Topological Completeness of S4

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Completeness: The Big Picture

Completeness of a logical system Σ in a language $\mathcal{L}$ roughly says

$$\text{Provable} \iff \text{Valid.}$$

Formally,

$$\Gamma \vdash_{\Sigma} \varphi \iff \Gamma \models_{\mathcal{L}} \varphi$$

for any consistent set Γ of formulas containing φ.

The Logic **S4**

S4 is given by the axioms For any well formed φ, ψ, the standard axiomatization of **S4** is given by

$$\begin{aligned} &\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) && (\mathbf{K}) \\ &\Box\varphi \to \Box\Box\varphi && (\mathbf{4}) \\ &\Box\varphi \to \varphi && (\mathbf{T}) \end{aligned}$$

with rules of inference

$$\begin{aligned} \{\varphi, \varphi \to \psi\} \vdash \psi && (\mathbf{MP}) \\ \vdash \varphi \implies \vdash \Box\varphi && (\mathbf{N}). \end{aligned}$$

The Logic **S4**, cont.

Alternatively, for any φ, ψ well formed **S4** is also given by

$$\Box\top \qquad (\mathbf{N})$$

$$\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi \qquad (\mathbf{R})$$

$$\Box\varphi \to \Box\Box\varphi \qquad (\mathbf{4})$$

$$\Box\varphi \to \varphi \qquad (\mathbf{T})$$

with rules of inference

$$\{\varphi, \varphi \to \psi\} \vdash \psi \qquad (\mathbf{MP})$$

$$\vdash \varphi \to \psi \implies \vdash \Box\varphi \to \Box\psi \qquad (\mathbf{M})$$

Fact: **S4** $= \mathbf{NR4T}$.

What is Topology?

Definition (Topology)

If X is a nonempty set, then a collection $\mathcal{T} \subseteq X$ is a *topology* on X if

1. $\varnothing, X \in \mathcal{T}$;
2. If $\{\mathcal{O}_i\}_{i\in I} \in \mathcal{T}$, then $\bigcup_{i\in I} \mathcal{O}_i \in \mathcal{T}$;
3. if $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} \mathcal{O}_n \in \mathcal{T}$.

We call the pair $(X, \mathcal{T})$ a *topological space*, $\mathcal{O} \in \mathcal{T}$ an *open* set, and a set $F \subseteq X$ in which $X \setminus F$ is open, is called *closed*.

For our purposes, this definition is rather difficult to work with.

Basis of a Topology

Definition (Basis of a Topology)

Let X be a nonempty set. A *basis* for a topology on X is a collection $\mathcal{B} \subseteq \mathcal{T}$ such that

1. For all $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B} \subseteq X$ satisfies the two items above, then the *topology $\mathcal{T}$ is generated by* $\mathcal{B}$ by iff A $U \subseteq X$ is open in X (i.e. $U \in \mathcal{T}$), if for all $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Example

If $X = \mathbb{R}^2$, then the set of open balls of radius $r > 0$, $B_r(x) = \{y \in \mathbb{R}^2 \mid |x - y| < r\}$ for a basis.

Interior and Closure of a set

Let X be a nonempty set, $A \subseteq X$, and let $(X, \mathcal{T})$ be a topological space.

Definition (Interior and Closure)

The *interior* of A is defined to be

$$A^\circ = \bigcup_{\substack{\mathcal{O} \subseteq A \\ \mathcal{O} \in \mathcal{T}}} \mathcal{O}.$$

$$x \in A^\circ \iff \exists \mathcal{O} \in \mathcal{T} \text{ s.t. } x \in \mathcal{O} \text{ and } \forall y \in \mathcal{O}, y \in A$$

The *closure* of A is defined to be

$$\overline{A} = \bigcap_{\substack{\mathcal{C} \subseteq A \\ X \setminus \mathcal{C} \in \mathcal{T}}} \mathcal{C}.$$

$$x \in \overline{A} \iff \forall \mathcal{O} \in \mathcal{T}, x \in \mathcal{O} \implies \exists y \in \mathcal{O}, \text{ s.t. } y \in A.$$

Topological Models

Definition

If $(X, \mathcal{T})$ is a topological space, then $M = ((X, \mathcal{T}), v)$ is a *topological model*, where $v : \mathcal{F} \to \mathcal{P}(X)$ is a valuation function, and $\mathcal{F}$ is the family of propositional letters of $\mathcal{L}$.

From this definition, we see that $v(p) \subseteq \mathcal{P}(X)$.

Definition

A formula φ is *true* in a model $M = ((X, \mathcal{T}), v)$ if φ is true at every $x \in X$. Moreover, φ is is valid in $(X, \mathcal{T})$ if φ is true in every model based on $(X, \mathcal{T})$.

Topological Semantics

Definition

Given a model $M = ((X, \mathcal{T}), v)$, a formula $\varphi \in$ **WFF** is said to be *true at a point* $x \in X$ if

- $M \models p[x] \iff w \in v(p)$,
- $M \models \neg\varphi[x] \iff M \not\models \varphi[x]$,
- $M \models (\varphi \wedge \psi)[x] \iff M \models \varphi[x]$ and $M \models \psi[x]$,
- $M \models \Box\varphi[x] \iff \exists\mathcal{O} \in \mathcal{T}$ s.t. $x \in \mathcal{O}$ and $\forall y \in \mathcal{O}, M \models \varphi[y]$
- $M \models \Diamond\varphi[x] \iff \forall\mathcal{O} \in \mathcal{T}$ s.t. $(x \in \mathcal{O} \implies \exists y \in \mathcal{O}$ s.t. $M \models \varphi[y])$.

The (Naive) Topological Connection

If $\|\varphi\|^M := \{x \in X \mid M \models \varphi[x]\}$, then $\|\Box\varphi\|^M = (\|\varphi\|^M)^\circ$.
Furthermore, for $A, B \subseteq X$, we have the correspondence

$$\begin{array}{lcc}
\Box\top & \leftrightarrows & X^\circ = X \\
\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi & \leftrightarrows & (A \cap B)^\circ = A^\circ \cap B^\circ \\
\Box\varphi \to \Box\Box\varphi & \leftrightarrows & A^\circ \subseteq (A^\circ)^\circ \\
\Box\varphi \to \varphi & \leftrightarrows & A^\circ \subseteq A \\
\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi & \leftrightarrows & A^\circ = X \setminus \overline{(X \setminus A)}
\end{array}$$

Soundness

With the semantics defined, soundness follows immediately.

Theorem (Soundness)

For any set of formulas Γ*,*

$$\Gamma \vdash_{\mathbf{S4}} \varphi \implies \Gamma \models_{\mathcal{L}} \varphi$$

Proof.

It is enough to verify that the axioms of **S4** are valid, and the rules of inference, namely, $(\mathbf{MP})$ and $(\mathbf{N})$ preserve validity. ■

Maximally Consistent Sets

Definition (Consistent set)

A set Γ is *consistent* relative to **S4** iff $\Gamma \nvdash_{\mathbf{S4}} \bot$.

We will call such sets **S4**-consistent.

Definition (Maximally consistent set)

A set Γ is *maximally* **S4**-*consistent* iff Γ is **S4**-consistent and for every φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$, but not both.

Lemma (Lindenbaum's Lemma)

For an arbitrary system Σ, if Γ is Σ-consistent, then there exists a maximally consistent set Δ extending Γ.

Note: This extension is consistent by its recursive definition.

Canonical Topological Space

Definition (Canonical Topological Space)

The *canonical topological space* is the pair $(X^{\mathcal{L}}, \mathcal{T}^{\mathcal{L}})$, where

- $X^{\mathcal{L}}$ is the set of all maximally consistent sets, Γ_{max};
- $\mathcal{T}^{\mathcal{L}}$ is a topology generated by base sets

$$B^{\mathcal{L}} = \{\widehat{\Box\varphi} \mid \varphi \text{ is any formula}\},$$

where

$$\widehat{\varphi} := \{\Gamma_{\text{max}} \in X^{\mathcal{L}} \mid \varphi \in \Gamma_{\text{max}}\}.$$

So all base sets are of the form

$$\mathcal{O}_\varphi = \{\Gamma_{\text{max}} \in X^{\mathcal{L}} \mid \Box\varphi \in \Gamma_{\text{max}}\}.$$

Intuition: Base sets are maximally consistent sets which contain $\Box\varphi$ for any φ. Think of $\Box$ as the interior operator acting on subsets of a topological space.

Canonical Model and Truth Lemma

Definition (Canonical Topological Model)

The *Canonical Topological Model* is the triple $M^{\mathcal{L}} = ((X^{\mathcal{L}}, \mathcal{T}^{\mathcal{L}}), v^{\mathcal{L}})$, where $(X^{\mathcal{L}}, \mathcal{T}^{\mathcal{L}})$ is the canonical topological space, and

$$v^{\mathcal{L}}(p) = \{\Gamma_{\mathsf{max}} \in X^{\mathcal{L}} \mid p \in \Gamma_{\mathsf{max}}\}.$$

Lemma (Truth lemma)

If $M^{\mathcal{L}}$ is a topological model, then for any modal formula φ,

$$M^{\mathcal{L}} \models \varphi[x] \iff x \in \widehat{\varphi}.$$

Intuition: "Truth=Membership". That is to say, a formula φ is true at a maximally consistent set, only if it is a member of that set.

Completeness

Theorem (Completeness)

For any set of formulas Γ *of* $\mathcal{L}$,

$$\Gamma \models_{\mathcal{L}} \varphi \implies \Gamma \vdash_{\mathbf{S4}} \varphi$$

Proof.

- By contraposition, suppose that $\Gamma \nvdash_{\mathbf{S4}} \varphi$.
- Then $\Delta = \Gamma \cup \{\neg\varphi\}$ is consistent.
- By Lindenbaum's lemma, there is a maximally consistent set Γ_{max} extending Δ.
- By the Truth lemma, for some model $M^{\mathcal{L}}$,

$$M^{\mathcal{L}} \models \neg\varphi[\Gamma_{\mathsf{max}}] \iff M^{\mathcal{L}} \not\models \varphi[\Gamma_{\mathsf{max}}]$$

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