ME 5250, RMC

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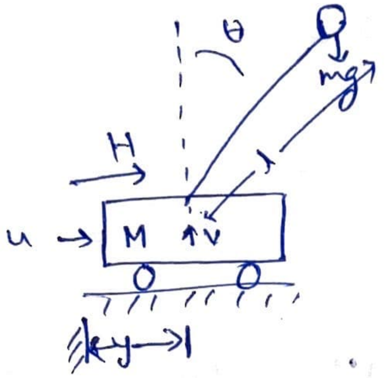
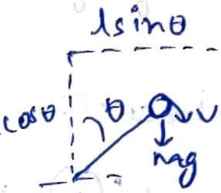


(a) FBD:



By lagrange's method,

$$T = \frac{1}{2} m (\dot{y})^2 + \frac{1}{2} m (R\dot{\theta} + \dot{y})^2$$

$$T = \frac{1}{2} m (\dot{y})^2 + \frac{1}{2} m (L^2\dot{\theta}^2 + \dot{y}^2 + 2L\dot{\theta}\dot{y}\cos\theta)$$

$$V = mgl\cos\theta$$

$$W = u\,dy$$

$$L = T - V$$

$$L = \frac{1}{2} m (\dot{y})^2 + \frac{1}{2} m \left[L^2\dot{\theta}^2 + \dot{y}^2 + 2L\dot{\theta}\dot{y}\cos\theta\right] - mgl\cos\theta$$

WKT,

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] - \frac{\partial L}{\partial \theta} = 0$$

$$\rightarrow ml^2\ddot{\theta} + m\ddot{y}l\cos\theta - m\dot{y}l\dot{\theta}\sin\theta + m\dot{y}l\dot{\theta}\sin\theta - mgl\sin\theta = 0$$

dividing by (ml) on both sides.

$$l\ddot{\theta} + \ddot{y}\cos\theta - g\sin\theta = 0$$

Thus, the dynamical eqn's:

$$(M+m)\ddot{y} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = u \quad \text{—— (1)}$$

$$ml^2\ddot{\theta} + m\ddot{y}\,l\cos\theta = mgl\sin\theta \quad \text{—— (2)}$$

→ These are non-linear as they got trigonometric terms.

(b) assuming θ is small,

$\cos\theta \simeq 1$ $\quad \dot{\theta}^2 = 0$

$\sin\theta \simeq \theta$

dynamical eq.^{n}s become:

$$(M+m)\ddot{y} + ml\ddot{\theta}(1) - ml(0)\theta = u$$

$$\boxed{(M+m)\ddot{y} + ml\ddot{\theta} = u}$$

$$ml^2\ddot{\theta} + ml\ddot{y}(1) = mgl\theta \Rightarrow ml^2\ddot{\theta} + ml\ddot{y} = mgl\theta.$$

→ We can come up with two coupled linear differential equations by assuming 'θ' is small.

(c) To find $\frac{\theta(s)}{u(s)}$: $(M+m)\ddot{y} + ml\ddot{\theta} = u$

$$s^2(M+m)y(s) + s^2 ml\,\theta(s) = u(s) \quad \text{—} ①$$

$$l\ddot{\theta} + \ddot{y} = g\theta$$

$$s^2 l\,\theta(s) + s^2 y(s) = g\,\theta(s)$$

$$(s^2 l - g)\,\theta(s) = y(s)\,[-s^2]$$

$$\theta(s) = \frac{-s^2 y(s)}{s^2 l - g} \quad \text{—} ②$$

$$y(s) = \frac{g - s^2 l\,\theta(s)}{s^2} \quad \text{—} ③$$

③ in ①,

$$\frac{s^2[M+m](g - s^2 l)\,\theta(s)}{s^2} + s^2 ml\,\theta(s) = u(s)$$

$$\frac{\theta(s)}{u(s)} = \frac{1}{g(M+m) - s^2 lM}$$

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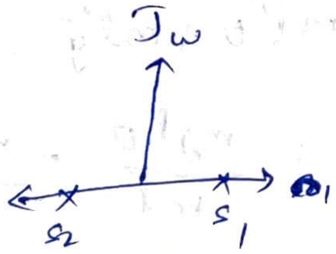
for poles,

$$g(M+m) - s^2 M l = 0$$



$$s^2 = \frac{g}{Ml}(M+m)$$

$$s_{12} = \pm\sqrt{\frac{(M+m)g}{Ml}}.$$

→ one of the poles is in Right hand plane, so the system is unstable.

→ for $y(s)/u(s)$:-

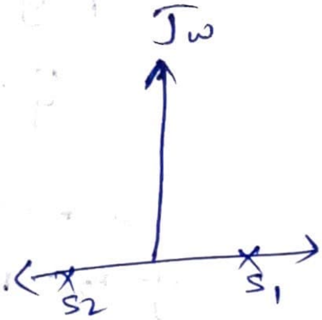
(2) in (1) :- $\dfrac{s^2(M+m)\,y(s) + s^2 m l(-s^2 y(s))}{s^2 l - g} = u(s)$

$$y(s)\left[s^4 l(M+m) - s^4 m l - s^2 g(M+m)\right] = (s^2 l - g)\,u(s)$$

$$\frac{y(s)}{u(s)} = \frac{s^2 l - g}{s^4 M l - s^2 g(M+m)}$$

for poles,

$$s^4 M l - s^2 g(M+m) = 0$$



$$s^2 = g/ML\,(M+m)$$

$$s_{12} = \pm\sqrt{\frac{g}{ML}(M+m)}$$

→ system is unstable (as one of the roots is in right half plane)

① (a) $\theta = x_1,\ \dot{\theta} = \dot{x}_1 = x_2,\ y = x_3,\ \dot{y} = \dot{x}_3 = x_4$

$$(M+m)\ddot{y} + ml\ddot{\theta} = f$$

$$ml\ddot{\theta} = \overset{M}{\cancel{m}}g\theta - m\ddot{y}$$

$$\ddot{\theta} = \frac{\overset{M}{\cancel{m}}g\theta - m\ddot{y}}{ml} \quad — ①$$

$$(M+m)\ddot{y} + mg\theta - m\ddot{y} = u$$

$$M\ddot{y} + mg\theta = u$$

$$\ddot{y} = \frac{u}{M} - \frac{mg\theta}{M} \quad — ②$$

② in ①,

$$\ddot{\theta} = \frac{\overset{M}{\cancel{m}}g\theta}{ml} - \frac{1}{l}\left[\frac{u}{M} - \frac{mg\theta}{M}\right]$$

$$\ddot{\theta} = \frac{-u}{Ml} + \frac{(M+m)g\theta}{Ml} \quad — ③$$

Also in ②,

$$\dot{x}_4 = \ddot{y} = \frac{u}{M} - \frac{mgx_1}{M}$$

$$\dot{x}_2 = \ddot{\theta} = \frac{-u}{ML} + \frac{(M+m)gx_1}{Ml}$$

$$\left.\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{M} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/ML \\ 0 \\ 1/M \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
\end{aligned}\right\} \text{state space eq}^{n}\text{s}$$

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1 e for state feedback control,

$$|\lambda I - (A + BK)| = \Delta d$$

here,

$$K = [-56.82 \quad -12.54 \quad -2.44 \quad -5.09]$$

↳ from MATLAB.

```
M = 2;
m = 0.1;
l = 0.5;
g = 9.81;

A = [0 1 0 0; (M+m)*g/(M*l) 0 0 0; 0 0 0 1; -(m*g)/M 0 0 0];
B = [0; -1/(M*l); 0; 1/M];
C = [1 0 0 0; 0 0 1 0];
D = 0;

p1 = -1;
p2 = -2;
p3 = -3;
p4 = -4;

K = place(A, B, [p1 p2 p3 p4])
```

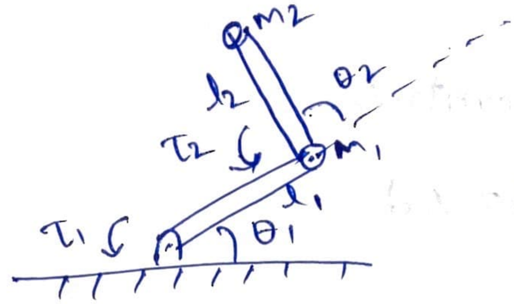
Command Window

```
>> rmc

K =

  -56.8242  -12.5484   -2.4465   -5.0968
```

(2)



(a) Jacobian $J = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix}$

case ①:

$\dot\theta_1 = 1$ rad/s ; $\dot\theta_2 = 3$ rad/s

$\theta_1 = 60^\circ$; $l_1 = 2m$

$\theta_2 = 30^\circ$; $l_2 = 3m$

$$\begin{bmatrix} \dot x \\ \dot y \end{bmatrix} = \begin{bmatrix} -2\sin 60^\circ - 3\sin 90^\circ & -3\sin 90^\circ \\ 2\cos 60^\circ + 3\cos 90^\circ & 3\cos 90^\circ \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} \dot x \\ \dot y \end{bmatrix} = \begin{bmatrix} -13.732 \\ 1 \end{bmatrix}$$

case ②:

$\theta_1 = 167^\circ$; $l_1 = 2m$

$\theta_2 = -156^\circ$; $l_2 = 3m$

$$\begin{bmatrix} \dot x \\ \dot y \end{bmatrix} = \begin{bmatrix} -2\sin(167^\circ) - 3\sin(-156^\circ) & -3\sin(-156^\circ) \\ 2\cos(167^\circ) + 3\cos(-156^\circ) & 3\cos(156^\circ) \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} \dot x \\ \dot y \end{bmatrix} = \begin{bmatrix} -1.0223 & -0.5724 \\ 0.9961 & 2.9449 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} \dot x \\ \dot y \end{bmatrix} = \begin{bmatrix} -2.7396 \\ 9.8308 \end{bmatrix}$$

(5) for $F = \begin{bmatrix} 30 \\ 20 \end{bmatrix}$

$Z = J^T F$

<u>case (1):</u> $Z = \begin{bmatrix} -4.732 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 30 \\ 20 \end{bmatrix}$

$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} -121.96 \\ -90 \end{bmatrix}$ Nm.

<u>case (2):</u>

$Z = \begin{bmatrix} -1.022 & 0.9961 \\ -0.5724 & 2.9449 \end{bmatrix} \begin{bmatrix} 30 \\ 20 \end{bmatrix}$.

$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} -10.738 \\ 41.7248 \end{bmatrix}$ Nm.

for $F = \begin{bmatrix} 30 \\ -20 \end{bmatrix}$; <u>case (1):</u> $Z = \begin{bmatrix} -3-\sqrt{3} & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 30 \\ -20 \end{bmatrix}$

$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} -161.9615 \\ -90 \end{bmatrix}$ Nm.

<u>case (2),</u>

$Z = \begin{bmatrix} -1.0223 & 0.9961 \\ -0.5724 & 2.9449 \end{bmatrix} \begin{bmatrix} 30 \\ -20 \end{bmatrix} = \begin{bmatrix} -50.59 \\ -76.07 \end{bmatrix}$ Nm.

→ We can conclude that, force at the tip depends on its arm design & varies ~~depe~~ ~~po~~ ~~d~~. according to its configuration.

ⓒ Dynamical eqⁿ is:

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

$$M(\theta) = \begin{bmatrix} l_2^2 m_2 + 2 l_1 l_2 m_2 c_2 + l_1^2 (m_1+m_2) & l_2^2 m_2 + l_1 l_2 m_2 c_2 \\ l_2^2 m_2 + l_1 l_2 m_2 c_2 & l_2^2 m_2 \end{bmatrix}$$

$$V(\theta,\dot{\theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_2 l_1 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1+m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}$$

by control law partioning, $\tau = M\ddot{\theta} + V\dot{\theta} + G\theta$

$$\tau = \alpha \tau' + \beta$$

let $\alpha = M$, $\beta = V\dot{\theta} + G\theta$; $\ddot{\theta} = \tau'$

for stabilizing, $\tau' = -k_v\dot{\theta} - k_p\theta$

$$\ddot{\theta} + k_v\dot{\theta} + k_p\theta = 0 \quad \text{—} \; (1)$$

$$k_v = 2\sqrt{k_p}$$

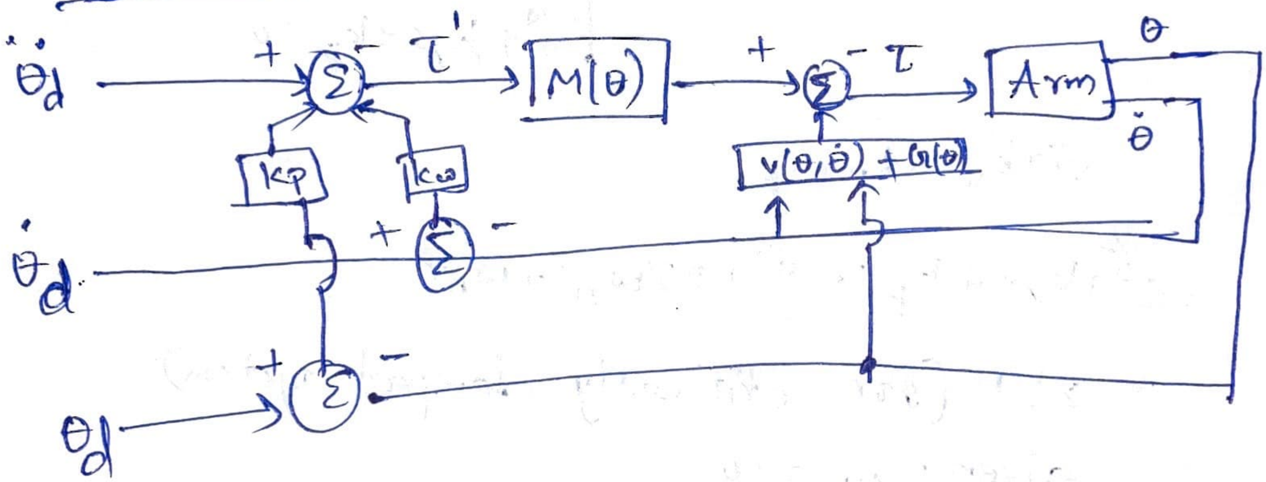
from (1), $s^2 + k_v s + k_p = s^2 + 2\omega_n s + \omega_n^2$

thus, $K_\omega = K_v = \begin{bmatrix} K_{\omega_1} & 0 \\ 0 & K_{\omega_2} \end{bmatrix}$

$K_p = K_\theta = \begin{bmatrix} K_{\theta_1} & 0 \\ 0 & K_{\theta_2} \end{bmatrix}$

for critical damping, $K_{\omega_i} = 2\sqrt{K_i}$

Block Diagram



(3) Given, $\tau = ml^2\ddot{\theta} + f\dot{\theta} + mgl\cos\theta$

(a) Given $\omega_n = 10$, $m = 1$, $l = 1$, $f = 7$, $g = 10$

$\Rightarrow \tau = \ddot{\theta} + 7\dot{\theta} + 10\cos\theta$

Method of Control law partioning:

(i) Model based portion: $\tau = \ddot{\theta} + 7\dot{\theta} + 10\cos\theta$

$\tau = \alpha\tau' + \beta$

$\ddot{\theta} + 7\dot{\theta} + 10\cos\theta = \alpha\tau' + \beta$

(ii) Servo portion:

$\alpha = m = 1$; $\beta = f\dot{x} + kx = 7\dot{\theta} + 10\cos\theta$

$\Rightarrow \ddot{\theta} + 7\dot{\theta} + 10\cos\theta = \tau' + 7\dot{\theta} + 10\cos\theta$

$$\boxed{\ddot{\theta} = \tau'}$$

$\tau' = -k_v\dot{\theta} - k_p\theta + \cancel{x}^0$

$\tau' + k_v\dot{\theta} + k_p\theta = 0 \Rightarrow \ddot{\theta} + k_v\dot{\theta} + k_p\theta = 0$

$s^2 + k_v s + k_p = 0$

$s^2 + k_v s + k_p = 0$

$\Rightarrow s^2 + k_v s + k_p \equiv s^2 + 2z\omega_n s + \omega_n^2$

$z = 1$ (for critically damped system)

given; $\omega_n = 10$

$\Rightarrow s^2 + k_v s + k_p \equiv s^2 + 2(1)(10)s + 10^2$

$$[k_v = 20;\ k_p = 100]$$

(b) For desired traj: $\theta_d = A\sin(2\pi t/T)$

$A = 0.1$ rad ; $T = 2$ sec ; $\tau' = \ddot{\theta}_d + k_v\dot{E} + k_p E$

error eq^n's : $E = \theta_d - \theta$; $\dot{E} = \dot{\theta}_d - \dot{\theta}$; $\ddot{E} = \ddot{\theta}_d - \ddot{\theta}$

$\theta_d = 0.1 \sin\left(\frac{2\pi t}{2}\right)$

PS no. 6

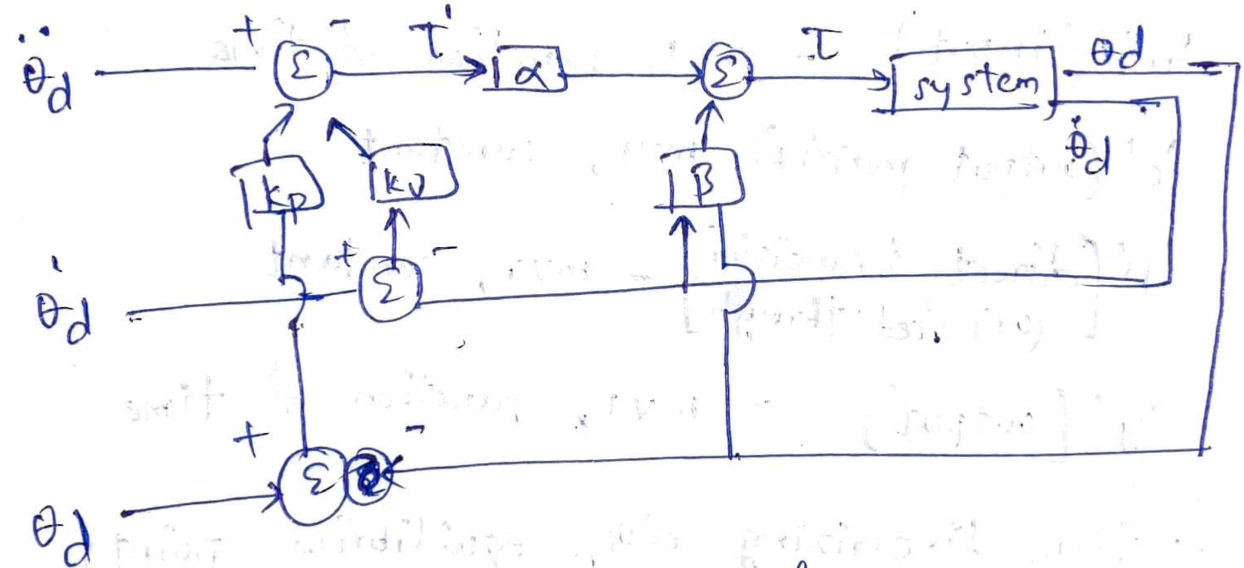
$$\dot{\theta}_d = 0.1\pi \cos(\pi t) = 0.314 \cos(\pi t)$$

$$\ddot{\theta}_d = -0.1\pi^2 \sin(\pi t) = -0.986 \sin(\pi t)$$

$$k_p = 100 \;;\; k_v = 20$$

$$\tau' = -0.1\pi^2 \sin(\pi t) + 20\dot{E} + 100E.$$

block diagram:



(c) The state space representation of a system replaces an n^{th} order differential eqn with a

(3) (c) $T = ml^2\ddot{\theta} + f\dot{\theta} + mgl\cos\theta$

given, $m=1, l=1, f=7, g=10.$

$T = \ddot{\theta} + 7\dot{\theta} + 10\cos\theta$

assume, $T = u$ & $\theta = y$

$u = \ddot{y} + 7\dot{y} + 10\cos y$

for state space eq^n, $x_1 = y$, $x_2 = \dot{y} = \dot{x}_1$, $\dot{x}_2 = \ddot{y}$

then, $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u - 7x_2 - 10x_1 \end{bmatrix} = f(x,u)$ — (1)

$y = x_1 = g(x,u)$ — (2)

finding equilibrium pt, (x_0, u_0) → let's take $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

to find $f(x_0, u_0) = 0$.

$f(x_0, u_0) = \begin{bmatrix} 0 \\ u_0 - 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{u_0 = 10}$

To find state space eq^{n}s,

$A = \frac{\partial f(x_0, u)}{\partial x} = \begin{bmatrix} 0 & 1 \\ 10\sin 0^\circ & -7 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -7 \end{bmatrix}$

$B = \frac{\partial f}{\partial u}(x_0, u_0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $C = \frac{\partial g}{\partial x}(x_0, u_0) = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$D = \frac{\partial g}{\partial u}(x_0, u_0) = [0]$

Pg no (8)

$$\Delta \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -7 \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u$$

$$x = x_0 + \Delta x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \Delta x$$

$u = u_0 + \Delta u = 10 + \Delta u; \quad y = y_0 + \Delta y = 0 + \Delta y = \Delta y$

from (1) ⇒ finding eigen values for stability,

$$(A - \lambda I) = 0$$

i.e,
$$\begin{bmatrix} \lambda & 1 \\ 0 & -7-\lambda \end{bmatrix} = 0.$$

$$\lambda = 0, \quad \lambda = -7.$$

→ the system is marginally ~~unstable~~ stable because there is a pole at '0'.