Math Notes

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Contents

1 Relations

1.1 Notions of Symmetry

Definition 1.1: Notions of Symmetry

Let $U \neq \emptyset$ and $\rho \subseteq U^2$ be a relation on U.

1. ρ is asymmetric $:\Leftrightarrow \forall x, y \in U : x \,\rho\, y \rightarrow y \not\rho\, x$
2. ρ is anti-symmetric $:\Leftrightarrow \forall x, y \in U : x \,\rho\, y \wedge y \,\rho\, x \rightarrow x = y$
3. ρ is not symmetric $:\Leftrightarrow \exists x, y \in U : x \,\rho\, y \wedge y \not\rho\, x$
4. ρ is symmetric $:\Leftrightarrow \forall x, y \in U : x \,\rho\, y \rightarrow y \,\rho\, x$

We will show that asymmetry implies both anti-symmetry and non-symmetry and is therefore a ”stronger” condition.

Claim 1.1

Let $U \neq \emptyset$ and $\rho \neq \emptyset$.

$$\rho \text{ is asymmetric } \Rightarrow \rho \text{ is anti-symmetric.}$$

Proof. Let ρ be asymmetric. Let $x, y \in U$. If $x \ \rho \ y$ then $x \not\rho \ y$ due to asymmetry of ρ. Hence $x \ \rho \ y \wedge x \not\rho \ y$ will never hold, making the implication in 2 of definition 1.1 vacuously true.

□

Claim 1.2

Let $U \neq \emptyset$ and $\rho \neq \emptyset$.

$$\rho \text{ is asymmetric } \Rightarrow \rho \text{ is not symmetric.}$$

Proof. Let ρ be asymmetric. Let $x, y \in U$. If $x \ \rho \ y$ then $x \not\rho \ y$ due to asymmetry of ρ. Hence, there exists $x, y \in U$ s.t. $x \ \rho \ y \wedge y \not\rho \ x$.

□

Claim 1.3: Symmetry of the Empty Relation

Let $U \neq \emptyset$ and $\rho = \emptyset$.

1. ρ is asymmetric.
2. ρ anti-symmetric.
3. ρ is symmetric.

Proof. As all three properties are defined as logical implications, their antedecents will never hold because:

$$x \ \rho \ y \Leftrightarrow (x, y) \in \rho = \emptyset$$

□

Let in the following $U = \{1, 2, 3\}$

- $\rho = \{(1, 2), (2, 1)\}$ is symmetric but not anti-symmetric.

- $\rho = \{(1,2)\}$ is anti-symmetric but not symmetric.
- $\rho = \{(1,1)\}$ is both anti-symmetric and symmetric.
- $\rho = \{(1,2),(2,1),(1,3)\}$ is neither anti-symmetric nor symmetric.