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Additional Lecture Notes

On

Jacobian

"Jacobian"

* Infinitesimal rotation about three axes

$$d\phi = \begin{bmatrix} d\phi_x \\ d\phi_y \\ d\phi_z \end{bmatrix}$$

→ Denote the infinitesimal end-effector translation by dX_e with respect to the base

→ Denote the infinitesimal end-effector rotation by $d\phi_e$ with respect to the base

$$dp = \begin{bmatrix} dX_e \\ d\phi_e \end{bmatrix}$$

$$\dot{p} = \frac{dp}{dt} = \begin{bmatrix} \frac{dX_e}{dt} \\ \frac{d\phi_e}{dt} \end{bmatrix} = \begin{bmatrix} V_e \\ \omega_e \end{bmatrix}$$

Similar to the two degree-of freedom case, in general for n degree of freedom we have the end-effector velocity and angular velocity as a function of joint velocities as

$$\boxed{\dot{p} = J\dot{q}}$$, where J is the Jacobian

q : d or θ

$$J = \left[\begin{array}{c|c|c|c} J_{L1} & J_{L2} & \cdots & J_{Ln} \\ J_{A1} & J_{A2} & & J_{An} \end{array}\right]$$

$\longrightarrow$ <u>Linear Velocity</u> : $v_e = J_{L1}\dot{q}_1 + \cdots + J_{Ln}\dot{q}_n$

Prismatic Joint $\Rightarrow$ Linear Velocity at the End-effector

$\longrightarrow$ <u>Angular Velocity</u> : $\omega_e = J_{A1}\dot{q}_1 + \cdots + J_{An}\dot{q}_n$

Revolute Joint $\Rightarrow$ Angular velocity at the End-effector

How fast the end-effector is moving in x direction

How fast the end-effector is rotating arround X axis

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$\dot{y}, \dot{z}, \omega_y, \omega_z$ are similarly defined

Total number of joints

q : θ, d

θ : <u>Angular Displacement</u> Representing the rotation of a revolute Joint

d : <u>Linear Displacement</u> Representing the extension of a prismatic Joint

Jacobian is a matrix that relates

<u>End-effector Velocities to joint Velocities</u>

" J is a $6 \times n$ matrix "

$\dot{q}$: $\dot{\theta}, \dot{d}$

$\dot{\theta}$: <u>Angular velocity</u> rad/sec

$\dot{d}$: <u>Linear velocity</u> m/sec

How to compute Jacobian ?

$$\dot{p} = J\dot{q}$$

Define the augmented velocity vector : $\gamma = \dot{p} = \begin{bmatrix} \overset{v_e}{v_n^0} \\ \underset{\omega_e}{\omega_n^0} \end{bmatrix}$

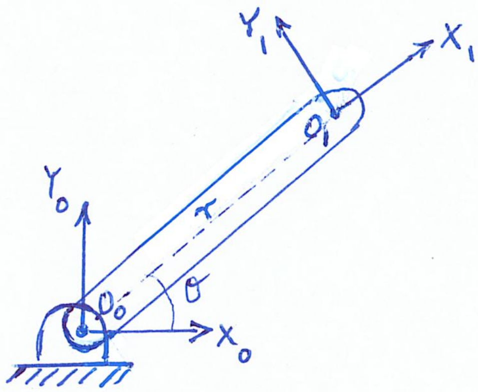
$$\gamma = \begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = \begin{bmatrix} \dot{x}_n^0 \\ \dot{y}_n^0 \\ \dot{z}_n^0 \\ \hline \omega_{x_n}^0 \\ \omega_{y_n}^0 \\ \omega_{z_n}^0 \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}$$

$\dot{q}$: $\dot{\theta}$ and/or $\dot{d}$

$v_n^0 = J_v \dot{q}$

$\omega_n^0 = J_\omega \dot{q}$

A simple observation



Think of linear velocity of O_1

$v = \omega \times r$

$\omega = \dot{\theta}$ angular velocity

$\uparrow$

Rotational velocity of frame 1

v is the linear velocity of frame 1

Jacobian consists of two parts J_v and J_ω

Each part is also consists of two parts, which can be determined as follows.

(1) Rotational Part of Jacobian : J_ω

For Prismatic Joint : $J_\omega = 0$

For Revolute Joint : $J_\omega = R_i^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ← Rotation is around Z axis

Rotation matrix from frame 0 to frame i

(2) Linear Part of Jacobian : J_v

For Prismatic Joint : $J_{v_i} = R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For Revolute Joint :

Also denoted by d_n^0 and d_{i-1}^0

$$J_{v_i} = R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(O_n^0 - O_{i-1}^0 \right)$$

position of joint

Position of endeffector

where $a \times b$ denotes the cross product, which can be obtained in several different ways. The simplest way is through the determinant as follows:

$$\triangleright \quad a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\, i + (a_3 b_1 - a_1 b_3)\, j + (a_1 b_2 - a_2 b_1)\, k$$

Table for Computing J

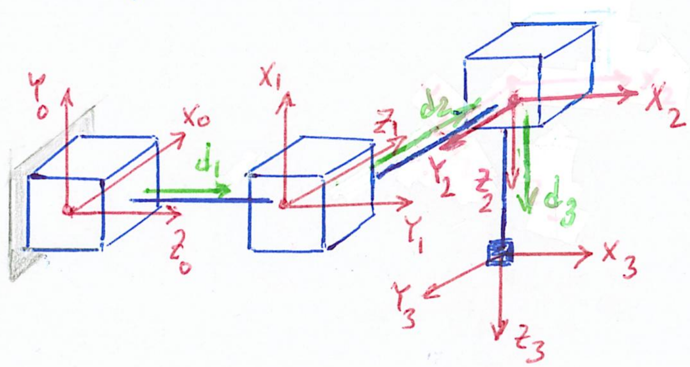
	Prismatic	Revolute
Linear	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}\times(d^0_n - d^0_{i-1})$
Rotational	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$

d^0_n means the distance from base to the end-effector

(d^0_n, d^0_{i-1}) or (O^0_n, O^0_{i-1})

O means w.r.t the origin of the coordinate system

Example 1: (All Prismatic)



$$R^0_0 = \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{I_3}, \quad R^0_1 = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

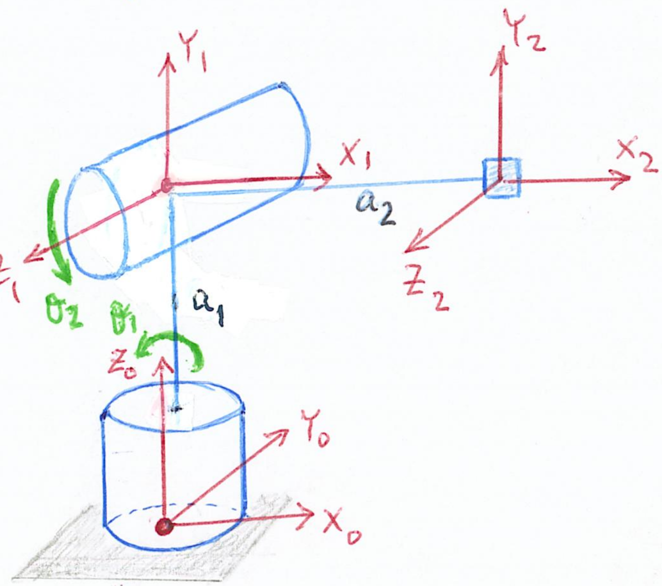
$$R^0_2 = R^0_1 R^1_2 = R^0_1\begin{bmatrix}0&0&-1\\1&0&0\\0&-1&0\end{bmatrix} = \begin{bmatrix}0&-1&0\\0&0&-1\\1&0&0\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{z}\\\hline \omega_x\\\omega_y\\\omega_z\end{bmatrix} = \left[\begin{array}{c|c|c} R^0_0\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_1\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_2\begin{bmatrix}0\\0\\1\end{bmatrix} \\ \hline 0&0&0\\0&0&0\\0&0&0\end{array}\right]\begin{bmatrix}\dot{d}_1\\\dot{d}_2\\\dot{d}_3\end{bmatrix} = \begin{bmatrix}0&1&0\\0&0&-1\\1&0&0\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\dot{d}_1\\\dot{d}_2\\\dot{d}_3\end{bmatrix}$$

$\Rightarrow$ $\dot{x} = \dot{d}_2, \; \dot{y} = -\dot{d}_3, \; \dot{z} = \dot{d}_1$ check the motion and verify ✓

$\omega_x = 0, \; \omega_y = 0, \; \omega_z = 0$

Example 2 : (All Revolute)



$$R_0^0 = I_3 \;,\quad R_1^0 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} X_1 & Y_1 & Z_1 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} X_0 \\ Y_0 \\ Z_0 \end{array} = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 \\ s\theta_1 & 0 & -c\theta_1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_2^0 = R_1^0 R_2^1 = R_1^0 \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Note } R_2^1 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Since } X_1, Y_1, Z_1 \\ \text{and } X_2, Y_2, Z_2 \\ \text{are parallel}}}$$

Homogeneous Transformations :

$$A_1^0 = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & 0 \\ s\theta_1 & 0 & -c\theta_1 & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} d_1^0 \;,\quad A_2^1 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} d_2^1 \;,\quad A_2^0 = A_1^0 A_2^1 = \begin{bmatrix} R_2^0 & d_2^0 \\ 0 & 1 \end{bmatrix}$$

Jacobian :

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \overset{J \Downarrow}{\begin{bmatrix} R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_2^0 - d_0^0) & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_2^0 - d_1^0) \\ R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

The rest is calculations and left as an exercize.

$$
\begin{array}{cc}
\quad\quad R_2^0 & d_2^0 \\
\multicolumn{2}{c}{
A_2^0 = \left[\begin{array}{ccc|c}
c\theta_1 c\theta_2 & -c\theta_1 s\theta_2 & s\theta_1 & a_2 c\theta_1 c\theta_2 \\
s\theta_1 c\theta_2 & -s\theta_1 s\theta_2 & -c\theta_1 & a_2 s\theta_1 c\theta_2 \\
0 & c\theta_2 & 0 & a_2 s\theta_2 + a_1 \\
\hline
0 & 0 & 0 & 1
\end{array}\right]}
\end{array}
$$

$$
J = \begin{bmatrix}
-a_2 s\theta_1 c\theta_2 & -a_2 c\theta_1 s\theta_2 \\
a_2 c\theta_1 c\theta_2 & -a_2 s\theta_1 s\theta_2 \\
0 & 2a_2 c\theta_2 \\
0 & s\theta_1 \\
0 & -c\theta_1 \\
1 & 0
\end{bmatrix}
$$

Further Discussion :

$$\dot{x} = -a_2 s\theta_1 c\theta_2 \dot{\theta}_1 - a_2 c\theta_1 s\theta_2 \dot{\theta}_2$$

$$\dot{y} = a_2 c\theta_1 c\theta_2 \dot{\theta}_1 - a_2 s\theta_1 s\theta_2 \dot{\theta}_2$$

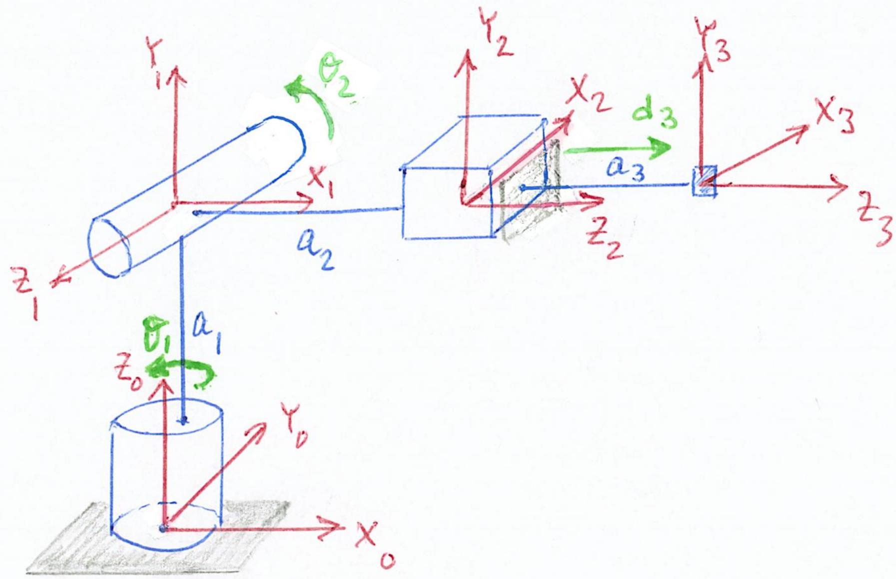
$$\dot{z} = a_2 c\theta_2 \dot{\theta}_2$$

$$\omega_x = s\theta_1 \dot{\theta}_2$$

$$\omega_y = -c\theta_1 \dot{\theta}_2$$

$$\omega_z = \dot{\theta}_1$$

Example 3: (Revolute and Prismatic)



Similar to example 2 with additional Prismatic joint.

Forward Kinematics : Homogeneous Transformations

$$A_1^0 = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & 0 \\ s\theta_1 & 0 & -c\theta_1 & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_2^1 = \begin{bmatrix} 0 & -s\theta_2 & c\theta_2 & a_2 c\theta_2 \\ 0 & c\theta_2 & s\theta_2 & a_2 s\theta_2 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_3+d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_0^0 = I_3$, R_1^0, R_2^1 and R_3^2 are easily identified from A_1^0, A_2^1 and A_3^2. Similarly one can identify d_1^0, d_2^1 and d_3^2

$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$$; define for simplicity $k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$J_\omega = \left[R_0^0 k \,\vdots\, R_1^0 k \,\vdots\, \underbrace{R_2^0 k}_{\text{prismatic}} \right] = \begin{bmatrix} 0 & s\theta_1 & c\theta_1 c\theta_2 \\ 0 & -c\theta_1 & s\theta_1 c\theta_2 \\ 1 & 0 & s\theta_2 \end{bmatrix}$$

Correction: Insert 0's in Yellow Column

Use it for Jv

$$\underbrace{R_1^0 R_2^1}_{\checkmark\, R_2^0} k = \begin{bmatrix} c\theta_1 c\theta_2 \\ s\theta_1 c\theta_2 \\ s\theta_2 \end{bmatrix}$$

$$J_v = \left[R_0^0 k \times (d_3^0 - d_0^0) \,\vdots\, R_1^0 k \times (d_3^0 - d_1^0) \,\vdots\, \underbrace{R_2^0 k}_{\text{prismatic}} \right]$$ ✓

, where $d_3^0 = \begin{bmatrix} a_2 c\theta_2 \\ a_2 s\theta_2 \\ a_1 + a_3 + d_3 \end{bmatrix}$

Needs Correction

* The rest is simple substitutions and calculations