Applications of Functional Dependence to Spatial and Network Econometrics

Zeqi Wu

Institute of Statistics and Big Data, Renmin University of China

Talk@Clubear

November 28th, 2025

References

- Wu, Zeqi, Wen Jiang, and Xingbai Xu (2024): "Applications of Functional Dependence to Spatial Econometrics," *Econometric Theory*, published online, 1–36.
  - the individuals are located in $\mathbb{R}^d$;
- Jiang, Wen, Yachen Wang, Zeqi Wu, and Xingbai Xu (2025): "Limit Theorems for Network Data without Metric Structure," *arXiv:2511.17928*.
  - Individuals need not locate in a metric space.

Motivation

- Law of large numbers (LLN) and central limit theorems (CLT) are indispensable for econometrics and statistics
- Spatial econometrics: LLN/CLT for spatially correlated data
- there are some LLN/CLT in the literature: linear-quadratic form, mixing, near-epoch dependence (NED)
- they are not convenient enough or some strong conditions are needed for some applications
- We aim to develop a weak spatial dependence concept that is more convenient to use than above concepts, especially NED

Tools for Spatial Econometrics: Linear-Quadratic Forms

- linear-quadratic forms: Kelejian and Prucha (1998, 2001); Lee (2004, 2007); Yu et al. (2008); ....
- $\epsilon_{i,n}$'s are independent, $\epsilon_n = (\epsilon_{1,n}, \cdots, \epsilon_{n,n})'$: $\epsilon_n' A \epsilon_n + b'\epsilon_n$
- useful for linear models (Spatial autoregressive (SAR) model)

$$Y_n = \lambda W_n Y_n + X_n \beta + \epsilon_n \Rightarrow Y_n = (I_n - \lambda W_n)^{-1}(X_n\beta + \epsilon_n)$$

- appears in the log-likelihood function, GMM, Moran's I test statistics. e.g., $(Y_n - \lambda W_n Y_n - X_n\beta)' P_n (Y_n - \lambda W_n Y_n - X_n\beta)$
- inconvenient for
  - many nonlinear estimators (quantile estimator, Huber estimator)
  - nonlinear spatial models (Tobit model)

Tools for Spatial Econometrics: Mixing

- mixing is widely used in time series and panel data
- however, spatial mixing (Jenish and Prucha, 2009) is not widely used in spatial econometrics, due to:
  - hard to establish, as it involves supremum over two σ-fields:

$$\alpha(\mathcal{A},\mathcal{B}) \equiv \sup_{A\in\mathcal{A},B\in\mathcal{B}} |P(A\cap B) - P(A)P(B)|$$

  - not preserved under infinity summations (e.g., $\sum_{j=1}^{n} w_{ij}y_j$)
- Xu and Lee (2023): the mixing property of linear spatial processes.
- For nonlinear processes, as far as we know, no work so far.

Tools for Spatial Econometrics: Near-Epoch Dependence

An Example: $y_t = \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \cdots$: y_t is mainly affected by ϵ_{t-s} for small s, and the contribution of all ϵ_{t-s} (large s) is small.

Definition. $\{Z_{i,n}, i \in D_n, n \geq 1\}$ is generated by $\{v_{i,n}, i \in D_n, n \geq 1\}$. $\{Z_{i,n}\}$ is said to be L^p-near-epoch dependent (NED) on $\{v_{i,n}\}$ if

$$\left\| Z_{i,n} - \mathbb{E}[Z_{i,n} | v_{j,n}, d_{ij} \leq s] \right\|_{L^p} \leq \psi(s) \to 0 \text{ as } s \to \infty.$$

Tools for Spatial Econometrics: Near-Epoch Dependence

- (spatial) near-epoch dependence (NED): Jenish and Prucha (2012), Xu and Lee (2015a, 2018), Qu and Lee (2015), Qu et al. (2017), Liu et al. (2022), Xu et al. (2022, quantile regression).
- Its applications are wide, but
  - sometimes strong moment conditions are needed to preserve NED
  - Limited to L^2-NED, as conditional expectation might not be easy to calculate

What's our work?

- We aim to develop a weak spatial dependence concept that is more convenient to use than NED
- We generalize the concept of functional dependence (Wu, W.B., 2005, PNAS) to the settings of spatial econometrics:
  - irregular lattice in $\mathbb{R}^d$, triangular arrays.
- We establish a set of theoretical tools for functional dependent data: inequalities, LLN, CLT, properties of functional dependence under various transformations.

Functional Dependence Measure

- Suppose there are some individuals (persons, cities, countries, etc, also called spatial units, nodes), and they are located in a lattice $D_n \subset \mathbb{R}^d$.
- For a cross-sectional data with n individuals, $|D_n| = n$.
- Let $\epsilon = \{\epsilon_{i,n}, i \in D_n, n \geq 1\}$ be independent $\mathbb{R}^{p_\epsilon}$-valued triangular array

$$Y_{i,n} = g_{i,n}(\epsilon_{1,n}, \cdots, \epsilon_{n,n}) = g_{i,n}(\epsilon_n), \tag{1}$$

Functional Dependence Measure

$$Y_{i,n} = g_{i,n}\left(\epsilon_n\right).$$

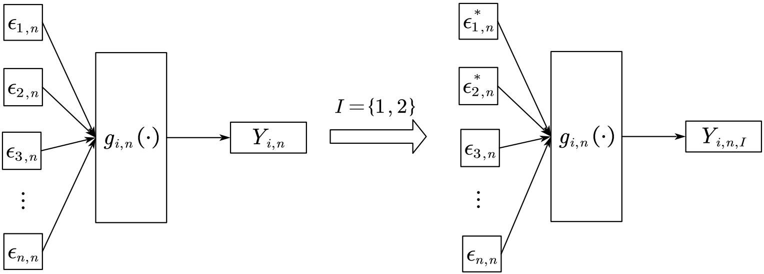
- $\forall i \in D_n$, let $\epsilon^*_{i,n}$ be an i.i.d. copy of $\epsilon_{i,n}$, and $\epsilon^*_{i,n}$ is independent to all $\epsilon_{j,n}, j \in D_n$.
- $\forall I \subset D_n$, define $\epsilon_{i,n,I} \equiv \epsilon^*_{i,n}$ if $i \in I$ and $\epsilon_{i,n,I} \equiv \epsilon_{i,n}$ if $i \notin I$; denote $\epsilon_{n,I} = (\epsilon_{i,n,I})_{i \in D_n}$.
- $Y_{i,n,I} = g_{i,n}\left(\epsilon_{n,I}\right)$ denotes a coupled version of $Y_{i,n}$ on I.

Definition. Let $p \geq 1$ be a constant. $\forall n \geq 1$ and $I \subset D_n$, the L^p spatial FDM is

$$\delta_p\left(i, I, n\right) \equiv \left\|Y_{i,n} - Y_{i,n,I}\right\|_{L^p}.$$

Functional Dependence Measure

The $\delta_p(i,j,n) \equiv \left\| Y_{i,n} - Y_{i,n,I} \right\|_{L^p}$ measures the influence of $\{\epsilon_{j,n} : j \in I\}$ on $Y_{i,n}$: if $\{\epsilon_{j,n} : j \in I\}$ is replaced by its i.i.d. copy $\{\epsilon^*_{j,n} : j \in I\}$, how much $Y_{i,n}$ will change under L^p-norm.

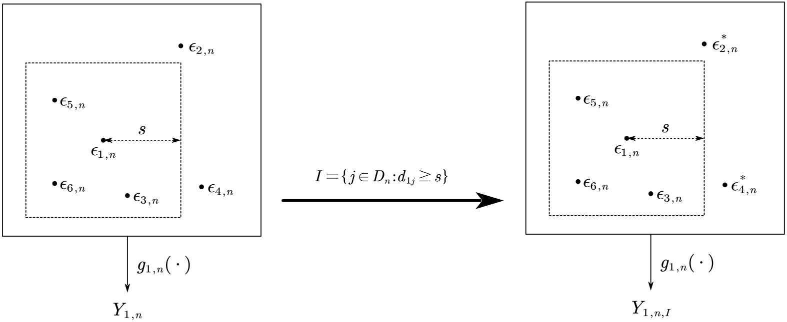


Functional Dependence Coefficient

Definition. $\left\{Y_{i,n}, i \in D_n, n \geq 1\right\}$ is said to be *L^p-functionally dependent* on $\epsilon = \left\{\epsilon_{i,n}, i \in D_n, n \geq 1\right\}$ if and only if the *L^p-functionally dependent coefficient*

$$\Delta_p(s) \equiv \sup_n \sup_{i \in D_n} \delta_p\left(i, \left\{j \in D_n : d_{ij} \geq s\right\}, n\right) \to 0 \quad \text{as } s \to \infty. \tag{2}$$

Functional Dependence Coefficient



Functional Dependence Coefficient

$$\Delta_p(s) \equiv \sup_n \sup_{i \in D_n} \delta_p\left(i, \left\{j \in D_n : d_{ij} \geq s\right\}, n\right) \to 0 \quad \text{as } s \to \infty.$$

- $\lim_{s\to\infty} \Delta_p(s) = 0$: the total impacts from individuals far away can be arbitrarily small uniformly in i and n.

- By Lyapunov's inequality, if $Y_{i,n}$ is L^p-FD, it is also L^q-FD for all $q \in [1, p]$.

- "monotonicity": $\Delta_p(s) \leq 3\Delta_p(\tilde{s})$ for any $s \geq \tilde{s}$.

Example: SAR Model

$$\begin{pmatrix} Y_{1,n} \\ \vdots \\ Y_{n,n} \end{pmatrix} = Y_n = F\left(\lambda W_n Y_n + \epsilon_n\right) = \begin{pmatrix} F\left(\lambda w_{1.,n} Y_n + \epsilon_{1,n}\right) \\ \vdots \\ F\left(\lambda w_{n.,n} Y_n + \epsilon_{n,n}\right) \end{pmatrix}, \tag{3}$$

- $W_n = (w_{ij,n})_{n \times n}$ is non-stochastic spatial weights matrix
  - $w_{i.,n}$: the ith row of W_n
- F is a Lipschitz function: $\left|F\left(e^{\bullet}\right) - F\left(e\right)\right| \leq L\left|e^{\bullet} - e\right|$
- $\zeta = L\left|\lambda\right| \sup_n \left\|W_n\right\|_{\infty} < 1$: Eq.(3) has a unique solution $\Rightarrow Y_{i,n} = Y_{i,n}(\epsilon_n)$

Example: SAR Model

$$Y_n = F\left(\lambda W_n Y_n + \epsilon_n\right).$$

- Let $M_n \equiv L(I_n - L\,|\lambda|\,|W_n|)^{-1}$, where $|W_n| \equiv (|w_{ij,n}|)_{n\times n}$.
- Then

$$|Y_{i,n}(\epsilon_n^{(1)}) - Y_{i,n}(\epsilon_n^{(2)})| \leq \sum_{j=1}^{n} M_{ij,n}|\epsilon_{j,n}^{(1)} - \epsilon_{j,n}^{(2)}|.$$

Example: SAR Model

$$|Y_{i,n}(\epsilon_n^{(1)}) - Y_{i,n}(\epsilon_n^{(2)})| \leq \sum_{j=1}^{n} M_{ij,n} |\epsilon_{j,n}^{(1)} - \epsilon_{j,n}^{(2)}|.$$

- the L^p-*FD coefficient*

$$\Delta_p(s) \leq \left(2 \sup_{j,n} ||\epsilon_{j,n}||_p\right) \sup_{i,n} \sum_{j \in D_n : d_{ij} \geq s} M_{ij,n}.$$

- Suppose $\epsilon_{i,n}$'s are independent over i and uniformly L^p-bounded.
  1. $w_{ij,n} \neq 0$ only if $d_{ij} < \bar{d}_0$:

$$\Delta_p(s) \leq C\zeta^{s/\bar{d}_0} \to 0 \quad \text{as } s \to \infty;$$

  2. $|w_{ij,n}| \leq c d_{ij}^{-\alpha}$ for some constants $c > 0$ and $\alpha > d$.

$$\Delta_p(s) \leq O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right) \to 0 \quad \text{as } s \to \infty.$$

SAR Models: Extensions

- In most of cases: $\epsilon_{i,n} = X_{i,n}'\beta + e_{i,n}$. We can allow $\{\epsilon_{i,n}\}$ to be dependent by
  - assuming that $\{\epsilon_{i,n}\}$ is functionally dependent on another random field $\{\eta_{i,n}\}$;
  - $\Delta_p(s) \leq O(s^{-(\alpha-d)}(\log s)^{\alpha-d}) + \Delta_p^{\epsilon}(s/2)$.
- We also calculate the FDM for
  - the threshold SAR models
  - functional-coefficient SAR models
  - smooth-coefficient SAR models
  - SAR models with stochastic weights matrices
  - ...

A Moment Inequality

Let $S_n \equiv \sum_{i \in D_n} Y_{i,n}$.

Theorem. For $p \geq 2$, if the L^p-FD coefficient $\Delta_p(s) = O\left(s^{-\kappa}\right)$ for some $\kappa > \frac{d}{2}$ as $s \to \infty$,

$$\|S_n - \mathbb{E}S_n\|_{L^p} \leq C\sqrt{n}.$$

- The rate is the same as the i.i.d. case;
- For SAR models, when $\left|w_{ij,n}\right| \leq c d_{ij}^{-\alpha}$, we have $\Delta_p(s) \leq O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right)$
- If $\alpha > 1.5d$, then $\Delta_p(s) = O\left(s^{-\kappa}\right)$

An Exponential Inequality

- Exponential inequality is useful in semi/nonparametric econometrics, high-dimensional statistics, machine learning.

Theorem. Assume $\{Y_{i,n}\}$ is L^p*-functionally dependent* on $\{\epsilon_{i,n}\}$ with $\Delta_p(s) \leq O(p^\nu)\, O(s^{-\kappa})$ for some $\kappa > \frac{d}{2}$ and $\nu \geq 0$. Denote $\alpha = \frac{2}{1+2\nu}$. Then $\forall \delta > 0$,

$$\mathbb{P}\left(|S_n - \mathbb{E}S_n| \geq n\delta\right) \leq C_1 \exp\left(-C_2 n^{\alpha/2} \delta^\alpha\right).$$

- SAR: $\Delta_p(s) \leq (2 \sup_{j,n} ||\epsilon_{j,n}||_p) \sup_{i,n} \sum_{j \in D_n : d_{ij} \geq s} M_{ij,n}$;
- $O(s^{-\kappa})$: $|w_{ij,n}| \leq c d_{ij}^{-\alpha}$ for some $\alpha > 1.5d$;
- $O(p^\nu)$:
  1. when $\epsilon_{i,n}$ is subexponential, $\nu = 1$, $n^{\alpha/2} = n^{1/3}$
  2. when $\epsilon_{i,n}$ is sub-Gaussian, $\nu = \frac{1}{2}$, $n^{\alpha/2} = n^{1/2}$
  3. when $\epsilon_{i,n}$ is uniformly bounded, $\nu = 0$, $n^{\alpha/2} = n$

An Exponential Inequality

$$\mathbb{P}\left(\left|S_n - \mathbb{E}S_n\right| \geq n\delta\right) \leq C_1 \exp\left(-C_2 n^{\alpha/2}\delta^{\alpha}\right).$$

- The proof is based on the moment inequality and Taylor formula for e^x
- Our result: $n^{\alpha/2} = n^{1/(1+2\nu)}$;
  - rate in Xu and Lee (2018): $n^{1/(2d+2+\nu)}$
- spatial FD: allow $w_{ij} \lesssim d_{ij}^{-\alpha}$;
  - rate in Xu and Lee (2018) requires: $w_{ij} \lesssim \exp(-d_{ij}^{\alpha})$
- When $\epsilon_{i,n}$ is uniformly bounded, $\alpha = 2$.
  - the decaying rate with respect to n is the same as the standard Hoeffding's inequality

Law of Large Numbers

Theorem. If (i) $\{Y_{i,n}\}$ is uniformly L^p-bounded for some $p > 1$, and (ii) $\{Y_{i,n}\}$ is L^1-FD on $\{\epsilon_{i,n}\}$, i.e., $\lim_{s\to\infty} \Delta_1(s) = 0$, then

$$\frac{1}{n}\left(S_n - \mathbb{E}S_n\right) \xrightarrow{L^1} 0.$$

Central Limit Theorem

Theorem. Denote $\sigma_n^2 = \mathrm{Var}\left(S_n\right)$, if
(i) $Y_{i,n}$ is uniformly L^p-bounded for some $p > 2$,
(ii) $\liminf_{n\to\infty} \sigma_n^2/n > 0$,
(iii) $\Delta_2(s) = O\left(s^{-\kappa}\right)$ for some $\kappa > \frac{d}{2}$ as $s \to \infty$,
then

$$\frac{S_n - \mathbb{E}S_n}{\sigma_n} \xrightarrow{d} N\left(0, 1\right).$$

- For SAR models with $w_{ij} \lesssim d_{ij}^{-\alpha}$:
  - FD CLT requires: $\alpha > 1.5d$;
  - the NED CLT requires: $\alpha > 2d$;
- By the Cramér–Wold device, we can generalize the CLT to the multivariate case.

Conditional Spatial Functional Dependence

- $(\Omega, \mathcal{F}, \mathbb{P})$: the underlying probability space; $\mathcal{C}$ is a sub-σ-field of $\mathcal{F}$.
- $\epsilon_{i,n}$'s conditionally independent given $\mathcal{C}$.
- conditional on $\mathcal{C}$, $\epsilon^*_{i,n}$ is an i.i.d. copy of $\epsilon_{i,n}$.
- For a set $I \subset D_n$, define $\epsilon_{i,n,I} \equiv \epsilon^*_{i,n}$ if $i \in I$, and $\epsilon_{i,n,I} \equiv \epsilon_{i,n}$ if $i \notin I$.
- $Y_{i,n} = g_{i,n}(\epsilon_n)$; $Y_{i,n,I} = g_{i,n}\left(\epsilon_{n,I}\right)$ is a coupled version of $Y_{i,n}$ on I.

Definition. For $p \geq 1, n \geq 1$ and $I \subset D_n$, define the conditional FDM

$$\delta_p^{\mathcal{C}}(i, I, n) \equiv \left\|Y_{i,n} - Y_{i,n,I}\right\|_{L^p,\mathcal{C}} \equiv \left(\mathbb{E}|Y_{i,n} - Y_{i,n,I}|^p|\mathcal{C}\right)^{1/p}.$$

And $\left\{Y_{i,n}, i \in D_n\right\}$ is $\mathcal{C}$-L^p-*functionally dependent* on $\epsilon = \left\{\epsilon_{i,n}, i \in D_n\right\}$ if and only if

$$\Delta_p^{\mathcal{C}}(s) \equiv \sup_{n\geq 1} \sup_{i \in D_n} \delta_p^{\mathcal{C}}\left(i, \left\{j : d_{ij} \geq s\right\}, n\right) \xrightarrow{a.s.} 0 \quad \text{as } s \to \infty.$$

Spatial Panel Data Model

- T periods, N spatial units in $D_N \subset \mathbb{R}^d$.
- for individual i at time t, we regard (i,t) as a point in $\mathbb{R}^{d+1}$:
$(i,t) \in D_{NT} \equiv \left\{(i,t) \in \mathbb{R}^{d+1} : i \in D_N, t = T, T-1, \ldots\right\}$

$$d_{it;j\tau} \equiv \left\|(i,t) - (j,\tau)\right\|_{\infty} \equiv \max\left\{\max_{1 \le k \le d} |i_k - j_k|, |t-\tau|\right\}.$$

- Spatial dynamic panel data (SDPD) model:

$$Y_{Nt} = \lambda W_N Y_{Nt} + \gamma Y_{N,t-1} + \rho W_N Y_{N,t-1} + X_{Nt}\beta + \mu_t l_N + \boldsymbol{\nu}_N + V_{Nt}$$

- We calculate the spatial FDM conditional on $(\mu_t, \boldsymbol{\nu}_N)$:
$\mathcal{C} \equiv \vee_{t=-\infty}^{\infty} \vee_{N=1}^{\infty} \sigma(\mu_t, \boldsymbol{\nu}_N)$

Spatial Panel Data Model

Assumptions:

1. $\left|w_{ij,N}\right| \leq c d_{ij}^{-\alpha}$ for some constants $c > 0$ and $\alpha > d$.
2. $\sup_N \|W_N\|_\infty \leq 1$ and $|\lambda| + |\gamma| + |\rho| < 1$. And $\zeta \equiv \frac{|\gamma|+|\rho|}{1-|\lambda|} < 1$.
3. $\sup_{N,T} \sup_{i,t} \|\epsilon_{it}\|_{L^p,\mathcal{C}} < \infty$ a.s. for some $p \geq 1$.
4. Conditional on $\mathcal{C}$, (x_{it}', v_{it})'s are independent over i and t.

Proposition. $\{y_{it} : (i,t) \in D_{NT}\}$ is $\mathcal{C}$-L^p-FD on $\{\epsilon_{it}\}$ with $\Delta_p^{\mathcal{C}}(s) = O\left(s^{-(\alpha-d)} (\log s)^{\alpha-d}\right)$ a.s. as $s \to \infty$.

Nonlinear Transformations in Spatial Econometrics

In spatial econometrics, we need to deal with a lot of nonlinear transformations about spatial random variables:

- In the log-likelihood function based on normal distribution: y^2
- censored data, binary data, quantile regression: $1(y > 0)$
- Least absolute deviation: $|y|$
- Spatial Tobit model: $\Phi(y)$, $\ln \Phi(y)$, $\frac{\phi(y)}{\Phi(y)}$, ...
- Huber estimator: $\max(0, x)$, or $\min(1, \max(0, x))$
- variance estimator: y^2 or yz

FDM under Nonlinear transformations

$H_{i,n}$ is a function:

$$\left\| H_{i,n}\left(y\right) - H_{i,n}\left(y^{\bullet}\right) \right\| \leq B_{i,n}\left(y, y^{\bullet}\right) \left\| y - y^{\bullet} \right\|$$

Denote $Z_{i,n} \equiv H_{i,n}\left(Y_{i,n}\right)$.

Proposition. If $B_{i,n}\left(y, y^{\bullet}\right) \leq C < \infty$, then $\delta_{Z,p}\left(i,j,n\right) \leq C\delta_{Y,p}\left(i,j,n\right)$ and $\Delta_{Z,p}\left(s\right) \leq C\Delta_{Y,p}\left(s\right)$.

- **Examples.** $H(x) = \phi(x)$, $H(x) = \max(x, 0)$, $H(x) = |x|$, $H(x) = \min(1, \max(0, x))$

FDM under Nonlinear transformations

Denote $Z_{i,n} \equiv H_{i,n}\left(Y_{i,n}\right)$

$$\left\|H_{i,n}\left(y\right)-H_{i,n}\left(y^{\bullet}\right)\right\| \leq B_{i,n}\left(y,y^{\bullet}\right)\left\|y-y^{\bullet}\right\|$$

Proposition. Suppose $B_{i,n}\left(y,y^{\bullet}\right) \leq C(\left\|y\right\|^{a}+\left\|y^{\bullet}\right\|^{a}+1)$ for some $a \geq 1$. The constants $p,q,r \geq 1$ satisfy $p^{-1}=q^{-1}+r^{-1}$. If $\{Y_{i,n}\}$ is L^{q}-FD on $\{\epsilon_{i,n}\}$ and $\sup_{n,i\in D_n}\left\|Y_{i,n}\right\|_{L^{ar}} < \infty$, then $\Delta_{p}^{Z}\left(s\right) \leq C_1\Delta_{q}^{Y}\left(s\right)$.

- Since $q > p$, this prop allows us to establish FD of lower order using FD of higher order. The order of $\Delta_{p}^{Z}\left(s\right)$ and $\Delta_{q}^{Y}\left(s\right)$ are the same.
- **Examples.** $H(x)=x^2$,
  $\left|\ln\Phi(x_1)-\ln\Phi(x_2)\right| \leq C(\left|x_1\right|+\left|x_2\right|+1)\left|x_1-x_2\right|$

FDM under Nonlinear transformations

Denote $Z_{i,n} \equiv H_{i,n}(Y_{i,n})$

$$\left\| H_{i,n}(y) - H_{i,n}(y^\bullet) \right\| \leq B_{i,n}(y, y^\bullet) \left\| y - y^\bullet \right\|$$

Proposition. Suppose $B_{i,n}(y, y^\bullet) \leq C\left(\|y\|^a + \|y^\bullet\|^a + 1 \right)$ for some $a \geq 1$ and $\sup_{n,i \in D_n} \left\| Y_{i,n} \right\|_{L^q} < \infty$ for some q satisfying $q > \max\{(a+1)p, \frac{ap}{p-1}\}$, where $p > 1$. Then

$$\Delta_p^Z(s) \leq C_2 \Delta_p^Y(s)^{(q-ap-p)/(pq-ap-p)}.$$

- This proposition allows us to calculate L^p FDM of Z using L^p FDM of Y, but the decreasing rate is slower, as $\frac{q-ap-p}{pq-ap-p} < 1$
- NED has a similar property

FDM under Nonlinear transformations

Proposition. Denote $Z_{i,n} \equiv 1(Y_{i,n} > 0)$. Suppose the probability density functions of $\{Y_{i,n}\}$ are uniformly bounded in i and n. Then there exists a constant $C > 0$ such that

$$\Delta_p^Z(s) \leq C \Delta_p^Y(s)^{1/(p+1)}.$$

Relationship between Spatial FD and NED

Definition. For some $p \geq 1$, $Z = \{Z_{i,n}, i \in D_n, n \geq 1\}$ said to be L^p-near-epoch dependent on $\epsilon = \{\epsilon_{i,n}, i \in D_n, n \geq 1\}$ if

$$\left\| Z_{i,n} - \mathbb{E}(Z_{i,n} | \epsilon_{j,n}, d_{ij} \leq s) \right\|_{L^p} \leq \psi(s)$$

with $\lim_{s\to\infty} \psi(s) = 0$.

- NED: every spatial unit is mainly affected by the $\epsilon_{j,n}$ of its close neighbors
- spatial FD: the impacts of the $\epsilon_{j,n}$'s faraway are small
- The ideas of these two concepts are similar.
- What's the relationship between FD and NED?

Relationship between Spatial FD and NED

Theorem. (1) When $\epsilon_{i,n}$'s independent, L^p-NED coefficient $\leq$ L^p-FD coefficient:

$$\psi_p(s) = \sup_{i,n} \left\| Z_{i,n} - \mathbb{E}\left(Z_{i,n} | \epsilon_{j,n}, d_{ij} \leq s\right) \right\|_{L^p} \leq \Delta_p(s).$$

(2) If $Y_{i,n} = \sum_{j \in D_n} w_{ij,n} \epsilon_{j,n}$, where $w_{ij,n}$'s are non-stochastic coefficients and $\epsilon_{i,n}$'s are independent, then

$$\Delta_p(s) \leq 2\psi_p(s).$$

- When $\epsilon_{i,n}$'s are NOT independent, the above conclusion might not hold.

Relationship between Spatial FD and NED

- Spatial FDM is more convenient to calculate than spatial NED, especially under nonlinear transformations and $p > 2$.
- It usually requires weaker conditions to establish a CLT and an exponential inequality by using FDM.
  - For CLT, it only requires $\alpha > 1.5d$ under FDM instead of $\alpha > 2d$ under NED.
  - The exponential inequality under FDM enjoys both less restrictive conditions and faster decaying rate.

Function Dependence in Network Data

- $1, 2, \ldots, n$ are n nodes in a network;
- i's might not be located in a Euclidean space $\mathbb{R}^d$;
  - e.g., financial and social networks
- the order of $1, 2, \ldots, n$ can be arbitrary;
- Let $\epsilon = \{\epsilon_{i,n} : 1 \leq i \leq n, n \geq 1\}$ be independent $\mathbb{R}^{p_\epsilon}$-valued triangular array

$$Y_{i,n} = g_{i,n}\left(\epsilon_{1,n}, \cdots, \epsilon_{n,n}\right) = g_{i,n}\left(\epsilon_n\right). \tag{4}$$

Definition

- Let $\epsilon^*_{j,n}$ be an i.i.d. copy of $\epsilon_{j,n}$, and $\epsilon^*_{j,n}$ is independent to $\epsilon_{i,n}$ for any $i \neq j$.
- Denote $Y_{i,n,j}$ as the coupled version of $Y_{i,n}$ with $\epsilon_{j,n}$ replaced by its i.i.d. copy $\epsilon^*_{j,n}$, i.e.,

$$Y_{i,n,j} \equiv g_{i,n}\left(\epsilon_{1,n}, \cdots, \epsilon_{j-1,n}, \epsilon^*_{j,n}, \epsilon_{j+1,n}, \cdots, \epsilon_{n,n}\right).$$

Definition. Let $p \geq 1$ be a constant. Define the functional dependence measure as

$$\delta_{p,n}(i,j) \equiv \left\|Y_{i,n} - Y_{i,n,j}\right\|_{L^p}. \tag{5}$$

It measures the impact of $\epsilon_{j,n}$ on $Y_{i,n}$.

Definition

- Recall

$$\delta_{p,n}(i,j) \equiv \left\| Y_{i,n} - Y_{i,n,j} \right\|_{L^p}.$$

Definition. For $p \geq 1$ and $q \geq 1$, $\{Y_{j,n}\}$ is said to be (L^p, q)-functionally dependent on $\{\epsilon_{i,n}\}$ if

$$\Delta_{p,q} \equiv \frac{1}{n^q} \sum_{j=1}^{n} \left[\sum_{i=1}^{n} \delta_{p,n}(i,j) \right]^q = o(1).$$

- A sufficient condition is

$$\frac{1}{n} \sum_{j=1}^{n} \left[\sum_{i=1}^{n} \delta_{p,n}(i,j) \right]^q = o(1)$$

for some $q > 1$.

Definition

- For SAR model, $\delta_{p,n}(i,j) \propto M_{ij,n}$, where $M_n \equiv L(I_n - L|\lambda W_n|)^{-1}$
- $\sum_{i=1}^{n} \delta_{p,n}(i,j) \propto \sum_{i=1}^{n} M_{ij,n}$ is the j^{th} column sum of M_n:
  - the total impact of $e_{j,n}$ on all $Y_{i,n}$'s
  - can be regarded as the "influence power" of j
- $\Delta_{p,q} = o(1)$ generalizes the condition $\sup_n \|M_n\|_1 < \infty$ in many spatial econometric papers.
  - We allow $\sup_{j,n} \sum_{i=1}^{n} \delta_{p,n}(i,j) = \infty$.
- However, $\Delta_{p,q} = o(1)$ excludes the case that all $Y_{i,n}$'s are mainly affected by the same very few $e_{j,n}$'s.
  - Consider: $Y_{i,n} = e_{1,n}\ \forall i = 1, \cdots, n$. Then $\sum_{i=1}^{n} \delta_{p,n}(i,1) \propto n$, and thus $\Delta_{p,q} \propto 1$.

Moment Inequality

Theorem. We have

$$\left\| \frac{1}{n} \sum_{i=1}^{n} (Y_{i,n} - \mathbb{E} Y_{i,n}) \right\|_{L^p} \leq C_p \left\{ \Delta_{p,\min\{p,2\}} \right\}^{1/\min\{p,2\}} .$$

- If $\sum_{i=1}^{n} \delta_{p,n}(i,j) < \infty$ for some $p \geq 2$, then

$$\Delta_{p,2} \equiv \frac{1}{n^2} \sum_{j=1}^{n} \left[\sum_{i=1}^{n} \delta_{p,n}(i,j) \right]^2 = O(n^{-1}).$$

- As a result, $\left\| \frac{1}{n} \sum_{i=1}^{n} (Y_{i,n} - \mathbb{E} Y_{i,n}) \right\|_{L^p} = O(\sqrt{n})$, same rate as the i.i.d. case.

Exponential Inequality

Theorem. Assume $\gamma_0 \equiv \sup_{p\geq 2}\sup_{n\geq 1} p^{-\nu}\sqrt{n}\Delta_{p,2}^{1/2} < \infty$ for some $\nu \geq 0$. Then $\forall t > 0$,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}(Y_{i,n} - \mathbb{E}Y_{i,n})\right| \geq t\right) \leq C_1 \exp\left(-C_2 n^{\frac{1}{1+2\nu}} t^{\frac{2}{1+2\nu}}\right).$$

- For SAR, recall $\delta_{p,n}(i,j) \leq 2\|\epsilon\|_{L^p} M_{ij,n}$. If

$$\frac{1}{n}\sum_{j=1}^{n}\left[\sum_{i=1}^{n} M_{ij,n}\right]^2 < \infty,$$

  then

$$\Delta_{p,2}^{1/2} \equiv \sqrt{\frac{1}{n^2}\sum_{j=1}^{n}\left[\sum_{i=1}^{n}\delta_{p,n}(i,j)\right]^2} \leq 2\|\epsilon\|_{L^p}\frac{1}{\sqrt{n}};$$

  1. when $\epsilon_{i,n}$ is subexponential: $\nu = 1$, $n^{\frac{1}{1+2\nu}} = n^{1/3}$
  2. when $\epsilon_{i,n}$ is sub-Gaussian: $\nu = \frac{1}{2}$, $n^{\frac{1}{1+2\nu}} = n^{1/2}$
  3. when $\epsilon_{i,n}$ is uniformly bounded: $\nu = 0$, $n^{\frac{1}{1+2\nu}} = n$

Central Limit Theorem

Theorem. Let $p > 2$ be a constant and $\sigma_n^2 \equiv \mathrm{Var}\left(\sum_{i=1}^n Y_{i,n}\right)$. Suppose that (i) $\liminf_{n\to\infty} \frac{\sigma_n^2}{n} > 0$ (ii)

$$\sup_{n,j} \sum_{i=1}^n \delta_{p,n}(i,j) < \infty, \tag{6}$$

and (iii)

$$\frac{1}{n^{\min\{2,p/2\}}} \sum_{i=1}^n \left\{ \sum_{j=1}^n \sum_{k=1}^n \min\left\{\delta_{p,n}(k,i), \delta_{p,n}(k,j)\right\} \right\}^{\min\{2,p/2\}} = o(1) \tag{7}$$

as $n \to \infty$. Then

$$\frac{\sum_{i=1}^n (Y_{i,n} - \mathbb{E}Y_{i,n})}{\sigma_n} \xrightarrow{d} N(0,1).$$

Central Limit Theorem

- $j(i)$: the index of the jth largest value of $\{\delta_{p,n}(i,j,\mathcal{C}_n) : j = 1, \ldots, n\}$.
- When $p \geq 4$, a sufficient condition for (7) is

$$\delta_{p,n}(i, j(i)) \leq C\{j(i)\}^{-\alpha}, \ \forall 1 \leq i \leq n, \tag{8}$$

  where $\alpha > 2$ and $C > 0$ are some constants.
- Existing weak dependence concepts: θ_s decreases as distance s increases;
  - $s \to \infty$ excludes networks with small diameters;
- Ours: no distance; applies to networks with small diameters.

Conclusion

- We generalize the concept of functional dependence proposed in Wu (2005) to spatial and network econometric settings: easy to verify, convenient to use
- We establish a moment inequality, an exponential inequality, LLN, and CLT
- We calculate the FDM for some models;
- spatial FDM: individuals are in $\mathbb{R}^d$;
- networks FDM: no metric space.

Thank You!

- Email: `wuzeqi@ruc.edu.cn`
- Homepage: https://zeqiwu1202.github.io