Analysis of 2nd Order System:

In polynomial form the T.F. of a Closed loop System may be written as,

$$\frac{C(s)}{R(s)} = \frac{s^m + a_1 s^{m-1} + a_2 s^{m-2} + \dots + a_m}{s^n + b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_n}$$

→ The roots of the numerator polynomial of a closed loop transfer function is called as Zeros.

→ Roots of Denominator polynomial is called as closed loop Zeros.

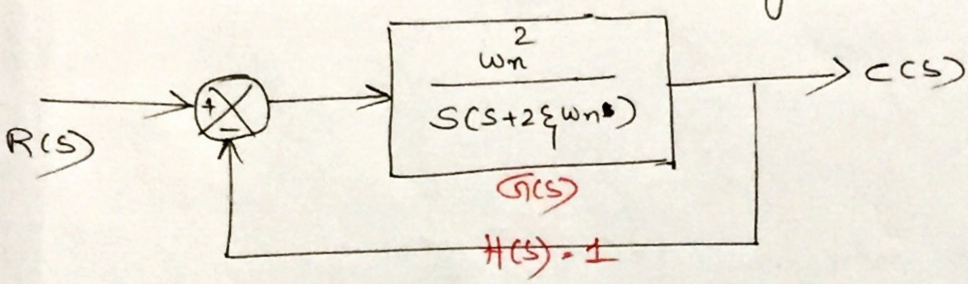
→ If we place these poles & Zeros in a complex s-plane, we get "Pole-Zero" plot.

→ In reality may physical systems are higher order Systems. But for the sake of simplicity only a 2nd order System is been discussed.

Higher systems (i.e order of the system greater than 2) are analysed using Computing tools like Mat-Lab.

→ If a person can understand the Underlying concepts of a 2nd order system, he can under higher Systems easily.

→ Canonical form of a 2nd order System:



Consider the block diagram of a standrad 2nd order system connected in Negative feedback manner.

The Closed loop tranfer function C.L.T.F

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{\frac{\omega_n^2}{S(S+2\xi\omega_n)}}{1+\frac{\omega_n^2}{S(S+2\xi\omega_n)}}$$

$$\boxed{\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}} = \frac{G(S)}{1+G(S)H(S)}$$

→ where ω_n = natural frequency of oscillation (rad/sec)

ξ = damping ratio (Unit less)

→ When the Denominator polynomial of $\frac{C(S)}{R(S)}$ is equated to zero, we get the Characteristic equation of the System.

→ Let the Characteristic equation be

$$\boxed{f(S) = S^2+2\xi\omega_n S+\omega_n^2 = 1+G(S)H(S)}$$

→ Roots of the Characteristic equation is called as closed loop poles of the system.

→ From Stability point of View, Characteristic equation is Very important.

→ Roots of Characteristic equation decides about Stability.

Stability may be defined as the ability of the system to return to normal working State, Once being distured by either changes in the input or in the Initial conditions.

Study of effect of ξ on 2nd order system performance:

Consider a unit step input $R(S) = 1/S$

$$\therefore \quad C(S) = R(S) \cdot \frac{\omega_n^2}{S^2 + 2\xi\omega_n S + \omega_n^2}$$

$$C(S) = \frac{\omega_n^2}{S(S^2 + 2\xi\omega_n S + \omega_n^2)}$$

→ find the roots of $S^2 + 2\xi\omega_n S + \omega_n^2 = 0$;

gives $S_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

Nature of these roots depend on damping ratio.

Consider :

$\xi = 0$: Undamped case

$S_{1,2} = \pm j\omega_n$; roots are purely imaginary

We have $C(t)$ for a typical 2nd order System (Underdamped) as

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta) \quad \text{——} \; (1)$$

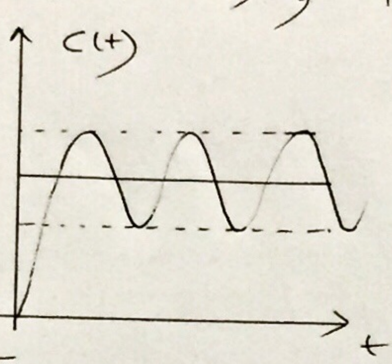
Subsituting $\xi = 0$ in eq (1) gives

$$C(t) = 1 - \sin(\omega_n t + \theta)$$

& $\omega_d = \omega_n \sqrt{1-\xi^2}$ as $\xi = 0$

$\omega_d = \omega_n$; System oscillates continuously with its natural frequency of oscillation.

$\therefore$ $C(t) = 1 - \sin(\omega_n t + \theta)$; Response is purely osillatory.



← System never settles to final value.

" Undamped System "

Case b: $1 > \xi > 0$: Underdamped Case :

here the roots of F(s) is complex conjugate with negative real parts

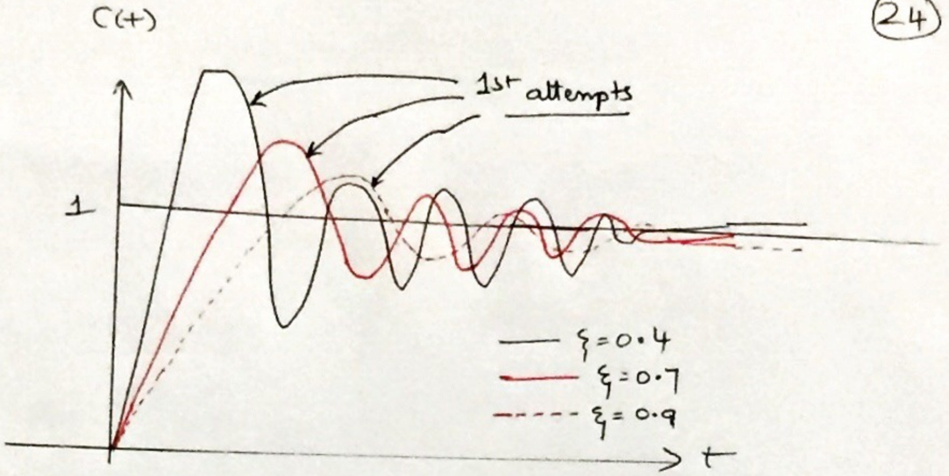
i.e $S_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

$$\therefore \quad C(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n + \omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

Solution of C(s) gives

$$\boxed{C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cdot \sin(\omega_d t + \theta)}$$

Response is exponentially decaying i.e oscillations will decay exponentially & finally settles to steady state value



We observe from graph is that as ξ starts increasing, System starts to take more time to reach Unity Value in the 1st attempt.

Case 6: $\xi = 1$: Critical damping :

When $\xi = 1$

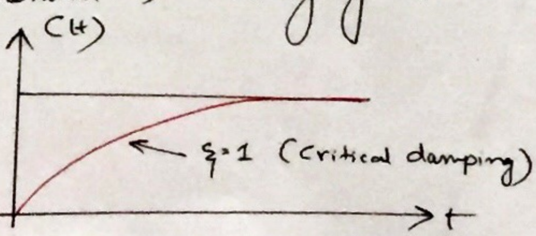
$S_{1,2} = -\xi\omega_n, -\xi\omega_n$; roots are repeative with negative real parts.

Then $C(s) = \dfrac{\omega_n^2}{S(S+\omega_n)(S+\omega_n)} = \dfrac{\omega_n^2}{S(S+\omega_n)^2}$

$\therefore C(s) = \dfrac{\omega_n^2}{s(s+\xi\omega_n)^2} = \dfrac{A}{S} + \dfrac{B}{(S+\omega_n)^2} + \dfrac{C}{(S+\omega_n)}$

$\therefore c(t) = \mathcal{L}^{-1}(C(s))$

$\boxed{c(t) = C_{ss} + Bt\cdot e^{-\omega_n t} + C e^{-\omega_n t}}$; response is purely exponential & decaying in nature.



Case d: When $\infty > \xi > 1$: Overdamped case :

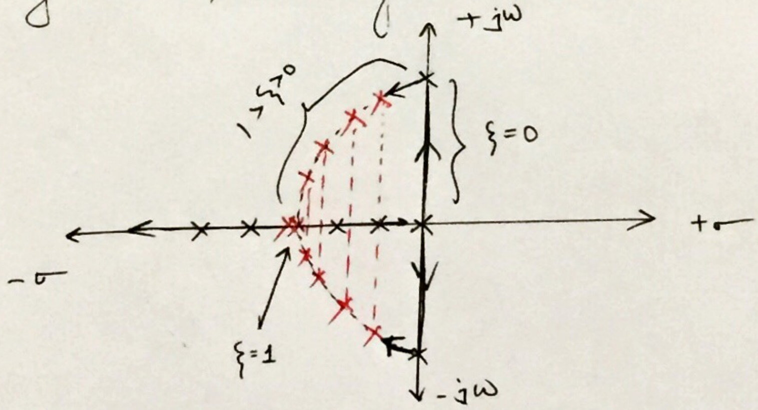
* When $\xi > 1$; the roots of $f(s)$ is given by

$$S_{1,2} = -\xi \omega_n \pm \omega_n \sqrt{\xi^2 - 1}$$

$\therefore$ $S_{1,2} = -K_1$ & $-K_2$ (say); roots are negative real but unequal.

$$\therefore C(s) = \frac{1}{S(S+K_1)(S+K_2)} = \frac{A}{S} + \frac{B}{S+K_1} + \frac{C}{S+K_2}$$

$\therefore$ $c(t) = C_{ss} + B e^{-K_1 t} + C e^{-K_2 t}$; The output is purely exponential consisting of steady state part C_{ss}. The damping is so high that, the system doesn't undergo oscillations.



Effect of ξ on the locations of closed loop poles.

ξ Range	Nature of Roots	types of response
$\xi = 0$	purely imaginary	* Undamped.
$1 > \xi > 0$	Complex conjugate with negative parts	* Underdamped
$\xi = 1$	negative real repeative	* Critically damped
$\infty > \xi > 1$	negative real unequal	* Overdamped.

XX Unit step response of a Standard 2nd order System

* **Under damped case :**

* This derivation is carried out for underdamped case, since P.F method for Critically damped & overdamped case is very simple.

* Consider $0 < \xi < 1$:

Then the roots of Characteristic equation become

$$S_{1,2} = -\xi \omega_n \pm j \omega_n \sqrt{1-\xi^2} = -\xi \omega_n \pm j \omega_d \quad \text{(rad/sec)}$$

where ω_n = natural — " — — " —
ω_d = damped frequency of oscillation (rad/sec)

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

$$S_{1,2} = -\alpha \pm j\omega_d \quad ; \text{ where } \alpha = \xi \omega_n$$

Then $$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad ; \quad R(s) = 1/s$$

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{A(s^2 + 2\xi\omega_n s + \omega_n^2) + s(Bs + C)}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

expanding the L.H.S of numerator

$$\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{As^2 + 2\xi\omega_n s A + \omega_n^2 A + Bs^2 + sC}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{s^2(A+B) + (2\xi\omega_n A + C)s + \omega_n^2 A}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

Comparing the Numerator co-efficients

$\cancel{\omega_n^2} = \cancel{\omega_n^2} A$; equating constant

$$\boxed{A = 1}$$

$A + B = 0$; equating co-efficient of s^2

$$\boxed{B = -A = -1}$$

$2\xi\omega_n A + C = 0$; equating co-efficient of s^1

$2\xi\omega_n + C = 0 =$

$\boxed{C = -2\xi\omega_n = -2\alpha}$ (as $\xi\omega_n = \alpha$)

$$\therefore C(s) = \frac{1}{s} + \frac{-s-2\alpha}{s^2+2\alpha s+\omega_n^2} = \frac{1}{s} - \frac{(s+2\alpha)}{s^2+2\alpha s+\omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{(s+2\alpha)}{(\underbrace{s^2+2\alpha s+\alpha^2}_{(s+\alpha)^2}+\omega_n^2-\alpha^2)}$$; adjusting the Denominator

$\omega_n^2-\alpha^2 = \omega_n^2 - \xi^2\omega_n^2$

$= \omega_n^2(1-\xi^2) = \omega_d^2$

$\boxed{\omega_d = \omega_n\sqrt{1-\xi^2}}$

$$C(s) = \frac{1}{s} - \frac{(s+2\alpha)}{(s+\alpha)^2+\omega_d^2}$$

adjusting C(s) such that,

$$C(s) = \frac{1}{s} - \left\{\frac{s+\alpha}{(s+\alpha)^2+\omega_d^2} + \frac{\alpha}{(s+\alpha)^2+\omega_d^2}\right\} = \frac{1}{s} - \left\{\frac{s+\alpha}{(s+\alpha)^2+\omega_d^2} + \frac{\alpha}{\omega_d}\cdot\frac{\omega_d}{(s+\alpha)^2+\omega_d^2}\right\}$$

taking Inverse Laplace Transform

$$C(t) = 1 - e^{-\alpha t}\cdot\cos(\omega_d t) - \frac{\alpha}{\omega_d}\cdot e^{-\alpha t}\sin(\omega_d t)$$

Substituting value of $\alpha = \xi\omega_n$ & $\omega_d = \omega_n\sqrt{1-\xi^2}$

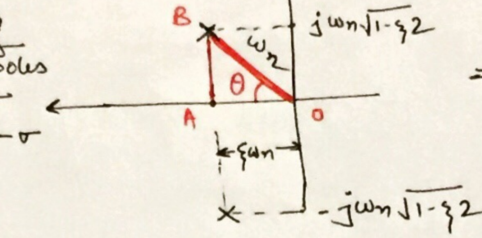
$$c(t) = 1 - e^{-\xi\omega_n t}\cdot\cos\omega_d t - \frac{\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}\cdot e^{-\xi\omega_n t}\cdot\sin\omega_d t$$

$$C(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\cdot\sin\omega_d t\right] = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cdot\cos\omega_d t + \xi\sin\omega_d t\right]$$

we have $\sin(\omega_d t+\theta) = \sin\omega_d t\cdot\cos\theta + \cos\omega_d t\cdot\sin\theta$

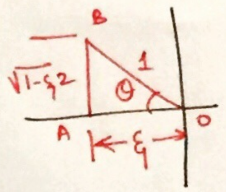
where $\sin\theta$ & $\cos\theta$ information can be extracted from Damping triangle $s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

(a) Location of Complex poles



$\Rightarrow$

(b) Damping Δle



from $\lfloor OAB$ = $\cos\theta = \dfrac{OA}{OB} = \dfrac{\xi\cancel{\omega_n}}{\cancel{\omega_n}} = \underline{\underline{\xi}}$

$\sin\theta = \dfrac{AB}{OB}$ $\qquad$ $\tan\theta = \dfrac{AB}{OA} = \dfrac{\omega_n\sqrt{1-\xi^2}}{\xi\cancel{\omega_n}} = \dfrac{\sqrt{1-\xi^2}}{\xi}$

$\boxed{\sin\theta = \sqrt{1-\xi^2}}$ $\quad \because OB = \sqrt{OA^2+AB^2} = \sqrt{\xi^2\omega_n^2+\omega_n^2(1-\xi^2)}$

$OB = \omega_n\sqrt{\cancel{\xi^2}+1-\cancel{\xi^2}} = \underline{\omega_n}$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin\theta\cdot\cos\omega_d t + \cos\theta\cdot\sin\omega_d t\right]$$

$$\boxed{c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta)}$$

Where $\underline{\underline{\omega_d = \omega_n\sqrt{1-\xi^2}}}$ $\qquad \theta = \tan^{-1}\left(\dfrac{\sqrt{1-\xi^2}}{\xi}\right)$

* Note: (1) Instead of Unit Step input, a step function with amplitude A is given, then

$$\boxed{c(t) = A\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\cdot\sin(\omega_d t+\theta)\right]}$$

(2) If $\dfrac{C(s)}{R(s)}$ is not expressed in standard form;

Say suppose $\dfrac{C(s)}{R(s)} = \dfrac{K}{s^2+2\xi\omega_n s+\omega_n^2}$

Then $\dfrac{C(s)}{R(s)}$ can be modified as $\dfrac{C(s)}{R(s)} = \dfrac{K}{\omega_n^2}\left[\dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\right]$

output $C(t)$ then may be written as:

$$C(t) = \frac{K}{\omega_n^2}\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)\right]$$

③ If $\frac{C(S)}{R(S)}$ is expressed as

$$\frac{C(S)}{R(S)} = \frac{P(S)}{S^2 + 2\xi\omega_n S + \omega_n^2} \quad \text{ex:} \quad \frac{S+2}{S^2+10S+4}$$

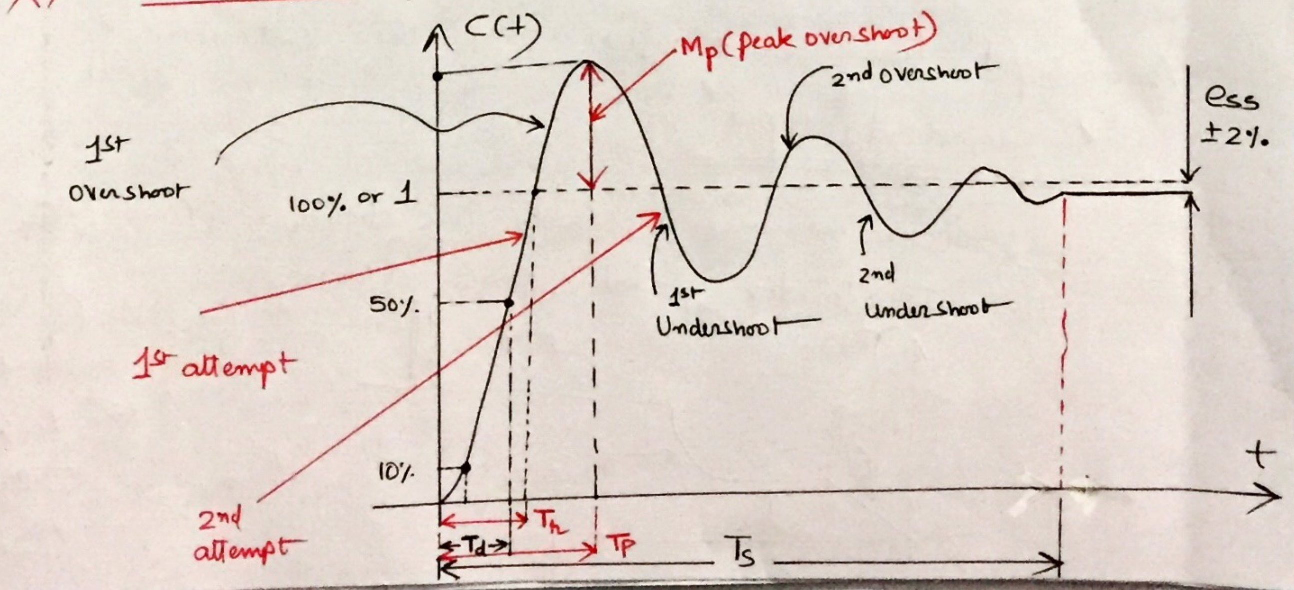
then the unit step response results can't be used to solve such problems.

However, the reader can solve these problems by applying actual partial fraction method. (not necessary in this course)

④ It is applicable to step input only.

for any other inputs, the results must be freshly calculated (not necessary in this course)

XX Transient Response Specifications for 2nd Order System



Derivation :

→ Delay Time : Time taken by the system to reach 50% of the final value in 1st attempt.

$$\boxed{T_d = \frac{1+0.7\xi}{\omega_n}}$$; no derivation.

→ Rise Time : T_r :

Under damped system : Time required for the response to rise from 0% to 100% of the final value in the 1st attempt.

Over damped system : Time required for the response to rise from 10% to 90% of the final value.

$$\boxed{T_r = \frac{\pi - \theta}{\omega_d}}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

→ Peak Time : T_p :

The time required for the response to reach its peak value

$$\boxed{T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$

→ Peak overshoot : M_p :

It is the amount by which the actual output exceeds the reference steady state value during the 1st overshoot.

$$\boxed{M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \times 100}$$

A

Derivations of Time Response specifications

① Rise Time (T_r):

→ It is defined as time taken for the response to reach 100% of the final value in the first attempt.

i.e at $t = t_r$, $C(t) = 1$ (reaches 100% value)

$$\cancel{1} = \cancel{1} - \frac{e^{-\xi \omega_n T_r}}{\sqrt{1-\xi^2}} \cdot \sin(\omega_d T_r + \theta)$$

$$-\frac{e^{-\xi \omega_n T_r}}{\sqrt{1-\xi^2}} \sin(\omega_d T_r + \theta) = 0$$

The above equation is satisfied when

$$\sin(\omega_d T_r + \theta) = 0$$

This is possible when $\omega_d T_r + \theta = n\pi$ $(n = 1, 2 \ldots)$

$$T_r = \frac{n\pi - \theta}{\omega_d} \quad (\because n = 1 = \text{first attempt})$$

$$\boxed{T_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n \sqrt{1-\xi^2}}}$$

B

2 Peak Time (T_p):

It is the time taken for the response to reach the peak value.

At peak time $C(t)$ = maximum; Applying principal of maxima theorem,

$$\left.\frac{dc(t)}{dt}\right|_{t=T_p} = 0$$

$$\frac{d}{dt}\left(1-\frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}}\cdot \sin(\omega_d T_p+\theta)\right)=0$$

$$0-\frac{1}{\sqrt{1-\xi^2}}\left(e^{-\xi\omega_n T_p}(-\xi\omega_n)\sin(\omega_d T_p+\theta)+e^{-\xi\omega_n T_p}\cos(\omega_d T_p+\theta)\cdot\omega_d\right)=0$$

$$\cancel{e^{-\xi\omega_n T_p}}(\xi\omega_n)\sin(\omega_d T_p+\theta)=\cancel{e^{-\xi\omega_n T_p}}\cdot\cos(\omega_d T_p+\theta)\cdot\omega_d$$

$$(\xi\cancel{\omega_n})\sin(\omega_d T_p+\theta)=\cancel{\omega_n}\sqrt{1-\xi^2}\cos(\omega_d T_p+\theta)$$

$$\frac{\sin(\omega_d T_p+\theta)}{\cos(\omega_d T_p+\theta)}=\frac{\sqrt{1-\xi^2}}{\xi}$$

$$\tan(\omega_d T_p+\theta)=\tan\theta$$

In trignometry it is possible when

$\omega_d T_p = n\pi$ i.e $\tan(n\pi+\theta)=\tan\theta$

$$T_p=\frac{n\pi}{\omega_d} \Rightarrow \boxed{T_p=\frac{\pi}{\omega_d}=\frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$

Maximum Overshoot :

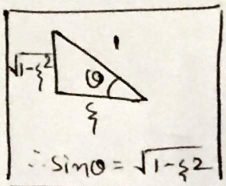
It is found mathematically as

$$M_p = C(t_p) - 1$$

$$M_p = \cancel{1} - \frac{e^{-\xi \omega_n T_p}}{\sqrt{1-\xi^2}} \cdot \sin(\omega_d T_p + \theta) - \cancel{1} \quad \text{--- (1)}$$

Substitute $T_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}}$ in eq (1)

$$M_p = \cancel{1} - \frac{e^{\frac{-\xi \cancel{\omega_n} \pi}{\cancel{\omega_n}\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \cdot \sin\left(\cancel{\omega_d} \times \frac{\pi}{\cancel{\omega_d}} + \theta\right) - \cancel{1}$$



$$M_p = \cancel{1} - \frac{e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \cdot \sin(\pi + \theta)$$

In tringnometry we have $\sin(\pi+\theta) = -\sin\theta$

$$\therefore \quad M_p = \frac{e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \cdot \sin\theta = \frac{e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}}{\cancel{\sqrt{1-\xi^2}}} \times \cancel{\sqrt{1-\xi^2}}$$

In terms of percentage

$$\boxed{\% M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100}$$

$\therefore$ M_p is the function of ξ alone. This shows that, M_p can be controlled by choosing appropriate value of ξ

4) Settling time (T_S):

Settling time is the time taken by the response to Settle down to ±2% of the final value.

i.e if the expected value is 1 (for unit step response in time domain);

$$C(t) = 1 \pm 2\% = 1.02 \text{ or } 0.98$$

Let us take say $C(t) = 0.98$

$$0.98 = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta) \quad \text{--- (1)}$$

But one needs to observe the fact that; after Settling time, the system output $C(t)$ will not have transient & oscillatory part.

Therefore in eq (1) Let us neglect the transient part of the output & consider only the exponential term.

i.e $0.98 = 1 - e^{-\xi\omega_n t}$

$$0.02 = e^{-\xi\omega_n t}$$

taking ~~gretedg~~ $\log_e$ one both side

$$\ln(0.02) = -\xi\omega_n T_s \Big|_{t=T_s}$$

$$-3.92 = -\xi\omega_n T_s \Rightarrow \boxed{T_s \approx \frac{4}{\xi\omega_n}}$$