Def 2.1 : ① A **vector**, is an object with **magnitude** and **direction**.

② an n-th dimensional vector $\vec{v}$ is a list of n real numbers, ★ordinal written as $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$.

we call the real numbers $\{v_i\}_{i=1}^{n}$ the standard coordinates of $\vec{v}$

③ $$\mathbb{R}^n = \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} : \begin{array}{l} \forall i \in \{j\}_1^n, \\ v_i \in \mathbb{R} \end{array} \right\}$$

(The set of all n-dimensional vector)

Def 2.2: Let $\vec{v}$ be n-th dim vec in $\mathbb{R}^n$.

We call

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

the standard coordinate representation.

Def 2.5: Let $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ be vectors in $\mathbb{R}^n$. Let $c \in \mathbb{R}$. Then we define

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}, \quad c \cdot \vec{v} = \begin{pmatrix} cv_1 \\ \vdots \\ cv_n \end{pmatrix}.$$

PROPOSITION 2.9.

Let $\vec{v}_1 = \begin{pmatrix} v_{1,1} \\ v_{2,1} \\ \vdots \\ v_{m,1} \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} v_{1,2} \\ v_{2,2} \\ \vdots \\ v_{m,2} \end{pmatrix}$ ... $\vec{v}_n = \begin{pmatrix} v_{1,n} \\ v_{2,n} \\ \vdots \\ v_{m,n} \end{pmatrix}$

$$\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$$

be n+1 m-dimensional vectors.

Then the vector equation, (based on Def 2.1 of vector addition and multipliation)

$$x_1 \vec{V_1} + x_2 \vec{V_2} + \cdots x_n \vec{V_n} = \vec{w}$$

has the same solution set as system of equations;

$$\begin{cases} V_{1.1} x_1 + V_{1.2} x_2 \cdots V_{1.n} x_n = w_1 \\ V_{2.1} x_1 + V_{2.2} x_2 \cdots V_{2.n} x_n = w_2 \\ \quad \vdots \qquad\qquad \ddots \qquad \vdots \qquad\qquad \vdots \\ V_{m.1} x_1 + V_{m.2} x_2 \cdots V_{m.n} x_n = w_m \end{cases}$$

represented by Augamented matrix

$$\left(\begin{array}{cccc|c} v_{1,1} & v_{1,2} & \cdots & v_{1,n} & w_1 \\ v_{2,1} & v_{2,2} & \cdots & v_{2,n} & w_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{m,1} & v_{m,2} & \cdots & v_{m,n} & w_m \end{array}\right)$$

$$\cong \left(\vec{V}_1 \ \vec{V}_2 \ \cdots \ \vec{V}_m \,\middle|\, \vec{w}\right)$$

Yes, this is also a matrix!!

Def 2.10. Let $\vec{V}_1, \vec{V}_2 \cdots \vec{V}_n$ be n vectors. Let $\{c_i\}_{i=1}^{n}$ be a seq of real numbers.

We call $\vec{W} = \sum_{i=1}^{n} c_i V_i$ a linear combination of vectors $\{\vec{V}_1, \vec{V}_2 \cdots \vec{V}_n\}$.

Def 2.11 Let $\vec{V_1}, \vec{V_2} \cdots \vec{V_n} \in \mathbb{R}^m$.

Span$(\vec{V_1}, \cdots \vec{V_n}) = \{c_1 \cdot \vec{V_1} + c_2 \vec{V_2} \cdots c_n \vec{V_n} : c_1, c_2 \cdots c_n \in \mathbb{R}\}$

Def 2.12 ① Let $m, n \in \mathbb{N}^+$.

Let $\{\vec{V_i}\}_1^n$ be n m-dimensional vectors. We say they're Linearly Dependent iff $\exists i \in \mathbb{N}$ s.t. $1 \leq i \leq n \wedge \vec{V_i} \in \text{Span}(\vec{V_1}, \vec{V_2} \cdots \vec{V_{i-1}}, \vec{V_{i+1}} \cdots \vec{V_n})$

i.e. $(\{c_j\}_{j=1}^{i-1} \cup \{c_j\}_{i+1}^n \subseteq \mathbb{R}$ s.t.

$$\vec{V_i} = \sum_1^{i-1} c_j \vec{V_j} + \sum_{i+1}^n c_j \vec{V_j}$$

② Linearly Dependent iff

$\exists \vec{C} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$ where $\{c_i\}_1^n \subseteq \mathbb{R}$ s.t.

$\vec{c} \neq \vec{0} \wedge (\vec{V_1}\ \vec{V_2} \cdots \vec{V_n}) \cdot \vec{c} = \vec{0}$

i.e. $c_1\vec{V_1} + c_2\vec{V_2} + \cdots c_n\vec{V_n} = \vec{0}$

① ⇒ ②: Assume ①. We have

$$\vec{V_i} = \sum_{1}^{i-1} c_j\vec{V_j} + \sum_{i+1}^{n} c_j\vec{V_j} \iff$$

$$c_1\vec{V_1} + c_2\vec{V_2} \cdots c_{i-1}\vec{V_{i-1}} - \vec{V_i} + c_{i+1}\vec{V_{i+1}} + \cdots c_n\vec{V_n} = \vec{0}$$

Hence we have solution

$\vec{c} = (c_1, c_2 .. c_i, \ldots c_n)$ where $c_i = -1 \neq 0$

thus $\vec{c} \neq \vec{0}$.

② ⇒ ① Assume ②.

We have $\vec{c} = (c_1, c_2 \ldots c_n) \in \mathbb{R}^n$

where $\exists i \{1, 2, \ldots n\}$ s.t. $c_i \neq 0$

and $(\vec{v}_1\ \vec{v}_2\ \vec{v}_3 \ldots \vec{v}_n) \cdot \vec{c} = \vec{0}$

i.e. $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_i\vec{v}_i + \cdots c_n\vec{v}_n = \vec{0}$

Hence $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1}$

$+ \cdots c_n\vec{v}_n = -c_i\vec{v}_i$

Since $c_i \neq 0$, $-c_i \neq 0$, Hence

$-\frac{c_1}{c_i}\vec{v}_1 - \frac{c_2}{c_i}\vec{v}_2 \cdots - \frac{c_{i-1}}{c_i}\vec{v}_{i-1} - \frac{c_{i+1}}{c_i}\vec{v}_{i+1} + \cdots$

$c_n\vec{v}_n = \vec{v}_i$

Hence $\vec{v}_i \in \{k_1\vec{v}_1 + \cdots k_{i-1}\vec{v}_{i-1} + k_{i+1}\vec{v}_{i+1} + \cdots k_n\vec{v}_n \mid k_1, k_2 \ldots k_{i-1}, k_{i+1} \ldots k_n \in \mathbb{R}\}$ Hence $v_i \in \text{Span}(v_1, v_2 \ldots v_{i-1}, v_{i+1} \ldots v_n)$

Thm 2.16 Let $\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n$ be n vectors in $\mathbb{R}^m$. Then $\text{rref}(\vec{v}_1\ \vec{v}_2\ \vec{v}_3 \ldots \vec{v}_n)$ has pivot in every column (P) $\Longleftrightarrow$ $\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n$ linearly independent (Q)

$P \Rightarrow Q$: Assume P. Let $A = (\vec{v}_1\ \vec{v}_2 \ldots \vec{v}_n \mid \vec{0})$ be the augumented matrix of a system. By (P) and the properties of the zero vector, rref(A) has no pivot in last column, and a pivot in every column of rref(C). Hence by Roché-Capalli theorem, the system only has a unique solution, By Prop 2.9, so does vector equation $c_1\vec{v_1} + c_2\vec{v_2} \cdots c_n\vec{v_n} = \vec{0}$. By observing the equation, we can see that $\vec{c} = (0, 0 \ldots 0) \in \mathbb{R}^n$ is always a solution.

Therefore, that's the unique solution.

By Def 2.12, $\vec{v_1}, \vec{v_2} \ldots \vec{v_n}$ are linearly indepedent

$Q \Rightarrow P$ $(\neg P \Rightarrow \neg Q)$: Assume $\neg P$.

That is, there is a column in $\text{rref}(\vec{v_1}\ \vec{v_2} \cdots \vec{v_n})$ without a pivot.

Therefore, $\text{rref}(A) = \text{rref}(\vec{v_1}\ \vec{v_2} \cdots \vec{v_n} \mid \vec{0})$ does not have a pivot in last column and has a non pivot column in rref(C). Therefore, the system represented by A as well as $c_1\vec{v_1} + c_2\vec{v_2} \cdots + c_n\vec{v_n} = \vec{0}$

has infinite solutions based on Roché Thm. This means there are other solutions then $\vec{0}$. So $\vec{V}_1, \vec{V}_2 \ldots V_n$ are linearly dependent.

Def: Let A be $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$.

We call $A\vec{x} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$

$$= \begin{pmatrix} a_{1.1}x_1 & a_{1.2}x_2 & \cdots & a_{1.n}x_n \\ a_{2.1}x_1 & a_{2.2}x_2 & & a_{2.n}x_n \\ \vdots & \vdots & & \vdots \\ a_{m.1}x_1 & a_{m.2}x_2 & & a_{m.n}x_n \end{pmatrix}$$

Obs: Let $\vec{a_1} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$, $\vec{a_2} = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}$ ...

$\vec{a_m} = \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}$ be m vectors in $\mathbb{R}^n$.

That is, $a_{11}, a_{21} \ldots a_{1m} \ldots a_{nm} \in \mathbb{R}$.

Let A be $n \times m$ matrix

$$(\vec{a_1}\ \vec{a_2}\ \vec{a_3} \cdots \vec{a_m}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

let $x_1, x_2 \ldots x_m \in \mathbb{R}$ and

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$.

Let $b_1, b_2 \ldots b_n \in \mathbb{R}$.

Let $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$ Hence all

the following system/equations have the same solution set:

① $$\begin{cases} a_{11}x_1 + a_{12}x_2 \cdots\cdots a_{1m}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 \cdots\cdots a_{2m}x_m = b_2 \\ \vdots \qquad\qquad \ddots \qquad\qquad \vdots \qquad\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 \cdots\cdots a_{nm}x_m = b_n \end{cases}$$

represented by augumented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & b_n \end{array}\right)$$

② $$\vec{a}_1 x_1 + \vec{a}_2 x_2 + \cdots \vec{a}_m x_m = \vec{b} \cong$$

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} x_2 + \cdots \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix} x_m = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

③ $A \cdot \vec{x} = \vec{b} \sim (\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_m)\vec{x} = \vec{b}$

$$\sim \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \vec{x} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \sim$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Def:

Basis of $\mathbb{R}^k$: B is a set of vectors $\{\vec{V_1}, \vec{V_2} \cdots \vec{V_n}\}$ where 1. $\text{Span}(\vec{V_1}, \vec{V_2} \ldots \vec{V_n}) = \mathbb{R}^k$

2. $\vec{V_1}, \vec{V_2} \cdots \vec{V_n}$ linearly independent

Obs: ① $|B| = k$

② not unique

Pf①: Let B be a basis in $\mathbb{R}^k$: $\{\vec{V_1} \cdots \vec{V_n}\}$

Hence $|B| = k$, and is called the dimension : $\dim(\mathbb{R}^k) = k$.

Assume case 1 : $|B| > k$. Hence $n > k$. and B linearly independent. We also have standard basis $E = \{\vec{e_1}, \vec{e_2} \ldots \vec{e_n}\}$.

We know $\vec{v}_1 = a_{1.1}\vec{e}_1 + a_{1.2}\vec{e}_2 + \cdots a_{1.k}\vec{e}_k$

$\vec{v}_2 = a_{2.1}\vec{e}_1 + a_{2.2}\vec{e}_2 + \cdots a_{2.k}\vec{e}_k$

$\vdots$

$\vec{v}_n = a_{n.1}\vec{e}_1 + a_{n.2}\vec{e}_2 + \cdots a_{n.k}\vec{e}_k$

Since B linear independent

- $b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots b_n\vec{v}_n = \vec{0}$ ~

- $b_1\sum_{i=1}^{k} a_{1.i}\vec{e}_i + b_2\sum_{i=1}^{k} a_{2.i}\vec{e}_i + \cdots b_n\sum_{i=1}^{k} a_{n.i}\vec{e}_i = \vec{0}$ ~

- $\vec{e}_1\sum_{i=1}^{n} b_i a_{i.1} + \vec{e}_2\sum_{i=1}^{n} b_i a_{i.2} \cdots \vec{e}_k\sum_{i=1}^{n} b_i a_{k.2} = \vec{0}$ ~

and $(b_1, b_2 \ldots b_n) = (0, 0 \ldots 0)$ △

- Since E is a basis,

$\vec{e}_1, \vec{e}_2 \ldots \vec{e}_k$ are linear independent.

Hence $\left(\sum_{i=1}^{n} b_i a_{i.1}, \sum_{i=1}^{n} b_i a_{i.2} \cdots \sum_{i=1}^{n} b_i a_{i.k}\right)$

$= (0, 0 \ldots\ldots 0)$

$$\begin{cases} a_{1.1} b_1 + a_{2.1} b_2 + a_{3.1} b_3 + \cdots a_{n.1} b_n = 0 \\ a_{1.2} b_1 + a_{2.2} b_2 + \cdots a_{n.2} b_n = 0 \\ \vdots \\ a_{1.k} b_1 + a_{2.k} b_2 + \cdots a_{n.k} b_n = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} a_{1.1} & a_{2.1} & \cdots & a_{n.1} & 0 \\ a_{1.2} & a_{2.2} & \cdots & a_{n.2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{1.k} & a_{2.k} & \cdots & a_{n.k} & 0 \end{array}\right)$$

augmented $k \cdot n$ matrix A

Since there are k rows and $n > k$, there are at most k pivots and at least 1 column without a pivot, so there are infinite solutions for $(b_1\ b_2 \dots b_n)$.

Contradicts $\triangle$. Hence $n \leq k$

Assume case 2: $|B| < k$:

$B = (\vec{v}_1, \vec{v}_2 \dots \vec{v}_n)$. linearly dependent.

Observe standard basis E

$= \{\vec{e}_1, \vec{e}_2 \cdots \vec{e}_k\}$, and vector equation

$a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots a_k\vec{e}_k = \vec{0}$. and $\vec{a} = \vec{0}$

We know $\exists\, b_{1.1}, b_{2.1} \dots b_{n.1} \in \mathbb{R}$ s.t.

$\vec{e}_1 = b_{1.1}\vec{v}_1 + b_{2.1}\vec{v}_2 + \cdots b_{n.1}\vec{v}_n$ — —

$$\vec{e}_k = b_{1.k}\vec{V}_1 + b_{2.k}\vec{V}_2 + \cdots b_{n.k}\vec{V}_n$$

Hence $a_1 b_{1.1}\vec{V}_1 + a_1 b_{2.1}\vec{V}_2 + \cdots a_1 b_{n.1}\vec{V}_n +$

$a_2 b_{1.2}\vec{V}_1 + a_2 b_{2.2}\vec{V}_2 + \cdots\cdots a_2 b_{n.2}\vec{V}_n +$

$\vdots$

$a_k b_{1.k}\vec{V}_1 + a_k b_{2.k}\vec{V}_2 + \cdots a_k b_{n.k}\vec{V}_n = \vec{0}$

$\sim$

$(a_1 b_{1.1} + a_2 b_{1.2} + \cdots a_k b_{1.k})\vec{V}_1 +$

$(a_1 b_{2.1} + a_2 b_{2.2} + \cdots\cdots a_k b_{2.k})\vec{V}_2 + \cdots$

$(a_1 b_{n.1} + a_2 b_{n.2} + \cdots\cdots a_k b_{n.k})\vec{V}_n = \vec{0}$

since B linearly independent,

$$\begin{cases} a_1 b_{1.1} + a_2 b_{1.2} + \cdots a_k b_{1.k} = 0 \\ a_1 b_{2.1} + a_2 b_{2.2} + \cdots a_k b_{2.k} = 0 \\ \vdots \\ a_1 b_{n.1} + a_2 b_{n.2} + \cdots a_k b_{n.k} = 0 \end{cases}$$

solve for $(a_1, a_2 \ldots a_k)$:

$$\left(\begin{array}{cccc|c} b_{1.1} & b_{1.2} & \cdots & b_{1.k} & 0 \\ b_{2.1} & b_{2.2} & \cdots & b_{2.k} & 0 \\ \vdots & & \ddots & & \vdots \\ b_{n.1} & b_{n.2} & \cdots & b_{n.k} & 0 \end{array}\right)$$

$k \times n$ augument matrix. Since $n < k$,

There's at least one column in rref(C) with no pivot and last column in rref(A)

Thus $(a_1, a_2 \ldots a_k)$ has ∞ solution other than $\vec{0}$. Controdicts

Obs1: For any span

$\text{Span}\{\vec{v}_1, \ldots \vec{v}_n\}$, $\vec{0} \in \text{Span}\{\vec{v}_1, \ldots \vec{v}_n\}$

And $\text{Span}\{\ \} = \{\vec{0}\}$

$$\text{col}(A) = \text{span}(\vec{a}_1 \cdots\cdots \vec{a}_m)$$

$$\text{null}(A) = \{\vec{x} : A\vec{x} = 0\}$$

$$A\vec{x} = \vec{b}$$

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

system consistent $\Longleftrightarrow$

matrix form has solution for $\vec{x}$ $\Longleftrightarrow$

$\vec{b}$ in the span of column vectors