it satisfying following conditions.

1. If $A^* \in K_F$ is acyclic, then $F(A^*)$ is.
2. Any $A^* \in K^+(A)$ is quasi-isomorphic to a complex in K_F.

If $\mathcal{I}_F$ is F-adapted, then $K^+(\mathcal{I}_F)$ is F-adapted.

Proposition 3.1.11

Supose A^*, B^* abelian category and $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ exact functor, and there exists an F-adpated class K_F. Then there exists a right derived functor RF satisfying

1. The following diagram commutes.

$$
\begin{tikzcd}
K^+(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "Q_\mathcal{A}"] & K^+(\mathcal{B}) \arrow[d, "Q_\mathcal{B}"] \\
K^+(\mathcal{A}) \arrow[r, "RF"] & K^+(\mathcal{B})
\end{tikzcd}
$$

2. (Universal property) Suppose $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is exact. Then $Q_B \circ K(F) \to G \circ Q_A$ factor through a unique morphism
$$RF \to G$$

Therefore, we can use flat resolution or other F-acyclic resolutions to do the computations.

We can check that

$$\mathrm{Ext}^i(A, -) := H^i \circ R\,\mathrm{Hom}(A, -)$$

For $A, B \in \mathcal{A}$ view as complex concentrated in degree 0.

Also, we have $\mathrm{Hom}^* : K^+(\mathcal{A})^{op} \times K^+(\mathcal{A}) \to K(\mathcal{A})$ defined by

$$\mathrm{Hom}^i(A^*, B^*) := \oplus \mathrm{Hom}(A^k, A^{k+i})$$

$$d(f) := d_B \circ f - (-1)^i f \circ d_A$$

And

$$\mathrm{Ext}^i(A^*, B^*) := H^i(R\,\mathrm{Hom}^*(A^*, B^*))$$

By above definition, we have

$$\mathrm{Ext}^i(A^*, B^*) \cong \mathrm{Hom}_{D(\mathcal{A})}(A^*, B^*[i])$$

Proposition 3.1.12

Let $F_1 : \mathcal{A} \to \mathcal{B}$ and $F_2 : \mathcal{B} \to \mathcal{C}$ left exact functor and adapted class $\mathcal{I}_{F_1} \subset \mathcal{A}$, $\mathcal{I}_{F_2} \subset \mathcal{B}$ such that $F_1(\mathcal{I}_{F_1}) \subset \mathcal{I}_{F_2}$, then there is a natural transformation

$$R(F_2 \circ F_1) \cong RF_2 \circ RF_1$$

♠

3.2 Some results about coherent sheaves

We have following comparison between complex algebraic geometry and complex analytic geometry.

Complex algebraic geometry	complex analytic geometry
scheme / variety	complex analytic space
affine scheme	$\{f(z^i) = 0\}$
regular function	holomorphic function
morphism	holomorphic morphism
locally free sheaves	vector bundles
Zariski topology	analytic topology

We now introduce two famous result to communicates betwee complex algebraic geometry and complex analytic geometry.

Theorem 3.2.1. Serre's GAGA [Ser56]

Given algebraic variety X, we have

$$X \to X^{an}$$

making X an analytic space. Moreover the coherent sheaves on X maps to coherent sheaves on X bijectively.

♡

Theorem 3.2.2. Chow's lemma [Cho49]

A compact analytic variety in $\mathbb{P}^n$ is an algebraic variety.

♡

So for projective variety, we can freely exchange the view of complex algebraic geometry and complex analytic geometry. By Jocobian crieteria, the irreducible smooth projective complex algebraic variety is a complex *manifold*.

By [Har08, Ex II.5.18, Ex III.6.8, 6.9], we have following result.

Proposition 3.2.3

We have one to one correspondence between locally free sheaves of rank n on Y and isomorphism classes of vector bundles of rank n over Y.

♠

Proposition 3.2.4

If X is Noetherian (for example, projective or affine), integral, separated, locally factorial scheme, then every coherent sheaf on X is a quotient of locally free sheaf. Moreover, the locally free resolution is of finite length.

♠

As regular schemes are locally factorial and regular is equivalent to smooth in characteristic 0. We can work in complex manifold with vector bundles if you wish to. Last, we rewrite Serre duality in the language of derived categories. We use $D(X)$ to denote derived category of coherent sheaves on X.

Theorem 3.2.5. Serre duality

We define

$$\begin{aligned} S_X : D^b(X) &\to D^b(X) \\ F^* &\to (\omega_X \otimes^L F^*)[n] \end{aligned}$$

where ω_X is the dualizing sheaf and $n = \dim X$. Then we have

$$\mathrm{Hom}(E^*, F^*) \cong \mathrm{Hom}(F^*, S_X(E))^\vee$$

♡

Remark 3.2.6

The category with a Serre functor is called a **Calabi-Yau category**. (For general definition of Serre functor, see [Huy06]) In [Kon95], Kontsevich make Calabi-Yau property as a sign for equivalence between derived category of coherent sheaves and derived category of Fukaya category.

♠

3.3 Examples of derived category of coherent sheaves

We now consider two examples of schems. First, we consider $X = \mathbb{A}^1 = \mathrm{Spec}\,\mathbb{C}[x]$. Two coherent sheave $\mathscr{O}_X$ trivial line bundle and skyscrapper sheaf $\mathscr{O}_a, a \in \mathbb{A}^1$ generates the category $\mathrm{Coh}(X)$ and therefore $D^b(X)$. We have $\mathrm{Coh}(\mathrm{Spec}\,A) = A$-mod. The morphism between generators are all easy to compute. For example, we compute $R\,\mathrm{Hom}(\mathscr{O}_a, \mathscr{O}_b)$, we need to take projective resolution for $\mathscr{O}_a$, that is

$$0 \to \mathbb{C}[x] \xrightarrow{\times(x-a)} \mathbb{C}[x] \to \mathbb{C}[x]/(x-a) \to 0$$

which is an example of locally free resolution. Then we have

$$R\,\mathrm{Hom}(\mathscr{O}_a, \mathscr{O}_b) = \begin{cases} 0 & \text{if } a \neq b \\ \mathscr{O}_a \xrightarrow{0} \mathscr{O}_a & \text{if } a = b \end{cases}$$

the second sequence started from index 0.

A similar result holds for cylinder $S^1 \times \mathbb{R} \cong_{top} \mathbb{C}^* = \mathrm{Spec}\,\mathbb{C}[x, x^{-1}]$.

Now we consider another example $D^b(\mathbb{P}^1)$.

Proposition 3.3.1

Let $M = \mathscr{O} \oplus \mathscr{O}(1)$. Then graded algebra $R\,\mathrm{Hom}(M, M)$ is concentrated in degree 0 and is the path algebra of Kronecker quiver $\bullet \rightrightarrows \bullet$.

♠

Proof

We have $\mathrm{Ext}^i(\mathscr{O}(l), \mathscr{O}(k)) = H^i(X, \mathscr{O}(k-l))$. For $i > 0$, this is nonzero unless $i = 1$ and $k - l = -2$ which is impossible. The equivalence between algebra is easy to see.

♣

Definition 3.3.2

A coherent sheaf T on X is called a **tilting sheaf** if

1. $A := \mathrm{End}_{\mathscr{O}_X}(T)$ has finite global dimension.
2. $\mathrm{Ext}^i_{\mathscr{O}_X}(T, T) = 0$ for $i > 0$.
3. T generates $D^b(X)$.

♣

Theorem 3.3.3

Let T be a tilting sheaf on a smooth projective scheme X, with tilting algebra $A = \mathrm{End}_{\mathscr{O}_X}(T)$. Then the functors

$$F(-) := \mathrm{Hom}_{\mathscr{O}_X}(T, -)$$
$$G(-) := - \otimes_A T$$

induces equivalence of triangulated categories

$$RF : D^b(X) \to D^b(A\text{-mod})^{op}$$

$$LG : D^b(A\text{-mod})^{op} \to D^b(X)$$

♡

Proof

First, by smoothness of X, coherent sheaves has finite length resolution and by property (1) Definition 3.3.2 A-mod^{op} has finite length resolution so we have RF, LG well defined as morphism between bounded categories.
By property (2) of Definition 3.3.2, we have $RF \circ LG(A) = RF(T) = A$. Hence it is equivalence on finitely generated projective A-module and therefore on all A-modules. By property (3) of Definition 3.3.2, we have the image of LG, the triangulated subcategory generated by T, is all of $D^b(X)$. So we for every F, we have $F = LG(M)$ for some M and therefore have $LG \circ RF(F) \cong LG \circ RF \circ LG(M) = LG(M) = F$.

♣

Theorem 3.3.4

$\mathscr{O} \oplus \mathscr{O}(1) \oplus \cdots \oplus \mathscr{O}(n)$ is a tilting sheaf on $X = \mathbb{P}^n$.

♡

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