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Lecture Notes on Differential Motion of Manipulators

Kinematics II : Differential Motion

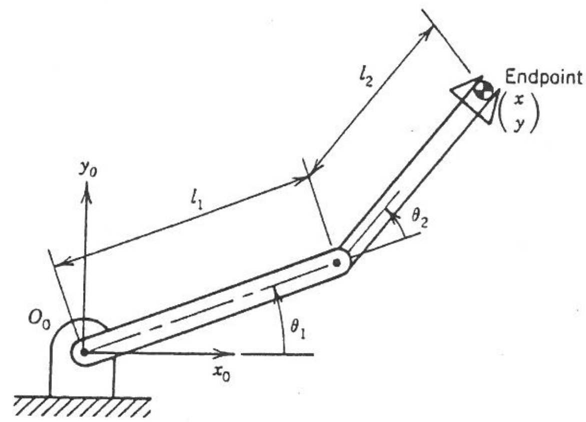
In the previous chapter, the position and orientation of the manipulator end-effector were evaluated in relation to joint displacements. The direct kinematic problem, namely the problem of finding the end-effector position and orientation for a given set of joint displacements was formulated and solved through the kinematic equation. The inverse kinematic problem, which requires the evaluation of joint displacements for given end-effector position and orientation was also formulated and solved by the kinematic equation.

In this chapter, we are concerned not only with the final location of the end-effector, but also with the velocity at which the end-effector moves. In order to move the end-effector in a specified direction at a specified speed, it is necessary to coordinate the motion of the individual joints. To do this, we need to derive the differential relationship between the joint displacements and the end-effector location, and then solve for the individual joint motions.

Kinematic Modeling of Instantaneous Motions

Differential Relationships

Let us begin by considering the two degree-of-freedom planar manipulator shown in the following figure.



The kinematic equations relating the end-effector position (x, y) to the joint displacements (θ_1, θ_2) are given by

$$x(\theta_1, \theta_2) = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y(\theta_1, \theta_2) = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

The differential relationship is obtained by simply differentiating the above equation,

$$dx = \frac{\partial x(\theta_1, \theta_2)}{\partial \theta_1} d\theta_1 + \frac{\partial x(\theta_1, \theta_2)}{\partial \theta_2} d\theta_2$$

$$dy = \frac{\partial y(\theta_1, \theta_2)}{\partial \theta_1} d\theta_1 + \frac{\partial y(\theta_1, \theta_2)}{\partial \theta_2} d\theta_2$$

or in vector form

$$\underline{\underline{dX = J\, d\theta}} \quad ; \quad dX = \begin{bmatrix} dx \\ dy \end{bmatrix}, \; d\theta = \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}, \; J = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} \end{bmatrix}$$

where the matrix J is referred to as the manipulator Jacobian and for the planar manipulator above is given by

$$J = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

Dividing both sides of differential relationship by infinitesimal time increment dt, we obtain

$$\frac{dx}{dt} = J \frac{d\theta}{dt}$$

or equivalently

$$v = J\omega \quad ; \quad v = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} , \quad \omega = \dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

where v represents the end-effector velocity vector and $\omega = \dot{\theta}$ is the joint angular velocity vector. Thus, the velocity relationship between the joints and the end-effector is determined by the manipulator Jacobian.

Let J_1 and J_2 be two 2×1 vectors consisting of the first and second columns of the Jacobian, respectively. Then the above relationship can be rewritten as

$$v = J_1 \dot{\theta}_1 + J_2 \dot{\theta}_2$$

where each term on the right hand side accounts for the end-effector velocity induced by the corresponding joint.

Infinitesmial Rotations

In the previous section we analyzed the infinitesimal translation and the linear velocity of the endpoint. To genralize the result, we need to include the infinitesimal rotation and angular velocity for a general spatial manipulator arm. Also, recall that in previous chapter we developed methods (rotation matrices and Euler angles) to represent rotations and orientations of finite angles. However, infinitesimal rotations or time derivatives of orientations are substantially different from finite angle rotations and orientations.

We begin by writing the 3X3 rotation matrix representing infinitesimal rotation $d\phi_x$ about the x axis:

$$R_x(d\phi_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(d\phi_x) & -\sin(d\phi_x) \\ 0 & \sin(d\phi_x) & \cos(d\phi_x) \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_x \\ 0 & d\phi_x & 1 \end{bmatrix}$$

Note that, since $d\phi_x$ is infinitesimal $\cos(d\phi_x) = 1$ and $\sin(d\phi_x) = d\phi_x$. For infinitesimal rotations about the y and z axes, similar matrix representations can be obtained. Let $R_y(d\phi_y)$ be the 3X3 infinitesimal rotation matrix about the y axis, then the result of consecutive rotations about x and y axes is given by

$$R_x(d\phi_x)R_y(d\phi_y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_x \\ 0 & d\phi_x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & d\phi_y \\ 0 & 1 & 0 \\ -d\phi_y & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d\phi_y \\ d\phi_x d\phi_y & 1 & -d\phi_x \\ -d\phi_y & d\phi_x & 1 \end{bmatrix}$$

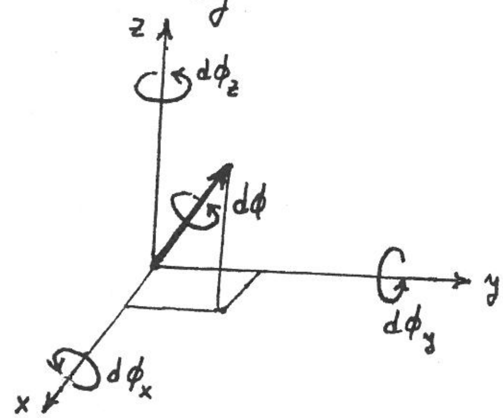
$$= \begin{bmatrix} 1 & 0 & d\phi_y \\ 0 & 1 & -d\phi_x \\ -d\phi_y & d\phi_x & 1 \end{bmatrix} \qquad \text{Note } d\phi_x d\phi_y \approx 0$$

The infinitesimal rotations do not depend on the order of rotations; in other words, they <u>commute</u>. For example, one can easily see that

$$R_x(d\phi_x)\,R_y(d\phi_y) = R_y(d\phi_y)\,R_x(d\phi_x)$$

In genral, infinitesimal rotations about the x, y, and z axes shown in Figure below can be represented by

$$R(d\phi_x, d\phi_y, d\phi_z) = \begin{bmatrix} 1 & -d\phi_z & d\phi_y \\ d\phi_z & 1 & -d\phi_x \\ -d\phi_y & d\phi_x & 1 \end{bmatrix}$$



The rotation matrix depends only on the three infinitesimal angles, but is independent of the order of rotations.

Let $R(d\phi_x, d\phi_y, d\phi_z)$ and $R(d\phi_x', d\phi_y', d\phi_z')$ be two infinitesimal rotation matrices, then the consecutive rotations of two yield

$$R(d\phi_x, d\phi_y, d\phi_z)\,R(d\phi_x', d\phi_y', d\phi_z') = \begin{bmatrix} 1 & (d\phi_z + d\phi_z') & -(d\phi_y + d\phi_y') \\ -(d\phi_z + d\phi_z') & 1 & (d\phi_x + d\phi_x') \\ (d\phi_y + d\phi_y') & -(d\phi_x + d\phi_x') & 1 \end{bmatrix}$$

$$= R(d\phi_x + d\phi_x',\ d\phi_y + d\phi_y',\ d\phi_z + d\phi_z')$$

where higher order quantities are neglected. Thus, the rotation resulting from two arbitrary infinitesimal rotations is simply given by the algebraic sum of the individual components for each axis (<u>additive</u> property)

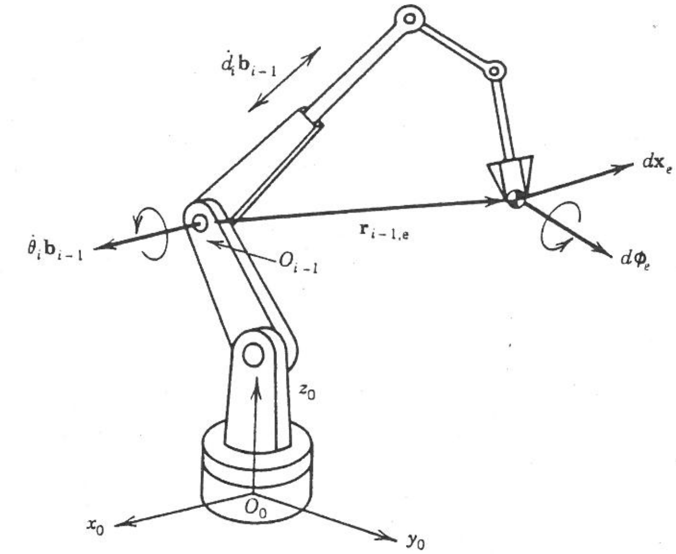
Let us write the infinitesimal rotations about the three axes in vector form :

$$d\boldsymbol{\phi} = \begin{bmatrix} d\phi_x \\ d\phi_y \\ d\phi_z \end{bmatrix}$$

In general, the infinitesimal rotations denoted by the above expression can be treated as a vector because it possesses all the properties that vectors in a vector field must satisfy. Geometrically, $d\boldsymbol{\phi}$ can be represented by an arrow, shown in the above figure. The direction of the arrow represents the axis of rotation, and the length represents the magnitude of the rotation. It should be noted that vector representation is not allowed for finite rotations, but is valid only for infinitesimal rotations.

Computation of the Manipulator Jacobian

In this section we deal with a general n degree-of-freedom manipulator arm and compute the manipulator Jacobian matrix associated with the infinitesimal translation and rotation of the end-effector. We denote the infinitesimal end-effector translation by a three-dimensional vector $d\mathbf{x}_e$ and we represent the infinitesimal end-effector rotation by a three-dimensional vector $d\boldsymbol{\phi}_e$. Both vectors are represented with reference to the base coordinate from $O_0-x_0y_0z_0$, as shown in the following figure.



For convenience, we combine the two vectors and define the following six-dimensional vector dp:

$$dp = \begin{bmatrix} dx_e \\ d\phi_e \end{bmatrix}$$

Dividing both sides by the infinitesimal time increment dt, we obtain the velocity and angular velocity of the end-effector:

$$\dot{p} = \begin{bmatrix} v_e \\ \omega_e \end{bmatrix}$$

Similar to the two degree-of-freedom, for the general n degree-of-freedom we can write the end-effector velocity and angular velocity as a function of joint velocities using the manipulator Jacobian:

$$\underline{\underline{\dot{p} = J\dot{q}}}$$

where $\dot{q} = [\dot{q}_1, \dots, \dot{q}_n]^T$ is the $n \times 1$ joint velocity vector. Recall that $q_i = \theta_i$ for a revolute joint and $q_i = d_i$ for a prismatic joint. The dimension of the Jacobian matrix is now $6 \times n$, the first three row vectors are associated with the linear velocity v_e, while the last three correspond to the angular velocity ω_e. Each column vector, on the other hand, represents the velocity and angular velocity generated by the corresponding individual joint. Let us determine each column vector of the Jacobian matrix as functions of link parameters and arm configuration. Let J_{Li} and J_{Ai} be 3×1 column vectors of the Jacobian matrix associated with the linear and angular velocities, respectively. Namely, we partition the Jacobian matrix so that

$$J = \left[\begin{array}{c|c|c|c} J_{L1} & J_{L2} & \dots & J_{Ln} \\ J_{A1} & J_{A2} & & J_{An} \end{array}\right]$$

Using the J_{Li}, we can write the <u>linear velocity</u> of the end-effector as

$$\rightarrow \; v_e = J_{L1}\dot{q}_1 + \dots + J_{Ln}\dot{q}_n$$

If joint i is <u>prismatic</u>, it produces a linear velocity at the end-effector in the same direction as the joint axis. Let b_{i-1} be the unit vector pointing along the direction of the joint axis i, as shown in the figure, and let $\dot{d}_i$ be the scalar joint velocity in this direction, then we obtain,

$$J_{Li}\dot{q}_i = b_{i-1}\dot{d}_i$$

Note that in the Denavit-Hartenberg notation the joint velocity $\dot{d}_i$ is measured along the z_{i-1} axis.

If the joint is revolute, it rotates the composite of distal links from links i to n at the angular velocity ω_i given by $\omega_i = b_{i-1}\dot{\theta}_i$. This angular velocity produces a linear velocity at the end-effector. Let $r_{i-1,e}$ be the position vector from O_{i-1} to the end-effector as shown in the figure, then the linear velocity generated by the angular velocity ω_i is given by

$$J_{Li}\dot{q}_i = \omega_i \times r_{i-1,e} = (b_{i-1} \times r_{i-1,e})\dot{\theta}_i$$

where $a \times b$ represents the vector product of two vectors a and b. Thus, the end-effector velocity is determined by either of the above two expressions.

Similarly, the angular velocity of the end-effector can be expressed as a linear combination of the column vectors J_{Ai} as follows,

$$\rightarrow \omega_e = J_{A1}\dot{q}_1 + \cdots + J_{An}\dot{q}_n$$

When joint i is a prismatic joint, it does not generate an angular velocity at the end-effector, hence

$$J_{A_i} \dot{q}_i = 0$$

If, on the other hand, the joint is a revolute joint, the angular velocity is given by

$$J_{A_i} \dot{q}_i = \omega_i = b_{i-1} \dot{\theta}_i$$

The above derived equations determine all the elements of the manipulator Jacobian. To summarize:

$$\begin{bmatrix} J_{L_i} \\ J_{A_i} \end{bmatrix} = \begin{bmatrix} b_{i-1} \\ 0 \end{bmatrix} \quad \text{"for a prismatic joint"}$$

and

$$\begin{bmatrix} J_{L_i} \\ J_{A_i} \end{bmatrix} = \begin{bmatrix} b_{i-1} \times r_{i-1,e} \\ b_{i-1} \end{bmatrix} \quad \text{"for a revolute joint"}$$

Vectors b_{i-1} and $r_{i-1,e}$ in the above expressions are functions of joint displacements. These vectors can be computed using the coordinate transformations discussed in the previous chapter. For the sake of completeness, let us provide the corresponding expressions for these vectors as well.

The direction of joint axis $i-1$ is represented by $\bar{b} = [0\ 0\ 1]^T$ with reference to coordinate frame $i-1$, because the joint axis is along the z_{i-1} axis. This vector $\bar{b}$ can be transformed to a vector which is defined with reference to the base frame, that is b_{i-1}, using 3X3 rotation matrices $R_j^{j-1}(q_j)$ as:

$$b_{i-1} = R_1^0(q_1) \ldots R_{i-1}^{i-2}(q_{i-1})\,\bar{b}$$

Position vector $r_{i-1,e}$ can be computed by using 4X4 matrices $A_j^{j-1}(q_j)$. Let $X_{i-1,e}$ be the 4X1 augmented vector of $r_{i-1,e}$ and let $\bar{X} = [0\ 0\ 0\ 1]^T$ be the augmented position vector representing the origin of its coordinate frame, then the position vector $r_{i-1,e}$ is derived from

$$X_{i-1,e} = A_1^0(q_1) \ldots A_n^{n-1}(q_n)\,\bar{X} - A_1^0(q_1) \ldots A_{i-1}^{i-2}(q_{i-1})\,\bar{X}$$

where the first term accounts for the position vector from the origin O_0 to the end-effector and the second term is from O_0 to O_{i-1}.

<u>Example</u> : Three Degree-of-Freedom Polar Coordinate Robot

The skeleton structure of a 3-degree-of-freedom manipulator is shown in the following figure. The joint displacements θ_1, θ_2 and d_3 defined in the figure are equivalent to polar coordinates.

To find the Jacobian matrix, we begin by determining the directions of the joint axes. From the figure these are given by

$$b_0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad b_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \qquad b_2 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix}$$

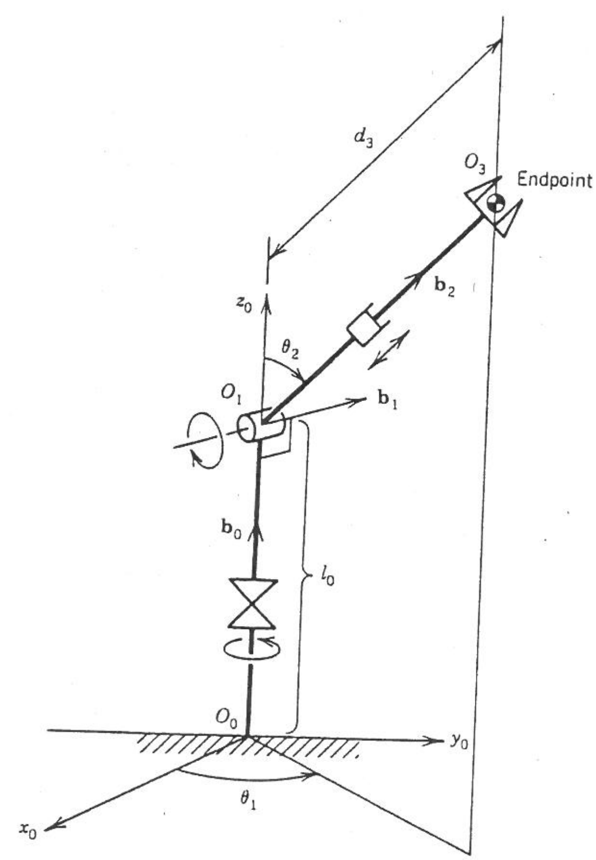
For revolute joints, we need to find position vector $r_{i-1,e}$

They are : $\quad r_{1,e} = d_3 b_2 \quad , \quad r_{0,e} = l_0 b_0 + d_3 b_2$

By simple substitution in the derived expressions, one can compute the following Jacobian matrix

$$J = \begin{bmatrix} -d_3 s_1 s_2 & d_3 c_1 c_2 & c_1 s_2 \\ d_3 c_1 s_2 & d_3 s_1 c_2 & s_1 s_2 \\ 0 & -d_3 s_2 & c_2 \\ 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

" Note that the elements of the manipulator Jacobian are functions of the joint displacements, hence the Jacobian is configuration-dependent. "



Inverse Instantaneous Kinematics

The instantaneous kinematic equation $\dot{P} = J\dot{q}$, as discussed in the previous section, provides the velocity and angular velocity of the end-effector as a function of joint velocities. Using this expression, we can formulate and solve the inverse problem as well. Namely, given a desired end-effector velocity, we find the corresponding joint velocities that cause the end-effector to move at the desired velocity.

As pointed out before, a manipulator arm must have at least six degrees of freedom in order to locate its end-effector at an arbitrary position with an arbitrary orientation. For a six degree-of-freedom manipulator, the Jacobian matrix J is a 6×6 square matrix. If this matrix is non-singular at the current arm configuration, the inverse matrix J^{-1} exists. We then can obtain the velocities required at the individual joints in order to obtain a given end-effector velocity $\dot{p}$ by

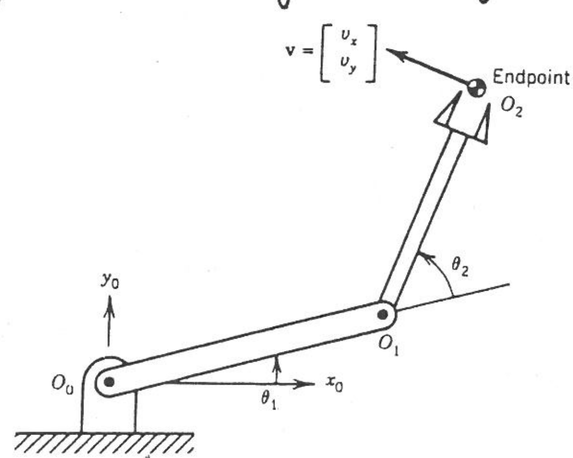
$$\dot{q} = J^{-1}\dot{p}$$

This is referred to as resolved motion rate control. Since the manipulator Jacobian varies with the arm configuration, it may become singular at certain configurations. In such cases the inverse Jacobian does not exist and the above solution is not valid. Therefore, at a singular configuration, there exists at least one direction in which the end-effector cannot be moved no matter how we choose joint velocities $\dot{q}_1$ through $\dot{q}_6$. In this case, the matrix J is not of full rank and does not span the whole six dimensional vector space $\dot{p}$.

For a general n degrees-of-freedom manipulator arm with m independent variables representing the infinitesimal displacement of the end-effector, the Jacobian matrix J is $m \times n$. Thus, one can discuss the solution of $\dot{P} = J\dot{q}$ similar to an algebraic equation $Ax = b$; where $J = A$, $\dot{q} = x$, and $\dot{P} = b$. This will not be repeated here.

<u>Example</u>:

Consider the two degree-of-freedom planar manipulator shown in the following figure. The length of each link is 1, and the endpoint velocity is denoted by $v = [v_x \ v_y]^T$



1. Given a desired endpoint velocity, find joint velocities.
2. Find singular configurations and discuss the solution.

1. The Jacobian matrix for this planar manipulator has been derived in equation $v = J\dot{\theta}$ (see previous section). Replacing l_1 and l_2 by 1, we obtain

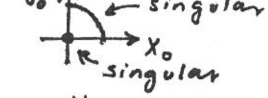
$$J = \begin{bmatrix} -\sin\theta_1 - \sin(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \cos\theta_1 + \cos(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

Inverting the Jacobian matrix and substituting in $\dot{q} = J^{-1}\dot{p}$, the joint velocities are obtained as

$$\dot{\theta}_1 = \frac{v_x \cos(\theta_1+\theta_2) + v_y \sin(\theta_1+\theta_2)}{\sin\theta_2}$$

$$\dot{\theta}_2 = -\frac{v_x[\cos\theta_1 + \cos(\theta_1+\theta_2)] + v_y[\sin\theta_1 + \sin(\theta_1+\theta_2)]}{\sin\theta_2}$$

2. Singularity occurs when the determinant of the manipulator Jacobian is zero, i.e. $\det(J) = \sin\theta_2 = 0$. Thus, singular configurations occur for $\theta_2 = 0$ or $\theta_2 = \pi$, i.e. when the arm is fully extended or fully contracted.



At the singular configuration, we have the following result for the endpoint velocity vector,

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} -2\sin\theta_1 \\ 2\cos\theta_1 \end{bmatrix}\dot{\theta}_1 + \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix}\dot{\theta}_2 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix}(2\dot{\theta}_1 + \dot{\theta}_2)$$

i.e. the two column vectors of the Jacobian matrix become parallel. The endpoint can then be moved only in the direction perpendicular to the arm links, but not in any other direction.