Parametric Inverse Simulation

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Abstract

We introduce the theory of *parametric inversion*, which generalizes function inversion to non-invertible functions. We define parametric inverses for several common functions, and present an algorithm to symbolically transform a composition of functions (i.e., a program) into a corresponding parametric inverse. We then formulate the Bayesian inference problem of sampling from a conditional distribution as a form of constrained parametric inversion.

Recent advances in probabilistic programming [3, 2] and deep-generative modeling [1, 4], have dramatically expanded the kinds of probabilistic models we can express and learn. Unfortunately, Bayesian inference procedures have not kept pace; there is a growing chasm between our ability to simulate complex probability distribution, and invert them for probabilistic inference.

In such complex probabilistic models it is often difficult to infer latent variables that are *consistent* with the observed values, let alone drawn from the correct posterior distribution. For example, we can view vision as inference of probable scenes (geometry, lighting, material, etc) that render to an observed image, but forward simulation of a prior distribution will in practice generate scenes which are simply inconsistent with our observations. An alternative approach is to execute the simulation in reverse, starting with the observations. Inferred latent values are then consistent by construction.

Random variables are functions, and hence at the heart of their inverse simulation lies the problem of function inversion. To define inverse simulation when a random variable is not invertible, as typically is the case, we present *parametric inversion*, which generalizes function inversion to non-injective functions. Non-injective function $f : X \to Y$ is not invertible because it lacks a unique right inverse, a function $f^{-1} : Y \to X$ which for all y satisfies:

$$f(f^{-1}(y)) = y \tag{1}$$

To generalize inversion, a parametric inverse represents a *set* of right-inverses parametrically, as a mapping from a parameter space to a function in a set.

Parametric inversion of random variables leads to a novel formulation of Bayesian inference by inverse simulation. In this abstract, we: (i) introduce the concept of parametric inversion, (ii) present an algorithm to construct a parametric inverse from a composition of these primitives, and (iii) outline its extension to random variables for conditional sampling.

1 Example: Bayesian Inference as Constrained Inversion

To demonstrate how Bayesian inference requires function inversion, consider the following model:

$$x \sim exponential(\lambda = 1) \qquad\qquad y \sim logistic(\mu = x, s = 1)$$

We can express this model as a pair of random variables (figure 1), i.e., transformations of $\omega \in [0, 1]$:

$$\begin{aligned} exponential(\omega; \lambda) &= -\ln(1-\omega)/\lambda & x(\omega_1) &= exponential(\omega_1, 1) \\ logistic(\omega; \mu, s) &= \mu + s\ \ln(\omega/(1-\omega)) & y(\omega_1, \omega_2) &= logistic(\omega_2, x(\omega_1), 1) \end{aligned}$$

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To conditionally sample a value of x which is consistent with observation $y = c$, let $(\omega_1^*, \omega_2^*) = y^{-1}(c)$, then evaluate $x(\omega_1^*)$. Hence, conditioning requires inverting y, but what if y is not invertible? How can we draw samples from x that are not only consistent, but from the posterior $P(x \mid y = c)$?

2 Parametric Inversion

Parametric inversion generalizes function inversion to non-invertible functions. A parametric inverse of a function $f : X \to Y$ is a parameterized function $f^{-1} : Y \times \Theta \to X$ which maps an element y to a single element of its preimage: $f(y; \theta) \in \{x \mid f(x) = y\}$. The parameter θ determines which element of the preimage is returned.

In contrast to a conventional inverse, a parametric inverse always exists. For example, while $f(x) = |x|$ is not invertible, we can define a parameter space $\theta \in \{-1, 1\}$ and parametric inverse $f^{-1}(y; \theta) = \theta \cdot y$, where θ determines whether the positive or negative inverse element is returned.

If a function takes multiple inputs, its parametric will return a tuple. For example, inverse multiplication (we denote $\times^{-1}$) maps a value z to two values whose product is z: $\times^{-1}(z; \theta) = (\theta, z/\theta)$, where $\theta \in \mathbb{R} \setminus 0$. Parametric inverses are not unique; it is equally valid to define $\times^{-1}(z; \theta) = (z/\theta, \theta)$.

Many functions map between a value and its preimage; we reserve the term parametric inverse for those which are sound and complete in this task.

Definition 1. $f^{-1} : Y \times \Theta \to X$ is a sound and complete parametric inverse of $f : X \to Y$ if $\forall y$:

$$\{f^{-1}(y; \theta) \mid \theta \in \Theta\} = \{x \mid f(x) = y\} \tag{2}$$

Condition 2 asserts that parametric inverses are (i) sound: for any θ, $f^{-1}(y; \theta)$ is an element of the preimage of y, and (ii) complete: there exists θ corresponding to every element of the preimage. Inverse addition as $+^{-1}(z; \theta_1, \theta_2) = (\theta_1, \theta_2)$ is complete but not sound; for any z, there exists a θ_1 and θ_2 which sum to z, but there are many more pairs which do not. $+^{-1}(z; \theta) = (0, z)$ is sound but not complete; there exists no θ such that for instance $+^{-1}(0; \theta) = (-2, 2)$, despite the fact that $-2 + 2 = 0$. Hence, neither of these are valid parametric inverses of addition.

A parametric inverse has a dual interpretation. If y is fixed, varying the parameter θ varies which element of the preimage of y is returned. If θ is fixed, f^{-1} is a function of only y and a right-inverse which satisfies equation 1. These two alternatives are useful for different applications.

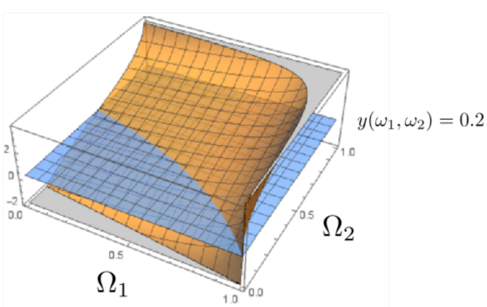

Figure 1: Bayesian parameter estimation. Random variable y is a function of sample space. Intersection of y with plane is subset of $\Omega_1 \times \Omega_2$ which is consistent with $y = 0.2$. Varying θ in parametric inverse $y^{-1}(0.2; \theta)$ covers this set

f	Θ	f^{-1}
$x + y$	$\mathbb{R}$	$(\theta, z - \theta)$
$x - y$	$\mathbb{R}$	$(z + \theta, \theta)$
$x \cdot y$	$\mathbb{R} \setminus 0$	$(z/\theta, \theta)$
x/y	$\mathbb{R}$	$(z \cdot \theta, \theta)$
x^y	$\mathbb{R}$	$(\theta, log_\theta(z))$
$log_x(y)$	$\mathbb{R} \setminus 0$	(θ, θ^z)
$min(x, y)$	$\mathbb{R}^+$	$(z, z + \theta)$
$max(x, y)$	$\mathbb{R}^+$	$(z, z - \theta)$
$sin(x)$	$\mathbb{Z}$	$asin(x) + 2\pi\theta$
$cos(x)$	$\mathbb{Z}$	$acos(x) + 2\pi\theta$

Figure 2: Primitive parametric inverses. Column f contains forward non-injective functions. Θ is the parameter space used for parametric inverses shown in column f^{-1}.

3 Parametric Inversion of Composite Functions

Arbitrarily complex functions can be found by composing functions from a small set of primitive functions. Given functions $f_2 : X \to Y$ and $f_1 : Y \to Z$, the composition $f = f_1 \circ f_2$ has signature $f : X \to Z$ and is defined as $f(x) = f_1(f_2(x))$.

To construct $f^{-1} : Z \times \Theta \to X$, a parametric inverse of f, we substitute f_1 and f_2 with their corresponding parametric inverses f_1^{-1} and f_2^{-1}, and reverse the order of composition. f^{-1} is then:

$$f^{-1}(z; \theta_1, \theta_2) = f_2^{-1}(f_1^{-1}(z, \theta_1), \theta_2) \tag{3}$$

For illustration, let $f = \sin \circ \cos$, then $f^{-1}(z) = \sin^{-1}(\cos^{-1}(z; \theta_1), \theta_2)$ with parameter space $\Theta = \Theta_1 \times \Theta_2$. In an extension of notation, we overload the composition operator $\circ$ to denote this inverse composition of parametric inverses as $\cos^{-1} \circ \sin^{-1}$.

By construction f^{-1} will be a parametric inverse of f on some subset of $\Theta_1 \times \Theta_2$. Continuing our example, for most θ, $\sin^{-1}(y; \theta)$ will return a value not within $[-1, 1]$, the domain of $\cos^{-1}$. f^{-1} is undefined for these parameter values, and hence a *partial*, not total parametric inverse.

Definition 2. If a function $f^{-1} : Y \times \Phi \to X$ does not satisfy condition 2, it is a partial parametric inverse if there exists a $\Theta \subset \Phi$ such that $f^{-1} : Y \times \Theta \to Y$ satisfies satisfies condition 2.

3.0.1 Composition Chains

The procedure to invert compositions of two functions extends naturally to longer chains of composed functions. To simplify algorithm 1, we first formalize composition of parametric inverses with the $\circ$ operator: if $f^{-1} = f_2^{-1} \circ f_1^{-1}$, then f^{-1} is defined as in equation 3 In a composition chain, algorithm 1 substitutes each primitive with a parametric inverse and reverses the order:

Algorithm 1. Given the composition $f = f_1 \circ f_2 \cdots \circ f_n$, substitute f_i with f_{n-i}^{-1} to construct $f^{-1} = f_n^{-1} \circ f_{n-1}^{-1} \circ \cdots \circ f_1^{-1}$, where f_i^{-1} is a parametric inverse of f_i. f^{-1} is a partial parametric inverse of f with a parameter space $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$, where Θ_i is the parameter space of primitive f_i^{-1}.

3.1 Totality conditions

Even if the forward function is total, algorithm 1 may produce a parametric inverse which is partial. Consider again $f = f_1 \circ f_2$, where $f_2 : X \to Y$ and $f_1 : Y \to Z$. f will be a total function provided that the image $f_2(X)$ is a subset of Y, the domain of f_1. This does not apply in the inverse direction however, as demonstrated in figure 3, and formalized in the following section:

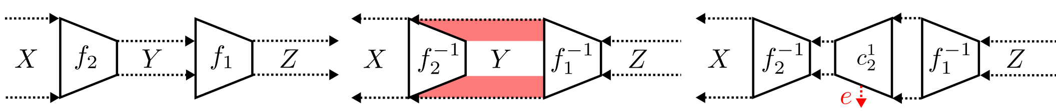

Figure 3: The forward function (left) is a total function. Its parametric inverse (center) is partial, because there exists parameter values which generate values (in red) on which f_2^{-1} is undefined. The solution used in approximate parametric inverses is to insert a restriction c between the two.

3.1.1 Conditions of Totality

Let $f^{-1} = f_n^{-1} \circ f_{n-1}^{-1} \circ \cdots \circ f_1^{-1}$ be an inverse composition with parameter space $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$, and Y_i be the domain of f_i^{-1}. f^{-1} is a total parametric inverse if for all i, $y_i \in Y_i$, $\theta_i \in \Theta_i$:

$$f_i^{-1}(y_i; \theta_i) \in Y_{i-1} \tag{4}$$

In the example of $\cos^{-1} \circ \sin^{-1}$, a simple restriction of the parameter space of cos^{-1} from $\mathbb{N}$ to $\{1\}$ will make the composition total. Unfortunately, in some cases automating this insight to fix arbitrary partial parametric inverse is beyond current methods of symbolic analysis. Instead, we approximate.

4 Approximate Parametric Inversion

Constraints on primitive parametric inverses in equation 4 lead naturally to a form of approximation.

Definition 3. An approximate parametric inverse is a function $\widetilde{f}^{-1} : Y \times \Theta \to X \times \mathbb{R}^+$, which returns an approximate inverse element $\tilde{x} \in X$ and a non-negative error term $e \in \mathbb{R}^+$. $\widetilde{f}^{-1}$ is an approximate parametric inverse of $f : X \to Y$ if $e = 0$ implies $\tilde{x} \in \{x \mid f(x) = y\}$.

An approximate parametric inverse is well defined on all values of its parameter space, but may return an incorrect inverse element unless its error value is zero. Hence, to transform a partial parametric inverse into an approximate we must make it: (1) well defined on all of its parameter space, and (ii) output an error term of 0 as appropriate.

To solve the first problem we insert a function c_2^1 between each pair of primitives f_{i-1}^{-1} (see figure 3) and f_i^{-1}, which takes any value in the domain of f_i^{-1} and restricts it to the domain of f_{i-1}^{-1}. For example, $\cos^{-1} \circ \sin^{-1}$ becomes $\cos^{-1} \circ clip_{-1,1} \circ \sin^{-1}$, where the $clip_{a,b}$ returns its input clamped to the closest point in the interval $[a, b]$, and the distance between its input and output as an error term:

$$clip_{a,b}(x) = (b, x - b) \text{ if } x \geq b \quad | \quad (a, a - x) \text{ if } x \leq a \quad | \quad (x, 0) \text{ otherwise}$$

The error term of the composition is then simply the sum of the error terms of its elements.

4.0.1 Approximate Composition Chain

Algorithm 2 formalizes this procedure and extends it to a chain of composed functions. Again, to simplify its presentation we first define the composition of approximate parametric inverses $\widetilde{f}_1^{-1}$ and $\widetilde{f}_2^{-1}$ to compose their inputs, concatenate their parameter values and sum the error terms:

Let $(\tilde{x}_1, e_1) = \widetilde{f}_1^{-1}(y; \theta_1)$ and $(\tilde{x}_2, e_2) = \widetilde{f}_2^{-1}(\tilde{x}_1; e_1)$, then $\widetilde{f}^{-1} = \widetilde{f}_1^{-1} \circ \widetilde{f}_2^{-1}$ is defined as:

$$\widetilde{f}^{-1}(y; \theta_1, \theta_2) = (\tilde{x}_2, e_1 + e_2) \tag{5}$$

Algorithm 2. Given a partial parametric inverse $f^{-1} = f_n^{-1} \circ f_{n-1}^{-1} \circ \cdots \circ f_1^{-1}$, substitute f_i^{-1} with $c_{i,j} \circ f_i^{-1}$ to construct $\widetilde{f}^{-1} = f_n^{-1} \circ c_n^{n-1} \circ f_{n-1}^{-1} \circ \cdots \circ f_2^{-1} \circ c_2^1 \circ f_1^{-1}$. $\widetilde{f}^{-1}$ is an approximate parametric inverse of f, where c_{i-1}^i restricts values in the codomain of f_i^{-1} to the domain of f_{i-1}^{-1}.

Often the model we wish to invert is not expressible in the form $f_i \circ f_{i+1} \circ \cdots \circ f_n$. However, the basic approach of algorithm 1 and 2 - to substitute each primitive operation with its parametric inverse and reverse the direction of information flow - extends to richer classes of computation.

5 Inversion of Random Variables

Sampling from a conditional probability distribution can be framed as form of constrained parametric inversion of random variables. If we view a probabilistic model as a collection of generative models which each map from some noise distribution, then sampling from a variable X conditioned on a variable Y means to run Y backwards to the noise.

In measure theoretic probability, a random variable is a function $X : \Omega \to T$ where Ω is sample space equipped with a measure $\mathbb{P} : \mathcal{P}(\Omega) \to [0, 1]$. Given a predicate $Y : \Omega \to \{0, 1\}$, to conditionally sample from the distribution $P(X \mid Y)$, we first construct an element $\omega \in A$ with probability $\mathbb{P}(\omega)/\mathbb{P}(A)$, where $A = \{\omega \mid \omega \in \Omega, Y(\omega) = 1\}$ is the conditioning event. A conditional sample from X is then simply $X(\omega)$.

From a conditioning predicate $Y : \Omega \to \{0, 1\}$ we can construct a parametric inverse $Y^{-1} : \{0, 1\} \times \Theta \to \Omega$. For any $\theta \in \Theta$, $X(Y^{-1}(1; \theta))$ is a value which is *consistent* with Y, but may well not be sampled from the correct distribution $P(X \mid Y)$. To sample correctly, we must impose a distribution on Θ. The challenging part is the inversion of Y, so we can ignore X entirely.

The simplest case arises if Y is discrete and its parametric inverse bijective. Let Ω be a finite set, $\mathbb{P}$ be a uniform measure ($\mathbb{P}(\omega) = 1/|\Omega|$) and Y^{-1} be a bijective parametric inverse. To sample $\omega \in \Omega$ with probability $\mathbb{P}(\omega)$, we sample uniformly from Θ with probability $1/|\Theta|$, and apply $Y^{-1}(\{1\}, \theta)$.

If the parametric inverse is not bijective we can impose a non-uniform measure over Θ. With respect to a random variable X, let $[\omega] = \{\omega' \mid X(\omega') = X(\omega)\}$ denote the equivalence class of ω. For any θ, let $\omega_\theta = Y^{-1}(\{1\}, \theta)$, then we should sample θ with probability $\mathbb{P}_2(\theta) = \mathbb{P}(\omega_\theta)/\mathbb{P}([\omega_\theta])$.

Rather than use a non-uniform measure, it's preferable to use a uniform measure and non-linear mapping. Let $Z : \Omega_2 \to \Theta$ be a random variable defined on a probability space $(\Omega_2, \mathbb{P}_2)$, where $\mathbb{P}_2$ is a uniform measure. Then Z must ensure that for all θ

$$\mathbb{P}_2(\{\omega_2 \in \Omega_2 \mid Z(\omega_2) = \theta\}) = \mathbb{P}(\omega_\theta)/\mathbb{P}([\omega_\theta])$$

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