Assignment 6

Computer Vision
Zeno Sambugaro 785367

October 17, 2019

Exercise 1

a. Let's compute the gradient of the least squares error with respect to the parameters of the transformation:

$$E = \sum_{i=1}^{n} \| \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \|^2 = \sum_{i=1}^{n} [(x' - M_1 x - M_2 y - t_1)^2 + (y' - M_3 x - M_4 y - t_2)^2] \tag{1}$$

Now we need to differentiate them with respect to each of the 6 parameters:

$$\frac{dE}{dt_1} = \sum_{i=1}^{n} 2(x'_i - M_1 x_i - M_2 y_i - t_1)(-1) \tag{2}$$

$$\frac{dE}{dt_2} = \sum_{i=1}^{n} 2(y'_i - M_3 x_i - M_4 y_i - t_2)(-1) \tag{3}$$

$$\frac{dE}{dM_1} = \sum_{i=1}^{n} 2(x'_i - M_1 x_i - M_2 y_i - t_1)(-x_i) \tag{4}$$

$$\frac{dE}{dM_2} = \sum_{i=1}^{n} 2(x'_i - M_1 x_i - M_2 y_i - t_1)(-y_i) \tag{5}$$

$$\frac{dE}{dM_3} = \sum_{i=1}^{n} 2(y'_i - M_3 x_i - M_4 y_i - t_2)(-x_i) \tag{6}$$

$$\frac{dE}{dM_4} = \sum_{i=1}^{n} 2(y'_i - M_3 x_i - M_4 y_i - t_2)(-y_i) \tag{7}$$

b. Let's define **S** as the matrix containing the coefficients of parameters of the gradient:

$$S = \begin{bmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i y_i & 0 & 0\sum_{i=1}^n x_i & 0 & 0 \\ \sum_{i=1}^n x_i y_i & \sum_{i=1}^n y_i^2 & 0 & 0\sum_{i=1}^n y_i & 0 & 0 \\ 0 & 0 & \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i y_i & 0 & \sum_{i=1}^n x_i \\ 0 & 0 & \sum_{i=1}^n x_i y_i & \sum_{i=1}^n y_i^2 & 0 & \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n y_i & 0 & 0 & n & 0 \\ 0 & 0 & 0 & \sum_{i=1}^n x_i & \sum_{i=1}^n y_i & n \end{bmatrix} \tag{8}$$

$$h = \begin{bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \\ \mathbf{M}_3 \\ \mathbf{M}_4 \\ t_1 \\ t_2 \end{bmatrix} \tag{9}$$

$$u = \begin{bmatrix} \sum_{i=1}^n x_i x_i' \\ \sum_{i=1}^n y_i x_i' \\ \sum_{i=1}^n x_i y_i' \\ \sum_{i=1}^n y_i y_i' \\ \sum_{i=1}^n x_i \\ \sum_{i=1}^n y_i \end{bmatrix} \tag{10}$$

c. In order to solve the equation we have to compute the all the summations contained in the matrix **S**:

- $$\sum_{i=1}^n x_i^2 = 1$$
- $$\sum_{i=1}^n y_i^2 = 1$$
- $$\sum_{i=1}^n y_i x_i = 0$$
- $$\sum_{i=1}^n x_i = 1$$
- $$\sum_{i=1}^n y_i = 1$$
- $$\sum_{i=1}^n x_i x_i' = 3$$

- $$\sum_{i=1}^{n} y_i x_i' = 1$$
- $$\sum_{i=1}^{n} x_i y_i' = 2$$
- $$\sum_{i=1}^{n} y_i y_i' = 4$$
- $$\sum_{i=1}^{n} x_i' = 5$$
- $$\sum_{i=1}^{n} y_i' = 8$$

So the equation:

$$h = S^{-1}u = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \tag{11}$$

$$t = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{12}$$

Exercise 2

a. Let's compute the vectors **v** and **v**':

$$v = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \tag{13}$$

$$v' = \begin{bmatrix} x'_2 - x'_1 \\ y'_2 - y'_1 \end{bmatrix} = s \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \tag{14}$$

These two vectors represent the direction of the lines that link the two points before and after the similarity transformation; now we can recover the angle between them in the following way, since the similarity transformation keeps the angle unvaried:

$$\cos\theta = \frac{\mathbf{v}' \cdot \mathbf{v}}{||\mathbf{v}'|| \cdot ||\mathbf{v}||} \tag{15}$$

$$= \frac{(x'_2 - x'_1)(x_2 - x_1) + (y'_2 - y'_1)(y_2 - y_1)}{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2)}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

So θ can be computed in the following way:

$$\theta = \cos^{-1} \frac{(x'_2 - x'_1)(x_2 - x_1) + (y'_2 - y'_1)(y_2 - y_1)}{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2)}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \tag{16}$$

b. The scale vector can be computed as the euclidean distance between the points in the transformed space and in the original one: Since the equation for mapping the points from the original space to the transformed one is:

$$\mathbf{v}' = s \cdot \mathbf{R} \cdot \mathbf{v} \tag{17}$$

The scaling will be not affected by the rotation of the vector in the transformed space. Thus we will able to compute s as:

$$s = \frac{||\mathbf{v}'||}{||\mathbf{v}||} = \frac{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \tag{18}$$

c. **t** can be easily computed starting from the equation:

$$\mathbf{x}' = s\mathbf{R}\mathbf{x} + \mathbf{t} \Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = s \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \tag{19}$$

$$x' = s\cos\theta x - s\sin\theta y + t_x \Rightarrow t_x = x' - s\cos\theta x + s\sin\theta y$$
$$y' = s\cos\theta x + s\sin\theta y + t_y \Rightarrow t_y = y' - s\sin\theta x - s\cos\theta y$$

d. Following the procedure explained in the previous point I will compute the transformation from the following point correspondences: $(\frac{1}{2}, 0) \rightarrow (0,0), (0, \frac{1}{2}) \rightarrow (-1,-1)$:

$$v = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \tag{20}$$

$$v' = \begin{bmatrix} x'_2 - x'_1 \\ y'_2 - y'_1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \tag{21}$$

Now let's compute the angle between them:

$$\theta = \cos^{-1} \frac{(x'_2 - x'_1)(x_2 - x_1) + (y'_2 - y'_1)(y_2 - y_1)}{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2)}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \tag{22}$$

$$= \cos^{-1} \frac{\frac{1}{2} - \frac{1}{2}}{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2)}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = \cos^{-1} 0 = \frac{\pi}{2}$$

The scaling will be computed as:

$$s = \frac{||\mathbf{v}'||}{||\mathbf{v}||} = \frac{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \tag{23}$$

$$= \frac{\sqrt{(-1)^2 + (-1)^2}}{\sqrt{(-\frac{1}{2})^2 + (\frac{1}{2})^2}} = \sqrt{\frac{2}{1/2}} = 2$$

$t_x and t_y$ will be computed as:

$$t_x = x' - s\cos\theta x + s\sin\theta y \tag{24}$$

$$= 0 - 2 * 0 * \frac{1}{2} + 2 * 0 * 1 = 0$$

$$t_y = y' - s\sin\theta x - s\cos\theta y \tag{25}$$

$$= 0 - 2 * 1 * \frac{1}{2} + 2 * 0 * = -1$$

So the final equation for the transformation will be:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = 2 \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{26}$$