

Foundations - type theory

November 4, 2025

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1 Deductive system

1.1 Contexts

Here are the basic principles for contexts in type theory.

Definition 0.1.1. We will note \cdot for the empty context

Axiom 0.1.1. empty derivation :

$$\text{rule_empty_derivation} : \epsilon \longrightarrow \text{empty_context ctx}$$

Axiom 0.1.2. context extension :

$$\text{rule_context_extension} : \Gamma \vdash A : \mathcal{U}_i, \quad x \text{ not in } \text{dom}(\Gamma) \longrightarrow \Gamma, x : A \text{ ctx}$$

Axiom 0.1.3. variable rule : $\text{rule_variable} : \Gamma \text{ ctx}, \quad x : A \text{ in } \Gamma \longrightarrow \Gamma \vdash x : A$

1.2 Judgemental equalities

We suppose that judgmental equality is an equivalence relation. Namely:

Axiom 0.2.1. Judgmental reflexivity :

$$\text{rule_judg_refl} : \Gamma \vdash a : A \longrightarrow \Gamma \vdash a \equiv a : A$$

Axiom 0.2.2. Judgmental symmetry :

$$\text{rule_judg_sym} : \Gamma \vdash a \equiv b : A \longrightarrow \Gamma \vdash b \equiv a : A$$

Axiom 0.2.3. Judgmental transitivity :

$$\text{rule_judg_trans} : \Gamma \vdash a \equiv b : A, \quad \Gamma \vdash b \equiv c : A \longrightarrow \Gamma \vdash a \equiv c : A$$

Axiom 0.2.4. Judgmental typing :

$$\text{rule_judg_typing} : \Gamma \vdash a : A, \quad \Gamma \vdash A \equiv B : \mathcal{U}_i \longrightarrow \Gamma \vdash a : B$$

Axiom 0.2.5. Judgmental typing equiv :

$$\text{rule_judg_typing_eq} : \Gamma \vdash a \equiv b : A, \quad \Gamma \vdash A \equiv B : \mathcal{U}_i \longrightarrow \Gamma \vdash a \equiv b : B$$

1.3 Type universe

Axiom 0.3.1. Universe intro :

$$\text{rule_univ_intro} : \Gamma \text{ ctx} \longrightarrow \Gamma \vdash \mathcal{U}_i : \mathcal{U}_{i+1}$$

Axiom 0.3.2. Universe cumul :

$$\text{rule_univ_cumul} : \Gamma \vdash A : \mathcal{U}_i \longrightarrow \Gamma \vdash A : \mathcal{U}_{i+1}$$

2 Type formers

2.1 Function type

Axiom 1.1.1. Function formation :

$$\text{rule_}\lambda\text{-form} : \Gamma \vdash A : \mathcal{U}_i, \quad \Gamma, a : A \vdash B : \mathcal{U}_i \longrightarrow \Gamma \vdash A \rightarrow B : \mathcal{U}_i$$

Axiom 1.1.2. Function introduction :

$$\text{rule_}\lambda\text{-intro} : \Gamma, a : A \vdash B : \mathcal{U}_i \longrightarrow \Gamma \vdash \lambda x : A.b : A \rightarrow B$$

2.2 Dependent function type

Axiom 1.2.1. Dependent function formation :

$$\text{rule_}\Pi\text{-form} : \Gamma \vdash A : \mathcal{U}_i, \quad \Gamma, a : A \vdash B : \mathcal{U}_i \longrightarrow \Gamma \vdash \Pi_{x:A}B : \mathcal{U}_i$$