Assignment 2

Computer Vision
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1 Exercise 1

Since $\triangle CZ\mathbf{x}_c$ and $\triangle Cp\mathbf{x}_p$ are similar triangles:

$$\frac{y_c}{z_c} = \frac{y_p}{f}, \quad y_p = f\frac{y_c}{z_c}$$

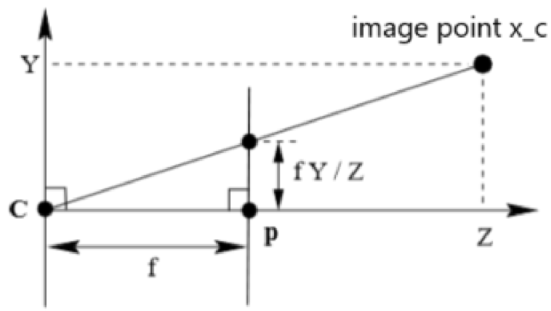

Figure 1: Similar triangles

The image plane is parallel to the xy-plane, thus similar triangles can be applied in the x-axis

$$\frac{x_c}{z_c} = \frac{x_p}{f}, \quad x_p = f\frac{x_c}{z_c}$$

2 Exercise 2

a. Since u and v axis are parallel to x and y axis
   The goal is to transform image coordinate into pixel coordinate,
   thus: $(x_p, y_p)^\intercal \mapsto (u, v)^\intercal$

   In order to perform this transformation we represent the image coordinates of the principal point be $[p_x, p_y]^\intercal$. The formula to transform the point x_p to pixel coordinates is:

$$u = m_u x_p + u_0, \quad v = m_v y_p + v_0$$

b. We are interested in obtaining the value of x' and y':

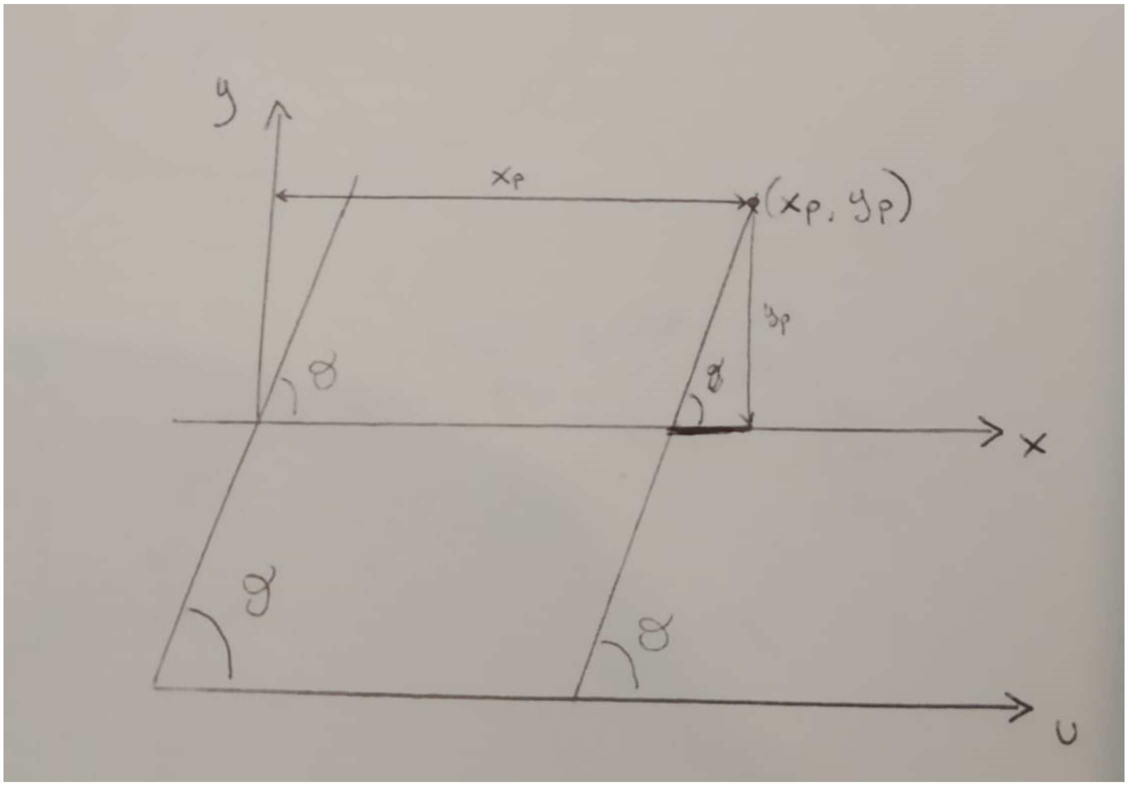

Figure 2: u axis parallel to x axis

$$\frac{y_p}{y'} = sin(\theta), \quad y' = \frac{1}{sin(\theta)} y_p$$
$$\frac{y_p}{x_p - x'} = \tan\theta, \quad x' = x_p - \frac{1}{tan(\theta)} y_p$$

Considering the rotation of the new coordinates, we are able to to obtain the new pixel coordinates modifying the scale and the image coordinates.

$$\begin{aligned} u &= m_u x' + u_0, \\ v &= m_v y' + v_0 \\ u &= m_u x_p - \frac{m_u}{tan(\theta)} y_p + u_0 \\ v &= m_v \frac{1}{sin(\theta)} y_p + v_0 \end{aligned} \tag{1}$$

3 Exercise 3

We want to use homogeneous coordinates in order to represent the results of the point (2.b) with a matrix $\mathbf{K}_{3\times3}$, also known as the camera's intrinsic calibration. The goal is to obtain a matrix thus:

$$\mathbf{K}_{3\times3}\begin{pmatrix}x_c\\y_c\\z_c\end{pmatrix}=\begin{pmatrix}u\\v\\1\end{pmatrix}$$

In order to obtain this matrix:

$$\begin{pmatrix}m_u & -\frac{m_u}{\tan\theta} & u_0\\0 & m_v(\frac{1}{\sin\theta}) & v_0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_p\\y_p\\1\end{pmatrix}=\begin{pmatrix}m_u & -\frac{m_u}{\tan\theta} & u_0\\0 & m_v(\frac{1}{\sin\theta}) & v_0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}fx_c\\fy_c\\z_c\end{pmatrix} \tag{2}$$

4 Exercise 4

Knowing the values of the matrix H for transforming points, we can derive:

$$\begin{pmatrix}u\\v\\1\end{pmatrix}=\mathbf{K}(\mathbf{R}\tilde{x}_w+t)=\mathbf{K}[\mathbf{R}|t]\begin{pmatrix}\tilde{x}_w\\1\end{pmatrix}=\mathbf{P}\begin{pmatrix}\tilde{x}_w\\1\end{pmatrix}$$

5 Exercise 5

a. Let **R** be the rotation matrix, **t** be a translation vector, which transorms a point **x** to **x'** $= Rx+t$. In addition let **R** be the rotation matrix, which rotates a vector x by the angle θ about the axis **u**, according to the Rodrigues formula:

$$Rx=\cos\theta x+\sin\theta u\times x+(1-\cos\Theta)(u\cdot x)u \tag{3}$$

The vector **u** can be decomposed into components parallel and perpendicular to the axis **u**.

$$\mathbf{v}=\mathbf{v}_{\parallel}+\mathbf{v}_{\perp} \tag{4}$$

where the parallel component to **u** is:

$$\mathbf{v}_{\parallel}=(\mathbf{v}\cdot\mathbf{k})\mathbf{k} \tag{5}$$

and the perpendicular component to **u** is:

$$\mathbf{v}_{\perp}=\mathbf{v}-\mathbf{v}_{\parallel}=-\mathbf{k}\times(\mathbf{k}\times\mathbf{v}) \tag{6}$$

Since **u** is a unit vector, the product $\mathbf{k}\times\mathbf{u}$ can be viewed as a copy of $\mathbf{v}_{\perp}$. The perpendicular component will thus maintain the same amplitude:

$$\mathbf{v}_{\perp rot}=|\mathbf{v}_{\perp}| \tag{7}$$

Since $\mathbf{k}$ and $v_{\parallel}$ are parallel their cross-product is zero so:

$$\mathbf{k} \times \mathbf{v}_{\perp rot} = \mathbf{k} \times (\mathbf{v} - v_{\parallel}) = \mathbf{k} \times \mathbf{v} \tag{8}$$

thus:

$$\mathbf{v}_{\perp rot} = \cos\theta \mathbf{v}_{\perp} + \sin\theta \mathbf{k} \times \mathbf{v} \tag{9}$$

The full rotated vector is:

$$\mathbf{v}_{rot} = \mathbf{v}_{\parallel rot} \times \mathbf{v}_{\perp rot} \tag{10}$$

Now, substituting the equations of $v_{\parallel rot}$ and $\mathbf{v}_{\perp rot}$ in the equations results in:

$$\mathbf{v}_{rot} = \mathbf{v}_{\parallel} + \cos\theta \mathbf{v}_{\perp} + \sin\theta \mathbf{k} \times \mathbf{v} = \cos\theta \mathbf{v} + (1 - \cos\theta)(\mathbf{k} \cdot v)\mathbf{k} + \sin\theta \mathbf{k} \times \mathbf{v} \tag{11}$$

b. Firstly, we represent the cross product $\mathbf{k} \times \mathbf{v}$ using a matrix notation:

$$\begin{bmatrix} (\mathbf{k} \times \mathbf{v})_x \\ (\mathbf{k} \times \mathbf{v})_y \\ (\mathbf{k} \times \mathbf{v})_z \end{bmatrix} = \begin{bmatrix} k_y v_z - k_z v_y \\ k_z v_x - k_x v_z \\ k_x v_y - k_y v_x \end{bmatrix} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \mathbf{K} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{12}$$

From definition we know that:

$$K^2 \mathbf{v} = \mathbf{k} \times (\mathbf{k} \times v) = v_{\perp} \tag{13}$$

and also:

$$\mathbf{v}_{\parallel} = \mathbf{v} - \mathbf{v}_{\perp} \tag{14}$$

Putting the definitions in the Rodrigues formula representing the cross product using matrix notation we obtain the following formulation:

$$\cos\theta \mathbf{v} + \sin\theta K \mathbf{v} + (1 - \cos\theta)(\mathbf{v} - K^2 v) \tag{15}$$

$$\cos\theta \mathbf{v} + \sin\theta K \mathbf{v} + \cos\theta \mathbf{v} - \cos\theta K^2 \mathbf{v} + \sin\theta K \mathbf{v}$$

$$\mathbf{v} + \sin\theta K \mathbf{v} + (1 - \cos\theta) K^2$$

Then substituting $v_\theta = \mathbf{R}\mathbf{v}$, we will retrieve R:

$$R = 1 + \sin\theta K + (1 - \cos\theta K^2) \tag{16}$$