Free Variables and Linear Systems

I'm confused about that proof of Lemma 3.10 in MAT223 notes ".

Since there is no pivot in the mth column of X, we know that the variable xm is

free. That is, we can set xm equal to any real number and obtain a solution to the

vector equation above. Setting cm = −1 gi". based on what assumption can we make the implication that Since there is no pivot in the mth column of X, we can set C_m equal to any real number and obtain a solution to the

vector equation above? is it that the shstem being consistent? and is it ever established explicitly in the textbook that consistent and "free column" => any value for the variable? how? whats the theoren?

Of course! You've asked an excellent and insightful question that gets to the very heart of how we understand solutions to linear systems. Your intuition is spot on. Let's break down the logic behind that step in the proof of Lemma 3.10.

You are correct; the reasoning relies on the system being **consistent** and the variable `cₘ` being a **free variable**. Here is a step-by-step explanation grounded in the provided `LectureNotesRecent.pdf`.

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1. The System is Homogeneous and Therefore Always Consistent

The proof of Lemma 3.10 begins by analyzing the vector equation:

$$c_1 v_1 + \cdots + c_m v_m = 0$$

This is a

homogeneous system of linear equations. A key property of any homogeneous system

`A**x** = **0**` is that it is **always consistent**, because it always has at least one solution: the trivial solution where all variables are zero (`x₁ = ... = xₘ = 0`).

So, we know from the start that a solution exists. The question then becomes: is the trivial solution the *only* solution, or are there others?

2. Free Variables and Infinite Solutions

The concept that connects a non-pivot column to the ability to choose any value for its corresponding variable is the distinction between

basic variables and **free variables**.

- **Basic Variable:** A variable that corresponds to a **pivot column** in the reduced row echelon form (RREF) of the coefficient matrix.
- **Free Variable:** A variable that corresponds to a **non-pivot column**.

When you use Gauss-Jordan elimination to find the `rref`, the resulting system of equations expresses each basic variable in terms of the free variables. Since the free variables are not constrained by a pivot, we are "free" to assign them any real number value. Once we choose values for the free variables, the values of the basic variables are uniquely determined.

This process of assigning arbitrary values to free variables is exactly how we

parametrize the solution set when there are infinitely many solutions.

3. The Rouché-Capelli Theorem (Theorem 1.26)

This is the specific theorem that formalizes the connection you're asking about. Let's look at the conditions for a system

`A**x** = **b**`:

1. **Inconsistent (No solutions):** The last column of `rref([A|**b**])` has a pivot.
2. **Unique solution:** The last column of `rref([A|**b**])` does not have a pivot, and *every* column of `rref(A)` *does* have a pivot.
3. **Infinitely many solutions:** The last column of `rref([A|**b**])` does not have a pivot, and `rref(A)` has a column *without* a pivot.

Now, let's apply this to the homogeneous system `A**c** = **0**` in the proof of Lemma 3.10.

- The augmented matrix is `[A|**0**]`. The last column is all zeros and will remain so through all row operations. Therefore, **the system is always consistent** because the last column can never have a pivot.
- The lemma's premise is that "the mth column of `rref(A)` does not have a pivot". This means the coefficient matrix has a non-pivot column, which corresponds to a free variable (c_m).
- This perfectly matches the condition for **infinitely many solutions** from part (3) of the Rouché-Capelli Theorem.

The existence of infinitely many solutions is precisely the consequence of having a free variable. Because c_m is a free variable, we can set it to *any* real number and still find corresponding values for the other (basic) variables to satisfy the equation.

The proof cleverly leverages this by choosing a convenient non-zero value,

$c_m = -1$, to show that a non-trivial solution exists, which in turn proves that v_m is a linear combination of the other vectors .

Summary of the Logic

To directly answer your question: "based on what assumption can we make the implication?"

The implication flows as follows:

1. The proof analyzes a **homogeneous** vector equation $c_1v_1 + ... + c_mv_m = 0$, which is always **consistent**.
2. The premise of the proof is that the mth column is a **non-pivot column**.
3. By definition, this means the corresponding variable, c_m, is a **free variable**.
4. The Rouché-Capelli Theorem (1.26) guarantees that a consistent system with a free variable has **infinitely many solutions**.
5. The *reason* it has infinitely many solutions is that you can assign **any real number** to the free variable c_m and then solve for the basic variables.
6. The proof makes the strategic choice $c_m = -1$ (one of the infinitely many possibilities) to complete its argument.

Since the free variables are not constrained by a pivot, we are "free" to assign them any real number value. that is not riorous. its my whole point. who says that a column wihout a pivot means a free variable? how donu prove it? does the linear alagebra done right book offet a nything more rigorous?

You're asking the perfect question to bridge the gap between a procedural course and a proof-based one. You are correct that "we are free to assign a value" sounds like a procedural step, not a rigorous mathematical justification. Let's break down how this is formally established, first using the MAT223 notes and then showing the more abstract (and perhaps more satisfying) justification offered by *Linear Algebra Done Right*.

In short, the term **"free variable" is a *name* we give to a variable whose column lacks a pivot in RREF**. The *proof* that we can assign it any value comes from the structure of the equations that the RREF represents.

The Rigor in the MAT223 Notes (The Procedural Justification)

The justification in a textbook like `LectureNotesRecent.pdf` is grounded in the algorithm of Gauss-Jordan elimination and the structure of a system in Reduced Row Echelon Form (RREF).

1. **Definition:** First, the text *defines* a free variable as a variable whose corresponding column in the RREF of the coefficient matrix does not contain a pivot. A basic variable corresponds to a pivot column.
2. **Structure of RREF:** By definition, a matrix in RREF has two key properties :
   - Every pivot is 1.
   - Every pivot is the *only* non-zero entry in its column.
3. **The Proof via Rearrangement:** Consider a homogeneous system `A**c** = **0**`. After row-reducing `[A|**0**]` to `[X|**0**]`, where `X` is the RREF of `A`, each row of the matrix represents an equation. Because of the structure of RREF, each equation will look like this: `(1) * (Basic Variable) + (coefficients) * (Free Variables) = 0` For example, in a system with variables c_1, c_2, c_3, c_4 where c_1 and c_3 are basic and c_2, c_4 are free, the RREF might look like: $c_1 + 3c_2 - 5c_4 = 0$ $c_3 + 2c_2 + 7c_4 = 0$ We can algebraically rearrange these equations to isolate the basic variables: $c_1 = -3c_2 + 5c_4$ $c_3 = -2c_2 - 7c_4$ **This rearrangement is the proof.** It shows that the values of the basic variables (c_1, c_3) are completely determined by

the values of the free variables (c_2, c_4). There are no other equations that place any constraints on c_2 or c_4. Therefore, we can choose any real numbers for c_2 and c_4, and these formulas will give us a valid solution. This process of describing the solution set in terms of arbitrary values for the free variables is called **parameterization**.

The Rigor in *Linear Algebra Done Right* (The Abstract Justification)

Linear Algebra Done Right (LADR) provides a more abstract, high-level justification that avoids the mechanics of row reduction entirely. It frames the problem in terms of linear maps and the dimensions of their null spaces and ranges.

1. **Reframing the Problem:** A system of `m` homogeneous equations in `n` variables `Ax = 0` can be seen as finding the **null space** of a linear map $T: F^n \to F^m$, where `T(x) = Ax`. The question "does a non-trivial solution exist?" is equivalent to asking "is `null T` different from the zero vector `{**0**}`?" .
2. **The Key Theorem:** The central result is the **Fundamental Theorem of Linear Maps** (3.21 in LADR): `dim(V) = dim(null T) + dim(range T)`
3. **The Connection:**
    - `dim(V)` is the dimension of the domain, which is `n` (the number of variables/columns).
    - `dim(range T)` is the dimension of the column space of `A`, which is precisely the **rank** of the matrix. The rank is equal to the number of linearly independent columns, which, after row reduction, is proven to be the **number of pivots** (basic variables).
    - Therefore, the equation becomes: `n = dim(null T) + rank(A)`.
4. **The Conclusion:** From this, we see that `dim(null T) = n - rank(A)`.
    - The number `n - rank(A)` is exactly the number of non-pivot columns, which MAT223 *defines* as the number of free variables. LADR gives this quantity a more abstract meaning: it is the **dimension of the null space**.

- If there is a non-pivot column (a free variable), it means `rank(A) < n`. This immediately implies that `dim(null T) > 0`.
- A subspace with a dimension greater than zero contains infinitely many vectors. Therefore, if a non-pivot column exists, the null space is non-trivial, and there are infinitely many solutions.

This approach proves that non-trivial solutions must exist based on a dimensional argument, without ever having to talk about "assigning values." The existence of infinite solutions is a direct consequence of the null space having a dimension greater than 0.