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Lecture Notes on Kinematics I

"Homogenous Transformation"

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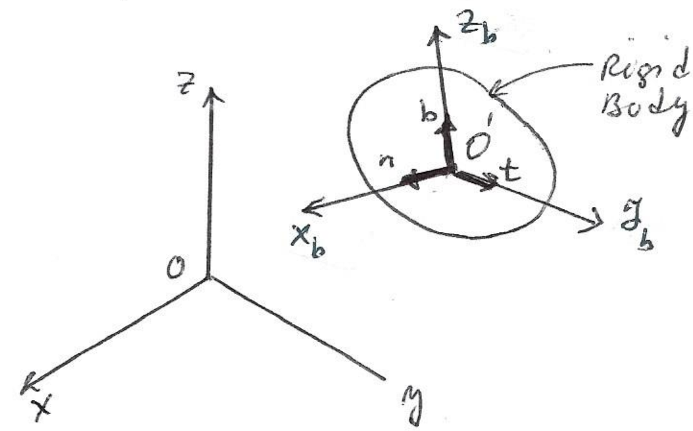
Kinematics I : Geometry

Position and Orientation of a Rigid Body

Let O-xyz be a coordinate frame fixed to the ground. Then the position of the rigid body (Point O′) is represented by

$$\mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$



To represent the orientation of the rigid body, three coordinate axes x_b, y_b, and z_b are attached to the rigid body, which form another coordinate frame O'-$x_b y_b z_b$ with n, t, and b as unit vectors on it. Thus, the orientation can be represented by a 3×3 matrix R:

$$R = [n, t, b] \qquad \text{"n, t, b are mutually orthogonal"}$$

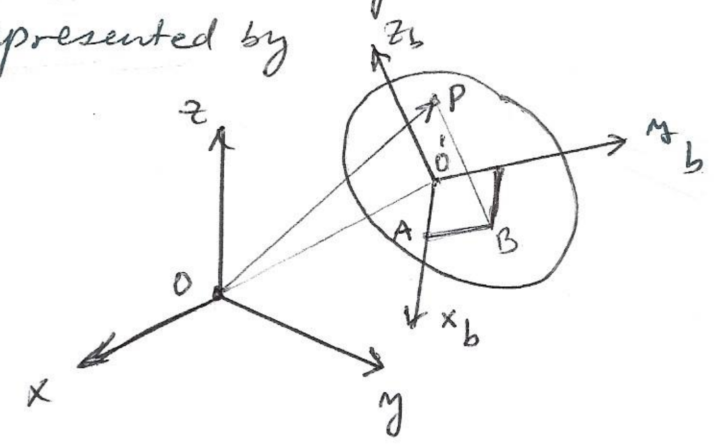
with properties $n^T t = 0$, $t^T b = 0$, and $b^T n = 0$;

and $|n| = 1$, $|t| = 1$, $|b| = 1$

One can see that the matrix R is an Orthogonal matrix, i.e. $R^T = R^{-1}$, $R^T R = I$.

Coordinate Transformation

Let P be an arbitrary point in space as shown in the figure below. The coordinate of P with respect to O xyz is represented by



$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The position of point P can also be represented with respect to O'-$x_b y_b z_b$ by

$$\mathbf{x}_b = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Now, we can write

$$\underset{\sim}{\overrightarrow{OP}} = \underset{\sim}{\overrightarrow{OO'}} + \underset{\sim}{\overrightarrow{O'A}} + \underset{\sim}{\overrightarrow{AB}} + \underset{\sim}{\overrightarrow{BP}}$$

$$\underline{\mathbf{x} = \mathbf{x}_o + u\mathbf{n} + v\mathbf{t} + w\mathbf{b}}$$

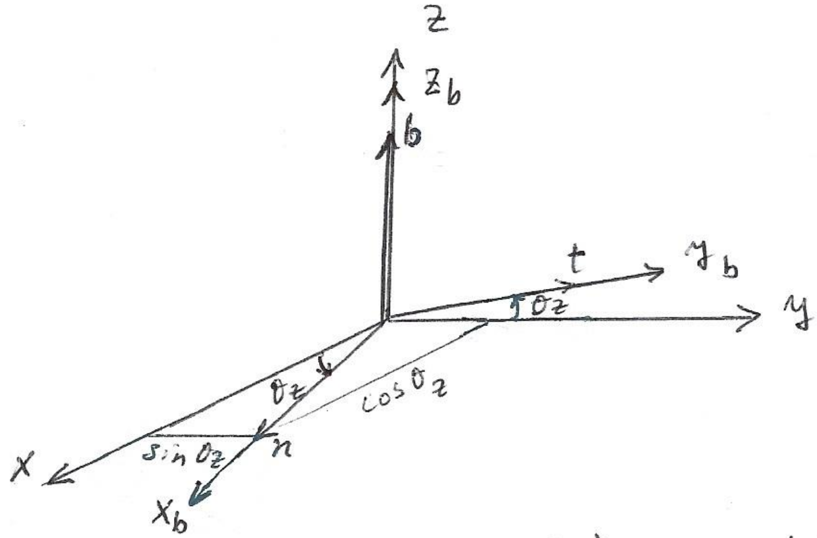
or simply

$$\mathbf{x} = \mathbf{x}_o + R\,\mathbf{x}_b \quad , \quad R^T R = I$$

Note that one can solve for $\mathbf{x}_b$ as

$$\mathbf{x}_b = -R^T \mathbf{x}_o + R^T \mathbf{x}$$

Example : (Rotation around z axis)



$$n = \begin{pmatrix} \cos\theta_z \\ \sin\theta_z \\ 0 \end{pmatrix}, \quad t = \begin{pmatrix} -\sin\theta_z \\ \cos\theta_z \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$R_z = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that one can similarily write rotation matrices with respect to X and Y axis as

$$R_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & -\sin\theta \\ 0 & \sin\theta_X & \cos\theta_X \end{bmatrix}, \quad R_Y = \begin{bmatrix} \cos\theta_Y & 0 & \sin\theta_Y \\ 0 & 1 & 0 \\ -\sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix}$$

Homogenous Transformation

$$x = x_o + R x_b \qquad (1)$$

Let

$$\underline{X} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \quad \underline{X}_b = \begin{bmatrix} u \\ v \\ w \\ 1 \end{bmatrix}$$

Then the homogenous transformation is defined by

$$\underline{X = A X_b} \qquad (2)$$

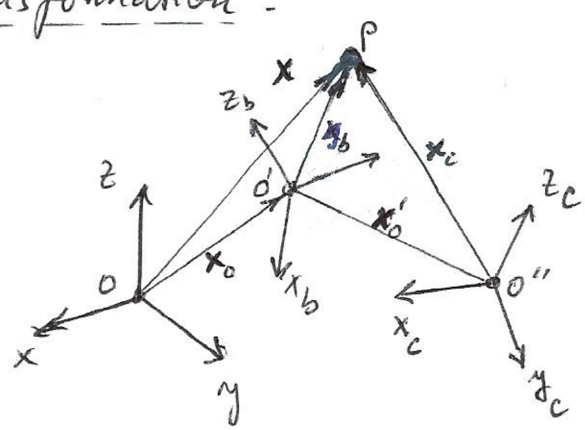
where

$$A = \left[\begin{array}{c|c} R & x_o \\ \hline 0 & 1 \end{array}\right], \quad A^{-1} = \left[\begin{array}{c|c} R^T & -R^T x_o \\ \hline 0 & 1 \end{array}\right]$$

Consecutive coordinate transformation :

$$x_b = x_o' + R' x_c$$

$$x = x_o + R x_o' + R R' x_c \quad \Leftarrow$$



General (for n chain) :

$$X_0 = A_1^0 A_2^1 A_3^2 \cdots A_n^{n-1} X_n$$

A_i^{i-1} : 4x4 Matrix Associated with Homog. Trafo. from frame i to frame i-1.