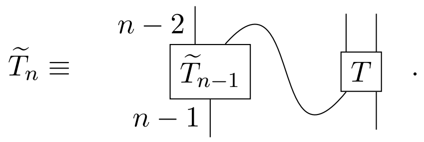
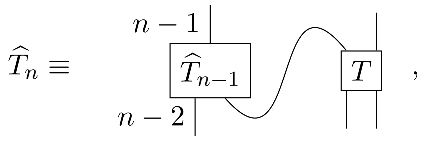
Constructing arbitrarily large perfect tangles from biunitaries

Let in the following $T \in TL_2$ be a perfect tangle, i.e. a (more general) biunitary, and we will consider things unshaded.

Definition 1. Let $\widetilde{T}_2 \equiv T$, and define inductively for all $n \geq 3$



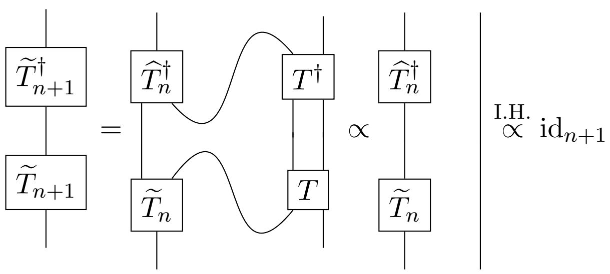
Also, with $\widehat{T}_2 \equiv T$, we define the flipped version of this:



which is basically the adjoint of the underlying tangle of $\widetilde{T}_n$, with T inserted into all boxes. Both of these are elements in TL_n. △

Lemma 2. *$\widetilde{T}_n \cdot \widetilde{T}_n^\dagger \propto \mathrm{id}_n$[1], and a similar result holds for $\widehat{T}_n$.*

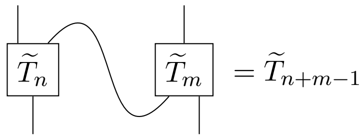
Proof. By induction. The base case $n = 2$ is clear, since T is perfect. Then suppose that it is true for $n \geq 2$. We get



□

[1]The convention for multiplication here is stacking the second (right) argument onto the first (left)

Also note the following:

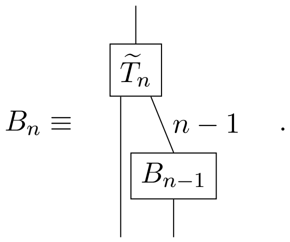


(1)

(here we are contracting a single strand) and similarly for $\widehat{T}_n$, for all $n, m \geq 2$.

Consider now the following construction:

Definition 3. Let $B_2 \equiv T = \widetilde{T}_2$, and define for $n \geq 3$

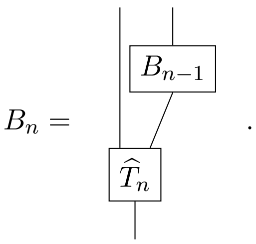


$\triangle$

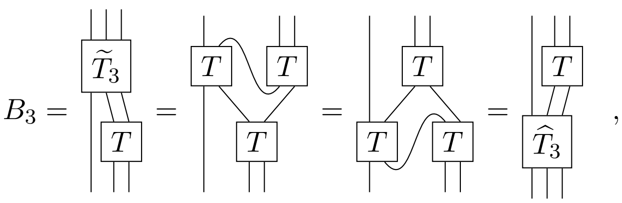
Our goal is to show that each $B_n \in TL_n$ is perfect. From Lemma 2 it is clear that $B_n \cdot B_n^\dagger \propto \mathrm{id}_n$. This is because Lemma 2 asserts that $\widetilde{T}$s occurring together with their adjoints give (something proportional to) identities until only one box is left (and some strings going straight from the top to the bottom), namely $B_2 = T$. But T is perfect, so this gives (something proportional to) the identity.

A basic yet important property of the B_n is now stated.

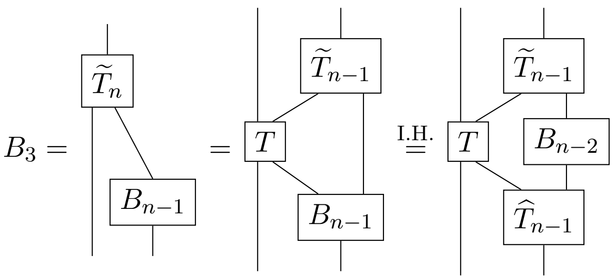
Lemma 4. *For all* $n \geq 3$

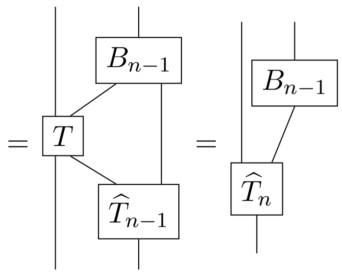


Proof. By induction. For the base case $n = 3$, first remember $B_2 = \widetilde{T}_2 = \widehat{T}_2 = T$. Then



as required. Now suppose that the claim is true up to $n-1$. Then we have





$\square$

This lemma allows us to switch between two ways of writing B_n, which ever way is more comfortable to work with in a given situation. The statement $B_n^\dagger \cdot B_n \propto \mathrm{id}_n$, for example, is now a simple corollary of Lemma 4 and the remark made after Definition 3.

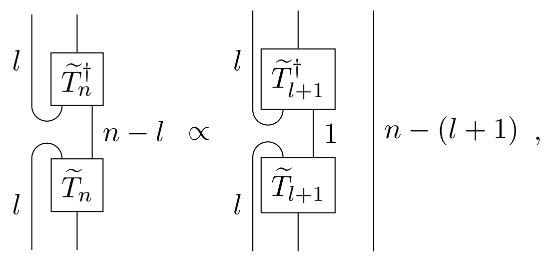
One thing we should note is that if we interpret $\widehat{\ }_n$ and $\widetilde{\ }_n$ as n-tangles with $n-1$ internal boxes, then

$$\widetilde{T}_n^\dagger = \widehat{T^\dagger}_n,$$

so that $\widetilde{T}_n \cdot \widehat{T^\dagger}_n \propto \mathrm{id}_n$ by Lemma 2.

To prove that the rotations of B_n are also 'unitary', we still need to record a few more facts, like the following lemma.

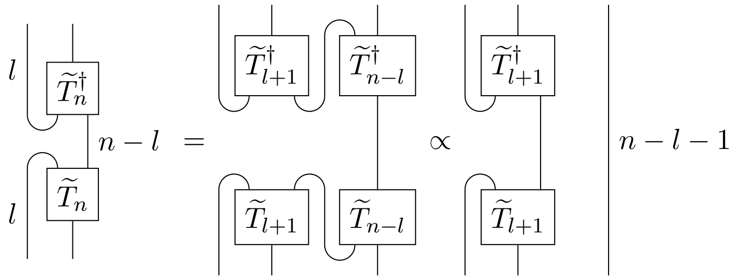
Lemma 5. *Let* $1 \leq l < n-1$, $n \geq 3$. *Then*



where a string with a 0 next to it is interpreted as no string at all.

Proof. A computation with careful consideration of the number of strings, and a trick involving equation 1, shows the result. We will not write the string count next to each string, so you should definitely do that to help you understand what's going on.

Fix any $n \geq 3$. Here is the calculation:

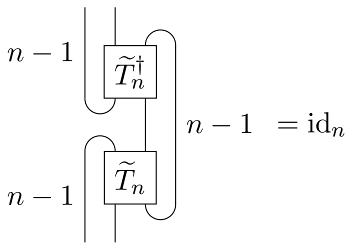


The first equality is the trick, the second (well, not equality but proportionality) uses Lemma 2. ☐

If you read the previous lemma carefully you will probably have noticed that we excluded the $l = n-1$ case. One reason for this is that we can postpone that part until we are in our main theorem, where we solve it by invoking Lemma 4. The other reason is that the formula would be ill-defined, since $n-(n-1) = 1$, but $\widetilde{T}_1$ is undefined.

The following proposition is straightforward, but the result is very useful in one of the last steps in the proof of the main theorem.

Proposition 6. *For all $n \geq 2$*



Proof. Omitted. An exercise in mathematical induction for the reader. ☐

We are now ready to prove our main result.

Theorem 7. *For all B_n the following are true.*

(i) *For all $1 \leq l < n$ we have*

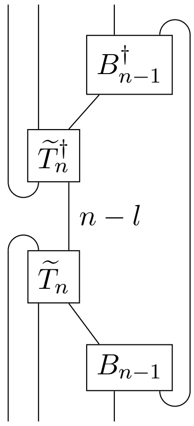
$$\mathrm{rot}^l B_n \cdot \mathrm{rot}^{-l} B_n^\dagger \propto \mathrm{id}_n.$$

(ii) *B_n is perfect.*

Proof. We will first show (i), then (i) $\Longrightarrow$ (ii). The second part is straigthforward.

For (i), first note that this is trivially true for B_2, and that proving it for B_3 is a basic computation. The base case is thus covered. Now suppose it is true for $n-1$, and consider first $l < n-1$. Then

$\mathrm{rot}^l B_n \cdot \mathrm{rot}^{-l} B_n^\dagger =$



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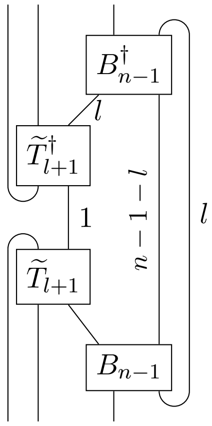

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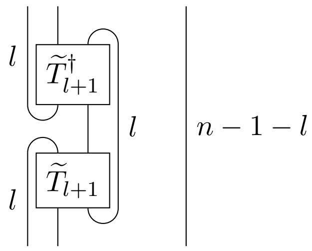

$\propto \mathrm{id}_n$

What's happening here is the following. The first step is writing out the definition. The second is applying Lemma 5, after which we are at the induction step: it's true that $\mathrm{rot}^l B_{n-1} \cdot \mathrm{rot}^{-l} B_{n-1}^\dagger \propto \mathrm{id}_{n-1}$. Note that the largest value of l here is $(n-1)-1$. Last but not least, we invoke Proposition 6, which gives the desired result.

For $l = n-1$ note that the rotation bends *all* legs of B_{n-1} up- and all legs of $B_{n-1}^\dagger$ downwards, thereby contracting the Bs along half of their legs. That is we have the product $B_{n-1}^\dagger \cdot B_{n-1}$, which, as we have argued before, is proportional of id_{n-1}. Then all that's left is applying Proposition 6 one time and the proof of part (i) is finished.

Part (ii) of the theorem follows from one of the lemmas in my thesis, namely the one for a k-tangle T, stating

$$0 \le l < k : \mathrm{rot}^{k+l} T \cdot \mathrm{rot}^{-(k+l)} T^\dagger \propto \mathrm{id}_k \qquad \Longleftrightarrow \qquad \mathrm{rot}^{-l} T^\dagger \cdot \mathrm{rot}^l T \propto \mathrm{id}_k,$$

and an argument about the symmetry of B_n shown in Lemma 4, as well as interpreting $\widetilde{}_n$ as a tangle with $n-1$ internal boxes. $\square$

These perfect tangles constructed in this section are extremely pretty. In Figure 1 we see a few examples which are in TL_n for $n = 3, 4, 5, 6$, respectively. The perfect tangle $T \in TL_2$ is represented as a vertex and should be interpreted as being in *standard form*, i.e. it is oriented so that the marked point is on the top left.

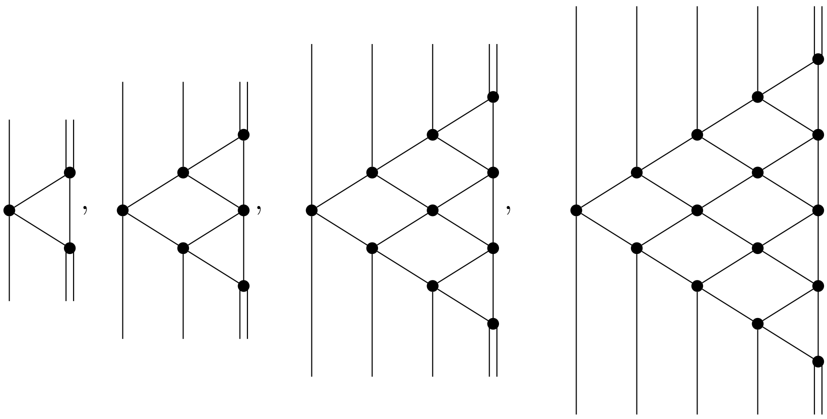

Figure 1: The perfect tangles B_3, B_4, B_5, B_6. Vertices represent the generating perfect 2-tangle T in standard form.

It is important to emphasize that $TL_n \subset \mathcal{C}_{2n}$ (the hom-space $\mathcal{C}[1, X^{\otimes 2n}]$). By construction, any B_n can thus be interpreted as a morphism in a trivalent category, which, in particular, is perfect. So this section not only gave us a ton of perfect tangles, but also more than enough perfect morphism!