Assignment 1

Computer Vision
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1 Exercise 1

a. Let **(x,y)** be a point in $\mathbb{R}^2$, mapping this point with homogeneous coordinates we obtain:

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{1}$$

We assume $\mathbf{l} = (a, b, c)^T$, we know that the point lies on line $\mathbf{l}$ if:

$$0 = ax + by + c = \mathbf{x}^T\mathbf{l} = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2}$$

b. We start from: $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$. As from the property of the scalar triple product we know that given three vectors **a**,**b** and **c**, then $(\mathbf{a} \times \mathbf{b})\mathbf{c} = 0$ if two of the three vectors are parallel. This means that the following formulas are both fulfilled:

$$\mathbf{x}^T\mathbf{l} = (\mathbf{l} \times \mathbf{l}')\mathbf{l} = 0 \tag{3}$$

$$\mathbf{x}^T\mathbf{l}' = (\mathbf{l} \times \mathbf{l}')\mathbf{l}' = 0 \tag{4}$$

Based on the a. section of this exercise, **x** lies both on **l** and **l**', so **x** is the intersection of the two lines.

c. We start from: **l = x × x'** . We consider the property of the scalar triple product, the following formulas are both fulfilled:

$$\mathbf{x}^T\mathbf{l} = \mathbf{x}(\mathbf{x} \times \mathbf{x}') = 0 \tag{5}$$

$$\mathbf{x}'^T\mathbf{l} = \mathbf{x}'(\mathbf{x} \times \mathbf{x}') = 0 \tag{6}$$

Based on the a. section of this exercise, both points lie on **l**, so it is actually the line that passes between the two points.

d. Let $\mathbf{y} = \alpha\mathbf{x} + (1-\alpha)\mathbf{x}'$. We assume that $\mathbf{y}$ lies on the line connecting $\mathbf{x}$ and $\mathbf{x}$', then the following equation will be fulfilled:

$$\mathbf{y}^T\mathbf{l} = (\alpha\mathbf{x} + (1-\alpha)\mathbf{x}')(\mathbf{x} \times \mathbf{x}') = 0 \tag{7}$$
$$(\alpha\mathbf{x})(\mathbf{x} \times \mathbf{x}') + ((1-\alpha)\mathbf{x}')(\mathbf{x} \times \mathbf{x}') = 0 \tag{8}$$
$$\alpha(\mathbf{x}(\mathbf{x} \times \mathbf{x}')) + (1-\alpha)(\mathbf{x}(\mathbf{x} \times \mathbf{x}')) = 0 \tag{9}$$

The property of the triple scalar product (the one used in b. and c. section) ensures that both scalar product will be equal to zero because of parallel vectors. As a consequence, whatever value of $\alpha \in \mathbb{R}$ is chosen, the point $\mathbf{y}$ will lie on $\mathbf{l}$.

2 Exercise 2

a. Translation:

$$\begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix}$$

Euclidean transformation:

$$\begin{bmatrix} \cos\theta & -\sin\theta & T_x \\ \sin\theta & \cos\theta & T_y \\ 0 & 0 & 1 \end{bmatrix}$$

Similarity transformation:

$$\begin{bmatrix} s\cos\theta & -s\sin\theta & T_x \\ s\sin\theta & s\cos\theta & T_y \\ 0 & 0 & 1 \end{bmatrix}$$

Affine transformation:

$$\begin{bmatrix} a_{11} & a_{12} & T_x \\ a_{21} & a_{22} & T_y \\ 0 & 0 & 1 \end{bmatrix}$$

Projective transformation:

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

b.
- Translation: 2 degrees of freedom, in the x axis and in the y axis translation.
- Euclidean transformation: 3 degrees of freedom, the angle theta x axis and y axis translation.

- Similarity transformation: 4 degrees of freedom: s, the angle theta and the 2 axis translation.
- Affine transformation: 6 degrees of freedom: the 4 coefficients and the 2 axis translation.
- Projective transformation: 8 degrees of freedom: the 8 ratios between the coefficients of the H matrix.

c. The number of degrees of freedom in the projective transformation is less than the number of elements because the matrix H is homogeneous. This is proved by the fact that it could be multiplied by a non-zero factor without altering the projective transformation. Consequently only the ratio of the matrix elements is significant, since there are eight independent ratios between the elements it follows that a projective transformation has eight degrees of freedom.

3 Exercise 3

a. Knowing the values of the matrix H for transforming points, we know:

$$\widetilde{l}H^{-1}H\widetilde{x} = 0 \tag{10}$$

Consequently all the points $H\widetilde{x}$ lie on the line $\widetilde{l}^T H^{-1}$. Thus the transformation is:

$$\widetilde{l} = H^{-1}l \tag{11}$$

b. As we have seen in the previous point the transformation in homogeneous coordinates for points is $\widetilde{x} = Hx$ and the transformation in homogeneous coordinate for lines is $\widetilde{l} = H^{-T}l$.

So applying these formulations to the given invariant we find out:

$$I = \frac{(\mathbf{l}_1'^T\mathbf{x}_1')(\mathbf{l}_2'^T\mathbf{x}_2')}{(\mathbf{l}_1'^T\mathbf{x}_2')(\mathbf{l}_2'^T\mathbf{x}_1')} = \frac{(\mathbf{l}_1^T\mathbf{H}^{-T}\mathbf{H}\mathbf{x}_1)(\mathbf{l}_2^T\mathbf{H}^{-T}\mathbf{H}\mathbf{x}_2)}{(\mathbf{l}_1^T\mathbf{H}^{-T}\mathbf{H}\mathbf{x}_2)(\mathbf{l}_2^T\mathbf{H}^{-T}\mathbf{H}\mathbf{x}_1)} = \frac{(\mathbf{l}_1^T\mathbf{x}_1)(\mathbf{l}_2^T\mathbf{x}_2)}{(\mathbf{l}_1^T\mathbf{x}_2)(\mathbf{l}_2^T\mathbf{x}_1)} \tag{12}$$

To prove that similar construction does not work with fewer number of points or lines, we multiply every point and line for a scalar, knowing that adding these scalars parameters the invariant will not change:

$$I = \frac{(\alpha\mathbf{l}_1'^T\gamma\mathbf{x}_1')(\beta\mathbf{l}_2'^T\delta\mathbf{x}_2')}{(\alpha\mathbf{l}_1'^T\delta\mathbf{x}_2')(\beta\mathbf{l}_2'^T\gamma\mathbf{x}_1')} = \frac{(\alpha\mathbf{l}_1^T\mathbf{H}^{-T}\mathbf{H}\gamma\mathbf{x}_1)(\beta\mathbf{l}_2^T\mathbf{H}^{-T}\mathbf{H}\delta\mathbf{x}_2)}{(\alpha\mathbf{l}_1^T\mathbf{H}^{-T}\mathbf{H}\delta\mathbf{x}_2)(\beta\mathbf{l}_2^T\mathbf{H}^{-T}\mathbf{H}\gamma\mathbf{x}_1)} = \frac{(\alpha\mathbf{l}_1^T\gamma\mathbf{x}_1)(\beta\mathbf{l}_2^T\delta\mathbf{x}_2)}{(\alpha\mathbf{l}_1^T\delta\mathbf{x}_2)(\beta\mathbf{l}_2^T\gamma\mathbf{x}_1)} \tag{13}$$

Removing one point or one line (for instance **x1**) will lead us to a new equation:

$$I = \frac{(\beta\mathbf{l}_2'^T\delta\mathbf{x}_2')}{(\alpha\mathbf{l}_1'^T\delta\mathbf{x}_2')} = \frac{(\beta\mathbf{l}_2^T\mathbf{H}^{-T}\mathbf{H}\delta\mathbf{x}_2)}{(\alpha\mathbf{l}_1^T\mathbf{H}^{-T}\mathbf{H}\delta\mathbf{x}_2)} = \frac{(\beta\mathbf{l}_2^T\delta\mathbf{x}_2)}{(\alpha\mathbf{l}_1^T\delta\mathbf{x}_2)} \tag{14}$$

Applying the simplifications we will have a term that cannot be simplified, so the equation does not satisfy the invariant conditions. The same result will be achieved removing the other point or a line.