PART 2

An Overview of State Feeback Control Design for Second Order Mechanical Systems and Its Connection to Method of Computed Torque of Robotic Manipulators

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Basic Control Method
for Robotic Manipulators

Note that the dynamical equation for robotic manipulator is given by vector-Matrix differential equation

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau \qquad (1)$$

Control of such a system is not a trivial task. However, there is a simple method that has been used for this purpose. Note that the above equation can be written as follows

$$M(q)\ddot{q} = \tau - \underbrace{V(q,\dot{q}) - G(q)}_{\text{Non Linear Term}}$$

or

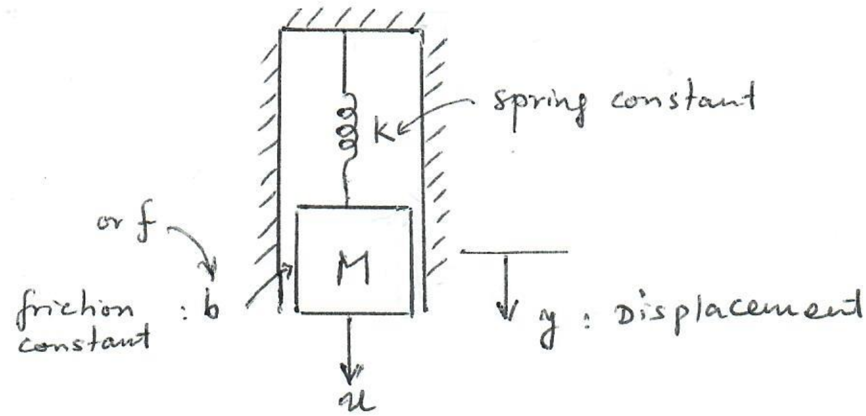
$$\ddot{q} = \underbrace{M^{-1}(q)\left[\tau - V(q,\dot{q}) - G(q)\right]}_{\text{Denote this by } \tau'}$$

Thus, we have

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \qquad \ddot{q} = \tau', \quad \text{where } q \text{ is a vector.} \qquad (2)$$

$$(\ddot{y} = u) \leftarrow \text{Recal Double Integrator System}$$

when q is scalar variable, the problem reduces to the control of a second order system. Consider the system



which can be described by the second order differential equation

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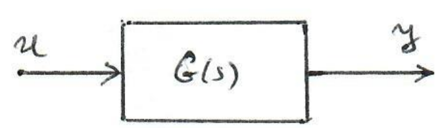
$$M\ddot{y} + b\dot{y} + ky = u \qquad (3)$$

So, we have a system with input u and output y.



where $G(s)$ represents the transfer function of the system, which can be obtained by taking the Laplace transform of (3), i.e.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ms^2 + bs + K} \qquad (4)$$

The poles of the system can be obtained by the roots of the denominator polynomial

$$Ms^2 + \overset{f}{b}s + k = 0$$

If the poles are in the left half of complex plane then the system is stable. On the other hand, if one or more poles are in the right half of complex plane, the system is unstable. Note that depending on the location of the poles, we have several defferent response shapes by applying an input signal. If the system is unstable, one can stabilize it by the so-called <u>state feedback control law</u> or some other control techniques. Even if the system is stable, one may improve the response shape by the same techniques.

The above differential equation can also be written as a set of two first order differential equation, known as state equation:

Define

$$y = x_1$$

$$\dot{y} = \dot{x}_1 = x_2$$

$$\ddot{y} = \dot{x}_2 = -\frac{k}{M}x_1 - \frac{b}{M}x_2 + \frac{1}{M}u$$

Then one can write the above equation in vector-matrix notation

f

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{b}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u \qquad \text{state equation}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{output equation} \tag{5}$$

The general notation for (5) is given by

$$\begin{cases} \dot{x} = Ax + Bu \\ y = cx \end{cases} \tag{6}$$

The poles or eigenvalues of the system can also be obtained by the roots of the <u>characteristic equation</u>

$$\det(\lambda I - A) = 0 \tag{7}$$

or f

<u>Example 1</u>: Let $M = 1$, $K = 2$, and $b = 3$. then we have

$$\det\left[\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\right] = \det\begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix}$$

$$= \lambda^2 + 3\lambda + 2 = 0 \implies \lambda_1 = -1,\ \lambda_2 = -2$$

Now, suppose we are interested to shift the eigenvalues (poles) to different locations in order to have faster response, for example $-3, -4$.

To achieve this goal, we can apply the state feedback control law to (6) as follows:

$$u = V + KX$$

$$\dot{X} = AX + B[V + KX]$$

$$\begin{cases} \dot{X} = (A + BK)X + BV \\ y = CX \end{cases} \quad \text{closed-loop system.} \qquad (8)$$

and we can set-up the following equation

$$\det[\lambda I - (\underset{\substack{\uparrow \\ \text{obtain}}}{A + BK})] \stackrel{!}{=} \underbrace{\Delta_d(\lambda)}_{\substack{\text{Desired} \\ \text{characteristic} \\ \text{equation}}} \qquad (9)$$

Since the desired eigenvalues are $-3, -4$, we have

$$\Delta_d(\lambda) = (\lambda + 3)(\lambda + 4) = \lambda^2 + 7\lambda + 12$$

and (9) reduces to

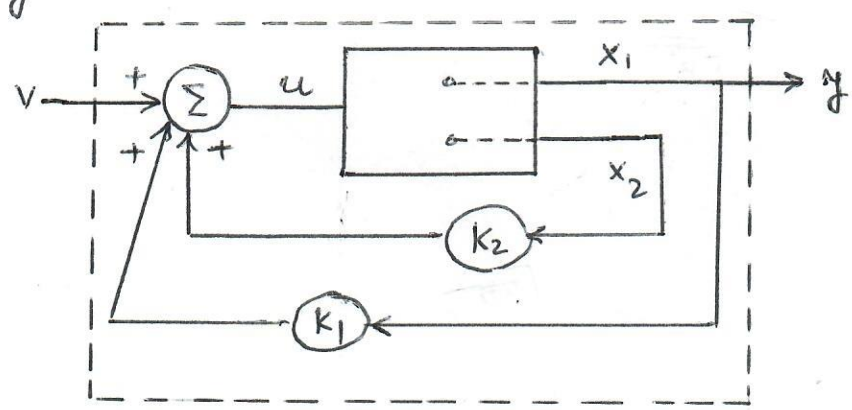
$$\det\left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}-\left[\begin{pmatrix}0 & 1\\ -2 & -3\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}(k_1\ \ k_2)\right]\right\} \stackrel{!}{=} \lambda^2+7\lambda+12$$

$$\det\begin{bmatrix}\lambda & -1\\ 2-k_1 & \lambda+3-k_2\end{bmatrix} \stackrel{!}{=} \lambda^2+7\lambda+12$$

$$\lambda^2+(3-k_2)\lambda+(2-k_1) \stackrel{!}{=} \lambda^2+7\lambda+12$$

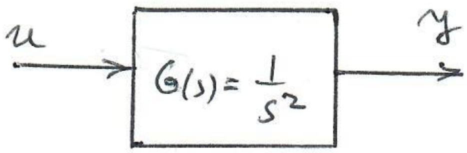
$$\begin{cases}3-k_2=7\\ 2-k_1=12\end{cases} \Rightarrow \quad \begin{matrix}k_1=-10\\ k_2=-4\end{matrix}$$

The closed-loop control system can be implemented by

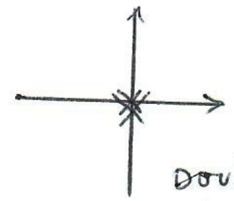


Next, we consider the second order system similar to the simplified equation of (2) when q is scalar variable.

Example 2 (Double Integrator Problem)



$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2}$$



Double poles at the origin "unstable"

$$s^2 Y(s) = U(s)$$

$$\ddot{y} = u$$

$$y = x_1$$

$$\dot{y} = \dot{x}_1 = x_2$$

$$\ddot{y} = \dot{x}_2 = u$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$\det(\lambda I - A) = 0 \Rightarrow \underbrace{\lambda_1 = \lambda_2 = 0}_{\text{eigenvalues (same as poles)}}$$

Let the desired eigenvalues to stabilize the system be -3 and -4, Then we have

$$\det[\lambda I - (A + BK)] \stackrel{!}{=} \Delta_d(\lambda)$$

$$\det\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(k_1 \; k_2)\right]\right\} \stackrel{!}{=} (\lambda + 3)(\lambda + 4)$$

$$\det\begin{bmatrix} \lambda & -1 \\ -k_1 & \lambda - k_2 \end{bmatrix} \stackrel{!}{=} \lambda^2 + 7\lambda + 12$$

$$\lambda^2 - k_2\lambda - k_1 \stackrel{!}{=} \lambda^2 + 7\lambda + 12 \Rightarrow \begin{array}{l} k_1 = -12 \\ k_2 = -7 \end{array}$$

Connection between Example 1 and 2 for better Understanding "the Method of Computed Torque" or Force Control u

Let us start again with example 1 and write (3) once more

Recall: If $M=1, k=2, b=3$; We get

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] x$$

$\det(\lambda I - A) = 0 \; \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -2 \end{cases}$

Desired eigenvalues $-3, -4$

Using $u = v + Kx$ leads $A + BK = A_d$ with $K = [-10 \quad -4]$

$$M\ddot{y} + b\dot{y} + ky = u \Rightarrow$$

which can also be written as

$$\Downarrow$$

$$\ddot{y} = \underbrace{-\frac{k}{M}y - \frac{b}{M}\dot{y} + \frac{u}{M}}_{\bar{u}}$$

$\ddot{y} = \bar{u}$ $\quad \frac{Y}{\bar{U}} = \frac{1}{s^2}$ $\quad \bar{u} = -2x_1 - 3x_2 + u$

$y = x_1$

$\dot{y} = \dot{x}_1 = x_2$

$\ddot{y} = \dot{x}_2 = \bar{u}$

$$\Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bar{u}$$

$$y = [1 \quad 0] x$$

$-4 \quad -3$

$$\bar{u} = v + \bar{K}x \quad \Rightarrow \quad \dot{x} = (A + B\bar{K})x + Bv$$

$\lambda_{1d} = -3 \quad \lambda_{2d} = -4$

$$A + B\bar{K} = \begin{bmatrix} 0 & 1 \\ \bar{k}_1 & \bar{k}_2 \end{bmatrix} \stackrel{!}{=} \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} = A_d \Leftarrow \lambda^2 + 7\lambda + 12 = 0$$

$\bar{k}_1 = -12$, $\bar{k}_2 = -7$ "As obtained in Example 2"

$$\bar{u} = v + \bar{k}_1 x_1 + \bar{k}_2 x_2 = v - 12x_1 - 7x_2$$

$$u = \bar{u} + 2x_1 + 3x_2 = v - 12x_1 - 7x_2 + 2x_1 + 3x_2$$

$$= v - 10x_1 - 4x_2$$

$k_1 = -10$, $k_2 = -4$ "As obtained in Example 1"

Conclusion: The method used to control manipulator governed by equation (1) or equivalently (2) is known as the method of computed torque. It is simply the generalization of what we showed above using state feedback control law. For more information on alternative derivation using control law partitioning, you should refer to the chapters 9 and 10 of your text-book. Both stabilization and trajectory-following control can be handeled by the same technique.