Newton’s Method of Fixed Point Iteration Extended to the Complex Plane

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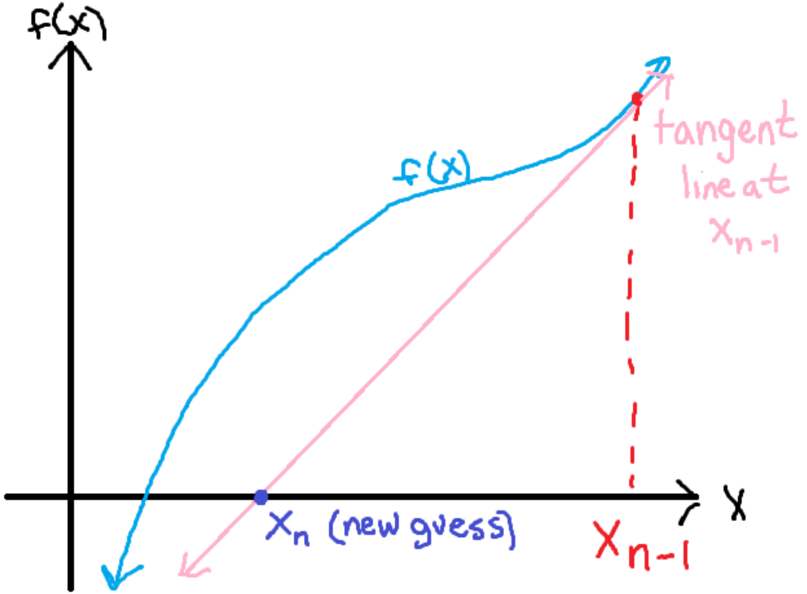
Refresher: How Newton's Method Works

- Use linear approximation of a function at a point to approximate root
- Use that approximation as new starting point; repeat
- Requirements: f(x) is continuous and twice differentiable; f'(x)!=0 on the interval chosen



Extension of Newton’s Method to the Complex Plane

- Complex numbers: a + bi
- Can be represented on a plane, where a is the x coordinate of a number and b is its y coordinate
- If Newton’s Method can be applied to some function f(x) on the real line, we can just start plugging in complex numbers and get f(z) in the complex plane

Basins in Newton’s Method

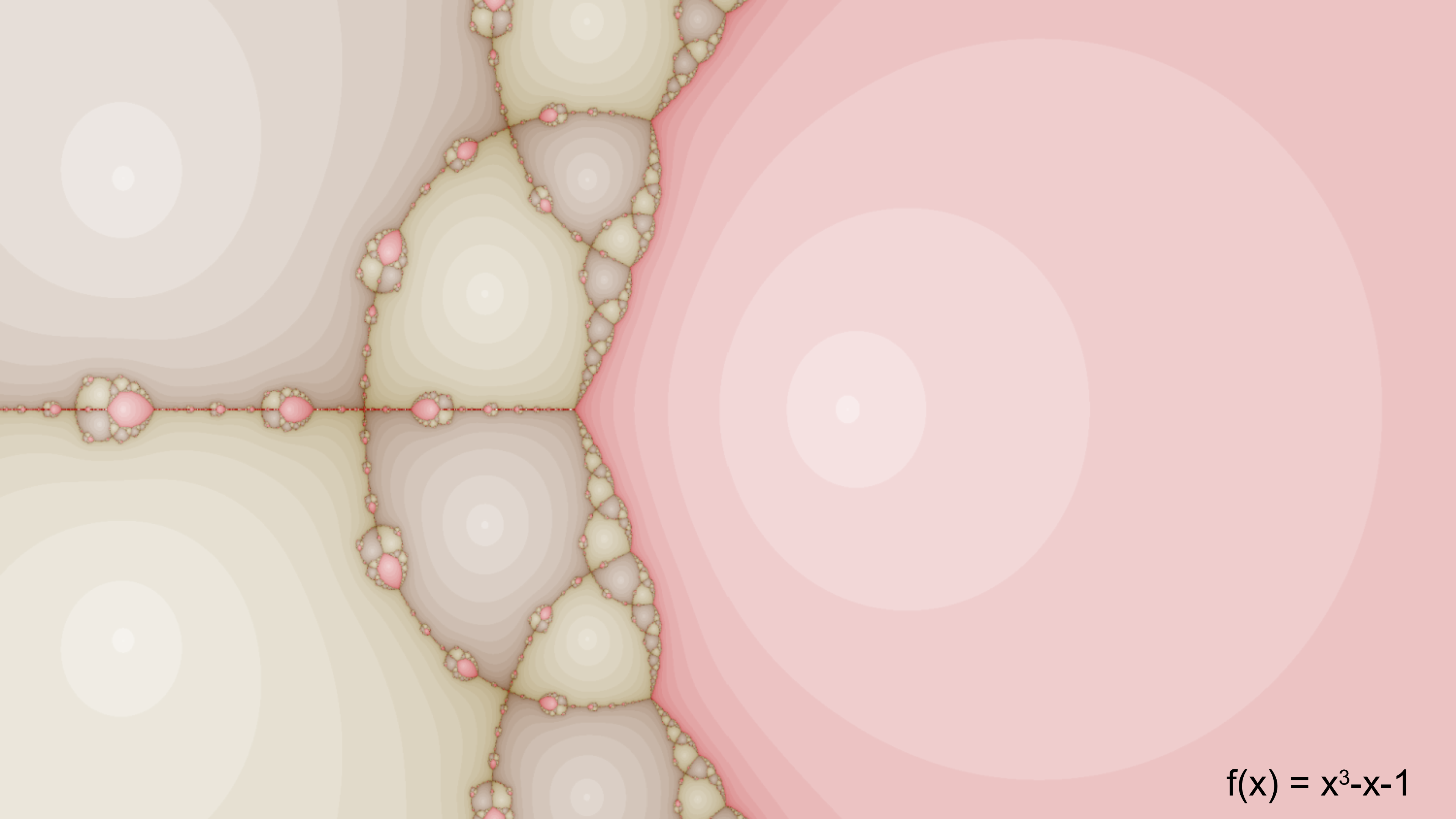
- For a function with more than one root, which root Newton’s Method converges to depends on the starting position
- The “basin” of a root is the set of initial conditions that converge to the root
- For some functions (such as functions with only one root), the basins are easy to figure out
- Other functions, like most cubic polynomials, are more complicated

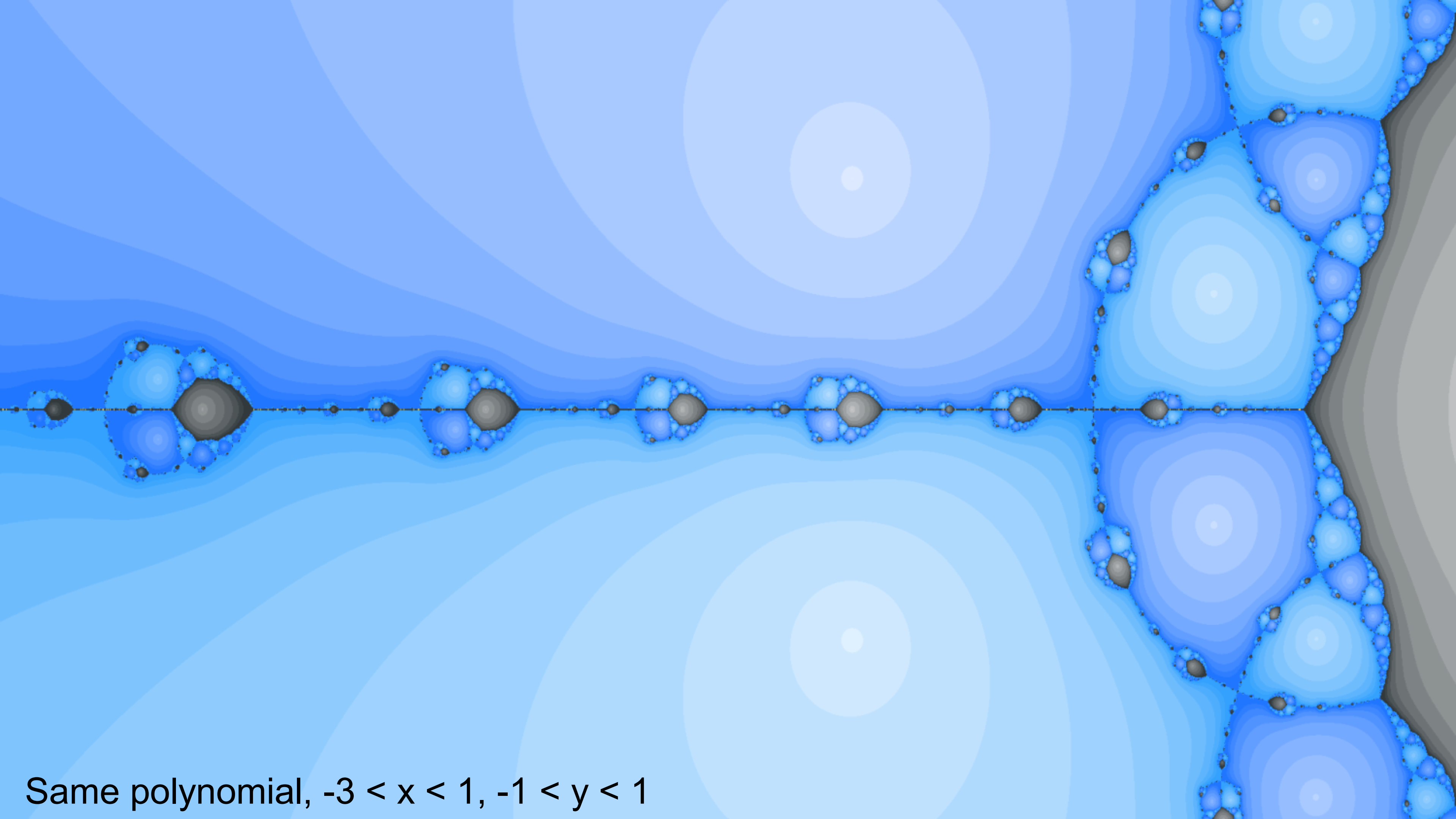
Basins in the Complex Plane

- Instead of being intervals on the real line, basins of attraction are now areas of the complex plane
- First investigated by Arthur Cayley in 1879, who figured out a rule governing the division of the basins for quadratic polynomials in the complex plane
- But he couldn't figure out the rule for cubic polynomials!

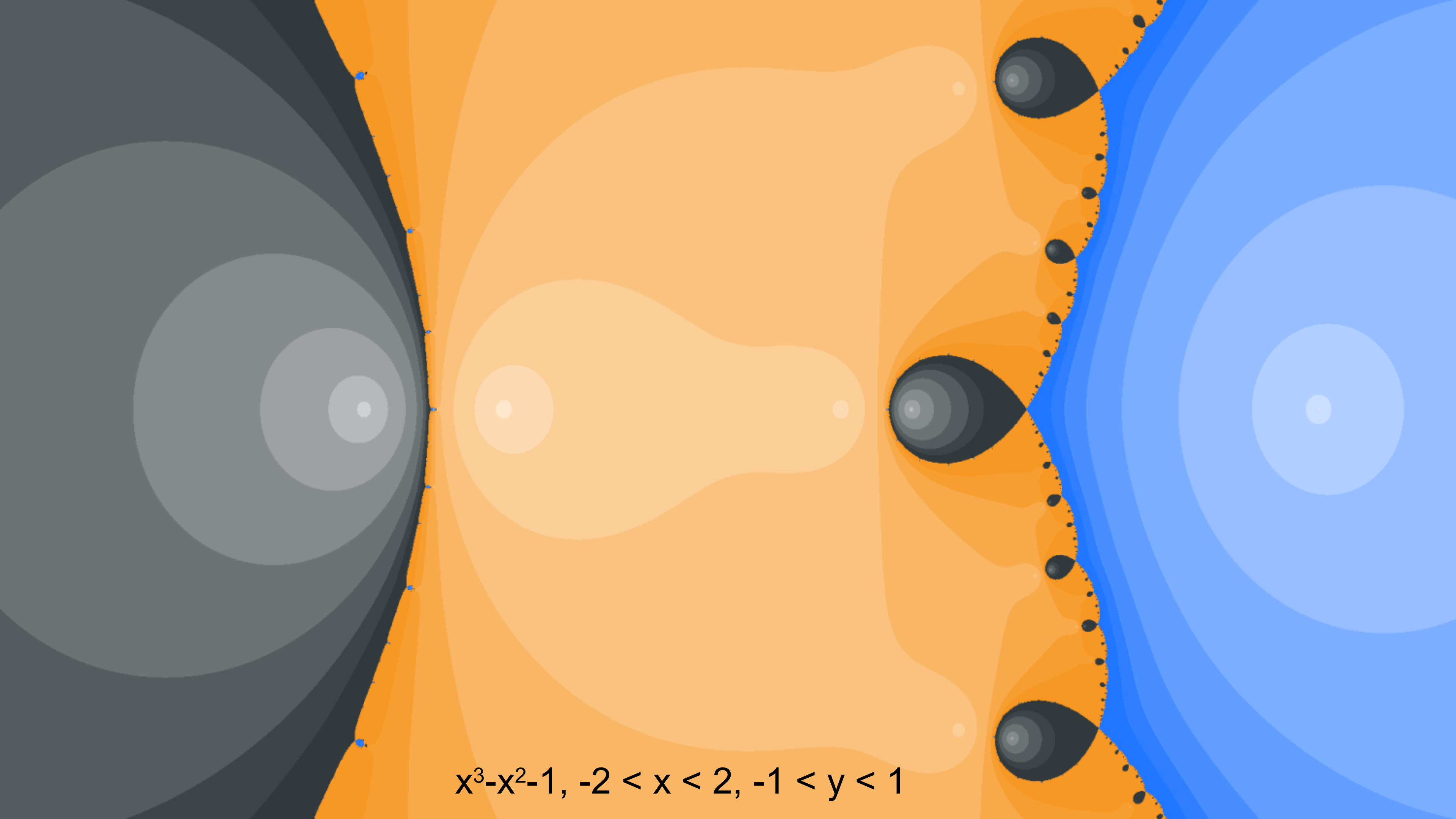
Code

- Complex object class: arithmetic operators, absolute value, a few trigonometric operators, and a toString that outputs “a + bi”
- Newton’s method class: static method that’s practically the same as the original, but with Complex objects substituted for doubles
- Application class that makes a picture
- (And a user-friendly, polynomials-only version, and Halley’s Method)


$f(x) = x^3-x-1$

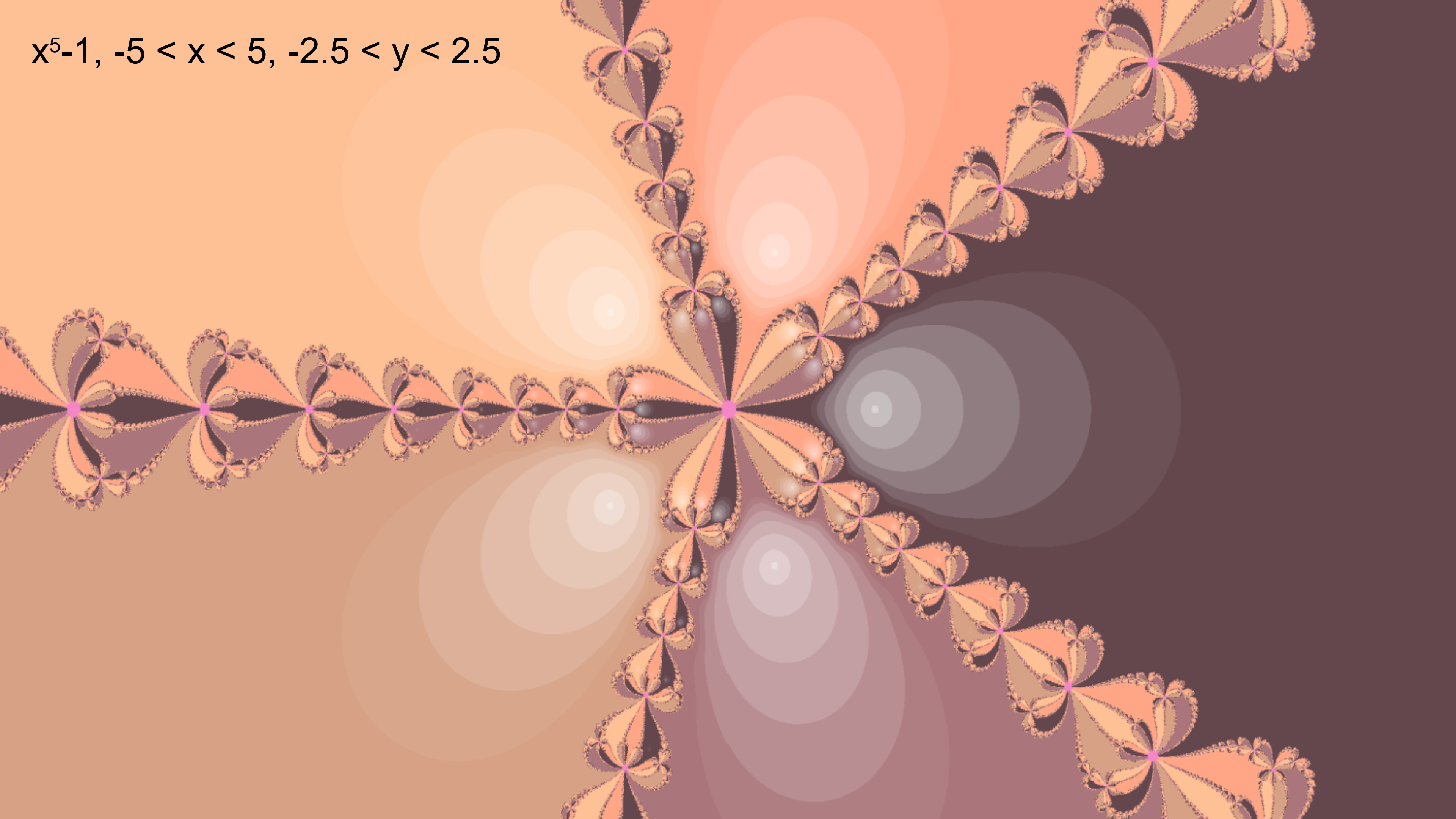


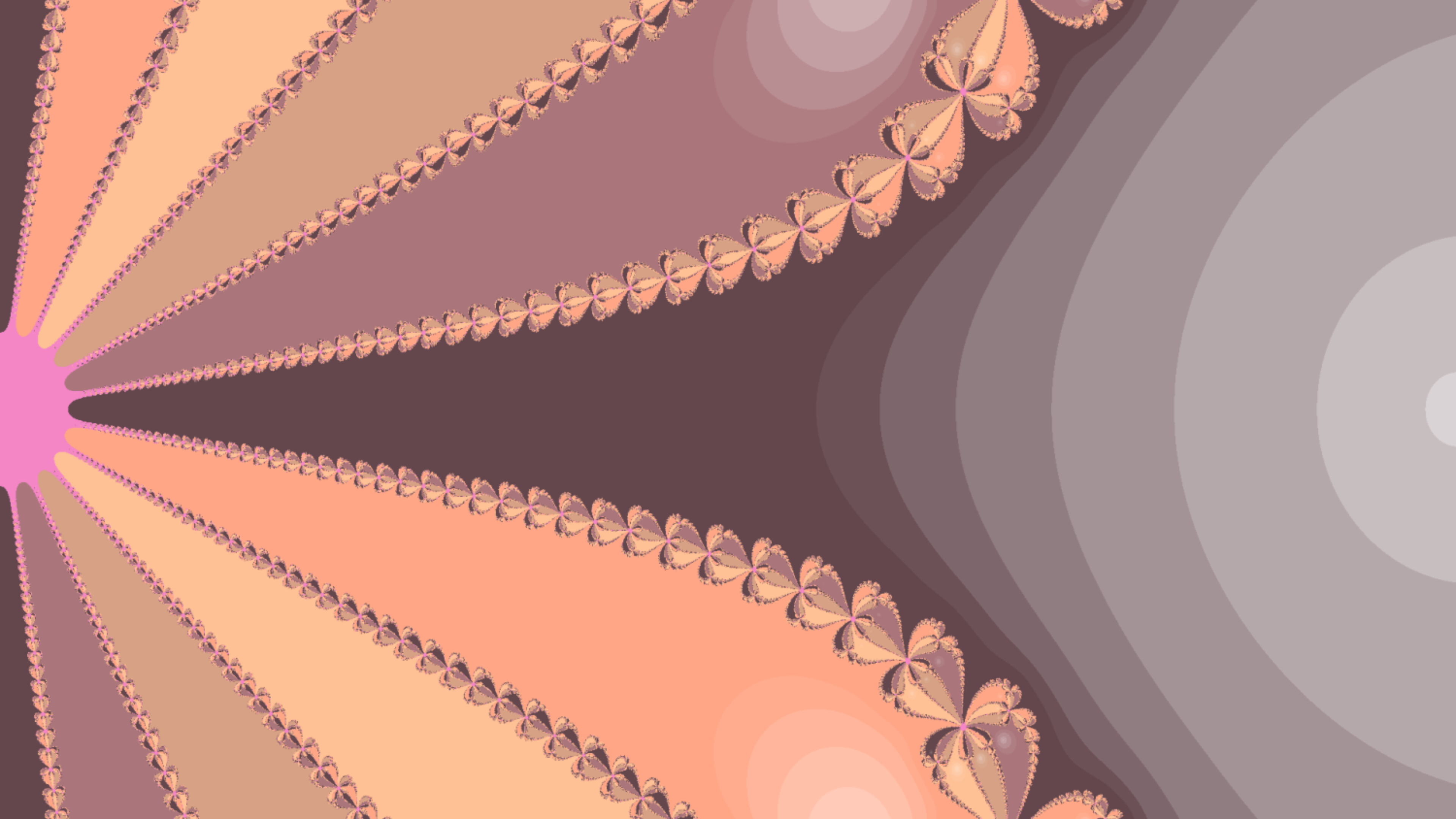
Same polynomial, -3 < x < 1, -1 < y < 1

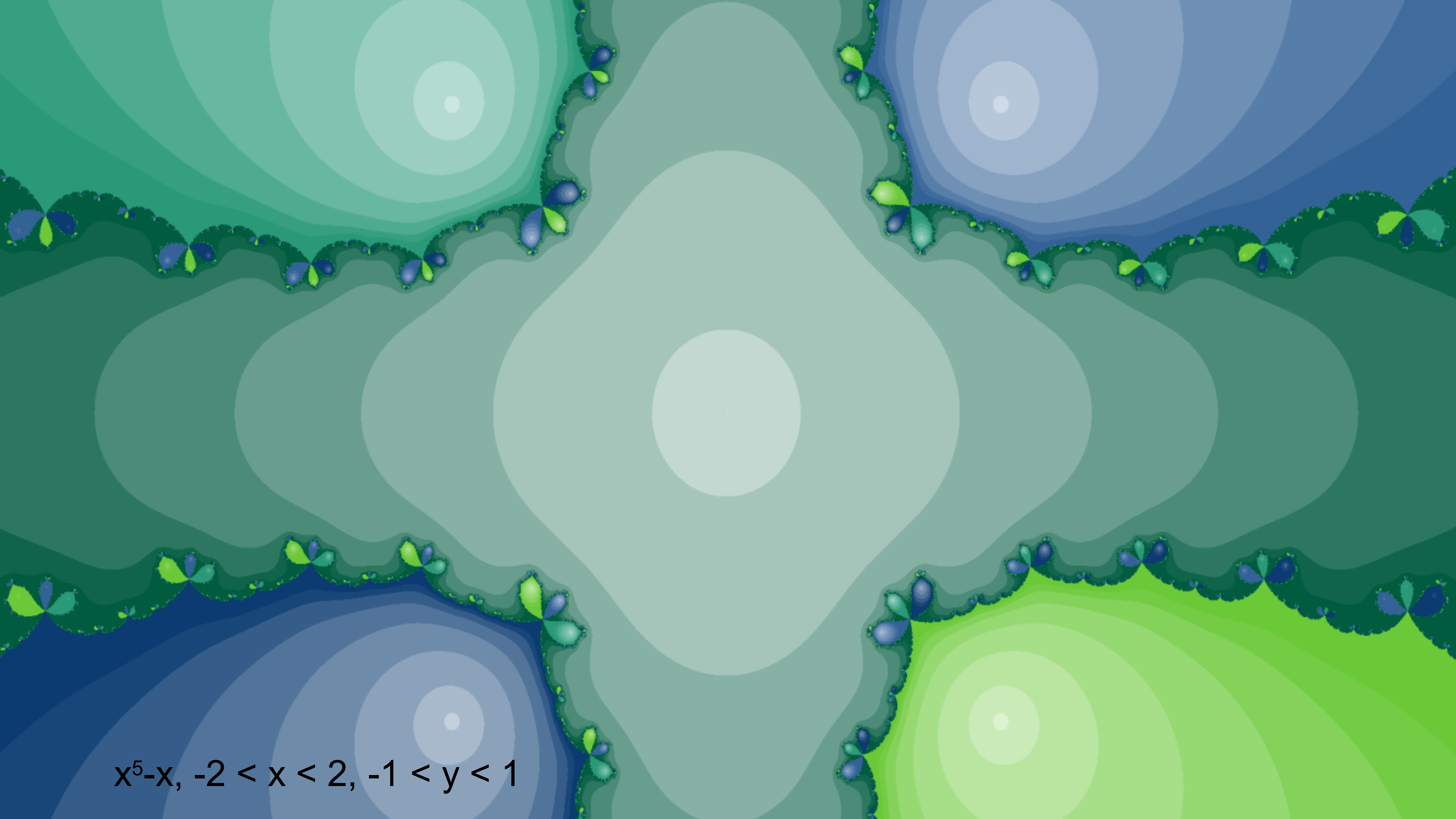


$x^3 - x^2 - 1$, $-2 < x < 2$, $-1 < y < 1$

$x^5-1, -5 < x < 5, -2.5 < y < 2.5$

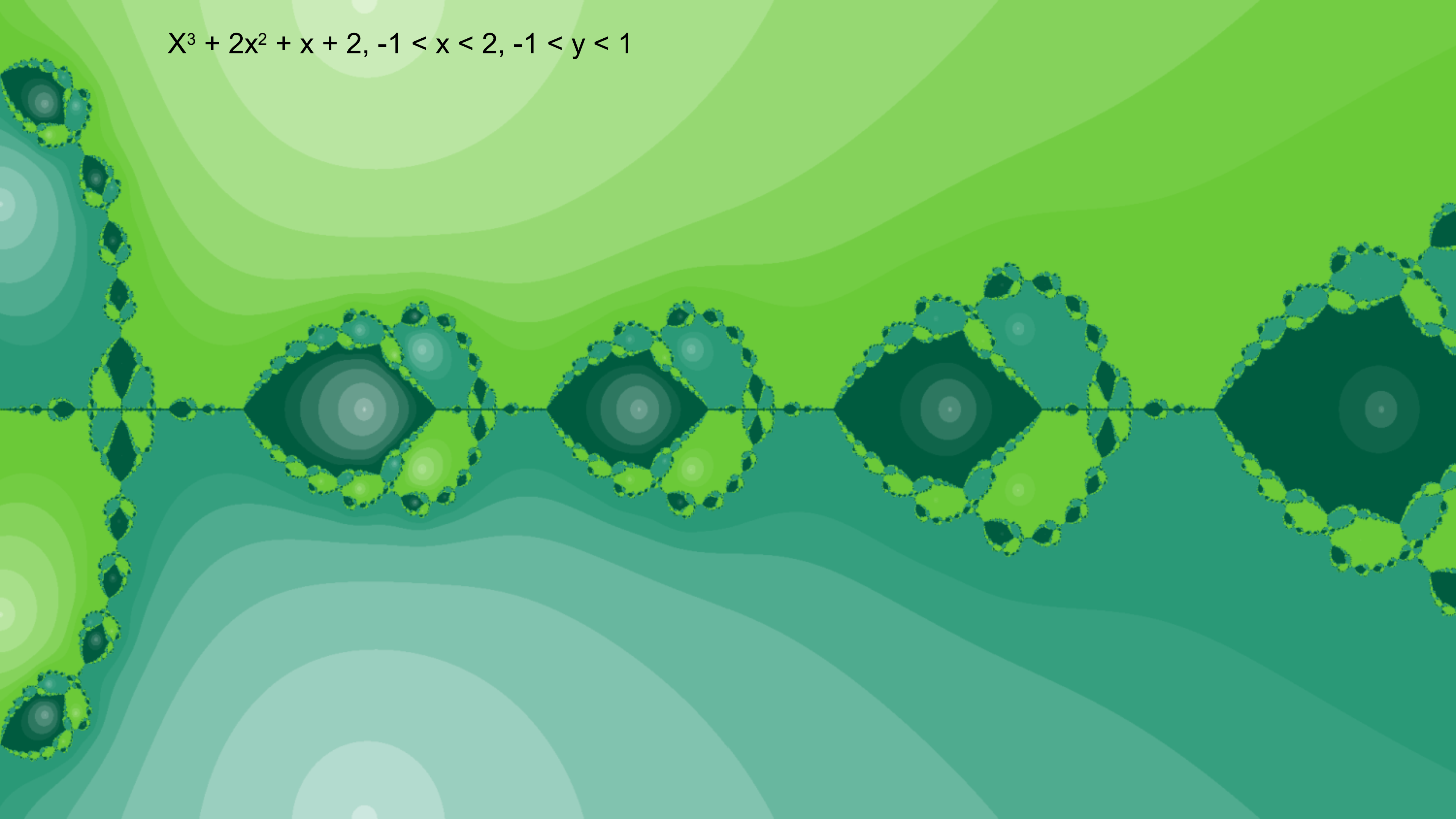






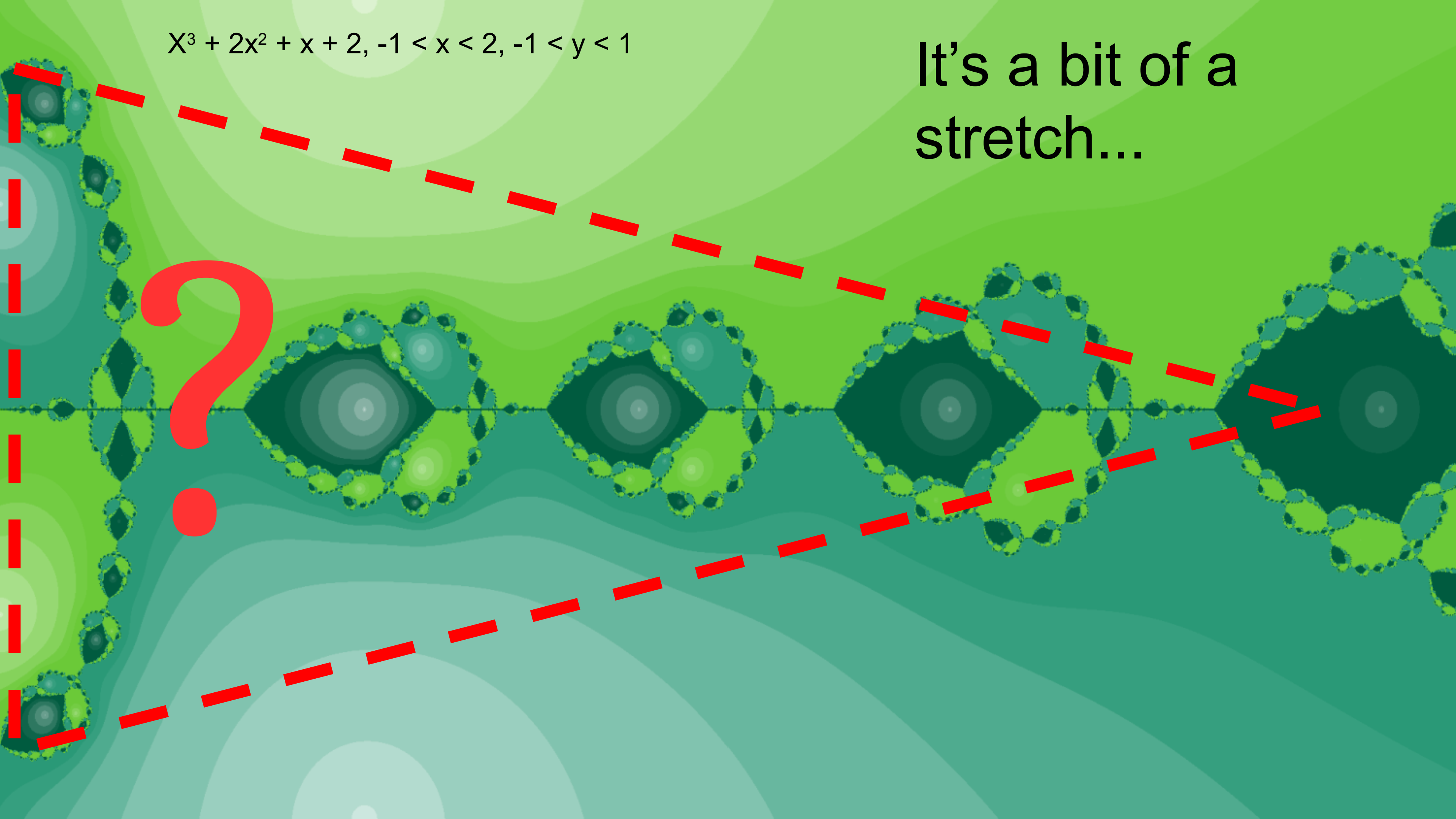
$x^5 - x$, $-2 < x < 2$, $-1 < y < 1$

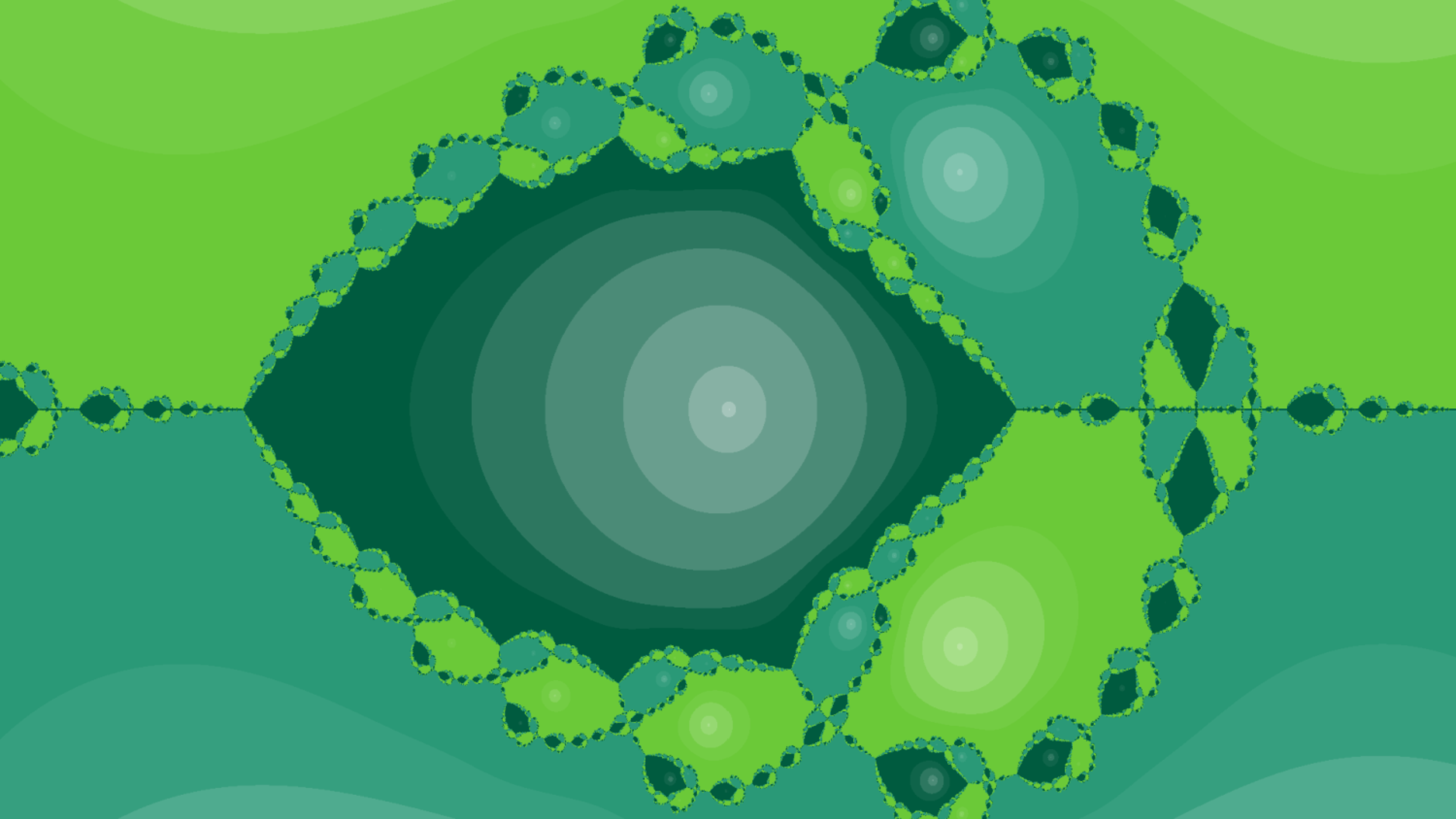
$X^3 + 2x^2 + x + 2$, $-1 < x < 2$, $-1 < y < 1$



$X^3 + 2x^2 + x + 2, -1 < x < 2, -1 < y < 1$

It’s a bit of a stretch...

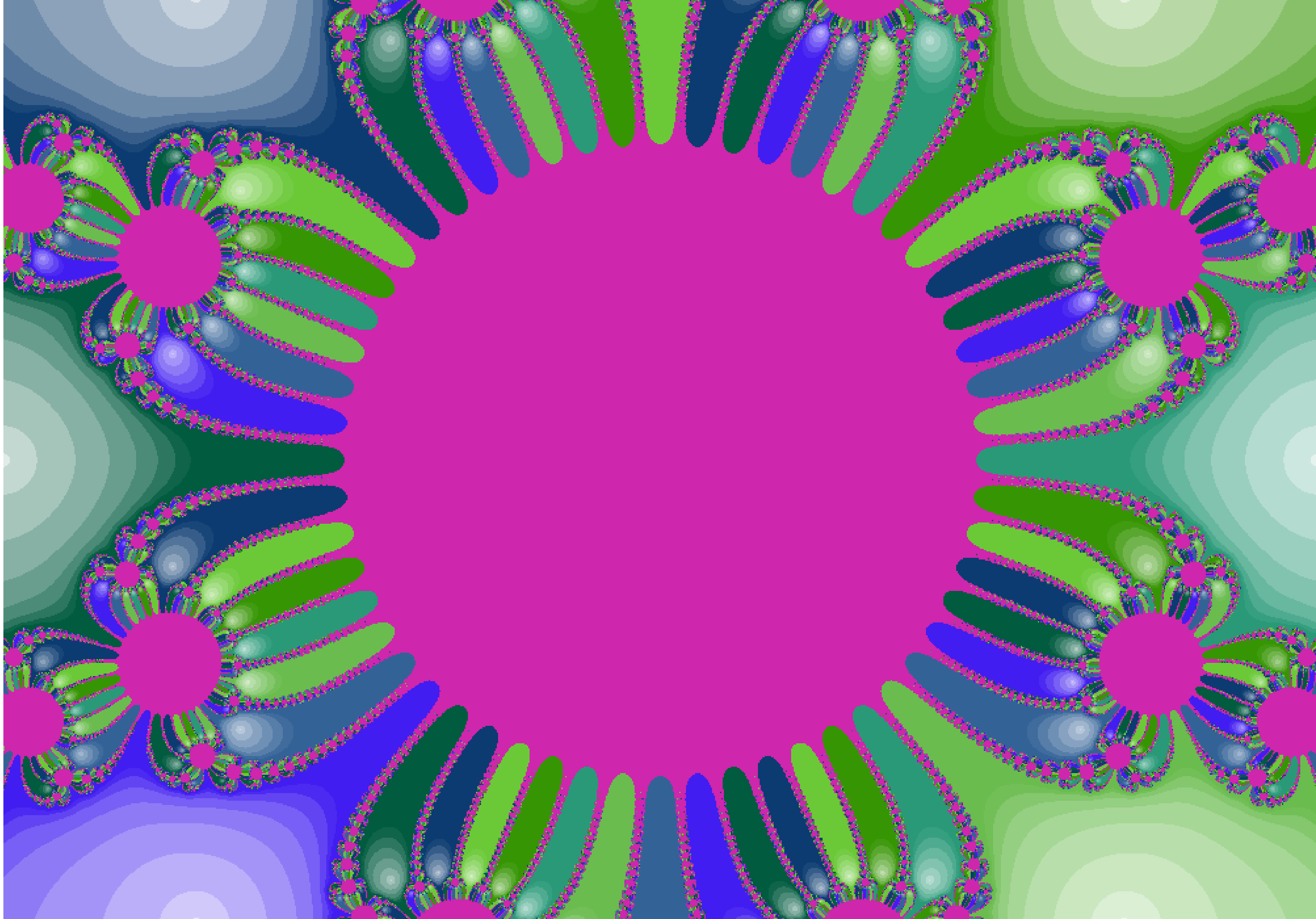


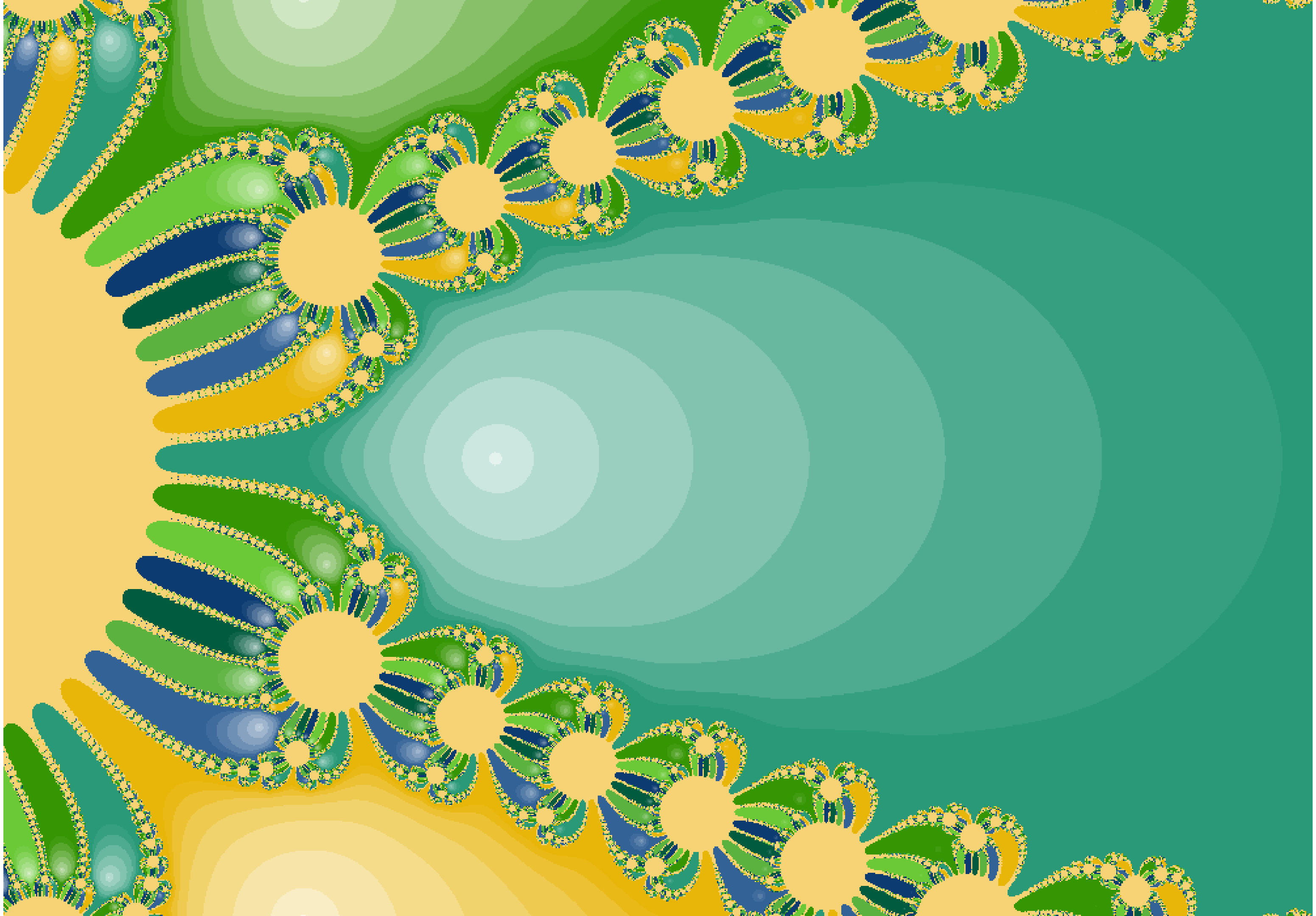


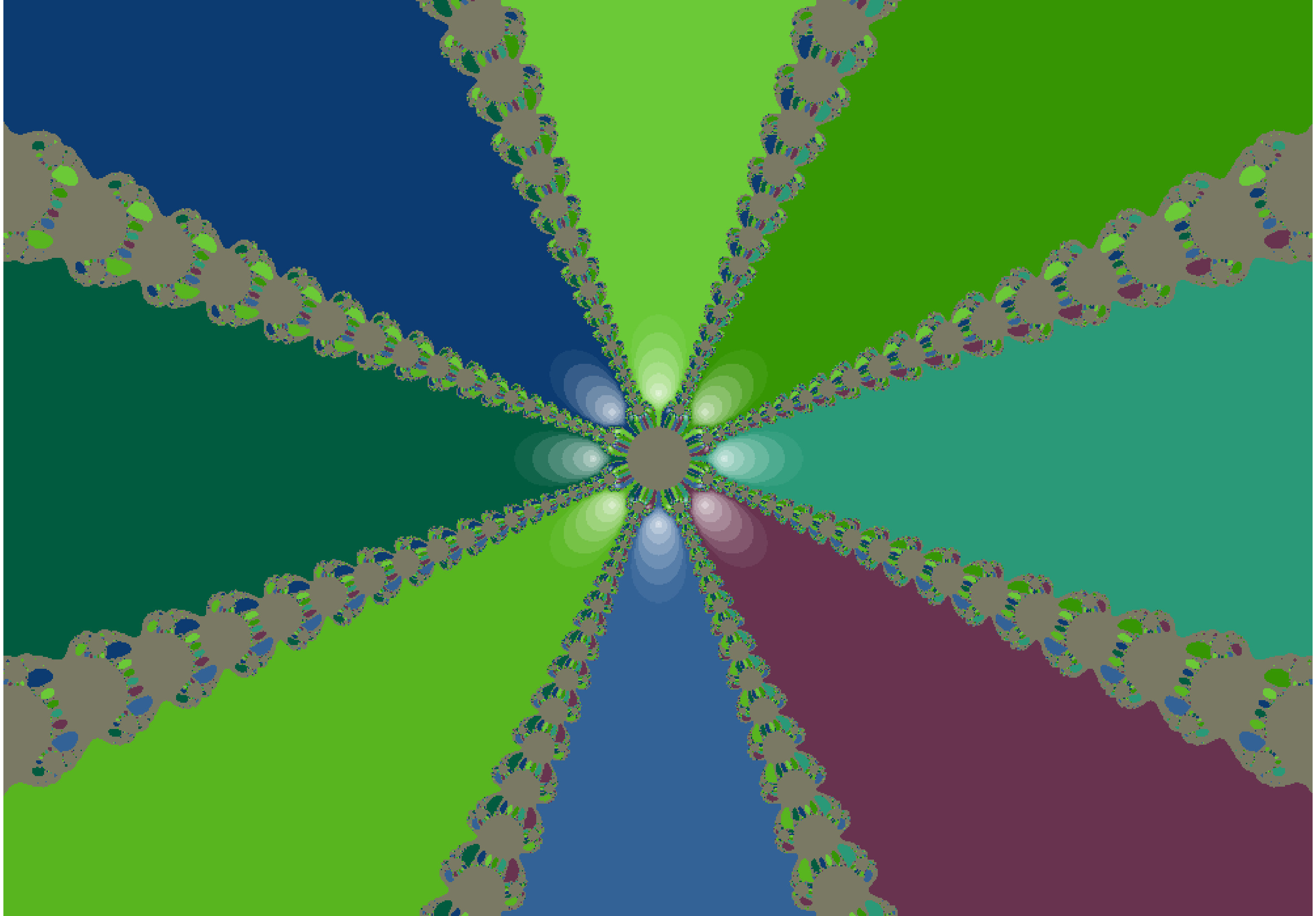
x^8-1

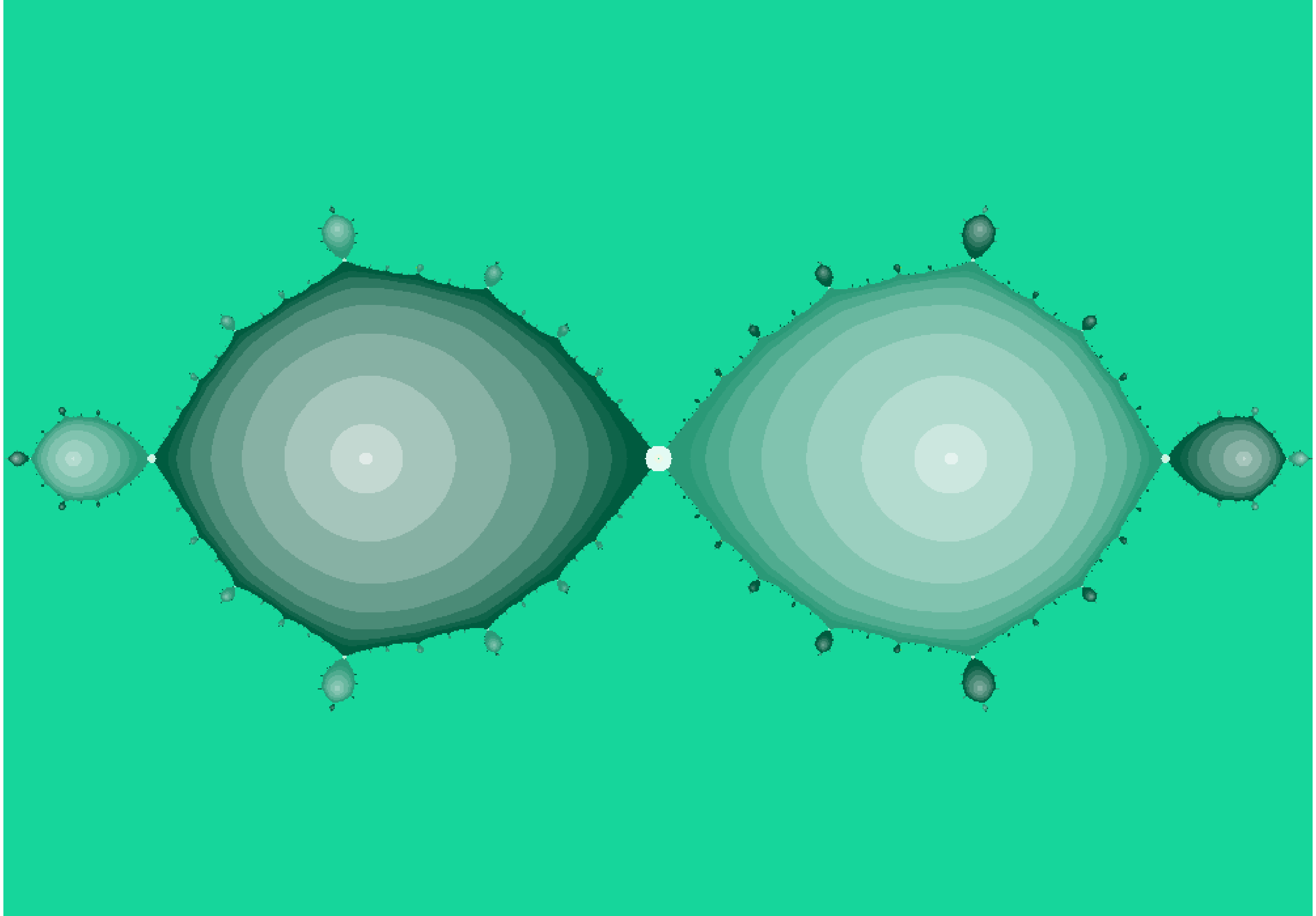
$-1 < x < 1$

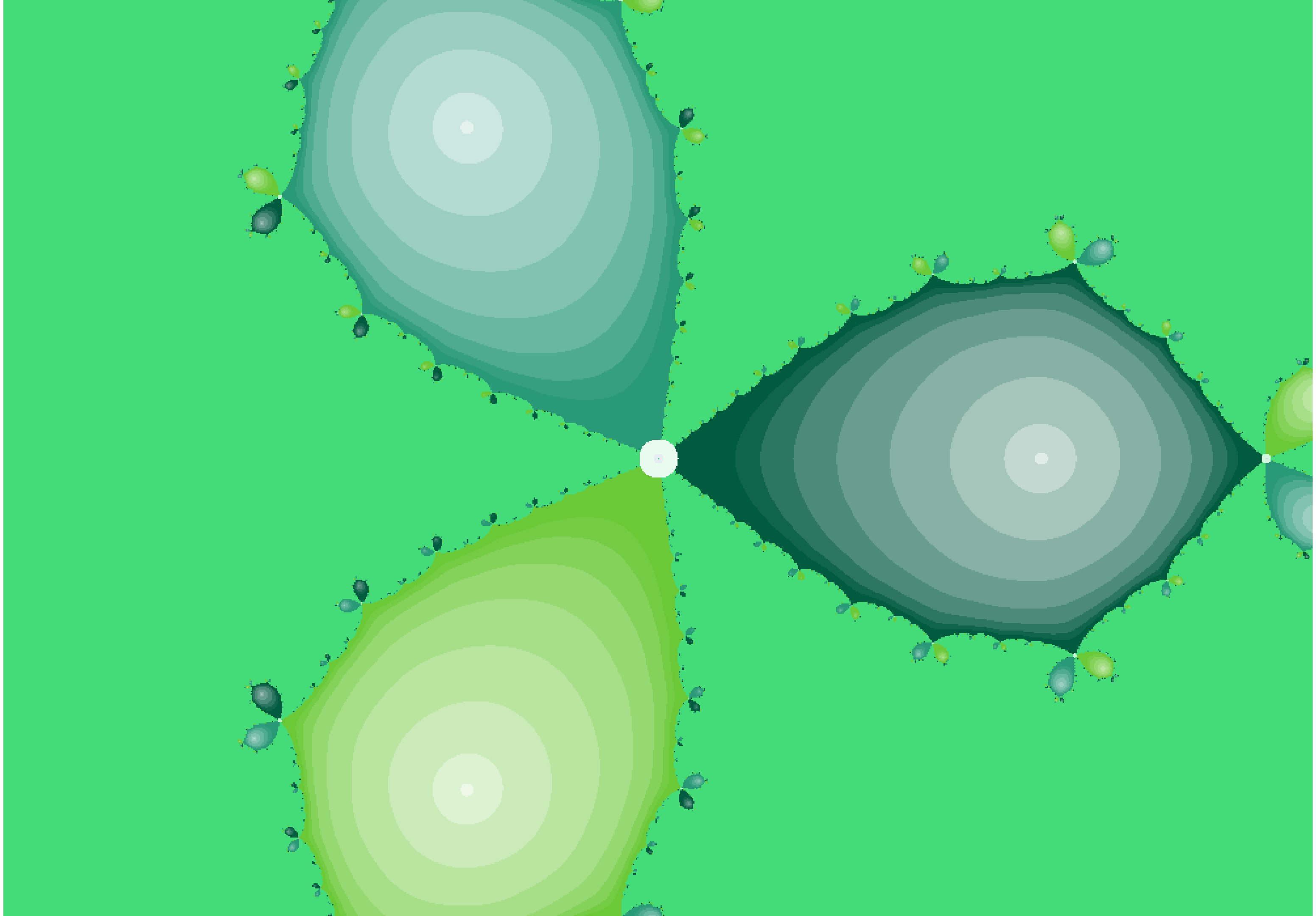
$-0.7 < y < 0.7$

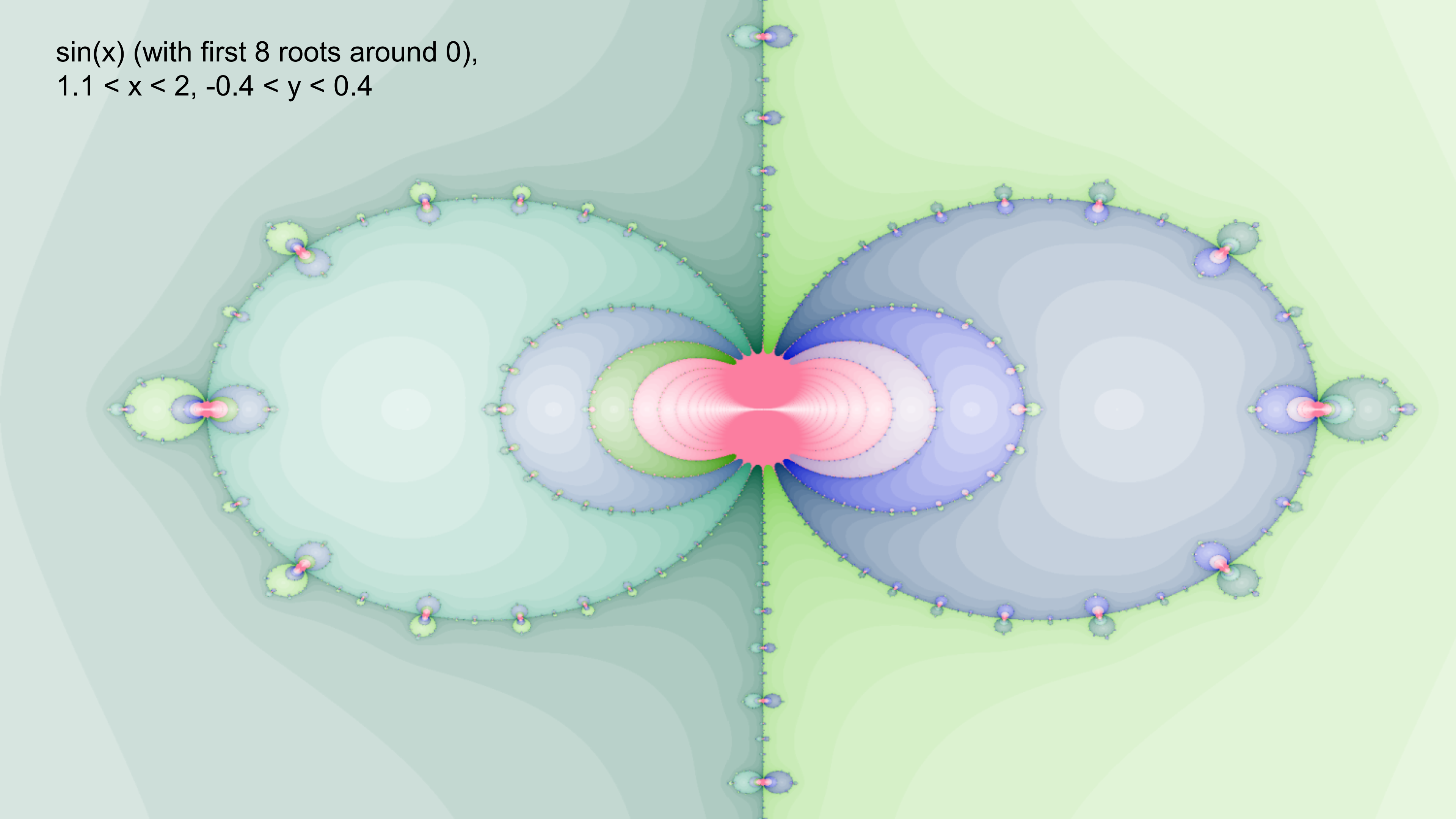











sin(x) (with first 8 roots around 0),
1.1 < x < 2, -0.4 < y < 0.4

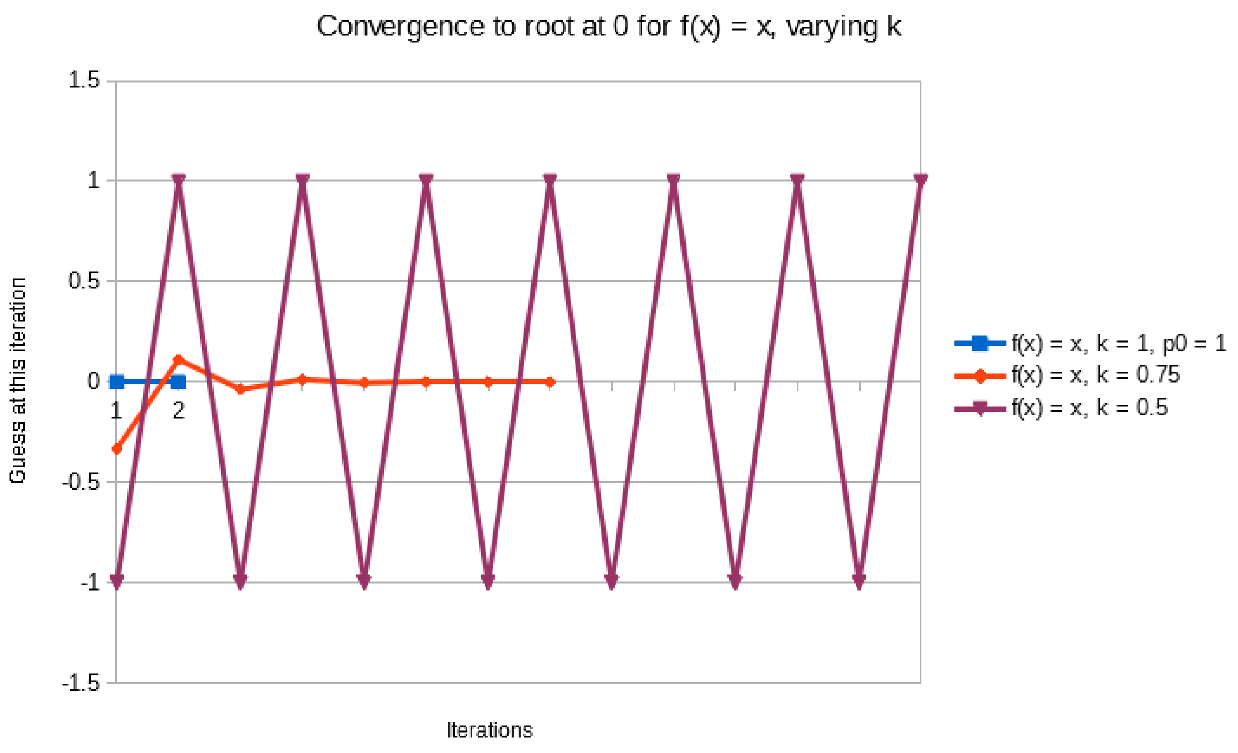
Let’s make this more interesting...

Part of the function that “causes” the fractal patterns: places near $f'(x) = 0$

How can we maximize these places?

What about taking the square root of the whole function?

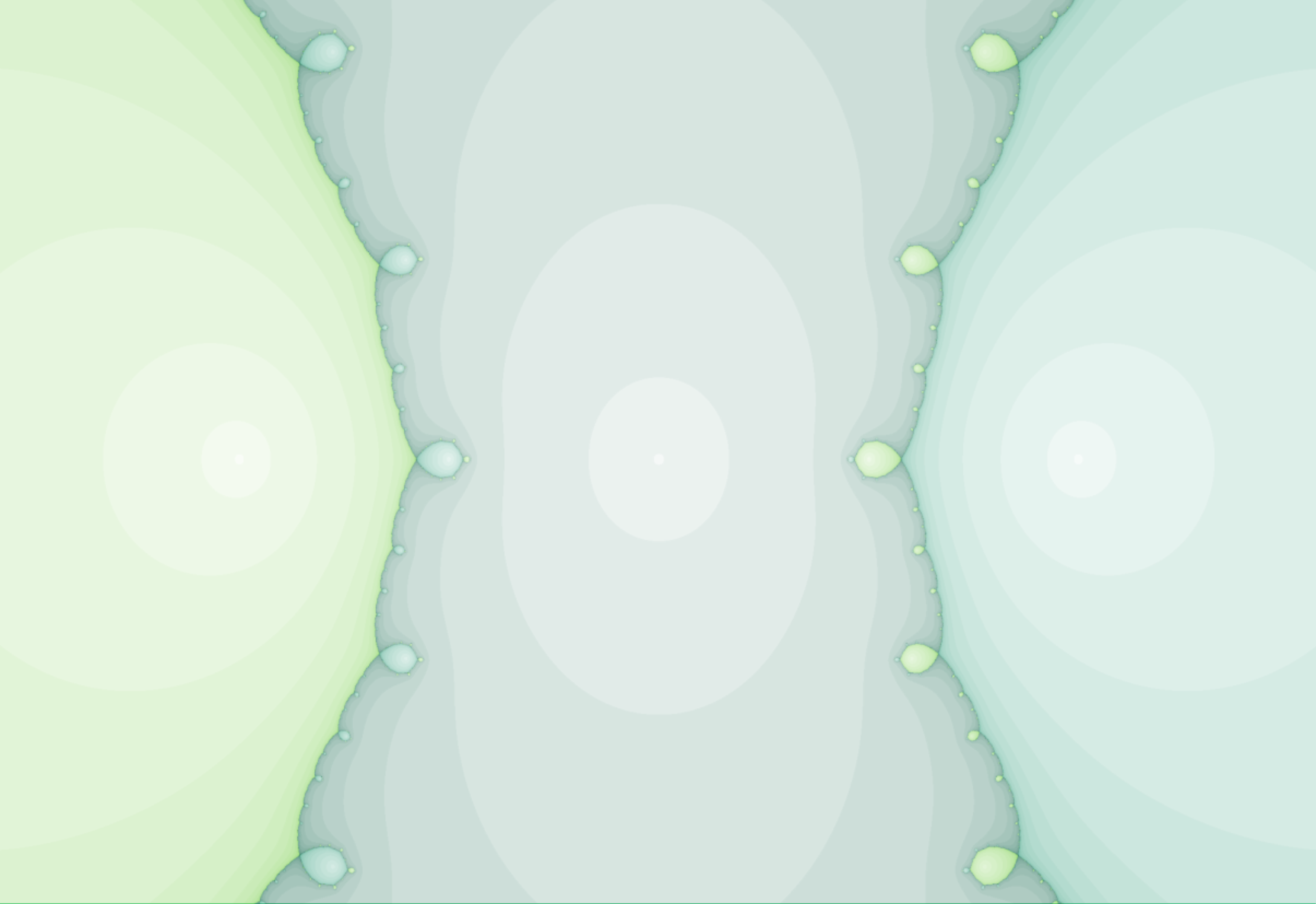
What this looks like for f(x) = x



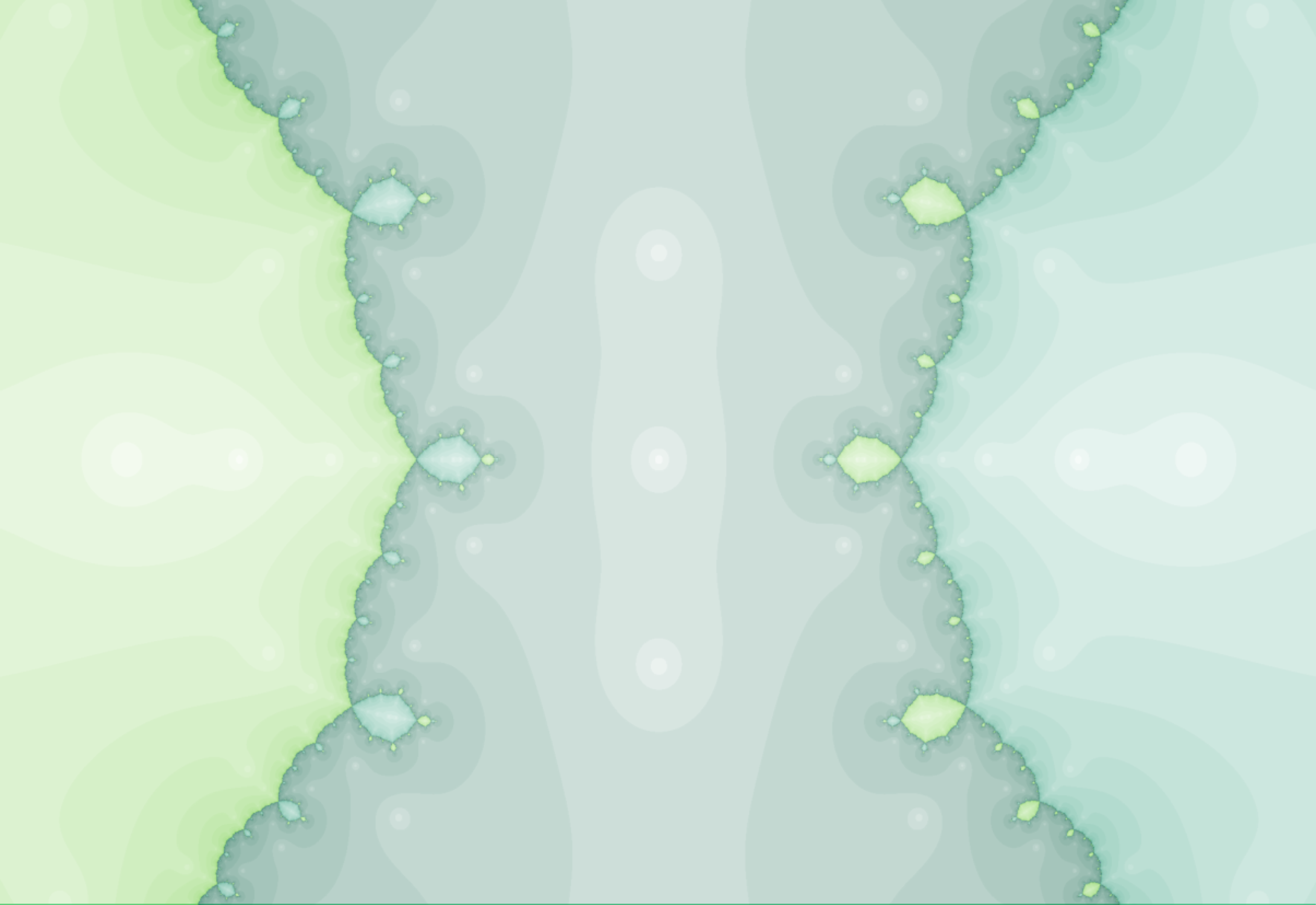
k = 1,

$f(x) = x^3 - x$,
$-\pi/2 < x < \pi/2$,
$-1 < y < 1$

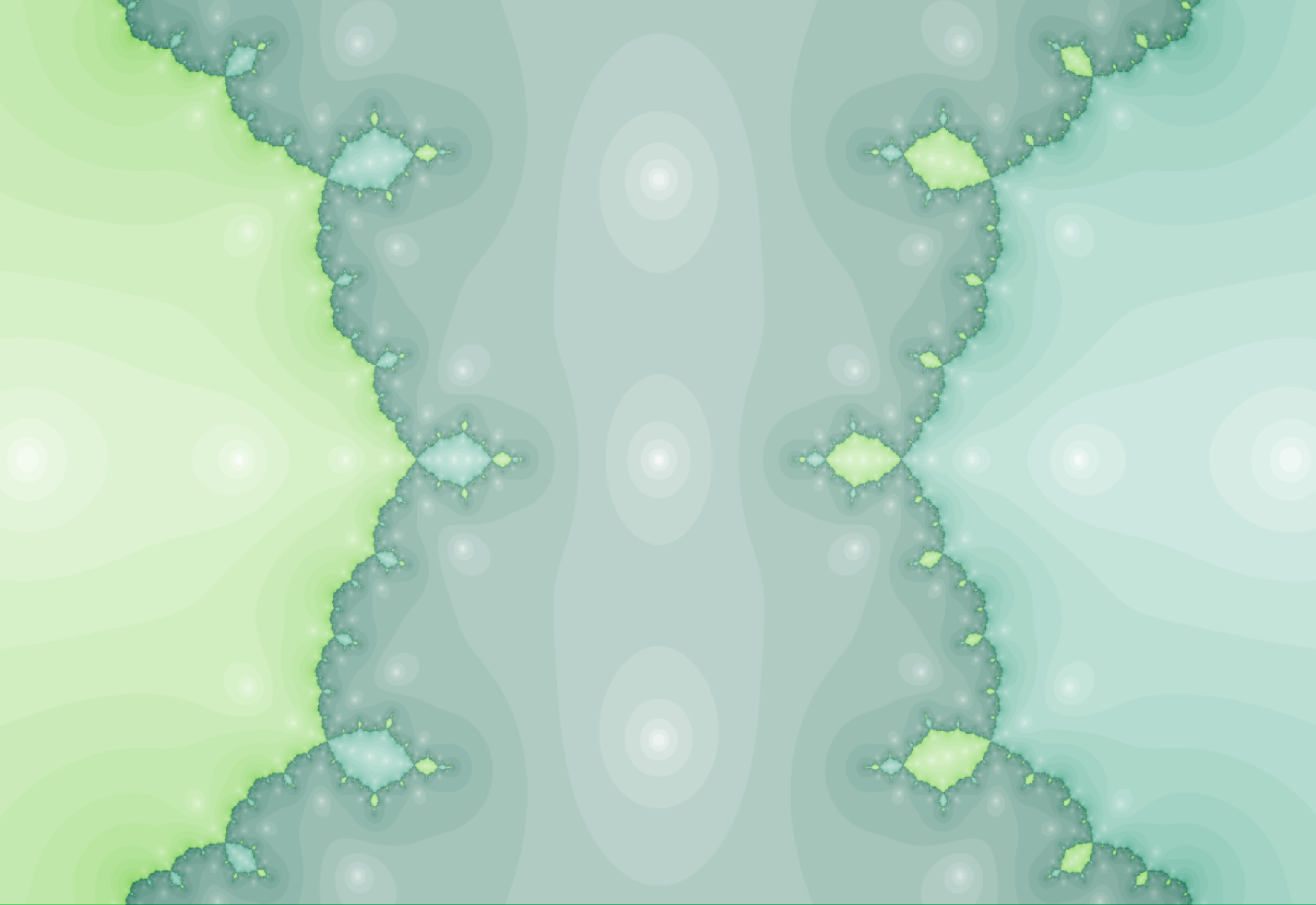
(Yes, I forgot to reset my bounds after playing with a trigonometric function...)



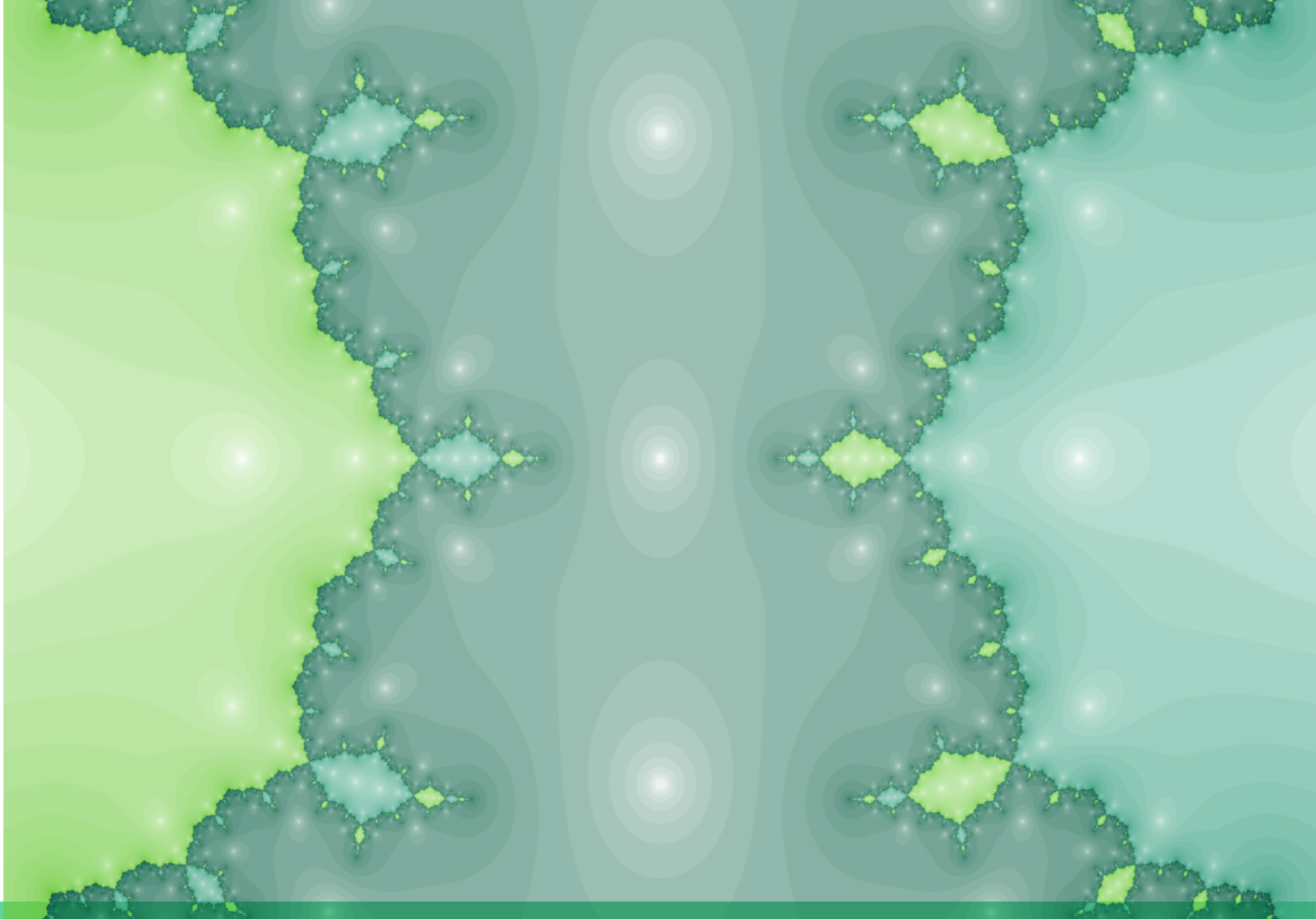
k = 0.75



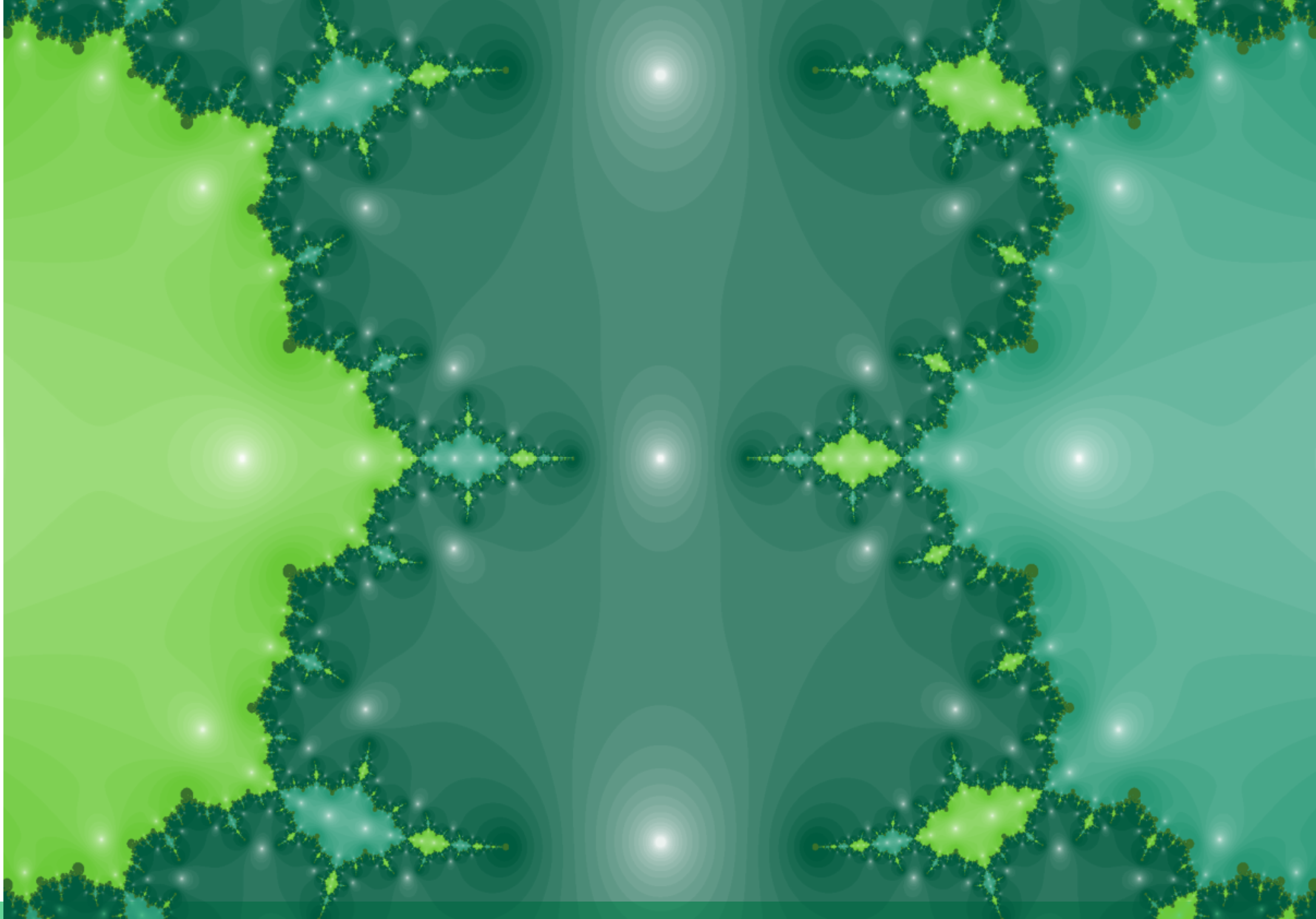
k = 0.65



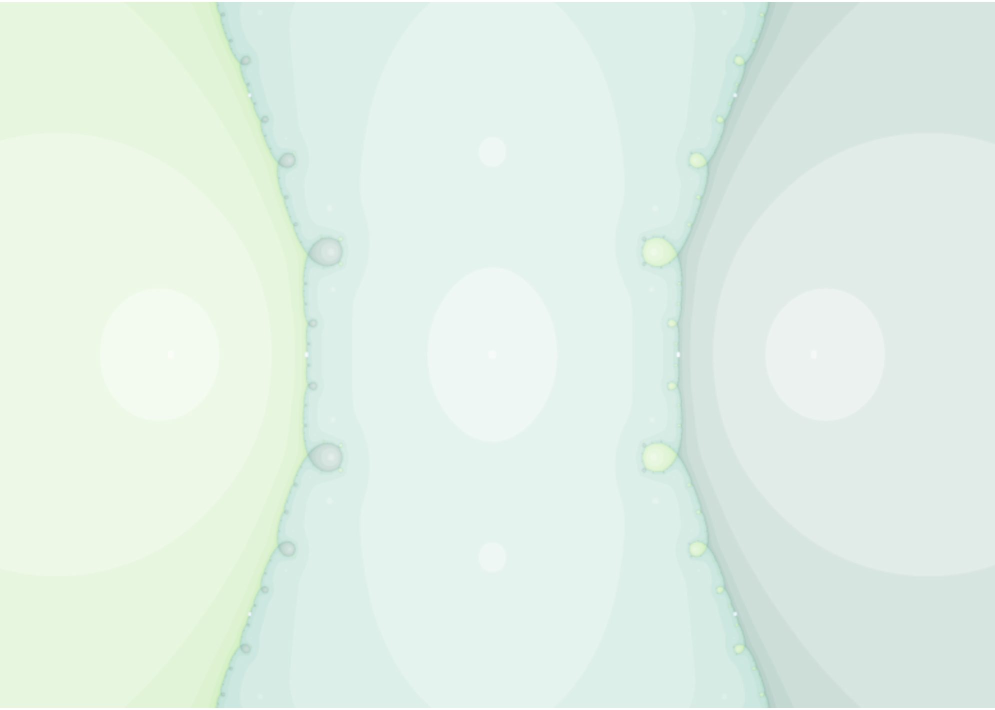
k = 0.6

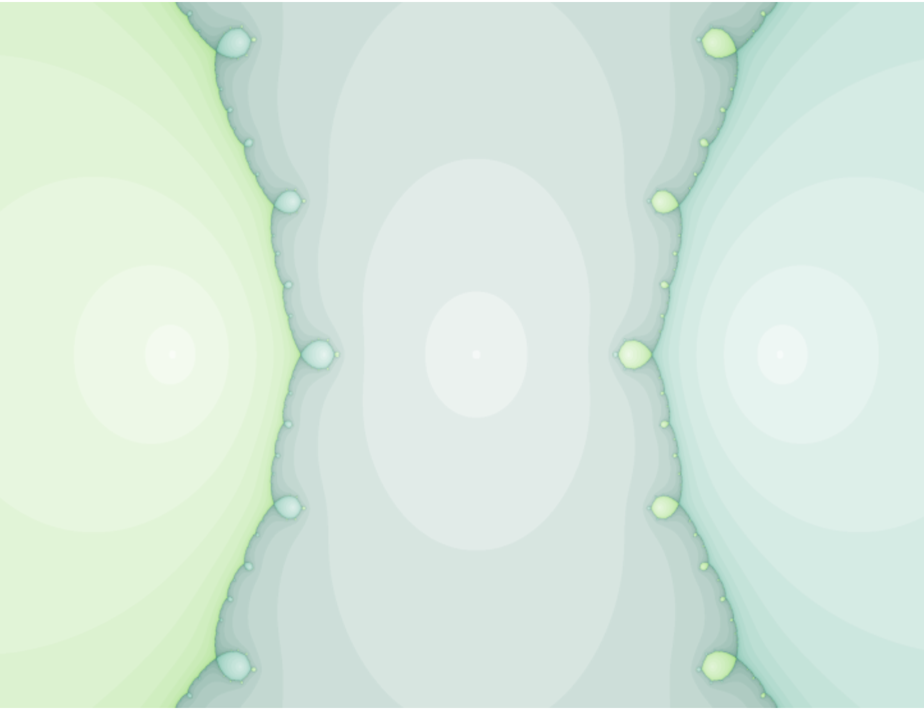


$k = 0.55$



Going in the other direction: Halley’s Method





Applications

- “Polynomiography” (fractal images generated by root-finding methods)
    - Math education
- Numerical Root-Finding
    - Orbital motion, area minimization, shooting projectiles, quantum mechanics
    - Chemistry
    - Computer algebra, algebraic optimization & algebraic geometry
- Fractal geometry
    - Flow of a viscous liquid through porous media
    - Economics: discontinuous, self-similar price variation in securities & commodities