Foundations - type theory

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Contents

1 Deductive system

1.1 Contexts

Here are the basic principles for contexts in type theory.

Definition 0.1.1. We will note . for the empty context

Axiom 0.1.1. empty derivation : $rule\_empty\_derivation : \epsilon$ —— $empty\_context\ ctx$

Axiom 0.1.2. context extension : $rule\_context\_extension : \Gamma \vdash A : \mathcal{U}_i, \quad x\ not\ in\ dom(\Gamma)$ —— $\Gamma,\ x : A\ ctx$

Axiom 0.1.3. variable rule : $rule\_variable : \Gamma\ ctx, \quad x : A\ in\ \Gamma$ —— $\Gamma \vdash x : A$

1.2 Judgemental equalities

We suppose that judgmental equality is an equivalence relation. Namely:

Axiom 0.2.1. Judgmental reflexivity : $rule\_judg\_refl : \Gamma \vdash a : A$ —— $\Gamma \vdash a \equiv a : A$

Axiom 0.2.2. Judgmental symmetry : $rule\_judg\_sym : \Gamma \vdash a \equiv b : A$ —— $\Gamma \vdash b \equiv a : A$

Axiom 0.2.3. Judgmental transitivity : $rule\_judg\_trans : \Gamma \vdash a \equiv b : A, \quad \Gamma \vdash b \equiv c : A$ —— $\Gamma \vdash a \equiv c : A$

Axiom 0.2.4. Judgmental typing : $rule\_judg\_typing : \Gamma \vdash a : A, \quad \Gamma \vdash A \equiv B : \mathcal{U}_i$ —— $\Gamma \vdash a : B$

Axiom 0.2.5. Judgmental typing equiv : $rule\_judg\_typing\_eq : \Gamma \vdash a \equiv b : A, \quad \Gamma \vdash A \equiv B : \mathcal{U}_i$ —— $\Gamma \vdash a \equiv b : B$

1.3 Type universe

Axiom 0.3.1. Universe intro : $rule\_univ\_intro : \Gamma\ ctx$ —— $\Gamma \vdash \mathcal{U}_i : \mathcal{U}_{i+1}$

Axiom 0.3.2. Universe cumul : $rule\_univ\_cumul : \Gamma \vdash A : \mathcal{U}_i$ —— $\Gamma \vdash A : \mathcal{U}_{i+1}$

1.4 Function type

Axiom 0.4.1. Function formation : $rule\_\lambda{-}form : \Gamma \vdash A : \mathcal{U}_i, \quad \Gamma, x : A \vdash B : \mathcal{U}_i$ —— $\Gamma \vdash A \to B : \mathcal{U}_i$

1.5 Dependent function type

Axiom 0.4.1. Dependent function formation : $rule\_\Pi - form : \Gamma \vdash A : \mathcal{U}_i, \quad \Gamma, x : A \vdash B : \mathcal{U}_i$ —— $\Gamma \vdash \Pi_{y:A} B : \mathcal{U}_i$