Fractals, Self-similarity and Hausdorff Dimension

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The URL for these Beamer Slides: *"Fractals: self similar fractional dimensional sets"*

`http://www.math.utah.edu/~treiberg/FractalSlides.pdf`

- Michael Barnsley, "Lecture Notes on Iterated Function Systems," in Robert Devaney and Linda Keen, eds., *Chaos and Fractals,* Proc. Symp. **39**, Amer. Math. Soc., Providence, 1989, 127–144.
- Gerald Edgar, *Classics on Fractals,* Westview Press, Studies in Nonlinearity, Boulder, 2004.
- Jenny Harrison, "An Introduction to Fractals," in Robert Devaney and Linda Keen, eds., *Chaos and Fractals,* Proc. Symp. **39**, Amer. Math. Soc., Providence, 1989, 107–126.
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- R. Clark Robinson, *An Introduction to Dynamical Systems: Continuous and Discrete,* Pearson Prentice Hall, Upper Saddle River, 2004.
- Shlomo Sternberg, *Dynamical Systems,* Dover, Mineola, 2010.

4. Outline.

- Fractals
  - Middle Thirds Cantor Set Example
- Attractor of Iterated Function System
  - Cantor Set as Attractor of Iterated Function System.
  - Contraction Maps
  - Complete Metric Space of Compact Sets with Hausdorff Distance
  - Hutchnson's Theorem on Attractors of Contracting IFS
  - Examples: Unequal Scaling Cantor Set, Sierpinski Gasket, von Koch Snowflake, Barnsley Fern, Minkowski Curve, Peano Curve, Lévy Dragon
- Hausdorff Measure and Dimension
  - Dimension of Cantor Set by Covering by Intervals
- Similarity Dimension
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  - Moran's Theorem
  - Similarity Dimensions of Examples
- Kiesswetter's IFS Construction of Nowhere Differentiable Function

A fractal is a set with fractional dimension. A fractal need not be self-similar. In this lecture we construct self-similar sets of fractional dimension. The most basic fractal is the Middle Thirds Cantor Set. One starts from an interval $I_1 = [0, 1]$ and at each successive stage, removes the middle third of the intervals remaining in the set.

$$\begin{aligned}
I_2 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\
I_3 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\
I_4 &= \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \\
&\quad \cup \left[\frac{2}{3}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{8}{27}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right] \\
&\quad \ldots
\end{aligned}$$

Then the Cantor Set is the limit $C = \bigcap_{n=1}^{\infty} I_n$.

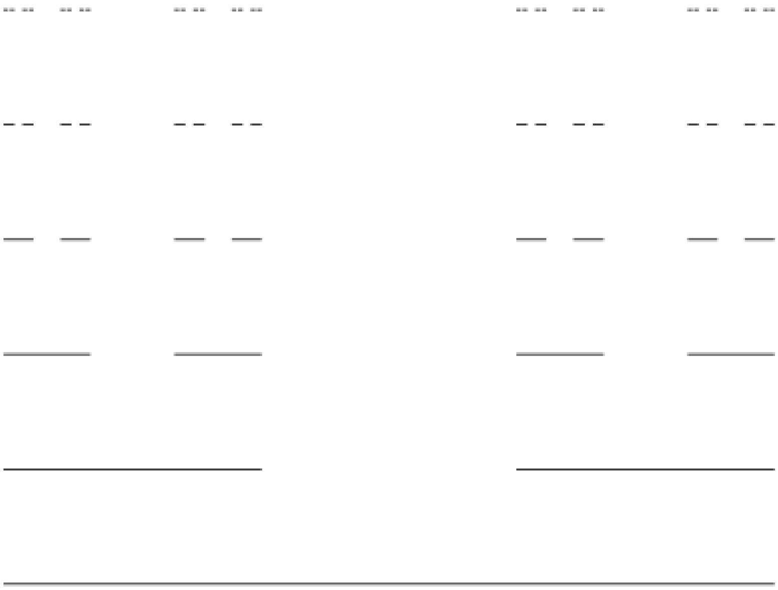
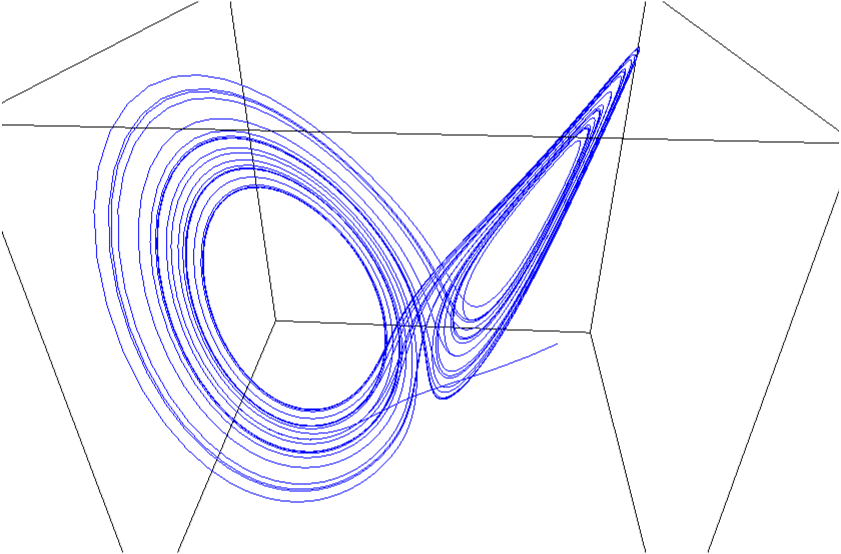


Figure: The sequence $\{I_n\}$ approximating the middle thirds Cantor Set.



"Butterfly" ODE limit set is a non self-similar fractal $1 < \dim_H(A) < 2$

The Cantor Set may be constructed using Iterated Function Systems. The IFS is given by two maps on the line, $\mathcal{F} = \{\ell, r\}$, where

$$\ell(x) = \frac{x}{3}; \qquad r(x) = \frac{x+2}{3}.$$

ℓ and r make two shrunken copies of the original interval located at the left and right ends. Define the induced union map taking compact sets $A \subset \mathbb{R}$ to new compact sets consisting of both shrunken copies

$$\mathcal{F}(A) = \ell(A) \cup r(A)$$

where $\ell(A) = \{\ell(x) : x \in A\}$. Consider the dynamical system of iterating the maps. We get the Cantor Set as its its attractor (limit)

$$I_2 = \mathcal{F}(I_1), \quad I_3 = \mathcal{F}(I_2), \quad \dots, \quad C = \lim_{n\to\infty} \mathcal{F}^{\circ n}(I_1)$$

where we define $\mathcal{F} \circ \mathcal{F}(A) = \mathcal{F}(\mathcal{F}(A))$ and

$$\mathcal{F}^{\circ n}(A) = \overbrace{\mathcal{F} \circ \mathcal{F} \circ \cdots \circ \mathcal{F}}^{n \text{ times}}(A)$$

Why does the sequence of sets converge? Let us put the structure of a metric space on the space of compact sets and do a little analysis.

For example, the distance function d on Euclidean Space $X = \mathbb{E}^n$ is

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}.$$

Euclidean Space has the structure of a metric space, namely, for all $x, y, z \in X$ we have

- $d(x, x) = 0$, $d(x, y) = d(y, x)$,
- $d(x, z) \leq d(x, y) + d(y, z)$ triangle inequality (which implies $d(x, x) \geq 0$),
- $d(x, y) = 0$ implies $x = y$.

$\{x_i\} \subset \mathbb{E}^n$ is a Cauchy Sequence if for every $\epsilon > 0$ there is an N such that

$$d(x_i, x_j) < \epsilon \qquad \text{whenever } i, j \geq N.$$

Euclidean Space is a complete metric space because all Cauchy Sequences converge. Namely, if $\{x_i\}$ is a Cauchy Sequence, then there is $z \in \mathbb{E}^n$ such that $x_i \to z$ as $i \to \infty$, *i.e.*, for all $\epsilon > 0$, there is $N > 0$ such that

$$d(x_i, z) < \epsilon \qquad \text{whenever } i > N.$$

A set K is compact if every sequence $\{x_i\} \subset K$ has a subsequence that converges to a point of K. In Euclidean Space, $K \in \mathbb{E}^n$ is compact if and only if it is closed and bounded (Heine Borel Theorem).

Surprisingly, the space $\mathcal{K}(\mathbb{E}^n)$ of all compact sets $\mathbb{E}^n$ and can be endowed with the structure of a complete metric space under the Hausdorff Metric.

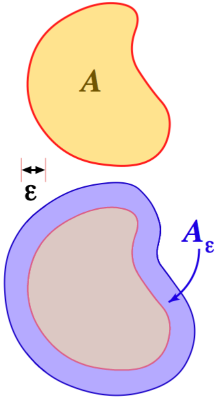


Figure: ϵ-Collar of A.

Let $\mathcal{K}(\mathbb{E}^n)$ denote the nonempty compact subsets. For any $A \in \mathcal{K}(\mathbb{E}^n)$ and $\epsilon \geq 0$ define the the ϵ-collar of A to be points within ϵ of A

$$A_\epsilon = \{x \in \mathbb{E}^n : d(x,y) \leq \epsilon \text{ for some } y \in A\}.$$

The distance of a point x to A is

$$d(x,A) = \inf_{y \in A} d(x,y).$$

It is zero if $x \in A$. The ϵ-collar may also be given

$$A_\epsilon = \{x \in \mathbb{E}^n : d(x,A) \leq \epsilon\}.$$

The infimum is achieved since A is compact. There is a $y \in A$ so that

$$d(x,y) = d(x,A).$$

Given compact sets $A, B \in \mathcal{K}(\mathbb{E}^n)$, if we let

$$d(A, B) = \max_{x \in A} d(x, B).$$

$d(A, B) \le \epsilon$ implies that $A \subset B_\epsilon$.

BUT $d(A, B)$ MAY NOT EQUAL $d(B, A)$ so it is not a metric. *e.g.*, $A = \{x \in \mathbb{E}^2 : |x| \le 1\}$, $B = \{(2, 0)\}$ then $d(B, A) = 1$ so $B \subset A_1$ but $d(A, B) = 3$ and $A \not\subset B_1$.

Hausdorff introduced

$$h(A, B) = \max\{d(A, B), d(B, A)\} = \inf\{\epsilon \ge 0 : A \subset B_\epsilon \text{ and } B \subset A_\epsilon\}$$

Theorem (Completeness of $\mathcal{K}(\mathbb{E}^n)$)

$\mathcal{K}(\mathbb{E}^n)$ with Hausdorff Distance h is a complete metric space. Furthermore, h satisfies for all $A, B, C, D \in \mathcal{K}(\mathbb{E}^n)$

$$h(A \cup B, C \cup D) \le \max\{h(A, C), h(B, D)\}$$

Proof. Symmetry ($h(A, B) = h(B, A)$) and positive definiteness ($h(A, B) \geq 0$ with $h(A, B) = 0 \iff A = B$) are obvious. To prove the triangle inequality it suffices to show

$$d(A, B) \leq d(A, C) + d(C, B).$$

This implies the triangle inequality for h:

$$\begin{aligned} h(A, B) &= \max\{d(A, B), d(B, A)\} \\ &\leq \max\{d(A, C) + d(C, B), d(B, C) + d(C, A)\} \\ &\leq \max\{h(A, C) + h(C, B), h(B, C) + h(C, A)\} \\ &= h(A, C) + h(C, B). \end{aligned}$$

Now to show $d(A, B) \leq d(A, C) + d(C, B)$,

$$\begin{aligned} d(a, B) &= \min_{b \in B} d(a, b) \\ &\leq \min_{c \in C} \min_{b \in B} (d(a, c) + d(c, b)) \\ &\leq \min_{c \in C} d(a, c) + \min_{c \in C} \min_{b \in B} d(c, b) \\ &\leq d(a, C) + \min_{c \in C} d(c, B) \\ &\leq d(a, C) + \min_{c \in C} d(C, B) \\ &\leq d(a, C) + d(C, B) \end{aligned}$$

Maximizing the right side over $a \in A$ gives

$$d(a, B) \leq d(A, C) + d(C, B)$$

Maximizing over $a \in A$,

$$d(A, B) \leq d(A, C) + d(C, B).$$

Sketch of completeness argument: suppose A_n is a Cauchy Sequence in $(\mathcal{K}(X), h)$. Define A_∞ to be the set of cluster points of sequences $\{x_n\}$ where $x_n \in A_n$. Thus $x \in A_\infty$ if and only if there is a subsequence of this type such that $x_{k_j} \to x$ as $j \to \infty$. Since the sets form a Cauchy Sequence, for every $\epsilon > 0$ there is an $R(\epsilon)$ so that $h(A_n, A_m) < \epsilon$ whenever $m, n \geq R(\epsilon)$. In particular, $A_m \subset (A_n)_\epsilon$ for all $m \geq n \geq R(\epsilon)$ so any sequence $x_m \in A_m$ is bounded and thus has a cluster point showing A_∞ is nonempty. Limits satisfy $A_\infty \subset (A_n)_\epsilon$ for all $n \geq R(\epsilon)$, hence A_∞ is bounded. A convergent sequence of cluster points is a cluster point, so A_∞ is closed, thus A_∞ is compact.

To show that $A_n \subset (A_\infty)_\epsilon$ whenever $n \geq R(\epsilon)$, pick $z_n \in A_n$. For $k \geq R(\epsilon)$, $h(A_n, A_k) < \epsilon$, so there is $x_k \in A_k$ so $d(x_k, z_n) < \epsilon$. Let $z \in A_\infty$ be a cluster point of $\{x_k\}$. For its converging subsequence $d(z, z_m) = \lim_{j\to\infty} d(x_{k_j}, z_m) \leq \epsilon$ so $z_m \in (A_\infty)_\epsilon$.

Putting the containments together shows $h(A_m, A_\infty) \leq \epsilon$ for all $m \geq R(\epsilon)$, thus A_m converges to A_∞ in the Hausdorff metric. $\square$

A mapping $f : \mathbb{E}^n \to \mathbb{E}^n$ is a λ-contraction if there is a constant $0 \le \lambda < 1$ such that

$$d(f(x), f(y)) \le \lambda d(x, y), \qquad \text{for all } x, y \in \mathbb{E}^n.$$

Lemma

If $f : \mathbb{E}^n \to \mathbb{E}^n$ is a λ-contraction, then the induced map on $\mathcal{K}(\mathbb{E}^n)$ is a contraction in the Hausdorff Metric with the same constant

$$h(f(A), f(B)) \le \lambda h(A, B), \qquad \text{for all } A, B \in \mathcal{K}(\mathbb{E}^n) \ .$$

Proof. Choose $A, B \in \mathcal{K}(\mathbb{E}^n)$.

$$d(f(A), f(B)) = \max_{a \in A} d(f(a), f(B)) \le \lambda \max_{a \in A} d(a, B) = \lambda d(A, B).$$

Similarly, $d(f(B), f(A)) \le \lambda d(B, A)$. Combining,

$$\begin{aligned} h(f(A), f(B)) &= \max\{d(f(A), f(B)), d(f(B), f(A))\} \\ &\le \lambda \max\{d(A, B), d(B, A)\} = \lambda h(A, B). \end{aligned}$$ □

Lemma (Hutchinson 1981)

Let $f_1, \dots, f_k : \mathbb{E}^n \to \mathbb{E}^n$ be an IFS of contractions with constants λ_k. Then the induced union map on $\mathcal{K}(\mathbb{E}^n)$ given for $A \in \mathcal{K}(\mathbb{E}^n)$ by

$$\mathcal{F}(A) = f_1(A) \cup f_2(A) \cup \cdots \cup f_k(A)$$

is a contraction with the constant $\lambda = \max\{\lambda_1, \dots, \lambda_k\}$.

Proof. Choose $A, B \in \mathcal{K}(\mathbb{E}^n)$. Since a point is closer to a union of sets than to any one set in the union,

$$d(\mathcal{F}(A),\mathcal{F}(B)) = d\left(\cup_{i=1}^k f_i(A), \cup_{j=1}^k f_j(B)\right) = \max_{1\le i\le k}\left\{d(f_i(A), \cup_{j=1}^k f_j(B))\right\}$$

$$\le \max_{1\le i\le k}\{d(f_i(A), f_i(B))\} \le \max_{1\le i\le k}\{\lambda_i d(A,B))\} \le \lambda d(A,B).$$

Similarly, $d(\mathcal{F}(B), \mathcal{F}(A)) \le \lambda d(B, A)$. Combining as before $h(\mathcal{F}(A), \mathcal{F}(B)) \le \lambda h(A, B)$. □

One of the ten basic facts every math major must know.

Theorem (Contraction Mapping)

Let (X, d) be a complete metric space and $f : X \to X$ be a contraction. Then there is a unique fixed point $x_\infty \in X$ such that $f(x_\infty) = x_\infty$.

In fact, x_∞ may be found by iteration. Starting from any $x_0 \in X$, define the sequence $x_1 = f(x_0)$, $x_2 = f(x_1)$, $\ldots$, $x_{n+1} = f(x_n)$, $\ldots$. Then one shows that the sequence converges to a unique point

$$x_\infty = \lim_{n\to\infty} x_n. \quad \square$$

Applying this to iterated function systems, if $\mathcal{F} : \mathcal{K}(\mathbb{E}^n) \to \mathcal{K}(\mathbb{E}^n)$ is a contraction then there is a unique invariant set $A_\infty \in \mathcal{K}(\mathbb{E}^n)$ such that $\mathcal{F}(A_\infty) = A_\infty$. It is found as the unique attractor for the dynamical system $\mathcal{F} : \mathcal{K}(\mathbb{E}^n) \to \mathcal{K}(\mathbb{E}^n)$. For any nonempty compact set S,

$$A_\infty = \lim_{n\to\infty} \mathcal{F}^{\circ n}(S).$$

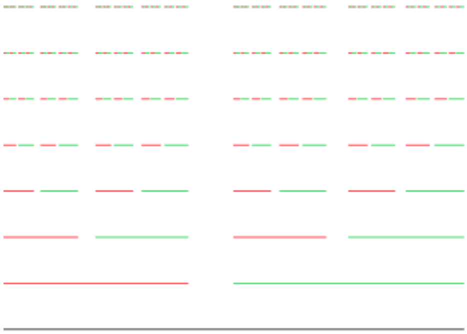


Figure: Cantor Set with Unequal Intervals

This Cantor set is obtained from IFS $\mathcal{F} = \{f_1, f_2\}$ on $\mathbb{R}$ where

$$f_1(x) = .4x,$$
$$f_2(x) = .5x + .5$$

Each f_i's are contractions with $\lambda_1 = .4$ and $\lambda_2 = .5$.

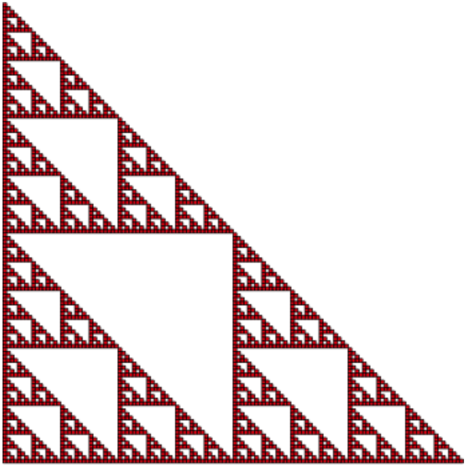


Figure: Sierpinski Gasket

The Sierpinski Gasket is obtained from IFS $\mathcal{F} = \{f_1, f_2, f_3\}$ where

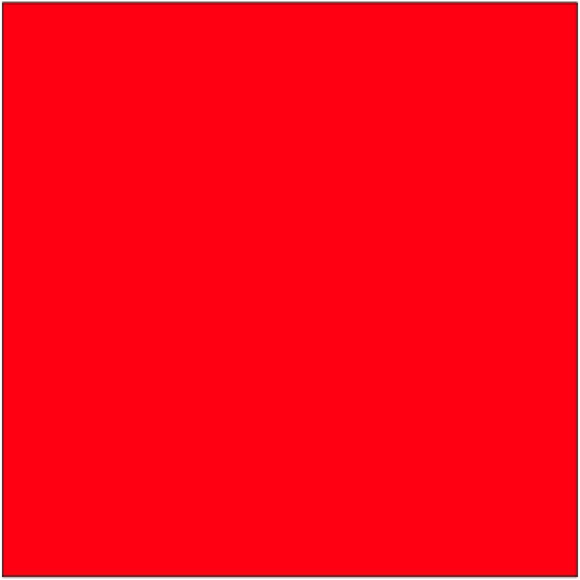
$$f_1(x) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$f_2(x) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix},$$

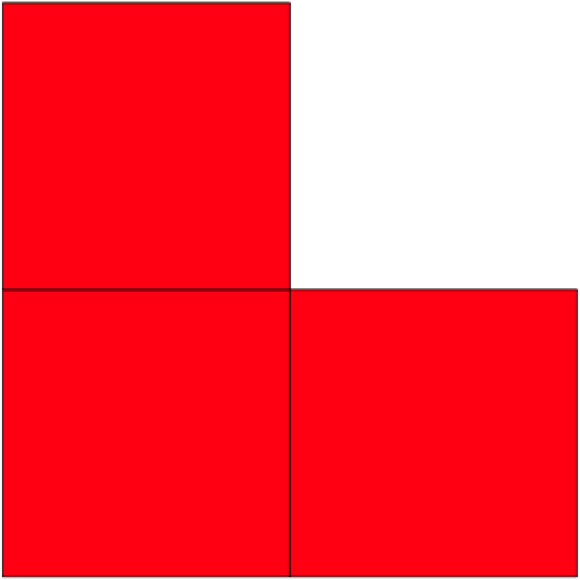
$$f_3(x) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}.$$

Each f_i is a contraction with $\lambda = \frac{1}{2}$.

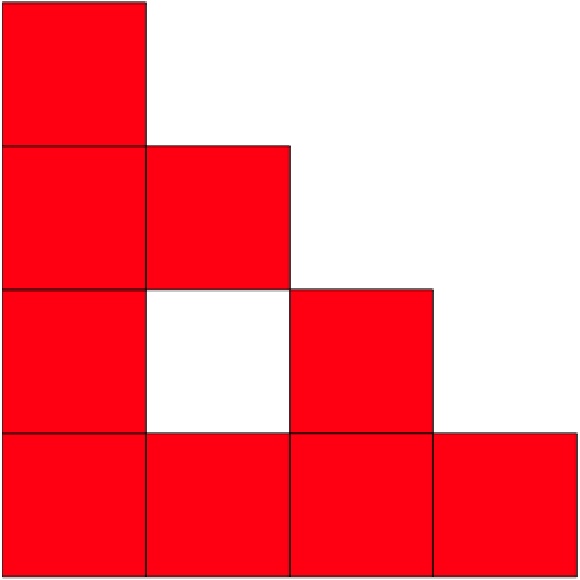
21. Sierpinski Gasket 0.



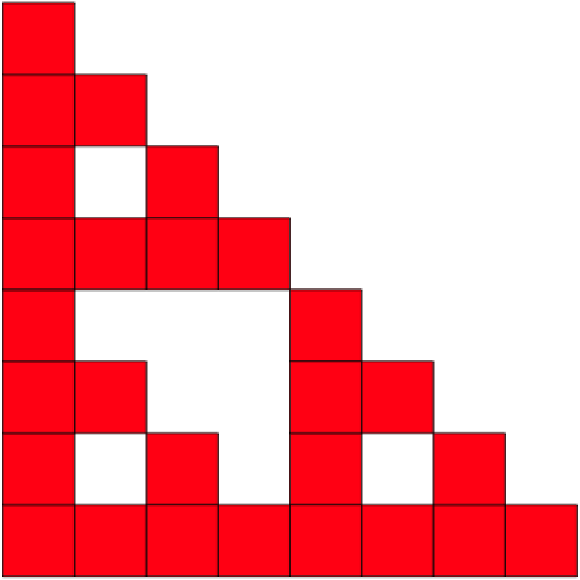
22. Sierpinski Gasket 1.



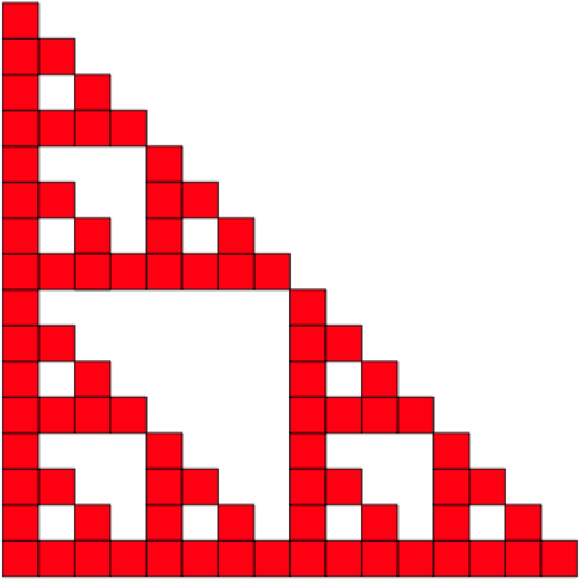
23. Sierpinski Gasket 2.



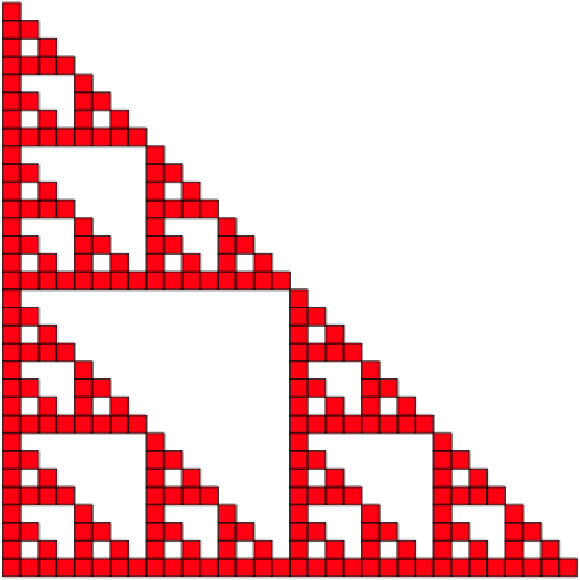
24. Sierpinski Gasket 3.



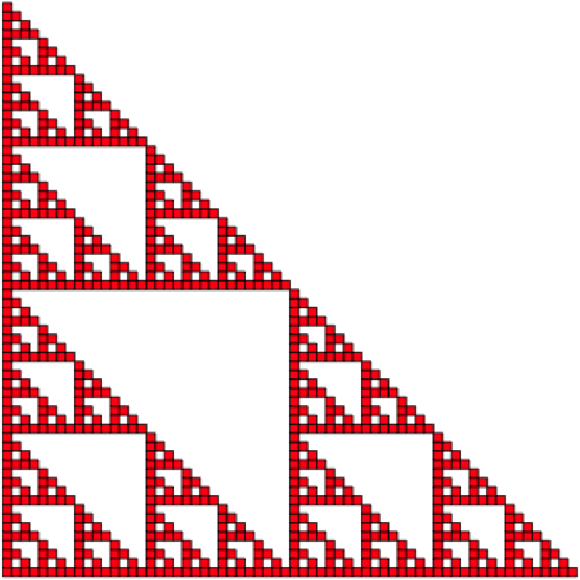
25. Sierpinski Gasket 4.



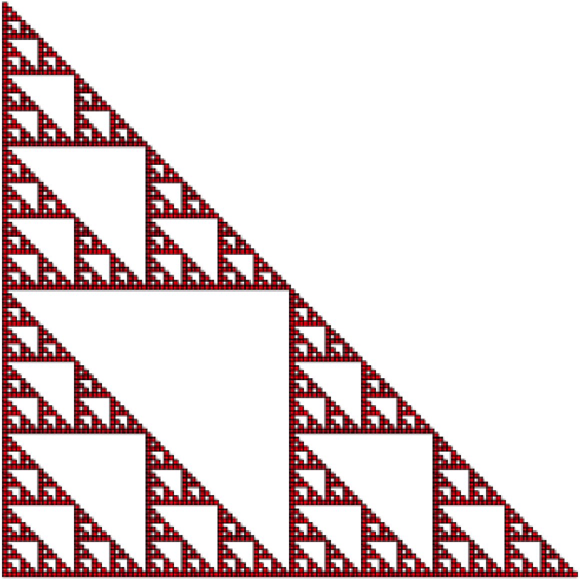
26. Sierpinski Gasket 5.



27. Sierpinski Gasket 6.



28. Sierpinski Gasket 7.



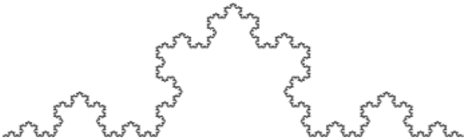


Figure: One of Three Sides of the Snowflake

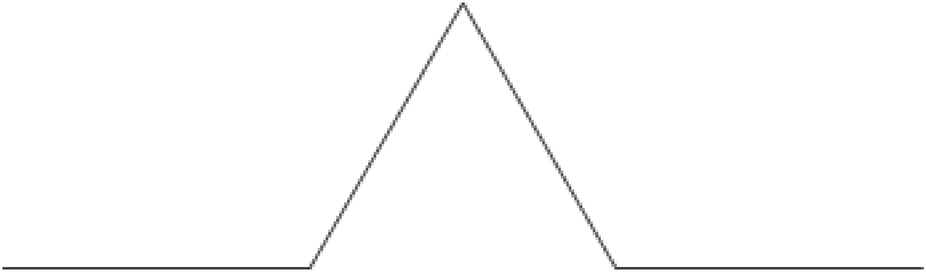
Helge von Koch (1870–1924) was a Swedish mathematician who studied systems of infinitely many linear equations. He used pictures and geometric language in the 1904 paper to construct his curve as an example of a non-differentiable curve. Weierstrass's 1872 description of such a curve used only formulas.

The von Koch Curve is obtained from IFS $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$ where in complex notation $z = x + iy$,

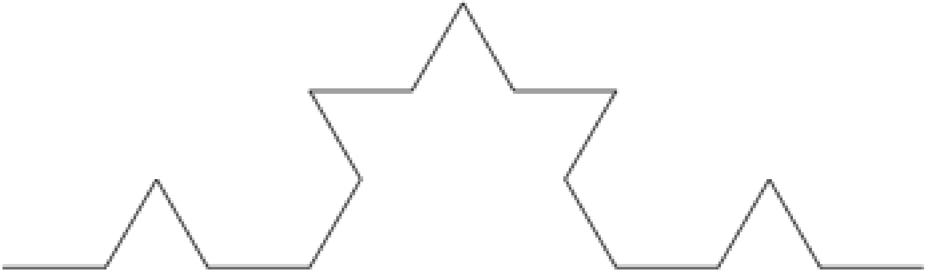
$$
\begin{aligned}
f_1(z) &= \frac{1}{3}z, \\
f_2(z) &= \frac{e^{\pi i/3}}{3}z + \frac{1}{3} \\
f_3(z) &= \frac{e^{-\pi i/3}}{3}z + \frac{e^{\pi i/3} + 1}{3} \\
f_4(z) &= \frac{1}{3}z + \frac{2}{3}.
\end{aligned}
$$

Each contraction has $\lambda = \frac{1}{3}$.

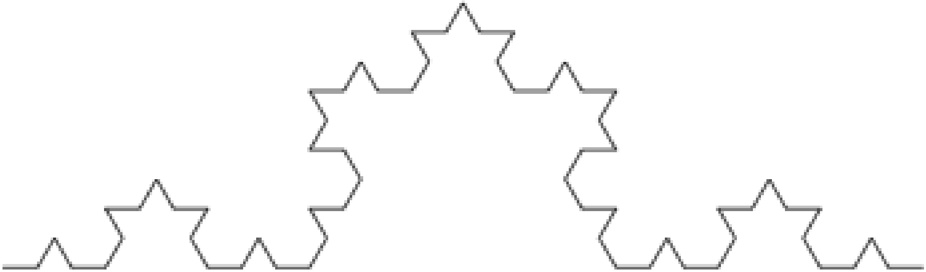
30. von Koch Curve 1.



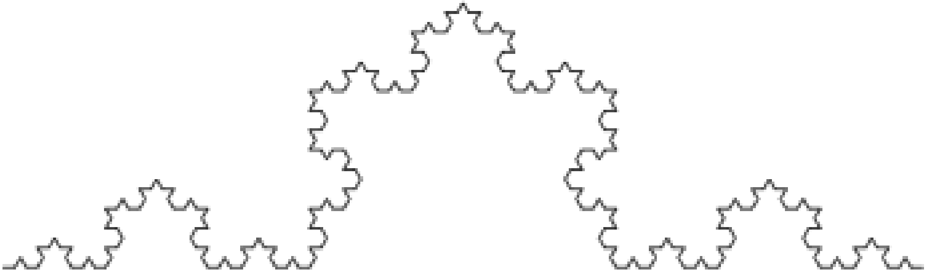
31. von Koch Curve 2.



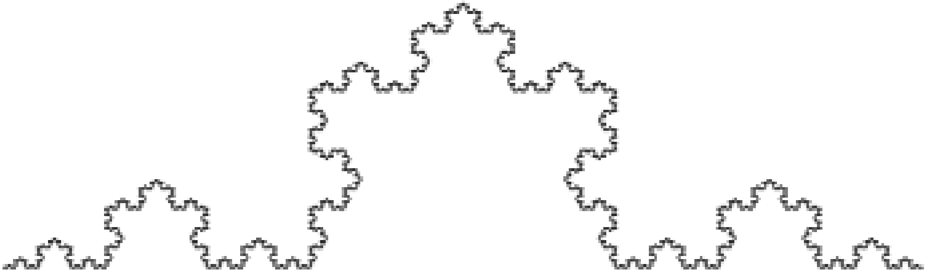
32. von Koch Curve 3.

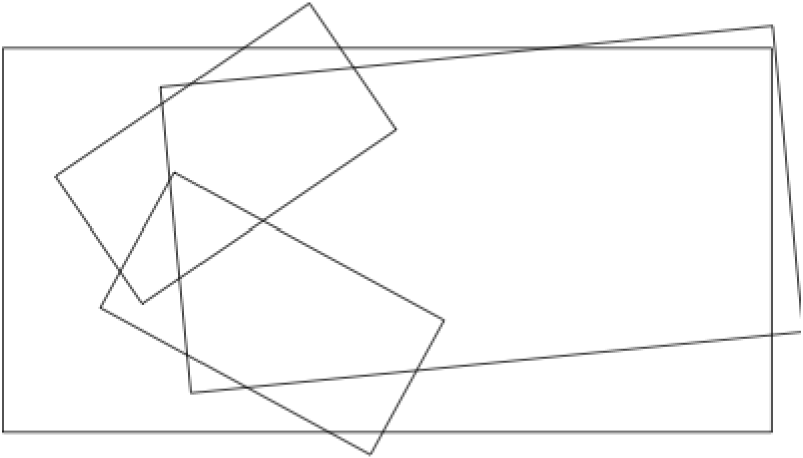


33. von Koch Curve 4.



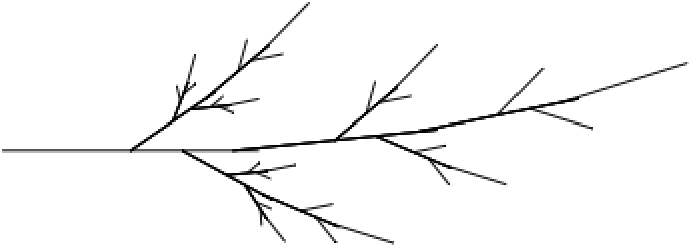
34. von Koch Curve 5.



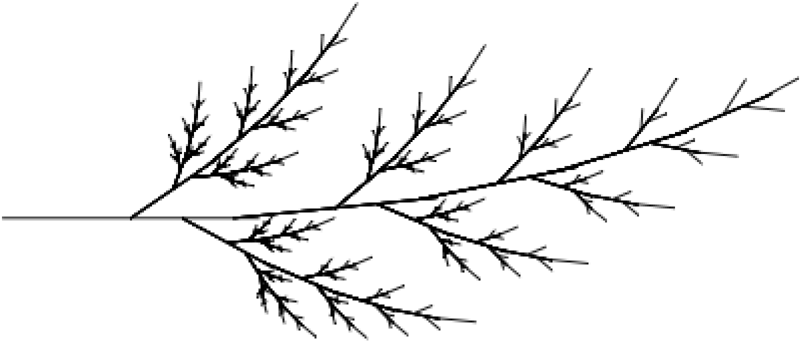


Images of Big Rectangle under $\mathcal{F} = \{f_1, f_2, f_3\}$.

36. Barnsley Fern $\mathcal{F}^{\circ 2}$



37. Barnsley Fern $\mathcal{F}^{\circ 4}$



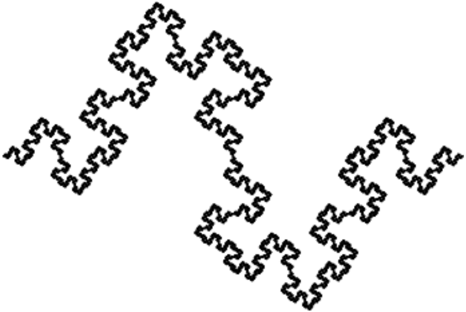


Figure: Minkowski Curve

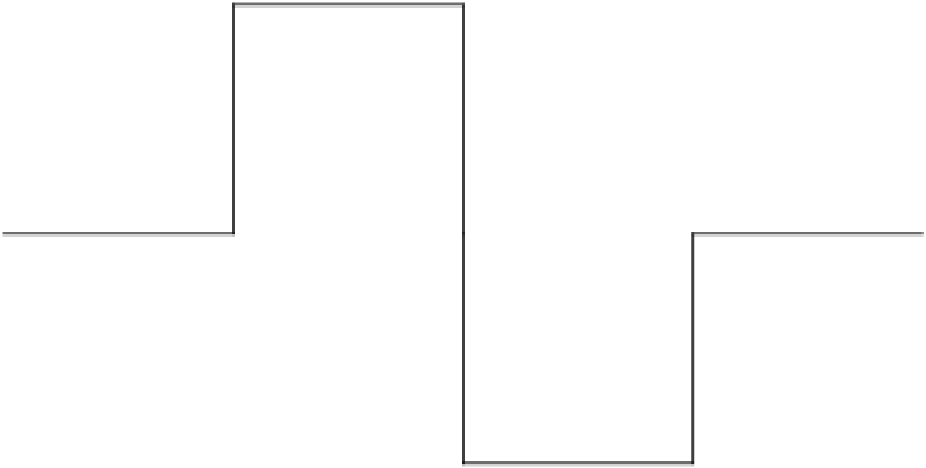
The downward line in the middle consists of two segments of length $\frac{1}{4}$.

The Minkowski Curve is obtained from IFS $\mathcal{F} = \{f_1, \dots, f_8\}$ where

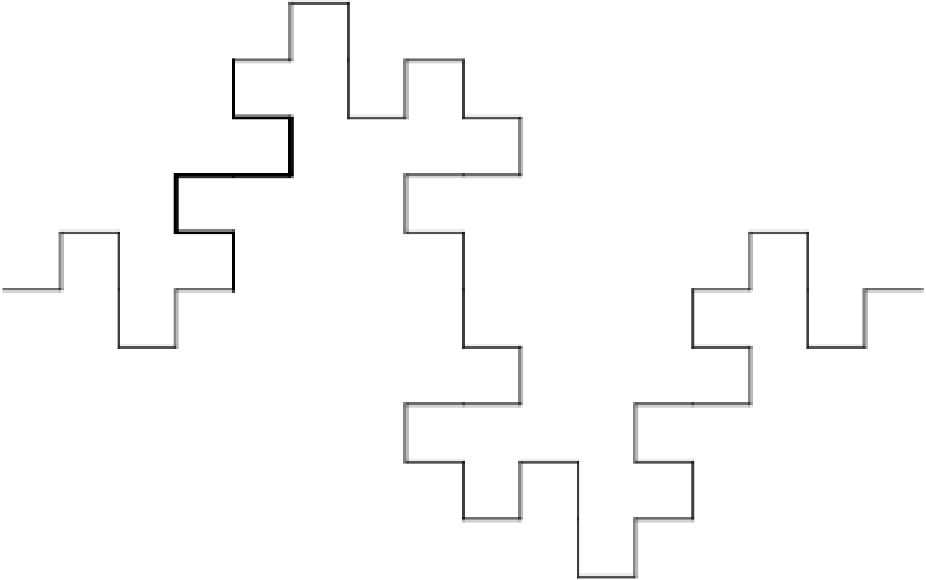
$$f_1(z) = \tfrac{1}{4}z,$$
$$f_2(z) = \tfrac{i}{4}z + \tfrac{1}{4}$$
$$f_3(z) = \tfrac{1}{4}z + \tfrac{1+i}{4}$$
$$f_4(z) = -\tfrac{i}{4}z + \tfrac{2+i}{4}$$
$$f_5(z) = -\tfrac{i}{4}z + \tfrac{1}{2}$$
$$f_6(z) = \tfrac{1}{4}z + \tfrac{2-i}{4}$$
$$f_7(z) = \tfrac{i}{4}z + \tfrac{3-i}{4}$$
$$f_8(z) = \tfrac{i}{4}z + \tfrac{3}{4}$$

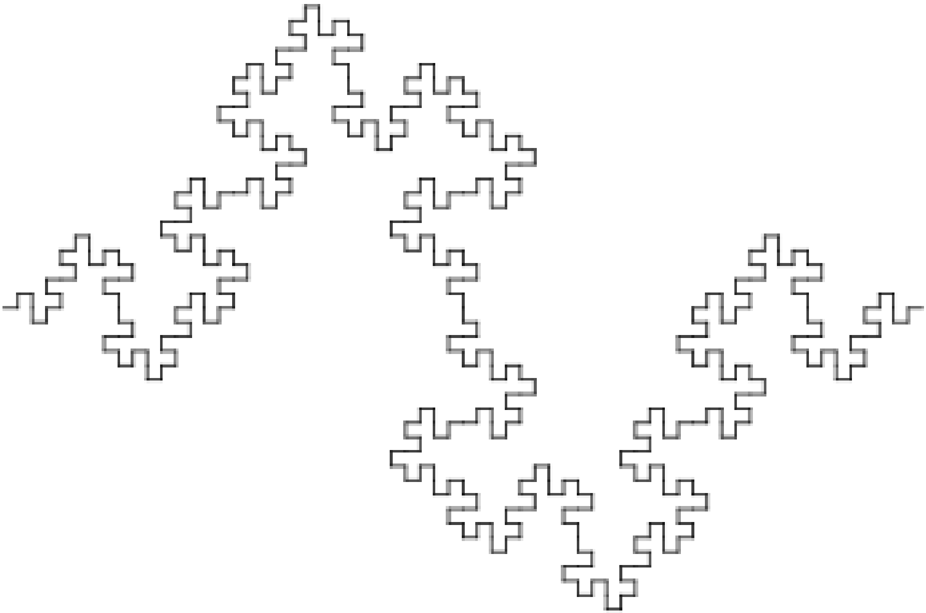
All $\lambda_i = \frac{1}{4}$.

39. Minkowski Curve 1.

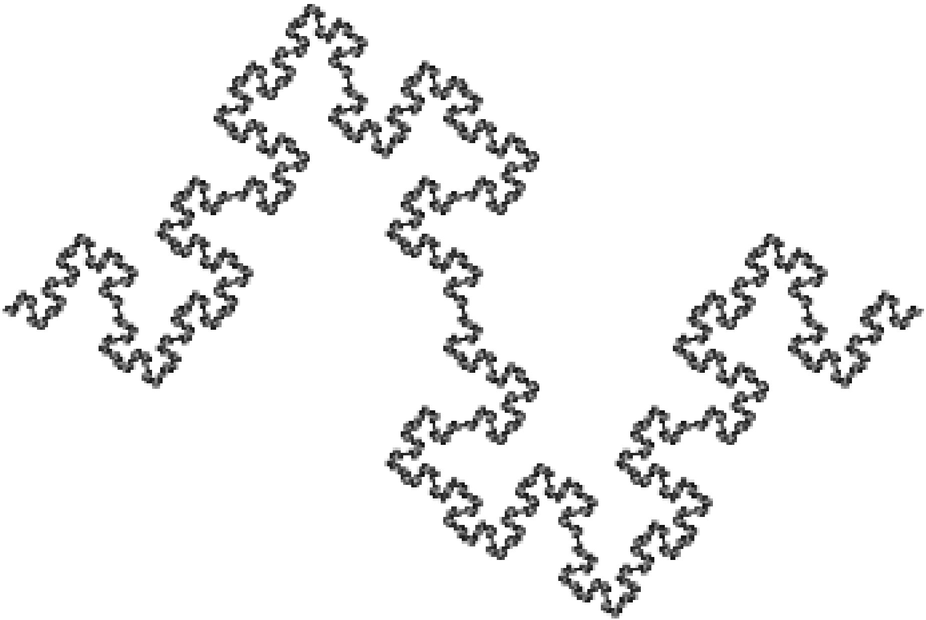


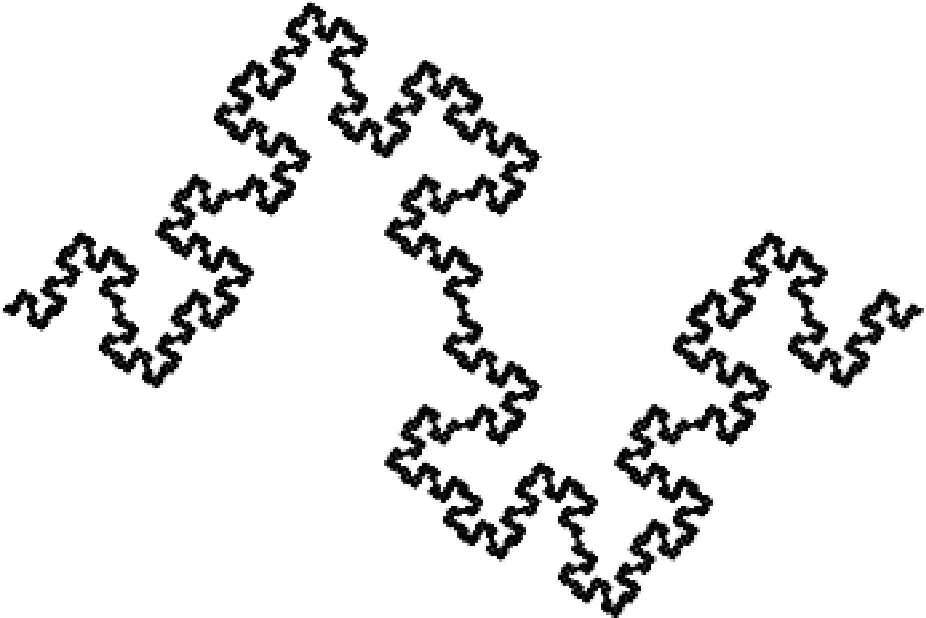
40. Minkowski Curve 2.





42. Minkowski Curve 4.





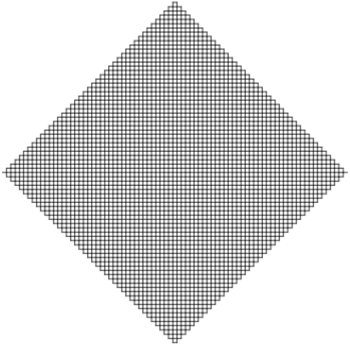


Figure: Peano Curve

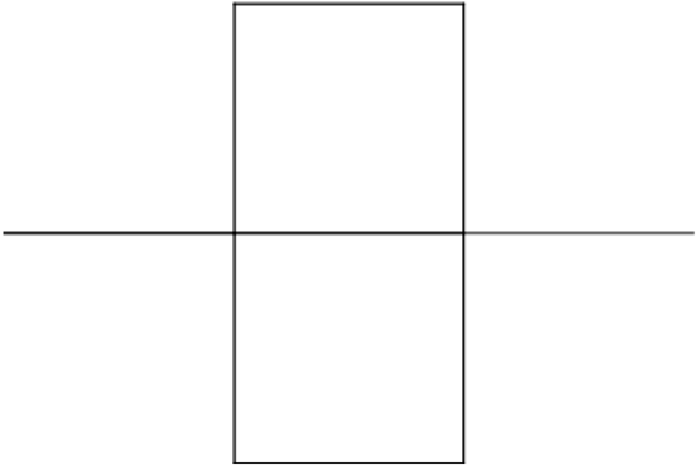
This is called a space filling curve. Every point of the diamond is on the curve. There are many self-intersection points.

The Peano Curve is obtained from IFS $\mathcal{F} = \{f_1, \dots, f_9\}$ where

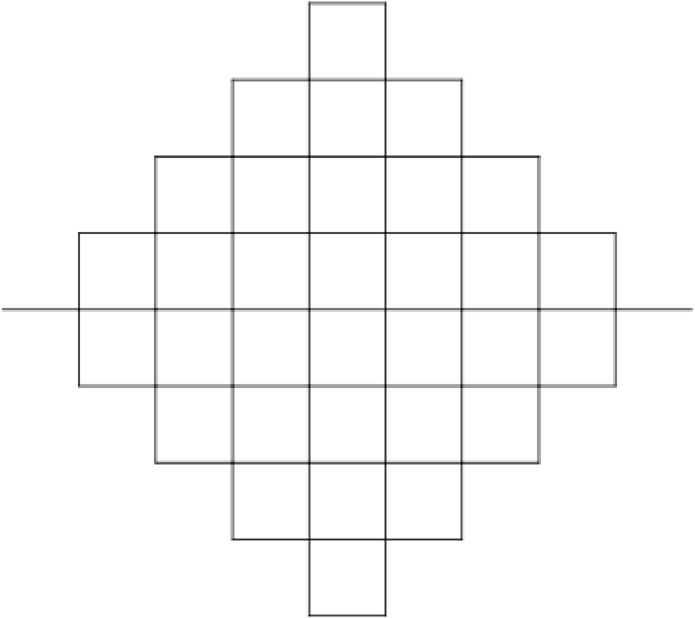
$$
\begin{aligned}
f_1(z) &= \tfrac{1}{3} z, \\
f_2(z) &= \tfrac{i}{3} z + \tfrac{1}{3} \\
f_3(z) &= \tfrac{1}{3} z + \tfrac{1+i}{3} \\
f_4(z) &= -\tfrac{i}{3} z + \tfrac{2+i}{3} \\
f_5(z) &= -\tfrac{1}{3} z + \tfrac{2}{3} \\
f_6(z) &= -\tfrac{i}{3} z + \tfrac{1}{3} \\
f_7(z) &= \tfrac{1}{3} z + \tfrac{1-i}{3} \\
f_8(z) &= \tfrac{i}{3} z + \tfrac{2-i}{3} \\
f_9(z) &= \tfrac{1}{3} z + \tfrac{2}{3}
\end{aligned}
$$

The contractions all have $\lambda_i = \frac{1}{3}$.

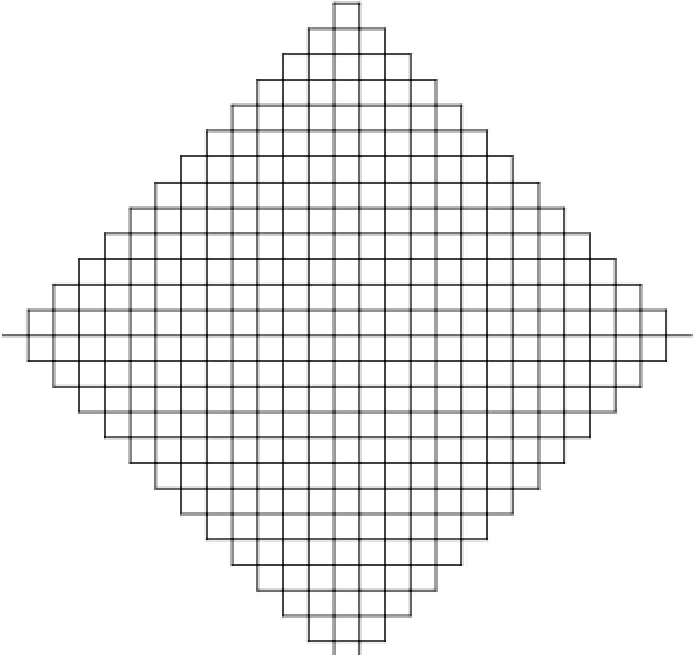
45. Peano Curve 1.



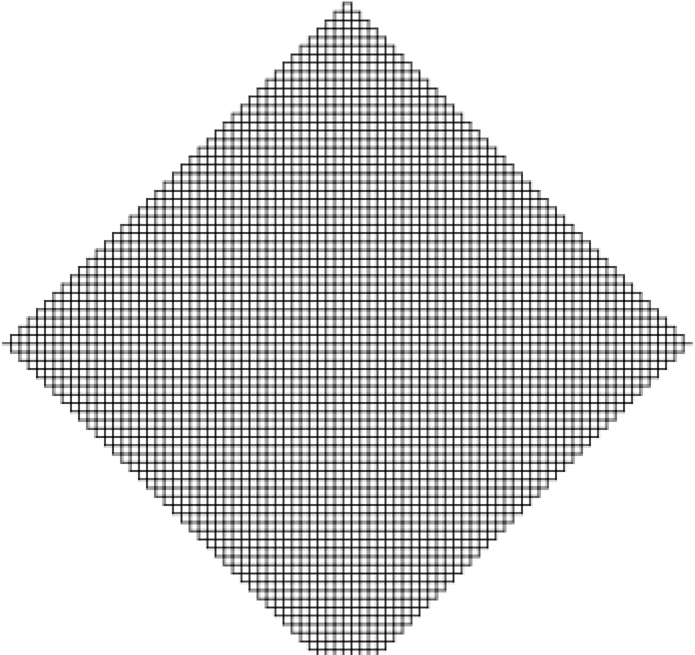
46. Peano Curve 2.



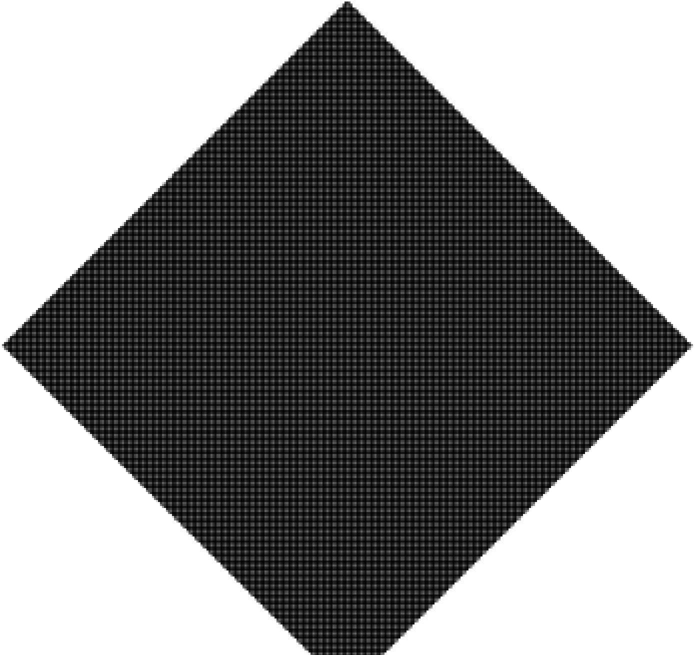
47. Peano Curve 3.



48. Peano Curve 4.



49. Peano Curve 5.



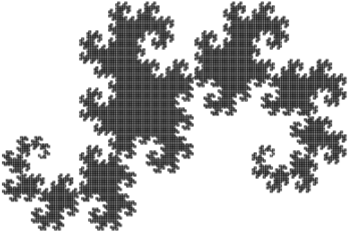


Figure: On of Many Levy's Dragons

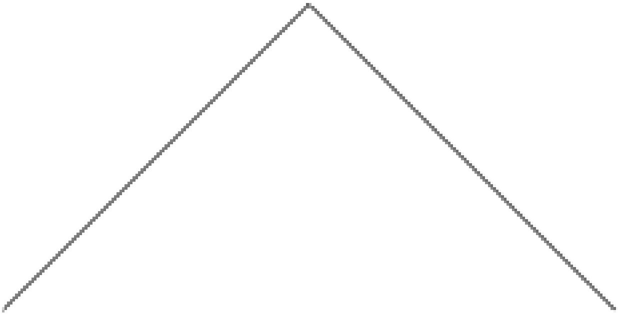
Paul Lévy (1886–1971) was first to exploit self-similarity. We reaserch focussed on probability theory.

Levy's Dragon Curve is obtained from IFS $\mathcal{F} = \{f_1, f_2\}$ where

$$f_1(z) = -\frac{1+i}{2}z + \frac{1+i}{2}$$
$$f_2(z) = \frac{1-i}{2}z + \frac{1+i}{2}$$

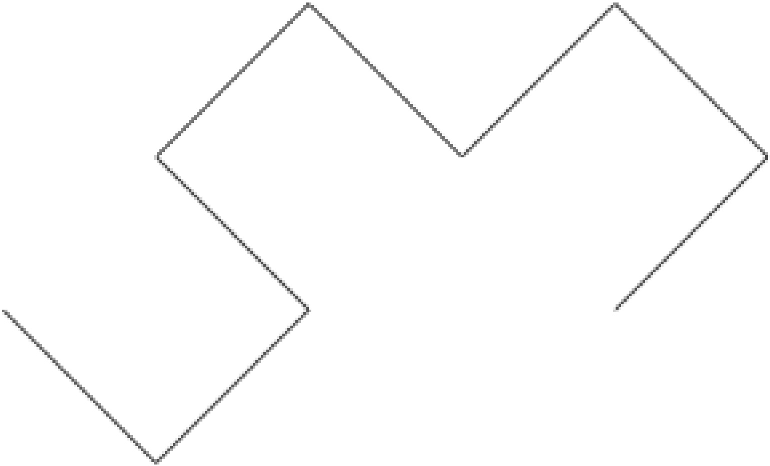
Both contractions have $\lambda_i = \frac{1}{\sqrt{2}}$. Note that f_1 sends the interval in the southwest direction to get the dragon to "snake."

51. Levy's Dragon 1.

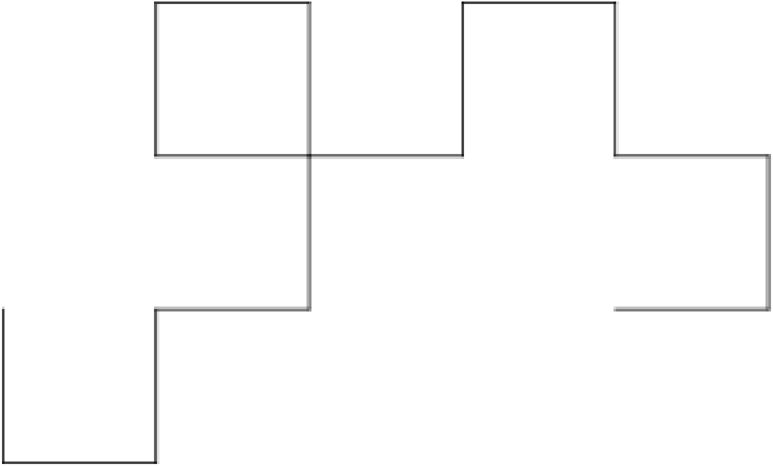


52. Levy's Dragon 2.

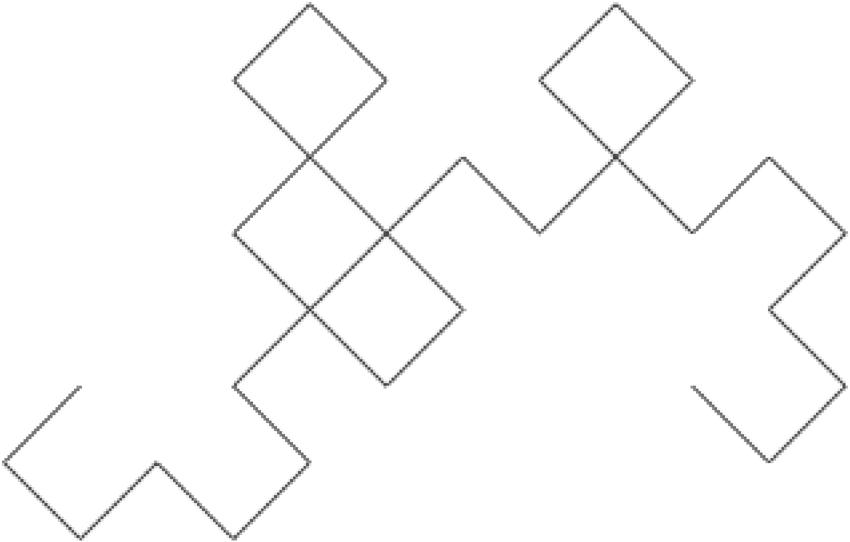
53. Levy's Dragon 3.



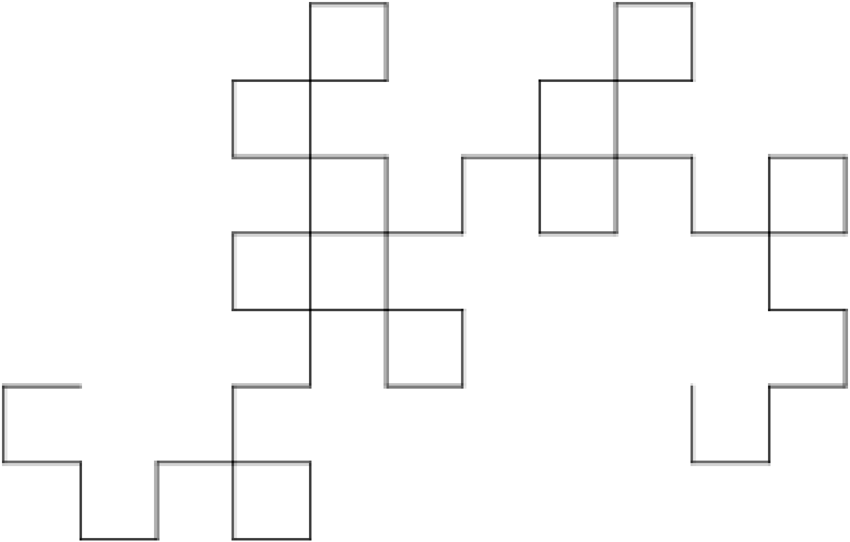
54. Levy's Dragon 4.

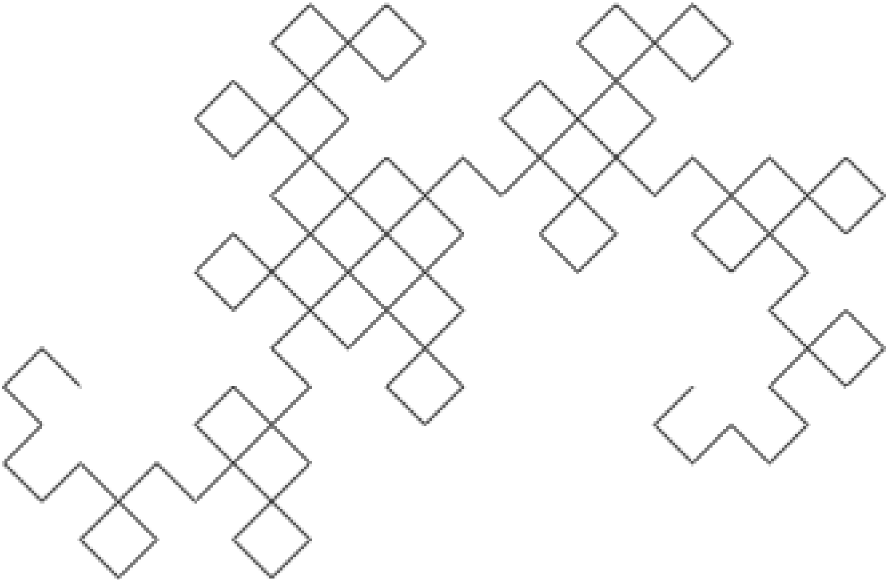


55. Levy's Dragon 5.

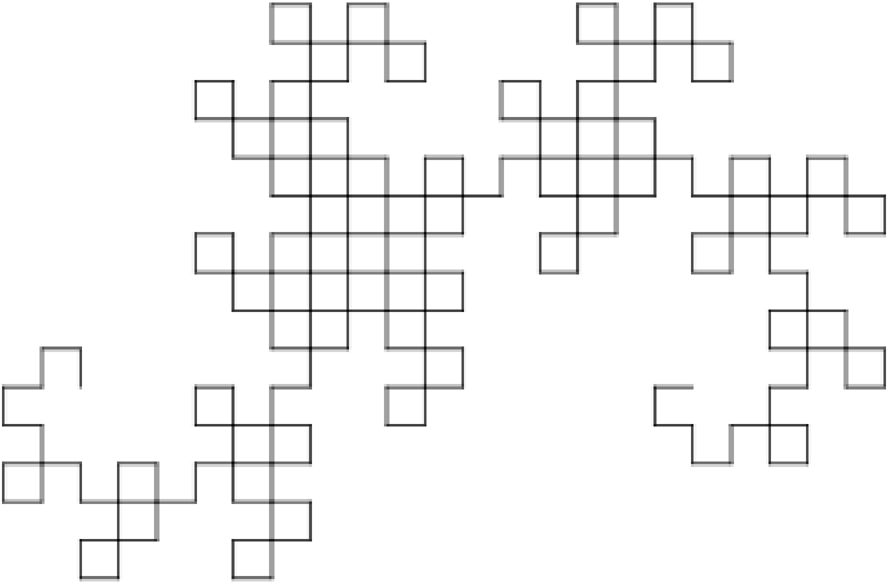


56. Levy's Dragon 6.

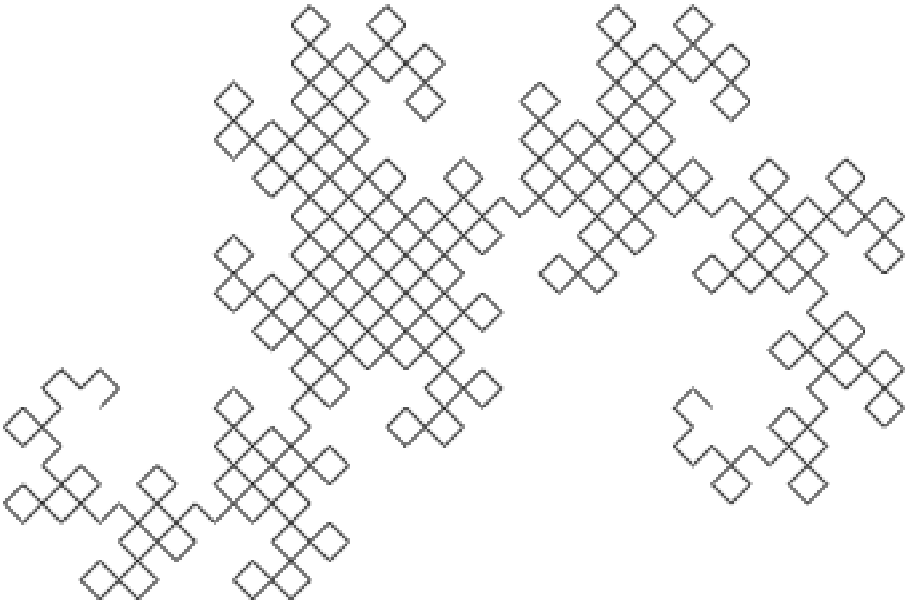


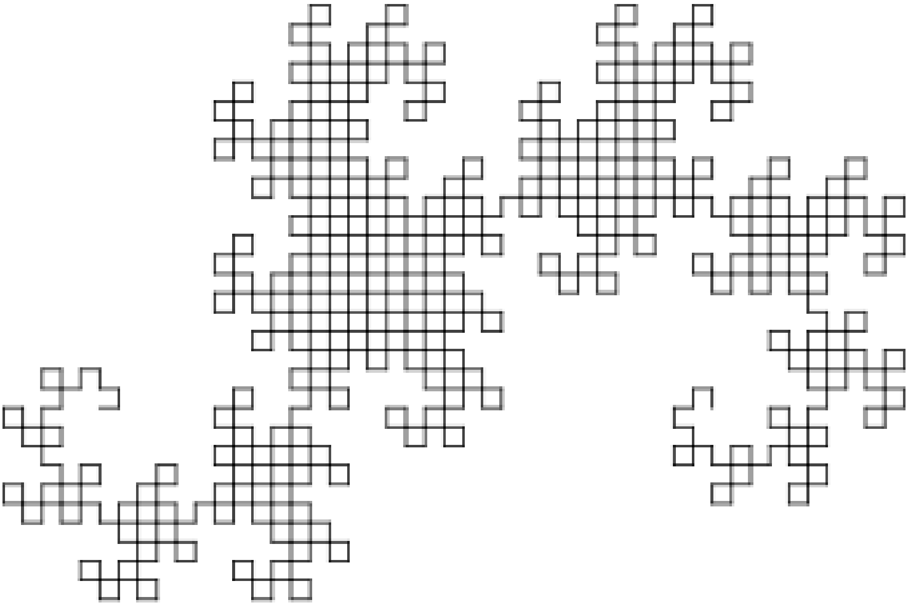


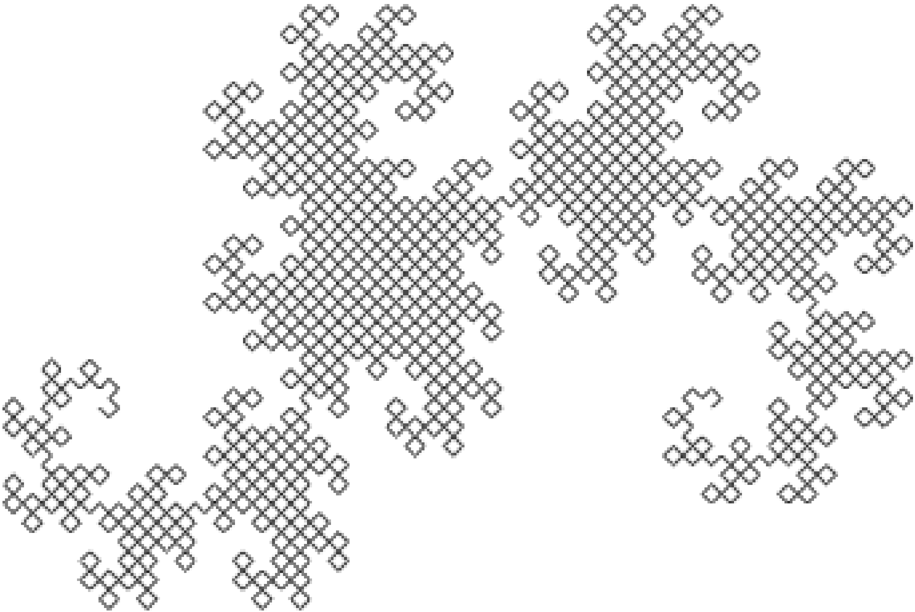
58. Levy's Dragon 8.

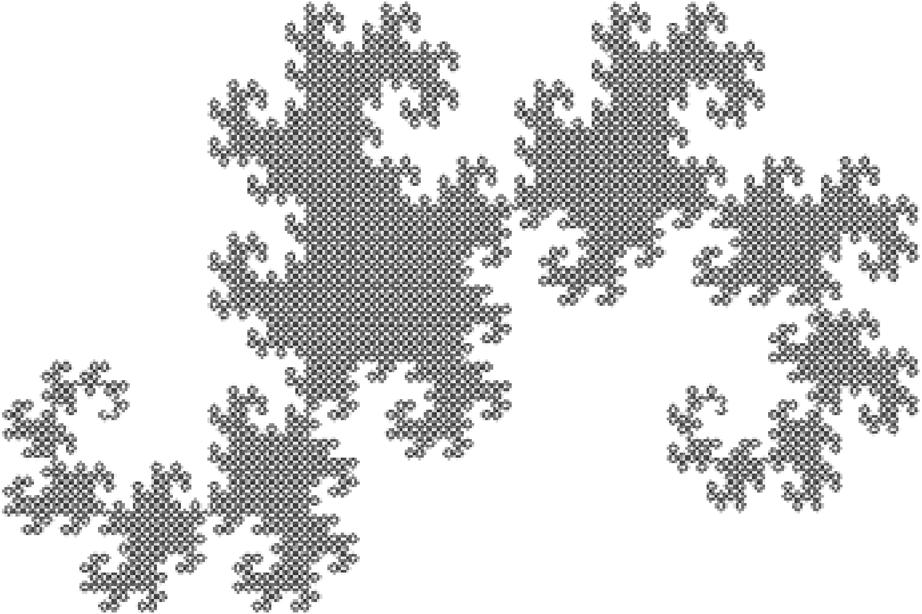


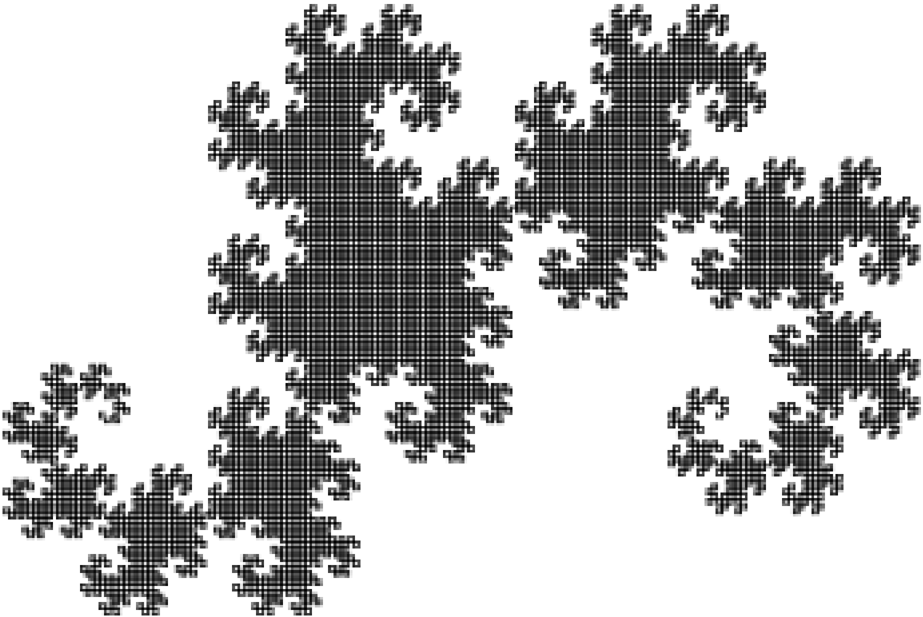
59. Levy's Dragon 9.



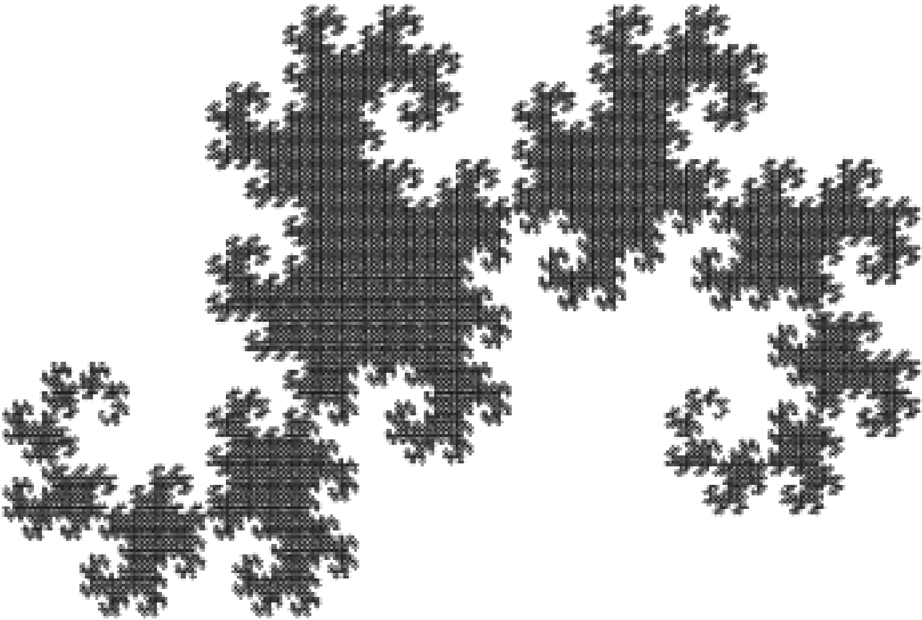








65. Levy's Dragon 15.



A similarity transformation in Euclidean space is a linear map for $x \in \mathbb{R}^d$

$$T(x) = \lambda Rx + b$$

where $\lambda \geq 0$ is a scaling factor, R is a rotation matrix and b is a translation vector. Reflections are also similarity transformations. In two dimensions, this is written in complex notation $z = x + iy$ by

$$T(z) = az + b, \qquad (\text{or } T(z) = a\bar{z} + b)$$

where $a = \lambda e^{i\theta} \in \mathbb{C}$, $\lambda = |a|$ is the norm and θ is the argument of a. T is thus dilation by λ followed by rotation by angle θ and then by translation of $b \in \mathbb{C}$.

A set $A \subset \mathbb{R}^d$ is self-similar if there is a similarity transformation T that identifies the a subset of $S \subset A$ with itself $T(S) = A$.

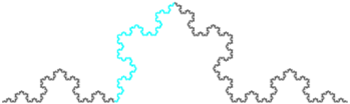


Figure: The von Koch Curve is self-similar. *e.g.*, the cyan subset is similar to the whole curve.

The von Koch curve A is the fixed set of the IFS $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$,

$$A = \mathcal{F}(A).$$

The cyan subset is $S = f_2(A)$, where

$$f_1(z) = \tfrac{1}{3}z,$$
$$f_2(z) = \frac{e^{\frac{\pi i}{3}}}{3}z + \frac{1}{3}$$
$$f_3(z) = \frac{e^{-\frac{\pi i}{3}}}{3}z + \frac{e^{\frac{\pi i}{3}}+1}{3}$$
$$f_4(z) = \tfrac{1}{3}z + \tfrac{2}{3}.$$

are all invertible similarity transformations. In particular

$$A = f_2^{-1}(S)$$

where the inverse is a similarity transformation

$$z = f_2^{-1}(w) = 3e^{-\frac{\pi i}{3}}w - e^{-\frac{\pi i}{3}}$$

The d-volume of a closed ball $\mathcal{B}_r(x) = \{y \in \mathbb{R}^d : |x - y| \leq r\}$ is $c_d r^d$, whose rate of growth is the dimension.

To measure the s-dimensional volume of $A \subset \mathbb{R}^n$, lets take an ϵ-cover $\mathcal{U}(\epsilon) = \{B_i\}$ of balls, namely $B_i = \mathcal{B}_{r_i}(x_i)$ with $r_i \leq \epsilon$ such that $A \subset \bigcup_i B_i$ and add their s-volumes. Then minimize over all such possible covers

$$m(A, s, \epsilon) = \inf_{\mathcal{U}(\epsilon)} \sum r_i^s$$

Since there are fewer sets in $\mathcal{U}(\epsilon)$ as ϵ decreases, the function $m(A, s, \epsilon)$ increases as ϵ decreases. So the refinement limit exists and we obtain the s-dimensional Hausdorff outer measure

$$m(A, s) = \lim_{\epsilon \to 0+} m(A, s, \epsilon)$$

For compact sets, this agrees with the Hausdorff measure.

Observe is that if T is a similarity transformation with factor $\lambda > 0$ then

$$m(T(A), s) = \lambda^s m(A, s)$$

Lemma

The set function $A \mapsto m(A, s)$ has the following properties

1. $m(\emptyset, s) = 0$ *for all* $s > 0$ *where* $\emptyset$ *is the empty set.*
2. $m(A_1, s) \leq m(A_2, s)$ *whenever* $A_1 \subset A_2$.
3. *(Subadditivity) For any finite or countable collection of subsets* A_i,

$$m\left(\bigcup_i A_i, s\right) \leq \sum_i m(A_i, s)$$

As a function of s, the function $m(A, s)$ is infinite for small values of s and zero for large values, Only for one s can $m(A, s)$ be something else.

Definition (Hausdorff Dimension)

$$\begin{aligned}\dim_H(A) &= \sup\{s \in [0, \infty) : m(A, s) = \infty\} \\ &= \inf\{s \in [0, \infty) : m(A, s) = 0\}\end{aligned}$$

Theorem

If $s \geq 0$ is such that $m(A, s) < \infty$ then $m(A, t) = 0$ for every $t > s$.

Proof.

$$m(A, t, \epsilon) = \inf_{\mathcal{U}(\epsilon)} \sum_i r_i^t = \inf_{\mathcal{U}(\epsilon)} \sum_i r_i^{t-s} r_i^s$$
$$\leq \inf_{\mathcal{U}(\epsilon)} \sum_i \epsilon^{t-s} r_i^s = \epsilon^{t-s} m(A, s, \epsilon).$$

Since $t - s > 0$ we have $\epsilon^{t-s} \to 0$ as $\epsilon \to 0+$. But $m(A, s, \epsilon) \leq m(A, s)$ because it is decreasing in ϵ, so

$$\lim_{\epsilon \to 0+} m(A, t, \epsilon) = 0. \qquad \square$$

Corollary

If $s \geq 0$ is such that $m(A, s) > 0$ then $m(A, t) = \infty$ for every $t < s$.

We find the dimension by covering with balls.

The IFS for the Cantor set is $\mathcal{F} = \{f_1, f_2\}$. If $I = [0, 1]$ then the k-th approximation to C is

$$\mathcal{F}^{\circ k}(I)$$

which consists of 2^k intervals which are balls of radius $\frac{1}{2 \cdot 3^k}$. If $\frac{1}{2 \cdot 3^k} \leq \epsilon$ this set of balls belongs to $\mathcal{U}(\epsilon)$ and for $s > 0$,

$$m(C, \epsilon) \leq \sum r_i^s = 2^k \left(\frac{1}{2 \cdot 3^k} \right)^s = \frac{1}{2^s} \left(\frac{2}{3^s} \right)^k$$

This quantity tends to zero as $\epsilon \to 0$ (same as $k \to \infty$) if $2 < 3^s$ or $s > \frac{\ln 2}{\ln 3}$. So $\dim_H(C) \leq \frac{\ln 2}{\ln 3} \cong .63$.

Show $\dim_H(C)$ is larger than $\frac{\ln 2}{\ln 3}$ is harder because we need to prove an inequality that holds for ALL covers $\mathcal{U}(\epsilon)$, but it is true.

We exploit the self-similarity to compute dimension of the Cantor Set.

Let's assume $s = \dim_H C$ and $0 < m(C, s) < \infty$. Because the IFS for the Cantor set consists of similarity transformations $\mathcal{F} = \{f_1, f_2\}$, with $\lambda_i = \frac{1}{3}$, the set is self-similar and $C = f_1(C) \cup f_2(C)$. By subadditivity and scaling for similarity transformations,

$$\begin{aligned} m(C, s) &= m(f_1(C) \cup f_2(C), s) \\ &\le m(f_1(C), s) + m(f_2(C), s) \\ &= \lambda^s m(C, s) + \lambda^s m(C, s) \end{aligned}$$

or

$$1 \le 2\left(\frac{1}{3}\right)^s.$$

Solving for s,

$$0 = \ln 1 \le \ln 2 - s \ln 3$$

so

$$s \le \frac{\ln 2}{\ln 3} \approx .63.$$

If A is the attractor of an IFS $\mathcal{F} = \{f_1, \ldots, f_k\}$ of similarity transformations with $0 < \lambda_i < 1$ and if the $f_i(A)$ are disjoint, then A is self similar. Assuming that $s = \dim_H(A)$ and $0 < m(A, s) < \infty$

$$m(A, s) = m\left(\bigcup_{i=1}^{k} f_i(C)\right) \leq \sum_{i=1}^{k} m(f_i(C)) = \sum_{i=1}^{k} \lambda_i^s m(A, s)$$

which implies

$$1 = \lambda_1^s + \cdots + \lambda_k^s = j(s)$$

Because the right side is a strictly decreasing function with $j(0) = k > 1$ and $\lim_{s \to \infty} j(s) = 0$, there is a unique solution $1 = j(s)$, called the similarity dimension, which is an upper bound for $\dim_H(A)$.

Because iterates may overlap, this may not be equal to $\dim_H(S)$. Moran's Theorem gives conditions so the similarity dimension equals the Hausdorff dimension.

Theorem (P. Moran, 1945)

Suppose that $A \subset R^d$ is a compact attractor of an IFS $\mathcal{F} = \{f_1, \dots, f_k\}$ of similarity transformations with $0 < \lambda_i < 1$. Assume that either $f_j(A)$ are disjoint for $j = 1, \dots, k$ or that A obtained in the following way: Suppose Ω_1 is an open bounded set and $\Omega_2^j = f_j(\Omega_1)$ be disjoint open sets for $j = 1, \dots, k$ contained in Ω_1. Similarly let $\Omega_2^{j\ell} = f_\ell(\Omega_1^j)$ for $\ell = 1, \dots, k$ be disjont in all j and so on. Suppose A is the intersection of

$$\overline{\Omega_1}, \quad \overline{\cup_j \Omega_2^i}, \quad \overline{\cup_{j\ell} \Omega_3^{j\ell}}, \quad \dots$$

Then $\dim_H(A)$ is the similarity dimension, namely, the unique $s > 0$ solving

$$1 = \lambda_1^s + \cdots + \lambda_k^s.$$

The theorem applies to Cantor sets in the line and the Sierpinski Gasket. It does not strictly apply to the von Koch curve. We'll compute several similarity dimensions.

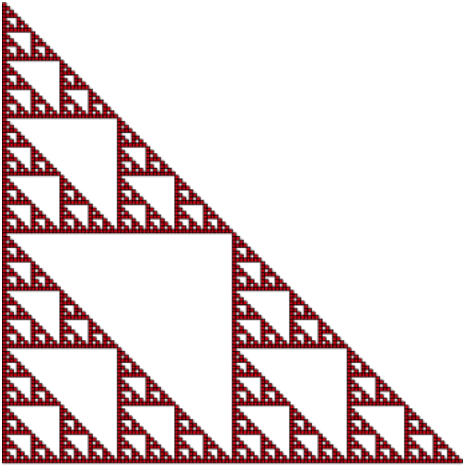


Figure: Sierpinski Gasket

The Sierpinski Gasket is obtained from IFS $\mathcal{F} = \{f_1, f_2, f_3\}$ where

$$f_1(z) = \tfrac{1}{2}z,$$
$$f_2(z) = \tfrac{1}{2}z + \tfrac{1}{2},$$
$$f_3(z) = \tfrac{1}{2}z + \tfrac{i}{2}.$$

Each f_i is a contraction with $\lambda = \frac{1}{2}$. Thus

$$1 = 3\left(\tfrac{1}{2}\right)^s$$

or $\dim_H(A) = \dfrac{\ln 3}{\ln 2} \cong 1.58$

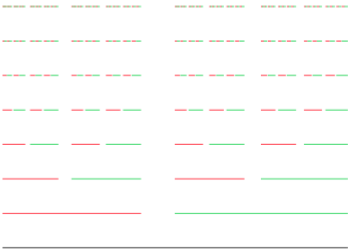


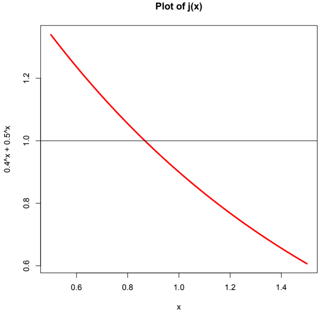
Figure: Cantor Set with Unequal Intervals

This Cantor set is obtained from IFS on $\mathbb{R}$

$$\mathcal{F} = \{.4x, .5x + .5\}$$

of contractions with $\lambda_1 = .4$ and $\lambda_2 = .5$.

$$1 = (.4)^s + (.5)^s = j(s).$$



Using a root finder, the solution is $\dim_H(C) = .867$.

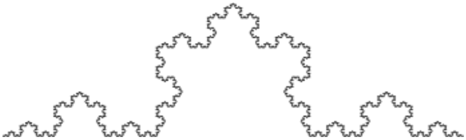


Figure: von Koch Curve

The von Koch Curve is obtained from IFS $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$ where in complex notation $z = x + iy$,

$$f_1(z) = \tfrac{1}{3}z,$$
$$f_2(z) = \tfrac{e^{\pi i/3}}{3}z + \tfrac{1}{3}$$
$$f_3(z) = \tfrac{e^{-\pi i/3}}{3}z + \tfrac{e^{\pi i/3}+1}{3}$$
$$f_4(z) = \tfrac{1}{3}z + \tfrac{2}{3}.$$

Each contraction has $\lambda = \frac{1}{3}$. Thus

$$1 = 4\left(\tfrac{1}{3}\right)^s$$

or $\dim_H(A) = \dfrac{\ln 4}{\ln 3} \cong 1.26$

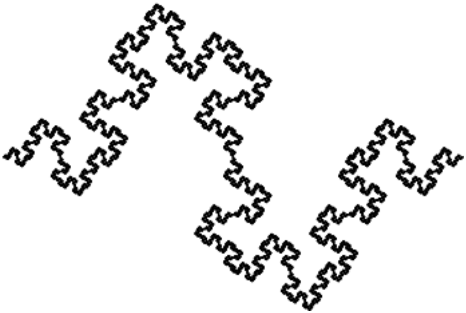


Figure: Minkowski Curve

$$f_1(z) = \tfrac{1}{4}z,$$
$$f_2(z) = \tfrac{i}{4}z + \tfrac{1}{4}$$
$$f_3(z) = \tfrac{1}{4}z + \tfrac{1+i}{4}$$

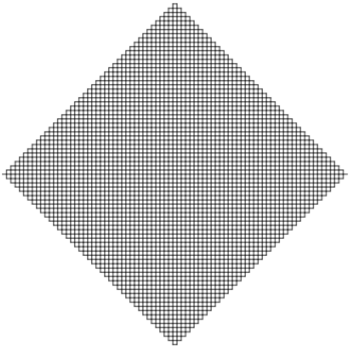
The Minkowski Curve is obtained from IFS $\mathcal{F} = \{f_1, \ldots, f_8\}$ where

$$f_4(z) = -\tfrac{i}{4}z + \tfrac{2+i}{4}$$
$$f_5(z) = -\tfrac{i}{4}z + \tfrac{1}{2}$$
$$f_6(z) = \tfrac{1}{4}z + \tfrac{2-i}{4}$$
$$f_7(z) = \tfrac{i}{4}z + \tfrac{3-i}{4}$$
$$f_8(z) = \tfrac{i}{4}z + \tfrac{3}{4}$$

All $\lambda_i = \frac{1}{4}$. Thus

$$1 = 8\left(\frac{1}{4}\right)^s$$

or $\dim_H(A) = \dfrac{\ln 8}{\ln 4} = 1.5.$



The Peano Curve is obtained from IFS $\mathcal{F} = \{f_1, \ldots, f_9\}$ where

$$f_1(z) = \tfrac{1}{3}z,$$
$$f_2(z) = \tfrac{i}{3}z + \tfrac{1}{3}$$
$$f_3(z) = \tfrac{1}{3}z + \tfrac{1+i}{3}$$
$$f_4(z) = -\tfrac{i}{3}z + \tfrac{2+i}{3}$$
$$f_5(z) = -\tfrac{1}{3}z + \tfrac{2}{3}$$
$$f_6(z) = -\tfrac{i}{3}z + \tfrac{1}{3}$$
$$f_7(z) = \tfrac{1}{3}z + \tfrac{1-i}{3}$$
$$f_8(z) = \tfrac{i}{3}z + \tfrac{2-i}{3}$$
$$f_9(z) = \tfrac{1}{3}z + \tfrac{2}{3}$$

The contractions all have $\lambda_i = \frac{1}{3}$. Thus

$$1 = 9\left(\frac{1}{3}\right)^s$$

or $\dim_H(A) = \dfrac{\ln 9}{\ln 3} = 2.$

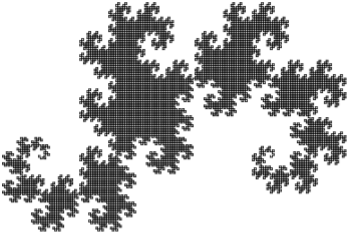


Figure: Levy Dragon

Levy's Dragon Curve is obtained from IFS $\mathcal{F} = \{f_1, f_2\}$ where

$$f_1(z) = -\frac{1+i}{2}z + \frac{1+i}{2}$$
$$f_2(z) = \frac{1-i}{2}z + \frac{1+i}{2}$$

Both contractions have $\lambda_i = \frac{1}{\sqrt{2}}$. Thus

$$1 = 2\left(\frac{1}{\sqrt{2}}\right)^s$$

or $\dim_H(A) = \dfrac{\ln 2}{\ln \sqrt{2}} = 2.$

Attractors of an IFS can be used to find relatively simple constructions of mathematically interesting objects. In 1872, Weierstrass first wrote a continuous nowhere differentiable function on $I = [0, 1]$

$$f(x) = \sum_{i=1}^{\infty} b^i \cos(a^i \pi x).$$

In 1916, Hardy sharpened conditions that it be continuous for $0 < b < 1$ and nowhere differentiable if also $a > 1$ and $ab \geq 1$.

von Koch' snowflake curve was contrived for the same purpose. But the easiest construction is due to Kiesswetter in 1966.

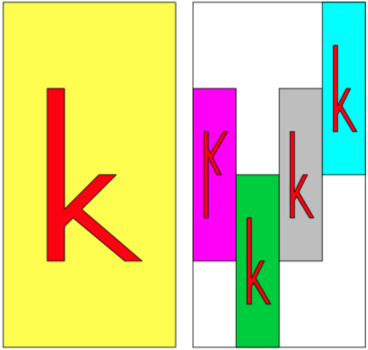


Figure: Yellow rectangle is mapped to four rectangles by $\mathcal{F}$

Kiesswetter considered the IFS

$$\mathcal{F} = \{f_1, f_2, f_3, f_4\}$$

on $[0,1] \times [-1,1]$ where

$$f_1(x) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$f_2(x) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{2} \end{pmatrix},$$

$$f_3(x) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$$

$$f_4(x) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{3}{4} \\ \frac{1}{2} \end{pmatrix}.$$

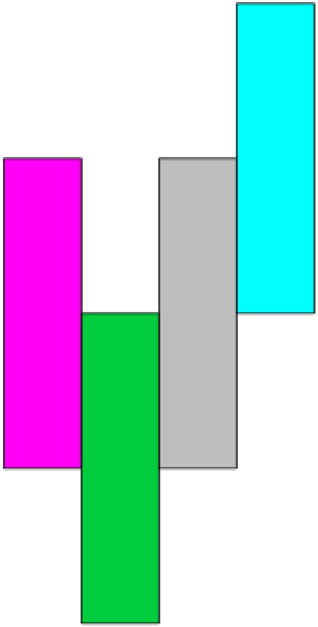
Each affine map shrinks horizontally by $\frac{1}{4}$ and vertically by $\frac{1}{2}$, thus has contraction constants $\lambda_i = \frac{1}{2}$.

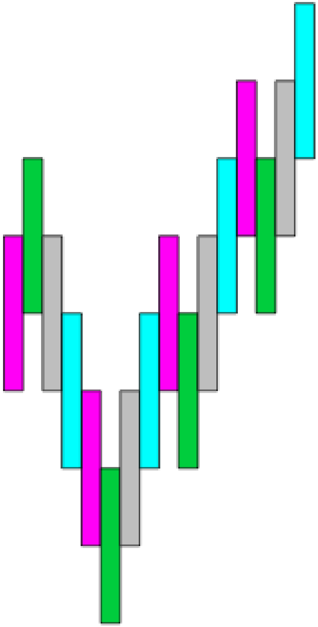
By Hutchinson's Theorem there is an attractor A for $\mathcal{F}$. Kiesswetter showed that A is the graph of a curve $A = \{(x, k(x)) : 0 \le x \le 1\}$ which is Hölder Continuous

$$|f(x) - f(y)| \le C|x - y|^{\frac{1}{2}} \qquad \text{for all } x, y \in [0, 1]$$

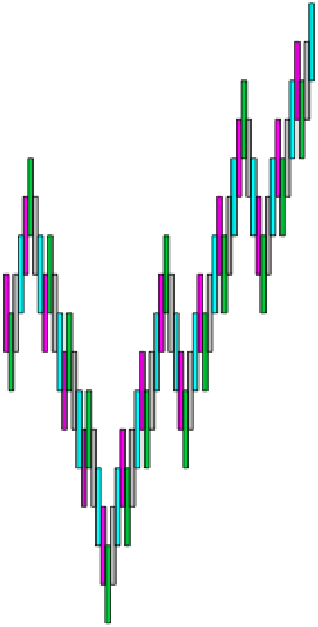
and that it is nowhere differentiable.

84. Kieswetter's Nondifferentiable Function 1.

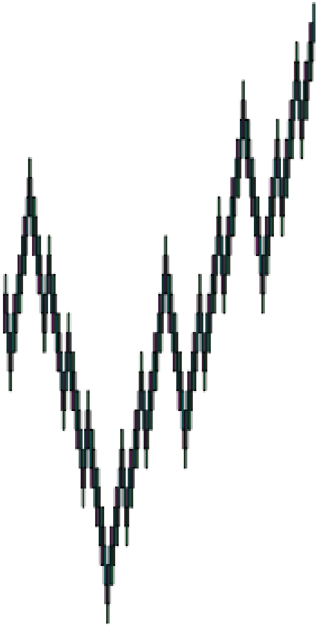




86. Kieswetter's Nondifferentiable Function 3.



87. Kieswetter's Nondifferentiable Function 4.



88. Kieswetter's Nondifferentiable Function 5.

Thanks!