Nachbesprechung Serie 12

Mittwoch, 15. Februar 2017 15:14

12.2. Berechnung von Integralen Berechnen Sie die folgenden unbestimmten Integrale:

(a) $\int \sin^2 t\, e^{-t}\, dt$; (b) $\int \frac{dt}{1+\cos t}$ $(\tan(t/2)=u)$;

(c) $\int \frac{t^3}{\sqrt{t^2+1}}\, dt$ $(t^2+1=u)$; (d) $\int \frac{\sqrt{1-t}}{\sqrt{t}-t}\, dt$ $(t=\sin^2 u)$;

(e) $\int \frac{dt}{\sqrt{1+e^t}}$.

a) <u>Beh:</u> $\sin^2 t = \frac{1-\cos(2t)}{2}$

$$\boxed{\begin{aligned} \underline{\text{Bew:}}\quad \cos(2t) = \cos(t+t) &\overset{\text{Add. Thm.}}{=} \cos^2 t - \sin^2 t \\ &= 1-\sin^2 t - \sin^2 t \\ &= 1 - 2\sin^2 t \\ \Longleftrightarrow \quad \sin^2 t &= \frac{1-\cos(2t)}{2} \qquad \square \end{aligned}}$$

$$\int \sin^2 t\, e^{-t}\, dt = \frac{1}{2}\int (1-\cos(2t))\cdot e^{-t}\, dt = \frac{1}{2}\underbrace{\int e^{-t}\, dt}_{①} - \frac{1}{2}\underbrace{\int \cos(2t) e^{-t}\, dt}_{②}$$

$$\boxed{\frac{d}{dt}(-e^{-t}) = e^{-t}}$$

① $= -e^{-t} + C_1$

② $$\begin{aligned} I := \int \cos(2t) e^{-t}\, dt &= -\cos(2t)e^{-t} - \int (-\sin(2t)\cdot 2)(-e^{-t})\, dt \\ &= -\cos(2t)e^{-t} - 2\int \sin(2t)\, e^{-t}\, dt \\ &= -\cos(2t)e^{-t} - 2\Big(-\sin(2t)e^{-t} - \int \cos(2t)\cdot 2\,(-e^{-t})\, dt\Big) \\ &= -\cos(2t)e^{-t} + 2\sin(2t)e^{-t} - 4\underbrace{\int \cos(2t) e^{-t}\, dt}_{=I} \end{aligned}$$

$$\Longleftrightarrow\ 5I = (2\sin(2t) - \cos(2t))e^{-t} + C_2$$

$$\Longleftrightarrow\ I = \frac{1}{5}(2\sin(2t) - \cos(2t))e^{-t} + C_3 \quad , \ C_3 := \frac{C_2}{5}$$

$$\Longrightarrow \int \sin^2 t\, e^{-t}\, dt = \frac{1}{2}(-e^{-t} + C_1) - \frac{1}{10}\Big[(2\sin(2t) - \cos(2t))e^{-t} + C_3\Big]$$

$$= \Big(\frac{1}{5}\cos(2t) - \frac{2}{5}\sin(2t) - 1\Big)\frac{1}{2}e^{-t} + \underbrace{\Big(\frac{1}{2}C_1 - \frac{1}{10}C_3\Big)}_{=:C}$$

b) $\int \frac{dt}{1+\cos t}$ $(\tan(t/2)=u)$;

$\tan(\frac{t}{2}) = u \iff t = 2\arctan(u)$

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$\Rightarrow \cos t = \cos(2\arctan(u)) \overset{\text{Add. Thm.}}{=} \cos^2(\arctan u) - \underbrace{\sin^2(\arctan u)}_{= 1 - \cos^2(\arctan u)}$

$= 2\cos^2(\arctan u) - 1$

$= 2 \cdot \frac{1}{1+\tan^2(\arctan u)} - 1 = \frac{2}{1+u^2} - 1$

$$\boxed{\begin{aligned} &\text{Es gilt: } 1+\tan^2 x = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \\ &\iff \cos^2 x = \frac{1}{1+\tan^2 x} \end{aligned}}$$

$\iff \cos t + 1 = \frac{2}{1+u^2}$

$\iff \frac{1}{\cos t + 1} = \frac{1+u^2}{2}$ (1)

weiter gilt: $\frac{dt}{du} = \frac{d}{du}(2\arctan(u)) = \frac{2}{1+u^2} \iff dt = \frac{2}{1+u^2}\,du$ (2)

Substitution $[u \leftarrow \tan(t/2)]$ liefert: $\int \frac{1}{\cos t + 1}\,dt = \int \left(\frac{1+u^2}{2}\right)\left(\frac{2}{1+u^2}\right) du = u + C$

(Rücksubst.) $= \tan(\frac{t}{2}) + C$

c) $\int \frac{t^3}{\sqrt{t^2+1}}\,dt \quad (t^2+1=u);$

$$\left.\begin{aligned} &t^2 = u-1 \\ &\frac{du}{dt} = 2t \iff \frac{1}{2t}\,du = dt \end{aligned}\right\} \Rightarrow I = \int \frac{(u-1)\cdot t}{\sqrt{u}} \cdot \frac{1}{2t}\,du$$

$= \int \frac{u-1}{2\sqrt{u}}\,du = \frac{1}{2}\left[\int u^{1/2}\,du - \int u^{-1/2}\,du\right]$

$= \frac{1}{2}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right] + C$

$= u^{1/2}\left(\frac{1}{3}u - 1\right) + C$

$= \sqrt{t^2+1}\left(\frac{1}{3}t^2 + \frac{1}{3} - 1\right) + C$

$= \sqrt{t^2+1}\,\frac{1}{3}(t^2-2) + C$

d) $\int \frac{\sqrt{1-t}}{\sqrt{t}-t}\,dt \quad (t = \sin^2 u);$

$\frac{dt}{du} = 2\sin u \cos u \iff dt = 2\sin u \cos u\,du$

$I = \int \frac{\sqrt{1-\sin^2 u}}{\sin u - \sin^2 u} \cdot 2\sin u \cos u\,du$

$= \int 2\cos^2 u \cdot \sin u$

$$\sqrt{\sin u - \sin^2 u}$$

$$= \int \frac{2\cos^2 u \cdot \sin u}{\sin u - \sin^2 u}\, du$$

$$= \int \frac{2\cos^2 u}{1-\sin u}\, du = 2\int \frac{1-\sin u^2}{(1-\sin u)}\, du = 2\int 1+\sin u \, du$$

$$= 2u - 2\cos u + C$$

$$t = \sin^2 u \iff \arcsin\sqrt{t} = u \qquad = 2u - 2\sqrt{1-\sin^2 u} + C$$

$$= 2\arcsin\sqrt{t} - 2\sqrt{1-t} + C$$

e) $$\int \frac{\mathrm{d}t}{\sqrt{1+\mathrm{e}^t}}$$

$$u = \sqrt{1+e^t} \implies u^2 - 1 = e^t$$

$$\frac{du}{dt} = \frac{1}{2\sqrt{1+e^t}} \cdot e^t \iff dt = \frac{2\sqrt{1+e^t}}{e^t}\, du = \frac{2u}{u^2-1}\, du$$

$$\implies I = \int \frac{1}{u} \cdot \frac{2u}{u^2-1}\, du = \int \frac{2}{u^2-1}\, du$$

$$= \int \frac{1}{u-1}\, du - \int \frac{1}{u+1}\, du$$

$$= \log|u-1| - \log|u+1| + C$$

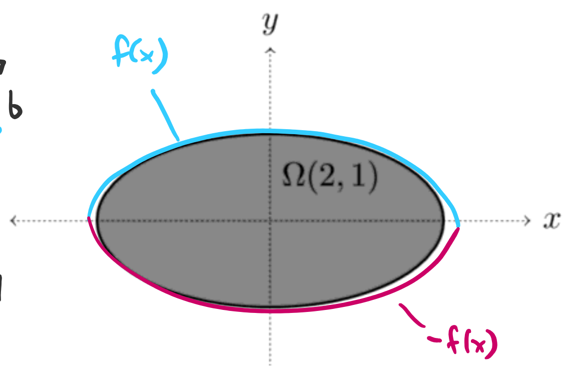
$$= \log\left|\frac{u-1}{u+1}\right| + C = \log\Bigg(\underbrace{\frac{\sqrt{1+e^t}-1}{\sqrt{1+e^t}+1}}_{>0}\Bigg) + C$$

12.3. Die Fläche einer Ellipse Eine Ellipse ist die Menge der Punkte $(x, y) \in \mathbb{R}^2$, die die Gleichung

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

erfüllen für fixierte Konstanten $a, b > 0$. Berechnen Sie die Fläche der Menge

$$\Omega(a, b) := \left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}.$$



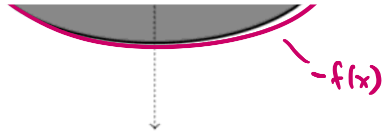
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \iff y^2 = \left(1 - \frac{x^2}{a^2}\right)b^2 \iff |y| = \underbrace{\sqrt{1-\left(\frac{x}{a}\right)^2}\, b}_{=: f(x)}$$

Definitionsbereich von f:

Es muss gelten: $1 - \left(\frac{x}{a}\right)^2 \overset{!}{\geq} 0 \iff 1 \geq \frac{|x|}{a} \iff a \geq |x|$

d.h. $f: [-a, a] \to \mathbb{R}, \; f(x) = \sqrt{1-\left(\frac{x}{a}\right)^2}\, b$

Es muss gelten: $1-\left(\frac{x}{a}\right)^2 \geq 0 \iff 1 \geq \frac{|x|}{a} \iff a \geq |x|$



d.h.: $f:[-a,a] \to \mathbb{R}$, $f(x)=\sqrt{1-\left(\frac{x}{a}\right)^2}\, b$

Sei $A = A(a,b)$ die Fläche der Ellipse.

Dann gilt: $A = 2\cdot \int_{-a}^{a} f\,dx$.

$$\int_{-a}^{a} f(x)\,dx = b\int_{-a}^{a}\sqrt{1-\left(\tfrac{x}{a}\right)^2}\,dx = ab\int_{-1}^{1}\sqrt{1-u^2}\,du = ab\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2 t}\,\cos t\,dt$$

$u = \frac{x}{a}$, $\frac{du}{dx} = \frac{1}{a} \iff a\,du = dx$

$u(a) = 1$, $u(-a) = -1$

$u = \sin t$, $\frac{du}{dt} = \cos t \iff du = \cos t\,dt$

$u(\pi/2) = 1$, $u(-\frac{\pi}{2}) = -1$

$$= ab\int_{-\pi/2}^{\pi/2}\cos^2 t\,dt$$

$$= ab\left[\frac{t}{2} + \frac{\cos t \sin t}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= ab\left(\frac{\pi}{4} + 0 - \frac{-\pi}{4} - 0\right) = ab\frac{\pi}{2}$$

$$\Longrightarrow A = 2\cdot ab\frac{\pi}{2} = \underline{\underline{ab\pi}}$$

12.4. Länge einer Kurve Sei $f : [-2, 2] \to \mathbb{R}^2$ die Kurve

$$f(t) = \begin{pmatrix} \frac{at^2}{2} \\ \frac{bt^2}{2} \end{pmatrix}.$$

für $a, b > 0$. Berechnen Sie die Länge von f.

Wir müssen berechnen: $I := \int_{-2}^{2} \|f'(t)\|_2\,dt$

$$\|f'(t)\|_2 = \left\|\begin{pmatrix} at \\ bt \end{pmatrix}\right\|_2 = \sqrt{(a^2+b^2)t^2} = \underbrace{\sqrt{a^2+b^2}}_{=:C}\cdot|t|$$

$$\Rightarrow I = C\int_{-2}^{2}|t|\,dt = C\left(\int_{-2}^{0}-t\,dt + \int_{0}^{2}t\,dt\right)$$

$$= 2\cdot C\cdot\int_{0}^{2}t\,dt = 2C\left[\frac{t^2}{2}\right]_{t=0}^{2}$$

$$= 2C\cdot 2$$

$$= \underline{\underline{4\sqrt{a^2+b^2}}}$$

12.5. Eine obere und untere Schranke Zeigen Sie, dass für $k, n \in \mathbb{N}$ gilt

$$\frac{n^{k+1}}{k+1} \leq \sum_{l=1}^{n} l^k \leq \frac{(n+1)^{k+1}-1}{k+1}.$$

Hinweise: Integrier!

Seien $f, g, h : [0,\infty) \to [0,\infty)$

$$f(t) = \lceil t \rceil^k \quad (k \in \mathbb{N}) \qquad \lceil t \rceil = \min\{n \in \mathbb{N} \mid n \geq t\}$$

$$g(t) = t^k$$

$$h(t) = (t+1)^k$$

Es gilt: $t \leq \lceil t \rceil \leq t+1 \overset{t \geq 0}{\Longrightarrow} t^k \leq \lceil t \rceil^k \leq (t+1)^k \quad (k \in \mathbb{N})$

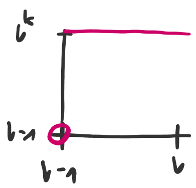
$$\Longleftrightarrow g(t) \leq f(t) \leq h(t) \quad (\forall t)$$

Monotonie d. Integrals $\Longrightarrow \underbrace{\int_0^n g\,dt}_{1} \leq \underbrace{\int_0^n f\,dt}_{2} \leq \underbrace{\int_0^n h\,dt}_{3} \quad (\forall t)$

$$2 = \sum_{l=1}^{n} \left(\int_{l-1}^{l} f\,dt \right) = \sum_{l=1}^{n} l^k \qquad 2 = \left[\frac{t^{k+1}}{k+1} \right]_{t=0}^{n} = \frac{n^{k+1}}{k+1}$$

$$\int_{l-1}^{l} \lceil t \rceil^k \, dt = l^k$$

$$3 = \int_1^{n+1} u^k \, du = \left[\frac{u^{k+1}}{k+1} \right]_{u=1}^{n+1} = \frac{(n+1)^{k+1}-1}{k+1} \qquad \square$$



$$\overline{S}(f,P^n) - \underline{S}(f,P^n) = \underbrace{l^k}_{= \sup_{[0,\frac{1}{n}]} f} \cdot \frac{1}{n} - \underbrace{(l-1)^k}_{\inf_{[0,\frac{1}{n})} f} \cdot \frac{1}{n} \xrightarrow{n \to \infty} 0 \Longrightarrow \int_{l-1}^{l} \lceil t \rceil \, dt \text{ existiert.}$$

Es gilt: $\lim_{n \to \infty} \overline{S}(f,P^n) = \lim_{n \to \infty} l^k = l^k = \int_{l-1}^{l} \lceil t \rceil \, dt$