PART II

★ STATISTICS

1

Theory of Probability

Some Definitions

1.1 Random Experiment and Events

When we toss a coin or throw a die or draw a card from a pack of playing cards, there are a number of possible results or *outcomes* which can occur but there is an uncertainty as to which one of them will actually occur. Such examples or experiments are called *random experiments.* Thus a **random experiment** *may be defined as an experiment which when repeated under essentially identical conditions does not give unique results but may result in any one of the several possible outcomes.* These outcomes are known as **events** or **cases**. Events are donoted as A, B, C etc.

1.2 Exhaustive Events

The total number of possible outcomes in a random experiment (or trial) are known as **exhaustive events.** For example,

(*i*) There are 2 exhaustive events viz. head (H) and tail (T) when we toss a coin.

(*ii*) There are 6 exhaustive events when we throw a die.

(*iii*) When we throw two dice, the exhaustive number of events is $6^2 = 36$ given as follows :

(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6) ; (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6) ; (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6) ; (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6) ; (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6) ; (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6).

1.3 Favourable Events

The events which cause the happening of a particular event A are called the **favourable events** *to the event A.* For example,

(*i*) There are three favourable events for the occurrence of an even number (or an odd number) in the throwing of a die.

(*ii*) There are 12 favourable cases for the drawing of a face card (4 kings, 4 queens, 4 jacks).

1.4 Mutually Exclusive Events

Such events, where the occurrence of one rules out the occurrence of the other, are called **mutually exclusive events.** For example,

There are two mutually exclusive events when we toss a coin, for if head comes in a trial, then tail cannot come in the same trial.

1.5 Equally Likely Events

The events are said to be ***equally likely*** *if none of them is expected to occur in preference to other.* For example, there are two equally likely events viz H and T when we toss a coin.

1.6 Mathematical or Classical or 'a Priori' Definition of Probability

If there are n exhaustive, mutually exclusive and equally likely events, out of which m are favourable to the happening of an event A, then the **probability of the happening of A,** *denoted by P(A), is defined as*

$$P(A)=\frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}}=\frac{m}{n}.$$

Remarks.

1. It is clear that if m cases out of n exhaustive cases favour the happening of an event A, then $n-m$ cases favour that the event A will not happen.

Thus *the probability that the event* **A will not happen,** *denoted by* $P(\bar{A})$, *is given by*

$$P(\bar{A})=\frac{n-m}{n}=1-\frac{m}{n}=1-P(A)$$

$$\therefore \quad P(A)+P(\bar{A})=1. \qquad \ldots(1)$$

2. Since $0 \leqslant m \leqslant n$, so $0 \leqslant \frac{m}{n} \leqslant 1$.

Hence $\mathbf{0 \leqslant P(A) \leqslant 1}$, for any event A.

If P(A)=0, then A is called an *impossible* or *null event*.

If P(A)=1, then A is called a *certain or sure event*.

3. The probability P(A) of the happening of an event A is also known as the *probability of success*, denoted by p, and the probability P $(\bar{A})$ of the non-happening of the event A is known as the *probability of failure*, denoted by q.

From (1), it follows that

$$\mathbf{p+q=1.} \quad (0 \leqslant p \leqslant 1,\ 0 \leqslant q \leqslant 1).$$

4. *If A and B are two events, then the probability of the happening of A or B (i.e., at least one of the two events) is denoted as P(A+B).*

The probability of the simultaneous occurrence of two events A and B is denoted by P(AB).

5. *The following results from permutations and combinations are very useful :*

(*i*) $n! = n(n-1)\ldots\ldots 3.2.1,\ n! = n(n-1)! = n(n-1)(n-2)!$ etc.

(*ii*) The number of permutations of n different things taken r at a time is

$$^{n}P_{r}=\frac{n!}{(n-r)!}.$$

(*iii*) The number of combinations of n different things taken r at a time is $^{n}C_{r}$, where

$$^{n}C_{r}=\frac{n!}{r!(n-r)!}.$$

For example,

$$^{6}C_{2}=\frac{6!}{2!\,4!}=\frac{6\,.\,5}{2\,.\,1}=15,\quad ^{10}C_{3}=\frac{10.9.8}{3.2.1}=120,$$

$$^{52}C_{4}=\frac{52.51.50.49}{4.3.2.1}=270725 \text{ etc.}$$

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Example 1. *Find the probability of getting an even number in a throw of a single die.*

Solution. A die has six faces bearing numbers 1 to 6.

Total number of exhaustive cases=6 *i.e.*, $n=6$.

Out of these six numbers, there are three even numbers viz. 2, 4 and 6.

Total number of favourable cases=3 *i.e.* $m=3$.

$$\text{Required probability}=\frac{m}{n}=\frac{3}{6}=\frac{1}{2}.$$

Example 2. *In a single throw with two dice, find the probability of getting a total of 10.*

Solution. Total number of cases=6×6 *i.e.*, $n=36$

The total of 10 on two dice can be obtained as follows :

(4, 6), (5, 5) and (6, 4).

Total number of favourable cases=3 *i.e.*, $m=3$.

$$\text{Required probability}=\frac{m}{n}=\frac{3}{36}=\frac{1}{12}.$$

Example 3. *Two cards are drawn at random from a well-shuffled pack of 52 cards. Find the probability of getting 2 aces.*

Solution. Total number of cases in which 2 cards can be drawn out of 52 cards is $n={}^{52}C_2$. There are 4 aces in a pack of 52 cards and out of 4 aces, 2 aces can be drawn in $m={}^{4}C_2$ ways.

$$\text{Required probability}=\frac{m}{n}=\frac{^{4}C_{2}}{^{52}C_{2}}=\frac{4.3}{52.51}=\frac{1}{221}.$$

Example 4. *From a pack of 52 cards, two cards are drawn at random. Find the chance that one is a king and the other a queen.*

Solution. Total number of cases is $n={}^{52}C_2$.

Since there are 4 kings and 4 queens, so the number of favourable cases is $m={}^4C_1\times{}^4C_1$.

$$\text{Required probability}=\frac{m}{n}=\frac{{}^4C_1\times{}^4C_1}{{}^{52}C_2}$$

$$=\frac{4.4}{52.51}\;.\;2.1=\frac{8}{663}.$$

Example 5. *From a pack of 52 cards, three are drawn at random. Find the chance that they are a king, a queen and a knave.*

Solution. Total number of cases$={}^{52}C_3$

A pack of cards contains 4 kings, 4 queens and 4 knaves. A king, a queen and a knave can each be drawn in ${}^4C_1$ ways. Since each way of drawing a king can be associated with each of the ways of drawing a queen and a knave, the total number of favourable cases$={}^4C_1\times{}^4C_1\times{}^4C_1$.

$$\text{Required probability}=\frac{{}^4c_1\times{}^4c_1\times{}^4c_1}{{}^{52}c_3}=\frac{4\times4\times4\times6}{52\times51\times50}=\frac{16}{5525}.$$

Example 6. *What is the chance the a leap year selected at random will contain 53 Sundays ?* (D.U., B.A.(P) 1984)

Solution. A leap year consisting of 366 days has 52 complete weeks and 2 days extra. The following are the possible combinations for these two extra days :

(*i*) Sunday and Monday, (*ii*) Monday and Tuesday, (*iii*) Tuesday and Wednesday, (*iv*) Wednesday and Thursday, (*v*) Thursday and Friday, (*vi*) Friday and Saturday, and (*vii*) Saturday and Sunday.

In order that a leap year selected at random should contain 53 Sundays, one of the two extra days must be Sunday. Since out of the above 7 possibilities, 2 viz., (*i*) and (*vii*), are favourable to this event, so

$$\text{Required probability}=\frac{2}{7}.$$

Example 7. *A bag contains 3 red, 6 white and 7 blue balls. What is the probab ility that two balls drawn are white and blue ?*

Solution. Total number of balls$=3+6+7=16$.

Out of 16 balls 2 ,balls can de brawn in ${}^{16}C_2$ ways.

$$\text{Total number of cases}={}^{16}C_2=\frac{16\times15}{2}=120.$$

Out of 6 white balls, 1 ball can be drawn in ${}^6C_1$ ways and out of 7 blue balls, one ball can be drawn in ${}^7C_1$ ways. Since each of the former cases can be associated with each of the latter cases, total number of favourable cases is ${}^6C_1\times{}^7C_1=6\times7=42$.

$$\text{Required probability}=\frac{42}{120}=\frac{7}{20}$$

Example 8. *A bag contains 7 white, 6 red and 5 black balls. Two balls are drawn at random. Find the probability that they will both be white.*

Solution. Total number of balls$=7+6+5=18$.

Out of 18 balls, 2 balls can be drawn in ${}^{18}C_2$ ways.

$$\text{Total number of cases}={}^{18}C_2=\frac{18\times17}{2\times1}=153.$$

Out of 7 white walls, 2 balls can be drawn in

$${}^7C_2=\frac{7\times6}{2\times1}=21 \text{ ways.}$$

$$\text{Required probability}=\frac{21}{153}=\frac{7}{51}.$$

Example 9. *A bag contains 6 white, 7 red and 5 black black alls. Find the chance that three balls drawn at random are all white.*

Solution. Total number of cases$={}^{18}C_3=\dfrac{18\times17\times16}{3\times2\times1}=816$.

$$\text{Favourable number of cases}={}^6C_3=\frac{6\times5\times4}{3\times2\times1}=20.$$

$$\text{Required probability}=\frac{20}{816}=\frac{5}{204}.$$

Example 10. *If n biscuits be distributed among N beggars, find chance that a particular beggar receives r $(<n)$ biscuits.* (D.U., B.Sc. (G) 1983)

Solution. Since one biscuit can be given in N ways, so the total number of ways in which n biscuits can be distributed among N^rbeggars$=N^n$.

Now r biscuits can be given to any particular beggar in ${}^nC_r$ ways. The remaining $(n-r)$ biscuits are to be distributed among the remaining $(N-1)$ beggars and this can be done in $(N-1)^{n-r}$ ways.

$\therefore$ Number of favourable cases$={}^nC_r.(N-1)^{n-r}$.

Hence required probability$=\dfrac{{}^nC_r(N-1)^{n-r}}{N^n}$.

Example 11. *Out of $(2n+1)$ tickets consecutively numbered, three are drawn at random. Find the chance that the numbers on them are in A.P.* (D.U., B.Sc. (G) 1993, 90, 84 ; B.A. (P) 87)

Solution. Out of $(2n+1)$ tickets, 3 tickets can be drawn in ${}^{2n+1}C_3$ ways,

$$\therefore \text{ Total number of cases} = {}^{2n+1}C_3 = \frac{(2n+1)\,2n\,(2n-1)}{3\,!}$$
$$= \frac{n(4n^2-1)}{3}.$$

Now we find the favourable number of cases for drawing tickets with their numbers in A.P. having the common difference 'd'.

If **d**$=$**1**, possible cases are as follows :

$$\left.\begin{array}{l} 1, 2, 3 \\ 2, 3, 4 \\ \vdots \\ 2n-1, n, 2n+1 \end{array}\right\}, \text{ i.e., } (2n-1) \text{ cases in all.}$$

If **d**$=$**2**, the possible cases are as follows :

$$\left.\begin{array}{l} 1, 3, 5 \\ 2, 4, 6 \\ \vdots \\ 2n-3, 2n-1, 2n+1 \end{array}\right\}, \text{ i.e., } (2n-3) \text{ cases in all.}$$

and so on.

If **d**$=$**n**$-$**1**, the possible cases are

$$\left.\begin{array}{l} 1, n, \quad 2n-1 \\ 2, n+1, 2n \\ 3, n+2, 2n+1 \end{array}\right\}, \text{ i.e., 3 cases in all.}$$

If **d**$=$**n**, there is only one case, *viz*., $(1, n+1, 2n+1)$.

Hence total number of favourable cases

$=(2n-1)+(2n-3)+\ldots+5+3+1=1+3+5+\ldots+(2n-1)$

which is a series in A.P. having n terms and $d=2$.

$$\therefore \text{ Number of favourable cases} = \frac{n}{2}\,[1+(2n-1)] = n^2.$$

$$\text{Required probability} = \frac{n^2}{n(4n^2-1)/3} = \frac{3n}{(4n^2-1)}.$$

1.7. Theorem of Total Probability

Statement. *If n events $A_1, A_2, \ldots, A_n$ are not mutually exclusive, then the probability of the happening of at least one of the events is the sum of the probabilities of the individual events. In symbols,*

$$P(A_1+A_2+\ldots+A_n)=P(A_1)+P(A_2)+\ldots+P(A_n).$$

(D.U., B.Sc. (G) 1993, 90, 84 ; B.A. (P) 1984)

Proof. Let N be the total number of mutually exclusive, exhaustive and equally likely cases out of which m_1 cases are favourable to A_1, m_2 cases are favourable to A_2 and so on.

$$\left.\begin{array}{l} \text{The probability of occurrence of the event } A_1 = P(A_1) = \dfrac{m_1}{N} \\ \text{The probability of occurrence of the event } A_2 = P(A_2) = \dfrac{m_2}{N} \\ \vdots \\ \text{The probability of occurrence of the event } A_n = P(A_n) = \dfrac{m_n}{N} \end{array}\right\} \quad \ldots(1)$$

The events being mutually exclusive and equally likely, the total number of cases favourable to the event A_1 or A_2 or...or A_n is

$$m=m_1+m_2+\ldots\ldots+m_n. \quad \ldots(2)$$

$$\therefore \quad P(A_1+A_2+\ldots\ldots+A_n)=\frac{m}{N}$$
$$=\frac{m_1+m_2+\ldots\ldots+m_n}{N}, \text{ using (2)}$$
$$=\frac{m_1}{N}+\frac{m_2}{N}+\ldots+\frac{m_n}{N}$$
$$=P(A_1)+P(A_2)+\ldots\ldots+P(A_n), \text{ using (1)}$$

Hence $P(A_1+A_2+\ldots\ldots+A_n)=P(A_1)+P(A_2)+\ldots\ldots+P(A_n)$.

Cor. *If an event A consists of n mutually exclusive forms $A_1, A_2, \ldots, A_n$, so that A happens whenever any one of these events happens and conversely, then*

$$A=A_1+A_2+\ldots\ldots+A_n \text{ and } P(A)=P(A_1)+P(A_2)+\ldots\ldots+P(A_n).$$

1.8. Theorem. *If two events A and B are not mutually exclusive, then* $P(A+B)=P(A)+P(B)-P(AB)$, *where $P(AB)$ denotes the probability of the simultaneous occurrence of the events A and B.*

Proof. Since AB and $A\overline{B}$ are two exhaustive and mutually exclusive ways in which A can occur, therefore, by the above cor.

$$P(A)=P(AB)+P(A\overline{B}).$$

Similarly $P(B)=P(AB)+P(\overline{A}B)$.

$$\therefore \quad P(A)+P(B)=P(AB)+[P(A\overline{B})+P(\overline{A}B)+P(AB)]. \quad \ldots(1)$$

It is clear that at least one of the events A and B can occur in the following mutually exclusive and exhausive ways :

$$A\bar{B},\ \bar{A}B,\ AB. \quad \ldots(2)$$

By the theorem of total probability,

$$P(A+B)=P(A\bar{B})+P(\bar{A}B)+P(AB).$$

From (1) and (2), we obtain

$$P(A+B)=P(A)+P(B)-P(AB).$$

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Example 12. *What is the chance of throwing a total of 5 or 11 with two dice ?*

Solution. Exhaustive number of cases$=6^2=36$.

Let P(A) denote the probability of getting a total of 5, and P(B) the probability of getting a total of 11.

Favourable cases of getting a total of 5 are

(1, 4) ; (4, 1) ; (2, 3) ; (3, 2).

$$\therefore \quad P(A)=\frac{4}{36}.$$

Favourable cases of getting a total of 11 are (5, 6) ; (6, 5).

$$\therefore \quad P(B)=\frac{2}{36}.$$

The probability of getting a total of 5 or 11 is

$$P(A+B)=P(A)+P(B)=\frac{4}{36}+\frac{2}{36}=\frac{6}{36}=\frac{1}{6}.$$

Example 13. *A bag contains 6 white, 5 black and 4 yellow balls. Find the chance of getting either a white or a black ball in a single draw.*

Solution. Let A denote the event of getting a white ball, and B denote the event of getting a black ball.

Total number of balls$=6+5+4=15$.

$$\therefore \quad P(A)=\frac{6}{15},\ P(B)=\frac{5}{15}.$$

Now $P(A+B)=P(A)+P(B)=\frac{6}{15}+\frac{5}{15}=\frac{11}{15}$.

Example 14. *A bag contains 6 white, 5 black and 4 yellow balls. Two balls are drawn from it. Find the probability of getting either 2 white balls or 2 yellow balls in a single draw.*

Solution. Two balls out of a total of $6+5+4=15$ balls can be drawn in ${}^{15}C_2$ ways. Then

$P(W)\equiv$probability of getting 2 white balls $={}^6C_2/{}^{15}C_2$

or $\quad P(W)=\frac{6\,.\,5}{15\,.\,14}=\frac{1}{7}.$

$P(Y)\equiv$probability of getting 2 yellow balls$={}^4C_2/{}^{15}C_2$

or $\quad P(Y)=\frac{4\,.\,3}{15\,.\,14}=\frac{2}{35}.$

Required probability$=P(W+Y)=P(W)+P(Y)$

$$=\frac{1}{7}+\frac{2}{35}=\frac{1}{5}.$$

1.9. Compound Events

Definition 1. The simultaneous occurrence of two or more events is called a **compound event**.

For example, drawing 5 white balls and then 3 black balls from an urn containing 10 white balls and 7 black balls is a *compound event*.

If A and B are two events, then AB denotes the simultaneous occurrence of A and B and P(AB) denotes the probability of the simultaneous occurrence of the two events A and B.

Definition 2. (Conditional Probability)

The probability of the happening of an event A when the event B has already happened is called the **conditional probability** and is denoted by P(A/B).

Similarly P(B/A) means the probability of the happening of an event B when the event A has already happened.

Definition 3. (Independent Events)

Two events are said to be **independent** if the probability of the happening of one does not depend on the happening or non-happening of the other.

For example, consider a bag containing balls of different colours. Suppose one ball is drawn from it and is not replaced back and then a second ball is drawn. Then the probability of the second ball certainly depends on that of the first ball. (Notice that the probability of the second ball is a conditional probability). However, if the ball is replaced after the first draw, then the second draw will be independent of the first draw. Two such draws are independent events.

1.10. Theorem of Compound Probability or Multiplicative Law of Probability

Statement. *If A and B are two events, then*

$$P(AB)=P(A)\,P(B/A).$$

OR

The probability of the simultaneous occurrence of two events is equal to the probability of the happening of one multiplied by the conditional probability of the other when the first has already happened. (D.U., B.Sc. (G) 1991, 86, 83 ; B.A. (P) 91)

Proof. Suppose there are n exhaustive, mutually exclusive and equally likely cases, out of which m cases are favourable to the happening of the event A.

$$\therefore \qquad P(A)=\frac{m}{n}.$$

Out of these m outcomes (which favour A), let m_1 outcomes be favourable to the happening of the event B.

The conditional probability of B, knowing that A has happened is

$$P(B/A)=\frac{m_1}{m}. \qquad \ldots(2)$$

Now out of n exhaustive, mutually exclusive and equally likely outcomes, m_1 are favourable to the happening of A *and* B.

The probability of the simultaneous occurrence of A and B is

$$P(AB)=\frac{m_1}{n}=\frac{m_1}{m}\times\frac{m}{n}=\frac{m}{n}\times\frac{m_1}{m}.$$

Hence $P(AB)=P(A)P(B/A)$, using (1) and (2)

Cor 1. Interchanging A and B in the above result, we get

$$P(BA)=P(B)P(A/B).$$

Hence $P(AB)=P(B)P(A/B)$. $\qquad [\because \; P(AB)=P(BA)]$

Cor 2. *If A and B are independent events, then*

$$P(AB)=P(A)\,P(B).$$

Proof. We have $P(B/A)=P(B)$, since A and B are independent.

Hence $P(AB)=P(A)P(B)$.

Cor 3. If $A_1, A_2, \ldots\ldots A_n$ are n independent events, then

$$P(A_1A_2\ldots A_n)=P(A_1)P(A_2)\ldots\ldots P(A_n).$$

Cor 4. If $p_1, p_2, \ldots\ldots, p_n$ are the probabilities that certain events happen, then the probabilities of their non-happening are $1-p_1, 1-p_2, \ldots\ldots, 1-p_n$. The probability of all the events not happening is $(1-p_1)(1-p_2)\ldots\ldots(1-p_n)$.

Hence *the probability that at least one of these events happens*

$$=1-(1-p_1)(1-p_2)\ldots\ldots(1-p_n).$$

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Example 15. *The probability of n independent events are $p_1, p_2, \ldots, p_n$. Find an expression for the probability that at least one of the events will happen. Use the result to find the chance of obtaining at least one 6 in a throw of four dice.* (D.U., B.Sc. (G) 1984)

Solution. By Cor. 4, the probability that at least one event will happen is

$$=1-(1-p_1)(1-p_2)\ldots(1-p_n). \qquad \ldots(1)$$

The probability of obtaining a 6 in a throw of dice is $\frac{1}{6}$.

Using (1), the probability of at least one 6 in a throw of four dice is

$$=1-\left(1-\frac{1}{6}\right)\left(1-\frac{1}{6}\right)\left(1-\frac{1}{6}\right)\left(1-\frac{1}{6}\right)$$

$$=1-\left(\frac{5}{6}\right)^4=\frac{671}{1296}.$$

Notice that a throw of four dice results in 4 independent events.

Example 16. *The probability that a teacher will give an unannounced test during any class meeting is 1/5. If a student is absent twice, what is the probability that he will miss at least one test?*

Solution. The student will not miss any test if on the two days when he is absent the teacher does not give any test.

The probability that the teacher will not give any test on the two days (when the student is absent)

$$=\left(1-\frac{1}{5}\right)\left(1-\frac{1}{5}\right)=\frac{16}{25}.$$

The probability that the student will miss at least one test

$$=1-\frac{16}{25}=\frac{9}{25}.$$

Example 17. *p is the probability that a man aged x will die within a year. Find the probability that out of five men A, B, C, D, E each aged x, A will die during the year and the first to die.* (D.U., B.Sc. (G) 1985)

Solution. Since the probability that a man aged x will die in a year is p, the probability that the man does not die in a year $=(1-p)$. The probability that none of the five men, each aged x, dies in a year $=(1-p)^5$. The probability that at least one of the 5 men will die during the year $=\{1-(1-p)^5\}$. It may be noted that one man may be any of A, B, C, D, E. The probability that A will be the first to die $=1/5$.

Hence the probability that A will die during the year and be the first to die

$$=\frac{1}{5}\{1-(1-p)^5\}.$$

Ex. *p is the probability that a man aged x will die in a year. Find the probability that out of n men $A_1, A_2, \ldots, A_n$ each aged x, A_1 will die in a year and be the first to die.* $\left[\textbf{Ans. } \frac{1}{n}\{1-(1-p)^n\}\right]$

Example 18. *It is 8 : 5 against a husband who is 55 years old living till he is 75 and 4 : 3 against his wife who is now 48, living till she is 68. Find the probability that (a) the couple will be alive 20 years hence, (b) one at least of these will be alive 20 years hence.*

Solution. Let A and B respectively denote the events of the husband and wife living 20 years hence. Then

$$P(A)=\frac{5}{8+5}=\frac{5}{13},\ P(B)=\frac{3}{4+3}=\frac{3}{7}$$

$$\Rightarrow\quad P(\bar{A})=\frac{8}{13}\ \text{ and }\ P(\bar{B})=\frac{4}{7}.$$

(*a*) Probability that the couple lives for 20 years

$$=P(AB)=P(A)\,P(B)=\frac{15}{91}.$$

(*b*) Probability that none of A and B lives $=P(\bar{A})P(\bar{B})$

$$=\frac{8}{13}\times\frac{4}{7}.$$

Hence the probability that at least one of A and B lives

$$=1-\frac{8}{13}\times\frac{4}{7}=\frac{59}{91}$$

Example 19. *The odds against A solving a certain problem are 8 to 6 and the odds in favour of B solving the same problem are 14 to 10. What is the probability that if both of them try, the problem would be solved?*

Solution. The probability that A cannot solve the problem

$$=\frac{8}{8+6}=\frac{4}{7}.$$

The probability that B cannot solve the problem

$$=\frac{10}{14+10}=\frac{5}{12}.$$

The probability that none of A and B can solve the problem

$$=\frac{4}{7}\times\frac{5}{12}=\frac{5}{21}.$$

Hence the probability that the problem will be solved

$$=1-\frac{5}{21}=\frac{16}{21}.$$

Example 20. *A problem in Statistics is given to the three students A, B and C whose chances of solving it are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that the problem is solved?*

(D.U., B.Sc. (G) 1983)

Solution. The probabilities that the problem is solved by the students A, B, C are respectively

$$P(A)=\tfrac{1}{2},\ P(B)=\tfrac{1}{3}\text{ and }P(C)=\tfrac{1}{4}.$$

Now $P(\bar{A})=\frac{1}{2}$, $P(\bar{B})=\frac{2}{3}$, $P(\bar{C})=\frac{3}{4}$.

The problem will be solved if at least one of them solves it.

The probability that none of them solves it

$$=P(\bar{A})\,P(\bar{B})\,P(\bar{C})=\tfrac{1}{2}\cdot\tfrac{2}{3}\cdot\tfrac{3}{4}=\tfrac{1}{4}.$$

Hence the probability that the problem is solved

$$=1-\tfrac{1}{4}=\tfrac{3}{4}.$$

Ex. *A problem in statistics is given to the three students A, B and C whose chances of solving it are $\frac{1}{2}$, $\frac{3}{4}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved?*

(D.U., B.A. (P) 1986)

[**Ans.** 29/32]

Example 21. *A can hit a target 3 times in 5 shots, B 2 times in 5 shots, and C 3 times in 4 shots. Find the probability of the target being hit when all of them try.*

Solution. Let E_1 be the event that A hits the target.

$$\therefore\quad P(E_1)=\frac{3}{5}\ \text{ and }\ P(\overline{E_1})=1-\frac{3}{5}=\frac{2}{5}.$$

Let E_2 be the event that B hits the target.

$$\therefore\quad P(E_2)=\frac{2}{5}\ \text{ and }\ P(\overline{E_2})=\frac{3}{5}.$$

Let E_3 be the event that C hits the target.

$$\therefore\quad P(E_3)=\frac{3}{4}\ \text{ and }\ P(\overline{E_3})=\frac{1}{4}.$$

The required probability that the target is hit when all of them try is

$$=P\,[\text{at least one of the three hits the target}]$$

$$=1-P\,[\text{none hits the target}]$$

$$=1-P(\overline{E_1})\,P(\overline{E_2})\,P(\overline{E_3})=1-\frac{2}{5}\cdot\frac{3}{5}\cdot\frac{1}{4}=\frac{47}{50}.$$

Ex. *A can hit a target 3 times in 5 shots, B 2 times in 5 shots, and C 3 times in 4 shots. They fire a volley. Find the chance that (a) 2 shots hit (b) at least two shots hit (c) exactly one of them hits.*

Hint. Refer to Example 21.

(*a*) Reqd. prob. $=P(E_1E_2\bar{E}_3)+P(E_1\bar{E}_2E_3)+P(\bar{E}_1E_2E_3)$

$$=\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{1}{4}+\frac{3}{5}\cdot\frac{3}{5}\cdot\frac{3}{4}+\frac{2}{5}\cdot\frac{2}{5}\cdot\frac{3}{4}=\frac{45}{100}.$$

(*b*) Reqd. prob. = prob. of two shots hit $+P(E_1E_2E_3)$

$$=\frac{45}{100}+\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{3}{4}=\frac{63}{100}.$$

(*c*) Reqd. prob. $=P(E_1\bar{E}_2\bar{E}_3)+P(\bar{E}_1E_2\bar{E}_3)+P(\bar{E}_1\bar{E}_2E_3)$

$$=\frac{3}{5}\cdot\frac{3}{5}\cdot\frac{1}{4}+\frac{2}{5}\cdot\frac{2}{5}\cdot\frac{1}{4}+\frac{2}{5}\cdot\frac{3}{5}\cdot\frac{3}{4}=\frac{31}{100}.$$

Example 22. *A speaks truth in 75% and B in 80% of the cases. In what percentages of cases are they likely to contradict each other in stating the same fact.*

Solution. Probabilities that A and B speak the truth are given by

$$P(A)=\frac{75}{100}=\frac{3}{4},\ P(B)=\frac{80}{100}=\frac{4}{5}.$$

Probabilities that they do not speak the truth are

$$P(\bar{A})=1-\frac{3}{4}=\frac{1}{4},\ P(\bar{B})=1-\frac{4}{5}=\frac{1}{5}.$$

Probability that they contradict each other is

$$=P(A)P(\bar{B})+P(B)P(\bar{A})$$
$$=\frac{3}{4}\times\frac{1}{5}+\frac{4}{5}\times\frac{1}{4}=\frac{7}{20}.$$

Hence A and B are likely to contradict each other in $\frac{7}{20}\times 100$ $=35\%$ of the cases.

Example 23. *A husband and wife appear in an interview for two vacancies in the same post. The probability of the husband's selection is 1/7 and that of wife's selection is 1/5. What is the probability that only one of them will be selected.* (D.U., B.Sc. (G) 1987)

Solution. The probability of the husband's selection is

$$P(H)=\frac{1}{7}\ \Rightarrow\ P(\bar{H})=1-\frac{1}{7}=\frac{6}{7}.$$

The probability of the wife's selection is

$$P(W)=\frac{1}{5}\ \Rightarrow\ P(\bar{W})=1-\frac{1}{5}=\frac{4}{5}.$$

The probability that only one of then will be selected

$$=P(H)P(\bar{W})+P(W)P(\bar{H})$$
$$=\frac{1}{7}\cdot\frac{4}{5}+\frac{1}{5}\cdot\frac{6}{7}=\frac{10}{35}=\frac{2}{7}$$

Example 24. *Four persons are chosen at random from a group containing 3 men, 2 women and 4 children. Show that the chance that exactly two of them will be children is 10/21.* (D.U., B.Sc. (G) 1987)

Solution. Four persons out of a total of 3+2+4=9 persons can be selected in ${}^9C_4$ ways.

Since out of 4 persons chosen exactly two are to be children, the remaining two persons will be chosen out of 3 men and 2 women *i.e.*, 5 persons. Now 2 children out of 4 can be selected in ${}^4C_2$ ways and 2 persons out of 5 persons can be selected in ${}^5C_2$ ways.

Required probability $=({}^4C_2\times{}^5C_2)/{}^9C_4=\frac{10}{21}.$

Example 25. *Probability that a boy will pass an examination is 3/5 and that for a girl is 2/5. What is the probability that at least one of the them passes?*

Solution. The probability that the boy passes is $P(A)=3/5$. The probability that the girl passes is $P(B)=2/5$. The probability that at least one of them passes is

$$=P(A+B)=P(A)+P(B)-P(AB)$$
$$=P(A)+P(B)-P(A)P(B)$$
$$=\frac{3}{5}+\frac{2}{5}-\frac{3}{5}\cdot\frac{2}{5}=\frac{19}{25}.$$

Example 26. *Find the probability of drawing either an ace or a spade or both from a pack of cards.*

Solution. Let A be the event of drawing an ace and B of drawing a spade. Then

$$P(A)=\frac{4}{52}=\frac{1}{13},\ P(B)=\frac{13}{52}=\frac{1}{4}.$$

Required probability

$$=P(A+B)=P(A)+P(B)-P(AB)$$
$$=P(A)+P(B)-P(A)P(B)$$
$$=\frac{1}{13}+\frac{1}{4}-\frac{1}{13}\times\frac{1}{4}=\frac{4}{13}.$$

Example 27. *The probability of occurrence of an event A is 0.7, the probability of non-occurrence of B is 0.5 and that of at least one of A and B not occurring is 0.6. Find the probability that one of A or B occurs.*

Solution. We have $P(A)=0.7 \Rightarrow P(\bar{A})=0.3$,

$$P(\bar{B})=0.5 \Rightarrow P(B)=0.5.$$

We are also given that $P(\bar{A}+\bar{B})=0.6$

i.e., $0.6=P(\bar{A}+\bar{B})=1-P(AB) \Rightarrow P(AB)=0.4.$

Hence the probability that one of A or B occurs is

$$=P(A+B)=P(A)+P(B)-P(AB)$$
$$=0.7+0.5-0.4=0.8.$$

Example 28. *Discuss and criticize the following :*

$$P(A)=\frac{2}{3}\quad P(B)=\frac{1}{4}\quad P(C)=\frac{1}{6}$$

for the probabilities of three mutually exclusive events A, B, C.

Solution. We have $P(A)=\frac{1}{6},\ \frac{2}{3}\quad P(B)=\frac{1}{6}$

$$\Rightarrow\quad P(B)=\frac{1}{4}\text{ and }\frac{1}{4}\quad P(C)=\frac{1}{6}\quad\Rightarrow\quad P(C)=\frac{2}{3}.$$

Now $P(A)+P(B)+P(C)=\frac{1}{6}+\frac{1}{4}+\frac{2}{3}=\frac{13}{12}>1,$

which is impossible, since probability is $\leqslant 1$.

Hence the statement is wrong.

Example 29. *If $P(A+B)=5/6$, $P(AB)=1/3$ and $P(\bar{B})=1/3$, find $P(A)$ and $P(B)$.*

Solution. $P(B)=1-P(\bar{B})=1-\frac{1}{3}=\frac{2}{3}.$

We know $P(A+B)=P(A)+P(B)-P(AB)$

or $\quad \frac{5}{6}=P(A)+\frac{2}{3}-\frac{1}{3} \Rightarrow P(A)=\frac{1}{2}.$

Example 30. *A bag contains 6 white and 9 black balls. Four balls are drawn at a time. Find the probability for the first draw to give 4 white and the second to give 4 black balls in each of the following cases :*

(*i*) *The balls are not replaced before the second draw.*

(*ii*) *The balls are replaced before the second draw.*

Solution. (*i*) We can draw 4 balls from a bag containing 6 white and 9 black balls results in ${}^{15}C_4$ ways.

Let A be the event that the first draw gives 4 white balls and B the event that the second draw gives 4 black balls. We can draw white balls out of 6 white balls in ${}^{6}C_4$ ways.

$$\therefore\quad P(A)=\frac{{}^{6}C_4}{{}^{15}C_4}.$$

Now if the drawn balls are not replaced, we can draw the other 4 balls in ${}^{11}C_4$ ways. The event B that the second draw results in 4 black balls (on the assumption that the first draw has given 4 white balls) has ${}^{9}C_4$ favourable cases.

$$\therefore\quad P(B\mid A)=\frac{{}^{9}C_4}{{}^{11}C_4}.$$

Hence $\quad P(AB)=P(A)\times P(B\mid A)$

$$=\frac{{}^{6}C_4}{{}^{15}C_4}\times\frac{{}^{9}C_4}{{}^{11}C_4}=\frac{3}{715}.$$

(*ii*) We have $P(A)=\frac{{}^{6}C_4}{{}^{15}C_4}$

Since the balls are replaced after the first draw, the probability of drawing 4 black balls in the second draw is ${}^{9}C_4/{}^{15}C_4$.

$\therefore\quad P(AB)=P(A)\,.\,P(B)$, as A and B are independent events.

Hence $\quad P(AB)=\frac{{}^{6}C_4}{{}^{15}C_4}\times\frac{{}^{9}C_4}{{}^{15}C_4}=\frac{3}{2963}.$

Example 31. *Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, and 1 girl and 3 boys. One child is selected at random from each group. Show that the chance that the three selected consist of 1 girl and 2 boys is 13/32.*

Solution. 1 girl and 2 boys can be selected from each group in the following mutually exclusive ways :

	Ist group	*2nd group*	*3rd group*
(*i*)	Girl	Boy	Boy
(*ii*)	Boy	Girl	Boy
(*iii*)	Boy	Boy	Girl

By the theorem of total probability,

required probability $=P(i)+P(ii)+P(iii)$. ...(1)

The probability of selecting a girl from the first group is 3/4, of selecting a boy from the second group is 2/4, and of selecting a boy from the third group is 3/4. Since the three events of selecting children from three groups are independent of each other, therefore

$$P(i)=\frac{3}{4}\times\frac{2}{4}\times\frac{3}{4}=\frac{9}{32}$$

Similarly

$$P(ii)=\frac{1}{4}\times\frac{2}{4}\times\frac{3}{4}=\frac{3}{32},$$

$$P(iii)=\frac{1}{4}\times\frac{2}{4}\times\frac{1}{4}=\frac{1}{32}.$$

Substituting in (1), the required probability

$$=\frac{9}{32}+\frac{3}{32}+\frac{1}{32}=\frac{13}{32}.$$

Example 32. *A coin is tossed three times. Find the chances of throwing (i) three heads (ii) two heads and one tail, (iii) head and tail alternately.*

Solution. If H denotes head, and T tail, then

$$P(H)=\tfrac{1}{2}=P(T).$$

(*i*) P (three heads)=P(HHH)=P(H)P(H)P(H)

($\because$ all the throws are independent)

$$=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}.$$

(*ii*) The event B of getting two heads and one tail can happen in the following mutually exclusive ways ;

HHT, HTH, THH

$$P(B)=P(HHT)+P(HTH)+P(THH)$$
$$=P(H)P(H)P(T)+P(H)P(T)P(H)+P(T)P(H)P(H)$$
$$=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

(*iii*) The probability of getting head and bail alternately is

$$=P(HTH)+P(THT)$$
$$=P(H)P(T)P(H)+P(T)P(H)P(T)$$
$$=\frac{1}{8}+\frac{1}{8}=\frac{1}{4}.$$

Example 33. *A and B toss a coin alternately on the understanding that the first to obtain head wins the toss. Show that their respective chances of winning are 2/3 and 1/3.*

Solution. Probabilities that A and B get head in a toss are given by $P(A)=P(B)=\tfrac{1}{2}$.

$$\therefore \quad P(\bar{A})=P(\bar{B})=1-\tfrac{1}{2}=\tfrac{1}{2}.$$

If A begins, he can win in following mutually exclusive ways :

$(A), (\bar{A}\,\bar{B}\,A), (\bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A), \ldots\ldots$

Therefore, by the theorem of total probability, probability that A wins

$$=P(A)+P(\bar{A}\,\bar{B}\,A)+P(\bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A)+\ldots\ldots$$
$$=P(A)+P(\bar{A})P(\bar{B})P(A)+\{P(\bar{A})P(\bar{B})\}^2\,P(A)+\ldots\ldots$$

$$=\frac{1}{2}+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^5+\ldots=\frac{a}{1-r},\ [a=\tfrac{1}{2},\ r=(\tfrac{1}{2})^2]$$
$$=\frac{\tfrac{1}{2}}{1-\tfrac{1}{4}}=\frac{2}{3}.$$

Since total probability is unity and one of the players is to win,

probability that B wins$=1-\dfrac{2}{3}=\dfrac{1}{3}$.

Example 34. *A and B take turns in throwing two dice, the first to throw 9 being awarded the prize. Show that their chance of winning are in the ratio 9 : 8.*

(DU., B. Sc. (G) 1991)

Solution. The total of 9 can be obtained as follows :

(3, 6), (4, 5), (5, 4) and (6, 3). No. of total cases$=6\times6=36$.

No. of favourable cases$=4$.

Then $P(A)=P(B)=\dfrac{4}{36}=\dfrac{1}{9}$.

Therefore $P(\bar{A})=P(\bar{B})=1-\dfrac{1}{9}=\dfrac{8}{9}$.

Since A begins, he can win in following mutually exclusive ways :

$(A), (\bar{A}\,\bar{B}\,A), (\bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A), \ldots\ldots$

Therefore, by the theorem of total probability, probability that A wins

$$=P(A)+P(\bar{A}\,\bar{B}\,A)+P(\bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A)+\ldots\ldots$$
$$=P(A)+P(\bar{A})P(\bar{B})P(A)+P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(A)+\ldots\ldots$$
$$=\frac{1}{9}\left\{1+\left(\frac{8}{9}\right)^2+\left(\frac{8}{9}\right)^4+\ldots\ldots\right\},$$

which is a G. P. with c.r.$=8/9$

$$=\frac{1}{9}\cdot\frac{1}{1-\frac{64}{81}}=\frac{9}{17}.$$

Since total probability is unity and one of two players is to win, probability that B wins$=1-\dfrac{9}{17}=\dfrac{8}{17}$.

Hence their chances of winning are in the ratio 9 : 8.

Example 35. (*Huyghen's Problem*). *A and B throw with a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he* *throws 7 before A throws 6. If A begins, show that his chance of winning is* $\frac{30}{61}$.

Solution. Probabilities that A gets 6 and B gets 7 with a pair of dice are given by

$$P(A)=\frac{5}{36} \text{ and } P(B)=\frac{6}{36}=\frac{1}{6},$$

Therefore $P(\bar{A})=1-\frac{5}{36}=\frac{31}{36}$ and $P(\bar{B})=1-\frac{1}{6}=\frac{5}{6}$.

Since A begins, he can win in following mutually exclusive ways :

$$(A), (\bar{A}\,\bar{B}\,A), (\bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A), \ldots\ldots$$

Therefore, by the theorem of total probability,

probability that A wins $=P(A)+P(\bar{A}\,\bar{B}\,A)+P(\bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A)+\ldots\ldots$

$$=P(A)+P(\bar{A})P(\bar{B})P(A)+P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(A)+\ldots\ldots$$

$$=\frac{5}{36}\left\{1+\left(\frac{31}{36}\cdot\frac{5}{6}\right)+\left(\frac{31}{36}\cdot\frac{5}{6}\right)^2+\ldots\ldots\right\}$$

$$=\frac{5}{36}\cdot\frac{1}{1-\frac{31}{36}\cdot\frac{5}{6}}=\frac{30}{61}.$$

Example 36. *A, B and C in order toss a coin. The first one to throw a head wins. What are their respective chances of winning assuming that the game may continue indefinitely ?*

Solution. The probabilities that A, B and C get a head in a toss are given by

$$P(A)=P(B)=P(C)=\frac{1}{2}.$$

$$\therefore \quad P(\bar{A})=P(\bar{B})=P(\bar{C})=1-\tfrac{1}{2}=\tfrac{1}{2}.$$

If A begins, he can win in following mutually exclusive ways :

$$(A), (\bar{A}\,\bar{B}\,\bar{C}\,A), (\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}\,\bar{B}\,\bar{C}\,A), \ldots\ldots$$

By the theorem of total probability, probability that A wins

$$=P(A)+P(\bar{A}\,\bar{B}\,\bar{C}\,A)+P(\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}\,\bar{B}\,\bar{C}\,A)+\ldots\ldots$$

$$=P(A)+P(\bar{A})P(\bar{B})P(\bar{C})P(A)+\{P(\bar{A})P(\bar{B})P(\bar{C})\}^2P(A)+\ldots\ldots$$

$$=\frac{1}{2}+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^7+\ldots\ldots$$

$$=\frac{1}{2}\left\{1+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^6+\ldots\ldots\right\}=\frac{1}{2}\cdot\frac{1}{1-(1/8)}=\frac{4}{7}.$$

Now B can win in following mutually exclusive ways :

$$(\bar{A}B), (\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}B), (\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}\,B), \ldots\ldots$$

The probability that B wins

$$=P(\bar{A}\,B)+P(\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}\,B)+P(\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}\,B)+\ldots\ldots$$

$$=P(\bar{A})P(B)+P(\bar{A})P(\bar{B})P(\bar{C})P(\bar{A})P(B)+\ldots\ldots$$

$$=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^5+\left(\frac{1}{2}\right)^8+\ldots\ldots=\frac{1/4}{1-1/8}=\frac{2}{7}.$$

The probability that C wins $=1-\frac{4}{7}-\frac{2}{7}=\frac{1}{7}$.

Example 37. *Cards are dealt one by one from a well-shuffled pack until an ace appears. Show that the probability that exactly n cards are dealt before the first ace appears is*

$$\frac{4(51-n)(50-n)(49-n)}{52\cdot51.50.49}.$$ [D.U., Physics (H) 1997]

(D.U., B.Sc. (G) 1992, 89, 85 ; B.A. (P) 90)

Solution. Probability that an ace does not appear in the first draw

$$=1-\frac{4}{52}=\frac{48}{52}.$$

Probability that the first ace does not appear in the second draw

$$=1-\frac{4}{51}=\frac{47}{51}.$$

Probability that the first ace does not appear in the third draw

$$=1-\frac{4}{50}=\frac{46}{50} \text{ and so on.}$$

Probability that the first ace does not appear in the $(n-1)$th draw

$$=1-\frac{4}{52-(n-2)}=\frac{50-n}{52-(n-2)}.$$

Probability that the first ace does not appear in the nth draw

$$=1-\frac{4}{52-(n-1)}=\frac{49-n}{52-(n-1)}.$$

Probability that the first ace appears in the $(n+1)$th draw

$$=\frac{4}{52-n}.$$

Hence the probability that the first ace appears in the $(n+1)$th draw

$$=\left[\frac{48}{52}\times\frac{47}{51}\times\frac{46}{50}\times\frac{45}{49}\times\frac{44}{48}\times\ldots\ldots\times\frac{(52-n)}{52-(n-4)}\times\frac{(51-n)}{52-(n-3)}\times\frac{(50-n)}{52-(n-2)}\times\frac{(49-n)}{52-(n-1)}\times\frac{4}{52-n}\right]$$

$$=\frac{4(49-n)(50-n)(51-n)}{52\times51\times50\times49}.$$

Ex. *Cards are dealt one by one from a full deck. What is the probability that 10 cards will precede the first ace.*

(D.U., B.Sc. (G) 1988)

[**Hint.** Take $n=10$ in Example 37].

Example 38. *A coin is tossed $(m+n)$ times $(m>n)$. Show that the probability of at least n consecutive heads is $\frac{n+2}{2^{m+1}}$.*

Solution. Since $m>n$, only one sequence of m consecutive heads is possible. This sequence may start either with the first toss or second toss or third toss, and so on, the last one will be starting with $(n+1)$th toss.

Let E_i denote the event that the sequence of m consecutive heads starts with ith toss. Then the required probability is

$$P(E_1)+P(E_2)+\ldots+P(E_{n+1}). \quad \ldots(1)$$

Now $P(E_1)=P$ [consecutive heads in the first m tosses]

$$\therefore \quad P(E_1)=\left(\frac{1}{2}\right)^m$$

$P(E_2)=P$ [tail in the first toss, followed by m consecutive heads]

$$\therefore \quad P(E_2)=\frac{1}{2}\left(\frac{1}{2}\right)^m=\frac{1}{2^{m+1}},$$

$P(E_r)=P$ [tail in the $(r-1)$th toss, followed by m consecutive heads]

$$\therefore \quad P(E_r)=\frac{1}{2}\left(\frac{1}{2}\right)^m=\frac{1}{2^{m+1}},\ r=2, 3, \ldots\ldots (n+1).$$

Putting in (1), required probability

$$=\frac{1}{2^m}+\frac{n}{2^{m+1}}=\frac{2+n}{2^{m+1}}.$$

Example 39. *A and B are two independent withnesses (i.e., there is no collusion between them) in a case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a certain statement. Show that the probability that this statement is true is*

$$\frac{xy}{1-x-y+2xy}$$

Solution Let A_1 be the event that A and B agree in a statement and A_2 be the event that their statement is correct.

Then $P(A_1)=xy+(1-x)(1-y)=1-x-y+2xy.$

Now $P(A_1A_2)=xy.$

By compound probability theorem,

$$P(A_1A_2)=P(A_1)P(A_2/A_1).$$

$$\therefore \quad P(A_2/A_1)=\frac{P(A_1A_2)}{P(A_1)}=\frac{xy}{1-x-y+2xy}.$$

Example 40. *A bag contains 50 tickets numbered 1, 2,... ,50. Five tickets are drawn at random and arranged in ascending order of magnitude : $(x_1<x_2<\ldots<x_{50})$. What is the probability that $x_3=30$.*

Solution. Five tickets out of 50 tickets can be drawn in ${}^{50}C_5$ ways. Since $x_3=30$, tickets x_1 and x_2 must be drawn from the set $\{1, 2, \ldots, 29\}$ and x_4 and x_5 from the set $\{31, 32, \ldots, 50\}$. The tickets x_1, x_2, x_4 and x_5 can be chosen in ${}^{29}C_2\times{}^{20}C_2$ ways. Hence

$$P(x_3=30)=\frac{{}^{29}C_2\times{}^{20}C_2}{{}^{50}C_5}.$$

Example 41. *A player tosses a coin and is to score one point for every head turned up and two for every tail. He is to play on until his score reaches or passes n. If p_n is the chance for attaining exactly n, show that $p_n=\frac{1}{2}(p_{n-1}+p_{n-2})$ and hence find the value of p_n.*

Solution. There are two ways of attaining exactly n :

(*i*) getting a tail when the score is $n-2$,

(*ii*) getting a head when the score is $n-1$.

Now $P(i)=\frac{1}{2}p_{n-2}$ and $P(ii)=\frac{1}{2}p_{n-1}$.

Since these events are mutually exclusive,

$$p_n=\tfrac{1}{2}(p_{n-1}+p_{n-2}). \quad \ldots(1)$$

We can write (1) as

$$p_n+\tfrac{1}{2}p_{n-1}=p_{n-1}+\tfrac{1}{2}p_{n-2}. \quad \ldots(2)$$

Putting $n=n-1, n-2, \ldots, 2$ in (2), we obtain

$$p_n+\tfrac{1}{2}p_{n-1}=p_{n-1}+\tfrac{1}{2}p_{n-2}=p_{n-2}+\tfrac{1}{2}p_{n-3}=\ldots=p_2+\tfrac{1}{2}p_1,$$

where $p_1=\frac{1}{2},\ p_2=P(HH)+P(T)=\frac{1}{4}+\frac{1}{2}=\frac{3}{4}.$

$$\therefore\ p_n+\tfrac{1}{2}p_{n-1}=p_2+\tfrac{1}{2}p_1=\tfrac{3}{4}+\tfrac{1}{4}=1=\tfrac{2}{3}+\tfrac{1}{2}\cdot\tfrac{2}{3} \qquad \ldots(3)$$

or $\quad p_n-\frac{2}{3}=-\frac{1}{2}(p_{n-1}-\frac{2}{3}).$

Putting $n=n-1, n-2, \ldots, 2$ in (3), we obtain

$$p_n-\tfrac{2}{3}=(-\tfrac{1}{2})(p_{n-1}-\tfrac{2}{3})=(-\tfrac{1}{2})^2(p_{n-2}-\tfrac{2}{3})=\ldots=(-\tfrac{1}{2})^{n-1}(p_1-\tfrac{2}{3})$$

$$\therefore\ p_n-\tfrac{2}{3}=(-\tfrac{1}{2})^{n-1}(\tfrac{1}{2}-\tfrac{2}{3})$$

Hence $p_n=\frac{2}{3}+\frac{1}{3}(-\frac{1}{2})^n.$

Example 42. *If m things are distributed among 'a' men and 'b' women, show that the probability that the number of things received by men is odd, is*

$$\frac{1}{2}\left[\frac{(b+a)^m-(b-a)^m}{(b+a)^m}\right]. \qquad \text{[D.U., B.Sc. (G) 1995]}$$

Solution. The probability p that a thing is received by a man

$$p=\frac{a}{a+b}.$$

The probability q that a thing is received by a woman

$$q=\frac{b}{a+b}.$$

The probability that out of m things, exactly x are received by men and the rest by women $={}^mC_x q^{m-x}p^x$.

The probability P that the number of things received by men is odd :

$$P={}^mC_1q^{m-1}p+{}^mC_3q^{m-3}p^3+{}^mC_5q^{m-5}p^5+\ldots\ldots$$

Now $(q+p)^m=q^m+{}^mC_1q^{m-1}p+{}^mC_2q^{m-2}p^2+{}^mC_3q^{m-3}p^3+\ldots\ldots$

and $\quad (q-p)^m=q^m-{}^mC_1q^{m-1}p+{}^mC_2q^{m-2}p^2-{}^mC_3q^{m-3}p^3+\ldots\ldots$

$$\therefore\ (q+p)^m-(q-p)^m=2[{}^mC_1q^{m-1}p+{}^mC_3q^{m-3}p^3+\ldots\ldots]=2P \qquad \ldots(1)$$

Now $\quad q+p=1$ and $q-p=\dfrac{b-a}{b+a}$. $\qquad \ldots(2)$

From (1) and (2), $1-\left(\dfrac{b-a}{b+a}\right)^m=2P.$

Hence $\quad P=\frac{1}{2}\left[\dfrac{(b+a)^m-(b-a)^m}{(b+a)^m}\right].$

Example 43. *Each of the n urn contains a white and b black balls. One ball is transferred from the first urn into the second, then one ball from the latter into third and so on. Finally one ball is taken from the last urn, what is the probability of its being white ?*

Solution. If U_k denotes the kth urn, a white ball from U_k can be drawn in the following two disjoint ways :

(*i*) A={Ball transferred from U_{k-1} to U_k is white},

$$P(A)=p_{k-1}.$$

(*ii*) B={Ball tranferred from U_{k-1} to U_k is black},

$$P(B)=1-p_{k-1}.$$

If p_k be the probability of drawing a white ball from U_k,

$$p_k=\frac{a+1}{a+b+1}p_{k-1}+\frac{a}{a+b+1}(1-p_{k-1})$$

or $\quad p_k=\dfrac{1}{a+b+1}p_{k-1}+\dfrac{a}{a+b+1}$

$\therefore\ p_k=c^{-1}p_{k-1}+c^{-1}a,\ (c=a+b+1).$

Replacing k by $k-1, k-2, \ldots, 3, 2$; we get

$$p_{k-1}=c^{-1}p_{k-2}+c^{-1}a,\ p_{k-2}=c^{-1}p_{k-3}+c^{-1}a,\ldots\ldots$$

$$p_3=c^{-1}p_2+c^{-1}a,\ p_2=c^{-1}p_1+c^{-1}a,\ p_1=a/(a+b)$$

From these relations, we obtain

$$p_k=(c^{-1})^{k-1}p_1+(c^{-1}a)[1+c^{-1}+(c^{-1})^2+\ldots+(c^{-1})^{k-2}]$$

$$=(c^{-1})^{k-1}p_1+(c^{-1}a)[1-(c^{-1})^{k-1}]/(1-c^{-1})$$

$$=\frac{a}{a+b}\cdot\frac{1}{(a+b+1)^{k-1}}+\frac{a}{a+b}\left[1-\frac{1}{(a+b+1)^{k-1}}\right]$$

$$[\because\ c^{-1}a/(1-c^{-1})=a/(c-1)]$$

$$=\frac{a}{a+b}$$

$p_k=a/(a+b)$, which is independent of k.

Hence $p_n=a/(a+b)$.

EXERCISES

1. If two balls are drawn from a bag containing 2 white, 4 red and 5 black balls. What is the chance that (*i*) both the balls are red (*ii*) one is red and the other black ?

Ans. (*i*) ${}^4C_2/{}^{11}C_2=\frac{6}{55}$, (*ii*) $4\times5/{}^{11}C_2=\frac{4}{11}$]

2. Find the chance of throwing a sum of 9 in a single throw of two dice. [**Ans.** 1/9]

3. What is the chance that a non-leap year selected at random will contain 53 Sundays ? [**Ans.** 1/7] (*D.U., B.A. (P) 1984*)

[**Hint.** In a non-leap year there are 52 weeks and 1 day (which can be any day of the week). There will be 53 Sundays if the remaining one day is Sunday whose probability is 1/7.

4. From a set of 17 cards, numbered 1, 2, 3, ..., 16, 17, one is drawn at random. What is the chance that (*i*) its number is a multiple of 3 or 7, (*ii*) its number is a multiple of 3 or 5 or both. [**Ans.** (*i*) 7/17, (*ii*) 7/17]

[**Hint.** (*i*) Favourable cases in (*i*) are 3, 6, 7, 9, 12, 14, 15 $(m=7)$

(*ii*) Favourable cases in (*ii*) are 3, 5, 6, 9, 10, 12, 15 $(m=7)$]

5. From a pack of 52 cards, two are drawn at random. Find the chance that one is a king and the other a queen.

[**Hint.** $({}^4C_1 \times {}^4C_1)/{}^{52}C_2 = \frac{8}{663}$.]

6. The chances of winning of two race horses are 1/3 and 1/6 respectively, what is the probability that at least one will win when the horses are running (*a*) in different races and (*b*) in the same race.

[**Hint.** (*i*) Reqd. prob. $=1-\left[\left(1-\frac{1}{3}\right)\left(1-\frac{1}{6}\right)\right]=\frac{4}{9}$.

(*ii*) Reqd. prob. $=\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$.]

7. The probability that a boy will pass an examination is 3/5 and that for a girl is 2/5. What is the probability that at least one of them passes. [**Ans.** 19/25.]

8. Three ships A, B and C sail from England to India. Odds in favour of their arriving safely are 2 : 5, 3 : 7 and 6 : 11 respectively. Find the chance that they all arrive safely.

[**Hint.** Reqd. prob. $=\frac{2}{7}\times\frac{3}{10}\times\frac{6}{17}=\frac{18}{595}$.]

9. From a group of 8 children, 5 boys and 3 girls, three children are selected at random. Calculate the probabilities that the selected group contains (*i*) no girl, (*ii*) only one girl, (*iii*) one particular girl, (*iv*) at least one girl, and (*v*) more girls than boys.

[**Ans.** (*i*) $\frac{5}{28}$, (*ii*) $\frac{15}{28}$, (*iii*) $\frac{5}{28}$, (*iv*) $\frac{23}{28}$, (*v*) $\frac{2}{7}$.]

[**Hint.** (*i*) ${}^5C_3/{}^8C_3$, (*ii*) ${}^3C_1\times{}^5C_2/{}^8C_3$, (*iii*) ${}^5C_2/{}^8C_3$.
(*iv*) $({}^3C_1\times{}^5C_2/{}^8C_3)+({}^3C_2\times{}^5C_1/{}^8C_3)+({}^3C_3/{}^8C_3)$.
(*v*) $({}^3C_2\times{}^5C_1/{}^8C_3)+({}^3C_3/{}^8C_3)$.]

10. There are n letters and n addressed envelopes. If the letters are placed in the envelopes at random, what is the probability that all the letters are not placed in the right envelopes.

[**Hint.** The total number of placing n letters in n envelopes is $n!$. Since all the letters can be placed correctly only in one way, its probability $=\frac{1}{n!}$.

The probability that all letters are sent wrongly $=1-\frac{1}{n!}$.]

11. A, B and C are independent witnesses of an event which is known to have occurred. A speaks the truth three times out of four, B four times out of five and C five times out of six. What is the probability that the occurrence will be reported truthfully by majority of three witnesses ?

[**Hint.** $P(A)=3/4$, $P(B)=4/5$, $P(C)=5/6$.

$P(\bar{A})=1/4$, $P(\bar{B})=1/5$, $P(\bar{C})=1/6$.

Reqd. prob. $=P(AB\bar{C})+P(A\bar{B}C)+P(\bar{A}BC)+P(ABC)$

$$=\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{1}{6}+\frac{3}{4}\cdot\frac{1}{5}\cdot\frac{5}{6}+\frac{1}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}+\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}.]$$

12. The odds that a book will be favourably reviewed by three independent critics are 5 to 2, 4 to 3 and 3 to 4 respectively. What is the probability that of the three reviews a majority will be favourable. [**Ans.** $\frac{209}{343}$]

[**Hint.** Similar to Ex. 11 above,

$P(A)=5/7$, $P(B)=4/7$, $P(C)=3/7$.]

13. An urn contains 3 red and 4 black balls. Two balls are taken out at random. Find the probabilities that the balls are of (*i*) different colours, (*ii*) black colour, (*iii*) red colour.

[**Hint.** (*i*) $({}^3C_1\times{}^4C_1)/{}^7C_2=\frac{4}{7}$, (*ii*) ${}^4C_2/{}^7C_2=\frac{2}{7}$,

(*iii*) ${}^3C_2/{}^7C_2=\frac{1}{7}$.]

14. Find the probability of throwing 15 with three unbiased dice. (*D.U.*, *B.A.* (*P*) *1989*) [**Ans.** 10/216]

[**Hint.** We can throw 15 in the following ways :

(6, 6, 3), (6, 3, 6), (3, 6, 6), (6, 5, 4), (6, 4, 5),
(5, 4, 6), (5, 6, 4), (4, 5, 6), (4, 6, 5), (5, 5, 5).

$m=10$, $n=6\times6\times6=216$.]

15. If two dice are thrown, what is the probability that the sum is (*i*) greater than 8, (*ii*) neither 7 nor 11. [**Ans.** 5/18, 7/9]

[**Hint.** (*i*) $P(9)+P(10)+P(11)+P(12)$
$=(4/36)+(3/36)+(2/36)+(1/36)$,

(*ii*) $1-[P(7)+P(11)]=1-(6/36)-2/36$.]

16. A person throws 3 coins. Find the probability that at least one head turns up. [Ans. 7/8]

[Hint. The probability of not throwing a head with 3 coins

$$=\left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)=\frac{1}{8}.$$

Reqd. prob. $=1-\frac{1}{8}=\frac{7}{8}$.]

17. Find the chance of throwing more than 15 in one throw with three dice. [Ans. 5/108]

[Hint. Reqd. prob. = P(16)+P(17)+P(18).

We can obtain the sum 16 as follows :

(6, 5, 5), (6, 4, 6), (6, 6, 4), (5, 6, 5), (5, 5, 6), (4, 6, 6).

$\Rightarrow P(16)=\frac{1}{216}$.

We can obtain the sum 17 as follows :

(6, 6, 5), (6, 5, 6), (5, 6, 6) $\Rightarrow$ $P(17)=\frac{3}{216}$.

and $P(18)=\frac{1}{216}$.]

18. Find the chance of throwing more than 14 with three symmetrical dice. [Ans. 5/54]

[Hint. Reqd. prob. = P(15)+P(16)+P(17)+P(18).

Use Ex. 14, 17].

19. A bag contains 4 red and 3 blue balls. Two drawings of 2 balls are made. Find the chance that the first drawing gives 2 red balls and the second drawing gives two blue balls (*a*) if the balls are returned to the bag after the first draw, (*b*) if the balls are not returned. [Ans. 2/49, 3/35]

[Hint. (*a*) $({}^4C_2/{}^7C_2)({}^3C_2/{}^7C_2)$. (*b*) $({}^4C_2/{}^7C_2)({}^3C_2/{}^5C_2)$].

20. A coin is tossed three times. Find the chance that head and tail will show alternately. [Ans. 1/4]

[Hint. P(HTH+THT) = 1/8 + 1/8 = 1/4].

21. A problem is given to five students A, B, C, D, E. Their chances of solving it are 1/2, 1/3, 1/4, 1/4, 1/5 respectively. What is the chance that the problem will be solved. [Ans. 17/20]

22. A and B take alternate turns in throwing a coin, the first to throw head being awarded a prize. Show that if A has the first throw, the chances of their winning are in the ratio 2 : 1. It is assumed that the game is continued till one of the players wins it.

[Hint. Similar to Example 33.]

23. A and B throw with one die for a stake of Rs. 44 which is to be won by the player who first throws 2. If A has the first throw, what are their respective expectations. [Rs. 24, Rs. 20].

[Hint. Proceed like Example 33, where $P(A)=1/6$, $P(\bar{A})=5/6$.

$$\text{Prob. that A wins}=\frac{1}{6}+\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}+\left(\frac{5}{6}\right)^4\cdot\frac{1}{6}+\ldots\ldots$$

$$=\frac{1}{6}\,\frac{1}{1-(5/6)^2}=\frac{6}{11}$$

$$\text{Expectation of A}=\frac{6}{11}\times 44=24.$$

24. Show that in a single throw with two dice, the chance of throwing more than seven is equal to that of throwing less than seven, each being 5/12. (*D.U., B.Sc. (G) 1987*)

25. The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. Find the chance of each.

[Hint. If p and p' be the chances of happening of two events, then $p=p'^2$. Also $(1-p)/p=\{(1-p')/p'\}^3$.

Solve for p and p' to get $p'=1/3$ and $p=1/9$].

1.11 Baye's Theorem

If an event E can only occur in combination with one of the mutually exclusive events $E_1, E_2, \ldots\ldots E_n$, then

$$P(E_k/E)=\frac{P(E_k)P(E/E_k)}{\sum_{i=1}^{n} P(E_i)P(E/E_i)},\qquad k=1, 2, \ldots n$$

[D.U., B.Sc. (G) 1995, 94]

Proof. Since the event E occurs only with the events $E_1, E_2, \ldots\ldots E_n$, the possible ways in which E can occur are :

$$EE_1,\ EE_2, \ldots\ldots EE_n.$$

These events are mutually exclusive, as the events E_i are mutually exclusive. By total probability theorem,

$$P(E)=P(EE_1)+P(EE_2)+\ldots\ldots+P(EE_n)$$

$$=\sum_{i=1}^{n} P(EE_i)=\sum_{i=1}^{n} P(E_i)P(E/E_i), \qquad \ldots(1)$$

(by compound probability theorem)

Again by compound probability theorem, we get

$$P(EE_k)=P(E)P(E_k/E)=P(E_k)P(E/E_k).$$

$$\therefore \quad P(E_k/E)=\frac{P(E_k)P(E/E_k)}{P(E)}.$$

Hence $\quad P(E_k/E)=\dfrac{P(E_k)P(E/E_k)}{\sum_{i=1}^{n} P(E_i)P(E/E_i)}$, using (1)

Remark 1. The probabilities $P(E_k)$ and $P(E_k/E)$ are known as *'priori' and 'posteriori' probabilities.*

Remark 2. The following particular case of Baye's theorem is very useful :

If an event E can occur only in combination with two mutually exclusive events E_1, E_2, then

$$P(E_i/E)=\frac{P(E_i)P(E/E_i)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)} \quad (i=1, 2).$$

EXAMPLES

Example 44. *The probability that a person can hit a target is 3/5 and the probability that another person can hit the same target is 2/5. But the first person can fire 8 shots in the time the second person fires 10 shots. They fire together. What is the probability that the second person shoots the target.*

Solution. Let E denote the event of shooting the target, E_1 and E_2 respectively denote the events that the first person and the second person shoot the target.

We are given $P(E/E_1)=3/5$ and $P(E/E_2)=2/5$.

The ratio of the shots of the first person to those of the second person in the same time is $\frac{8}{10}=\frac{4}{5}$. Thus $P(E_1)=\frac{4}{5}\ P(E_2)$. By Baye's theorem, we get

$$P(E_2/E)=\frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$$

$$=\frac{P(E_2)\cdot\frac{2}{5}}{\frac{4}{5}\ P(E_2)\cdot\frac{3}{5}+P(E_2)\cdot\frac{2}{5}}$$

$$=\frac{1}{(6/5)+1}=\frac{5}{11}.$$

Example 45. *Three urns A_1, A_2, A_3 contain respectively 3 red, 4 white, 1 blue ; 1 red, 2 white, 3 blue ; 4 red, 3 white, 2 blue balls. One urn is chosen at random and a ball is withdrawn. It is found to be red. Find the probability that it came from urn A_2.*

Solution. If A_i denotes the ith urn chosen and R denotes the event of withdrawing the red ball, then

$$P(A_1)=P(A_2)=P(A_3)=1/3.$$

Now $P(R/A_1)=3/8$, $P(R/A_2)=1/6$, $P(R/A_3)=4/9$.

By Baye's theorem, we have

$$P(A_2/R)=\frac{P(A_2)P(R/A_2)}{\sum_{i=1}^{3} P(A_i)P(R/A_i)}$$

$$=\frac{(1/3)(1/6)}{(1/3)(3/8)+(1/3)(1/6)+(1/3)(4/9)}$$

$$=\frac{1}{18}\times\frac{216}{71}=\frac{12}{71}.$$

Ex. *Urns A, B, C have the following coloured balls :*

A : 6 red, 4 white ; B : 2 red, 6 white ; C : 1 red, 8 white. An urn is chosen at random ; a ball drawn turns out to be red. Find the chance that urn A is chosen. [**Ans.** 108/173]

[**Hint.** $P(R)=P(A)P(R/A)+P(B)P(R/B)+P(C)P(R/C)$

$$=\frac{1}{3}\left[\frac{6}{10}+\frac{2}{8}+\frac{1}{9}\right]=\frac{173}{540}.$$

By Baye's Theorem,

$$P(A/R)=P(A)P(R/A)/P(R)=\left(\frac{1}{3}\cdot\frac{6}{10}\right)\Big/\frac{173}{540}=\frac{108}{173}$$

Example 46. *In a bolt factory machines A, B, C manufacture respectively 25, 35 and 40 percent of the total. Out of their out put 5, 4 and 2 percent are defective bolts. A bolt is drawn from the produce and is found defective. What are the probabilities that is was manufactured by A, B and C ?* [D.U., Physics (H) 2000]

Solution. Let E be the event that the bolt is defective and E_1, E_2, E_3 the events that the bolt is produced by A, B C respectively.

We have $P(E_1)=0{\cdot}25$, $P(E_2)=0{\cdot}35$, $P(E_3)=0{\cdot}40$,

$P(E/E_1)=0{\cdot}05$, $P(E/E_2)=0{\cdot}04$ and $P(E/E_3)=0{\cdot}02$.

It is required to find $P(E_1/E)$, $P(E_2/E)$ and $P(E_3/E)$.

By Baye's theorem, we have

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)+P(E_3)P(E/E_3)}$$

$$=\frac{(0{\cdot}25)(0{\cdot}05)}{(0{\cdot}25)(0{\cdot}05)+(0{\cdot}35)(0{\cdot}04)+(0{\cdot}4)(0{\cdot}02)}$$

$$\therefore \quad P(E_1/E)=\frac{125}{345}=\frac{25}{69}.$$

Similarly,

$$P(E_2/E)=\frac{0{\cdot}35 \cdot 0{\cdot}04}{0{\cdot}25\times 0{\cdot}05+0{\cdot}35\times 0{\cdot}04+0{\cdot}40\times 0{\cdot}02}=\frac{140}{345}=\frac{28}{69},$$

and $$P(E_3/E)=1-[P(E_1/E)+P(E_2/E)]=1-\frac{25}{69}-\frac{28}{69}=\frac{16}{69}.$$

Example 47. *The contents of urns I, II and III are as follows :*

1 white, 2 red and 3 black balls,

2 white, 3 red and 1 black ball, and

3 white, 1 red and 2 black balls.

One urn is chosen at random and two balls drawn. They happen to be white and red. What is the probability that they come from urns I, II or III ?

Solution. Let E_1, E_2 and E_3 denote the events that the urns I, II and III are chosen respectively, and let E be the event that the two balls taken from the selected urn are white and red. Then

$$P(E_1)=P(E_2)=P(E_3)=1/3.$$

$$P(E/E_1)=\frac{{}^1C_1\times{}^2C_1}{{}^6C_2}=\frac{2}{15},\ P(E/E_2)=\frac{{}^2C_1\times{}^3C_1}{{}^6C_2}=\frac{6}{15},$$

and $$P(E/E_3)=\frac{{}^3C_1\times{}^1C_1}{{}^6C_2}=\frac{3}{15}.$$

By Baye's theorem, we get

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{\sum_{i=1}^{3} P(E_i)P(E/E_i)}$$

$$=\frac{(1/3)(2/5)}{(1/3)(2/5)+(1/3)(6/15)+(1/3)(3/15)}=\frac{2}{11}.$$

Similarly,

$$P(E_2/E)=\frac{(1/3)\times(6/15)}{(1/3)\,.\,(2/15)+(1/3)\,.\,(6/15)+(1/3)\,.\,(3/15)}$$

$$=\frac{6}{11},$$

and $$P(E_3/E)=\frac{(1/3)\,.\,(3/15)}{(1/3)\ (2/15)+(1/3)\,.\,(6/15)+(1/3)\,.\,(3/15)}$$

$$=\frac{3}{11}.$$

Example 48. *Urn A contains 2 white and 2 black balls. Urn B contains 3 white and 2 black balls. One ball is transferred from A to B and then one ball is drawn out of B. Find the chance that this ball is white. If this ball turns out to be white, find the probability that the transferred ball was white.*

Solution. Let E_1 and E_2 denote the events that the transferred ball from A to B is white and black respectively.

Let W denote the event that a white ball is drawn from B. Then

$$P(E_1)=2/4,\ P(E_2)=2/4,\ P(W/E_1)=4/6,\ P(W/E_2)=3/6.$$

$$\therefore\ P(W)=P(E_1)P(W/E_1)+P(E_2)P(W/E_2)$$

$$=(2/4)(4/6)+(2/4)(3/6)=7/12.\ \text{Thus } P(W)=7/12. \quad \ldots(1)$$

By Baye's theorem, we have

$$P(E_1/W)=P(E_1)P(W/E_1)/P(W)=\frac{(2/4)(4/6)}{7/12}=\frac{4}{7}.$$

Example 49. *An urn contains a white balls and b black balls another contains c white and d black balls. One ball is transferred from the first urn into the second: and then a ball is drawn from the latter. What is the probability that it will be a white ball ?*

Solution. Refer to the notations of Example 48.

$$P(E_1)=\frac{a}{a+b},\ P(E_2)=\frac{b}{a+b},$$

$$P(W/E_1)=\frac{c+1}{(c+1)+d},\ P(W/E_2)=\frac{c}{c+(d+1)}.$$

From (1), $$P(W)=\frac{a}{a+b}\cdot\frac{c+1}{c+d+1}+\frac{b}{a+b}\cdot\frac{c}{c+d+1}$$

Hence $$P(W)=\frac{ac+bc+a}{(a+b)(c+d+1)}.$$

Example 50. *An urn contains 10 white and 3 black balls, while another urn contains 3 white and 5 black balls. Two are drawn from the first urn and put into the second urn and then a ball is drawn from the latter. What is the probobility that it is a white ball ?*

Solution. The two balls drawn from the first urn may be : (*i*) both white or (*ii*) both black or (*iii*) one white and one black.

Let these events be denoted by A, B C respectively. Then

$$P(A)=\frac{{}^{10}C_2}{{}^{13}C_2}=\frac{10\times 9}{13\times 12}=\frac{15}{26},$$

$$P(B)=\frac{{}^{3}C_2}{{}^{13}C_2}=\frac{3\times 2}{13\times 12}=\frac{1}{26},$$

$$P(C)=\frac{{}^{10}C_1\times{}^{3}C_1}{{}^{13}C_2}=\frac{10\times3\times2}{13\times12}=\frac{10}{26}.$$

When two balls are transferred from the first urn to the second urn, the second urn will contain (*i*) 5 white and 5 black balls, (*ii*) 3 white and 7 black balls, (*iii*) 4 white and 6 black balls.

Let W denote the event of drawing a white ball from the second urn in the above three cases.

Then $P(W/A)=\frac{5}{10}$, $P(W/B)=\frac{3}{10}$, $P(W/C)=\frac{4}{10}$.

$\therefore \quad P(W)=P(A).P(W/A)+P(B).P(W/B)+P(C).P(W/C)$

Hence $P(W)=\frac{15}{26}\cdot\frac{5}{10}+\frac{1}{26}\cdot\frac{3}{10}+\frac{10}{26}\cdot\frac{4}{10}=\frac{59}{130}$.

EXERCISES

1. A bag contains 3 white and 2 black balls, another contains 5 white and 3 black balls. If a bag is chosen at random and a ball is drawn from it, what is the chance that it is white? **[Ans.** 49/80]

Hint. Reqd. prob. $=\frac{1}{2}\times\frac{{}^{3}C_1}{{}^{5}C_1}+\frac{1}{2}\times\frac{{}^{5}C_1}{{}^{8}C_1}=\frac{49}{80}$.

2. In a factory, machine A produces 40% of the output and machine B produces 60%. On the average 9 item in 1,000 produced by A are defective and 1 item in 250 produced by B is defective. An item drawn at random from a day's output is defective. What is the probability that it was produced by A or by B?

3. There are two identical boxes containing respectively 4 white and 3 red balls, 3 white and 7 red balls. A box is chosen at random and a ball is drawn from it. Find the probability that the ball is white. If the ball is white, what is the probability that it is from first box? **[Ans.** 61/140, 40/61]

[Hint. E_1=event that it is a first box $\Rightarrow P(E_1)=\frac{1}{2}$,

E_2=event that it is a second box $\Rightarrow P(E_2)=\frac{1}{2}$,

E=event that the ball is white. $P(E/E_1)=4/7$, $P(E/E_2)=3/10$.

(*i*) $P(E)=P(E_1)P(E/E_1)+P(E_2)P(E/E_2)=(1/2)(4/7)+(1/2)(3/10)$

$\therefore$ $P(E)=61/140$. By Baye's theorem,

(*ii*) $P(E_1/E)=\{P(E_1)P(E/E_1)\}/P(E)=\frac{1}{2}\times\frac{4}{7}\times\frac{140}{61}=\frac{40}{61}$.]

4. Suppose 5 men out of 100 and 25 women out of 10,000 are colour blind. A colour blind person is chosen at random. What is the probability of his being male? (**Assume male and female to be in equal numbers.)**

[Hint. $E_1\equiv$ Person is a male, $E_2\equiv$ Person is a female.

$E\equiv$ Person is colour blind.

Then $P(E_1)=P(E_2)=\frac{1}{2}$, $P(E/E_1)=0.05$, $P(E/E_2)=0.0025$. Find $P(E_1/E)$, by Baye's theorem.]

5. Three boxes contain galvanometers, some of which are defective. The proportion of defectives in boxes A, B and C are respectively 1/3, 2/9 and 1/6. A box is selected at random and a galvanometer drawn from it. If the selected galvanometer is found to be defective, what is the probability that the box B was selected? [D.U., Physics (H) 1998]

[Hint. $P(D\mid A)=\frac{1}{3}$, $P(D\mid B)=\frac{2}{9}$, $P(D\mid C)=\frac{1}{6}$; D denotes defective.

$P(A)=P(B)=P(C)=\frac{1}{3}$. Required probability is

$$P(B\mid D)=\frac{P(B)\,P(D\mid B)}{P(A)\,P(D\mid A)+P(B)\,P(D\mid B)+P(C)\,P(D\mid C)}$$

$$=\frac{\frac{1}{3}\cdot\frac{2}{9}}{\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{2}{9}+\frac{1}{3}\cdot\frac{1}{6}}=\frac{4}{13}.$$

6. State Baye's theorem. A and B are two weak students of Statistics and their chances of solving a problem correctly are 1/8 and 1/12 respectively. If the probability of their making a common mistake a 1/1001 and they obtain the same answer, prove that the probability that their answer is correct is 13/14. [D.U., Physics (H) 1999]

7. There are five urns and they are numbered 1 to 5. Each urn contains 10 balls, urn i has i defective balls and $10-i$ non-defective balls ($i=1,2,3,4,5$). An urn to selected at random, and then a ball is selected at random from the selected urn. What is the probability that a defective ball will be selected? If the ball is defective, what is the probability that it came from urn 5?

[Hint & Answer. Let E denote the event that a defective ball is selected and E_i the event that urn i is selected.

$$P(E_i)=\frac{1}{5},\ P(E\mid E_i)=\frac{i}{10},\ i=1,2,3,4,5.$$

Reqd. prob. $=P(E_5\mid E)=\dfrac{P(E_5)\,P(E\mid E_5)}{\sum_{i=1}^{5}P(E_i)\,P(E\mid E_i)}=\dfrac{\frac{1}{5}\cdot\frac{5}{10}}{\sum_{i=1}^{5}\frac{1}{5}\cdot\frac{i}{10}}=\dfrac{3}{10}$.]

8. Urn A contains 2 white, 1 black and 3 red balls. Urn B contains 3 white, 2 black and 4 red balls. Urn C contains 4 white, 3 black and 2 red balls. An urn is chosen at random and 2 balls are drawn. They happen to be red and black. What is the chance that both balls came from urn B.

[Hint. Similar to Example 43.]

MISCELLANEOUS EXAMPLES

Example 1. *Find the probability of throwing 10 exactly in one throw with three dice.* [D.U., B.Sc. (G) 1996]

Solution. We can throw 10 with three dice in the following ways :

(1, 3, 6), (1, 4, 5), (1, 5, 4), (1, 6, 3)
(2, 2, 6), (2, 3, 5), (2, 4, 4), (2, 5, 3), (2, 6, 2),
(3, 1, 6), (3 2, 5), (3, 3, 4), (3, 4, 3), (3, 5, 2), (3, 6, 1)
(4, 1, 5), (4, 2, 4), (4, 3, 3), (4, 4, 2), (4, 5, 1)
(5, 1, 4), (5, 2, 3), (5, 3, 2), (5, 4, 1)
(6, 1, 3), (6, 2, 2), (6, 3, 1).

These favourable cases are $m = 27$.

Total number of exhaustive cases is $n = 6 \times 6 \times 6 = 216$.

Hence the required probability$= \frac{27}{216} = \frac{1}{8}$.

Example 2. *It is 8 : 5 against a person who is now 40 years old living till he is 70 and 4 : 3 against a person now 50 living till he is 80. Find the probability that at least one of these persons will be above 30 years hence.* [D.U., B.A. (P) 1995]

Solution. Refer to Example 18.

Probability that the first person lives 30 years hence is $P(A) = \frac{5}{13}$.

Probability that the second person lives 30 years hence is $P(A) = \frac{3}{7}$.

$$\therefore \quad P(\bar{A}) = 1 - \frac{5}{13} = \frac{8}{13} \text{ and } P(\bar{B}) = 1 - \frac{3}{7} = \frac{4}{7}.$$

Probability that none of the two persons lives is

$$= P(\bar{A})\,P(\bar{B}) = \frac{8}{13} \times \frac{4}{7} = \frac{32}{91}.$$

Hence the probability that at least one of the two persons lives 30 years hence $= 1 - \frac{32}{91} = \frac{59}{91}$.

Example 3. *The odds against student A solving a problems are 8 : 6 and odds in favour of B solving the problem are 7 : 8. What is the probability that the problem will be solved if they both try ?* [D.U., Physics (H) 1995]

Solution. The probability that A cannot solve the problem is

$$p_1 = \frac{8}{14} = \frac{4}{7}.$$

The probability that B cannot solve the problem is

$$p_2 = \frac{8}{15}.$$

The probability that none can solve the problem is

$$p_1 p_2 = \frac{4}{7} \times \frac{8}{15} = \frac{32}{105}.$$

Hence the probability that the problem will be solved if they both try

$$= 1 - \frac{32}{105} = \frac{73}{105}.$$

Example 4. *An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter, car and a truck is 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver ?* [D.U., B.A. (P) 1996]

Solution. Let E_1, E_2, E_3 denote the events that a scooter driver, car driver, truck driver is chosen respectively. Then

$$P(E_1) = \frac{2000}{12000} = \frac{1}{6},\ P(E_2) = \frac{4000}{12000} = \frac{1}{3},\ P(E_3) = \frac{6000}{12000} = \frac{1}{2}.$$

Let A denote the event the any of the drivers meets with an accident. The probability that a scooter driver meets with an accident is

$P(A \mid E_1) = 0.01$. Similarly, $P(A \mid E_2) = 0.03$; $P(A \mid E_3) = 0.15$.

The probability that the insured person is a scooter driver is given by

$$P(E_1 \mid A) = \frac{P(E_1)\,P(A \mid E_1)}{P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2) + P(E_3)\,P(A \mid E_3)}$$

$$= \frac{\frac{1}{6}(0.01)}{\frac{1}{6}(0.01) + \frac{1}{3}(0.03) + \frac{1}{2}(0.15)}$$

$$= \frac{0.01}{0.01 + 2(0.03) + 3(0.15)} = \frac{0.01}{0.52} = \frac{1}{52}.$$

Example 5. *Three machines A, B, C with capacities proportional to 2 : 3 : 4 (actual output 20000, 30000, 40000) are producing bullets. The probabilities that these machines produce defectives are 0.1, 0.2, 0.1 respectively. One bullet is taken from day's production and found to be defective. What is the probability that this bullet came from machine A or B ?*

Hint. Similar to Example 4 above. Reqd. probability is

$$\frac{\frac{2}{9}(0.1)}{\frac{2}{9}(0.1) + \frac{3}{9}(0.2) + \frac{4}{9}(0.1)} + \frac{\frac{3}{9}(0.2)}{\frac{2}{9}(0.1) + \frac{3}{9}(0.2) + \frac{4}{9}(0.1)}.$$

Example 6. *Suppose that in answering a question in a multiple choice test, an examinee knows the answer with probability p, or he guesses with probability 1 − p. Assume that the probability of answering a question correctly is unity for an examinee who knows the answer and 1/m for the* *examinee who guesses, where m is the number of multiple choice alternatives. Show that the probability that an examinee knows the answer to a problem, given that he has correctly answered it, is* $\dfrac{mp}{1+(m-1)p}$.

Solution. Let E_1 denote the event when the examinee knows the answer, E_2 : he guesses the answer and E_3 : he answers correctly. Then

$$P(E_1)=p,\ P(E_2)=1-p,\ P(A\mid E_1)=1 \text{ and } P(A\mid E_2)=\frac{1}{m}.$$

By Baye's theorem, the probability that an examinee knows the answer to problem is given by

$$P(E_1\mid A)=\frac{P(E_1)\,P(A\mid E_1)}{P(E_1)\,P(A\mid E_1)+P(E_2)\,P(A\mid E_2)}$$

$$=\frac{p\,.\,1}{p\,.\,1+(1-p)\,.\,\frac{1}{m}}=\frac{mp}{mp+1-p}=\frac{mp}{1+(m-1)\,p}.$$

Example 7. *In answering a question on a multiple choice test, a student either knows the answer (with probability p) or he guesses (with probability 1 − p). If the probability of answering the question correctly be 1, for a student who knows the answer ; and 1/4, when he guesses (4 being the number of multiple choice alternatives). Find the conditional probability that a student knows the answer to a question given that he answered it correctly.* [D.U., Physics (H) 1997]

Solution. Taking $m=4$ in Example 6, we obtain

$$P(E_1\mid A)=\frac{p\,.\,1}{p\,.\,1+(1-p)\,\frac{1}{4}}=\frac{4p}{1+3p}.$$

Example 8. *A box contains 6 red, 4 white and 5 black balls. A person draws 4 balls from the box at random. Find the probability that among the balls drawn there is at least one ball of each colour.*

Solution. When 4 balls are drawn from the box, we shall have at least one ball of each colour in the following ways :

I : 1 red, 1 white, 2 black

II : 1 white, 1 black, 2 red

III : 1 black, 1 red, 2 white.

The required probability $= P(I)+P(II)+P(III)$

$$=\frac{6_{c_1}\times 4_{c_1}\times 5_{c_2}}{15_{c_4}}+\frac{4_{c_1}\times 5_{c_1}\times 6_{c_2}}{15_{c_4}}+\frac{5_{c_1}\times 6_{c_1}\times 4_{c_2}}{15_{c_4}}$$

$$=\frac{1}{15_{c_4}}(6\times 4\times 10+4\times 5\times 15+5\times 6\times 6)$$

$$=\frac{4\times 3\times 2}{15\times 14\times 13\times 12}(240+300+180)=\frac{17280}{32760}=\frac{48}{91}.$$

Example 9. *Out of 3n consecutive numbers, 3 are selected at random. Find the chance that their sum is divisible by 3.*

Solution. Total number of exhaustive cases in which 3 numbers can be selected out of a total of $3n$ numbers is

$$(3n)_{c_3}=\frac{3n\,(3n-1)\,(3n-2)}{3\,.\,2\,.\,1}=\frac{n}{2}(9n^2-9n+2).$$

Let $3n$ consecutive numbers be

$$x, x+1, x+2, \ldots, x+3n-1.$$

We arrange these numbers in three rows, each consisting of n numbers as follows :

R_1 : $x, x+3, x+6, \ldots, x+3n-3$

R_2 : $x+1, x+4, x+7, \ldots, x+3n-2$

R_3 : $x+2, x+5, x+8, \ldots, x+3n-1$.

It is clear that the sum of three numbers is divisible by 3 if (*i*) all three numbers are from the same row or (*ii*) one each from R_1, R_2, R_3. Now

$$P(i)=n_{c_3}+n_{c_3}+n_{c_3}=3\,(n_{c_3})=3\left[\frac{n\,(n-1)\,(n-2)}{3\,.\,2\,.\,1}\right]$$

$$P(ii)=n_{c_1}\,.\,n_{c_1}\,.\,n_{c_1}=n^3.$$

The total number of favourable cases is

$$P(i)+P(ii)=\frac{1}{2}n\,(n-1)\,(n-2)+n^3=\frac{1}{2}n\,(3n^2-3n+2).$$

$$\text{The required probability }=\frac{\frac{1}{2}n\,(3n^2-3n+2)}{\frac{1}{2}n\,(9n^2-9n+2)}=\frac{3n^2-3n+2}{9n^2-9n+2}.$$

Example 10. *If n people are seated at a round table, what is the chance that two named individuals will be next to each other ?*

Solution. The total number of ways in which n persons can sit at a round table is $(n-1)\,!$. If two named individuals sit next to each other, the remaining $n-2$ persons can sit in $(n-2)\,!$ ways. These two persons can sit together in 2 ! ways, by interchanging their seats, without any change in the seating arrangement for the others. Hence the probability that two specified persons always sit together is

$$\frac{2\,(n-2)\,!}{(n-1)\,!}=\frac{2}{n-1}.$$