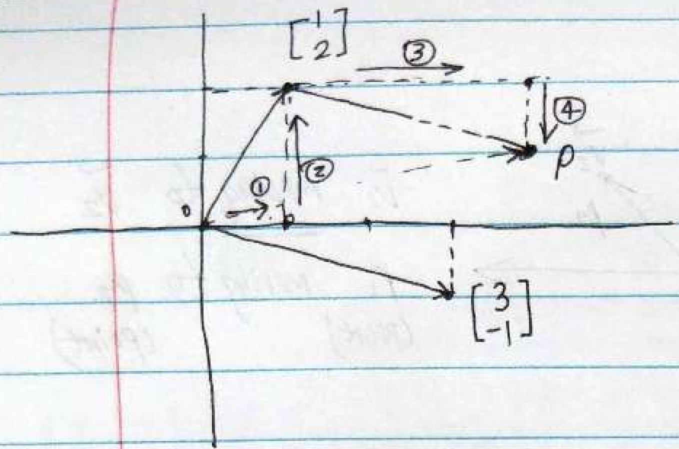
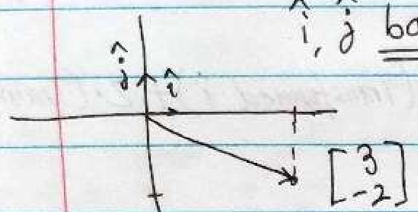
Linear Algebra (Essence of linear algebra)



$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}3\\-1\end{bmatrix}$

4步走到p点,
① 先x方向 1
② 后y方向 2
③ 后x方向 3
④ 后y方向 -1

或者
① 先x方向 1
② 后x方向 3
③ 后y方向 2
④ 后y方向 -1

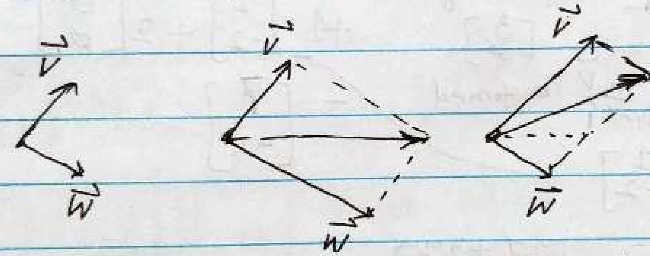


$\hat{i}, \hat{j}$ basis vectors

$\boxed{3\hat{i}+(-2)\hat{j}}$

"Linear combination" of $\vec{v}$ and $\vec{w}$

$= a\vec{v}+b\vec{w}$ (a, b, scalars)



The "span" of $\vec{v}$ and $\vec{w}$ is the set of all their linear combinations.

$a\vec{v}+b\vec{w}$

⇒ the whole plane.

linear combination of $\vec{v}$, $\vec{w}$ and $\vec{u}$.

$a\vec{v}+b\vec{w}+c\vec{u}$

"Linear dependent"

$\vec{u}=a\vec{v}+b\vec{w}$. $\vec{u}$ belongs to "span" of $\vec{v}$ and $\vec{w}$.

The basis of a vector space is a set of linearly independent vectors that span the full space.

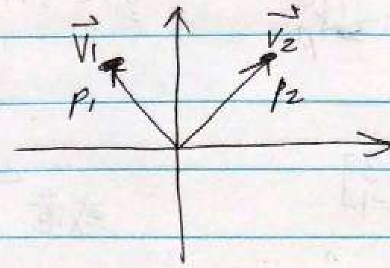
$\vec{u}\neq a\vec{v}$ for all 'a'.

$\vec{u}\neq a\vec{v}+b\vec{w}$ for all 'a'.

Linear Transformation → "function"

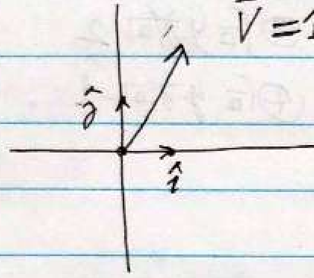
(2 properties)
- linear ⇒ line
- origin fixed

input vector
⇓
$f(\vec{v})$
⇓
output vector



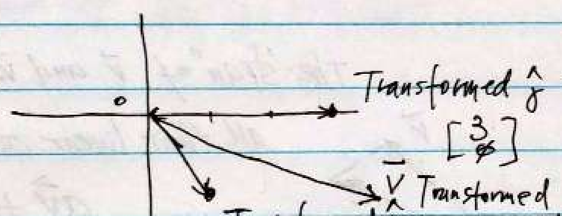
$\vec{v}_1$ moving to $\vec{v}_2$

p_1 (point) moving to p_2 (point)



⇒ transformed $\vec{v} = 1 \cdot (\text{Transformed } \hat{i}) + 2 \cdot (\text{Transformed } \hat{j})$

⇓



$$+1 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$\underset{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\hat{i}} \to \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \underset{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\hat{j}} \to \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \to x \begin{bmatrix} 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix}$$

$$= \underset{\hat{i} \quad \hat{j}}{\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

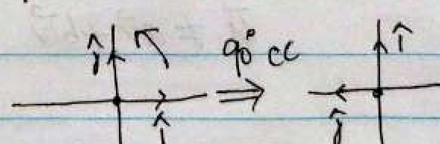
adding scales to the new "basis"

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}$$

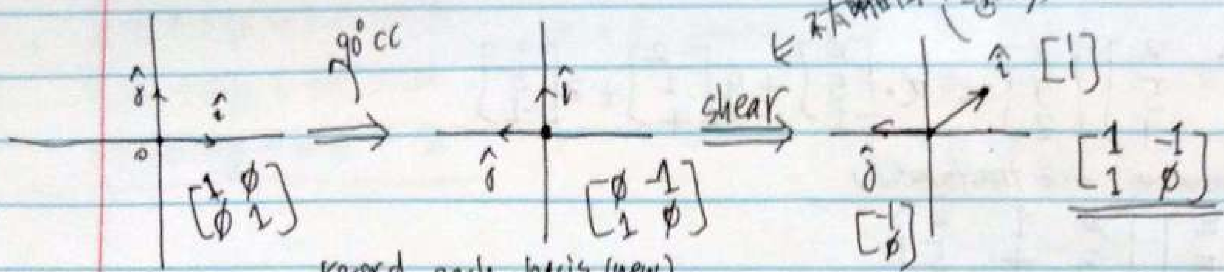
$$= \begin{bmatrix} xa + yb \\ xc + yd \end{bmatrix}$$

90° rotation counterclockwise



rotation matrix: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

"combination" of a rotation and a shear

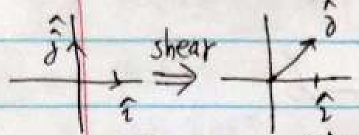


$$\begin{bmatrix} 1 & \emptyset \\ \emptyset & 1 \end{bmatrix} \xrightarrow{90^\circ cc} \begin{bmatrix} \emptyset & -1 \\ 1 & \emptyset \end{bmatrix} \xrightarrow{\text{shear}} \begin{bmatrix} 1 & -1 \\ 1 & \emptyset \end{bmatrix}$$

$\hat{i} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\hat{j} \begin{bmatrix} -1 \\ \emptyset \end{bmatrix}$

record each basis (new)

$$\underbrace{\begin{bmatrix} 1 & 1 \\ \emptyset & 1 \end{bmatrix}}_{\text{shear}} \left(\underbrace{\begin{bmatrix} \emptyset & -1 \\ 1 & \emptyset \end{bmatrix}}_{\text{rotation}} \begin{bmatrix} x \\ y \end{bmatrix} \right) = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & \emptyset \end{bmatrix}}_{\text{composition}} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$\overbrace{\begin{bmatrix} \emptyset & 2 \\ 1 & \emptyset \end{bmatrix}}^{M_2} \overbrace{\begin{bmatrix} 1 & -2 \\ 1 & \emptyset \end{bmatrix}}^{M_1} = \begin{bmatrix} 2 & \emptyset \\ 1 & -2 \end{bmatrix}$$

$\hat{i}$ $\hat{j}$

$$1 \cdot \begin{bmatrix} \emptyset \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ \emptyset \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad -2 \cdot \begin{bmatrix} \emptyset \\ 1 \end{bmatrix} + \emptyset \begin{bmatrix} 2 \\ \emptyset \end{bmatrix} = \begin{bmatrix} \emptyset \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ea+gb & fa+gb \\ ec+gd & fc+gd \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ g \end{bmatrix} = e \begin{bmatrix} a \\ c \end{bmatrix} + g \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ea+gb \\ ec+gd \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = f \begin{bmatrix} a \\ c \end{bmatrix} + g \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} fa+gb \\ fc+gd \end{bmatrix}$$

Associative

$AB \neq BA$

$$ABC = A(BC) = (AB)C$$

$$\begin{bmatrix} 0 & -2 & 2 \\ 5 & 1 & 5 \\ 1 & 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\cdot\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} + y\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + z\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$

second transformation first transformation

$$\begin{bmatrix} 0 & -2 & 2 \\ 5 & 1 & 5 \\ 1 & 4 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

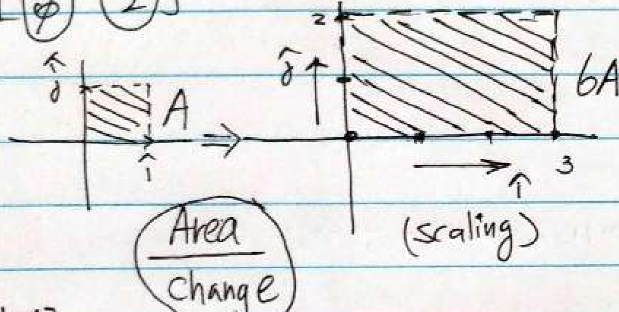
$$\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \end{array}$$

transformation

$$\boxed{\det(M_2M_1) = \det(M_2)\cdot\det(M_1)}$$

The "determinant" of a transformation

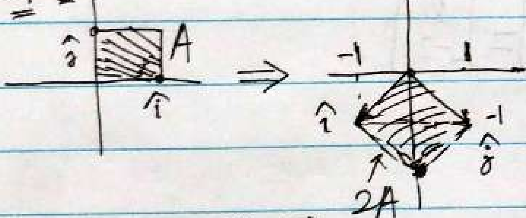
$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$



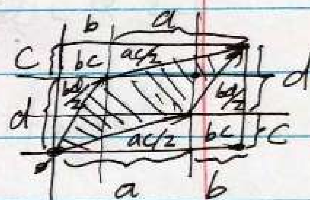
$$\det\left(\begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix}\right) = 6$$

$$\det\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = a\cdot d - 0\cdot 0 = a\cdot d$$

$$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

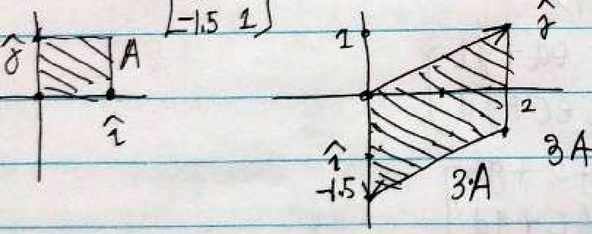


$$\det\left(\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\right) = 2$$



$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a\cdot d - bc = (a+b)(c+d) - ac - bd - 2bc$$

$$\begin{bmatrix} 0 & 2 \\ -1.5 & 1 \end{bmatrix}$$

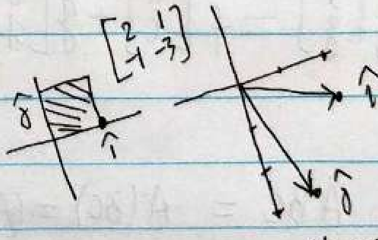


(新的basis不一定要垂直)

$$\det\left(\begin{bmatrix} 0 & 2 \\ -1.5 & 1 \end{bmatrix}\right) = 3$$

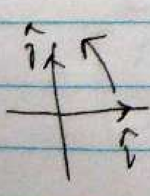
$$\det\left(\begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}\right) = -5$$

$$\begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}$$



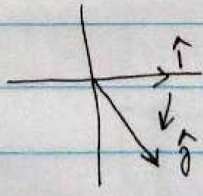
$$\det\left(\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\right) = 0 \Rightarrow \text{变成一条线}$$

orientation flipped



$\hat{i}$在$\hat{j}$的右边.

$\hat{j}$在$\hat{i}$的逆时针方向.



$\hat{i}$在$\hat{j}$的左边

$\hat{j}$在$\hat{i}$的顺时针方向.

3D cube 体积单元

Linear system of equations

$$2x+5y+3z=-3$$
$$4x+0y+8z=0$$
$$1x+3y+0z=2$$

x, y, z unknowns

$$\begin{array}{ccc} \text{coeffients} & \text{variables} & \text{constants} \\ \begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix} & \begin{bmatrix} x \\ y \\ z \end{bmatrix} & = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \\ A & \vec{x} & \vec{v} \end{array}$$

$$\boxed{A\vec{x}=\vec{v}}$$

▶ 向量$\vec{x}$经过A变换后变成$\vec{v}$.

$A\vec{x}=\vec{v} \Rightarrow$ if found A^{-1}

$A^{-1}A\vec{x}=\cancel{A}^{-1}\vec{v}$ $\quad (A^{-1}A=I$ (identity))

$$\boxed{\vec{x}=A^{-1}\vec{v}}$$

▶ 找到逆变换A^{-1},使得$\vec{v}$经过A^{-1}变换后变成$\vec{x}$.

(inverse transformation)

如果 A变换没有将$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$变成低dimensional 向量,

例如 $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ($90°$ CC变换) 的inverse为 $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ($90°$ C变换)

A^{-1}存在 $\Leftarrow$ $\det(A) \neq 0$. (单元体的体量没变成0).

则线性系统将有唯一解. $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$可由$\vec{v}$经过$A^{-1}$变换回来.

$\det(A)=0. \Rightarrow$ 可能线性系统仍然有解,因为$\vec{v}$正好在$A\vec{x}$变换(降维)后的span中. 比如同条线. 同个点..

rank: number of dimensions in the output $(\vec{v})$ ($\overset{\text{rank}}{\text{transformation}}$)

set of all possible outputs of $A\vec{v}$. $\Rightarrow$ "column space" of A

$$\underbrace{\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}}_{\text{span of columns}}$$

$\Updownarrow$

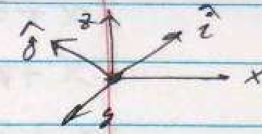
"column space"

non-squared matrix

$$\begin{array}{cc} \hat{i} & \hat{j} \end{array}$$
$$\underbrace{\begin{bmatrix} 2 & 0 \\ -1 & 1 \\ -2 & 1 \end{bmatrix}}_{\text{span of columns}}$$

input ⟹ output

$\underline{2D} \Longrightarrow 3D$ map



$\hat{i}$ $\hat{j}$ 在3D空间中，作为basis

span of columns

⇕

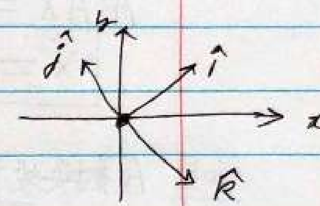
"column space" (full rank) ⟹ number of columns $\overset{(2)}{}$ = input dimensions $\overset{(2)}{}$

3 basis vectors

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$$

2 coordinates for each landing spots

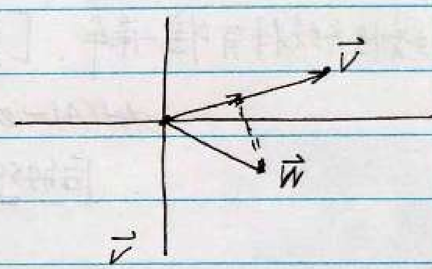
map 3D(input) to 2D(output)



dot product

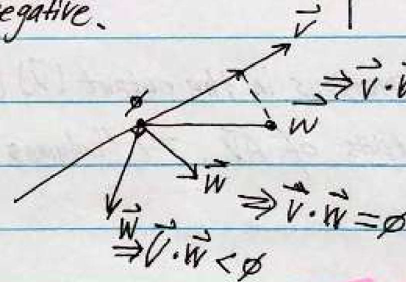
$\begin{bmatrix} 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ = (length of projected $\vec{w}$)(length of $\vec{v}$)

$\vec{v}$ $\vec{w}$



if (project is opposite direction)
dot product will be negative.

(次序互换也是对的)



$\begin{bmatrix} 4 \\ 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 4*2 + 1\cdot(-1) = 7$

⇓

$\underline{\begin{bmatrix} 4 & 1 \end{bmatrix}}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix}$

⇓ 线性变换，将 2D $\vec{v}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ 降维到 1D, (scalar).

(M 只有1 row, full rank)

1×2 matrix ⟺ 2d vector $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

[4, 1]

$\hat{i}$ $\hat{j}$ (仍然为 basis)

故可以认为它是往 $\vec{v}$ 作为basis 投影 (project)

Matrix-vector product

⇕

dot product

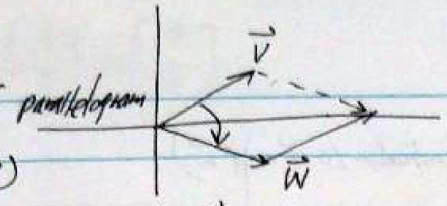
(duality)

$\underline{\begin{bmatrix} u_x & u_y \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix} = u_x \cdot x + u_y \cdot y$

$\begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = u_x \cdot x + u_y \cdot y$

Cross Product

$\vec{v} \times \vec{w} = -$Area of parallelogram (negative)

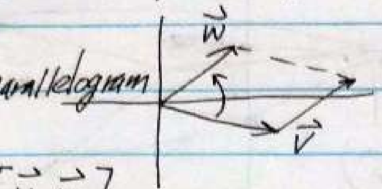


$\vec{v} \times \vec{w} = \vec{p}$ (vector)

(length = area of parallelogram)

(Perpendicular to parallelogram)

$\vec{v} \times \vec{w} =$ Area of parallelogram



(右手right rule)

$\|$

$\boxed{\text{determinat}}$ of $[\vec{v}\ \vec{w}]$

↳ linear transformation of basis $\hat{i}, \hat{j}$,

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \det\left(\begin{bmatrix} \hat{i} & v_1 & w_1 \\ \hat{j} & v_2 & w_2 \\ \hat{k} & v_3 & w_3 \end{bmatrix}\right) = \hat{i}(v_1 w_3 - w_1 v_3) + \hat{j}(v_3 \cdot w_1 - v_1 w_3) + \hat{k}(v_1 w_2 - v_2 w_1)$$

(notation trick)

$$= \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

$\rightarrow \begin{bmatrix} v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}$ 划掉row1

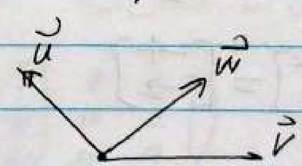
$\rightarrow \begin{bmatrix} v_3 & w_3 \\ v_1 & w_1 \end{bmatrix}$ 划掉row2

$\rightarrow \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ 划掉row3.

(Duality)

- Define a 3d-to-1d linear transformation in terms of $\vec{v}$ and $\vec{w}$
- Find its dual vector

$\vec{p} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$ length of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ project to $\vec{p}$ * length of $\vec{p}$.



$\underline{\vec{u}} \times \underline{\vec{v}} \times \underline{\vec{w}}$ (3d空间的向量)

(形成一个长方体)

(determinant就是长方体的体积)

$$= \det\left(\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}\right)$$

$\vec{p} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$

$\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = p_1 x + p_2 y + p_3 z$

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \overset{(未知)}{=} \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right) \Rightarrow [?\ ?\ ?] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

$\boxed{\text{linear}}$

⇓ 不停varing

$\underline{\vec{v}}$ $\vec{w}$ → fixed

因为这个function是linear transformation.

从3d-to-1d

这个det展开以后会得到下面的形式

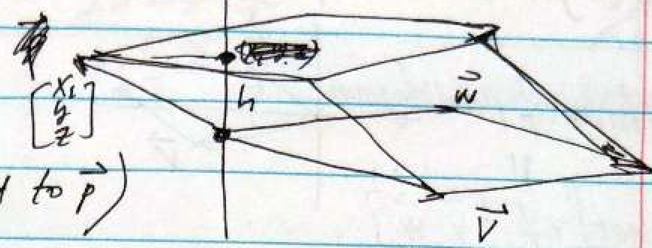
$x(\ ) \rightarrow p_1$

$+y(\ ) \rightarrow p_2$

$+z(\ ) \rightarrow p_3$

(Area of Parallelogram) × (component of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ perpendicular to $\vec{v}$ and $\vec{w}$)

体积 = 底面积 × 高)



$$\vec{p} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(\text{length of } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ projected to } \vec{p}\right) \cdot \left(\text{length of } \vec{p}\right)$$

$$\vec{p} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\underline{\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}}\right)$$

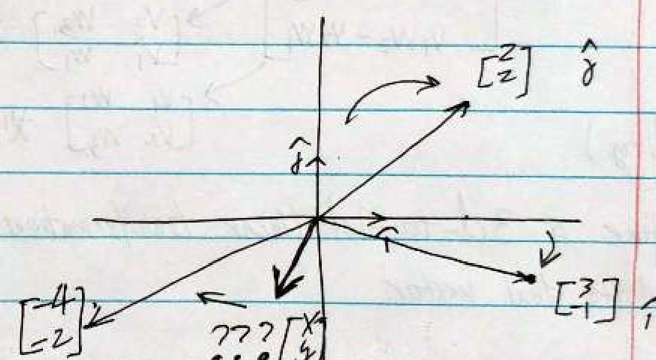
⇒ $\vec{p}$ 方向为垂直方向 ($\vec{v}$ 和 $\vec{w}$ 形成的plane).
$\vec{p}$ 长度为 $\vec{v}$, $\vec{w}$ 形成平面的 area.

3向量形成 volumn, det就是其体积.

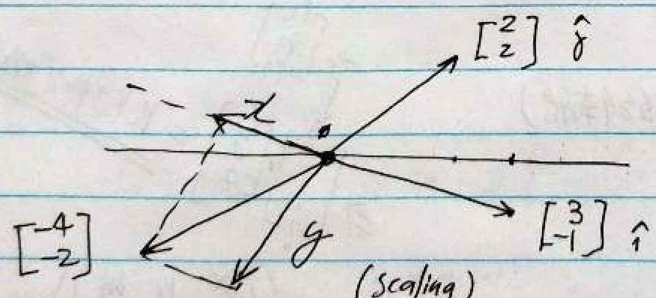
$$3x + 2y = -4$$
$$-1x + 2y = -2$$

$$\begin{bmatrix} +3 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

$$x \begin{bmatrix} +3 \\ -1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

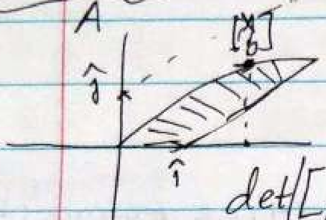


⇒ 在经过 $[\hat{i}, \hat{j}]$ 变换成 $\begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$ 后
变换成了 $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$.

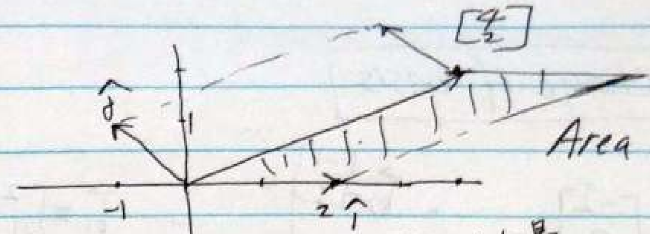


determinant is the amout of area change
not new area
if original area is 1, the det is same of new area.

$$\underbrace{\begin{bmatrix} 2 & -1 \\ \cancel{0} & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



$\Longrightarrow$



$$\det\left(\begin{bmatrix} 1 & x \\ \cancel{0} & y \end{bmatrix}\right) = y.$$

$$\det\begin{bmatrix} x & \cancel{0} \\ y & 1 \end{bmatrix}) = x.$$

$\begin{bmatrix} 2 & 4 \\ \cancel{0} & 2 \end{bmatrix}$ 2个向量.

$$Area = \det(A) \cdot \frac{\underline{Original\_Area}}{y}$$

$$Area = \det(A) \cdot y$$

$$\underline{y} = \frac{Area}{\det(A)} = \frac{Area}{\det\left(\begin{bmatrix} 2 & -1 \\ \cancel{0} & 1 \end{bmatrix}\right)} = \frac{\det\left(\begin{bmatrix} 2 & 4 \\ \cancel{0} & 2 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 2 & -1 \\ \cancel{0} & 1 \end{bmatrix}\right)}$$

$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ 2个向量

$$Area = \det(A) \cdot \frac{\underline{Original\_Area}}{x}$$

$$= \det(A) \cdot x \Longrightarrow \underline{x} = \frac{Area}{\det(A)} = \frac{\det\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 2 & -1 \\ \cancel{0} & 1 \end{bmatrix}\right)}.$$

所以求解 $\underline{\begin{bmatrix} x \\ y \end{bmatrix}}$ 得看 det(A) 存在与否 det(A) 在分母

$\boxed{\text{"Cramer's Rule"}}$

$\boxed{3d}$

$$x = \det\left(\begin{bmatrix} x & \cancel{0} & \cancel{0} \\ y & 1 & \cancel{0} \\ z & \cancel{0} & 1 \end{bmatrix}\right)$$

$$y = \det\left(\begin{bmatrix} 1 & x & \cancel{0} \\ \cancel{0} & y & \cancel{0} \\ \cancel{0} & z & 1 \end{bmatrix}\right)$$

$$z = \det\left(\begin{bmatrix} 1 & \cancel{0} & x \\ \cancel{0} & 1 & y \\ \cancel{0} & \cancel{0} & z \end{bmatrix}\right).$$

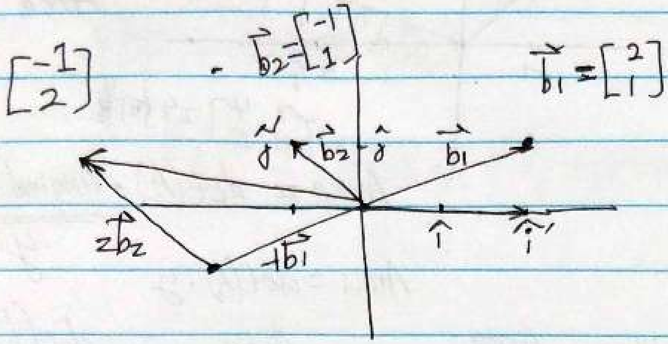
$$\begin{bmatrix} -4 & 2 & 3 \\ -1 & \cancel{0} & 2 \\ -4 & 6 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} +7 \\ -8 \\ 3 \end{bmatrix}$$

$$x = \frac{\det\left(\begin{bmatrix} 7 & 2 & 3 \\ -8 & \cancel{0} & 2 \\ 3 & 6 & -9 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} -4 & 2 & 3 \\ -1 & \cancel{0} & 2 \\ -4 & 6 & -9 \end{bmatrix}\right)}$$

$$y = \frac{\det\left(\begin{bmatrix} -4 & 7 & 3 \\ -1 & -8 & 2 \\ -4 & 3 & -9 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} -4 & 2 & 3 \\ -1 & \cancel{0} & 2 \\ -4 & 6 & -9 \end{bmatrix}\right)}$$

$$z = \frac{\det\left(\begin{bmatrix} -4 & 2 & 7 \\ -1 & \cancel{0} & -8 \\ -4 & 6 & 3 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} -4 & 2 & 3 \\ -1 & \cancel{0} & 2 \\ -4 & 6 & -9 \end{bmatrix}\right)}.$$

Change of basis



在 $[\vec{b_1}\ \vec{b_2}]$ 坐标系统中的 $\begin{bmatrix}-1\\2\end{bmatrix}$ 向量，在我们的 $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ 系统中是如何表示的。

注意，这个向量在空间中的方向和length不变，只是表示系统变换..

$$\begin{bmatrix}?\\?\end{bmatrix} = -1\cdot\begin{bmatrix}2\\1\end{bmatrix} + 2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-4\\1\end{bmatrix}$$

$\Rightarrow$ 某个坐标系统中的 $\begin{bmatrix}-1\\2\end{bmatrix}$ 表示向量在我们标准坐标系统中表示为 $\begin{bmatrix}-4\\1\end{bmatrix}$.

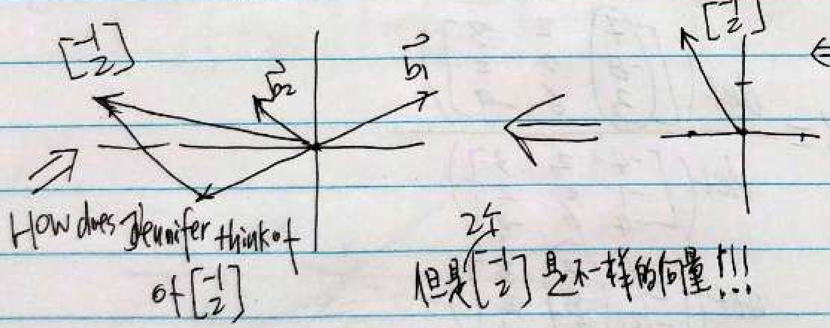
$$= \begin{bmatrix}2&-1\\1&1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}$$

$$\Downarrow \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

our grid $\Longrightarrow$ Jennifer's grid

$$\boxed{\begin{bmatrix}2&-1\\1&1\end{bmatrix}}$$

Our language $\Longleftarrow$ Jennifer's language



$\Leftarrow$ How do we think of $\begin{bmatrix}-1\\2\end{bmatrix}$

$\Rightarrow$ How does Jennifer think of $\begin{bmatrix}-1\\2\end{bmatrix}$

但是 $\begin{bmatrix}-1\\2\end{bmatrix}$ 是不一样的向量!!! (2个)

Jennifer's basis vector in grid of our coordinate system.

$A\vec{x} = \vec{y}$ ← our representation

Jennifer's representation

$$\begin{bmatrix}2&-1\\1&1\end{bmatrix}^{-1} = \begin{bmatrix}1/3&1/3\\-1/3&2/3\end{bmatrix} \quad (\text{inverse of } \begin{bmatrix}2&-1\\1&1\end{bmatrix})$$

our language $\Longrightarrow$ Jennifer's language

如果要求 $\begin{bmatrix}-1\\2\end{bmatrix}$ 或者 $\begin{bmatrix}3\\2\end{bmatrix}$ 在 Jennifer's system 如何表示？

(同一个向量)

$$\begin{bmatrix}1/3&1/3\\-1/3&2/3\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}$$

$$\begin{bmatrix}1/3&1/3\\-1/3&2/3\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}5/3\\1/3\end{bmatrix}$$

How to translate a matrix

如何在Jennifer's 坐标系统中对一个vector做90°-cc rotation??

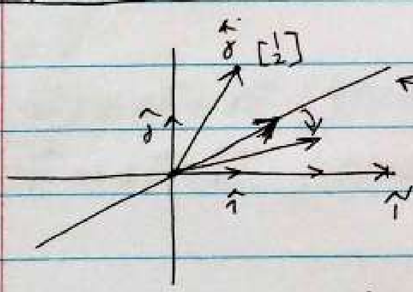
$\begin{bmatrix}2 & -1\\ +1 & +1\end{bmatrix}$ 是 Jennifer's basis vector in our grid

$$\underbrace{\underbrace{\underbrace{\begin{bmatrix}2 & -1\\ +1 & 1\end{bmatrix}^{-1}}_{\substack{\Downarrow\\ \text{inverse}\\ \text{change of}\\ \text{basis}\\ \text{matrix}}}\underbrace{\underbrace{\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}}_{\substack{\text{在our language 描述}\\ 90^{\circ}\text{-rotation cc}}}\overbrace{\underline{\begin{bmatrix}2 & -1\\ +1 & 1\end{bmatrix}}\underline{\begin{bmatrix}-1\\ 2\end{bmatrix}}}^{\substack{\text{same vector}\\ \text{in our language}}}}_{\substack{\text{Transformed vector}\\ \text{in our language.}}}}_{\substack{\text{transformed vector}\\ \text{in her language}}}}_{\text{(Tranformation matrix in her language)}}$$

$\begin{bmatrix}-1\\ 2\end{bmatrix}$ ↘ vector in her language ($\vec{v}$)

$$\boxed{A^{-1}MA}$$

$\boxed{\text{Eigenvectors and Eigenvalues}}$



$\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}$

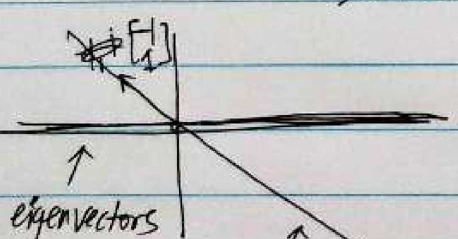
← span of this vector

after change (linear transformation)

⇓

get knocked off the span

Some vectors may still remain on it's own span.



← 这条 linear transformation 后的仍然满足.

← 这条也是,满足条件

$$\boxed{A\vec{v} = \lambda\vec{v}} \quad \left(\lambda \begin{smallmatrix}>0\\ <0\end{smallmatrix}\right)$$

$\vec{v}$ eigen vectors

λ associated eigenvalue

each eigenvectors have associated eigen value

($\underline{\underline{\text{scaling value}}}$ during linear transformation)

$$A\vec{v} = \lambda\vec{v} = \underbrace{(\lambda I)\vec{v}}_{\vec{v}} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A\vec{v} - (\lambda I)\vec{v} = \vec{0}$$

$(A-\lambda I)\vec{v} = \vec{0}$ 对 $\vec{v}$ 作线性变换，变成 $\vec{0}$（原点）

而且 由于 $\vec{v}$ 是 eigen vector，变换过程中保持在span中

$\Downarrow$ 压缩一 basis 面积为 ϕ.

$$\det(A-\lambda I) = 0$$

$\Downarrow$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 1 & 5-\lambda & 9 \\ 2 & 6 & 5-\lambda \end{bmatrix}$$

求解 λ (eigenvalues)

将 λ 代入 $\underline{(A-\lambda I)}\vec{v} = \vec{0}$，求解 $\underline{\vec{v}}$（线性方程组求解）

有些 transformation 并没有 eigenvectors.

比如 rotation $90°$-cc. 每个 vector 都进行了 $90°$-cc rotation，不可能保持在原来的 span of vector 中.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right)$$

$$= \lambda^2 + 1 = 0 \ (??)$$

no λ 存在

A quick trick to compute eigen value

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \frac{a+d}{2} = \frac{\lambda_1+\lambda_2}{2} = m$$

$$\det = ad - bc = \lambda_1\lambda_2 = p$$

$$\boxed{\lambda_1, \lambda_2 = m \pm \sqrt{m^2 - p}}$$

Formal of definition of Linearity

Additivity: $L(\vec{v}+\vec{w}) = L(\vec{v}) + L(\vec{w})$

Scaling: $L(c\vec{v}) = cL(\vec{v})$

- Linear transformations preserve addition and scalar multiplication.

$\Rightarrow$ derivative is Linear

$$\frac{d}{dx}(x^2+x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x)$$

$$\frac{d}{dx}(4x^3) = 4\frac{d}{dx}(x^3)$$

对不同polynomial 级数 $X^n.\ X^{n-1}\ldots\ X^2.\ X^1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

构造个M (线性变换矩阵). apply这个M到任一个polynomial 使其做 Linear 变换(求导).

Rules for vectors ~~and~~ addition and scaling

① $\vec{u} + (\vec{v}+\vec{w}) = (\vec{u}+\vec{v}) + \vec{w}$ (结合率 associative)

② $\vec{v}+\vec{w} = \vec{w}+\vec{v}$ (交换率 cummulative)

③ There is a vector $\vec{0}$ such that $\vec{0}+\vec{v} = \vec{v}$ for all $\vec{v}$

④ For every vector $\vec{v}$ there is a vector $-\vec{v}$, so that $\vec{v}+(-\vec{v}) = \vec{0}$

⑤ $a(b\cdot\vec{v}) = (a\cdot b)\vec{v}$

⑥ $1\vec{v} = \vec{v}$

$\boxed{\text{Axioms}}$

⑦ $a(\vec{v}+\vec{w}) = a\vec{v} + a\vec{w}$ (分配率 distributive)

⑧ $(a+b)\vec{v} = a\vec{v} + b\vec{v}$. (distributive).