① Set: A collection of objects

$\{\ \}$. No repeated objects

(1) $\mathbb{N}$: {Non-negative integers}

(2) $\mathbb{Z}$: {Integers}

(3) $\mathbb{Q}: \{\frac{p}{q} \mid\mid p, q \in \mathbb{Z} \land q \neq 0\}$

(4) $\mathbb{R} \approx$ {convergent sequences w/ elements from $\mathbb{Q}$}

② Notations

$$A \subseteq B \Rightarrow \{\forall x \in \mathbb{R}, x \in A \Rightarrow x \in B\}$$

$$A \cap B : \{x \mid\mid x \in A \land x \in B\}$$

$$A \cup B : \{x \mid\mid x \in A \lor x \in B\}$$

$$A \setminus B = \{x \mid\mid x \in A \land x \notin B\}$$

$$\mathbb{Q} \setminus \mathbb{Z} = \{\frac{p}{q} \mid\mid q \neq 0, p, q \in \mathbb{Z}, (\forall k \in \mathbb{Z}\ p \neq kq)\}$$

③ Thm De Morgan's Laws

Let A, B, C be sets, then

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

Pf. (a) $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$

Let $x \in A \setminus (B \cap C)$. Hence

$x \in A \wedge (x \notin B$ or $x \notin C)$

wlog, Assume $x \in A \wedge x \notin B$

Hence $x \in A / B$

Hence $x \in A/B \cup A/C$

(b) $A \setminus (B \cap C) \supseteq (A \setminus B) \cup (A \setminus C)$

Assume $x \in (A \setminus B) \cup (A \setminus C)$,

Hence $x \in A \setminus B$ or $x \in A \setminus C$

Wlog assume $x \in A \setminus B$

Hence $x \in A \wedge x \notin B$.

Since $B \cap C \subseteq B$, Δ

$x \notin B \cap C$. Since $x \in A$,

$x \in A \setminus (B \cap C)$. Δ δ?

__Thm2__

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

(a) $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$

Let $x \in A \setminus (B \cup C)$.

Hence $\underbrace{x \in A}_{(1)} \wedge \underbrace{x \notin B}_{(2)} \wedge \underbrace{x \notin C}_{(3)}$.

By (1)(2), $x \in A \setminus B$. By (1)(3), $x \in A \setminus C$.

Hence $x \in (A \setminus B) \cap (A \setminus C)$

(b) $A \setminus (B \cup C) \supseteq (A \setminus B) \cap (A \setminus C)$

Let $x \in (A \setminus B) \cap (A \setminus C)$.

Hence $x \in A \setminus B \wedge x \in A \setminus C$

Hence $x \in A \wedge x \notin B \wedge x \notin C$

Since $x \in B \cup C \iff x \in B \vee x \in C$,

$x \notin B \wedge x \notin C \implies x \notin B \cup C$

Hence $x \in A \wedge x \notin B \cup C$

Hence $x \in A \setminus B \cup C$.

④ A system of Linear Equations

in vars $x_1 \ldots x_n$ is a collection of linea equations in var $x_1 \ldots x_n$ (same variables)

eg
$$\begin{cases} (1)\ a_{1,1}x_1 + a_{1,2}x_2 + \cdots a_{1,n}x_n = D_1 \\ (2)\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots a_{2,n}x_n = D_2 \\ \quad \vdots \\ (m)\ a_{m,1}x_1 + \cdots a_{m,n}x_n = D_m \end{cases}$$

constant coeficients

A tuple $(s_1, \ldots s_n)$ is a solution to the system if its a solution to every equation in it.

$m \times n$ MATRIX : m rows, n columns

$$\left(\begin{array}{ccc|c} a_{1,1} & \cdots & a_{1,n} & D_1 \\ & \vdots & & \\ a_{m,1} & \cdots & a_{m,n} & D_m \end{array}\right)$$

augumented coefficient matrix

coefficient matrix

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \vdots & \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}$$

$$\begin{cases} 3x+9y=0 \\ 2x+4y=0 \end{cases} :$$

$$\left[\begin{array}{cc|c} 3 & 9 & 0 \\ 2 & 4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

⑥ Def. System are called equivalent if they have the same solution set. non-empty

Rmk. Performing elementary operations on system yield equivalent system.

Def. Two matrix are row equivalent if one can be obtained from another through a seq of elementary observations.

Thm. If 2 matrix are row equivalent
Then the systems they represent are
equivalent.

④ Def. Given a row in a matrix,
the 1st non-zero entry counting
from left to right is a matrix

Def 1. A matrix is in REF if:
(1) all zero rows are in the bottom
(2) each pivot occurs to the right of
the pivot of the preceding row.

Def 2 A matrix in RREF if:
(1) It's in REF
(2) All pivots are 1
(3) Each pivot is the only non-zero entry
in the column

⑤ Thm Gauss-Jordan $\forall$ matrix A,

(1) $\exists$ matrix B s.t. B is in RREF and (2) A equivalent to B.

(3) B is unique (It's the only matrix that is rref and $B \sim A$)

(4) B can be obtained by Gaussian elimination. we call $B = \mathrm{rref}(A)$

⑥ any system represented by a augumented matrix in ref $\vee$ rref can be solved (easily) with back substitution.

Alg : To solve a system of linear equations:

1. write down the corresponding aug matrix A.

2. find $B = \mathrm{rref}(A)$ through Gaussian Elimination (Thm ⑤)

3. find the solution of the system represented by B (6)

4. Since A & B are equivalent (Thm 5), we also found sol to the original system represented by A.

Def 7 Let A be a matrix.

Let $X = \text{rref}(A)$.

1. Given ith column of A, we say it's a pivot column IFF the ith column of X has a pivot.

2. We say x_i is a basic variable of the system represented by A if ith column is pivot column to A.

3. We say x_i is a free variable of the system represented by A if ith column is pivot column to A.

This is informal!!!!

4. If a system has a free variable, then the system has infinite solutions, and the solution set can be represented using the free var as a

parameter.

e.g. $\left[\begin{array}{ccc|c} 1 & 0 & a & b \\ 0 & 1 & c & d \end{array}\right] \rightarrow \begin{cases} x_1 \quad + a \cdot x_3 = b \\ x_2 + c \cdot x_3 = d \end{cases}$

sol: $\{(b-az, d-cz, z) \mid z \in \mathbb{R}\}$

Def ⑧ system has ≥ 1 solution $\Leftrightarrow$ syst consistent

system has 0 solutions $\Leftrightarrow$ syst inconsistent

Thm ⑨: Rouché-Capalli,

Let A be the augumented matrix, and C be the coefficient matrix of a system.

(1) system inconsistent IFF last column of rref(A) has a pivot

(2) system has one solution IFF last column of rref(A) has 0 pivots and every column of rref(C) has a pivot

(3) System has ∞ solutions IFF last column of rref(A) has 0 pivots and ≥ 1 columns of rref (C) has 0 pivots!

Pf. Let C be $m\times n$ thus A $m\times(n+1)$ matrix

(1).a $\Leftarrow$ Suppose last column of rref (A) has a pivot. Since pivot is the leftmost non-zero entry, that row would like this by definition:

$$[0, 0 \underbrace{\cdots\cdots}_{\text{all } 0s} 0\ 0 \mid 1]$$

This represents equation

$0\cdot x_1 + 0\cdot x_2 + \cdots 0\cdot x_n = 1$, which has solution set $\emptyset$. So the solution set to system is also $\emptyset$.

(1). b $\Rightarrow$ suppose controposative that last column of rref (A) has no pivot. Then for any row, it's either zero or has a pivot in the coefficient submatrix.

Suppose there are r non-zero rows. Let i be an arbitarary non zero row $(1 \leq i \leq r)$ with a pivot in i column $(1 \leq i \leq n)$.

Hence the row represents equation

reprsents equation

$$x_i + \sum_{l=i+1}^{n} r_{j,l} x_l = a_j$$

$$x_i = a_j - \sum_{l=i+1}^{n} r_{j,l} x_l$$

ALWAYS has a real value.
so every variable has ≥ 1 real value so system consistent.

We showed:
last column of rref(A) no pivot
$\Rightarrow$ system consistent.

Thus system inconsistent $\Rightarrow$ last column of rref(A) has a pivot.

(2). a ($\Leftarrow$) Suppose last column of rref(A) has 0 pivot and every column of rref (C) has pivot.

Thus rref(C) has n pivots. By P1.1, rref(C) has most min (m, n) pivots. Hence $n \leq \min(m,n) \leq m$.

Since $m \geq n$, and each row has 0 or 1 pivots, and rref(C) has n pivots, it must be the case, that rref(A) has $m-n$ rows with 0 pivots.

Therefore, rref(A) must look like:

$$
\begin{array}{c}
\\
m\left\{ \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right. \\
m-n\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right.
\end{array}
\overbrace{\begin{array}{ccccc|c}
1 & 0 & 0 & \cdots & 0 & a_1 \\
0 & 1 & 0 & \cdots & 0 & a_2 \\
0 & 0 & 1 & \cdots & 0 & a_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & & 1 & a_n \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0
\end{array}}^{n}
$$

Hence we can solve the system it represents using back substitution and get:

$$
\begin{cases}
x_1 = a_2 \\
x_2 = a_2 \\
\vdots \\
x_n = a_n
\end{cases}
$$

as a <u>unique</u> solution.

(2). b ($\Rightarrow$) suppose system has unique solution $(a_1, a_2 \ldots a_n)$ so the row

Equivalent to system $\begin{cases} x_1 = a_2 \\ x_2 = a_2 \\ \vdots \\ x_n = a_n \end{cases}$ which

has augumented matrix A_2:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & & & 0 & a_1 \\ 0 & 1 & 0 & \cdots & & 0 & a_2 \\ 0 & 0 & 1 & \ddots & & 0 & \vdots \\ \vdots & & & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & & 1 & a_n \end{array}\right].$$

Hence $A \sim A_2$.

Since A_2 is in rref $\wedge$ $A \sim A_2$,
by uniqueness of rref (A) in
Gauss - Jordan Thm, $A_2 = \text{rref}(A)$.
Notice it has no pivots in last column
and 1 pivot in any other column.
So no pivot in last column of rref (A)
and 1 pivot in every column of rref(C).

(3). a $\Leftarrow$) suppose last column in
rref (A) has no pivot and jth column
in rref (C) has no pivot.

By Δ, system is consistent (has ≥ 1 solution).
By ☆, we can show by back substitution that x_j is a free variable that can take any value. so there are infinite solutions.

b (=> Suppose infinite solutions.
Since the system is consistent, by (1) a. Last column has no pivot.
Since ∞ solutions, there must be ≥ 1 free variable, thus ≥ 1 non-pivot column in rref(C).