PART 1

Basic Review of Linear Systems Analysis and Control Design

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The Study of Systems Consists of 4 Parts

1. <u>Modeling</u> → <u>Analytical</u> "KVL, KCL, Newton"
   → Experimental "Systems ID"

2. <u>Mathematical Description</u>
   - Diff. Eq.
   - Transfr Function
   - State Equation

3. <u>Analysis</u> → Quantitative "Response"
   → Qualitative "stability"

4. <u>Design</u>

"Feedback Control"

B. Shafai

Basic Review of Linear Systems and Control Design

1. Modeling and Mathematical Descriptions

A dynamical system can be described by differential equation. For linear time invariant system one can write it as

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y =$$

$$b_n \frac{d^n u}{dt^n} + b_{n-1} \frac{d^{n-1} u}{dt^{n-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u \qquad (1)$$

to represent the relationship between input u and output y of the system.

▷ <u>Input-Output Description</u>

Consider the LTI system shown below,

If the LTI system is causal and initially relaxed at $t_0 = 0$, then one can represent the input-output relationship by the convolution integral

$$y(t) = \int_{t_0=0}^{t} g(t-\tau)\, u(\tau)\, d\tau \qquad (2)$$

where $g(\cdot)$ is called the <u>impulse response</u>.

It is not difficult to show that the Laplace transform of (2) is given by

$$L\{y(t)\} = Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt$$

$$= \int_0^\infty \left(\int_0^\infty g(t-\tau)\, u(\tau)\, d\tau \right) e^{-st}\, dt$$

$$= \int_0^\infty \left(\int_0^\infty g(\underbrace{t-\tau}_{v})\, e^{-s(t-\tau)}\, dt \right) u(\tau)\, e^{-s\tau}\, d\tau$$

$$= \underbrace{\int_0^\infty g(v)\, e^{-sv}\, dv}_{G(s)} \; \underbrace{\int_0^\infty u(\tau)\, e^{-s\tau}\, d\tau}_{U(s)}$$

$$\Rightarrow \underline{Y(s) = G(s)\, U(s)} \qquad (3)$$

where $G(s)$ is the <u>transfer function</u> of the system which is the Laplace transform of the impulse response $g(t)$.

<u>Example 1</u>: Suppose the differential equation of an LTI mechanical or electrical system is given by

$$\ddot{y} + \alpha \dot{y} + \beta y = u$$

Then by taking the Laplace transform of the above equation, assuming all initial conditions equal to zero, we get the transfer function $G(s)$ as follows.

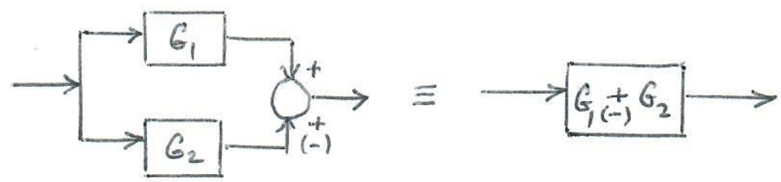
$$s^2 Y(s) + \alpha s\, Y(s) + \beta\, Y(s) = U(s)$$

$$\Rightarrow G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + \alpha s + \beta}$$

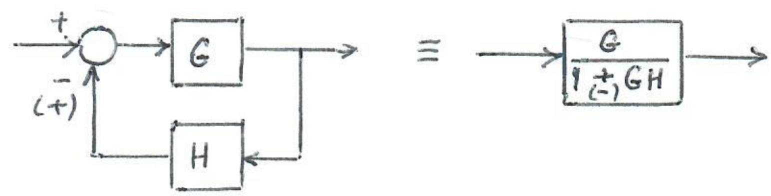
Basic Building Blocks :

Cascade :

Parallel :

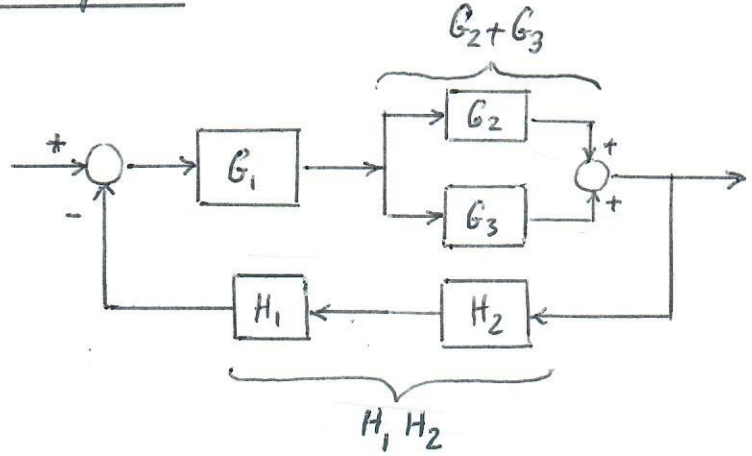


Feedback :



More complex systems can be represented by a combination of the above basic building blocks and can be simplified to a single block using several techniques such as Mason's Gain Formula. Once the transfer function of each block is specified, MATLAB can be used to obtain the overall transfer function of the complex system.

Example 2 :



"Given Expressions of G_i's and H_i's One Can Substitute them in $G(s)$."

Total Transfer Function : $G(s) = \dfrac{G_1(G_2+G_3)}{1 + G_1(G_2+G_3)H_1 H_2}$

▷ State Variable Description

Any n^{th} order differential equation can be written as a set of n first order differential equation called state equation

Example 3 :

consider again example 1 with

$$\ddot{y} + \alpha \dot{y} + \beta y = u$$

Let us define $y = x_1$ as the state variable, Then we can also define

$$\dot{y} = \dot{x}_1 = x_2$$

$$\ddot{y} = \dot{x}_2 = -\beta x_1 - \alpha x_2 + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{"state Equation"}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{"output Equation"}$$

General Notation for nth order system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad (4)$$

Relationship between state variable description and transfer function:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$sX(s) - \cancelto{0}{x(0)} = AX(s) + BU(s)$$

$$X(s) = (sI - A)^{-1} B\, U(s)$$

$$Y(s) = C\, X(s) + D\, U(s)$$

$$Y(s) = \left[\underbrace{C(sI - A)^{-1} B + D}_{G(s)}\right] U(s)$$

Given $\{A, B, C, D\}$ one can obtain $G(s) = C(sI - A)^{-1} B + D$. This process is <u>unique</u>. However, if $G(s)$ is given one can find different state variable descriptions. Consequently $G(s) \rightarrow \{A, B, C, D\}$ is "<u>not unique</u>" and this process is known as <u>realization</u>.

Analysis of Linear Systems

Response : Input-output $\qquad G(s) = \frac{Y(s)}{U(s)}$ given u, one can obtain y

Ex. Let $G(s) = \frac{1}{s+1}$ and $u(t) = 1 \;\circ\!\!-\!\!\bullet\; U(s) = \frac{1}{s}$

Then $Y(s) = G(s)\, U(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$

$$y(t) = 1 - e^{-t}$$

In time domain $y(t) = \int_0^t g(t-\tau)\, u(\tau)\, d\tau$

Response : State-space

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

Methods: $e^{At} = \mathcal{L}^{-1}[sI - A]^{-1}$ or use Cayley-Hamilton

$$y(t) = Cx(t) + Du(t) = \ldots$$

Stability : Input-output

BIBO stability (Bounded-Input Bounded Output) stability

$g(t)$ Impulse response must be absolutely integrable.
For causal proper systems : $\lim_{t \to \infty} g(t) = 0$

$$G(s) = \frac{N(s)}{D(s)} \leftarrow$$

$D(s) = 0$ poles

system is BIBO stable iff all poles lie in strict left half of complex plane

Stability : State-space $\qquad \dot{x} = Ax + Bu$

- $\det(\lambda I - A) = 0$ eigenvalues $\operatorname{Re}\{\lambda_i\} < 0$

or

Lyapunov
- $A^T P + PA = -Q$ ← for any $Q > 0$

$P > 0$: P must be positive definite

<u>Controllability</u> : $\dot{x} = Ax + Bu$

$\{A, B\}$ Controllable pair

rank $\rightarrow \rho[U] = \rho[B \; AB \; A^2B \; \cdots \; A^{n-1}B] = n$

$\uparrow$ Controllability matrix

<u>Observability</u> : $\dot{x} = Ax + Bu$

$y = Cx + Du$

$\{A, C\}$ observable pair

rank $\rightarrow \rho[V] = \rho \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

$\uparrow$ Observability matrix

<u>Equivalent Transformation:</u>

$$\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array} \xrightarrow[\bar{x} = Px]{} \begin{array}{l} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} + \bar{D}u \end{array}$$

$$\Rightarrow \bar{A} = PAP^{-1}, \quad \bar{B} = PB, \quad \bar{C} = CP^{-1}, \quad \bar{D} = D$$

<u>Transformation to Controllable Canonical form</u>

Given $\{A, B\} \xrightarrow{P?} \bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & & & \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

for SISO Systems

The required transformation is :

$P^{-1} = U\bar{U}^{-1}$ $\rightarrow$ U is the Controllability matrix

$$\bar{U}^{-1} = \begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & 1 \\ \alpha_{n-2} & & & \\ \vdots & & \mathbf{0} & \\ 1 & & & \end{bmatrix}$$

where $\det(\lambda I - A) = \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n$

Design of Classical PID Controller for Linear Systems Represented by Transfer function

- Design Based On Root_locus

- Design Based on Frequency Response Maps
  - Bode Plot
  - Nyquist Diagram
  - Nichols Chart

We only consider this case. Design based on the frequency response maps can also be performed.

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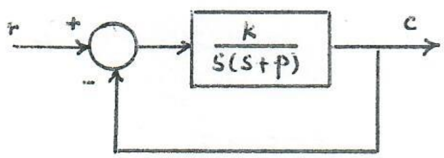
The Design of Feedback Control Systems Based on The Root Locus Technique

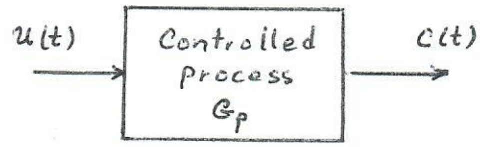
1. Introduction

It is often possible to adjust the system parameters in order to provide the desired system response.
A typical example would be to obtain some parameters of a system such that certain conditions on design specification, such as maximum overshoot, rise time, settling time etc., to be satisfied.
Recall the following example:
Find k, p of the following system for an overshoot of less than 5% and a settling time of less than 4 sec.



We simply obtained the parameters k and p to be 2 and 2. (see homework problems). This is a parameter design Problem.
Sometimes we are not able to simply adjust system parameters and thus obtain the desired performance. Rather we are forced to reconsider the structure of the system and redesign the system in order to obtain a suitable one. In general, the process under control or

The controlled process can be represented by the following block diagram representation :

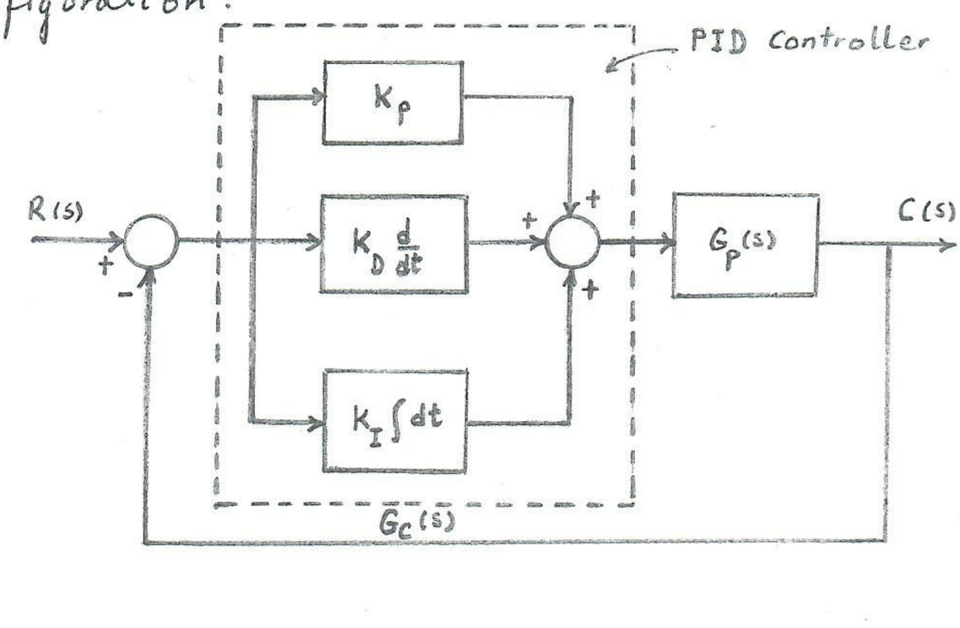


The Problem is to find the control signal $u(t)$ such that, The output signal $C(t)$ behave as desired ; or simply, the design specifications are all satisfied.

The Controller or Compensator which causes to produce the Control u has to be designed. <u>This is a structure design problem.</u>

If The controller is placed in series with the process Then the configuration is referred to as series or Cascade compensation. In the following we consider Cascade compensation for the discussion of feedback Control system design.

If the above process is connected with the controller, which in general consists of Proportional-, Integral- and Derivative Control, Then we have the following configuration.



The transfer function of a PID controller can be written as

$$G_c(s) = K_p + K_D s + \frac{K_I}{s}$$

and the design problem is to determine the values of the constants K_p, K_D and K_I so that the performance of the system is as prescribed.

In order to have a feeling for design problem consider the simple example below

let the process transfer function of a system be $G_p(s) = \frac{1}{s+1}$. The step response of this system is given by $1 - e^{-t}$. The time constant of the system is 1 sec. and the settling time is $4\tau = T_s = 4$ sec.

It is desired to have a response such that the T_s is 1 sec.

suppose in the figure last page we introduce only a proportional controller K_p and ignoring the rest. Then $G_c(s) = K_p$ and the Transfer function of the closed loop system is given by:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G_c(s)\,G_p(s)}{1 + G_c(s)\,G_p(s)} = \frac{K_p \cdot \frac{1}{s+1}}{1 + K_p \cdot \frac{1}{s+1}} = \frac{K_p}{s + K_p + 1}$$

The step response of the system is now $\frac{K_p}{K_p+1}\left[1 - e^{-(K_p+1)t}\right]$

The Time Constant is $\frac{1}{K_p+1}$ and $T_s = \frac{4}{K_p+1}$ which must be set equal 1 so that $\frac{4}{K_p+1} = 1$ and $K_p = 3$ satisfies the desired specification.

In section 8.2 The effects of PD and PI controllers are well discribed.

We can conclude from all these discussion that:

The first step in design is to use a variable amplifier as a compensator
(Proportional controller : K_p)
If the performance specifications are not met, we design our compensator when the following cases occurs. (**Generic Rules**)

1) The given system is unstable → use PD
2) The given system is stable, but the transient behaviour is not satisfactory → use PD
3) The given system is stable, but the steady state behaviour is not satisfactory → use PI
4) The given system is stable, but the transient and steady state behaviours are not satisfactory → use PID

Note also that:

- The <u>stability</u> can be discussed by Routh criterion or root locus technique.
- The <u>steady state behaviour</u> is given by steady state error with respect to step, ramp and parabola inputs.
- The <u>transient behaviour</u> is given by T_r, T_p, M_p, T_s.

2. Lead Compensation based on the root locus technique

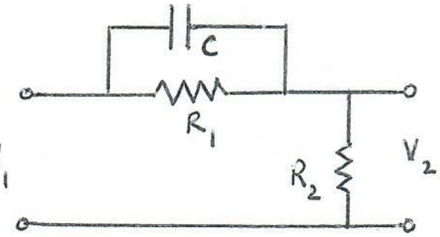
The root locus approach to design is very powerful when the specifications are given in terms of time domain quantities, such as maximum overshoot, rise time, settling time, damping ratio, and undamped natural frequency of the desired dominant closed loop poles.

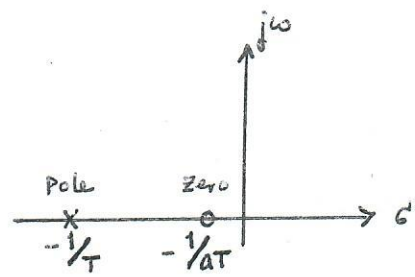
Phase lead controller :

The Transfer function of a simple controller that can be realized by passive resistive - capacitive network elements is

$$G_c(s) = \frac{s + z_1}{s + p_1}$$

The controller is high pass or phase lead if $p_1 > z_1$. A network realization of the phase lead controller along with its pole zero configuration is given below.





$$\frac{V_2}{V_1} = \frac{R_2}{R_1 + R_2} \cdot \frac{1 + R_1 C s}{1 + \frac{R_1 R_2}{R_1 + R_2} C s}$$

let $a = \frac{R_1 + R_2}{R_2} > 1$, $T = \frac{R_1 R_2}{R_1 + R_2} C$

Then $$\frac{V_2}{V_1} = \frac{s + 1/aT}{s + 1/T}$$

Method of Design :

1. List the system specifications and translate these specifications into a desired root location for the dominant roots.
2. Sketch the uncompensated root locus and determine whether the desired root locations can be realized with an uncompensated system.
3. If the compensator is necessary, place the zero of the phase lead network directly below the desired root location.
4. Determine the pole location so that the total angle at the desired root location is $\pm 180^{\circ}$ and therefore is on the compensated root locus.
5. Evaluate the total system gain at the desired root location and then calculate the error constant.
6. Repeat the steps if the error constant is not satisfactory.

Example :

Let the open loop Transfer function of an uncompensated system be $G_p(s) = \frac{K_1}{s^2}$

The characteristic equation of the uncompensated system is

$$1 + G_p(s) = 1 + \frac{K_1}{s^2} = 0$$

and the root locus is simply the $j\omega$-axis. No matter what K_1 is, the system is unstable and according to our rule we need a PD controller. Suppose the system must satisfy the following specifications

settling Time, $T_s \leq 4$ sec

percent overshoot for a step input $\leq 30\%$

using the method of Design outlined above we first translate the given specifications into a desired root location for the dominant roots.

1. The damping ratio should be $\zeta \geq 0.35$. The settling time requirement is $T_s = \frac{4}{\zeta\omega_n} = 4$ and Therefore $\zeta\omega_n = 1$. Thus we will choose a desired dominant root location as

$$r_1,\ \hat{r}_1 = -1 \pm j2 \qquad ; \qquad \zeta = 0.45$$

2. Figure below shows the uncompensated root locus and the desired root location r_1. ($\hat{r}_1$ is not shown)

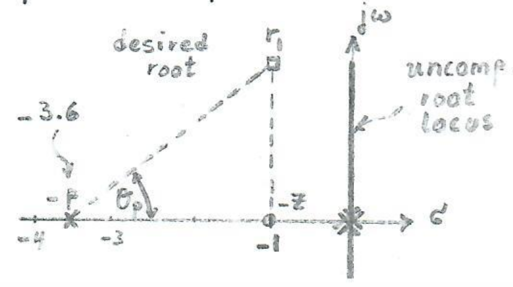
3. and 4. Now, we place the zero of the compensator directly below the desired location at $s = -z = -1$ as shown in the figure. Then, measuring the angle at the desired root, we have

$$\phi = -2(116^\circ) + 90^\circ = -142^\circ$$

Therefore, in order to have a total of 180° at the desired root, we evaluate the angle from the undetermined pole, θ_p, as

$$-180^\circ = -142^\circ - \theta_p \quad \Rightarrow \theta_p = 38^\circ$$

Then a line is drawn at an angle $\theta_p = 38^\circ$ intersecting the desired root and real axis.



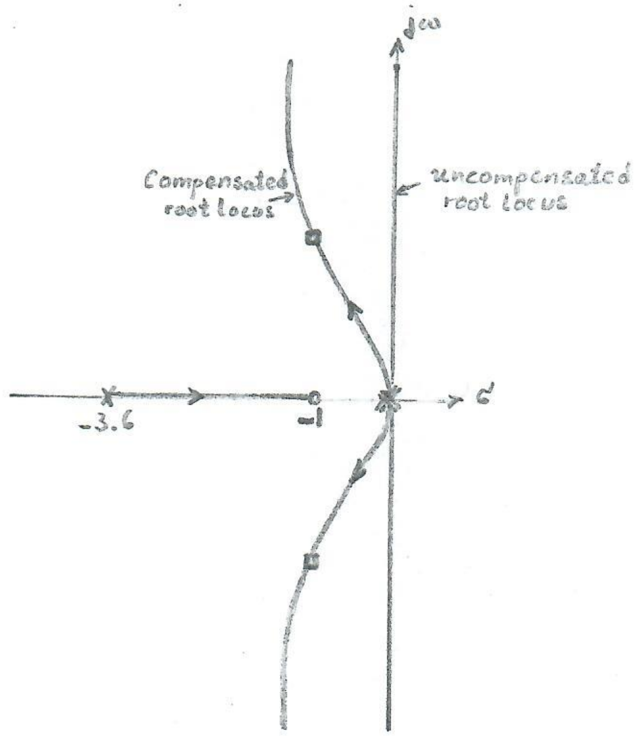
Therefore, The compensator is

$$G_c(s) = \frac{s+1}{s+3.6}$$

and the compensated Transfer function for the system is

$$G_p(s)\,G_c(s) = \frac{K_1\,(s+1)}{s^2\,(s+3.6)}$$

The root locus of the compensated system is also drawn below.



5. The gain K_1 is evaluated by measuring the vector lengths from the poles and zeros to the root location
Hence $$K_1 = \frac{(2.23)^2\,(3.25)}{2} = 8.1$$

The acceleration constant is found to be

$$K_a = \frac{8.1}{3.6} = 2.25$$

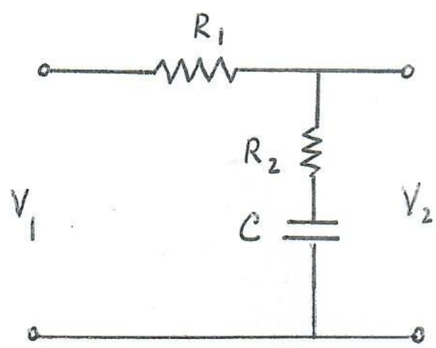
3. lag Compensation based on the root locus Technique

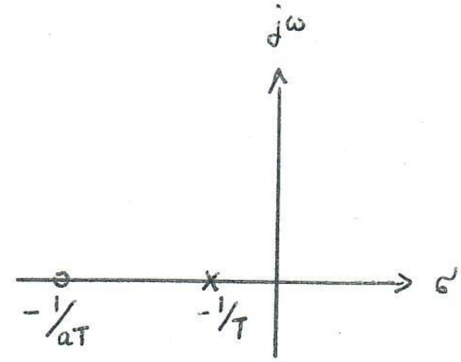
Phase lag controller :

The compensator transfer function as we know has the following form

$$G_c(s) = \frac{s + z_1}{s + p_1}$$

The controller is low pass or phase lag if $z_1 > p_1$.
A network realization of the phase lag controller along with its pole zero configuration is given below.





$$\frac{V_2}{V_1} = \frac{1 + R_2 C s}{1 + (R_1 + R_2) C s}$$

let $a = \frac{R_2}{R_1 + R_2} < 1$, $T = \frac{R_2 C}{a}$

Then $$\frac{V_2}{V_1} = \frac{1 + aTs}{1 + Ts} = a \cdot \frac{s + 1/aT}{s + 1/T}$$

we take care of this factor by using proportional controller constant K_p.

Method of Design :

1. Sketch the root loci of the characteristic equation of the uncompensated system.
2. Determine on these root loci where the desired eigenvalues should be located to achieve the desired relative stability of the system. Find the value of K that corresponds to these eigenvalues.
3. Compare the value of K required for steady state performance and the K found in the last step. The ratio of these two K's is a $(a<1)$, which is the ratio between the pole and the zero of the phase lag controller.
4. the exact value of T is not critical as long as it is relatively large. We may choose the value of $1/aT$ to be many orders of magnitudes smaller than the smallest pole of the process transfer function.

Example :

consider the uncompensated open loop Transfer function of a system to be

$$G_p(s) = \frac{K}{s(s+2)}$$

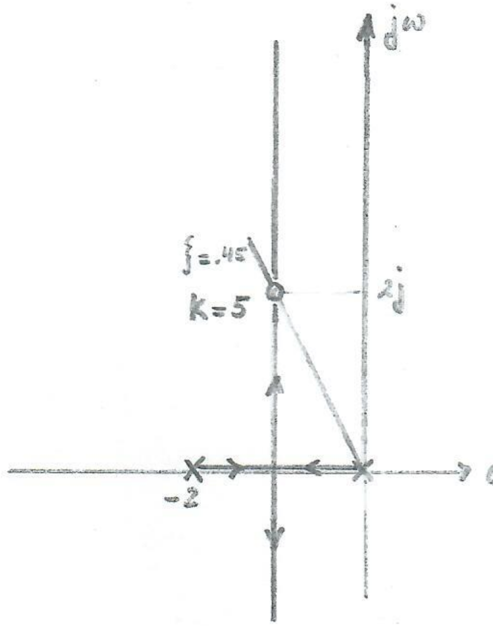
it is required that the damping ratio of the dominant complex roots is 0.45, while a system velocity constant equal to 20 is attained.

The uncompensated root locus is a vertical line at $s = -1$ and results in a root on the $\zeta = 0.45$ line at $s = -1 \pm j2$ as shown in the figure below. Measuring the gain at this root, we have :

$$K = (2.24)^2 = 5 .$$

Therefore the velocity constant of the uncompensated system is

$$K_v = \frac{K}{2} = \frac{5}{2} = 2.5$$



Thus The ratio of the zro to the pole of the compensator is

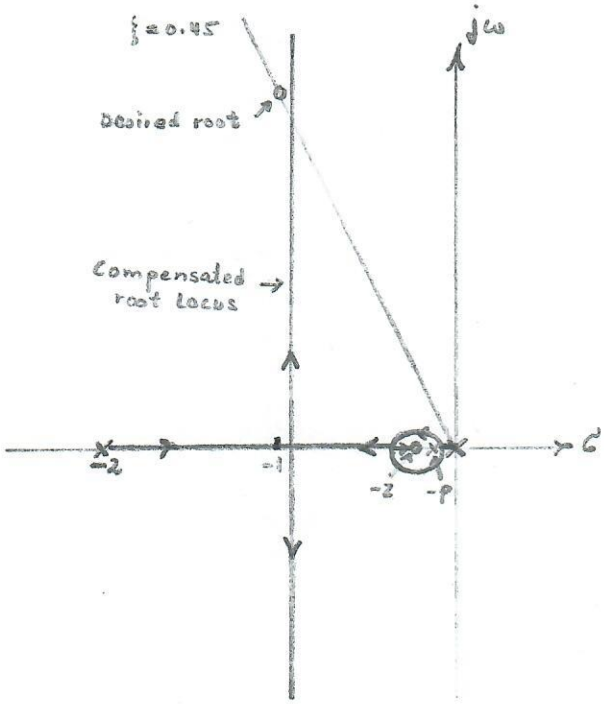
$$\left| \frac{z}{p} \right| = 1/a = \frac{K_{v_{comp}}}{K_{v_{uncomp}}} = \frac{20}{2.5} = 8$$

according to the procedure we might set $z = -0.1$ and then $p = -0.1/8$. The difference of the angles from p and z at the desired root is approximately one degree, and therefore, $s = -1 \pm j2$

is still the location of the dominant roots.
A sketch of the compensated root locus is shown in figure below. The compensated Transfer function is

$$G_c(s)\ G_p(s) = \frac{5(s+0.1)}{s(s+2)(s+0.0125)}$$

where $\frac{K}{a} = 5$ or $K = 40$ in order to account for the attenuation of the lag network.



Note:

In both cases it is easy to evaluate the network elements as soon as the pole and zero of the compensated network are found. Then the phase lead or phase lag network can be realized.

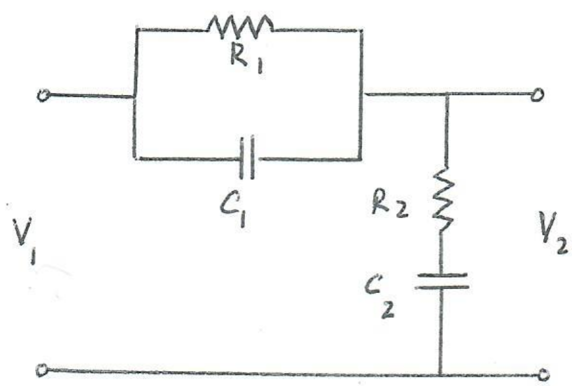
4. Lag - lead compensation based on the root locus Technique.

lag-lead controller :

The compensator Transfer function is given by

$$G_c(s) = \underbrace{\frac{1+aT_1 s}{1+T_1 s}}_{\text{lead}} \cdot \underbrace{\frac{1+bT_2 s}{1+T_2 s}}_{\text{lag}} \qquad a>1 \quad b<1$$

The network realization would be



$$G_c(s) = \frac{V_2}{V_1} = \frac{(1+R_1C_1 s)(1+R_2C_2 s)}{1+(R_1C_1+R_1C_2+R_2C_2)s+R_1R_2C_1C_2 s^2}$$

By comarison we have

$$aT_1 = R_1C_1$$
$$bT_2 = R_2C_2$$
$$T_1T_2 = R_1R_2C_1C_2$$

From the above equation we get $ab = 1$ and it means that a and b cannot be specified in-dependently.

Example :

As an example consider a process with the specifications given below.

$$G_p(s) = \frac{K}{s(1+0.1s)(1+0.2s)}$$

$$K_v = 100$$

$$\zeta = 0.707$$

A phase lag controller design for this example gives the following results :

For $K_v = 100$, $K = 100$ which corresponds to an unstable system. However, when $K = 1.63$, the uncompensated characteristic equation roots are at -11.118 , $-1.91 + j1.91$ and $-1.91 - j1.91$ which corresponds to a relative damping ratio of 0.707, as desired for ζ specification. using the design procedure we get $b = \frac{1.63}{100} = 0.0163$

let T to be arbitrarily large at 100 , Then the phase lag controller is given by

$$G_c(s) = \frac{1+1.635s}{1+100s}$$

The open loop transfer function of the compensated system is

$$G(s) = G_c(s)\,G_p(s) = \frac{1.63K(s+0.613)}{s(s+5)(s+10)(s+0.01)}$$

Where $K = 100$.

Of course, the root locus of uncompensated and compensated system can be drawn. When $k = 100$ the roots of compensated characteristic equation are at -11.13, -1.198, $-1.341 \pm j1.397$. The relative damping ratio of the complex root is slightly less than 0.7. This can be improved by selecting a smaller value for a or a larger T.

A phase lag-lead controller for this example would have the following transfer function

$$G_c(s) = \left(\frac{1+aT_1 s}{1+T_1 s}\right)\cdot\left(\frac{1+1.635s}{1+100s}\right)$$

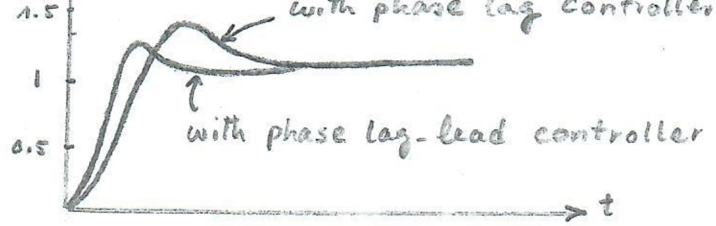
Let us choose $a = 10$ and $T_1 = 0.01$ Then the open loop transfer function of the compensated system is

$$G(s) = \frac{1630\,(s+0.6135)\,(s+10)}{s(s+5)(s+10)(s+100)(s+0.01)}$$

The characteristic equation roots are :

-100.2, -10, -0.761, $-2.04+j2.993$ and $-2.04-j2.993$

$\therefore$ Figure below shows the unit step response of the system with the lag controller and lag-lead controller. It is clear that the system with lag-lead controller has an improved rise time and overshoot over the system with the phase lag controller.



Design of Feedback Controller for Linear Systems Represented by state equation

- State Feedback
- static output Feedback
- Dynamic output Feedback

We only consider this case.

Other more advanced techniques can be employed.

Design of State Feedback Controller for linear Systems Represented by

$$\dot{x} = Ax + Bu \qquad \text{open-loop system}$$

$$u = V + KX \qquad \text{Control law}$$

$$\dot{x} = (A+BK)X + BV \qquad \text{Closed-loop System}$$

Suppose A is unstable and one would like to stabilize the systems with the above control law.

Assumption: The system must be controllable i.e

$$\rho[U] = \rho[B \; AB \; \ldots \; A^{n-1}B] = n$$

Then, one can find K such that $A+BK$ is stable, i.e.

$$\det[\lambda I - (A+BK)] \stackrel{!}{=} \Delta_d(\lambda)$$

closed-loop characteric polynomial   Desired characteristic polynomial

This leads to coefficient matching, which is not desirable due to nonlinear equations that need to be solved.

Systematic Methods

Indirect Method $\qquad\qquad$ Direct Method

1. <u>Indirect Method for SISO systems</u>

Indirect Method is based on transformation to controllable canonical form.

<u>Example</u>:

Consider $\dot{x} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u$

$$\det(\lambda I - A) = 0 \Rightarrow \lambda^3 - 9\lambda + 2$$

$\alpha_1 \rightarrow (0)\lambda^2 \qquad \alpha_2 \qquad \alpha_3$

$$P^{-1} = U\bar{U}^{-1} = [B \; AB \; A^2B] \begin{bmatrix} \alpha_2 & \alpha_1 & 1 \\ \alpha_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 16 \\ 1 & 6 & 8 \\ 1 & 2 & 12 \end{bmatrix} \begin{bmatrix} -9 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 & 2 \\ -1 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\dot{\bar{X}} = \bar{A}\bar{X} + \bar{B}u \qquad P = Q^{-1} = \begin{bmatrix} -0.125 & 0 & 0.25 \\ -0.125 & 0.25 & 0 \\ 0.625 & -0.5 & 0.25 \end{bmatrix}$$

$$\bar{A} = PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 9 & 0 \end{bmatrix}, \quad \bar{B} = PB = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Desired $\lambda_i = -1, -2, -3$

$$\bar{A} + \bar{B}\bar{K} = \bar{A}_d \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2+\bar{k}_1 & 9+\bar{k}_2 & \bar{k}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$\bar{k}_1 = -4$, $\bar{k}_2 = -20$, $\bar{k}_3 = -6 \qquad \bar{K} = [-4 \;\; -20 \;\; -6]$

$$K = \bar{K}P = [-0.75 \;\; -2 \;\; -2.5] \checkmark$$

2. Direct Methods

A. Ackermann Formula

SISO System

$$\dot{x} = Ax + Bu \qquad B \in R^{n \times 1}$$

$$u = v + Kx \quad : \quad K = [k_1 \ k_2 \cdots k_n]$$

$$\dot{x} = (A + BK)x + Bv$$

$$K = -[0 \ 0 \ \dots \ 1] \ U^{-1} p(A)$$

where

$U = [B \ AB \ \dots \ A^{n-1}B]$ controllability matrix

$$p(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1} A + \alpha_n I$$

$p(A)$ is a polynomial matrix constructed by the desired characteristic polynomial

$$\Delta_d(\lambda) = \det(\lambda I - A_d) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n$$

Ex.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{Given } \lambda_{1d}, \lambda_{2d} : -1 \pm j$$

$$\Delta_d(\lambda) = \lambda^2 + 2\lambda + 2$$

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad p(A) = A^2 + 2A + 2I = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$$

$$K = -[0 \ \ 1] \ U^{-1} p(A) = [-2 \ \ -2]$$

$A + BK$

B. Entire Eigenstructure Assignment

$\dot{x} = Ax + Bu$ system

$u = r + Kx$ Control law

$\dot{x} = (A + BK)x + Br$ closed-loop system

$\sigma(A + BK) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ Desired Spectrum

$v(A + BK) = \{v_1, v_2, \ldots, v_n\}$ Associated eigenvectors

$(A + BK)v_i = \lambda_i v_i$; $(A - \lambda_i I)v_i + \underbrace{BKv_i}_{q_i} = 0$

$$\underbrace{[A - \lambda_i I \quad B]}_{S(\lambda_i)} \begin{bmatrix} v_i \\ q_i \end{bmatrix} = 0 , \quad q_i = K v_i \quad \forall i = 1, \ldots, n$$

$[v_i^T \; q_i^T]^T$ must lie in the kernel or nullspace of $S(\lambda_i)$.

Construct the vectors $[v_i^T \; q_i^T]^T$ for all $i = 1, \ldots, n$; then form

$$[q_1 \; q_2 \; \cdots \; q_n] = [Kv_1 \; Kv_2 \; \cdots \; Kv_n]$$

$$\Rightarrow \quad K = [q_1 \; q_2 \; \cdots \; q_n][v_1 \; v_2 \; \cdots \; v_n]^{-1}$$

C. Optimal LQR Control

$$\dot{x} = Ax + Bu \qquad , \; x(t_0) = x_0$$

$$J = \int_{t_0}^{\infty} (x^T Q x + u^T R u) \, dt$$

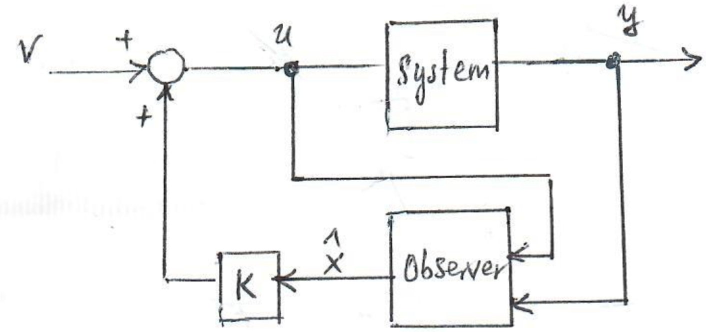
$$u_{opt}(t) = \underbrace{-R^{-1} B^T P}_{K_{opt}} \, x(t)$$

where

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

Observer Design

If the states of the system is not available for implementation, then one can use the Luenberger observer to estimate the states. Then the feedback control law can be applied.



System : $\dot{x} = Ax + Bu$ $\qquad$ $\{A, C\}$ must be observable

$y = Cx$

Luenberger Observer

$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu$$

Error Dynamics

$$e = \hat{x} - x$$

$$\dot{e} = \dot{\hat{x}} - \dot{x} = (A - LC)\hat{x} + Ly + \cancel{Bu} - Ax - \cancel{Bu}$$

$$= (A - LC)e$$

choose L such that $A - LC$ is stable.

$$\lim_{t \to \infty} e(t) \to 0$$

It can be shown that the overall observer-based state feedback system with $u = V + K\hat{x}$ is stable.