CS 4610/5335: Robotic Science and Systems (Spring 2023) Robert Platt
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HW 1: Transformations and Kinematics

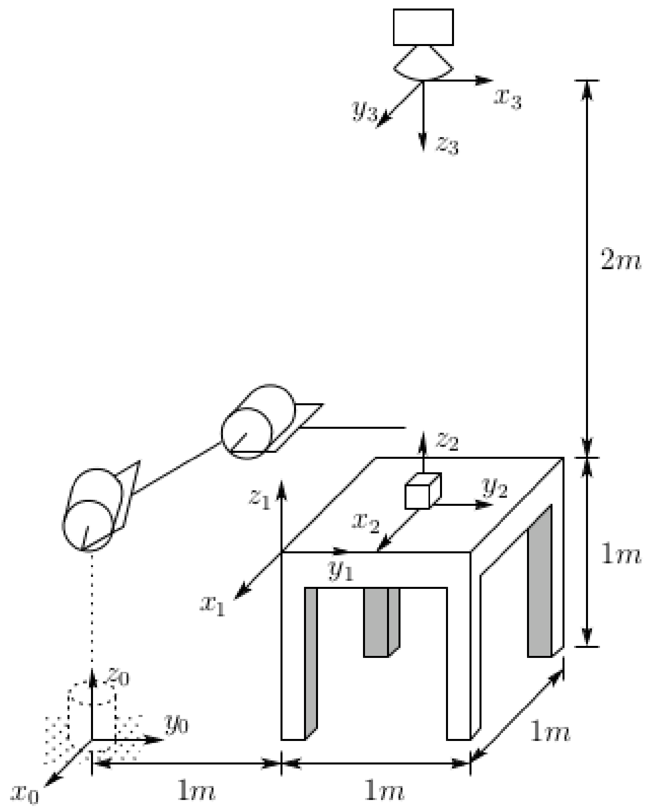
Please remember the following policies:

- Submissions should be made electronically via the Canvas. Please ensure that your solutions for both the written and/or programming parts are present and zipped into a single file.
- Solutions may be handwritten or typeset. For the former, please ensure handwriting is legible.
- You are welcome to discuss the programming questions (but *not* the written questions) with other students in the class. However, you must understand and write all code yourself. Also, you must list all students (if any) with whom you discussed your solutions to the programming questions.

1. **2 points.** Consider a frame $\{B\}$, which is obtained from frame $\{A\}$ by first rotating about the X_A axis by an angle of θ, followed by a translation of $[1, 2, 3]^\top$ (in the $\{A\}$ frame).
   (a) Find the homogeneous transformation matrix ${}^AT_B$.
   (b) Compute the homogeneous transformation matrix ${}^BT_A$.
   (c) Given $\theta = \frac{\pi}{4}$ and ${}^Ap = [4, 5, 6]^\top$, compute ${}^Bp$.
   (d) Sketch the frames for $\theta = \frac{\pi}{4}$ and ${}^Ap = [4, 5, 6]^\top$, and check that your answer in part (c) seems correct.

2. **2 points.**
   (a) Consider a frame $\{A\}$. It is first rotated about the Y_A axis by an angle θ to form frame $\{B\}$, and then rotated about the Z_B axis by an angle ϕ to form frame $\{C\}$. Determine the 3×3 rotation matrix, ${}^AR_C$, which will transform the coordinates of a position vector from ${}^Cp$, its value in frame $\{C\}$, into ${}^Ap$, its value in frame $\{A\}$.
   (b) Recall that there are multiple interpretations of the rotation matrix. Suppose ${}^AR_B = R_y(\theta)$, the rotation matrix obtained by rotating about the Y_A axis.
      - When viewed as a *transformation*, multiplying the rotation matrix ${}^AR_B$ by a vector ${}^Bp$ results in the vector ${}^Ap = {}^AR_B{}^Bp$, specifying the *same* point in space in a *different* coordinate frame.
      - When viewed as an *operator*, multiplying the same rotation matrix $R = R_y(\theta)$ by a vector ${}^Ap$ results in the vector ${}^Ap' = R_y(\theta){}^Ap$, specifying a *different* point in space in the *same* coordinate frame. The new point ${}^Ap'$ is the result of rotating the original point ${}^Ap$ about the Y_A axis by an angle of θ.

   Interpreting the rotation matrix from part (a) as an operator, explain why it is equivalent to first rotating by an angle ϕ about the Z_A axis, followed by rotating by an angle θ about the Y_A axis.

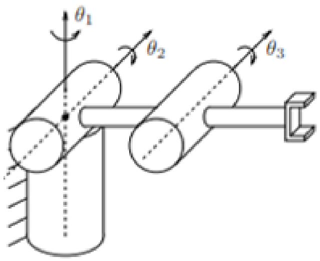


3. **2 points.** (Spong, Problem 2-37) Consider the diagram above. A robot is set up 1m from a table. The table is 1m long on each side. A frame $\{1\} : x_1, y_1, z_1$ is fixed to the edge of the table as shown. A cube measuring 20cm on a side is placed on top of the table and at the center of the table with frame $\{2\} : x_2, y_2, z_2$ established at the center of the cube's bottom face as shown. A camera is situated directly above the center of the block 2m above the table top with frame $\{3\} : x_3, y_3, z_3$ attached as shown.

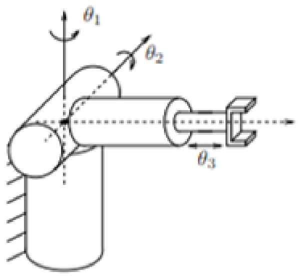
   (a) Find the homogeneous transformations relating each of these frames to the base frame $\{0\}$.

   (b) Find the homogeneous transformation relating the cube's frame $\{2\}$ to the camera frame $\{3\}$.

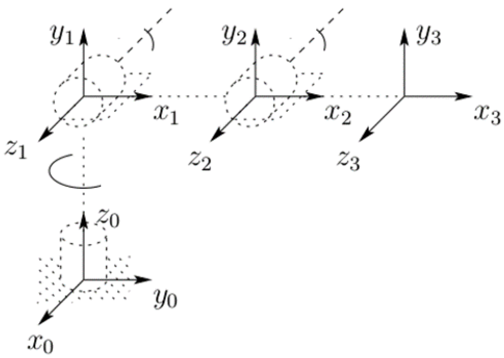
   (c) Suppose the robot pushes the cube and the camera now detects the cube's origin at $[0.2, -0.3, 2]^\top$. Use the above transformations to determine the position of the cube in the base frame $\{0\}$.



(a)



(b)

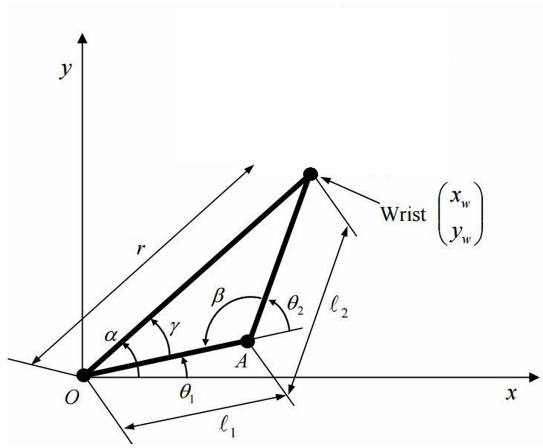


4. **2 points.** For the three degree-of-freedom (3-DOF) manipulators shown above, find the forward kinematics map from the robot base (center of the vertical cylinder's bottom face) to the end-effector. For rotational joints, the manipulators are shown in their zero configuration, i.e., the diagram illustrates the arms at $\theta_1 = 0, \theta_2 = 0$, and $\theta_3 = 0$ for part (a). The link lengths are given by L_1, L_2, L_3, as follows:

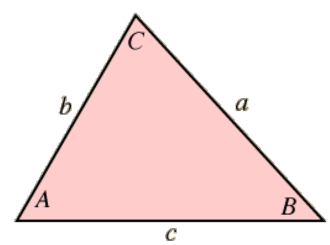
   (a) L_1 is the vertical height of the first joint, L_2 is the length of the arm segment after $\theta_{1,2}$ and before θ_3, and L_3 is the length of the arm after θ_3.

   (b) L_1 and L_2 are the same as in part (a), and L_3 is determined by the translational joint, θ_3, which determines the additional amount by which the hand extends beyond the L_2 part of the arm.

   Hint: Decompose the forward kinematics map into a series of simple transformations. A suggested set of intermediate coordinate frames for part (a) are shown in the diagram above. You are free to choose the intermediate coordinate frames – it should not affect the answer.



$$c^2 = a^2 + b^2 - 2ab\cos C$$



5. **2 points.** Consider the 2-DOF planar arm shown above with two rotational joints, specified by θ_1 and θ_2, and link lengths ℓ_1 and ℓ_2 respectively. We are interested in solving the inverse kinematics problem for this arm, i.e., given a desired wrist position $[x_w, y_w]^\top$, find a joint configuration $[\theta_1, \theta_2]^\top$ that achieves the position, if a solution exists. In this problem, we will derive a *geometric* solution to this problem by finding α, β, γ.

   (a) Find an expression for $\cos\beta$ using the *law of cosines*, illustrated on the right.

   (b) Using your answer from part (a), find an expression for θ_2.

   (c) Assuming you now have θ_2, find an expression for γ.

   (d) Find an expression for α, and therefore also for θ_1.

   (e) Depending on $[x_w, y_w]^\top$, there may be exactly 0, 1, or 2 solutions. Identify the conditions for when each of these cases occurs, and for the two-solution case, sketch and find the other solution.
   Hint: The diagram shows a wrist position with two inverse kinematics solutions.
   The expressions for θ_1 and θ_2 for the other solution should be very similar to what you have found above.