Additional Lecture Notes
for
Robot Mechanics and Control ME 5250

Mathematical Analysis

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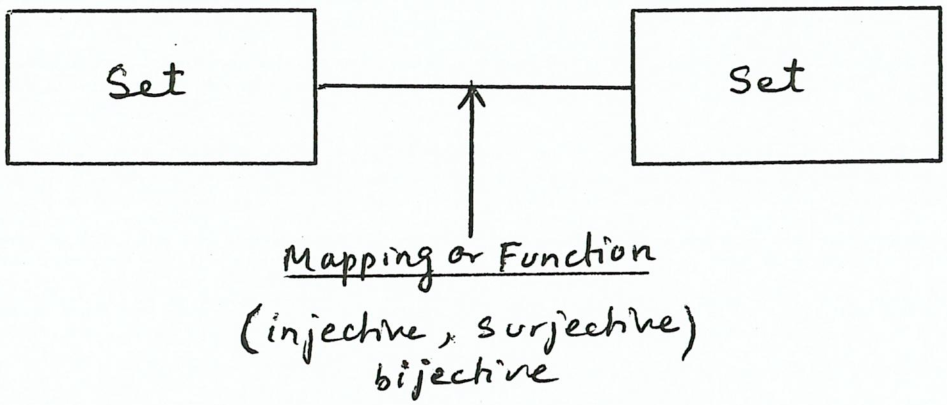
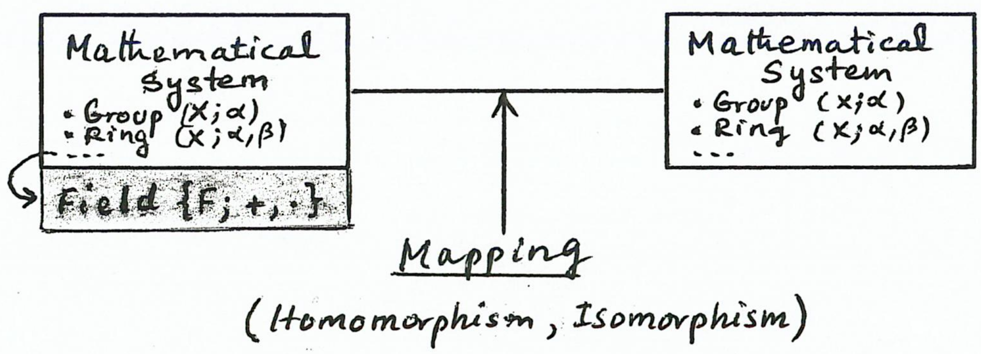
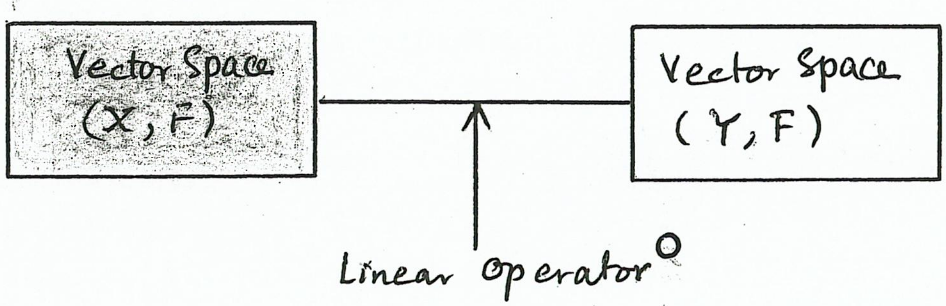
Summary: After learning fundamental concepts of Set, Mathematical systems, and composite Mathematical systems with their properties, one can relate them as follows:



Mathematical Systems or Algebraic Structures



Composite Mathematical Systems



Representations:

▨ Vector Space $\Longrightarrow$ Vector

○ Linear Operator $\Longrightarrow$ Matrix

Field : A field consists of a set, denoted by F, of elements called **scalars** and two operations "$+$" and "$\cdot$", written as $\{F;+,\cdot\}$, such that the following conditions are satisfied:

1. To every pair of elements $\alpha,\beta \in F$, there is an element $(\alpha+\beta)\in F$ called the sum and an element $(\alpha\cdot\beta)\in F$ called product of α and β.
2. Addition and Multiplication are commutative
$$\alpha+\beta=\beta+\alpha \quad , \quad \alpha\cdot\beta=\beta\cdot\alpha$$
3. Addition and Multiplication are associative
$$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma) \quad , \quad (\alpha\cdot\beta)\cdot\gamma=\alpha\cdot(\beta\cdot\gamma)$$
4. Multiplication is distributive with respect to addition
$$\alpha\cdot(\beta+\gamma)=(\alpha\cdot\beta)+(\alpha\cdot\gamma)$$
5. F contains an element 0 and an element 1 such that
$$\alpha+0=\alpha \quad , \quad 1\cdot\alpha=\alpha$$
6. To every $\alpha\in F$, there are $\beta,\gamma\in F$ such that
$$\alpha+\beta=0 \quad \text{Additive Inverse } \beta \quad , \quad \alpha\cdot\gamma=1 \; (\alpha\neq 0) \quad \text{Multiplicative Inverse } \gamma$$

EX. $\{R;+,\cdot\}$: Field of real numbers, $\{C;+,\cdot\}$: Field of Complex numbers.

EX. Consider the set $X=\{0,1\}$. Then X does not form a field if we use the usual definition of addition and multiplication, because the element $1+1=2\notin X$. However, if we define $0+0=0$, $1+1=0$, $1+0=1$ and $0.1=0.0=0$, $1.1=1$, then X forms a field. It is called the field of binary numbers.

Vector Space: A vector space over a field, denoted by (X, F), consists of a set, denoted by X, of elements called vectors, a field F, and two operations called vector addition and scalar multiplication such that the following conditions are satisfied:

1. To every pair of vectors $x_1, x_2 \in X$, there is a vector $(x_1 + x_2) \in X$ and to every $\alpha \in F$, and every $x \in X$, there is a vector $\alpha x \in X$
2. Addition is commutative $x_1 + x_2 = x_2 + x_1$
3. Addition is associative $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$
4. Scalar multiplication is associative $\alpha(\beta x) = (\alpha\beta)x$ and it is distributive with respect to vector addition $\alpha(x_1 + x_2) = \alpha x_1 + \alpha x_2$.
5. X contains an element 0 and F contains an element 1 such that $0 + x = x$ and $1x = x$.
6. To every $x \in X$, there is a vector $\bar{x} \in X$ such that $x + \bar{x} = 0$

EX. A field forms a vector space over itself with the vector addition and scalar multiplication defined as the corresponding operations in the field. For example (R, R), (C, C) are vector spaces.

EX. Given a field F, let F^n be all n-tuples of scalars written as columns

$$x_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{bmatrix}$$

to denote different vectors in F^n. If the vector addition and scalar multiplication are defined in the following way

$$x_i + x_j = \begin{bmatrix} x_{1i} + x_{1j} \\ x_{2i} + x_{2j} \\ \vdots \\ x_{ni} + x_{nj} \end{bmatrix} \qquad \alpha x_i = \begin{bmatrix} \alpha x_{1i} \\ \alpha x_{2i} \\ \vdots \\ \alpha x_{ni} \end{bmatrix}$$

Then (F^n, F) is a vector space.

If $F = R$, then (R^n, R) is called the n-dimensional real vector space, and if $F = C$, then (C^n, C) is called the n-dimensional complex vector space.

EX. The set of all solution (x, y) to the equation $ax + by = 0$, where $a, b, x, y \in R$, forms a vector space over R if both addition and scalar multiplication are done componentwise i.e. in usual sense.

<u>Subspace</u> : Let (X, F) be a vector space and let Y be a subset of X. Then (Y, F) is said to be a subspace of (X, F) if under the operations of (X, F), Y itself forms a vector space over F.

Ex. In the two-dimensional real vector space (R^2, R), every straight line passing through the origin is a subspace of (R^2, R).

Ex. (R^n, R) is a subspace of (C^n, C)

Ex. Consider again (R^2, R) then $X = \{(x, 0) : x \in R\}$ forms a subspace over R.

<u>Linear Independence</u> : A set of vectors $x_1, x_2, \dots, x_n$ in a linear vector space over a field, (X, F), is said to be <u>linearly dependent</u> if and only if there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ in F, <u>not all zero</u>, such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$$

If the only set of α_i is zero, then the set of vectors x_i is said to be linearly independent

Ex. Obviously the set of three vectors $(1\ 1\ 1)'$, $(1\ {-1}\ 0)'$, $(1\ 0\ 0)'$ is linearly independent.

<u>Dimension</u> : The maximal number of linearly independent vectors in a linear vector space (X, F) is called the dimension of the vector space.

<u>Basis</u> : A set of linearly independent vectors of a linear vector space (X, F) is said to be a basis of X if every vector in X can be expressed as a unique linear combination of these vectors.

<u>Theorem</u> : In an n-dimensional vector space, any set of n linearly independent vectors qualifies as a basis.

In an n-dimensional vector space (X, F), if a basis $\{e_1, e_2, \dots e_n\}$ is chosen, then every vector x in X can be uniquely written in the following form

$$x = [e_1\ e_2\ \cdots\ e_n] \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}}_{\triangleq \beta}$$

where β is called the <u>representation</u> of x with respect to the basis.

Change of Basis : Let the representations of a vector x in $(\mathcal{X}, F)$ with respect to $[e_1 \; e_2 \; \cdots \; e_n]$ and $[\bar{e}_1 \; \bar{e}_2 \; \cdots \; \bar{e}_n]$ be β and $\bar{\beta}$, respectively ; that is

$$x = [e_1 \; e_2 \; \cdots \; e_n]\,\beta = [\bar{e}_1 \; \bar{e}_2 \; \cdots \; \bar{e}_n]\,\bar{\beta}$$

and let

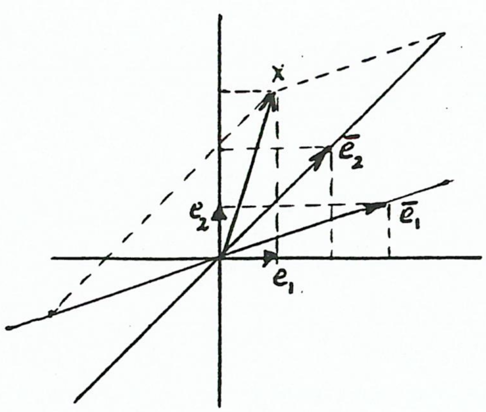
$$e_i = [\bar{e}_1 \; \bar{e}_2 \; \cdots \; \bar{e}_n] \begin{bmatrix} p_{1i} \\ p_{2i} \\ \vdots \\ p_{ni} \end{bmatrix}$$

Then

$$x = [\bar{e}_1 \; \bar{e}_2 \; \cdots \; \bar{e}_n]\, P\beta \overset{!}{=} [\bar{e}_1 \; \bar{e}_2 \; \cdots \; \bar{e}_n]\,\bar{\beta}$$

$$\Rightarrow \quad \boxed{\bar{\beta} = P\beta} \qquad P = \begin{bmatrix} i\text{th column is the} \\ \text{representation of} \\ e_i \text{ with respect to} \\ [\bar{e}_1 \; \bar{e}_2 \; \cdots \; \bar{e}_n] \end{bmatrix}$$

EX.



The representations of x with respect to the bases $[e_1 \; e_2]$ and $[\bar{e}_1 \; \bar{e}_2]$ are given by the figure as

$$x = [e_1 \;\; e_2] \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and

$$x = [\bar{e}_1 \;\; \bar{e}_2] \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

One can construct P, which relates β and $\bar{\beta}$, as follows

$$\bar{e}_1 = [e_1 \;\; e_2] \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad , \quad \bar{e}_2 = [e_1 \;\; e_2] \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$P^{-1} = Q = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1/2 & -1/2 \\ -1/4 & 3/4 \end{bmatrix}$$

Linear Operators and Their Representations

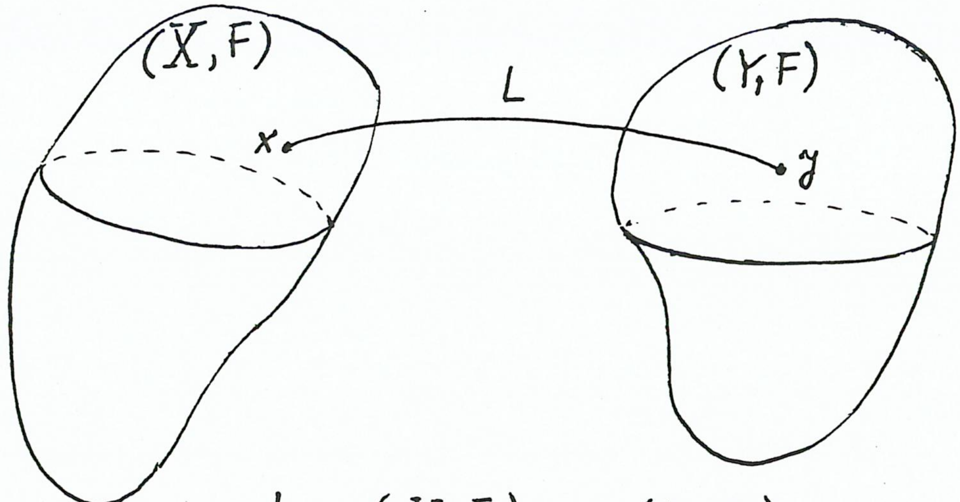
Linear Operator : A function L that maps (X, F) into (Y, F) is said to be a linear operator if and only if

$$L(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 L x_1 + \alpha_2 L x_2$$

for any vectors $x_1, x_2 \in X$ and any scalars $\alpha_1, \alpha_2 \in F$.

Remark:

Similar to the mapping concept between two sets, we may also consider various types of mapping (e.g., homomorphism and isomorphism etc.) between two mathematical systems



$$L : (X, F) \longrightarrow (Y, F)$$

Matrix Representation of a Linear Operator

Theorem: Let (X, F) and (Y, F) be n- and m-dimensional vector spaces, respectively, over the same field. Let $x_1, x_2, \ldots, x_n$ be a set of linearly independent vectors in X. Then the linear operator $L : (X, F) \rightarrow (Y, F)$ is uniquely determined by the n pairs of mappings $y_i = L x_i$, for $i = 1, 2, \ldots, n$. Furthermore, with respect to the basis $[x_1\ x_2 \cdots x_n]$ of X and a basis $[u_1\ u_2 \cdots u_m]$ of Y, L can be represented by an $m \times n$ matrix A with its coefficients in the field F. The ith column of A is the representation of y_i with respect to the basis $[u_1\ u_2 \cdots u_m]$.

Proof : Let x be an arbitrary vector in X. Since $x_1, x_2, \ldots, x_n$ are linearly independent, the set of vectors qualifies as a basis. Consequently

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

and by linearity

$$\begin{aligned} Lx &= \alpha_1 L x_1 + \alpha_2 L x_2 + \cdots + \alpha_n L x_n \\ &= \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n \end{aligned}$$

which implies that for any $x \in X$, Lx is uniquely determined by $y_i = Lx_i$, for $i = 1, 2, \ldots, n$. This proves the first part of the theorem.

For the second part of the theorem we write

$$y_i = [u_1\ u_2 \cdots u_m] \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix} \qquad i = 1, 2, \ldots, n$$

and

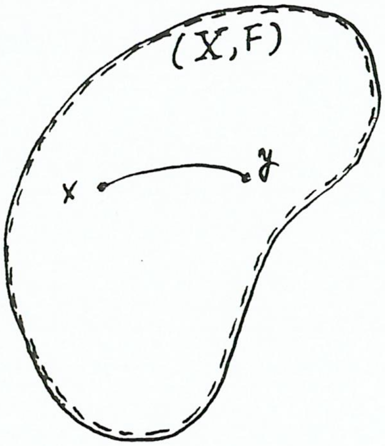
$$\underbrace{L[x_1\ x_2 \cdots x_n]}_{*} = [y_1\ y_2 \cdots y_n] = [u_1\ u_2 \cdots u_m]\, A$$

Now, let the representation of $y \in Y$ be β then $y = Lx$ can be written as

$$[u_1\ u_2 \cdots u_m]\, \beta = \underbrace{L\,[x_1\ x_2 \cdots x_n]}_{*}\, \alpha = [u_1\ u_2 \cdots u_m]\, A\alpha$$

$$\Rightarrow \quad \boxed{\beta = A\alpha}$$

<u>Special Case</u> $L : (X, F) \mapsto (X, F)$



Independent of Basis

<u>Basis</u>

$[e_1 \; e_2 \cdots e_n]$

$[\bar{e}_1 \; \bar{e}_2 \cdots \bar{e}_n]$

$$\begin{tikzcd}
x \arrow[r, "L"] \arrow[d, dashed] & y \; (= Lx) \arrow[d, dashed] \\
\alpha \arrow[r, "A"] \arrow[d, shift right, "P"'] & \beta \; (= A\alpha) \arrow[d, shift right, "P"'] \\
\bar{\alpha} \arrow[r, "\bar{A}"] \arrow[u, shift right, "Q"'] & \bar{\beta} \; (= \bar{A}\bar{\alpha}) \arrow[u, shift right, "Q = P^{-1}"']
\end{tikzcd}$$

$\therefore \quad A = \begin{bmatrix} i\text{th column is} \\ \text{the representation} \\ \text{of } Le_i \text{ with respect} \\ \text{to } [e_1 \; e_2 \cdots e_n] \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} i\text{th column is} \\ \text{the representation} \\ \text{of } L\bar{e}_i \text{ with respect} \\ \text{to } [\bar{e}_1 \; \bar{e}_2 \cdots \bar{e}_n] \end{bmatrix}, \quad P = \begin{bmatrix} i\text{th column is} \\ \text{the representation} \\ \text{of } e_i \text{ with respect} \\ \text{to } [\bar{e}_1 \; \bar{e}_2 \cdots \bar{e}_n] \end{bmatrix}$

Relationship :

$$\bar{\beta} = \bar{A}\bar{\alpha}$$

$$P\beta = \bar{A} P\alpha$$

$$PA\alpha = \bar{A} P\alpha \quad \Longrightarrow \quad \boxed{\bar{A} = PAP^{-1}}$$

<u>Simple Case</u> $L = A$

Independent of Basis

<u>Basis</u>

Note : $An_i = a_i \quad \Leftarrow \quad [n_1 \; n_2 \cdots n_n]$

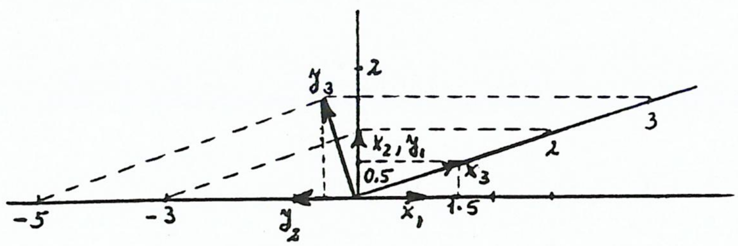
$[q_1 \; q_2 \cdots q_n]$

$$\begin{tikzcd}
x \arrow[r, "L=A"] \arrow[d, dashed] & y \; (= Ax) \arrow[d, dashed] \\
\alpha \arrow[r, "A"] & \beta \\
\bar{\alpha} \arrow[r] \arrow[u, "Q"] & \bar{\beta} \arrow[u, "Q"']
\end{tikzcd}$$

$\bar{A} = Q^{-1} A Q = \begin{bmatrix} i\text{th column is the} \\ \text{representation of} \\ Aq_i \text{ with respect to} \\ [q_1 \; q_2 \cdots q_n] \end{bmatrix}, \quad Q = [q_1 \; q_2 \cdots q_n]$

Ex. (Special Case)

Consider the transformation that rotates a point in a geometric plane counterclockwise 90° with respect to the origin as shown below. Given any two vectors in the plane, it is easy to verify that the vector that is the sum of the two vectors after rotation is equal to the rotation of the vector that is the sum of the two vectors before rotation. Hence the transformation is a linear transformation. The spaces (X, F) and (Y, F) of this example are all equal to (R^2, R).



If we choose $[x_1\ x_2]$ as a basis, then

$$y_1 = Lx_1 = [x_1\ x_2]\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad y_2 = Lx_2 = [x_1\ x_2]\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence the representation of L with respect to the basis $[x_1\ x_2]$ is

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The representation x_3 is $[1.5\ \ 0.5]'$. It is easy to verify that the representation of y_3 with respect to $[x_1\ x_2]$ is given by

$$\beta = A\alpha = [-0.5\ \ 1.5]'$$

If instead of $[x_1\ x_2]$ we choose $[x_1\ x_3]$ as a basis, then

$$y_1 = Lx_1 = [x_1\ x_3]\begin{bmatrix} -3 \\ 2 \end{bmatrix} \quad \text{and} \quad y_3 = Lx_3 = [x_1\ x_3]\begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

and

$$\bar{A} = \begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix} \qquad \text{check} \quad \bar{A} = PAP^{-1}$$