Note on Poisson Process

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1 Stochastic Process

Definition 1.1 (Stochastic Process)

A stochastic process $X = \{X(t, w), t \in T\}$ *is a collection of random variables. We often interpret* t *as time and call* $X(t)$ *the state of the process at time* t.

Definition 1.2 (Sample Path)

Any realization of X *is called a sample path.*

For example, when t *is given, then you get a random variable* $X(w)$*, which characterizes the nature of stochastic. When* w *is given, then you get a sample path, you get a constant at every point of* t*, which characterizes the nature of process.*

Definition 1.3 (Independent Increments)

A continuous stochastic process $\{X(t), t \in T\}$ *is said to have independent increments if for all* $t_0 < t_1 < ... < t_n$*, the random variables*

$$X(t_1) - X(t_0), ..., X(t_n) - X(t_{n-1})$$

are independent.

This means the changes in its value over nonoverlapping time intervals are independent.

Definition 1.4 (Stationary increments)

A continuous stochastic process $\{X(t), t \in T\}$ *is said to possess stationary increments if* $X(t+s) - X(t)$ *has the same distribution for all* t.

This means the distribution of the change in value between any two points depends only on the distance between those points.

Definition 1.5 (Counting Process)

A stochastic process $\{N(t), t \geq 0\}$ *is said to be a counting process if* $N(t)$ *represents the total number of "events" that have occurred up to time* t*. A counting process* $N(t)$ *must satisfy*

- $N(t) \geq 0$

- *$N(t)$ is integer valued*
- *If $s < t$, then $N(s) \leq N(t)$*
- *For $s < t$, $N(t) - N(s)$ equals the number of events occured in the interval $(s, t]$*

2 Poisson Process

Definition 2.1 (Poisson Process from events in interval 1)

A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if

- $N(0) = 0$
- *The process has independent increments*
- *The number of events in any interval of length t is Poisson distributed with mean λt, i.e., for all $s, t \geq 0$,*

$$P\{N(t+s) - N(s) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1 \ldots$$

Remark The condition 3 also reflects that a Poisson process has stationary increments and $E[N(t)] = \lambda t$, which explains why λ is called the rate of the process.

Proof **SP_HW1** On the basis of Erlang distribution, we can derive $P\{N(t) = n\}$

$$P\{N(t) = n \mid S_n = \tau\} = P\{X_{n+1} > t - \tau\} = e^{-\lambda(t-\tau)}$$

$$P\{N(t) = n\} = \int_0^t P\{N(t) = n \mid S_n = \tau\} f_{S_n}(\tau) ds = \int_0^t e^{-\lambda(t-\tau)} \cdot \lambda e^{-\lambda\tau}\frac{(\lambda\tau)^{n-1}}{(n-1)!} \cdot d\tau$$
$$= e^{-\lambda t}\int_0^t \lambda\frac{(\lambda\tau)^{n-1}}{(n-1)!} d\tau = e^{-\lambda t}\int_0^t d\frac{(\lambda\tau)^n}{n!} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

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Definition 2.2 (Poisson Process from events in interval 2)

A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if

- $N(0) = 0$
- *the process has stationary and independent increments*
- $P\{N(h) = 1\} = \lambda h + o(h)$
- $P\{N(h) \geq 2\} = o(h)$

Where a function f is said to be $o(h)$ if $lim_{h\to 0}\frac{f(h)}{h} = 0$. The last two conditions imply

$$P\{N(h) = 0\} = 1 - \lambda h + o(h)$$

Theorem 2.1 (Definition 2.1 and 2.2 are equivalent)

Just prove the third condition of definition 2.1 is equal to the last two conditions of definition 2.2.

When it comes to $1 \to 2$, just set $t = h, s = 0, n = 0, 1$, and expand it by Taylor's formula.

When it comes to $2 \to 1$, just imagine an interval $[0, t]$ which is subdivided into k equal parts where k is very large. Hence, $N(t)$ equal to the number of subintervals in which an event occurs. By stationary and independent increments, this number will have a binomial distribution with $k, p = \lambda t/k + o(t/k)$, and this binomial distribution converges to a Poisson distribution with parameter λ as $n \to \infty$.

3 Interarrival and Waiting time distribution

Theorem 3.1 (Sequence of interarrival times in Poisson process)

Consider a Poisson process, and let X_1 denote the time of the first event. Further, for $n \geq 1$, letX_n denote the time between the $(n-1)$st and the nth events. The sequence $\{X_n, n \geq 1\}$ is called the sequence of interarrival times.
Particularly, $X_n, n = 1, 2...$ are independent identically distributed exponential random variables having mean $1/\lambda$.

Proof At first, we prove that X_1 has an exponential distribution with mean $1/\lambda$.

$$P\{X_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

Then we prove that X_2 is an exponential random variable with mean $1/\lambda$ too.

$$\begin{aligned}
&P\{X_2 > t \mid X_1 = s\} \\
=&P\{0 \text{ events in } (s, s+t] \mid X_1 = s\} \\
=&P\{0 \text{ events in } (s, s+t]\} \quad \text{independent increments} \\
=&P\{N(t) = 0\} = e^{-\lambda t} \quad \text{stationary increments}
\end{aligned}$$

$$\begin{aligned}
P\{X_2 > t\} &= \int_s P\{X_2 > t \mid X_1 = s\} f_{X_1}(s) ds \quad \text{Lemma ??} \\
&= \int_s e^{-\lambda t} f_{X_1}(s) ds = e^{-\lambda t}
\end{aligned}$$

Next we prove that X_2 is independent of X_1.

$$\begin{aligned}
P\{X_1 > t_1, X_2 > t_2\} &= \int_S P\{X_1 > t_1, X_2 > t_2 \mid X_1 = s\} f_{X_1}(s) ds && \text{Lemma ??} \\
=& \int_{s=t_1}^{\infty} P\{X_1 > t_1, X_2 > t_2 \mid X_1 = s\} f_{X_1}(s) ds && \text{Trim the integration range} \\
=& \int_{s=t_1}^{\infty} P\{X_2 > t_2 \mid X_1 = s\} f_{X_1}(s) ds \\
=& \int_{s=t_1}^{\infty} e^{-\lambda t_2} f_{X_1}(s) ds \\
=&P\{X_1 > t_1\} e^{-\lambda t_2} \\
=&P\{X_1 > t_1\} P\{X_2 > t_2\}
\end{aligned}$$

Repeating the same argument yields the desired result. ■

Definition 3.1 (Poisson Process from waiting time distribution)

Consider a sequence $\{X_n, n \geq 1\}$ of independent identically distributed exponential random variables each having mean $1/\lambda$. Define a counting process such that the nth event of this process occurs at time S_n, where

$$S_n = X_1 + \ldots + X_n$$

The resultant counting process $\{N(t), t \geq 0\}$ is Poisson with rate λ.

Remark S_n is referred to as the arrival time of the nth event or the waiting time until the nth event, and has an Erlang or gamma distribution with parameters n and λ, thus we can get its density function simply, or we can deduce it as follows.

$$S_n \leq t \iff N(t) \geq n$$

$$\begin{aligned} P\{S_n \leq t\} = P\{N(t) \geq n\} &= 1 - \sum_{j=0}^{n-1} P\{N(t) = j\} \\ &= 1 - e^{-\lambda t} - \sum_{j=1}^{n-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \end{aligned}$$

$$\begin{aligned} f(t) &= \lambda e^{-\lambda t} - \sum_{j=1}^{n-1} \left(-\lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!} + \lambda e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!} \right) \\ &= \lambda e^{-\lambda t} + \sum_{j=1}^{n-1} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!} - \sum_{j=1}^{n-1} \lambda e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!} \\ &= \sum_{j=0}^{n-1} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!} - \sum_{j=0}^{n-2} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!} = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \end{aligned}$$

4 Arrival Times

Definition 4.1 (Order statistics)

Let $Y_1, ..Y_n$ be n random variables. We say that $Y_{(1)}, ...Y_{(n)}$ are the order statistics corresponding to $Y_1, ...Y_n$ if $Y_{(k)}$ is the kth smallest value among $Y_1, ..Y_n, k = 1, ...n$. If Y_i are i.i.d continuous random variables with probability density f, then the joint density of the order statistics $Y_{(1)}, ...Y_{(n)}$ is given by

$$f_{\text{os}}(y_1, y_2, \ldots, y_n) = n! \prod_{i=1}^{n} f(y_i), \quad y_1 < y_2 < \cdots < y_n$$

Remark Note that $n!$ comes from: $Y_{(1)}, ...Y_{(n)} = (y_1, ...y_n) \iff Y_1, ..Y_n \quad$ has a permutation of $y_1, ...y_n$.

Theorem 4.1 (Uniform arrival time)

Given that $N(t) = n$, the n arrival times $S_1, ..., S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$. The joint density of the order statistics $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ is

$$f_{os}\left(y_1, y_2, \ldots, y_n\right) = \frac{n!}{t^n}, \quad 0 < y_1 < y_2 < \cdots < y_n < t$$

Proof Firstly, we show that $P\{X_1 < s | N(t) = 1\} = \frac{s}{t} \quad \forall 0 \leq s \leq t$ is uniformly distributed over $[0, t]$.

$$S_1 | N(t) = 1 \iff X_1 | N(t) = 1$$

$$\begin{aligned}
P\left\{X_1 < s \mid N(t) = 1\right\} &= \frac{P\left\{X_1 < s, N(t) = 1\right\}}{P\{N(t) = 1\}} \\
&= \frac{P\{1 \text{ event in } [0, s), 0 \text{ events in } [s, t)\}}{P\{N(t) = 1\}} \\
&= \frac{P\{1 \text{ event in } [0, s)\} P\{0 \text{ events in } [s, t)\}}{P\{N(t) = 1\}} \\
&= \frac{\lambda s e^{-\lambda s} \cdot e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} = \frac{s}{t}
\end{aligned}$$

Let $0 = t_0 < ... < t_n < t$. And then we choose $t_1^0, ...t_{n+1}^0$ such that $0 = t_0 \leq t_1^0 < ... < t_n^0 < t_n < t_{n+1}^0 = t$.

$$\begin{aligned}
& P\left\{t_i^0 < S_i \leq t_i, i = 1, 2, \ldots, n \mid N(t) = n\right\} \\
= & \frac{P\left\{\begin{array}{l} \text{exactly } 1 \text{ event in } \left(t_i^0, t_i\right], i = 1, \ldots, n, \\ \text{no events in } \left(t_{i-1}, t_i^0\right], i = 1, \ldots, n+1 \end{array}\right\}}{P(N(t) = n)} \\
= & \frac{\prod_{i=1}^n \left(e^{-\lambda\left(t_i - t_i^0\right)} \lambda\left(t_i - t_i^0\right)\right) \prod_{i=1}^{n+1} e^{-\lambda\left(t_i^0 - t_{i-1}\right)}}{e^{-\lambda t}(\lambda t)^n / n!} \\
= & \frac{n!}{t^n} \cdot \prod_{i=1}^n \left(t_i - t_i^0\right) \cdot \exp\left(\lambda t - \lambda \sum_{i=1}^n \left(t_i - t_i^0\right) - \lambda \sum_{i=1}^{n+1} \left(t_i^0 - t_{i-1}\right)\right) \\
= & \frac{n!}{t^n} \prod_{i=1}^n \left(t_i - t_i^0\right)
\end{aligned}$$

By differentiating it with respect to $t_1, ...t_n$, we obtain the conditional density of $S_1, ..S_n$ given that $N(t) = n$ is as follows for any $0 < t_1 ... < t_n < t$.

$$\begin{aligned}
f\left(t_1, \ldots, t_n\right) &= \frac{\partial^n}{\partial t_1 \partial t_2 \cdots \partial t_n} P\left\{t_i^0 < S_i \leq t_i, i = 1, 2, \ldots, n \mid N(t) = n\right\} \\
&= \frac{\partial^n}{\partial t_1 \partial t_2 \cdots \partial t_n} \frac{n!}{t^n} \prod_{i=1}^n \left(t_i - t_i^0\right) = \frac{\partial^n}{\partial t_2 \cdots \partial t_n} \frac{n!}{t^n} \prod_{i=2}^n \left(t_i - t_i^0\right) \\
&= \frac{\partial^n}{\partial t_3 \cdots \partial t_n} \frac{n!}{t^n} \prod_{i=3}^n \left(t_i - t_i^0\right) = \cdots = \frac{n!}{t^n}
\end{aligned}$$

■

Example 4.1Expectation of travelers' waiting times Suppose that travelers arrive with a Poisson process with rate λ. If the train departs at time t, compute the expected sum of waiting times of travelers $E[\sum_{i=1}^{N(t)}(t-S_i)]$.

Solution

$$
\begin{aligned}
E\left[\sum_{i=1}^{N(t)}(t-S_i) \mid N(t)=n\right] &= E\left[\sum_{i=1}^{n}(t-S_i) \mid N(t)=n\right] \\
&= nt - E\left[\sum_{i=1}^{n} S_i \mid N(t)=n\right]
\end{aligned}
$$

$$
\begin{aligned}
E\left[\sum_{i=1}^{n} S_i \mid N(t)=n\right] &= E\left[\sum_{i=1}^{n} U_{(i)}\right] \quad \textit{by Theorem 4.1} \\
&= E\left[\sum_{i=1}^{n} U_i\right] \\
&= \frac{nt}{2}
\end{aligned}
$$

$$
E\left[\sum_{i=1}^{N(t)}(t-S_i) \mid N(t)=n\right] = \frac{nt}{2}
$$

$$
\begin{aligned}
E\left[\sum_{i=1}^{N(t)}(t-S_i)\right] &= E\left[E\left[\sum_{i=1}^{N(t)}(t-S_i) \mid N(t)\right]\right] \\
&= \sum_{n=0}^{\infty} E\left[\sum_{i=1}^{N(t)}(t-S_i)\, |N(t)=n|P\{N(t)=n\}\right. \\
&= \sum_{n=0}^{\infty} \frac{nt}{2} P\{N(t)=n\} = \frac{t}{2}E[N(t)] = \frac{\lambda t^2}{2}
\end{aligned}
$$

Alternatively, we have

$$
E\left[\sum_{i=1}^{N(t)}(t-S_i) \mid N(t)=n\right] = \frac{nt}{2} \rightarrow E\left[\sum_{i=1}^{N(t)}(t-S_i) \mid N(t)\right] = \frac{N(t)t}{2}
$$

$$
E\left[\sum_{i=1}^{N(t)}(t-S_i)\right] = E\left[E\left[\sum_{i=1}^{N(t)}(t-S_i) \mid N(t)\right]\right] = E\left[\frac{N(t)t}{2}\right] = \frac{\lambda t^2}{2}
$$

Example 4.2Distribution of S_n Let E denote the event that exactly n questions by time 1, given the event E, what is the pdf of S_n?

Solution *Conditioning on E, S_n has the same distribution as* $\max\{U_1, \ldots, U_n\}$*, where* $U_1, \ldots, U_n$ *are iid uniform distribution random variables in* $[0,1]$.

$$
P(S_n \le y \mid E) = \prod_{i=1}^{n} P(U_i \le y) = y^n
$$

5 Split or Merge

Theorem 5.1 (Split a Poisson Process)

Suppose that each event of a Poisson process with rate λ is classified as being either a type-I or type-II event. And the event occurs at time s will be classified as type-I with probability $P(s)$ and type-II with probability $1 - P(s)$.

If $N_i(t)$ represents the number of type-i events that occur by time $t, i = 1, 2$, then $N_1(t)$ and $N_2(t)$ are independent Poisson random variables having respective means λtp and $\lambda t(1-p)$, where

$$p = \frac{1}{t}\int_0^t P(s)ds$$

Proof

$$\begin{aligned}
&P\{N_1(t) = n, N_2(t) = m\} \\
=&\sum_{k=0}^{\infty} P\{N_1(t) = n, N_2(t) = m \mid N(t) = k\} P\{N(t) = k\} \\
=&P\{N_1(t) = n, N_2(t) = m \mid N(t) = n+m\} P\{N(t) = n+m\}
\end{aligned}$$

Consider an event occurs at time s, the probability that it would be a type-I event would be $P(s)$. By theorem 4.1 this event will have occured uniformly distributed on $(0, t)$. It follows that the probability that it would be a type-I event is p independently of the other events.

$$p = \frac{1}{t}\int_0^t P(s)ds$$

Thus we can see $P\{N_1(t) = n, N_2(t) = m \mid N(t) = n+m\}$ as the probability of n success and m failures in $n + m$ independent trials.

$$\begin{aligned}
&P\{N_1(t) = n, N_2(t) = m\} \\
=&P\{N_1(t) = n, N_2(t) = m \mid N(t) = n+m\} P\{N(t) = n+m\} \\
=&\frac{(n+m)!}{n!m!} p^n (1-p)^m \cdot e^{-\lambda t} \frac{(\lambda t)^{n+m}}{(n+m)!} \\
=&e^{-\lambda pt} \frac{(\lambda pt)^n}{n!} \cdot e^{-\lambda(1-p)t} \frac{(\lambda(1-p)t)^m}{m!}
\end{aligned}$$

$$\begin{aligned}
P\{N_1(t) = n\} &= \sum_m P\{N_1(t) = n, N_2(t) = m\} \\
&= \left(e^{-\lambda pt}\frac{(\lambda pt)^n}{n!}\right) \sum_m \left(e^{-\lambda(1-p)t}\frac{(\lambda(1-p)t)^m}{m!}\right) \\
&= e^{-\lambda pt}\frac{(\lambda pt)^n}{n!}
\end{aligned}$$

Similarly, we show that $N_1(t)$ is Poisson with mean λpt, $N_2(t)$ is Poisson with mean $\lambda(1-p)t$, and $N_1(t)$, $N_2(t)$ are independent. ■

Theorem 5.2 (Merger)

Merging of independent Poisson processes is Poisson.

Proof

■

6 Compound Poisson Process

Definition 6.1 (Compound Poisson Random variable)

Let $X_1, X_2, \ldots$ be a sequence of iid random variables having distribution F, and suppose that this sequence is independent of N, a Poisson random variable with mean λ. The random variable

$$W = \sum_{i=1}^{N} X_i$$

is said to be a compound Poisson random variable with Poisson parameter λ and component distribution F.

Definition 6.2 (Compound Poisson Process)

A stochastic process $\{X(t), t \geq 0\}$ is said to be a compound Poisson process if it can be represented, for $t \geq 0$, by

$$X(t) = \sum_{i=1}^{N(t)} X_i$$

where $\{N(t), t \geq 0\}$ is a Poisson process, and $\{X_i, i = 1, 2, \ldots\}$ is a family of iid random variables that is independent of the process $\{N(t), t \geq 0\}$. Thus, if $\{X(t), t \geq 0\}$ is a compound Poisson process then $X(t)$ is a compound Poisson random variable.

Lemma 6.1

SP_HW1 *Suppose for a Poisson process with rate λ, an event occurring at time s contributes a random amount having distribution $F_s, s \geq 0$. Let W denote the sum of the contributions up to time t, i.e., $W = \sum_{i=1}^{N(t)} X_i$. Then W is a compound Poisson random variable, with the same distribution as $\sum_{i=1}^{N(t)} \tilde{X}_i$, where $\tilde{X}_i$ is independent of $N(t)$ and are iid with $F(x) = \frac{1}{t}\int_0^t F_s(x)ds$.*

7 Conditional Poisson Process

Definition 7.1 (Conditional Poisson process)

Let Λ be a positive random variable having distribution G and let $\{N(t), t \geq 0\}$ be a counting process such that, given that $\Lambda = \lambda$, $\{N(t), t \geq 0\}$ is a Poisson process having rate λ. The process $\{N(t), t \geq 0\}$ is then called a conditional Poisson process.

Remark Note that a conditional Poisson process still possess stationary increment, but do not possess independent increment.

Lemma 7.1 (Property of Conditional Poisson process)

$$\begin{aligned} P\{N(t+s) - N(s) = n\} &= E[P\{N(t+s) - N(s) = n \mid \Lambda\}] \\ &= \int_0^\infty P\{N(t+s) - N(s) = n \mid \Lambda = \lambda\} dG(\lambda) \\ &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} dG(\lambda) \end{aligned}$$

The conditional distribution of Λ can be calculated by

$$\begin{aligned} P\{\Lambda \leq x, N(t) = n\} &= E[P\{\Lambda \leq x, N(t) = n \mid \wedge\}] \\ &= \int_{\lambda=0}^\infty P\{\Lambda \leq x, N(t) = n \mid \Lambda = \lambda\} dG(\lambda) \\ &= \int_{\lambda=0}^x P\{N(t) = n \mid \Lambda = \lambda\} dG(\lambda) \\ &= \int_{\lambda=0}^x e^{-\lambda t} (\lambda t)^n / n! dG(\lambda) \end{aligned}$$

$$\begin{aligned} P\{\Lambda \leq x \mid N(t) = n\} &= \frac{P\{\Lambda \leq x, N(t) = n\}}{P\{N(t) = n\}} = \frac{\int_{\lambda=0}^x e^{-\lambda t} (\lambda t)^n / n! dG(\lambda)}{\int_{\lambda=0}^\infty e^{-\lambda t} (\lambda t)^n / n! dG(\lambda)} \\ &= \frac{\int_{\lambda=0}^x e^{-\lambda t} (\lambda t)^n dG(\lambda)}{\int_{\lambda=0}^\infty e^{-\lambda t} (\lambda t)^n dG(\lambda)} \end{aligned}$$