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Solution to Homework #4

1. (a) $L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = u$, $\frac{1}{C}\int i\,dt = y$

$LsI(s) + RI(s) + \frac{1}{C}\cdot\frac{I(s)}{s} = U(s)$ $\quad \frac{1}{C}\frac{I(s)}{s} = Y(s)$

$\rightarrow I(s) = CsY(s)$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{LCs^2 + RCs + 1}$$

$LC\ddot{y} + RC\dot{y} + y = u$ or $\ddot{y} + \frac{R}{L}\dot{y} + \frac{1}{LC}y = \frac{u}{LC}$

$y = x_1$

$\dot{y} = \dot{x}_1 = x_2$

$\ddot{y} = \dot{x}_2 = -\frac{1}{LC}x_1 - \frac{R}{L}x_2 + \frac{1}{LC}u$

$$\Rightarrow \boxed{\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/LC \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}}$$

(b) $\frac{1}{C_1}\int (i_1 - i_2)\,dt + R_1 i_1 = u$

$\frac{1}{C_1}\int (i_2 - i_1)\,dt + R_2 i_2 + \underbrace{\frac{1}{C_2}\int i_2\,dt}_{y} = 0$

$\frac{1}{C_2}\int i_2\,dt = y$

Write the equations in Laplace domain, eliminate $I_1(s)$, and write $U(s)$ in terms of $I_2(s)$. Then obtain $Y(s) = G(s)\,U(s)$

$$\Rightarrow G(s) = \frac{1}{\underbrace{R_1C_1R_2C_2}_{\alpha}s^2 + \underbrace{(R_1C_1 + R_2C_2 + R_1C_2)}_{\beta}s + 1} = \frac{Y(s)}{U(s)}$$

$\alpha\ddot{y} + \beta\dot{y} + y = u$ or $\ddot{y} + \frac{\beta}{\alpha}\dot{y} + \frac{1}{\alpha}y = \frac{u}{\alpha}$

$$\begin{cases} y = x_1 \\ \dot{y} = \dot{x}_1 = x_2 \\ \ddot{y} = \dot{x}_2 = -\frac{1}{\alpha}x_1 - \frac{\beta}{\alpha}x_2 + \frac{1}{\alpha}u \end{cases}$$

$$\Rightarrow \boxed{\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1/\alpha & -\beta/\alpha \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/\alpha \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}}$$

(b) Continued : Replacing C_1 by L and denoting $C_2 = C$ we get

$$R_1 i_1 + L\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) = u$$

$$R_2 i_2 + \frac{1}{C}\int i_2\, dt + L\left(\frac{di_2}{dt} - \frac{di_1}{dt}\right) = 0$$

$$\frac{1}{C}\int i_2\, dt = y$$

Taking the above equations in Laplace domain and apply similar algebraic manipulations lead to

$$G(s) = \frac{Y(s)}{U(s)} = \frac{Ls}{\underbrace{LC(R_1+R_2)}_{\alpha}s^2 + \underbrace{(R_1R_2C+L)}_{\beta}s + R_1}$$

$$Y(s) = \frac{(Ls)\,U(s)}{\underbrace{\alpha s^2 + \beta s + R_1}_{Z(s)}} \quad \left\{ \frac{Z(s)}{U(s)} = \frac{1}{\alpha s^2 + \beta s + R_1} \right. , \quad Y(s) = Ls\, Z(s)$$

$$\alpha \ddot{z} + \beta \dot{z} + R_1 z = u \quad \text{and} \quad y = L\dot{z} \quad \Leftarrow$$

$$\begin{cases} z = x_1 \\ \dot{z} = \dot{x}_1 = x_2 \\ \ddot{z} = \dot{x}_2 = -\frac{R_1}{\alpha}x_1 - \frac{\beta}{\alpha}x_2 + \frac{1}{\alpha}u \\ y = L\dot{z} = Lx_2 \end{cases} \Rightarrow \boxed{\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -\frac{R_1}{\alpha} & -\frac{\beta}{\alpha} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{\alpha} \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & L \end{bmatrix} x \end{aligned}}$$

2. $$m_1 \ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + u$$

$$m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$

Taking both equations in Laplace domain, one can solve $X_1(s)$ and $X_2(s)$ in terms of $U(s)$. Then, it easy to get

$$\frac{X_1(s)}{U(s)} = \frac{m_2 s^2 + bs + k_2 + k_3}{(m_1 s^2 + bs + k_1 + k_2)(m_2 s^2 + bs + k_2 + k_3) - (bs + k_2)^2}$$

$$\frac{X_2(s)}{U(s)} = \frac{bs + k_2}{(m_1 s^2 + bs + k_1 + k_2)(m_2 s^2 + bs + k_2 + k_3) - (bs + k_2)^2}$$

3. (a) $\dot{x} = Ax + Bu$ $\quad A = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \end{bmatrix}$

$y = Cx$

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \cdots$$

$$= \frac{12s + 59}{s^2 + 6s + 8}$$

(b) $\dot{x} = Ax + Bu$ $\quad A = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \end{bmatrix}$

$y = Cx$

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau \qquad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}[sI - A]^{-1} = \mathcal{L}^{-1}\begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{s+1}{(s+2)(s+4)} & \frac{-1}{(s+2)(s+4)} \\ \frac{3}{(s+2)(s+4)} & \frac{s+5}{(s+2)(s+4)} \end{bmatrix}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{-1/2}{s+2} + \frac{3/2}{s+4} & \frac{-1/2}{s+2} + \frac{1/2}{s+4} \\ \frac{3/2}{s+2} - \frac{3/2}{s+4} & \frac{3/2}{s+2} - \frac{1/2}{s+4} \end{bmatrix}$$

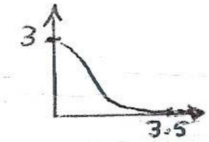
$$= \begin{bmatrix} -\frac{1}{2}e^{-2t} + \frac{3}{2}e^{-4t} & -\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-4t} \\ \frac{3}{2}e^{-2t} - \frac{3}{2}e^{-4t} & \frac{3}{2}e^{-2t} - \frac{1}{2}e^{-4t} \end{bmatrix}$$

Let us find the zero input response of the system

$$x(t) = e^{At} x(0) = e^{At}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-e^{-2t}+2e^{-4t}\\ 3e^{-2t}-2e^{-4t}\end{bmatrix}, \quad y(t) = Cx(t) = 5e^{-2t}-2e^{-4t}$$

Use MATLAB to find the general response as an exerceze.

ZIR :



Note :

1. For zero input response use "initial" command.

2. For zero state response or response due to input ($u(t)=1$) use "step" command.

3. For general response use 1 + 2 : $y_t = y_i + y_s$
or use " lsim (sys, u, t, x0) ", where
sys = ss (a, b, c, d) , u = ones (length (t)) ; x0 = [1 ; 1]
t = linspace (0, 5, 10) or t = 0:0.5:5
t = 0:0.5555:5 matches with linspace

(C) $\det(\lambda I - A) = \det\begin{bmatrix}\lambda+5 & 1\\ -3 & \lambda+1\end{bmatrix} = \lambda^2 + 6\lambda + 8$

$$\lambda_{1,2} = -3 \pm \sqrt{9-8} = -3 \pm 1 \begin{cases} -2 \\ -4 \end{cases}$$ stable eigenvalues

Zero input response is asymptotically stable

$$A(\delta) = \begin{bmatrix}-5+\delta & -1\\ 3 & -1\end{bmatrix}$$

$$\det[\lambda I - A(\delta)] = \lambda^2 + (6-\delta)\lambda + 8 - \delta$$

$$6-\delta > 0 , \quad 8-\delta > 0 \Rightarrow \delta < 6 , \ \delta < 8$$

Condition for remaining stable : $\delta < 6$

4. $$G(s) = \frac{k\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \zeta = 0.4\,;\ \omega_n = 5\,,\ k = 1$$

$$= \frac{25}{s^2 + 4s + 25}$$

$$R(s) = \frac{1}{s} \longrightarrow \boxed{G(s)} \longrightarrow C(s)$$

$$G(s) = \frac{C(s)}{R(s)} \Rightarrow C(s) = R(s)\,G(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

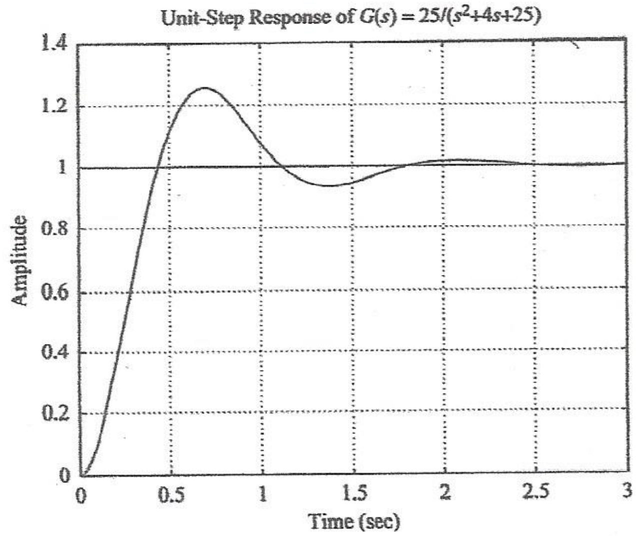
$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t\,, \qquad \omega_d = \sqrt{1-\zeta^2}$$

$$= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

MATLAB

```
% Enter the numerator and denominator of G(s)
num = [0 0 25]
den = [1 4 25]
% Step response command
step(num, den)
grid
title('unit-step response of G(s) = 25/(s^2+4s+25)')
```

Unit-Step Response of $G(s) = 25/(s^2+4s+25)$



5.

(a) Using sections 6.5 – 6.8, it is not difficult to rederive the dynamical equation of two-links planar manipulator (6.58), which can compactly be written as

$$M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) = \tau$$

where

$$M(\theta) = \begin{bmatrix} l_2^2 m_2 + 2 l_1 l_2 m_2 c_2 + l_1^2(m_1+m_2) & l_2^2 m_2 + l_1 l_2 m_2 c_2 \\ l_2^2 m_2 + l_1 l_2 m_2 c_2 & l_2^2 m_2 \end{bmatrix}$$

$$V(\theta,\dot{\theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1+m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}$$

Obviously, the above dynamical equation represent a nonlinear vector-matrix differential equation

Note that equations in (6.58) can also be solved in terms of $\ddot{\theta}_1$ and $\ddot{\theta}_2$ after long derivation so that

$$\ddot{\theta}_1 = f_1(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, m_1, m_2, l_1, l_2, g)$$

$$\ddot{\theta}_2 = f_2(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, m_1, m_2, l_1, l_2, g)$$

which are 2 coupled nonlinear differential equations.

(b) A convenient way to write the above equations for control purpose based on method of computed torque is

$$\ddot{\theta} = \underbrace{M^{-1}(\theta)[\tau - V(\theta,\dot{\theta}) - G(\theta)]}_{\bar{\tau}}$$

or

$$\ddot{\theta}_1 = \bar{\tau}_1$$

$$\ddot{\theta}_2 = \bar{\tau}_2$$

Defining $\theta_1 = x_1$, $\dot{\theta}_1 = x_2$, $\theta_2 = x_3$, $\dot{\theta}_2 = x_4$ we have

$$\theta_1 = x_1 \qquad \theta_2 = x_3$$

$$\dot{\theta}_1 = \dot{x}_1 = x_2 \qquad \dot{\theta}_2 = \dot{x}_3 = x_4$$

$$\ddot{\theta}_1 = \dot{x}_2 = \bar{\tau}_1 \qquad \ddot{\theta}_2 = \dot{x}_4 = \bar{\tau}_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\tau}_1 \\ \bar{\tau}_2 \end{bmatrix}$$

A more convenient way to write this equation is by reordering the state variables to get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_3 \\ \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\tau}_1 \\ \bar{\tau}_2 \end{bmatrix}$$

which can compactly be written as

$$\dot{x} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{\tau}$$

" Note that $\bar{\tau}$ contains the nonlinear terms "

6. Using the section 6.5 and applying the equations (6.45) – (6.53) after long derivation we get

$$M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) = \tau$$

where

$$M(\theta) = \begin{bmatrix} M_1 L_1^2 + M_2(L_1 + L_2 C_2)^2 & 0 \\ 0 & M_2 L_2^2 \end{bmatrix}$$

$$V(\theta,\dot{\theta}) = \begin{bmatrix} -2(L_1 + L_2 C_2) M_2 L_2 S_2 \dot{\theta}_1 \dot{\theta}_2 \\ (L_1 + L_2 C_2) M_2 L_2 S_2 \dot{\theta}_1 \end{bmatrix}$$

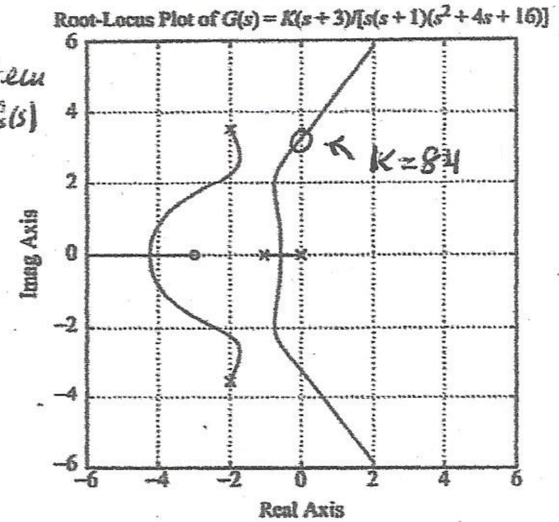
$$G(\theta) = \begin{bmatrix} 0 \\ M_2 g L_2 C_2 \end{bmatrix}$$

Simulations of problems 5. and 6. are not performed since they were not required as it was announed. However, if you perform the simulations you observe the unstable behavior of the systems. This means that one needs feedback control system to achieve stability and performance.

7. (a) Root-locus of closed-loop system can be obtained based on G(s)

MATLAB Command

```
num = [0 0 0 1 3]
den = [1 5 20 16 0]
rlocus(num, den)
v = [-6 6 -6 6]
axis(v); axis('square')
grid
title('Root-locus Plot of G(s) = k(s+3)/[s(s+1)(s^2+4s+16)]')
```



8. $G_c(s) = \dfrac{K(s+z)}{s+p}$ $\quad G_p(s) = \dfrac{1}{(s-1)^2}$

$$T(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} = \frac{K(s+z)}{\underbrace{s^3+(p-2)s^2+(k-2p+1)s+p+kz}}$$

Design by simple choice:

$G_c(s) = 10\dfrac{s+1}{s+4}$

or set it to desired characteristic polynomial

$\Delta_d(s) = s^3 + \alpha s^2 + \beta s + \gamma$

Many solutions possible

If we select poles at $-2, -3, -4$

then $\Delta_d(s) = s^3 + 9s^2 + 26s + 24$

$\Rightarrow p = 11,\ K = 47,\ z = 13/47 = 0.2765$

State space solution:

$G_p(s) = \dfrac{Y(s)}{U(s)} = \dfrac{1}{(s-1)^2} \Longrightarrow \dfrac{Y(s)}{U(s)} = \dfrac{1}{s^2-2s+1}$

$Y(s)[s^2-2s+1] = U(s)$ $\bullet\!\!-\!\!\circ$ $\ddot{y} - 2\dot{y} + y = u$

$y = x_1,\ \dot{y} = \dot{x}_1 = x_2,\ \ddot{y} = -x_1 + 2x_2 + u$

$\dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad , \quad u = v + KX$

$\dot{x} = (A + BK)x + Bv$

Let desired stable eigenvalues be $-2, -3$:

$(\lambda+2)(\lambda+3) = \underbrace{\lambda^2 + 5\lambda + 6}$

$A_d = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$

$\underline{K_1 = -5,\ K_2 = -7}$ $\left\{ A + BK = \begin{bmatrix} 0 & 1 \\ \underbrace{-1+k_1} & \underbrace{2+k_2} \end{bmatrix} \overset{!}{=} A_d = \begin{bmatrix} 0 & 1 \\ \underbrace{-6} & \underbrace{-5} \end{bmatrix} \right.$