4. If $u^3 + xv^2 - uy = 0,\ u^2 + xyv + v^2 = 0$, find $\dfrac{\partial u}{\partial x}$ Ans. $\dfrac{xyv^2 - 2v^3}{2xyu^2 - xy^2 + 6u^2v - 2vy - 4xuv}$

5. If $u^2 + xv^2 = x + y,\ v^2 + yu^2 = x - y$, find $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial y}$ Ans. $\dfrac{1 - x - v^2}{2u(1-xy)}, \dfrac{1 + y + u^2}{-2v(1-xy)}$

6. If $u = xyz,\ v = x^2 + y^2 + z^2,\ w = x + y + z$ find $\dfrac{\partial x}{\partial u}$ Ans. $\dfrac{1}{(x-y)(x-z)}$

7. If $u = x^2 + y^2 + z^2,\ v = xyz$, find $\dfrac{\partial x}{\partial u}$ Ans. $\dfrac{x}{2(2x^2 - y^2)}$

1.26 TAYLOR'S SERIES OF TWO VARIABLES

If $f(x, y)$ and all its partial derivatives upto the nth order are finite and continuous for all points (x, y), where

$$a \le x \le a + h,\ b \le y \le b + k$$

Then $$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \ldots$$

Proof . Suppose that $f(x + h, y + k)$ is a function of one variable only, say x where y is assumed as constant. Expanding by Taylor's Theorem for one variable, we have

$$f(x + \delta x, y + \delta y) = f(x, y+\delta y) + \delta x \frac{\partial f(x, y+\delta y)}{\partial x} + \frac{(\delta x)^2}{2!}\frac{\partial^2 f(x, y+\delta y)}{\partial x^2} + \ldots\ldots$$

Now expanding for y, we get

$$= \left[f(x,y) + \delta y\frac{\partial f(x,y)}{\partial y} + \frac{(\delta y)^2}{2!}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots..\right] + \delta x \cdot \frac{\partial}{\partial x}\left[f(x,y) + \delta y \frac{\partial}{\partial y}\frac{f(x,y)}{\partial y} + \ldots\ldots\right]$$
$$+ \frac{(\delta x)^2}{2!}\frac{\partial^2}{\partial x^2}\left[f(x, y) + \delta y \frac{\partial}{\partial y} f(x, y) + \ldots..\right] + \ldots..$$

$$= \left[f(x, y) + \delta y \frac{\partial f(x, y)}{\partial y} + \frac{(\delta y)^2}{2!} \cdot \frac{\partial^2 f(x, y)}{\partial y^2} + \ldots..\right] +$$
$$+ \delta x\left[\frac{\partial f(x, y)}{\partial x} + \delta y . \frac{\partial^2}{\partial x \partial y} f(x, y)\right] + \frac{(\delta x)^2}{2!}\left[\frac{\partial^2 f(x, y)}{\partial x^2} + \ldots\right] + \ldots$$

$$= f(x, y) + \left[\delta x \frac{\partial f(x, y)}{\partial x} + \delta y . \frac{\partial f(x, y)}{\partial y}\right] + \frac{1}{2!}\left[(\delta x)^2 \frac{\partial^2 f(x, y)}{\partial x^2} + 2\,\delta x . \delta y . \frac{\partial^2 f(x, y)}{\partial x \partial y}\right.$$
$$\left. + (\delta y)^2 . \frac{\partial^2 f(x, y)}{\partial y^2}\right] + \ldots$$

$$\Rightarrow f(a + h, b + k) = f(a, b) + \left[h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right] + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots..$$

$$\Rightarrow f(a + h, b + k) = f(a, b) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]f + \frac{1}{2!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f + \ldots..$$

On putting $a = 0,\ b = 0,\ h = x,\ k = y$, we get

$$f(x, y) = f(0, 0) + \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots..$$

Example 78. *Expand $e^x . \sin y$ in powers of x and y, $x = 0, y = 0$ as far as terms of third degree.*

Solution.

		$x = 0, y = 0$
$f(x, y)$	$e^x \sin y,$	0
$f_x(x, y)$	$e^x \sin y,$	0
$f_y(x, y)$	$e^x \cos y,$	1
$f_{xx}(x, y)$	$e^x \sin y,$	0
$f_{xy}(x, y)$	$e^x \cos y,$	1
$f_{yy}(x, y)$	$-e^x \sin y,$	0
$f_{xxx}(x, y)$	$e^x \sin y,$	0
$f_{xxy}(x, y)$	$e^x \cos y,$	1
$f_{xyy}(x, y)$	$-e^x \sin y,$	0
$f_{yyy}(x, y)$	$-e^x \cos y,$	-1

By Taylor's theorem

$$f(x, y) = f(0, 0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0, 0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0, 0)$$

$$+ \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0, 0) + \ldots$$

$$= f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{x^2}{2!} f_{xx}(0, 0) + \frac{2xy}{2!} f_{xy}(0, 0) + \frac{y^2}{2!} f_{yy}(0, 0)$$

$$+ \frac{1}{3!} x^3 f_{xxx}(0, 0) + \frac{3x^2 y}{3!} f_{xxy}(0, 0) + \frac{3}{3!} x y^2 f_{xyy}(0, 0) + \frac{1}{3!} y^3 f_{yyy}(0, 0) + \ldots$$

$$e^x \sin y = 0 + x(0) + y(1) + \frac{x^2}{2}(0) + x y(1) + \frac{y^2}{2}(0) + \frac{x^3}{6}(0) + \frac{3x^2 y}{6}(1) + \frac{3xy^2}{6}(0) + \frac{y^3}{6}(-1) + \ldots$$

$$= y + x y + \frac{x^2 y}{2} - \frac{y^3}{6} + \ldots\ldots$$

Ans.

Example 79. *Find the expansion for cos x cos y in powers of x, y upto fourth order terms.*

Solution.

		$x = 0, y = 0$
$f(x, y)$	$\cos x \cos y,$	1
f_x	$-\sin x \cos y,$	0
f_y	$-\cos x \sin y,$	0
f_{xx}	$-\cos x \cos y,$	-1
f_{xy}	$\sin x \sin y,$	0
f_{yy}	$-\cos x \cos y,$	-1
f_{xxx}	$\sin x \cos y,$	0
f_{xxy}	$\cos x \sin y,$	0
f_{xyy}	$\sin x \cos y,$	0
f_{yyy}	$\cos x \sin y,$	0

f_{xxxx}	$\cos x \cos y,$	1
f_{xxxy}	$-\sin x \sin y,$	0
f_{xxyy}	$\cos x \cos y,$	1
f_{xyyy}	$-\sin x \sin y,$	0
f_{yyyy}	$\cos x \cos y,$	1

By Taylor's Series

$$f(x,y) = f(0,0) + x\,f_x(0,0) + y\;f_y(0,0) + \frac{1}{2!}\left[x^2 f_x^2(0,0) + 2xy\,f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$+\frac{1}{3!}\left[x^3 f_x^3(0,0) + 3x^2 y\,f_{xy}^2(0,0) + 3xy^2 f_{xy}^{\;2}(0,0) + y^3 f_y^3(0,0)\right]$$

$$+\frac{1}{4!}\left[x^4 f_x^4(0,0) + 4x^3 y\,f_{xy}^3(0,0) + 6x^2y^2 f_{xy}^{2\,2}(0,0) + 4\,xy^3 f_{xy}^{\;3}(0,0) + y^4 f_y^4(0,0)\right] + \ldots$$

$$\cos x \cos y = 1 + 0 + 0 + \frac{1}{2}(-x^2 + 0 - y^2) + \frac{1}{6}(0+0+0+0) + \frac{1}{24}(x^4 + 0 + 6x^2y^2 + 0 + y^4)$$

$$= 1 - \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^4}{24} + \frac{x^2y^2}{4} + \frac{y^4}{24} + \ldots$$

Example 80. *Find the first six terms of the expansion of the function $e^x \log(1+y)$ in a Taylor's series in the neighbourhood of the point* (0, 0).

Solution.

Taylor's series is

$$f(x,y) = f(0,0) + \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)$$

$$+\frac{1}{2!}\left(x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots$$

$$\Rightarrow \quad e^x \log(1+y) = 0 + (x \times 0 + y \times 1)$$

$$+\frac{1}{2!}[x^2 \times (0) + 2xy \times 1 + y^2 \times (-1)] + ..$$

$$\Rightarrow \quad e^x \log(1+y) = y + xy - \frac{y^2}{2} \qquad \textbf{Ans.}$$

		$x=0, y=0$
$f(x,y)$	$e^x \log(1+y)$	0
$\frac{\partial f}{\partial x}$	$e^x \log(1+y)$	0
$\frac{\partial f}{\partial y}$	$\frac{e^x}{1+y}$	1
$\frac{\partial^2 f}{\partial x^2}$	$e^x \log(1+y)$	0
$\frac{\partial^2 f}{\partial y^2}$	$-\frac{e^x}{(1+y)^2}$	-1
$\frac{\partial^2 f}{\partial x \partial y}$	$\frac{e^x}{(1+y)}$	0

EXERCISE 1.16

1. Expand $e^x \cos y$ at (0, 0) upto three terms. **Ans.** $1 + x + \frac{1}{2}(x^2 - y^2) + \ldots\ldots$
2. Expand $z = e^{2x}\cos 3y$ in power series of x and y upto quadratic terms. **Ans.** $1 + 2x + 2x^2 - \frac{9}{2}y^2 + \ldots$
3. Show that $e^y \log(1+x) = x + xy - \dfrac{x^2}{2}$ approximately.
4. Verify $\sin(x+y) = x + y - \dfrac{(x+y)^3}{3} + \ldots\ldots$

Example 81. *Expand* sin (xy) *in powers of* $(x-1)$ *and* $\left(y-\frac{\pi}{2}\right)$ *as far as the terms of second degree.*

(Nagpur University, Summer 2003)

Solution. We have, $f(x,y) = \sin(xy)$

Here $\left[\begin{array}{l} a+h = x \text{ and } h = x-1 \\ \Rightarrow a+(x-1) = x \Rightarrow a = 1 \end{array}\right.$

$\left[\begin{array}{l} b+k = y \text{ and } k = y-\frac{\pi}{2} \\ \Rightarrow b+y-\frac{\pi}{2} = y \Rightarrow b = \frac{\pi}{2} \end{array}\right.$

		$x=0, y=\frac{\pi}{2}$
$f(x,y)$	$\sin(xy)$	1
$f_x(xy)$	$y\cos(xy)$,	0
$f_y(x,y)$	$x\cos(xy)$,	0
$fxx(x,y)$	$-y^2\sin(xy)$,	$-\frac{\pi^2}{4}$
$f_{xy}(x,y)$	$\cos(xy) - xy\sin(xy)$	$-\frac{\pi}{2}$
$f_{yy}(x,y)$	$-x^2\sin(xy)$,	-1

By Taylor's theorem for a function of two variables, we have

$$f(a+h, b+k) = f(a,b) + hf_x(a,b) + kf_y(a,b)$$
$$+\frac{1}{2!}\left\{h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)\right\}$$

$$\Rightarrow f(x,y) = f\left(1,\frac{\pi}{2}\right) + (x-1) f_x\left(1,\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right) f_y\left(1,\frac{\pi}{2}\right)$$
$$+\frac{1}{2!}\left\{(x-1)^2 f_{xx}\left(1,\frac{\pi}{2}\right) + 2(x-1)\left(y-\frac{\pi}{2}\right) f_{xy}\left(1,\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)^2 f_{yy}\left(1,\frac{\pi}{2}\right)\right\}$$

$$\Rightarrow \sin(xy) = 1 + (x-1).0 + \left(y-\frac{\pi}{2}\right).0 +$$
$$\frac{1}{2!}\left\{(x-1)^2\left(-\frac{\pi^2}{4}\right) + 2(x-1)\left(y-\frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)^2(-1)\right\} + \ldots$$

$$\Rightarrow \sin(xy) = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right) - \frac{1}{2}\left(y-\frac{\pi}{2}\right)^2 + \ldots$$ **Ans.**

Example 82. *Expand* e^x *cos y near the point* $\left(1,\frac{\pi}{4}\right)$ *by Taylor's Theorem.*

(U.P., I Semester Dec. 2007)

Solution. $f(x+h, y+k) = f(x,y) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \ldots$$

$$e^x\cos y = f(x,y) = f\left[1+(x-1), \frac{\pi}{4} + \left(y-\frac{\pi}{4}\right)\right]$$

where $h = x-1$, $k = y-\frac{\pi}{4}$ $= f\left(1+h, \frac{\pi}{4}+k\right)$

		$x=0, y=\frac{\pi}{4}$
$f(x,y)$	$e^x\cos y$	$\frac{e}{\sqrt{2}}$
$\frac{\partial f}{\partial x}$	$e^x\cos y$,	$\frac{e}{\sqrt{2}}$
$\frac{\partial f}{\partial y}$	$-e^x\sin y$,	$\frac{-e}{\sqrt{2}}$
$\frac{\partial^2 f}{\partial x^2}$	$e^x\cos y$,	$\frac{e}{\sqrt{2}}$
$\frac{\partial^2 f}{\partial y^2}$	$-e^x\cos y$,	$\frac{-e}{\sqrt{2}}$
$\frac{\partial^2 f}{\partial x\,\partial y}$	$-1\,e^x\sin y$,	$\frac{-e}{\sqrt{2}}$

Putting these values in Taylor's Theorem, we get

$$e^x\cos y = \frac{e}{\sqrt{2}} + \left[(x-1)\frac{e}{\sqrt{2}} + \left(y-\frac{\pi}{4}\right)\left(\frac{-e}{\sqrt{2}}\right)\right]$$

$$+\frac{1}{2!}\left[(x-1)^2\frac{e}{\sqrt{2}}+2(x-1)\left(y-\frac{\pi}{4}\right)\left(\frac{-e}{\sqrt{2}}\right)+\left(y-\frac{\pi}{4}\right)^2\left(\frac{-e}{\sqrt{2}}\right)\right]+\ldots.$$

$$=\frac{e}{\sqrt{2}}\left[1+(x-1)-\left(y-\frac{\pi}{4}\right)+\frac{(x-1)^2}{2}-(x-1)\left(y-\frac{\pi}{4}\right)-\left(y-\frac{\pi}{4}\right)^2+\ldots.\right]$$ **Ans.**

Example 83. *If $f(x, y) = \tan^{-1}(x\,y)$, compute an approximate value of $f(0.9, -1.2)$.*

Solution. We have,

$$f(x, y) = \tan^{-1}(x\,y)$$

Let us expand $f(x, y)$ near the point $(1, -1)$

$$f(0.9, -1.2) = f(1-0.1, -1-0.2)$$

$$= f(1,-1)+\left[(-0.1)\frac{\partial f}{\partial x}+(-0.2)\frac{\partial f}{\partial y}\right]+\frac{1}{2!}\left[(-0.1)^2\frac{\partial^2 f}{\partial x^2}\right.$$

$$\left.+2(-0.1)(-0.2)\frac{\partial^2 f}{\partial x\,\partial y}+(-0.2)^2\frac{\partial^2 f}{\partial y^2}\right]+\ldots \qquad \ldots(1)$$

		$x = 1,\ \ y = -1$
$f(x, y)$	$\tan^{-1}(xy)$	$-\frac{\pi}{4}$
$\frac{\partial f}{\partial x}$	$\frac{y}{1+x^2y^2}$,	$-\frac{1}{2}$
$\frac{\partial f}{\partial y}$	$\frac{x}{1+x^2y^2}$,	$\frac{1}{2}$
$\frac{\partial^2 f}{\partial x^2}$	$-\frac{(2x)y}{(1+x^2y^2)^2}$,	$\frac{1}{2}$
$\frac{\partial^2 f}{\partial y\,\partial x}$	$\frac{1+x^2y^2-x(2x\,y^2)}{(1+x^2y^2)^2}=\frac{1-x^2y^2}{(1+x^2y^2)^2}$	0
$\frac{\partial^2 f}{\partial y^2}$	$\frac{-x\,(2x^2y)}{(1+x^2y^2)^2}$,	$\frac{1}{2}$

Substituting the values of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ etc. in (1), we get

$$f(0.9,-1.2) = -\frac{\pi}{4}+(-0.1)\left(-\frac{1}{2}\right)+(-0.2)\left(\frac{1}{2}\right)+\frac{1}{2}\left[(-0.1)^2\left(\frac{1}{2}\right)\right.$$

$$\left.+2(-0.1)(-0.2)\,0+(-0.2)^2\left(\frac{1}{2}\right)\right]+\ldots$$

$$= -\frac{22}{28}+0.05-0.1+\frac{1}{2}(0.005+0.02)$$

$$= -0.786+0.05-0.1+0.0125 = -0.8235$$ **Ans.**

Example 84. *Obtain Taylor's expansion of $\tan^{-1}\frac{y}{x}$ about (1, 1) upto and including the second degree terms. Hence compute $f(1.\ 1, 0.9)$.* (*U.P., I Sem. Winter 2005*)

Solution.

		$x = 1, y = 1$
$f(x, y)$	$\tan^{-1}\dfrac{y}{x}$	$\dfrac{\pi}{4}$
$\dfrac{\partial f}{\partial x}$	$\dfrac{1}{1+\dfrac{y^2}{x^2}}\left(-\dfrac{y}{x^2}\right) = \dfrac{-y}{x^2+y^2},$	$-\dfrac{1}{2}$
$\dfrac{\partial f}{\partial y}$	$\dfrac{1}{1+\dfrac{y^2}{x^2}}\left(\dfrac{1}{x}\right) = \dfrac{x}{x^2+y^2},$	$\dfrac{1}{2}$
$\dfrac{\partial^2 f}{\partial x^2}$	$\dfrac{y\,(2x)}{(x^2+y^2)^2} = \dfrac{2xy}{(x^2+y^2)^2},$	$\dfrac{1}{2}$
$\dfrac{\partial^2 f}{\partial y^2}$	$\dfrac{-x\,(2y)}{(x^2+y^2)^2} = \dfrac{-2xy}{(x^2+y^2)^2},$	$-\dfrac{1}{2}$
$\dfrac{\partial^2 f}{\partial y \partial x}$	$\dfrac{(x^2+y^2)-(x)(2x)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2},$	0

By Taylor's Theorem

$$f(x, y) = f(a, b) + \left[(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right] + \frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2} + 2\,(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots$$

Here, $a = 1, b = 1$

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{4} + (x-1)\left(-\frac{1}{2}\right) + (y-1)\frac{1}{2} + \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + 2\,(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right] + \ldots$$

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + \ldots \quad \ldots(1)$$

Putting $(x-1) = 1.1 - 1 = 0.1$, $(y-1) = 0.9 - 1 = -0.1$ in (1), we get

$$f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(0.1) + \frac{1}{2}(-0.1) + \frac{1}{4}(0.1)^2 - \frac{1}{4}(-0.1)^2$$

$$= 0.786 + 0.05 - 0.05 + 0.0025 - 0.0025 = 0.686$$

Ans.

Example 85. *Expand* $\dfrac{(x+h)(y+k)}{x+h+y+k}$ *in powers of h, k upto and inclusive of the second degree terms.*

Solution.

$$f(x+h, y+k) = \frac{(x+h)(y+k)}{x+h+y+k}$$

$$f(x, y) = \frac{x\,y}{x+y}$$

$$\frac{\partial f}{\partial x} = \frac{(x+y)\,y - x\,y}{(x+y)^2} = \frac{y^2}{(x+y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x+y)\,x - x\,y}{(x+y)^2} = \frac{x^2}{(x+y)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-2y^2}{(x+y)^3}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{(x+y)^2\, 2x - 2\,(x+y)\,x^2}{(x+y)^4} = \frac{(x+y)\,2x - 2x^2}{(x+y)^3} = \frac{2x\,y}{(x+y)^3}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2x^2}{(x+y)^3}$$

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y) + \ldots$$

$$\frac{(x+h)\,(y+k)}{x+h+y+k} = \frac{x\,y}{x+y} + h\,\frac{y^2}{(x+y)^2} + k\,\frac{x^2}{(x+y)^2}$$

$$+ \frac{h^2}{2!}\frac{(-2y^2)}{(x+y)^3} + \frac{1}{2!}\,2h\,k\,\frac{2x\,y}{(x+y)^3} + \frac{1}{2!}\,k^2\,\frac{(-2x^2)}{(x+y)^3} + \ldots$$

$$= \frac{x\,y}{x+y} + \frac{h\,y^2}{(x+y)^2} + \frac{kx^2}{(x+y)^2} - \frac{h^2y^2}{(x+y)^3} + \frac{2h\,kxy}{(x+y)^3} - \frac{k^2x^2}{(x+y)^3} + \ldots \quad \textbf{Ans.}$$

Example 86. *Expand $x^2y + 3y - 2$ in powers of $x - 1$ and $y + 2$ using Taylor's Theorem.*

Solution. $f(x, y) = x^2y + 3\,y - 2$

Here $\quad a + h = x$ and $h = x - 1$, so $a = 1$

$\quad b + k = y$ and $k = y + 2$ so $b = -2$

		$x = 1, y = -2$
$f(x, y)$	$x^2y + 3y - 2$,	-10
$f_x(x, y)$	$2xy$,	-4
$f_y(x, y)$	$x^2 + 3$,	4
$f_{xx}(x, y)$	$2\,y$,	-4
$f_{xy}(x, y)$	$2\,x$,	2
$f_{yy}(x, y)$	$2\,x$,	0
$f_{xxx}(x, y)$	0,	0
$f_{xxy}(x, y)$	2,	2
$f_{xyy}(x, y)$	0,	0
$f_{yyy}(x, y)$	0,	0

Now Taylor's Theorem is

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{(a,b)} + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2h\,k\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right]_{(a,b)}$$

$$+\frac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3}+3h^2k\frac{\partial^3 f}{\partial x^2\partial y}+3hk^2\frac{\partial^2 f}{\partial x\partial y^2}+k^3\frac{\partial^3 f}{\partial y^3}\right)+\ldots$$

Putting the values of $f(a, b)$ etc. in Taylor's Theorem, we get

$$x^2y+3y-2=-10+[(x-1)(-4)+(y+2)(4)]$$
$$+\frac{1}{2!}[(x-1)^2(-4)+2(x-1)(y+2)(2)+(y+2)^2(0)]$$
$$+\frac{1}{3!}[(x-1)^3(0)+3(x-1)^2(y+2)(2)+3(x-1)(y+2)^2(0)+(y+2)^3(0)]$$
$$x^2y+3y-2=-10-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2)+(x-1)^2(y+2)$$ **Ans.**

EXERCISE 1.17

1. Expand e^{xy} at (1, 1) upto three terms.

Ans. $e\left[1+(x-1)+(y-1)+\frac{1}{2!}[(x-1)^2+4(x-1)(y-1)+(y-1)^2]\right]$

2. Expand y^x at (1, 1) upto second term **Ans.** $1+(y-1)+(x-1)(y-1)+\ldots\ldots$
3. Expand $e^{ax}\sin by$ in powers of x and y as far as the terms of third degree. (*U.P. I sem. Jan 2011*)

Ans. $by+abxy+\frac{1}{3!}\left(3a^2bx^2y-b^3y^3\right)+\ldots$

4. Expand $(x^2y+\sin y+e^x)$ in powers of $(x-1)$ and $(x-\pi)$.

Ans. $\pi+e+(x-1)(2\pi+e)+\frac{1}{2}(x-1)^2(2\pi+e)+2(x-1)(y-\pi)$.

5. Expand $(1+x+y^2)^{1/2}$ at (1, 0). **Ans.** $\sqrt{2}\left[1+\frac{x-1}{4}-\frac{(x-1)^2}{32}+\frac{y^2}{4}+\ldots\right]$

6. Obtain the linearised form $T(x, y)$ of the function $f(x,y)=x^2-xy+\frac{1}{2}y^2+3$ at the point (3, 2), using the Taylor's series expansion. Find the maximum error in magnitude in the approximation $f(x,y)\equiv T(x,y)$ over the rectangle R: $|x-3|<0.1, |y-2|<0.1$.

Ans. $8+4(x-3)-(y-2)$., Error 0.04.

7. Expand $\sin(x+h)(y+k)$ by Taylor's Theorem.

Ans. $\sin xy+h(x+y)\cos xy+hk\cos xy-\frac{1}{2}h^2(x+y)^2\sin xy+\ldots$

8. Fill in the blank:

$f(x,y)=f(2,3)+\ldots\ldots\ldots\ldots$ **Ans.** $\left[(x-2)\frac{\partial}{\partial x}+(y-3)\frac{\partial}{\partial y}\right]f+\frac{1}{2!}\left[(x-2)\frac{\partial}{\partial x}+(y-3)\frac{\partial}{\partial y}\right]^2 f+\ldots$

9. If $f(x)=f(0)+kf_1(0)+\frac{k^2}{2!}f_2(\theta k), 0<\theta<1$ then the value of θ when $k=1$ and $f(x)=(1-x)^{3/2}$ is given as ………. (*U.P. Ist Semester, Dec 2008*)

1.27 MAXIMUM VALUE

A function $f(x, y)$ is said to have a maximum value at $x=a, y=b$, if there exists a small neighbourhood of (a, b) such that,

$$f(a,b)>f(a+h,b+k)$$

Minimum Value. A function $f(x, y)$ is said to have a minimum value for $x=a, y=b$, if there exists a small neighbourhood of (a, b) such that

$$f(a,b)<f(a+h,b+k)$$