Data Structure and Algorithms HW-1

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1. Prove $2n + \Theta(n^2) = \Theta(n^2)$

   By definition of O-notation,

$$\begin{aligned}2n + \Theta(n^2) &= O(n^2) + \Theta(n^2)\\ &= \Theta(n^2).\end{aligned}$$

2. Prove $\Theta(g(n)) \bigcap o(g(n)) = \emptyset$

   If $lim_{n\to\infty} g(n) = 0$, $o(g(n))$ will be an empty set.

   Assume the limitation is not zero, $\forall f(n) \in o(g(n))$, by definition, we have

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

   If $f(n) \in \Theta(g(n))$, we have

$$\exists c_1, c_2, n_0 \in \mathbb{R}^+, s.t. \forall n \geq n_0, c_1 g(n) \leq f(n) \leq c_2 g(n)$$

   which means

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c_0 \neq 0$$

   By contradiction, we know that $f(n) \notin \Theta(g(n))$. Hence, $\Theta(g(n)) \bigcap o(g(n)) = \emptyset$.

3. Prove $\Theta(g(n)) \bigcup o(g(n)) \neq O(g(n))$

   Assume $dom(f) = \mathbb{R}^+$, let

$$f(n) = \begin{cases} n, & n \in \mathbb{N}^+ \\ n^2, & n \in \mathbb{R}^+ \backslash \mathbb{N}^+ \end{cases}$$

   Let $g(n) = n^2$, we can find that $f(n) \in O(g(n))$.

   But $f(n) \neq \Theta(g(n))$ and $f(n) \neq o(g(n))$ by definition, so $f(n) \notin \Theta(g(n)) \bigcup o(g(n))$.

   Hence, we find a case and the inequation is proved.

4. Prove $\max\{f(n), g(n)\} = \Theta(f(n) + g(n))$

   In this problem we should assume $f(n) \geq 0$ and $g(n) \geq 0$.

   Otherwise, let $f(n) = n$ and $g(n) = -n$, $f(n)+g(n) = 0$, the equation is not satisfied.

   If $f(n), g(n) \geq 0$, we have

$$\frac{1}{2}(f(n) + g(n)) \leq \max\{f(n), g(n)\} \leq f(n) + g(n)$$

   By definition of Θ-notation, $\max\{f(n), g(n)\} = \Theta(f(n) + g(n))$.

5. Solve the recurrence $T(n) = 2T(\sqrt{n}) + 1$

   Let $n = 2^m$, we have

$$T(2^m) = 2T(2^{m/2}) + 1$$

   Let $S(m) = T(2^m)$, we have

$$S(m) = 2S(\frac{m}{2}) + 1$$

   By the master method, $a = 2, b = 2, f(m) = 1$,

$$m^{\log_b a} = m^{\log_2 2} = m$$

   Let $\epsilon = 1$, the equation

$$1 = f(m) = O(m^{1-\epsilon}) = O(1)$$

   holds.

   So, $T(n) = T(2^m) = S(m) = \Theta(m) = \Theta(\log n)$.

6. Solve the recurrence $nT(n) = (n-2)T(n-1) + 2$

   Let $n = 2$, we have

$$\begin{aligned} 2T(2) &= 0T(1) + 2 = 2 \\ T(2) &= 1 \end{aligned}$$

   By basic arithmetic,

$$\begin{aligned} \frac{T(n)}{n-2} &= \frac{T(n-1)}{n} + \frac{2}{n(n-2)} \\ &= \frac{T(n-1)}{n} + \frac{1}{n-2} - \frac{1}{n} \end{aligned}$$

   Rearrange the equation,

$$\begin{aligned} \frac{T(n)-1}{n-2} &= \frac{T(n-1)-1}{n} \\ \frac{T(n)-1}{T(n-1)-1} &= \frac{n-2}{n} \end{aligned}$$

So we have

$$\begin{aligned}
T(n) &= \frac{T(n)-1}{T(n-1)-1} * \frac{T(n-1)-1}{T(n-2)-1} * \cdots * \frac{T(3)-1}{T(2)-1} * (T(2)-1) + 1 \\
&= \frac{n-2}{n} * \frac{n-3}{n-1} * \cdots * \frac{1}{3} * (2-1) + 1 \\
&= \frac{2}{n * (n-1)} + 1 \\
&= \Theta(1)
\end{aligned}$$