Module 3

Time Response Analysis

Introduction:

→ What do you mean by Time Response Analysis?

In simple terms, it is nothing but the behaviour of the output w.r.t. time.

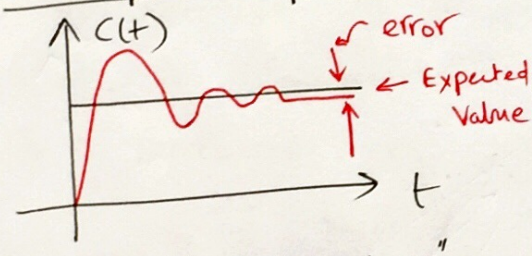
Generally time is taken as independant variable.

→ Why it is required?

It is necessary to find the time response of a system, to know how the output is going to approach the expected value.

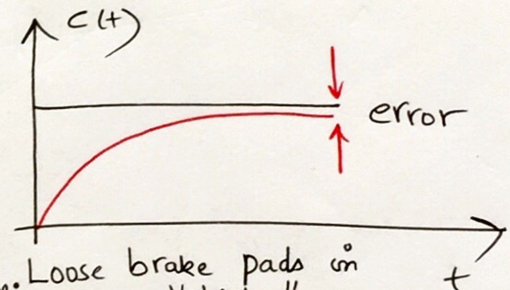
There are 3 possiblites for an output to reach the expected value.

① Undamped Response:



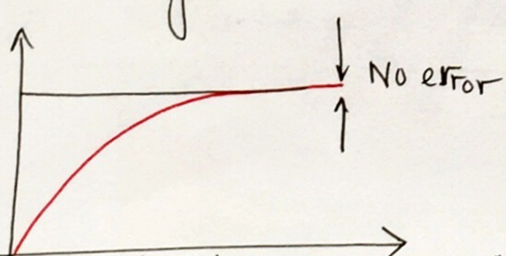
ex: "Most of the physical systems"
Musical Instruments, elevators, Diving Board.

② Overdamped Response



ex: Loose brake pads in Vehicle"

③ Critically damped

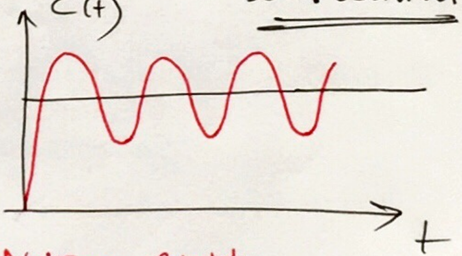


automatic door closing mechanism

ex: Sergical Robots; Precision tool Cutting m/c

Undamped System

ex: Oscillator



"Neither Stable Nor Unstable"

Note: Most of the systems are underdamped type.

→ Time Response is consisting of two parts $C(t)$

1. Transient response $C_{tr}(t)$
2. Steady state response (DC response) $C_{ss}(t)$

1. Transient response:

It is that part of the time response, at which the output is varrying w.r.t time.
i.e output is dependent on time.

→ represented as $C_{tr}(t)$.

ex: The inrush current of an induction motor while starting.

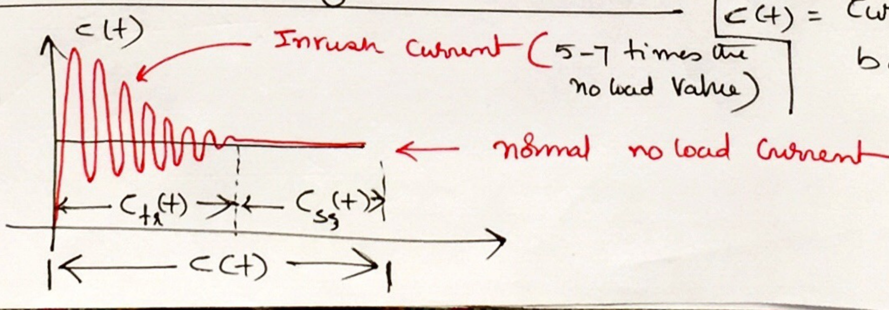
2. Steady state Response:

It is that part of the time response, at which output is independent time.

→ It is represented by $C_{ss}(t)$.

ex: The normal noload current drawn by the induction motor once the transient period is over.

⇒ graphical representation of inrush current.



$C(t)$ = No load current drawn by I.m.

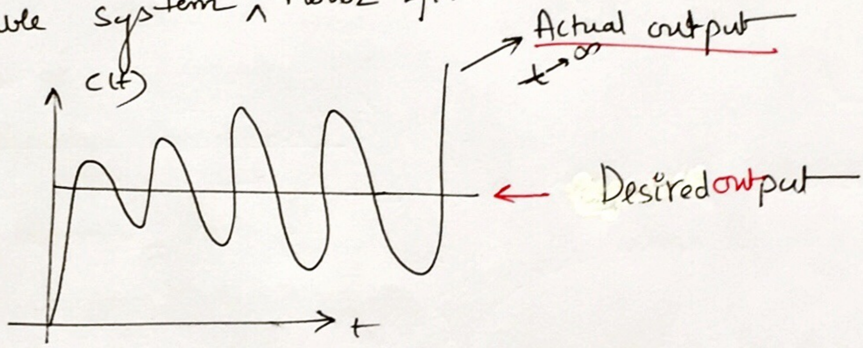
Summary:

- Transient response persists for a small duration of time. for a good system it should quickly decays to zero.

  i.e $\lim_{t \to \infty} c_{tr}(t) = 0$

- Once the transient response period is elapsed, the output settles down to steady state value.
- If the output settels to a final value, which is very close to the desired value, then the system may be regarded as a stable system.
- The difference between the desired value and the final value is called as steady state error.
- An unstable system can never find the final value.

  ex:



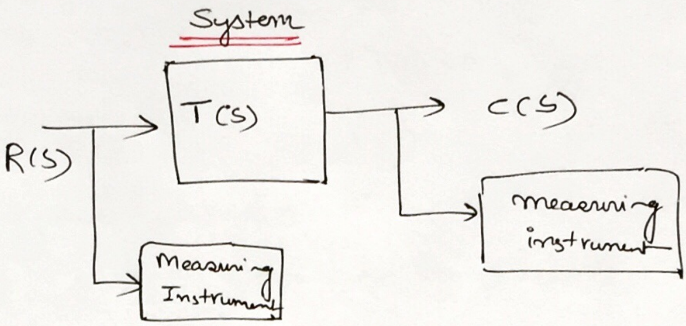
- Steady state response determines the accuracy of the system
- transient response determines the quickness or sluggishness of the system.

- total response of a system is given by

$$\boxed{c(t) = c_{tr}(t) + c_{ss}(t)}$$

→ **Test inputs : Standard test inputs :**

A typical test setup :



Where R(s) = Reference input ; T(s) = Transfunction of the System.

C(s) = output or actual output

What should be the input? i.e DC, AC, linear, random etc. etc.

→ Should it be a steady one?

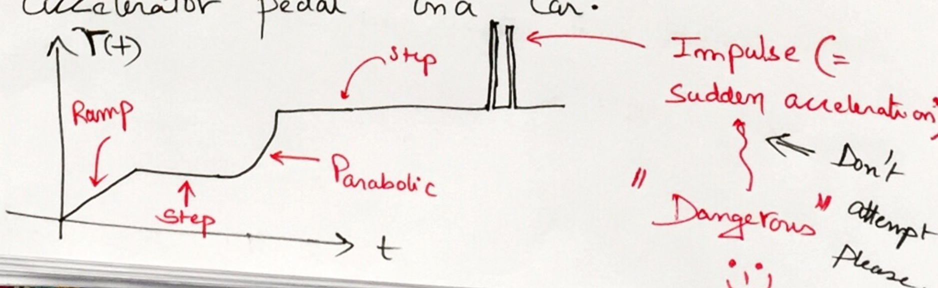
— " — rapidly increasing type?

— " — Sudden Jerk ?

— " — Sudden Shock ?

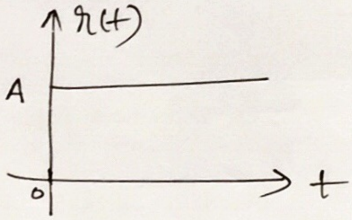
Any input can be brought under the category of "Standard input"

· example : The manner in which the driver presses the accelerator pedal in a car.



Standard inputs :

(a) Step input / (Position function) :



$r(t)$ = input in time domain

$r(t) = A$ for $t \geq 0$

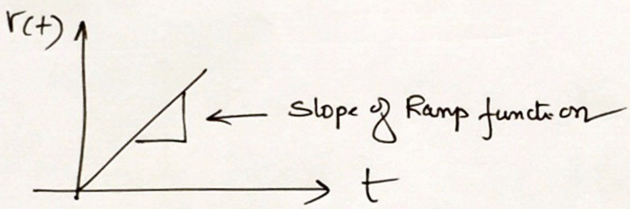
$r(t) = 0$ for $t < 0$

- In laplace domain

$R(s) = A/s$ where A = Amplitude of Step function

$\boxed{\text{When } A = 1 \text{ ; } R(s) = 1/s \text{ ; unit step function}}$

(b) Ramp input / Velocity function :



$r(t) = At$ for $t \geq 0$

$= 0$ for $t < 0$

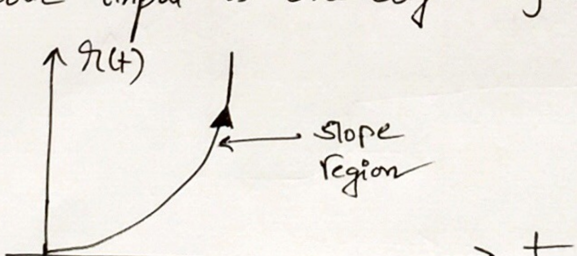
When $A = 1$; It is called as unit ramp function

- In Laplace domain

$$R(s) = \frac{A}{s^2}$$

(c) Parabolic Input (Acceleration function) :

- Parabolic input is one degree faster than ramp input.



$r(t) = \frac{A}{2}t^2$; for $t \geq 0$

$r(t) = 0$; $t < 0$

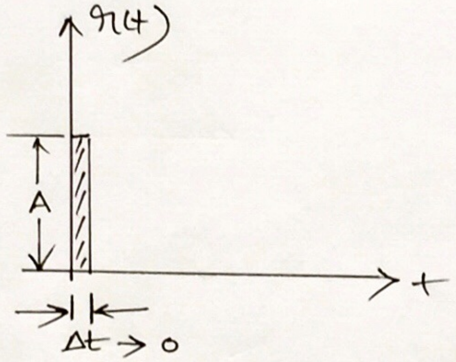
- In laplace domain

$$R(s) = A/s^3$$

When $A=1$; It is called as Unit ramp function

(d) <u>Impulse function:</u>

- The input exists for a very small duration of time (ms, μs, ns).
- The amplitude of input is very large
  ex: energy carried by a lightning boult (mJ's of energy)
- In testing scenario, normally $A = 1$; hence it is called as <u>"Unit Ramp"</u> function.



$r(t) = A$, for $t=0$

$r(t) = 0$, for $t \neq 0$

- Laplace Domain
  $R(s) = 1$; unit impulse function

- Normally it is denoted by $\delta(t)$

→ <u>Summary:</u>

Input type	time domain	Laplace domain
Step	$r(t) = A$	$R(s) = \frac{A}{s}$
Ramp	$r(t) = At$	$R(s) = A/s^2$
Parabolic	$r(t) = A\frac{t^2}{2}$	$R(s) = A/s^3$
Impulse	$r(t) = A$	$R(s) = A$

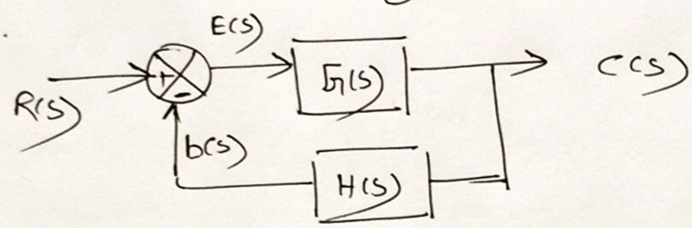
Steady State Analysis:

- It mainly consists of the following two information
  - → How much time a system has taken to reach the final value ("settling time ts)
    Settling time is found from transient response.
  - → How far is the output from its desired value? which is called as steady state error.

"Under steady state analysis, our aim is to find steady state error."

Expression for steady state error:

- Consider the block diagram of a standard negative feedback system.



where $E(S)$ = Error Signal.

$$E(S) = R(S) - b(S)$$

but $b(S) = C(S)\,H(S)$

$C(S) = E(S)\,G(S)$

$$E(S) = R(S) - H(S)\,E(S)\,G(S)$$

$$E(S) + E(S)\,G(S)\,H(S) = R(S)$$

$$E(S) = \frac{R(S)}{1+G(S)\,H(S)}$$; for a non-unity feedback (1)

$$E(S) = \frac{R(S)}{1+G(S)}$$; for unity feedback $(\because H(S)=1)$

→ But in reality we ~~so~~ want to find the error in time domain.

→ So that we will be able to find the steady state error of the system when $t \to \infty$

∴ Steady state error $e_{ss} = \lim_{t \to \infty} e(t)$ — (2)

- The above expression can be related to laplace domain by using final value theorem (F.V.T) which states that

$$\boxed{\lim_{t \to \infty} F(t) = \lim_{s \to 0} sF(s)} \quad \text{— (3)}$$

where $F(s) = \mathcal{L}(F(t))$

$$\boxed{\begin{aligned} e_{ss} &= \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \\ &\text{where } E(s) \text{ is } \mathcal{L}(e(t)) \end{aligned}} \quad \text{— (4)}$$

Substituting $E(s)$ from eq (1) in eq (4)

$$\boxed{e_{ss} = \lim_{s \to 0} \frac{s\,R(s)}{1 \pm G(s)H(s)}} \quad \text{— (5)}$$

$-$ sign for positive feedback system

$+$ sign for negative feedback system

From expression (5) it is clear that steady state error depends on

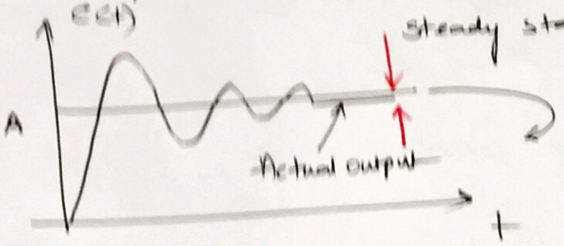
(i) $R(s)$; the reference input

(ii) $G(s)H(s)$; open loop transfer function

(iii) Non linearities present if any.

Static Error Co-efficients

(a) Reference input is step input with magnitude A



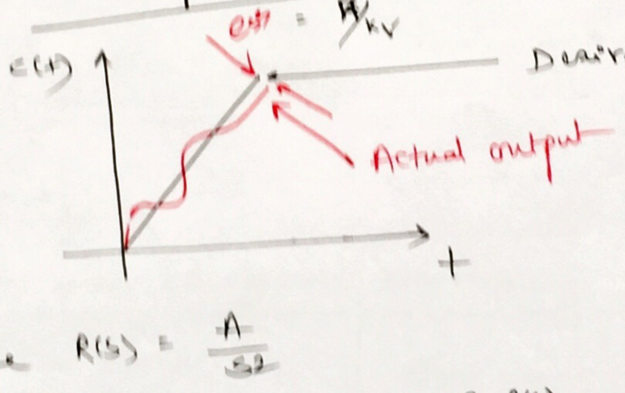
$R(s) = \frac{A}{s}$

$\therefore e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s\,R(s)}{1+G(s)H(s)} = \lim_{s \to 0} \frac{\cancel{s} \times A/\cancel{s}}{1+G(s)H(s)}$

$$\boxed{\therefore e_{ss} = \frac{A}{1+\lim_{s \to 0} G(s)H(s)} = \frac{A}{1+K_p}}$$

where $K_p = \lim_{s \to 0} G(s)H(s)$; positional error constant.

(b) Reference input is Ramp with a magnitude A



take $R(s) = \frac{A}{s^2}$

$\therefore e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s\,R(s)}{1+G(s)H(s)} = \lim_{s \to 0} \frac{\cancel{s}\; A/s^2}{1+G(s)H(s)}$

$e_{ss} = \lim_{s \to 0} \frac{A}{s + s\,G(s)H(s)} = \frac{A}{\lim_{s \to 0} s\,G(s)H(s)}$

where $\lim_{s \to 0} s\,G(s)H(s) = K_v$; Velocity error constant / coefficient

$$\boxed{\therefore\; e_{ss} = \frac{A}{K_v}}$$

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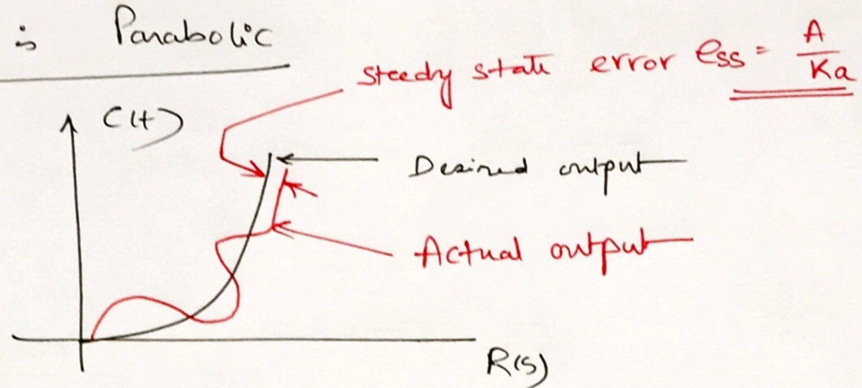
(3) <u>Input is Parabolic</u>



Here $R(s) = \frac{A}{s^3}$

$$\therefore\; e_{ss} = \lim_{s \to 0} s\,E(s) = \lim_{s \to 0} \frac{s\,R(s)}{1+G(s)H(s)} = \lim_{s \to 0} \frac{s \times A/s^3}{1+G(s)H(s)} = \lim_{s \to 0} \frac{A}{s^2 + s^2 G(s)H(s)}$$

$$\therefore\; e_{ss} = \frac{\lim A}{\lim_{s \to 0} \cancel{s^2} + \lim_{s \to 0} G(s)H(s) \times s^2} \Rightarrow e_{ss} = \frac{A}{\lim_{s \to 0} s^2 G(s)H(s)}$$

$$\boxed{e_{ss} = \frac{A}{K_a}}$$

where K_a = acceleration error constant / co-efficient.

<u>Summary:</u>

Static error co-efficients	Corresponding steady state error
$K_p = \lim_{s \to 0} G(s)H(s)$	$e_{ss} = \frac{A}{1+K_p}$
$K_v = \lim_{s \to 0} s\,G(s)H(s)$	$e_{ss} = \frac{A}{K_v}$
$K_a = \lim_{s \to 0} s^2 G(s)H(s)$	$e_{ss} = \frac{A}{K_a}$

Type and Order of a System

- Type of a system is obtained, by identifying number of poles located at the origin.
- To find type & order of a system, we must have the system description interms of transfer function.

A transfer function can be expressed in 3 different ways.

① Numerator and Denominator polynomial form

$$G(S)H(S) = \frac{S^{m} + b_1 S^{m-1} + b_2 S^{m-2} + b_3 S^{m-3} + \cdots + b_m S^{0}}{S^{n} + a_1 S^{n-1} + a_2 S^{n-2} + a_3 S^{n-3} + \cdots + a_n S^{0}}$$

The highest power of S in the denominator polynomial is called as order of a system

ex: ① $G(S)H(S) = \dfrac{S^2 + 2S + 1}{S^4 + 2S^3 + 8S^2 + 4S + 1}$

In example ① the highest power of S in the denominator polynomial is 4. Therefore order = 4

"It is a 4th order system"

ex ②: $G(S)H(S) = \dfrac{1}{S}$; Power of S in the denominator polynomial is ①

"Therefore it is called as 1st order system.

(2) Pole Zero format :

→ Pole zero form of a transfer function can be obtained by finding out the roots of the Nuemerator Polynomid and denominator polynomial.

→ Roots of Numerator polynomial is called as "Zeros" of the system.

→ Roots of Denominator polynomial is called as "Poles" of the system

Pole zero form is given below :

$$G(S)\,H(S) = \frac{K\,(S+Z_1)\,(S+Z_2)\,(S+Z_3)\cdots(S+Z_m)}{(S+P_1)\,(S+P_2)\,(S+P_3)\cdots(S+P_n)}$$

where $Z_1, Z_2 \cdots Z_m$; Zero's of the system

$P_1, P_2, \cdots P_n$; Poles of the system

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(3) Time constant form :

- To identify Type of a system, the transfer function must be expressed in time constant form ;

$$G(S)\,H(S) = \frac{K\,(1+ST_1)\,(1+ST_2)\,(1+ST_3)\cdots(1+ST_m)}{S^n\,(1+ST_{a1})\,(1+ST_{a2})\,(1+ST_{a3})\cdots(1+ST_{an})}$$

where S^n are the poles of the system present at origin. Power of S is type of the system.

→ In other words we can say that; Number integrators give Type of a system.

ex 1: $G(s)H(s) = \dfrac{K(1+s)(1+0.5s)}{s^3(1+2s)(1+4s)}$

Here power of s at the origin is (3)

Hence it is a type - 3 system

Analysis of Type 0, Type 1 & Type 2 Systems

(1) Let us assume Type—0 system

Remarks: for a type—0 system no of poles present at the origin is zero.

$$\therefore G(s)H(s) = \frac{K(1+sT_1)(1+sT_2)\cdots}{s^0(1+sT_a)(1+sT_b)\cdots}$$

For a Step input

Then K_p = Positional error constant = $\lim\limits_{s \to 0} G(s)H(s)$

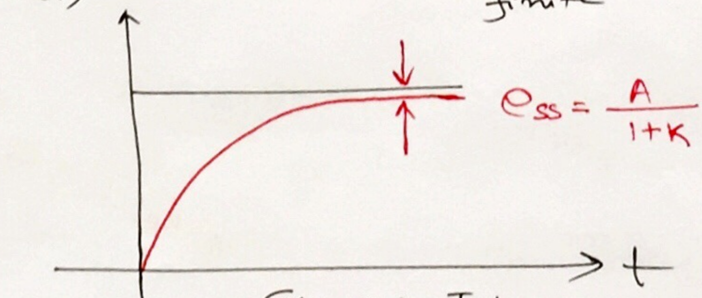
$\boxed{\text{S.S error } e_{ss} = \frac{A}{1+K_p}}$

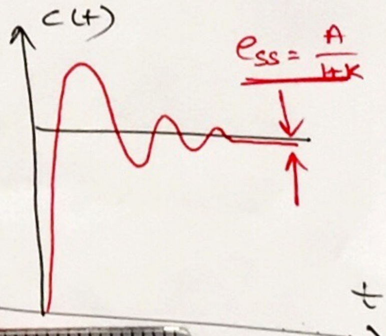
where

$$K_p = \lim_{s \to 0} \frac{K(1+\cancel{sT_1})(1+\cancel{sT_2})\cdots}{(1+\cancel{sT_a})(1+\cancel{sT_b})\cdots} = \frac{K(1)(1)}{(1)(1)} = K$$

$\therefore K_p = K$

Then the steady state error $e_{ss} = \dfrac{A}{1+K}$ which is finite





Graphical Interpretation

→ Where the factor K could be the gain of amplifier in the forward path.

→ By increasing the K value, steady state error (e_{ss}) can be reduced.

→ But there is a limitation on increament of K value. Because it poses challenge for system stability. As we start increasing the value of K, a higher order system may become unstable!!! Which is Bad ← This will be discussed in (Module 4)

(b) Type of input is Ramp:

For a ramp input steady state error is

$$e_{ss} = \frac{A}{K_V} \; ; \quad \text{Where } K_V = \lim_{s \to 0} s\,G(s)\,H(s)$$

$$\therefore K_V = \lim_{s \to 0} \frac{s\, K (1+sT_1)(1+sT_2)\cdots}{(1+sT_a)(1+sT_b)} = \frac{0 \cdot K \cdot (1)(1)}{1 \cdot 1} = \underline{\underline{0}}$$

$$\boxed{\therefore e_{ss} = \frac{A}{K_V} = \frac{A}{0} = \infty}$$

* So a ramp input gives ∞ steady state error, i.e. The actual output will never be equal to close to the desired value.

* System becomes "Unstable"

Note: We don't have to test a type 0 system against Ramp input. Because the response is known beforehand only which is "Unstable"

(c) Parabolic Input :

For a parabolic input $e_{ss} = \dfrac{A}{K_a}$

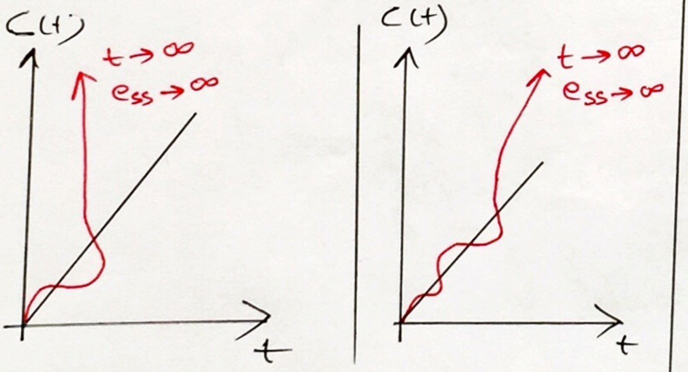
where $K_a = \lim\limits_{s \to 0} s^2\, G(s)H(s) = \lim\limits_{s \to 0} \left[\dfrac{\cancelto{0}{s^2}\, K\, (1+\cancelto{0}{sT_1})(1+\cancelto{0}{sT_2}) \cdots}{(1+\cancelto{0}{sT_a})(1+\cancelto{0}{sT_b}) \cdots} \right]$

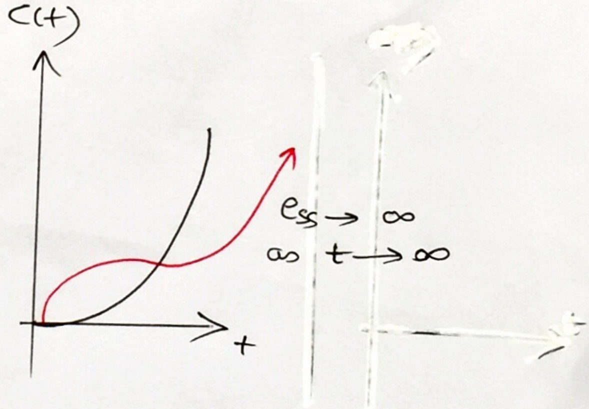
$\therefore K_a = \dfrac{0 \cdot K \cdot 1 \cdot 1}{1 \cdot 1 \cdot 1} = \underline{\underline{0}}$

$$\boxed{\therefore e_{ss} = \frac{A}{K_a} = \frac{A}{0} = \infty}$$

* Just like ramp input, a parabolic input gives ∞ Steady state error. which bad in system point of view (Unstable system)

Response of type 0 system for Ramp input	Response of type 0 system for parabolic input





Analysis of Type -1 System :

Type 1 System

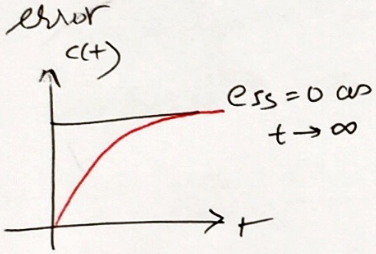
$$G(S)\,H(S) = \frac{K\,(1+ST_1)\,(1+ST_2)\cdots}{S\,(1+STa)\,(1+S\,Tb)\cdots}$$

(i) For Step input :

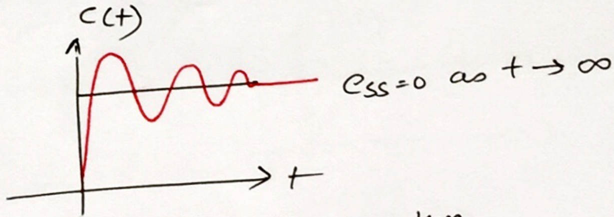
$$e_{ss} = \frac{A}{1+K_P} \;; \quad K_P = \lim_{S\to 0} G(S)\,H(S)$$

$$K_P = \lim_{S\to 0} \frac{K\,(1+\cancel{ST_1})\,(1+\cancel{ST_2})\cdots}{\cancel{S}\,(1+\cancel{STa})\,(1+\cancel{STb})\cdots} = \frac{K\,(1)\,(1)}{0\,(1)\,(1)} = \underline{\underline{\infty}}$$

$$\therefore\; e_{ss} = \frac{A}{1+\infty} = \frac{A}{\infty} = 0 \;;$$ Zero steady state



Critically damping



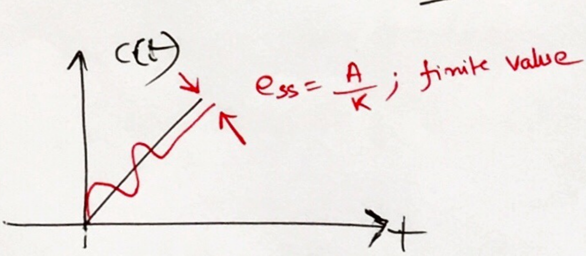
Under damped system

(ii) For Ramp input :

$$e_{ss} = \frac{A}{K_V} \;; \quad K_V = \lim_{S\to 0} S\,G(S)\,H(S)$$

$$K_V = \lim_{S\to 0} \frac{\cancel{S}\,K\,(1+\cancel{ST_1})\,(1+\cancel{ST_2})\cdots}{\cancel{S}\,(1+\cancel{STa_1})\,(1+\cancel{STa})\cdots} = \underline{\underline{K}}\,.$$

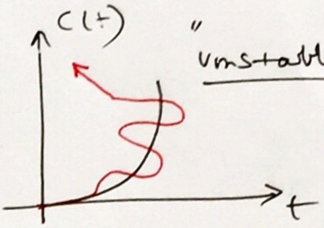
$$\therefore\; e_{ss} = \frac{A}{K} \;;$$ <u>finite</u> Value



(iii) For acceleration/ parabolic input :

The steady error when a parabolic input is given, can be found by;

$$e_{ss} = \frac{A}{K_a} \; ; \quad K_a = \lim_{s \to 0} K s^2 G(s) H(s)$$



"unstable"

$$\therefore K_a = \lim_{s \to 0} \frac{K s^2 (1+sT_1)(1+sT_2)\cdots}{(1+sT_{a1})(1+sT_{a2})\cdots}$$

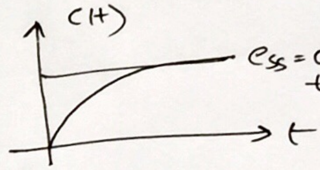
$$\therefore K_a = \frac{K(0)(1)\quad(1)}{1 \quad 1} = 0$$

$\therefore e_{ss} = \frac{A}{0} = \infty$. Parabolic input can't be given to a type-1. The s.s. error becomes ∞ as $t \to \infty$

Analysis of Type 2 System:

(i) Positional / Step input

For type 2 system we have



~~G(s) H(s)~~

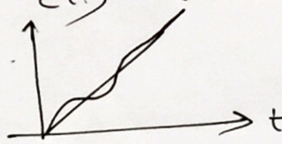
$$e_{ss} = \frac{A}{1+K_p}$$

also $K_p = \lim_{s \to 0} G(s) H(s)$

$$\therefore K_p = \lim_{s \to 0} \frac{K(1+s\cancel{T_1})(1+s\cancel{T_2})\cdots}{s^2(1+s\cancel{T_{a1}})(1+s\cancel{T_{a2}})\cdots} = \frac{K(1)(1)}{0(1)(1)} = \infty$$

$$\therefore e_{ss} = \frac{A}{1+\infty} = \frac{A}{\infty} = \underline{\underline{0}}$$

$\therefore$ A type 2 system gives Zero s.s. error when excited by a step input.



(ii) Ramp input :

we have s.s. error as $e_{ss} = \frac{A}{K_v}$

$$K_v = \lim_{s \to 0} s\,G(s) H(s) = \lim_{s \to 0} \frac{\cancel{s} K (1+s\cancel{T_1})(1+s\cancel{T_2})\cdots}{\cancel{s^2}(1+s\cancel{T_{a1}})(1+s\cancel{T_{a2}})\cdots} = \frac{K(1)}{0}$$

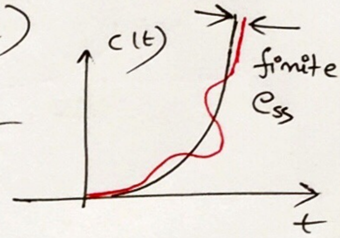
$K_v = \infty \Rightarrow \boxed{e_{ss} = \frac{A}{\infty} = 0; \text{ which is Zero s.s. error}}$

(iii) Parabolic / Acceleration input :

For an acceleration input S.S. error is given by

$$e_{ss} = \frac{A}{K_a} \; ; \quad K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$K_a = \lim_{s \to 0} \frac{\cancel{s^2} K(1+sT_1)(1+sT_2)\cdots}{\cancel{s^2}(\quad)(1+sT_{a2})\cdots} = \frac{K(1)}{(1)} = K.$$



$$\boxed{\therefore e_{ss} = \frac{A}{K}} \; ; \text{ finite S.S. error}$$

Summary :

System type	Error co-eff			S.S. error e_{ss}		
	K_p	K_v	K_a	K_p	K_v	K_a
Type 0	K	0	0	$\frac{A}{1+K}$	∞	∞
Type 1	∞	K	0	0	A/K	∞
Type 2	∞	∞	K	0	0	A/K

—x—x—

Problems :

① A unity feedback C.S has $G(s) = \frac{20(1+s)}{s^2(2+s)(4+s)}$,

Calculate (1) Steady state error co-efficients & error when the applied input is $r(t) = 40 + t + 5t^2$

Solution :

→ Compare the given input with the standard form,

$$r(t) = \underbrace{A}_{\text{Step}} + \underbrace{At}_{\text{Ramp}} + \underbrace{\frac{At^2}{2}}_{\text{Parabolic}}$$

We get for Step input → $A = 40$

for Ramp input → $A = 1$

for Parabolic input → $\frac{A}{2} = 5 \Rightarrow \underline{\underline{A = 10}}$

From the T.F of we see that no. of poles at the origin is 2. Therefore it is a type 2 System.

→ ∴ $K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{20(1+\cancel{s})}{\cancel{s^2}(2+\cancel{s})(4+\cancel{s})} = \infty$

∴ $e_{ss_1} = \frac{A}{1+K_p} = \frac{40}{1+\infty} = \underline{\underline{0}}$

→ Velocity / Ramp input :

$K_V = \lim_{s \to 0} s\,G(s)H(s) = \lim_{s \to 0} \cancel{s} \frac{20(1+\cancel{s})}{\cancel{s^2}(2+\cancel{s})(4+\cancel{s})} = \underline{\underline{\infty}}$

∴ $e_{ss_2} = \frac{A}{K_V} = \frac{A}{\infty} = \underline{\underline{0}}$

→ Parabolic function :

$K_a = \lim_{s \to 0} s^2 G(s)H(s) = \lim_{s \to 0} \frac{\cancel{s^2}\, 20(1+s)}{\cancel{s^2}(2+s)(4+s)}$

$K_a = \frac{\cancel{20}^{10}}{\cancel{2} \times 4} = \underline{\underline{2.5}} \Rightarrow e_{ss_3} = \frac{A}{K_a} = \frac{10}{2.5} = \underline{\underline{4}}$

∴ Steady State error $= e_{ss_1} + e_{ss_2} + e_{ss_3}$

$e_{ss} = \underline{\underline{0 + 0 + 4 = 4}}$