B. Shafai, ME5250 Solution to set # 2

Chapter 2

Spatial Descriptions and Transformations

Exercises

2.1) $R = \text{rot}(\hat{x}, \phi)\,\text{rot}(\hat{z}, \theta)$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & -S\phi \\ 0 & S\phi & C\phi \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta & -S\theta & 0 \\ C\phi S\theta & C\phi C\theta & -S\phi \\ S\phi S\theta & S\phi C\theta & C\phi \end{bmatrix}$$

2.2) $R = \text{rot}(\hat{x}, 45^\circ)\,\text{rot}(\hat{y}, 30^\circ)$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & .707 & -.707 \\ 0 & .707 & .707 \end{bmatrix} \begin{bmatrix} .866 & 0 & .5 \\ 0 & 1 & 0 \\ -.5 & 0 & .866 \end{bmatrix}$$

$$= \begin{bmatrix} .866 & 0 & .5 \\ .353 & .707 & -.612 \\ -.353 & .707 & .612 \end{bmatrix}$$

2.3) Since rotations are performed about axes of the frame being rotated, these are Euler-Angle style rotations:

$R = \text{rot}(\hat{z}, \theta)\,\text{rot}(\hat{x}, \phi)$

We might also use the following reasoning:

$${}^A_BR(\theta, \phi) = {}^B_AR^{-1}(\theta, \phi)$$

$$= [\text{rot}(\hat{x}, -\phi)\,\text{rot}(\hat{z}, -\theta)]^{-1}$$

$$= \text{rot}^{-1}(\hat{z}, -\theta)\,\text{rot}^{-1}(\hat{x}, -\phi)$$

$$= \text{rot}(\hat{z}, \theta)\,\text{rot}(\hat{x}, \phi)$$

$$\Rightarrow \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

$$= \begin{bmatrix} c\theta & -s\theta c\phi & s\theta s\phi \\ s\theta & c\theta c\phi & -c\theta s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

Yet another way of viewing the same operation:

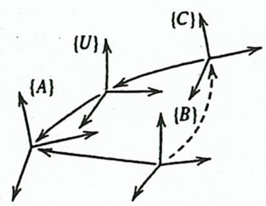
1st rotate by $\text{rot}(\hat{z}, \theta)$

2nd rotate by $\text{rot}(\hat{z}, \theta)\,\text{rot}(\hat{x}, \phi)\,\text{rot}^{-1}(z, \theta)$

2.4) (This is the same as 2.3 only with numbers

$$R = \text{rot}(\hat{z}, \theta)\,\text{rot}(\hat{x}, \phi) = \begin{bmatrix} 0.866 & -0.353 & 0.353 \\ 0.5 & 0.612 & -0.612 \\ 0 & 0.707 & 0.707 \end{bmatrix}$$

2.13) By just following arrows, and reversing (by inversion) where needed, we have:



$$ {}^{B}_{C}T = {}^{B}_{A}T\ {}^{U}_{A}T^{-1}\ {}^{C}_{U}T^{-1} $$

Inverting a transform is done using eq. (2.40) in book. Rest is boring.

2.12) Velocity is a "free vector" and only will be affected by rotation, and not by translation:

$$ {}^{A}V = {}^{A}_{B}R\,{}^{B}V = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} $$

$$ {}^{A}V = [-1.34 \quad 22.32 \quad 30.0]^{T} $$

2.14) This rotation can be written as:

$$ {}^{A}_{B}T = \mathrm{trans}({}^{A}\hat{P}, |{}^{A}P|)\ \mathrm{rot}(\hat{K}, \theta)\mathrm{trans}(-{}^{A}\hat{P}, |{}^{A}P|) $$

Where $\mathrm{rot}(\hat{K}, \theta)$ is written as in eq. (2.~~77~~80).

$$ \text{And} \quad \mathrm{trans}({}^{A}\hat{P}, |{}^{A}P|) = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \text{And} \quad \mathrm{trans}(-{}^{A}\hat{P}, |{}^{A}P|) = \begin{bmatrix} 1 & 0 & 0 & -P_x \\ 0 & 1 & 0 & -P_y \\ 0 & 0 & 1 & -P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Multiplying out we get:

$$ {}^{A}_{B}T = \left[\begin{array}{ccc:c} R_{11} & R_{12} & R_{13} & Q_x \\ R_{21} & R_{22} & R_{23} & Q_y \\ R_{31} & R_{32} & R_{33} & Q_z \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] $$

where the R_{ij} are given be eq. (2.~~77~~80). And:

$$ Q_x = P_x - P_x(K_x^2 V\theta + C\theta) - P_y(K_x K_y V\theta - K_z S\theta) $$
$$ - P_z(K_x K_z V\theta + K_y S\theta) $$

$$ Q_y = P_y - P_x(K_x K_y V\theta + K_z S\theta) - P_y(K_y^2 V\theta + C\theta) $$
$$ - P_z(K_y K_z V\theta + K_x S\theta) $$

$$ Q_z = P_z - P_x(K_x K_z V\theta - K_y S\theta) - P_y(K_y K_z V\theta + K_x S\theta) $$
$$ - P_z(K_z^2 V\theta + C\theta) $$

2.27) $${}^{A}_{B}T = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.28) $${}^{A}_{C}T = \begin{bmatrix} 0 & -0.5 & 0.866 & 3 \\ 0 & 0.866 & 0.5 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Use direction Cosines (see 2.3) for Rotation matrix ${}^{A}_{C}R$.
Note that frame{C} is rotated relative to frame {A}. The translation is obvious

2.29) $${}^{B}_{C}T = \begin{bmatrix} 0 & 0.5 & -0.866 & 0 \\ 0 & -0.866 & -0.5 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

, ${}^{B}_{C}T = {}^{A}_{B}T^{-1}\,{}^{A}_{C}T = {}^{B}_{A}T\,{}^{A}_{C}T$

2.30) $${}^{C}_{A}T = \begin{bmatrix} 0 & 0 & -1 & 2 \\ -0.5 & 0.866 & 0 & 3*0.5 \\ 0.866 & +0.5 & 0 & -3*0.866 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

, ${}^{C}_{A}T = {}^{A}_{C}T^{-1}$
Note that frame {A} is rotated and translated relative to frame {C}.

2.31) $${}^{A}_{B}T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.32) $${}^{A}_{C}T = \begin{bmatrix} 0.866 & 0.5 & 0 & -3 \\ 0.5 & -0.866 & 0 & 4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.33) $${}^{B}_{C}T = \begin{bmatrix} -0.866 & -0.5 & 0 & 3 \\ 0 & 0 & +1 & 0 \\ -0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.34) $${}^{C}_{A}T = \begin{bmatrix} 0.866 & 0.5 & 0 & +3*.86 - 2 \\ 0.5 & -0.866 & 0 & +4*.86 + 1.5 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

0.59
4.95
2