Note on Game Theory

Zepeng CHEN

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Contents

Chapter 1 Introduction: Theory of Game

This note introduces the theory of game according to the classifications via Gibbons (1992).

- Static games of complete information: We define Nash equilibrium for this kind of games.
- Static games of incomplete information: Incomplete information means the lack of information about the state of the world. We define Bayesian Nash equilibrium for this kind of games.
- Dynamic games of complete information: Dynamic games may introduce imperfect information, which means the lack of information about the actions that other players have previously taken. In this kind of game with perfect information: first player 1 moves, then player 2 observes player 1's move, then player 2 moves and the game ends. We define the backwards-induction outcome for this kind of games. In this kind of game with imperfect information: first player 1 moves, then other players simultaneously move and the game ends. We define the Subgame-perfect equilibrium for this kind of games.
- Dynamic games of incomplete information: One way to study this kind of game is using Harsanyi transformation and change the game with incomplete information into that with imperfect information. We define Perfect bayesian equilibrium for this kind of game.

Definition 1.1 (Harsanyi Transformation (Kurizaki, 2015))

- *Transform the game of incomplete information into a game of imperfect information.*
- *Introduce a prior move by Nature that determines Player 1's type (i.e., its cost).*
  - *Player 1 observes Nature's move but Player 2 can't*
  - *But Player 2 knows the probability of Nature's move*
- *Player 2's incomplete information about player 1's type becomes Player 2's imperfect information about Nature's move.*

In this book, we introduce different solution concepts and some are the refinement of others. More specifically, their relationships are summarized as follows.

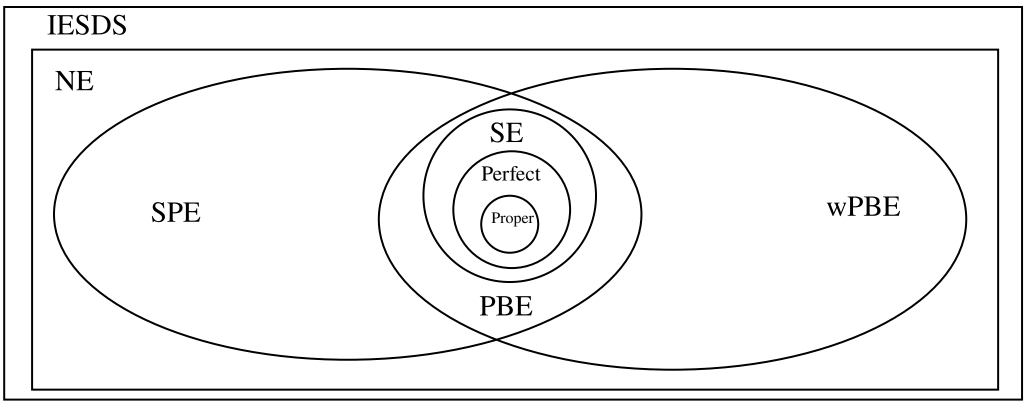


Figure 1.1: Correlations of solution concepts

Generally, solutions derived via maximin strategy coincides with NE in strictly competitive games. And we can say the former is weaker than NE in most cases (Munoz-Garcia, 2017g, p. 48).

Chapter 2 (Non-cooperative) Static game with complete information

2.1 Strategy, Information and Game 1

2.1.2 Game and Assumptions

Definition 2.1 (Common knowledge vs. Mutual knowledge)

1. *A property Y is said to be* ***mutual knowledge*** *if all players know Y (but don't necessarily know that others know it).*
2. *A property Y is* ***common knowledge*** *if everyone knows Y, everyone knows that everyone knows Y, everyone knows that everyone know that everyone knows Y, ..., ad infinitum. Clearly, common knowledge implies mutual knowledge but not vice-versa.*

Definition 2.2 (Normal (Strategic) Form of Game)

1. *Player Set, e.g.* $N = \{1, 2, 3, 4, \cdots, n\}$.
2. *Set of strategy profiles, e.g.* $\prod_{j \in N} S_j$*, where* S_j *is the strategy set for player* j.
3. *Payoff function for each player:* $U_i(s_1, s_2, \ldots, s_n)$.

2.1.2 Rationality, Dominance and Best Response

Definition 2.3 (Set of k-rationalizable strategies (Duozhe, 2022))

Set $\Sigma_i^0 = \Sigma_i$*, and recursively define*

$\Sigma_i^n = \{\sigma_i \in \Sigma_i^{n-1} : \exists \sigma_{-i} \in \times_{j \neq i} \left(\hat{\Sigma}_j^{n-1}\right) \text{ s.t.} U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}) \, \forall \sigma_i' \in \Sigma_i^{n-1}\}$,

where $\hat{\Sigma}_j^{n-1}$ *is the convex hull of* Σ_j^{n-1}.

Note on Convex hull *Here convex full can capture mixed strategies.*

Note on Interpretation (Kartik, 2009) *Here* $\sigma_i \in \sum_i$ *is a k-rationalizable strategy* ($k \geq 2$)*for player* i *if it is a best response to some strategy profile* $\sigma_{-i} \in \sum_{-i}$ *such that each* σ_i *is (k-1)-rationalizable for player* $j \neq i$.

Definition 2.4 (Set of rationalizable strategies (Duozhe, 2022))

The set of rationalizable strategies for player i *is*

$$R_i = \cap_{n=0}^{\infty} \Sigma_i^n.$$

Note on Interpretation (Kartik, 2009) *Here* $\sigma_i \in \sum_i$ *is rationalizable for player* i *if it is* k*-rationalizable for all* $k \geq 1$.

Definition 2.5 (Rational player and Rationalizable strategy profile (Duozhe, 2022))

1. *A rational player will not play a strategy that is never a best response.*
2. *The set of strategies that survive iterated elimination of strategies that are never a best response are rationalizable.*
3. *A strategy profile is rationalizable if the strategy prescribed for each player is rationalizable.*

Theorem 2.1

The set of rationalizable strategies R_i for each player i is nonempty and contains at least one pure strategy. Further each $\sigma_i \in R_i$ is a best response to an element of $\times_{j \neq i}$ convex hull (R_j).

Note on Nonempty *Each period you compare all strategies to find best responses, you can always find at least one strategy.*

Definition 2.6 (Strict and Weak Dominance)

1. *A strategy s_i is strictly dominated for player i if there exists $\sigma_i' \in \sum_i$ such that*
$$u_i(s_i, s_{-i}) < u_i(\sigma_i', s_{-i}) \quad \forall s_{-i} \in S_{-i}.$$
2. *A strategy s_i is weakly dominated by a_i' if $u_i(s_i, s_{-i}) \leq u_i(\sigma_i', s_{-i})$ for $\forall s_{-i} \in S_{-i}$, and the inequality is strict for some s_{-i}.*

Definition 2.7 (Best response mapping / correspondence)

Given an n-player game, player i's best response (mapping: one to multiple) to the strategies x_{-i} of the other players is the strategy x_i^* that maximizes players i's payoff $\pi_i(x_i, x_{-i})$:
$$x_i^*(x_{-i}) = \arg\max_{x_i} \pi_i(x_i, x_{-i}).$$
When the best response is not unique, we use the definition of correspondence, i.e. $n \to n$.

Note on BR *Best response may not be unique.*

Lemma 2.1 (Derivative of BR)

Given payoff $u_i(q_i, x)$ and best response $r_i(q_j) : \frac{\partial u_i(r(q_j), q_j)}{\partial q_i} = 0$, then the derivative is
$$\frac{\partial^2 u_i}{\partial q_i^2}\frac{\partial r}{\partial q_j} + \frac{\partial^2 u_i}{\partial q_i \partial q_j} = 0 \to \frac{dr}{dq_j} = -\frac{\partial^2 u_i}{\partial q_i \partial q_j} / \frac{\partial^2 u_i}{\partial q_i^2}$$

2.1.3 From Pure Strategy to Mixed Strategy

Note that in business area, we focus on pure strategy applications. Thus, we omit further discussions of mixed strategy in later sections.

Definition 2.8 (Pure or Mixed Strategy)

1. *A pure strategy is the action which a player chooses for sure.*
2. *A mixed strategy σ_i of player i is a probability distribution over pure strategies. The set of mixed strategies is denoted by Σ_i ($\Sigma \equiv \times_{i\in N}\Sigma_i$). The expected payoff is*

$$U_i(\sigma) = \sum_{a\in A} \left(\Pi_{j=1}^{N}\sigma_j\left(a_j\right)\right) u_i(a) \tag{2.1}$$

$$= \sum_{a_i\in A_i} \left[\sigma_i(a_i)U_i(a_i, \sigma_{-i})\right] \tag{2.2}$$

Note on Mixed strategy *Here randomization is independent, and we relax this assumption in correlated equilibrium.*

Note on Expected payoff (2.1) *captures the expected payoff via summing all weighted expected payoff given strategy profiles.* (2.2) *captures the expected payoff via summing all weighted expected payoff given strategies with nonnegative probability.*

2.2 Solution Concept 1: Minmax Theorem (Leyton-Brown, 2008a) in Strictly Competitive Games (Munoz-Garcia, 2017g)

Definition 2.9 (Strictly competitive game)

A two-player, strictly competitive game is a two-player game with the property that, for every two strategy profiles s and s',

$$u_1(s) \geq u_1\left(s'\right) \text{ and } u_2(s) \leq u_2\left(s'\right)$$

Note on Special case *Constant-sum games are special cases of strictly competitive games, and zero-sum games are special cases of constant-sum games.*

Definition 2.10 (Maxmin strategy and Maxmin value)

The maxmin strategy for player i is a strategy that maximize i's worst-case payoff

$$\arg\max_{s_i}\min_{s_{-i}} u_i\left(s_1, s_2\right)$$

and the maxmin value for player i is the payoff guaranteed by the maxmin strategy.

$$\max_{s_i}\min_{s_{-i}} u_i\left(s_1, s_2\right)$$

Definition 2.11 (Minmax strategy and Minmax value)

The minmax strategy for player i is a strategy that minimizes the other player $-i$'s best-case payoff

$$\arg\min_{s_i}\max_{s_{-i}} u_{-i}\left(s_1, s_2\right)$$

and the minmax value for player i is the payoff achieved by the minmax strategy.

$$\min_{s_i} \max_{s_{-i}} u_{-i}(s_1, s_2)$$

Theorem 2.2 (Minmax theorem (von Neumann))

In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

Remark In geometry, it means a saddle point.

2.3 Solution Concept 2: Iterated Strict Dominance Equilibrium

These strategy profiles are usually derived via iterated elimination of strictly dominated strategies (IESDS).

Definition 2.12 (iterated strict dominance equilibrium)

Given strong rationality and common knowledge assumption, repeatedly applying strict dominance to derive this equilibrium.

Note on Uniqueness *If we do elimination of strictly dominated strategies, then the equilibrium is unique, otherwise the uniqueness can not be ensured.*

Note on Condition *Some (very few though) games can be "solved" by iterated elimination of strictly dominated strategies.*

Note on Order independence *An important feature of iterated elimination of strictly dominated strategies is that the order of the elimination has no effect on the set of strategies that remain in the end.*

Note on Weakly dominated *A strategy that is weakly dominated cannot be ruled out based only on principles of rationality. Iterated elimination of weakly dominated strategies is order dependent, for example,*

	L	C	R
T	$1,1$	$1,1$	$0,0$
B	$0,0$	$1,2$	$1,2$

.

Note on Mixed strategy in IESDS

1. *If there is a mixed strategy σ_i' dominate all pure strategies, then it dominate all mixed strategies.*
   - *Since mixed strategy is convex combination of pure strategy.*
2. *A pure strategy may be dominated by a mixed strategy, and vice versa.*
   - *This property is very useful to prove some mixed strategies is dominated.*
   - *For example, 0.5T+0.5M is dominated by B, while T and M are not dominated by B. Using this property, we con show that all mixed strategies of T and M is dominated, by*

the new strategy we construct. For example, for 0.9T+0.1M=0.8T+0.2(0.5T+0.5M), we does know whether the former is dominated, but the latter is dominated by 0.8T+0.2*B.*

	L	R
T	3	0
M	0	3
B	2	2

- *For example, there are no strictly dominated pure strategies for player 1 nor for player 2. However, if player 1 mixes between B (with prob q) and C (with prob 1-q), he obtains an expected utility that exceeds that from selecting F. Thus we can eliminate F from the matrix, since it is strictly dominated by a randomization between B and C.*

	F	C	B
F	$0, 5$	$2, 3$	$2, 3$
$\mathrm{C}(1-\mathrm{q})$	$2, 3$	$0, 5$	$3, 2$
$\mathrm{B}(\mathrm{q})$	$5, 0$	$3, 2$	$2, 3$

3. *A mixed strategy that assigns positive probability to a (strictly) dominated pure strategy must be (strictly) dominated.*
    - *This property is very useful to eliminate dominated strategies and get a smaller game.*

Definition 2.13 (Dominant Solvable (Dean, 2017))

A game G is dominant solvable if we can derive a unique equilibrium through eliminating strictly dominated strategies.

Proposition 2.1 (iterated dominance equilibrium and NE (Gibbons, 1992, p. 12))

1. *In the n-player normal-form game* $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, *if iterated elimination of strictly dominated strategies eliminates all but the strategies* $(s_1^*, ..., s_n^*)$, *then these strategies are the unique Nash equilibrium of the game.*
2. *In the n-player normal-form game* $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, *if the strategies* $(s_1^*, ..., s_n^*)$ *are a Nash equilibrium, then they survive iterated elimination of strictly dominated strategies.*

Note on *Note that NE is stronger than IDSDS, actually, if we cannot eliminate strictly dominated strategies, all strategy profiles survive the application of IDSDS (Munoz-Garcia, 2017d).*

Proof We prove (2) by contradiction, assuming that $(s_1^*, ..., s_n^*)$ is a NE, and there exists s_i^* such that s_i^* is strictly dominated. Then it means s_i^* is not best response and contradicts the definition of NE. This proof actually shows that NE is stronger than iterated strict dominance equilibrium, that is, every NE survives iterated elimination of strictly dominated strategies. (1) can be proved by contradiction too. ■

Lemma 2.2

Rationalizability and iterated strict dominance coincide in two-player games.

Proof This is equivalent to show that

$$\sigma_i \text{ is strictly dominated } \iff \sigma_i \text{ is never a best response.}$$

■

Note on *With more than two players, the equivalence between being strictly dominated and never a best response breaks down. The reason is that mixed strategies require that player's randomizations are independent. For example, D is not a best response to player 3, while it is also not dominated.*

	L	R
U	3	0
D	0	0

Figure 2.1: A

	L	R
U	0	3
D	3	0

Figure 2.2: B

	L	R
U	0	0
D	0	3

Figure 2.3: C

	L	R
U	2	0
D	0	2

Figure 2.4: D

2.4 Solution Concept 3: Nash Equilibrium

2.4.1 Nash Equilibrium

Definition 2.14 (Pure vs Mixed Nash Equilibrium)

A mixed-strategy profile σ^* is a Nash equilibrium if, for all players i,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \quad \forall s_i \in S_i.$$

A pure-strategy Nash equilibrium is a pure-strategy profile that satisfies the same conditions.

Lemma 2.3 (NE as mutual best response)

An outcome $(x_1^*, ...x_n^*)$ is a Nash equilibrium of the game if x_i^* is a best response to x_{-i}^* for all $i = 1, ..n$.

Note on Computation *This lemma can be used to find NE. For example in Matching pennies.*

		2	
		H	T
1	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

Suppose player 1 chooses H with probability r and T with $1-r$, player 2 chooses H with q and T with $1-q$. Then player 1's expected utility is $u_1 = -(2q-1)(2r-1)$. And FOC implies that $\frac{\partial U_1}{\partial r} = -2(2q-1)$: >0 if $q<\frac{1}{2}$, $=0$ if $q=\frac{1}{2}$, <0 if $q>\frac{1}{2}$. Thus the best response correspondence for player 1 and 2 are

$$\beta_1(q) = \begin{cases} 1 & q<\frac{1}{2} \\ [0,1] & q=\frac{1}{2} \\ 0 & q>\frac{1}{2} \end{cases} \qquad \beta_2(r) = \begin{cases} 0 & r<\frac{1}{2} \\ [0,1] & r=\frac{1}{2} \\ 1 & r>\frac{1}{2} \end{cases}$$

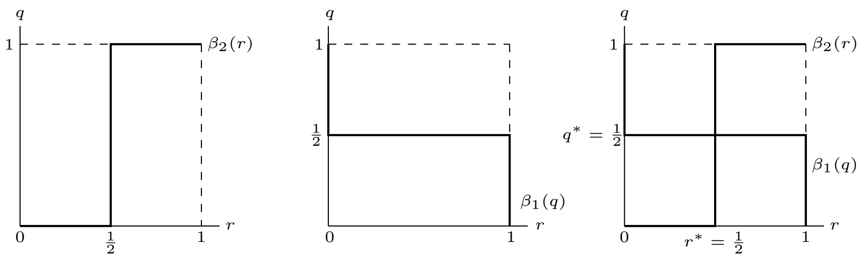


Figure 2.5: MNE for Matching Pennies

Note on Mixed NE

1. *A pure-strategy NE is a degenerate mixed-strategy NE, and there may exist no pure-strategy NE. For example, there is only a unique mixed NE where both chooses Head with probability 1/2.*

	Head	*Tail*
Head	$\underline{1}, -1$	$-1, \underline{1}$
Tail	$-1, \underline{1}$	$\underline{1}, -1$

2. *Mixed strategy means to randomize to confuse your opponent.*
3. *Mixed strategies are a concise description of what might happen in repeated (infinite) play.*
4. *Dominated strategies are never used in mixed Nash equilibria, even if they are dominated by another mixed strategy.*

Note on Infinite mixed NE *There can be infinite mixed NE. For example, this game has two pure NEs: (In, Accept) and (Out, Fight). When player 1 chooses Out, 2 is indifferent between two strategies. Thus there is a continuum of mixed NE: player 1 chooses "Out", while 2 chooses "Fight" with a probability at least 1/3. Note that the pure NE (Out, Fight) belongs to this family of mixed NE.*

	Accept	Fight
Out	$0, 4$	$\underline{0}, \underline{4}$
In	$\underline{1}, \underline{2}$	$-2, 1$

Note on Condition *In a Nash equilibrium,*

1. *Each player's strategy is required to be optimal given his belief about other players' strategies.*
2. *Each player's belief about other players' strategies is required to be correct.*

The latter requirement is strong and somewhat unreasonable.

Note on Interpretation of mixed NE *Mixed strategies can be viewed as pure strategies in a perturbed game, and a mixed NE can be viewed as a steady state. A mixed strategy profile can be viewed as players' beliefs about each other's strategic choices.*

Note on Rationlizable in mixed NE (Duozhe, 2022) *Every strategy used with positive probability in some mixed strategy NE is rationalizable.*

Lemma 2.4 (Property of Mixed NE)

In a mixed-strategy NE, each player is indifferent among all those pure strategies that he chooses with positive probability, that is, $E[u_i(s_i, p_{-i})]$ *must be the same for all such strategies.*

Note on Interpretation and Proof *This property can be directly seen and proved from Equation 2.2.*

Note on Condition and Computation and Geometry Interpretation *This property is very useful to compute and find a mixed NE, however, we should pay attention to its condition of* ***positive probability****. For those with zero probability, we do not require their expected payoff to be the same. For example, if we denote the probability of L, M and R as* q_1, q_2 *and* $1 - q_1 - q_2$, *and compute mixed NE by this property, then there is no mixed NE. The correct way to find mixed NE is next verify the combinations* $\{L, M\}$, $\{L, R\}$ *and* $\{M, R\}$.

	L	M	R
T	$3, 4$	$1, 3$	$3, 0$
B	$0, 1$	$2, 3$	$0, 4$

Lemma 2.5 (Oddness Theorem (Wilson 1971))

Almost *every finite game has an odd number of NE's in mixed strategies.*

Note on Almost *Actually, there is a counter example. This game has two pure NEs: (T,L) and (B,R), but no mixed NE.*

	L	R
T	$1,1$	$0,0$
B	$0,0$	$0,0$

Lemma 2.6 (Fixed points and NE)

Nash equilibrium must satisfy $\frac{\partial \pi_i}{\partial x_i} = 0$ *for all player* i*, let* $f_i(x_1, ..x_n) = \frac{\partial \pi_i}{\partial x_i} + x_i$*, then we can find NE by fixed points theorem.*

$$f_i(x_1^*, ..x_n^*) = x_i^* \to \frac{\partial \pi_i(x_1^*, ...x_n^*)}{\partial x_i} = 0, \quad \forall i$$

2.4.2 Existence of NE

Theorem 2.3 (Nash's Theorem)

Every finite strategic-form game has a mixed-strategy equilibrium.

Note on Condition *If the game is not finite, then the existence can not be guaranteed.*

Proof P10 Li duozhe's note. ■

Theorem 2.4 (Debreu 1952)

Consider a strategic-form game whose strategy spaces S_i *are nonempty compact*[a] *convex subsets of an Euclidean space. If the payoff functions* u_i *are continuous in* s *and quasi-concave in* s_i*, there exists a pure-strategy Nash equilibrium.*

[a] Strategy space is compact if it is closed and bounded

Corollary 2.1

Suppose that a game is symmetric, and for each player, the strategy space is compact and convex and the payoff function is continuous and quasiconcave with respect to each player's own strategy. Then, there exists at least one symmetric pure strategy NE in the game.

Theorem 2.5 (Glicksberg 1952)

Consider a strategic-form game whose strategy spaces S_i *are nonempty compact subsets of a metric space. If the payoff functions* u_i *are continuous then there exists a Nash equilibrium in mixed strategies.*

Note on Comparison *Compared to Theorem 2.3, this theorem generalizes the condition to infinite strategy space. Compared to Theorem 2.4, this theorem relaxes the condition of quasi-concavity, however, this theorem does not guarantee that the NE is in pure strategies.*

2.4.3 Uniqueness of NE

2.4.3.1 Contraction Mapping Argument

Definition 2.15 (Contraction Mapping Argument)

Mapping $f(x) : R^n \to R^n$ *is a contraction iff* $||f(x_1) - f(x_2)|| \leq \alpha||x_1 - x_2|| \quad \forall x_1, x_2, \alpha < 1$.

For example, in Figure 2.6, contraction mapping can derive a converged iterated series, which satisfies $|f'(x)| < 1$; otherwise, the series will not converge.

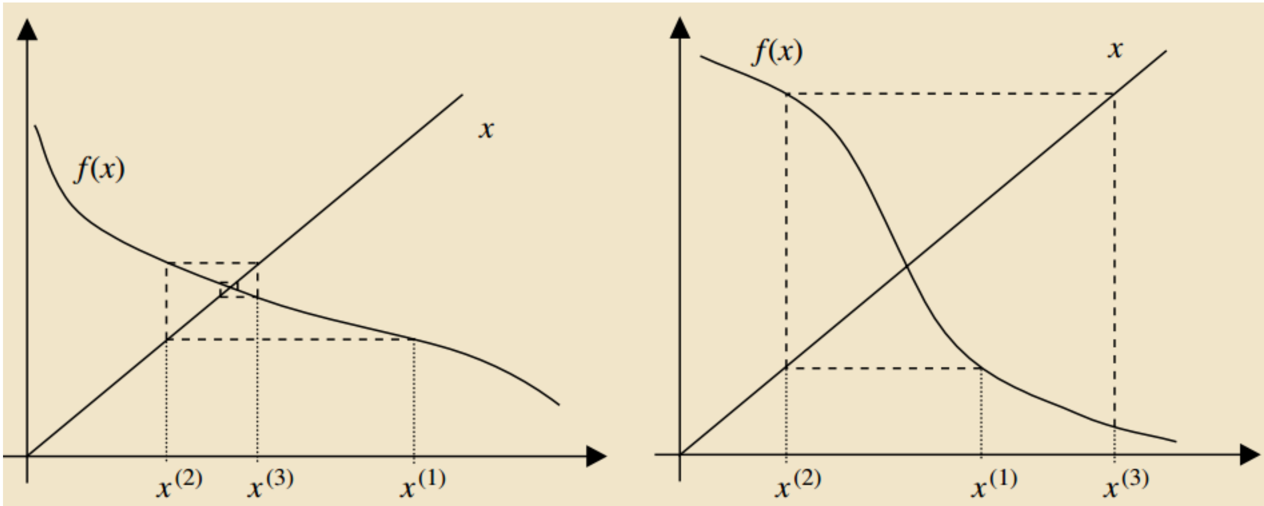


Figure 2.6: Converging (left) and diverging (right) iterations

Theorem 2.6 (contraction and NE)

If the best response mapping is a contraction on the entire strategy space, there is a unique NE in the game.

Theorem 2.6 connects contraction mapping with NE, though not clarifies which type of game satisfy the condition of contraction mapping. Suppose n players with strategy x_i and BR $x_i = f_i(x_{-i})$, then define A as

$$A = \begin{bmatrix} 0 & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & 0 & \dots & \frac{\partial f_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & 0 \end{bmatrix}.$$

The spectral radius is defined as $\rho(A) = \{\max |\lambda| : Ax = \lambda x, x \neq 0\}$.

Theorem 2.7 (spectral radius rule)

The mapping $f(x) : R^n \to R^n$ *is a contraction if and only if* $\rho(A) < 1$ *everythere.*

Note on Problems *There are two problems of Theorem 2.7:*

- *Eigenvalue is not easy to calculate, however, it is enough to verify* $||A|| < 1$ *via the largest* *eigenvalue, that is, ensure that the sum of every row or column is smaller than 1.*

$$\sum_{i=1}^{n} |\frac{\partial f_k}{\partial x_i}| < 1 \quad or \quad \sum_{i=1}^{n} |\frac{\partial f_i}{\partial x_k}| < 1, \quad \forall k$$

In the case of two players, the condition can be rewritten as

$$|\frac{\partial f_1}{\partial x_2}| < 1 \quad and \quad |\frac{\partial f_2}{\partial x_1}| < 1$$

- *BR is not easy to find sometimes. However, via the implicit function theorem, Theorem 2.7 can be rewritten as* ***diagonal dominace****, that is, every elements on the diagonal should be greater than the sum of other elements in its row.*

$$\sum_{i=1, i\neq k}^{n} |\frac{\partial^2 \pi_k}{\partial x_k \partial x_i}| < |\frac{\partial^2 \pi_k}{\partial x_k^2}|, \quad \forall k$$

2.4.3.2 Univalent Mapping Argument

Theorem 2.8 (univalent and unique)

Suppose the strategy space of the game is convex and all equilibria are interior. Then, if the determinant $|H|$ is negative quasidefinite (i.e. if the matrix $H + H^T$ is negative definite) on the players' strategy set, there is a unique NE.

Note on *Note that the condition of univalent mapping argument is weaker than that of argument. In the case of two players, this theorem can be rewritten as*

$$|\frac{\partial^2 \pi_2}{\partial x_2 \partial x_1} + \frac{\partial^2 \pi_1}{\partial x_1 \partial x_2}| \leq 2\sqrt{\frac{\partial^2 \pi_1}{\partial x_1^2}\frac{\partial^2 \pi_2}{\partial x_2^2}}, \quad \forall x_1, x_2.$$

2.4.3.3 Index Theory Approach

Theorem 2.9 (index-theory and unique)

Suppose the strategy space of the game is convex and all payoff functions are quasiconcave. Then, if $(-1)^n|H|$ is positive whenever $\frac{\partial \pi_i}{\partial x_i} = 0$, all i, there is a unique NE.

Note on *Theorem 2.9 is also weaker than Theorem 2.6. However, Theorem 2.9 only requires the condition holds at equilibrium. In case of two players, Theorem 2.9 can be rewritten as*

$$\begin{vmatrix} \frac{\partial^2 \pi_1}{\partial x_1^2} & \frac{\partial^2 \pi_1}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi_2}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi_2}{\partial x_2^2} \end{vmatrix} > 0, \quad \forall x_1, x_2 : \frac{\partial \pi_1}{\partial x_1} = 0, \frac{\partial \pi_2}{\partial x_2} = 0$$

This can also be interpreted as that the product of the slopes of BR cannot be greater than 1, i.e.,

$$\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} < 1 \quad at x_1^*, x_2^*$$

2.4.4 Other properties of NE: Pareto optimality, Stability and Interiority

Definition 2.16 (Pareto frontier set)

The set of strategies such that each player can be made better off only if some other player is made worse off.

Definition 2.17 (Pareto optimal and Pareto inferior)

A set of strategies is Pareto optimal if they are on the Pareto frontier, otherwise, a set of strategies is Pareto inferior.

Definition 2.18 (stable equilibrium)

An equilibrium is considered stable (for simplicity we will consider asymptotic stability only) if the system always returns to it after small disturbances. If the system moves away from the equilibrium after small disturbances, then the equilibrium is unstable.
The sufficient condition for asymptotically stable is

$$|\frac{dr_1}{dq_2}||\frac{dr_2}{dq_1}| < 1 \quad or \quad \frac{\partial^2 u_1}{\partial q_1 \partial q_2}\frac{\partial^2 u_2}{\partial q_1 \partial q_2} < \frac{\partial^2 u_1}{\partial q_1^2}\frac{\partial^2 u_2}{\partial q_2^2}$$

Note on *For example, B, C and D are all intersects of BR, and thus all NE. However, C is not stable, unless the status is C at the beginning, otherwise the status will move to B and D.*

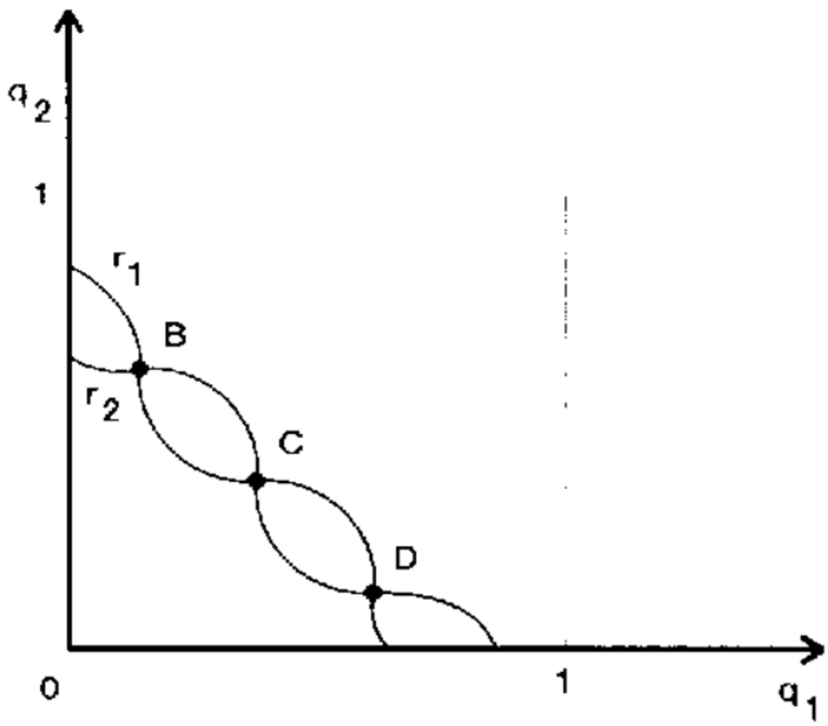


Figure 2.7: Example of stable and unstable NE

Definition 2.19 (Interior equilibrium and Boundary equilibrium)
Interior equilibrium is the one in which first-order conditions hold for each player. The alternative is boundary equilibrium in which at least one of the players select the strategy on the boundary of his strategy space.

2.5 Examples of finding NE

2.5.1 Hotelling Model

Consumers are uniformly distributed on $[0, 1]$ and each consumer buy one product from the closest vendor (minimizing transportation cost). Two vendors choose their locations simultaneously. In the unique Nash equilibrium, both vendors choose the midpoint of the boardwalk.

Note on Product differentiation *In the competition for market share, product differentiation is minimized in equilibrium. Similarly, in electoral competitions under bipartisan system, candidates tend to propose similar policies or express similar views on sensitive issues.*

Note on Multiple vendors *No pure NE is found currently.*

Note on Real examples *This model can be used to capture many real examples: 1 tax rate; 2 coco percent of chocolate bar. Actually, location means market share here.*

2.5.2 Cournot Model (Quantity Competition)

Two firms produce an identical product, and they set output levels (q_1, q_2) simultaneoulsy. Total output is $Q = q_1 + q_2$, and the market clearing price is $P = P(Q)$, each firm's utility is $u_i(q_1, q_2) = q_i P(Q) - C_i(q_i)$. Here we consider a special case where the inverse market demand is linear $p(Q) = a - bQ$, and the cost functions are also linear, $c(q_i) = c_i q_i$. This game thus can solve by BR equations via FOC. For example,

$$\beta_1(q_2) = \begin{cases} \frac{a-c_1}{2b} - \frac{q_2}{2} & q_2 \leq \frac{a-c_1}{b} \\ 0 & q_2 > \frac{a-c_1}{b} \end{cases}.$$

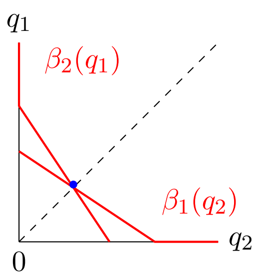


Figure 2.8: Case: $c_1 = c_2$

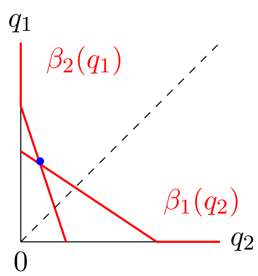


Figure 2.9: Case: $c_1 < c_2$

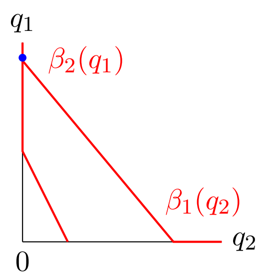


Figure 2.10: Corner Solution

IESDS. Cournot model can also be solved by iterated deletion of strictly dominated strategies. We first define firm i's best response function $r_i : [0, \infty) \to [0, \infty)$, and by FOC we have $r(q_{-i}) = \frac{a-c}{2b} - \frac{q_{-i}}{2}$. At the beginning of iteration, our strategy space is $S_i^0 = [0, \infty)$. In the first period, any strategy above $r(0)$ is strictly dominated for each firm, then the updated strategy space is $S_i^1 = [0, r(0)]$. Similarly in the second period, any strategy below $r(r(0)) = r^2(0)$ is strictly dominated, and our updated strategy space is $S_i^2 = [r^2(0), r(0)]$. Following this track, it converge to $\frac{a-c}{3b}$.

$$r^n(0) = -\frac{a-c}{b} \sum_{k=1}^{n} (-\frac{1}{2})^k$$

Multiple firms. When extending to n firms with linear inverse market demand, the NE is $q_i^* = \frac{a-c}{b(n+1)}$.

Corner Solution (Munoz-Garcia, 2017c). For example, in Figure 2.10, if $\frac{a-c_2}{b} < \frac{a-c_1}{2b}$, i.e., the vertical intercept of $\beta_1(q_2)$ is greater than the horizontal intercept of $\beta_2(q_1)$, then we have a corner solution.

Substitution and Complementation. There are three firms, A, B and C, in the market, in which A and B provides product 1, C provides product 2. $d \in [-1, 1]$ denotes the relationship between product 1 and 2, where $d = 0$ means no relationship, $d = 1$ means complement and $d = -1$ means substitute. The margin cost are c_A, c_B, c_C, and each firm decides the quantity q_A, q_B, q_C, the price function are

$$p_i = A - q_i - dq_j, \quad i, j = 1, 2$$

By FOC, we have,

$$\begin{aligned} q_A^*(d) &= \frac{1}{2}(c_B - c_A) + \frac{2A - c_A - c_B - d(A - c_C)}{6 - 2d^2} \\ q_B^*(d) &= \frac{1}{2}(c_A - c_B) + \frac{2A - c_A - c_B - d(A - c_C)}{6 - 2d^2} \\ q_C^*(d) &= \frac{3(A - c_C) - d(2A - c_A - c_B)}{6 - 2d^2} \end{aligned}$$

2.5.3 Bertrand Model (Price Competition)

Firm 1 and 2 choose price p_1 and p_2, their demand functions are follows, where $b > 0$ reflects substitution. There are not fixed costs of production, and marginal costs are constant at $c < a$. Finally we have $p_1^* = p_2^* = \frac{a+c}{2-b}$.

$$q_i(p_i, p_j) = a - p_i + bp_j$$

2.5.4 The Tragedy of the Commons

n farmers in a village chooses the number g_i of goats. Assume that the margin costs for each goat is c, and the value of each goat is $v(G)$, where $G = g_1 + ... + g_n$ is the total number of goats. Due to the limit of land resource, there exists $G_{max} : v(G) > 0 \quad \forall G < G_{max}, v(G) = 0 \quad \forall G \geq G_{max}$, and $v'(G) < 0, v''(G) < 0 \quad \forall G < G_{max}$. The payoff function is

$$g_i v(g_1 + \cdots + g_{i-1} + g_i + g_{i+1} + \cdots + g_n) - cg_i.$$

Let G^* denotes the total number in equilibrium when they make separate decisions, and summarizing all FOC conditions, we have

$$v\left(G^*\right)+\frac{1}{n}G^*v'\left(G^*\right)-c=0.$$

However, the optimal quantity for the whole society $G^E<G^*$.

$$v\left(G^E\right)+G^Ev'\left(G^E\right)-c=0$$

2.5.5 A Model of Sales (Varian, 1980)

There are N firms and two types of consumers (informed I and uniformed U) in the market, and the population of consumers is 1 ($I+U=1$). Saying that the willingness to pay v is the same for all consumers, and firms do bertrand competition with marginal cost c. The payoff function for firm i given charging $p_i\in[c,v]$ is the following, where k is the number of firms charging the lowest price.

$$\pi_i\left(p_i,p_{-i}\right)=\begin{cases}\left(p_i-c\right)\frac{U}{N} & \text{if } p_i>\min_{j\neq i}p_j\\ \left(p_i-c\right)\left(\frac{U}{N}+\frac{l}{k}\right) & \text{if } p_i=\min_{j\neq i}p_j\end{cases}$$

Lemma 2.7

There is no pure Nash equilibrium in this game, and a unique symmetric equilibrium such that

$$\begin{aligned}&\bar{p}(F)=v\\&(\underline{p}(F)-c)\left(\frac{U}{N}+I\right)=(v-c)\frac{U}{N}\\&(p-c)\left[\frac{U}{N}+(1-F(p))^{n-1}I\right]=(v-c)\frac{U}{N},\forall p\in[\underline{p}(F),\bar{p}(F)]\end{aligned}$$

Here $F(p),f$ means the cdf,pdf of strategy p and $\bar{p}=\inf\{p\mid F(p)=1\}$, $\underline{p}=\sup\{p\mid F(p)=0\}$. This MNE means

1. *The highest possible price is v.*
2. *The lowest price give the same profit for highest price.*
3. *Charging any price in between gives the same profit.*

Chapter 3 (Non-cooperative) Static game with incomplete information

3.1 Information and Knowledge (Levin, 2006)

3.1.1 A Model of Knowledge

Definition 3.1 (Model of Knowledge)

1. *A set of states* Ω, *one of which is true.*
2. *For each state* $\omega \in \Omega$, *and a given agent* i, *there is a set of information function* $h_i(\omega)$.
3. *An event is a set of states* $E \subseteq \Omega$.
4. *An agent knows* E *if* E *obtains at all the states that the agent believes are possible.*

Definition 3.2 (Information function)

Given a set of states Ω, *an information function associates every state* $\omega \in \Omega$ *with a nonempty subset* $h(\omega)$ *of* Ω.

Note on $h(\omega)$ $h(\omega)$ *is the set of states the agent believes to be possible at* ω.

Definition 3.3 (Partitional information function 1)

An information function is partitional if there is some partition of Ω *such that for any* $\omega \in \Omega$, $h(\omega)$ *is the lement of the partition that contians* ω.

Definition 3.4 (Partitional information function 2)

An information function is partitional iff it satisfies P1 and P2.

P1 $\omega \in P(\omega)$ *for every* $\omega \in \Omega$.

P2 *If* $\omega' \in P(\omega)$ *then* $P(\omega') = P(\omega)$.

Note on Interpretation

1. *Property P1 says that, given state* ω, *the agent is not convinced that the state is not* ω.
2. *Property P2 says that if* ω' *is also deemed possible, then the set of states that would be deemed possible were the state actually* ω' *must be the same as those currently deemed possible at* ω.

Example 3.1 Given state $\Omega = \{a, b\}$, the information funciton $P(a) = \{a\}, P(b) = \{a, b\}$ satisfies P1 but not P2.

Definition 3.5 (Knowledge function 1)

The agent's knowledge function of event E is the set of states at which the agent knows E:

$$K(E) = \{\omega \in \Omega : h(\omega) \subseteq E\}.$$

Note on Interpretation *If $h(\omega) \subset E$, then in state ω, the agent views $\neg E$ as impossible. Hence we say that the agent knows E.*

Example 3.2 Suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $H = \{\{\omega_1, \omega_2\}, \{\omega_3\}\{\omega_4\}\}$, then $K(\{\omega_3, \omega_4\}) = \{\omega_3, \omega_4\}$ and $K(\{\omega_1, \omega_3\}) = \{\omega_3\}$.

Lemma 3.1 (Knowledge function's property)

The knowledge function derived from any information function satisfies:

K1 (Axiom of Awareness) $K(\Omega) = \Omega$.

K2 $K(E) \cap K(F) = K(E \cap F)$.

K3 If $E \subseteq F$, then $K(E) \subseteq K(F)$.

If P1 is satisfied, then

K4 (Axiom of knowledge) $K(E) \subseteq E$.

If $P(.)$ is partitional, then

K5 (Axiom of transparency) $K(E) \subseteq K(K(E))$.

K6 (Axiom of wisdom) $\neg K(E) \subseteq K(\neg K(E))$, where $\neg K(E) = \Omega \backslash K(E)$.

Note on Interpretation

1. *K1: Regardless of the actual state, the agent knows that he is in some state.*
2. *K2: If the agent knows E and knows F, then he knows $E \cap F$.*
3. *K3: If F occurs whenever E occurs, then knowing F means knowing E as well. This property can be derived directly from K2.*
4. *K4: If the agent knows E, then E must have occurred.*
5. *K5: If the agent knows E, then he knows that he knows E. Moreover, if $P(.)$ is partitional, then we can say $K(E) = E$ (and also $K(E) = K(K(E))$).*
6. *K6: If the agent doesn't know E, then he knows that he doesn't know E.*

Example 3.3 Puzzle of hats Each of three individuals is wearing a hat that is either black or white. Each can see others' hats, but not his own. The states of the world are

	a	b	c	d	e	f	g	h
1	B	B	B	B	W	W	W	W
2	B	B	W	W	B	B	W	W
3	B	W	B	W	B	W	B	W

.

And the information partitions are

$$\begin{aligned} P_1 &= \{\{a,e\},\{b,f\},\{c,g\},\{d,h\}\} \\ P_2 &= \{\{a,c\},\{b,d\},\{e,g\},\{f,h\}\}. \\ P_3 &= \{\{a,b\},\{c,d\},\{e,f\},\{g,h\}\} \end{aligned}$$

Suppose that all the hats are black. If you ask any one whether he can identify the color of his own hat, the answer is always negative. Now, if you tell them that there is at least one black hat, the answer may change. The event that i knows the color of his own hat is $K_c^i := \{w : P_i(w) \in E_W^i \quad \text{or} \quad P_i(W) \in E_B^i\}$, and at the begining, $K_C^i = \emptyset$. Firstly, we can exclude h.

$$P_1 = \{\{a, e\}, \{b, f\}, \{c, g\}, \{d, \cancel{h}\}\}$$
$$P_2 = \{\{a, c\}, \{b, d\}, \{e, g\}, \{f, \cancel{h}\}\}$$
$$P_3 = \{\{a, b\}, \{c, d\}, \{e, f\}, \{g, \cancel{h}\}\}$$

Then for player 1, we have $K_c^1 = \{d\}$ and 1 knows the answer if state is d, if 1 say no, it means the case cannot be d.

$$P_1 = \{\{a, e\}, \{b, f\}, \{c, g\}, \{d, \cancel{h}\}\}$$
$$P_2 = \{\{a, c\}, \{b, \cancel{d}\}, \{e, g\}, \{f, \cancel{h}\}\}$$
$$P_3 = \{\{a, b\}, \{c, \cancel{d}\}, \{e, f\}, \{g, \cancel{h}\}\}$$

Then for player 2, 2 knows the answer if state is b or f, if player 2 say no, it cannot be b or f.

$$P_1 = \{\{a, e\}, \{b, f\}, \{c, g\}, \{d, \cancel{h}\}\}$$
$$P_2 = \{\{a, c\}, \{b, \cancel{d}\}, \{e, g\}, \{f, \cancel{h}\}\}$$
$$P_3 = \{\{a, \cancel{b}\}, \{c, \cancel{d}\}, \{e, \cancel{f}\}, \{g, \cancel{h}\}\}$$

3.1.2 Common Knowledge

Definition 3.6 (Common knowledge 1)

Let K_1 and K_2 be the knowledge functions of individuals 1 and 2 for the set Ω of states. An event $E \subseteq \Omega$ is common knowledge between 1 and 2 in the state $\omega \in \Omega$ if ω is a member of every set in the infinite sequence $K_1(E), K_2(E), K_1(K_2(E))$, ...

Note on Interpretation *That is, at the state ω, 1 knows E, 2 knows E, 1 knows that 2 knows E,* ...

Definition 3.7 (Self-evident event)

Let P_1 and P_2 be the information functions of individuals 1 and 2 for the set Ω of states. An event $F \subseteq \Omega$ is self-evident between 1 and 2 if for all $\omega \in F$ we have $P_i(\omega) \subseteq F$ for $i = 1, 2$.

Note on $\neg F$ *If F is self-evident, then $\neg F$ is also self-evident.*

Note on Ω *The entire space is always self-evident and common knowledge.*

Definition 3.8 (Common knowledge 2)

An event $E \subseteq \Omega$ is common knowledge between 1 and 2 in the state $\omega \in \Omega$ if there is a self-evident event F for which $\omega \in F \subseteq E$.

Lemma 3.2 (Equivalence of Self-evident)

The following are equivalent:

1. $K_i(E) = E$ *for all* i,
2. E *is self-evident,*
3. *For all* i, E *is a union of members of the partition induced by* h_i.

Lemma 3.3

Definition 3.1.2 and Definition 3.1.2 are equivalent.

Note on Example *Let* $\Omega = \{a, b, c, d, e, f\}$ *and* $E = \{a, b, c, d\}$, *and*

$$\begin{aligned} P_1 &= \{\{a, b\}, \{c, d, e\}, \{f\}\} \\ P_2 &= \{\{a\}, \{b, c, d\}, \{e\}, \{f\}\} \end{aligned}.$$

Then we have

$$\begin{aligned} K_1(E) &= \{a, b\} \\ K_2(E) &= E \\ K_1(K_2(E)) &= \{a, b\} \\ K_2(K_1(E)) &= \{a\} \\ K_1(K_2(K_1(E))) &= \phi \end{aligned}$$

Thus E *cannot be common knowledge between 1 and 2.*

Definition 3.9 (Posterior belief)

Suppose that individuals 1 and 2 have the same prior belief $\rho(.)$ *on* Ω *and partitional information functions* $P_i(.)$. *Then in some states* $\omega^* \in \Omega$, *individual* i*'s posterior belief that some state in the event* E *has occurred is*

$$\rho(E \mid P_i(\omega^*)) = \frac{\rho(E \cap P_i(\omega^*))}{\rho(P_i(\omega^*))}$$

Definition 3.10 (Knowledge function 2)

Individual i *knows event* E *in state* ω ($P_i(\omega) \subseteq E$) *is equivalent to*

$$\rho(E \mid P_i(\omega)) = 1$$

and then the knowledge function can be defined as

$$K_i(E) = \{\omega \in \Omega : \rho(E \mid P_i(\omega)) = 1\}.$$

Note on *More generally, we can define the event that individual* i *assigns probability* q_i *to event* E *as*

$$\{\omega \in \Omega : \rho(E \mid P_i(\omega)) = q_i\}.$$

Note on Example *Let* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\rho(\omega_1) = \rho(\omega_4) = 1/6$ *and* $\rho(\omega_2) = \rho(\omega_3) =$

1/3.

$$P_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$$
$$P_2 = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$$

Let $E = \{\omega_2, \omega_3\}$. It is easy to see that

$$\rho\left(E \mid P_i\left(\omega_s\right)\right) = \frac{\rho(w_2)}{\rho(w_1) + \rho(w_2)} = 2/3$$
$$\{\omega \in \Omega : \rho\left(E \mid P_i(\omega)\right) = 2/3\} = \Omega.$$

3.1.3 The Agreement Theorem

Theorem 3.1 (Agreeing to Disagree 1 (Aumann, 1976))

If two people have the same priors, and their posteriors for an event A are common knowledge, then these posteriors are equal.

Theorem 3.2 (Agreeing to Disagree 2 (Aumann, 1976))

Suppose two agents have the same prior belief over a finite set of states Ω. If each agent's information function is partitional and it is common knowledge in some state $\omega \in \Omega$ that agent 1 assigns probability η_1 to some event E and agent 2 assigns probability η_2 to E, then $\eta_1 = \eta_2$.

Note on Interpretation *Could two individuals who share the same prior over agree to disagree? That is, if i and j share a common prior over states, could a state arise at which it was commonly known that i assigned probability η_i to some event, j assigned probability η_j to that some event and $\eta_i \neq \eta_j$. This theorem concluded that this sort of disagreement is impossible.*

Note on Agreement Theorem to No-trade Theorem *Consider a trade where two agents bet on a coin, and this trade holds only when agent believe that* $\Pr($ *Heads* $) > 1/2$ *and agent believe that* $\Pr($ *Heads* $) < 1/2$. *However, Aumann's theorem says the bet cannot happen since these opposing beliefs would then be common knowledge!*

Proof Let p be a probability measure on Ω – interpreted as the agent's prior belief. For any state ω and event E, let $p\left(E \mid h_i(\omega)\right)$ denote i's posterior belief, then the event that "i assigns probability η_i to E" is $\{\omega \in \Omega : p\left(E \mid h_i(\omega) = \eta_i\right\}$. By the assumption of common knowledge, there is some self-evident event F with $\omega \in F$ such that:

$$F \subset \left\{\omega' \in \Omega : p\left(E \mid h_1\left(\omega'\right) = \eta_1\right\} \cap \left\{\omega' \in \Omega : p\left(E \mid h_2\left(\omega'\right) = \eta_2\right\}$$

By Lemma 3.1.2, F is a union of members of i's information partition, i.e. $F = \cup_k A_k = \cup_k B_k$ (Ω is finite), where A and B are 1's and 2's information partition functions. Now, for any nonempty disjoint sets C, D with $p(E \mid C) = \eta_i$ and $p(E \mid D) = \eta_i$, since

$$p(E \mid C) = \frac{p(E \cap C)}{p(C)} = p(E \mid D) = \frac{p(E \cap D)}{p(D)} = \eta_i$$
$$p(E \mid C \cup D) = \frac{p(E \cap (C \cup D))}{p(C \cup D)} = \frac{p(E \cap C) + p(E \cap D)}{p(C) + p(D)} = \frac{\eta_i(p(C) + p(D))}{p(C) + p(D)},$$

we have $p(E \mid C \cup D) = \eta_i$. Then for each k, we have $p(E \mid F) = p(E \mid A_k) = \eta_1 = \eta_1$ and $p(E \mid F) = p(E \mid B_k) = \eta_2$. ■

Note on Example *Let* $\Omega = \{a, b, c, d\}$ *and each state occurs with prob 1/4, and*

$$P_1 = \{\{a, b\}, \{c, d\}\} \text{ and } P_2 = \{\{a, b, c\}, \{d\}\}.$$

Let $E = \{a, d\}$*, since E is not common knowledge, at a we have*

$$\begin{aligned} \eta_1(E) &= \rho(E \mid \{a, b\}) = 1/2 \\ \eta_2(E) &= \rho(E \mid \{a, b, c\}) = 1/3 \end{aligned}.$$

Player 1 knows $\eta_2(E)$*, player 2 knows* $\eta_1(E)$*, but player 2 does not know what player 1 thinks of* $\eta_2(E)$.

3.1.4 The No-Trade Theorem

Let Ω be a set of states and X a set of consequences (trading outcomes). A contingent contract is a function mapping Ω into X. Let A be the space of contracts. Each agent has a utility function $u_i : X \times \Omega \to \mathbb{R}$. Let $U_i(a) = u_i(a(\omega), \omega)$ denote i's utility from contract a – $U_i(a)$ is a random variable that depends on the realization of ω. Let $\mathbb{E}[U_i(a) \mid H_i]$ denote i's expectation of $U_i(a)$ conditional on his information H.

Theorem 3.3 (No-Trade Theorem 1 (Milgrom and Stokey (1982))

If a contingent contract b is ex ante efficient, then it cannot be common knowledge between the agents that every agent prefers contract a *to contract b.*

Theorem 3.4 (No-Trade Theorem 2 (Milgrom and Stokey (1982))

If ex ante allocation is Pareto optimal, then even after the players receive their private information, it cannot be common knowledge that they all expect to gain from trade.

3.1.5 E-mail Game (Rubinstein, 1989)

Even if each player is quite certain about the game being play, even small uncertainty about other's information can eliminate equilibria that exist when payoffs are common knowledge. Formally, the fact that small perturbations of the information structure can eliminate Nash equilibria occurs because the Nash equilibrium correspondence is not lower semi-continuous.

Consider the following Bayesian game, where $L > M > 1$. Player 1 is informed about the true game, 2 is not. The unique bayesian nash equilibrium is ((A,A),A).

	A	B
A	M, M	$1, -L$
B	$-L, 1$	$0, 0$

Figure 3.1: G_1 (1-p)

	A	B
A	$0, 0$	$1, -L$
B	$-L, 1$	M, M

Figure 3.2: G_2 (p<1/2)

If player 1 can communicate with player 2 in such a way that the true game becomes common knowledge, then there is a BNE in which both choose A in G_1, and both choose B in G_2. If the communication is imperfect, e.g. in G_2 player 1 sends a message to 2, and 2 sends back a confirmation message. With probability $\varepsilon > 0$, a message is not received, and each player sees the number of messages that he sent.

Formally, this bayesian game are

1. $\Omega = \{(k_1, k_2) : k_1 = k_2 \text{ or } k_1 = k_2 + 1\}$;
2. Signal function $\tau_i : \tau_i(k_1, k_2) = k_i$;
3. Common prior on Ω:

$$
\begin{aligned}
P_i(0,0) =& 1-p \\
P_i(1,0) =& p\varepsilon \\
P_i(1,1) =& p\varepsilon(1-\varepsilon), \\
& \ldots \\
P_i(k+1,k) =& p\varepsilon(1-\varepsilon)^{2k} \\
P_i(k+1,k+1) =& p\varepsilon(1-\varepsilon)^{2k+1}
\end{aligned}
$$

Lemma 3.4

The e-mail game has a unique BNE, in which both players always choose A.

Proof

■

3.2 Solution Concept 4: Bayesian Nash Equilibrium

Definition 3.11 (Complete vs. Incomplete Information)

A complete information game is one where all players' payoff functions (and all other aspects of the game) are common knowledge.

Definition 3.12 (Common Belief)

All players share the same belief: $p_i = p(t) = p(t_1, t_2, \ldots, t_n)$ *for all* $i \in N$.

Definition 3.13 (Reduced form Bayesian Game (Duozhe, 2022))

$$G = \{A_i; T_i; p_i; u_i\}_{i=1}^n$$

A reduced-form Bayesian game is a list as above, where

1. A_i *is the action space of* i,
2. T_i *is the type space of player* i, *and a strategy of a player* i *is any function* $s_i : T_i \to A_i$,
3. $p_i(t_{-i} \mid t_i)$ *is* i*'s belief about the other players,*

4. *$u_i\left(a_1, \ldots, a_n ; t_1, \ldots, t_n\right)$ is i's payoff function.*

Definition 3.14 (Reduced form Bayesian Nash equilibrium (Duozhe, 2022))

A strategy profile $s^*=\left(s_1^*, \ldots, s_n^*\right)$ is a Bayesian Nash equilibrium (BNE) if for all t_i, $s_i^*\left(t_i\right)$ is a best response to s_{-i}^*, i.e., $s_i^*\left(t_i\right)$ solves

$$\max _{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} p_i\left(t_{-i} \mid t_i\right) \cdot u_i\left(a_i, s_{-i}^*\left(t_{-i}\right) ; t\right) .$$

Theorem 3.5 (Purification Theorem (Harsanyi 1973))

The mixed-strategy NE of a strategic game can be viewed as the limit of pure-strategy BNE in the slightly perturbed games.

Example 3.4 The stag hunt game has a mixed NE (p=q=1/2).

	Hare	Stag
Hare	$1,1$	$2,0$
Stag	$0,2$	$3,3$

Consider a perturbed game, where x and y are i.i.d. with uniform on $[-\varepsilon, \varepsilon]$, x and y are privately known by 1 and 2 respectively.

	Hare	Stag
Hare	$1+x, 1+y$	$2+x, 0$
Stag	$0, 2+y$	$3,3$

Then there is a pure strategy BNE $\left(s_1(x), s_2(y)\right)$ and beliefs:

$$\begin{aligned}
& s_1(x)=H \quad \text{iff } x>0 \quad \text{and} \quad \operatorname{Pr}\left(s_2(y)=H \mid x\right)=\operatorname{Pr}(y>0)=1/2 \\
& s_2(y)=H \quad \text{iff } y>0 \quad \text{and} \quad \operatorname{Pr}\left(s_1(x)=H \mid y\right)=\operatorname{Pr}(x>0)=1/2
\end{aligned}$$

And we can verify that the expected payoff of player 1 from choosing H is higher iff $x>0$, so as player 2.

$$u_1(H \mid x)=\frac{1}{2}(1+x)+\frac{1}{2}(2+x)>u_1(S \mid x)=\frac{3}{2}$$

Definition 3.15 (Bayesian Game (Duozhe, 2022))

A bayesian game consists of

1. *a finite set of players N, for each player $i \in N$,*
   - *a set of actions A_i,*
   - *a finite set of signals T_i (type space) and a signal function $\tau_i: \Sigma \rightarrow T_i$,*
   - *a probability distribution (prior belief) p_i on Σ, for which $p_i\left(\tau_i^{-1}\left(t_i\right)\right)>0$ for all $t_i \in T_i$,*
   - *a vNM preference $\succeq_i$ over the set of lotteries over $A \times \Sigma$ (or equivalently, a Bernoulli payoff function $u_i(a ; w)$);*
2. *a finite set of states Σ.*

Note on Strategy *In a Bayesian game, a strategy of a player i is any function $s_i : T_i \to A_i$.*

Definition 3.16 (Bayesian Nash equilibrium (Duozhe, 2022))

A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium (BNE) if for all i and t_i, $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{\omega \in \Omega} P_i(\omega \mid t_i) \cdot u_i\left(a_i, s_{-i}^*\left(\tau_{-i}(\omega)\right); \omega\right),$$

where the posterior belief is given by Bayes' law:

$$P_i(\omega \mid t_i) = \frac{p_i(\omega)}{p\left(\tau_i^{-1}(t_i)\right)} \text{ if } \omega \in \tau_i^{-1}(t_i)$$

$$P_i(\omega \mid t_i) = 0 \text{ if } \omega \notin \tau_i^{-1}(t_i)$$

Definition 3.17 (Separating, Pooling equilibrium)

3.3 Examples of finding BNE

3.3.1 Cournot Game with Incomplete information

3.3.2 Battle of the Sexes with Incomplete information

Suppose player 2 has two types and player 1 does not know player 2's type, but holds a belief about it. Then the expected payoffs are as follows, and the pure strategy NE is (F,(F,M)).

	F	M
F	$2, 1$	$0, 0$
M	$0, 0$	$1, 2$

Figure 3.3: Prob 1/2: 2 wishes to meet 1

	F	M
F	$2, 0$	$0, 2$
M	$0, 1$	$1, 0$

Figure 3.4: Prob 1/2: 2 wishes to avoid 1

	F, F	F, M	M, F	M, M
F	$\underline{2}, \frac{1}{2}$	$\underline{1}, \underline{\frac{3}{2}}$	$\underline{1}, 0$	$0, 1$
M	$0, \frac{1}{2}$	$1/2, 0$	$1/2, \underline{\frac{3}{2}}$	$\underline{1}, 1$

Note that in this example (reduced form), we do not elaborate the signal function, that is, we assume that player 2 also knows player 1 knows the state 1 with prob 1/2 and state 2 with prob 1/2. This assumption will be relaxed in the following sections.

3.3.3 Public good provision

Two players decide simultaneously whether or not to contribute to a public good. Each player i derives a commonly known value v if at least one of them contributes and 0 if none of them does. Player i's cost of contribution is c_i, only known to himself. It is common knowledge that c_1 and c_2 are $i.i.d.$ on $[c_L, c_H]$, and the continuous distribution function is $F(.)$. The reduced-form Bayesian game is:

1. Two players, $i = 1, 2$.
2. Player i's action space: $\{0, 1\}$, where 1 stands for "contribute", and 0 stands for "don't".
3. Player i's type space: $[c_L, c_H]$.
4. Player i's belief: $\Pr(c_j \leq c | c_i) = F(c)$.
5. Player i's payoff function:

$$u_i(a_1, a_2; c_1, c_2) = \begin{cases} 0 & \text{if } a_1 = a_2 = 0 \\ v - c_i & \text{if } a_i = 1 \\ v & \text{if } a_i = 0 \text{ and } a_j = 1 \end{cases}$$

Symmetric BNE: Assume that $v = 2$, c_1 and c_2 are uniformly distributed on $[1, 3]$. There is a symmetric BNE, in which $s_i^*(c_i) = 1$ iff $c_i \leq c^*$. To find c^*, note that BNE exhibits a monotonicity property and player i of type c^* must be indifferent between two actions. If $s_i(c_i) = 1$, his payoff is $2 - c^*$; if $s_i(c_i) = 0$, then his expected payoff is $\Pr(c_j \leq c^*) \cdot v = \frac{c^*-1}{3-1} \cdot 2 = c^* - 1$. Solve this equation, we obtain $c^* = 3/2$. The insight here is the game becomes inefficient if $c_i > 3/2$, even if both players find it profitable, there is no public good. Because both players have incentive to free ride.

$$2 - c^* = c^* - 1$$

Two Asymmetric BNE: in which one player never contributes and the other player contributes for all $c \leq v = 2$. The existence of such asymmetric equilibria depends on the common value v and the distribution of c_i. More specifically, asymmetric equilibria exist when

$$vF(v) \geq v - c_L$$

that is, it is optimal for a player with the lowest cost not to contribute if he believes that the other player contributes whenever that player's cost does not exceed v. For example, if $v = 1$ and $c_1, c_2 \sim U[0, 2]$, then there does not exist such asymmetric equilibria.

3.3.4 BoS with incomplete information and uncommon belief

	F	M
F	$2, 1$	$0, 0$
M	$0, 0$	$1, 2$

Figure 3.5: Prob 1/2: 2 wishes to meet 1

	F	M
F	$2, 0$	$0, 2$
M	$0, 1$	$1, 0$

Figure 3.6: Prob 1/2: 2 wishes to avoid 1

When player 2 wishes to meet player 1, she believes that with probability 2/3 player 1 knows it and with probability 1/3 he does not know; when player 2 wishes to avoid player 1, she believes that with probability 1/3 player 1 knows it and with probability 2/3 he does not know. Everything above is common knowledge between 1 and 2. The bayesian game thus can be formally formulated as below:

1. The set of players: $\{1, 2\}$;
2. The set of actions for each player i: $\{\mathrm{F}, \mathrm{M}\}$;
3. The set of states $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, where
   - in ω_1, player 2 wishes to meet player 1 and player 1 knows it,
   - in ω_2, player 2 wishes to meet player 1 and player 1 does not know it,
   - in ω_3, player 2 wishes to avoid player 1 and player 1 knows it,
   - in ω_4, player 2 wishes to avoid player 1 and player 1 does not know it.
4. For each player i, a set of signals T_i and a signal function $\tau_i : \Omega \to T_i$,
   - $T_1 = \{t_{11}, t_{12}, t_{13}\} : \tau_1(\omega_1) = t_{11}, \tau_1(\omega_3) = t_{12}$, and $\tau_1(\omega_2) = \tau_1(\omega_4) = t_{13}$.
   - $T_2 = \{t_{21}, t_{22}\} : \tau_2(\omega_1) = \tau_2(\omega_2) = t_{21}$, and $\tau_2(\omega_3) = \tau_2(\omega_4) = t_{22}$.
5. The common prior belief p on Ω: $p(\omega_1) = 1/3, p(\omega_2) = 1/6, p(\omega_3) = 1/6, p(\omega_4) = 1/3$.
6. Updated beliefs after receiving signals:
   - Player 1: since only when the state is ω_1, player 1 can receive signal t_{11}, player 1's updated belief after receiving t_{11} is:
     $$P_1(\omega_1 \mid t_{11}) = \frac{p(\omega_1)}{p\left(\tau_1^{-1}(t_{11})\right)} = \frac{p(\omega_1)}{p(\omega_1)} = 1 \quad \text{and} \quad P_1(\omega_s \mid t_{11}) = 0 \text{ for } \omega_s \neq \omega_1$$
     similarly, player 1's updated belief after receiving t_{12} is
     $$P_1(\omega_3 \mid t_{12}) = 1 \quad \text{and} \quad P_1(\omega_s \mid t_{12}) = 0 \text{ for } \omega_s \neq \omega_3$$
     and after receiving t_{13}, player 1 knows that the state is either ω_2 or ω_4, and using the Bayes rule, his updated belief is: $P_1(\omega_1 \mid t_{13}) = P_1(\omega_3 \mid t_{13}) = 0$, and
     $$P_1(\omega_2 \mid t_{13}) = \frac{p(\omega_2)}{p\left(\tau_1^{-1}(t_{13})\right)} = \frac{p(\omega_2)}{p(\omega_2) + p(\omega_4)} = \frac{1}{3}, P_1(\omega_4 \mid t_{13}) = \frac{2}{3}.$$
   - Player 2: after receiving t_{21}, player 2 knows that the state is either ω_1 or ω_2, and using the Bayes rule, her updated belief is: $P_2(\omega_3 \mid t_{21}) = P_2(\omega_4 \mid t_{21}) = 0$, and
     $$P_2(\omega_1 \mid t_{21}) = \frac{p(\omega_1)}{p\left(\tau_2^{-1}(t_{21})\right)} = \frac{p(\omega_1)}{p(\omega_1) + p(\omega_2)} = \frac{2}{3}, P_2(\omega_2 \mid t_{21}) = \frac{1}{3};$$
     similarly, after receiving t_{22}, her updated belief is: $P_2(\omega_1 \mid t_{22}) = P_2(\omega_2 \mid t_{22}) = 0$,
     $$P_2(\omega_3 \mid t_{22}) = \frac{p(\omega_3)}{p\left(\tau_2^{-1}(t_{22})\right)} = \frac{p(\omega_3)}{p(\omega_3) + p(\omega_4)} = \frac{1}{3}, P_2(\omega_4 \mid t_{22}) = \frac{2}{3}.$$
7. The vNM payoff functions:
   - Player 1: for all $\omega_s \in \Omega$,
     $$u_1((F, F); \omega_s) = 2, u_1((M, M); \omega_s) = 1, u_1((F, M); \omega_s) = u_1((M, F); \omega_s) = 0$$
   - Player 2:
     $$u_2((M, M); \omega_1) = u_2((M, M); \omega_2) = 2, u_2((F, F); \omega_1) = u_2((F, F); \omega_2) = 1$$
     $$u_2((F, M); \omega_1) = u_2((F, M); \omega_2) = u_2((M, F); \omega_1) = u_2((M, F); \omega_2) = 0$$

and

$$u_2\left((F,M);\omega_3\right)=u_2\left((F,M);\omega_4\right)=2, u_2\left((M,F);\omega_3\right)=u_2\left((M,F);\omega_4\right)=1$$

$$u_2\left((F,F);\omega_3\right)=u_2\left((F,F);\omega_4\right)=u_2\left((M,M);\omega_3\right)=u_2\left((M,M);\omega_4\right)=0$$

8. A strategy for player i is a function $s_i : T_i \to \{\mathrm{F}, \mathrm{M}\}$. That is, the strategy specifies the actions that player i chooses after receiving each possible signal.
9. Bayesian Nash equilibrium: a strategy profile s^* is a BNE if for each player i and each $t_i \in T_i$, $s_i^*(t_i)$ solves
$$\max_{a_i\in\{F,M\}} \sum_{\omega_s\in\Omega} P_i\left(\omega_s \mid t_i\right)\cdot u_i\left(a_i, s_j^*\left(\tau_2\left(\omega_s\right)\right);\omega_s\right)$$
That is, after receiving a signal $t_i \in T_i$, the action $s_i^*(t_i) \in \{\mathrm{F}, \mathrm{M}\}$ gives player i the highest expected payoff calculated using his updated beliefs.

Note that player 1's strategy consists of three actions and player 2's strategy consists of two actions, one action for each signal that the player may receive. Thus, player 1 has eight pure strategies and player 2 has four pure strategies. In our analysis below, we consider the possibility that each of player 2's pure strategies is used in a BNE.

1. FF: If player 2 always chooses F in a BNE, then player 1's strategy in such a BNE must be choosing F regardless of the signal he receives. However, if player 1 always chooses F, it is not optimal for player 2 to choose F after receiving signal t_{22}, i.e. player 2 knows that the state is either ω_3 or ω_4, in which she receives a higher payoff from action profile (F,M) than from (F,F). Thus FF can not be player 2's strategy in a BNE.
2. MM: Similarly, MM cannot be player 2's strategy in a BNE.
3. MF, i.e. choosing M (F resp.) after receiving t_{21} (t_{22} resp.). It is easy to see that player 1's best response to this strategy is to choose M after receiving t_{11}, choose F after receiving t_{12}, and choose F after receiving t_{13}. Then we need to check whether MF is player 2's best response to player 1's strategy MFF. When player 2 receives t_{22}, her updated belief is that with probability 1/3, the state is ω_3, in which player 1 would receive signal t_{12} and choose F; with probability 2/3, the state is ω_4, in which player 1 would receive signal t_{13} and also choose F. Clearly, it cannot be optimal for player to choose F. Thus, MF cannot be player 2's strategy in a BNE.
   - To identify player 1's optimal action after receiving t_{13}, we need to calculate player 1's expected payoff from using either action. Note that after receiving t_{13}, player 1's update belief is that with probability 1/3, the state is ω_2, in which player 2 would receive signal t_{21} and choose M; with probability 2/3, the state is ω_4, in which player 2 would receive signal t_{22} and choose F. Thus, player 1's expected payoff from choosing F is $2\times 2/3 = 4/3$, his expected payoff from choosing M is $1\times 1/3 = 1/3$, and it is optimal to choose F.
4. FM, i.e., choosing F (M resp.) after receiving t_{21} (t_{22} resp.) : It is easy to see that player 1's best response to this strategy is to choose F after receiving t_{11}, choose M after receiving

t_{12}, and player 1 is indifferent between F and M after receiving t_{13}, both FMM and FMF are player 1's best responses to player 2's strategy FM. Next we want to check whether MF is player 2's best response to FMM or FMF, and only (FMF, FM) is a BNE.

- Note that after receiving t_{13}, player 1's update belief is that with probability 1/3, the state is ω_2, in which player 2 would receive signal t_{21} and choose F; with probability 2/3, the state is ω_4, in which player 2 would receive signal t_{22} and choose M. Thus, player 1's expected payoff from choosing F is $2 \times 1/3 = 2/3$, his expected payoff from choosing M is $1 \times 2/3 = 2/3$, and it is optimal to choose F.
- FMM: After receiving t_{22}, player 2's updated belief is that the state is either $\omega(1/3)$ or $\omega_4(2/3)$, in which player 1 receives either t_{12} or t_{13}. Since player 1 will choose M after receiving t_{12} or t_{13}, it is not optimal for player 2 to choose M after receiving t_{22}. Thus, (FMM, FM) is not a BNE.

5. FMF: After receiving t_{21}, player 2's updated belief is that the state is either $\omega_1(2/3)$ or $\omega_2(1/3)$, player receives either t_{11} or t_{13}. Since player 1 would choose F after receiving t_{11} or t_{13}, it is optimal for player 2 to choose F after receiving t_{21}. After receiving t_{22}, player 2's updated belief is that the state is either $\omega_3(1/3)$ or $\omega_4(2/3)$, player 1 receives either t_{12} or t_{13}, and choose F (M resp.) with probability 2/3 (1/3 resp.), it can be verified that M is indeed the optimal action for player 2 after receiving t_{22}. Thus, (FMF, FM) is a BNE.

To sum up, there is a unique pure-strategy BNE:

$$s_1^*(t_{11}) = F, s_1^*(t_{12}) = M, s_1^*(t_{13}) = F; s_2^*(t_{21}) = F, s_2^*(t_{22}) = M.$$

Note on Existence *In general, a pure-strategy BNE may not exist, but Nash existence theorem still applies, that is, a finite Bayesian game always has a BNE in pure or mixed strategies.*

Note on Reduced form *This game can also be analyzed in its reduced form the same as we do under the state-space formulation. While the later gives us a better understanding of the concept of "type".*

1. *The set of players:* $\{1, 2\}$*;*
2. *The set of actions for each player* i*:* $\{\mathrm{F}, \mathrm{M}\}$*;*
3. *For each player* i*, a set of signals* T_i *and a signal function* $\tau_i : \Omega \to T_i$*,*
   - $T_1 = \{t_{11}, t_{12}, t_{13}\} : \tau_1(\omega_1) = t_{11}, \tau_1(\omega_3) = t_{12}$*, and* $\tau_1(\omega_2) = \tau_1(\omega_4) = t_{13}$*.*
   - $T_2 = \{t_{21}, t_{22}\} : \tau_2(\omega_1) = \tau_2(\omega_2) = t_{21}$*, and* $\tau_2(\omega_3) = \tau_2(\omega_4) = t_{22}$*.*
4. *Beliefs*
   - *Player 1*

$$P_1(t_{21} \mid t_{11}) = 1, P_1(t_{22} \mid t_{11}) = 0$$
$$P_1(t_{21} \mid t_{12}) = 0, P_1(t_{22} \mid t_{12}) = 1$$
$$P_1(t_{21} \mid t_{13}) = \frac{1}{3}, P_1(t_{22} \mid t_{13}) = \frac{2}{3}$$

- *Player 2*

$$P_2\left(t_{11} \mid t_{21}\right)=\frac{2}{3}, P_2\left(t_{12} \mid t_{21}\right)=0, P_2\left(t_{13} \mid t_{21}\right)=\frac{1}{3}$$
$$P_2\left(t_{11} \mid t_{21}\right)=0, P_2\left(t_{12} \mid t_{21}\right)=\frac{1}{3}, P_2\left(t_{13} \mid t_{21}\right)=\frac{2}{3}$$

5. *The vNM payoff functions:*
   - *Player 1: for all type profile $\mathbf{t} \in T_1 \times T_2$, i.e., player 1's payoff depends only on the action profile, not on the types.*

     $$u_1((F,F);\mathbf{t})=2, u_1((M,M);\mathbf{t})=1, u_1((F,M);\mathbf{t})=u_1((M,F);\mathbf{t})=0$$
   - *Player 2: for all type profile $\mathbf{t}_{21} \in T_1 \times \{t_{21}\}$*

     $u_2\left((M,M);\mathbf{t}_{21}\right)=2, u_2\left((F,F);\mathbf{t}_{21}\right)=1, u_2\left((F,M);\mathbf{t}_{21}\right)=u_2\left((M,F);\mathbf{t}_{21}\right)=0$

     and for all type profile $\mathbf{t}_{22}=T_1 \times \{t_{22}\}$

     $u_2\left((F,M);\mathbf{t}_{22}\right)=2, u_2\left((M,F);\mathbf{t}_{22}\right)=1, u_2\left((F,F);\mathbf{t}_{22}\right)=u_2\left((M,M);\mathbf{t}_{22}\right)=0$

3.4 Auction Theory

3.4.1 Auction Mechanism

Definition 3.18 (Open ascending auction (English auction))

Definition 3.19 (Open descending auction (Dutch auction))

Definition 3.20 (First-Price (Sealed Bid) Auction)

Definition 3.21 (Second-Price (Sealed Bid) Auction)

Lemma 3.5 (Strategic equivalence of different auctions)

Definition 3.22 (Auction Mechanism)

An auction is a mechanism with well-defined allocation rule and payment rule.

1. *Allocation rule:*
2. *Payment rule:*

Definition 3.23 (Assumptions for Auction)

1. *Independent Private Value: v_i are independent and private.*
2. *Symmetry: same distribution F for v_i*
3. *Zero-one support: F has a support of $[0,1]$*
4. *Linear payoff: u(win)=$v_i - p_i$*

5. *Risk Neutrality:*

Definition 3.24 (Bidding function)

Bidder i's bid is a function from $[0, 1]$ to non-negative number: $\beta_i : [0, 1] \to \mathbb{R}_+$.

Definition 3.25 (Bidding Strategy)

Bidding strategy β_i can be represented by a graph.

1. *Truthful bidding:*
2. *Overbidding:*
3. *Underbidding (Bid-shading):*

Figure

3.4.2 Second-price auction (Kartik, 2009)

There are I bidders, with value $0 \leq v_1 \leq ... \leq v_I$, and their values are common knowledge. All bidders simultaneously bid $s_i \in [0, \infty]$, the highest one wins the auction and pays the second-high price. Define $W(s) = k : \forall j, s_k \geq s_j$ as the set of highest bidders, then bidder i's utility is

$$u_i(s_i, s_{-i}) = \begin{cases} v_i - max_{j \neq i} s_j & \text{if } s_i > max_{j \neq i} s_j \\ \frac{1}{|W(s)|}(v_i - s_i) & \text{if } s_i = max_{j \neq i} s_j \\ 0 & \text{if } s_i < max_{j \neq i} s_j \end{cases}$$

Note on Another mechanism under equality *When more than one bidder submits the highest bid, each gets the object with equal probability by a lottery, and the payment is equal to the highest bid in this case. Note that this does not change our results.*

Lemma 3.6 (BNE for Second-price Auction)

Everyone chooses to bid their real valuation $s_i = v_i$, and this strategy is weakly dominant.

Proof Let $m(s_{-i}) = max_{j \neq i} s_j$. Supposes $s_i > v_i$, then for any strategy profile s_{-i}, if $m(s_{-i}) > s_i$, then $u_i(s_i, s_{-i}) = 0$; if $m(s_{-i}) \leq v_i$, then $u_i(s_i, s_{-i}) = u_i(v_i, s_{-i}) \geq 0$; otherwise if $m(s_{-i}) \in (v_i, s_i]$, then $u_i(s_i, s_{-i}) < 0$. Thus when $s_i > v_i$, $s_i = v_i$ is weakly dominant. The case of $s_i < v_i$ can be proved similarly. ■

3.4.3 First-price Auction

Lemma 3.7 (BNE for Uniform First-price Auction (Greenwald, 2021))

In a first-price auction, assuming values are i.i.d. uniformly distributed on $[0, \bar{v}]$, the bidding strategy $b_i = \left(\frac{n-1}{n}\right) v_i$ comprises a symmetric Bayesian Nash equilibrium.

Proof Fix a bidder i. We assume that all bidders chooses b_i according to linear strategy

$s_i = \alpha v_i + \beta$, and argue that bidder i should do the same to find the value of α and β. Note that the probability of bidder i winning the auction is $Pr(s_i > s_j, j \neq i)$, and the expected utility is

$$E[s_i] = (\frac{s_i - \beta}{\alpha \bar{v}})^{N-1}(v_i - s_i)$$

By the FOC, we have $s_i = \frac{(N-1)v_i + \beta}{N}$. Combine the condition $s_i = \alpha v_i + \beta$, we have $\alpha = \frac{N-1}{N}$ and $\beta = 0$, and the bidding strategy $s_i^* = \frac{N-1}{N} v_i$ comprises a symmetric BNE. ■

3.4.4 Revenue Equivalence (Greenwald, 2022)

Definition 3.26 (k**th-order statistic)**

The kth-order statistic, denoted $X_{(k)}$, is the kth-largest value among n draws of a random variable X.

Lemma 3.8 ($E[X_{(1)}]$ **for** $X \sim U[0,1]$)

$$E[X_{(1)}] = \int_0^1 x f_{X_{(1)}}(x) dx = \frac{n}{n+1}$$

Proof

$$\begin{aligned} F_{X_{(1)}}(x) &= \Pr\left(X_{(1)} \leq x\right) \\ &= \prod_n U(x) \\ &= x^n. \end{aligned}$$

■

Lemma 3.9 ($E[X_{(2)}]$ **for** $X \sim U[0,1]$)

$$E[X_{(2)}] = \int_0^1 x f_{X_{(2)}}(x) dx = \frac{n-1}{n+1}$$

Proof

$$F_{X_{(2)}}(x) = \Pr\left(X_{(2)} \leq x\right) = x^n + nx^{n-1}(1-x)$$

■

Theorem 3.6 (Revenue Equivalence)

If bidder's values are uniform i.i.d., then the expected revenue of the first-price auction is equal to that of the second-price auction, assuming bidders behave according to their respective equilibrium strategies.

Proof The support of the uniform distribution does not matter; we choose $[0, 1]$ for convenience. Let R_1 and R_2 denote the expected revenue of the first- and second-price auctions, respectively.

$$R_2 = \frac{n-1}{n+1}$$

$$R_1 = \mathbb{E}\left[\left(\frac{n-1}{n}\right) v_{\max}\right] = \left(\frac{n-1}{n}\right) \mathbb{E}\left[v_{\max}\right] = \frac{n-1}{n+1}$$

■

3.4.5 Double Auction

Definition 3.27 (Double Auction)

Definition 3.28 (BNE for Double Auction)

Lemma 3.10 (Single Price BNE)

Lemma 3.11 (Linear BNE)

3.5 From Bayesian Game to Mechanism Design: The Revelation Principle

Theorem 3.7 (The Revelation Principle)

Any Bayesian Nash equilibrium of any Bayesian game can be represented by an incentive-compatible direct mechanism.

Chapter 4 (Non-cooperative) Dynamic game with complete information

4.1 Strategy and Information in Extensive Game

Definition 4.1 (Extensive Form of Game)

1. *Player Set, e.g.* $N = \{1, 2, 3, 4, \cdots, n\}$.
2. *Set of History* H*, terminal history set* Z.
3. *Player function* P*, action set* A_i.
4. *Payoff functions* U_i.

Definition 4.2 (Perfect Information)

A game has perfect information if all information sets are singletons. Otherwise, it has imperfect information.

Definition 4.3 (Pure strategy)

A pure strategy of player i *in an extensive form game with perfect information is a complete list of actions, one action for each decision node that player* i *is entitled to move.*

Example 4.1 For example, Ann has 8 strategies in this game. Liduozhe P4.

Definition 4.4 (Strategy profile)

A strategy profile s *(one strategy* s_i *for each player* i*) determines a sequence of actions leading to a terminal node, namely, a path of play. We refer to this path of play as the outcome of* s*.*

Lemma 4.1

A finite game of perfect information has a pure strategy Nash equilibrium.

Note on Is PSNE reasonable? *However, some of PSNE are more reasonable than those with incredible threats.*

Proof We use backward induction to solve this game, and from the terminal node to the starting node, we must be able to find a PSNE. ■

Definition 4.5 (Sequential rationality)

A player is sequentially rational iff, at each node he is to move, he maximizes his expected utility conditional on that he is at the node – even if this node is precluded by his own strategy.

Note on Backward induction outcome *In a finite game of perfect information, common* *knowledge of sequential rationality yields the backward induction outcome.*

Note on Is Backward induction outcome reasonable? *For example, centipede game.*

For example, chain store paradox.

4.2 Solution Concept 5: Subgame Perfect Equilibrium

Definition 4.6 (Imperfect information)

The simultaneity of moves means that these games have imperfect information.

Note on *We define the subgame-perfect outcome of such games, which is the natural extension of backwards induction to these games. Here the subgame-perfect is different from backward induction outcome since we solve a real game in the 1st step rather than solving a single-person optimization problem.*

Definition 4.7 (Information set)

An information set for a player is a collection of decision nodes in which he has to move.

Note on *In a perfect information game, each information set contains a single decision node.*

Note on *In each information set: a player knows he/she is one of nodes from the information set, however, he/she does not know which node exactly.*

Definition 4.8 (History)

A sequence of decision nodes starting from the initial decision node and connected by actions taken by the players is often referred to as a history.

Note on Terminal history *A terminal history is simply a complete path of play (an outcome), a nonterminal history ends with a decision node where one player is to move.*

Definition 4.9 (Pure strategy in imperfect information)

A pure strategy of player i in an extensive form game is a complete list of actions, one action for each information set that player i is entitled to move.

Note on Number *If player i has K information sets, and at the n^{th} information set, the number of actions is A_n, then the number of pure strategies is*

$$\#S_i = A_1 \times A_2 \times \ldots \times A_K.$$

Definition 4.10 (Equivalent pure strategy)

A player's two pure strategies are equivalent if they lead to the same outcome for every pure strategy profile of other players.

Note on Outcome *Note that outcome means not only payoff, but also the path of play.*

Definition 4.11 (Reduced strategic form)

The reduced strategic form of an extensive game is obtained by eliminating all but one member of each equivalent class.

Note on *Be very careful, this is just a simplification.*

Example 4.2 Example.

Definition 4.12 (Perfect recall)

No players ever forgets any information he once knew, including his past actions.

Note on Perfect recall vs. information *The condition of perfect information is stronger, perfect information means perfect recall, but not vice versa.*

Definition 4.13 (Behavioral strategy)

A behavioral strategy of player i specifies a probability distribution on the set of actions at each information set of player i.

Note on Mixed vs. Behavioral Strategy *A mixed strategy σ_i generates a unique behavioral strategy b_i, and a behavioral strategy b_i can be generated by one or more mixed strategies. σ_i and b_i are equivalent if for any pure strategy profile s_{-i}, (σ_i, s_{-i}) and (b_i, s_{-i}) induce same probability distribution over terminal histories.*

Lemma 4.2

In a game of perfect recall, every mixed strategy is equivalent to the behavioral strategy it generates, and every behavioral strategy is equivalent to each mixed strategy that generates it.

Note on Imperfect recall example

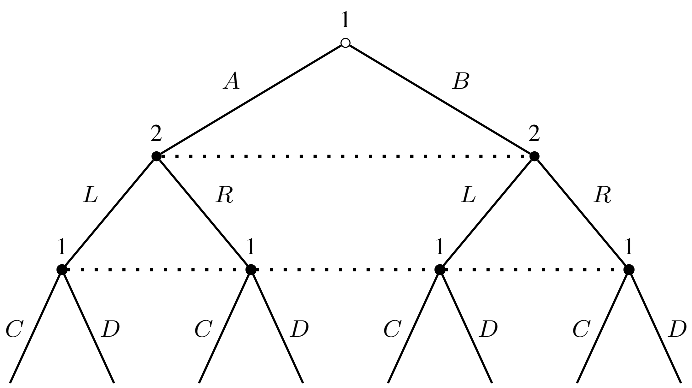


Figure 4.1: Example of imperfect recall

In this game tree, player 1 forgets whether he chooses A or B before. Here player 1's set of pure strategies is $S^1 = \{AC, AD, BC, BD\}$, consider a mixed strategy $\sigma_1 = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$, it can generate the behavioral strategy $b_1 = \left\{\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$.

Note that b_1 is not equivalent to σ_1. Let player 2 choose L. Then (σ_1, L) induces a probability *of 1/2 to path (A,L,C) and (B,L,D) respectively, but* (b_1, L) *induces a probability of 1/4 to each of the four paths: (A,L,C), (A,L,D), (B,L,C) and (B,L,D).*

Definition 4.14 (Subgame)

A subgame is a part of the original game tree with following properties:

- *it begins with an information set containing a single decision node;*
- *it contains all the successor nodes, their information sets, and connecting branches, up to all the relevant terminal nodes.*

Note on *If a subgame contains one node in an information set, it must contain all the nodes in that information set.*

Definition 4.15 (Subgame Perfect Equilibrium (Gibbons, 1992, p. 95))

A strategy profile is a SPE if it induces a NE in every subgame.

Note on *Subgame-perfect Nash equilibrium is a refinement of Nash equilibrium.*

Note on Existence and Uniqueness *Every finite game of perfect information has a pure strategy SPE. Moreover, if no player is indifferent at any two terminal nodes, then there is a unique SPE, which can be derived by backward induction.*

Note on Find SPE

1. *Identify all NEs of the final subgames.*
2. *Select one NE in each final subgame, and replace the subgame with a terminal node with the payoffs of the selected NE.*
3. *Go backwards until a strategy profile of the original game is determined.*

Note on Unreasonable SPE *For example, (Out, Fight) becomes a SPE, which seems unreasonable, Since Fight is not rational for Incumbent. Thus we introduce other solution concepts later.*

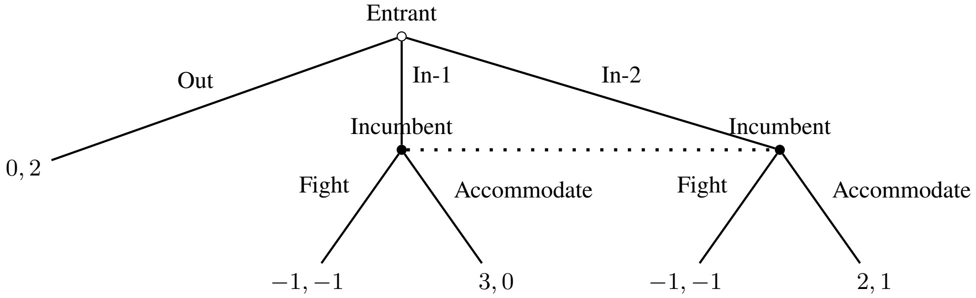


Figure 4.2: Unreasonable SPE

Lemma 4.3 (One-step deviation principle)

A strategy profile is subgame perfect iff no player can gain by deviating from the strategy profile in a single information set and conforming to it thereafter.

Note on Nature *The logic here is, if there is no profitable one-step deviation, then there is no* *profitable (multiple-steps) deviation everywhere.*

Example 4.3 Suppose there is no profitable one-step deviation in this game, and the subgame perfect decision is to choose C_1 for player 1 in each period. And we want to show that there is no profitable (multiple-steps) deviation too. The condition actually means $u_1 \geq u_3$, $u_3 \geq u_5$ and $u_5 \geq u_7$, that is, $u_1 \geq u_7$, even though player 1 changes decisions in three periods to C_2 cannot improve his payoff.

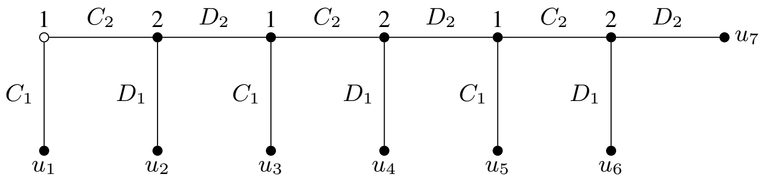


Figure 4.3: Example of one-step deviation

Theorem 4.1 (Condition for subgame perfect in finite horizon)

In a finite-horizon extensive game, a strategy profile s^* is subgame perfect iff there is no player i and no strategy $\hat{s}_i$ that agrees with s_i^* at all but one but one of player i's information sets, such that $\hat{s}_i$ is a better response to s_{-i}^* than s_i^* conditional on that information set being reached.

Definition 4.16 (Continuous at infinity)

A infinite-horizon extensive game is continuous at infinity if for each player i the payoff function u_i satisfies

$$\sup_{h, \tilde{h} \text{ s.t. } h^t = \tilde{h}^t} \left| u_i(h) - u_i(\tilde{h}) \right| \to 0 \text{ as } t \to \infty$$

where h denotes an infinite-horizon history.

Note on Example *Discounted future payoffs is a example continuous at infinity, consider two history which is the same from 1 to T, and different from $T+1$, e.g. 10 and 100 respectively, the payoff difference between these two history becomes negligible when T is large enough.*

Example 4.4 The following game is not continuous at infinity. The only SPE is to choose C at every decision node, the strategy of choosing D at every node does not have any profitable one-step deviation, but is not subgame perfect; thus, one-step deviation principle fails here.

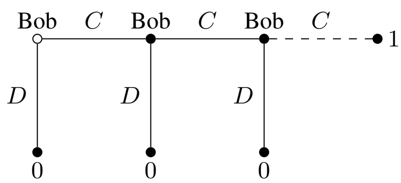


Figure 4.4: Example of non-continuous at infinity

Theorem 4.2 (Condition for subgame perfect in infinite horizon)

In an infinite-horizon extensive game which is continuous at infinity, a strategy profile s^* is subgame perfect iff there is no player i and no strategy $\hat{s}_i$ that agrees with s_i^* at all but one of player i's information sets, such that $\hat{s}_i$ is a better response to s_{-i}^*, than s_i^* conditional on that information set being reached.

Note on From finite to infinite *With the property of continuous at infinity, suppose there is an infinite-step deviation with profit improvement ε, then you can always find a large T period deviation to get improvement such as $\frac{\varepsilon}{2}$, $\frac{\varepsilon}{3}$ and so on, and finally you can find a one-step profitable deviation.*

4.3 Examples of finding SPE

4.3.1 Stackelberg model

Firm 1 (leader) chooses q_1 first, and then firm 2 chooses q_2 after observing q_1. Say that $p(q) = 100 - q$, and player i's utility is $u_i(q_1, q_2) = [100 - (q_1 + q_2)]q_i$. This game can be solved by backward induction. Given q_1, firm 2's optimal decision is $\frac{100-q_1}{2}$. In the first period, firm 1 foresees firm 2's choice, and his optimal decision is $q_1 = 50$. And the SPE strategy profile is $q_1^* = 50, q_2^* = \frac{100-q_1}{2}$, the SPE outcome is $q_1 = 50, q_2 = 25, p = 25, \pi_1 = 1250, \pi_2 = 625$.

Insights: Information makes palyer 2 worse off, if player 2 cannot see q_1, then this is a simultaneous game (cournot game), and player 2 enjoys higher payoff.

4.3.2 Ultimatum Rubinstein Bargaining (Munoz-Garcia, 2017a)

One-period The only SPNE is that: the proposer makes an offer $x^* = 0$, and the responder accepts any offer $x \geq 0$. Actually, the latter is a prediction what the responder would do after receiving any offer.

Two-period The SPNE is that

1. Player 1 offers $x_1 = \delta_2$ in period $t = 1$, and accepts any offer $x_2 \geq 0$ in $t = 2$, and
2. Player 2 offers $x_2 = 0$ in period $t = 2$, and accepts any offer $x_1 \geq \delta$ in $t = 1$.

Now go to infinite periods, let us start from three periods. Note that players are impatient: they discount payoffs received in later periods by the factor δ per period, where $0 < \delta < 1$.

1. At the beginning of the first period, player 1 proposes to take a share s_1 of the dollar, leaving $1 - s_1$ for player 2. Player 2 either accepts the offer or rejects the offer.
2. At the beginning of the 2nd period, player 2 proposes that player 1 take a share s_2 of the dollar, leaving $1 - s_2$ for player 2. Player 1 either accept the offer or rejects the offer.
3. Player 1 get s, and player 2 get $1 - s$.

Three-period:

(iii) player 1 and 2 get s and $1 - s$ respectively.

(ii) player 1 accepts the offer only when $s_2 > \delta s$, and player 2 faces a trade-off between $1-\delta s$ and $\delta(1-s)$, obviously $1-\delta s$ is better, thus the optimal decision for player 2 is $s_2^* = \delta s$.

(i) player 2 accepts the offer only when $1 - s_1 \geq \delta\left(1-s_2^*\right)$ $\left(s_1 \leq 1-\delta\left(1-s_2^*\right)\right)$. Player 1 faces a trade-off between $1-\delta\left(1-s_2^*\right)$ and δs_2^*, obviously $1-\delta\left(1-s_2^*\right)$ is better, thus $s_1^* = 1-\delta(1-\delta s)$.

- The backwards outcome is player 1 offers $\left(s_1^*, 1-s_1^*\right)$, and player 2 accepts it.

Infinite-period: Suppose there is a backwards induction outcome where player 1 and 2 get s and $1-s$ respectively, then we can use the equilibrium result $(f(s), 1-f(s))$ in the two-period to derive the new backwards-induction outcome, here $f(s) = 1-\delta(1-\delta s)$. Let s_H be the highest payoff in these backwards-induction outcome that palyer 1 can achieve, imagine it is player 1's third period profit, and backwards to the first period, player 1's profit is $f(s_H)$. And $f(s)$ increases in s, thus $f(s_H)$ is the higher one, by assumption we have $f(s_H) = s_H$, and also $f(s_L) = s_L$. The only solution for $f(s) = s$ is $s^* = s_H = s_L \frac{1}{1+\delta}$. That is, this game has a unique backwards-induction outcome, where player 1 offers $\left(s_1^*, 1-s_1^*\right)$ in the first period, and player 2 accepts it.

Interpretation: In bargaining games, patience works as a measure of bargaining power: To be continue

Multilateral bargaining

4.3.3 Strategy Pre-commitment (Munoz-Garcia, 2017f)

Note on Example: Advertising and Competition

Note on Example: Entry-deterrence game

Definition 4.17 (Top dog, puppy dog ploy, lean and hungry look, fat cat strategy)

4.3.4 Imperfect information: Tournament

1. The boss choose wage w_H and w_L to maximize payoff $y_1 + y_2 - w_H - w_L$.
2. Two worker chooses effort e_i to maximizes their payoff $u(w, e) = w - g(e)$, where $g(e)$ is increasing and convex (i.e., $g'(e) > 0, g''(e) > 0$).
3. Worker's output realize as $y_i = e_i + \varepsilon_i$, where ε_i is noise and iid to $f(\varepsilon)$ with zero mean. The winner earns w_H while the loser earns w_L.

Here we ignore the possibilities of asymmetric equilibria and an equilibrium given by the corner solution $e_1 = e_2 = 0$.

Second period: Suppose w_H and w_L are given, then NE $\left(e_1^*, e_2^*\right)$ should satisfy the following conditions:

$$\begin{aligned}
\max_{e_i \geq 0} \quad & w_H \operatorname{Prob}\left\{y_i\left(e_i\right) > y_j\left(e_j^*\right)\right\} + w_L \operatorname{Prob}\left\{y_i\left(e_i\right) \leq y_j\left(e_j^*\right)\right\} - g\left(e_i\right) \\
& = \left(w_H - w_L\right) \text{ Prob } \left\{y_i\left(e_i\right) > y_j\left(e_j^*\right)\right\} + w_L - g\left(e_i\right)
\end{aligned}$$

where

$$\begin{aligned}
\text{Prob}\left\{y_i\left(e_i\right) > y_j\left(e_j^*\right)\right\} &= \text{Prob}\left\{\varepsilon_i > e_j^* + \varepsilon_j - e_i\right\} \\
&= \int_{\varepsilon_j} \text{Prob}\left\{\varepsilon_i > e_j^* + \varepsilon_j - e_i \mid \varepsilon_j\right\} f\left(\varepsilon_j\right) d\varepsilon_j \\
&= \int_{\varepsilon_j} \left[1 - F\left(e_j^* - e_i + \varepsilon_j\right)\right] f\left(\varepsilon_j\right) d\varepsilon_j
\end{aligned}$$

and the FOC is

$$\begin{aligned}
&\left(w_H - w_L\right) \frac{\partial \,\text{Prob}\left\{y_i\left(e_i\right) > y_j\left(e_j^*\right)\right\}}{\partial e_i} = g'\left(e_i\right) \\
\Rightarrow &\left(w_H - w_L\right) \int_{\varepsilon_j} f\left(e_i^* - e_i + \varepsilon_j\right) f\left(\varepsilon_j\right) d\varepsilon_j = g'\left(e_i\right) \\
\Rightarrow &\left(w_H - w_L\right) \int_{\varepsilon_j} f\left(\varepsilon_j\right)^2 d\varepsilon_j = g'\left(e^*\right) \quad (SymmetricNashequilibrium)
\end{aligned}$$

There are two insights: (i) a bigger prize $(w_H - w_L)$for winning induces more effort, (ii) it is not worthwhile to work hard when output is very noisy, because the outcome of the tournament is likely to be determined by luck rather than effort.

Suppose worker's alternative employment opportunity is U_a, then there is a constraint $\frac{1}{2}w_H + \frac{1}{2}w_L - g\left(e^*\right) \geq U_a$, and in optimality we have bounded $w_L = 2U_a + 2g\left(e^*\right) - w_H$. And the boss's objective is to maximize $2e^* - w_H - w_L$, replace it we have $2e^* - 2U_a - 2g\left(e^*\right)$, which is equivalent to choose e^* to maximize $e^* - g(e^*)$, where $g'(e^*) = 1$. That is, the optimal decision should satisfy

$$\left(w_H - w_L\right) \int_{\varepsilon_j} f\left(\varepsilon_j\right)^2 d\varepsilon_j = 1$$

4.3.5 Secret Price Cut and Imperfect monitoring (Abreu, Pearce, and Stacchetti, 1986)

4.4 Open-loop and Closed-loop Equilibria

Definition 4.18 (Open-loop, closed-loop (Fudenberg and Levine, 1988))

In the open-loop model, players cannot observe the play of their opponents; in the closed-loop model, all past play is common knowledge at the beginning of each stage.

Note on Nature *Open-loop and closed-loop equilibria are then the perfect equilibria corresponding to the two information structures.*

Note on Open vs. Closed *Open-loop equilibria are more tractable than closed-loop equilibria, because players need not consider how their opponents would react to deviations from the equilibrium path. This is why sometimes economists prefer the open-loop, even though the closed-loop is more practical.*

4.5 Repeated Games

4.5.1 From Finite Period to Infinite Period

Proposition 4.1 (Finitely repeated game's equilibrium (Gibbons, 1992, p. 84))

If the stage game G has a unique Nash equilibrium then, for any finite T, the repeated game $G(T)$ has a unique subgame-perfect outcome: the Nash equilibrium of G is played in every stage (Backward Induction).

Note on Multiple equilibria case *In a stage game with multiple equilibria, its SPE may contains other outcomes (i.e., non-NE) in periods $t < T$. For example, the stage game has two NEs: (C,M), (B,R). In any SPE, the outcome at the last period must be either (C,M) OR (B,R). Suppose $T = 2$, there is a SPE, where in period 1, player 1 (2) chooses T (L), and in period 2, player 1 (2) chooses C (M) if the outcome of the first period is (T,L), chooses B (R) otherwise. Since for each player, payoffs from ((T,L), (C,M)) is better than payoffs from ((C,L), (B,R)), i.e., $8 + 4 > 9 + 1$.*

	L	M	R
T	$8, 8$	$0, 9$	$0, 0$
C	$9, 0$	$4, 4$	$0, 0$
B	$0, 0$	$0, 0$	$1, 1$

The nature of this phenomenon is the threat is severe enough to deter other's deviation. Follow the similar track, we can incorporate (T,M) into SPE in a three-period game. This phenomenon becomes more obvious in infinitely repeated game, even though there is a unique NE in each period, SPE may contains other outcome (i.e., non-NE) in periods.

Definition 4.19 (Feasible payoffs (Gibbons, 1992, p. 96))

We call the payoffs $(x_1, ..., x_n)$ feasible in the stage game G if they are a convex combination of the pure-strategy payoffs of G.

Definition 4.20 (Average payoff (Gibbons, 1992, p. 97))

Given the discount factor δ, the average payoff of the infinite sequence of payoffs $\pi_1, \pi_2, \pi_3 ...$ is $C = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \pi_t$.

Note on Normalization *By this definition, we can treat the sequence $\pi_1, \pi_2, \pi_3 ...$ as a constant sequence $C, C, C ...$.*

Definition 4.21 (History and Outcome path)

A history (up to period t) h_t is a sequence of past observed outcomes in the stage game, i.e., $h_t = (a^0, a^1, \ldots, a^{t-1})$. Initial history is written as h_0. An outcome path $\mathbf{a} = (a^0, a^1, \ldots)$ is an infinite history.

Definition 4.22 (Pure strategy and Behavioral strategy)

A pure strategy s_i of player i specifies an action $s_i(h_t) \in A_i$ for every h_t. A behavioral strategy σ_i specifies a randomization over A_i for every h_t, i.e., $\sigma_i(h_t) \in \Delta(A_i)$.

Lemma 4.4 (One-step deviation principle)

A strategy profile σ is a SPE of $G^{\infty}(\delta)$ iff it passes the following test: after any history h_t, every player i, assuming that all but himself will play according to σ at t, and that all (including himself) will follow σ at $t+1$ and thereafter, does not have an incentive to deviate from $\sigma_i(h_t)$ at t.

Theorem 4.3 (Infinitely repeated game's equilibrium (Gibbons, 1992, p. 97))

Let G be a finite, static game of complete information. Let $(e_1, ..., e_n)$ denote the payoffs from a Nash equilibrium of G, and let $(x_1, ..., x_n)$ denote any other feasible payoffs from G. If $x_i > e_i$ for every player i and if δ is sufficiently close to one, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game $G(\infty, \delta)$ that achieves $(x_1, ..., x_n)$ as the average payoff.

Note on Motivation *The motivation to model an infinitely repeated game is, though no one lives forever in reality, but people don't know when the game ends, in other words, the time period is stochastic.*

4.5.2 Strategy in Infinite Period

Next we show some strategies by the infinitely repeated prisoners' dilemma. First of all, both players choose D regardless of the history is a SPE.

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$

Definition 4.23 (Grim Trigger Strategy)

Choose C in the first period, stay with C if the opponent has never defected, otherwise switch to D forever.

Note on NE? *When δ is sufficiently large (close to 1), the strategy pair (grim trigger, grim trigger) is a NE. The payoff stream for this strategy pair is (3,3,...), and the discounted average payoff is 3; if he deviates to D, the payoff stream will become (4,1,1...), the discounted average payoff is $4-3\delta$, and $3 \geq 4-3\delta$ when $\delta \geq \frac{1}{3}$.*

$$(1-\delta)\left(4+\frac{\delta}{1-\delta}\right) = 4-3\delta$$

Note on SPE? *This strategy pair is not a SPE. Consider the subgame following the outcome (C,D) in the first period:*

1. *Suppose that player 1 adheres to the grim trigger and chooses D in the second period. It is optimal for player 2 to switch to D too, which is not consistent with the grim trigger.*
2. *Suppose that player 2 adheres to the grim trigger and chooses C in the second period. When $\delta \geq \frac{1}{3}$, it is optimal for player 1 not to switch to D.*

Definition 4.24 (Tit-for-tat strategy)

Choose C at $t = 1$, then do whatever the other player did in the previous period.

Note on NE? *The strategy pair (tit-for-tat, tit-for-tat) is a NE when $\delta \geq \frac{1}{3}$. The payoff stream for this strategy pair is (3,3,...), and the discounted average payoff is 3; if a player deviates to D in period t, then he may either alternate between D and C, or chooses D at every period.*

1. *Alternate between D and C: his payoff stream is (4,0,4,0...), the discounted average payoff is*
$$(1-\delta)\frac{4}{(1-\delta^2)} = 4/(1+\delta).$$
2. *Chooses D at every period: his payoff stream is (4,1,1...), the discounted average payoff is*
$$(1-\delta)\left(4+\frac{\delta}{1-\delta}\right) = 4-3\delta.$$

Note on SPE? *The strategy pair (tit-for-tat, tit-for-tat) is a SPE only when $\delta = \frac{1}{3}$.*

1. *History ending in (C,D):*
2. *History ending in (D,C):*
3. *History ending in (D,D):*

Definition 4.25 (Modified grim trigger strategy)

Start with C, and stay with C iff both have been choosing C before (in other words, switch to D forever iff someone, including himself, has defected before).

Note on SPE *When δ is sufficiently large, i.e. $\delta \geq \frac{1}{3}$, the pair of modified grim trigger strategy is a SPE. To see this,*

Note on Another example (Munoz-Garcia, 2017e) *It would be better to rewrite this section and supplemtns the visualization.*

Definition 4.26 (Limited punishment)

Start with C, switch to D for k periods whenever someone (including himself) defects, and then switch back to C.

Note on SPE *When δ and k are sufficiently large, the strategy pair in which each player uses the k-period punishment strategy is a SPE. To be continue,*

Note on *In the infinitely repeated PD, cooperation can be sustained in a SPE when players are sufficiently patient.*

4.5.3 Folk Theorem

Theorem 4.4 (Folk theorem with Trigger strategy)

If a^* is a NE of the stage game of an infinitely repeated game, then any action profile Pareto dominates (improves all players' payoff) is a SPE by using a^*-trigger strategy if discount factor is sufficiently large.

Remark In the prisoner's dilemma example, (D,D) is an NE but (C,C) Pareto dominates (D,D) so we can use (D,D) to support (C,C) when players are sufficiently patient (δ_i are large).

Definition 4.27 (Minmax payoff and action profile)

The minmax payoff of player i:

$$\underline{v}_i = \min_{\alpha_{-i} \in \Pi_{j \neq i} \Delta(A_j)} \max_{a_i \in A_i} u_i(a_i, \alpha_{-i}).$$

The minmax action profile for i4:

$$m^i = (m_1^i, \ldots, m_n^i)$$

Note on *The minmax payoff of your opponent actually denotes the worst punishment you can impose to it.*

Note on Example

Definition 4.28 (Individually rational payoff and profile)

1. *A payoff profile v is (strictly) individually rational if $(v_i > \underline{v}_i)$ $v_i \geq \underline{v}_i$ for all i.*
2. *An action profile a is (strictly) individually rational if $u(a)$ is (strictly) individually rational.*

Note on Example

Lemma 4.5

Every Nash equilibrium payoff profile of the stage game G is individually rational.

Theorem 4.5 (Folk theorem 1(Fudenberg and Maskin 1986))

Let V be the set of feasible payoff profiles. Assume either

1. *dim $V \geq N-1$ or*
2. *projection of V to any two player is two dimensional.*

Then, for any strictly individually rational and feasible v, there is $\bar{\delta} < 1$ such that for any $\delta \in (\bar{\delta}, 1)$, there is a SPE s of $G^\infty(\delta)$ with $U(s) = v$.

Theorem 4.6 (Folk theorem 2)

Let a^* be a strictly individually rational action profile of G. Assume that minmax strategy profile m^i is pure for each i. Assume that there is a collection $(a^i)_{i \in N}$ of strictly individually rational action profiles of G such that for every $i \in N$, $u_i(a^*) > u_i(a^i)$ and

$u_i\left(a^j\right) > u_i\left(a^i\right)$ for $j \neq i$. Then there is $\bar{\delta} < 1$ such that for every $\delta \in (\bar{\delta}, 1)$, there is a SPE of $G^{\infty}(\delta)$ in which a^* is played on the equilibrium path.

Proof

■

Note on "mutual minimaxing" method in two-player game *For two-player games, we can adopt a simpler SPE strategy profile: after a deviation by either player, each player minimaxes the other for a certain number of periods, after which they return to the original path; if a further deviation occurs during the punishment phase, the phase is begun again.*

However, this method of "mutual minimaxing" does not extend to games with three or more players. This is because with three players, there may not exist an action profile in whichevery player is minimaxed by the other two. This is why we need the dimensionality condition in the folk theorem. For example, p59.

Note on Interpretation *Actually, folk theorem shows that any point on the edge or interior of the feasible individually rational region can be supported as a SPNE of the infinitely-repeated game as long as the discount factor δ is close enough to 1, i.e., players care about the future.*

Note on Partial cooperation and example

Note on Advantaegs and disadvantages

4.5.4 Examples of Repeated Game

4.5.4.1 Repeated Cournot Game

Original cournot game: Two firms, market clearing price $P(Q) = a - q_1 - q_2$, each firms has a marginal cost c and no fixed costs, there is a unique equilibrium $q_{NE} = \frac{a-c}{3}$ and $p_{NE} = \frac{a+2c}{3}$. And in the monopoly case, we have $q_M = \frac{a-c}{2}$ and $p_M = \frac{a+c}{2}$.

Define a trigger strategy: each firm produces $q_M/2$, if each of them has done so in all previous periods; otherwise produce q_{NE}. Easy to see profit under cooperation is $\pi_C = \pi_M/2 = \frac{(a-c)^2}{8}$, and the profit under punishment is $\pi_{NE} = \frac{(a-c)^2}{9}$. Suppose one firm intends to deviates from cooperation, his profit will be $\pi_D = \max_q \left(a - q - \frac{q_M}{2} - c\right) q = \frac{9(a-c)^2}{64}$.

- Payoff of cooperation: $\sum_{t=0}^{\infty} \delta^t \pi_C = \frac{1}{1-\delta}\pi_C$.
- Payoff of deviation: $\pi_D + \sum_{t=1}^{\infty} \delta^t \pi_{NE} = \pi_D + \frac{\delta}{1-\delta}\pi_{NE}$.

The condition for cooperation in NE is $\frac{1}{1-\delta}\pi_C \geq \pi_D + \frac{\delta}{1-\delta}\pi_{NE}$, that is, $\delta \geq \frac{9}{17}$. Thus only when the discount factor is large enough, will firms keep cooperation in the long run. The question is, what happen when $\delta < \frac{9}{17}$?

Suppose now the trigger strategy is: each firm produces q_C, if each of them has done so in all previous periods; otherwise produce q_{NE}. Easy to see profit under cooperation is $\pi_C = (a - 2q_C - c)\, q_C$, and the profit under punishment is $\pi_{NE} = \frac{(a-c)^2}{9}$. Suppose one firm intends to deviates from cooperation, his profit will be $\pi_D = \max_q (a - q - q_C - c)\, q = \frac{(a-q_C-c)^2}{4}$.

The condition for cooperation in NE is $\frac{1}{1-\delta}\pi_C \geq \pi_D + \frac{\delta}{1-\delta}\pi_{NE}$, and by finding the largest q_C we have $q_C = \frac{9-5\delta}{3(9-\delta)}(a-c)$. Note that $\frac{dq_C}{d\delta} < 0$, when $\delta \to 9/17$ we have $q_C \to q_M/2$, when $\delta \to 0$ we have $q_C \to q_{NE}$.

4.5.4.2 Efficiency Wages

1. The firm offers the worker a wage w.
2. The worker accepts or rejects the firm's offer with outside wage w_0. If accept, the worker chooses either to supply effort (which entails disutility e) or to shirk. Note that the effort decision is not observed by the firm. Output can either high $y > 0$ or low 0. Here with effort the output is sure to be high, otherwise it is high with probability p and low with probability $1-p$.

4.5.4.3 Time-Consistent Monetary Policy

1. Employers form an expectation of inflation π^e.
2. The monetary authority observes this expectation and chooses actual inflation pi. The payoff to employers is $-(\pi-\pi^e)^2$, and employers simply want to anticipate inflation correctly, they achieve their maximum payoff (zero) when $\pi = \pi^e$. The monetary authority would like inflation to be zero but output y to be at its efficient level y^*, its payoff is $U(\pi, y) = -c\pi^2 - (y-y^*)^2$, where $c > 0$ reflects the tradeoff between its two goals. And the auctual output is $y = by^* + d(\pi - \pi^e)$, where $b < 1$ reflects the presence of monopoly power in product markets and $d > 0$ measures the effect of surprise inflation on output.

4.5.4.4 Discrete Cournot Game (Penal code) (Abreu, 1988)

4.5.4.5 Fluctuating Demand and Perfect monitoring (Rotemberg and Saloner, 1986)

4.5.4.6 Quid Pro Quo

Definition 4.29 (Quid Pro Quo)

Chapter 5 (Non-cooperative) Dynamic game with incomplete information

5.1 Solution Concept 6: Perfect Bayesian Equilibrium and Sequential Equilibrium

Definition 5.1 (Assessment)

An assessment (σ, μ) in an extensive game consists of a behavioral strategy profile and a belief system, where beliefs μ at a given information set is a probability distribution on the information set.

Definition 5.2 (Sequential Rationality (Imperfect Information))

A player is sequentially rational iff, at each of his information sets, he maximizes his expected payoff given his beliefs.

Definition 5.3 (Weak Consistency)

Given any strategy profile s and any information set I on the path of play of s, a player's beliefs at I is weakly consistent with s iff the beliefs are derived using the Bayes' rule and s.

Note on Example *In this example, assume that the first and second information sets are on the path of play, and the third is off the path of play. Thus weak consistency requires that*

$$x = \frac{pq}{pq + (1-p)r} \quad \text{and}$$
$$y = \frac{p(1-q)}{p(1-q) + (1-p)(1-r)}$$

and it does not put any restriction on z.

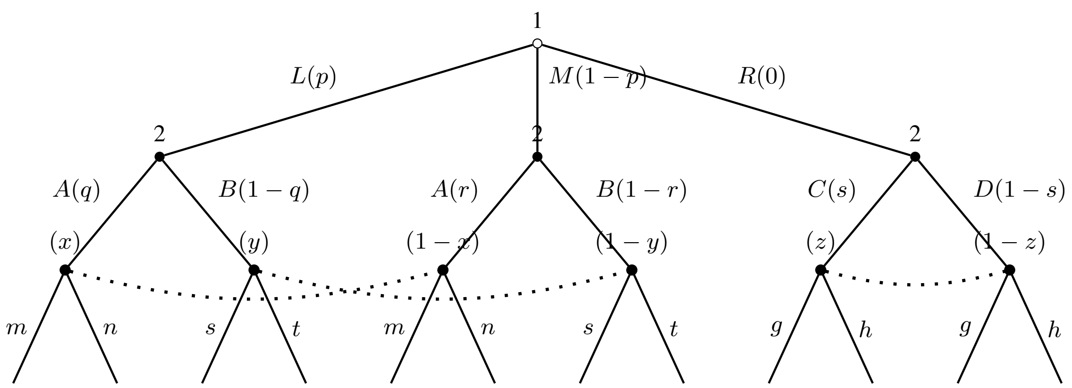


Figure 5.1: Example of weak consistency

Definition 5.4 (wPBE)

An assessment (σ, μ) in an extensive game is a weak perfect Bayesian equilibrium if it satisfies both sequential rationality and weak consistency.

Note on wPBE vs. NE *wPBE is also referred to as weak sequential equilibrium. A wPBE is a NE, but not every NE is a wPBE. Note that (σ, μ) is a NE if sequential rationality is satisfied on information sets on the path of play and beliefs are weakly consistent. However, (σ, μ) is a wPBE requires sequential rationality on all information sets.*

Note on Weak *The "weak" in wPBE is because of the weak consistency, we have no restrictions on beliefs at information sets that are off the path of play.*

Note on How to find wPBE *The beliefs are consistent with the strategies, which are optimal given the beliefs. Due to this circularity, wPBE cannot be determined by backward induction. To find all wPBEs, we first find all NEs, and then for each NE strategy profile σ, check whether there is a system of belief μ such that (σ, μ) satisfies both sequential rationality and weak consistency.*

Note on wPBE vs. SPE *A SPE may not be a wPBE, and a wPBE need not be a SPE.*

For example, the strategy profile of a SPE need not be a wPBE. Here (O,F) is a NE and SPE, but not a wPBE. Li duozhe p25.

For example, in Figure 5.2, the strategy profile of a wPBE need not be a SPE. Here the only SPNE is ((In,In-2),A), while there are xx wPBE: .

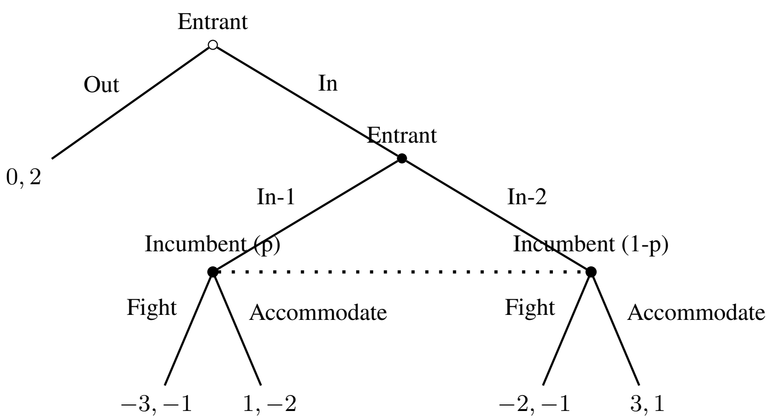


Figure 5.2: Example of wPBE (no SPE)

Definition 5.5 (PBE (Banal-Estanol, 2006))

A Perfect Bayesian Equilibrium is a WPBE that induces a WPBE in every subgame.

Note on *PBE still does not place much restrictions on out-of-equilibrium beliefs.*

Definition 5.6 (Consistency)

An assessment (σ, μ) is consistent if there is a sequence (σ^k, μ^k) of assessment s.t.

(i) Each σ^k is completely mixed.

(ii) Each μ^k is derived from σ^k using Bayes' rule.

(iii) $(\sigma^k, \mu^k) \to (\sigma, \mu)$.

Note on Weak consistency vs. Consistency *Weak consistency has no restriction on the beliefs at information sets that are off the path of play. And consistency provides reasonable restrictions about the beliefs at the information sets which are off the path of play.*

Note on Interpretation *(i) means that players imaging how they make mistakes and allocate positive probability to all actions. For example, player 1 plans to choose action 1, but action 2 and 3 may be chosen by mistake. Based on (i), (ii) show that we are able to compute the probability for information sets out of the path of play. (iii) further require the strategy profile and the belief should both converge to wPBE (i.e. the chance of making mistakes becomes arbitrary small), otherwise this wPBE is not a SE.*

Definition 5.7 (SE)

An assessment (σ, μ) *is a sequential equilibrium if it satisfies both sequential rationality and consistency.*

Note on SE vs. SPE vs. wPBE *A SE is both a SPE and a wPBE, and SE is nearly to PBE. For example,*

Note on How to find SE *First we find all wPBE, and check whether they satisfy consitency.*

Example 1*: we first find all NEs in this game. The set of NE is ((Out,In-1), Fight), ((Out,In-2), Fight) and ((In,In-2), Accommodate).*

	Fight	Accommodate
$(\text{Out}, \text{In}-1)$	$\underline{0}, \underline{2}$	$0, 2$
$(\text{Out}, \text{In}-2)$	$\underline{0}, \underline{2}$	$0, 2$
$(\text{In}, \text{In}-1)$	$-3, -1$	$1, -2$
$(\text{In}, \text{In}-2)$	$-2, -1$	$\underline{3}, \underline{1}$

Note that ((Out,In-2), Fight) requires $p > \frac{2}{3}$ *to be a wPBE, that is, the expected utility of choosing fight is larger than the expected utility of choosing accommodate:*

$$(-1)p + (-1)(1-p) > (-2)p + 1(1-p).$$

However, ((Out,In-2), Fight) with $p > \frac{2}{3}$ *does not satisfy consistency and is not a SE. To see this, let* $\sigma_1^k(In) = \varepsilon_k$ *and* $\sigma_2(In-1) = \delta_k$*, i.e., the entrant chooses In with probability* ε_k*, and chooses In-1 with probability* δ_k*. Then according to Bayes' rule,* $p = \frac{\sigma_2^k(In-1)}{\sigma_2^k(In-1)+\sigma_2^k(In-2)} = \frac{\delta_k \varepsilon_k}{\delta_k \varepsilon_k + (1-\delta_k)\varepsilon_k} = \delta_k$*, and when* $\sigma^k \to$ *((Out,In-2), Fight), we have* $p = \delta_k \to 0$*, which contradicts* $p > \frac{2}{3}$.

Example 2, the set of NE is (b,d,e) and (b,c,f).

And the set of wPBE is (b,d,e | $\alpha = \beta = 0$*) and (b,c,f |* $\alpha = 0, \beta \geq \frac{2}{3}$*). However, (b,c,f |* $\alpha = 0, \beta \geq \frac{2}{3}$*) is not a SE because consistency is not satisfied. Suppose player 1 makes a mistake and chooses a with probability* ε_k *and player 2 chooses d with probability* δ_k*. Note that*

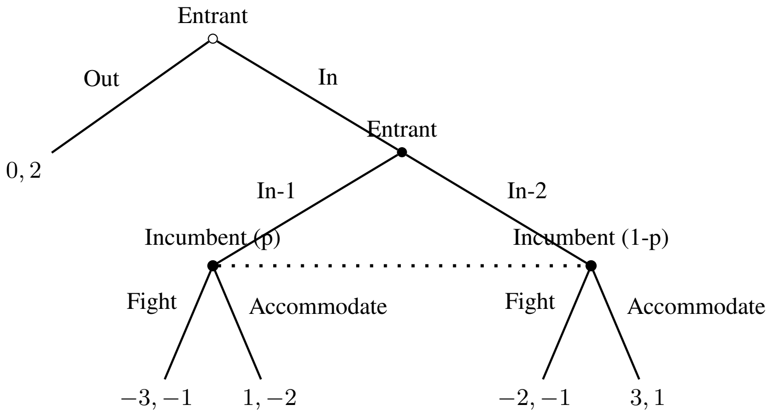


Figure 5.3: Example of finding SE

	c	d
a	$1, 2, 0$	$-1, 1, 0$
b	$2, 1, 3$	$\underline{0}, \underline{3}, \underline{2}$

Figure 5.4: e

	c	d
a	$1, 2, 0$	$0, 3, 1$
b	$\underline{2}, \underline{1}, \underline{3}$	$1, -1, 0$

Figure 5.5: f

both nodes for player 3 are chosen with infinitely small probability, but the right node is more possible. Thus it should be $\beta \to 0$.

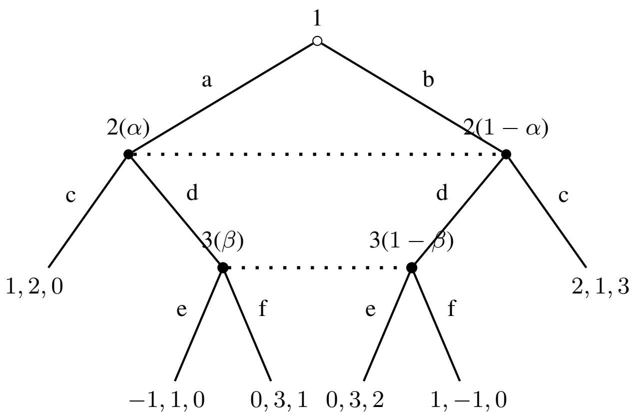


Figure 5.6: Example of finding SE 2

Note on Invariancy *SE is not invariant, i.e., insensitive to inessential transformation (preserving the reduced strategic form) on the game tree. For example, Li duozhe p29.*

5.2 Signaling Game

Theorem 5.1 (Intuitive criterion)

If, under some ongoing equilibrium, a non-equilibrium signal is received which a equilibrium-dominated for some types but not others, then beliefs cannot place positive probability weight on the former set of types.

Note on *Intuitive criterion shows that no poooling equilibrium can survive intuitive criterion.*

The following papers show some interesting insights about signaling.

- Akerlof (1970): it shows that a market may function badly if the informed party has no way to signal the quality of the good it is selling.
- Spence (1973): the signal that is sent by the informed party has a cost that depends on its type so that higher types are more likely to send higher signals. This signal may then help the uninformed party to discriminate among the different types.
- Crawford-Sobel (1982): even if the signal is purely extrinsic (if it has no cost for the informed party) and thus constitutes cheap talk, both parties may still coordinate on equilibria that reveal some information.
- There are multiple equilibria in Spence's and Crawford-Sobel's models, however, equilibria in Spence's can be refined by PBE, while the latter cannot.

5.2.1 The market for lemons (Akerlof, 1978)

Suppose there are two types of car in the market: q proportion of "plums" (value g for seller and $G > g$ for buyer) and $1 - q$ proportion of "lemons" (value l for seller and $L > l$ for buyer). Here we also assumes that $G > L, g > l$. Suppose that only seller knows the type.

Pooling equilibrium. Case 1 $L \geq g$: The seller sell all cars at $qG + (1 - q)L \geq g$.

Separating equilibrium. Case 2 $L < g$: Since the seller sells "plums" at a price higher than g, thus, the seller knows the car are "lemons" if $p \leq g$. When there are two types of cars in the market, the buyer must consider that a car is worth $qG + (1 - q)L < g$ and will not purchase them. Thus, the only equilibrium is that the seller sells "lemons" at $p = L < g$ and only lemons are sold.

5.2.2 Signal with cost (Spence, 1978)

Definition 5.8 (Spence-Mirrlees single-crossing property)

Employee side:

- Private information in productivity $\theta \in \{\theta_1, \theta_2\}$.
- Earn utility $u(w) - C(e, \theta)$ if studies for e years and is hired at wage w.

- Productivity does not depend on education, but more costly if he is by nature not productive. This means they can use diploma (public information) to signal their type (private information).

$$u' > 0, u'' < 0, \frac{\partial C}{\partial e} > 0, \frac{\partial C}{\partial \theta} > 0, \frac{\partial^2 \theta}{\partial e^2} > 0, \frac{\partial^2 C}{\partial e \partial \theta} < 0$$

Employer side:

- Provide wage $w(e) = \mu(e)\theta_1 + (1 - \mu(e))\theta_2$ if employers think that the candidate is θ_1 with probability $\mu(e)$.
- μ_0: the priori of employers on the worker's productivity.

The perfect Bayesian equilibria should consists of a vector of strategies (e_1^*, e_2^*, w^*) and a system of beliefs μ^* as follows:

$$\forall i = 1, 2 \quad e_i^* \in \arg\max_e (u(w^*(e)) - C(e, \theta_i)) \tag{5.1}$$

$$w^*(e) = \mu^*(e)\theta_1 + (1 - \mu^*(e))\theta_2 \tag{5.2}$$

$$For \quad e_1^* \neq e_2^* \qquad if \quad e = e_1^*, then \quad \mu^*(e) = 1 \tag{5.3}$$

$$if \quad e = e_2^*, then \quad \mu^*(e) = 0 \tag{5.4}$$

$$For \quad e_1^* = e_2^* \qquad if \quad e = e_1^* = e_2^*, then \quad \mu^*(e) = \mu_0 \tag{5.5}$$

Separating equilibrium. Low type chooses e_1^*, and high type chooses $e_2^* > e_1^*$. Low type get wages which equal to θ_1, thus, it is useless for this type to invest in study, i.e., $e_1^* = 0$. Hype type get wages θ_2. To ensure the existence of separating equilibrium, there is no incentive for low type to deviate, i.e., $u(\theta_1) - C(0, \theta_1) \geq u(\theta_2) - C(e_2^*, \theta_1)$, this characterizes a lower bound $\underline{e}$ for e_2^*; there is no incentive for high type to deviate too, i.e., $u(\theta_2) - C(e_2^*, \theta_2) \geq u(\theta_1) - C(0, \theta_2)$, this characterizes a upper bound $\overline{e}$ for e_2^*.

Pooling equilibrium. Both types chooses e^*, thus, the employer provides the same wage $\mu_0\theta_1 + (1 - \mu_0)\theta_2$ to all employee.

There is a threshold between separating and pooling equilibrium, in which low type gets the same in separating and pooling equilibrium, i.e., $\mu(\mu_0\theta_1 + (1 - \mu_0)\theta_2) - C(\overline{\overline{e}}, \theta_1) = u(\theta_1) - C(0, \theta_1)$.

Hybrid equilibrium. Suppose there are p proportion of low type, and $1 - p$ proportion of high type. Then in a hybrid equilibrium, low type chooses $e = 0$, while high type chooses $e = 0$ with prob q and e with prob $1 - q$. In employer's opinion, the posterior for $e = 0$ to be high type is $\frac{qp}{qp+(1-p)}$, thus, empolyer offers $\frac{qp}{qp+(1-p)}\theta_1 + \frac{1-p}{qp+(1-p)}\theta_2$. The condition for high type's indifference between $e = 0$ and e is

$$\frac{qp}{qp + (1 - p)}\theta_1 + \frac{1 - p}{qp + (1 - p)}\theta_2 = w(e).$$

5.2.3 Signal without cost (Cheap Talk) (Crawford and Sobel, 1982)

There are N villagers with private costs $c_i \sim [0, 1+\varepsilon]$ for hunting. Suppose all of them chooses hunting, they get 1, respectively. However, if anyone does not opt for hunting, everyone gets 0. Suppose every villagers chooses hunting with prob π, then the expected payoff of choosing hunting is π^{N-1}. And a villager chooses hunting only when c_i is smaller than π^{N-1}, i.e., π is the probability where c_i is smaller than π^{N-1}. In equilibrium, $\pi = c = 0$ and no one opts for hunting.

$$\pi = \frac{c}{1+\varepsilon} = \frac{\pi^{N-1}}{1+\varepsilon}$$

Now we change this game to a sequence game, in which villagers say yes or no in the first period, and opts for hunting in the second period. Thus, if all villagers say yes, then they go for hunting together; otherwise, they all stay at home. However, there still exists babbling equilibrium in simultaneous game.

5.2.4 Limit Pricing (Munoz-Garcia, 2017b)

Consider an entry game with an incumbent monopolist (Firm 1) and an entrant (Firm 2) who analyzes whether or not to enter the market. The incumbent's marginal costs are either high or low, i.e., $c_1^H > c_1^L > 0$. Let us consider a two-stage game here

1. In the first stage, the incumbent has monopoly power and selects an output level q.
2. In the second stage, a potential entrant decides whether or not to enter. If entry occurs, Cournot competition with x_1 and x_2, otherwise, firm 1 monopolizes the market.

Complete Information

We can apply backward induction to find the SPE. **Second period, No entry.** The incumbent chooses x_1 to maximizes $\bar{M}_1^K$, and $x_1^{K,m}$ is the profit-maximizing output, where $K = \{H, L\}$.

$$\bar{M}_1^K \equiv \max_{x_1} p\left(x_1\right) x_1 - c_1^K x_1$$

Second period, Entry. Both firms do Cournot competition, here $c_2 = c_1^H$ represents the entrant's marginal cost (can be relaxed), and F denotes the fixed entry cost. To make the entry decision interesting, assume that entry occurs only when the incumbent's costs are low, i.e., $D_2^L < 0 < D_2^H$ for all q.

$$D_1^K \equiv \max_{x_1} p(x_1 + x_2)x_1 - c_1^K x_1 \quad \text{and} \quad D_2^K \equiv \max_{x_2} p(x_1 + x_2)x_2 - c_2 x_2 - F$$

First period. With c_L, entry does not occur, the incumbent chooses q to maximizes profits, here $\delta \in (0, 1)$ denotes the discount factor, let $q^{L,Info}$ denotes the optimal solution.

$$\max_{q} p(q)q - c_1^L q + \delta \bar{M}_1^L$$

When the incumbent's cost is high, the incumbent chooses q to maximizes profits, and let

$q^{H,Info}$ denotes the optimal solution.

$$\max_q p(q)q - c_1^H q + \delta D_1^H.$$

Importantly, under complete information, the high-cost incumbent cannot deter entry, and the low-cost incumbent doesn't need to deviate from $q^{L,Info}$ to deter entry.

Incomplete Information

The game is redesigned as follows.

1. The incumbent privately observes the realization of marginal costs, c_H with $p \in (0,1)$ and c_L with $1-p$. The incumbent chooses first-period output level q.
2. Observing q, the entrant forms beliefs $\mu\left(c_1^K \mid q\right)$ about the incumbent's marginal costs. Given these posterior beliefs, the entrant decides whether or not to enter the market.
3. With entry, they do Cournot competition, otherwise, the incumbent monopolizes the market.

Separating equilibrium. Assume this equilibrium is that the incumbent selects q^H with c_H and q^L with c_L. Entrant's equilibrium beliefs are $\mu\left(c_1^H \mid q^H\right) = 1$ and $\mu\left(c_1^H \mid q^L\right) = 0$, for simplicity we assume off-the-equilibrium beliefs are $\mu\left(c_1^H \mid q\right) = 1$ for all $q \neq q^H \neq q^L$, and the entrant enters only when it infers a high type. Now we can analyze the conditions for the existence of separating equilibrium.

High-cost incumbent. The incumbent should have not incentive to deviate to q^L, that is

$$M_1^H\left(q^{H,\text{ Info}}\right) + \delta D_1^H = \max_q M_1^H(q) + \delta D_1^H \geq M_1^H\left(q^L\right) + \delta \bar{M}_1^H \tag{C1}$$

Low-cost incumbent. The incumbent should have not incentive to deviate to q^H, that is

$$M_1^L\left(q^L\right) + \delta \bar{M}_1^L \geq \max_q M_1^L(q) + \delta D_1^L = M_1^L\left(q^{L,\text{Info}}\right) + \delta D_1^L \tag{C2}$$

By solving these quadratic constraints about q^L, we can derive a feasible region $q^L \in [q^A, q^B]$.

Proposition 5.1 (Separating PBEs)

A separating strategy profile can be sustained as a Perfect Bayesian Equilibria in the signaling game where:

1. *In the first period, the high-cost incumbent selects $q^{H,Info}$ and the low-cost chooses $q^L \in [q^A, q^B]$, where q^A solves condition $C1$ with equality, and $q^A > q^{L,\text{ Info}}$; wheras q^B solves condition $C2$ with equality.*
2. *The entrant enters only after observing $q^{H,Info}$, given equilibrium beliefs $\mu\left(c_1^H \mid q^{H,\text{Info}}\right) = 1$ and $\mu\left(c_1^H \mid q^L\right) = 0$ after observing any $q^L \in \left[q^A, q^B\right]$. For every off-the-equilibrium output level $q \neq q^{H,\text{ Info}} \neq q^L$, entrant's beliefs are $\mu\left(c_1^H \mid q\right) = 1$; and*
3. *In the second period of the game, the incumbent selects an output $x_1^{K,m}$ if entry does not occur, and every firms $i = \{1,2\}$ chooses $x_i^{K,d}$ if entry occurs.*

Proposition 5.2 (Separating PBEs survives Intuitive Criterion)

Separating equilibrium. Assume both types select the same output level q, and equilibrium beliefs are $\mu\left(c_1^H \mid q\right)=p$ and $\mu\left(c_1^L \mid q\right)=1-p$, and for simplicity we say off-the-equilibrium beliefs are $\mu\left(c_1^H \mid q'\right)=1$ for any $q' \neq q$.

Entrant's response. After observing q, the entrant enters iff $pD_2^H+(1-p)D_2^L \geq 0$, that is, $p \geq \frac{-D_2^L}{D_2^H-D_2^L} \equiv \bar{p}$. We hence conclude that the entrant enters if $p \geq \bar{p}$, and stays out otherwise. In particular, it must be that $p<\bar{p}$, otherwise with entry, the incumbent must deviates to $q^{K,Info}$, and since $q^{H, \text{ Info }} \neq q^{L, \text{ Info }}$, this strategy profile cannot be a pooling equilibrium.

Incumbent's IC constraints. Again, check its IC constraints. A high-cost incumbent does not deviate from q if (C1), and a low-cost incumbent does not deviate from q if (C2). By solving these quadratic constraints about q, we can derive a feasible region $q \in [q^C, q^D]$.

$$M_1^H(q)+\delta \bar{M}_1^H \geq \max_q M_1^H(q)+\delta D_1^H=M_1^H\left(q^{H, \text{ Info }}\right)+\delta D_1^H \tag{C1}$$

$$M_1^L(q)+\delta \bar{M}_1^L \geq \max_q M_1^L(q)+\delta D_1^L=M_1^L\left(q^{L,\text{Info}}\right)+\delta D_1^L \tag{C2}$$

Proposition 5.3 (Pooling PBEs)

The following strategy profiles can be sustained as pooling PBE:

1. *In the first period, both types of incumbent select the same first-period output $q \in [q^C, q^D]$, where q^C solves condition (C2) with equality, while q^D solves (C1) with equality.*
2. *The entrant does not enter after observing the equilibrium output $q \in [q^C, q^D]$, but enters after observing off-the-equilibrium output $q' \neq q$, given beliefs $\mu\left(c_1^H \mid q^{L, \text{ Info }}\right)=p<\bar{p}$ and $\mu\left(c_1^H \mid q'\right)=1$; and*
3. *In the second period of the game, the incumbent selects $x_1^{K,m}$ if entry does not occur, and every firm $i=\{1,2\}$ chooses $x_i^{K,d}$ if entry occurs.*

Proposition 5.4 (Pooling PBEs survives Intuitive Criterion)

5.3 Sorting Game

5.4 Screening Game

5.5 Moroal Hazard

5.6 Solution Concept 7: Treambling-hand Perfect Equilibrium

Definition 5.9 (Trembling-Hand perfect equilibrium)

A strategy profile σ is a trembling-hand perfect equilibrium if there exists a sequence of totally mixed strategy profiles $\sigma^n \to \sigma$ such that, for all i,

$$u_i\left(\sigma_i, \sigma_{-i}^n\right) \geq u_i\left(a_i, \sigma_{-i}^n\right) \text{ for all } a_i \in A_i.$$

Note on Example *In G_1, NE (B,R) is not trembling-hand perfect. In G_2, both NE (A,A) and (C,C) are not perfect.*

	L(ε)	R($1-\varepsilon$)
T	$\underline{1}, \underline{1}$	$0,0$
B	$0,0$	$\underline{0}, \underline{0}$

Figure 5.7: G_1

	A(ε)	B(ε)	C($1-2\varepsilon$)
A	$\underline{0}, \underline{0}$	$0,0$	$0,0$
B	$0,0$	$\underline{1}, \underline{1}$	$2,0$
C	$0,0$	$0,2$	$\underline{2}, \underline{2}$

Figure 5.8: G_2

Note on Weakly dominated strategy *Trembling-hand perfection rules out the use of weakly dominated strategies. In two-player games, any NE in which neither player uses a weakly dominated strategy is trembling-hand perfect, but it is not true for games with more than two players. NE (D, L, A) is undominated, since given (R,B), D is better than U. But it is not trembling-hand perfect. To see this, say that player 2 and 3 may make mistakes in (D,L,A), then the utility for choosing U is greater than for choosing D, even though the difference is infinitely small.*

$$(1-\varepsilon_k)(1-\delta_k)+0+0+1\varepsilon_k\delta_k < (1-\varepsilon_k)(1-\delta_k)+0++\delta_k(1-\varepsilon_k)+\varepsilon_k(1-\delta_k)$$

	$L(1-\delta_k)$	$R(\delta_k)$
U	$1,1,1$	$1,0,1$
D	$1,1,1$	$0,0,1$

$A(1-\varepsilon_k)$

	$L(1-\delta_k)$	$R(\delta_k)$
U	$1,1,0$	$0,0,0$
D	$0,1,0$	$1,0,0$

$B(\varepsilon_k)$

Lemma 5.1

Every finite strategic game has a trembling-hand perfect equilibrium.

Definition 5.10 (Strictly perfect equilibrium)

A strategy profile σ is a strictly perfect equilibrium if each player's strategy is optimal against all possible (not just one) sequences of perturbations.

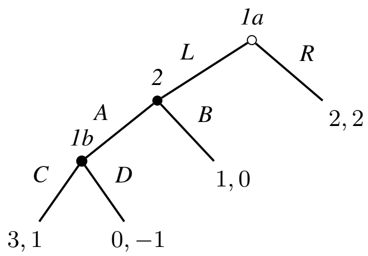
Note on Existence *Such an equilibrium might not exist. See, for example, both (T,L) and (T,R) are THBE, e.g.* $(\sigma_1^k(T) = 1 - \varepsilon_k - \varepsilon_k^2, \sigma_1^k(C) = \varepsilon_k, \sigma_1^k(B) = \varepsilon_k^2)$ *for (T,L) and* $(\sigma_1^k(T) = 1 - \varepsilon_k - \varepsilon_k^2, \sigma_1^k(C) = \varepsilon_k^2, \sigma_1^k(B) = \varepsilon_k)$ *for (T,R). But these two NEs can not be THPE for both two sequences, thus they are not strictly perfect equilibrium.*

	L	R
T	$3,2$	$2,2$
C	$1,1$	$0,0$
B	$0,0$	$1,1$

Definition 5.11 (Agent-strategic form (Selten, 1975))

Each information set is manned by a different "agent", and all agents of the same player have the same payoff.

Note on Origin *For an extensive game, trembling-hand perfection in its strategic form is not totally satisfactory. In the strategic form, (R,B) is trembling-hand perfect, but the unique SPE of the game is (LC,A).*



extensive form

	A	B
R	$2,2$	$2,2$
LC	$3,1$	$1,0$
LD	$0,-1$	$1,0$

strategic form

Selten (1975) considers the agent-strategic form, and here the only THPE of the agent-strategic form is the unique SPE of the extensive form. Actually, here D is weakly dominated by C for agent 1b.

	A	*B*
R	$2,2,2$	$2,2,2$
L	$3,1,3$	$1,0,1$

Agent 1b: C

	A	B
R	$2,2,2$	$2,2,2$
L	$0,-1,0$	$1,0,1$

Agent 1b: D

Definition 5.12 (Perfect equilibrium)

For an extensive game, the THPE of the agent-strategic form is referred to as perfect equilibrium.

Note on Perfect vs. Sequential *A perfect equilibrium must be sequential, but the converse is not true; for generic games the two concepts coincide. That is, perfect is stronger than sequential.*

For example, (B,D) is sequential but not perfect. Actually, we do not require σ_i^n to be BR to σ_{-i}^n in SE, we just require σ to be BR to σ_{-i} (in convergence).

	C	D
A	$1,1$	$0,0$
B	$0,0$	$0,0$

Note on THPE in agent-strategic form and strategic form *The set of THPE of the agent-strategic form of an extensive game is NOT a subset of the set of THPE of the corresponding strategic form. For example, p35.*

Note on Invariancy *Perfect equilibrium is not invariant. For example, p35.*

Theorem 5.2

In finite games, at least one perfect equilibrium exists.

5.7 Solution Concept 8: Proper Equilibrium

Definition 5.13 (Proper equilibrium (Myerson 1978))

An ε-proper equilibrium is a totally mixed strategy profile σ^ε such that, if

$$u_i\left(a_i, \sigma_{-i}^{\varepsilon}\right) < u_i\left(a_i', \sigma_{-i}^{\varepsilon}\right)$$

then $\sigma_i^\varepsilon\left(a_i\right) \leq \varepsilon \sigma_i^\varepsilon\left(a_i'\right)$. A proper equilibrium σ is any limit of ε-proper equilibria σ^ε as ε tends to 0.

Note on Interpretation *The basic idea is that players are less likely to make "mistakes" that are more costly.*

Note on Proper vs. perfect *A proper equilibrium must be perfect. For example, there are three NEs: (U,L), (M,C) and (D,R). T-H perfection rules out (D,R), but not (M,C), but properness rules out (M,C). Assume that player 2 chooses L,C,R with probability ε_L, $1-\varepsilon_L-\varepsilon_R$ and ε_R, then when $\sigma^k \to$ (M,C), the payoff of choosing U is greater than choosing D, then it requires $\varepsilon_D \leq \varepsilon \delta_U$. Similarly, for player 2, it requires $\varepsilon_R \leq \varepsilon \varepsilon_L$.*

	L(ε_L)	C$(1-\varepsilon_L-\varepsilon_R)$	R(ε_R)
U(δ_U)	$1,1$	$0,0$	$-9,-9$
M$(1-\varepsilon_U-\varepsilon_D)$	$0,0$	$0,0$	$-7,-7$
D(δ_D)	$-9,-9$	$-7,-7$	$-7,-7$

A proper equilibrium of a strategic-form game need not be a treambling-hand perfect equilibrium in the agent-strategic form of every extensive game with the given (reduced) strategic form. For example, LA is proper in the strategic form, but is not perfect in the extensive form. p39.

Note on Proper vs. backward induction outcome *Proper equilibrium yields backward induction outcome without the use of the agent-strategic form. For example, p37.*

Note on Invariancy *Proper equilibrium is invariant. More precisely, every proper equilibrium of a strategic-form game is sequential in every extensive game with the given (reduced) strategic form. For example, p38.*

Definition 5.14 (Fully reduced strategic form)

Eliminating any pure strategy that is equivalent to a mixed strategies with support excluding it.

Note on Fully invariancy *Proper equilibrium is not fully invariant.*

Theorem 5.3

Every finite strategic game has a proper equilibrium.

Chapter 6 Advanced Topics

6.1 Supermodular Game

Definition 6.1 (supermodular game)

A twice continuously differentiable payoff function $\pi_i(x_1,..x_n)$ is supermodular (submodular) iff $\frac{\partial^2 \pi_i}{\partial x_i \partial x_j} \geq 0 (\leq 0)$ for all x and all $j \neq i$. The game is called supermodular if the players' payoffs are supermodular.

Note on *Theorem*

Note on From supermodularity to nondecrease *Theorem 6.1 points out the importance of nondecreasing for the existence of equilibria. Here we show that how supermodularity ensures nondecreasing. For any continuous function, the derivative of BR can be calculated via implicit function theorem. If x_i^* is BR, then $\frac{\partial^2 \pi_i}{\partial x_i^2} < 0$ must hold. Thus, the condition for BR to be nondecreasing is $\frac{\partial^2 \pi_i}{\partial x_i \partial x_j} > 0$.*

$$\frac{\partial x_i^*}{\partial x_j} = -\frac{\partial^2 \pi_i}{\partial x_i \partial x_j} / \frac{\partial^2 \pi_i}{\partial x_i^2}$$

However, the implicit function theorem cannot apply to non-continuous function directly. To be continue.

Theorem 6.1 (Topkis 1998)

In a supermodular game, there exists at least one NE.

Note on *This means that the existence of NE only requires that BR is nondecreasing, and does not require the quasi-concavity of utility function. In addition, BR can be non-continuous. For example, when $f(x)$ is nondecreasing, even though $f(x)$ is non-continuous, there exists an intersect between BR and $y = x$.*

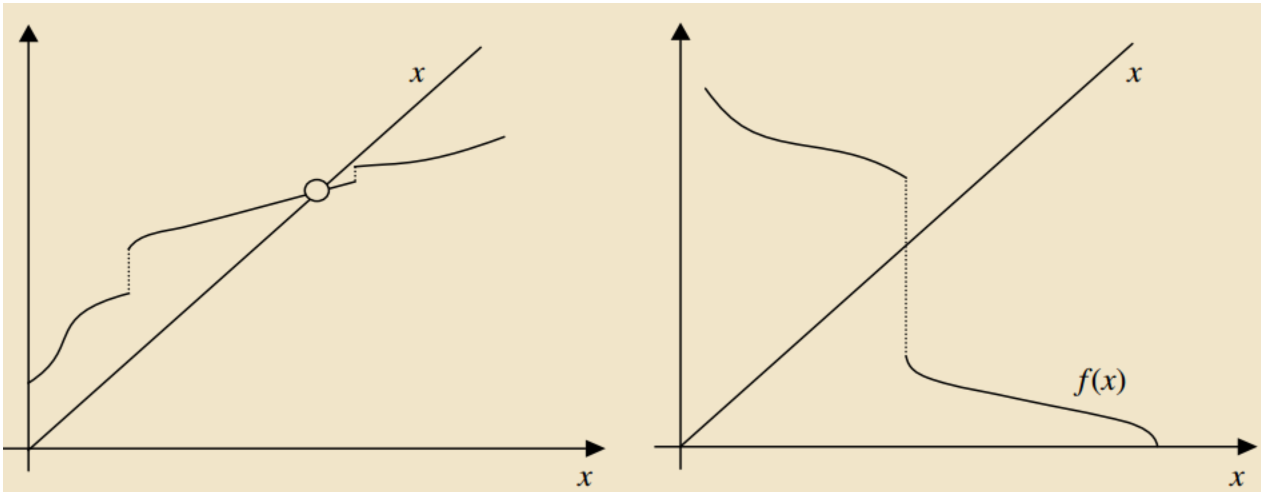


Figure 6.1: Increasing (left) and decreasing (right) mappings

6.2 Bargaining

6.3 Games with information externalities

6.4 Reputation in Repeated Interaction

6.5 Theory of fairness

6.6 Higher order beliefs

6.7 Relational Incentive Contracts

6.8 Learning in Games

6.9 Wars of Attrition

Chapter 7 Other solution concepts

7.1 Correlated equilibrium

Definition 7.1 (Correlating device)

A correlating device $(\Omega, p, \{H_i\})$ *is*

1. Ω *is a (finite) state space,*
2. p *is a probability measure on* Ω *(same for all* i*),*
3. H_i *is the information partition for player* i*:*
    - $\omega \in h_i(\omega) \subset \Omega$,
    - *posterior belief are given by Bayes'law.*

Note on Example

1. $\Omega : \{A, B, C\}$ *with* $p = 1/3$*;*
2. $H_1 = \{\{A\}, \{B, C\}\}$, $H_2 = \{\{A, B\}, \{C\}\}$*;*

Definition 7.2 (Strategy)

A strategy of a player i *is any function* $s_i : \Omega \to A_i$.

Note on $h(\omega)$ *Here* ω *is the event of* Ω*, and* $h(\omega)$ *is the signal which covers multiple* ω *events. If* $\omega' \in h_i(\omega)$*, then* $s_i(\omega') = s_i(\omega)$.

Note on Ω **and** A *The space* Ω *does not need to be equal to the space* A.

Example 7.1

	L	R
U	$5,1$	$0,0$
D	$4,4$	$1,5$

Flip a coin, (U,L) if head, and (D,R) if tail. The expected payoffs (3,3) is greater than the mixed-strategy NE. Furthermore, consider a randomization device with three equally likely state: A, B and C. Player 1 (2) can only observe whether or not state A (C) occurs. Player 1 (2) chooses U (R) iff state A (C) occurs. That is, if A, player 1 chooses U, if B or C, player 1 chooses D, and (U,L), (D,L) and (D,R) are chosen with probability 1/3 each, and the expected payoffs are (10/3,10/3). No one wants to deviate from such an arrangement.

To show that this is an equilibrium, we can show that these strategies are their best responses respectively. Given player 2 chooses L under signal A and B, and R under signal C, observing A, player 1 should choose U; observing B or C, player 1 knows that player 2 chooses L and R with probability 1/2 respectively, then player 1 is indifferent from U and D. Similarly, this strategy is player 2's best response.

Note on Geometric Interpretation *Figures.*

Definition 7.3 (Correlated equilibrium 1)

A strategy profile s^* is a correlated equilibrium (CE) relative to the information structure $(\Omega, p, \{H_i\})$ if for every player i and every h_i with $p(h_i) > 0$, $s_i^*(\omega)$ solves

$$\max_{a_i \in A_i} \sum_{\{\omega : h_i(\omega) = h_i\}} p(\omega \mid h_i)\, u_i\left(a_i, s_{-i}^*(\omega)\right)$$

Note on *This definition means, given specific signal h, the strategy of CE can maximize utility among all states ω.*

Definition 7.4 (Correlated equilibrium 2)

A CE is any probability distribution $p(.)$ over $A = \times_{i \in N} A_i$ such that, for every player i and every a_i^* with $p(a_i^*) > 0$, a_i^* solves

$$\max_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} p(a_{-i} \mid a_i^*)\, u_i(a_i, a_{-i})$$

Note on How to find CE *PS1.4*

Note on *This definition means the strategy of CE can maximize utility among all other players' strategies a. Though this definition is more abstract, but it is more useful to find a CE. Actually, CE is easier to compute than NE, since the computation on NE involves nonlinear constraint (Leyton-Brown, 2008b).*

Lemma 7.1

Definition 7.1 and Definition 7.1 are equivalent.

Theorem 7.1 (CE vs. NE)

Any NE is a CE. The set of CE is convex. It is at least the convex hull of the NE. Non-public correlating device can lead to equilibria outside the convex hull of the NE.

Note on Example of non-public correlating device *For example, the unique NE is (D,L,A). Suppose we flip a coin in front of 1 and 2, and not 3. Then we have a CE such that $\frac{1}{2}$(U,L), $\frac{1}{2}$(D,R) and B. Interestingly, not knowing the information makes player 3 better.*

	L	R
U	$0,1,3$	$0,0,0$
D	$1,1,1$	$1,0,0$

Figure 7.1: A

	L	R
U	$2,2,2$	$0,0,0$
D	$2,2,0$	$2,2,2$

Figure 7.2: B

	L	R
U	$0,1,0$	$0,0,0$
D	$1,1,0$	$1,0,3$

Figure 7.3: C

Note on CE vs. mixed NE

1. *All mixed NE are correlated, so CE exist.*
2. *All convex combinations of mixed NE are also correlated.*

Definition 7.5 (Subjective correlated equilibrium)

Note on Example *For example, let player 1 chooses according to* $\{A, B\} \to U$ *and* $\{C, D\} \to D$*, and player 2 chooses according to* $\{A, C\} \to L$ *and* $\{B, D\} \to R$*. Then this subjective CE generates expected payoffs* $(1/3, 1/3)$.

	L	R
U	$1, -1$	$-1, 1$
D	$-1, 1$	$1, -1$

Figure 7.4: Payoff

ω	A	B	C	D
1's prior	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
2's prior	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

Figure 7.5: Prior belief

Note on How to find Subjective CE *PS1.5*

7.2 Quantal Response Equilibrium ()

7.3 Epsilon equilibrium

7.4 Coalition-proof equilibrium

7.5 Markov Perfect Equilibrium

Definition 7.6 (Markov Perfect Equilibrium)

Chapter 8 Cooperative game

Definition 8.1 (Shapley value)

Chapter 9 Algorithmic game theory

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