Additional Lecture Notes on

Position and Orientation of Rigid Body

and

Homogeneous Transformation

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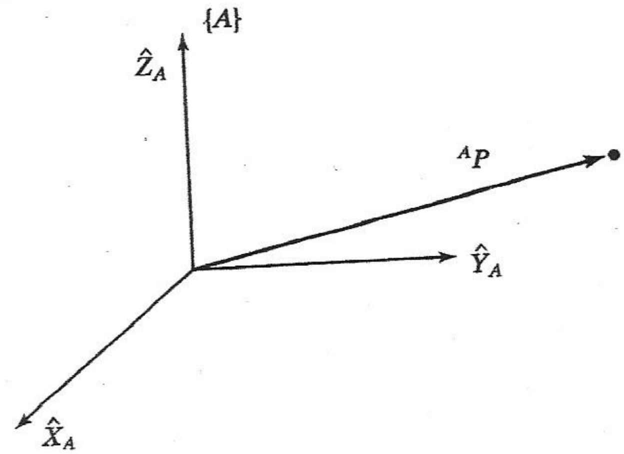
Description of Position, Orientation, and Frame

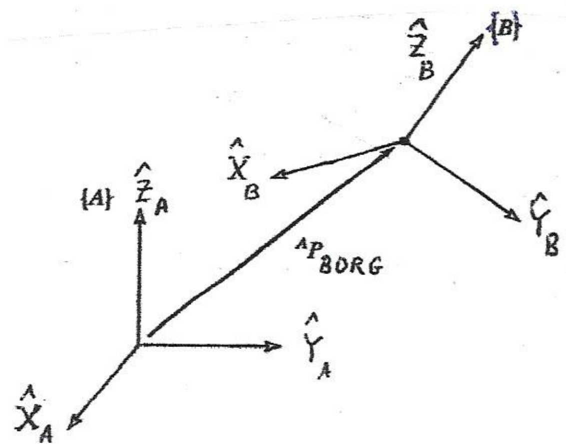
"Homogeneous Transformation"

1. Description of Position



$$^A P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

2. Description of Orientation



Rotation Matrix :

$$^A_B R = [\, ^A\hat{X}_B \;\; ^A\hat{Y}_B \;\; ^A\hat{Z}_B \,] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

We can give expressions for the scalars r_{ij} by using the dot product for each pair of unit vectors. The dot product of two unit vectors yields the cosine of the angle between them. So, it is clear why the components of rotation matrices are referred to as <u>direction Cosines</u>.

$$ {}^{A}_{B}R = [\, {}^{A}\hat{X}_{B} \;\; {}^{A}\hat{Y}_{B} \;\; {}^{A}\hat{Z}_{B} \,] = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}. $$

Rotation Matrix is an orthogonal matrix and we have

$$ {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T} $$

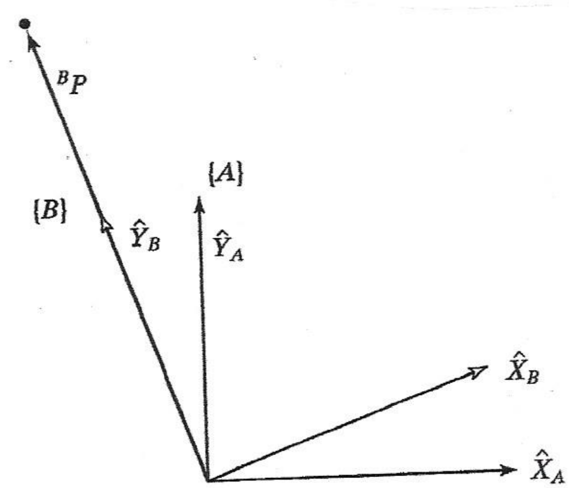
3. <u>Description of Frame</u>

A frame can be used as a description of one coordinate system relative to another. Denoting ${}^{A}P_{BORG}$ as the vector that locates the origin of the frame $\{B\}$; then frame $\{B\}$ is described by ${}^{A}_{B}R$ and ${}^{A}P_{BORG}$

$$ \{B\} = \{ {}^{A}_{B}R ,\ {}^{A}P_{BORG} \} $$

Example 1.

Figure below shows a frame {B} that is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees. Obviously, $\hat{Z}_A$ and $\hat{Z}_B$ coincide and called $\hat{Z}$, which is pointing out of the page.



Using the cosine direction formula developed before we can write

$$ {}^A_B R = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

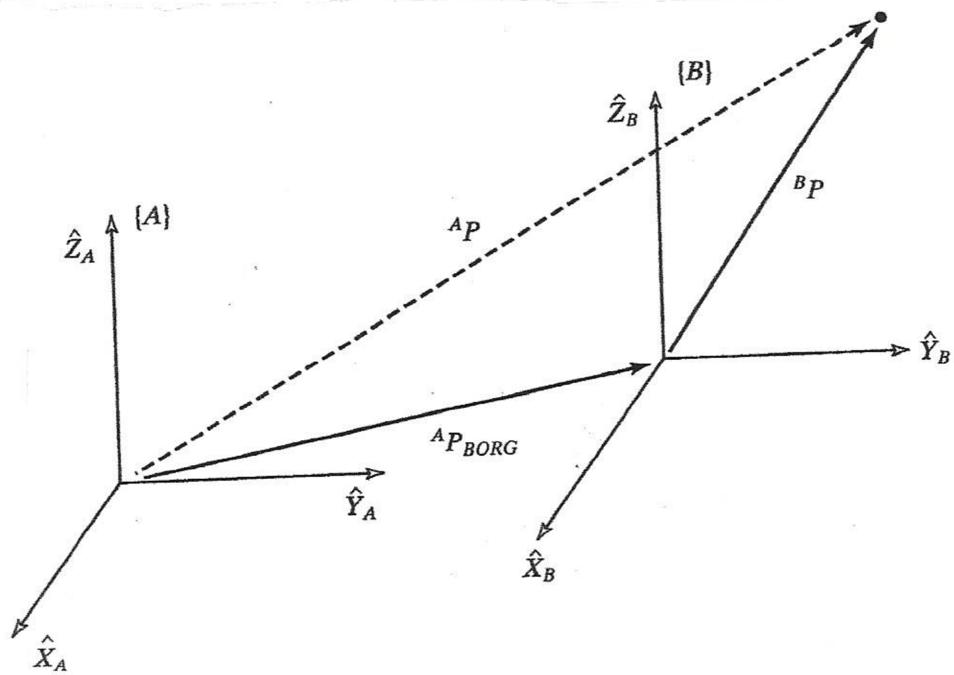
Given $\quad {}^BP = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

we can calculate ^AP as

$$ {}^AP = {}^A_B R \; {}^BP = \begin{bmatrix} -1 \\ 1.732 \\ 0 \end{bmatrix} $$

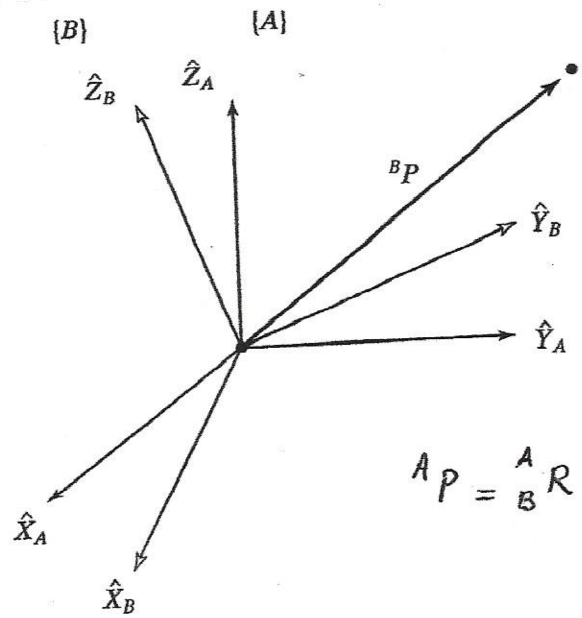
* Development for Homogeneous Transformation

1. Mappings involving translated frames



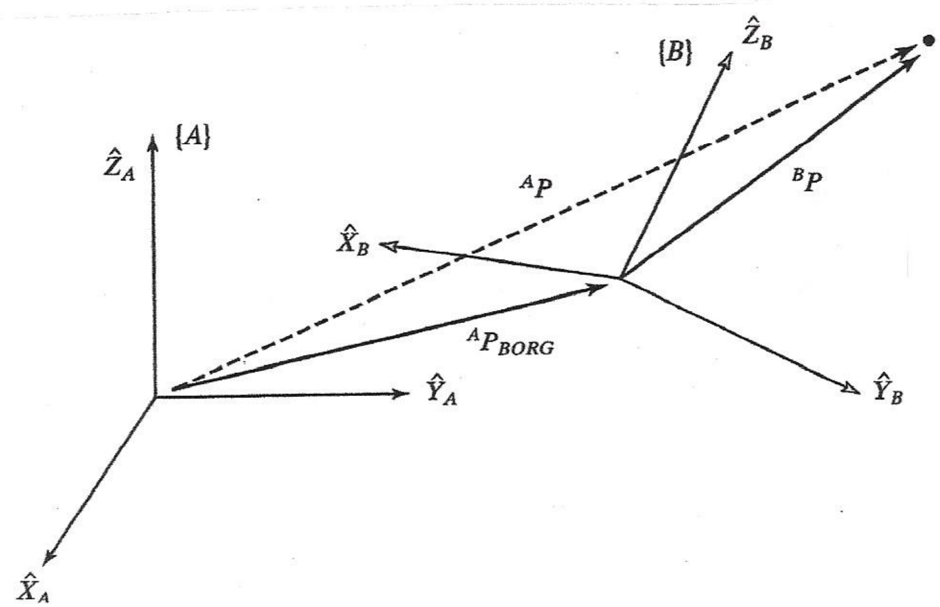
It is evident that $\quad {}^AP = {}^BP + {}^AP_{BORG}$

2. Mappings involving rotated frames



$$^AP = {}^A_BR\,{}^BP$$

3. Mappings involving general frames



Combining mappings involving translated and rotated frames, we have

$$ {}^A P = {}^A_B R \, {}^B P + {}^A P_{BORG} $$

The above form is not as appealing when a sequence of mappings are involved. By a simple mathematical trick we can write the above expression as

$$ \underbrace{\begin{bmatrix} {}^A P \\ \hline 1 \end{bmatrix}}_{{}^A \hat{P}} = \underbrace{\left[\begin{array}{c|c} {}^A_B R & {}^A P_{BORG} \\ \hline \underbrace{0\ 0\ 0}_{0} & 1 \end{array}\right]}_{{}^A_B T} \underbrace{\begin{bmatrix} {}^B P \\ \hline 1 \end{bmatrix}}_{{}^B \hat{P}} \qquad (*) $$

${}^A_B T$ is called <u>Homogeneous Transformation</u>.

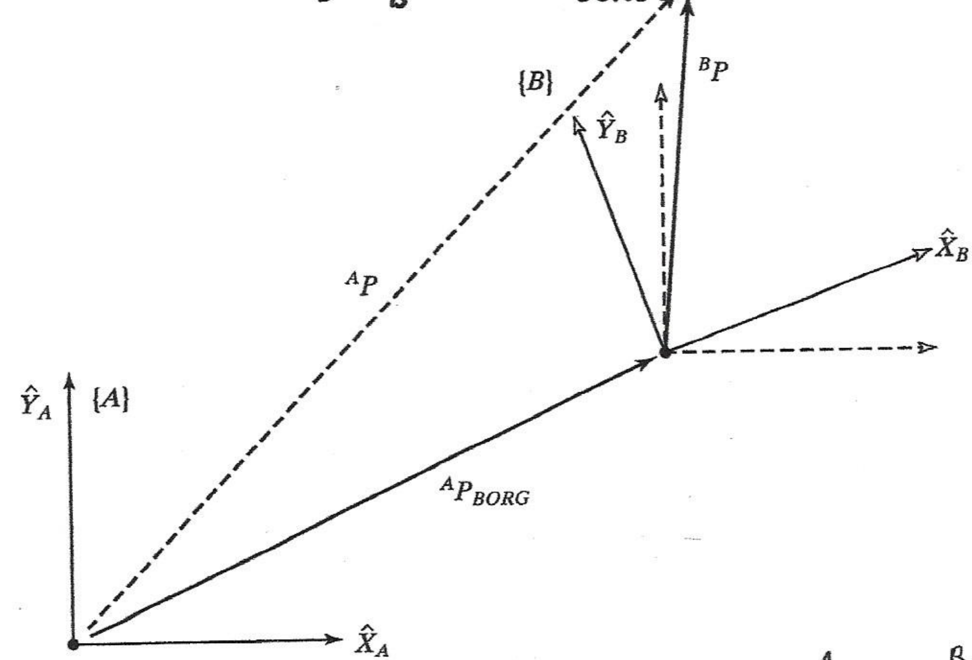
It is easy to see that (*) can be written as

$$ \begin{cases} {}^A P = {}^A_B R \, {}^B P + {}^A P_{BORG} & \checkmark \\ 1 = 1 & \checkmark \end{cases} $$

Example 2

Figure below shows a frame $\{B\}$, which is rotated relative to frame $\{A\}$ about $\hat{Z}$ by 30 degrees, translated 10 units in $\hat{X}_A$, and translated 5 units in $\hat{Y}_A$. Find ${}^{A}P$, where ${}^{B}P = [3 \;\; 7 \;\; 0]^T$.

$${}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG}$$



$${}^{A}\hat{P} = {}^{A}_{B}T\;{}^{B}\hat{P} = \left[\begin{array}{ccc|c} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \overset{{}^{B}P}{\begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix}} = \overset{{}^{A}P}{\begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix}}$$

${}^{A}P_{BORG}$

${}^{A}_{B}R$

General: A sequence of homogeneous transformation can be computed easily as follows. For simplicity consider two of them, then we have

$${}^{A}_{B}T \cdot {}^{B}_{C}T = {}^{A}_{C}T$$

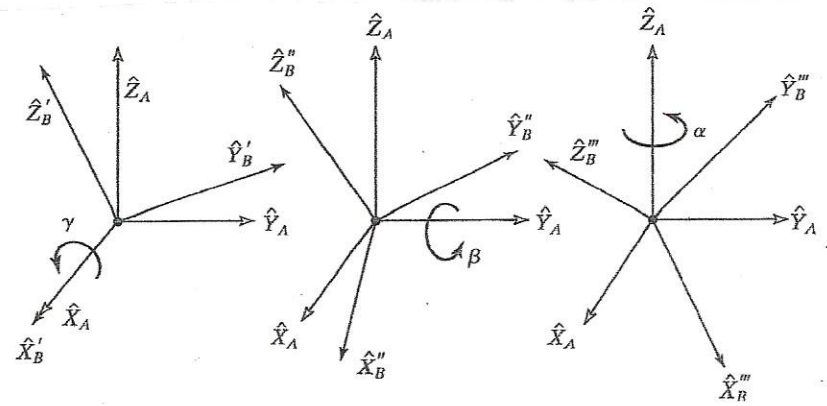
Conventions Used for Rotating Frames

1. X–Y–Z Fixed Angles (Roll, Pitch, Yaw Angles)

One method of describing the orientation of a frame $\{B\}$ is as follows:

Start with the frame coincident with a known reference frame $\{A\}$. Rotate $\{B\}$ first about $\hat{X}_A$ by an angle γ, then about $\hat{Y}_A$ by an angle β, and finally, about $\hat{Z}_A$ by an angle α

Note that the word "fixed" refers to the fact that the orientation are specified about the fixed (nonmoving) reference frame.



$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma)$$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$s\alpha = \sin\alpha$

$c\alpha = \cos\gamma$

etc.

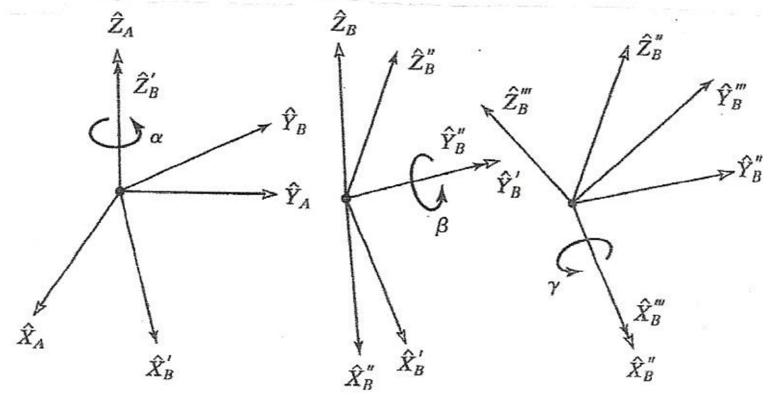
$$= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

2. Z-Y-X Euler Angles

Another possible description of a frame $\{B\}$ is as follows:

> Start with the frame coincident with a known frame $\{A\}$. Rotate $\{B\}$ first about $\hat{Z}_B$ by an angle α, then about $\hat{Y}_B$ by an angle β, and finally, about $\hat{X}_B$ by an angle γ.

Note that in this representation each rotation is performed about an axis of the moving system $\{B\}$ rather than one of the fixed reference $\{A\}$.



$$
{}^A_B R_{ZYX}(\alpha,\beta,\gamma) = R_Z(\alpha)\,R_Y(\beta)\,R_X(\gamma)
$$

$$
= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}
$$

$$
= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
$$

$$
= {}^A_B R_{XYZ}(\gamma,\beta,\alpha)
$$