Solutions to Exercises from *Fourier Analysis*

Brad Velasquez
bvelasquez@ucdavis.edu

July 22, 2017

Chapter 1

The Genesis of Fourier Analysis

1.1 Exercise 3

(Part a)

Proof. For the sake of contradiction, suppose that the complex sequence (w_n) converges and has two limits w and w'. Let $\varepsilon = \frac{|w-w'|}{2} > 0$. Then there exists a $N_1 \in \mathbb{N}$ s.t.

$$|w_n - w| < \frac{|w-w'|}{2}, \quad \text{for } n \geq N_1.$$

Furthermore, there exists a $N_2 \in \mathbb{N}$ s.t.

$$|w_n - w'| < \frac{|w-w'|}{2}, \quad \text{for } n \geq N_2.$$

Choose $N = \max\{N_1, N_2\}$. Then applying the triangle inequality yields

$$\begin{aligned} |w - w'| = |(w_n - w) - (w_n - w')| &\leq |w_n - w| + |w_n - w'| \\ &< 2 \cdot \frac{|w-w'|}{2} = |w - w'|, \end{aligned}$$

a contradiction. Therefore, the limit is unique. ■

(Part b)
Claim: A sequence of complex numbers (z_n) converges iff (z_n) is Cauchy.

Proof. ($\Rightarrow$) Let $\varepsilon > 0$ and Suppose that $(z_n) \to w$. Then by definition there exists a $N \in \mathbb{N}$ s.t. $|z_n - w| < \varepsilon$, for all $n \geq N$.
For $n, m \geq N$, by the uniqueness of the limit

$$|z_n - w| < \frac{\varepsilon}{2}$$

$$|z_m - w| < \frac{\varepsilon}{2}.$$

Then

$$|z_n - z_m| \leq |z_n - w| + |z_m - w| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

($\Leftarrow$)
Let (z_n) be as sequence of complex numbers s.t. $\Re(z_n) = (x_n)$ and $\Im(z_n) = (y_n)$. To prove (z_n) is Cauchy implies convergent, we must prove the following string of implications

$$(z_n) \text{ Cauchy} \implies (x_n) \text{ and } (y_n) \text{ Cauchy} \implies (z_n) \text{ convergent.}$$

Essentially, our goal is to reduce the convergence of (z_n) to the convergence of the sum of it's respective real sequences.
Let $\varepsilon > 0$ and suppose that (z_n) is Cauchy. Since (z_n) is Cauchy, there exists and $N \in \mathbb{N}$ s.t. if $n, m > N$ then $|z_n - z_m| < \varepsilon$. Then we must show that $|x_n - x_m| < \varepsilon$. So

$$|x_n - x_m| = \sqrt{(x_n - x_m)^2} < \sqrt{(x_n - x_m)^2 + (y_n + y_m)^2} = |w_n - w_m| < \varepsilon.$$

A similar argument shows that $|y_n - y_m| < \varepsilon$.
Thus, (x_n) and (y_n) are Cauchy.
Next, since the sum of Cauchy are sequences is Cauchy, and real Cauchy sequences are convergent we have that

$$\lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = \lim_{n\to\infty} (x_n + iy_n) = \lim_{n\to\infty} z_n.$$

Therefore, (z_n) converges.

$\blacksquare$

(Part c)

Proof. Let $\varepsilon > 0$. Let (a_n) be a sequence of non-negative real numbers s.t. $\sum_n a_n$ converges and let (z_n) be a sequence of complex numbers satisfying $|z_n| \leq a_n$ for all $n \in \mathbb{N}$. Denote the partial sums by

$$S_N = \sum_{n=1}^{N} z_n, \qquad A_N = \sum_{n=1}^{N} a_n.$$

Then for $n > m \geq N$,

$$\begin{aligned}
|S_n - S_m| &= |z_{m+1} + z_{m+2} + z_{m+3} + \ldots + z_n| \\
&\leq |z_{m+1}| + |z_{m+2}| + |z_{m+3}| + \ldots + |z_n| \\
&\leq a_{m+1} + a_{m+2} + a_{m+3} + \ldots + a_n \\
&= |a_{m+1} + a_{m+2} + a_{m+3} + \ldots + a_n| \\
&= |A_n - A_m| < \varepsilon
\end{aligned}$$

Where $|a_{m+1} + a_{m+2} + a_{m+3} + \ldots + a_n| = a_{m+1} + a_{m+2} + a_{m+3} + \ldots + a_n$, since (a_n) is a sequence of non-negative terms. The last inequality follows from the *Cauchy Criterion*. Hence, the series $\sum_{n=1}^{\infty} z_n$ converges. ∎

1.2 Exercise 4

(Part a) We want to show that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges for every $z \in \mathbb{C}$. Since $a_n = \frac{z^n}{n!} \neq 0$ for any $n \in \mathbb{N}$, we may use the ratio test to verify the claim.

Proof. Let $z \in \mathbb{C}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{z^{n+1}}{(n+1)!} \frac{n!}{z^n}\right| \\
&= \left|\frac{z}{n+1}\right| = 0 < 1.
\end{aligned}$$

Therefore, the series converges absolutely. ∎

Claim: e^z converges uniformly on any bounded $A \subseteq \mathbb{C}$ First we define what it means for a subset of $\mathbb{C}$ to be bounded.

Definition 1.2.1. (Stein and Shakarchi, Complex Analysis) A subset $A \subseteq \mathbb{C}$ is *bounded* if there is some $r > 0$ s.t. $A \subseteq \overline{D}(z_0, r)$, where

$$\overline{D}(z_0, r) = \{z \in \mathbb{C} | \, |z - z_0| \leq r\},$$

i.e. a closed disc in the complex plane with radius r.

Furthermore, we use the following theorem

Theorem 1.2.2 (Weierstraß M-test)**.** *Let (f_k) be a sequence of complex valued functions on a set $A \subseteq \mathbb{C}$. Suppose that $|f_n(z)| \leq M_n$ for all $z \in A$. If $\sum_{n=0}^{\infty} M_n < +\infty$, then the series $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly on A.*

We now begin the proof of the claim.

Proof. Let A be a disc of radius $r \in \mathbb{R}$ centered about the origin. Then for any $z \in A$,

$$\left|\frac{z^n}{n!}\right| = \frac{|z|^n}{n!} \leq \frac{r^n}{n!}.$$

Pick $M_n = r^n/n!$. Then by the ratio test we have that

$$\lim_{n\to\infty} \left|\frac{r^{n+1}}{(n+1)!} \frac{n!}{r^n}\right| = 0 < 1.$$

Therefore, by the M-test $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges uniformly. ■

(Part b)

Proof. First note the following definition.

Theorem 1.2.3. *(Apostol, theorem 8.4.5) Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ that converge absolutely, define*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad n \in \mathbb{N}.$$

The series $\sum_{n=0}^{\infty} c_n$ is called the Cauchy product of $\sum a_n$ and $\sum b_n$.

Remark. I think there is something to be said of absolute convergence of the series above.

Let $z_1, z_2 \in \mathbb{C}$, then by part a we have that

$$e^{z_1}e^{z_2} = \Big(\sum_{n=0}^{\infty} \frac{z_1^n}{n!}\Big)\Big(\sum_{m=0}^{\infty} \frac{z_2^m}{m!}\Big).$$

Then by definition 1.2.1,

$$\begin{aligned}
\Big(\sum_{m=0}^{\infty} \frac{z_1^m}{m!}\Big)\Big(\sum_{n=0}^{\infty} \frac{z_2^n}{n!}\Big) &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} \frac{z_1^k}{k!}\frac{z_2^{n-k}}{(n-k)!}\Big) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\sum_{k=0}^{n} n!\frac{z_1^k}{k!}\frac{z_2^{n-k}}{(n-k)!}\Big) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}z_1^k z_2^{n-k}\Big) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}\Big) \\
&= \sum_{n=0}^{\infty}\frac{(z_1+z_2)^n}{n!} = e^{z_1+z_2}.
\end{aligned}$$

■

(Part c) Claim: If $\Re(z) = 0$, then $e^{iy} = \cos(y) + i\sin(y)$ First recall that the power series expansions for sin and cos are

$$\begin{aligned}
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots.
\end{aligned}$$

Proof. Noting that $i^2 = -1, i^3 = -i, i^4 = 1 = i^0$, we have

$$\begin{aligned}
\cos(y) + i\sin(y) &= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \dots + (ix - \frac{iy^3}{3!} + \frac{iy^5}{5!} - \frac{iy^7}{7!} + \dots) \\
&= (1i)^0 + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \dots \\
&= \sum_{n=0}^{\infty}\frac{(iy)^n}{n!} = e^{iy}.
\end{aligned}$$

■

(Part d) Claim: $e^{x+iy} = e^x(\cos(y) + i\sin(y))$

Proof. By part b of *exercise 4* we know that for any $z_1, z_2 \in \mathbb{C}$, $e^{z_1}e^{z_2} = e^{z_1+z_2}$. Let $\Re(z_1) = x, \Im(z_1) = 0, \Re(z_2)$ and $\Im(z_2) = y$. Then $e^{z_1}e^{z_2} = e^x e^{iy} = e^{x+iy}$. Then by the result of part c we have that $e^{x+iy} = e^x(\cos(y) + i\sin(y))$. ∎

Claim: If $x, y \in \mathbb{R}$, show that

$$|e^{x+iy}| = e^x.$$

Proof. We will use the following theorem.

Theorem (Exercise 1, part c)**.** *If $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$, then $|\lambda z| = |\lambda||z|$, where $|\lambda|$ is the standard absolute value for $\mathbb{R}$.*

By the result of part b we have that $|e^{x+iy}| = |e^x e^{iy}|$. Then, $|e^x e^{iy}| = |e^x||e^{iy}|$. Since $e^x > 0$ for all $x \in \mathbb{R}$, we have $|e^x e^{iy}| = e^x|e^{iy}|$. So it is enough to show that $|e^{iy}| = 1$.
By Euler's identity $|e^{iy}| = |\cos(y) + i\sin(y)|$. Then by definition of $|\cdot|$

$$|e^{iy}| = |\cos(y) + i\sin(y)| = \sqrt{\cos^2(y) + \sin^2(y)} = 1.$$

Hence $|e^{x+iy}| = e^x$

∎

(Part e)

Proof. Let $z \in \mathbb{C}$. Then $z = x + iy$ for some $x, y \in \mathbb{R}$.
By part (d) we have that

$$e^{x+iy} = e^x(\cos(y) + i\sin(y)).$$

So $e^x(\cos(y) + i\sin(y)) = 1$ yields the following system of equations

$$\begin{cases} e^x\cos(y) & = 1 \\ e^x\sin(y) & = 0 \end{cases}.$$

Since $e^x > 0$ for all $x \in \mathbb{R}$ we have that $\sin(y) = 0$, which implies that $y = k\pi$ for some $k \in \mathbb{Z}$. Substituting $y = k\pi$ into the other equation yields

$$\begin{aligned} &\cos(y) = e^x \cos(k\pi) = 1 \\ \iff\ & 1 = e^x \cos(k\pi) = \begin{cases} e^x, & \text{if } k = 2\ell, \text{for } \ell \in \mathbb{Z} \\ -e^x, & \text{if } k = 2\ell+1, \text{for } \ell \in \mathbb{Z} \end{cases} \end{aligned}$$

Since $e^x \neq 0$ for all $x \in \mathbb{R}$, if k is odd, then there is no solution. However, if k is even, then there is a solution if $\Re(z) = 0$. Hence, e^z iff $z = 0 + 2\ell\pi i$, for some $\ell \in \mathbb{Z}$. ∎

(Part f)

Proof. Let $z \in \mathbb{C}$. Then $z = x + iy$ for some $x, y \in \mathbb{R}$. Changing to polar coordinates then yields $z = r(\cos(\theta) + i\sin(\theta))$ for $r \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. Then by Euler's identity we have that $z = re^{i\theta}$. ∎

We now check

$$r = |z| \quad \text{and} \quad \theta = \arctan(y/x).$$

Let $x = \cos(\theta)$ and $y = i\sin(\theta)$. Then

$$r = \sqrt{(\cos\theta)^2 + (i\sin\theta)^2} = \sqrt{x^2 + y^2} = |z|.$$

and similarly

$$y/x = \frac{i\sin\theta}{\cos\theta} = \tan(\theta) \Rightarrow \arctan(y/x) = \theta$$

which is well defined for all $(x, y) \in \mathbb{R}^2$ s.t. $x \neq 0$. Furthermore, since we are limiting $r \geq 0$ and $\theta \in [0, 2\pi)$ the representation is unique.

(Part g) Let $i = e^{i\pi/2}$. Consider

$$\begin{aligned} zi = re^{i\theta} i &= re^{i\theta} e^{i\pi/2} \\ &= re^{i(\theta+\pi/2)} \\ &= r(\cos(\theta + \pi/2) + i\sin(\theta + \pi/2)). \end{aligned}$$

Therefore, we have rotated the point z by 90^o anti-clockwise about the origin on the unit circle; or shifted the period of the functions sin and cos by $\pi/2$. In general, if we multiply by i we rotate the point z about the boundary of the closed disc with radius $|z|$.

(Part h) Claim: Given $\theta \in \mathbb{R}$, show

$$\frac{e^{i\theta} + e^{i\theta}}{2} = \cos(\theta) \qquad \frac{e^{i\theta} - e^{i\theta}}{2i} = \sin(\theta)$$

Proof. By Euler's identity we have that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$. Adding the above then yields

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta) \iff \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos(\theta)$$

and subtracting yields

$$e^{i\theta} - e^{-i\theta} = 2i\sin(\theta) \iff \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin(\theta).$$

■

(Part i) Claim: $\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$

Proof. Let $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{i\varphi} = \cos\varphi + i\sin\varphi$. Then

$$\begin{aligned} e^{i\theta}e^{i\varphi} &= (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi) \\ &= \cos\theta\cos\varphi + i\sin\theta\cos\varphi + i\sin\varphi\cos\theta - \sin\theta\sin\varphi \\ &= \cos\theta\cos\varphi - \sin\theta\sin\varphi + i(\sin\theta\cos\varphi + \sin\varphi\cos\theta). \end{aligned}$$

Furthermore, since $e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$ we have that

$$\begin{aligned} e^{i(\theta+\varphi)} &= \cos(\theta + \varphi) + i\sin(\theta + \varphi) \\ &= \cos\theta\cos\varphi - \sin\theta\sin\varphi + i(\sin\theta\cos\varphi + \sin\varphi\cos\theta). \end{aligned}$$

Then taking the real and imaginary parts, respectively, yields

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

and

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \sin\varphi\cos\theta.$$

■

Claim: $2\sin\theta\sin\varphi = \cos(\theta - \varphi) - \cos(\theta + \varphi)$ and $2\sin\theta\cos\varphi = \sin(\theta + \varphi) + \sin(\theta - \varphi)$.

Proof. Since $\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$, we have

$$\cos(\theta - \varphi) - \cos(\theta + \varphi) = \cos\theta\cos(-\varphi) = \sin\theta\sin(-\varphi) - (\cos\theta\cos\varphi - \sin\theta\sin\varphi).$$

Moreover, since cos is an even function and sin is an odd function

$$\begin{aligned} \sin\theta\sin(-\varphi) - (\cos\theta\cos\varphi - \sin\theta\sin\varphi) &= \cos\theta\cos\varphi + \sin\theta\sin\varphi - \cos\theta\cos\varphi + \sin\theta\sin\varphi \\ &= 2\sin\theta\sin\varphi. \end{aligned}$$

A similar argument shows that $2\sin\theta\cos\varphi = \sin(\theta + \varphi) + \sin(\theta - \varphi)$

■

1.3 Exercise 5

Claim: $f(x) = e^{inx}$ is 2π-periodic.

Proof. Let $x = \theta + 2\pi$. Then

$$e^{inx} = e^{in(\theta+2\pi)} = \cos(n\theta + 2\pi n) + i\sin(n\theta + 2\pi n).$$

Where the last equality follows from Euler's identity. By the identities proven in the last exercise we have

$$\begin{aligned}\cos(n\theta + 2\pi n) + i\sin(n\theta + 2\pi n) &= \cos(n\theta)\cos(2\pi n) - \sin(n\theta)\sin(2\pi n)\\ &\quad + i(\sin(n\theta)\cos(2\pi n) + \sin(2\pi n)\cos(n\theta))\\ &= \cos(n\theta) + i\sin(n\theta),\end{aligned}$$

since $\cos(2\pi n) = 1$ and $\sin(2\pi n) = 0$ for all $n \in \mathbb{Z}$ ∎

I have verified the calculations of the given integrals using Maple. Below is the code and output for each of the integrals, along with some edits to make the output more readable.
We begin by calculating

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,dx.$$

```
> ans1 := (int((cos(n*x)+I*sin(n*x))/(2*Pi), x = -Pi..Pi,AllSolutions)
        assuming n::integer;

                    ans1:=piecewise(1, if n = 0; 0, otherwise)
```

Calculating

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx\,dx$$

for $n, m \geq 1$ yields

```
> ans2 := int(cos(n*x)*cos(m*x)/Pi, x = -Pi..Pi, AllSolutions)
        assuming [n::posint , m::posint];
                ans2:= piecewise(1, if -n + m = 0; 0, otherwise).
```

Computing

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx\,dx$$

for $n, m \geq 1$ yields

```
> ans3 := int(sin(n*x)*sin(m*x)/Pi, x = -Pi..Pi, AllSolutions)
       assuming [n::posint, m::posint];
             ans3:= piecewise(1, if -n + m = 0; 0, otherwise).
```

Computing

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx$$

for any n, m yields

```
> ans4 := ([int(sin(n*x)*cos(m*x), x = -Pi..Pi, AllSolutions)],
       assuming [n::integer, m::integer];
            ans4:= 0.
```

1.4 Exercise 8

Since F is a function with two continuous derivatives, by the Fundamental Theorem of Calculus we have that

$$F(x+h) - F(x) = \int_x^{x+h} F'(y)\, dy.$$

Then Taylor expanding F' about x yields

$$\begin{aligned}\int_x^{x+h} F'(y)\, dy &= \int_x^{x+h} \big[F'(x) + (y-x)F''(x) + (x-y)\psi(y-x)\big]\, dy \\ &= \int_x^{x+h} F'(x)\, dy + \int_x^{x+h} (y-x)F''(x)\, dy + \int_x^{x+h} (x-y)\psi(y-x)\, dy.\end{aligned}$$

For $\int_x^{x+h} (x-y)\psi(y-x)\, dy$, if $u = y - x$, then $du = dy$ and $[x, x+h] \mapsto [0, h]$. Then evaluating yields

$$\int_x^{x+h} F'(x)\, dy + \int_x^{x+h} (y-x)F''(x)\, dy + \int_0^h (u)\psi(u)\, du =$$
$$F'(x)h + F''(x)[\frac{y^2}{2} - xy]_x^{x+h} + \int_0^h u\psi(u)\, du,$$

and since ψ is continuous, by the Mean Value theorem for integrals, there exists a $\xi \in (0, h)$ s.t. $\int_0^h u\psi(u)\, du = \psi(\xi) \int_0^h u\, du$. So

$$F'(x)h + F''(x)[\frac{y^2}{2} - xy]_x^{x+h} + \int_0^h u\psi(u)\, du = F'(x) + F''(x)\frac{h^2}{2} + \psi(\xi)\frac{h^2}{2}.$$

Returning to our original equation we have $F(x+h)-F(x) = F'(x)+F''(x)\frac{h^2}{2}+\psi(\xi)\frac{h^2}{2}$. Rearranging this equations yields,

$$\frac{F(x+h)-F(x)+F(x-h)-F(x)}{h^2} - F''(x) = F'(x)h + \psi(\xi)\frac{h^2}{2},$$

where $F'$and ψ are continuous, so $F'(x)h+\psi(\xi)\frac{h^2}{2} \to 0$ as $h \to 0$. Furthermore, since F is continuous by hypothesis,

$$\frac{F(x+h)+F(x-h)-2F(x)}{h^2} \to F''(x), \quad \text{as } h \to 0.$$

1.5 Exercise 9

Let

$$f(x) = \begin{cases} \dfrac{xh}{p}, & x \in [0,p] \\ \dfrac{h(\pi - x)}{\pi - p}, & x \in [p,\pi] \end{cases}.$$

We then wish to compute

$$A_n = \frac{2}{\pi}\Big[\int_0^p \frac{xh}{p}\sin(nx)\,dx + \int_p^\pi \frac{h(\pi-x)}{\pi-p}\sin(nx)\,dx\Big],$$

where in the latter integral, if $u = \pi - x$, then

$$\frac{h}{(\pi-p)}\int_p^\pi (\pi-x)\sin(nx)\,dx = \frac{h}{(\pi-p)}\int_0^{\pi-p} u\sin(nu)\,du.$$

Integrating by parts we have

$$\begin{aligned}\int_0^p x\sin(nx)\,dx &= \frac{-(x)\cos(nx)}{n} + \frac{1}{n}\int_0^p (x)'\cos(nx)\,dx \\ &= \frac{-(x)\cos(nx)}{n} + \frac{1}{n^2}\sin(nx)\Big]_0^p \\ &= \frac{-p}{n}\cos(np) + \frac{1}{n^2}\sin(np).\end{aligned}$$

Similarly,

$$\int_0^{\pi-p} u\sin(nu)\,du = \frac{-(\pi-p)}{n}\cos(n(\pi-p)) + \frac{1}{n^2}\sin(n(\pi-p)).$$

So we have that

$$A_n = \Big[\frac{2h(-p)}{\pi pn}\cos(np) + \frac{2h}{n^2 p\pi}\sin(n(\pi - p))\Big] + \Big[\frac{-(\pi-p)2h}{\pi n(\pi-p)} + \frac{2h}{n^2\pi(\pi-p)}\sin(n(\pi-p))\Big]$$

and since cos and sin are 2π-periodic,

$$\begin{aligned} A_n &= \frac{-2h}{\pi n}\Big[\cos(np) - \cos(np)\Big] + \frac{2h}{\pi n^2}\Big[\frac{\sin(np)}{\pi - p} + \frac{\sin(np)}{p}\Big] \\ &= 0 + \frac{2h}{n^2(\pi-p)p}\sin(np). \end{aligned}$$

Hence, $A_n = \frac{2h}{n^2(\pi-p)p}\sin(np)$.

Our next goal is to determine for which positions of p the second, fourth. . . , and third, sixth, . . . harmonics are missing. Formally, we want to know what values of p satisfy $A_{2k} = 0$ and $A_{3k} = 0$ for $k \in \mathbb{Z}$.
We see that $A_{2k} = 0$ iff $\sin(2kp) = 0$ iff $p = \frac{\pi}{2}$ for $k \in \mathbb{Z}$, and similarly, $A_{3k} = 0$ iff $\sin(3kp) = 0$ iff $p = \frac{\pi}{3}$ for $k \in \mathbb{Z}$.

1.6 Exercise 10

Proof. Let $u(r, \theta) = f(r\sin\theta, r\cos\theta)$. Then by the chain rule and product rules

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\ \frac{\partial^2 u}{\partial r^2} &= \frac{\partial^2 f}{\partial x^2}\cos\theta + \frac{\partial^2 f}{\partial y^2}\sin\theta + 2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta \\ \frac{\partial u}{\partial \theta} &= r\frac{\partial f}{\partial x}(-\sin)\theta + r\frac{\partial f}{\partial y}\cos\theta \\ \frac{\partial^2 u}{\partial \theta^2} &= -r\frac{\partial f}{\partial x}\cos\theta - r\frac{\partial f}{\partial y}\sin\theta + r^2\frac{\partial^2 f}{\partial x^2}\sin^2\theta + r\frac{\partial^2 f}{\partial y^2}\cos^2\theta - 2r^2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta. \\ &d \end{aligned}$$

If $r \neq 0$, summing the above yields

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

■

Claim:
$$\Big|\frac{\partial u}{\partial x}\Big|^2 + \Big|\frac{\partial u}{\partial r}\Big|^2 = \Big|\frac{\partial u}{\partial r}\Big|^2 + \frac{1}{r^2}\Big|\frac{\partial u}{\partial \theta}\Big|^2.$$

Proof. Suppose that u is a complex valued function. Then

$$\Big|\frac{\partial u}{\partial x}\Big|^2 = \Big|\frac{\partial u}{\partial r}\Big|\cos^2\theta + \frac{1}{r^2}\Big|\frac{\partial u}{\partial \theta}\Big|^2 \sin^2\theta + \frac{2}{r}\Re\Big(\overline{\frac{\partial u}{\partial r}}\frac{\partial u}{\partial \theta}\cos\theta\sin\theta\Big)$$

and

$$\Big|\frac{\partial u}{\partial y}\Big|^2 = \Big|\frac{\partial u}{\partial r}\Big|\sin^2\theta + \frac{1}{r^2}\Big|\frac{\partial u}{\partial \theta}\Big|^2 \cos^2\theta - \frac{2}{r}\Re\Big(\overline{\frac{\partial u}{\partial r}}\frac{\partial u}{\partial \theta}\cos\theta\sin\theta\Big).$$

Therefore,
$$\Big|\frac{\partial u}{\partial x}\Big|^2 + \Big|\frac{\partial u}{\partial r}\Big|^2 = \Big|\frac{\partial u}{\partial r}\Big|^2 + \frac{1}{r^2}\Big|\frac{\partial u}{\partial \theta}\Big|^2$$

∎

1.7 Problem 1

As the text suggest, we look for a solution to the heat equation which is of the form
$$u(x,y) = F(x)G(y).$$

Applying the Laplacian yields

$$\Delta u = u_{xx} + u_{yy} = F''(x)G(y) + F(x)G''(y) = 0 \iff \frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = \lambda$$

for $\lambda \in \mathbb{R}$. This yields the following system of equations

$$\begin{cases} F''(x) - \lambda F(x) = 0 \\ G''(y) + \lambda G(y) = 0. \end{cases}$$

Recall that we must satisfy the the boundary conditions $u(0,y) = u(\pi,y) = 0$. Thus, F must satisfy $F(0) = F(\pi)$. Therefore, we claim that $F(x) = c\sin(mx)$ is a possible solution, if $\lambda = -m^2$.
This can be easily verified since

$$\frac{F''(x)}{F(x)} = \frac{-c\,m^2\sin(mx)}{c\sin(mx)} = -m^2.$$

Furthermore, if $\lambda = -m^2$, then $G(y)$ is a linear combination of e^{my} and e^{-my} by results demonstrated in 22B. Thus, choosing the basis $\{\sinh m(1-y), \sinh(my)\}$ we have

$$G(y) = a \sinh m(1-y) + b \sinh(mx).$$

Thus,

$$u_m(x,y) = (a_m \sinh m(1-y) + b_m \sinh(my)) \sin(mx),$$

since $u(0,y) = u(\pi,y) = 0$. Then taking the linear combination u, we have $u(x,y) = \sum_{m=1}^{\infty} u_m(x,y)$.
Lastly, we much satisfy the horizontal boundary conditions, namely, $u(x,y) = f_0(x)$ and $u(x,1) = f_1(x)$ are satisfied.
So

$$u(x,0) = \sum a_m \sinh m \sin(mx) = \sum A_m \sin(mx) \implies a_m = \frac{A_m}{\sinh(m)}.$$

and similiarly

$$u(x,1) = \sum b_m \sinh(m) \sin(mx) = \sum B_m \sin(mx) \implies b_m = \frac{B_m}{\sinh(m)}.$$

Therefore,

$$u(x,y) = \sum_{m=1}^{\infty} (\frac{\sinh m(1-y)}{\sinh m} A_m + \frac{\sinh(mx)}{\sinh m} B_m) \sin(mx).$$

is a solution to the Dirichlet problem in a rectangle.

Chapter 2

Basic Properties of Fourier Series

2.1 Exercise 3

Proof. It's enough to show that $f(x) = \sum A_m \sin(mx)$ is uniformly convergent, so that we can apply corollary 2.3.
Since $\sin(mx) \leq 1$,

$$\begin{aligned}\sum_{m=1}^{\infty} |A_m \sin(mx)| &\leq \sum_{m=1}^{\infty} |A_m| \\ &\leq C \sum_{m=1}^{\infty} \frac{1}{m^2}\end{aligned}$$

where $C = \frac{2h}{p(\pi - p)}$.
Since the last sum is a convergent p-series, by the comparison theorem $\sum_{m=1}^{\infty} A_m \sin(mx)$ converges absolutely.
Then by corollary 2.3, since the Fourier series is absolutely convergent and f is continuous function defined on the circle, the Fourier series converges to f. ∎

2.2 Exercise 4

(Part a) See figure 2.1 below.

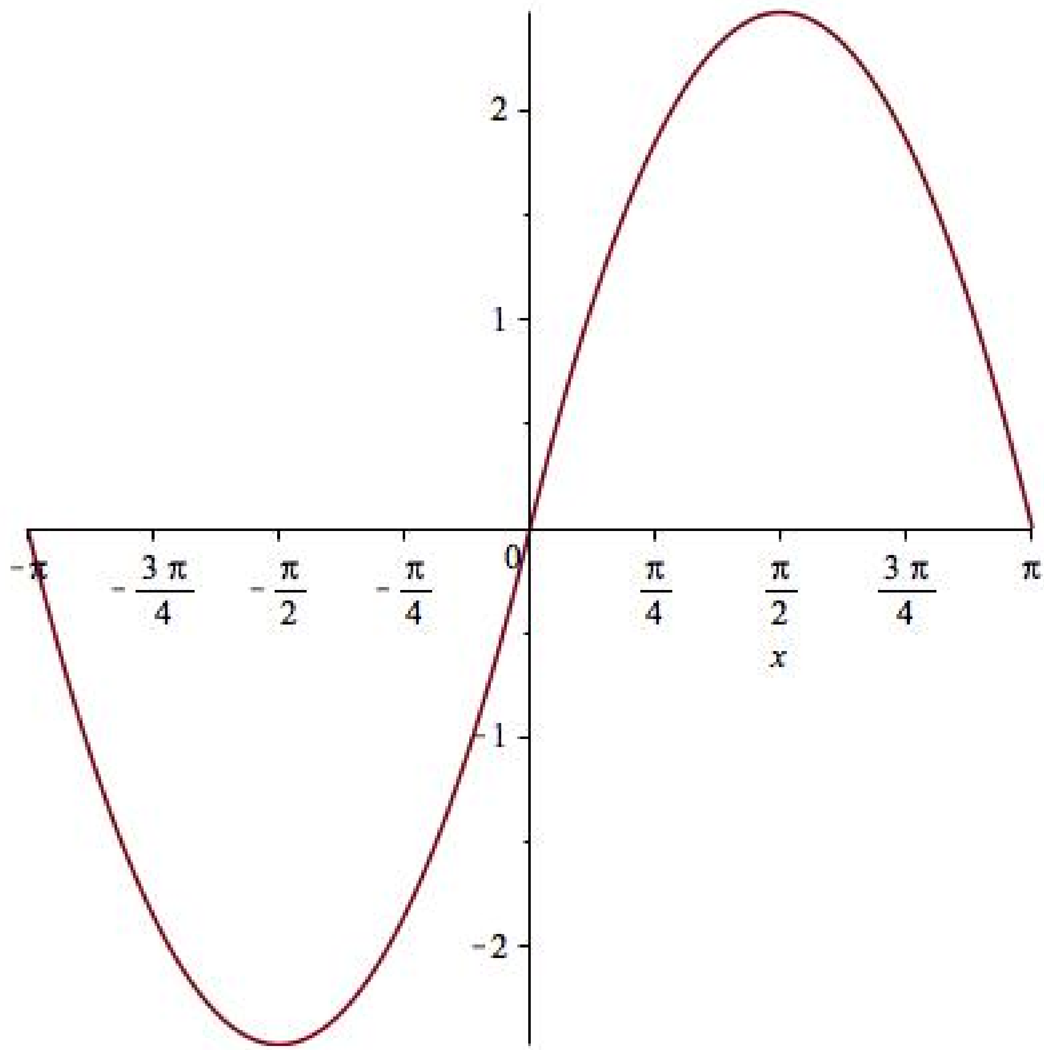

Figure 2.1: $f(\theta) = \theta(\pi - \theta)$

(Part b) By exercise 2 part c, since f is assumed to be an odd function, we calculate the sine Fourier series. So

$$\hat{f}(n) = \frac{2}{\pi}\int_0^{\pi} \theta(\pi-\theta)\sin(n\theta)\,d\theta$$

Computing in Maple yields

```
ans := 2*(int(x*(Pi-x)*sin(n*x), x = 0 .. Pi))/Pi
                -4((-1)+(-1)^n)/Pi n^3
```

Thus,

$$\hat{f}(n) = -\frac{4(-1+(-1)^n)}{\pi n^3}.$$

However, for even n the Fourier coefficients are 0. Thus,

$$\hat{f}(n) = \frac{8}{\pi n^3}.$$

Therefore, the Fourier series of f is given by

$$f(\theta) = \frac{8}{\pi}\sum_{k\,\text{odd}\geq 1} \frac{\sin(n\theta)}{n^3}.$$

2.3 Exercise 7

Let $B_k = \sum_{n=1}^{k} b_n$ denote the partial sums of the series $\sum b_n$.
Claim:

$$\sum_{n=M}^{N-1} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1}(a_{n+1}-a_n)B_n.$$

Proof.

$$\begin{aligned}
\sum_{n=M}^{N} a_n b_n &= \sum_{n=M}^{N} a_n (B_n - B_{n-1}) \\
&= \sum_{n=M}^{N} a_n B_n - \sum_{n=M}^{N} a_n B_{n-1} \\
&= \sum_{n=M}^{N} a_n B_n - \sum_{n=M-1}^{N-1} a_{n+1} B_n \\
&= a_N B_N + \sum_{M}^{N-1} a_n B_n - a_M B_{M-1} - \sum_{M}^{N-1} a_{n+1} B_n \\
&= a_N B_N - a_M B_{M-1} - \sum_{M}^{N-1} (a_{n+1} - a_n) B_n.
\end{aligned}$$

Note that in the fourth line we pulled out the last term and first term of the last and first sums respectively.

Theorem 2.3.1 (Dirichlet's test)**.** *If the partial sums of the series $\sum b_n$ are bounded, and (a_n) is a sequence of real numbers that decreases monotonically to 0, then $\sum a_n b_n$ converges.*

Suppose $|B_k| \leq B$ for all $k \in \mathbb{N}$. Let $\varepsilon > 0$. Since $(a_n) \to 0$ monotonically, there exists an $N \in \mathbb{N}$ s.t. for $n > N$, $a_n < \frac{\varepsilon}{2B}$.

$$\begin{aligned}
\Big|\sum_{M}^{N} a_n b_n\Big| &= |a_N B_N - a_M B_{M-1} - \sum_{M}^{N-1} (a_{n+1} - a_n) B_n| \\
&\leq B a_N + B a_M + B \sum_{M}^{N-1} (a_n - a_{n+1}) \\
&= B a_N + B a_M + B(a_M - a_N) \\
&= 2B a_M \\
&< \frac{2B\varepsilon}{2B} = \varepsilon
\end{aligned}$$

whenever $m > n \geq N$ Thus, by the Cauchy criterion the series $\sum a_n b_n$ converges. ■

2.4 Exercise 8

We must verify that the Fourier series of

$$f(x) = \begin{cases} -\frac{\pi}{2} - \frac{x}{2}, & x \in (-\pi, 0) \\ \frac{\pi}{2} - \frac{x}{2}, & x \in (0, \pi) \end{cases}$$

where $f(0) = 0$.
First we compute the Fourier coefficients given by

$$\hat{f}(n) = \underbrace{\frac{1}{\pi}\int_{-\pi}^{0}(\frac{-\pi}{2} - \frac{x}{2})e^{-2inx}\,dx}_{I_1} + \underbrace{\frac{1}{\pi}\int_{0}^{\pi}(\frac{\pi}{2} - \frac{x}{2})e^{-2inx}\,dx}_{I_2}.$$

Beginning with I_1,

$$\begin{aligned} I_1 &= \frac{1}{\pi}\Big[-\frac{(\frac{-\pi}{2} - \frac{x}{2})}{2in}e^{-2inx} + \frac{1}{2in}\int_{-\pi}^{0} -e^{-2inx}\,dx\Big] \\ &= \frac{1}{\pi}\Big[-\frac{(\frac{-\pi}{2} - \frac{x}{2})}{2in}e^{-2inx} - \frac{1}{4n}e^{-2inx}\,dx\Big]_{-\pi}^{0} \\ &= \frac{1}{\pi}\Big[(\frac{\pi}{4in} - \frac{1}{4n}) - (0 - \frac{1}{4n})e^{-2\pi in}\Big] \\ &= \frac{1}{\pi}\Big[\frac{\pi}{4in} - \frac{1}{4n} + \frac{1}{4n}\Big] \\ &= \frac{1}{4in}. \end{aligned}$$

Then computing I_2 yields

$$\begin{aligned} I_2 &= \frac{1}{\pi}\Big[-\frac{(\frac{\pi}{2} - \frac{x}{2})}{2in}e^{-2inx} + \frac{1}{2in}\int_{0}^{\pi} e^{-2inx}\,dx\Big] \\ &= \frac{1}{\pi}\Big[-\frac{(\frac{\pi}{2} - \frac{x}{2})}{2in}e^{-2inx} + \frac{1}{4n}e^{-2inx}\Big]_{0}^{\pi} \\ &= \frac{1}{\pi}\Big[(0 + \frac{1}{4n}) - (\frac{-\pi}{4in} + \frac{1}{4n})\Big] \\ &= \frac{1}{4in}. \end{aligned}$$

Thus, $\hat{f}(n) = I_1 + I_2 = \frac{1}{2in}$. So the Fourier series of f is given by

$$f(x) \sim \frac{1}{2i}\sum_{n\neq 0}\frac{1}{n}e^{inx}.$$

We now verify that the series converges for all x. We will verify this by using Dirichlet's test.
Clearly $(\frac{1}{n}) \to 0$ as $n \to \infty$. So it remains to show that $|F_N| = |\sum e^{inx}/2i| \leq M$. We may then break up the sum partial sums of $\sum e^{inx}/2i$ in terms of negative and positive integers. Thus,

$$\sum_{n=1}^{N} \frac{e^{inx}}{2i} + \sum_{n=-N}^{-1} \frac{e^{inx}}{2i} = \sum_{n=1}^{N} \frac{e^{inx} - e^{-inx}}{2i}.$$

If $\omega = e^{ix}$, then we identify the above is simply a finite geometric series, which we can evaluate. Hence,

$$\begin{aligned}
\frac{1}{2i}\sum_{n=1}^{N} \omega^n - \omega^{-n} &= \frac{1}{2i}\Big[\sum_{n=1}^{N}\omega^n - \sum_{n=1}^{N}\omega^{-n}\Big] \\
&= \frac{1}{2i}\Big[\frac{\omega(1-\omega^N)}{1-\omega} + \frac{\omega^{-N}-1}{1-\omega}\Big] \\
&= \frac{1}{2i}\Big[\frac{\omega - \omega^{N+1} + \omega^{-N} - 1}{1-\omega}\Big] \\
&= \frac{1}{2i}\frac{\omega^{-1/2}}{\omega^{-1/2}}\Big[\frac{\omega - \omega^{N+1} + \omega^{-N} - 1}{1-\omega}\Big] \\
&= \frac{1}{2i}\Big[\frac{\omega^{1/2} - \omega^{N-1/2} + \omega^{-(N+1/2)} - \omega^{-1/2}}{\omega^{-1/2} - \omega^{1/2}}\Big] \\
&= \frac{1}{2i}\Big[\frac{(\omega^{N} - \omega^{-N})(\omega^{(N+1)} - \omega^{-(N+1)})}{\omega^{-1/2} - \omega^{1/2}}\Big] \\
F_N &= \Big[\frac{\sin(Nx)\sin((N+1)x/2)}{\sin(x/2)}\Big].
\end{aligned}$$

Then since $|\sin(y)| \leq 1$, we have $|F_N| \leq \frac{1}{\sin(|x|/2)}$ for $|x| < \pi$ and $x \neq 0$. Therefore, by Dirichlet's test $\frac{1}{2i}\sum_{n\neq 0}\frac{1}{n}e^{inx}$ converges for $x \neq 0$. Moreover, at $x = 0$ we see that sum is symmetric, so $(F_N) \to 0$ at $x = 0$. Therefore, the series for all x.

2.5 Exercise 10

Proof. Since f is 2π periodic and $f \in \mathcal{C}^k$, we may integrate by parts k-times. Recall the Fourier coefficient are given by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)\, dx.$$

$$
\begin{aligned}
2\pi\hat{f}(n) &= \int_{-\pi}^{\pi} e^{-inx} f(x)\,dx \\
&= \underbrace{\left[\frac{1}{(-in)} e^{-inx} f(x)\right]_{\pi}^{\pi}}_{=0} - \frac{1}{(-in)} \int_{-\pi}^{\pi} e^{-inx} f'(x)\,dx \\
&= \underbrace{\left[\frac{1}{(-in)^2} e^{-inx} f'(x)\right]_{\pi}^{\pi}}_{=0} - \frac{1}{(-in)^2} \int_{-\pi}^{\pi} e^{-inx} f''(x)\,dx \\
&= \underbrace{\left[\frac{1}{(-in)^3} e^{-inx} f''(x)\right]_{\pi}^{\pi}}_{=0} - \frac{1}{(-in)^3} \int_{-\pi}^{\pi} e^{-inx} f'''(x)\,dx \\
&\vdots \\
&= \underbrace{\left[\frac{1}{(-in)^{(k-1)}} e^{-inx} f^{(k-2)}(x)\right]_{\pi}^{\pi}}_{=0} - \frac{1}{(-in)^{(k-1)}} \int_{-\pi}^{\pi} e^{-inx} f^{(k-1)}(x)\,dx \\
&= \underbrace{\left[\frac{1}{(-in)^{(k)}} e^{-inx} f^{(k-1)}(x)\right]_{\pi}^{\pi}}_{=0} - \frac{1}{(-in)^{(k)}} \int_{-\pi}^{\pi} e^{-inx} f^{(k)}(x)\,dx \\
\hat{f}(n) &= \frac{-1}{2\pi(-in)^{(k)}} \int_{-\pi}^{\pi} e^{-inx} f^{(k)}(x)\,dx.
\end{aligned}
$$

Then, since $f \in \mathcal{C}^k$, the integral $\int \exp(-inx) f^k(x)\,dx$ is bounded, $|\int_{-\pi}^{\pi} e^{-inx} f^{(k)}(x)\,dx| \leq M2\pi$. Taking the absolute value of our Fourier coefficient yields

$$
\begin{aligned}
|\hat{f}(n)| &= \left|\frac{-1}{2\pi(-in)^{(k)}} \int_{-\pi}^{\pi} e^{-inx} f^{(k)}(x)\,dx\right| \\
&\leq \left|\frac{-1}{2\pi(-in)^{(k)}}\right| \left|\int_{-\pi}^{\pi} e^{-inx} f^{(k)}(x)\,dx\right| \\
&\leq \left|\frac{-1}{2\pi(-in)^{(k)}}\right| 2\pi M \\
&= \frac{M}{|n|^k}.
\end{aligned}
$$

Therefore, $|\hat{f}(n)| = O(1/|n|^k)$ as $|n| \to \infty$. ■

2.6 Exercise 11

Proof. Let $(f_k)_{k=1}^{\infty}$ is a sequence of Riemann integrable functions on the interval $[0,1]$ s.t.

$$\int_0^1 |f_k(x) - f(x)|\,dx \to 0, \quad \text{as } k \to \infty.$$

Let $\varepsilon > 0$. Then for $k > N \in \mathbb{N}$ we have

$$\begin{aligned}
|\hat{f}_k(n) - \hat{f}(n)| &= \Big|\int_0^1 f_k(x)e^{-2\pi inx}\,dx - \int_0^1 f(x)e^{-2\pi inx}\,dx\Big| \\
&= \Big|\int_0^1 (f_k(x) - f(x))e^{-2\pi inx}\,dx\Big| \\
&\le \int_0^1 |f_k(x) - f(x)||e^{-2\pi inx}|\,dx \\
&= \int_0^1 |f_k(x) - f(x)|\,dx < \varepsilon.
\end{aligned}$$

Thus, $\hat{f}_k(n) \to \hat{f}(n)$ uniformly as $k \to \infty$. ■

2.7 Exercise 12

Proof. Let $s_n = \sum_{k=1}^{n} c_k$ where c_k is a sequence of complex numbers. Let $\sigma_N = \frac{s_1+s_2+\ldots+s_N}{N}$ denote the N^{th} Cesàro Mean. Suppose that $(s_n) \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$. Choose $N_s \in \mathbb{N}$ s.t. if $n \ge N_s$, then $|s_n| \le \frac{\varepsilon}{2}$. Furthermore, choose $N' > N_s$ s.t.

$$\Big|\frac{\sum_{k=1}^{N_s} s_k}{n}\Big| < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}
|\sigma_n| &= \Big|\frac{s_1 + s_2 + s_3 + \ldots + s_n}{n}\Big| \\
&\le \frac{|s_1| + |s_2| + \ldots + |s_n|}{n} \\
&= \frac{|s_1| + |s_2| + \ldots + |s_{N_s}|}{n} + \frac{|s_{N_s+1}| + |s_{N_s+2}| + \ldots + |s_n|}{n} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Note that it is sufficent to prove the that $(s_n) \to 0$, since we may set $t_n = s_n - s$ and use the same bounds as above. ■

2.8 Exercise 13

(Part a)

Proof. Suppose that $s_n = \sum_{n=1}^{n} c_n$ is a series of complex numbers that converges to 0. We claim that this series is Abel summable to s. Let $\varepsilon > 0$. Since $(s_n) \to 0$, then $|s_n| < \varepsilon$ for $n \leq N$. Moreover, $|s_n| \leq M$ for some $0 < M$.
Note that if c_0=0, then $(s_n - s_{n-1}) = c_n$, for $n \geq 1$.
Then since $0 \leq r < 1$ and s_n is convergent the following are absolutely convergent,

$$\begin{aligned}\sum_{n=1}^{N} r^n c_n &= \sum_{n=1}^{N} r^n (s_n - s_{n-1}) \\ &= \sum_{n=1}^{N} r^n s_n - \sum_{n=1}^{N} r^n s_{n-1} \\ &= \sum_{n=1}^{N} r^n s_n - \sum_{n=1}^{N} r^{n+1} s_n \\ &= (1-r)\sum_{n=1}^{N} r^n s_n.\end{aligned}$$

Then

$$\begin{aligned}\left|(1-r)\sum_{n=1}^{\infty} r^n s_n\right| &\leq (1-r)\sum_{n=1}^{\infty} r^n |s_n| \\ &= (1-r)\sum_{n=1}^{N-1} r^n |s_n| + (1-r)\sum_{n=N}^{\infty} r^n |s_n| \\ &< (1-r)\sum_{n=1}^{N-1} r^n M + \varepsilon r^N.\end{aligned}$$

Then note $\limsup_{r\to 1^-}(1-r)\sum_{n=1}^{N-1} r^n M = 0$. Thus, $\left|(1-r)\sum_{n=1}^{\infty} r^n s_n\right| < \varepsilon$. ■

(Part b)

Proof. Consider $c_n = (-1)^n$, then

$$\lim_{r\to 1^-}\sum_{n=1}^{\infty}(-1)^n r^n = \lim_{r\to 1^-}\sum_{n=1}^{\infty}(-r)^n = \lim_{r\to 1^-}\frac{-r}{1-r} = -\frac{1}{2},$$

But $\sum_{n=1}^{\infty}(-1)^n$ diverges. ■

(Part c)

Proof. Suppose that $\sum_{n=1}^{\infty} c_n$ is Cesàro summable to σ. We claim that $\sum_{n=1}^{\infty} c_n$ is Abel summable to σ.
Let $\varepsilon > 0$ and $0 \le r < 1$. Suppose that $\sigma = 0$. Since $\sum c_n$ is Cesàro summable, $|\sigma_n| < \varepsilon$ for all $n \in \mathbb{N}$. Furthermore, $|\sigma_n| \le M$ for some $0 < M$. Then following the text's hint

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} c_n r^n\right| &= \left|(1-r)^2\sum_{n=1}^{\infty} n\sigma_n r^n\right| \\
&= \left|(1-r)^2\sum_{n=1}^{N-1} n\sigma_n r^n + (1-r)^2\sum_{n=N}^{\infty} n\sigma_n r^n\right| \\
&\le (1-r)^2\sum_{n=1}^{N-1} n|\sigma_n| r^n + (1-r)^2\sum_{n=N}^{\infty} n|\sigma_n| r^n \\
&\le (1-r)^2 M\sum_{n=1}^{N-1} n r^n + (1-r)^2\varepsilon\sum_{n=N}^{\infty} n r^n \\
&\le (1-r)^2 M\sum_{n=1}^{N-1} n r^n + r^N\varepsilon(N - rN + r).
\end{aligned}$$

Where in the expression on the right, $\sum_{n=N}^{\infty} nr^n = r^N(N - rN + r)/(1-r)^2$, as verified in Maple. Then taking the limit as $r \to 1^-$, the last expression converges to ε.
Thus,

$$\left|\sum_{n=1}^{\infty} c_n r^n\right| = \left|(1-r)^2\sum_{n=1}^{\infty} n\sigma_n r^n\right| < \varepsilon.$$

■

(Part d)

Proof. Now we provide a series which is Abel summable, but not Cesàro summable.
Let $c_n = (-1)^n n$. Then

$$A(r) = \sum_{n=1}^{\infty}(-1)^n n r^n = \sum_{n=1}^{\infty}(-r)^n n = \lim_{r\to 1^-} \frac{-r}{(r+1)^2} = -\frac{1}{4}.$$

As hinted at in the text, if $\sum c_n$ is Cesàro summable, then $\lim_{n\to\infty} c_n/n \to 0$. However, $\frac{c_n}{n} = (-1)^n$, which does not converge as $n \to \infty$. ■

2.9 Exercise 14

(Part a)

Proof. Let $\varepsilon > 0$. Suppose that $s_n = \sum c_n$ is Cesàro summable to σ and $c_n = o(1/n)$. Then there exists an $N_\sigma \in \mathbb{N}$ s.t. for all $n > N_\sigma$, $|\sigma_n - \sigma| < \varepsilon$. Furthermore,

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{s_1 + s_2 + \ldots + s_n}{n} && (2.1)\\ &= \frac{ns_n - (s_1 + s_2 + \ldots + s_n)}{n} && (2.2)\\ &= \frac{(s_n - s_1) + (s_n - s_2) + \ldots + (s_n - s_n)}{n} && (2.3)\\ &= \frac{(n-1)c_n + (n-2)c_{n-1} + \ldots + (n-(n-2))c_3 + c_2}{n}. && (2.4)\end{aligned}$$

Let $\tau_n = (n-1)c_n$, then substituting τ_n into the expression on line 2.4, we have

$$s_n - \sigma_n = \frac{\tau_n + \tau_{n-1} + \ldots + \tau_2}{n} \to 0, \quad \text{as } n \to \infty.$$

Which goes to zero since $c_n = o(1/n)$.

Thus, for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ s.t. if $n > N$, then $|s_n - \sigma_n| < \frac{\varepsilon}{2}$.

So

$$|s_n - \sigma| = |s_n - \sigma_n + \sigma_n - \sigma| \leq |s_n - \sigma_n| + |\sigma_n - \sigma| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

■

(Part b)

Proof. Suppose that $\sum c_n$ is Abel summable and $c_n = o(1/n)$. Let $r = (1 - \frac{1}{N})$. Let $\varepsilon > 0$.
Since $c_n = o(1/n)$, $|c_n n| < \varepsilon$ for $n \geq N$.

We next establish the following lemma

Lemma 2.9.1.

$$(1 - \frac{1}{N})^n \geq (1 - \frac{n}{N}),$$

for all $0 \leq n \leq N$.

Proof. **Base Case:** If $n = 0$, then $(1 - \frac{1}{N})^0 = 1 = 1 - 0$. So the claim holds.
Inductive Step: Suppose that the claim holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
\left(1 - \frac{(n+1)}{N}\right) &\leq \left(1 - \frac{n}{N}\right)\left(1 - \frac{n+1}{N}\right) \\
&\leq \left(1 - \frac{n+1}{N}\right)^2 \\
&\leq \left(1 - \frac{1}{N}\right)\left(1 - \frac{n}{N}\right) \\
&\leq \left(1 - \frac{1}{N}\right)^2 \\
&\leq \left(1 - \frac{1}{N}\right)^{n+1}.
\end{aligned}$$

∎

Next we establish the following bound as per the texts suggestion. For $n \geq$

$N > 0$,

$$\begin{aligned}
\Big|\sum_{n=1}^{N} c_n - \sum_{n=1}^{N} c_n r^n\Big| &= \Big|\sum_{n=1}^{N} c_n(1-(1-\frac{1}{N})^n)\Big| \\
&\leq \sum_{n=1}^{N} |c_n||1-(1-\frac{1}{N})^n)| \\
&\leq \sum_{n=1}^{N} \frac{|c_n n|}{N} \\
&< \sum_{n=1}^{N} \frac{\varepsilon}{N} \\
&= \frac{\varepsilon N}{N} \\
&= \varepsilon
\end{aligned}$$

Thus, $\Big|\sum_{n=1}^{N} c_n - \sum_{n=1}^{N} c_n r^n\Big| < \varepsilon$. ∎

2.10 Exercise 17

(Part a)

Proof. Suppose that f is integrable on the circle and that f has a jump discontinuity at θ.
Let

$$A_r(f)(\theta) = (f * P_r)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x) f(\theta - x)\, dx,$$

and recall $P_r(\theta) = \frac{1-r^2}{1-2r\cos(\theta)+r^2}$.
Since $P_r(\theta) = P_r(-\theta)$, we have $\int_{-\pi}^{0} P_r(\theta)\, d\theta = \int_{0}^{\pi} P_r(\theta)\, d\theta = 1/2$.

Let $\varepsilon > 0$ and pick a $\delta > 0$ s.t. $|f(\theta+h) - f(\theta^+)| < \varepsilon$ and $|f(\theta-h) - f(\theta^-)| < \varepsilon$

for $0 < h < \delta$. Furthermore, since f is integrable, $|f(x)| \leq M$ for $M > 0$. Then

$$\begin{aligned}
&\Big|A_r(f)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2}\Big| = \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x) f(\theta - x)\,dx - \frac{f(\theta^+) + f(\theta^-)}{2}\Big| \\
&= \Big|\frac{1}{2\pi}\int_{-\pi}^{0} P_r(x) f(\theta - x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} P_r(x) f(\theta - x)\,dx - \frac{f(\theta^+) + f(\theta^-)}{2}\Big| \\
&\leq \Big|\frac{1}{2\pi}\int_{-\pi}^{0} P_r(x) f(\theta - x) - \frac{f(\theta^+)}{2}\,dx\Big| + \Big|\frac{1}{2\pi}\int_{0}^{\pi} P_r(x) f(\theta - x) - \frac{f(\theta^-)}{2}\,dx\Big| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{0} |P_r(x)||f(\theta - x) - f(\theta^+)|\,dx + \frac{1}{2\pi}\int_{0}^{\pi} |P_r(x)||f(\theta - x) - f(\theta^-)|\,dx. \\
&\leq \frac{1}{2\pi}\int_{-\delta < x} |P_r(x)||f(\theta - x) - f(\theta^+)|\,dx + \frac{1}{2\pi}\int_{x < \delta} |P_r(x)||f(\theta - x) - f(\theta^-)|\,dx \\
&+ \frac{1}{2\pi}\int_{\delta < x \leq \pi} |P_r(x)| 2M\,dx \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \int_{\delta < x \leq \pi} |P_r(x)|\frac{M}{\pi}\,dx.
\end{aligned}$$

Then since $P_r(x)$ is a good kernel, as $r \to 1$, $\int_{\delta < x \leq \pi} |P_r(x)|\,dx < \varepsilon$.
Therefore, for some constant $C > 0$

$$\Big|A_r(f)(\theta) - \frac{f(\theta^+) + f(\theta^-)}{2}\Big| < C\varepsilon.$$

■

(Part b)

Proof. The proof is very similar to part a.
However, we use the fact that the N^{th} Cesàro mean is given by

$$\sigma_N(f)(x) = (f * F_N)(x)$$

and if $\{D_n(x)\}_{n=0}^{N-1}$ denotes the family of Dirichlet kernels, then

$$F_N(x) = \frac{D_0(x) + \ldots + D_{N-1}(x)}{N}$$

is the Fejér kernel which is a good kernel.
Furthermore, since $F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}$, $F_N(x) = F_N(-x)$. Therefore, $\int_{-\pi}^{0} F_N(x)\,dx =$

$\int_0^\pi F_N(x)\,dx = 1/2$, and by the third property of good kernels, $\int_{\delta\le|x|\le\pi} |F_N(x)|\,dx \to 0$, as $N \to \infty$.

Thus, for all $\varepsilon > 0$, there exist a $\delta > h > 0$ s.t. $|f(\theta+h) - f(\theta^+)| < \varepsilon$ and $|f(\theta-h) - f(\theta^-)| < \varepsilon$ implies

$$\left|\sigma_N(f)(x) - \frac{f(\theta^+) + f(\theta^-)}{2}\right| < C'\varepsilon.$$

■

2.11 Exercise 20

Let $u(r,\theta) = \sum c_n(r)e^{in\theta}$, where $c_n(r) = A_n r^n + B_n r^{-n}, n \neq 0$. Set $f(\theta) \sim \sum a_n e^{in\theta}$ and $g(\theta) \sim \sum b_n e^{in\theta}$.
Furthermore,

$$u(r,\theta) = \sum_{n\neq 0} \frac{((\rho/r)^n - (r/\rho)^n)}{(\rho^n - \rho^{-n})} a_n e^{in\theta} + \frac{(r^n - r^{-n})}{(\rho^n - \rho^{-n})} b_n e^{in\theta} + a_0 + (b_0 - a_0)\frac{\log r}{\log \rho}.$$

We wish to show that

$$u(r,\theta) - (P_r * f)(\theta) \to 0$$

as $r \to 1$ uniformly in θ and

$$u(r,\theta) - (P_{p/r} * g)(\theta) \to 0$$

as $r \to \rho$ uniformly in θ. We begin with the first statement.

$$\begin{aligned} u(r,\theta) - (P_r * f)(\theta) &= \sum_{n=1}^{\infty} c_n(r)e^{in\theta} - \sum_{n=1}^{\infty} r^n a_n e^{in\theta} \\ &= \sum_{n=1}^{\infty} (c_n(r) - a_n r^n)e^{in\theta}. \end{aligned}$$

Then we see that

$$c_n(r) - a_n r^n = \underbrace{\frac{1}{\rho^n - \rho^{-n}}\Big((\rho/r)^n - (r/\rho)^n\Big)a_n - a_n r^n}_{(1)} + \underbrace{\frac{r^n - r^{-n}}{\rho^n - \rho^{-n}} b_n}_{(2)}.$$

We now proceed by splitting this sum into two cases, since if $u(r,\theta) - (P_r * f)(\theta) \to 0$, then both (1) and (2) both go to zero.
We make the following key observations: $\frac{(r^n - r^{-n})}{(\rho^n - \rho^{-n})} = \frac{(\rho/r)^{2n} - 1}{\rho^{2n} - 1}(\rho/r)^n$, and since a_n is a Fourier coefficient, it follows that $|a_n| \leq A$ for some $A > 0$.
So

$$\begin{aligned}
\sum_{n=1}^{\infty} \Big| \left(\frac{(\rho/r)^n - (r/\rho)^n}{\rho^n - \rho^{-n}} - r^n \right) a_n \Big| &= \sum_{n=1}^{\infty} \Big| \frac{(\rho/r)^{2n} - 1}{\rho^{2n} - 1} + 1 \Big| a_n r^n \\
&\leq A \sum_{n=1}^{\infty} \Big| \frac{(\rho/r)^{2n} - \rho^{2n}}{\rho^2 n - 1} r^n \Big| \\
&= A \sum_{n=1}^{\infty} \frac{(\rho/r)^{2n} - \rho^{2n}}{1 - \rho^{2n}} r^n \\
&= \frac{A}{1-\rho^2} \sum_{n=1}^{\infty} (\rho/r)^n - (\rho^2 r)^n \\
&= \frac{A}{1-\rho^2} \left(\frac{\rho^2/r}{1-\rho^2/r} - \frac{\rho^2 r}{1-\rho^2 r} \right) \to 0, \text{ as } r \to 1 \ .
\end{aligned}$$

Note that for the last line we identified a geometric series. Then for part (2) we see that $\lim_{r\to 1} \frac{r^n - r^{-n}}{\rho^n - \rho^{-n}} b_n = 0$. Thus, $u(r,\theta) - (P_r * f)(\theta) \to 0$ uniformly in θ.
A similar argument shows that $u(r,\theta) - (P_{p/r} * g)(\theta) \to 0$ as $r \to \rho$ uniformly in θ.

Chapter 3

Convergence of Fourier Series

3.1 Exercise 6

Let $(a_n)_{n\in\mathbb{Z}}$ be defined by

$$a_n = \begin{cases} 1/n, & \text{if } k \geq 1 \\ 0, & \text{if } k \leq 0 \end{cases}$$

For the sake of contradiction, suppose that a_n are the Fourier coefficients of a Riemann integrable function f. Since f is integrable, f is bounded. Let $M = \sup_{-\pi\leq\theta\leq\pi} |f(\theta)|$. Then since the Poisson kernel, $P_r(\theta)$, is an even function and $P_r(\theta) > 0$ we have

$$\begin{aligned} |A_r(f)(0)| &= \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)P_r(-\theta)\,d\theta\Big| \\ &= \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)P_r(\theta)\,d\theta\Big| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|P_r(\theta)\,d\theta \\ &\leq \frac{M}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta \\ &\leq M. \end{aligned}$$

However,

$$\lim_{r\to 1} A_r(f)(0) = \lim_{r\to 1}\sum_{n=1}^{\infty}\frac{r^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

which diverges, a contradiction.
Therefore, no such Riemann integrable function with Fourier coefficients a_n exists.

3.2 Exercise 7

Claim: : $\sum_{n\geq 2} \frac{1}{\log(n)} \sin(nx)$ converges for all x, but is not the Fourier series of a Riemann integrable function.

Proof. We first show convergence.
Let $a_n = 1/\log n$, then clearly a_n is monotonically decreasing to 0.
Let $s_n = \sum_{n=1}^{N} \sin(nx)$. Rewriting $\sin(nx)$ as $\Im(e^{inx})$ and if $t = e^{ix}$, then

$$\sum_{n=1}^{N} \sin(nx) = \sum_{n=1}^{N} t^n = \frac{t(1-t^N)}{1-t} = \Im\Big(\frac{e^{ix}(1-e^{iNx})}{1-e^{ix}}\Big).$$

Then rewriting $1 - e^{ix}$, we have

$$1 - e^{ix} = e^{ix/2}(e^{-ix/2} - e^{ix/2}) = -2ie^{ix/2}\sin(x/2).$$

So

$$\begin{aligned}\frac{e^{ix}(1-e^{iNx})}{1-e^{ix}} &= i\Big(\frac{e^{ix/2} - e^{i(N+1/2)x}}{2\sin(x/2)}\Big)\\ &= \frac{\cos(x/2) - \cos((N+1/2)x)}{2\sin(x/2)}\end{aligned}$$

where we took the real part of the second expression. Therefore, $\sum_{n=1}^{N} \sin(nx) = \frac{\cos(x/2)-\cos((N+1/2)x)}{2\sin(x/2)}$. So we see that $|s_n| = |\sum_{n=1}^{N}\sin(nx)| \leq 1/|\sin(x/2)|$, whenever $\sin(x/2) \neq 0$.
So by Dirichlet's test,

$$\sum_{n\geq 2} \frac{1}{\log(n)} \sin(nx)$$

converges for all x.

For the sake of contradiction, suppose that $\sum_{n\geq 2}\frac{1}{\ln(n)}\sin(nx)$ is the Fourier series of a Riemann integrable function. Then by Parseval's identity,

$$\sum_{n\geq 2}\left|\frac{1}{\log(n)}\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2\,d\theta.$$

The Cauchy condensation test tells us $\sum a_n$ converges iff $\sum 2^n a_{2^n}$ converges. So

$$\sum_{n\geq 2}\frac{2^n}{|\log(2^n)|^2} = \sum_{n\geq 2}\frac{2^n}{n^2\log^2(2)} \geq \sum_{n=1}^{\infty}\frac{1}{n}$$

since $2^n > n^2$ for large n. Since the latter sum diverges, it follows that $\sum_{n\geq 2}\frac{1}{|\log(n)|^2}$ diverges, contradicting Parseval's identity.

■

3.3 Exercise 8

(Part a)

Proof. We will first show that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Let $f(x) = x^2$ for $x \in [-\pi,\pi]$. We then compute the Fourier coefficients corresponding to the function. Hence,

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}x^2e^{-inx}\,dx = \frac{2(-1)^n}{n^2}$$
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\,dx = \frac{\pi^2}{3}.$$

By Parseval's identity we have that

$$\frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{2\pi}\int_{-\pi}^{\pi}x^4\,dx$$

Thus,

$$\begin{aligned}\frac{\pi^2}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} &= \frac{\pi^4}{5}\\ \sum_{n=1}^{\infty}\frac{1}{n^4} &= \frac{\pi^4}{8}\left[\frac{1}{5}-\frac{1}{9}\right] = \frac{\pi^4}{90}.\end{aligned}$$

Next we show that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

Let $f(\theta) = |\theta|$ for $x \in [-\pi, \pi]$.
By the results of chapter 3 exercise 6, the Fourier coefficients of $|\theta|$ are given by

$$\hat{f}(n) = \begin{cases} \frac{\pi}{2}, & \text{if } n = 0 \\ \frac{-1+(-1)^n}{\pi n^2}, & \text{if } n \neq 0 \end{cases}.$$

So by Parseval's identity

$$\frac{\pi^2}{4} + \frac{4}{\pi^2}\sum_{n \text{ odd } \geq 1}\frac{2}{n^4} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta^2\, d\theta = \frac{\pi^2}{3}$$

So

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

Note: For $\sum \frac{1}{n^4}$ we could have derived the value if we expressed the series as the sum of the odd and even integers, and made several algebraic manipulations, rather than defining a new function and repeating the Parseval argument. ■

(Part b)

Proof. We now show that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6} = \frac{\pi^6}{960}$$

and consequently

$$\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}.$$

Let $f(\theta) = \theta(\pi - \theta)$ be a 2π-periodic odd function defined on $[-\pi, \pi]$. as shown in chapter 3 exercise 4, the Fourier series is given by

$$f(\theta) = \frac{8}{\pi} \sum_{k \text{ odd} \geq 1} \frac{\sin(n\theta)}{n^3} \iff f(\theta) = \frac{-i4}{\pi} \sum_{k \text{ odd} \geq 1} \frac{e^{in\theta} - e^{-in\theta}}{n^3}.$$

Thus,

$$a_n = \begin{cases} \frac{-i4}{\pi n^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Applying Parseval's identity yields

$$\frac{16}{\pi^2} \sum_{n=0}^{\infty} \frac{2}{n^6} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta^2(\pi - \theta)^2 = \frac{\pi^4}{30}.$$

So

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960}.$$

We then see that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} + \frac{1}{2^6} \sum_{n=1}^{\infty} \frac{1}{n^6}$$

Solving for $\sum_{n=1}^{\infty} \frac{1}{n^6}$ yields

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

■

3.4 Exercise 11

(Part a)

Proof. Let f be T-periodic, continuous and piecewise $\mathcal{C}^1$ with $\int_0^T f(t)\, dt = 0$. If $n \neq 0$, then since f is T-periodic and continuous, we may integrate by parts. Thus,

$$\hat{f}(n) = \frac{1}{T} \int_0^T f(t) e^{-in\frac{2\pi}{T}}\, dt = \frac{1}{Tin2\pi} \int_0^T f'(t) e^{-in\frac{2\pi}{T}}\, dt = \frac{T}{in2\pi} \hat{f}(n).$$

So,

$$
\begin{aligned}
\int_0^T |f(t)|^2\,dt &= T\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \\
&= T\sum_{n\neq 0} |\hat{f}(n)|^2 \\
&= \frac{T^3}{4\pi^2}\sum_{n\neq 0} |\frac{\hat{f}(n)}{n}|^2 \\
&\leq \frac{T^3}{4\pi^2}\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \\
&= \frac{T^2}{4\pi^2}\int_0^T |f'(t)|^2\,dt.
\end{aligned}
$$

It then follows that we have equality iff $\hat{f}(n) = 0$, which occurs whenever $|n| = 1$. This means that $f(t) = \hat{f}(1)e^{\frac{2i\pi}{T}t} + \hat{f}(-1)e^{\frac{-2i\pi}{T}t}$ which occupies the same space as $f(t) = A\sin(\frac{2\pi t}{T}) + B\cos(\frac{2\pi t}{T})$. ■

(Part b)

Proof. Let f be as above, and let g be $\mathcal{C}^1$ and T-periodic. Then by Cauchy-Schwarz and part a we have

$$
\begin{aligned}
\left|\int_0^T \bar{f}(t)g(t)\,dt\right|^2 &\leq \Big(\int_0^T |f(t)||g(t)|\,dt\Big) \\
&\leq \Big(\int_0^T |f(t)|^2|\Big)\Big(\int_0^T |g(t)|^2\,dt\Big) \\
&\leq \frac{T^2}{4\pi^2}\int_0^T |f(t)|^2\,dt \int_0^T |g(t)|^2\,dt
\end{aligned}
$$

■

(Part c)

Proof. Define $F(x) = f(x+a)$ to be a $2(b-a)$-periodic, $\mathcal{C}^1$, odd function with the conditions $f(0) = f(a) = 0$ and $f(b-a) = f(b) = 0$. Furthermore, since

F is odd, F' is even. Then applying part a, we have

$$\begin{aligned}
\int_a^b |f(t)|^2\,dt &= \frac{1}{2}\int_0^{2(b-a)} |F(t)|^2\,dt \\
&\le \frac{2(b-a)^2}{2\pi^2}\int_0^{2(b-a)} |F'(t)|^2\,dt \\
&= \frac{(b-a)^2}{\pi^2}\int_0^{2(b-a)} |F'(t)|^2\,dt \\
&= \frac{(b-a)^2}{\pi^2}\int_a^b |f'(t)|^2\,dt.
\end{aligned}$$

■

3.5 Exercise 14

Proof. Suppose that f is continuously differentiable.
Recall that $\hat{f'}(n) = in\hat{f}(n)$. So

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |\hat{f}(n)| &= \Big(\sum_{n\neq 0}\frac{1}{n^2}\Big)^{1/2}\Big(\sum_{n\in\mathbb{Z}} |\hat{f'}(n)|\Big)^{1/2} \\
&\le \Big(\frac{\pi^2}{3}\Big)^{1/2}\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(t)|^2\,dt.
\end{aligned}$$

Since f is continuously differentiable, $\int_{-\pi}^{\pi} |f'(t)|\,dt \le M$ for some $M > 0$. Thus,

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| \le M\Big(\frac{\pi^2}{3}\Big)^{1/2}$$

So the Fourier series is absolutely convergent. ■

3.6 Exercise 15

(Part a)

Proof. Let f be $2\pi-$periodic and Riemann integrable on $[-\pi, \pi]$. Then

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + \pi/n) e^{-inx}\, dx.$$

By change of variables, if $u = x + \pi/n$, then $dx = du$ and $[-\pi, \pi] \mapsto [-\pi + \pi/n, \pi + \pi/n]$. So

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi} \int_{-\pi+\pi/n}^{\pi+\pi/n} f(u) e^{-inu}\, du \\
&= \frac{-1}{2\pi} \int_{-\pi}^{\pi} f(u) e^{-inu}\, du \\
&= \frac{-1}{2\pi} \int_{-\pi}^{\pi} f(x + \pi/n) e^{-inx}\, dx.
\end{aligned}$$

Then applying part a to the function $f(x) - f(x + \pi/n)$ we have

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{4\pi} \int_{-\pi}^{\pi} f(x) - f(x + \pi/n)\, dx \\
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx - \frac{1}{4\pi} \int_{-\pi}^{\pi} f(x + \pi/n)\, dx \\
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx - \frac{1}{2} \hat{f}(n) \\
\frac{1}{2} \hat{f}(n) &= \frac{1}{4\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \\
\hat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.
\end{aligned}$$

So the identity holds. ■

(Part b)

Proof. Suppose that f satisfies the Hölder condition of order α. Then

$$\begin{aligned}
|\hat{f}(n)| &= \Big| \frac{1}{4\pi} \int_{-\pi}^{\pi} f(x) - f(x + \pi/n) e^{-inx}\, dx \Big| \\
&\leq \frac{1}{4\pi} \int_{-\pi}^{\pi} |f(x) - f(x + \pi/n)|\, dx \\
&\leq \frac{1}{4\pi} 2\pi C \left(\frac{\pi}{n} \right)^{\alpha} \\
&= C' \frac{1}{n^{\alpha}}
\end{aligned}$$

Thus, $\hat{f}(n) = O(1/|n|^{\alpha})$. ■

3.7 Problem 2

(Part a)

Proof. Let

$$L_n(x) = \frac{d^n}{dx^n}(x^2-1)^n$$

for $n \in \mathbb{N}$ and suppose f is indefinitely differentiable on $[-1,1]$. Then we may integrate by parts n times, where upon each integration by parts we pick up an extra (-1) factor. Thus,

$$\int_{-1}^{1} L_n(x) f(x)\, dx = (-1)^n \int_{-1}^{1} (x^2-1)^n f^n(x)\, dx.$$

Furthermore, if $m < n$, then

$$\int_{-1}^{1} L_n(x) x^m \, dx = 0,$$

since for the mth derivative of x^m vanishes. ■

(Part b) Next, as hinted at in the text

$$\begin{aligned} ||L_n||^2 &= \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^n}{dx^n}(x^2-1)^n \, dx \\ &= (-1)^n \int_{-1}^{1} (x^2-1)^n \frac{d^{2n}}{dx^{2n}}(x^2-1)^n \, dx \\ &= (-1)^n (2n)! \int_{-1}^{1} (x^2-1)^n \, dx \\ &= (-1)^n (2n)! \int_{-1}^{1} (x-1)^n (x+1)^n \, dx. \end{aligned}$$

Noting that it doesn't matter whether we choose $(x+1)^n$ or $(x-1)^n$ to differentiate when integrating by parts, we will integrate $(x+1)^n$ and differentiate

$(x-1)^n$. So integrating by parts yields

$$\begin{aligned}\int_{-1}^{1}(x-1)^n(x+1)^n\,dx &= -\frac{n}{n+1}\int_{-1}^{1}(x-1)^{n-1}(x+1)^{n+1}\,dx\\ &\vdots\\ &= \frac{(-1)^n n(n-1)(n-2)\dots(2)(1)}{(n+1)(n+2)\dots(n+n)}\int_{-1}^{1}(x+1)^{2n}\,dx\\ &= \frac{n!^2 2^{2n+1}}{(2n)!(2n+1)}\end{aligned}$$

Note that for all $n = 1, 2, \dots$, the boundary terms vanish. Hence,

$$||L_n||^2 = \frac{(n!)^2 2^{2n+1}}{2n+1}.$$

(Part c)

Proof. Let $P(x)$ be an nth degree polynomial orthogonal to $1, x, \dots, x^{n-1}$. Let C be a constant s.t. $P(x) - CL_n(x)$ is degree $n-1$ or less.
Then since the degree is $n-1$, $P(x) \perp CL_n(x)$ by part a, thus

$$P(x) - CL_n(x) = 0 \iff P(x) = CL_n(x).$$

■

(Part d)

Proof. Let $\mathcal{L}_n = \frac{L_n}{||L_n||}$. We claim that $\{\mathcal{L}\}$ forms an orthonormal basis in which every Riemann integrable function f on $[-1,1]$ has a Legendre expansion

$$\sum_{n=1}^{\infty}\langle f|\mathcal{L}_n\rangle|\mathcal{L}_n\rangle.$$

To show this we will apply to the the Gram-Schmidt process to $\mathcal{B} = \{1, x, x^2, x^3, \dots x^n, \dots\}$. We begin with $v_0 = 1$ by normalizing v_1 and denoting it's normalization by v_0'. So $||v_0||^2 = \int_{-1}^{1}\,dx = 2$, which implies $v_1' = 1/\sqrt{2}$. Then

$$\begin{aligned}v_1 &= x - \langle v_0'|x\rangle|v_0'\rangle\\ &= x - \frac{1}{2}\Big(\int_{-1}^{1} x\,dx\Big)\\ &= x\end{aligned}$$

and

$$||x||^2 = \int_{-1}^{1} x^2 = \frac{2}{3} \Rightarrow ||x|| = \sqrt{\frac{2}{3}}.$$

So $v_1' = \sqrt{\frac{2}{3}}x$. Next,

$$\begin{aligned} v_2 &= x^2 - \langle \frac{1}{\sqrt{2}} | x^2 \rangle | \frac{1}{\sqrt{2}} \rangle \\ &= x^2 - \frac{1}{2}\Big(\int_{-1}^{1} x^2 \, dx \Big) \\ &= x^2 - \frac{1}{2}(\frac{x^3}{3})\Big|_{-1}^{1} \\ &= x^2 - 1/3 \end{aligned}$$

Then

$$||x^2 - \frac{1}{3}||^2 = \int_{-1}^{1} (x^4 - \frac{2}{3}x^2 + \frac{1}{9}) \, dx = \frac{8}{45}$$

So

$$v_2' = \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3}).$$

Continuing this process, we see that $v_i' = \mathcal{L}_i$ for all $i \in \mathbb{N}$. Thus, $\{\mathcal{L}_n\}$ forms an orthonormal basis by applying the Gram-Schmidt process to $\mathcal{B}$.

Moreover, since $\{\mathcal{L}_n\}$ is an orthonormal basis, it follows that any $f \in \mathcal{R}([-1,1])$ can be expressed in this basis and we have convergence of

$$||f||^2 = \sum_{k=1}^{\infty} |\langle f | \mathcal{L}_n \rangle|^2$$

by Parseval's identity.

■

Chapter 4

Some Applications of Fourier Series

In order to compare sums to their corresponding integrals, we will use Euler's summation formula in the following exercises.

Theorem 4.0.1 (Euler's Summation Formula)**.** *(Apostol, theorem 7.1.3) If f has a continuous derivative f' on $[a, b]$, then*

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)\langle x\rangle\,dx + f(a)\langle a\rangle - f(b)\langle b\rangle$$

where $\langle x\rangle = x - \lfloor x\rfloor$. When a and b are integers, this becomes

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)\big(\langle x\rangle - 1/2\big)\,dx + \frac{f(a)+f(b)}{2}.$$

4.1 Exercise 8

Proof. Applying Euler's summation formula, we have

$$\begin{aligned}
\left|\sum_{n=1}^{N} e^{2\pi i b n^\sigma}\right| &= \left|\int_1^N e^{2\pi i b x^\sigma}\,dx + \int_1^N (e^{2\pi i b x^\sigma})'(\langle x\rangle - 1/2)\,dx + \frac{e^{2\pi i b}}{2} + \frac{e^{2\pi i b N^\sigma}}{2}\right| \\
&\leq \left|\int_1^N e^{2\pi i b x^\sigma}\,dx\right| + \left|\int_1^N (e^{2\pi i b x^\sigma})'(\langle x\rangle - 1/2)\,dx\right| + \left|\frac{e^{2\pi i b}}{2}\right| + \left|\frac{e^{2\pi i b N^\sigma}}{2}\right| \\
&= \left|\int_1^N e^{2\pi i b x^\sigma}\,dx\right| + \left|(e^{2\pi i b N^\sigma} - e^{2\pi i b})(\langle x\rangle - 1/2)\right| + \left|\frac{e^{2\pi i b}}{2}\right| + \left|\frac{e^{2\pi i b N^\sigma}}{2}\right| \\
&\leq \left|\int_1^N e^{2\pi i b x^\sigma}\,dx\right| + \left|(e^{2\pi i b N^\sigma} - e^{2\pi i b})\right| + \left|\frac{e^{2\pi i b}}{2}\right| + \left|\frac{e^{2\pi i b N^\sigma}}{2}\right|
\end{aligned}$$

First note that we used the fact that $|\langle x\rangle - 1/2| \leq 1$.
Then once we divide by N, we see that the last three terms go to 0, as $N \to \infty$.
So it's sufficient to show that the first integral go to 0 when divided by N as $N \to \infty$.
Beginning with the first integral, we have

$$\left|\int_1^N e^{2\pi i b x^\sigma}\,dx\right| = \left|\int_1^N \frac{2\pi i b\sigma}{2\pi i b\sigma}\frac{x^{1-\sigma}}{x^{1-\sigma}} e^{2\pi i b x^\sigma}\,dx\right|.$$

Then by change of variable, if $u = \exp(2\pi i b x^\sigma)$, then $du = 2\pi i x^{\sigma-1}\sigma\exp(2\pi i b x^\sigma)\,dx$. So $\frac{1}{2\pi b i x^{\sigma-1}}\,du = \exp(2\pi i x^\sigma b)\,dx$. Therefore, integration by parts yields

$$\begin{aligned}
\left|\int_1^N e^{2\pi i b x^\sigma}\,dx\right| &= \left|\int_1^N \frac{1}{2\pi i b\sigma x^{\sigma-1}}\,du\right| \\
&= \left|\left.\frac{x^{1-\sigma}}{2\pi i b\sigma} e^{2\pi i b x^\sigma}\right]_1^N - \int_1^N \frac{(1-\sigma)x^{-\sigma}}{2\pi i b\sigma} e^{2\pi i b x^\sigma}\,dx\right| \\
&\leq \left|\frac{N^{1-\sigma}e^{2\pi i b N^\sigma} - e^{2\pi i b}}{2\pi i b\sigma}\right| + \int_1^N \frac{(1-\sigma)x^{-\sigma}e^{2\pi i b x^\sigma}}{|2\pi i b\sigma|}\,dx \\
&\leq \frac{N^{1-\sigma}+1}{|2\pi i b\sigma|} + \frac{1-\sigma}{|2\pi i b\sigma|}\int_1^N x^{-\sigma}\,dx \\
&= \frac{N^{1-\sigma}+1}{|2\pi i b\sigma|} + \frac{1}{|2\pi i b\sigma|}(N^{1-\sigma} - 1) \\
&= O(N^{1-\sigma}).
\end{aligned}$$

Since $\sigma \in [0,1)$, when divided by N the term becomes $O(N^{\sigma-1})$ and goes to 0 as $N \to \infty$. ■

4.2 Exercise 9

Proof. Suppose the contrary, that $\langle a\log(n)\rangle$ is equidistributed. Then by Weyl's criteria

$$\frac{1}{N}\sum_{n=1}^{\infty} e^{2\pi ib\log(n)} \to 0, \quad \text{as } N \to \infty.$$

Applying Euler's summation yields

$$\sum_{n=1}^{N} e^{2\pi ib\log(n)} = \frac{Ne^{2\pi ib\log(N)}-1}{2\pi ib+1} + \int_a^b \frac{e^{2\pi bi\log(x)}2\pi ib}{x}\big(\langle x\rangle - 1/2\big)\,dx + \frac{e^{2\pi i\log(N)}+1}{2}.$$

Then dividing both sides by N we see that the last term goes to zero, as does the middle term since

$$\begin{aligned}\left|\frac{1}{N}\int_1^N \frac{2\pi ibe^{2\pi ib\log(x)}}{x}(\langle x\rangle - 1/2)\,dx\right| &\le \frac{1}{N}\int_1^N \left|\frac{2\pi ibe^{2\pi ib\log(x)}}{x}\right|\big|(\langle x\rangle - 1/2)\big|\,dx \\ &\le \frac{2\pi b}{N}\int_1^N \frac{1}{x}\,dx \\ &= 2\pi b\frac{\log(N)}{N} \to 0, \quad \text{as } N \to \infty \text{ by L'Hospital's rule.}\end{aligned}$$

Note that since $\langle x\rangle \in [0,1)$, $|\langle x\rangle - 1/2| \le 1$.
However,

$$\begin{aligned}\frac{Ne^{2\pi ib\log(N)}-1}{N(2\pi ib+1)} &= \frac{e^{2\pi ib\log(N)}}{2\pi ib+1} - \frac{1}{N(2\pi bi+1)} \\ &= \frac{N^{2\pi bi}}{2\pi ib+1} - \frac{1}{N(2\pi bi+1)},\end{aligned}$$

which clearly diverges. This contradicts the assumption that

$$\frac{1}{N}\sum_{n=1}^{\infty} e^{2\pi ib\log(n)} \to 0, \quad \text{as } N \to \infty.$$

So $\langle a\log(n)\rangle$ is not equidistributed. ■

The figures below demonstrate what an equidistributed sequences looks like compared to a non-equidistributed sequence, respectively. The Maple code for the graphs is also given below.

```
N:=1000:
f:=n ->evalf(frac(n*Pi)):
g:=n ->evalf(frac(n*sqrt(2))):
h:=n ->evalf(frac(ln(n))):
X:=Vector(N,i->i):
Yf:=Vector(N,f):
Yg:=Vector(N,g):
Yh:=Vector(N,h):
```

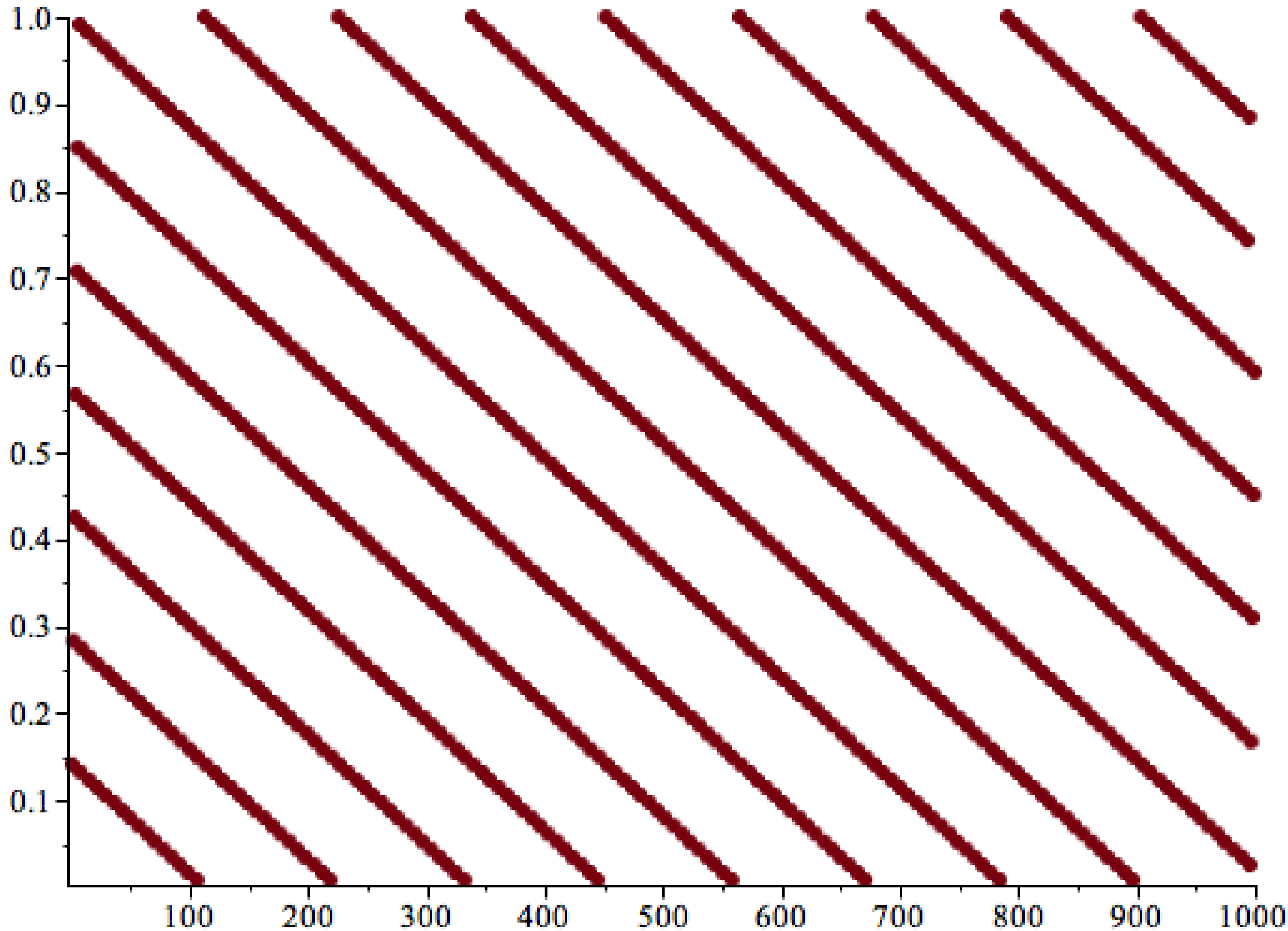

Figure 4.1: $\langle \pi n \rangle$

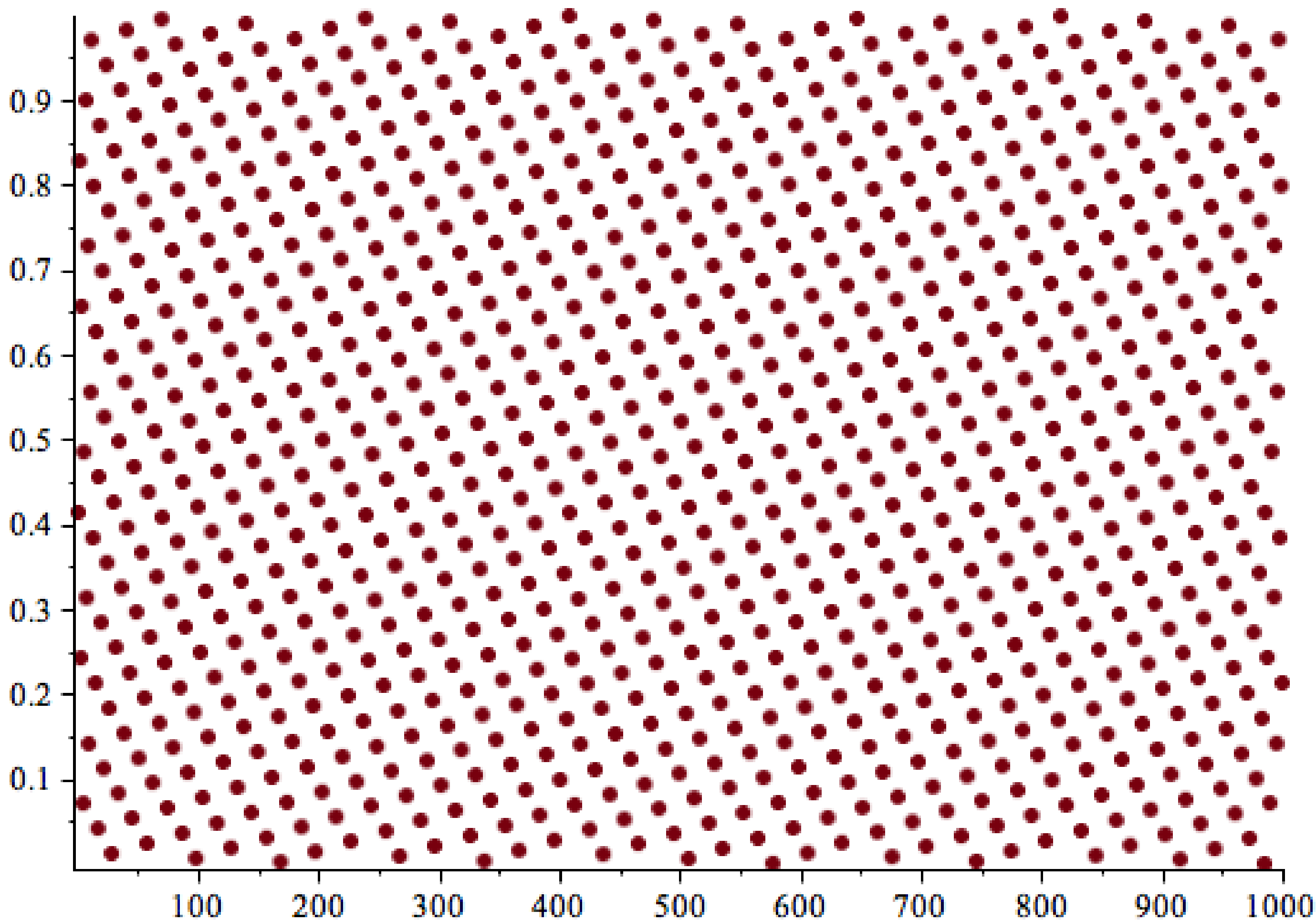

Figure 4.2: $\langle\sqrt{2}n\rangle$

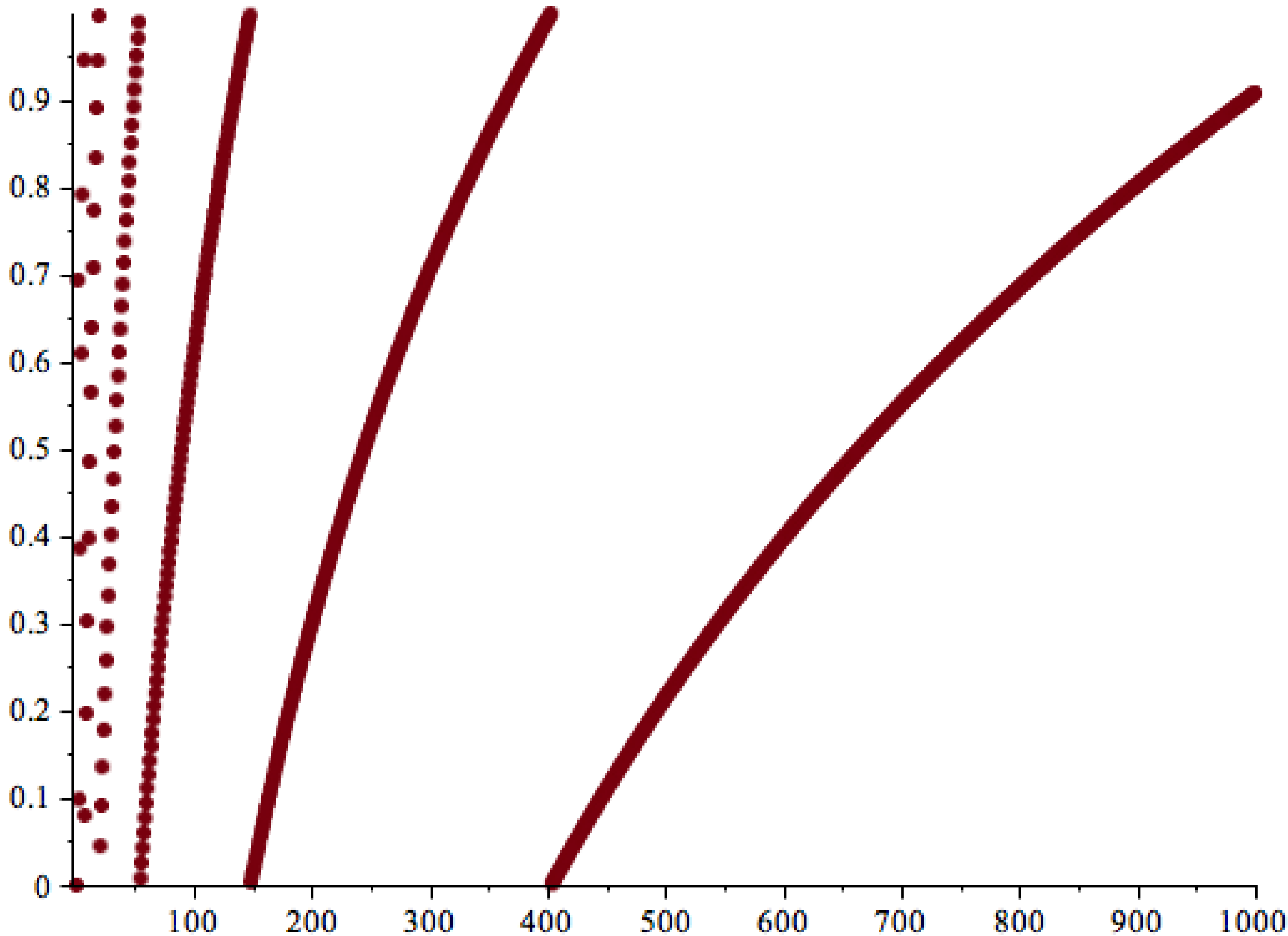

Figure 4.3: $\langle \log(n) \rangle$

Chapter 5

The Fourier Transform on $\mathbb{R}$

5.1 Exercise 8

Proof. Let $f \in \mathcal{M}(\mathbb{R})$ and suppose that

$$\int_{-\infty}^{\infty} e^{-y^2} e^{2xy}\, dy = 0$$

for all $x \in \mathbb{R}$.
We claim that $f = 0$.
Consider $(f * K)(x)$, where $K(x) = e^{-x^2}$. Then,

$$\begin{aligned} (\widehat{f * e^{-x^2}})(x) &= \int_{-\infty}^{\infty} \hat{f}(y) e^{-(x-y)^2}\, dy \\ &= e^{-x^2} \int_{-\infty}^{\infty} \hat{f}(y) e^{2xy} e^{-y^2}\, dy \\ &= e^{-x^2} \hat{f}(x) = 0. \end{aligned}$$

Since $e^{-x^2} \neq 0$ for any $x \in \mathbb{R}$, $\hat{f} = 0$. Furthermore, $f \in \mathcal{M}(\mathbb{R})$, so f is continuous. Therefore, $f(x) = 0$ for all $x \in \mathbb{R}$. ■

5.2 Exercise 15

(Part a)

Proof. By exercise 2 we have that

$$g(x) = \begin{cases} 1 - |x|, & \text{if } |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\hat{g}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2.$$

Since $g \in \mathcal{S}(\mathbb{R})$, $\hat{g} \in \mathcal{S}(\mathbb{R})$. So if $f = \hat{g}$, then by Poisson summation

$$\sum_{n\in\mathbb{Z}} \hat{g}(x+n) = \sum_{n\in\mathbb{Z}} g(n)e^{2\pi inx}.$$

Splitting up the left hand side and the right hand side we have that

$$\text{RHS} = \sum_{n\in\mathbb{Z}} g(n)e^{2\pi inx} = \sum_{n=-1}^{1} g(n)e^{2\pi inx} = g(0)e^0 = 1$$

and

$$\text{LHS} = \sum_{n\in\mathbb{Z}} \hat{g}(x+n) = \sum_{n\in\mathbb{Z}} \frac{\sin^2(\pi x + \pi n)}{\pi^2(x+n)^2} = \frac{1}{\pi^2}\sum_{n\in\mathbb{Z}} \frac{(-1)^{2n}\sin^2(\pi x)}{(x+n)^2}.$$

If $x = \alpha \in \mathbb{R}\backslash\mathbb{Z}$, then

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{\sin^2 \pi\alpha}.$$

■

(Part b)
Claim: If $a \in \mathbb{R}\backslash\mathbb{Z}, then$

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+a)} = \frac{\pi}{\tan \pi a}$$

Proof. Let $a \in (0,1)$, then integrating the LHS expression from part a yields

$$\int_0^a \frac{1}{(n+x)^2} + \frac{1}{(-n+x)^2}\, dx = -\Big(\frac{1}{-n+a} + \frac{1}{n+a}\Big)$$

Moreover, integrating the RHS of part a yields

$$\begin{aligned}\int_0^a \frac{\pi^2}{(\sin \pi x)^2} - \frac{1}{x^2}\,dx &= \frac{\pi}{\tan(\pi x)} + \frac{1}{x}\Big|_0^a \\ &= \frac{-\pi}{\tan(\pi a)} - \frac{1}{a},\end{aligned}$$

since $\frac{-\pi}{\tan(\pi x)} \to 0$ as $x \to 0$. Thus,

$$-\frac{1}{a} + \sum_{n=1}^{\infty}\left(\frac{1}{(n+a)} + \frac{1}{(-n+a)}\right) = \frac{\pi}{\tan(\pi a)}.$$

Then evaluating at $a = 1/2$ we have

$$\begin{aligned}-2 + \sum_{n=1}^{\infty}\left(\frac{1}{(n+2)} + \frac{1}{(-n+2)}\right) &= \frac{\pi}{\tan(\pi 2)} \\ \sum_{n=1}^{\infty}\left(\frac{1}{(n+2)} + \frac{1}{(-n+2)}\right) &= 2 \\ \sum_{n=1}^{\infty}\left(\frac{2}{(2n-1)} - \frac{2}{(2n+1)}\right) &= 2 \\ \left(\frac{2}{1} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{5}\right) + \left(\frac{2}{5} - \frac{2}{7}\right) + \ldots &= 2\end{aligned}$$

So the summation is valid.
So

$$\sum_{n\in\mathbb{Z}} \frac{1}{n+a} = \frac{\pi}{\tan(a\pi)}$$

where $a \in (0, 1)$. ■

5.3 Exercise 17

(Part a)

Proof. Let

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}\,dx = \lim_{\delta\to 0}\int_\delta^1 e^{-x}x^{s-1}\,dx + \lim_{A\to\infty}\int_1^A e^{-x}x^{s-1}\,dx.$$

We will first show that for $\delta > 0$

$$\lim_{\delta \to 0} \int_\delta^1 e^{-x} x^{s-1}\, dx$$

exists.
Since $x \in [0,1]$, we see that $e^{-x} \leq 1$ on $[0,1]$. So $e^{-x}x^{s-1} \leq x^{s-1}$. So

$$\int_\delta^1 e^{-x} x^{s-1}\, dx \leq \int_\delta^1 x^{s-1}\, dx = \frac{1}{s} - \frac{\delta^s}{s} \to \frac{1}{s}, \quad \text{as } \delta \to 0.$$

So the first integral converges by the comparison test.

Next, we use the following lemma to show convergence of the latter integral.

Lemma 5.3.1. *Let $x \in [0, A)$ and $s > 0$. Then*

$$\lim_{x \to \infty} x^{s-1} e^{-x/2} = 0$$

Proof. We begin by rewriting $x^{s-1}e^{-x/2} = (xe^{-(x/2(s-1))})^{(s-1)}$. Then, by L'Hospital

$$\lim_{x\to\infty} \left(\frac{x}{e^{\frac{x}{2(s-1)}}} \right)^{(s-1)} = \left(\lim_{x\to\infty} \frac{1}{\frac{1}{2(s-1)} e^{\frac{x}{2(s-1)}}} \right)^{(s-1)} = 0.$$

■

Moreover, by the definition of convergence

$$\left| x^{s-1} e^{-x/2} \right| = x^{s-1} e^{-x/2} \leq \varepsilon = 1 \iff x^{s-1} e^{-x} \leq e^{-x/2}.$$

Therefore, we have that

$$\int_1^A e^{-x} x^{s-1}\, dx \leq \int_1^A e^{-x/2}\, dx = \lim_{A\to\infty} -2e^{-x/2}\Big|_1^A = \frac{2}{\sqrt{e}}.$$

So by the comparison test the integral converges. Therefore,

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx$$

is well defined for $s > 0$. ■

(Part b)

Proof. Integrating by parts yields

$$\Gamma(s+1) = \underbrace{-e^{-x}x^s\Big|_0^\infty}_{0} + s\int_0^\infty e^{-x}x^{s-1} = s\Gamma(s).$$

In particular, if $n \in \mathbb{N}$, then

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) \\
&= n(n-1)\Gamma(n-1) \\
&= n(n-1)(n-2)\Gamma(n-2) \\
&\vdots \\
&= n(n-1)(n-2)\dots 2\Gamma(1) \\
&= n(n-1)(n-2)\dots 2\int_0^\infty xe^{-x}\,dx \\
&= n(n-1)(n-2)\dots(2)(1) = n!.
\end{aligned}$$

■

(Part c) We will first show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and then $\Gamma(\frac{3}{2})$ will follow from the properties of the gamma function.

Proof. Let $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}\,dx$. By change of variables if $u^2 = x$, then $2du = dx$. Thus,

$$\Gamma\Big(\frac{1}{2}\Big) = 2\int_0^\infty e^{-u^2}x^{2(\frac{1}{2})-1}\,du = 2\int_0^\infty e^{-u^2}\,du,$$

since at $s = 1/2$, $u^{2s-1} = u^{1-1} = 1$
So

$$\Big[\Gamma\Big(\frac{1}{2}\Big)\Big]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

Then changing to polar coordinates, we have that for $\theta \in [0, \pi/2)$ and $r \in [0, \infty)$,

$$\Big[\Gamma\Big(\frac{1}{2}\Big)\Big]^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,dr\,d\theta$$

Changing variables once again, if $w = -r^2$, then $\frac{-1}{2}\,dw = r\,dr$, then

$$\begin{aligned}\left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta \\ &= 4\int_0^{\pi/2}\int_0^{\infty} -\frac{1}{2}e^u\,du\,d\theta \\ &= 4\int_0^{\pi/2} -\frac{1}{2}e^u\Big|_0^{\infty}\,du\,d\theta \\ &= 2\int_0^{\pi/2} d\theta \\ &= \pi.\end{aligned}$$

Therefore,

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi \iff \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Since $\Gamma(s+1) = s\Gamma(s)$ for all $s > 0$, we have

$$\Gamma\left(\frac{3}{2}\right) = \left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

■

5.4 Exercise 23

(Part a)

Proof. Let

$$L(f) = -\frac{d^2 f}{dx^2} + x^2 f \quad \text{and} \quad \langle f|g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx.$$

Suppose that $\langle f|f\rangle = 1$. Integrating by parts yields,

$$\begin{aligned}
\langle Lf|f\rangle &= -\langle f''|f\rangle + \langle x^2 f|f\rangle \\
&= \langle f'|f'\rangle + \langle x^2 f|f\rangle \\
&= \int_{\mathbb{R}} |f'|^2 + \int_{\mathbb{R}} x^2|f|^2 \\
&= \int_{\mathbb{R}} 4\pi^2\xi^2|\hat{f}|^2 + \int_{\mathbb{R}} x^2|f|^2 \\
&= \int_{\mathbb{R}} \big(2\pi||\xi\hat{f}|| - ||xf||\big)^2 \\
&= 4\pi^2\Big(\int_{\mathbb{R}} \xi^2|\hat{f}|^2\Big)^{1/2}\Big(\int_{\mathbb{R}} x^2|f|^2\Big)^{1/2} \\
&\geq 4\pi\frac{1}{4\pi} = 1
\end{aligned}$$

■

(Part b) Let

$$A(f) = \frac{df}{dx} + xf \quad \text{and} \quad A^*(f) = -\frac{df}{dx} + xf.$$

Let $f, g \in \mathcal{S}(\mathbb{R})$. Then the following hold

1. $\langle Af|g\rangle = \langle f|A^*g\rangle$
2. $\langle Af|Af\rangle = \langle A^*A|f\rangle \geq 0$
3. $A^*A = L - I$

Proof. Beginning with 1 and integrating by parts we have that

$$\begin{aligned}
\langle Af|g\rangle &= \int_{\mathbb{R}} \Big(\frac{d}{dx} + x\Big) f\bar{g}\,dx \\
&= f(x)\overline{g(x)}\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} f\frac{d}{dx}\bar{g}\,dx + \int_{\mathbb{R}} xf\bar{g}\,dx \\
&= \int_{\mathbb{R}} f\Big(-\frac{d}{dx} + x\Big)\bar{g}\,dx \\
&= \langle f|A^*g\rangle.
\end{aligned}$$

Note that $f(x)\overline{g(x)} \to 0$ as $x \to \infty$ and $x \to -\infty$, since $f, g \in \mathcal{S}(\mathbb{R})$.
Then for part 2, we have.

$$\begin{aligned}
\langle Af|Af\rangle &= \int_{\mathbb{R}} \Big(-\frac{d}{dx}f + xf\Big)\overline{\Big(\frac{d}{dx}f + xf\Big)}\,dx \\
&= \int_{\mathbb{R}} (f' + xf)\overline{(f' + xf)}\,dx \\
&= \int_{\mathbb{R}} (f')^2 - ff'x + x^2f^2\,dx \\
&= \int_{\mathbb{R}} \Big(-\frac{d}{dx} + x\Big)\Big(\frac{d}{dx} - x\Big)f\,dx
\end{aligned}$$

Furthermore, by the exercise below we see that $\langle A^*A|f\rangle \geq 0$, since both $\int_{\mathbb{R}}(f')^2\,dx \geq 0$ and $\int_{\mathbb{R}} f^2\,dx \geq 0$.
For part 3, since

$$\frac{d}{dx}(xf) = f + x\frac{df}{dx} \Rightarrow \frac{d}{dx}x = 1 + x\frac{d}{dx}$$

we have

$$\begin{aligned}
A^*A &= \Big(-\frac{d}{dx} + x\Big)\Big(\frac{d}{dx} + x\Big) \\
&= -\frac{d^2}{dx^2} - \frac{d}{dx}x + x\frac{d}{dx} + x^2 \\
&= -\frac{d^2}{dx^2} - 1 - x\frac{d}{dx} + x\frac{d}{dx} + x^2 \\
&= L - 1
\end{aligned}$$

■

(Part c) Let

$$A_t(f) = \frac{df}{dx} + txf \quad \text{and} \quad A_t^*(f) = -\frac{df}{dx} + txf.$$

Proof. Suppose that $\int_{\mathbb{R}} |f|^2\,dx = 1$.
Applying part b and thinking of $\langle A_t^*A_tf|f\rangle$ as a polynomial, we have that

$$t^2\langle x^2f|f\rangle - t\langle f|f\rangle - \langle f''|f\rangle = t^2\langle x^2f|f\rangle - t + \langle f'|f'\rangle \geq 0, \forall t \in \mathbb{R}.$$

Then

$$1 - 4\langle x^2f|f\rangle\langle f'|f'\rangle \leq 0 \iff \langle x^2f|f\rangle\langle f'|f'\rangle \geq \frac{1}{4}.$$

■

Note: I asked for clarificiation about this problem on math.stackexchange and the post can be found at `http://math.stackexchange.com/q/2007409/225477`.

5.5 Problem 7

Define the Hermite function, $h_k(x)$ by it's generating identity

$$\sum_{k=0}^{\infty} h_k(x)\frac{t^k}{k!} = e^{-(x^2/2-2tx+t^2)}.$$

(Part a)

Proof. Let $h_k(x) = e^{x^2/2}e^{-(x-t)^2}$, then Taylor expanding we have that

$$\begin{aligned} h_k(x) &= \left(\frac{d}{dt}\right)^k e^{x^2/2}e^{-(x-t)^2}\bigg|_{t=0} \\ &= e^{x^2/2}\left(\frac{d}{dt}\right)^k e^{-(x-t)^2}\bigg|_{t=0}. \end{aligned}$$

Then if $u = x - t$, then $t = x - u$ and $dt = -du$, so $\frac{d}{dt} = -\frac{d}{du}$. Thus,

$$h_k(x) = e^{x^2/2}\left(-\frac{d}{du}\right)^k e^{-u^2}\bigg|_{u=x} = e^{x^2/2}\left(-\frac{d}{dx}\right)^k e^{-x^2} = e^{x^2/2}(-1)^k\left(\frac{d}{dx}\right)^k e^{-x^2}.$$

Moreover, we see that

$$\begin{aligned} \left(\frac{d}{dx}\right) e^{-x^2} &= -2xe^{-x^2} \\ \left(\frac{d}{dx}\right)^2 e^{-x^2} &= 4x^2e^{-x^2} - 2e^{-x^2} \\ \left(\frac{d}{dx}\right)^3 e^{-x^2} &= -8^3e^{-x^2} + 12xe^{-x^2} \\ \left(\frac{d}{dx}\right)^4 e^{-x^2} &= -32x^5e^{-x^2} + 160x^3e^{-x^2} - 120xe^{-x^2} \\ &\vdots \end{aligned}$$

Hence,

$$\left(\frac{d}{dx}\right)^k e^{-x^2} = P_k(x)e^{-x^2}$$

where $P_k(x)$ is a k-th degree polynomial. In particular, $P_k(x)$ is odd when k is odd, and even when k is even. So,

$$h_k(x) = P_k(x)e^{x^2/2}e^{-x^2} = P_k(x)e^{-x^2/2}.$$

■

(Part b)

Proof. Let $f \in \mathcal{S}(\mathbb{R})$ and suppose that $\langle f|h_k\rangle = 0$, for all $k \in \mathbb{N}$. Then

$$\begin{aligned}
\langle f|h_k\rangle &= \int_{\mathbb{R}} f(x)h_k(x)\,dx \\
&= \int_{\mathbb{R}} f(x)e^{-(x^2/2-2tx+t^2)}\,dx \\
&= \langle f|\sum_{k=0}^{\infty} h_k(x)\frac{t^k}{k!}\rangle = 0.
\end{aligned}$$

Then by exercise 8, $f = 0$. ■

Part c
Let $h_k^*(x) = h_k(\sqrt{2\pi}x)$.
Claim: $\widehat{h_k^*}(\xi) = (-i)^k h_k^*(\xi)$.

Proof. Evaluating the Hermite function at $\sqrt{2\pi}x$ yields

$$\sum_{k=0}^{\infty} h_k(\sqrt{2\pi}x)\frac{t^k}{k!} = \exp(-\pi x^2 + 2tx\sqrt{2\pi} + t^2).$$

Then taking the Fourier transform of both sides, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \widehat{h_k^*}(\xi)\frac{t^k}{k!} &= \exp(-\pi\xi^2 - 2it\xi\sqrt{2\pi} + t^2) \\
&= \exp(-(\pi\xi^2 - 2(-it)\xi\sqrt{2\pi} - (it)^2)) \\
&= \sum_{k=0}^{\infty} h_k^*(\xi)\frac{(-it)^k}{k!},
\end{aligned}$$

which implies that $\widehat{h_k^*}(\xi) = (-i)^k h_k^*(\xi)$.

■

(Part d)

Proof. Let $L = -\frac{d^2}{dx^2} + x^2$. Then

$$\begin{aligned}
Lh_k(x) &= \Big(-\frac{d^2}{dx^2} + x^2\Big)h_k(x) && (5.1)\\
&= (1 - (-x+2t)^2 + x^2)e^{-x^2/2+2xt-t^2} && (5.2)\\
&= (1 + 4xt - 4t^2)e^{-x^2/2+2xt-t^2} && (5.3)\\
&= e^{-x^2/2+2xt-t^2} + (4xt - 4t^2)e^{-x^2/2+2xt-t^2} && (5.4)\\
&= \sum_{k=0}^{\infty} h_k(x)\frac{t^k}{k!} + (4xt - 4t^2)e^{-x^2/2+2xt-t^2} && (5.5)\\
&= \sum_{k=0}^{\infty} h_k(x)\frac{t^k}{k!} + 2t\frac{d}{dt}e^{-x^2/2+2xt-t^2} && (5.6)\\
&= \sum_{k=0}^{\infty} h_k(x)\frac{t^k}{k!} + \sum_{k=0}^{\infty} 2kh_k(x)\frac{t^k}{k!} && (5.7)\\
&= \sum_{k=0}^{\infty}(2k+1)h_k(x)\frac{t^k}{k!}. && (5.8)
\end{aligned}$$

Where on line 5.6 we used the fact that $\frac{d}{dt}e^{-x^2/2+2xt-t^2} = (2x-2t)e^{-x^2/2+2xt-t^2}$.
Hence $Lh_k(x) = (2k+1)h_k(x)$.
Relating h_k back to the results of exercise 23, $L = 1 + A^*A$. So, $h_k(x)$ are mutually orthogonal for the L^2 inner product.
More precisely, $(2k+1)\langle h_k|h_\ell\rangle = (2\ell+1)\langle h_\ell|h_k\rangle$ iff $\ell = k$. ■

(Part e)
Claim:

$$\int_{\mathbb{R}} [h_k(x)]^2\,dx = \sqrt{\pi}2^k k!.$$

Proof. We begin by squaring the generating function,

$$\Big[\sum_{k=0}^{\infty} h_k(x)\frac{t^k}{k!}\Big]^2 = \Big[e^{-(x^2/2-2tx+t^2)}\Big]^2.$$

Hence,

$$\Big[\sum_{k=0}^{\infty} h_k(x)\frac{t^k}{k!}\Big]^2 = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} h_k(x)h_j(x)\frac{t^k}{k!}\frac{t^j}{j!}$$

and by the previous exercise, h_k and h_j are orthogonal if $h = j$. So

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} h_k^2(x)h_j(x)\frac{t^k}{k!}\frac{t^j}{j!} = \sum_{k=0}^{\infty} h_k(x)\frac{t^{2k}}{(2k)!}.$$

Then integrating both sides we have

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_{\mathbb{R}} h_k^2(x)\,dx = \int_{\mathbb{R}} e^{-(x^2/2-2tx+t^2)}\,dx.$$

Moreover, integrating the left hand side by parts and using the alternate formulation of the Hermite polynomial from part a,

$$\begin{aligned}\int_{\mathbb{R}} e^{x^2/2}e^{-(x-t)^2}\,dx &= \int_{\mathbb{R}} e^{-x^2/2}e^{2tx-t^2}\,dx\\ &= \int_{\mathbb{R}} (-1)^{2k}e^{x^2/2}\left(\frac{d}{dx}\right)^k e^{2xt-t^2}\,dx\\ &= \int_{\mathbb{R}} (-1)^{2k}2^k t^k e^{-(x-t)^2}\,dx.\end{aligned}$$

Putting the LHS together with the RHS yields

$$2^k t^k \int_{\mathbb{R}} e^{-(x-t)^2}\,dx = \iff \int_{\mathbb{R}} [h_k(x)]^2\,dx = 2^k\sqrt{\pi}k!.$$

■

Chapter 6

Interlude: Prof. Tracy's Random Walker problem

Consider a random walk on the integer lattice $\mathbb{Z}$ in continuous time. When a particle is at an *even* site the probability of hopping to the right is p_e and probability $q_e = 1 - p$ to the left. On an *odd* site the right jumping probability is p_o and the left jumping probability is $q_o = 1 - p_o$. Let $\mathbb{P}(x,t)$ denote the probability that the position of the walker at time t is at lattice site x. The differential equations that describe this process are

$$\begin{aligned}\frac{d\mathbb{P}(2x,t)}{dt} &= p_o P(2x-1,t) + q_o P(2x+1,t) - P(2x,t)\\ \frac{d\mathbb{P}(2x+1,t)}{dt} &= p_e P(2x,t) + q_e P(2x+2,t) - P(2x+1,t).\end{aligned}$$

Use Fourier analysis to reduce the problem of finding $P(x,t)$ to definite integrals.

Proof. First note that 1-step translational variance is broken by the two different site probabilities, that is to say, in order to recover an even site probability we must move either two steps to the left or two steps to the right, and similarly for an odd site. In order to recover this we consider the following vector valued function

$$\mathcal{P}(x) = \begin{pmatrix} P(2x) \\ P(2x+1) \end{pmatrix}$$

with the initial condition that $\mathcal{P}(0) = 0$. Then differentiating $\mathcal{P}$ yields

$$\begin{aligned}
\frac{d\mathcal{P}(x)}{dt} &= \frac{d}{dt}\begin{pmatrix} P(2x) \\ P(2x+1) \end{pmatrix} \\
&= \begin{pmatrix} p_o P(2x-1,t) + q_o P(2x+1,t) - P(2x,t) \\ p_e P(2x,t) + q_e P(2x+2,t) - P(2x+1,t) \end{pmatrix} \\
&= \begin{pmatrix} 0 & p_0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P(2x-2) \\ P(2x-1) \end{pmatrix} + \begin{pmatrix} -1 & q_o \\ p_e & -1 \end{pmatrix}\begin{pmatrix} P(2x) \\ P(2x+1) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ q_e & 0 \end{pmatrix}\begin{pmatrix} P(2x+2) \\ P(2x+3) \end{pmatrix} \\
\frac{d\mathcal{P}(x)}{dt} &= A_{-1}P(x-1) + A_0 P(x) + A_1 P(x+1).
\end{aligned}$$

Then multiplying and by $e^{2\pi i x\theta}$ and summing over all x we have the following Fourier series

$$\begin{aligned}
\frac{d\widehat{\mathcal{P}}}{dt} &= \sum_{x\in\mathbb{Z}} e^{2\pi i x\theta}\Big[A_{-1}P(x-1) + A_0 P(x) + A_1 P(x+1)\Big] \\
&= e^{2\pi i\theta}A_{-1}\widehat{P}(\theta) + A_0\widehat{P}(\theta) + e^{-2\pi i\theta}A_1\widehat{P}(\theta) \\
&= M(\theta)\widehat{P}(\theta)
\end{aligned}$$

where $M(\theta) = A_{-1}e^{2\pi i\theta} + A_0 + A_1 e^{-2\pi i\theta}$. Finally we see that $P(\theta, t) = e^{tM(\theta)}P(\theta, 0)$ which can be solved by techniques from *Math 22b*. ■

Chapter 7

The Fourier Transform on $\mathbb{R}^d$

7.1 Exercise 6

Proof. Let A be a $n \times n$ positive definite symmetric matrix with real coefficients.
Claim: $\int_{\mathbb{R}^n} \exp(-\pi\langle x|Ax\rangle)\, d^n x = \det(A)^{-1/2}$.
Since A is positive definite, we have that

$$\int_{\mathbb{R}^n} \exp(-\pi\langle x|Ax\rangle)\, d^n x = \int_{\mathbb{R}^n} \exp(-\pi x^T Ax)\, d^n x.$$

By the spectral theorem, we may diagonalize A. Hence, $A = RDR^T$, where D is diagonalized and consists of the eigenvalues, λ_i, of A.
So,

$$\int_{\mathbb{R}^n} \exp(-\pi x^T Ax)\, d^n x$$

If we let $z = R^T x$, then

$$\int_{\mathbb{R}^n} \exp(-\pi z^T Dz)\, d^n z = \prod_i^n \int_{\mathbb{R}} \exp(-\pi\lambda_i z_i^2)\, dz_i \tag{7.1}$$

$$= \prod_i^n \sqrt{\frac{1^n \pi}{\lambda_i \pi}} \tag{7.2}$$

$$= \prod_i^n \sqrt{\frac{1}{\lambda_i}} \tag{7.3}$$

$$= \sqrt{\frac{1}{\det(A)}}. \tag{7.4}$$

Where in line 7.2 we used an identity for computing the Gaussian integrals that was demonstrated in class. Therefore, the claim holds ■

Chapter 8

Dirichlet's Theorem

8.1 Exercise 10

Claim: If $\gcd(\ell, q) = 1$, then

$$\sum_{p\equiv\ell(q)} \frac{1}{p^s} = \frac{1}{\varphi(q)} \log(\frac{1}{s-1}) + O(1) \quad \text{as } s \to 1^+.$$

Proof. Suppose that $\gcd(\ell, q) = 1$ and the results of exercise 8 hold; $\sum_p \frac{1}{p^s} = \log(\frac{1}{s-1}) + O(1)$. Then by (4) on page 255 we have that

$$\sum_{p\equiv\ell(q)} \frac{1}{p^s} = \frac{1}{\varphi(q)} \sum_p \frac{\chi_0(p)}{p^s} + \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} \overline{\chi(\ell)} \sum_p \frac{\chi(p)}{p^s}.$$

Since $\gcd(\ell, q) = 1$ and $p \equiv \ell(q)$, by definition of the Dirichlet characters $\chi_0(p) = 1$. So

$$\sum_{p\equiv\ell(q)} \frac{1}{p^s} = \frac{1}{\varphi(q)} \sum_p \frac{1}{p^s} + \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} \overline{\chi(\ell)} \sum_p \frac{\chi(p)}{p^s}$$

and by exercise 8 it follows that

$$\sum_{p\equiv\ell(q)} \frac{1}{p^s} = \frac{1}{\varphi(q)} \big[\log(\frac{1}{s-1}) + O(1)\big] + \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} \overline{\chi(\ell)} \sum_p \frac{\chi(p)}{p^s}.$$

It remains to show that the second term on the right hand side of the sum is $O(1)$.

Beginning with $\sum_{\chi\neq\chi_0}\overline{\chi(\ell)}$, we see that by construction of Dirichlet characters mod q, there are at most $\varphi(q)-1$ terms in this sum, and $\varphi(q)$ terms if we were to include the trivial character. Moreover, if $\chi\neq\chi_0$, then $\sum_p \frac{\chi(p)}{p^s}$ is also finite. Let Q denote the largest value of this sum s.t.

$$\sum_p \frac{\chi(p)}{p^s} \leq |Q|.$$

Putting these results together we have that

$$\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0}\overline{\chi(\ell)}\sum_p \frac{\chi(p)}{p^s} \leq \frac{1}{\varphi(q)}[\varphi(q)-1]|Q| = O(1)$$

and the claim follows. ∎

Note: I asked for clarificiation about this problem on math.stackexchange and the post can be found at `http://math.stackexchange.com/q/2029096/225477`.

8.2 Exercise 14

Claim: If $x\neq 0 \mod (2\pi)$, then

$$E(x)=\sum_{n=1}^{\infty}\frac{e^{inx}}{n}$$

converges. Moreover,

$$E(x)=\frac{1}{2}\log\left(\frac{1}{2-2\cos x}\right)+\frac{i}{2}F(x)$$

where

$$F(x)=\begin{cases} i(-\pi-x) & \text{if } x\in(-\pi,0)\\ i(\pi-x) & \text{if } x\in(0,\pi).\end{cases}$$

Proof. Beginning with convergence of $E(x)$, we consider the Abel means

$$\begin{aligned}A(r)&=\sum_{n=1}^{\infty}\frac{r^n e^{inx}}{n}, \quad r<1\\ &=\log\frac{1}{1-re^{ix}}.\end{aligned}$$

Since $x \neq 0$, as $r \to 1^-$ the series is Abel summable. So we may replace r with 1. Note that $\frac{e^{inx}}{n} = O(1/n)$. Then by Litttlewood's Tauberian theorem, the series converges.

We next show the identity.
First recall that any complex number can be written in the form $z = re^{ix}$ where $0 \leq r < \infty$ is unique and $x \in \mathbb{R}$.
Furthermore, $\log z = \log |r| + ix$.

If $z = \frac{1}{1-e^{ix}}$, then $\log(z) = |\frac{1}{1-e^{ix}}| + ix$. Since $1 - e^{ix} = 1 - \cos x - i \sin x$, we have $|1 - e^{ix}| = \sqrt{2(1 - \cos x)}$. Moreover, as demonstrated in problem session,

$$\begin{aligned} \frac{z}{\bar{z}} &= e^{2\pi\theta} \\ &= \frac{1 - e^{-i\theta}}{1 - e^{i\theta}} \\ &= e^{-i\theta} e^{\pm \pi i}. \end{aligned}$$

This implies that we must choose both angles that approach a branch in the complex plane. Therefore we are left with

$$E(x) = \frac{1}{2} \log \left(\frac{1}{2 - 2\cos x} \right) + \frac{i}{2} F(x).$$

■

8.3 Exercise 15

(Part a)
Let $a_n = a_m$ if $n = m \mod (q)$, $\sum_{n=1}^{q} a_n = 0$ and define

$$A(m) = \sum_{n=1}^{q} a_n \zeta^{-mn}, \quad E(\theta) = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}$$

where $\zeta = e^{2\pi i/q}$.

Claim:

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = \frac{1}{q} \sum_{m=1}^{q-1} A(m) E(2\pi m/q).$$

Proof. First note that a_n are the Fourier coefficients of $A(m)$. Applying Fourier inversion on $\mathbb{Z}(q)$ yields

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a_n}{n} &= \frac{1}{q}\sum_{n=1}^{\infty} \frac{1}{n}\frac{1}{q}\sum_{m=1}^{q} A(m)e^{2\pi imn/q} \\
&= \frac{1}{q}\sum_{m=1}^{q} A(m)\sum_{n=1}^{\infty} \frac{e^{2\pi imn/q}}{n} \\
&= \frac{1}{q}\sum_{m=1}^{q-1} A(m)E(2\pi m/q).
\end{aligned}$$

Note that $A(q) = 0$ since $\sum_{n=1}^{q} a_n = 0$ by our assumption and $e^{2\pi iq/q} = 1$. Therefore, the identity holds. ∎

(Part b)
Claim: If $\{a_m\}$ is odd, $m \in \mathbb{Z}$, $a_0 = a_q = 0$ then

$$A(m) = \sum_{1 \le n < \frac{q}{2}} a_n(\zeta^{-mn} - \zeta^{mn})$$

Proof. We begin by breaking up the sum into two pieces and make a change of variables. So

$$A(m) = \sum_{n=1}^{q-1} a_n\zeta^{-mn} = \sum_{1 \le n < q/2} a_n\zeta^{-mn} + \sum_{q/2 \le n < q-1} a_n\zeta^{-mn}.$$

Letting $n' = q - n$, we have that $n = q - n'$. By our assumptions $a_n = a_{q-n'} = a_{-n'} = -a_{n'}$. So substituting n' into the second term in the equation above we have

$$\begin{aligned}
A(m) &= \sum_{1 \le n < q/2} a_n\zeta^{-mn} + \sum_{1 \le n' < q/2} a_{q-n'}\zeta^{-m(q-n')} \\
&= \sum_{1 \le n < q/2} a_n\zeta^{-mn} - \sum_{1 \le n < q/2} a_n\zeta^{mn} \\
&= \sum_{1 \le n < q/2} a_n(\zeta^{-mn} - \zeta^{mn}).
\end{aligned}$$

∎

(Part c)
Claim: If $\{a_m\}$ is odd, then

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = \frac{1}{2q} \sum_{m=1}^{q-1} A(m)F(2\pi m/q).$$

Note that by part b we have that $A(-m) = -A(m)$. Furthermore, from part a we have that

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = \frac{1}{q} \sum_{m=1}^{q-1} A(m)E(2\pi m/q).$$

Proof. Let $m' = -m$. Then

$$\begin{aligned}
\sum_{m=1}^{q-1} A(m)E(2\pi m/q) &= \sum_{m'=-1}^{-(q-1)} A(-m')E(-2\pi m'/q) \\
&= \sum_{m'=-1}^{-(q-1)} -A(m')E(-2\pi m'/q) \\
&= -\sum_{m=1}^{q-1} A(m)E(-2\pi m/q).
\end{aligned}$$

Then adding $\sum_{m=1}^{q-1} A(m)E(2\pi m/q)$ and $-\sum_{m=1}^{q-1} A(m)E(-2\pi m/q)$ and dividing by 2 we have

$$\frac{1}{q} \sum_{m=1}^{q-1} A(m)E(2\pi m/q) + \sum_{m=1}^{q-1} A(m)E(-2\pi m/q)$$

$$= \frac{1}{2q} \sum_{m=1}^{q-1} A(m)(E(2\pi m/q) - E(-2\pi m/q)).$$

Next we claim that

$$E(\theta) - E(-\theta) = \frac{1}{2}F(\theta) - \frac{1}{2}F(-\theta) = F(\theta).$$

Once we demonstrate this, the claim follows. Recall that

$$F(\theta) = \sum_{|n| \neq 0} \frac{e^{in\theta}}{n}.$$

So

$$
\begin{aligned}
F(\theta) = \sum_{|n|\neq 0} \frac{e^{in\theta}}{n} &= \sum_{n=1}^{N} \frac{e^{in\theta}}{n} + \sum_{n=-N}^{-1} \frac{e^{in\theta}}{n} \\
&= \sum_{n=1}^{N} \frac{e^{in\theta}}{n} + \sum_{n=1}^{N} \frac{e^{-in\theta}}{-n} \\
&= \sum_{n=1}^{N} \frac{e^{in\theta}}{n} - \sum_{n=1}^{N} \frac{e^{-in\theta}}{n}.
\end{aligned}
$$

Then

$$
\begin{aligned}
F(\theta) - F(-\theta) &= \Big(\sum_{n=1}^{N} \frac{e^{in\theta}}{n} - \sum_{n=1}^{N} \frac{e^{-in\theta}}{n}\Big) - \Big(\sum_{n=1}^{N} \frac{e^{-in\theta}}{n} - \sum_{n=1}^{N} \frac{e^{in\theta}}{n}\Big) \\
&= 2\sum_{n=1}^{N} \frac{e^{in\theta}}{n} - 2\sum_{n=1}^{N} \frac{e^{-in\theta}}{n} \\
&= 2E(\theta) - 2E(-\theta), \quad \text{as } N \to \infty
\end{aligned}
$$

Hence,

$$
F(\theta) = \frac{1}{2}F(\theta) - \frac{1}{2}F(-\theta) = E(\theta) - E(-\theta)
$$

Therefore,

$$
\frac{1}{2q}\sum_{m=1}^{q-1} A(m)(E(2\pi m/q) - E(-2\pi m/q))
$$

■

8.4 Exercise 16

For the next two problem, in order to compute $F(\theta)$, we will use it's Fourier series which is defined to be

$$
F(\theta) = \begin{cases} i(-\pi - \theta) & \text{if } x \in (-\pi, 0) \\ i(\pi - \theta) & \text{if } x \in (0, \pi). \end{cases}
$$

Claim:

$$
\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} \dots
$$

which is $L(1, \chi)$ for the non-trivial odd Dirichlet character modulo 3.

Proof. The character table is given by

$$\begin{array}{c||c|c} & 1 & 2 \\ \hline \chi_0 & 1 & 1 \\ \chi_1 & 1 & -1. \end{array}$$

Then by our equation from 15 part c we have that if $q = 3$, then

$$\frac{1}{6}\sum_{m=1}^{2} A(m)F(2\pi m/3) = \frac{1}{6}\Big[A(1)F(2\pi/3) + A(2)F(-2\pi/3)\Big].$$

Calculating the $F's$ first we see $F(2\pi/3) = \frac{\pi i}{3}$ and $F(4\pi/3) = F(-2\pi/3) = -\frac{\pi i}{3}$.
Recall that a_n corresponds to the Dirichlet characters, then computing A yields

$$\begin{aligned} A(1) &= \sum_{n=1}^{2} a_n \zeta^{-n} \\ &= a_1\zeta^{-1} + a_2\zeta^{-2} \\ &= a_1 e^{-2\pi/3} - a_2 e^{2\pi/3} \\ &= e^{-2\pi/3} - e^{2\pi/3} \\ &= -2i\sin(2\pi/3). \end{aligned}$$

Similarly for $A(2)$, we have

$$\begin{aligned} A(2) &= \sum_{n=1}^{2} a_n \zeta^{-2n} \\ &= a_1\zeta^{-2} + a_2\zeta^{-4} \\ &= a_1 e^{2\pi/3} + a_2 e^{-2\pi/3} \\ &= e^{2\pi/3} - e^{-2\pi/3} \\ &= 2i\sin(2\pi/3). \end{aligned}$$

Then computing the sum in Maple yields

$$\frac{1}{6}\Big[A(1)F(2\pi/3) + A(2)F(-2\pi/3)\Big] = \frac{\pi}{3\sqrt{3}}.$$

∎

8.5 Problem 1

Claim: For the non-trivial Dirichlet character modulo 6,

$$L(1,\chi) = \frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \dots$$

Proof. The character table for the Dirichlet character modulo 6 is given by

	1	5
χ_0	1	1
χ_1	1	-1.

Then

$$\frac{1}{12}\sum_{m=1}^{5} A(m)F(2\pi m/6) = \frac{1}{12}\Big[A(1)F(2\pi/6) + A(2)F(4\pi/6) + A(3)F(\pi) + A(4)F(-4\pi/6) + A(5)F(-2\pi/6)\Big].$$

Beginning with the $F's$ again, we have

$$\begin{aligned} F(2\pi/6) &= \frac{2\pi i}{3} \\ F(4\pi/6) &= \frac{\pi i}{3} \\ F(\pi) &= 0 \\ F(-4\pi/6) &= \frac{-\pi i}{3} \\ F(-2\pi/6) &= \frac{-2\pi i}{3} \end{aligned}$$

Then the $A's$ are given by

$$\begin{aligned} A(1) &= \sum_{n=1}^{5} a_n \zeta^{-n} \\ &= a_1\zeta^{-1} + a_5\zeta^{-5} \\ &= e^{-\pi i/3} - e^{-5\pi i/3} \\ &= e^{-\pi i/3} - e^{\pi i/3} \end{aligned}$$

where a_2, a_3, a_4 are 0 since $2, 3$ and 4 are not relatively prime to 6.
Since the other computations are similar, I have chosen to omit them. The rest of the $A's$ are given by

$$\begin{aligned}
A(1) &= e^{-\pi i/3} - e^{\pi i/3} \\
A(2) &= e^{-2\pi i/3} - e^{2\pi i/3} \\
A(3) &= 0 \\
A(4) &= -e^{-2\pi i/3} + e^{2\pi i/3} \\
A(5) &= -e^{-\pi i/3} + e^{\pi i/3}
\end{aligned}$$

Then computing the sum in maple we have that

$$\begin{aligned}
\frac{1}{12}\sum_{m=1}^{5} A(m)F(2\pi m/6) &= \frac{1}{12}\Big[A(1)F(2\pi/6) + A(2)F(4\pi/6) + A(3)F(\pi) \\
&\quad + A(4)F(-4\pi/6) + A(5)F(-2\pi/6)\Big] \\
&= \frac{\pi}{2\sqrt{3}}.
\end{aligned}$$

∎

Chapter 9

References

Stephen Abbott, *Understanding Analysis*. Second Edition. Springer, 2015.

Tom M. Apostol, *Mathematical Analysis* . Second Edition. Pearson, 1974.

Elias Stein and Rami Shakarchi, *Complex Analysis*. First Edition. Princeton, 2003