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Additional Lecture Notes

on

Velocities, Jacobian, and Static Forces

Transformation of a Free vector

"Linear and Angular velocity vectors"

A velocity vector written in $\{B\}$ as ${}^{B}V$, it will be written in $\{A\}$ as

$$ {}^{A}V = {}^{A}_{B}R \, {}^{B}V \leftarrow {}^{A}P = {}^{A}_{B}R \, {}^{B}P \quad (*) $$

Similarily, a moment vector in $\{B\}$ written as ${}^{B}N$, it will be represented in $\{A\}$ by

$$ {}^{A}N = {}^{A}_{B}R \, {}^{B}N $$

Linear Velocity vector

- Differentiation of position vector

The velocity of a position vector can be thought of as the linear velocity of the point in space represented by the position vector.

Derivative of a vector Q Relative to Frame B

$$ {}^{B}V_{Q} = \frac{{}^{B}d}{dt} \, {}^{B}Q = \lim_{\Delta t \to 0} \frac{{}^{B}Q(t+\Delta t) - {}^{B}Q(t)}{\Delta t} \qquad (1) $$

The velocity vector (1), when expressed in terms of frame $\{A\}$ will be written as

$$ {}^{A}({}^{B}V_{Q}) = \frac{{}^{A}d}{dt} \, {}^{B}Q \qquad (2) $$

$$ {}^{B}({}^{B}V_{Q}) = {}^{B}V_{Q} \qquad (3) $$

So, using (*) we have

$$ {}^{A}\left({}^{B}V_{Q}\right) = {}^{A}_{B}R \; {}^{B}V_{Q} \qquad (4) $$

Special case : velocity of the origin of a frame relative to a universe frame is written in shorthand notation

$$ v_C = {}^{U}V_{CORG} \qquad (5) $$

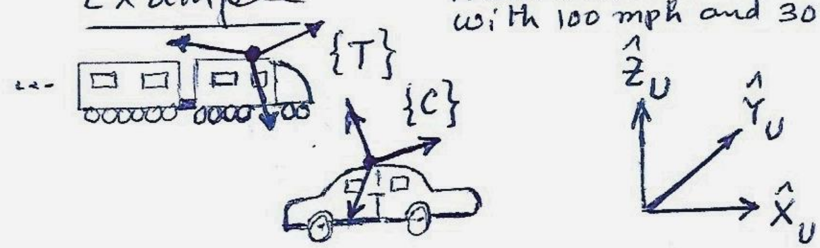
Refrence frame — origin of the frame C

Now, if we want to represent the velocity of the origin of frame $\{C\}$ expressed in terms of frame $\{A\}$, then we have ${}^{A}v_C$

" Note that the original derivative is taken w.r.t. the universe frame $\{U\}$."

Example

Train and car heading in $\hat{X}$ direction of $\{U\}$ with 100 mph and 30 mph respectively



Given ${}^{U}_{T}R$ and ${}^{U}_{C}R$

$$ \frac{{}^{U}d}{dt}\, {}^{U}P_{CORG} = {}^{U}V_{CORG} = {}^{U}v_C = 30\hat{X} $$

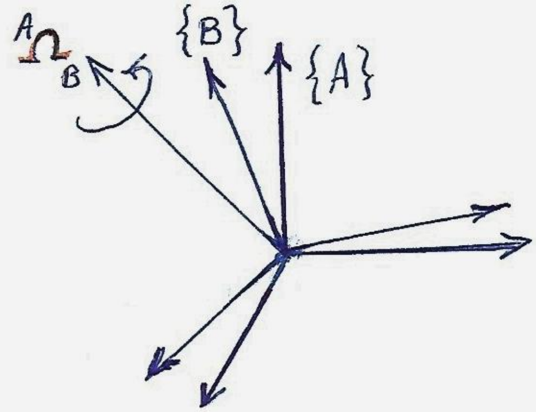
$$ {}^{C}\left({}^{U}V_{TORG}\right) = {}^{C}v_T = {}^{C}_{U}R\, {}^{U}v_T = {}^{C}_{U}R\,(100\hat{X}) = {}^{U}_{C}R^{-1}\, 100\hat{X} $$

$$ {}^{C}\left({}^{T}V_{CORG}\right) = {}^{C}_{T}R\; {}^{T}V_{CORG} = {}^{C}_{U}R\; {}^{U}_{T}R\,(-70\hat{X}) $$

$$ = {}^{U}_{C}R^{-1} \cdot {}^{U}_{T}R\,(-70\hat{X}) $$

Angular Velocity Vector

We can think of angular velocity as describing the rotational motion of a frame.
In the following figure ${}^{A}\Omega_{B}$ describes the rotation of frame $\{B\}$ relative to $\{A\}$.



At any instant, the <u>direction</u> of ${}^{A}\Omega_{B}$ indicates the <u>instantaneous axis of rotation</u> of $\{B\}$ relative to $\{A\}$, and the <u>magnitude</u> of ${}^{A}\Omega_{B}$ indicates the <u>speed of rotation</u>

Similar to linear velocity vector, ${}^{C}({}^{A}\Omega_{B})$ represents the angular velocity of frame $\{B\}$ relative to $\{A\}$ expressed in terms of frame $\{C\}$.

$$ {}^{C}({}^{A}\Omega_{B}) = {}^{C}_{A}R\ {}^{A}\Omega_{B} \quad ! \qquad (6) $$

Special case
$\omega_c = {}^{U}\Omega_c$

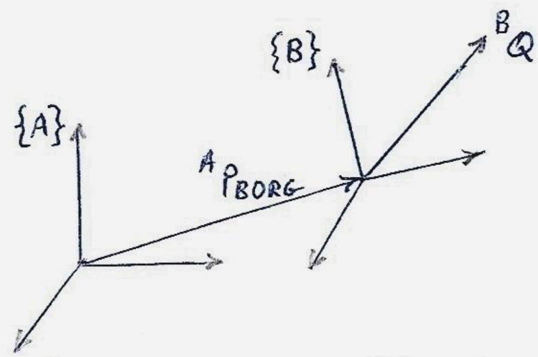
Here, ω_C is the angular velocity of frame $\{C\}$ relative to a universe reference frame $\{U\}$.
For example, ${}^{A}\omega_C$ represents the angular velocity of frame $\{C\}$ expressed in terms of $\{A\}$: Noting that the angular velocity is with respect to $\{U\}$.

Linear and Rotational velocities of Rigid Bodies

Linear velocity

Consider a frame {B} attached to a rigid body.
Frame {B} is located relative to fixed frame {A} described by ${}^{A}P_{BORG}$ and a rotation matrix ${}^{A}_{B}R$.

Assuming the orientation ${}^{A}_{B}R$ is not changing with time i.e. the motion of point Q relative to {A} is due to ${}^{A}P_{BORG}$ or ${}^{B}Q$ changing in time.



" Frame {B} is translating with velocity ${}^{A}V_{BORG}$ relative to {A} "

Recall from chapter 2 "Homogeneous Transformation"

$$ {}^{A}Q = {}^{A}P_{BORG} + {}^{A}_{B}R\,{}^{B}Q \qquad \text{or} \qquad {}^{A}Q = {}^{A}_{B}T\,{}^{B}Q \ ✓ $$

The motion of ${}^{B}Q$ relative to frame {A} can be regarded as an extension of translation and orientation of rigid body to the time-varying case.

So, the linear velocity of point Q in terms of {A} can easily be written

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} \qquad (7) $$

$$ \frac{{}^{A}d\,{}^{A}Q}{dt} = {}^{A}V_{Q} $$

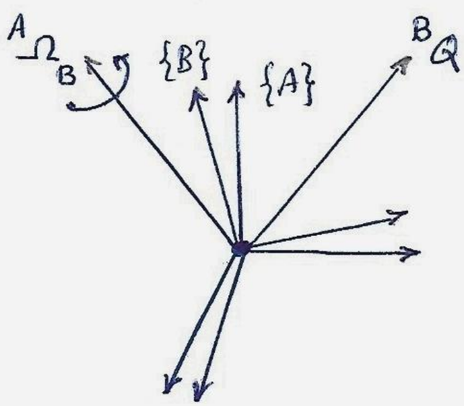
$$ \frac{{}^{B}d\,{}^{B}Q}{dt} = {}^{B}V_{Q} $$

$$ \frac{d\,{}^{A}P_{BORG}}{dt} = {}^{A}V_{BORG} $$

Rotational Velocity

Now let us consider two frames with coincident origins and with zero linear relative velocity. Their origins will remain coincident for all time and one or both could be attached to the rigid bodies.

The orientation of frame {B} with respect to frame {A} is changing in time. So, rotational velocity of {B} relative to {A} is described by the vector ${}^{A}\underline{\Omega}_B$. We also have a vector ${}^{B}Q$ that locates a point fixed in {B}.



" Vector ${}^{B}Q$, fixed in frame {B}, is rotating with respect to frame {A} with angular velocity ${}^{A}\underline{\Omega}_B$ "

Question : How does a vector change with time as viewed from {A} when it is fixed in {B} and the systems are rotating?

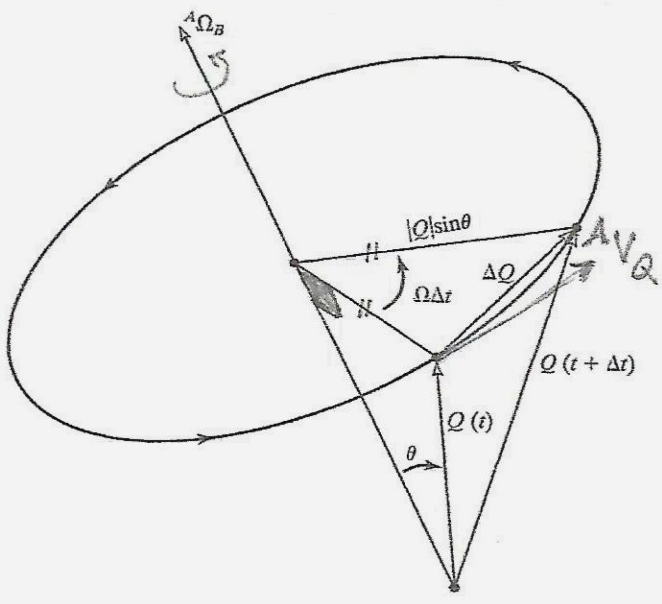
Let us consider vector Q to be constant as viewed from frame $\{B\}$, i.e.

$$ {}^{B}V_{Q} = 0 \qquad (8) $$

Even though it is constant relative to $\{B\}$; it is clear that point Q will have a velocity as seen from $\{A\}$ due to the rotational velocity ${}^{A}\Omega_{B}$.

To solve for the velocity of point Q relative to $\{A\}$ i.e. ${}^{A}V_{Q}$ we show two instants of time a vector Q rotates around ${}^{A}\Omega_{B}$

${}^{A}\Omega_{B} \perp \Delta Q$
so,
${}^{A}V_{Q} \perp {}^{A}\Omega_{B}$
$\perp {}^{A}Q$



" The Velocity of a point Q due to angular velocity ${}^{A}\Omega_{B}$ "

This implies → Cross Product

$$ |\Delta Q| = (|{}^{A}Q|\sin\theta)(|{}^{A}\Omega_{B}|\Delta t) \qquad (9) \qquad \frac{\Delta Q}{\Delta t} $$

$$ {}^{A}V_{Q} = {}^{A}\Omega_{B} \times {}^{A}Q \qquad (10) \qquad \frac{dQ}{dt} = V_{Q} $$

General Case

If the vector Q is also changing with respect to frame {B} i.e. we have ${}^{B}V_Q$. Then

$$ {}^{A}V_Q = {}^{A}({}^{B}V_Q) + {}^{A}\Omega_B \times {}^{A}Q \qquad (11)$$

$$ = {}^{A}_{B}R\, {}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\, {}^{B}Q \qquad (12)$$

Simultaneous Linear and Rotational Velocity

If the origins are not coincident, then we should also add the linear velocity of the origin to (12). Thus, the general formula for velocity of a vector fixed in frame {B} as seen from frame {A} is given by

$$ \Rightarrow \boxed{ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\, {}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\, {}^{B}Q } $$

(13)

Alternative Derivation of (10) Using a Mathematical Approach

The geometric approach of deriving (10) in previous page was transparent. However, one can also use an elegant way to derive (10) by a mathematical appraach. First, we need a useful property on derivative of orthogonal matrices.

A Property of the Derivative of an Orthogonal Matrix

We know that for an orthogonal matrix R, one can write

$$R R^T = I \qquad R \in \mathbb{R}^{n \times n} \tag{14}$$

Here we have $n=3$. Taking the derivative of (14) yields

$$\dot{R} R^T + R \dot{R}^T = 0 \tag{15}$$

or

$$\dot{R} R^T + (\dot{R} R^T)^T = 0 \tag{16}$$

Define

$$S = \dot{R} R^T \tag{17}$$

Then (16) becomes

$$S + S^T = 0 \tag{18}$$

which shows that S is a skew-symmetric matrix and we can also write (17) as

$$S = \dot{R} R^{-1} \tag{19}$$

Velocity of a Point Due to Rotating Reference Frame

Consider a fixed vector ${}^{B}P$ unchanging with respect to frame $\{B\}$. Its description in another frame $\{A\}$ is given by

$$ {}^{A}P = {}^{A}_{B}R \, {}^{B}P \qquad (20) $$

If frame $\{B\}$ is rotating (the derivative ${}^{A}_{B}\dot{R}$ is nonzero), then ${}^{A}P$ will be changing even though ${}^{B}P$ is constant, i.e.

$$ \underbrace{{}^{A}\dot{P}} = {}^{A}_{B}\dot{R} \, {}^{B}P \qquad (21) $$

or

$$ {}^{A}V_{P} = {}^{A}_{B}\dot{R} \, {}^{B}P \qquad (22) $$

Substitute ${}^{B}P$ from (20) to get

$$ {}^{A}V_{P} = \underbrace{{}^{A}_{B}\dot{R} \, {}^{A}_{B}R^{-1}} \, {}^{A}P \qquad (23) $$

Using Derived Relation in (19), we have

$$ {}^{A}V_{P} = \underbrace{{}^{A}_{B}S \, {}^{A}P}_{\substack{\text{Abreviate} \\ \text{This Product} \\ \text{as } SP}} \qquad (24) $$

${}^{A}_{B}S$ is also called angular-velocity matrix

Now if we assign the elements of a skew-symmetric matrix S as

$$S = \begin{bmatrix} 0 & -\Omega_x & \Omega_y \\ \Omega_x & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \qquad (25)$$

and define

$$\Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}, \quad P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} \qquad (26)$$

Then it is easy to verify that

$$SP = \Omega \times P \qquad (27)$$

So, (24) can be written as

Note (8) ${}^{B}V_p = 0$

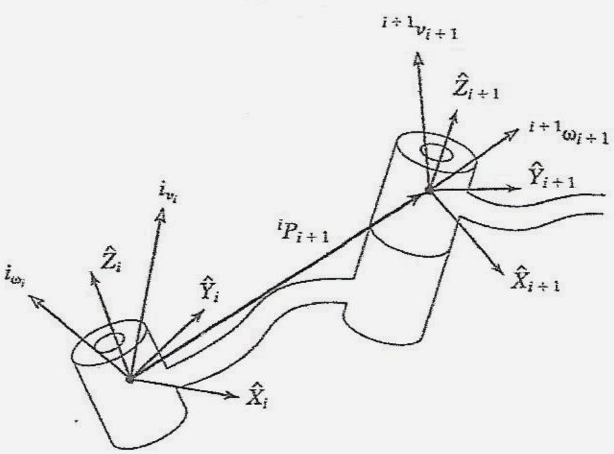
$$\begin{aligned} {}^{A}V_p &= {}^{A}_{B}S\,{}^{A}P \\ &= {}^{A}\Omega_B \times {}^{A}P \end{aligned} \qquad (28)$$

which is an equivalent derivation of (10) as an alternative way of geometric approach using an elegant mathematical approach.

Similar to (10), (28) means that **point P** will have a velocity as seen from $\{A\}$ due to ${}^{A}\Omega_B$

Velocity Propagation From Link To Link

We now consider the problem of calculating the linear and angular velocities of the manipulator links. We think of each link as a rigid body with linear and angular velocity vectors describing its motion. We will express these velocities with respect to the link frame itself. Figure below shows link i and $i+1$, along with their velocity vectors.



" The velocity of link i is given by vectors v_i and ω_i "

- Rotational velocities can be added when both ω vectors are written with respect to the same frame. Therefore, the angular velocity of link $i+1$ is the same as that of link i plus a new component caused by rotational velocity at joint $i+1$. This can be written in terms of frame $\{i\}$ as

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_i + {}^{i}_{i+1}R\, \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} \qquad (29)$$

where

$$ \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} \qquad (30)$$

By premultiplying both sides of (29) by ${}^{i+1}_{i}R$ we can find the description of the angular velocity of link $i+1$ with respect to frame $\{i+1\}$:

$$\underline{{}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \, {}^{i}\omega_i + \dot{\theta}_{i+1} \, {}^{i+1}\hat{Z}_{i+1}} \qquad (31)$$

▷ <u>The Linear velocity</u> of the origin of frame $\{i+1\}$ is the same as that of the origin of frame $\{i\}$ plus a new component caused by rotational velocity of link i. This is exactly the situation described by (13), with one term vanishing because ${}^{i}P_{i+1}$ is constant in frame $\{i\}$. Thus, we have

$${}^{i}V_{i+1} = {}^{i}V_i + {}^{i}\omega_i \times {}^{i}P_{i+1} \qquad (32)$$

premultiplying both sides by ${}^{i+1}_{i}R$, we get

$$\underline{{}^{i+1}V_{i+1} = {}^{i+1}_{i}R \left({}^{i}V_i + {}^{i}\omega_i \times {}^{i}P_{i+1} \right)} \qquad (33)$$

The corresponding relationships for the case that joint $i+1$ is <u>Prismatic</u> are

$$\underline{{}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \, {}^{i}\omega_i} \qquad (34)$$

$$\underline{{}^{i+1}V_{i+1} = {}^{i+1}_{i}R \left({}^{i}V_i + {}^{i}\omega_i \times {}^{i}P_{i+1} \right) + \dot{d}_i \, {}^{i+1}\hat{Z}_{i+1}}$$

(35)

Example

A two-link manipulator with rotational joints is shown below. Calculate the velocity of the tip of the arm as a function of joint rates. Give the answer in two forms - in terms of frame {3} and also in terms of frame {0}.

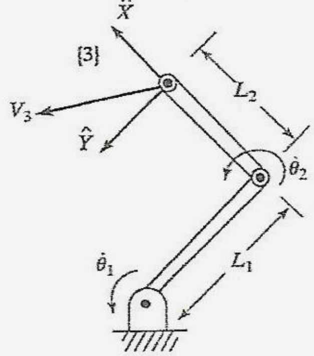
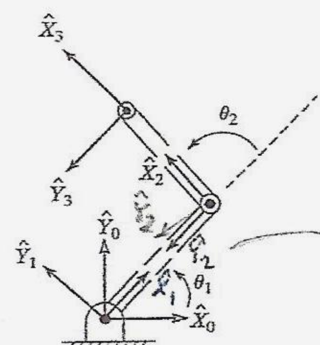


FIGURE 1 : A two-link manipulator.



In general $\hat{Y}_2$ may not lie along the link 1 if link 2 was positioned with $\theta_2 < 90$

FIGURE 2 : Frame assignments for the two-link manipulator.

We will use (31) and (33) to compute the velocity of the origin of each frame starting from the base frame {0}, which has zero velocity.

Computing Jacobian from Derived Velcities

As discussed before, we generally use Jacobian that relate joint velocities to the end-effecter velocities. In terms of notation used in previous discussion we have

$$ {}^0V = {}^0J(q)\,\dot{q} \qquad q : \theta, d$$

where

$$ {}^0V = \begin{bmatrix} {}^0v \\ {}^0\omega \end{bmatrix}$$

Example (Continued)

In the case of two-link arm of our example, we can obtain Jacobian ${}^0J(q)$ as follows

$$ {}^0V_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} \qquad q : \theta_1, \theta_2$$

$$ = \left[\begin{array}{cc} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

${}^0J_v$ ${}^0J_\omega$

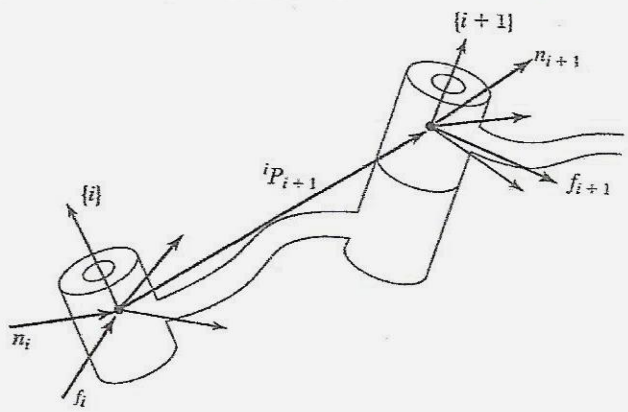
Also, we can obtain ${}^3J(q)$ from ${}^3V_3$ as

$$ {}^3V_3 = \left[\begin{array}{cc} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

${}^3J_v$ ${}^3J_\omega$

Static Forces in Manipulators

In this section we consider each link of the robotic manipulator and write a force-moment balance relationship in terms of the link frames. We compute what static torque must be acting about the joint axis in order for the manipulator to be in static equilibrium. In this way, we solve for the set of joint torques needed to support a static load acting at the end-effector. Here, we will not consider the force on the links due to gravity. The static forces and torques at the joints are those caused by a static force or torque (or both) acting on the last link — for example as when the manipulator has its end-effector in contact with the environment. Figure below shows the static forces and moments acting on link i.



We define

f_i = force exerted on link i by link $i-1$

n_i = moment & Torque exerted on link i by link $i-1$

A. Static Force-Moment Balance Equation for a Single Link

$$ {}^{i}f_{i} - {}^{i}f_{i+1} = 0 \qquad (36)$$

$$ {}^{i}n_{i} - {}^{i}n_{i+1} - {}^{i}P_{i+1} \times {}^{i}f_{i+1} = 0 \qquad (37)$$

If we start with a description of the force and moment applied by the hand, we can calculate the force and moment applied by each link, working from the last link down to the base. To do this, we write (36) and (37) such that they specify iterations from higher numbered links to lower numbered links. The result is identical written as

$$ {}^{i}f_{i} = {}^{i}f_{i+1} \qquad (38)$$

$$ {}^{i}n_{i} = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1} \qquad (39)$$

and by including the rotation matrix describing frame $\{i+1\}$ relative to frame $\{i\}$ we have

$$ {}^{i}f_{i} = {}^{i}_{i+1}R \, {}^{i+1}f_{i+1} \qquad (40)$$

$$ {}^{i}n_{i} = {}^{i}_{i+1}R \, {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i} \qquad (41)$$

Next, we compute the required torques at the joints in order to balance the reaction forces and moments.

Revolute joint $\qquad {}^{i}n_i \cdot \hat{Z}_i$

- To find the required joint torque to maintain the static equilibrium, the dot product of the joint axis vector with the moment vector acting on the link is computed:

$$\tau_i = {}^{i}n_i^{T}\, {}^{i}\hat{Z}_i \tag{42}$$

- In the case that joint i is prismatic, we compute the joint actuator force as the dot product ${}^{i}f_i \cdot {}^{i}\hat{Z}_i$ too

$$\tau_i = {}^{i}f_i^{T}\, {}^{i}\hat{Z}_i \tag{43}$$

Note that we are using the symbol τ even for a linear joint force.

Example (continued)

The two link manipulator is applying a force vector ${}^{3}F$ with its end-effector. Consider this force to be acting at the origin of {3}. Find the required joint torques as a function of the applied force and the structure parameters.

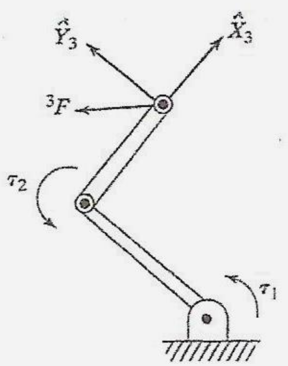


FIGURE 3 : A two-link manipulator applying a force at its tip.

Since the joints are not prismatic, we apply (40)-(42) starting from the last link and going toward the base of the robot.

$$ {}^2f_2 = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}, \quad {}^2n_2 = l_2\hat{X}_2 \times \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} $$

$$ {}^1f_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 f_x - s_2 f_y \\ s_2 f_x + c_2 f_y \\ 0 \end{bmatrix}, $$

$$ {}^1n_1 = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} + l_1 \hat{X}_1 \times {}^1f_1 = \begin{bmatrix} 0 \\ 0 \\ l_1 s_2 f_x + l_1 c_2 f_y + l_2 f_y \end{bmatrix} $$

Therefore, we have

$$ \tau_1 = l_1 s_2 f_x + (l_1 c_2 + l_2) f_y $$

$$ \tau_2 = l_2 f_y $$

which can be written as

$$ \tau = \underbrace{\begin{bmatrix} l_1 s_2 & l_2 + l_1 c_2 \\ 0 & l_2 \end{bmatrix}}_{J^T} \begin{bmatrix} f_x \\ f_y \end{bmatrix} $$

Note that this is the transpose of the Jacobian obtained before ${}^3J_v^T$

B. Derivation of Jacobian in the Force Domain

In general, work is the dot product of a vector force or torque and a displacement vector. Thus, we have

$$F \cdot \delta x = \tau \cdot \delta q \qquad q: \theta, d \qquad (44)$$

where F is a 6×1 cartesian force-moment vector acting at the end-effector, δx is a 6×1 infinitesimal cartesian displacement of the end-effector, τ is a 6×1 torque vector at the joints, and δq is a 6×1 vector representing infinitesimal joint displacements.

Equation (44) can be written as

$$F^T \delta x = \tau^T \delta q \qquad (45)$$

Using the definition of Jacobian

$$\delta x = J \delta q \qquad (46)$$

(45) becomes $F^T J \delta q = \tau^T \delta q$ which reduces to

$$\underline{\tau = J^T F} \qquad (47)$$

When the Jacobian is written with respect to frame $\{0\}$ we write

$$\tau = {}^0J^T \, {}^0F \qquad (48)$$

C. Cartesian Transformation of velocities and Static Forces

Let

$$v = \begin{bmatrix} V \\ \omega \end{bmatrix} \qquad (49)$$

be 6X1 general representation of body velocity and

$$\mathcal{F} = \begin{bmatrix} F \\ N \end{bmatrix} \qquad (50)$$

be 6X1 general representation of force vectors where F is a 3X1 force vector and N is a 3X1 moment vector. We are interested to perform the transformations that map these quantities from one frame to another. Although we have already done the propagation of velocities and forces from link to link, here we perform the matrix transformations corresponding to (31),(32) and (40),(41), respectively.

▷ Velocity Transformation

We write (31),(32) in matrix operator form to transform general velocity vectors from frame {B} to frame {A} or vice versa.

The two frames involved here are rigidly connected, so $\dot{\theta}_{i+1}$ in (31) is set equal to zero.

So, we have

$$\begin{bmatrix} {}^{B}v_{B} \\ {}^{B}\omega_{B} \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & -{}^{B}_{A}R \; {}^{A}P^{*}_{BORG} \\ 0 & {}^{B}_{A}R \end{bmatrix} \begin{bmatrix} {}^{A}v_{A} \\ {}^{A}\omega_{A} \end{bmatrix} \tag{51}$$

where

$$P^{*} = \begin{bmatrix} 0 & -P_x & P_y \\ P_x & 0 & -P_x \\ -P_y & P_x & 0 \end{bmatrix}$$

which can compactly be written as

$$\underline{{}^{B}\mathcal{V}_{B} = {}^{B}_{A}T_{v} \; {}^{A}\mathcal{V}_{A}} \tag{52}$$

Inverting (51) or equivalent (52) we get the velocity in terms of $\{A\}$, given the velocity in $\{B\}$ i.e.

$$\begin{bmatrix} {}^{A}v_{A} \\ {}^{A}\omega_{A} \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P^{*}_{BORG} \; {}^{A}_{B}R \\ 0 & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}v_{B} \\ {}^{B}\omega_{B} \end{bmatrix} \tag{53}$$

or compactly as

$$\underline{{}^{A}\mathcal{V}_{A} = {}^{A}_{B}T_{v} \; {}^{B}\mathcal{V}_{B}} \tag{54}$$

▷ <u>Force-Moment Transformation</u> "From (40), (41)"

$$\begin{bmatrix} {}^{A}F_{A} \\ {}^{A}N_{A} \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ {}^{A}P^{*}_{BORG} \; {}^{A}_{B}R & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}F_{B} \\ {}^{B}N_{B} \end{bmatrix} \tag{55}$$

or compactly

$$\underline{{}^{A}\mathcal{F}_{A} = {}^{A}_{B}T_{f} \; {}^{B}\mathcal{F}_{B}} \tag{56}$$