B. Shafai, ME 5250 Solution to set #3

Chapter 3
Manipulator Kinematics

Exercises

3.1)

α_{i-1}	a_{i-1}	d_i
0	0	0
0	L_1	0
0	L_2	0

$$ {}^{0}_{1}T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} C_2 & -S_2 & 0 & L_1 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{2}_{3}T = \begin{bmatrix} C_3 & -S_3 & 0 & L_2 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{3}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T = \begin{bmatrix} C_{123} & -S_{123} & 0 & L_1C_1 + L_2C_{12} \\ S_{123} & C_{123} & 0 & L_1S_1 + L_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

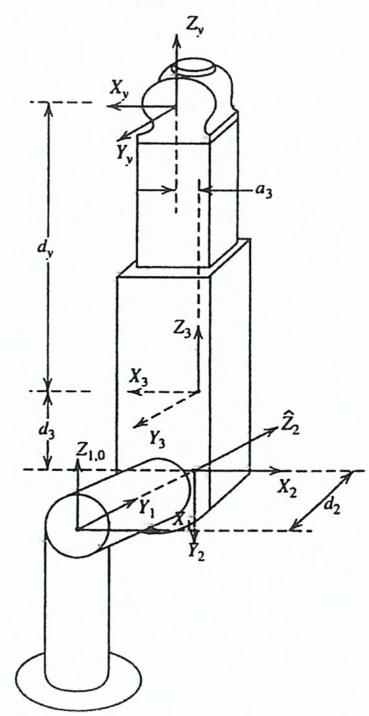
where:

$C_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$

$S_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$, etc.

3.2)

α_{i-1}	a_{i-1}	d_i	θ_i
0	0	0	θ_1
$-90°$	0	d_2	θ_2
$90°$	0	d_3	$180°$
0	a_3	d_4	θ_4
90	0	0	θ_5
-90	0	0	θ_6



When $d_3 = 0$ the origins of frames 2 and 3 coincide. Frame 3 is fixed to link 3.

$$ {}^{0}_{1}T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} C_2 & -S_2 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ -S_2 & -C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{2}_{3}T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{4}T = \begin{bmatrix} C_4 & -S_4 & 0 & a_3 \\ S_4 & C_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{4}_{5}T = \begin{bmatrix} C_5 & -S_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ S_5 & C_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

3.2 (*Continued*)

$$
{}^5_6T = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -S_6 & -C_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^0_6T = {}^0_3T\,{}^3_6T
$$

$$
{}^0_3T = \begin{bmatrix} -C_1C_2 & S_1 & C_1S_2 & -d_2S_1 + d_3C_1S_2 \\ -S_1C_2 & -C_1 & S_1S_2 & d_2C_1 + d_3S_1S_2 \\ S_2 & 0 & C_2 & d_3C_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^3_6T = \begin{bmatrix} C_4C_5C_6 - S_4S_6 & -(C_4C_5S_6 + S_4C_6) & -C_4S_5 & a_3 \\ (S_4C_5C_6 + C_4S_6) & -S_4C_5S_6 + C_4C_6 & -S_4S_5 & 0 \\ S_5C_6 & -S_5S_6 & C_5 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0_6T = \begin{bmatrix} R_{11} & R_{12} & R_{13} & P_x \\ R_{21} & R_{22} & R_{23} & P_y \\ R_{31} & R_{32} & R_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where:

$$R_{11} = -C_1C_2C_4C_5C_6 + C_1C_2S_4S_6 + S_1S_4C_5C_6 + S_1C_4S_6 + C_1S_2S_5S_6$$

$$R_{12} = C_1C_2C_4C_5S_6 + C_1C_2S_4C_6 - S_1S_4C_5S_6 + S_1C_4C_6 - S_1S_2S_5S_6$$

$$R_{13} = C_1C_2C_4S_5 - S_1S_4S_5 + C_1S_2C_5$$

$$R_{21} = -S_1C_2C_4C_5C_6 + S_1C_2S_4S_6 - C_1S_4C_5C_6 - C_1C_4S_6 + S_1S_2S_5C_6$$

$$R_{22} = S_1C_2C_4C_5S_6 + S_1C_2S_4C_6 + C_1S_4C_5S_6 - C_1C_4C_6 - S_1S_2S_5S_6$$

$$R_{23} = S_1C_2C_4S_5 + C_1S_4S_5 + S_1S_2C_5$$

$$R_{31} = S_2C_4C_5C_6 - S_2S_4S_6 + C_2S_5C_6$$

$$R_{32} = -S_2C_4C_5S_6 - S_2S_4C_6 - C_2S_5S_6$$

$$R_{33} = -S_2C_4C_5 + C_2C_5$$

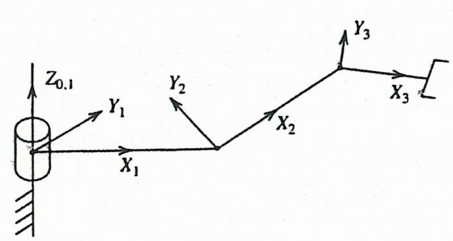
$$P_x = -d_2S_1 + (d_3 + d_4)C_1S_2 - a_3C_1C_2$$

$$P_y = d_2C_1 + (d_3 + d_4)S_1S_2 - a_3S_1C_2$$

$$P_z = (d_3 + d_4)C_2 + a_3S_2$$

3.3)

α_{i-1}	a_{i-1}	d_i
0	0	0
90°	L_1	0
0	L_2	0



$$
{}^0_1T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1_2T = \begin{bmatrix} C_2 & -S_2 & 0 & L_1 \\ 0 & 0 & -1 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

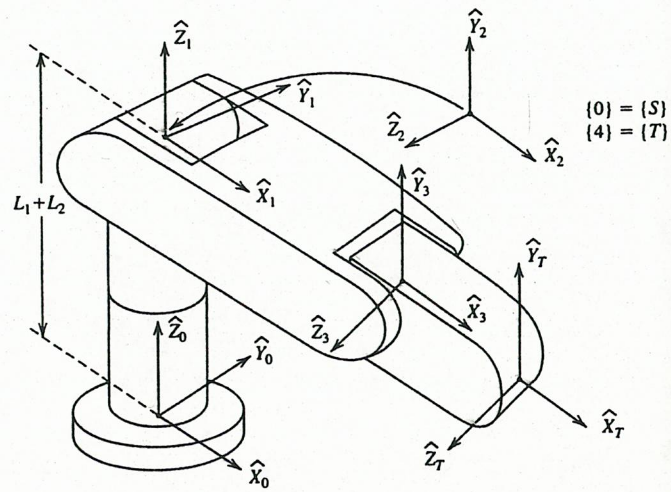
3.3) *(Continued)*

$$
{}^{2}_{3}T = \begin{bmatrix} C_3 & -S_3 & 0 & L_2 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{B}_{W}T = {}^{0}_{3}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T
$$

$$
{}^{B}_{W}T = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 & L_1C_1 + L_2C_1C_2 \\ S_1C_{23} & -S_1S_{23} & -C_1 & L_1S_1 + L_2S_1C_2 \\ S_{23} & C_{23} & 0 & L_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

3.4)

α_{i-1}	a_{i-1}	d_i	θ_i
0	0	$L_1 + L_2$	θ_1
90°	0	0	θ_2
0	L_3	0	θ_3
0	L_4	0	0

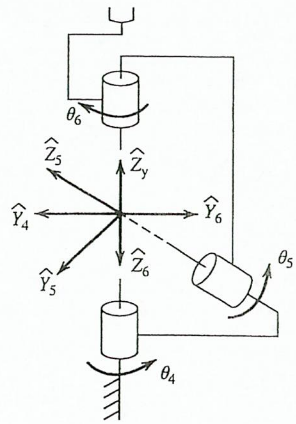


$$
{}^{0}_{1}T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & L_1 + L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

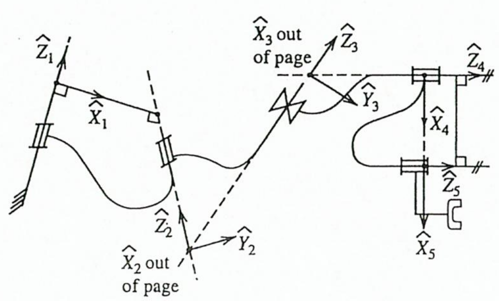
$$
{}^{1}_{2}T = \begin{bmatrix} C_2 & -S_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}T = \begin{bmatrix} C_3 & -S_3 & 0 & L_3 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

3.11) Mechanism lies in page as drawn. All $\hat{X}$-axes are normal to page.



3.13)

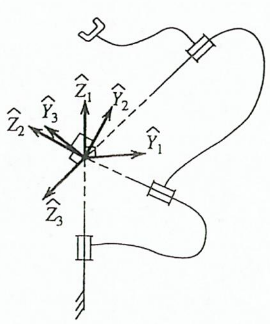


3.8) When $\{G\} = \{T\}$ we have:

$${}^B_W T\, {}^W_T T = {}^B_S T\, {}^S_G T$$

so, $${}^W_T T = {}^B_W T^{-1}\, {}^B_S T\, {}^S_G T$$

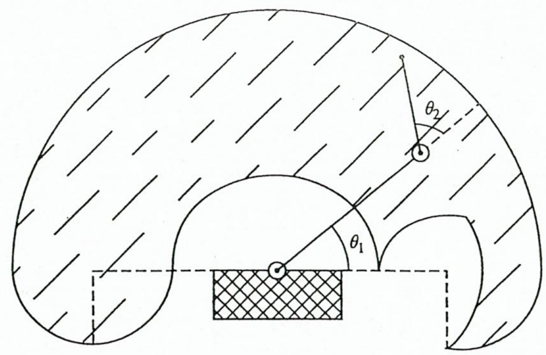
3.15)



Chapter 4

Inverse Manipulator Kinematics

4.9) This is slightly trickier than it looks at first. Approximately:



4.2) This problem can have different solutions depending how it is interpreted. I intended that a goal is specified which includes a desired orientation of the last link. In this case, the solution is fairly easy.

${}^S_TT$ is given, so compute:

$${}^B_WT = {}^B_ST\,{}^S_TT\,{}^W_TT^{-1}$$

Now ${}^B_WT = {}^0_3T$ which we write out as:

$${}^0_3T = \begin{bmatrix} R_{11} & R_{12} & R_{13} & P_x \\ R_{21} & R_{22} & R_{23} & P_y \\ R_{31} & R_{32} & R_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the solution of exercise 3 from chapter 3 we have:

$${}^0_3T = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 & C_1(C_2L_2+L_1) \\ S_1C_{23} & -S_1S_{23} & -C_1 & S_1(C_2L_2+L_1) \\ S_{23} & C_{23} & 0 & S_2L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Equate elements $(1,3)$: $S_1 = R_{13}$

Equate elements $(2,3)$: $-C_1 = R_{23}$

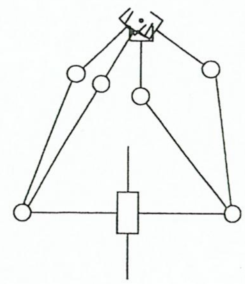
$$\therefore \quad \boxed{\theta_1 = \operatorname{atan2}(R_{13}, -R_{23})}$$

If both $R_{13} = 0$ and $R_{23} = 0$ the goal is unattainable.

Equate elements $(1,4)$: $P_x = C_1(C_2L_2 + L_1)$

Equate elements $(2,4)$: $P_y = S_1(C_2L_2 + L_1)$

4.2) *(Continued)*



If $C_1 \neq 0$ then $C_2 = \frac{1}{L_2}\left(\frac{P_x}{C_1} - L_1\right)$

Else $C_2 = \frac{1}{L_2}\left(\frac{P_y}{S_1} - L_1\right)$

Equate Elements (3,4): $P_z = S_2 L_2$

so, $\boxed{\theta_2 = \text{atan2}\left(\frac{P_z}{L_2}, C_2\right)}$

Equate elements (3, 1): $S_{23} = R_{31}$

Equate elements (3, 2): $C_{23} = R_{32}$

so, $\boxed{\boldsymbol{\theta_3 = \text{atan2}(R_{31}, R_{32}) - \theta_2}}$

If both R_{31} and R_{32} are zero, the goal is unattainable.

A second interpretation of the problem is that only a desired position is given (no orientation). In this there may be up to four solutions:

Assume ${}^3P_{\text{tool}} = L_3\hat{X}_3$, then

$$^0P_{\text{tool}} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} L_1C_1 + L_2C_1C_2 + L_3C_1C_{23} \\ L_1S_1 + L_2S_1C_2 + L_3S_1C_{23} \\ L_2S_2 + L_3S_{23} \end{bmatrix}$$

First,

$$S_1 = \frac{P_y}{L_1 + L_2C_2 + L_3C_{23}} \qquad C_1 = \frac{P_x}{L_1 + L_2C_2 + L_3C_{23}}$$

so, $\boxed{\boldsymbol{\theta_1 = \text{atan2}(P_y, P_x) \text{ or } \text{atan2}(-P_y, -P_x)}}$

Since the sign of the "$L_1 + L_2C_2 + L_3C_{23}$" term may be + or −.

Next, define:

$$\alpha = \begin{cases} \frac{P_x}{C_1} - L_1 & \text{if } C_1 \neq 0 \\ \frac{P_y}{S_1} - L_1 & \text{if } S_1 \neq 0 \end{cases}$$

And we have:

$$L_2C_2 + L_3C_{23} = \alpha$$

$$L_2S_2 + L_3S_{23} = P_z$$

Square and add these two equations to get:

$$L_2^2 + L_3^2 + 2L_2L_3C_3 = \alpha^2 + P_z^2$$

$$C_3 = \frac{1}{2L_2L_3}(\alpha^2 + P_z^2 - L_2^2 - L_3^2)$$

$S_3 = \pm\sqrt{1 - C_3^2}$; $\boxed{\theta_3 = \text{atan2}(S_3, C_3)}$