The Uniqueness of the Einstein Field Equations in a Four-Dimensional Space

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Abstract

The Euler-Lagrange equations corresponding to a Lagrange density which is a function of g_{ij} and its first two derivatives are investigated. In general these equations will be of fourth order in g_{ij}. Necessary and sufficient conditions for these Euler-Lagrange equations to be of second order are obtained and it is shown that in a four-dimensional space the Einstein field equations (with cosmological term) are the only permissible second order Euler-Lagrange equations. This result is false in a space of higher dimension. Furthermore, the only permissible third order equation in the four-dimensional case is exhibited.

1. Introduction

Multiple integral theories in the calculus of variations are used extensively in theoretical physics, almost all field equations being obtainable from Euler-Lagrange equations with suitably chosen Lagrangian L. Mathematically this is expressed in the following fashion:

if $\Psi^A = \Psi^A(x^i)$, $A = 1, \ldots, m$, are m quantities defined over an n-dimensional space with coordinates x^i $(i = 1, \ldots, n)$ then with any given Lagrangian

$$L = L(\Psi^A, \Psi^A{}_{,i_1}, \Psi^A{}_{,i_1 i_2}, \ldots, \Psi^A{}_{,i_1 i_2 \ldots i_r}) \tag{1.1}$$

where

$$\Psi^A{}_{,i_1 \ldots i_p} \equiv \partial^p \Psi^A / \partial x^{i_1} \ldots \partial x^{i_p}, \qquad 1 \leqq p \leqq r,$$

we may associate the Euler-Lagrange equations[1]

$$\frac{\partial L}{\partial \Psi^A} + \sum_{p=1}^{r} (-1)^p \frac{\partial^p}{\partial x^{i_1} \partial x^{i_2} \ldots \partial x^{i_p}} \left(\frac{\partial L}{\partial \Psi^A{}_{,i_1 i_2 \ldots i_p}} \right) = 0. \tag{1.2}$$

In general these will represent m partial differential equations which are of order $2r$ in Ψ^A, and are to be solved for Ψ^A subject to suitable boundary conditions. In physical terms these equations would represent the field equations for Ψ^A.

In the general theory of relativity, where the components g_{ij} of the (symmetric) metric tensor play the role of Ψ^A, a particular Lagrangian of the type (1.1) is used with $r = 2$. However, the resulting Euler-Lagrange equations (1.2) are, in this case, of *second* order in g_{ij} (the usual Einstein field equations *in vacuo*)

[1] The summation convention is used throughout this paper.

and not of fourth order as one might expect. In view of the importance of the Einstein equations it seems advisable to investigate under what conditions the Euler-Lagrange equations (1.2) degenerate in this sense *i.e.* have order lower than $2r$.

In this paper we assume that the Lagrange function L is a scalar density, thereby ensuring that the corresponding Euler-Lagrange equations are tensorial in character (*e.g.* [1]). Furthermore, we restrict ourselves to the case which is of importance in the theory of general relativity, *viz.*[2]

$$L = L(g_{ij}, g_{ij,k}, g_{ij,kh}). \tag{1.3}$$

Necessary and sufficient conditions for the fourth order Euler-Lagrange equations to degenerate into third and second order equations are obtained in § 2. It is remarkable that these conditions involve the components of only one tensor density. In § 3 we show that *if the dimension n of the space is 2 or 3 then there are no genuine third order Euler-Lagrange equations, whereas for $n=4$ there is one such third order equation.* Furthermore, it is also shown that *for 2-, 3- and 4-dimensional spaces the only permissible second order Euler-Lagrange equations are precisely the Einstein equations with the so-called cosmological term.* However, for $n>4$ this is not the case.

2. Degenerate Lagrange Densities in *n*-Dimensions

We restrict our considerations to transformations of the type

$$\bar{x}^i = \bar{x}^i(x^j)$$

which are arbitrary except that they be of class C^3 and that

$$\det \left\| \frac{\partial x^i}{\partial \bar{x}^j} \right\| > 0.$$

As pointed out in § 1 we further confine our attention to Lagrange densities

$$L = L(g_{ij}, g_{ij,k}, g_{ij,kh}) \tag{2.1}$$

where g_{ij} are components of a symmetric tensor of covariant valency two (with non-vanishing determinant g) and where

$$g_{ij,k} \equiv \partial g_{ij} / \partial x^k, \quad \text{etc.}.$$

We make no assumptions relating to the signature of g_{ij}. It is assumed that L is differentiable in all its variables to any required order. We define the quantities

$$\Lambda^{ij,kh} \equiv \partial L / \partial g_{ij,kh},$$

$$\Lambda^{ij,k} \equiv \partial L / \partial g_{ij,k}, \tag{2.2}$$

and

$$\Lambda^{ij} \equiv \partial L / \partial g_{ij},$$

[2] Certain other cases have been investigated and discussed in a preliminary report [2].

of which only the first is a tensor density ([1] p. 254). The Euler-Lagrange equations associated with (2.1) are

$$E^{hk}=0, \tag{2.3}$$

where

$$E^{hk}\equiv\frac{\partial}{\partial x^i}\left[\Lambda^{hk,i}-\frac{\partial}{\partial x^j}\Lambda^{hk,ij}\right]-\Lambda^{hk}. \tag{2.4}$$

In general, from (2.1) and (2.4), the Euler-Lagrange expression (2.4) will contain derivatives of g_{ij} up to the *fourth* order, *viz.*

$$E^{hk}=E^{hk}(g_{ij}, g_{ij,r}, g_{ij,rs}, g_{ij,rst}, g_{ij,rstu}). \tag{2.5}$$

Consequently, the Euler-Lagrange equations (2.3) are in general fourth order partial differential equations in g_{ij}. However, in the general theory of relativity, where a particular Lagrange density of the type (2.1) is used, the resulting Euler-Lagrange equations are only of second order, and our main concern in this section is to determine necessary and sufficient conditions under which (2.5) degenerates to an expression independent of all terms involving $g_{ij,rstu}$ and $g_{ij,rst}$, *i.e.*

$$E^{hk}=E^{hk}(g_{ij}, g_{ij,r}, g_{ij,rs}). \tag{2.6}$$

Since under these circumstances both L and E^{hk} will in general be functions of the same variables *we shall call a Lagrange density which has the property* (2.6) *L-degenerate.*

In view of the fact that L is a scalar density, three invariance identities must be satisfied [1] two of which are

$$\Lambda^{ij,kh}+\Lambda^{ih,jk}+\Lambda^{ik,hj}=0 \tag{2.7}$$

and

$$-\Lambda^{hk,i}=\Gamma^i_{jm}\Lambda^{hk,jm}+2\Gamma^k_{jm}\Lambda^{hj,im}+2\Gamma^h_{jm}\Lambda^{kj,im}, \tag{2.8}$$

where Γ^i_{jm} are the usual Christoffel symbols of the second kind, *viz.*

$$\Gamma^i_{jm}=\tfrac{1}{2}g^{ih}(g_{hj,m}+g_{mh,j}-g_{jm,h}).$$

From (2.7) it can be shown that

$$\Lambda^{ij,kh}=\Lambda^{kh,ij}. \tag{2.9}$$

We shall first find necessary and sufficient conditions for (2.5) to be independent of $g_{ij,rstu}$ in which case the corresponding Euler-Lagrange equation (2.3) will be at most of *third* order in g_{ij}. We introduce two tensor densities which will occur repeatedly in the subsequent analysis, *viz.*

$$\Lambda^{ij,kh;rs,tu}\equiv\partial\Lambda^{ij,kh}/\partial g_{rs,tu}, \tag{2.10}$$

and

$$\chi^{ij,kh;rs,tu}\equiv\Lambda^{ij,kh;rs,tu}+\Lambda^{ij,ku;rs,ht}+\Lambda^{ij,kt;rs,uh}. \tag{2.11}$$

In view of (2.2) and (2.9), $\Lambda^{ij,kh;rs,tu}$ enjoys certain symmetry properties, *e.g.*

$$\Lambda^{ij,kh;rs,tu}=\Lambda^{rs,tu;ij,kh}=\Lambda^{ij,kh;tu,rs}, \tag{2.12}$$

together with a cyclic identity in (rs, tu) (see (2.7)).

We may write (2.4) in the form

$$E^{hk} = -\Lambda^{hk,ij;rs,tu} g_{rs,tuji} + O^{hk}, \tag{2.13}$$

where

$$O^{hk} = O^{hk}(g_{ij}, g_{ij,r}, g_{ij,rs}, g_{ij,rst}).$$

By taking into account the symmetry properties of $g_{rs,tuji}$ we have, from (2.13),

Lemma 1. *A necessary and sufficient condition for*

$$\partial E^{hk}/\partial g_{rs,tuji} = 0$$

is that

$$\chi^{hk,im;rs,tu} = -\chi^{rs,im;hk,tu}. \tag{2.14}$$

Thus Lemma 1 states that the Euler-Lagrange equations (2.3) will be at most of third order in g_{ij} if and only if (2.14) is satisfied.

The dependence of E^{hk} on $g_{ij,rst}$ will be contained entirely in the O^{hk} term of (2.13). In order to determine conditions under which $g_{ij,rst}$ does not contribute to E^{hk} we require a detailed analysis of the structure of O^{hk}. This involves a lengthy and uninteresting calculation, and to maintain continuity we have relegated these details to Appendix 1, where it is shown that

$$\begin{aligned} O^{hk} = \{ -\tfrac{2}{3}(\chi^{hk,mi;rs,tu})_{|m} + \Lambda^{hk,mi;rs,tu;ab,cd} g_{ab,cdm} \\ + \tfrac{1}{3}\Gamma^p_{jm}\alpha_p^{i,hk,mj;rs,tu}\} g_{rs,tui} + P^{hk}, \end{aligned} \tag{2.15}$$

where

$$\Lambda^{hk,mi;rs,tu;ab,cd} \equiv \partial\Lambda^{hk,mi;rs,tu}/\partial g_{ab,cd},$$

$$\begin{aligned} \alpha_p^{i,hk,mj;rs,tu} \equiv \delta^i_p(\chi^{hk,mj;rs,tu} + \chi^{rs,mj;hk,tu}) \\ + \delta^t_p(\chi^{hk,mj;rs,ui} + \chi^{rs,mj;hk,ui}) + \delta^u_p(\chi^{hk,mj;rs,ti} + \chi^{rs,mj;hk,ti}) \\ + \delta^r_p(\chi^{hk,mi;js,tu} + \chi^{js,mi;hk,tu} + \chi^{hk,ji;ms,tu} + \chi^{sm,ji;hk,tu}) \\ + \delta^s_p(\chi^{hk,mi;jr,tu} + \chi^{jr,mi;hk,tu} + \chi^{hk,ji;mr,tu} + \chi^{rm,ji;hk,tu}), \end{aligned} \tag{2.16}$$

$$P^{hk} = P^{hk}(g_{ij}, g_{ij,r}, g_{ij,rs}),$$

and where the vertical bar denotes the corresponding covariant derivative.

From (2.13) and (2.15) we can thus prove

Lemma 2. *A necessary and sufficient condition for*

$$\frac{\partial E^{hk}}{\partial g_{ij,rst}} = 0$$

is that

$$2\chi^{hk,mi;rs,tu}{}_{|m} - \Gamma^p_{jm}\alpha_p^{i,hk,mj;rs,tu} = 0. \tag{2.17}$$

A comparison of Lemmas 1 and 2 leads to the observation that *it is remarkable that both* (2.14) *and* (2.17) *are conditions on the same tensor density, viz.* $\chi^{ij,kh;rs,tu}$. In order that E^{hk} be at most of second order in g_{ij} it is evident that (2.14) together with (2.17) must be satisfied by L. From (2.16) we see that (2.14) implies

$$\alpha_p^{i,hk,mj;rs,tu}=0,$$

in which case we have proved

Theorem 1. *A necessary and sufficient condition for the Lagrange Density*

$$L=L(g_{ij}, g_{ij,k}, g_{ij,kh})$$

to be L-degenerate is that

$$\chi^{hk,im;rs,tu}=-\chi^{rs,im;hk,tu}$$

together with

$$\chi^{hk,im;rs,tu}{}_{|i}=0. \tag{2.18}$$

We shall briefly digress to consider Lemma 2 in more detail. It states that

$$E^{hk}=E^{hk}(g_{ij}, g_{ij,r}, g_{ij,rs}, -, g_{ij,rstu}) \tag{2.19}$$

if and only if (2.17) is valid. However, by using a transformation of the form

$$\bar{x}^i=\bar{x}^i(x^j) \tag{2.20}$$

it is in general possible to introduce into (2.19) third order derivatives in the new coordinate system. This is closely associated with the fact that (2.17) is a non-tensorial condition. Nevertheless if we demand that (2.19) be valid in all coordinate systems then the left hand side of (2.17) must vanish in all coordinate systems. This leads immediately to

Lemma 3. *Third order derivatives of g_{ij} will be absent from E^{hk} under arbitrary transformations of the type* (2.20) *if and only if*

$$\chi^{hk,im;rs,tu}{}_{|i}=0$$

and

$$\alpha_p^{i,hk,mj;rs,tu}=0.$$

We remark that it is not clear whether the vanishing of $\alpha_p^{i,hk,mj;rs,tu}$ implies condition (2.14).

3. Dimensionality Restrictions

Up to the present our analysis has been independent of the dimension n of the space. In this section we appeal to the dimension and immediately find

Lemma 4. *If $n=2$ or $n=3$ and $\chi^{ij,kh;rs,tu}$ satisfies* (2.14) *then*

$$A^{ij,kh;rs,tu}=0.$$

The proof of this lemma is based on the fact that $A^{ij,kh;rs,tu}$ has eight indices. For $n=2$ or $n=3$ many of these indices must be equal and, by using (2.7), (2.9) and (2.14), the lemma is established in an elementary fashion.

As a consequence of Lemma 4 we note that

$$\chi^{ij,kh;rs,tu}=0,$$

i.e. for $n=2$ or $n=3$ there exist no third order Euler-Lagrange equations. In fact we are in a position to prove a far stronger result than this, *viz.*

Theorem 2. *If $n=2$ or $n=3$ then the only second order Euler-Lagrange expressions are obtainable from*[3]

$$L=a\sqrt{g}\,R+b\sqrt{g}, \tag{3.1}$$

where a and b are constants and R is the curvature scalar.

Proof. From Lemma 4 and equation (2.10) we see that

$$\Lambda^{ij,hk}=\Lambda^{ij,hk}(g_{rs}, g_{rs,t}). \tag{3.2}$$

However, it has been shown by du Plessis ([3] p. 53) that irrespective of its valency or weight any tensorial function of g_{rs}, $g_{rs,t}$, $g_{rs,tu}$ which is independent of $g_{rs,tu}$ is also independent of $g_{rs,t}$. Thus (3.2) implies that

$$\Lambda^{ij,hk}=\Lambda^{ij,hk}(g_{rs}).$$

Furthermore, it can be shown (see Appendix Lemma A 4) that under these circumstances

$$\Lambda^{ij,kh}=\tfrac{1}{2}a\sqrt{g}\,(g^{ik}g^{jh}+g^{ih}g^{jk}-2g^{ij}g^{kh})$$

where a is a constant. Integration of the latter gives rise to

$$L=\tfrac{1}{2}a\sqrt{g}\,(g^{ik}g^{jh}+g^{ih}g^{jk}-2g^{ij}g^{hk})\,g_{ij,kh}+\phi(g_{rs}, g_{rs,t}),$$

which may be expressed in the form

$$L=\tfrac{1}{3}a\sqrt{g}\,(g^{ik}g^{jh}+g^{ih}g^{jk}-2g^{ij}g^{hk})\,R_{kijh}+\psi(g_{rs}, g_{rs,k}),$$

where

$$R_{kijh}=g_{ir}\left[\frac{\partial}{\partial x^h}\Gamma^r_{kj}-\frac{\partial}{\partial x^j}\Gamma^r_{kh}+\Gamma^t_{kj}\Gamma^r_{th}-\Gamma^t_{kh}\Gamma^r_{tj}\right]$$

is the usual Riemann curvature tensor. We thus see that

$$L=a\sqrt{g}\,R+b\sqrt{g},$$

where we have again made use of du Plessis' result and Lemma A 1 of the Appendix and where R is the curvature scalar

$$R\equiv g^{ih}g^{jk}R_{kijh}.$$

The Euler-Lagrange equations (2.3) corresponding to (3.1) are

$$a(R_{ij}-\tfrac{1}{2}g_{ij}R)+\tfrac{1}{2}b\,g_{ij}=0, \tag{3.3}$$

where

$$R_{ij}\equiv g^{hk}R_{ihjk}.$$

However, for $n=2$ the coefficient of a is identically zero, so that in this case we do not obtain a genuine second order equation. For $n=3$ it is easily seen that (3.3) implies that the space is isotropic.

[3] Without loss of generality we may assume that $g>0$.

Guided by the 2- and 3-dimensional cases discussed above, we now turn our attention to $n=4$. In the first instance we seek the counterpart of Lemma 4 and subsequently a generalisation of Theorem 2.

In order to proceed in this direction we introduce three tensor densities:

$$A^{ij,kh;rs,tu;ab,cd} \equiv \frac{\partial A^{ij,kh;rs,tu}}{\partial g_{ab,cd}}, \tag{3.4}$$

$$A^{ij,kh;rs,tu;ab,cd;pq,lm} \equiv \frac{\partial A^{ij,kh;rs,tu;ab,cd}}{\partial g_{pq,lm}}, \tag{3.5}$$

and

$$\varepsilon^{ij,kh;rs,tu;ab,cd} \equiv \sum_{cd} \sum_{ab} \sum_{tu} \sum_{rs} \sum_{kh} \sum_{ij} \varepsilon^{ikrt} \varepsilon^{jhac} \varepsilon^{usbd}/g, \tag{3.6}$$

where ε^{ikrt} is the 4-dimensional permutation symbol[4] and the symbol $\sum_{ij}$ denotes symmetrisation with respect to ij, *e.g.* for any quantity $A^{ij}:::$

$$\sum_{ij} A^{ij}::: \equiv A^{ij}::: + A^{ji}:::.$$

Consequently the right hand side of (3.6) represents the sum of 64 terms.

In Appendix 3 we show that although $A^{ij,kh;rs,tu;ab,cd}$ has 4^{12} components for $n=4$, only one of these is independent if condition (2.14) is satisfied. In fact we prove

Lemma 5. *If $n=4$ and if $\chi^{ij,kh;rs,tu}$ satisfies* (2.14) *then*

$$A^{ij,kh;rs,tu;ab,cd} = A\, \varepsilon^{ij,kh;rs,tu;ab,cd} \tag{3.7}$$

where

$$A = A(g_{ij}, g_{ij,k}, g_{ij,kh})$$

is a scalar.

However, we cannot integrate (3.7) without a greater knowledge of A, and this is furnished by

Lemma 6. *If $n=4$ and if $\chi^{ij,kh;rs,tu}$ satisfies* (2.14) *then*

$$A^{ij,kh;rs,tu;ab,cd;pq,lm} = 0\,. \tag{3.8}$$

In order to maintain continuity we again relegate this proof to Appendix 3. From (3.7) and (3.8) we conclude that

$$A = A(g_{ij})$$

where we have once more used DU PLESSIS' result. Furthermore, from Lemma A 1 of the Appendix

$$A = \text{const.}.$$

Integration of (3.7) immediately yields

$$A^{ij,kh;rs,tu} = \tfrac{2}{3} A\, \varepsilon^{ij,kh;rs,tu;ab,cd} R_{cabd} + \lambda^{ij,kh;rs,tu}, \tag{3.9}$$

[4] The permutation symbol $\varepsilon^{i_1 \cdots i_n}$ is antisymmetric in every pair of indices and has the value $+1$ if $i_1, \ldots i_n$ is an even permutation of $1, \ldots, n$.

where

$$\lambda^{ij,kh;rs,tu}=\lambda^{ij,kh;rs,tu}(g_{ab}, g_{ab,c})$$

is a tensor density with the same symmetry properties as $\Lambda^{ij,kh;rs,tu}$. However, since $\lambda^{ij,kh;rs,tu}$ is a tensor density it must be independent of $g_{ab,c}$ and Lemma A 5 of the Appendix is applicable.

By integrating (3.9) twice and using similar arguments we find

$$L=\tfrac{2}{27}A\,\varepsilon^{ij,kh;rs,tu;ab,cd}R_{cabd}R_{trsu}R_{kijh}$$
$$+\tfrac{2}{9}\lambda^{ij,kh;rs,tu}R_{trsu}R_{kijh}+\mu^{ij,kh}R_{kijh}+\phi$$

where

$$\mu^{ij,kh}=\mu^{ij,kh}(g_{ab})$$

is a tensor density with the same symmetry properties as $\Lambda^{ij,kh}$ and

$$\phi=\phi(g_{ab})$$

is a scalar density. From Lemmas A 1, A 4, A 5 and A 6 of the Appendix it is not difficult to prove

Theorem 3. *The most general Lagrange density*

$$L=L(g_{ij}, g_{ij,k}, g_{ij,kh})$$

which satisfies (2.14) *for* $n=4$ *is*

$$L=\alpha *R^{ij}{}_{kh}*R^{kh}{}_{rs}*R^{rs}{}_{ij}/g+\beta\sqrt{g}(R^2-4R_{ij}R^{ij}+R_{ijkh}R^{ijkh})$$
$$+\gamma *R^{ij}{}_{kh}R^{kh}{}_{ij}+a\sqrt{g}\,R+b\sqrt{g},$$

where α, β, γ, *a and b are constants* and

$$*R^{ij}{}_{kh}=\varepsilon^{ijrs}R_{rskh}.$$

It has been shown by Lanczos [4] that[5] the coefficients of β and γ satisfy the Euler-Lagrange equations identically for $n=4$. Theorem 3 thus gives rise to

Theorem 4. *If* $n=4$ *the only third order Euler-Lagrange equations arising from a scalar density*

$$L=L(g_{ij}, g_{ij,k}, g_{ij,kh})$$

are

$$\frac{6\alpha}{g}\,[*R^{it}{}_{mh|k}**R^{mhkj}{}_{|t}+R_{hk}*R^{ih}{}_{mt}**R^{mtkj}$$
$$-R_m{}^h{}_{tr}(*R^{ir}{}_{hk}**R^{mktj}+*R^{jr}{}_{hk}**R^{mkti})$$
$$+\frac{1}{12}g^{ij}(*R^{rs}{}_{tu}*R^{tu}{}_{hk}*R^{hk}{}_{rs})]+a\sqrt{g}\left(R^{ij}-\frac{1}{2}g^{ij}R\right)-\frac{b}{2}\sqrt{g}\,g^{ij}=0$$

where

$$**R^{ijkt}=\varepsilon^{ijrs}\varepsilon^{ktmh}R_{rsmh}.$$

[5] For a different approach to this type of identity see [5].

If we now impose condition (2.18) on Theorem 3 we find

$$\alpha=0.$$

Consequently we have proved[6]

Theorem 5. *If $n=4$ the only second order Euler-Lagrange equations arising from a scalar density*

$$L=L(g_{ij}, g_{ij,k}, g_{ij,kh})$$

are

$$a(R^{ij}-\tfrac{1}{2}g^{ij}R)+b\,g^{ij}=0,$$

i.e. *Einstein's equation with cosmological constant.*

This Theorem is false for $n>4$ since in that case

$$L=\sqrt{g}(R^2-4R_{ij}R^{ij}+R_{ijkc}R^{ijkc})$$

also gives rise to a second-order Euler-Lagrange equation.

Finally we remark that the above analysis does not exclude the possibility of Lagrange densities of the type

$$L=L(g_{ij}, g_{ij,k_1}, g_{ij,k_1k_2}, \dots, g_{ij,k_1k_2\dots k_m})$$

from giving rise to either second or third order Euler-Lagrange equations.

Appendices

Appendix 1. In this appendix we wish to show that the third order terms in (2.13) are of the form (2.15). We introduce the following notation: if $F_{:::}$ is any quantity then

$$F_{:::}^{\;\;;ij}\equiv\frac{\partial F_{:::}}{\partial g_{ij}},$$

$$F_{:::}^{\;\;;ij,k}\equiv\frac{\partial F_{:::}}{\partial g_{ij,k}},$$

$$F_{:::}^{\;\;;ij,kh}\equiv\frac{\partial F_{:::}}{\partial g_{ij,kh}},$$

so that, for example,

$$A^{ij,kh;rs,t}=\frac{\partial A^{ij,kh}}{\partial g_{rs,t}}=\frac{\partial^2 L}{\partial g_{ij,kh}\,\partial g_{rs,t}}=A^{rs,t;ij,kh}.$$

[6] It is possible to prove this theorem from a different point of view, and to generalize it for arbitrary n [6].

The third order terms in (2.4) are easily seen to be

$$O^{hk}-P^{hk}\equiv -\frac{\partial}{\partial x^r}(\Lambda^{hk,rm;ab,cd})\, g_{ab,cdm}-\Lambda^{hk,rm;ab,c}\, g_{ab,cmr}$$
$$-\Lambda^{hk,mc;ab;rs,tu}\, g_{ab,m}\, g_{rs,tuc}-\Lambda^{hk,em;ab,c;rs,tu}\, g_{rs,tue}\, g_{ab,cm}$$
$$+\Lambda^{hk,r;ab,cd}\, g_{ab,cdr}.$$

The third and fourth terms on the right hand side can be expressed in terms of the first, so that

$$O^{hk}-P^{hk}=-2\frac{\partial}{\partial x^r}(\Lambda^{hk,rm;ab,cd})\, g_{ab,cdm}+\Lambda^{hk,em;ab,cd;rs,tu}\, g_{ab,cdm}\, g_{rs,tue}$$
$$-\Lambda^{hk,rm;ab,c}\, g_{ab,cmr}+\Lambda^{hk,r;ab,cd}\, g_{ab,cdr}.$$

By virtue of (2.8) it is possible to express the final two quantities in terms of $\Lambda^{ij,kh;rs,tu}$. From (2.11) we thus find

$$O^{hk}-P^{hk}=-\frac{2}{3}\frac{\partial}{\partial x^r}(\chi^{hk,rm;ab,cd})\, g_{ab,cdm}$$
$$+\Lambda^{hk,im;ab,cd;rs,tu}\, g_{ab,cdm}\, g_{rs,tui}$$
$$-\Gamma^m_{ji}\Lambda^{hk,ji;ab,cd}\, g_{ab,cdm}-\frac{2}{3}\Gamma^k_{ji}\chi^{hj,im;ab,cd}\, g_{ab,cdm}$$
$$-\frac{2}{3}\Gamma^h_{ji}\chi^{kj,im;ab,cd}\, g_{ab,cdm}+\Gamma^c_{ji}\Lambda^{ab,ji;hk,md}\, g_{ab,cdm}$$
$$+\frac{2}{3}\Gamma^a_{ji}\chi^{bj,ic;hk,md}\, g_{ab,cdm}+\frac{2}{3}\Gamma^b_{ji}\chi^{aj,ic;hk,md}\, g_{ab,cdm}.$$

Noting that $\chi^{hk,im;ab,cd}$ is a tensor density we express its partial derivative in terms of its covariant derivative. The result (2.15) follows with a little further calculation.

Appendix 2. This appendix deals with tensorial quantities which are functions of g_{ij} alone.

Lemma A 1. *If $\phi=\phi(g_{ij})$ is a scalar density then*

$$\phi=c\sqrt{g}$$

where c is a constant.[7]

Proof. Since ϕ is a scalar density we may define a scalar ψ by

$$\psi=\phi/\sqrt{g}.$$

This scalar ψ must be such that

$$\psi(B^h_i B^k_j g_{hk})=\psi(g_{rs})$$

[7] This result can also be found in [7], p. 169.

is an identity in

$$B_i^h \equiv \partial x^h / \partial \bar{x}^i.$$

Differentiation of this condition with respect to B_t^p leads to

$$g_{ht} \frac{\partial \psi}{\partial g_{hp}} = 0,$$

or

$$\psi = \text{constant}.$$

Lemma A 2. *If $\phi_{ij} = \phi_{ij}(g_{rs})$ is a tensor then*

$$\phi_{ij} = c\, g_{ij} + \delta_2^n\, b \sqrt{g}\, \varepsilon_{ij}$$

where b and c are constants and ε_{ij} is the two-dimensional Levi-Civita symbol.

Proof. Since ϕ_{ij} is a tensor it follows that

$$\phi_{rs}(B_k^t B_h^u g_{ut}) = B_s^h B_r^k \phi_{kh}(g_{ij}),$$

from which it can be shown that

$$2 \frac{\partial \phi_{rs}}{\partial g_{hk}} = \delta_r^h g^{tk} \phi_{ts} + \delta_s^h g^{tk} \phi_{rt}. \tag{A 1}$$

The left hand side of (A 1) is symmetric in (hk) so we have

$$\delta_r^h g^{tk} \phi_{ts} + \delta_s^h g^{tk} \phi_{rt} = \delta_r^k g^{th} \phi_{ts} + \delta_s^k g^{th} \phi_{rt}.$$

From this, with $h = r$, we find

$$(n-1)\, \phi_{ts} + \phi_{st} = \lambda\, g_{st} \tag{A 2}$$

where

$$\lambda = g^{st} \phi_{st}.$$

If $n \neq 2$ then (A 2) implies that

$$\phi_{st} = \frac{1}{n} \lambda\, g_{st},$$

which, when substituted in (A 1) leads to

$$\partial \lambda / \partial g_{st} = 0.$$

This establishes the lemma for $n \neq 2$. If, however, $n = 2$ then (A 2) yields

$$\phi_{ts} + \phi_{st} = \lambda\, g_{st},$$

so that

$$\phi_{st} = \tfrac{1}{2} \lambda\, g_{st} + \tfrac{1}{2}(\phi_{st} - \phi_{ts}).$$

Since $n = 2$ the last term on the right hand side is proportional to ε_{st}, in which case

$$\phi_{st} = \tfrac{1}{2} \lambda\, g_{st} + b \sqrt{g}\, \varepsilon_{st}.$$

Substitution of the latter in (A 1) gives rise to

$$\lambda = \text{constant},$$

$$b = \text{constant},$$

which proves the lemma.

Lemma A 3. *If $\phi_{ijkh} = \phi_{ijkh}(g_{rs})$ is a tensor then for $n > 2$,*

$$\phi_{ijkh} = a\, g_{ij}\, g_{kh} + b\, g_{ik}\, g_{jh} + c\, g_{ih}\, g_{jk} + \delta^n_4\, \beta \sqrt{g}\, \varepsilon_{ijkh}$$

where a, b, c and β are constants and ε_{ijkh} is the four-dimensional Levi-Civita symbol.

Proof. Since ϕ_{ijkh} is a tensor and a function of g_{rs} it follows that

$$\begin{aligned} 2\frac{\partial \phi_{ijkm}}{\partial g_{rs}} &= \delta^r_i\, g^{hs}\, \phi_{hjkm} + \delta^r_j\, g^{hs}\, \phi_{ihkm} + \delta^r_k\, g^{hs}\, \phi_{ijhm} + \delta^r_m\, g^{hs}\, \phi_{ijkh} \\ &= \delta^s_i\, g^{hr}\, \phi_{hjkm} + \delta^s_j\, g^{hr}\, \phi_{ihkm} + \delta^s_k\, g^{hr}\, \phi_{ijhm} + \delta^s_m\, g^{hr}\, \phi_{ijkh} \end{aligned} \tag{A 3}$$

from which we deduce that

$$\begin{aligned} &\delta^r_i\, \phi_{tjkh} + \delta^r_j\, \phi_{itkh} + \delta^r_k\, \phi_{ijth} + \delta^r_h\, \phi_{ijkt} \\ &\quad = g_{it}\, g^{sr}\, \phi_{sjkh} + g_{jt}\, g^{sr}\, \phi_{iskh} + g_{kt}\, g^{sr}\, \phi_{ijsh} + g_{ht}\, g^{sr}\, \phi_{ijks}. \end{aligned} \tag{A 4}$$

In the latter we sum over (r, i) and replace t by i to find

$$\begin{aligned} &(n-1)\, \phi_{ijkh} + \phi_{kjih} + \phi_{hjki} + \phi_{jikh} \\ &\quad = g_{ij}\, g^{rs}\, \phi_{rskh} + g_{ik}\, g^{rs}\, \phi_{rjsh} + g_{ih}\, g^{sr}\, \phi_{rjks}. \end{aligned} \tag{A 5}$$

From Lemma A 2 (with $n > 2$) we see that

$$g^{rs}\, \phi_{rskh} = \lambda\, g_{kh},$$

$$g^{rs}\, \phi_{rjsh} = \mu\, g_{jh},$$

and

$$g^{rs}\, \phi_{rjhs} = \rho\, g_{jh},$$

where λ, μ and ρ are constants. Furthermore it is easily shown that

$$g^{rs}\, \phi_{khrs} = \lambda\, g_{kh},$$

$$g^{rs}\, \phi_{jrhs} = \mu\, g_{jh},$$

and

$$g^{rs}\, \phi_{jrsk} = \rho\, g_{jk}.$$

Consequently (A 5) reduces to

$$(n-1)\, \phi_{ijkh} + \phi_{jikh} + \phi_{kjih} + \phi_{hjki} = \lambda\, g_{ij}\, g_{kh} + \mu\, g_{ik}\, g_{jh} + \rho\, g_{ih}\, g_{jk}. \tag{A 6}$$

We return to (A 4) and obtain three similar equations by summing in turn over (r,j), (r,k) and (r,h) and replacing t by j, k and h respectively, *viz.*

$$(n-1)\phi_{ijkh}+\phi_{jikh}+\phi_{ikjh}+\phi_{ihkj}=\lambda g_{ij}g_{kh}+\mu g_{ik}g_{jh}+\rho g_{ih}g_{jk}, \quad (A\ 7)$$

$$(n-1)\phi_{ijkh}+\phi_{kjih}+\phi_{ikjh}+\phi_{ijhk}=\lambda g_{ij}g_{kh}+\mu g_{ik}g_{jh}+\rho g_{ih}g_{jk}, \quad (A\ 8)$$

$$(n-1)\phi_{ijkh}+\phi_{hjki}+\phi_{ihkj}+\phi_{ijhk}=\lambda g_{ij}g_{kh}+\mu g_{ik}g_{jh}+\rho g_{ih}g_{jk}. \quad (A\ 9)$$

The sum of (A 6) and (A 7) is subtracted from the sum of (A 8) and (A 9) to yield

$$\phi_{ijkh}=\phi_{jihk},$$

which, when substituted in the difference between (A 7) and (A 8), gives rise to

$$\phi_{ihkj}=\phi_{kjih}=\phi_{hijk}. \quad (A\ 10)$$

It is easily seen that (A 6) and (A 10) imply (A 6), (A 7), (A 8) and (A 9).

From (A 6) we derive three equivalent equations by interchanging i with j, k and h respectively, *viz.*

$$(n-1)\phi_{jikh}+\phi_{ijkh}+\phi_{kijh}+\phi_{hikj}=\lambda g_{ij}g_{kh}+\rho g_{ik}g_{jh}+\mu g_{ih}g_{jk}, \quad (A\ 11)$$

$$(n-1)\phi_{kjih}+\phi_{jkih}+\phi_{ijkh}+\phi_{hjik}=\rho g_{ij}g_{kh}+\mu g_{ik}g_{jh}+\lambda g_{ih}g_{jk}, \quad (A\ 12)$$

$$(n-1)\phi_{hjki}+\phi_{jhki}+\phi_{kjhi}+\phi_{ijkh}=\mu g_{ij}g_{kh}+\lambda g_{ik}g_{jh}+\rho g_{ih}g_{jk}. \quad (A\ 13)$$

From the combination $[(n-1)\ (A\ 6)+2(A\ 11)-(A\ 12)-(A\ 13)]$, and by virtue of (A 10), we see that

$$(n-1)^2\phi_{ijkh}+3(n-1)\phi_{jikh}=\alpha g_{ij}g_{kh}+\beta g_{ik}g_{jh}+\gamma g_{ih}g_{jk}, \quad (A\ 14)$$

where

$$\alpha=(n+1)\lambda-\rho-\mu,$$

$$\beta=(n-2)\mu-\lambda+2\rho,$$

$$\gamma=(n-2)\rho-\lambda+2\mu.$$

In (A 14) we interchange i and j and eliminate ϕ_{jikh} to find

$$(n-1)^2(n-4)(n+2)\phi_{ijkh}=a\, g_{ij}g_{kh}+b\, g_{ik}g_{jh}+c\, g_{ih}g_{jk},$$

which establishes the lemma for $n>2$ and $n\neq 4$.

If $n=4$, (A 14) reduces to

$$9(\phi_{ijkh}+\phi_{jikh})=\alpha g_{ij}g_{kh}+\beta(g_{ik}g_{jh}+g_{ih}g_{jk}). \quad (A\ 15)$$

In order to prove the lemma in this case we consider the tensor A_{ijkh} defined by

$$\begin{aligned}A_{ijkh}=&\phi_{ijkh}-\phi_{ijhk}-\phi_{ikjh}+\phi_{ikhj}-\phi_{ihkj}+\phi_{ihjk}\\&-\phi_{jikh}+\phi_{jihk}+\phi_{jkih}-\phi_{jkhi}+\phi_{jhki}-\phi_{jhik}\\&-\phi_{kjih}+\phi_{kjhi}+\phi_{kijh}-\phi_{kihj}+\phi_{khij}-\phi_{khji}\\&-\phi_{hjki}+\phi_{hjik}+\phi_{hikj}-\phi_{hijk}+\phi_{hkji}-\phi_{hkij}.\end{aligned} \quad (A\ 16)$$

Since A_{ijkh} is antisymmetric in every pair of indices, it has only one independent component in a four-dimensional space and is therefore proportional to the Levi-Civita symbol, *viz.*

$$A_{ijkh}=\psi(g_{rs})\,\varepsilon_{ijkh}. \tag{A 17}$$

However, by virtue of (A 10), (A 16) reduces to

$$A_{ijkh}=4(\phi_{ijkh}-\phi_{jikh}+\phi_{kijh}+\phi_{hikj}-\phi_{kjih}-\phi_{hjki}).$$

The right hand side of the latter can be further reduced since the final four quantities are expressible in terms of the first two (see (A 6) and (A 11)) while ϕ_{jikh} is given in terms of ϕ_{ijkh} by (A 15). We thus obtain

$$\phi_{ijkh}=a\,g_{ij}\,g_{kh}+b\,g_{ik}\,g_{jh}+c\,g_{ih}\,g_{jk}+\tfrac{1}{24}\,\psi(g_{rs})\,\varepsilon_{ijkh}.$$

However, from (A 3), it is easily seen that

$$\psi=\beta\sqrt{g},$$

which completes the proof.

Lemma A 4. *If $\phi_{ijkh}=\phi_{ijkh}(g_{rs})$ is a tensor and*

$$\phi_{ijkh}=\phi_{jikh}=\phi_{ijhk}$$

together with

$$\phi_{ijkh}+\phi_{ihjk}+\phi_{ikhj}=0$$

then for $n\geqq 2$

$$\phi_{ijkh}=a\left[g_{ij}\,g_{kh}-\tfrac{1}{2}(g_{ik}\,g_{jh}+g_{ih}\,g_{jk})\right]$$

where a is a constant.

Proof. For $n>2$, the result follows from Lemma A 3.

For $n=2$, (A 5) is still valid. The last three terms on the left hand side of this equation do not contribute when taken together, while the right hand side can be rewritten using Lemma A 2. The lemma follows in an elementary fashion.

Remark. The quantity $g(\varepsilon_{ik}\varepsilon_{jh}+\varepsilon_{ih}\varepsilon_{jk})$ has all the properties of ϕ_{ijkh} for $n=2$ in Lemma A 4. However, this does not contradict the lemma in view of the identity

$$2\,g_{ij}\,g_{kh}-g_{ih}\,g_{jk}-g_{ik}\,g_{jh}=g(\varepsilon_{ik}\varepsilon_{jh}+\varepsilon_{ih}\varepsilon_{jk}).$$

The following two lemmas are proved in a manner similar to that used in Lemma A 3.

Lemma A 5. *If $\phi_{ij,kh;rs,tu}=\phi_{ij,kh;rs,tu}(g_{ab})$ is a tensor and*

$$\phi_{ij,kh;rs,tu}=\phi_{rs,tu;ij,kh}=\phi_{ji,kh;rs,tu}=\phi_{ij,hk;rs,tu},$$

together with

$$\phi_{ij,kh;rs,tu}+\phi_{ih,jk;rs,tu}+\phi_{ik,hj;rs,tu}=0$$

and

$$\phi_{ij,kh;rs,tu}+\phi_{ij,ku;rs,ht}+\phi_{ij,kt;rs,uh}+\phi_{rs,kh;ij,tu}+\phi_{rs,ku;ij,ht}+\phi_{rs,kt;ij,uh}=0$$

then, for $n>3$,

$$\phi_{ij,kh;rs,tu} R^{kijh} R^{trsu}$$
$$=[(2n-5)\alpha_{ij,kh;rs,tu}+2\alpha_{ij,rs;kh,tu}] R^{kijh} R^{trsu}/\{(2n-3)(n-3)\},$$

where

$$\alpha_{ij,kh;rs,tu}=g^{lm}[g_{ij}\phi_{ml,kh;rs,tu}+g_{ik}\phi_{mj,lh;rs,tu}$$
$$+g_{ih}\phi_{mj,kl;rs,tu}+g_{ir}\phi_{mj,kh;ls,tu}+g_{is}\phi_{mj,kh;rl,tu}$$
$$+g_{it}\phi_{mj,kh;rs,lu}+g_{iu}\phi_{mj,kh;rs,tl}].$$

Lemma A 6. *Under the conditions of Lemma* A 5

$$g^{tu}\phi_{hj,ke;rs,tu}=\frac{1}{n-2}\beta_{hj,ke;rs},$$

and

$$g^{tu}\phi_{kt,ih;lu,rs}=-\tfrac{1}{2}g^{tu}\phi_{ih,kl;rs,tu}$$
$$+n\mu\delta^4_n\sqrt{g}\,(g_{rh}\varepsilon_{ikls}+g_{sh}\varepsilon_{iklr}+g_{ri}\varepsilon_{hkls}+g_{si}\varepsilon_{hklr}),$$

where

$$\beta_{hj,ke;rs}=\lambda(g_{jh}g_{kers}-\tfrac{1}{2}g_{kh}g_{jers}-\tfrac{1}{2}g_{eh}g_{jkrs}-\tfrac{1}{2}g_{rh}g_{kejs}-\tfrac{1}{2}g_{sh}g_{kejr}),$$
$$g_{kers}=g_{ke}g_{rs}-\tfrac{1}{2}(g_{ks}g_{er}+g_{kr}g_{es}),$$

and λ, μ *are constants.*

Appendix 3. In this appendix we investigate quantities $\lambda^{ij,kh;rs,tu;ab,cd}$ which satisfy the following conditions:

$$\lambda^{ij,kh;rs,tu;ab,cd}=\lambda^{rs,tu;ij,kh;ab,cd}=\lambda^{ab,cd;rs,tu;ij,kh},$$
$$\lambda^{ij,kh;rs,tu;ab,cd}=\lambda^{ji,kh;rs,tu;ab,cd}=\lambda^{ij,hk;rs,tu;ab,cd}, \qquad \text{(A 18)}$$

$$\lambda^{ij,kh;rs,tu;ab,cd}+\lambda^{ih,jk;rs,tu;ab,cd}+\lambda^{ik,hj;rs,tu;ab,cd}=0, \qquad \text{(A 19)}$$

and

$$\lambda^{ij,kh;rs,tu;ab,cd}+\lambda^{ij,ku;rs,ht;ab,cd}+\lambda^{ij,kt;rs,uh;ab,cd}$$
$$+\lambda^{rs,kh;ij,tu;ab,cd}+\lambda^{rs,ku;ij,ht;ab,cd}+\lambda^{rs,kt;ij,uh;ab,cd}=0. \qquad \text{(A 20)}$$

As usual (A 18) and (A 19) imply that

$$\lambda^{ij,kh;rs,tu;ab,cd}=\lambda^{kh,ij;rs,tu;ab,cd}.$$

We shall adopt the following notation:

$$\lambda^{ij,kh;rs,tu;ab,cd}\equiv ij,kh;rs,tu;ab,cd,$$

and we shall henceforth not use the summation convention. Furthermore, we will restrict our considerations to the case $n=4$, which implies that some of the twelve indices coincide. We shall show that only one of ij, kh; rs, tu; ab, cd is independent.

If 5 or more of the indices $ijkh\,rs\,tu\,ab\,cd$ are equal then $ij, kh; rs, tu; ab, cd = 0$. This result follows because at least 2 of these indices (say i) must occur in two of the three groups $(ij\,kh)(rs\,tu)$ and $(ab\,cd)$. [If 3 or 4 are equal in the same group then (A 19) ensures it is zero]. (A 19) can be used to group them in the form $.., ii; .., ii; i.,..$ which will vanish by (A 20).

If 4 of the indices $ijkh\,rs\,tu\,ab\,cd$ are equal then a similar (but much longer) analysis shows that these will also not contribute to $ij, kh; rs, tu; ab, cd$.

If 3 of $ijkh\,rs\,tu\,ab\,cd$ are equal then there must be four groups of 3 equal indices. A detailed analysis shows in fact that only the following (and their obvious equalities) survive:

$$ii,ll;ik,jl;kk,jj, \quad ii,ll;ik,jj;lj,kk,$$

$$ii,jk;ij,ll;jl,kk, \quad ii,jk;ij,kl;jk,ll,$$

and

$$jk,il;jk,il;ij,kl.$$

However, even these are not independent since

$$ii,ll;ik,jj;lj,kk = -ii,ll;ik,jl;kk,jj,$$

$$ii,jk;ij,ll;jl,kk = -\tfrac{1}{2}\,ii,ll;ik,jl;kk,jj,$$

$$ii,jk;ij,kl;jk,ll = -\tfrac{1}{4}\,ii,ll;ik,jl;kk,jj,$$

and

$$jk,il;jk,il;ij,kl = \tfrac{1}{8}\,ii,ll;ik,jl;kk,jj.$$

Furthermore, $ii, ll; ik, jl; kk, jj$ is antisymmetric under interchange of any two groups of three equal indices. Thus, for $n=4$, $\lambda^{ij,kh;rs,tu;ab,cd}$ has only one independent component.

It is not difficult to show that

$$\varepsilon^{ij,kh;rs,tu;ab,cd},$$

defined by (3.6), has exactly the same symmetry properties as $\lambda^{ij,kh;rs,tu;ab,cd}$ so we may express the latter in the form

$$\lambda^{ij,kh;rs,tu;ab,cd} = B\,\varepsilon^{ij,kh;rs,tu;ab,cd}$$

where

$$B = \tfrac{1}{16}\,\lambda^{11,22;33,44;14,23}.$$

We now apply these results to $\Lambda^{ij,kh;rs,tu;ab,cd}$ and we see that if $\chi^{ij,kh;rs,tu}$ satisfies (2.14) then

$$\Lambda^{ij,kh;rs,tu;ab,cd} = A\,\varepsilon^{ij,kh;rs,tu;ab,cd}$$

where

$$A = A(g_{ij}, g_{ij,k}, g_{ij,kl})$$

is a scalar. This is Lemma 5.

Finally we consider

$$\Lambda^{ij,kh;rs,tu;ab,cd;pq,lm}$$

where $\chi^{ij,kh;rs,tu}$ satisfies (2.14). In view of the above analysis the only possible surviving components for $n=4$ will be of the form

$$\Lambda^{ii,ll;kk,jj;ik,jl;ij,kl} \quad \text{and} \quad \Lambda^{ii,ll;kk,jj;ik,jl;il,kj}.$$

However, when (2.14) is applied to $(ik, jl; ij, kl)$ and $(ik, jl; il, kj)$ we immediately find that

$$\Lambda^{ij,kh;rs,tu;ab,cd;pq,lm}=0.$$

This is Lemma 6.

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