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Lecture Notes on Static and Dynamics

of Manipulators

Statics

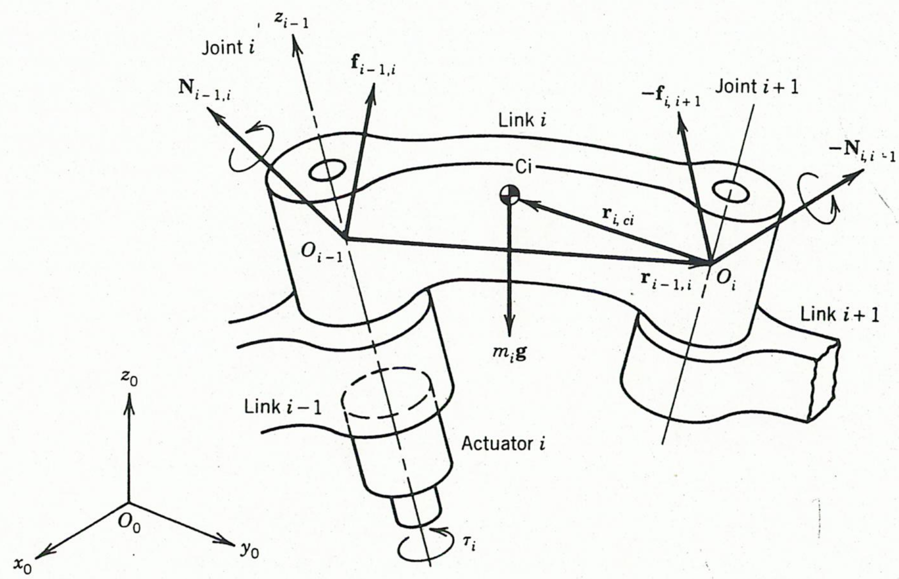
The contact between a manipulator and its environment results in interactive forces and moments, which should be studied in statics. Each joint of a manipulator arm is driven by an individual actuator. The corresponding input joint torques are transmitted through the arm linkage to the end-effector, where the resultant force and moment act upon the environment. The relationship between the actuator drive torques and the resultant force and moment applied at the manipulator endpoint is one of the major topic discussed in this chapter.

Force and Moment Analysis

We begin by considering the free body diagram of an individual link incorporated in an open kinematic chain. Figure below shows the forces and moments acting on link i, which is connected to link $i-1$ and link $i+1$ by joint i and joint $i+1$, respectively. the linear force acting at point O_{i-1}, that is the origin of the coordinate frame $O_{i-1}-x_{i-1}\, y_{i-1}\, z_{i-1}$, is denoted by force $f_{i-1,i}$, where the force is exerted by the link of the first subscript and acts upon the link of the second subscript. The vector $f_{i,i+1}$, therefore, represents the force applied to link $i+1$ by link i. The force applied to link i by link $i+1$ is given by $-f_{i,i+1}$. The gravity force acting at the centroid C_i is denoted by $m_i g$, where m_i is the mass of link i and g is the 3×1 vector representing the acceleration of gravity. The balance of linear forces is given by

$$f_{i-1,i} - f_{i,i+1} + m_i g = 0 , \quad i = 1, \ldots, n$$

Note that all the vectors are defined with reference to the base coordinate frame $O_0 - x_0 y_0 z_0$.



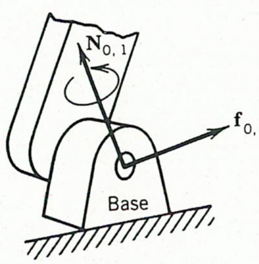
Next, we derive the balance of moments. The <u>moment</u> applied to link i by link $i-1$ is denoted by $N_{i-1,i}$, and therefore the moment applied to link i by link $i+1$ is $-N_{i,i+1}$. Further, the linear forces $f_{i-1,i}$ and $-f_{i,i+1}$ also cause moments about the centroid C_i. The balance of moments with respect to the centroid C_i is thus given by,

$$N_{i-1,i} - N_{i,i+1} - (r_{i-1,i} + r_{i,C_i}) \times f_{i-1,i} + (-r_{i,C_i}) \times (-f_{i,i+1}) = 0 ,$$

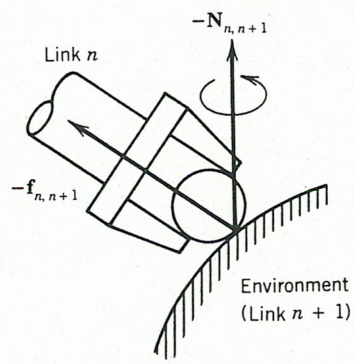
$$i = 1, \ldots, n$$

where $r_{i-1,i}$ is the 3×1 position vector from the point O_{i-1} to point O_i with reference to the base coordinate frame, and r_{i,C_i} represents the position vector from the point O_i to C_i.

The force $f_{i-1,i}$ and moment $N_{i-1,i}$ are called the coupling force and moment between the adjacent links i and $i-1$. When $i=1$, the coupling force and moment becomes $f_{0,1}$ and $N_{0,i}$ These are interpreted as the reaction force and moment applied to the base link to which the arm linkage is fixed. When $i=n$, on the other hand, the above coupling force and moment become $f_{n,n+1}$ and $N_{n,n+1}$. When the endeffector (i.e. link n) contacts the environment, the reaction force and moment act on the last link. For convenience we regard the environment as an additional link, numbered $n+1$, and we represent the reaction force and moment by $-f_{n,n+1}$ and $-N_{n,n+1}$, respectively. Figures below show the forces and moments exerted by the base link and the enviroment.



$i=1$



$i=n$

The above two equations can be derived for all the link members except the base link, $i=1,\ldots,n$. The total number of vector equations that we can derive is then $2n$, wheras the number of coupling forces and moments involved is $2(n+1)$. Therefore two of the coupling forces and moments must be specified; otherwise the equations cannot be solved. The final coupling force and moment, $f_{n,n+1}$ and $N_{n,n+1}$, are the force and moment that the manipulator arm applies to the environment. To perform a task successfully, the manipulator arm needs to accommodate this force and moment. Thus, we specify this coupling force and moment, and solve the simultaneous equations. For convenience we combine the force $f_{n,n+1}$ and the moment $N_{n,n+1}$ and define the following six-dimensional vector,

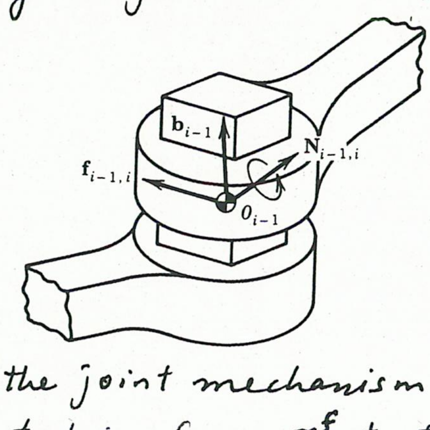
$$F = \begin{bmatrix} f_{n,n+1} \\ N_{n,n+1} \end{bmatrix}$$

We call F the endpoint force and moment vector, or simply the <u>endpoint force</u>.

Equivalent Joint Torques

In this section we derive the functional relationship between the input torques excerted by the actuators and the resultant endpoint force. We assume that each joint is driven by an individual actuator that excerts a <u>drive torque or force</u> between the adjacent links. Let τ_i be the drive torque or force excerted by the ith actuator driving joint i.

For a <u>prismatic joint</u>, the drive force τ_i is a linear force excerted along the joint axis $i-1$, as shown below



Assuming that the joint mechanism is frictionless, we can relate the joint drive force τ_i^f to the linear coupling force $f_{i-1,i}$ between links $i-1$ and i by

$$\tau_i^f = b_{i-1}^T \cdot f_{i-1,i}$$

where b_{i-1} represents the unit vector pointing in the direction of the joint axis and $a^T b$ represents the inner product of vectors a and b. The above equation implies that the actuator must bear only the component of $f_{i-1,i}$ which is in the direction of the joint axis, and that the components in all the other directions are supported by the joint structure. These components of the coupling force are internal constraint forces, which do not produce work.

For a revolute joint, τ_i^N represents a drive torque rather than a drive force. This drive torque is balanced with the coupling torque component of $N_{i-1,i}$ which is the direction of its joint axis:

$$\tau_i^N = b_{i-1}^T \cdot N_{i-1,i}$$

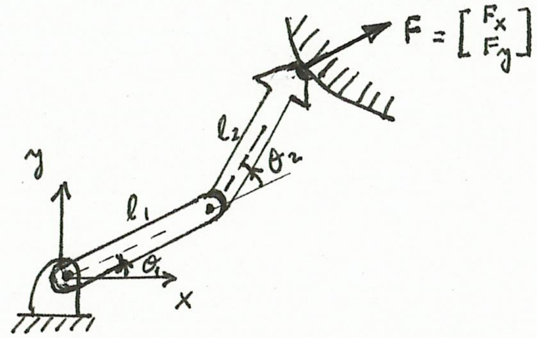
Other components of the coupling torque $N_{i-1,i}$ are borne by the joint structure. They are workless constraint moments. We can combine all the joint drive force and torques together to define the n-vector given by $\tau = [\tau_1 \tau_2 \ldots \tau_n]^T$ and simply call it the joint torques. The joint torques represent the actuators' inputs to the arm linkage. The relationship between the joint torques τ and the endpoint force vector F is stated in the following Theorem:

Theorem: Assume that the joint mechanism are frictionless, then the joint torques τ that are required to bear an arbitrary endpoint force F are given by

$$\tau = J^T F$$

where J is the $6 \times n$ manipulator Jacobian relating infinitesimal joint displacements dq to infinitesimal end-effector displacements dp: $dp = J dq$ as we discussed before.

Example : Find the equivalent joint torques $\tau = [\tau_1 \ \tau_2]^T$ corresponding to the endpoint force $F = [F_x \ F_y]^T$ for a two degree-of-freedom planar manipulator.



The Jacobian has been derived for the two degree-of-freedom planar manipulator previously. Thus,

$$\tau = J^T F$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) \\ -l_2 \sin(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

Stiffness

In this section we analyze the stiffness of a manipulator arm. When a force is applied at the endpoint of a manipulator arm, the endpoint will deflect by an amount which depends on the stiffness of the arm and the force applied. The stiffness of the arm's endpoint determines the strength of the manipulator arm and, more importantly, the positioning accuracy in the presence of disturbance forces and loads. Stiffness is also an important control variable which allows a robot to perform complex task. With the appropriate stiffness, the robot can accommodate endpoint forces with acceptable displacements. Consider an industrial robot consisting of a number of arms connected by joints. Each joint is driven by an individual actuator through a reducer and transmission mechanisms. When a drive force or torque is transmitted, each member involved may deflect. Also, the actuator itself has a limited stiffness determined by its feedback control system, which generates the drive torque based on the discrepancy between the reference position and the actual measured position. The stiffness of the drive system is then dependent on the loop gain of the feedback system. We model the stiffness of the drive system combined with the stiffness of the reducer and transmissions by a spring constant k_i that relates the deflection at joint i to the force or torque transmitted. Namely,

$$\tau_i = k_i \, \Delta q_i$$

where τ_i is the joint torque and Δq_i is the deflection at the joint axis.

To derive the endpoint stiffness from the individual joint stiffness, we denote the endpoint force and moment by F and the resultant deflection by Δp as before. When we neglect gravity and friction at the joints, the following relationship is valid (see previous section)

$$\tau = J^T F$$

The relationship $\tau_i = k_i \, \Delta q_i$ can also be written for all joints in vector matrix notation as

$$\boldsymbol{\tau} = K \, \Delta q$$

where K is a diagonal matrix with diagonal elements K_i; $i = 1, \ldots, n$. The individual joint deflections Δq produce endpoint deflection Δp according to

$$\Delta p = J \, \Delta q$$

When the individual joint drive systems are active and the stiffness are non-zero, the matrix K is invertible. Thus, using the above relationships we have

$$\Delta p = C \, F$$

where

$$C = J K^{-1} J^T$$

The matrix C is called the <u>compliance matrix</u> of the arm endpoint, which relates deflection at the endpoint Δp to the endpoint force F. If the manipulator Jacobian is a square matrix and of full rank, the manipulator has a nonsingular compliance matrix and we have

$$F = C^{-1} \Delta p$$

The inverse of the compliance matrix is called the <u>stiffness matrix</u> of the arm endpoint.

Dynamics

The dynamic behavior of manipulator arms is described by a set of differential equations, called equations of motion, which relates joint torques to joint displacements. Two methods can be used in order to obtain the equations of motion: the Newton-Euler formulation, and the Lagrangian formulation. The Newton-Euler formulation is derived by the direct interpretation of Newton's second law of motion, whereas the Lagrangian methods is based on the energy principle. The dynamic equations of a manipulator can be represented by two equations: one describes the translational motion of the centroid (or center of mass), while the other describes the rotational motion about the centroid. Using the Newton-Euler formulation, we derive the equations of motion by simple adjustment of static balance equations of forces as derived in previous chapter.

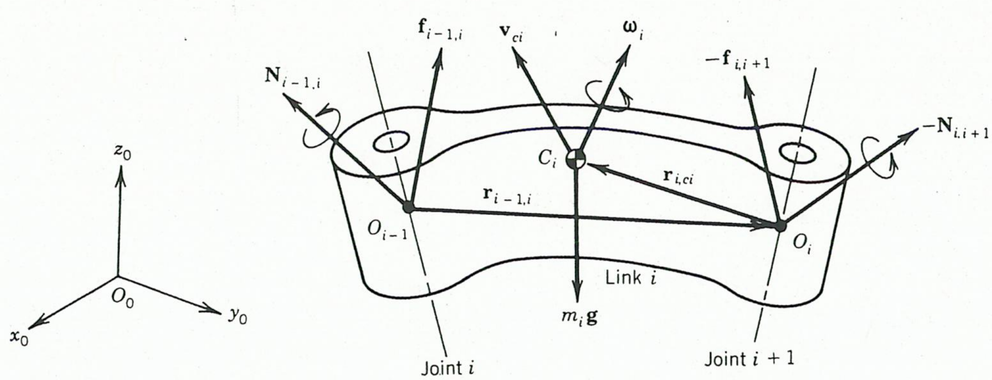
Let v_{ci} be the linear velocity of the centroid of link i with reference to the base coordinate frame $O_0 - x_0 y_0 z_0$. The inertial force is then given by $-m_i \dot{v}_{ci}$, where m_i is the mass of the link i and $\dot{v}_{ci}$ is the acceleration. Considering the figure shown below, the equation of motion is then obtained by adding the inertial force to the static balance of forces as

$$f_{i-1,i} - f_{i,i+1} + m_i g - m_i \dot{v}_{c_i} = 0 \quad , \quad i = 1, \ldots, n$$

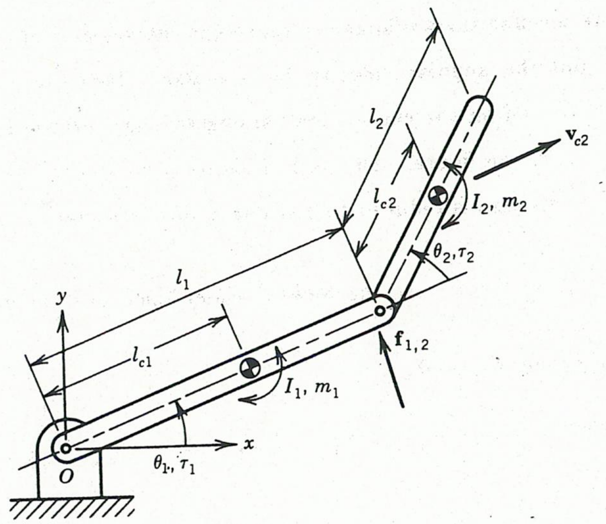
where $f_{i-1,i}$ and $-f_{i,i+1}$ are the coupling forces applied to link i by link $i-1$ and $i+1$, respectively, and g is the acceleration of gravity.

The rotational motions are described by Euler's equations. In the same way as for translational motions, the dynamic equations are derived by adding "inertial torques" to the static balance of moments. The mass properties of manipulator arms are represented by an inertia tensor, which is a 3X3 symmetric matrix I. The inertial torque acting on link i is given by the time rate of change of the angular momentum of the link at that instant. Let ω_i be the angular velocity vector and I_i be the centroidal inertia tensor of link i, then the angular momentum is given by $I_i \omega_i$. Since the inertia tensor varies as the orientation of the link changes, the time derivative of the angular momentum includes not only the angular acceleration term $I_i \dot{\omega}_i$, but also a gyroscopic torque term $\omega_i \times (I_i \omega_i)$ resulting from changes in inertia tensor. Thus,

$$N_{i-1,i} - N_{i,i+1} + r_{i,c_i} \times f_{i,i+1} - r_{i-1,c_i} \times f_{i-1,i} - I_i \dot{\omega}_i - \omega_i \times (I_i \omega_i) = 0 \qquad i = 1, \ldots, n$$



Example : Consider the two degree-of-freedom planar manipulator that we discussed in the previous chapter. Using the Newton-Euler formulation, obtain the equations of motion and write them in the vector-matrix form.



Since the link mechanism is planar, we represent the velocity of the centroid of each link by a 2-vector $\mathbf{v}_{c_i}$ and the angular velocity by a scalar ω_i. We assume that the centroid of link i is located on the center line passing through adjacent joints at a distance l_{c_i} from joint i, as shown in the figure. The axis of rotation does not vary for the planar linkage. The inertia tensor in this case is reduced to a scalar moment of inertia denoted by I_i.

The Newton-Euler equations for link 1 are given by

$$f_{0,1} - f_{1,2} + m_1 g - m_1 \dot{\mathbf{v}}_{c1} = 0 \qquad (1)$$

$$N_{0,1} - N_{1,2} + r_{1,c_1} \times f_{1,2} - r_{0,c_1} \times f_{0,1} - I_1 \dot{\omega}_1 = 0 \qquad (2)$$

Note that all vectors are 2x1, so that $N_{i-1,i}$ and the vector products are scalar quantities. Similarly, for link 2

$$f_{1,2} + m_2 g - m_2 \dot{v}_{c2} = 0 \qquad (3)$$

$$N_{1,2} + r_{1,c_2} \times f_{1,2} - I_2 \dot{\omega}_2 = 0 \qquad (4)$$

To obtain closed-form dynamic equations, we first eliminate the constraint forces and separate them from the joint torques, so as to explicitly involve the joint torques in the dynamic equations. For the planar manipulator, the joint torques τ_1 and τ_2 are equal to the coupling moments :

$$N_{i-1,i} = \tau_i \qquad (5)$$

substituting (5) into (3) and (4), and eliminating $f_{1,2}$, we obtain

$$\tau_2 - r_{1,c2} \times m_2 \dot{v}_{c2} + r_{1,c2} \times m_2 g - I_2 \dot{\omega}_2 = 0 \qquad (6)$$

similarly, eliminating $f_{0,1}$ yields

$$\tau_1 - \tau_2 - r_{0,c1} \times m_1 \dot{v}_{c1} - r_{0,1} \times m_2 \dot{v}_{c2} + r_{0,c1} \times m_1 g + r_{0,1} \times m_2 g - I_1 \dot{\omega}_1 = 0 \qquad (7)$$

Next, we rewrite v_{ci}, ω_i, and $r_{i,i+1}$ using joint displacements θ_1 and θ_2, which are independent variables. Note that ω_2 is the angular velocity relative to the base coordinate frame, while $\dot{\theta}_2$ is measured relative to link 1. Then we have

$$\omega_1 = \dot{\theta}_1 \qquad \omega_2 = \dot{\theta}_1 + \dot{\theta}_2 \qquad (8)$$

The linear velocities can be written as

$$v_{c1} = \begin{bmatrix} -l_{c1}\dot{\theta}_1 \sin\theta_1 \\ l_{c1}\dot{\theta}_1 \cos\theta_1 \end{bmatrix} \qquad v_{c2} = \begin{bmatrix} -(l_1\sin\theta_1 + l_{c2}\sin(\theta_1+\theta_2))\dot{\theta}_1 - l_{c2}\sin(\theta_1+\theta_2)\dot{\theta}_2 \\ (l_1\cos\theta_1 + l_{c2}\cos(\theta_1+\theta_2))\dot{\theta}_1 + l_{c2}\cos(\theta_1+\theta_2)\dot{\theta}_2 \end{bmatrix} \quad (9)$$

substituting (8), (9) along with their time derivatives into equations (6) and (7), we obtain the closed-form dynamic equations in terms of θ_1 and θ_2 :

$$\begin{aligned} \tau_1 &= H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 - h\dot{\theta}_2^2 - 2h\dot{\theta}_1\dot{\theta}_2 + G_1 \\ \tau_2 &= H_{22}\ddot{\theta}_2 + H_{12}\ddot{\theta}_1 + h\dot{\theta}_1^2 + G_2 \end{aligned} \qquad (10)$$

where

$$\begin{aligned} H_{11} &= m_1 l_{c1}^2 + I_1 + m_2[l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos\theta_2] + I_2 \\ H_{22} &= m_2 l_{c2}^2 + I_2 \\ H_{12} &= m_2 l_1 l_{c2}\cos\theta_2 + m_2 l_{c2}^2 + I_2 \\ h &= m_2 l_1 l_{c2}\sin\theta_2 \\ G_1 &= m_1 l_{c1} g\cos\theta_1 + m_2 g(l_{c2}\cos(\theta_1+\theta_2) + l_1\cos\theta_1) \\ G_2 &= m_2 l_{c2} g\cos(\theta_1+\theta_2) \end{aligned} \qquad (11)$$

The scalar g represents the acceleration of gravity along the negative y axis.

More generally, the closed form dynamic equations of n degree-of-freedom manipulator can be written as

$$\tau_i = \sum H_{ij}\ddot{q}_i + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}\dot{q}_j\dot{q}_k + G_i \qquad i=1,\dots,n \quad (12)$$

or in vector-matrix form as

$$H\ddot{q} + h = \tau \qquad (13)$$

where $H = H(q)$ is manipulator inertia matrix and the nonlinear term $h = h(q,\dot{q},t)$ contains centrifugal, Coriolis, and gravitational forces.

An alternative notation for (13) is given by

$$M(q)\,\ddot{q} + V(q,\dot{q}) + G(q) = \tau \qquad (14)$$

where $H(q)$ is denoted by $M(q)$, which represents the inertia matrix, and $h(q,\dot{q})$ is decomposed into two terms $V(q,\dot{q})$ and $G(q)$ representing Coriolis/centrifugal vector, and gravity vector respectively.

Finally if we assume that the link masses of the planar manipular in the example are concentrated at the ends of the links, we obtain the following vector-matrix equation:

$$\begin{bmatrix} (m_1+m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 \cos\theta_2 & m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 \\ m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -m_2 l_1 l_2 (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\sin\theta_2 \\ m_2 l_1 l_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix} + \begin{bmatrix} (m_1+m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos(\theta_1+\theta_2) \\ m_2 g l_2 \cos(\theta_1+\theta_2) \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

(15)