MAT137Y1 - Calculus!
Test 2. — November 26th, 2014
Time: 110 minutes

**Please, fill in this information with ALL CAPITAL LETTERS:**

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Instructions: (READ CAREFULLY!)

- Do not write or draw anything on the QR code on the top corner of any page.
- This exam booklet contains 11 pages including this one. It consists of 8 questions. The maximum score is 30 points.
  - Questions 1–3 are short computational questions. For each question, write your final answer in the box provided, along with any justifications, explanations, or calculations you need underneath. In order to get any points, you need your final answer to be correct and you need a justification.
- If you need scratch paper, use the back of the pages. **We will only read and grade what you write on the front of each page.**
- **If you need extra space for a question, you may use Page 11 for this purpose.** If you do so, clearly indicate it on the corresponding problem page.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be in a bag underneath your chair.
- Do not turn over this page until the invigilators instruct you to do so. Good luck!

1. *[4 points]*

(a) Given the function $P(x) = x^{10} - 2x^2 + 5$, calculate $P''(0)$ and $P^{(10)}(7)$.

| Your answers: $P''(0) = -4$ | $P^{(10)}(7) = 10!$ |
|---|---|

The first two derivatives are computed to be $P'(x) = 10x^9 - 4x$ and $P''(x) = 90x^8 - 4$, for which $P(0) = -4$ follows immediately. For $P^{(10)}$ we notice that any monomial whose degree is less than 10 will not survive the differentiation process, and that that $10^{\text{th}}$ derivative of $x^{10}$ is simply $10!$, so $P^{(10)}(x) = 10!$ is a constant function.

(b) Given the function $f(x) = \dfrac{1-x^2}{3+x^2}$, compute $f'(2)$.

| Your answer: $f'(2) = -\dfrac{16}{49}$ |
|---|

Using the quotient rule, we get

$$f'(x) = \frac{(-2x)(3+x^2) - (2x)(1-x^2)}{(3+x^2)^2} = \frac{-6x - 2x^3 - 2x + 2x^3}{(3+x^2)^2} = \frac{-8x}{(3+x^2)^2}.$$

Substituting $x = 2$ we get $f'(2) = -\dfrac{16}{49}$.

2. [2 points] Let $u$ and $v$ be positive, differentiable functions. We define a new function $w$ by

$$w(x) = u(x)^{u(x)} + v(x)^{v(x)}$$

Find a formula for $w'(x)$ in terms of $u(x)$, $u'(x)$, $v(x)$, and $v'(x)$.

**Your answer:** $w'(x) = u(x)^{u(x)}u'(x)\left[\log u(x) + 1\right] + v(x)^{v(x)}v'(x)\left[\log v(x) + 1\right]$

Since the derivative of the sum is the sum of the derivatives, it suffices to differentiate each term separately. We start by setting $y = u(x)^{u(x)}$ so that

$$\log(y) = \log\left[u(x)^{u(x)}\right] = u(x)\log u(x).$$

Differentiating implicitly yields

$$\frac{1}{y}\frac{dy}{dx} = u'(x)\log u(x) + \frac{u(x)}{u(x)}u'(x) = u'(x)\left[\log u(x) + 1\right].$$

Multiplying both sides by $y = u(x)^{u(x)}$ gives

$$\frac{dy}{dx} = u(x)^{u(x)}u'(x)\left[\log u(x) + 1\right].$$

The derivative of $v(x)^{v(x)}$ is similarly $v(x)^{v(x)}v'(x)\left[\log v(x) + 1\right]$. Adding these together gives the desired solution.

3. [4 points] Calculate the derivatives of the following functions.

(a) $g(x) = e^2 + xe^{\sin^2(x^3)}$

**Your answer:** $g'(x) = e^{\sin^2(x^3)}\left[1 + 6x^3\sin(x^3)\cos(x^3)\right]$

Set $\alpha(x) = \sin^2(x^3)$ and $\beta(x) = e^x$ so that $g(x) = e^2 + x\beta(\alpha(x))$. We can compute

$$\alpha'(x) = 2\sin(x^3)\left[\frac{d}{dx}\sin(x^3)\right] = 6x^2\sin(x^3)\cos(x^3), \quad \beta'(x) = \beta(x) = e^x,$$

and

$$g'(x) = \beta(\alpha(x)) + x\beta'(\alpha(x))\alpha'(x).$$

Putting this all together gives

$$g'(x) = e^{\sin^2(x^3)} + 6x^3\sin(x^3)\cos(x^3)e^{\sin^2(x^3)}.$$

(b) $h(x) = \cot x$.

**Your answer:** $h'(x) = -\csc^2(x)$

Writing $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$ and applying the quotient rule, we have

$$\begin{aligned}\frac{d}{dx}\cot(x) &= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}\\ &= -\frac{1}{\sin^2(x)}\\ &= -\csc^2(x)\end{aligned}$$

4. *[3 points]*

Use implicit differentiation to **obtain and prove** a formula for the derivative of $H(x) = \operatorname{arccot} x$. Simplify your answer as much as possible.

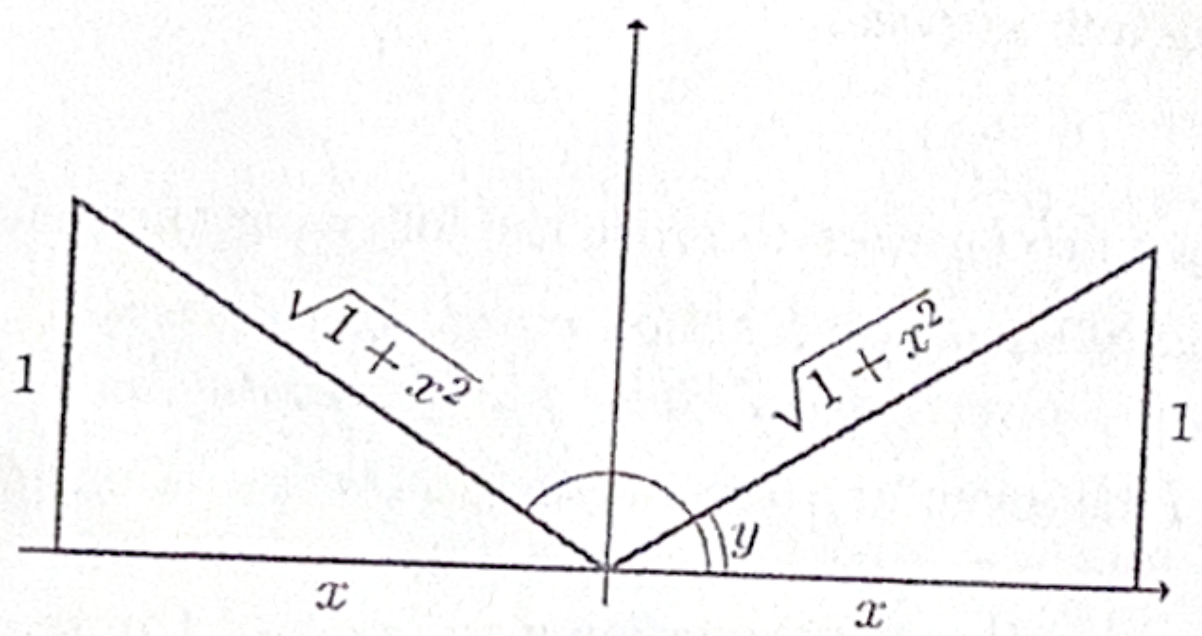
**Your answer:** $H'(x) = -\dfrac{1}{1+x^2}$

*Hint:* You probably want to use your answer to Question 3b.

We know that $\cot : (0, \pi) \to \mathbb{R}$ and so by setting $y = \operatorname{arccot}(x)$ we know that $x \in \mathbb{R}$ and $y \in (0, \pi)$. By definition of the inverse function, this means that $\cot(y) = x$. Differentiating implicitly we have

$$-\csc^2(y)\frac{dy}{dx} = 1 \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{1}{\csc^2(y)} = -\sin^2(y).$$

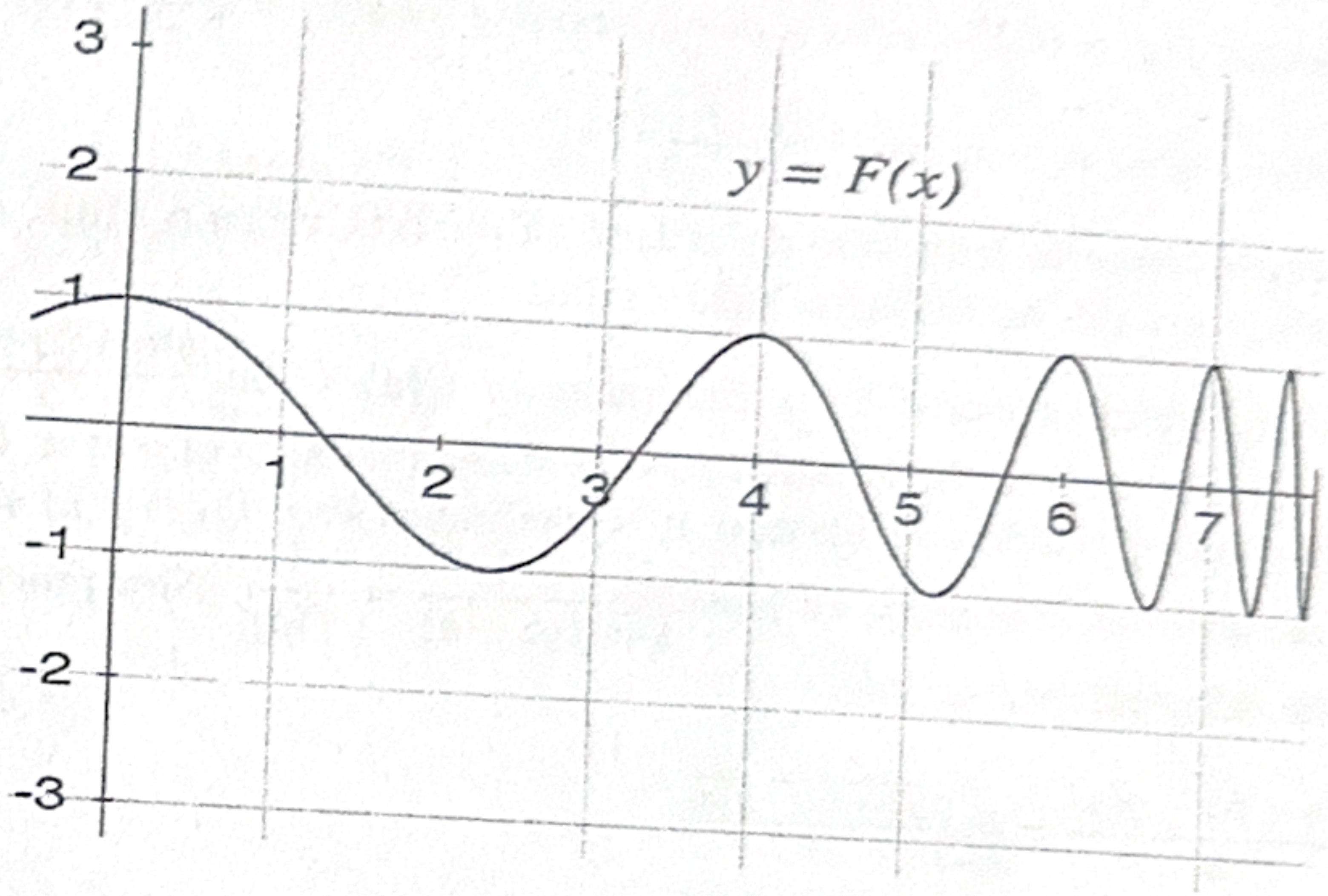
We need to determine the value of $\sin(y) = \sin(\operatorname{arccot}(x))$. Notice that since $y \in (0, \pi)$ this value will always be positive, and since $\cot(y) = x$ we have the following diagram:



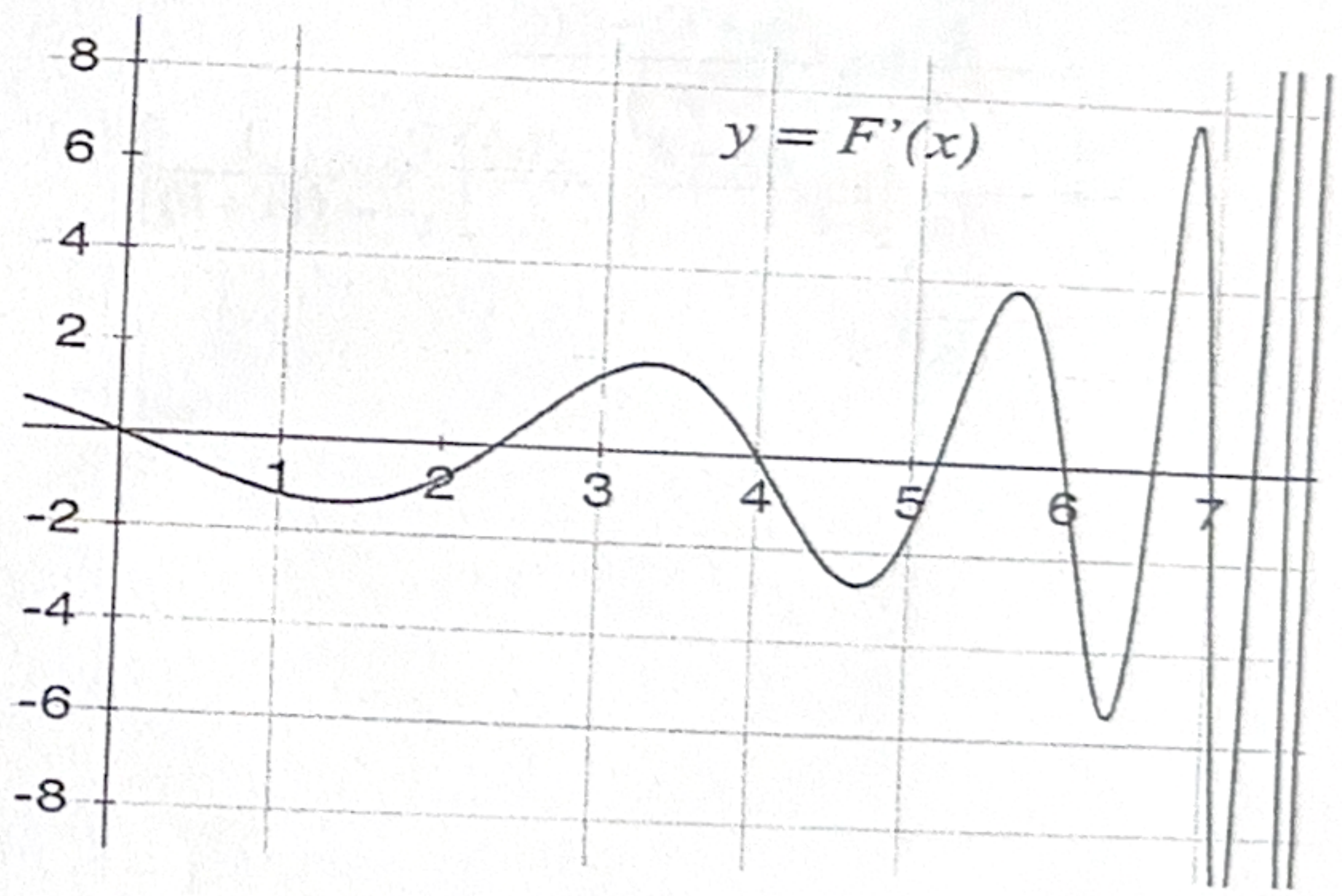

Hence $\sin(y) = \frac{1}{\sqrt{1+x^2}}$, so

$$\frac{d}{dx}\operatorname{arccot}(x) = -\sin^2(y) = -\frac{1}{1+x^2}.$$

5. *[3 points]* Here is the graph of a function $F$.




Sketch the graph of its derivative.




You do not need to justify your answer to this question.

6. *[3 points]* Let $a \in \mathbb{R}$. Let $f$ be a function which is differentiable at $a$ and assume that $f(a) > 0$. We define a new function $g$ by $g(x) = \dfrac{1}{f(x)}$. Prove that $g$ is differentiable at $a$ and $g'(a) = \dfrac{-f'(a)}{f(a)^2}$.

**Do a proof *directly* from the definition of derivative as a limit, without using any of the differentiation rules.**

We will determine a closed form expression for $g'(a) = \lim_{h\to 0} \dfrac{g(a+h)-g(a)}{h}$ which will give us both the formula, and the differentiability of $g$ at $a$. Notice first that since $f$ is differentiable at $a$, it is continuous at $a$. Since $f(a) \neq 0$, by the limit laws and continuity we have $\lim_{h\to 0} \dfrac{1}{f(a+h)} = \dfrac{1}{f(a)}$. Now proceeding with the differentiation we get

$$\begin{aligned}
\lim_{h\to 0} \frac{g(a+h)-g(a)}{h} &= \lim_{h\to 0} \frac{\frac{1}{f(a+h)} - \frac{1}{f(a)}}{h} \\
&= -\lim_{h\to 0} \frac{\frac{f(a)-f(a+h)}{f(a)f(a+h)}}{h} && \text{common denominator} \\
&= -\frac{1}{f(a)} \lim_{h\to 0} \frac{f(a+h)-f(a)}{hf(a+h)} \\
&= -\frac{1}{f(a)} \left[\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right] \left[\lim_{h\to a} \frac{1}{f(a+h)}\right] && \text{limit laws} \\
&= -\frac{f'(a)}{f(a)^2}
\end{aligned}$$

7. *[7 points]*

(a) State the Mean Value Theorem.

If $f$ is a function which is continuous on $[a,b]$ and differentiable on $(a,b)$, then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

(b) Define what it means for a function $f$ to be decreasing on an interval $I$.

We say that $f$ is decreasing on the interval $I$ if whenever $x_1, x_2 \in I$ satisfy $x_1 < x_2$ then $f(x_1) > f(x_2)$.

**Note:** Some books call the above definition "*strictly decreasing*" and define decreasing instead to mean $x_1 < x_2 \implies f(x_1) \geq f(x_2)$. As long as you are clear and consistent (use the same definition in all parts of this question), we will accept your answer.

(c) Use your above two answers to prove the following theorem:

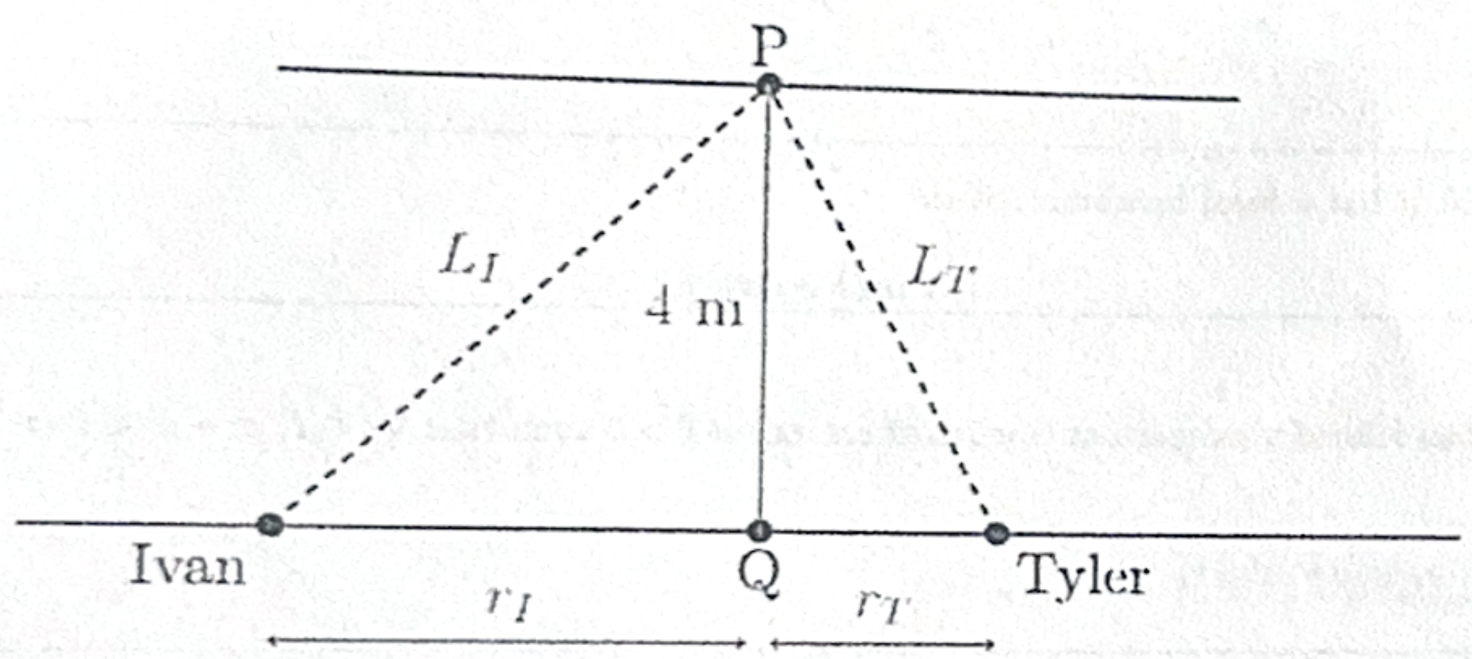
*Let $f$ be a differentiable function on an open interval $I$.*
*If $f'(x) < 0$ for every $x \in I$, then $f$ is decreasing on $I$.*

Let $x_1, x_2 \in I$ be given and assume that $x_1 < x_2$. Consider the interval $[x_1, x_2] \subseteq I$, and note that since $f$ is differentiable on the latter, it is continuous on $[x_1, x_2]$ and differentiable on $(x_1, x_2)$. Consequently, we may apply the Mean Value Theorem and conclude that there exists a $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

By assumption we have that $f'(c) < 0$, and since $x_1 < x_2$ we know that $x_2 - x_1 > 0$. It must follow that $f(x_2) - f(x_1) < 0$, or equivalently, $f(x_1) > f(x_2)$.

8. *[4 points]* Tyler and Ivan are tied at opposite ends of a rope of fixed length. The rope is fully stretched; it starts at Ivan's foot, then passes through a fixed point $P$ on the ceiling, and then continues to Tyler's foot. The rope is the dashed line in the picture below. The point $P$ is on the ceiling, $4m$ directly above the point $Q$ on the floor. Ivan is running away from Tyler while Tyler is just being dragged along the floor. How fast is Tyler moving at the time when Ivan is $4m$ away from point $Q$, the distance between Ivan and Tyler is $6m$, and Ivan's velocity is $1m/s$?




Let $r_I$ be the distance between Ivan and $Q$, while $r_T$ denotes the distance between Tyler and $Q$. Hence at our moment of interest, we have that $r_I = 4$m and $r_T = 2$m, and we are interested in solving for $\frac{dr_T}{dt}$. Let the total length of the rope be some positive constant $L$ (which we will not care about) and take $L_I, L_T$ to be as indicated in the diagram, so that $L_I + L_T = L$. In particular, by differentiating this equation we have that

$$\frac{dL_I}{dt} = -\frac{dL_T}{dt}. \qquad (1)$$

Now the triangles (Tyler)$QP$ and (Ivan)$QP$ are right-triangles, so by the Pythagorean theorem we have $r_T^2 + 16 = L_T^2$ and $r_I^2 + 16 = L_I^2$; in particular, when $r_T = 2$ we have $L_T = 2\sqrt{5}$ and when $r_I = 4$ we have $L_I = 4\sqrt{2}$.

Differentiating the previous equations, we get

$$2r_T\frac{dr_T}{dt} = 2L_T\frac{dL_T}{dt}, \qquad 2r_I\frac{dr_I}{dt} = 2L_I\frac{dL_I}{dt}. \qquad (2)$$

Solving for $\frac{dr_T}{dt}$ and using Equation (1) we get

$$\begin{aligned}\frac{dr_T}{dt} &= \frac{L_T}{r_T}\frac{dL_T}{dt} \\ &= -\frac{L_T}{r_T}\frac{dL_I}{dt} && \text{since } \tfrac{dL_T}{dt} = -\tfrac{dL_I}{dt} \\ &= -\frac{L_T}{r_T}\frac{r_I}{L_I}\frac{dr_I}{dt}. && \tfrac{dL_I}{dt} = \tfrac{r_I}{L_I}\tfrac{dr_I}{dt} \text{ by (2)}\end{aligned}$$

We know all of these quantities; namely, $r_I = 4, r_T = 2, L_I = 4\sqrt{2}, L_T = 2\sqrt{5}, \frac{dr_I}{dt} = 1$, yielding

$$\frac{dr_T}{dt} = \sqrt{\frac{5}{2}}.$$

MAT137Y Test 2. Dec 3, 2021
X. Cui, V. Dimitrov, A. Malusà, B. Khesin, Z. Qian.

1. For each question below, only your final answer will be graded. **No justification is necessary.**

(1a) (2 points) Let $f(x) = e^{-x}(\ln x)^2$ and calculate $f'(x)$.

**Final Answer**

$f'(x) = -e^{-x}(\ln x)^2 + e^{-x} \cdot \frac{2\ln x}{x}$

(1b) (2 points) Find the equation of the line tangent to the graph of $f$ defined by $f(x) = \frac{x+1}{2x-3}$ at the point with $x$-coordinate 1.

**Final Answer**

$y + 2 = -5(x-1)$ or $y = -5x + 3$

(1c) (2 points) Let $h(x) = (\arctan x)^{x^2}$ and calculate $h'(x)$.

**Final Answer**

$h'(x) = (\arctan x)^{x^2}\left(2x\ln(\arctan(x)) + \frac{x^2}{(x^2+1)\arctan(x)}\right)$

(1d) (2 points) Let $F(x) = \cos(2x)$ and calculate $F^{(102)}(0)$.

**Final Answer**

$F^{(102)}(0) = -2^{102}$

2. For each question below, only your final answer will be graded. **No justification is necessary.**

Let $f$ be a function defined on an interval $I$ and $a \in I$. Write the definition of the following concepts.

(2a) (1 point) $f$ has a maximum at $a$.

**Final Answer**

We say $f$ has a maximum at $a$ if $\forall x \in I, f(x) \leq f(a)$

(2b) (2 points) $f$ has a local maximum at $a$.

**Final Answer**

We say that $f$ has a local maximum at $a$ when there exists $\delta > 0$ such that $\forall x \in I, |x-a| < \delta \Rightarrow f(x) \leq f(a)$

State the following theorems. Make sure to specify all assumptions.

(2c) (2 points) Intermediate Value Theorem.

**Final Answer**

Let $a, b \in \mathbb{R}$ with $a < b$. Let $f$ be a function defined on the interval $[a, b]$.
Let $M \in \mathbb{R}$.IF $f(a) < M < f(b)$ and $f$ is continuous on $[a, b]$
THEN there exists $c \in (a, b)$ such that $f(c) = M$.
OR
If $f(a) < 0, f(b) > 0$ and $f$ is continuous on $[a, b]$
THEN there exists $c \in (a, b)$ such that $f(c) = 0$.
OR
If $f$ is continuous on $[a, b]$
THEN $f$ takes all the values between $f(a)$ and $f(b)$.

3. For each question below, only your final answer will be graded. **No justification is necessary.**

(3a) (1 point) Approximate $\ln(1.01)$ using a linear approximation.

**Final Answer**

0.01

(3b) (2 points) Let $f$ and $g$ be differentiable functions. We know $f(1) = \frac{\pi}{4}, f'(1) = 2$. We also know that $f$ and $g$ satisfy the equation $\cos^2(f(x)) + 6g(x) = g(x^2)$ when $x$ is close to 1. Calculate $g'(1)$.

**Final Answer**

$\frac{1}{2}$

(3c) (2 points) Let $f$ be a differentiable, one-to-one function with domain $\mathbb{R}$. Assume $f'$ is never 0. The following table gives us some values for $f$ and $f'$:

| $x$ | $f(x)$ | $f'(x)$ |
|---|---|---|
| 1 | 4 | -6/5 |
| 2 | 3 | -4/5 |
| 3 | 5/2 | -1/2 |
| 4 | 2 | -11/10 |

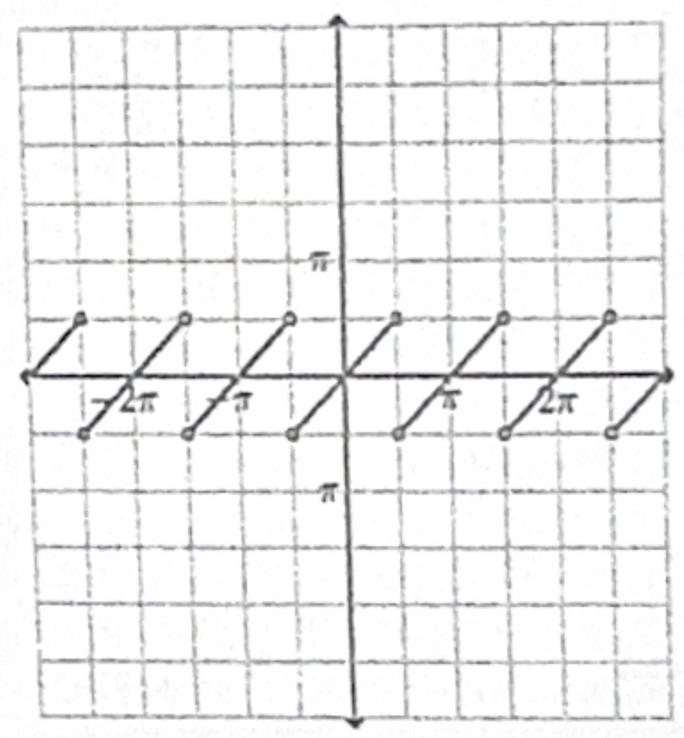
Let $g(x) = \ln(f^{-1}(x))$. $f^{-1}(x)$ is the inverse function of $f$. Calculate $g'(4)$. Enter your answer as a fraction.

**Final Answer**

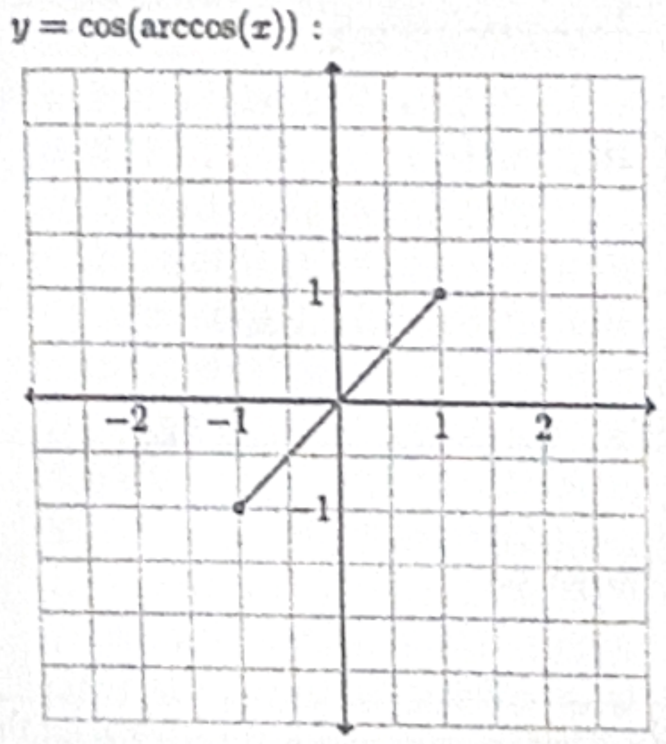
$-\frac{5}{6}$

(3d) (4 points) Sketch the following two graphs. Make sure to label your axes and that you have the correct domain.

$y = \arctan(\tan(x))$ :



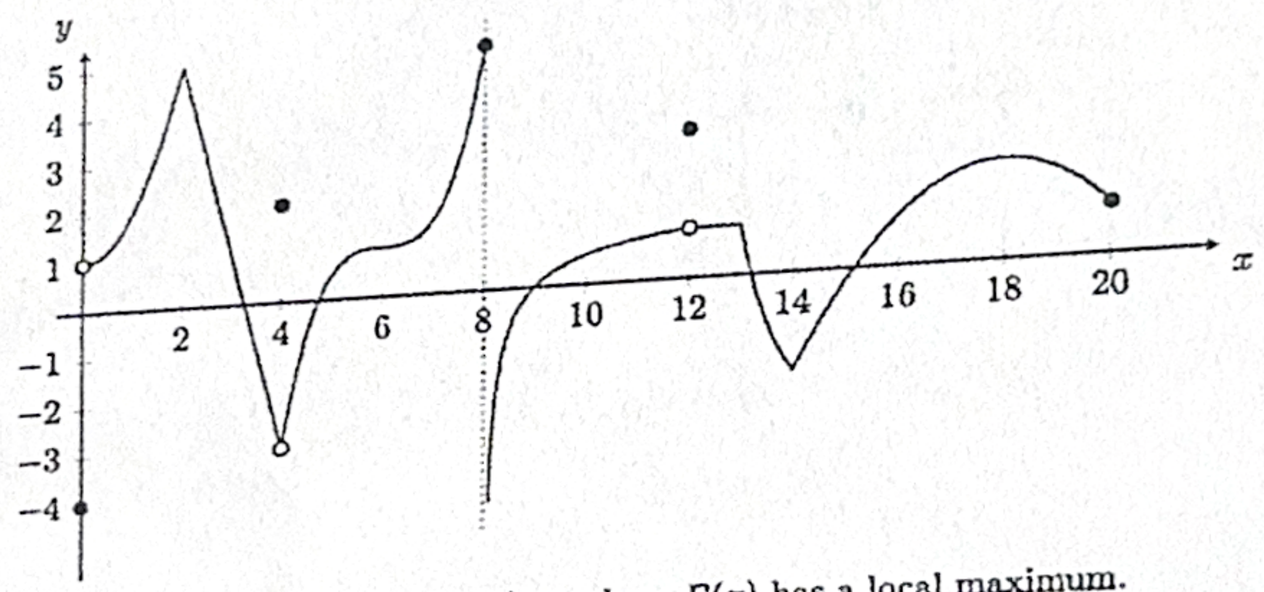

$y = \cos(\arccos(x))$ :



(3e) (3 points) Let $I$ be an OPEN interval. Let $f$ be a function with domain $I$. Let $c \in I$. For each statement, determine whether it is TRUE or FALSE. Circle the correct answer. Again, no justification is necessary.

| | | |
|---|---|---|
| i)IF $f'$ is continuous at $c$ THEN $f$ is differentiable at $c$. | TRUE | FALSE |
| ii)IF $c$ is a critical point of $f$ THEN $f$ has a local extremum at $c$. | TRUE | FALSE |
| iii)Let $a < b$ and $(a, b) \subset I$. IF $f$ is increasing on $(a, b)$ THEN $\forall x \in (a, b), f'(x) > 0$. | TRUE | FALSE |

4. Consider the graph of $y = F(x)$ for $0 \leq x \leq 20$. Note that $F(x) = 1$ when $12 < x \leq 13$.




(4a) (1 point) Identify all values of $x \in (0, 20)$ at where $F(x)$ has a local maximum.

**Final Answer**

2, 4, 8, [12, 13], 18

(4b) (1 point) Identify all critical points of $F$ on $(0, 20)$.

**Final Answer**

2, 4, 6, 8, [12, 13], 14, 18

(4c) (2 points) Consider the restriction of $F$ to each of the following intervals, and CIRCLE all those on which it admits a minimum.

i) $\boxed{[0, 4]}$

ii) $[4, 8]$

iii) $[6, 10]$

iv) $\boxed{[11, 15]}$

University of Toronto
Faculty of Arts and Science

# MAT137Y Test 2

X. Cui, I. Gaiur, B. Khesin, T. Kojar, M. Pedreira, J. Siefken, A. Zalloum

December 2, 2022
Duration: 110 minutes
No Aids Permitted

Last Name: ______________________

First Name: ______________________

UToronto email: ______________________@mail.utoronto.ca

UTORid: ______________________

Student Number: ______________________

Please make sure your information above are correct.

This exam contains 10 one-sided pages (including this cover page) and is printed on 10 sheets of paper. There are 8 problems.

Instructions

- Keep this booklet closed, until an invigilator announces that the test has begun. However, you may fill out your information in this page before the test starts.
- Please place your student ID in a location on your desk that is easy for an invigilator to check without disturbing you during the test.
- Turn your cell phone completely off (not just in vibrate mode) and store it with your other belongs.
- No aids are permitted on this examination. Examples of illegal aids include but are not limited to textbooks, notes, calculators, cellphones, or any electronic device.
- If you need space for your draft work, you can use the back of each page as your scratch papers. Note that we will not scan the back of each page.
- Write clearly and concisely by using dark pencil or pen. Work scattered all over the page without a clear ordering will receive very little credit.
- For questions with a boxed area, ensure your answer is completely inside the box.
- If you need more space, use the blank page at the end of the exam and clearly indicate on the question page when you have done this. Do not tear any pages off this exam.

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Points: | 8 | 5 | 3 | 4 | 6 | 4 | 6 | 5 | 41 |
| Score: | | | | | | | | | |



**Do not tear this page off. This page is a formula sheet and can be used for rough work only. It will not be graded under any circumstances.**

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1+\tan^2\theta = \sec^2\theta \qquad 1+\cot^2\theta = \csc^2\theta$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned} \qquad \sin 2\theta = 2\sin\theta\cos\theta \qquad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

$$\sin A \sin B = \frac{1}{2}\big(\cos(A-B) - \cos(A+B)\big) \qquad \sin(A+B) = \sin A\cos B + \cos A \sin B$$

$$\cos A \cos B = \frac{1}{2}\big(\cos(A-B) + \cos(A+B)\big) \qquad \cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\sin A \cos B = \frac{1}{2}\big(\sin(A+B) + \sin(A-B)\big) \qquad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

**You may use the rest of this formula sheet page for rough work. It will not be graded. If you need more space for a solution, please use one of the blank pages at the back of the test.**

1. For each question below, only your final answer will be graded. **No justification is necessary.**

(1a) (2 points) Let $f(x) = e^{\sin(x\sin(x))}$ and calculate $f'(x)$.

Final Answer

$f'(x) =$

(1b) (2 points) The function $f(x) = x^3 + x + 3$ is one-to-one. Calculate $f^{-1}(5)$ and $(f^{-1})'(5)$.

Final Answer

$f^{-1}(5) =$

Final Answer

$(f^{-1})'(5) =$

(1c) (2 points) Let $h(x) = \arcsin(\sqrt{\frac{1+x}{2}})$ and calculate $h'(0)$.

Final Answer

$h'(0) =$

(1d) (2 points) Let $F(x) = x^{(\ln x)^3}$ and calculate $F'(e)$.

Final Answer

$F'(e) =$



For each question below, only your final answer will be graded. **No justification is necessary.**

2. Let $f(x) = \sqrt{x}$ with domain $D = [0, \infty)$. For every point $a \in D$, let $L_a : D \to \mathbb{R}$ be the lin approximation to $f$ at the point $a$, if such an approximation is defined.

(2a) (2 points) Find an expression for $L_1(x)$ and $L_4(x)$.

Final Answer

$L_1(x) =$

Final Answer

$L_4(x) =$

(2b) (3 points) For $x \in D$, we would like to approximate $f(x)$. Let $C$ be the set of $x \in D$ such that $L_1(x)$ is a *strictly better* approximation than $L_4(x)$ (i.e., has less error).

1. We can claim that 1.1 is in $C$.

○ True ○ False ○ Not enough information to determine.

2. Express $C$ in interval notation.

Final Answer

$C =$

For each question below, only your final answer will be graded. **No justification is necessary.**

3. Determine if the following statements are True or False for all differentiable functions $f$ with domain $\mathbb{R}$.

(3a) (1 point) If $f$ has a local maximum at 2, then $f'(2) = 0$.

○ True ○ False

(3b) (1 point) If $f(2) = 2$ and $f(4) = 2$, then there is a $c \in (2, 4)$ such that $f$ has a local extrema at $c$.

○ True ○ False

(3c) (1 point) If $f(1) = 4$ and $f'(x) \geq 4$ for all $x \in \mathbb{R}$, then $f(3) \geq 10$.

○ True ○ False

4. (4a) (2 points) Let $f$ be a function defined on an interval $I$.
Write down the definition of "$f$ is (strictly) decreasing on $I$".

**Final Answer**

(4b) (2 points) State the Extreme Value Theorem. Make sure to specify all assumptions.

**Final Answer**



5. Let $f$ and $g$ be functions with domain $\mathbb{R}$ and define the function $h$ by

$$h(x) = \begin{cases} f(x) & \text{if } x > 0 \\ g(x) & \text{if } x \leq 0 \end{cases}$$

For each of the following, give an example of $f$ and $g$ such that $h$ has the desired property or explain why no such example exists. You should give $f$ and $g$ explicitly in math expressions.

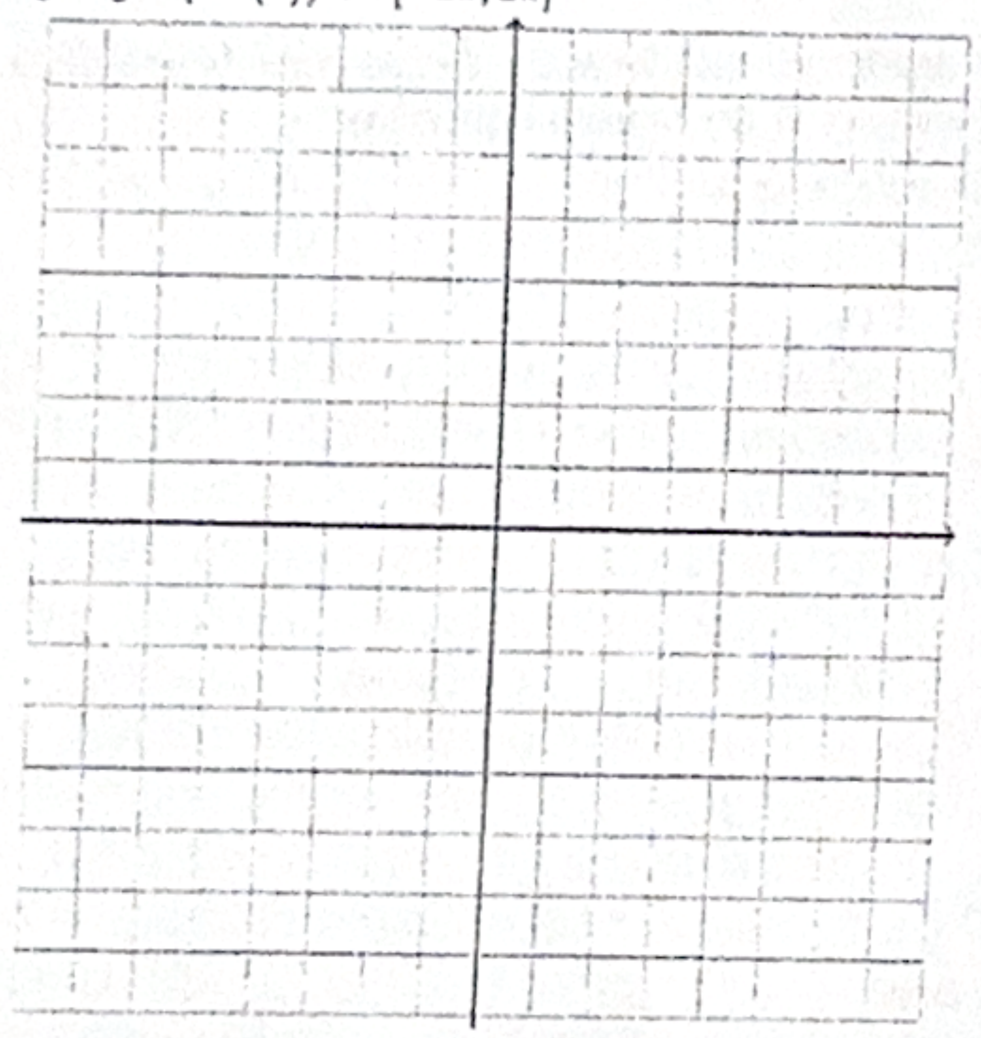
(5a) (2 points) $h$ is continuous at 0 but not differentiable at 0.

(5b) (2 points) $h'(0)$ exists and $h''(0)$ does not exist.

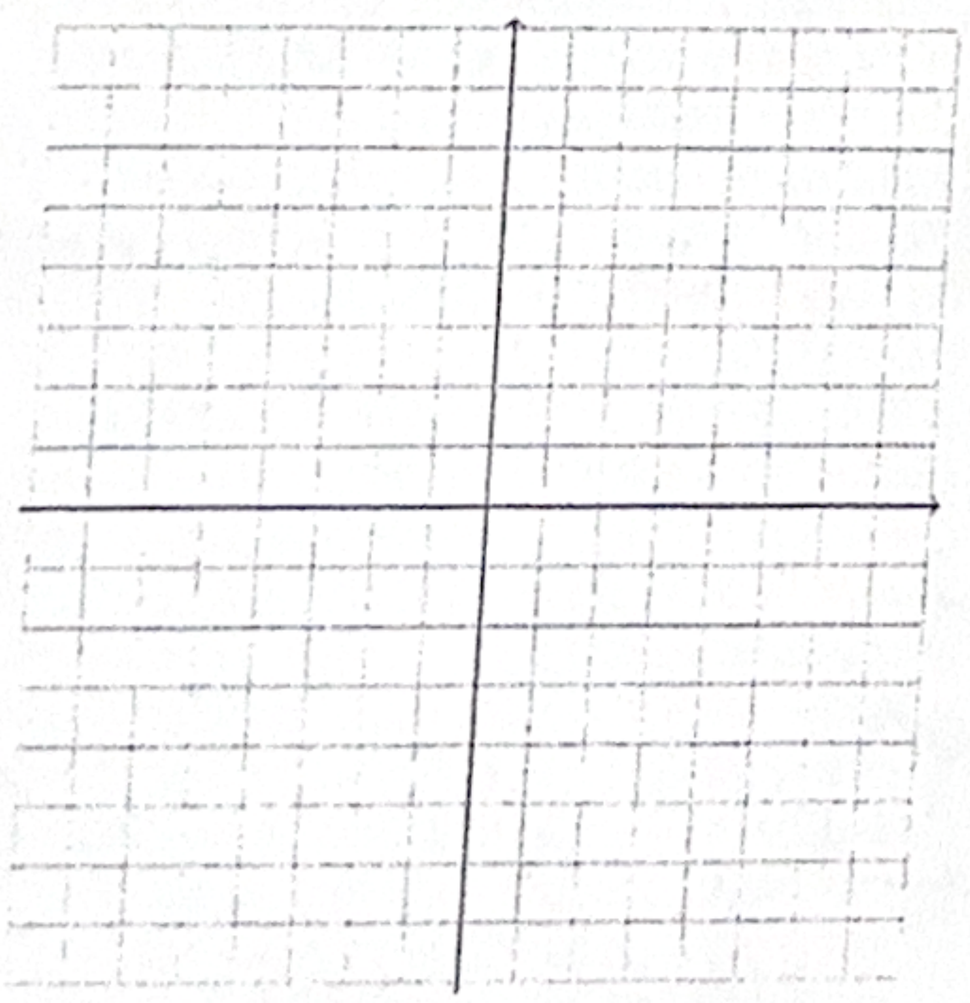
(5c) (2 points) $h''(0)$ exists and $h'(0)$ does not exist.

6. (4 points) Let $g : [\pi, 2\pi] \to [-1, 1]$ be the restriction of $\cos x$ to the interval $[\pi, 2\pi]$.
Define $g^{-1}$ to be the inverse function of $g$. Sketch the following graphs on the given intervals. Make sure to label your axes.

$y = g^{-1}(\cos(x))$ on $[-2\pi, 2\pi]$



$y = \cos(g^{-1}(x))$ on $[-1, 1]$





7. Let $a, b \in \mathbb{R}$. Let $g$ be a function with domain $\mathbb{R}$.

(7a) (2 points) Write the definition "$g$ is differentiable everywhere".

(7b) (4 points) We know the following:

- $g(0) = a$
- $g$ is differentiable at 0 and $g'(0) = b$
- for every $x, y \in \mathbb{R}$, $g(x + y) = g(x) + g(y) + 2xy - a$

Prove that $g$ is differentiable everywhere and find a formula for $g'$.

Final Answer

$\forall x \in \mathbb{R}, g'(x) =$

8. Let $f$ and $g$ be two differentiable functions with domain $\mathbb{R}$. Suppose that for all $x \in \mathbb{R}$

$$f'(x) = -2g(x), g'(x) = 2f(x)$$

(8a) (3 points) Prove that there is a constant $C$ such that $f^2(x) + g^2(x) = C$ for all $x \in \mathbb{R}$.

(8b) (1 point) Suppose that $f(0) = 0$ and $g(0) = -3$.What is $C$?

Final Answer

$C =$

(8c) (1 point) (Bonus problem) Give an example of $f$ and $g$ which satisfy the following properties:

- $f$ and $g$ are differentiable with domain $\mathbb{R}$
- $f(0) = 0$, $g(0) = -3$
- $\forall x \in \mathbb{R}$, $f'(x) = -2g(x)$, $g'(x) = 2f(x)$

Final Answer

$f(x) =$

Final Answer

$g(x) =$



**Do not tear this page off. This page is for additional work and will not be marked, unless you clearly indicate it on the original question page.**