

# Foundations - type theory

November 4, 2025

## Contents

<b>1</b>	<b>Deductive system</b>	<b>1</b>
1.1	Contexts . . . . .	1
1.2	Judgemental equalities . . . . .	1
1.3	Type universe . . . . .	2
<b>2</b>	<b>Type formers</b>	<b>2</b>
2.1	Function type . . . . .	2
2.2	Dependent function type . . . . .	2

## 1 Deductive system

### 1.1 Contexts

Here are the basic principles for contexts in type theory.

**Definition 0.1.1.** We will note  $.$  for the empty context

**Axiom 0.1.1.** empty derivation :

$$rule\_empty\_derivation : \epsilon \longrightarrow empty\_context\ ctx$$

**Axiom 0.1.2.** context extension :

$$rule\_context\_extension : \Gamma \vdash A : \mathcal{U}_i, \quad x \text{ not in } dom(\Gamma) \longrightarrow \Gamma, x : A\ ctx$$

**Axiom 0.1.3.** variable rule :  $rule\_variable : \Gamma\ ctx, \quad x : A \text{ in } \Gamma \longrightarrow \Gamma \vdash x : A$

### 1.2 Judgemental equalities

We suppose that judgmental equality is an equivalence relation. Namely:

**Axiom 0.2.1.** Judgmental reflexivity :

$$rule\_judg\_refl : \Gamma \vdash a : A \longrightarrow \Gamma \vdash a \equiv a : A$$

**Axiom 0.2.2.** Judgmental symmetry :

$$rule\_judg\_sym : \Gamma \vdash a \equiv b : A \longrightarrow \Gamma \vdash b \equiv a : A$$

**Axiom 0.2.3.** Judgmental transitivity :

$$rule\_judg\_trans : \Gamma \vdash a \equiv b : A, \quad \Gamma \vdash b \equiv c : A \longrightarrow \Gamma \vdash a \equiv c : A$$

**Axiom 0.2.4.** Judgmental typing :

$$rule\_judg\_typing : \Gamma \vdash a : A, \quad \Gamma \vdash A \equiv B : \mathcal{U}_i \longrightarrow \Gamma \vdash a : B$$

**Axiom 0.2.5.** Judgmental typing equiv :

$$rule\_judg\_typing\_eq : \Gamma \vdash a \equiv b : A, \quad \Gamma \vdash A \equiv B : \mathcal{U}_i \longrightarrow \Gamma \vdash a \equiv b : B$$

## 1.3 Type universe

**Axiom 0.3.1.** Universe intro :

$$rule\_univ\_intro : \Gamma \text{ ctx} \longrightarrow \Gamma \vdash \mathcal{U}_i : \mathcal{U}_{i+1}$$

**Axiom 0.3.2.** Universe cumul :

$$rule\_univ\_cumul : \Gamma \vdash A : \mathcal{U}_i \longrightarrow \Gamma \vdash A : \mathcal{U}_{i+1}$$

# 2 Type formers

## 2.1 Function type

**Axiom 1.1.1.** Function formation :

$$rule\_ \lambda - form : \Gamma \vdash A : \mathcal{U}_i, \quad \Gamma, a : A \vdash B : \mathcal{U}_i \longrightarrow \Gamma \vdash A \rightarrow B : \mathcal{U}_i$$

**Axiom 1.1.2.** Function introduction :

$$rule\_ \lambda - intro : \Gamma, a : A \vdash B : \mathcal{U}_i \longrightarrow \Gamma \vdash \lambda x : A. b : A \rightarrow B$$

## 2.2 Dependent function type

**Axiom 1.2.1.** Dependent function formation :

$$rule\_ \Pi - form : \Gamma \vdash A : \mathcal{U}_i, \quad \Gamma, a : A \vdash B : \mathcal{U}_i \longrightarrow \Gamma \vdash \Pi_{x:A} B : \mathcal{U}_i$$