TOPOLOGICAL COMPLETENESS AND COMPACTNESS OF S4

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Abstract. Given a logical system, completeness and soundness are arguably the most desirable properties of such a system. In essence, soundness tells us that provable statements are valid, and completeness tells us that all valid statements are provable. In this paper we present a brief outline of the topological semantics of the modal logic **S4** and prove its completeness, soundness and compactness. None of this work is original, however we have tried to make the paper accessible to mathematics and philosophy students alike.

Contents

1. Introduction: The Big Picture

Before diving into the details, we wish to give the reader a gentle outline of the proof of the completeness theorem. We begin by providing the necessary topological background, including theorems and definitions to be used in developing topological semantics. However, in the lemmas and theorems to come, we will shy away from using abstract topological spaces. Instead, we will primarily use the basis of a topology, since typically, topological bases are easier to work with. Several examples of Euclidean space are given to guide the reader.

Next, we provide an alternate axiomatization for **S4** which makes the topological characterization abundantly clear. We then set out into more logical territory, wherein maximally consistent sets of formulas are defined and Lindenbaum's lemma is stated, which guarantees the existence of a maximally consistent set of formulas. Moving onto the pre-finale, we define the canonical topology and canonical model, which are constructed in such a way to refute any non-theorems of **S4**. Finally, the Truth lemma characterizes the valuation of a proposition at a maximally consistent set to be true if it is a member of said maximally consistent set.

Once the Truth lemma has been proven, the Completeness theorem follows immediately. We conclude with a short proof showing the canonical topological space of **S4** is compact.

Date: July 20, 2018.

2. What is Topology?

Before we present the semantics of **S4**, some necessary background in topology must be established. We presuppose the reader has some experience in elementary set theory and a basic understanding of the real numbers.

Definition 2.1 (Topology)**.** If X is a nonempty set, then a collection $\mathcal{T} \subseteq X$ is a *topology* on X if

(1) $\varnothing, X \in \mathcal{T}$;
(2) If $\mathcal{O}_i \in \mathcal{T}$ for all $i \in I$, then $\bigcup_{i\in I} \mathcal{O}_i \in \mathcal{T}$;
(3) if $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} \mathcal{O}_n \in \mathcal{T}$.

We call the pair $(X, \mathcal{T})$ a *topological space*, $\mathcal{O} \in \mathcal{T}$ an *open* set, and a set $F \subseteq X$ in which $X \setminus F$ is open, is called *closed*.

Example 2.2. Let $X = \mathbb{R}^2$. A set $\mathcal{O} \subset \mathbb{R}^2$ is open iff for $x \in \mathbb{R}^2$ there exist a ball of radius $r > 0$, denoted by $B_r(x) = \{y \in \mathbb{R}^2 \mid |x - y| < r\}$, s.t. $B_r(x) \subset \mathcal{O}$.

Example 2.3. Let X be a nonempty set. The set $\{\varnothing, X\}$ is a topology on X, and is called the *trivial topology.* Then the set of all subsets of X, denoted by $\mathcal{P}(X)$, is a topology on X, called the *discrete topology* on X.

Often times the topology on a set X is too large and unwieldy to work with explicitly. However, it is enough to define a topology in terms of a smaller collection of subsets of X called a basis.

Definition 2.4 (Basis of a Topology)**.** Let X be a nonempty set. A *basis* for a topology on X is a collection $\mathcal{B} \subseteq \mathcal{T}$ such that

(1) For all $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$.
(2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $c \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B} \subseteq X$ satisfies the two items above, then the *topology* $\mathcal{T}$ *is generated by* $\mathcal{B}$ by iff A $U \subseteq X$ is open in X (i.e. $U \in \mathcal{T}$), if for all $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Example 2.5. Let $X = \mathbb{R}$, then $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}\}$ is a basis for the topology on $\mathbb{R}$. So, a set $V \subset \mathbb{R}$ is open iff for all $x \in V$, $r > 0$, $B_r(x) = \{y \in \mathbb{R} \mid |x - y| < r\} \subset V$.

Example 2.6. If $X = \mathbb{R}^2$, then the set of balls $B_r(x) = \{y \in \mathbb{R}^2 \mid |x - y| < r\}$ forms a basis of $\mathbb{R}^2$.

Example 2.7. If X is a topological space with the discrete topology then $\mathcal{B} = \{\{x\} \mid x \in X\}$ is a base.

A necessary set structure for our study will be the interior and closure operators of a set in a topological space.

Definition 2.8 (Interior)**.** Let X be a nonempty set, $A \subseteq X$, and let $(X, \mathcal{T})$ be a topological space. Then the *interior* of A is defined to be

$$A^\circ = \bigcup_{\substack{\mathcal{O}\subseteq A \\ \mathcal{O}\in\mathcal{T}}} \mathcal{O}.$$

and furthermore,

$$x \in A^\circ \iff \exists \mathcal{O} \in \mathcal{T} \text{ s.t. } x \in \mathcal{O} \text{ and } \forall y \in \mathcal{O}, y \in A$$

Definition 2.9 (Closure)**.** Let X be a nonempty set, $A \subseteq X$, and let $(X, \mathcal{T})$ be a topological space. Then the *closure* of A is defined to be

$$\overline{A} = \bigcap_{\substack{X \setminus \mathcal{C} \subseteq A \\ X \setminus \mathcal{C} \in \mathcal{T}}} \mathcal{C}.$$

and furthermore,

$$x \in \overline{A} \iff \forall \mathcal{O} \in \mathcal{T}, x \in \mathcal{O} \implies \exists y \in \mathcal{O}, \text{ s.t. } y \in A.$$

By the definitions above it's easy to see that A° is the largest open set containing A and $\overline{A}$ is the smallest closed set containing A.
From these definitions we can derive the following consequences

Theorem 2.10. *Let $(X, \mathcal{T})$ be a topological space, and $A, B \subseteq X$. Then*

$$A^\circ \subseteq A \subseteq \overline{A}$$

$$\begin{aligned} X^\circ &= X & \overline{\varnothing} &= \varnothing \\ (A \cap B)^\circ &= A^\circ \cap B^\circ & \overline{(A \cup B)} &= \overline{A} \cup \overline{B} \\ A^\circ &\subseteq (A^\circ)^\circ & \overline{\overline{A}} &\subseteq \overline{A} \\ A^\circ &= X \setminus \overline{(X \setminus A)} & \overline{A} &= X \setminus (X \setminus A)^\circ \end{aligned} \tag{2.11}$$

Remark 2.12. The duality beteween the closure and interiro will play a key part in the semantics of the modal logic **S4** we develop later on.

3. The Modal Logic S4

Before presenting the topological semantics for the modal logic **S4**, we quickly review the language of basic propositional modal logic, **S4** axioms and present an alternate axiomatization of **S4**.

Definition 3.1. The *language $\mathcal{L}$ of propositional modal logic* consists of the following:

1. A countable family $\mathcal{F} = \{p_0, p_1, p_2, \ldots\}$ of propositional constants;
2. The propositional constant $\perp$;
3. The Boolean connectives $\wedge, \neg$;
4. The modal operator $\Box$.

Definition 3.2. If **WFF** denotes the set of well formed formulas of $\mathcal{L}$, then $\varphi \in \mathbf{WFF}$ iff

$$\varphi ::= p \mid \perp \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi$$

for any $p \in \mathcal{F}$.

By the two definitions above we can define $\Diamond := \neg\Box\neg$, $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, and $\varphi \to \psi := (\neg\varphi \vee \psi)$ for any $\varphi, \psi \in \mathbf{WFF}$.

For any $\varphi, \psi \in \mathbf{WFF}$, the standard axiomatization of **S4** is given by

$$\begin{aligned} &\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) && (\mathbf{K}) \\ &\Box\varphi \to \Box\Box\varphi && (\mathbf{4}) \\ &\Box\varphi \to \varphi && (\mathbf{T}) \end{aligned}$$

with rules of inference

$$\{\varphi, \varphi \to \psi\} \vdash \psi \qquad (\mathbf{MP})$$
$$\vdash \varphi \implies \vdash \Box\varphi \qquad (\mathbf{N})$$

for any $\varphi, \psi \in \mathbf{WFF}$.

In order to highlight the similarity between the modal operators and the interior and closure operations in a topological space, we express **S4** as follows:

$$\Box\top \qquad (\mathbf{N})$$
$$\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi \qquad (\mathbf{R})$$
$$\Box\varphi \to \Box\Box\varphi \qquad (\mathbf{4})$$
$$\Box\varphi \to \varphi \qquad (\mathbf{T})$$

with rules of inference

$$\{\varphi, \varphi \to \psi\} \vdash \psi \qquad (\mathbf{MP})$$
$$\vdash \varphi \to \psi \implies \vdash \Box\varphi \to \Box\psi \qquad (\mathbf{M})$$

for any $\varphi, \psi \in \mathbf{WFF}$.

4. Topological Semantics for S4

With the alternate axioms of **S4** presented, we now introduce the notion of validity in a topological model, which is quite different than the Kripke notion of validity, and the topological semantics in all their glory.

Definition 4.1. If $(X, \mathcal{T})$ is a topological space, then $M = ((X, \mathcal{T}), v)$ is a *topological model*, where $v : \mathcal{F} \to \mathcal{P}(X)$ is a valuation function, and $\mathcal{F}$ is the family of propositional letters of $\mathcal{L}$.

Definition 4.2. A formula φ is *true* in a model $M = ((X, \mathcal{T}), v)$ if φ is true at every $x \in X$. Moreover, φ is is *valid* in $(X, \mathcal{T})$ if φ is true in every model based on $(X, \mathcal{T})$.

Definition 4.3. Given a topological model $M = ((X, \mathcal{T}), v)$, a formula $\varphi \in \mathbf{WFF}$ is said to be *true at a point* $x \in X$ if

- $M \models p[x] \iff x \in v(p)$,
- $M \models \neg\varphi[x] \iff M \not\models \varphi[x]$,
- $M \models (\varphi \wedge \psi)[x] \iff M \models \varphi[x]$ and $M \models \psi[x]$,
- $M \models \Box\varphi[x] \iff \exists \mathcal{O} \in \mathcal{T}$ s.t. $x \in \mathcal{O}$ and $\forall y \in \mathcal{O}, M \models \varphi[y]$
- $M \models \Diamond\varphi[x] \iff \forall \mathcal{O} \in \mathcal{T}$ s.t. $(x \in \mathcal{O} \implies \exists y \in \mathcal{O}$ s.t. $M \models \varphi[y])$.

From the definition provided above, observe that the semantics of the $\Box$ operator coincide precisely with the definition of the interior operator acting on a subset of a set, and the semantics of $\Diamond$ coincide with the closure operator.

To be more precise, if $A \subseteq X$, where X is a nonempty topological space, then recall by definition

$$x \in A^\circ \iff \exists \mathcal{O} \in \mathcal{T} \text{ s.t. } x \in \mathcal{O} \text{ and } \forall y \in \mathcal{O}, y \in A$$

and

$$x \in \overline{A} \iff \forall \mathcal{O} \in \mathcal{T}, x \in \mathcal{O} \implies \exists y \in \mathcal{O}, \text{ s.t. } y \in A.$$

To further this analogy, let $\|\varphi\|^M = \{x \in X \mid M \models \varphi[x]\}$. Then φ is true in a topological model $M = ((X, \mathcal{T}), v)$ only if:

$$M \models \varphi \iff \|\varphi\| = X,$$

To illustrate this notion of topological truth, consider the following examples.

Example 4.4. Let $X = [0, 1]$, and let $v(p) = \|p\| = [0, 1)$. Given $M = ((X, \mathcal{T}), v)$, where $\mathcal{T}$ is the standard Euclidean topology on $\mathbb{R}$, as given in *example 2.5* we claim that

$$M \models \Box(p \vee \neg p).$$

Using the notation above, we have

$$\|\Box(p \vee \neg p)\| = (\|p\| \cup X \setminus \|p\|)^\circ = ([0, 1) \cup \{1\})^\circ = [0, 1].$$

Therefore, $M \models \Box(p \vee \neg p)$.

Example 4.5. Let X, p and M be as in the previous example. Then

$$M \not\models \Box p \vee \Box \neg p,$$

since

$$\|\Box p \vee \Box \neg p\| = \|p\|^\circ \cup (X \setminus \|p\|)^\circ = (0, 1) \cup \{1\}^\circ = (0, 1) \cup \varnothing = (0, 1) \neq [0, 1].$$

Thus, $M \not\models \Box p \vee \Box \neg p$.

By the definition above we see that topological duality between interior and closure captures the semantic duality between the modal operators, if we interpet φ as a subset of the topological space X.

Informally, we can give topological interpretation the axioms given in section 3 as follows:

$\Box\top \rightsquigarrow$ The space is open in itself

$\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi \rightsquigarrow$ Finite intersection of open sets is open

$\Box\varphi \to \Box\Box\varphi \rightsquigarrow$ Interior operator is idempotent.

$\Box\varphi \to \varphi \rightsquigarrow$ Interior of a set is contained in that set

With the topological semantics defined, soundness follows immediately.

Theorem 4.6 (Soundness)**.** *For any set of formulas* Γ*,*

$$\Gamma \vdash_{\mathbf{S4}} \varphi \implies \Gamma \models_{\mathcal{L}} \varphi$$

Proof. It is enough to show that all axioms of **S4** are valid, and the rules of inference preserve validity in a topological model. We will use the first given axiomatization of **S4**.
Let $(X, \mathcal{T})$ be an arbitrary topological space, and $M = ((X, \mathcal{T}), v)$ a topological model. Without loss of generality, let the formulas $\varphi = p, \psi = q \in \mathcal{F}$.

We begin by showing (**K**) is valid. Suppose that for $x \in X$, $M \models \Box(p \to q)[x]$ and $M \models \Box p[x]$. Then by definition of $\Box$, there are $\mathcal{O}, \mathcal{U} \in \mathcal{T}$ such that $M \models (p \to q)[y]$ and $M \models p[z]$ for all $y \in \mathcal{O}$ and $z \in \mathcal{U}$. Now Consider $\mathcal{O} \cap \mathcal{U}$. Certainly, $\mathcal{O} \cap \mathcal{U} \in \mathcal{T}$ and $x \in \mathcal{O} \cap \mathcal{U}$, so $M \models (p \to q)[w]$ and $M \models p[w]$ for all $w \in \mathcal{O} \cap \mathcal{U}$. Therefore, $M \models q[w]$ for all $w \in \mathcal{O} \cap \mathcal{U}$, so $M \models \Box q[x]$.

Moving onto (**T**), suppose that $x \in X$ and $M \models \Box p[x]$. Then there is an open set $\mathcal{O} \ni x$ such that $M \models p[y]$ for all $y \in \mathcal{O}$. Since $x \in \mathcal{O}$, it follows that $M \models p[x]$. Therefore, (**T**) is valid.

Finally for (**4**), suppose that $x \in X$ and $M \models \Box p[x]$. Then there is an $\mathcal{O} \in \mathcal{T}$ such that $M \models p[y]$ for all $y \in \mathcal{O}$. Furthermore, $M \models \Box p[y]$ for all $y \in \mathcal{O}$, which implies $M \models \Box\Box p[x]$.

It's easy to see that (**MP**) is valid, so we move onto showing (**N**) preserves validity. Suppose that φ is an arbitrary formulay. By contraposition, suppose that $\Box\varphi$ is not valid. Then there is a model $M = ((X, \mathcal{T}), v)$ with $x \in X$ such that $M \not\models \Box\varphi[x]$. So there is a $y \in X$ such that $M \not\models \varphi[y]$. ∎

5. Completeness of S4

We now provide the main argument for the completeness theorem following a sequence of lemmas.

Definition 5.1. A set Γ is *consistent* relative to **S4** iff $\Gamma \not\vdash_{\mathbf{S4}} \bot$.

We will call such sets **S4**-consistent.

Definition 5.2 (Maximally consistent set)**.** A set Γ is *maximally **S4**-consistent* iff Γ is **S4**-consistent and for every φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$, but not both.

Lemma 5.3 (Lindenbaum's Lemma)**.** *If* Γ *is a* Σ*-consistent set of formulas, then there is a maximally consistent set* Γ_{max} *such that* $\Gamma \subseteq \Gamma_{max}$.

While the proof of Lindenbaum's lemma is instructive, it leads us too far astray from our current topic. See [3], [5], or [8] for a complete proof.
To further meld the topological and logical notion, we now define a topological space which consists of maximally consistent sets.

Definition 5.4 (Canonical Topological Space)**.** The *canonical topological space* is the pair $(X^{\mathcal{L}}, \mathcal{T}^{\mathcal{L}})$, where

- $X^{\mathcal{L}}$ is the set of all maximally consistent sets, $\Gamma_{\max}$;
- $\mathcal{T}^{\mathcal{L}}$ is a topology generated by base sets
$$B^{\mathcal{L}} = \{\widehat{\Box\varphi} \mid \varphi \text{ is any formula}\},$$
where
$$\widehat{\varphi} := \{\Gamma_{\max} \in X^{\mathcal{L}} \mid \varphi \in \Gamma_{\max}\}.$$

The definition above implies that the basis sets are of the form
$$\mathcal{O}_{\varphi} = \{\Gamma_{\max} \in X^{\mathcal{L}} \mid \Box\varphi \in \Gamma_{\max}\}.$$
Observe that from this definition, if $\Gamma_{\max}$ is a maximally consistent set and $\Box\varphi \in \Gamma_{\max}$, then $\Gamma_{\max} \in \widehat{\Box\varphi}$. Therefore, the points of the canonical topological space are maximally consistent sets. The construction of this space allows us to build models in such a way to refute any non-theorem of **S4**. In that respect, this model is canonical to **S4**.

Before proceeding we verify that the canonical topological space is in fact a topology. To do so, it suffices to show that $B^{\mathcal{L}}$ is a basis for the topology.

Claim: $B^{\mathcal{L}}$ is a basis for the topology.

Proof. We want to show that

(1) For any $\mathcal{O}_\varphi, \mathcal{O}_\psi \in B^{\mathcal{L}}$, and any $\Gamma_{\max} \in \mathcal{O}_\varphi \cap \mathcal{O}_\psi$, there is a $\mathcal{O}_\lambda \in B$ such that $\Gamma_{\max} \in \mathcal{O}_\lambda \subseteq \mathcal{O}_\varphi \cap \mathcal{O}_\psi$, and
(2) For any $\Gamma_{\max} \in X^{\mathcal{L}}$, there is a $\mathcal{O}_\varphi \in B^{\mathcal{L}}$ such that $\Gamma_{\max} \in \mathcal{O}_\varphi$.

By axiom (**N**), $\Box\top \in \Gamma_{\max}$ for any maximally consistent set. Therefore $X = \widehat{\Box\top}$, so item two is satisfied.
Observe that by (**R**),

$$\begin{aligned}\Box\widehat{(\varphi \wedge \psi)} &= \{\Gamma_{\max} \in X^{\mathcal{L}} \mid \Box(\varphi \wedge \psi) \in \Gamma_{\max}\} \\ &= \{\Gamma_{\max} \in X^{\mathcal{L}} \mid \Box\varphi \wedge \Box\psi \in \Gamma_{\max}\} \\ &= \widehat{\Box\varphi} \cap \widehat{\Box\psi}.\end{aligned}$$

So if $\Gamma_{\max} \in \mathcal{O}_\varphi \cap \mathcal{O}_\psi \neq \varnothing$, then there is $\lambda \in \Gamma_{\max}$ such that $\Gamma_{\max} \in \mathcal{O}_\lambda \subseteq \mathcal{O}_\varphi \cap \mathcal{O}_\psi$.
Therefore, $B^{\mathcal{L}}$ is a base. ■

What we have shown is that $(X^{\mathcal{L}}, \mathcal{T}^{\mathcal{L}})$ is a topology generated by maximally consistent sets which contain $\Box\varphi$ for any formula φ.

Definition 5.5 (Canonical Topological Model)**.** The *Canonical Topological Model* is the triple $M^{\mathcal{L}} = ((X^{\mathcal{L}}, \mathcal{T}^{\mathcal{L}}), v^{\mathcal{L}})$, where $(X^{\mathcal{L}}, \mathcal{T}^{\mathcal{L}})$ is the canonical topological space, and

$$v^{\mathcal{L}}(p) = \{\Gamma_{\max} \in X^{\mathcal{L}} \mid p \in \Gamma_{\max}\}.$$

The valuation function $v^{\mathcal{L}}$ defined this way says that a proposition p is true at a maximally consistent set, only if it is a member of that maximally consistent set.
To make sense of this definition for arbitrary formula, we introduce the Truth lemma, which says that a formula is true at a maximally consistent set, only if it is a member of that set, and conversely, any formula of a maximally consistent set is true.

Lemma 5.6 (Truth lemma)**.** *If $M^{\mathcal{L}}$ is a topological model, then for any modal formula φ,*

$$M^{\mathcal{L}} \models \varphi[x] \iff x \in \widehat{\varphi}.$$

In order to prove the theorem, we must appeal to the principle of induction on formulas.

Theorem 5.7 (Principle of Induction On Formulas)**.** *If a property F holds for all propositional variables and it holds for $\Box\varphi, \neg\varphi$ and $\varphi \wedge \psi$ whenever it holds for φ and ψ, then F holds for all well formed formulas.*

For a proof and examples of the theorem see [8].

Proof. We proceed by induction on the complexity of φ.
Base Case: Suppose that $\varphi = p$. Let $M^{\mathcal{L}}$ be a canonical model. Then by definition of canonical model,

$$v^{\mathcal{L}}(p) = \{\Gamma_{\max} \in X^{\mathcal{L}} \mid p \in \Gamma_{\max}\}$$

so

$$M^{\mathcal{L}} \models p[w] \iff w \in \widehat{p}.$$

Inductive Step: Suppose that for some φ, ψ, and for some $w \in X$, that

$$M^{\mathcal{L}} \models \varphi[w] \iff w \in \widehat{\varphi}$$

and

$$M^{\mathcal{L}} \models \psi[w] \iff w \in \widehat{\psi}$$

Case 1: $\neg\varphi$.
Observe that $\widehat{\neg\varphi} = X \setminus \widehat{\varphi}$. Therefore, by the induction hypothesis

$$M^{\mathcal{L}} \models \neg\varphi[w] \iff M^{\mathcal{L}} \not\models \neg\varphi[w] \iff w \in X \setminus \widehat{\varphi}.$$

Case 2: $\varphi \wedge \psi$.
Since $\widehat{\varphi \wedge \psi} = \widehat{\varphi} \cap \widehat{\varphi}$, by the induction hypothesis it follows that

$$M^{\mathcal{L}} \models (\varphi \wedge \psi)[w] \iff w \in \widehat{\varphi} \cap \widehat{\psi}.$$

Case 3: $\Box\varphi$.
$(\Longleftarrow)$
Suppose that $w \in \widehat{\Box\varphi}$. Then $\widehat{\Box\varphi}$ is a base set containing w. Furthermore, $\widehat{\Box\varphi} \subseteq \widehat{\varphi}$ by axiom (**T**). So by definition of $\widehat{\Box\varphi}$, for any $v \in \widehat{\Box\varphi}$, $v \in \widehat{\varphi}$, and by our induction hypothesis, $M^{\mathcal{L}} \models \varphi[v]$.
$(\Longrightarrow)$
Suppose that $M^{\mathcal{L}} \models \Box\varphi[w]$. Then there is a base set $\widehat{\Box\psi} \in B^{\mathcal{L}}$ such that $w \in \widehat{\Box\psi}$ and for any $v \in \widehat{\Box\psi}$, $M \models \Box\psi[v]$. By our induction hypothesis, $v \in \widehat{\varphi}$ for all $v \in \widehat{\Box\psi}$, which is to say $\widehat{\Box\psi} \subseteq \widehat{\varphi}$. Applying the box operator yields $\widehat{\Box\Box\psi} \subseteq \widehat{\Box\varphi}$. Moreover, $\widehat{\Box\psi} \subseteq \widehat{\Box\Box\psi} \subseteq \widehat{\Box\varphi}$. Therefore, $w \in \widehat{\Box\varphi}$. ■

We now have enough tools to give a very quick proof of our main result.

Theorem 5.8 (Completeness)**.** *For any set of formulas* Γ *of* $\mathcal{L}$*,*

$$\Gamma \models_{\mathcal{L}} \varphi \implies \Gamma \vdash_{\boldsymbol{S4}} \varphi$$

Proof. By contraposition, suppose that $\Gamma \not\vdash_{\mathbf{S4}} \varphi$. We now want to show that there is some model in which φ is not true. Since $\Gamma \not\vdash_{\mathbf{S4}} \varphi$, $\Delta = \Gamma \cup \{\neg\varphi\}$ is consistent. By Lindenbaum's lemma, there is a maximally consistent set $\Gamma_{\max}$ extending Δ. So by the Truth lemma, for some model $M^{\mathcal{L}}$,

$$M^{\mathcal{L}} \models \neg\varphi[\Gamma_{\max}] \iff M^{\mathcal{L}} \not\models \varphi[\Gamma_{\max}],$$

which is the desired countermodel. ■

6. Compactness of S4

When given a topological space several natural questions arise considering its structure, i.e. is its basis countable, is distance definable, is the space connected, etc. One such structure that is ubiquitous to mathematics is the notion of compactness. As a motivating example, a famous theorem of Heine and Borel states that compactness in Euclidean spaces is the same as a space being "closed" and "bounded". So in $\mathbb{R}$, all compact sets are of the form of closed intervals $[a, b]$, $a, b \in \mathbb{R}$. An interested reader can investigate further in [7].

Definition 6.1 (Cover)**.** Let I be an index set. Given a family $\{A_i\}_{i\in I}$ of subsets of a space X, $\{A_i\}_{i\in I}$ is a *cover* of X if

$$X \subseteq \bigcup_{i\in I} A_i.$$

If A_i is open for all $i \in I$, then we the cover an *open cover.*

Definition 6.2 (Compact)**.** A space X is *compact* if every open cover has a finite subcover.

Theorem 6.3. *The canonical topological space* $(X^{\mathcal{L}}, \mathcal{T}^{\mathcal{L}})$ *is compact.*

Proof. For the sake of contradiction, suppose that $X^{\mathcal{L}}$ is not compact. Then there is a family of base sets $\{\widehat{\Box\psi_i}\}_{i\in I}$ such that $\bigcup_{i\in I}\widehat{\Box\psi_i} = X^{\mathcal{L}}$, but there is no finite subcover. Let $\Gamma = \{\neg\Box\psi_i\}_{i\in I}$. We claim that Γ is consistent. Suppose otherwise. Then there are formulas $\neg\Box\psi_1, \ldots, \neg\Box\psi_n \in \Gamma$ such that

$$\mathbf{S4} \vdash \neg(\neg\Box\psi_1 \wedge \ldots \wedge \neg\Box\psi_n) \iff \mathbf{S4} \vdash \Box\psi_1 \vee \ldots \vee \Box\psi_n.$$

However, this implies that $\bigcup_{i=1}^{n}\widehat{\Box\psi_i} = X^{\mathcal{L}}$, contradicting the assumption that $X^{\mathcal{L}}$ has no finite subcover. Therefore, Γ is consistent.

By Lindenbaum's lemma, Γ can be extended to a maximally consistent set $\Gamma_{\max}$ so that $\neg\Box\psi_i \in \Gamma_{\max}$ for all $i \in I$. Since $\neg\Box\psi_i \in \Gamma_{\max}$, it follows that $\Gamma_{\max} \in \widehat{\neg\Box\psi_i} = X^{\mathcal{L}} \setminus \widehat{\Box\psi_i}$ for all $i \in I$. Therefore, $\Gamma_{\max} \in X^{\mathcal{L}} \setminus \bigcup_{i\in I}\widehat{\Box\psi_i}$. But $X^{\mathcal{L}} = \bigcup_{i\in I}\widehat{\Box\psi_i}$, so $\Gamma_{\max} \in \varnothing$, a contradiction. Therefore, $X^{\mathcal{L}}$ is compact. ■

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