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Lecture Notes on Kinematics II

"Denavit-Hartenberg Formulation"

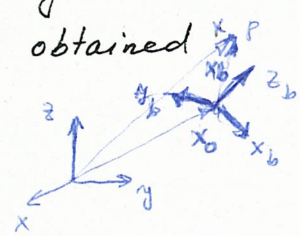
Kinematic Modeling of Manipulator Arms

Open Kinematic chains

Recall that we derived the relationship

$$X = X_0 + R\,X^b$$

which provides the desired coordinate transformation from the body coordinates X^b to the fixed coordinates X. The matrix R completely describes the orientation of the rigid body with reference to the fixed frame and X_0 represents the position of the rigid body. A useful method for representing the coordinate transformations in a compact form is obtained by



$$X = A\,X^b$$

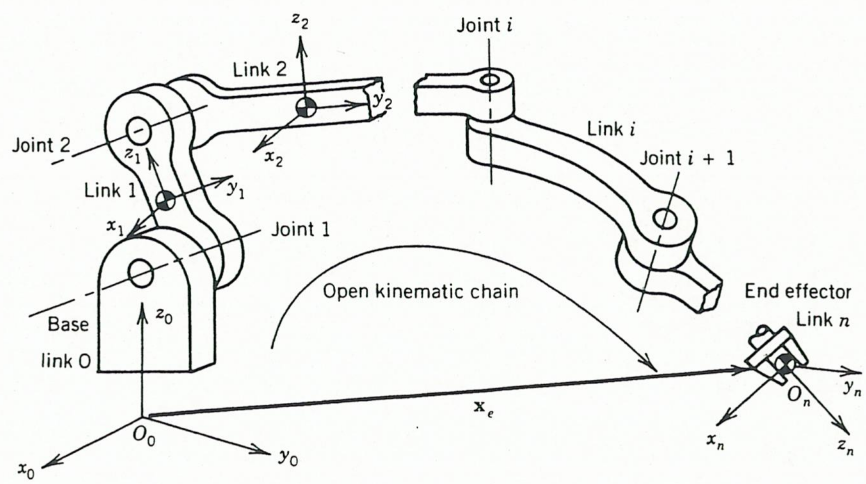
where

$$X = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \quad A = \left[\begin{array}{c|c} R & X_0 \\ \hline 0 & 1 \end{array}\right], \quad X^b = \begin{bmatrix} u \\ v \\ w \\ 1 \end{bmatrix}$$

This alternative representation is known as homogeneous transformation. Consider n consecutive transformation from frame n back to frame 0. Let A_i^{i-1} be the 4×4 matrix associated with the homogenous transformation from frame i to frame $i-1$, then a position vector X^n in frame n is transformed to X^0 in frame 0 by

$$X^0 = A_1^0\, A_2^1 \cdots A_n^{n-1}\, X^n$$

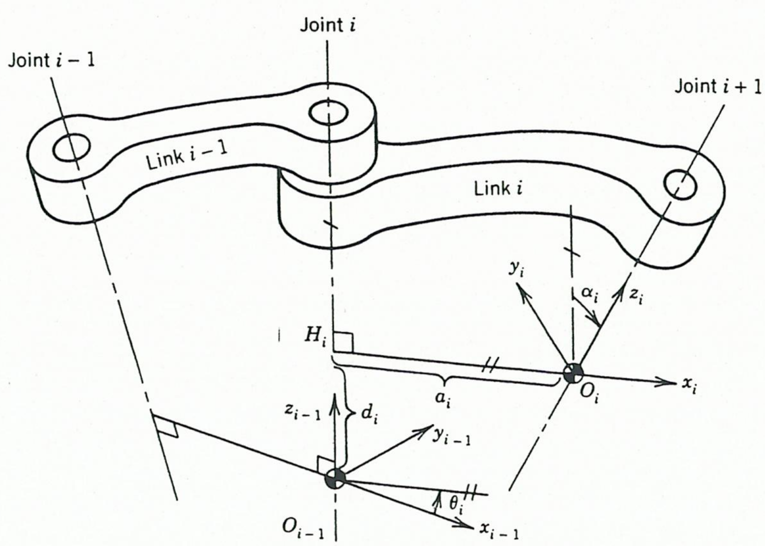
The above mathematical tools are now applied to the kinematic modeling of manipulator arms. A manipulator arm is basically a series of rigid bodies in a kinematic structure. Figure below shows a manipulator arm modeled as a serial linkage of rigid bodies represented by an "open loop" structure known as "open kinematic chain". We attach a coordinate frame to each of the links, namely frame $O_i - x_i y_i z_i$ to link i. We describe the position and orientation of frame $O_i - x_i y_i z_i$ relative to the previous frame $O_{i-1} - x_{i-1} y_{i-1} z_{i-1}$ by using the 4X4 matrix describing the homogenous transformation between these frames. In this way the end-effector position and orientation is obtained by the consecutive homogenous transformations from the last frame $(O_n - x_n y_n z_n)$ back to the base frame $(O_0 - x_0 y_0 z_0)$.



Open kinematic chain.

The Denavit-Hartenberg Notation and Kinematic Equations

The Denavit-Hartenberg notation is used as a systematic method of describing the kinematic relationship between a pair of adjacent links involved in an open kinematic chain. The following figure shows a pair of adjacent links, link $i-1$ and link i, and their associated joints $i-1$, i and $i+1$. Line $H_i O_i$ is the common normal to joint axes i and $i+1$. The relationship between the two links is described by the relative position and orientation of the two coordinate frames attached to the two links. In the Denavit-Hartenberg notation, the origin of the i-th coordinate frame O_i is located at the intersection of joint axis $i+1$ and the common normal between joint axes i and $i+1$.



The Denavit-Hartenberg notation.

Note that the frame of link i is at joint $i+1$ rather than at joint i. The x_i axis is directed along the extention line of the common normal, while the z_i axis is along the joint axis $i+1$. Finally, the y_i axis is chosen such that the resultant frame $O_i - x_i y_i z_i$ forms a right-hand coordinate system. The relative location of the two frames can be completely determined by the following four parameters:

a_i the length of the common normal

d_i the distance between the origin O_{i-1} and the point H_i

α_i the angle between the joint axis i and the z_i axis in the right-hand sense.

θ_i the angle between the x_{i-1} axis and the common normal $H_i O_i$ measured about z_{i-1} axis in the right hand sense.

The parameters a_i and α_i are constant depending the geometry of the link. One of the other two parameters d_i and θ_i varies depending on the joint structure (its motion). For a revolute joint, parameter θ_i is the variable that represents the joint displacement, while parameter d_i is constant. On the other hand, for a prismatic joint, d_i is variable and θ_i is constant.

Let us now formulate the kinematic relationship between the adjacent links using 4x4 matrices. Figure below shows once more the two coordinate frames $O_i - x_i y_i z_i$ and $O_{i-1} - x_{i-1} y_{i-1} z_{i-1}$ along with the intermediate coordinate frame $H_i - x'_i y'_i z'_i$ attached at point H_i. Let X^i, X' and X^{i-1} be 4x1 position vectors corresponding to each coordinate frame. From the figure the coordinate transformation from X^i to X' is given by

$$X' = A_i^{int} X^i$$

where

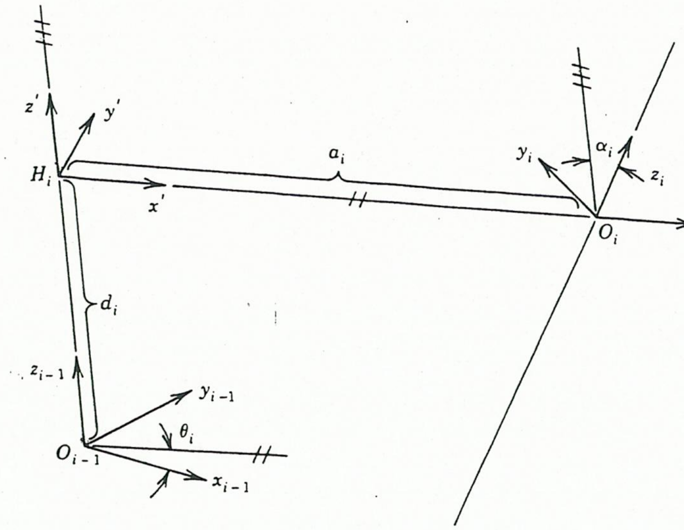
$$A_i^{int} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

similarly the transformation from X' to X^{i-1} is given by

$$X^{i-1} = A_{int}^{i-1} X'$$

where

$$A_{int}^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



The relationship between adjacent coordinate frames in the Denavit-Hartenberg notation.

Combining the above equations leads to

$$X^{i-1} = A_i^{i-1} X^i$$

where

$$A_i^{i-1} = A_{int}^{i-1} A_i^{int} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cos\alpha_i & \sin\theta_i \sin\alpha_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cos\alpha_i & -\cos\theta_i \sin\alpha_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix A_i^{i-1} represents the position and orientation of frame i relative to frame $i-1$.

Kinematic Equations

Using the Denavit_Hartenberg notation we express the position and orientation of the end-effector as a function of joint displacements. The displacement of each joint is either angle θ_i or distance d_i, depending on the type of joint. In general we denote the joint displacement by q_i, which is defined as

$$q_i = \theta_i \qquad \text{for a revolute joint}$$

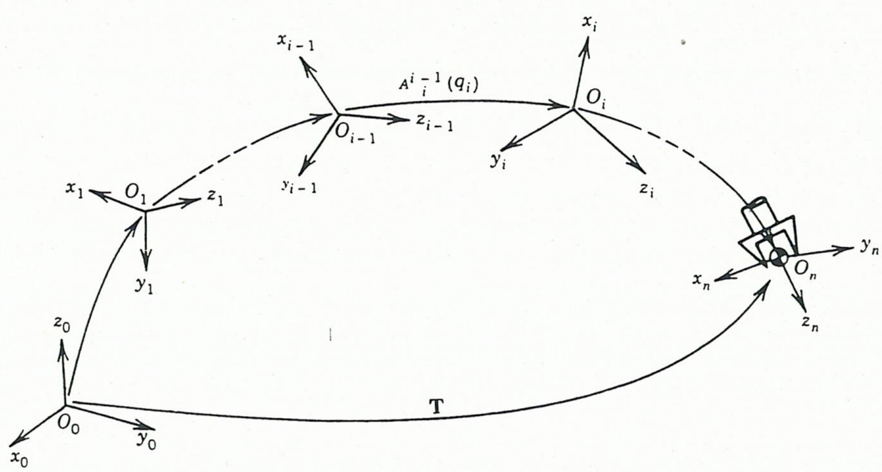
$$q_i = d_i \qquad \text{for a prismatic joint}$$

The position and orientation of link i relative to link $i-1$ is then described as a function of q_i using the 4X4 matrix $A_i^{i-1}(q_i)$.

As shown in the figure, the manipulator arm consists of $n+1$ links from the base to the tip. Considering the n consecutive coordinate transformations along the serial linkage, we can derive the end-effector location viewed from the base frame. From our previous derivation of two adjacent links, the position and orientation of the last link relative to the base frame is easily obtained by

$$T = A_1^0(q_1)\, A_2^1(q_2) \cdots A_n^{n-1}(q_n)$$

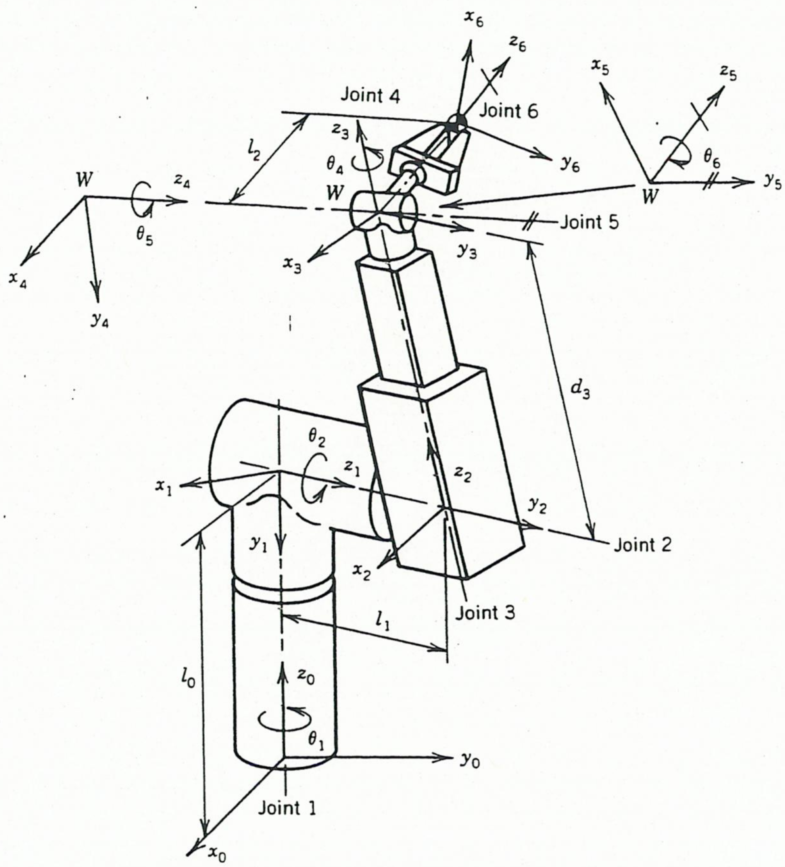
where T is a 4x4 matrix representing the kinematic equation of the manipulator arm



The representation of the end-effector location by a 4x4 matrix.

<u>Example</u> : The Kinematic Model of a 5R1P Manipulator Arm

Figure below shows a six-degree-of-freedom manipulator arm with 5 revolute joints and one prismatic joint. The Denavit-Hartenberg parameters are also given in the table.



5-R-1-P manipulator.

Link parameters for the 5-R-1-P manipulator.

link number	α_i	a_i	d_i	θ_i
1	-90°	0	l_0	θ_1
2	+90°	0	l_1	θ_2
3	0	0	d_3	0
4	-90°	0	0	θ_4
5	+90°	0	0	θ_5
6	0	0	l_2	θ_6

The 4x4 matrix $A_i^{i-1}(q_i)$ can be determined by using the parameters given in the table as follows:

$$A_1^0(\theta_1) = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_2^1(\theta_2) = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_3^2(d_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4^3(\theta_4) = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_5^4(\theta_5) = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad A_6^5(\theta_6) = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$.

The kinematic equation of this manipulator arm is then given by

$$T = A_1^0(\theta_1)\, A_2^1(\theta_2)\, A_3^2(d_3)\, A_4^3(\theta_4)\, A_5^4(\theta_5)\, A_6^5(\theta_6)$$

Thus, the end-effector position and orientation T is represented as a function of joint displacements $\theta_1, \theta_2, d_3, \theta_4, \theta_5$, and θ_6.

Inverse Kinematic

The problem of finding the end-effector position and orientation for a given set of joint displacements is referred to as direct kinematic problem, which was discussed in previous section. In this section the inverse kinematic problem is discussed, namely the problem of finding the joint displacements that lead the end-effector to the specified position and orientation. In the direct problem, the end-effector location is determined uniquely for a given set of joint displacements. On the other hand, the inverse problem is more complex in the sense that kinematic equation must be solved in certain special way and the fact that multiple solutions may exist for the same end-effector locations and arm structures. Furthermore, since the kinematic equation is comprised of nonlinear equations with many trigonometric functions, it is not always possible to derive a closed-form solution.

Let us solve the kinematic equation for the 5-R-1-P example discussed in previous section. For this example closed-form solution exist for an arbitrary end-effector location T. Recall that the kinematic equation was given by

$$T = A_1^0 A_2^1 A_3^2 A_4^3 A_5^4 A_6^5$$

To set-up and solve this equation for the inverse problem, let us postmultiply both sides of the equation by the inverse of A_6^5 i.e.

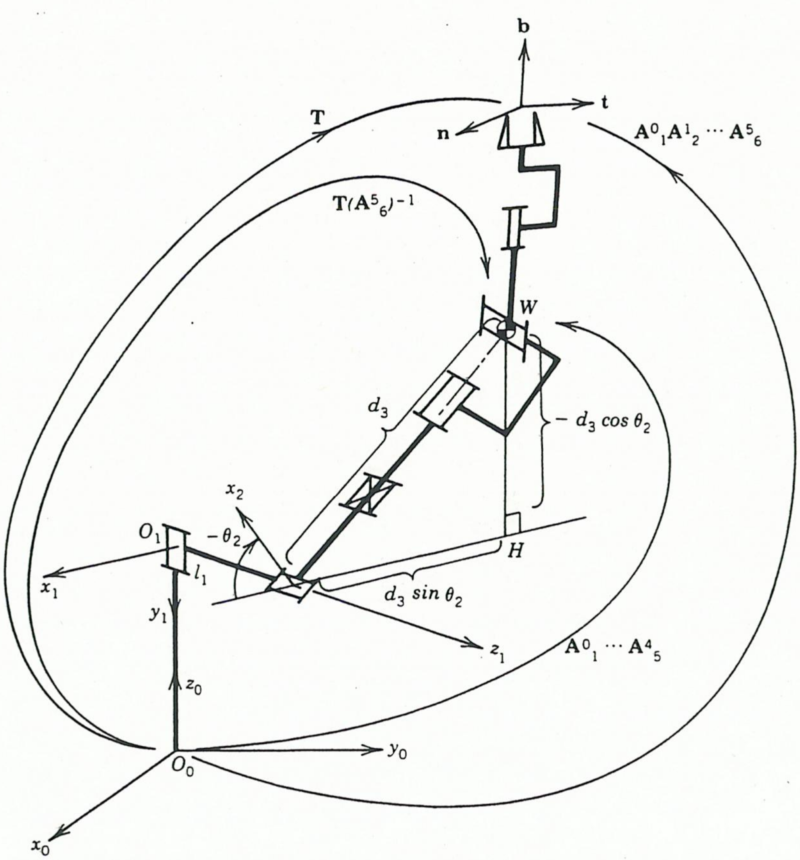
$$T\,(A_6^5)^{-1} = A_1^0\,A_2^1\,A_3^2\,A_4^3\,A_5^4 \qquad (*)$$

Further premultiplying both sides by $(A_1^0)^{-1}$ we get

$$(A_1^0)^{-1}\,T\,(A_6^5)^{-1} = A_2^1\,A_3^2\,A_4^3\,A_5^4 \qquad (**)$$

Note that the left hand side of the equation (*) is only a function of θ_6, while the right hand side involves all the other joint displacements. Similarly, equation (**) has θ_1 and θ_6 on the left-hand side and the remaining joint displacements on the right-hand side.
To this end, let us interpret the physical meanings of different expressions by using the following figure, which shows the skeleton structure of the 5-R-1-P manipulator. Each arc in the figure represents the relationship between the two coordinate frames, and the 4x4 matrix on the arc gives the position and orientation of the frame viewed from the frame at the origin of the arc. The product of multiple matrices represents the position and orientation of the final frame viewed from

the initial frame along the path of the arcs associated with the matrices. The left-hand side of the original kinematic equation represents the end-effector position and orientation viewed from the base frame directly, while the right-hand side represents the same end-effector position and orientation through another path along the arm linkage. Both sides of equation (*) represent the position and orientation of the frame attached to link 5 with reference to the base frame through two different paths reaching the same frame. The origin of the coordinate frame 5 is at point W, the coordinates of which are represented by the fourth column of the 4×4 matrices in (*).



Skeleton structure of the 5-R-1-P manipulator.

Note, however, that the position of W in the figure depends only on the first three joints, and is independent of the last three joints. Therefore, if one compares the fourth column vectors of the matrices on both sides of (*), simultaneous equations with only three unknowns should be obtained. Further, a more convenient form of simultaneous equations can be derived by evaluating the fourth columns of equation (**). The fourth column vector of the right-hand side of (**) represents the position of W with respect to the first coordinate frame through the arm linkage, as shown in the figure above, and is simply given by

$$x'_W = \begin{pmatrix} d_3 s_2 \\ -d_3 c_2 \\ \ell_1 \end{pmatrix}$$

the left-hand side of (**) describes the same position, now reached through the base frame and the end-effector. Thus, writing the desired end-effector position and orientation T in the form

$$T = \begin{bmatrix} n_x & t_x & b_x & p_x \\ n_y & t_y & b_y & p_y \\ n_z & t_z & b_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and substituting into the left-hand side of (**), we obtain another expression of x'_W, namely

$$x'_W = \begin{bmatrix} p_x^* c_1 + p_y^* s_1 \\ -p_z^* + \ell_0 \\ -p_x^* s_1 + p_y^* c_1 \end{bmatrix} \quad \text{where} \quad \begin{aligned} p_x^* &= p_x - \ell_2 b_x \\ p_y^* &= p_y - \ell_2 b_y \\ p_z^* &= p_z - \ell_2 b_z \end{aligned}$$

Equating both expressions for X'_w we get

$$d_3 s_2 = p_x^* c_1 + p_y^* s_1$$
$$-d_3 c_2 = -p_z^* + l_0$$
$$l_1 = -p_x^* s_1 + p_y^* c_1$$

To solve the last equation, we let:

$$t = \tan\left(\frac{\theta_1}{2}\right)$$

so that

$$c_1 = \cos\theta_1 = \frac{1-t^2}{1+t^2}, \quad s_1 = \sin\theta_1 = \frac{2t}{1+t^2}$$

Substituting into the last equation, we obtain

$$(l_1 + p_y^*)t^2 + 2p_x^* t + l_1 - p_y^* = 0$$

$$\Rightarrow \theta_1 = 2\tan^{-1}\left[\frac{-p_x^* \pm \sqrt{p_x^{2*} + p_y^{2*} - l_1^2}}{l_1 + p_y^*}\right]$$

Dividing both sides of the first equation by the corresponding sides of the second equation, we obtain

$$\theta_2 = \tan^{-1}\left[\frac{p_x^* c_1 + p_y^* s_1}{p_z^* - l_0}\right]$$

Furthermore, d_3 can be obtained by taking the sum of the squares of the first two equations as

$$d_3 = \pm\sqrt{(p_x^* c_1 + p_y^* s_1)^2 + (p_z^* - l_0)^2}$$