Appendix A
Nonnegative Matrices

In this appendix, we present some of the results of the nonnegative matrix theory, which are used in the main part of the book. A detailed presentation of this theory can be found in Gantmacher (1959) and Horn and Johnson (1985) (see also Bellman 1970; Lancaster 1969; Marcus and Minc 1992).

Recall that a matrix is called *nonnegative (positive)* if all its elements are nonnegative (positive). Here we consider only nonnegative square matrices of order n, i.e., matrices that have n rows and n columns.

In nonnegative matrix theory, the most easily formulated and proven statements are those that concern positive matrices. The main result that deals with the properties of the eigenvalues of such matrices was obtained by O. Perron in 1907.

Theorem A.1 (Perron's Theorem). *Let A be a positive matrix; then A has a positive eigenvalue λ_A such that*

(a) λ_A is a simple root of the characteristic equation of A, and
(b) the value of λ_A is strictly greater than the absolute value of any other eigenvalue of A.

The eigenvalue λ_A corresponds to a unique (with an accuracy of up to a scalar factor) positive eigenvector x_A.

The set of nonnegative matrices has a special subset. Matrices from that subset, called *irreducible (indecomposable)* matrices, have many of the properties of positive matrices. The set of irreducible matrices is defined as the complement of the set of *reducible (decomposable)* matrices. Given a matrix that has zero elements, its (ir)reducibility is determined by the positions of those elements. Let us recall a formal definition of a reducible matrix of order n (which is not necessarily nonnegative). In this definition, N denotes the index set $\{1, \ldots, n\}$.

Definition A.1. Let A be a square matrix of order n, $n \geqslant 2$, formed from elements a_{ij}. The matrix A is called a reducible matrix if there exists a subset S of the index set N such that $a_{ij} = 0$ for all $j \in S$ and $i \notin S$.

A.P. Abramov, *Balanced and Cyclical Growth in Models of Decentralized Economy*, Lecture Notes in Economics and Mathematical Systems 672,
DOI 10.1007/978-3-319-07917-2,

The index set S that appears in this definition is called *isolated* (Ashmanov 1984). If A is a reducible matrix such that $S = \{1, \ldots, k\}$, then A has the following form:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{A.1}$$

where A_{11} and A_{22} are square submatrices of order k and $n-k$, respectively. Note that these submatrices can also contain zero elements. Moreover, these submatrices can, in turn, be reducible matrices. Any reducible matrix can be put in form (A.1) by simultaneously permuting its rows and columns.

It is easy to see that, in the Leontief model, if the technology matrix of an economic system is reducible, the group of sectors $S = \{1, \ldots, k\}$ does not need the commodities produced by the group of sectors $S' = \{k+1, \ldots, n\}$ and forms an isolated, independent economic subsystem.

If A is a reducible matrix and S is its isolated subset, then its transpose A^T is also reducible and the isolated subset of A^T is the set S' of A.

Example A.1. Let us consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 7 \end{bmatrix}, \quad \tilde{A^T} = \begin{bmatrix} 7 & 6 & 3 \\ 0 & 5 & 2 \\ 0 & 4 & 1 \end{bmatrix},$$

where $\tilde{A^T}$ is the matrix A^T put in the canonical form for reducible matrices. For these matrices, the sets S are as follows, respectively: $\{1, 2\}$, $\{3\}$, and $\{1\}$. □

As the following example shows, the matrix product of two irreducible matrices can be a reducible matrix and, similarly, a power of an irreducible matrix can be reducible.

Example A.2.

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad \square$$

The example below shows that the matrix product of two reducible matrices can be either reducible or irreducible.

Example A.3.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \qquad AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad BA = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Here the matrix BA is reducible, whereas AB is irreducible. □

Note that, in the above example, the matrices AB and BA have the same *spectrum* (set of eigenvalues), which consists of the numbers 0 and 2. Therefore, the spectrum of a matrix does not tell us whether or not the matrix is irreducible.

It is clear that any positive matrix is irreducible. Any matrix of order 1 is considered to be reducible if and only if its (only) element equals zero.

We now list some of the properties of irreducible matrices. In those properties where a matrix is multiplied by a vector, that vector is a column vector. The products of row vectors and irreducible matrices have the same properties.

Given a matrix A and a natural number m, let $a_{ij}^{(m)}$ denote the element of the matrix A^m at position (i, j) (i.e., the position where row i intersects column j).

Theorem A.2. *Suppose a nonnegative square matrix A of order n is irreducible. Let x be a nonnegative vector in $\mathbf{R}^n$ and I_n be the identity matrix of order n. Then the following holds:*

(a) *A has no zero rows or columns,*
(b) *if all the component of x are positive, then all the components of the vector Ax are also positive,*
(c) *if x is a nonzero vector, but some of its coordinates equal zero, then the vector Ax has at least one positive coordinate, whose index corresponds to the index of a zero coordinate of x,*
(d) *if x has m positive coordinates and $0 < m < n$, then the number of positive coordinates of the vector $(I_n + A)x$ is strictly greater than m,*
(e) $(I_n + A)^{n-1} > 0$,
(f) *if $x \geqslant 0$ and $x \neq 0$, then it follows from the inequality $Ax \leqslant \alpha x$, where α is a scalar, that $\alpha > 0$ and $x > 0$,*
(g) *for any pair of indices (i, j), there exists a natural number m such that $a_{ij}^{(m)} > 0$,*
(h) *for any natural number m, the matrix A^m does not have zero rows or columns,*
(i) *if $a_{ij} = 0$, then there exists a sequence of indices $\{i, k, l, \ldots, r, s, j\}$ such that $a_{ik} > 0$, $a_{kl} > 0$, $\ldots$, $a_{rs} > 0$, $a_{sj} > 0$.*

Note that property (i) is a sufficient condition for a matrix A to be irreducible.

G. Frobenius generalized Perron's theorem to irreducible matrices.

Theorem A.3 (Frobenius' Theorem). *Let A be an irreducible nonnegative matrix; then A has a positive eigenvalue λ_A such that*

(a) *λ_A is a simple root of the characteristic equation of A, and*
(b) *the value of λ_A is not less than the absolute value of any other eigenvalue of A.*

The eigenvalue λ_A corresponds to a unique (with an accuracy of up to a scalar factor) positive eigenvector x_A.

Suppose A has m eigenvalues $\lambda_1, \ldots, \lambda_{m-1}, \lambda_m = \lambda_A$ such that the absolute value of any of them equals λ_A. Then all these numbers are distinct and the set of these numbers, when considered as a system of points in the complex plane, goes over into itself under a rotation of the plane by the angle $2\pi/m$, i.e., these numbers *are the roots of the equation* $\lambda^m - (\lambda_A)^m = 0$. *If* $m > 1$, *by simultaneously permuting its rows and columns, A can be put in the following "cyclic" form:*

$$A = \begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \cdots & A_{m-1,m} \\ A_{m1} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where there are zero square blocks along the main diagonal.

The vector x_A and the number λ_A that appear in this theorem are called the *Frobenius vector* and the *Frobenius eigenvalue* of A, respectively. When it is important to stress that x_A is a column vector, it is referred to as the *right Frobenius vector* of A. It is clear that, given an irreducible nonnegative matrix A, its eigenvalue λ_A also corresponds to a unique (with an accuracy of up to a scalar factor) positive left eigenvector (row vector) p_A: $p_A A = \lambda_A p_A$. The vector p_A is called the *left Frobenius vector* of A.

Theorem A.4. *Let A be an irreducible nonnegative matrix; then the Frobenius eigenvalue of A is the only eigenvalue of A that corresponds to a nonnegative eigenvector.*

Proof. Suppose to the contrary that, for some other eigenvalue λ, its eigenvector x is nonnegative. Multiplying the equality $Ax = \lambda x$ by the left Frobenius vector p_A, we obtain $\lambda_A \langle p_A, x \rangle = \lambda \langle p_A, x \rangle$. Since $p_A > 0$, we have $\langle p_A, x \rangle > 0$. This yields that, $\lambda = \lambda_A$. □

If we waive the requirement that A be irreducible in the Frobenius theorem, then the following statement holds.

Theorem A.5. *Let A be a nonnegative matrix; then A has a nonnegative eigenvalue* λ_A *such that the value of* λ_A *is not less than the absolute value of any other eigenvalue of A. The eigenvalue* λ_A *corresponds to a nonnegative eigenvector* x_A.

The vector x_A and the number λ_A that appear in this theorem are also called the Frobenius vector and the Frobenius eigenvalue of A, respectively.

Let us show that a nonnegative reducible matrix can have zero as the Frobenius eigenvalue.

Example A.4.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \lambda_A = 0, \qquad x_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad \square$$

On the other hand, a nonnegative matrix A can have a positive eigenvector even if A is reducible.

Example A.5.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},\ \lambda_A = 1,\ x_A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ \lambda_B = 1,\ x_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Here both A and B are reducible matrices. However, A has a positive eigenvector, whereas B does not. □

We now provide bounds for the intervals that contain the Frobenius eigenvalue of any nonnegative matrix. Given a nonnegative matrix A, let r_i denote the sum of the elements of its row i and let s_j denote the sum of the elements of its column j. Next, let

$$r = \min_{1\leqslant i\leqslant n} r_i, \quad R = \max_{1\leqslant i\leqslant n} r_i, \quad s = \min_{1\leqslant j\leqslant n} s_j, \quad S = \max_{1\leqslant j\leqslant n} s_j.$$

Theorem A.6. *Let A be a nonnegative matrix; then the following estimates hold for the Frobenius eigenvalue λ_A of A:*

$$r \leqslant \lambda_A \leqslant R, \qquad s \leqslant \lambda_A \leqslant S.$$

If A is irreducible, then all the above inequalities are strict, except the case where $r = R$ and $s = S$.

Theorem A.7. *For any nonnegative matrix A and any positive vector $x \in \mathbf{R}^n$, the following holds:*

$$\min_{1\leqslant i\leqslant n} \frac{1}{x_i} \sum_{j=1}^{n} a_{ij}x_j \leqslant \lambda_A \leqslant \max_{1\leqslant i\leqslant n} \frac{1}{x_i} \sum_{j=1}^{n} a_{ij}x_j, \tag{A.2}$$

$$\min_{1\leqslant j\leqslant n} x_j \sum_{i=1}^{n} \frac{a_{ij}}{x_i} \leqslant \lambda_A \leqslant \max_{1\leqslant j\leqslant n} x_j \sum_{i=1}^{n} \frac{a_{ij}}{x_i}.$$

Theorem A.8. *If a nonnegative matrix A has a positive eigenvector, then*

$$\lambda_A = \max_{x>0} \min_{1\leqslant i\leqslant n} \frac{1}{x_i} \sum_{j=1}^{n} a_{ij}x_j = \min_{x>0} \max_{1\leqslant i\leqslant n} \frac{1}{x_i} \sum_{j=1}^{n} a_{ij}x_j.$$

Theorem A.9. *Let A and B be two nonnegative matrices of the same order. If $A \leqslant B$, then the following holds for the Frobenius eigenvalues of A and B: $\lambda_A \leqslant \lambda_B$. If we have the strict inequality, namely, $A < B$, then the inequality between the Frobenius eigenvalues of A and B is also strict: $\lambda_A < \lambda_B$.*

The following result concerns the convergence of the power series of irreducible matrices.

Theorem A.10. *Let A be an irreducible nonnegative matrix, I_n the identity matrix of order n. Suppose the Frobenius eigenvalue λ_A of A satisfies the inequality $\lambda_A < 1$; then we have*

$$\lim_{t\to\infty} A^t = 0 \tag{A.3}$$

and

$$\sum_{t=0}^{\infty} A^t = (I_n - A)^{-1}. \tag{A.4}$$

Proof. For the Frobenius eigenvector x_A, we have

$$\lim_{t\to\infty} A^t x_A = \lim_{t\to\infty} \lambda_A^t x_A,$$

Then, since $x_A > 0$ and $A^t \geqslant 0$, we obtain (A.3).

Formula (A.4) follows immediately from (A.3) and the following relation:

$$(I_n - A)(I_n + A + A^2 + \cdots + A^{t-1}) = I_n - A^t. \quad \square$$

Irreducible nonnegative matrices are classified by the number of eigenvalues whose absolute value is maximum. If a matrix A has only one such eigenvalue ($m = 1$ in the formulation of Theorem A.3), then A is called a *primitive* matrix. If $m > 1$, then A is called an *imprimitive* matrix. The value m is called the *index of imprimitivity* of the matrix A.

Note that the primitivity of any given nonnegative matrix depends only on zero elements' positions and not on the absolute values of its positive elements.

We now recall a criterion that allows us to check whether or not a matrix is primitive without calculating its eigenvalues.

Theorem A.11. *A nonnegative matrix A is primitive if and only if we have $A^t > 0$ for some $t \geqslant 1$.*

There exists an upper bound on the powers to be calculated when using this criterion.

Theorem A.12. *Let A be a nonnegative matrix of order n. If A is primitive, then $A^t > 0$ for some positive integer $t \leqslant (n-1)n^n$.*

H. Wielandt obtained a precise upper estimate.

Theorem A.13 (Wielandt's Theorem). *Let A be a nonnegative matrix of order n; then A is primitive if and only if $A^{n^2-2n+2} > 0$.*

The form of the main diagonal of a given matrix can also tell us whether or not the matrix is primitive.

Theorem A.14. *Let A be an irreducible nonnegative matrix of order n. If the main diagonal of A is positive, then $A^{n-1} > 0$.*

In Example A.2, an irreducible matrix has a reducible power. By contrast, any power of a primitive matrix is also primitive.

Theorem A.15. *Let A be a primitive nonnegative matrix; then the matrix A^t is nonnegative, irreducible, and primitive for all $t = 1, 2, \ldots$.*

Suppose A is a primitive matrix. Let us now consider the behavior of its powers A^t as $t \to \infty$.

Theorem A.16. *Let A be a primitive nonnegative matrix; then*

$$\lim_{t\to\infty} \left(\frac{A}{\lambda_A}\right)^t = L, \tag{A.5}$$

where the square matrix L of order n is the product of (positive) left and right Frobenius vectors of A: $L = x_A p_A$. The scalar product of these vectors must satisfy the following condition: $\langle p_A, x_A\rangle = 1$.

Let us show that in this theorem, the hypothesis of primitivity is essential.

Example A.6. Consider the following imprimitive irreducible matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

whose Frobenius eigenvalue λ_A equals one. For integer $t \geqslant 1$ we have:

$$A^t = \begin{cases} A, & \text{for odd } t; \\ I_2, & \text{for even } t, \end{cases}$$

i.e., the limit specified in the theorem does not exist. □

However, the normalized powers of any imprimitive matrix do *converge in mean*.

Theorem A.17. *Let A be an irreducible nonnegative matrix; then we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \left(\frac{A}{\lambda_A}\right)^t = L, \tag{A.6}$$

where the matrix L is as in Theorem A.16.

We now estimate the rate of convergence in (A.5) and (A.6). Recall that, when we need to measure the "size" (Horn and Johnson 1985) of a vector or matrix, we use a uniquely defined nonnegative number called a *norm*. The function that returns this number must satisfy a number of axioms that reflect the usual properties of the length of line segments in a plane. For example, the following functions are often chosen as norms for column vectors $x = [x_1, \dots, x_n]^T$ in the n-dimensional space $\mathbf{R}^n$:

$$\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2} \quad \text{(the } \textit{Euclidean norm}\text{, or the } l_2 \textit{ norm}\text{)},$$
$$\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i| \quad \text{(the } l_\infty \textit{ norm}\text{)}. \tag{A.7}$$

For a square matrix A of order n, its *Euclidean norm*, or l_2 *norm*, is defined as

$$\|A\|_{l_2} = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2} \tag{A.8}$$

and its l_∞ *norm*, is defined as

$$\|A\|_{l_\infty} = \max_{1 \leqslant i,j \leqslant n} |a_{ij}|. \tag{A.9}$$

Although the argument of function (A.9) is a matrix, this norm is a *vector norm*. For a norm to be called a *matrix norm*, it must additionally be *submultiplicative*: $\|AB\| \leqslant \|A\|\|B\|$. Norm (A.8) is a matrix norm. By contrast, norm (A.9) is not a matrix norm, because it is not submultiplicative, as the following example shows:

$$A = B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{where} \quad \|A^2\|_{l_\infty} = 2, \quad \|A\|_{l_\infty} = 1.$$

Note that, if we consider square matrices of order n as linear operators on $\mathbf{R}^n$, we can use any vector norm $\|\cdot\|_v$ on $\mathbf{R}^n$ to obtain an *induced (operator)* matrix norm $\|\cdot\|_{ind}$, as follows:

$$\|A\|_{ind} = \max_{\|x\|_v = 1} \|Ax\|_v.$$

For example, the l_∞ norm given by (A.7) induces a matrix norm $\|\cdot\|_\infty$ called the *maximum row sum matrix norm* and defined as follows:

$$\|A\|_\infty = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |a_{ij}|.$$

Recall that, for any irreducible matrix A, its left Frobenius vector (row vector) p_A is positive. When dealing with irreducible matrices, it is often convenient to use a norm based on this vector. We can define this norm on the space $\mathbf{R}^n$ as follows:

$$\|x\|_A = \langle p_A, |x| \rangle, \tag{A.10}$$

where $|x| = [|x_1|, \ldots, |x_n|]^T$. The row vector p_A is usually chosen so that $\|p_A^T\|_A = 1$.

We can now write out the rate of convergence in (A.5). Given a primitive matrix A, let λ_{n-1} denote the second largest (in terms of its absolute value) eigenvalue of A, i.e., for any eigenvalue λ of A such that $\lambda \neq \lambda_A$, the following holds: $|\lambda| \leqslant |\lambda_{n-1}|$.

Theorem A.18. *Let A be a primitive nonnegative matrix. Let r be any number that satisfies the following inequality: $|\lambda_{n-1}|/\lambda_A < r < 1$. Then there exists a constant $C = C(r, A)$ such that*

$$\left\| \left(\frac{A}{\lambda_A} \right)^t - L \right\|_{l_\infty} \leqslant C r^t \tag{A.11}$$

for all $t = 1, 2, \ldots$.

For the normalized powers of irreducible matrices, the rate of convergence in mean given by (A.6) is as follows:

$$\left\| \frac{1}{N} \sum_{t=1}^{N} \left(\frac{A}{\lambda_A} \right)^t - L \right\|_{l_\infty} \leqslant \frac{C}{N},$$

where $C = C(A)$ is some positive constant.

References

Ashmanov SA (1984) Vvedenie v Matematicheskuju Ekonomiku (Introduction to mathematical economics). Nauka, Moscow
Bellman RE (1970) Introduction to matrix analysis, 2nd edn. McGraw-Hill, New York
Gantmacher FR (1959) The theory of matrices. Chelsea Publishing, Chelsea/New York
Horn RA, Johnson CR (1985) Matrix analysis. Cambridge University Press, Cambridge
Lancaster P (1969) Theory of matrices. Academic, New York
Marcus M, Minc H (1992) A survey of matrix theory and matrix inequalities. Dover, New York

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A.P. Abramov, *Balanced and Cyclical Growth in Models of Decentralized Economy*,
Lecture Notes in Economics and Mathematical Systems 672,
DOI 10.1007/978-3-319-07917-2,