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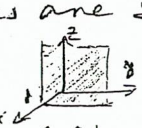
Solution to the Homework I on Mathematical Background

1. (a) Linearly independent with respect to (R^3, R).
   (b) Linearly independent with respect to (R^2, R), however, they are linearly dependent with respect to (C^2, C).

2. A subspace of a vector space is defined in such a way that under operations of vector addition and scalar multiplication the subspace itself form a vector space. Thus, a subspace of vector space (R^3, R) is a subset that satisfies two requirements: (1) If we add two vectors in the subspace, their sum should still be in the subspace, (2) If we multiply any vector in the subspace by a scalar, the resulting vector must also lie in the subspace.

   (a) The plane of vectors with first component $x_1 = 0$ is a subspace since 

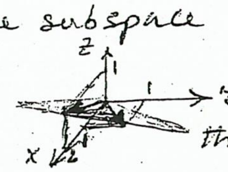
   the subspace is yz plane and the above two conditions are satisfied.

   (b) It is not a subspace since 

   the addition of two vectors in the subspace lie outside of the subspace

   (c) It is not a subspace since 

   the sum of two vectors lie outside of the subspace

   (d) It is a subspace since 

   the plane generated by these vectors passes through the origin and satisfy both conditions.

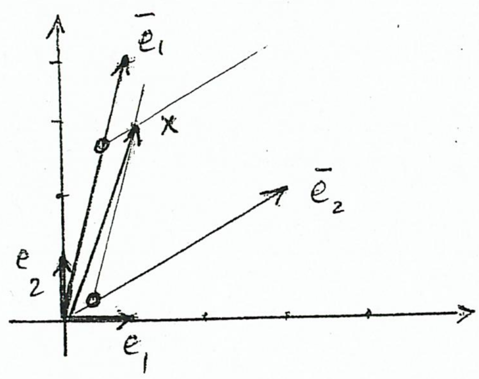
   (e) It is a subspace since $3x_1 - x_2 + x_3 = 0$ is a plane that passes through the origin.

3. $x = [e_1 \;\; e_2]\begin{bmatrix}1\\3\end{bmatrix} \leftarrow \beta \qquad \bar{e}_1 = [e_1 \;\; e_2]\begin{bmatrix}1\\4\end{bmatrix}, \quad \bar{e}_2 = [e_1 \;\; e_2]\begin{bmatrix}3\\2\end{bmatrix}$

$$Q = \begin{bmatrix}1 & 3\\4 & 2\end{bmatrix}$$

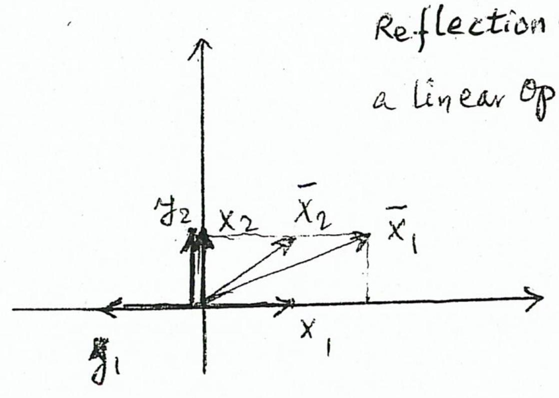
$$P = Q^{-1} = \frac{-1}{10}\begin{bmatrix}2 & -3\\-4 & 1\end{bmatrix}$$

$$\bar{\beta} = P\beta = \begin{bmatrix}0.7\\0.1\end{bmatrix}$$



4.

Reflection with respect to y axis is a linear operator $y = L[x]$



Matrix Representation of Linear Operator

$\Downarrow$

$$y_1 = L[x_1] = [x_1 \;\; x_2]\begin{bmatrix}-1\\0\end{bmatrix}$$

$$y_2 = L[x_2] = [x_1 \;\; x_2]\begin{bmatrix}0\\1\end{bmatrix}$$

$$A = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$$

$$\bar{x}_1 = [x_1 \;\; x_2]\begin{bmatrix}2\\1\end{bmatrix}$$

$$\bar{x}_2 = [x_1 \;\; x_2]\begin{bmatrix}1\\1\end{bmatrix}$$

$$Q = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix} \Rightarrow P = Q^{-1} = \begin{bmatrix}1 & -1\\-1 & +2\end{bmatrix}$$

$$\bar{A} = PAP^{-1} = \begin{bmatrix}-3 & -2\\4 & 3\end{bmatrix}$$

5.

$$L:(R^3,R)\longrightarrow(R^2,R)$$

$$L[x]=\begin{bmatrix}x_1-x_2\\ x_2-x_3\end{bmatrix} \qquad v_1=\begin{bmatrix}1\\0\\0\end{bmatrix}\quad v_2=\begin{bmatrix}1\\0\\1\end{bmatrix}\quad v_3=\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$L[v_1]=\begin{bmatrix}1\\0\end{bmatrix}=y_1 \qquad w_1=\begin{bmatrix}1\\1\end{bmatrix}\quad w_2=\begin{bmatrix}1\\-1\end{bmatrix}$$

$$L[v_2]=\begin{bmatrix}1\\-1\end{bmatrix}=y_2$$

$$y_i=L[v_i]=[w_1\ w_2]\,a_i \qquad i=1,2,3$$

$$L[v_3]=\begin{bmatrix}-1\\2\end{bmatrix}=y_3$$

$$\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1&1\\1&-1\end{bmatrix}a_1 \qquad a_1=\begin{bmatrix}1&1\\1&-1\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1/2\\1/2\end{bmatrix}$$

$$\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1&1\\1&-1\end{bmatrix}a_2 \qquad a_2=\begin{bmatrix}1&1\\1&-1\end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1&1\\1&-1\end{bmatrix}a_3 \qquad a_3=\begin{bmatrix}1&1\\1&-1\end{bmatrix}^{-1}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1/2\\-3/2\end{bmatrix}$$

$$\Rightarrow \qquad A=[a_1\ a_2\ a_3]=\begin{bmatrix}1/2&0&1/2\\1/2&1&-3/2\end{bmatrix}$$

"Matrix Representation of Linear Operator"

6. One can solve this problem by applying row operations and transform A to echelon form. Then, using the steps of solving basic variables in terms of free variables, one can obtain the basis for the null space as the solution of $AX = 0$.

We can also solve the problem as follows:

It is easy to see that the last three columns of

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 & -1 \\ 1 & 2 & 3 & 4 & -1 \\ 2 & 0 & 2 & 0 & 2 \end{bmatrix}$$

are linearly dependent on the first two columns of A. Hence the rank of A, $\rho(A) = 2$. Let $X = [x_1\ x_2\ x_3\ x_4\ x_5]^T$ and write

$$AX = x_1\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + x_4\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$= (x_1 + x_2 + x_5)\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (x_2 + x_3 + 2x_4 - x_5)\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0$$

Since the indicated vectors are linearly independent, we conclude that $AX = 0$ if and only if

$$(*) \begin{cases} x_1 + x_2 + x_5 = 0 \\ x_2 + x_3 + 2x_4 - x_5 = 0 \end{cases}$$

$(*)$ has 2 equations and five unknowns: hence three of the unknown can be arbitrarily selected.

let $x_3 = 1$, $x_4 = 0$, $x_5 = 0$ then $x_1 = -1$ and $x_2 = -1$.

let $x_3 = 0$, $x_4 = 1$, $x_5 = 0$ then $x_1 = 0$ and $x_2 = -2$

let $x_3 = 0$, $x_4 = 0$, $x_5 = 1$ then $x_1 = -1$ and $x_2 = 1$

Thus, the basis for the null space is specified by three vectors v_1, v_2, v_3:

$[-1\ -1\ 1\ 0\ 0]^T$, $[0\ -2\ 0\ 1\ 0]^T$, $[-1\ 1\ 0\ 0\ 1]^T$

The general solution is a linear combination of the above basis vectors: $x = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$.

Regarding $Ax = b$, since $\rho(A) = 2$, b must be in the range space of A in order to have a solution. Thus, $Ax = b$ does not have solution for any b.

7.

(a) $$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}}_{\text{This matrix is nonsingular}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} \Rightarrow x = A^{-1} b = \begin{bmatrix} 5/4 \\ -3 \\ 3/2 \end{bmatrix}$$

(b) First consider the homogenous part $Ax = 0$ and transform A to echelon form by row operations

$$A x_h = 0 \rightarrow \tilde{A} x_h = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix} = 0$$

- x_1, x_3 basic variables
- x_2, x_4 free variables

$$\Rightarrow \left.\begin{array}{l} \tilde{x}_1 + 3\tilde{x}_2 + 3\tilde{x}_3 + 2\tilde{x}_4 = 0 \\ 3\tilde{x}_3 + \tilde{x}_4 = 0 \end{array}\right\} \begin{array}{l} \tilde{x}_3 = -1/3\, x_4 \\ \tilde{x}_1 = -3x_2 - x_4 \end{array}$$

$$x_h = \begin{bmatrix} -3\tilde{x}_2 - \tilde{x}_4 \\ \tilde{x}_2 \\ -1/3\, \tilde{x}_4 \\ \tilde{x}_4 \end{bmatrix} = \tilde{x}_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \tilde{x}_4 \begin{bmatrix} -1 \\ 0 \\ -1/3 \\ 1 \end{bmatrix}$$

The system $Ax = b$ can be written as $\tilde{A} x_p = \tilde{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and a particular solution is obtained as $x_p = [-2\ 0\ 1\ 0]^T$

so, the general solution is $x = x_h + x_p$.

(c) $\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{b}$

Apply row operations to reduce $Ax = b$ to $\tilde{A}x = \tilde{b}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \qquad \rho(\tilde{A}) = 2 \quad \vartheta(\tilde{A}) = 1$$

The homogenous part is $\tilde{A}x_h = 0$, which leads to the solution of the homogenous part

$$\underbrace{\begin{aligned} \tilde{x}_1 + 2\tilde{x}_2 + 3\tilde{x}_3 &= 0 \\ -3\tilde{x}_2 - 6\tilde{x}_3 &= 0 \end{aligned}}_{\tilde{x}_3 \text{ is free variable}}$$

$$x_h = \begin{bmatrix} \tilde{x}_3 \\ -2\tilde{x}_3 \\ \tilde{x}_3 \end{bmatrix} = \tilde{x}_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

A particular solution is obtained from the above equation:

$$x_p = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

So, the general solution is given by

$$x = x_h + x_p = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \alpha + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

or $\tilde{x}_3$
free parameter

8 (b) $f(x) = C + Dx + Ex^2$ Using the given points we get

$$\underbrace{\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \\ E \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{bmatrix}}_{b} \qquad x^* = (A^TA)^{-1}A^Tb = \begin{bmatrix} 3.5714 \\ 1.7 \\ 0.2143 \end{bmatrix}$$

(a) Set-up the equation as $Ax = b$

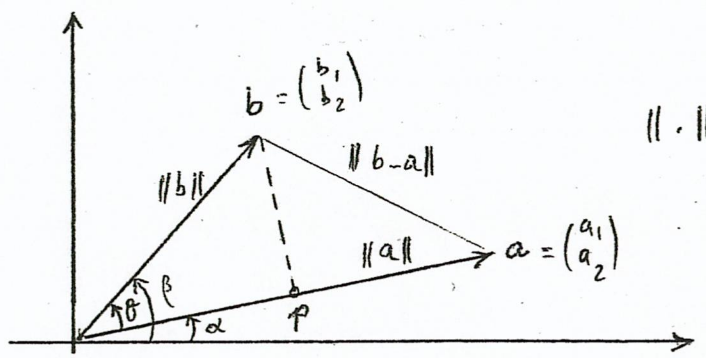
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x^* = A^T(AA)^T b = \begin{bmatrix} 0.2844 \\ 0.0711 \\ 0.1611 \\ 0.1611 \\ 0.3223 \end{bmatrix}$$

(c) $i = v_1/R + v_2/R + v_3/R$ or $\frac{1}{R}[1\ 1\ 1]v = i$ ⊛

Total energy dissipated per unit time is $\dot{\mathcal{E}} = \frac{1}{R}[v_1^2 + v_2^2 + v_3^2] = \frac{1}{R}\|v\|^2$

⊛ $$v = \frac{1}{R}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \left\{ \frac{1}{R^2}(1\ 1\ 1)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}^{-1} i = \underline{\underline{R/3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} i}}$$

9.



$\|\cdot\|$: represents norm or the length of the vector

Method 1:

Law of cosine : $\|b - a\|^2 = \|b\|^2 + \|a\|^2 - 2\|b\|\,\|a\| \cos\theta$

or $(b-a)^T(b-a) = b^Tb + a^Ta - 2\|b\|\,\|a\|\cos\theta$

$\cancel{b^Tb} - 2a^Tb + \cancel{a^Ta} = \cancel{b^Tb} + \cancel{a^Ta} - 2\|b\|\,\|a\|\cos\theta$

$$\cos\theta = \frac{a^Tb}{\|a\|\,\|b\|}$$

Method 2: $\sin\alpha = \frac{a_2}{\|a\|}$, $\cos\alpha = \frac{a_1}{\|a\|}$, $\sin\beta = \frac{b_2}{\|b\|}$, $\cos\beta = \frac{b_1}{\|b\|}$

$$\Rightarrow \cos\theta = \cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$$

$$= \cdots = \frac{a_1b_1 + a_2b_2}{\|a\|\,\|b\|} = \frac{a^Tb}{\|a\|\,\|b\|}$$

projection point $p = \bar{x}a$

$\underbrace{(b - \bar{x}a)}_{bp} \perp a$ or $a^T(b - \bar{x}a) = 0 \Rightarrow \bar{x} = \frac{a^Tb}{a^Ta}$

$$\therefore p = \bar{x}a = \frac{a^Tb}{a^Ta}\cdot a$$

10.

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(b) The answer is not true unless $x_1, x_2, \ldots, x_m$ are linearly independent

11. The third column is

$$\pm \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Orthogonal means $a^T b = 0$

So, $q_1^T q_2 = 0$, $q_1^T q_3 = 0$, $q_2^T q_3 = 0$: q_1, q_2, q_3 are columns of Q.

12. $Q^T Q = (I - 2uu^T)(I - 2uu^T) = I - 4uu^T + 4u\underbrace{u^T u}_{1}u^T$

$= I$

An example is constructed in problem 11. "check"