On Perfect Planar Tangles

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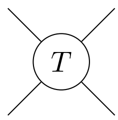
Abstract. We propose the concept of *perfect planar tangles*, which can serve as a generalization of biunitaries as well as perfect tensors.

1 Introduction

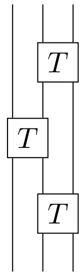
A tensor



is called *perfect* if any balanced bipartition of its legs yields a unitary transformation, up to a nonzero scalar. This notion was introduced in [PYHP15], but for the first part of that paper it is trivially true that a weaker requirement still gives rise to such things as the Ryu-Takayanagi formula for a connected bipartite boundary region. This weaker form shall be called a *planarly perfect tensor*, implying that we only take balanced planar bipartitions into account — that is, partitions of the indices into two sets of the same size so that all indices have at least one 'neighbor' in their respective set.

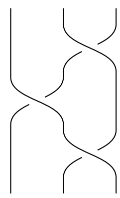
In this paper we argue that a construction of larger planarly perfect tensors from smaller ones exists. A complete proof of this particular construction can be found in the master's thesis of the first author, [Ber17], where planarly perfect tensors were formalized in the more general setting of Jones' *planar algebras* [Jon99].

We will show this using a simple example that exhibits the basic idea behind the construction. More concretely, it is easy to see that



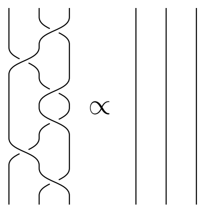
also has this *planar multi-unitarity* property, given that T itself satisfies it. We will show the calculations in detail.

Before doing so, we want to quickly exhibit where the intuition comes from. In the Temperley-Lieb algebra on 2 strands with loop parameter q, $TL_2(q)$, the *perfect* elements give rise to a representation of the braid group. We can thus make sense of diagrams such as



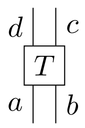
by simply substituting a perfect element for each crossing. This diagram is then really nothing but the previous tensor!

And indeed, since the adjoint is given by a *horizontal flip*, and because representations of the braid group satisfy the Yang-Baxter equations, we see that all balanced planar bipartitions — which manifest themselves in 'rotations' — yield something proportional to a *unitary* (whatever that means in this setting):



2 A Simple Example

Consider a tensor

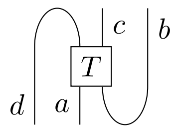
$T =$  ,

read from bottom to top. We can then also represent this as the linear map

$$T \doteq T_{abcd} \left| cd \right\rangle \left\langle ab \right|$$

where summation is implied.

A natural question to ask is then: What do the rotations look like? To answer this, let us first draw the rotation and then read it off of the picture.

$\operatorname{rot} T =$  , so that

$$\operatorname{rot} T \doteq T_{abcd} \left|bc\right\rangle \left\langle da\right| ,$$

and we see that rotation is really just cyclic permutation of the indices.

We now restrict to the simple case where all indices range over the same finite set of values, and where T and all its rotations are unitary. In that case, rotating twice is realized in the matrix representation as nothing else than taking the transpose. As an example, we shall consider the qutrit *controlled not*-gate $CNOT_a$[1], which is given by

$$\begin{aligned} CNOT_a = \quad & \left|00\right\rangle \left\langle 00\right| + \left|01\right\rangle \left\langle 01\right| + \left|02\right\rangle \left\langle 02\right| \\ & + \left|10\right\rangle \left\langle 12\right| + \left|11\right\rangle \left\langle 10\right| + \left|12\right\rangle \left\langle 11\right| \\ & + \left|20\right\rangle \left\langle 21\right| + \left|21\right\rangle \left\langle 22\right| + \left|22\right\rangle \left\langle 20\right| , \end{aligned}$$

or in matrix form (only nonzero entries are shown)

$$CNOT_a = \left(\begin{array}{ccc:ccc:ccc} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ \hdashline & & & & & 1 & & & \\ & & & 1 & & & & & \\ & & & & 1 & & & & \\ \hdashline & & & & & & & 1 & \\ & & & & & & & & 1 \\ & & & & & & 1 & & \end{array} \right) .$$

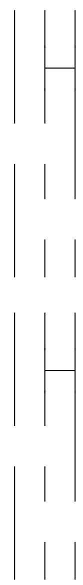
This is clearly unitary. Finding the first rotation and seeing that it is also unitary is trivial, because

$$\operatorname{rot} CNOT_a = (CNOT_b)^T ,$$

and we are done showing that this is planarly perfect.

[1] [ÇKG16]

Trying to get the LHS of yang-baxter in cubic cat



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