Closed Model

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Example

A cargo plane has three compartments for storing cargo: front, center, and back. These compartments have capacity limits on both weight and space as summarized below:

Compartment	Weight Capacity (Tons)	Space Capacity (Cubic Feet)
Front	12	7,000
Center	18	9,000
Back	10	5,000

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the airplane.

Example

The following four cargoes have been offered for shipment on an upcoming flight as space is available:

Cargo	Weight (Tons)	Volume (Cubic Feet/Ton)	Profit ($/Ton)
1	20	500	320
2	16	700	400
3	25	600	360
4	13	400	290

Any portion of these cargoes can be accepted. The objective is to determine how much (if any) of each cargo should be accepted and how to distribute each among the compartments to maximize the total profit for the flight.

Alternative

- Formulate this problem in closed form as an LP to maximize profit.
- Does the model meet the four assumptions of LP?
- Solve the model with Gams.

Open Form:Decision variables

x_{11}=tones of cargo 1 put in to Front
x_{12}=tones of cargo 1 put in to Center
x_{13}=tones of cargo 1 put in to Back
x_{21}=tones of cargo 2 put in to Front
x_{22}=tones of cargo 2 put in to Center
x_{23}=tones of cargo 2 put in to Back
x_{31}=tones of cargo 3 put in to Front
x_{32}=tones of cargo 3 put in to Center
x_{33}=tones of cargo 3 put in to Back
x_{41}=tones of cargo 4 put in to Front
x_{42}=tones of cargo 4 put in to Center
x_{43}=tones of cargo 4 put in to Back

Open Form:Objective Function

$$Max z = 320(x_{11} + x_{12} + x_{13}) + 400(x_{21} + x_{22} + x_{23})$$

$$360(x_{31} + x_{32} + x_{33}) + 290(x_{41} + x_{42} + x_{43})$$

Open Form:Capacity Constraints

the space capacity of each compartment can not exceed

$$500x_{11} + 700x_{21} + 600x_{31} + 400x_{41} \leq 7000$$

$$500x_{12} + 700x_{22} + 600x_{32} + 400x_{42} \leq 9000$$

$$500x_{13} + 700x_{23} + 600x_{33} + 400x_{43} \leq 5000$$

Open Form:Capacity Constraints

the weight capacity of each compartment can not exceed

$$x_{11} + x_{21} + x_{31} + x_{41} \leq 12$$

$$x_{12} + x_{22} + x_{32} + x_{42} \leq 18$$

$$x_{13} + x_{23} + x_{33} + x_{43} \leq 10$$

Open Form:Availability Constraints

can not packed more than available tones of cargoes

$$x_{11} + x_{12} + x_{13} \leq 20$$

$$x_{21} + x_{22} + x_{23} \leq 16$$

$$x_{31} + x_{32} + x_{33} \leq 25$$

$$x_{41} + x_{42} + x_{43} \leq 13$$

Open Form:Balance Constraints

the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the plane

$$\frac{x_{11}+x_{21}+x_{31}+x_{41}}{12}=\frac{x_{12}+x_{22}+x_{32}+x_{42}}{18}$$

$$\frac{x_{12}+x_{22}+x_{32}+x_{42}}{18}=\frac{x_{13}+x_{23}+x_{33}+x_{43}}{10}$$

Open Form:Sign Constraints

the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the plane

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}, x_{41}, x_{42}, x_{43} \geq 0$$

Closed Form

- Sets
- Parameters
- Decision variables
- Closed mathematical model

Sets

I=$\{1,2,3,4\}$ cargoes

J=$\{1,2,3\}$ compartments (Front,Center,Back)

Parameters

w_j:weight capacity of compartment j

$$
\begin{aligned}
x_{11} + x_{21} + x_{31} + x_{41} &\leq 12 \\
x_{12} + x_{22} + x_{32} + x_{42} &\leq 18 \\
x_{13} + x_{23} + x_{33} + x_{43} &\leq 10
\end{aligned}
$$

$$
\mathrm{w} = \begin{bmatrix} 12 \\ 18 \\ 10 \end{bmatrix}
$$

Parameters

s_j:space capacity of compartment j

$$500x_{11} + 700x_{21} + 600x_{31} + 400x_{41} \leq \boxed{7000}$$

$$500x_{12} + 700x_{22} + 600x_{32} + 400x_{42} \leq \boxed{9000}$$

$$500x_{13} + 700x_{23} + 600x_{33} + 400x_{43} \leq \boxed{5000}$$

$$s = \begin{bmatrix} 7000 \\ 9000 \\ 5000 \end{bmatrix}$$

Parameters

v_i:volume of cargo i

$$\boxed{500x_{11} + 700x_{21} + 600x_{31} + 400x_{41}} \leq 7000$$

$$500x_{12} + 700x_{22} + 600x_{32} + 400x_{42} \leq 9000$$

$$500x_{13} + 700x_{23} + 600x_{33} + 400x_{43} \leq 5000$$

$$v = \begin{bmatrix} 500 & 700 & 600 & 400 \end{bmatrix}$$

Parameters

a_i:available tones of cargo i

$$
\begin{aligned}
x_{11} + x_{12} + x_1 &\le 20\\
x_{21} + x_{22} + x_2 &\le 16\\
x_{31} + x_{32} + x_3 &\le 25\\
x_{41} + x_{42} + x_4 &\le 13
\end{aligned}
$$

$$
a = \begin{bmatrix} 20 \\ 16 \\ 25 \\ 13 \end{bmatrix}
$$

Parameters

p_i:profit of cargo i

$$Max z = 320(x_{11} + x_{12} + x_{13}) + 400(x_{21} + x_{22} + x_{23}+)$$

$$360(x_{31} + x_{32} + x_{33}) + 290(x_{41} + x_{42} + x_{43})$$

$$\text{p} = \begin{bmatrix} 320 & 400 & 360 & 290 \end{bmatrix}$$

Decision variables

x_{ij}= of tones cargo i is put into compartment j

Open Form:Objective Function

$$Maxz = 320(x_{11} + x_{12} + x_{13}) + 400(x_{21} + x_{22} + x_{23})$$

$$360(x_{31} + x_{32} + x_{33}) + 290(x_{41} + x_{42} + x_{43})$$

$$\mathrm{p} = \begin{bmatrix} 320 & 400 & 360 & 290 \end{bmatrix}$$

Open Form:Objective Function

$$Max z = 320(x_{11} + x_{12} + x_{13}) + 400(x_{21} + x_{22} + x_{23})$$

$$360(x_{31} + x_{32} + x_{33}) + 290(x_{41} + x_{42} + x_{43})$$

$$\mathrm{p} = \begin{bmatrix} 320 & 400 & 360 & 290 \end{bmatrix}$$

$$Max z = p_1(x_{11} + x_{12} + x_{13}) + p_2(x_{21} + x_{22} + x_{23})$$

$$p_3(x_{31} + x_{32} + x_{33}) + p_4(x_{41} + x_{42} + x_{43})$$

Open Form:Objective Function

$$Max z = 320(x_{11} + x_{12} + x_{13}) + 400(x_{21} + x_{22} + x_{23})$$

$$360(x_{31} + x_{32} + x_{33}) + 290(x_{41} + x_{42} + x_{43})$$

$$\mathrm{p} = \begin{bmatrix} 320 & 400 & 360 & 290 \end{bmatrix}$$

$$Max z = p_1(x_{11} + x_{12} + x_{13}) + p_2(x_{21} + x_{22} + x_{23})$$

$$p_3(x_{31} + x_{32} + x_{33}) + p_4(x_{41} + x_{42} + x_{43})$$

$$Max z = p_1 \sum_{j=1}^{3} x_{1j} + p_2 \sum_{j=1}^{3} x_{2j} + p_3 \sum_{j=1}^{3} x_{3j} + p_4 \sum_{j=1}^{3} x_{4j}$$

Objective Function

$$Max z = \sum_{i=1}^{4} \sum_{j=1}^{3} p_i x_{ij}$$

Open Form:Capacity Constraints

the space capacity of each compartment can not exceed

$$500x_{11} + 700x_{21} + 600x_{31} + 400x_{41} \leq 7000$$

$$500x_{12} + 700x_{22} + 600x_{32} + 400x_{42} \leq 9000$$

$$500x_{13} + 700x_{23} + 600x_{33} + 400x_{43} \leq 5000$$

$$\mathrm{v} = \begin{bmatrix} 500 & 700 & 600 & 400 \end{bmatrix}$$

$$\mathrm{s} = \begin{bmatrix} 7000 \\ 9000 \\ 5000 \end{bmatrix}$$

Open Form:Capacity Constraints

$$500x_{11} + 700x_{21} + 600x_{31} + 400x_{41} \leq 7000$$

$$500x_{12} + 700x_{22} + 600x_{32} + 400x_{42} \leq 9000$$

$$500x_{13} + 700x_{23} + 600x_{33} + 400x_{43} \leq 5000$$

$$\mathrm{v} = \begin{bmatrix} 500 & 700 & 600 & 400 \end{bmatrix}$$

$$\mathrm{s} = \begin{bmatrix} 7000 \\ 9000 \\ 5000 \end{bmatrix}$$

$$v_1x_{11} + v_2x_{21} + v_3x_{31} + v_4x_{41} \leq s_1$$

$$v_1x_{12} + v_2x_{22} + v_3x_{32} + v_4x_{42} \leq s_2$$

$$v_1x_{13} + v_2x_{23} + v_3x_{33} + v_4x_{43} \leq s_3$$

Open Form:Capacity Constraints

$$v_1 x_{11} + v_2 x_{21} + v_3 x_{31} + v_4 x_{41} \leq s_1$$

$$v_1 x_{12} + v_2 x_{22} + v_3 x_{32} + v_4 x_{42} \leq s_2$$

$$v_1 x_{13} + v_2 x_{23} + v_3 x_{33} + v_4 x_{43} \leq s_3$$

$$\sum_{i=1}^{4} v_i x_{i1} \leq s_1$$

$$\sum_{i=1}^{4} v_i x_{i2} \leq s_2$$

$$\sum_{i=1}^{4} v_i x_{i3} \leq s_3$$

Capacity Constraints

the space capacity of each compartment can not exceed

$$\sum_{i=1}^{4} v_i x_{ij} \leq s_j, \quad \forall j$$

Open Form:Capacity Constraints

$$x_{11} + x_{21} + x_{31} + x_{41} \leq 12$$

$$x_{12} + x_{22} + x_{32} + x_{42} \leq 18$$

$$x_{13} + x_{23} + x_{33} + x_{43} \leq 10$$

$$\mathrm{w} = \begin{bmatrix} 12 \\ 18 \\ 10 \end{bmatrix}$$

Open Form:Capacity Constraints

$$x_{11} + x_{21} + x_{31} + x_{41} \leq w_1$$

$$x_{12} + x_{22} + x_{32} + x_{42} \leq w_2$$

$$x_{13} + x_{23} + x_{33} + x_{43} \leq w_3$$

$$\text{w} = \begin{bmatrix} 12 \\ 18 \\ 10 \end{bmatrix}$$

$$\sum_{i=1}^{4} x_{i1} \leq w_1$$

$$\sum_{i=1}^{4} x_{i2} \leq w_2$$

$$\sum_{i=1}^{4} x_{i3} \leq w_3$$

Capacity Constraints

the weight capacity of each compartment can not exceed

$$\sum_{i=1}^{4} x_{ij} \leq w_j, \quad \forall j$$

Open Form:Availability Constraints

can not packed more than available tones of cargoes

$$x_{11} + x_{12} + x_{13} \leq 20$$

$$x_{21} + x_{22} + x_{23} \leq 16$$

$$x_{31} + x_{32} + x_{33} \leq 25$$

$$x_{41} + x_{42} + x_{43} \leq 13$$

$$\mathrm{a} = \begin{bmatrix} 20 \\ 16 \\ 25 \\ 13 \end{bmatrix}$$

Open Form:Availability Constraints

can not packed more than available tones of cargoes

$$a = \begin{bmatrix} 20 \\ 16 \\ 25 \\ 13 \end{bmatrix}$$

$$\sum_{j=1}^{3} x_{1j} \leq a_1$$

$$\sum_{j=1}^{3} x_{2j} \leq a_2$$

$$\sum_{j=1}^{3} x_{3j} \leq a_3$$

$$\sum_{j=1}^{3} x_{4j} \leq a_4$$

Availability Constraints

can not packed more than available tones of cargoes

$$\sum_{j=1}^{3} x_{ij} \leq a_i, \quad \forall i$$

Balance Constraints

can not packed more than available tones of cargoes

$$\frac{\sum_{i=1}^{4} x_{i1}}{12} = \frac{\sum_{i=1}^{4} x_{i2}}{18}$$

$$\frac{\sum_{i=1}^{4} x_{i2}}{18} = \frac{\sum_{i=1}^{4} x_{i3}}{10}$$

Sign Constraints

can not packed more than available tones of cargoes

$$x_{ij} \geq 0, \quad \forall (i, j)$$

Mathematical Model

$$Max z = \sum_{i}^{4} \sum_{j}^{3} p_i x_{ij}$$

s.t

$$\sum_{i=1}^{4} v_i x_{ij} \leq s_j, \quad \forall j$$

$$\sum_{i=1}^{4} x_{ij} \leq w_j, \quad \forall j$$

$$\sum_{j=1}^{3} x_{ij} \leq a_i, \quad \forall i$$

$$\frac{\sum_{i=1}^{4} x_{i1}}{12} = \frac{\sum_{i=1}^{4} x_{i2}}{18}$$

$$\frac{\sum_{i=1}^{4} x_{i2}}{18} = \frac{\sum_{i=1}^{4} x_{i3}}{10}$$

$$x_{ij} \geq 0, \quad \forall (i,j)$$

Assumptions of LP

- Proportionality
- Additivity
- Divisibility
- Certanity

Assumptions of LP

Proportionality: the contribution of each decision variable in any one equation is proportional to a constant.

Additivity: Every equation in the model is the sum of the individual contributions of the respective activities.

Divisibility:each cargo can be split into whatever proportions/fractions

Certanity:all the parameters are known